Fridtjov Irgens

Continuum Mechanics

With 279 Figures and 4 Tables

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Preface

Students of engineering and engineering science are early in their studies exposed to Applied Mechanics: Statics and Dynamics of rigid bodies, Strength of Materials and Fluid Mechanics. As a teacher of these subjects for many years at the Norwegian University of Science and Technology, and for students of most special branches of engineering, I have found that it is highly relevant and important that students who want to obtain a better understanding of the mechanics of materials or of the mechanics of fluids, should take a course in Continuum Mechanics, a discipline that synthesizes the basic concepts of the mechanics of solids and fluids.

The pioneers in natural science were experts in many different disciplines of physics and mechanics. Leonhard Euler [1707–1783] is an example of a scientist who has made permanent contributions both in Solid Mechanics: the Euler theory of column buckling, and Fluid Mechanics: the Euler equations of inviscid fluids. Later came times when special fields like the Theory of Elasticity, Hydrodynamics, and Gas Dynamics dominated. Classical Theory of Elasticity is the mechanics of isotropic and linearly elastic materials, for which the behavior is based on extensions of Hooke's law, and where deformations are assumed to be small. Hydrodynamics is the mechanics of liquids, and Gas Dynamics is the thermomechanics of gasses. These two disciplines are joined in the science of Fluid Mechanics, the term fluid being a common name for liquids and gasses.

Although these separate parts of mechanics still exist as important special fields, a need for unification of the mechanics of solids and fluids became important in modern times, let us say in the last 60 years or so. New and complex materials have been invented and made available, and a better utilization of the materials has become necessary or important. The fundamental distinctions between the thermomechanical behavior of liquids and solid materials are no longer clear. Solid materials behave fluid-like in many forming processes, although they still are in the solid phase, and these forming processes need to be analyzed and modelled. Modern engineering materials may exhibit linear or non-linear elastic, viscous, plastic, viscoelastic, viscoplastic, or visco-elasto-plastic behavior.

Another reason for the importance of understanding properly the thermomechanical behavior of advanced materials is the development of numerical methods and the existence of highly advanced and speedy computers, which may be used to provide solutions of problems involving complex geometries and very complex thermomechanical material behavior.

The study of distributions of energy, matter, and other physical quantities under circumstances where their discrete nature is unimportant, and where they may be regarded as continuous functions of position and time, is called Classical Field Theory or Continuum Physics. This branch of Physics presumes nothing directly regarding the structure of the materials, although the structure of a material may be reflected in the way the physical quantities are described. When the physical quantities are purely of mechanical or thermodynamical nature, the subject is called Continuum Mechanics.

Continuum Mechanics is based on the continuum hypothesis: Matter is continuously distributed throughout the space occupied by the matter. Regardless of how small volume elements the matter is subdivided into, every element will contain matter. The basis for the hypothesis is how we directly experience matter and its macroscopic properties, and furthermore on how physical quantities we use, as for example pressure, temperature, and velocity, are measured macroscopically.

The material in this book has evolved from my lecture notes produced during many years of teaching undergraduate courses and graduate courses in Mechanics of materials, Continuum Mechanics, Tensor Analysis, Rheology, and Biomechanics. The book is a textbook designed for classroom teaching or self-study. Most of the chapters conclude with exercise problems.

The references at the end of the book present the major sources during my study of the subject of Continuum Mechanics. Extensive bibliographies from the time before 1965 may be found in the important treatises by Truesdell and Toupin [54] and Truesdell and Noll [53].

A short presentation of the contents of the book. Chapter 1 presents some of the basic concepts of macromechanical properties of materials when they are considered to be continuous matter: normal stress, shear stress, longitudinal strain and shear strain. The simplest equations relating these quantities are discussed, e.g. Hooke's law relating normal stress and longitudinal strain in an elastic material, and Newton's law of fluid friction relating shear stress and shear strain rate.

Chapter 2 presents some basic mathematical concepts: matrix algebra, vector algebra and vector analysis. Vectors are presented using index notation, i.e. through vector components in a Cartesian coordinate system, by collecting the components in vector matrices, i.e. the index free notation, and finally by the coordinate invariant and bold face notation.

Chapter 3 contains the basic Continuum Mechanics: Kinematics, Kinetics with the equations of motion for bodies and material points, and General Stress Analysis, common to all continuous materials. The concept of tensors is introduced by defining the stress tensor as a coordinate invariant quantity.

