Outline

- Reynolds transport theorem
- Conservation of mass
- Euler's axioms
- Equations of motion

Governing equations in mechanics

- The equations
  - Conservation of mass
  - Equations of motion
  - Energy equation
- Formulated for material bodies in motion (i.e. V(t))
- In fluid mechanics it is more convenient to work with control volumes, i.e. a volume independent of time (V=const)
- Reynolds transport theorem is called for

Extensive and intensive properties

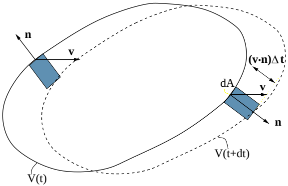
- Extensive property
  - A physical property
  - A function of the volume or mass of a body
- Intensive property
  - Physical property independent of body volume or mass
  - Density when given per unit volume
  - Specific property when given per unit mass
- Examples
  - Linear momentum: $\mathbf{p} = \int_V \mathbf{v}\, \rho\, dV$
  - Kinetic energy: $K = \int_V \frac{\mathbf{v} \cdot \mathbf{v}}{2}\, \rho\, dV$
  - Internal energy: $E = \int_V \epsilon\, \rho\, dV$
- General form for *extensive quantity*: $B(t) = \int_{V(t)} \beta\, \rho\, dV$
- Specific intensive property $\beta(\mathbf{r}, t)$
- $\beta(\mathbf{r}, t)$ is a property per unit mass

Derivation Reynolds Transport Theorem

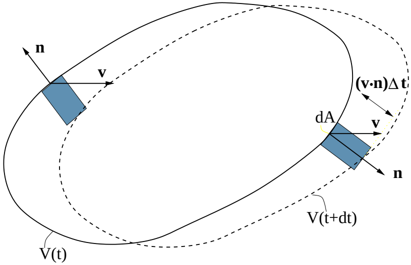
- Density b of a property B:
  $B(t) = \int\limits_{V(t)} b(\mathbf{r}, t)\, dV$
- Canonical definition of a derivative

$$\dot{B} = \lim_{\Delta t \to 0} \frac{B(t + \Delta t) - B(t)}{\Delta t}$$

- A closer look at $B(t + \Delta t)$



A closer look at $B(t + \Delta t)$



$$
\begin{aligned}
B(t + \Delta t) &= \int_{V(t+\Delta t)} b(\mathbf{r}, t + \Delta t)\, dV \\
&= \int_{V(t)} b(\mathbf{r}, t + \Delta t)\, dV + \int_{\Delta V} b(\mathbf{r}, t + \Delta t)\, dV
\end{aligned}
$$

$$
\int_{\Delta V} b(\mathbf{r}, t + \Delta t)\, dV = \int_{A(t)} b(\mathbf{r}, t + \Delta t)\, (\mathbf{v} \cdot \mathbf{n}\, \Delta t)\, dA
$$

A closer look at $B(t + \Delta t)$ (cont)

By collection terms:

$$\frac{B(t + \Delta t) - B(t)}{\Delta t} = \int\limits_{V(t)} \frac{b(\mathbf{r}, t + \Delta t) - b(\mathbf{r}, t)}{\Delta t} \, dV + \int\limits_{A(t)} b(\mathbf{r}, t + \Delta t) \, (\mathbf{v} \cdot \mathbf{n}) \, dA$$

By letting $\Delta t \to 0$ the Reynolds Theorem is obtained

$$\dot{B} = \int\limits_{V(t)} \frac{\partial b}{\partial t} \, dV + \int\limits_{A(t)} b \, (\mathbf{v} \cdot \mathbf{n}) \, dA$$

Reynolds Transport Theorem

$$\dot{B} = \int\limits_{V(t)} \frac{\partial b}{\partial t}\, dV + \int\limits_{A(t)} b\, (\mathbf{v} \cdot \mathbf{n})\, dA$$

- As the derivative is inside the integral sign we may now assume
  $V(t) = V = \text{constant}$
- An equivalent representation is therefore

$$\dot{B} = \frac{d}{dt} \int\limits_{V} b\, dV + \int\limits_{A} b\, (\mathbf{v} \cdot \mathbf{n})\, dA$$

