Anisotropic Materials

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Elasticity

- A material is (Cauchy) elastic if

$$\mathbf{T} = \mathbf{T}(\mathbf{E}, \mathbf{r})$$

- which is a *constitutive* or *material* equation
- Homogenous if the elastic properties are the same *in every particle*

$$\mathbf{T} = \mathbf{T}(\mathbf{E})$$

- Isotropic if the elastic propreties are the same *in every direction*
- Linear elastic if stress is linear function of strain

Isotropic linear elastic material

- Uniaxial stress ($\sigma_1 \neq 0, \sigma_2 = \sigma_3 = 0$)

$$\epsilon_1 = \frac{\sigma_1}{\eta}, \quad \epsilon_2 = \epsilon_2 = -\nu \frac{\sigma_1}{\eta}$$

η modulus of elasticity
ν Poisson's ratio

- Superposition valid due to isotropic and linear stress/strain relationship

$$\epsilon_i = \frac{1+\nu}{\eta}\,\sigma_i - \frac{\nu}{\eta}\,(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1+\nu}{\eta}\,\sigma_i - \frac{\nu}{\eta}\ \mathrm{Tr}\mathbf{T}$$

The generalized Hooke's law

- Matrix representation in Ox-system with base vectors $\parallel$ to principal directions

$$\epsilon_i \, \delta_{ij} = \frac{1+\nu}{\eta} \, \sigma_i \, \delta_{ij} - \frac{\nu}{\eta} \; \mathrm{Tr}\mathbf{T} \, \delta_{ij}$$

- In an arbitrary Ox-system

$$E_{ij} = \frac{1+\nu}{\eta} \, T_{ij} - \frac{\nu}{\eta} \, T_{kk} \, \delta_{ij}$$

- Tensor representation

$$\mathbf{E} = \frac{1+\nu}{\eta} \mathbf{T} - \frac{\nu}{\eta} \, \mathrm{Tr}\mathbf{T} \, \mathbf{1}$$

Equivalent forms of Hooke's law

- Strain on LHS

$$\begin{aligned} E_{ij} &= \frac{1+\nu}{\eta}\, T_{ij} - \frac{\nu}{\eta}\, T_{kk}\, \delta_{ij} \\ \mathbf{E} &= \frac{1+\nu}{\eta} \mathbf{T} - \frac{\nu}{\eta}\, \mathrm{Tr} \mathbf{T}\, \mathbf{1} \end{aligned}$$

- Stress on LHS

$$\begin{aligned} T_{ij} &= \frac{\eta}{1+\nu} \left(E_{ij} + \frac{\nu}{1-2\nu}\, E_{kk}\, \delta_{ij} \right) \\ \mathbf{T} &= \frac{\eta}{1+\nu} \left(\mathbf{E} + \frac{\nu}{1-2\nu}\, E_{kk}\, \mathbf{1} \right) \end{aligned}$$

Notation for anisotropic materials

- Angular momentum $\Rightarrow \mathbf{T} = \mathbf{T}^T$
- Green's strain $\mathbf{E} \equiv \mathbf{E}^T \Leftarrow \mathbf{E} \equiv \mathbf{H} + \mathbf{H}^\mathbf{T} + \mathbf{H}^\mathbf{T}\mathbf{H}$
- Only 6 distinct values for $\mathbf{T}$ and $\mathbf{E}$
- Special notation for coordinate stresses and strains

$$T = \begin{bmatrix} T_1 & T_6 & T_5 \\ \vdots & T_2 & T_4 \\ \ldots & \ldots & T_3 \end{bmatrix} \qquad E = \begin{bmatrix} E_1 & E_6 & E_5 \\ \vdots & E_2 & E_4 \\ \ldots & \ldots & E_3 \end{bmatrix}$$

Anisotropic constitutive equation

- A fully anisotropic, linearly elastic constitutive equation

$$T_\alpha = S_{\alpha\,\beta} E_\beta, \qquad \{\alpha, \beta\} \in \{1 \dots 6\}$$

- T_α and E_β is 6×1 vector matrices
- $S_{\alpha\,\beta}$ is a 6×6 *elasticity* or *stiffness* matrix
- We will show $S = S^T$
- i.e. only 21 independent stiffnesses for full anisotropy
- Alternative formulation

$$E_\alpha = K_{\alpha\,\beta} T_\beta, \qquad K = S^{-1}$$

- $K_{\alpha\,\beta}$ is a 6×6 *compliance* or *flexibility* matrix

Stiffness matrix for Hookean materials

- Isotropic, linarly elastic material

$$S = \frac{\eta}{2(1+\nu)(1-2\nu)}$$
$$\begin{bmatrix} 2(1-\nu) & 2\nu & 2\nu & 0 & 0 & 0 \\ 2\nu & 2(1-\nu) & 2\nu & 0 & 0 & 0 \\ 2\nu & 2\nu & 2(1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu) \end{bmatrix}$$