Chapter 4 presents tensors as coordinate invariant quantities through multilinear scalar-valued functions of vectors, alternatively expressed by their components in Cartesian coordinates. The formal presentation of tensor analysis expressed by components in general curvilinear coordinates is postponed to Chap. 12.

The analysis of small and large deformations and strains is the subject of Chap. 5. The analysis of strain rates, rates of deformation, and rates of rotations, of major importance in Fluid Mechanics and in describing rate dependent materials, is also included in this chapter.

Chapter 6 discusses the concepts work, power, and energy. The mechanical energy balance equation and the thermal energy balance equation, which is based on the first law of thermodynamics, are presented. The second law of thermodynamics is given a brief presentation. Some consequences of this law in constitutive modelling are given in Chap. 11.

Chapter 7 is an extensive exposition of the classical theory of isotropic, linearly elastic materials. Many important applications in two-dimensional theory are treated, and the chapter gives a comprehensive discussion on elastic waves. A section on anisotropic elastic materials having different degrees of symmetries serves as an introduction to fiber composite materials. Finite elasticity is briefly presented.

Chapter 8 has sections on perfect fluids, linearly viscous fluids, potential flow, and non-Newtonian fluid models. Advanced non-Newtonian fluids are presented in Chap. 9 and in Chap. 11.

Chapter 9 presents the theory of viscoelasticity for both solids and fluids.

Chapter 10 contains some basic parts of the theory of plasticity. The criteria for yielding and the yield laws or flow rules are presented for the most commonly used material models. The important theorems for upper and lower bound solutions for elastoplastic materials are presented. Viscoplasticity, which combines plastic behaviour with viscous response at yielding, is given a brief presentation. Finally the yield line theory is presented.

Principles and theories for the mathematical modelling of materials is the main theme in Chap. 11. An extension of the deformation analysis from Chap. 5 and the concepts of objective tensors and corotational time derivative are presented and discussed. Non-linear materials with elastic, viscous, viscoelastic, plastic, viscoplastic and viscoelastic-plastic responses are discussed.

Chapter 12 presents the components of tensors in general curvilinear coordinate systems in the 3-dimensional physical Euclidean space E_3. The concept of tensors is the same as in Chap. 4, but is now extended to more general coordinate systems.

Chapter 13 provides the general field equations of Continuum Mechanics in curvilinear coordinates. Concepts like convected coordinates and the related deformation analysis are introduced. The concept of convected time derivatives is given geometrically definitions.

The present book was originally presented in Norwegian and in three parts: Part I Continuum Mechanics, Part II Mechanics of Materials (Constitutive Modelling), and Part III Tensor Analysis (Curvilinear coordinates in continuum mechanics). Part I was published by Tapir Akademisk Forlag in 1974, and Part III was published in 1982 by the same publishers. Part II was only presented in a compendium form. The reason for this was that I found that the extension of the theme and the continued developments within the science of constitutive modelling made it difficult to choose and limit the content properly. Part I and II have since gone through many and extensive revisions and have been presented as departmental compendiums in a number of editions. In 2000–2001 the compendiums were translated into English.

Each of the three parts of the book was meant to cover the course material in advanced semester courses. Part I contained: The Foundation of Continuum Mechanics using tensors in Cartesian coordinate systems, Classical Theory of Elasticity, and Fluid Mechanics. Part II had chapters on Viscoelasticity, Plasticity, Principles of Constitutive Modelling, and Thermodynamics. Part III presented Tensor Analysis in curvilinear coordinates and the fundamental equations of Continuum Mechanics in these coordinates.

In the present English version of the book the three parts are put together in one volume. The intentions of the first version of the book have been retained, which means that the book may be used as a text book in more than one course. I will suggest four different courses:

Course 1. An introductory course in Continuum Mechanics. Contents: Fundamental equations of Kinematics and Kinetics applied in Continuum Mechanics, Analysis of Stress, Small Deformations, and Rates of Deformation, Tensor Analysis using Cartesian coordinates, Linear Elasticity, Fluid Mechanics. The course may be based upon the material in Chaps. 1, 2, 3, Sect. 4.1–3, 5.1–4, 6.1.1, 7.1–3, 7.5, 7.7.1–3, 8.1, 8.3 (except 8.3.1–3), 8.4.1–2.

Course 2. Advanced Mechanics of Materials. This course is a continuation of Course 1 and includes: Anisotropic Elasticity, Composite Materials, Viscoelasticity, and Plasticity. The course may be based upon the materials in Sect. 7.8 and 7.9, Chaps. 9 and 10.