Material derivative of an extensive property

- The governing equations contain material derivatives of extensive properties
- It may be shown based on conservation of mass that

$$\dot{B} = \frac{d}{dt} \int\limits_{V(t)} \beta \, \rho \, dV = \int\limits_{V(t)} \dot{\beta} \, \rho \, dV$$

- Transform equations to apply for a *fixed region in space* in fluid mechanics for convenience
- A *control volume* is such a fixed region

Conservation of mass

- The conservation of mass equation is obtained by setting $b = \rho$ in Reynolds Transport Theorem
$$\dot{m} = \frac{d}{dt} \int\limits_{V(t)} \rho(\mathbf{r}, t)\, dV = \int\limits_{V} \frac{\partial \rho}{\partial t}\, dV + \int\limits_{A} \rho\, (\mathbf{v} \cdot \mathbf{n})\, dA = 0$$

- By using the Gauss' theorem for the surface integral:
$$\dot{m} = \int\limits_{V} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})\, dV = 0$$

Differential form for mass conservation

- The integral form (control volume) formulation

$$\dot{m} = \int\limits_V \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})\, dV = 0$$

- Must be valid for arbitrary V thus the integrand must be zero

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

- On component form

$$\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i} = 0$$

Mass conservation for incompressible flow

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

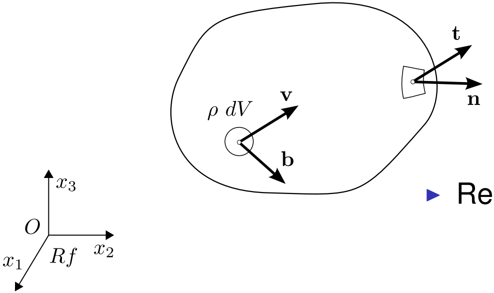
- $\rho =$ constant

$$\nabla \cdot \mathbf{v} = 0 \quad \Leftrightarrow \quad v_{i,i} = 0$$

Equations of motion

- Newton's laws are valid for the motion of a mass particle
- Cannot directly be transferred to a body of continuously distributed matter
- Two axioms are postulated in Continuum Mechanics
  - Euler's first axiom: Law of balance of linear momentum
  - Euler's second axiom: Law of balance of angular momentum

Forces and moments



- Body forces **b** per unit mass
- Contact force **t** per unit area
- Particle velocity **v**

- Resultant force

$$\mathbf{f} = \int_A \mathbf{t}\, dA + \int_V \mathbf{b}\, \rho\, dV$$

- Resultant moment $\mathbf{m}_O$

$$\mathbf{m}_O = \int_A \mathbf{r} \times \mathbf{t}\, dA + \int_V \mathbf{r} \times \mathbf{b}\, \rho\, dV$$

Law of balance of linear momentum

- Linear momentum $\mathbf{p} = \int_V \mathbf{v}\rho \, dV$
- Euler's first axiom: *The rate of change of the linear momentum for a continuum volume equals the resultant force*

$$\dot{\mathbf{p}} \equiv \int_V \dot{\mathbf{v}}\rho \, dV = \mathbf{f}$$

- Equivalent form

$$\int_V \dot{\mathbf{v}}\rho \, dV = \int_A \mathbf{t} \, dA + \int_V \mathbf{b} \, \rho \, dV$$

Relation to Newton's second law

- Center of mass $\mathbf{r}_C = \frac{1}{m} \int_V \mathbf{r} \, \rho \, dV$
- Velocity and acceleration of the center of mass

$$
\begin{aligned}
\mathbf{v}_C &= \dot{\mathbf{r}}_C = \frac{1}{m} \int_V \dot{\mathbf{r}} \, \rho \, dV = \frac{1}{m} \int_V \mathbf{v} \, \rho \, dV \\
\mathbf{a}_C &= \dot{\mathbf{v}}_C = \ddot{\mathbf{r}}_C = \frac{1}{m} \int_V \dot{\mathbf{v}} \, \rho \, dV = \frac{1}{m} \int_V \mathbf{a} \, \rho \, dV
\end{aligned}
$$

