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Exam in TKT4150 Biomechanics

Friday December 9, 2011
Duration: kl. 09.00-13.00

No printed or handwritten aids are permitted (D). Approved calculators are permitted.

Deadline for examination results: Week 2 2012

Exercise 1

A typical left ventricular pressure-volume loop is presented schematically in figure 1(a). With every heart beat a full loop is trajected.

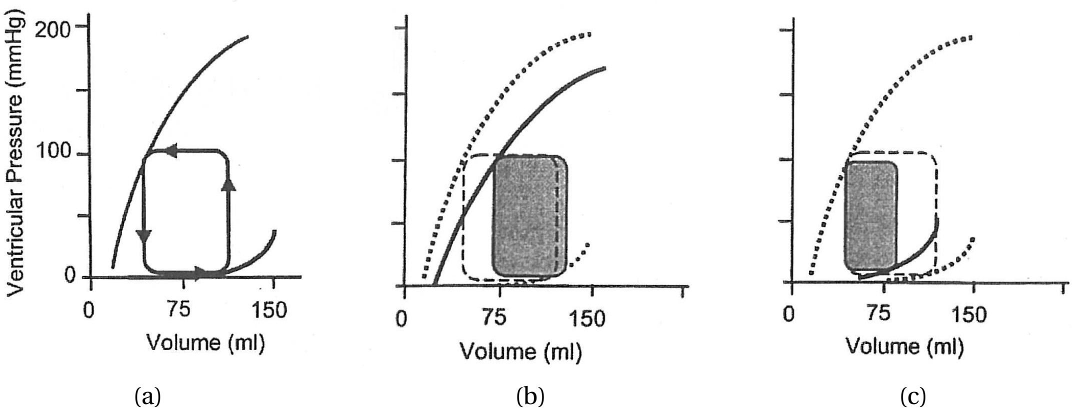


Figure 1: A schematic illustration of ventricular pressure-volume relations.

1. a) Describe a normal left ventricular pressure-volume loop by relating it to the various phases of the heart cycle.
   b) Explain the Frank-Starling law for the effect of left ventricular filling.
2. The figures 1(b) and 1(c) illustrate ventricular pressure-volume loops for normal and dysfunctional conditions.
   a) Mark normal and diseased conditions in your own figures.
   b) Explain which dysfunctional condition pertains to the two cases and on the consequences for the cardiac function.

3. a) Explain how changes in ventricular filling may be used to assess cardiac contractility.

   b) What is considered to be the best measure for contractility for both healthy and diseased states, and for various mammals? Explain why.

Exercise 2

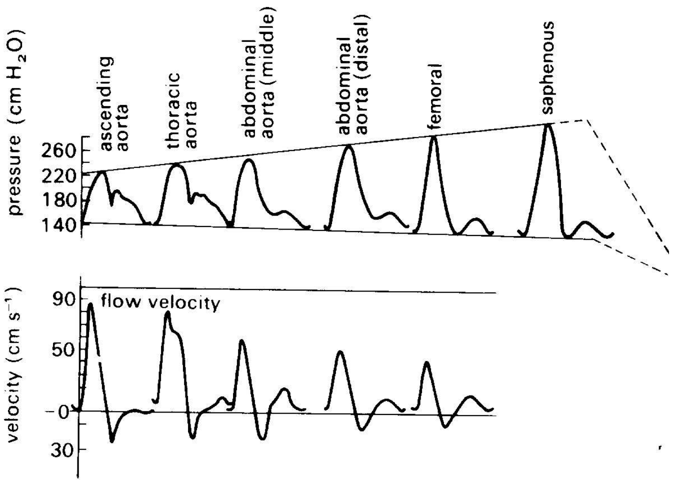


Figure 2: Spatial evolution of pressure (upper panel) and velocity (lower panel) waves from the heart to the periphery.

