Suggested solution:
PROBLEM SET 11

TKT4150 Biomechanics

Main topics: Arts heart model.

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(1) Arts model (exam 2011)

We consider a simplified model of left ventricular dynamics: the Arts model. In this model, the left ventricle is assumed to be a thick-walled cylinder of thickness consisting of many thin-walled cylinders of thickness dr. Let's assume that the stresses in a cylindrical surface of radius r can be expressed in the cylindrical coordinate system ($\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$), where the z-direction is parallel to the axis of symmetry of the left ventricle, as:

$$\sigma_r = -p \tag{1}$$
$$\sigma_\theta = -p + \sigma \cos^2 \alpha \tag{2}$$
$$\sigma_z = -p + \sigma \sin^2 \alpha \tag{3}$$

where σ is the stress in the myocardial muscle fiber oriented in the direction $\mathbf{n} = cos\alpha\mathbf{e}_\theta + sin\alpha\mathbf{e}_z$. The model is shown in Figure 1. Assume that we consider one such thin-walled cylinder, with thickness dr. Two equilibrium sketches are given in Figure 2, corresponding to a thin-walled container. Hint: when $d\theta$ is small, $d\theta = sin d\theta$.

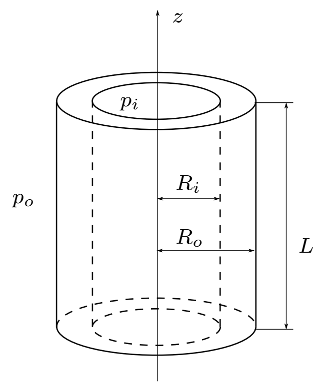

Figure 1: *The left ventricle is assumed a thick-walled cylinder in Arts model, consisting of several thin-walled cylinders. The outer and inner radii R_o and R_i; the outer and inner pressures p_o and p_i; and the length of the cylinder L, are denoted on the figure.*

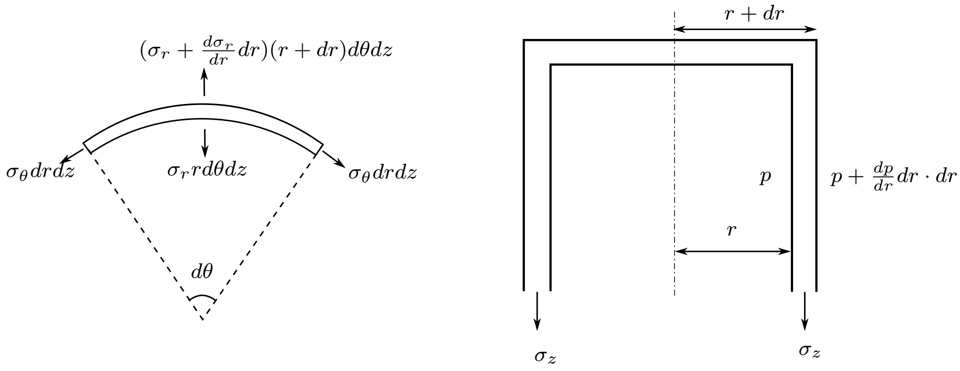

Figure 2: *Equilibrium sketches of thinwalled piece.*

a) Write the equilibrium in the r-direction and show that:

$$\frac{dp}{dr} = \frac{-\sigma \cos^2 \alpha}{r} \tag{4}$$

By considering the left part of Figure 2, we get:

$$-2\sigma_\theta \frac{d\theta}{2} drdz - \sigma_r r d\theta dz + \sigma_r r d\theta dz + \sigma_r dr d\theta dz + \frac{d\sigma_r}{dr} dr r d\theta dz + \frac{d\sigma_r}{dr} dr^2 d\theta dz = 0 \tag{5}$$

Here, it is used that the r-component of $\sigma_\theta drdz$ is equal to $\sin(d\theta/2) = d\theta/2$. By neglecting terms with higher order differentials and performing additions this reduces to:

$$-\sigma_\theta d\theta drdz + \sigma_r dr d\theta dz + \frac{d\sigma_r}{dr} dr d\theta dz \cdot r = 0 \tag{6}$$

$$\Downarrow$$

$$-\sigma_\theta + \sigma_r + \frac{d\sigma_r}{dr} r = 0 \tag{7}$$

By using Equations 1 and 2, we get:

$$-(-p + \sigma \cos^2 \alpha) + (-p) + r \cdot (-\frac{dp}{dr}) = 0 \tag{8}$$

which results in the final result (as asked for):

$$\frac{dp}{dr} = \frac{\sigma - \cos^2 \alpha}{r} \tag{9}$$

b) Write the equilibrium in the z-direction and show that:

$$\frac{dp}{dr} = \frac{-2\sigma \sin^2 \alpha}{r} \tag{10}$$

By considering the right part of Figure 2, and using the necessary formulas for areas the pressures and stresses are acting on, we get:

$$-2\sigma_z \pi r dr - \left(p + \frac{dp}{dr} dr\right)(r+dr)^2 \pi + p\pi r^2 = 0 \tag{11}$$

