Problem set 3 for TKT4150 Biomechanics

AUTHOR:

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Exercise 1: Principal and equivalent stress in femur bone of running human

The running human from the previous exercise is investigated further. The mechanics of the femur (the thigh bone) are in the spotlight this time. Fracture stress and density of bone are both given in Figure 1. Experiments reveal that point P has the largest stress and that the yield strength of the femur is $\sigma_{\text{max}} = 130\text{MPa}$. Preliminary calculations and experiments suggest the following stress in point P:

$$T = \begin{bmatrix} -50 & 35 & 0 \\ 35 & -70 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa} \tag{1}$$

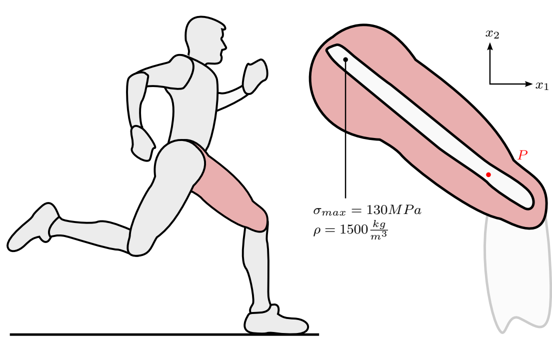

Figure 1: Running human.

a) Calculate the principal stresses in this point, and sketch them together with their corresponding angles relative to the x_1,x_2,x_3 coordinate system.

Solution. Have the following stress matrix:

$$\mathbf{T} = \begin{bmatrix} -50 & 35 & 0 \\ 35 & -70 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa} \tag{2}$$

We see that all stress components T_{ij} with $i = 3$ or $j = 3$ are equal to zero, *i.e.* the stress state is plane. Therefore, the problem can be solved as a 2D problem (even though this is not necessary, no problem doing all this in full 3D):

$$\mathbf{T}_{2D} = \begin{bmatrix} -50 & 35 \\ 35 & -70 \end{bmatrix} \text{MPa} \tag{3}$$

The principal stresses are found by defining

$$\mathbf{Tn} = \sigma \mathbf{n} \tag{4}$$

where σ is the normal stress acting out of the plane defined by the normal vector $\mathbf{n}$. This implies that the traction (remember Cauchy stress theorem) in a plane defined by $\mathbf{n}$ (LHS) equals the normal stress along the normal (RHS). In other words, we are searching for a plane and a corresponding stress magnitude where the traction can be expressed as a pure normal stress.

This leads to

$$(\mathbf{T} - \sigma \mathbf{1})\mathbf{n} = \mathbf{0} \tag{5}$$

which for non-trivial solutions requires that

$$det(\mathbf{T} - \sigma \mathbf{1}) = 0 \tag{6}$$

Combined with Equation (3), this gives

$$det \begin{bmatrix} -50 - \sigma & 35 \\ 35 & -70 - \sigma \end{bmatrix} = 0 \tag{7}$$

$$\Rightarrow (-50 - \sigma) \cdot (-70 - \sigma) - 35 \cdot 35 = 0 \tag{8}$$

$$\Rightarrow \sigma^2 + 120\sigma + 2275 = 0 \tag{9}$$

This results in

$$\sigma = \frac{-120}{2} \pm \frac{\sqrt{120^2 - 4 \cdot 1 \cdot 2275}}{2} \quad \text{(MPa)} \tag{10}$$

$$\sigma = -60 \pm 36.4 \quad \text{(MPa)} \tag{11}$$

If the principal stresses are sorted from largest to smallest, and the zero normal stress in x_3-direction is interpreted as a principal stress, we get

$$\sigma_1 = 0 \tag{12}$$
$$\sigma_2 = -23.6 \ \text{MPa} \tag{13}$$
$$\sigma_3 = -96.4 \ \text{MPa} \tag{14}$$

The corresponding angles are found by inserting the found values for σ into Equation (6):

$$\begin{bmatrix} -50-\sigma & 35 \\ 35 & -70-\sigma \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{15}$$

