INSTITUTT FOR KONSTRUKSJONSTEKNIKK NTNU

	Dato	Side
	Sign	

PROBLEM 1 : hyperelasticity.

a) $S = 2\frac{\partial \Psi}{\partial C} = 2\left[\frac{\partial \Psi}{\partial I_1}\frac{\partial I_1}{\partial C} + \frac{\partial \Psi}{\partial J}\frac{\partial J}{\partial C}\right]$

$\underline{S = 2c_1 \mathbb{1} - pJC^{-1}}$

b) $F_{ij} = \frac{\partial x_i}{\partial X_j}$ $\quad \det F = 1 = \lambda_1\lambda_2\lambda_3 \Rightarrow \lambda_3 = \frac{1}{\lambda_1\lambda_2}$

$F = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1/\lambda_1\lambda_2 \end{bmatrix}$ $\quad C = F^T F = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \frac{1}{\lambda_1^2\lambda_2^2} \end{bmatrix}$ $\quad B = FF^T = C$

c) $T = \frac{1}{J} F S F^T \quad , J = 1$

$\underline{T = 2c_1 B - p\mathbb{1}}$ $\quad F C^{-1} F^T = F(F^{-1}F^{-T})F^T = \mathbb{1}$.

d) plane stress condition $\Rightarrow T_{i3} = 0$ for $i = 1..3$.

using $T_{33} = 0 \Rightarrow 0 = 2c_1 B_{33} - p \quad p = 2c_1 B_{33}$

b) $\Rightarrow$ $\underline{p = \frac{2c_1}{\lambda_1^2\lambda_2^2}}$

e) $T = 2c_1 \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \frac{1}{(\lambda_1\lambda_2)^2} \end{bmatrix} + 2k_1(I_4 - 1)\exp\left(k_2(I_4-1)^2\right) \begin{bmatrix} \lambda_1^2\cos\theta & \lambda_1\lambda_2\cos\theta\sin\theta & 0 \\ \lambda_1\lambda_2\cos\theta\sin\theta & \lambda_2^2\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{2c_1}{(\lambda_1\lambda_2)^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$T_{13} = T_{23} = 0 \Rightarrow$
1 and 2 are principal direction of stress if $T_{12} = 0$
(according to the principal stress theorem)

$\Rightarrow$ if $\sin\theta\cos\theta = 0 \iff \underline{\theta = n\frac{\pi}{2}}$ n is an integer

f) $\lambda_1 = \lambda \quad , \lambda_2 = \lambda_3 \quad \lambda_1\lambda_2\lambda_3 = 1 \Rightarrow \lambda_2 = \lambda_3 = \frac{1}{\sqrt{\lambda}}$

$\theta = 0 \Rightarrow e = [1 \quad 0 \quad 0]^T$

$a = Fe = \lambda_1 e = [\lambda_1 \quad 0 \quad 0]^T = [\lambda \quad 0 \quad 0]^T$

$I_4 = e_i C_{ij} e_j = [1 \quad 0 \quad 0]\begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \lambda_1^2 = \underline{I_4 = \lambda^2}$

$$T = 2c_1 \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & \frac{1}{\lambda} \end{bmatrix} + 2k_1(\lambda^2-1)\exp\left[k_2(\lambda^2-1)^2\right] \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - 2\frac{c_1}{\lambda} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_{11} = 2c_1\lambda^2 + 2k_1(\lambda^2-1)\exp\left[k_2(\lambda^2-1)^2\right]\lambda^2 - 2\frac{c_1}{\lambda}$$

$$\boxed{T_{11} = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right) + 2k_1\lambda^2(\lambda^2-1)\exp\left[k_2(\lambda^2-1)^2\right]}$$