This leads to:

$$-2\sigma_z \pi r dr - \left(p + \frac{dp}{dr}dr\right)(r^2 + 2rdr + dr^2) + \pi r^2 = 0 \tag{12}$$

By neglecting terms with higher order differentials and performing additions, this reduces to:

$$-2\sigma_z \pi r dr - pr^2\pi - p2rdr\pi - pdr^2\pi - \tag{13}$$

$$\frac{dp}{dr}drr^2\pi - \frac{dp}{dr}dr2rdr\pi - \frac{dp}{dr}drdr^2\pi + p\pi r^2 = 0 \tag{14}$$

$$\Downarrow$$

$$-2\sigma_z \pi r dr - 2prdr\pi - \frac{dp}{dr}drr^2\pi = 0 \tag{15}$$

By dividing this by $drr\pi$, we get:

$$-2\sigma_z - 2p - \frac{dp}{dr}r = 0 \tag{16}$$

By introducing the expression for σ_z given in Equation 3 we get:

$$\frac{dp}{dr} = \frac{-2\sigma_z - 2p}{r} = \frac{2(p - \sigma \sin^2\alpha) - 2p}{r} = \frac{-2\sigma \sin^2\alpha}{r} \tag{17}$$

c) Express $\frac{dp}{dr}$ with respect to σ and r only. Equations 4 and 10 are weighted and summed:

$$(\mathrm{a}) + \frac{1}{2}(\mathrm{b}) = \frac{dp}{dr} + \frac{1}{2}\frac{dp}{dr} = \frac{3}{2}\frac{dp}{dr} \tag{18}$$

This results in the following expression for the pressure gradient:

$$\frac{dp}{dr} = -\frac{2}{3}\frac{\sigma}{r}\underbrace{(\cos^2\alpha + \sin^2\alpha)}_{=1} = -\frac{2}{3}\frac{\sigma}{r} \tag{19}$$

d) Let's introduce the following relations:

$$\text{Left ventricular pressure:} \quad p_{LV} = p_o - p_i \tag{20}$$

$$\text{Volume of left ventricular wall:} \quad V_W = \pi L(R_o^2 - R_i^2) \tag{21}$$

$$\text{Volume of left ventricular cavity:} \quad V_{LV} = \pi L R_i^2 \tag{22}$$

Integrate the result found in c) from R_i to R_o and express a relationship between p_{LV}, σ, V_W and V_{LV}.

The pressure difference can be found as:

$$p_{LV} = p_o - p_i = \int_{R_i}^{R_o} dp = \int_{R_i}^{R_o} \frac{dp}{dr}dr \tag{23}$$

$$= \int_{R_i}^{R_o} -\frac{2}{3}\frac{\sigma}{r}dr = -\frac{2}{3}[\ln r]_{R_i}^{R_o} \tag{24}$$

$$= -\frac{2}{3}\sigma(\ln R_o - \ln R_i) = -\frac{2}{3}\sigma \ln\frac{R_o}{R_i} \tag{25}$$

We have the following:

$$
\begin{aligned}
V_W &= \pi L(R_o^2 - R_i^2) && (26)\\
V_{LV} &= \pi L R_i^2 && (27)\\
&\Downarrow && (28)\\
V_W + V_{LW} &= \pi L R_o^2
\end{aligned}
$$

This results in the following expressions for R_i and R_o:

$$
\begin{aligned}
R_i &= \sqrt{\frac{V_{LV}}{\pi L}} && (29)\\
R_o &= \sqrt{\frac{V_W + V_{LV}}{\pi L}} && (30)
\end{aligned}
$$

By inserting this into Equation 25, we get:

$$p_{LV} = -\frac{2}{3}\sigma \ln \frac{R_o}{R_i} = -\frac{2}{3}\sigma \ln \left(\sqrt{\frac{V_W + V_{LV}}{V_{LV}}} \right) \tag{31}$$