The governing equations for a compliant tube are given by:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{1a}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) = -\frac{A}{\rho}\frac{\partial p}{\partial x} + \frac{\pi D}{\rho}\tau \tag{1b}$$

1. a) Assign physical meaningful names to the two equations.

   b) Close the system of equations (1) by introducing the compliance: $C = \partial A/\partial p$, argue for appropriate simplifications, and show that:

$$p = p_0\, f(x - ct) + p_0^*\, g(x + ct) = p_f + p_b \tag{2a}$$

$$Q = Q_0\, f(x - ct) + Q_0^*\, g(x + ct) = Q_f + Q_b \tag{2b}$$

   are general solutions to the simplified equations for wave propagation in compliant vessels.

   c) Present an expression for c and describe its physical meaning based on equation (2).

2. a) Find an expression for p_0/Q_0 and p_0^*/Q_0^* by substitution of equation (2) into (1).

   b) What is the common name for this ratio?

c) What is the physical meaning of this ratio?

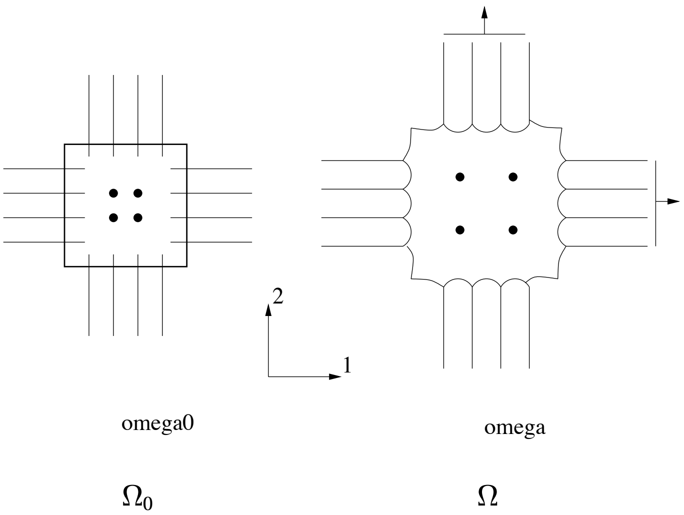
d) How is p_0/Q_0 related to p_f/Q_f?

3. In Fig. 2, the spatial evolution of pressure and velocity waves are plotted from the heart to the periphery.

   a) Based on the equations and simplifications above, what can you deduce from the pressure and velocity waveforms?

   b) Use the ratio derived in question 2 and equation (2a) to find expressions for the forward and backward components of pressure and flow (p_f, p_b, Q_f, Q_b).

Exercise 3



A square piece of tissue cut out of a blood vessel is mechanically tested by means of in-plane biaxial stretching test. Away from the attachement sites (i.e. in the middle of the specimen) the stress and strain fields are homogeneous. We consider a particle originally at location (X_1, X_2, X_3) in the unloaded reference configuration Ω_0. The deformations of this particle in the middle of the specimen may be expressed as:

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3,$$

where (x_1, x_2, x_3) are the coordinates of the particle in the current configuration Ω. The tissue is assumed to be incompressible ($\det\mathbf{F} = 1$). 1 and 2–directions are in–plane and the 3–direction is the out of plane direction.

1. Express the components of the deformation gradient $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$, the right Cauchy Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and the Green strain tensor $\mathbf{E} = 1/2(\mathbf{C} - \mathbf{1})$ in the (1,2,3) orthogonal coordinate system. Express the λ_3 as a function of λ_1 and λ_2. Determine the principal directions of $\mathbf{F}$, $\mathbf{E}$, and $\mathbf{C}$ in the middle of the specimen.

2. Give the value of the Cauchy stress in the 3-direction and explain why?

   Now, $\lambda_1 = \lambda_2 = 1.2$ in Ω.

3. H=1 mm is the thickness in the middle of the specimen in the reference configuration Ω_0. Calculate h the thickness in the current configuration Ω.

