Fluid mechanics

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Outline

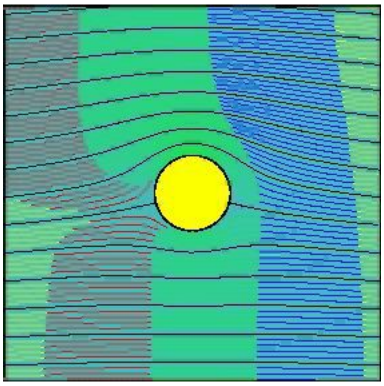
Introduction

- A fluid is a material that deforms continuously when subjected to anisotropic stress
- Anisotropy implies shear stresses
- The fluid may be at rest for isotropic stress states

$$\mathbf{T} = -p\,\mathbf{1}, p = p(\rho, \theta)$$

- Eulerian coordinates due to large displacements and chaotic motion of the individual fluid particles
- The velocity vector $\mathbf{v}(\mathbf{r}, t)$ is the primary kinetic property

Fundamental concepts in fluid mechanics



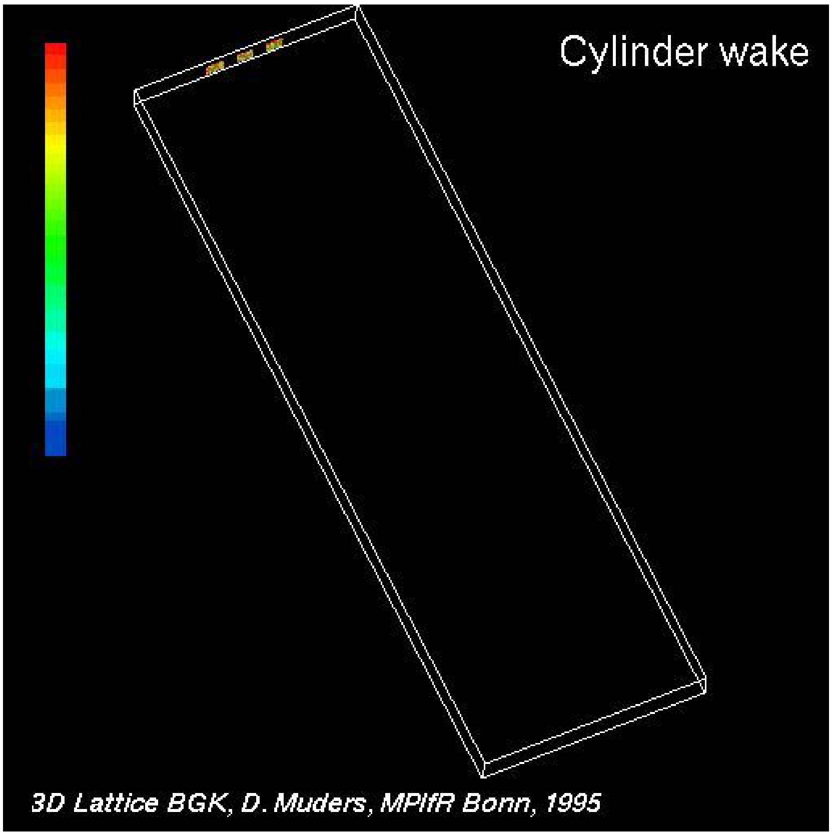
- *Stream lines*
  - Vector lines to the velocity field
  - Changes with time for non-steady flow $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$
  - Coincide with particle trajectories (path lines) for *steady flow* ($\mathbf{v} = \mathbf{v}(\mathbf{r})$)
  - $d\mathbf{r} \times \mathbf{v}(\mathbf{r}, t) = 0$
- Path lines (particle trajectories)

$$\dot{\mathbf{r}} = \mathbf{v}(\mathbf{r}, t)$$

- Vorticity field $\mathbf{c} = \nabla \times \mathbf{v}$
- Potential flow

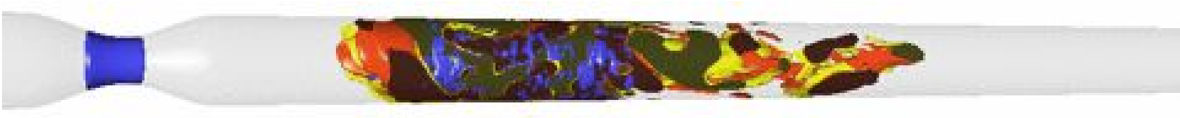
$$\mathbf{v} = \nabla\phi, \quad \phi = \phi(\mathbf{r}, t) \quad \Leftarrow \nabla \times \mathbf{v} = 0$$

Cylinder wake simulation



Stenosis

- Stenosis 75% occlusion
- Reynolds number 400, Womersley number 15.85
- Vortex rings that are blown out of the stenosis with each pulse, tilt slightly forward, then backward, on successive periods
- Self-induction and wall interaction of the vorticity in the ring, this tilting proceeds rapidly to an energetic breakdown.



