Exercise 1: Flow in a compliant tube

Blood flow in arteries may be thought of as blood flowing through compliant tubes, i.e. the artereis expand or contract based on the pressure gradient across the arterial wall.

a) Using basic principles, derive the governing equations for a compliant tube

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0$$
$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) = -\frac{A}{\rho}\frac{\partial p}{\partial x} + \frac{\pi D}{\rho}\tau$$

Hint. Note the key principles are conservation of momentum, and conservation of mass. Consider using the average velocity, $\bar{v}$, over the cross sectional area. For simplicity assume a flat velocity profile, i.e $v(\mathbf{r}) = \bar{v}$.

Solution. Mass conservation implies that

$$\rho\dot{V} = \rho(Q_i - Q_o), \tag{1}$$

where Q_i is the flow into the control volume, and Q_o is the flow out. We define our control volume as the region between $x_1 = x$ and $x_2 = x + dx$, and moving with the vessel walls. Then

$$Q_i = A(x_1)v(x_1) \tag{2}$$

and

$$Q_o = A(x_2)v(x_2). \tag{3}$$

Now we may approximate $A(x_2) = A(x_1) + \frac{\partial A}{\partial x}dx$, and similarly $v(x_2) = \frac{\partial v}{\partial x}dx$. We the change in volume $\dot{V}$ along the length dx is simply

$$\dot{V} = \frac{\partial A}{\partial t}dx. \tag{4}$$

Substituting into (1)

$$\rho\frac{\partial A}{\partial t}dx = \rho(A(x_1)v(x_1) - (A(x_1) + \frac{\partial A}{\partial x}dx)), \tag{5}$$

which simplifies to

$$\frac{\partial A}{\partial x} = -\frac{\partial A}{\partial x}. \tag{6}$$

We approach conservation of momentum using Reynolds transport theorem to derive an expression for the time derivative of the extensive quantity of momentum. Reynolds transport theorem says that for $B = \int_V \beta\rho dV$ then

$$\dot{B} = \frac{d}{dt}\int_{V_c(t)} bdV + \int_{A_c} b(\mathbf{v} - \mathbf{v}_c)\cdot\mathbf{n}dA. \tag{7}$$

Since $\mathbf{v} = \mathbf{v}_c$ on along the vessel wall, the surface integral $\int_{A_c} b(\mathbf{v} - \mathbf{v}_c) \cdot \mathbf{n}) dA$ accounts for momentum entering the control volume from the left, P_i, and momentum leaving from the right P_o. P_i is simply the amount of momentum carried into the control volume by the flow and is thus $(m(x_1)v(x_1)) \times v(x_1)$, where $m = A\rho$, similarly $P_o = (m(x_2)v(x_2)) \times v(x_2)$. Thus we have

$$\dot{P} = \frac{d}{dt} \int_{V_c} \rho v dV + \rho A(x_1) v^2(x_1) - \rho A(x_2) v^2(x_2). \tag{8}$$

Substituting

$$\dot{P} = \frac{d}{dt} \int_{V_c} \rho v dV + \rho (A(x_1) v^2(x_1) - (A(x_1) + \frac{\partial A}{\partial x} dx)(v(x_1) + \frac{\partial v}{\partial x} dx)^2). \tag{9}$$

Dropping x_1, $(A + \frac{\partial A}{\partial x} dx)(v + \frac{\partial v}{\partial x} dx)^2 = (A + \frac{\partial A}{\partial x} dx)(v^2 + 2v \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial x}^2 dx^2)$ we ignore quadratic terms in dx and have $(A + \frac{\partial A}{\partial x} dx)(v^2 + 2v \frac{\partial v}{\partial x} dx) = Av^2 + v^2 \frac{\partial A}{\partial x} dx + 2Av \frac{\partial v}{\partial x} + 2v \frac{\partial v}{\partial x} \frac{\partial A}{\partial x} dx^2$ Once again we drop quadratic terms in dx leaving $Av^2 + v^2 \frac{\partial A}{\partial x} dx + 2Av \frac{\partial v}{\partial x} dx$. Substituting this in we have

$$\rho(Av^2 - (Av^2 + v^2 \frac{\partial A}{\partial x} dx + 2Av \frac{\partial v}{\partial x} dx) = -\rho dx (v^2 \frac{\partial A}{\partial x} + 2Av \frac{\partial v}{\partial x}) \tag{10}$$

and now in the full equation

$$\dot{P} = \frac{d}{dt} \int_{V_c} \rho v dV - \rho dx (v^2 \frac{\partial A}{\partial x} + 2Av \frac{\partial v}{\partial x}) \tag{11}$$

We now note that $Q = Av$ and $Q^2 = A^2 v^2$ now consider

$$\frac{\partial (Av^2)}{\partial x} = 2Av \frac{\partial v}{\partial x} + v^2 \frac{\partial A}{\partial x} \tag{12}$$

which is equivalent to term inside parenthesis in (12). Thus

$$\dot{P} = \frac{d}{dt} \int_{V_c} \rho v dV - \rho dx \frac{\partial}{\partial x} \left(\frac{Q^2}{A} \right) \tag{13}$$

Further

$$\frac{d}{dt} \int_{V_c} \rho v dV = \int_{x_1}^{x_2} \frac{d}{dt} \int_{A(t)} \rho v dA dx = \int_{x_1}^{x_2} \frac{d(A(t) v \rho)}{dt} dx. \tag{14}$$