Course 3. A graduate course in Continuum Mechanics. Contents: Foundation of Continuum Mechanics, Analysis of Stress, Small and Large Deformations, and Rates of Deformation, Tensor Analysis using Cartesian coordinates, Linear and Non-linear Theory of Elasticity, Fluid Mechanics, Non-Newtonian Fluids, Viscoelasticity, and Plasticity, Principles of Constitutive Modelling. The course may be based upon the materials in Chaps. 1–11.

Course 4. Tensor Analysis. The course presents tensors as applied in Continuum Mechanics. The tensors are defined as invariant quantities with respect to coordinate systems but presented by components in both Cartesian systems and in curvilinear coordinate systems. The course may be based upon material from Chaps. 1, 2, 3, 4, 5, Sect. 6.1, 7.1–2, 7.5, 7.6, 7.7.1–3, 8.1, 8.3 (except Sect. 8.3.1–3), 8.4.1–2, and Chaps. 12, and 13.

During my study of general Mechanics and Continuum Mechanics I have received many comments and suggestions from colleagues at Department of Applied Mechanics, Thermodynamics, and Fluid Dynamics and Department of Structural Engineering, both at Norwegian University of Science and Technology. In particular I will thank professor Arvid Gustafson, professor Jan B. Aarseth, professor Henry Øiann, and professor Leif Rune Hellevik, and from my former students Dr. Geir Berge and professor Per Olaf Tjelflaat for their valuable contributions.

As visiting professor at University of Connecticut and University of Colorado I have received many impulses from American colleagues. In particular I will like to thank professor Kaspar Willam and professor Stein Sture for their hospitality during my sabbaticals in 1990–91 and in 2000 at Department of Civil, Environmental, and Architectural Engineering, University of Colorado at Boulder.

Siv.ing. Lars Haga and stud. techn. Marthe Sæther have produced most of the figures in the book, and I thank them for their contributions.

Finally, thanks to my wife Eva, who has supported and encouraged me.

Bergen, October, 2007 Fridtjov Irgens

Contents

Chapter 1
Introduction

1.1 The Continuum Hypothesis

It is a classical concept that matter may take three aggregate forms or phases: solid, liquid, and gaseous. A body of solid matter, called *solid*, has a definite volume and a definite form, both of which only dependent on the temperature and the forces that the body is subjected to. A body of liquid matter, called a *liquid*, has a definite volume, determined by the temperature in the body and forces on the body, but not a definite form. A liquid in a container is formed by the container and may attain a free horizontal surface. A body of gaseous matter, called a *gas*, has neither a definite volume nor form determined only by temperature and forces. The gas fills any container it is poured into.

Matter is made of atoms and molecules. The molecules contain usually many atoms, bound together by *interatomic forces*. The molecules interact through *intermolecular forces*, which in the liquid and gaseous phases are considerably weaker than the interatomic forces.

In the liquid phase the molecular forces are too weak to bind the molecules to definite equilibrium positions in space, but the forces will keep the molecules from departing to far from each other. This explains why volume changes are relatively small for a liquid.

The distances between the molecules in a gas are so large that the intermolecular forces play a minor role. The molecules move about each other with high velocities and interact through elastic impacts. The molecules will disperse throughout the vessel containing the gas. The pressure against the vessel walls is a consequence of molecular impacts.

In the solid phase there is no longer a clear distinction between molecules and atoms. In the equilibrium state the atoms vibrate about fixed positions in space. The solid phase is realized in either of two ways: In the *amorphous state* the molecules are not arranged in any definite pattern. In the *crystalline state* the molecules are arranged in rows and planes within certain subspaces called crystals. A crystal may have different physical properties in different directions, and we say that the crystal has macroscopic *structure*. Solid matter in crystalline state usually consists of a disordered collection of crystals, denoted grains. The solid matter is then polycrystalline. From a macroscopic point of view polycrystalline materials may have structure, resulting in *anisotropic properties*, or they may be treated as having *isotropic properties*, which means that the physical properties are the same in all directions.

Continuum mechanics is a special branch of Physics in which matter, regardless of phase or structure, is treated by the same theory until the special macroscopic properties are described through *material equations* or *constitutive equations*, as they are called. The constitutive equations represent macromechanical models for the real materials. The simplest constitutive equation is given by Hooke's law, represented by formula (1.2.7) below, and which expresses the linear relationship between force and deformation in an elastic material.

Continuum Mechanics is based on *the continuum hypothesis*:

Matter is continuously distributed throughout the space occupied by the matter. Regardless of how small volume elements the matter is subdivided into, every element will contain matter. The matter may have a finite number of discontinuous surfaces, for instance fracture surfaces or yield surfaces, but material curves that do not intersect such surfaces, retain their continuity during the motion and deformation of the matter.

