Velocity profiles for straight pipe flow

Leif Rune Hellevik

Department of Structural Engineering
Norwegian University of Science and Technology
Trondheim, Norway

September 18, 2017

Outline

- Nondimensionalized version of the general NS equations
- Nondimensionalized equations for straight pipe flow
- Womersley number
  - Friction dominated (α small)
  - Inertia dominated (α large)
  - Womersley profiles for arbitrary α

Dimensionless Navier-Stokes equations

- The incompressible NS equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \mathbf{v} + \mathbf{b}$$

- Characteristic constant scales

$$\begin{aligned} \mathbf{x}^* &= \mathbf{x}/L, & \mathbf{v}^* &= \mathbf{v}/V, \\ t^* &= t/\theta, & p^* &= p/\rho V^2 \end{aligned}$$

Dimensionless Navier-Stokes equations

- The incompressible NS equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \mathbf{v} + \mathbf{b}$$

- Characteristic constant scales

$$\begin{aligned} \mathbf{x}^* &= \mathbf{x}/L, \qquad & \mathbf{v}^* &= \mathbf{v}/V, \\ t^* &= t/\theta, \qquad & p^* &= p/\rho V^2 \end{aligned}$$

- Dimensionless NS-equations

$$Sr\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{1}{Re}\nabla^2 \mathbf{v} + \frac{1}{Fr^2}\mathbf{b}$$

Dimensionless Navier-Stokes equations

- The incompressible NS equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \mathbf{v} + \mathbf{b}$$

- Characteristic constant scales

$$\mathbf{x}^* = \mathbf{x}/L, \qquad \mathbf{v}^* = \mathbf{v}/V,$$
$$t^* = t/\theta, \qquad p^* = p/\rho V^2$$

- Dimensionless NS-equations

$$Sr\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{1}{Re}\nabla^2 \mathbf{v} + \frac{1}{Fr^2}\mathbf{b}$$

- Strouhal
  - $Sr = \frac{L}{\theta V}$
  - Periodic flows
- Reynolds
  - $Re = \frac{\rho V L}{\mu}$
  - All viscous flows
- Froude
  - $Fr = \frac{V}{\sqrt{gL}}$
  - Free-surface flows

Fully developed flow in straight pipes

- Momentum equation

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)$$

- Characteristic scales
  - Length: $r^* = r/a, \quad z^* = z/a$
  - Time: $t^* = t\omega$
  - Velocity: $v^* = v/V$

Fully developed flow in straight pipes

- Momentum equation

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)$$

- Characteristic scales
  - Length: $r^* = r/a, \quad z^* = z/a$
  - Time: $t^* = t\omega$
  - Velocity: $v^* = v/V$
- Dimensionless form

$$\left(a^2\frac{\omega}{\nu}\right)\frac{\partial v^*}{\partial t^*} = -\left(\frac{a}{\rho\nu V}\right)\frac{\partial p}{\partial z^*} + \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial v^*}{\partial r^*}\right)$$

Fully developed flow in straight pipes

- Momentum equation

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)$$

- Characteristic scales
  - Length: $r^* = r/a, \quad z^* = z/a$
  - Time: $t^* = t\omega$
  - Velocity: $v^* = v/V$
- Dimensionless form

$$\left(a^2\frac{\omega}{\nu}\right)\frac{\partial v^*}{\partial t^*} = -\left(\frac{a}{\rho\nu V}\right)\frac{\partial p}{\partial z^*} + \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial v^*}{\partial r^*}\right)$$

- Characteristic pressure $p^* = p/(\rho\nu V/a)$

Dimensionless straight tube equations

$$\alpha^2 \frac{\partial v^*}{\partial t^*} = -\frac{\partial p^*}{\partial z^*} + \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial v^*}{\partial r^*}\right)$$

- Womersley parameter $\alpha = a\sqrt{\frac{\omega}{\nu}}$

Dimensionless straight tube equations

$$\alpha^2 \frac{\partial v^*}{\partial t^*} = -\frac{\partial p^*}{\partial z^*} + \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial v^*}{\partial r^*}\right)$$

- Womersley parameter $\alpha = a\sqrt{\frac{\omega}{\nu}}$

- Inertia dominated
  - $\alpha \to \infty$
  - large vessels
  - high frequency
  - aorta $\alpha \geq 20$

Dimensionless straight tube equations

$$\alpha^2 \frac{\partial v^*}{\partial t^*} = -\frac{\partial p^*}{\partial z^*} + \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial v^*}{\partial r^*}\right)$$

- Womersley parameter $\alpha = a\sqrt{\frac{\omega}{\nu}}$

- Inertia dominated
  - $\alpha \to \infty$
  - large vessels
  - high frequency
  - aorta $\alpha \geq 20$
- Friction dominated
  - $\alpha \to 0$
  - small vessels
  - low frequency
  - capillaries $\alpha = 10^{-2}$

Straight tube velocity profiles

- Momentum equation

$$\alpha^2 \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)$$

- Remove $*$ for simplicity
- Linear in $v \Rightarrow$ superposition of harmonics OK

[1] Note time is dimensionless.

