NTNU
Dept. Structural Engineering

Exam in TKT4150 Biomechanics

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Duration: 09:00-13:00

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No printed or handwritten aids are permitted (D). Approved calculators are permitted.

Exercise 1 — The motion of a continuum is given by the position vector $x_i(X,t)$. The Green strain tensor has the components:

$$E_{ij} = \frac{1}{2}\left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right) \tag{1}$$

where δ_{ij} is the Kronecker delta. A material line element in the direction $\mathbf{e}$ and of length ds_0 in the reference configuration K_0 has in the present configuration K the length ds. Then:

$$ds^2 - ds_0^2 = 2e_i E_{ij} e_j\, ds_0^2 \tag{2}$$

1. Show that the longitudinal strain and the stretch in the line element become, respectively:
$$\epsilon = \sqrt{1+2e_i E_{ij} e_j} - 1, \qquad \lambda = \epsilon + 1 \tag{3}$$
2. Determine the expression for the longitudinal strain and the stretch in a line element in the direction of the x_i-axis.

Answer (Exercise 1) — 1. The longitudinal strain ϵ ant the stretch λ in a line element can be expressed as:

$$\epsilon = \frac{ds - ds_0}{ds_0} \tag{4}$$
$$\lambda = \frac{ds}{ds_0}. \tag{5}$$

From equation 2, we find,

$$\frac{ds}{ds_0} = \sqrt{1+2e_i E_{ij} e_j}, \tag{6}$$

hence, the longitudinal strain is expressed by the following relation:

$$\epsilon = \frac{ds}{ds_0} - 1 = \sqrt{1+2e_i E_{ij} e_j} - 1. \tag{7}$$

From equations 5, 6 and 7, the stretch can be expresssed by:

$$\lambda = \epsilon + 1 \tag{8}$$

2. If the material line element in the undeformed configuration K_0 is parallel to the x_i-direction, the corresponding longitudinal strain ϵ_{ii} is found from the relation 7 with $\mathbf{e}$ chosen to be $\mathbf{e}_i$:

$$\epsilon_{ii} = \sqrt{1+2E_{ii}} - 1. \tag{9}$$

Then, the stretch λ_i of the line element parallel to the x_i-direction is expressed as:

$$\lambda_i = \sqrt{1+2E_{ii}}. \tag{10}$$

Exercise 2 — 1D flow in a compliant pipe.

1. Derive the governing equations:

$$\begin{aligned} \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} &= 0 \\ \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) &= -\frac{A}{\rho}\frac{\partial p}{\partial x} + \frac{\pi D}{\rho}\tau \end{aligned} \tag{11}$$

for 1D flow in a compliant pipe.

2. The characteristic impedance Z_c, is defined as the ratio of pressure and flow for a unidirectional wave. Use the linearized, inviscid, governing equations for 1D flow in a compliant pipe to show that:

$$Z_c = \frac{\rho c}{A} \tag{12}$$

3. Outline two methods for practical estimation of Z_c.
4. The solutions to the linearized, inviscid, governing equations for 1D flow in a compliant pipe may be written:

$$p = p_f + p_b \qquad \wedge \qquad Q = Q_f + Q_b \tag{13}$$

Introduce the characteristic impedance Z_c and present expressions for the separated waves p_f and p_b.

5. Based on wave propagation, discuss the evolution due to age in Fig. 1.

Answer (Exercise 2) — 1. Mass conservation may be expressed by:

$$\rho\,\dot{V} = \rho(Q_i - Q_o) \tag{14}$$

The flux at the outlet is expressed by the relation:

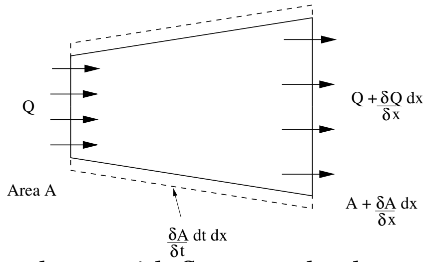
$$Q_o = Q_i + \frac{\partial Q}{\partial x}\,\mathrm{dx} \tag{15}$$