When $\sigma = \sigma_2 = -23.6$ MPa and $\sigma = \sigma_3 = -96.4$ MPa, the first and second row in the equation system above will be linearly dependent - which implies that one row can be reduced to a zero row (check this yourself!). Therefore, only the first row in the equation system is considered:

$$(-50+23.6)n_1 + 35n_2 = 0 \Rightarrow \mathbf{n}_2 = \begin{Bmatrix} 1 \\ \frac{26.4}{35} \end{Bmatrix} \quad (\sigma_2) \tag{16}$$
$$(-50+96.4)n_1 + 35n_2 = 0 \Rightarrow \mathbf{n}_3 = \begin{Bmatrix} -1 \\ \frac{46.4}{35} \end{Bmatrix} \quad (\sigma_3) \tag{17}$$

The vectors are normalized (length 1):

$$\mathbf{n}_2 = \begin{Bmatrix} 0.798349 \\ 0.602196 \end{Bmatrix} \tag{18}$$
$$\mathbf{n}_3 = \begin{Bmatrix} -0.602196 \\ 0.798349 \end{Bmatrix} \tag{19}$$

The stresses and normal vectors are illustrated in Figure (2).

Further, the angles become

$$\alpha_2 = \arctan\left(\frac{0.602196}{0.798349}\right) = 37\,\text{deg} \tag{20}$$
$$\alpha_3 = \arctan\left(\frac{0.798349}{-0.602196}\right) = -53\,\text{deg} + 180\,\text{deg} = 127\,\text{deg} \tag{21}$$

b) One way of evaluating whether a material is stressed beyond its strength is to compare the equivalent, also called von Mises, stress to the yield strength ($\sigma_{\max} = 130$MPa). The equivalent (von Mises) stress is given by

$$\sigma_{eq} = \sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2\right]} \tag{22}$$

where σ_1, σ_2, and σ_3 are the **ordered** principal stresses.

Calculate the equivalent stress (von Mises). Does the given stress matrix lead to fracture in the femur?

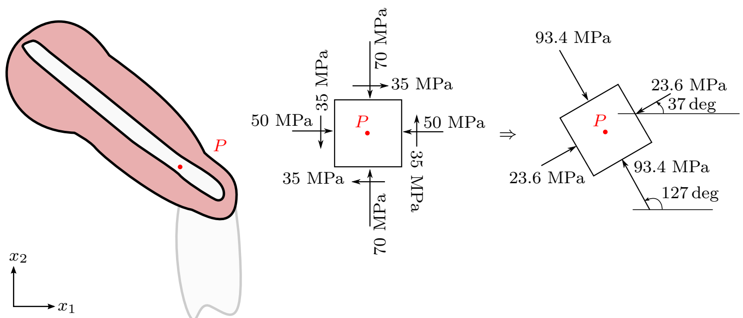

Figure 2: Principal stresses.

Solution. The equation (22) gives

$$\sigma_{eq} = 87.03 \text{ MPa} \tag{23}$$

which clearly does not exceed the fracture stress.

Exercise 2: Deformation measures

Consider the homogenous deformation

$$x_1 = X_1 + aX_2$$
$$x_2 = (1 + a)X_2$$

where $a = 0.1$.

Draw a figure to show how the square with corners $A = (0, 0)$, $B = (1, 0)$, $C = (1, 1)$, $D = (0, 1)$ deforms.

Calculate the following deformation measures based on their geometric deformation and the figure you have drawn.

a) The stretch λ of a line element is defined as the ratio of its stretched length to unstretched length. What is the stretch for a line element along AC? What about along BD?

Solution. The unstretch lenghts are simply the diagonals of the square, i.e. $\sqrt{2}$, the deformed lengths on the other hand are $\bar{AC}' = \sqrt{(1.1 - 0)^2 + (1.1 - 0)^2}$ and $\bar{BD}' = \sqrt{(1.1 - 0)^2 + (0.1 - 1.0)^2}$. Thus the stretch $\lambda AC = \frac{\sqrt{2.42}}{\sqrt{2}} = 1.1$ and $\lambda_{BD} = \frac{\sqrt{2.02}}{\sqrt{2}} = 1.00499$

b) The shear strain γ is defined as the change in angle between two line elements which are perpendicular in the reference configuration, i.e. $\gamma = \frac{\pi}{2} - \alpha$ where α is the angle bewteen the deformed line elements. What is the shear between $\mathbf{e}_1$ and $\mathbf{e}_2$?