Straight tube velocity profiles

- Momentum equation

$$\alpha^2 \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)$$

- Remove $*$ for simplicity
- Linear in $v \Rightarrow$ superposition of harmonics OK
- Driving force[1]: $\frac{\partial p}{\partial z} = \frac{\partial \hat{p}}{\partial z} e^{it}$
- Velocity $v = \hat{v} e^{it}$

[1] Note time is dimensionless.

Straight tube velocity profiles

- Momentum equation

$$\alpha^2 \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)$$

- Remove $*$ for simplicity
- Linear in $v \Rightarrow$ superposition of harmonics OK
- Driving force[1]: $\frac{\partial p}{\partial z} = \frac{\partial \hat{p}}{\partial z} e^{it}$
- Velocity $v = \hat{v} e^{it}$
- By substitution into momentum equation

$$i\alpha^2 \hat{v}(r) = -\frac{\partial \hat{p}}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{v}}{\partial r}\right)$$

[1] Note time is dimensionless.

Friction dominated straight pipe flow

- Small Womersley parameter ($\alpha \to 0$)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{v}}{\partial r}\right) = \frac{\partial \hat{p}}{\partial z}$$

Friction dominated straight pipe flow

- Small Womersley parameter ($\alpha \to 0$)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{v}}{\partial r}\right) = \frac{\partial \hat{p}}{\partial z}$$

- Solution

$$\hat{v} = -\frac{1}{4}\frac{\partial \hat{p}}{\partial z}\left(1 - r^2\right)$$

Friction dominated straight pipe flow

- Small Womersley parameter ($\alpha \to 0$)

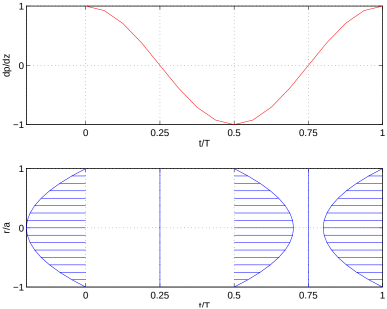
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{v}}{\partial r}\right) = \frac{\partial \hat{p}}{\partial z}$$

- Solution

$$\hat{v} = -\frac{1}{4}\frac{\partial \hat{p}}{\partial z}\left(1 - r^2\right)$$

- In the time domain

$$v(r,t) = Re\left(-\frac{1}{4\mu}\frac{\partial p}{\partial z}(a^2 - r^2)\right)$$



Inertia dominated straight pipe flow

- Large Womersley parameter

$$i\alpha^2 \hat{v}(r) = -\frac{\partial \hat{p}}{\partial z}$$

Inertia dominated straight pipe flow

- Large Womersley parameter

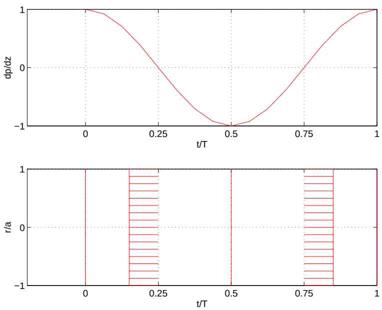
$$i\alpha^2\hat{v}(r) = -\frac{\partial \hat{p}}{\partial z}$$

- Solution

$$\hat{v}(r) = \frac{i}{\alpha^2}\frac{\partial \hat{p}}{\partial z}$$

- In the time domain

$$v(t) = Re\left(\frac{i}{\rho\omega}\frac{\partial p}{\partial z}\right)$$



A first guess on velocity profile for arbitrary α

- First guess

$$\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial z} \quad + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)$$

$$\mathcal{O}(\omega V) \qquad\qquad\qquad\qquad \mathcal{O}\left(\frac{\nu V}{\delta^2}\right)$$

A first guess on velocity profile for arbitrary α

- First guess

$$\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial z} \quad + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)$$

$$\mathcal{O}(\omega V) \qquad\qquad \mathcal{O}\left(\frac{\nu V}{\delta^2}\right)$$

- Instationary boundary layer thickness

$$\mathcal{O}(V\omega) = \mathcal{O}\left(\frac{\nu V}{\delta^2}\right) \Rightarrow \delta = \mathcal{O}\left(\sqrt{\frac{\nu}{\omega}}\right)$$

A first guess on velocity profile for arbitrary α

- First guess

$$\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial z} \quad + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)$$