And the volume change may be estimated by:

$$\dot{V} \approx \frac{\partial A}{\partial t}\,\mathrm{dx} \tag{16}$$

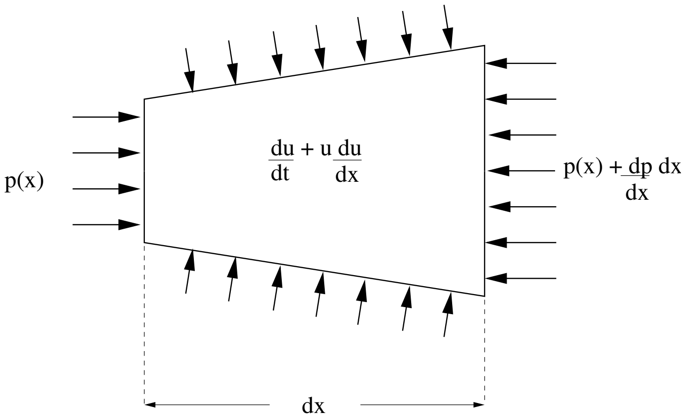
From the above we get the mass equation:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{17}$$



Control volume with fluxes and volume change.

For the momentum equation, we consider the forces first.



On the lhs we have: pA, whereas on the rhs we have:

$$-\left(p+\frac{\partial p}{\partial x}\,\mathrm{dx}\right)\left(A+\frac{\partial A}{\partial x}\,\mathrm{dx}\right) \tag{18}$$

The forces acting on the circumferential surface must also be accounted for:

$$p\frac{\partial A}{\partial x}\,\mathrm{dx} \tag{19}$$

as well as friction

$$F=\tau\pi D\,\mathrm{dx} \tag{20}$$

By summing the forces an expression for the net force is obtained:

$$F_{tot}=-A\frac{\partial p}{\partial x}\,\mathrm{dx}+\tau\pi D\,\mathrm{dx} \tag{21}$$

We use *Reynold's transport theorem*

$$\frac{d\mathbf{p}}{dt}=\frac{\partial}{\partial t}\int_{CV}\rho\mathbf{u}\,\mathrm{dV}+\int_{CS}\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,\mathrm{dA} \tag{22}$$

and for a 1D flow field with constant ρ:

$$\begin{aligned}\frac{d\mathbf{p}}{dt} &= \rho A\,\mathrm{dx}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)\\ &= \rho\left(\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right)\right)\end{aligned}$$

The rate of change of linear momentum must be balanced by the net force, and if the flow rate formulation of the momentum equation is preferred we get:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) = -\frac{A}{\rho}\frac{\partial p}{\partial x} + \frac{\pi D}{\rho}\tau \tag{23}$$

Thus, the governing equations for 1D flow in a compliant pipe, comprised by the mass and momentum equations, are given by:

$$\begin{aligned} \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} &= 0 \\ \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) &= -\frac{A}{\rho}\frac{\partial p}{\partial x} + \frac{\pi D}{\rho}\tau \end{aligned}$$

2. The area may be eliminated from the mass equation by introducing an area compliance $\partial_t A = C\partial_t p$, and thus the linearized, inviscid governing equations for 1D flow in compliant pipe are:

$$\begin{aligned} C\frac{\partial p}{\partial t} &= -\frac{\partial Q}{\partial x} \\ \frac{\partial Q}{\partial t} &= -\frac{A}{\rho}\frac{\partial p}{\partial x} \end{aligned}$$

with corresponding solutions:

$$\begin{aligned} p &= p_0 f(x-ct) + p_0^* g(x+ct) \\ Q &= Q_0 f(x-ct) + Q_0^* g(x+ct) \end{aligned}$$

By substitution of the solutions into the momentum equation and collection of terms we get:

$$\left(\frac{A}{\rho}p_0 - cQ_0\right)f' + \left(\frac{A}{\rho}p_0^* - cQ_0^*\right)g' = 0 \tag{24}$$

The expressions in the parentheses must be zero as Eq. (24) must hold for arbitrary f' and g' and thus:

$$Z_c = \frac{p_0}{Q_0} = \frac{p_0^*}{Q_0^*} = \frac{\rho c}{A}$$

3.
   - Average of Z_i
     - Higher frequencies cancel and are damped
     - Average between 4th and 10th harmonic
   - Slope of p and Q
     - In early part of systole/ejection phase

$$Z_c = \frac{\Delta p/\Delta t}{\Delta Q/\Delta t}$$

   - Both methods rely on the fact that Z_c is a p-Q relation in absence of reflections

