Impedance and Windkessels

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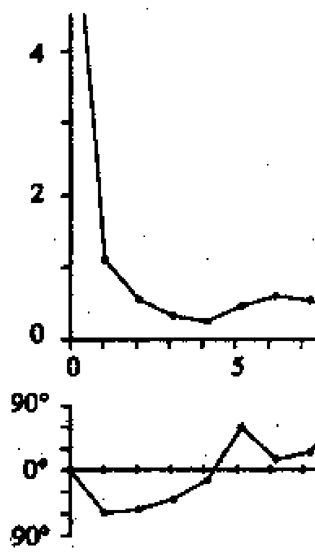
Outline

Arterial input impedance



- Provides a (complete) and comprehensive description of the arterial system [a]
- Impedance
  - A measure of oppostion to flow
  - Frequency dependent resistance
  - Resistance for non-oscillatory or steady motion
- Definition
  - The ratio of harmonic terms of pressure and corresponding harmonic terms of flow

$$p = |\hat{p}|e^{j(\omega t+\phi)}, \quad q = |\hat{q}|e^{j(\omega t+\beta)}$$
$$Z_i = \frac{|\hat{p}|}{|\hat{q}|}e^{j\theta}, \qquad \theta = \phi - \beta$$

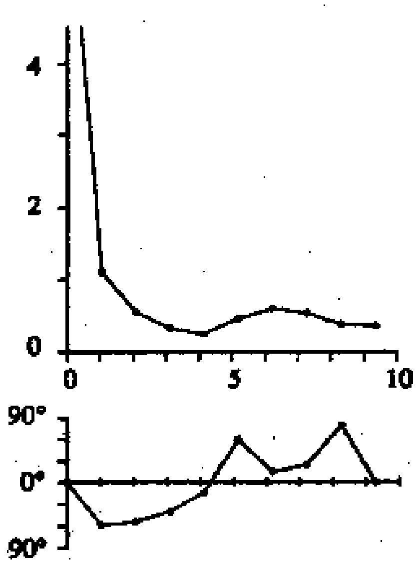
[a] Snapshots of hemodynamics

Applicability of Fourier analysis

- Used to related hemodynamic variables such as pressure and flow
- Not meaningful to related time signals
  - Diastolic pressure and flow
  - Division of pressure by zero flow does not provide meaningful information
- Produce a mean and multiples of the heart rate
- Each harmonic has an amplitude and a phase angle
- Impedance
  - Relate pressure and flow
  - Ohm's law is applied for each frequency
  - Only valid for a linear relation between pressure and flow
- Aortic input impedance
  - Venous pressure may be neglected
  - Aortic pressure and flow gives a sufficiently accurate approximation of the input impedance

Limitations to the use of Fourier analysis

- May only be used for periodic signals. The value of the signal at start and end should be the same.
- The relation of two signals should be linear.
- OK in many cases for pressure and flow despite the nonlinear relation
- The scatter in modulus and phase has been attributed to nonlinearities
- High frequency information should be considered with care.



Reflection factor

- Occur at any point where there is an abrupt change in characteristic impedance (mismatch in impedance)
- Oscillations at origin: the reflections will mix with the original pulse
- Spatial variations in amplitude and different wave pattern in flow and pressure are indicators for reflections
- Wave separation

$$p = p_f + p_b$$

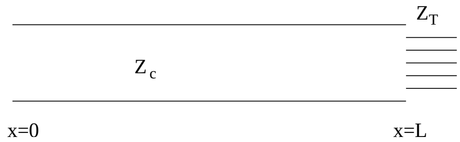
- Reflection factor

$$\Gamma \equiv \frac{p_b}{p_f} = -\frac{Q_b}{Q_f}$$

- Easy to show

$$\Gamma = \frac{Z_{in} - Z_c}{Z_{in} + Z_c}$$

The quarter wavelength formula



- Forward waves

$$p_f = p_0\ e^{j\omega t}, \quad Q_f = \frac{p_0}{Z_c}\ e^{j\omega t}$$

- Reflected waves ($\Gamma = 1$)

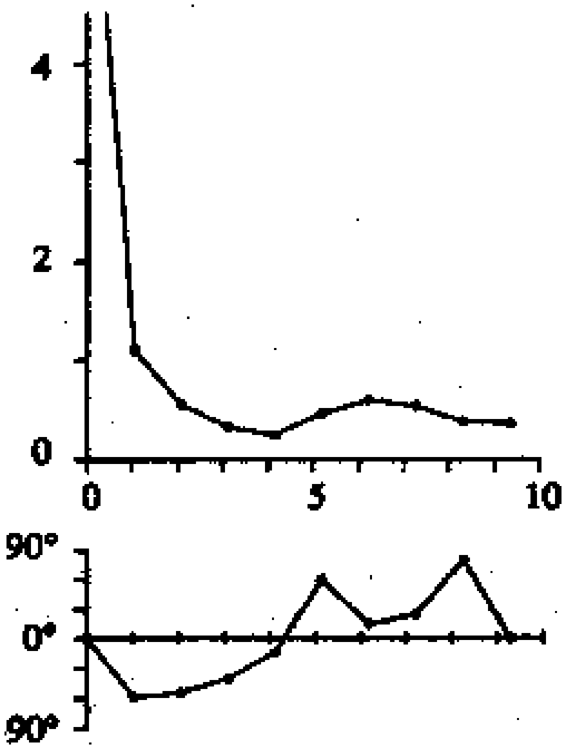
$$p_b = p_0\ e^{j\omega(t-2L/c)}, \quad Q_b = -\frac{p_0}{Z_c}\ e^{j\omega(t-2L/c)}$$