4. Assume an isotropic material and compare the Cauchy stresses in 1 and 2–directions in Ω.

Exercise 4

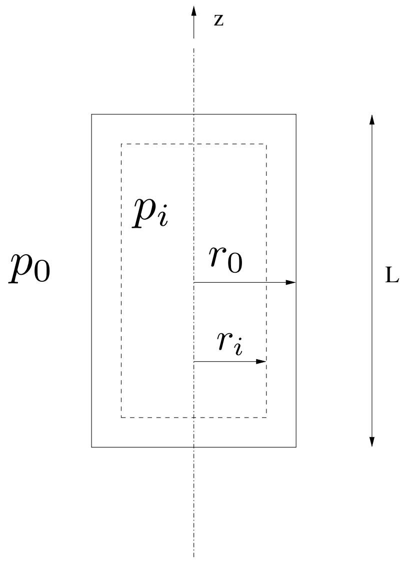


Figure 3: Arts Model, left ventricle as a thickwalled cylinder, r_i and r_0 are the inner and outer radii, respectively

We now consider a simplified model of left ventricular dynamics: the Arts model. In this model, the left ventricle is assumed to be a thickwalled cylinder of thickness consisting of many thinwalled cylinders of thickness dR.

Let's assume that the stresses in a cylindrical surface of radius R can be expressed in the cylindrical coordinate system ($\mathbf{e}_R$,$\mathbf{e}_\theta$,$\mathbf{e}_z$), where the z-direction is parallel to the axis of symmetry of the left ventricle, as:

$$\begin{aligned} \sigma_R &= -p \\ \sigma_\theta &= -p + \sigma \cos^2 \alpha \\ \sigma_z &= -p + \sigma \sin^2 \alpha, \end{aligned}$$

where σ is the stress in the myocardial muscle fiber oriented in the direction $\mathbf{n} = \cos\alpha\, \mathbf{e}_\theta + \sin\alpha\, \mathbf{e}_z$. Now, we consider a thinwalled cylinder of thickness dR.

1. Write the equilibrium in the R-direction and show that:

$$\frac{dp}{dR} = \frac{-\sigma \cos^2 \alpha}{R}.$$

2. Write the equilibrium in the z-direction and show that:

$$\frac{dp}{dR} = \frac{-2\sigma \sin^2 \alpha}{R}.$$

   ($d\theta$ is small: $\sin d\theta \approx d\theta$)

3. Express $\frac{dp}{dR}$ with respect to σ and R only.

4. Integrate the result found in question 3) from r_i to r_0 and express a relationship between p_{LV}, σ, V_W and V_{LV}.
   $p_{LV} = p_0 - p_i$ is the left ventricular pressure.
   $V_W = \pi r_0^2 L - \pi r_i^2 L$ is the volume of the left ventricular wall (endocardium).
   $V_{LV} = \pi r_i^2 L$ is the volume of the left ventricular cavity.