The basis for the hypothesis is how we directly experience matter and its macroscopic properties, and furthermore on how physical quantities we use, as for example pressure, temperature, and velocity, are measured macroscopically. Such measurements are performed with instruments that give average values over small volume elements of the material. The probe of the instrument may be small enough to give a local value, or an intensive value of the quantity, but always so extensive that it registers the action of a very large number of atoms or molecules.

1.2 Elasticity, Plasticity, and Fracture

A solid material is tested for strength and stiffness by subjecting a test specimen to axial tension or compression. Figure 1.2.1 shows a test specimen used in a tensile test. The cross-sections at the ends of the specimen are enlarged to give a more even transmission of the forces N from the testing machine to the specimen. We consider now a homogeneous part of the specimen of length L_o and diameter D_o. When the specimen is subjected to tension, it will elongate and the homogeneous part obtains the length L. The diameter decreases to the value D. Change of length per unit length of material elements axially and in the direction of the diameter may be given by the *longitudinal strains*:

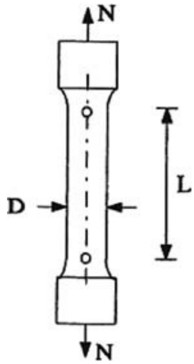


Fig. 1.2.1 Tensile specimen with circular cross-section

$$\varepsilon = \frac{L - L_o}{L_o}, \ \varepsilon_t = \frac{D - D_o}{D_o} \tag{1.2.1}$$

ε is the *axial strain*, while ε_t is called *transverse strain*. The longitudinal strains defined by (1.2.1), i.e. as change of length per unit length of original length, L_o or D_o, is also called *conventional, nominal* or *engineering strain*. Strains are dimensionless quantities. Another measure of longitudinal strain, especially used for large strains in experimental situations, is *logarithmic strain* or *true strain:*

$$\varepsilon_l = \ln\frac{L}{L_o} = \ln(1 + \varepsilon) \tag{1.2.2}$$

This strain measure is defined based on the strain increment:

$$d\varepsilon = \frac{dL}{L} \tag{1.2.3}$$

which expresses incremental change of length dL per unit of the present length L, i.e. the length of the specimen prior to the increment. The logarithmic strain represents a sum of the strain increments (1.2.3):

$$\varepsilon_l = \int_0^{\varepsilon_l} d\varepsilon = \int_{L_o}^{L} \frac{dL}{L} = \ln L - \ln L_o = \ln\frac{L}{L_o}$$

We find that:

$$\varepsilon = \exp\varepsilon_l - 1 \tag{1.2.4}$$

For small strains, say $\varepsilon < 0.01$, the logarithmic strain is approximately equal to the conventional strain: $\varepsilon_l \approx \varepsilon$.

The axial force in the test specimen in Fig. 1.2.1 is N, and it is assumed that N is transmitted through the homogeneous part of the specimen as an evenly distributed force over the cross-section of the specimen and perpendicular to the cross-section. The intensity of the force or force per unit cross-sectional area is the *normal stress*

$\sigma = N/A$, where $A = \pi D^2/4$ is the cross-sectional area. It is customary to use the expressions:

$$\sigma = \frac{N}{A} \quad \text{true normal stress, or Cauchy stress} \tag{1.2.5}$$

$$\sigma_o = \frac{N}{A_o} \quad \text{nominal normal stress, or engineering stress} \tag{1.2.6}$$

where $A_o = \pi D_o^2/4$ is the cross-sectional area of the undeformed specimen. In many practical cases the change in cross-sectional area is so small that we need not distinguish between true and nominal stress. True stress is also called *Cauchy stress*, see Sect. 3.2.4. Other names for nominal stress are *engineering stress* and *Piola-Kirchhoff stress*, see Sect. 5.6. In the homogeneous part of the tensile specimen we shall find that material planes parallel to the axis of the specimen are free of forces. We say that the this part of the specimen is a *uniaxial stress state*.

In order to analyze how a particular material behaves for different levels of loading under normal stress, a stress-strain diagram is created. Figure 1.2.2 shows a sketch of such a diagram for mild steel, also known as ordinary structural steel. We shall briefly present this diagram because mild steel clearly demonstrate some of the most important mechanical properties for solid materials.