- Note: $S = S^T$

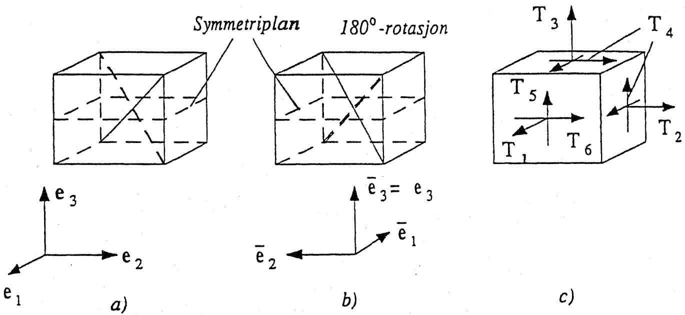
Symmetry of stiffness matrix

- Hyperelastic material
$$\mathbf{T} = \frac{\partial \phi}{\partial \mathbf{E}} \Leftrightarrow T_{ij} = \frac{\partial \phi}{\partial E_{ij}}$$
$$\Leftrightarrow T_\alpha = \frac{\partial \phi}{\partial E_\alpha}$$
  - The stress tensor $\mathbf{T}$
  - Elastic energy ϕ per unit volume
- Linearly anisotropic material
$$T_\alpha = S_{\alpha\beta} E_\beta$$
- Hyperelastic linearly anisotropic material
$$\frac{\partial T_\alpha}{\partial E_\beta} = S_{\alpha\beta} = \frac{\partial^2 \phi}{\partial E_\alpha \partial E_\beta} \equiv \frac{\partial^2 \phi}{\partial E_\beta \partial E_\alpha} = S_{\beta\alpha}$$

$$\Rightarrow S_{\alpha\beta} = S_{\beta\alpha} \Leftrightarrow S = S^T$$

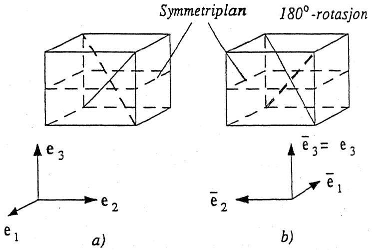
- Independent stiffnesses reduces from 36 to 21

Materials with one plane of symmetry



- Structure symmetric with respect to a plane through the particle
- The mirror image of the structure is identical to the structure itself
- The number of stiffnesses is reduced from 21 to 13

Materials with one plane of symmetry (contd)



- The fig a) shows a plane of symmetry normal to $\mathbf{e}_3$
- The fig b) shows 180° rotation about the $\mathbf{e}_3$-axis
- A state of strain E will produce the
- $\Rightarrow \bar{S} = S$
- The number of stiffnesses is reduced from 21 to 13

π-rotation about the $\mathbf{e}_3$-axis

$$T = \begin{bmatrix} T_1 & T_6 & T_5 \\ \vdots & T_2 & T_4 \\ \cdots\cdots & & T_3 \end{bmatrix} \qquad E = \begin{bmatrix} E_1 & E_6 & E_5 \\ \vdots & E_2 & E_4 \\ \cdots\cdots & & E_3 \end{bmatrix}$$

Transformation matrix

$$Q = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q_\pi \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{T} = Q^T T Q = \begin{bmatrix} T_1 & T_6 & -T_5 \\ \vdots & T_2 & -T_4 \\ \cdots\cdots & & T_3 \end{bmatrix}, \quad \bar{E} = Q^T E Q = \begin{bmatrix} E_1 & E_6 & -E_5 \\ \vdots & E_2 & -E_4 \\ \cdots\cdots & & E_3 \end{bmatrix}$$

Plane symmetric anisotropic constitutive equations

- $T_\alpha = S_{\alpha\,\beta} E_\beta$
- $\bar{T}_\alpha = \bar{S}_{\alpha\,\beta} \bar{E}_\beta$
- Plane symmetry $\Rightarrow \bar{S}_{\alpha\,\beta} = S_{\alpha\,\beta}$
- Notation λ= 4 and 5, and $\rho,\ \gamma$ = 1 $\ldots$ 3 and 6
- From the constitutive equations

$$
\begin{aligned}
T_\lambda &= S_{\lambda\rho} E_\rho + S_{\lambda 4} E_4 + S_{\lambda 5} E_5 \\
-T_\lambda &= S_{\lambda\rho} E_\rho + S_{\lambda 4}(-E_4) + S_{\lambda 5}(-E_5) \\
T_\gamma &= S_{\gamma\rho} E_\rho + S_{\gamma 4} E_4 + S_{\gamma 5} E_5 \\
T_\gamma &= S_{\gamma\rho} E_\rho + S_{\gamma 4}(-E_4) + S_{\gamma 5}(-E_5)
\end{aligned}
$$

- By comparison $S_{\lambda\,\rho} = 0$ and $S_{\gamma\lambda} = 0$

Summary

- Anisotropic formulations
- Stiffness matrix
- Number of material parameters
- Isotropy