- Linear momentum: $\mathbf{p} = \int_V \mathbf{v} \rho \, dV = m \mathbf{v}_C$
- Euler's first axiom yields Newton's second law for the center of mass

$$\mathbf{f} = m \mathbf{a}_C$$

Law of balance of angular momentum

- Angular momentum $\mathbf{l}_O = \int_V \mathbf{r} \times \mathbf{v} \rho \, dV$
- Euler's second axiom: *The rate of change of the angular momentum for a continuum volume equals the resultant moment*

$$\dot{\mathbf{l}}_O = \int_V \mathbf{r} \times \dot{\mathbf{v}} \rho \, dV$$

- Equivalent form:

$$\int_V \mathbf{r} \times \dot{\mathbf{v}} \rho \, dV = \int_A \mathbf{r} \times \mathbf{t} \, dA + \int_V \mathbf{r} \times \mathbf{b} \, \rho \, dV$$

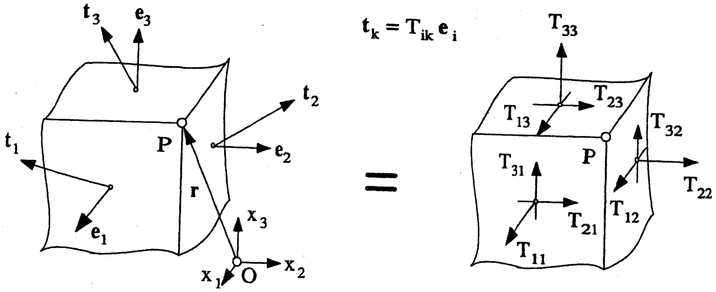
Vector relation for simplification in angular momentum expression

- Outset
$$\mathbf{r} \dot{\times} \mathbf{v} = \dot{\mathbf{r}} \times \mathbf{v} + \mathbf{r} \times \dot{\mathbf{v}}$$
- We have
$$\dot{\mathbf{r}} = \mathbf{v} \quad \text{and} \quad \mathbf{v} \times \mathbf{v} = 0$$
- Thus
$$\mathbf{r} \dot{\times} \mathbf{v} = \mathbf{r} \times \dot{\mathbf{v}} = \mathbf{r} \times \mathbf{a}$$

Coordinate stresses

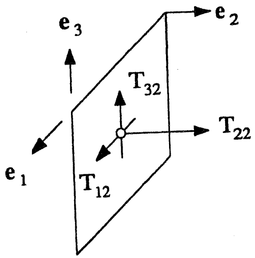
- Stress $\mathbf{t}_k$ on surfaces with unit normals $\mathbf{e}_k$
- Components of the stress vectors are denoted T_{ik}

$$\mathbf{t}_k = T_{ik}\mathbf{e}_i \quad \Leftrightarrow \quad \mathbf{e}_i \cdot \mathbf{t}_k = T_{ik}$$



Coordinate stresses

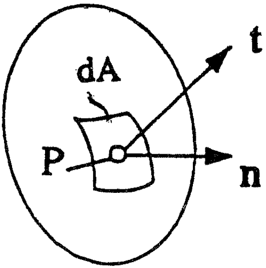
- T_{ik} is the coordinate stresses in the particle P
- T_{ik} are elements of the stress matrix T in P with respect to Ox
- Normal stresses: T_{11}, T_{22}, T_{33}
- Shear stresses: T_{ik}, when $i \neq k$
- A positive coordinate stress acts in the direction of the positive coordinate axis on that side of the surface facing the positive direction of a coordinate axis



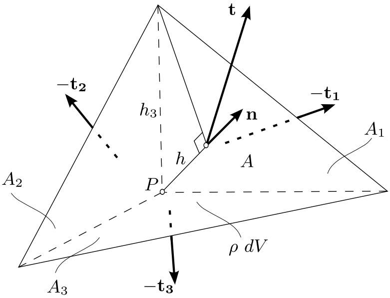
Cauchy's stress theorem and the stress tensor