Solution. Since the line elements along x_1 are not rotated we can simply calculate the angle between x_2 and $\bar{AD}'$. $\tan\gamma = \frac{0.1}{1.1}$. Thus $\gamma = 0.09066$ or in degrees $\gamma = 5.194$

c) Compute $\mathbf{F}$ and $\mathbf{E}$ for the given deformation. Use the formulas for longitudinal and shear strain to compare to your previous answers.

Solution. $F_{ij} = x_{i,j}$ so

$$\mathbf{F} = \begin{bmatrix} 1 & a \\ 0 & 1+a \end{bmatrix} \tag{24}$$

$$\begin{aligned} \mathbf{E} &= \frac{1}{2}\left(\mathbf{F}^\top \mathbf{F} - \mathbf{1}\right) \\ &= \begin{bmatrix} 0 & 0.5a \\ 0.5a & a + a^2 \end{bmatrix} \end{aligned}$$

Thus $\lambda_{AC} = \sqrt{1 + 2\mathbf{n}_{AC}\mathbf{E}\mathbf{n}_{AC}}$ and $\mathbf{n}_{AC} = \frac{1}{\sqrt{2}}\left[1\&1\right]^\top$. Thus

$$\begin{aligned} \lambda_{AC}^2 = 1 + 2\frac{1}{\sqrt{2}}^2 \begin{bmatrix} 1 & 1 \end{bmatrix}^\top \begin{bmatrix} 0 & 0.5a \\ 0.5a & a + a^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}^\top \\ = 1 + a^2 + 2a \\ = 1.21 \end{aligned}$$

And similarly you can check that λ_{BD} and

$$\gamma_{12} = \sin\gamma = \frac{\boldsymbol{e_1} \cdot \boldsymbol{C} \cdot \boldsymbol{e_2}}{(1+\epsilon)(1+\bar{\epsilon})} \tag{25}$$

from the fromulas equal those derived geometrically.

d) Show that for any deformation, the longitudinal strain and the stretch in the line element aligned with the direction vector e are, respectively:

$$\varepsilon = \sqrt{1 + 2e_i E_{ij} e_j} - 1 \tag{26}$$

$$\lambda = \varepsilon + 1 \tag{27}$$

where longitudinal strain is defined as:

$$\epsilon^l = \frac{ds - ds_0}{ds_0} = \frac{ds}{ds_0} - 1 \tag{28}$$

and the stretch ratio λ is defined as

$$\lambda = \frac{ds}{ds_0} \tag{29}$$

where ds is the deformed length and ds_0 is the reference length of the material element.

Hint. Reference the section in the compendium titled "The Green strain tensor".

Solution. Green strain:

$$2 \cdot \epsilon^G = \frac{ds^2 - ds_0^2}{ds_0^2} = 2e_i E_{ij} e_j \tag{30}$$

This gives

$$\frac{ds^2}{ds_0^2} - 1 = 2e_i E_{ij} e_j \tag{31}$$

$$\Rightarrow \frac{ds^2}{ds_0^2} = 2e_i E_{ij} e_j + 1 \tag{32}$$

$$\Rightarrow \frac{ds}{ds_0} = \sqrt{1 + 2e_i E_{ij} e_j} \tag{33}$$

The definition of longitudinal strain given in Equation (28) gives

$$\epsilon^l = \frac{ds}{ds_0} - 1 = \sqrt{1 + 2e_i E_{ij} e_j} - 1 \tag{34}$$

Further, the stretch is defined

$$\lambda = \frac{ds}{ds_0} \tag{35}$$

This therefore implies that

$$\lambda = \frac{ds}{ds_0} = \epsilon^l + 1 \tag{36}$$

e) Use (26) determine the general expression for the longitudinal strain and the stretch ratio in a line element aligined with the x_k-axis of the coordinate system where k may be 1, 2, or 3.

Hint. Try to do it for the direction $\mathbf{e} = \mathbf{e}_1 = \begin{pmatrix} 1\&0 & 0 \end{pmatrix}^\mathsf{T}$ first, then generalize the formula for arbitrary k.