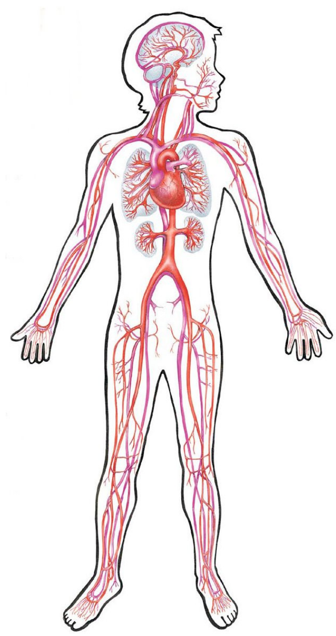
- Input impedance

$$Z_{in} = \frac{p_f + p_b}{Q_f + Q_b} = Z_c\ \frac{e^{j\omega t} + e^{j\omega(t-2L/c)}}{e^{j\omega t} - e^{j\omega(t-2L/c)}}$$

- $Z_{in} = 0 \Rightarrow \dfrac{2\omega L}{c} = \pi \Rightarrow L = \frac{\lambda}{4}$

Example: The quarter wavelength formula and Z_{in}





min $|Z_{in}|$ at f = 3.8 Hz
$\Rightarrow L \approx 0.33m$

Characteristic impedance

- Governing equations

$$
\begin{aligned}
C\frac{\partial p}{\partial t} &= -\frac{\partial Q}{\partial x} \\
\frac{\partial Q}{\partial t} &= -\frac{A}{\rho}\frac{\partial p}{\partial x}
\end{aligned}
$$

- Solutions

$$
\begin{aligned}
p &= p_0\, f(x-ct) + p_0^*\, g(x+ct) \\
Q &= Q_0\, f(x-ct) + Q_0^*\, g(x+ct)
\end{aligned}
$$

- By subst in momentum equation and collection of terms

$$\left(\frac{A}{\rho} p_0 - cQ_0\right) f' + \left(\frac{A}{\rho} p_0^* + cQ_0^*\right) g' = 0$$

- Must hold for arbitrary f' and g'

$$Z_c = \frac{p_0}{Q_0} = -\frac{p_0^*}{Q_0^*} = \frac{\rho c}{A}$$

Practical estimation of Z_c

- Average of Z_i
  - Higher frequencies cancel and are damped
  - Average between 4th and 10th harmonic
- Slope of p and Q
  - In early part of systole/ejection phase

$$Z_c = \frac{\Delta p/\Delta t}{\Delta Q/\Delta t}$$

- Both methods rely on the fact that Z_c is a p-Q relation in absence of reflections
- Reflections are small in early systole and at high frequencies

Explanations of the input impedance

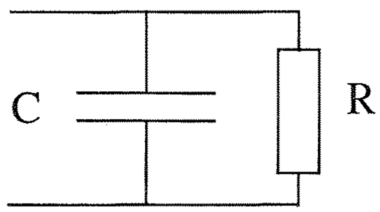
- The Windkessels
  - Two-element Windkessel
    - The original
    - Peripheral resistance R_p
    - Total arterial compliance C
  - Three-element Windkessel
    - Aortic charateristic impedance Z_c
    - $\bar{p}/\bar{q} = R_p + R_c \neq R_P$
    - Leads to errors for estimates of C
  - Four-element Windkessel
    - Corrects for the above shortcomings
    - Total arterial inertance L
- Wave transmission
  - $Z_i = Z_c$ for reflectionless system
  - $Z_i \neq Z_c$ in the aorta for low frequencies
  - Reflections at bifurcations and locations of impedance mismatch
  - Waves with high frequencies return out of phase and cancel
  - Damping is also stronger for high frequencies
  - Apparently no reflections for high frequencies

The Windkessel model[1]

- Peripheral resistance: R_p
- Total arterial compliance: $C = \frac{\partial V}{\partial p}$
- Flow split
  - $Q_a = \frac{\partial V}{\partial p}\frac{\partial p}{\partial t} = C\,\frac{\partial p}{\partial t}$
  - $Q_p = \frac{p}{R_p}$
- Mass: $Q = C\frac{\partial p}{\partial t} + \frac{p}{R_p}$
- In diastole

$$\frac{\partial p}{\partial t} = -\frac{p}{RC}$$

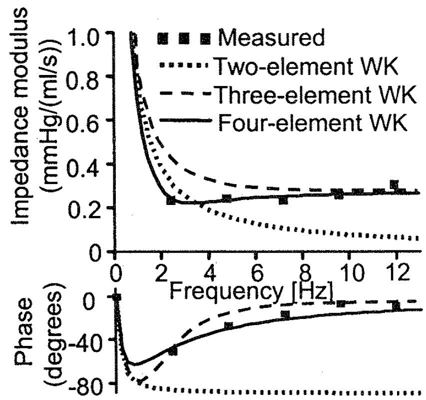
$$p = p_0\, e^{-t/RC}, \qquad R = \frac{\bar{p}}{\bar{Q}}$$