- HM Blackburn, Monash University, Australia

The Reynolds number

- Solutions to the governing equations may not be unique
- Increasing velocities in a pipe will eventually produce chaotic unsteady flow
- The Reynolds number predicts transition from *laminar* to *turbulent* flow

$$Re = \frac{\rho \bar{v} d}{\mu}, \quad \bar{v} = \frac{Q}{A}$$

  - ρ density
  - μ viscosity
  - d pipe diameter
  - Q flow rate $[m^3/s]$
  - A pipe cross section
- Turbulent flow when $Re > 2000$

Illustration of blood cells in vessel

Reynolds Transport Theorem

$$\dot{B} = \int\limits_{V(t)} \frac{\partial b}{\partial t}\, dV + \int\limits_{A(t)} b\,(\mathbf{v}\cdot\mathbf{n})\, dA$$

- As the derivative is inside the integral sign we assume $V(t) = V = \text{constant}$
- An equivalent representation is therefore

$$\dot{B} = \frac{d}{dt}\int\limits_{V} b\, dV + \int\limits_{A} b\,(\mathbf{v}\cdot\mathbf{n})\, dA$$

Conservation of mass

- Derived with $b = \rho$ in Reynolds Transport Theorem

$$\dot{m} = \frac{d}{dt}\int\limits_{V(t)} \rho(\mathbf{r},t)\, dV = \int\limits_{V} \frac{\partial \rho}{\partial t}\, dV + \int\limits_{A} \rho\,(\mathbf{v}\cdot\mathbf{n})\, dA = 0$$

- Field equation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

- On component form

$$\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i} = 0$$

Equivalent equations for conservation of mass

- Material derivative
  - $\dot{f} = \frac{\partial f}{\partial t} + f_{,i} v_i = \frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i}$
  - Velocity $\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial \mathbf{r}(X,t)}{\partial t}$ with $\mathbf{r} = [x_1, x_2, x_3]$
- Conservative formulation

$$\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i} = 0 \quad \Leftrightarrow \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

- Non-conservative formulation

$$\frac{\partial \rho}{\partial t} + \rho_{,i} v_i + \rho v_{i,i} = 0 \quad \Leftrightarrow \quad \dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0$$

Mass conservation for incompressible flow

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

- $\rho =$ constant

$$\nabla \cdot \mathbf{v} = 0 \quad \Leftrightarrow \quad v_{i,i} = 0$$

Eulerian fluid = perfect fluid

- Shear stresses due to a viscous fluid may be neglected in many situations
- Boundary layer (BL) analysis near the solid surfaces may be sufficient
- Outside the BL, the fluid may be taken as inviscid
- Constitutive equation (material)
  - $\mathbf{T} = -p\,\mathbf{1}, \quad T_{ij} = -p\,\delta_{ij}$
  - $p = p(\rho, \theta)$
  - $\rho(\mathbf{r}, t)$ is the fluid density
  - $\theta(\mathbf{r}, t)$ is the fluid temperature
- Thermoelastic material as $p = p(\rho, \theta)$

Incompressible Eulerian fluids

- Compressibility may often be disregarded
- Liquids are rarely considered compressible
- Gases may also often be rendered incompressible ($v < c/3$)
- The pressure is no longer a state variable: $p = p(\mathbf{r}, t)$
- Pressure must be found from boundary conditions

Linear momentum with Reynolds and Gauss

$$\begin{aligned}
\frac{d}{dt}\int_V v_i\rho\, dV &= \int_V \frac{\partial \rho v_i}{\partial t} + \frac{\partial}{\partial x_k}(\rho v_i v_k)\, dV \\
&= \int_V \rho\frac{\partial v_i}{\partial t} + v_i\frac{\partial \rho}{\partial t} + v_i\frac{\partial}{\partial x_k}(\rho v_k) + \rho v_k\frac{\partial v_i}{\partial x_k} \\
&= \int_V \rho\underbrace{\left(\frac{\partial v_i}{\partial t} + v_k\frac{\partial v_i}{\partial x_k}\right)}_{\dot{v}_i} + v_i\underbrace{\left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho v_k)\right)}_{=0 \quad \text{masscons}} dV \\
&= \int_V \dot{v}_i\, \rho dV
\end{aligned}$$