Solution. Longitudinal strain in direction k:

$$\epsilon_k = \sqrt{1 + 2e_i E_{ij} e_j} - 1 \tag{37}$$

where

$$e_i = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases} \tag{38}$$

(Remember that here e_i represents component i of the chosen direction vector $\mathbf{e}$.) This yields the following formula where there is no summation over k

$$\epsilon_k = \sqrt{1 + 2E_{kk}} - 1 \tag{39}$$

$$\lambda_k = \sqrt{1 + 2E_{kk}} \tag{40}$$

Exercise 3: Laplace's law for membranes

Laplace's law states

$$\frac{\sigma_1}{r_1} + \frac{\sigma_2}{r_2} = \frac{p}{t} \tag{41}$$

where σ_i is the stress along x_i, r_i the radius of the shell in x_i-direction, p the internal pressure and t the thickness of the membrane. For a thin-walled sphere, the following equation holds:

$$\sigma = \sigma_\theta = \sigma_\phi = \frac{r}{2t} p \tag{42}$$

For a thin-walled cylinder with capped ends, we have the following equations:

$$\sigma_z = \frac{r}{2t} p, \quad \sigma_\theta = \frac{r}{t} p \tag{43}$$

a) Use Laplace's law, given in Equation (41), to derive the formula for the membrane stress in Equation (42), for a spherical membrane.

Solution. For a spherical membrane we have $r_1 = r_2 = r$ and $\sigma_1 = \sigma_2 = \sigma$. Hence, from Laplace's law, we see that

$$\sigma = \sigma_1 = \sigma_2 = \frac{rp}{2t} \tag{44}$$

$$\sigma = \sigma_\theta = \sigma_\phi = \frac{rp}{2t} \tag{45}$$

b) Use Laplace's law to derive the membrane stress σ_θ in Equation (43), for a cylindrical membrane.

Solution. For a cylindrical membrane we have that $r_1 = r$ and $r_2 \to \infty$. Again from Laplace's law, we see that in the case of a cylindrical membrane reduces to

$$\sigma = \sigma_\theta = \frac{rp}{t} \tag{46}$$

c) A thin-walled cylindrical container is subjected to an internal pressure p_i. The stress σ_z, on a plane perpendicular to the axis of the cylinder is given in Equation (43). Sketch a suitable free-body-diagram of the container, and derive the formula for σ_z, by requiring equilibrium of the free body.

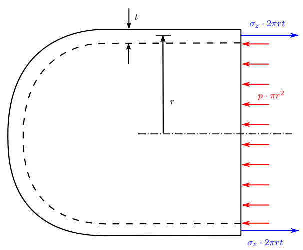

Figure 3: Free body diagram.

Solution. From Figure 3, we see that equilibrium yields

$$\sigma_z(2\pi rt) - p\pi r^2 = 0 \tag{47}$$

$$\sigma_z = \frac{rp}{2t} \tag{48}$$

Exercise 4: Linear Algebra and Matrix Analysis

Calculate the quantities listed for the following matrices and vectors.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \tag{49}$$

$$\mathbf{a} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\intercal \tag{50}$$

$$\mathbf{b} = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}^\intercal \tag{51}$$

a) Calculate tr$A$, det$A$, the Frobenius norm $\|A\|$, $A\mathbf{a}$, $\mathbf{a}^\intercal\mathbf{b}$ and $\mathbf{b}^\intercal\mathbf{a}$.

Solution.

$$\text{tr}A = A_{ii} = A_{11} + A22 + A33 = 1 + 5 + 9 = 15 \tag{52}$$

$$\det A = 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \tag{53}$$

$$= 1 \cdot (5 \cdot 9 - 6 \cdot 8) - 2 \cdot (4 \cdot 9 - 6 \cdot 7) + 3 \cdot (4 \cdot 8 - 5 \cdot 7) \tag{54}$$

$$= -3 + 12 - 9 = 0 \tag{55}$$

or, by using index notation (see Irgens), where the permutation symbol e_{ijk} is defined

$$e_{ijk} = \begin{cases} 0 & \text{two or three indices are equal} \\ 1 & \text{indices form a cyclic permutation of numbers 123} \\ -1 & \text{indices form a cyclic permutation of numbers 321} \end{cases} \tag{56}$$