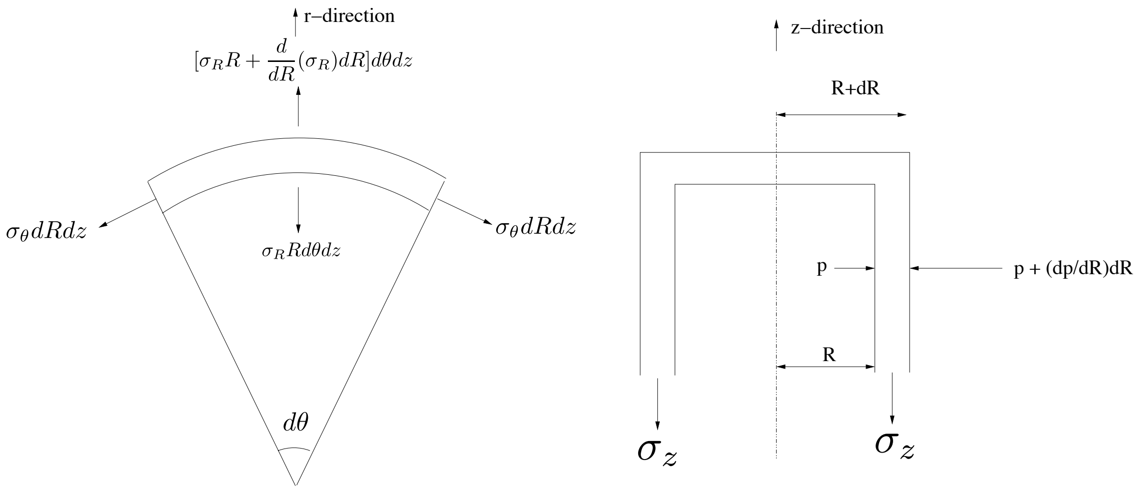


Figure 4: questions 1 and 2, equilibrium in R and z-directions

Suggested solutions

Exercise 1, suggested solution

1. a) The pressure-volume loop:
   - Starting at end diastole, the first part of the loop is the isovolumic contraction.
   - When the aortic valve opens , ejection begins and during the ejection period LV volume decreases and pressure changes realtively little.
   - After aortic valve closure, the isovolumic relaxation follows.
   - When the mitral valve (between the left atrium and left ventricle) opens, fillings starts and the LV volume increases with a very small increase in the left ventricular pressure, until the end diastolic volume is reached.

   b) Frank-Starling law
   - Frank: increased diastolic filling for isovolumic (non-ejecting) hearts yields increased systolic (maximum) pressure.
   - Starling: Increased diastolic filling for ejecting hearts against constant resistance (Starling resistor), increase stroke volume.

2. a) Thick, solid lines correspond to diseased conditions.

   b) Figure 1(b) corresponds to systolic dysfunction, whereas figure 1(c) corresponds to diastolic dysfunction.

3. a) Changes in ventricular filling yields an estimate of the end systolic pressure volume ratio (ESPVR) or E_{max}, which is a measure of cardiac contractility.

   b) The ratio $E_{\text{max}}/E_{\text{min}}$ is considered to be the best measure of cardiac contractility, as it is independent of body size.

Exercise 2, suggested solution

1. a) Equation (1a) is the mass equation and equation (1b) is the momentum equation.

   b) Introduce the compliance:

$$C = \frac{\partial A}{\partial p} \tag{3}$$

which allows for closure of the equations (1) by elimination of the area:

$$\frac{\partial A}{\partial t} = \frac{\partial A}{\partial p}\frac{\partial p}{\partial t} = C\frac{\partial p}{\partial t} \tag{4}$$

The convective term will be small subject to the assumption of a straight vessel (pipes), and for fairly large vessels the viscous losses may be neglected too. Additonally, by assuming:

$$A\frac{\partial p}{\partial x} \approx \bar{A}\frac{\partial p}{\partial x} \tag{5}$$

the linearized and invicid form of the equations (1) may be presented:

$$C\frac{\partial p}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{6a}$$

$$\frac{\partial Q}{\partial t} + \frac{\bar{A}}{\rho}\frac{\partial p}{\partial x} = 0 \tag{6b}$$

A differential equation for p (or Q) may be obtained by cross-derivation and substraction of equation (6):

$$C\frac{\partial^2 p}{\partial t^2} - \frac{\bar{A}}{\rho} = 0 \tag{7}$$

which may be reformulated as:

$$\frac{\partial^2 p}{\partial t^2} - c^2\frac{\bar{A}}{\rho} = 0 \tag{8}$$

where the pulse wave velocity c, has been introduced as:

$$c^2 = \frac{\bar{A}}{\rho C} \tag{9}$$

Equation (8) is a classical wave equation An identical wave equation for Q may be obtained by elimination of p in equation (6). The wave equations have solutions on the form:

$$p = p_0 f(x - ct) + p_0^* g(x + ct) = p_f + p_b \tag{10a}$$

$$Q = Q_0 f(x - ct) + Q_0^* g(x + ct) = Q_f + Q_b \tag{10b}$$

The functions f and g are functions of the (Lagrangian) variables $z_1 = x - ct$ and $z_2 = x + ct$, respectively. For simplicity be introduce the notation:

$$f_{,z1} = \frac{df}{dz_1} \quad \text{and} \quad g_{,z2} = \frac{dg}{dz_2} \tag{11}$$