$$T_{22} = 2c_1\frac{1}{\lambda} - 2c_1\frac{1}{\lambda} = 0$$

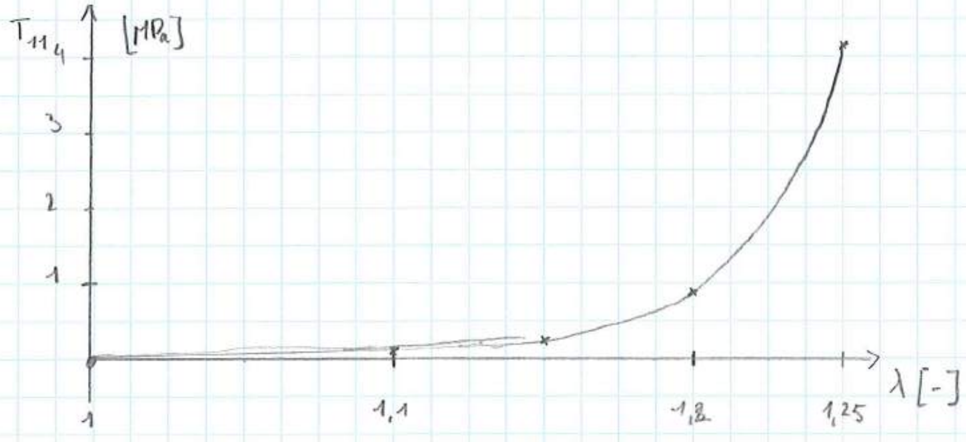
$$T_{11} = 0{,}02\left(\lambda^2 - \frac{1}{\lambda}\right) + 0{,}2\lambda^2(\lambda^2-1)\exp\left[10(\lambda^2-1)^2\right]$$

$T_{11}(\lambda=1) = 0$ MPa

$T_{11}(\lambda=1{,}1) = 0{,}085$ MPa $\qquad T_{11}(\lambda=1{,}15) = 0{,}25$ MPa

$T_{11}(\lambda=1{,}2) = 0{,}89$ MPa

$T_{11}(\lambda=1{,}25) = 4{,}17$ MPa



PROBLEM 2

Exercise ~~1~~ 2: Flow in a compliant tube

The governing equations for flow in a compliant tube are

$$\begin{aligned}\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}&=0\\ \frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right)&=-\frac{A}{\rho}\frac{\partial p}{\partial x}+\frac{\pi D}{\rho}\tau\end{aligned} \tag{1}$$

a) What basic principles do the two equations represent physically? What does each variable and parameter represent and what are their dimensions?

Solution. The first equation represents conservation of mass, and the second conservation of momentum.

- A Area $[m^2]$ of a compliant vessel
- D Diameter $[m]$
- Q Flow rate $[m^3/s]$.
- p Pressure $[Pa]$.
- ρ Fluid density $[kg\, m^{-3}]$.
- τ Wall shear stress of the compliant vessel wall on fluid $[Pa]$.

b) If $C=\frac{\partial A}{\partial p}$ one may show that

$$\begin{aligned}p&=p_0 f(x-ct)+p_0^* g(x+ct)\\ Q&=Q_0 f(x-ct)+Q_0^* g(x+ct)\end{aligned} \tag{2}$$

are general solutions to the governing equations under suitable assumptions.

1. List the necessary assumptions and show that the general solutions satisfy the governing equations under these assumptions.
2. What is the physical meaning of C?
3. Provide an expression for c and explain the physical meaning of c.
4. Explain how the pressure and flow in (2) should be interpreted with respect the pressure and flow in the orignal equations (1)

Solution. We assume invsicid flow and that nonlinear terms are small. We drop nonlinear terms and eliminate the friction term due to the assumption of in-viscid flow.

This leaves

$$\frac{\partial A}{\partial t} = -\frac{\partial Q}{\partial x}$$
$$\frac{\partial Q}{\partial t} = -\frac{A}{\rho}\frac{\partial p}{\partial x}.$$

We may close the system imposing a linear area- pressure relationship

$$A(p) = A_0 + C(p - p_0) \tag{3}$$

and thus

$$\frac{\partial A}{\partial t} = C\frac{\partial p}{\partial t}. \tag{4}$$

Substituting this in we may show the equations reduce to

$$C\frac{\partial p}{\partial t} = -\frac{\partial Q}{\partial x},$$
$$\frac{\partial Q}{\partial t} = -\frac{A}{\rho}\frac{\partial p}{\partial x}.$$

Further we may differentiate the first with respect to t and the second with respect to x to produce

$$C\frac{\partial^2 p}{\partial t^2} = -\frac{\partial^2 Q}{\partial x \partial t},$$
$$\frac{\partial^2 Q}{\partial t \partial x} = -\frac{A}{\rho}\frac{\partial^2 p}{\partial x^2}.$$

From this we see that p satisfies the wave equation

$$\frac{\partial^2 p}{\partial t^2} = \frac{A}{C\rho}\frac{\partial^2 p}{\partial x^2}.$$

