Problem set 3 for TKT4150 Biomechanics

Exercise 1: Principal and equivalent stress in femur bone of running human

The running human from the previous exercise is investigated further. The mechanics of the femur (the thigh bone) are in the spotlight this time. Fracture stress and density of bone are both given in Figure 1. Experiments reveal that point P has the largest stress and that the yield strength of the femur is $\sigma_{\max} = 130\text{MPa}$. Preliminary calculations and experiments suggest the following stress in point P:

$$T = \begin{bmatrix} -50 & 35 & 0 \\ 35 & -70 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa} \tag{1}$$

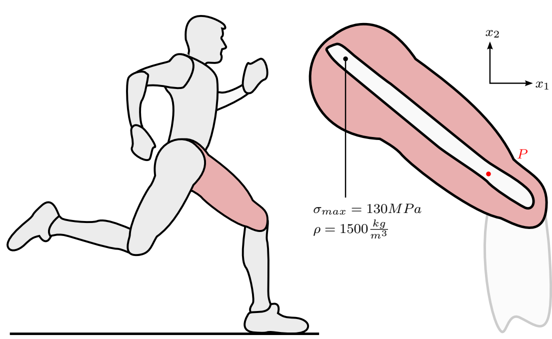

Figure 1: Running human.

a) Calculate the principal stresses in this point, and sketch them together with their corresponding angles relative to the x_1,x_2,x_3 coordinate system.

b) One way of evaluating whether a material is stressed beyond its strength is to compare the equivalent, also called von Mises, stress to the yield strength ($\sigma_{\max} = 130$MPa). The equivalent (von Mises) stress is given by

$$\sigma_{eq} = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\right]} \tag{2}$$

where σ_1, σ_2, and σ_3 are the **ordered** principal stresses.
Calculate the equivalent stress (von Mises). Does the given stress matrix lead to fracture in the femur?

Exercise 2: Deformation measures

Consider the homogenous deformation

$$x_1 = X_1 + aX_2$$
$$x_2 = (1 + a)X_2$$

where $a = 0.1$.
Draw a figure to show how the square with corners $A = (0, 0)$, $B = (1, 0)$, $C = (1, 1)$, $D = (0, 1)$ deforms.
Calculate the following deformation measures based on their geometric deformation and the figure you have drawn.

a) The stretch λ of a line element is defined as the ratio of its stretched length to unstretched length. What is the stretch for a line element along AC? What about along BD?

b) The shear strain γ is defined as the change in angle between two line elements which are perpendicular in the reference configuration, i.e. $\gamma = \frac{\pi}{2} - \alpha$ where α is the angle bewteen the deformed line elements. What is the shear between $\mathbf{e}_1$ and $\mathbf{e}_2$?

c) Compute $\mathbf{F}$ and $\mathbf{E}$ for the given deformation. Use the formulas for longitudinal and shear strain to compare to your previous answers.

d) Show that for any deformation, the longitudinal strain and the stretch in the line element aligned with the direction vector e are, respectively:

$$\varepsilon = \sqrt{1 + 2e_i E_{ij} e_j} - 1 \tag{3}$$
$$\lambda = \varepsilon + 1 \tag{4}$$

where longitudinal strain is defined as:

$$\epsilon^l = \frac{ds - ds_0}{ds_0} = \frac{ds}{ds_0} - 1 \tag{5}$$

and the stretch ratio λ is defined as

$$\lambda = \frac{ds}{ds_0} \tag{6}$$

where ds is the deformed length and ds_0 is the reference length of the material element.

Hint. Reference the section in the compendium titled "The Green strain tensor".

e) Use (3) determine the general expression for the longitudinal strain and the stretch ratio in a line element aligined with the x_k-axis of the coordinate system where k may be 1, 2, or 3.

Hint. Try to do it for the direction $\mathbf{e} = \mathbf{e}_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^\mathsf{T}$ first, then generalize the formula for arbitrary k.

Exercise 3: Laplace's law for membranes

Laplace's law states

$$\frac{\sigma_1}{r_1} + \frac{\sigma_2}{r_2} = \frac{p}{t} \tag{7}$$

where σ_i is the stress along x_i, r_i the radius of the shell in x_i-direction, p the internal pressure and t the thickness of the membrane. For a thin-walled sphere, the following equation holds:

$$\sigma = \sigma_\theta = \sigma_\phi = \frac{r}{2t}p \tag{8}$$

For a thin-walled cylinder with capped ends, we have the following equations:

$$\sigma_z = \frac{r}{2t}p, \quad \sigma_\theta = \frac{r}{t}p \tag{9}$$

a) Use Laplace's law, given in Equation (7), to derive the formula for the membrane stress in Equation (8), for a spherical membrane.

b) Use Laplace's law to derive the membrane stress σ_θ in Equation (9), for a cylindrical membrane.

c) A thin-walled cylindrical container is subjected to an internal pressure p_i. The stress σ_z, on a plane perpendicular to the axis of the cylinder is given in Equation (9). Sketch a suitable free-body-diagram of the container, and derive the formula for σ_z, by requiring equilibrium of the free body.

Exercise 4: Linear Algebra and Matrix Analysis

Calculate the quantities listed for the following matrices and vectors.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \tag{10}$$

$$\mathbf{a} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\intercal \tag{11}$$

$$\mathbf{b} = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}^\intercal \tag{12}$$

a) Calculate $\text{tr}A$, $\det A$, the Frobenius norm $\|A\|$, $A\mathbf{a}$, $\mathbf{a}^\intercal\mathbf{b}$ and $\mathbf{b}^\intercal\mathbf{a}$.

b) A and B are 3x3 matrices. $\mathbf{a}$ and b are 3x1 matrices (i.e. vectors). Prove the following implications:

- If $\mathbf{a}^\intercal A\mathbf{b} = 0$ for all $\mathbf{a}$ and $\mathbf{b}$, then $A = 0$
- $A^\intercal = -A \Leftrightarrow \mathbf{a}^\intercal A\mathbf{a} = 0$ for all $\mathbf{a}$.
- If $\mathbf{a}^\intercal A\mathbf{a} = a^\intercal B\mathbf{a}$ for all $\mathbf{a}$, then $A + A^\intercal = B + B^\intercal$.

Hint. If something is true for all $\mathbf{a}$ then it is true for the unit vectors in each of the coordinate directions e.g. $\mathbf{e}_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^\intercal$.