From the solutions in equation (10) we may calculate:

$$\frac{\partial Q}{\partial t} = -cQ_0 f_{z_1} + cQ_0^* g_{z_2}, \qquad \frac{\partial p}{\partial x} = p_0 f_{z_1} + p_0^* g_{z_2} \tag{12a}$$

$$\frac{\partial Q}{\partial x} = Q_0 f_{z_1} + Q_0^* g_{z_2}, \qquad \frac{\partial p}{\partial t} = -cp_0 f_{z_1} + cp_0^* g_{z_2} \tag{12b}$$

Substitution of equation (12b) into equation (6b) yields:

$$-cQ_0 f_{z_1} + cQ_0^* g_{z_2} + \frac{\bar{A}}{\rho} p_0 f_{z_1} + \frac{\bar{A}}{\rho} p_0^* g_{z_2} = 0 \tag{13}$$

Colletion of terms with f_{z_1} and g_{z_2} yields:

$$\left(\frac{\bar{A}}{\rho} p_0 - cQ_0 +\right) f_{z_1} + \left(\frac{\bar{A}}{\rho} p_0^* + cQ_0^*\right) g_{z_2} \tag{14}$$

As equation (14) must hold for arbitrary f (i.e. $f_{,z_1}$) and g, the expressions in the parantheses must vanish, a fact yielding:

$$\frac{p_0}{Q_0} = \frac{\rho c}{\bar{A}} = -\frac{p_0^*}{Q_0^*} = Z_c \tag{15}$$

Thus, equations (10) are general solutions to the system of partial differantial equations (6), provided equation (15) is satisfied.

c) From equation (9) we have:

$$c = \sqrt{\frac{\bar{A}}{\rho C}} \tag{16}$$

and c is the pulse wave velocity, i.e. the velocity with which a pressure (or flow) increase (perturbation) travels with in a compliant pipe.

2. a) The ratios p_0/Q_0 and p_0^*/Q_0^* are found by substitution of the generic solutions in equations (10) into the linearzied, inviscid momentum equation (6b) (see equation (15) in 1b).

   b) Z_c is called the characteristic impedance.

   c) Z_c is the ratio of the forward propgagating pressure component and the forward propgagating flow component. Or equivalently minus the ratio of the backward propgagating pressure component and the backward propgagating flow component.

d) From equation (2) we have:

$$p_f = p_0 f(x - ct) \qquad \text{and} \qquad Q_f = Q_0 f(x - ct) \tag{17}$$

and thus:

$$Z_c \equiv \frac{p_0}{Q_0} = \frac{p_f}{Q_f} \tag{18}$$

as the f-function cancels, i.e. the ratios are identical.

3. a) From equation (2) we see that in the case of unidirectional flow (i.e. waves travelling in one direction only), the p and Q curves will differ by a factor only (Z_c), and thus their forms will be the same. The forms or appearance of the pressure and flow waves in figure 2 are not the same, thus both forward and backward (reflected) waves from the periphery must be present.

b) We start with the algebraic relations:

$$Q = Q_f + Q_b, \qquad Z_c = \frac{p_f}{Q_f} = -\frac{p_b}{Q_b} \tag{19}$$

$$p = p_f + p_b \tag{20}$$

To the expressions for p_f and p_b:

$$Q = Q_f + Q_b = \frac{p_f - p_b}{Z_c} \tag{21}$$

We isolate p_f by substitution of $p_b = p - p_f$ into equation (21), yielding:

$$Z_c Q = p_f - p_b = p_f - p + p_f \tag{22}$$

and thus:

$$p_f = \frac{p + Z_c Q}{2} \tag{23}$$

An expression for p_b is obtained by substution of $p_b = p - p_f$ into equation (21):

$$p_b = \frac{p - Z_c Q}{2} \tag{24}$$

For expressions for Q_f and Q_b:

$$p = Z_c(Q_f - Q_b), \qquad Q = Q_f + Q_b \tag{25}$$

$$\frac{p}{Z_c} = Q_f - Q + Q_f, \qquad \frac{p}{Z_c} = Q - Q_b - Q_b \tag{26}$$

which yields:

$$Q_f = \frac{Q + p/Z_c}{2}, \qquad Q_b = \frac{Q - p/Z_c}{2} \tag{27}$$