- Problems
  - Cannot fit exp decay to whole diastole
  - When $c\uparrow$ for old, 2-elt WK does better

[1] Otto Frank 1899

Impedance for the Windkessel model



- Mass: $Q = C\,\frac{\partial p}{\partial t} + \frac{p}{R_p}$
- Fourier: $p = \hat{p}\,e^{j\omega t}, \quad Q = \hat{Q}\,e^{j\omega t}$
- By substitution

$$\hat{Q} = j\omega C\,\hat{p} + \frac{\hat{p}}{R_p}$$

- Impedance

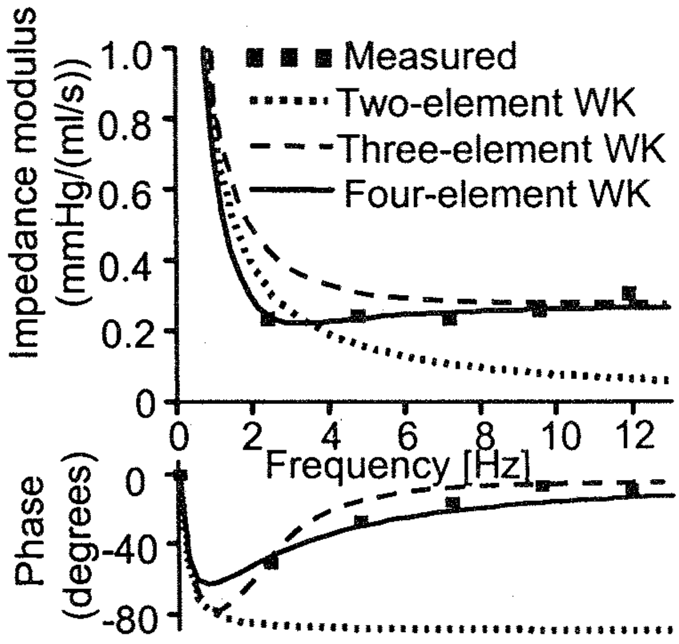
$$Z = \frac{\hat{p}}{\hat{Q}} = \frac{R_p}{1 + j\omega R_p C} = \frac{R_p}{1 + (\omega R_p C)^2}(1 - j\omega R_p C)$$

$$|Z| = \frac{R_p}{1 + (\omega R_p C)^2}\sqrt{1 + (\omega R_p C)^2} = \frac{R_p}{\sqrt{1 + (\omega R_p C)^2}}$$

$$\angle Z = -\arctan\omega R_p C$$

Limitations of the Windkessel

- At high frequencies $|Z| \to 0$, not Z_c
- At high frequencies $\angle Z \to -90^\circ$, not 0
- High frequency information not captured

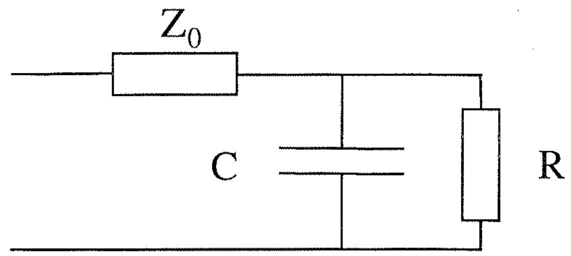


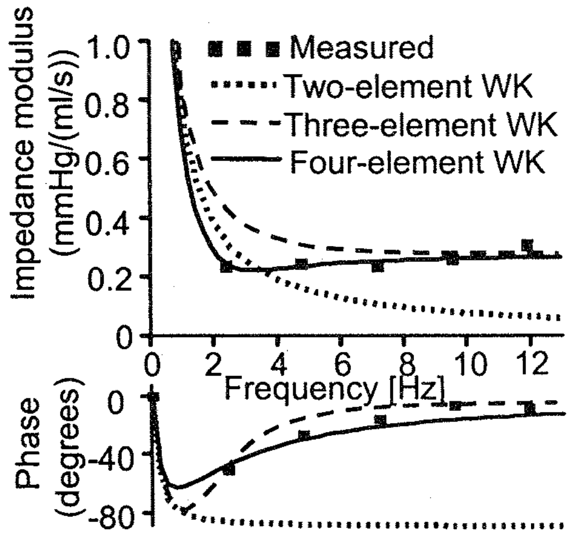
The Westkessel model[2] (3-elt WK)

- To correct for high frequency problems
- Impedance

$$Z = Z_c + \frac{R_p}{1 + j\omega R_p C}$$

- Pros
  - Good high freqency $|Z|$
  - Good high freqency $\angle Z$
  - Good fit of p and Q
- Cons
  - Bad compliance estimates
  - Bad low frequency est
  - Only monotonous decay in $|Z|$





[2] Westerhof

Summary