From this, only 6 combinations of i, j, k give non-zero values:

$$e_{123} = 1 \qquad e_{231} = 1 \qquad e_{321} = 1 \tag{57}$$
$$e_{321} = -1 \qquad e_{213} = -1 \qquad e_{132} = -1 \tag{58}$$

This gives

$$\begin{aligned}
\det A &= e_{ijk} A_{i1} A_{j2} A_{k3} && (59)\\
&= \underbrace{e_{ijk} A_{11} A_{j2} A_{k3}}_{I} + \underbrace{e_{2jk} A_{21} A_{j2} A_{k3}}_{II} + \underbrace{e_{3jk} A_{31} A_{j2} A_{k3}}_{III} && (60)\\
&= e_{123} A_{11} A_{22} A_{33} + e_{132} A_{11} A_{32} A_{23} \qquad (I) && (61)\\
&+ e_{231} A_{21} A_{32} A_{13} + e_{213} A_{21} A_{12} A_{33} \qquad (II) && (62)\\
&+ e_{312} A_{31} A_{12} A_{23} + e_{321} A_{31} A_{22} A_{13} \qquad (III) && (63)\\
&= A_{11} A_{22} A_{33} - A_{11} A_{32} A_{23} + A_{21} A_{32} A_{13} && (64)\\
&- A_{21} A_{12} A_{33} + A_{31} A_{12} A_{23} - A_{31} A_{22} A_{13} && (65)\\
&= 1 \cdot 5 \cdot 9 - 1 \cdot 8 \cdot 6 + 4 \cdot 8 \cdot 3 && (66)\\
&- 4 \cdot 2 \cdot 9 + 7 \cdot 2 \cdot 9 - 7 \cdot 5 \cdot 3 = 0 && (67)
\end{aligned}$$

$$\begin{aligned}
\text{norm} A = \|A\| &= \sqrt{\text{tr}(AA^\intercal)} = \sqrt{A_{ij} A_{ij}} && (68)\\
&= \sqrt{A_{i1} A_{i1} + A_{i2} A_{i2} + A_{i3} Ai3} && (69)\\
&= \sqrt{A_{11}^2 + A_{21}^2 + A_{31}^2 + A_{12}^2 + A_{22}^2 + A_{23}^2 + A_{13}^2 + A_{23}^2 + A_{33}^2} && (70)\\
&= \sqrt{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2} && (71)\\
&= \sqrt{285} \approx 16.88 && (72)
\end{aligned}$$

$$A\mathbf{a} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} = \begin{Bmatrix} 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 \\ 1 \cdot 4 + 2 \cdot 5 + 6 \cdot 3 \\ 7 \cdot 1 + 8 \cdot 2 + 9 \cdot 3 \end{Bmatrix} = \begin{Bmatrix} 14 \\ 32 \\ 50 \end{Bmatrix} \tag{73}$$

$$
\begin{aligned}
\mathbf{a}\cdot\mathbf{b} = \mathbf{a}^\intercal\mathbf{b} = \mathbf{b}^\intercal\mathbf{a} &= a_i b_j(\mathbf{e}_i\cdot\mathbf{e}_j) &(74)\\
&= a_1 b_j(\mathbf{e}_1\cdot\mathbf{e}_j) + a_2 b_j(\mathbf{e}_2\cdot\mathbf{e}_j) + a_3 b_j(\mathbf{e}_3\cdot\mathbf{e}_j) &(75)\\
&= a_1 b_1(\mathbf{e}_1\cdot\mathbf{e}_1) + a_1 b_2\underbrace{(\mathbf{e}_1\cdot\mathbf{e}_2)}_{=0} + a_1 b_3\underbrace{(\mathbf{e}_1\cdot\mathbf{e}_3)}_{=0} &(76)\\
&+ a_2 b_1\underbrace{(\mathbf{e}_2\cdot\mathbf{e}_1)}_{=0} + a_2 b_2(\mathbf{e}_2\cdot\mathbf{e}_2) + a_2 b_3\underbrace{(\mathbf{e}_2\cdot\mathbf{e}_3)}_{=0} &(77)\\
&+ a_3 b_1\underbrace{(\mathbf{e}_3\cdot\mathbf{e}_1)}_{=0} + a_3 b_2\underbrace{(\mathbf{e}_3\cdot\mathbf{e}_2)}_{=0} + a_3 b_3(\mathbf{e}_3\cdot\mathbf{e}_3) &(78)\\
&= a_1 b_1 + a_2 b_2 + a_3 b_3 = 1\cdot 4 + 2\cdot 5 + 3\cdot 6 = 32 &(79)
\end{aligned}
$$

b) A and B are 3x3 matrices. **a** and b are 3x1 matrices (i.e. vectors). Prove the following implications:

- If $\mathbf{a}^\intercal A\mathbf{b} = 0$ for all **a** and **b**, then $A = 0$
- $A^\intercal = -A \Leftrightarrow \mathbf{a}^\intercal A\mathbf{a} = 0$ for all **a**.
- If $\mathbf{a}^\intercal A\mathbf{a} = a^\intercal B\mathbf{a}$ for all **a**, then $A + A^\intercal = B + B^\intercal$.

Hint. If something is true for all **a** then it is true for the unit vectors in each of the coordinate directions e.g. $\mathbf{e}_1 = \begin{pmatrix}1\&0 & 0\end{pmatrix}^\intercal$.

Solution. $\mathbf{a}^\intercal\mathbf{Ab} = 0$ is written $a_i b_j A_{ij}$ using index notation. This implies $\sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_j A_{ij}$. From this, we may show that for arbitrary a_i and b_j, $A_{ij} \equiv 0$. Since $a_i b_j A_{ij} = 0$ for arbitrary **a** and **b**, it holds for $\mathbf{a} = \mathbf{e}_k$ so we have $\delta_{ki} b_j A_{ij} = 0$. Thus $b_j A_{kj} = 0$ for any **b**, so $A_{kj} = 0$, thus $\mathbf{A} = \mathbf{0}$

$A^\intercal = -A$ results in $A_{ij} = -A_{ji}$ on index notation. Furthermore, $\mathbf{a}^\intercal A\mathbf{a}$ gives $a_i a_j A_{ij}$. By combining these expressions, we get $a_i a_j A_{ij} = a_i a_j(-A_{ji})$. This can be rewritten $a_i a_j A_{ij} = -a_i a_j A_{ij}$, and thus we can conclude that $a_i a_j A_{ij} = 0$, as $a_i a_j A_{ij} + a_i a_j A_{ij} = 2a_i a_j A_{ij} = 0$.

Going the other way we assume $a_i a_j A_{ij} = 0$, which means we may also write $a_j a_i A_{ji} = 0$. Combining these we have $a_i a_j A_{ij} + a_j a_i A_{ji} = 0$, which is equivalent to $a_i a_j(A_{ij} + A_{ji}) = 0$. Since this is true for any a it is true for $\mathbf{e}_k$ yielding $\delta_{ki}\delta_{kj}(A_{ij} + A_{ji}) = 0$, where k is a specific value (No summation over k). This reduces to $(A_{ij} + A_{ji}) = 0$, and finally $A_{ij} = -A_{ji}$.

We have $\mathbf{a}^\intercal A\mathbf{a} = \mathbf{a}^\intercal B\mathbf{a}$.

In index notation we have

$$a_i a_j A_{ij} = a_i a_j B_{ij} \tag{80}$$

Since the summation indices are arbitrary, note that $a_i a_j A_{ij} = a_i a_j A_{ji}$ and $a_i a_j B_{ij} = a_i a_j B_{ij}$.

Thus we have the following equivalent equation

$$a_i a_j A_{ji} = a_i a_j B_{ij},$$

which may be rearranged as

$$a_i a_j A_{ji} - a_i a_j B_{ij} = 0$$

or finally which may be rearranged as

$$a_i a_j (A_{ji} - B_{ij}) = 0$$

This is equivalent to $\mathbf{a}^\intercal C \mathbf{a} = 0$ for all $\mathbf{a}$, where $C = A^\intercal - B$.

Applying the result from part 2 we have $C = -C^\intercal$ and thus

$$A^\intercal - B = -A + B^\intercal.$$

Collecting A and B we find

$$A^\intercal + A = B + B^\intercal.$$