- Consider an abitrary plane through P
- Normal vector $\mathbf{n} = n_k \mathbf{e}_k$
- Stress vector $\mathbf{t} = t_i \mathbf{e}_i$
- Coordinate stresses T_{ik}
- Cauchy's stress theorem

$$t_i = T_{ik} n_k \Leftrightarrow \mathbf{t} = \mathbf{T}\mathbf{n}$$



Proof of Cauchy's stress theorem



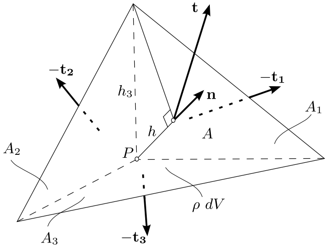
- Balance of linear momentum

$$\int_V \mathbf{a}\rho \, dV = \sum_{k=1}^{3} \int_{A_k} -\mathbf{t}_k \, dA + \int_A \mathbf{t} \, dA + \int_V \mathbf{b}\rho \, dV$$

- Let $\mathbf{t}_k$, $\mathbf{t}$, $\mathbf{b}\rho$ and $\mathbf{a}\rho$ represent mean values

$$-\mathbf{t}_k A_k + \mathbf{t}A + \mathbf{b}\rho V = \mathbf{a}\rho V$$

Geometric relations



- h_k denote edges $\parallel$ to base vectors $\mathbf{e}_k$
- As $\mathbf{n}$ is a unit vector: $n_k = \frac{h}{h_k}$
- The volume V may be expressed

$$V = \frac{1}{3}Ah = \frac{1}{3}A_1 h_1 = \frac{1}{3}A_2 h_2 = \frac{1}{3}A_3 h_3$$

- Which together yields:
  $A_k = An_k \quad \text{and} \quad V = \frac{1}{3}Ah$

Substitution in to averaged equation

- Substitution of $A_k = An_k \quad \text{and} \quad V = \frac{1}{3}Ah$

$$
\begin{aligned}
-\mathbf{t}_k A_k + \mathbf{t}A + \mathbf{b}\rho V &= \mathbf{a}\rho V \\
-\mathbf{t}_k n_k + \mathbf{t} + \mathbf{b}\rho\frac{h}{3} &= \mathbf{a}\rho\frac{h}{3}
\end{aligned}
$$

- Let $h \to 0 \quad \Rightarrow \quad \mathbf{t} = \mathbf{t}_k n_k$
- From before $\mathbf{t}_k = T_{ik}\mathbf{e}_i$
- Which yields

$$
\mathbf{t} = T_{ik} n_k \mathbf{e}_i \Leftrightarrow t_i = T_{ik} n_k
$$

Cauchy's stress theorem

- The stress vector $\mathbf{t}$ on a surface is uniquely determined by the stress tensor $\mathbf{T}$ and the unit normal $\mathbf{n}$

$$\mathbf{t} = \mathbf{T} \cdot \mathbf{n}$$

- Normal stress $\sigma = \mathbf{n} \cdot \mathbf{t} = \mathbf{n} \cdot \mathbf{T} \cdot \mathbf{n} = n_i T_{ik} n_k$
- Shear stress $\tau = |\mathbf{n} \times \mathbf{t}| = \sqrt{\mathbf{t} \cdot \mathbf{t} - \sigma^2}$
- Stress tensor T and its components in a general Cartesian coordinate system Ox

$$T_{ik} = \mathbf{e}_i \cdot \mathbf{T} \cdot \mathbf{e}_k$$

Cauchy equations of motion

- Euler's axioms are equations of motion for a *body* of continuous material
- From these laws we derive field equations valid for *particles* in a continuum

Derivation of Cauchy's equations of motion

- The balance of linear momentum

$$\int_V \dot{\mathbf{v}}\rho \, dV = \int_A \mathbf{t} \, dA + \int_V \mathbf{b} \, \rho \, dV$$

- Cauchy's stress theorem $\mathbf{t} = \mathbf{T} \cdot \mathbf{n}$ and Gauss' integration theorem

$$\int_V \dot{\mathbf{v}}\rho \, dV = \int_V \nabla \cdot \mathbf{T} + \mathbf{b}\rho \, dV$$

- Integrand must be zero for validity for abitrary V

$$\rho a_i = T_{ik,k} + \rho b_i$$

Symmetry of stress tensor

- From balance of angluar/linear momentum one may show:

$$T_{ij} = T_{ji} \quad \Leftrightarrow \quad \mathbf{T}^T = \mathbf{T}$$

Summary

- Reynolds transport theorem
- Conservation of mass
- Euler's axioms
- Equations of motion