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Exam in TKT4150 Biomechanics and TTK 4170 Modelling and identification of biological systems

Tuesday December 11, 2012
Duration: kl. 09.00-13.00

No printed or handwritten aids are permitted (D). Approved calculators are permitted.

Deadline for examination results: January 11, 2013

Exercise 1

Assume stationary, laminar blood flow through an artery with internal diameter $d = 4$ mm. The pressure drop over a length $L = 30$ mm is 200 Pa. The density of the blood is $\rho = 1.05 \cdot 10^3$ kg/m^3 and the dynamic viscosity $\mu = 4.2 \cdot 10^{-3}$ Pa s.

1. Use the Hagen-Poiseuille-formula for a volumetric flow:

$$\frac{dp}{dx} = -\frac{8\pi\mu}{A^2} Q \tag{1}$$

   to compute the volumetric flow and the average velocity over the cross section of the artery.

2. How can you check the validity of the laminar flow assumption?

3. In addition to the assumption of local Poiseuille-flow (equation (1)), assume a linear constitutive model:

$$A(p) = A_0 + C\,(p - p_0) \tag{2}$$

   (a) Use equation (2) to eliminate the pressure p from equation (1), integrate and express the area $A(x)$ at a given location x as a function of the inlet area $A(0)$, and μ, C, Q and x.

   (b) Illustrate and discuss how the pressure (area) flow relationship in a compliant vessel differs from a rigid vessel.

Exercise 2

The Cauchy equation is given by:

$$\rho\, \dot{v}_i = T_{ik,k} + \rho\, b_i \tag{3}$$

and the mathematical representation of a Hookean material may be presented:

$$T_{ij} = \frac{\eta}{1+\nu}\left(E_{ij} + \frac{\nu}{1-2\nu} E_{kk}\,\delta_{ij}\right) \tag{4}$$

1. Identify all the symbols in equations (3) and (4).

2. For convenience introduce:

$$\mu = \frac{\eta}{2(1+\nu)}, \qquad \text{and} \qquad E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{5}$$

   (a) What is a common name for μ?
   (b) Use equations (5), (4), and (3) to show that:

$$u_{i,kk} + \frac{1}{1-2\nu} u_{k,ki} + \frac{\rho}{\mu}(b_i - \ddot{u}_i) = 0 \tag{6}$$

   (c) What is the name of equation (6)?

3. Assume plane waves $u_i = u_i(x_3, t)$.

   (a) Argue for necessary simplifications and show that equations (6) reduce to wave equations subject to the plane wave assumption.
   (b) Express the distinct waves speeds as functions of ρ, η, and ν. Explain the physical significance of the waves.
   (c) Show that the squared ratio of the two wave speeds is given by:

$$\frac{1-2\nu}{2(1-\nu)} \tag{7}$$

   and explain what range the ratio will have for typical materials.

Exercise 3

1. Derive the two-element Windkessel model for the systemic arterial tree from first principles and show that it has the mathematical representation:

$$\frac{\partial p}{\partial t} + \frac{1}{RC} p = \frac{q(t)}{C} \tag{8}$$

2. Explain the meaning of all the symbols in equation (8).

3. Give a physical interpretation of the combined quantity RC. (Hint: use the homogenous solution of equation (8)).

4. In fig.(1) an electrical circuit diagram of the three-element Windkessel model is shown. Derive the differential equation for three-element Windkessel (Westkessel model) and present it on the form:

$$\frac{dp}{dt} = -\frac{1}{RC} p + \left(\frac{1}{C} + \frac{Z_c}{RC}\right) q + Z_c \frac{dq}{dt}. \tag{9}$$

5. Compare with equation (8) and explain the meaning of the parameters p, q, C, Z_c and R.

6. (a) Find the expressions for the impedances for both the two- and three-element Windkessels. Illustrate the impedances (modulus and phase) schematically.
   (b) List pros and cons for the two models of the systemic arterial tree.

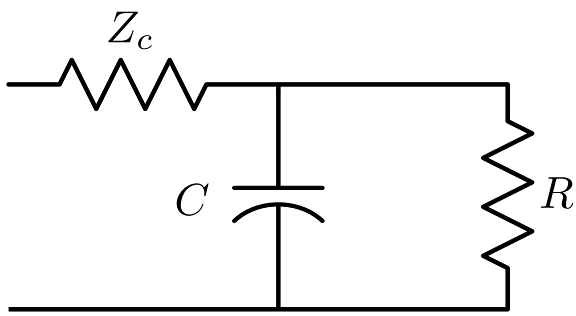

Figure 1: Three-element Windkessel model.

Exercise 4

1. Use first order Taylor expansions, discretize differential equation (9) for the three-element Windkessel and show that the result can be written on the form

$$p_{k+1} = \alpha_1 p_k + \alpha_2 q_k + \alpha_3 q_{k+1}$$

   where k and $k+1$ denotes different time steps in the discretization. Define the parameters α_1, α_2 and α_3 in relation to the parameters R, C and Z_c.

2. (a) Explain the difference between *step-wise* and *ballistic* estimation methods.
   (b) How can the two be efficiently combined?

3. You have now performed a set of measurements, $\mathbf{p} = [p_1, \cdots, p_N]^T$ and $\mathbf{q} = [q_1, \cdots, q_N]^T$ and wish to perform a parameter fit of the model values to the measurements. Assume that the measurements are cyclic, $p_{N+1} = p_1$ and $q_{N+1} = q_1$.

   Based on the results in b) find a model of the form:

$$\hat{\mathbf{p}} = A\alpha,$$

   where $\hat{\mathbf{p}}$ is the model pressure at time $k+1$ and α is the model's parameter vector.

   (a) Find the least squares estimate of the parameter vector, $\hat{\alpha}$.
   (b) Is this estimation method *step-wise* or *ballistic*?

4. Suggest an iterative ballistic estimation method which minimizes a functional where perturbation of the different parameters are arguments of the functional.

5. (a) What is the *normalized* sensitivity matrix?
   (b) What information can be extracted from the matrix?