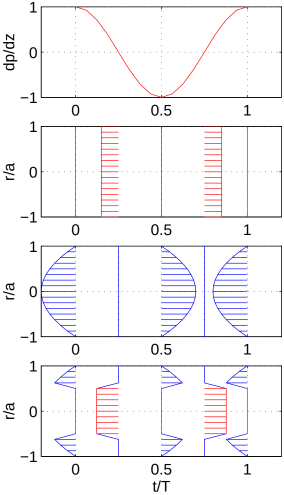
$$\mathcal{O}(\omega V) \qquad\qquad \mathcal{O}\left(\frac{\nu V}{\delta^2}\right)$$

- Instationary boundary layer thickness

$$\mathcal{O}(V\omega) = \mathcal{O}\left(\frac{\nu V}{\delta^2}\right) \Rightarrow \delta = \mathcal{O}\left(\sqrt{\frac{\nu}{\omega}}\right)$$

- Expressed by α

$$\delta = \mathcal{O}\left(\frac{a}{\alpha}\right), \quad \alpha > 1$$



Velocity profile for arbitrary α

- Momentum equation in frequency domain

$$i\omega\alpha^2 \hat{v} = -\frac{\partial \hat{p}}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{v}}{\partial r}\right)$$

Velocity profile for arbitrary α

- Momentum equation in frequency domain

$$i\omega\alpha^2 \hat{v} = -\frac{\partial \hat{p}}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{v}}{\partial r}\right)$$

- Substitute $s = i^{3/2}\alpha r$

Velocity profile for arbitrary α

- Momentum equation in frequency domain

$$i\omega\alpha^2\hat{v} = -\frac{\partial\hat{p}}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\hat{v}}{\partial r}\right)$$

- Substitute $s = i^{3/2}\alpha r$
- Bessel equation $n = 0$

$$\frac{\partial^2\hat{v}}{\partial s^2} + \frac{1}{s}\frac{\partial\hat{v}}{\partial s} + (1 - \frac{n^2}{s^2})\, v = \frac{i}{\rho\omega}\,\frac{\partial\hat{p}}{\partial z}$$

Velocity profile for arbitrary α

- Momentum equation in frequency domain

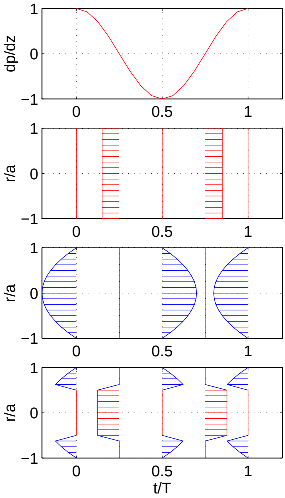
$$i\omega\alpha^2\hat{v} = -\frac{\partial\hat{p}}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\hat{v}}{\partial r}\right)$$

- Substitute $s = i^{3/2}\alpha r$
- Bessel equation $n = 0$

$$\frac{\partial^2\hat{v}}{\partial s^2} + \frac{1}{s}\frac{\partial\hat{v}}{\partial s} + (1 - \frac{n^2}{s^2})\, v = \frac{i}{\rho\omega}\frac{\partial\hat{p}}{\partial z}$$

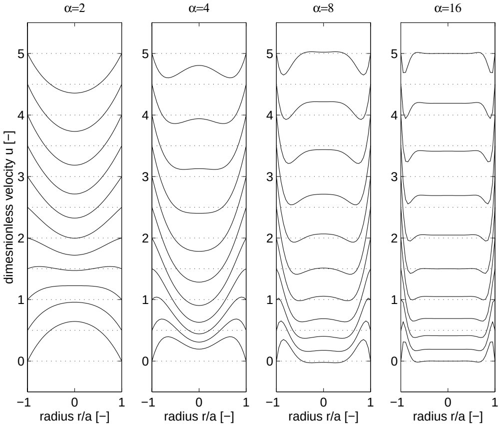
- Solution

$$\hat{v}(r) = \frac{i}{\rho\omega}\frac{\partial\hat{p}}{\partial z}\left(1 - \frac{J_0(i^{3/2}\alpha r/a)}{J_0(i^{3/2}\alpha)}\right)$$



Womersley profiles for straight pipe flow

$$v(r,t) = Re\left(\frac{i}{\rho\omega}\,\frac{\partial p}{\partial z}\left(1 - \frac{J_0(i^{3/2}\alpha r/a)}{J_0(i^{3/2}\alpha)}\,e^{i\omega t}\right)\right)$$



Summary

- Nondimensionalized version of the general NS equations
- Nondimensionalized equations for straight pipe flow
- Womersley number
  - Friction dominated (α small)
  - Inertia dominated (α large)
  - Womersley profiles for arbitrary α