Similarly we may show that

$$\frac{\partial^2 Q}{\partial t^2} = \frac{A}{C\rho}\frac{\partial^2 Q}{\partial x^2}.$$

These are the wave equations with wave speed $c = \sqrt{\frac{A}{C\rho}}$, which we show the general forms of the solutions suggested satisfy.

$$\frac{\partial^2 p}{\partial x^2} = p_0 f''(x-ct) + p_0^* g''(x+ct)$$
$$\frac{\partial^2 p}{\partial t^2} = c^2 p_0 f''(x-ct) + c^2 p_0^* g''(x+ct)$$

Thus $c^2 \frac{\partial^2 p}{\partial x^2} = \frac{\partial^2 p}{\partial t^2}$, satisfying the differential equation.

c is called the pulse wave velocity and represents how fast a perturbation in pressure or flow propagates along the tube

C is called the compliance and is a measure of how easily the artery expands with increased pressure.

The pressure and flow in (2) may be thought of a perturbation about a steady value, i.e. if the pressure in (1) is $p = p_0 + p'$ then the pressure in (2) is simply p'.

c) The characteristic impedance Z_c is defined as the ratio between pressure and flow in a unidirectional wave. Show that $Z_c = \frac{\rho c}{A}$ for waves in a compliant tube.

Solution. By introducing (2) into (1) we obtain:

$$\begin{aligned} -Q_0\, c\, f' + Q_0^*\, c\, g' &= -\frac{A}{\rho}\left(p_o f' + p_0^* g'\right) \\ f'\left(p_0 \frac{A}{\rho} - Q_0 c\right) + g'\left(p_0^* \frac{A}{\rho} + Q_0^* c\right) &= 0 \end{aligned} \tag{5}$$

As (5) must hold for arbitrarily chosen f and g, an expression for the *characteristic impedance* Z_c is obtained:

$$Z_c \equiv \frac{p_0}{Q_0} = \frac{\rho\, c}{A} = -\frac{p_0^*}{Q_0^*} \tag{6}$$

d) Describe how wave speed c could be estimated using the following measurements:

1. ultrasound measurements of area A and flow Q (Hint: Consider how changes in flow ΔQ could be related to a deviation in volume observed $\Delta V = \Delta x \Delta A$ in a section of artery with a length Δx.)
2. pressure measurments p in the arm and leg.

Discuss the differences and limitations of these two methods.

When would one be better than the other in assessing a person's health?

Solution. Using ultrasound we could measure the flow velocity and cross sectional area over a pulse and we can consider the perturbation in volume of some length of artery Δx, the corresponding change in area ΔA may be used to calculate the change in volume $\Delta V = \Delta A \Delta x$. The volume perturbation propagates and thus Δx changes in length allowing us to calculate $c = \frac{\Delta x}{\Delta t} = \frac{\Delta V}{\Delta A \Delta t} = \frac{\Delta Q}{\Delta A}$. Thus we use this formula with the incremental changes in measured flow and cross sectional area to estimate c.

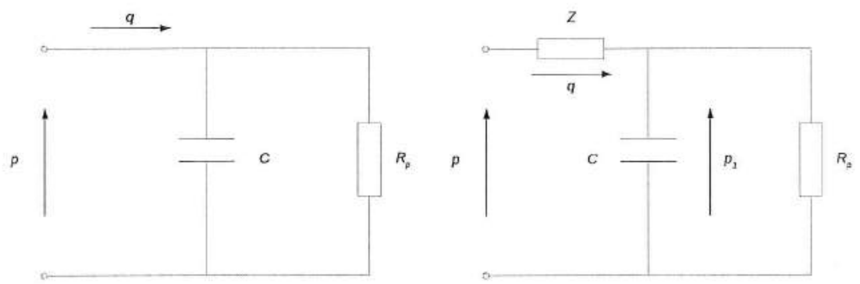
To measure the velocity with pressure measurments one would simply record the pressure wave in two locations simultaneously, and the time difference between peaks or troughs of the signals could be calculated, t_{arm} and t_{leg}. Combined with an estimate of the distance between measurement points $L$the wave speed could be estimated as $\frac{L}{t_{arm} - t_{leg}}$.

Using ultrasound provides a way to look at the wave speed in a particular vessel, which could be useful for assesing how stiff certain arteries are, while the arm-leg measurements provide more information about all the arteries between the two measurement points. Thus they give more global measure of stiffness.