Since there is no flux at the boundaries of the cross sectional area this may be simplified to

$$\frac{d}{dt} \int_{V_c} \rho v dV = \int_{x_1}^{x_2} \frac{\partial \bar{v} A(x)}{\partial t} \rho dx = \int_{x_1}^{x_2} \frac{d(A(t) v \rho)}{dt} dx, \tag{15}$$

where $\bar{v} = \frac{1}{A} \int_A v dA$ (For the remainder of this derivation we assume a flat velocity profile such that $\bar{v} = v$.)

$$\dot{P} = \int_{x_1}^{x_2} \frac{\partial \rho A v}{dt} dx - \rho dx \frac{\partial}{\partial x} \left(\frac{Q^2}{A} \right). \tag{16}$$

Now the change in momentum must equal the net applied force. At the inlet x_1 the pressure applies a force $F_i = Ap$, and at the outlet the pressure applies a force $F_o = -(A+\frac{\partial A}{\partial x}dx)(p+\frac{\partial p}{\partial x}dx)$. Along the vessel boundary pressure generates a force in the positive direction over the differential change in circumferential area $F_b = p\frac{\partial A}{\partial x}$. Thus we have a net force

$$F_{\text{net}} = pA + p\frac{\partial A}{\partial x} - (A + \frac{\partial A}{\partial x}dx)(p + \frac{\partial p}{\partial x}dx) \tag{17}$$

or expanding and dropping quadratic terms in dx

$$\begin{aligned} F_{\text{net}} =& Ap + p\frac{\partial A}{\partial x}dx - Ap - A\frac{\partial p}{\partial x}dx - p\frac{\partial A}{\partial x}dx - \frac{\partial A}{\partial x}\frac{\partial p}{\partial x}dx^2 && (18) \\ =& - A\frac{\partial p}{\partial x}dx && (19) \end{aligned}$$

Thus combining the results

$$\int_{x_1}^{x_2} \frac{\partial \rho v}{dt} A dx - \rho dx \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) = -A\frac{\partial p}{\partial x}dx \tag{20}$$

or rearranging Thus combining the results

$$\frac{\int_{x_1}^{x_2} \frac{\partial \rho v}{dt} A dx}{dx} = \rho\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) - A\frac{\partial p}{\partial x} \tag{21}$$

Taking the limit $\lim_{dx\to 0}$ we have

$$\frac{\partial \rho v}{dt} A = \rho\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) - A\frac{\partial p}{\partial x} \tag{22}$$

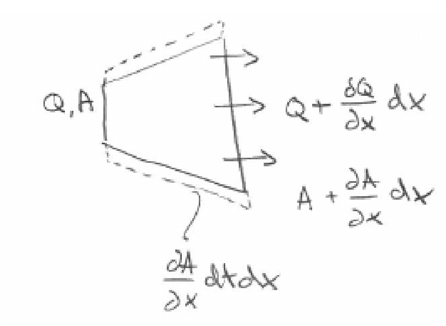

Figure 1: Fluid between two parallel plates.

b) Show that

$$\begin{aligned} p &= p_0 f(x-ct) + p_0^* g(x+ct) && (23) \\ Q &= Q_0 f(x-ct) + Q_0^* g(x+ct) && (24) \end{aligned}$$

are general solutions to the linearized and inviscid forms of the 1D governing equations for wave propagation in linearly-compliant vessels.

Hint. What components of the equations derived above may be ignored by assuming linear and inviscid flow? Recall that for a linearly compliant tube $A(p) = A_0 + C(p - p_0)$.

Solution. We drop nonlinear terms and eliminate the friction term due to the assumption of in-viscid flow. This leaves

$$\frac{\partial A}{\partial t} = -\frac{\partial Q}{\partial x}$$
$$\frac{\partial Q}{\partial t} = -\frac{A}{\rho}\frac{\partial p}{\partial x}.$$

We may close the system imposing a linear area- pressure relationship

$$A(p) = A_0 + C(p - p_0) \tag{25}$$

and thus

$$\frac{\partial A}{\partial t} = C\frac{\partial p}{\partial t}. \tag{26}$$

Substituting this in we may show the equations reduce to

$$C\frac{\partial p}{\partial t} = -\frac{\partial Q}{\partial x},$$
$$\frac{\partial Q}{\partial t} = -\frac{A}{\rho}\frac{\partial p}{\partial x}.$$

Further we may differentiate the first with respect to t and the second with respect to x to produce

$$C\frac{\partial^2 p}{\partial t^2} = -\frac{\partial^2 Q}{\partial x \partial t},$$
$$\frac{\partial^2 Q}{\partial t \partial x} = -\frac{A}{\rho}\frac{\partial^2 p}{\partial x^2}.$$

From this we see that p satisfies the wave equation

$$\frac{\partial^2 p}{\partial t^2} = \frac{A}{C\rho}\frac{\partial^2 p}{\partial x^2}.$$

Similarly we may show that

$$\frac{\partial^2 Q}{\partial t^2} = \frac{A}{C\rho}\frac{\partial^2 Q}{\partial x^2}.$$

These are the wave equations with wave speed $c = \frac{A}{C\rho}$, which we show the general forms of the solutions suggested satisfy.

$$\frac{\partial^2 p}{\partial x^2} = p_0 f''(x - ct) + p_0^* g''(x + ct)$$
$$\frac{\partial^2 p}{\partial t^2} = c^2 p_0 f''(x - ct) + c^2 p_0^* g''(x + ct)$$

Thus $c^2 \frac{\partial^2 p}{\partial x^2} = \frac{\partial^2 p}{\partial t^2}$, satisfying the differential equation.