- Reflections are small in early systole and at high frequencies

4. From solution of linearized wave equations the pressure and flow waves are split into forward and backward propagating components:

$$p = p_f + p_b \tag{25}$$
$$Q = Q_f + Q_b \tag{26}$$

The characteristic impedance is defined by:

$$Z_c \equiv \frac{p_f}{Q_f} = -\frac{p_b}{Q_b} \tag{27}$$

From the definition of the characteristic impedance and Eq. (26) we get:

$$p_b = -Z_c Q_b = -Z_c(Q - Q_f) = -Z_c Q + p_f \tag{28}$$

and by substitution of Eq. (28) into Eq. (25) we get:

$$p_f = \frac{p + Z_c Q}{2} \tag{29}$$

The forward propagating component of Eq. (25) may be eliminated by proceeding in the same manner to yield:

$$p_b = \frac{p - Z_c Q}{2} \tag{30}$$

5. As a human being gets older, the blood vessel become stiffer, corresponding to higher pulse wave velocities. This increased pulse wave velocities induce faster return of reflected waves, as can bee seen from Fig. 1, resulting in a higher pressure in the aorta *before* the aortic valve closure, denoted by the dicrotic notch in the pressure profile. The increased aortic pressure, while the aortic valve is still open, corresponds to a higher resistance to heart, which in turn might lead to hypertension and hypertrophy.

Exercise 3 — Consider the 1D momentum equation for fully developed flow in straight pipes:

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) \tag{31}$$

1. Introduce appropriate characteristic scales in Eq. (31) and present a dimensionless form of the equation.
2. Identify the Womersley number.
3. Illustrate the time-varying velocity profiles for extreme values of the Womersley number, i.e. $\alpha \to 0$ and $\alpha \to \infty$. Relate the velocity profiles to a time varying (sinusoidal) pressure gradient.

Answer (Exercise 3) — 1. Characteristic scales may be introduced by:

- Length: $r^* = r/a, \quad z^* = z/a$

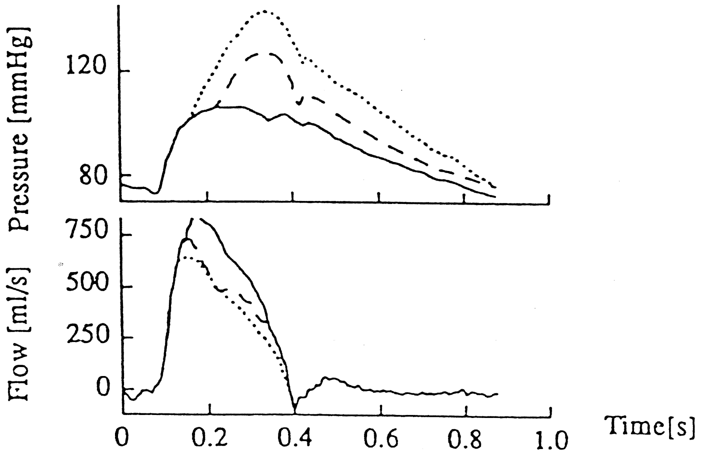


Figure 1: Aortic pressure (upper panel) and flow (lower panel) waveforms for three normal subjects of different age: 28 years (cont), 52 years (dashed), and 68 years (dotted).

- Time: $t^* = t\omega$
- Velocity: $v^* = v/V$

Which by substitution in Eq. (31) and some rearrangement yields the dimensionless form

$$\left(a^2\frac{\omega}{\nu}\right)\frac{\partial v^*}{\partial t^*} = -\left(\frac{a}{\rho\nu V}\right)\frac{\partial p}{\partial z^*} + \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial v^*}{\partial r^*}\right) \tag{32}$$

A characteristic pressure $p^* = p/(\rho\nu V/a)$ may be introduced and then the dimensionless straight tube equations are obtained:

$$\alpha^2\frac{\partial v^*}{\partial t^*} = -\frac{\partial p^*}{\partial z^*} + \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial v^*}{\partial r^*}\right) \tag{33}$$

2. Womersley parameter $\alpha = a\sqrt{\frac{\omega}{\nu}}$.
   Flow regimes with $\alpha \rightarrow \infty$ are inertia dominated, and corresponds to flow in large vessels and/or with high frequencies ($\alpha \geq 20$ for the aorta). Flows with $\alpha \rightarrow 0$ are friction dominated, and corresponds to flow in small vessels and/or with low frequencies ($\alpha = 10^{-2}$ for capillaries) Friction dominated
3. For small Womersley numbers ($\alpha \rightarrow 0$) Eq. (31) reduces to:

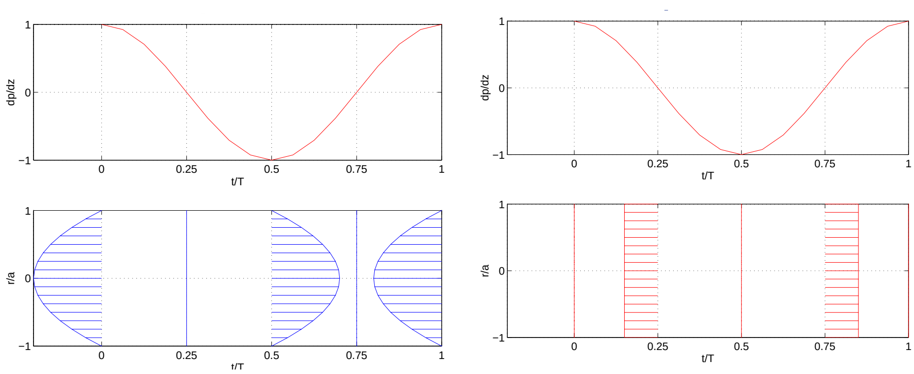
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\hat{v}}{\partial r}\right) = \frac{\partial\hat{p}}{\partial z} \tag{34}$$

with solution

$$\hat{v} = \frac{\partial\hat{p}}{\partial z}\left(1 - r^2\right) \tag{35}$$

and in the time domain:

$$v(r,t) = Re\left(-\frac{1}{4\mu}\frac{\partial p}{\partial z}(a^2 - r^2)\right) \tag{36}$$



Velocity profile for friction dominated straight pipe flow (lower left panel) and inertia dominated flow (lower right panel). Imposed pressure gradient (upper panel).

For large Womersley numbers ($\alpha \to \infty$) Eq. (31) reduces to:

$$\hat{v}(r) = \frac{i}{\omega\alpha^2}\frac{\partial \hat{p}}{\partial z} \tag{37}$$

with solution

$$i\omega\alpha^2 \hat{v}(r) = -\frac{\partial \hat{p}}{\partial z} \tag{38}$$

and in the time domain

$$v(t) = Re\left(\frac{i}{\rho\omega}\frac{\partial p}{\partial z}\right) \tag{39}$$

Exercise 4 — The Windkessel-model

1. Derive the Windkessel-model for the cardiovascular system.
2. Derive the impedance for the Windkessel-model (modulus and phase).
3. Compare the Windkessel-model impedance with the arterial input impedance.

Answer (Exercise 4) — 1. In the Windkessel model the flow in the aorta is assumed to have two parallel components: one which distends the arteries Q_a and another which flows to the periphery Q_p. The total arterial compliance is:

$$C = \frac{\partial V}{\partial p} \tag{40}$$

The flow which distends the arteries is then:

$$Q_a = \frac{\partial V}{\partial p}\frac{\partial p}{\partial t} = C\frac{\partial p}{\partial t} \tag{41}$$

whereas the peripheral flow is:

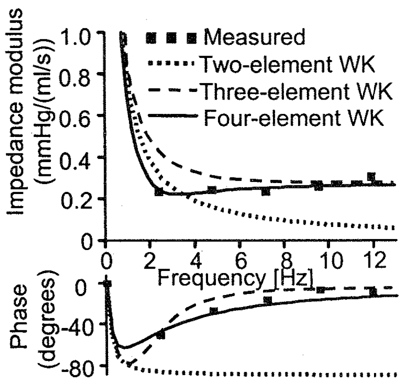
$$Q_p = \frac{p}{R_p} \tag{42}$$

where R_p denote the peripheral resistance. From conservation of mass we get:

$$Q = Q_a + Q_p = C\frac{\partial p}{\partial t} + \frac{p}{R_p} \tag{43}$$

2. The impedance of the Windkessel is obtained by introducing Fourier-components for pressure and flow.
$$p = \hat{p}\, e^{j\omega t}, \quad Q = \hat{Q}\, e^{j\omega t} \tag{44}$$
which by substitution in Eq. (43) yield
$$\hat{Q} = j\,\omega C\,\hat{p} + \frac{\hat{p}}{R_p} \tag{45}$$
from which we find the impedance:
$$Z = \frac{\hat{p}}{\hat{Q}} = \frac{R_p}{1 + j\,\omega R_p C} = \frac{R_p}{1 + (\omega R_p C)^2}\,(1 - j\,\omega R_p C) \tag{46}$$
$$|Z| = \frac{R_p}{1 + (\omega R_p C)^2}\sqrt{1 + (\omega R_p C)^2} = \frac{R_p}{\sqrt{1 + (\omega R_p C)^2}} \tag{47}$$
$$\angle Z = -\arctan \omega R_p C \tag{48}$$

3. Limitations of the Windkessel:
   - At high frequencies $|Z| \to 0$, not Z_c
   - At high frequencies $\angle Z \to -90°$, not 0
   - High frequency information not captured



Impedance for the aorta and various lumped models.