Contact person: Stian Solberg
Tlf: 95475537

PROBLEM

~~Exercise~~ 3: Modelling and parameter estimation (30%)



(a) Two element model (b) Three element model

Figure 1: Windkessel models

a) Figure 1a and 1b shows the two element Windkessel model (2WK) and the three element Windkessel model (3WK) respectively. Explain the parameters in the models.

Answer:
For the 2WK model the parameters are:

p Aortic pressure

q Inflow to the aorta

C Total arterial compliance

$\mathbf{R_p}$ Total peripheral resistance

—

For the 3WK model the parameters are:

p Pressure at the aortic root

$\mathbf{p_1}$ Pressure at a more distal position in the aorta

q Inflow to the aorta

Z Combined effects of compliance and inertance of the very proximal aorta

C Total arterial compliance

$\mathbf{R_p}$ Total peripheral resistance

b) How does the 3WK model improve upon the 2WK model, i.e. how does the model input impedances compare to the measured input impedance?

Answer:
The 3WK impedance is a closer match to measured impedance for higher frequencies, as can be seen in figure 2. (figure not required in answer.)

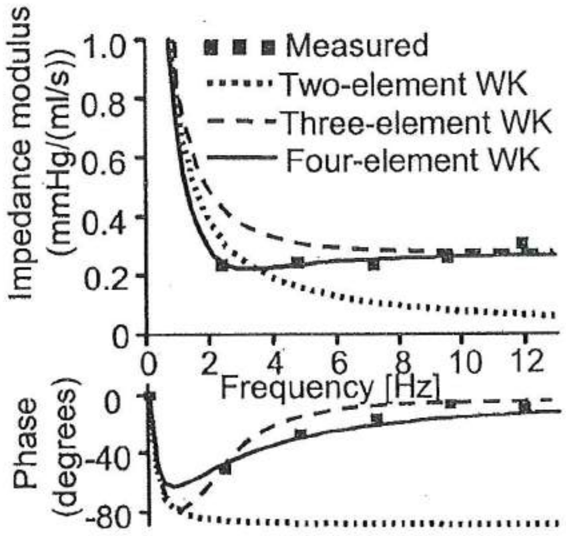


Figure 2: A comparison of the input impedances of the two-element, three-element and four element Windkessel models

c) As humans grow older, their arteries get stiffer. Calculate the input impedance of the 2WK model, and use this to discuss how age influences the pressure-flow ratio.

Answer:
The impedance of the 2WK model is given by

$$Z_{2WK} = \frac{R_p}{1 + i\omega RC} \tag{1}$$

We also have that the impedance is defined as

$$Z_{2WK} = \frac{p(\omega)}{q(\omega)} \tag{2}$$

Stiffer arteries means that the compliance C is lower, and we can see from (1) that this causes $|Z_{2WK}|$ to be higher, and thus we get a higher pressure to flow ratio.

d) The 3WK model is described by the following set of equations:

$$\frac{\partial p_1(t)}{\partial t} = -\frac{1}{RC}p_1(t) + \frac{1}{C}q(t) \tag{3}$$

$$p(t) = p_1(t) + Zq(t) \tag{4}$$

Show that a discretization of $p(t)$ kan be written as

$$p_{k+1} = \alpha_1 p_k + \alpha_2 q_k + \alpha_3 q_{k+1} \tag{5}$$

And further in matrix form as

$$\hat{\mathbf{p}} = \mathbf{A}\alpha \tag{6}$$

where

$$\alpha_1 = 1 - \frac{\Delta t}{RC} \tag{7}$$

$$\alpha_2 = (\frac{1}{C} + \frac{Z}{RC})\Delta t - Z \tag{8}$$

$$\alpha_3 = Z \tag{9}$$

Answer:
Inserting (4) into (3) we get

$$\frac{dp}{dt} = -\frac{1}{RC}p + \left(\frac{1}{C} + \frac{Z}{RC}\right) q + Z\frac{dq}{dt}. \tag{10}$$