- Consequently:

$$\frac{d}{dt}\int_V \mathbf{v}\rho\, dV = \int_V \dot{\mathbf{v}}\rho\, dV$$

Equations of motion for Eulerian fluids

- From Cauchy's equations of motion

$$\frac{d}{dt}\int_V \mathbf{v}\rho\, dV = \int_V \dot{\mathbf{v}}\rho\, dV = \int_V \nabla\cdot\mathbf{T} + \mathbf{b}\rho\, dV$$

- General field equations

$$\dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{1}{\rho}\nabla\cdot\mathbf{T} + \mathbf{b}$$

- The constitutive equation for a perfect fluid $T_{ij} = -p\delta ij$

$$T_{ik,k} = -\frac{\partial p\delta_{ik}}{\partial x_k} = -\frac{\partial p}{\partial x_k}\delta_{ik} = -p_{,i}$$

- The Euler equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{b}$$

The governing equations for Eulerian fluids

- The Euler equations (momentum equations)

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{b}$$

- Conservation of mass (continuity)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

- 4 equations with 6 unknowns $\mathbf{v}, p, \rho, \theta$
- Close equation system with an energy equation and a state equation $p = p(\rho, \theta)$

Equations of state

- Ideal gas $p = R\rho\theta$
- Polytropic process

$$p = p_0 \left(\frac{\rho}{\rho_0}\right)^{\alpha}$$

  - α constant
  - p_0 and ρ_0 are reference values in K_0
- Various processes
  - Isobaric process $\Leftrightarrow$ constant pressure $(\alpha = 0)$
  - Isothermal process $\Leftrightarrow$ constant temperature $(\alpha = 1)$
  - Isentropic process $\Leftrightarrow$ constant entropy $(\alpha = \kappa = c_p/c_\mu)$
  - Isochoric process $\Leftrightarrow$ $\nabla \cdot \mathbf{v} = 0$ and $(\alpha = \infty)$

Elastic fluid

- Barotropic if $p = p(\rho)$ and $\rho = \rho(p)$ are one-to-one relations
- Elastic fluid if barotropic and inviscid
- An elastic fluid is hyperelastic

Sound waves

- Sound propagate as elastic waves
- The elastic waves correspond to small variations in pressure
- Threshold of pain is 28 Pa
- Threshold of hearing: $2 \cdot 10^{-5}$ Pa
- Atmospheric pressure: $p_o = 1.01 \cdot 10^5$ Pa
- Mathematically described by the governing equations for Eulerian fluids and $p = p(\rho)$

Sound wave equations

- Velocity of sound (wave speed) $c^2 = \frac{dp}{d\rho}|_{\rho=\rho_0}$
- Introduce perturbations in governing equations for perfect fluids

$$\begin{aligned} p &= p_o + \tilde{p} = p_0 + c^2\,\tilde{\rho} \\ \rho &= \rho_0 + \tilde{\rho} \end{aligned}$$

- Linearized Euler equations

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} &= -\frac{1}{\rho_0}\nabla \tilde{p} \\ \frac{\partial \tilde{\rho}}{\partial t} &= -\rho_0 \nabla \cdot \mathbf{v} \end{aligned}$$

Sound wave equations II

- From perturbations: $c^2 = \frac{\tilde{p}}{\tilde{\rho}}$
- Linearized Euler equations

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} &= -\frac{1}{\rho_0}c^2\nabla \tilde{\rho} \\ \frac{\partial^2 \tilde{\rho}}{\partial t^2} &= -\rho_0 \nabla \cdot \frac{\partial \mathbf{v}}{\partial t} = \nabla \cdot \left(c^2 \nabla \tilde{\rho}\right) \end{aligned}$$

- The canonical linear wave equation results

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} = c^2 \nabla^2 \tilde{\rho}$$

- In air c = 340 m/s, in water c = 1460 m/s

Summary

- Fluid deforms continuously when subjected to anisotropic stress
- Governing equations for invisicd fluids
- Sound waves