Approximating the differential through a through a first order Taylor expansion, $\frac{dp}{dt} \approx \frac{p_{k+1}-p_k}{\Delta t}$, we can discretize (10) as

$$\frac{p_{k+1} - p_k}{\Delta t} = -\frac{1}{RC}p_k + \left(\frac{1}{C} + \frac{Z}{RC}\right) q_k + Z\frac{q_{k+1} - qk}{\Delta t}. \tag{11}$$

which can be rewritten as

$$p_{k+1} = \left(1 - \frac{\Delta t}{RC}\right) p_k + \left(\left(\frac{1}{C} + \frac{Z}{RC}\right) \Delta t - Z\right) q_k + Zq_{k+1} \tag{12}$$

$$= \alpha_1 p_k + \alpha_2 q_k + \alpha_3 q_{k+1} \tag{13}$$

Further defining

$$\hat{\mathbf{p}} = \begin{bmatrix} p_2 \\ p_3 \\ \vdots \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} p_1 & q_1 & q_2 \\ p_2 & q_2 & q_3 \\ \vdots & \vdots & \vdots \end{bmatrix} \qquad \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \tag{14}$$

we can write the discretized equation in matrix form as requested

$$\hat{\mathbf{p}} = \mathbf{A}\alpha \tag{15}$$

e) Assume that you have a set of measurements $\mathbf{p} = [p_1, p_2, \ldots, p_N]$ and $\mathbf{q} = [q_1, q_2, \ldots, q_N]$. Find the (one-step) least squares estimate of the parameter vector α from equation 6. Show your calculations.

Answer:
For a one-step least square estimation, we want to minimize the following functional with respect to α:

$$J(\alpha) = [\mathbf{Y} - \mathbf{A}\alpha]^T [\mathbf{Y} - \mathbf{A}\alpha] \tag{16}$$

where $\mathbf{Y}$ and $\mathbf{A}$ is constructed from the *measurements* as

$$\mathbf{Y} = \begin{bmatrix} p_2 \\ p_3 \\ \vdots \\ p_N \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} p_1 & q_1 & q_2 \\ p_2 & q_2 & q_3 \\ \vdots & \vdots & \vdots \\ p_{N-1} & q_{N-1} & q_N \end{bmatrix} \tag{17}$$

Differentiating the functional with respect to α and setting it equal to zero to find the minimum, we get

$$\frac{\partial J}{\partial \alpha} = 2\mathbf{A}^T\mathbf{A}\alpha - 2\mathbf{A}^T\mathbf{Y} = 0 \tag{18}$$

which gives the optimal parameters as

$$\alpha = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{Y} \tag{19}$$

f) Define observabillity and robustness of parameter estimation. What is the sensitivity matrix (S-matrix) of a model? How is the elements of the S-matrix defined? How can this matrix be used to analyse observability and robustness of the parameter estimation?

Answer:
Observabillity means that it is possible to determine all the parameters in the model from the measurements. Robustness means that the estimation is not sensitive to errors in the measurements.

The S-matrix is the gradient of the model vector with respect to its parameters. The elements are defined as:

$$S_{ij} = \frac{\partial \hat{Y}_i(\alpha)}{\partial \alpha_j} \tag{20}$$

If we normalize the S-matrix with respect to its columns, we can construct the *normalized* sensitivity matrix, $\mathbf{s}^\mathbf{T}\mathbf{s}$. The diagonal elements of this matrix will all be 1. Observabillity and robustness can be analysed by looking at the off-diagonal elements. For observabillity, we require that no off-diagonal elements are 1. For robustness, we require that the absolute value of the off-diagonal elements are $<< 1$.

g) Given the matrix

$$\begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}$$

where a is always in the range [0, 1]. Calculate the eigenvalues and the ratio between the largest and the smallest eigenvalue. What is the requirement on this ratio if we want a robust inverse?

Answer:
The eigenvalues are found by solving the following equation

$$\left| \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix} - \lambda \mathbf{I} \right| = 0 \tag{21}$$

$$(1-\lambda)(1-\lambda) - a^2 = 0 \tag{22}$$

$$\lambda^2 - 2\lambda + (1-a^2) = 0 \tag{23}$$

which has the solutions

$$\lambda_1 = 1 + a \tag{24}$$

$$\lambda_2 = 1 - a \tag{25}$$

Since we know that a is in the range [0, 1], we have that $\lambda_1 > \lambda_2$, and the ratio of the largest to lowest eigenvalue becomes

$$\frac{1 + a}{1 - a} \tag{26}$$

For a robust inverse of matrices of this type, we need $a << 1$. This means that the ratio of the highest to lowest eigenvalue needs to be small for the inverse to be robust.