Mild steel is "perfectly" elastic for stresses below a limit f_y, called the *yield limit* or the *yield stress*. This implies that the material retains its original shape after it has been unloaded, as long as the normal stress satisfies the condition: $\sigma_{\max} < f_y$. The yield stress varies with the steel quality and usually is found within the range: 200–400 MPa (megapascal). The unit *pascal* [Pa] is equal to $1\,\mathrm{N/m^2}$. An unloading curve from a stress level $\sigma < f_y$ will follow the loading curve. Both curves are linear, and the relationship between stress and strain may be expressed by the linear law:

$$\sigma = \eta\varepsilon \tag{1.2.7}$$

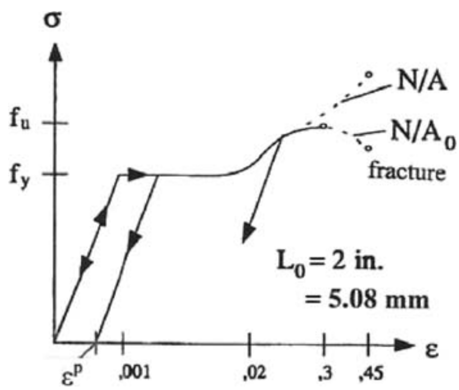


Fig. 1.2.2 Stress-strain diagram for mild steel

The coefficient of proportionality η is called the *modulus of elasticity* or *Young's modulus* and has the same dimension as stress, i.e. force per unit area. The standard symbol for the modulus of elasticity is E. However, in this book the letter E is reserved for another quantity, the strain matrix. In Chap. 7 Theory of Elasticity both symbols, η and E, will be used for the modulus of elasticity. The modulus of elasticity varies somewhat with the steel quality. A reference value for most structural steels may be 210 GPa (gigapascal). Figure 1.2.2 shows that the linear relation (1.2.7) only applies when the strains are less than approximately 0.001.

For a normal stress at the yield limit f_y, or slightly above, the test specimen experiences an increasing elongation at a constant load N, until the material hardens and the axial load N must be increased to increase the elongation. At a nominal stress equal to f_u, called the *tensile strength*, the *ultimate stress*, or the *fracture stress*, the limit load capacity of the specimen is reached, and fracture has started to develop. The tensile strength varies with the steel quality and usually lies within the region 350–700 MPa.

At unloading from a stress level $\sigma > f_y$ to zero stress, the specimen retains a residual strain called *plastic strain* ε^p. The unloading curves shown in Fig. 1.2.2, are nearly straight and parallel to the loading curve from $\sigma = 0$ to $\sigma = f_y$. A new loading procedure will give a curve that approximately follows the unloading curve to the point from which the unloading started. Thereafter the curve follows the original loading curve towards the ultimate stress f_u.

Figure 1.2.3 shows stress-strain diagrams from tensile tests of four materials. The data for the curves are obtained from Calladine [5]. The figure to the right represents a part of the left figure.

A material that experience plastic strains is called a *ductile material*. Many ductile materials do not show a clearly defined yield stress, see Fig. 1.2.3. For such materials a yield limit f_y is defined as the stress $\sigma_{0.2}$ at which the plastic strain after unloading is $\varepsilon^P = 0.002 (= 0.2\%)$.

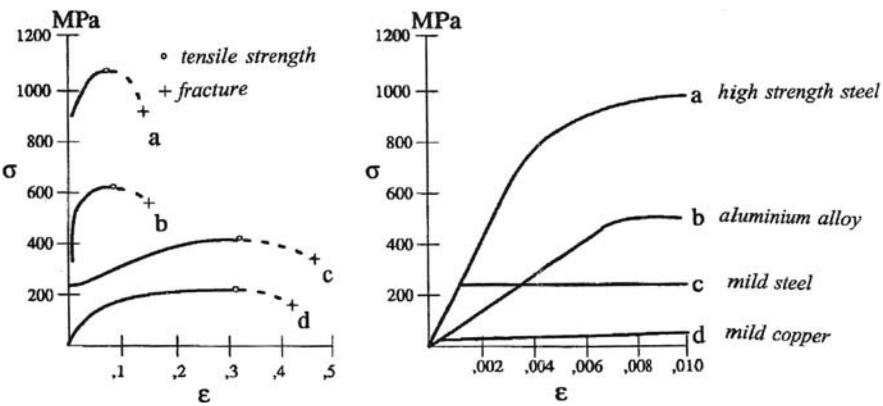


Fig. 1.2.3 Tensile tests of four materials

Experiments with specimens of a ductile material in compression show that the yield stress is $-f_y$ and that Hooke's law applies with the same modulus of elasticity as in the tensile test with the same material. Compression tests are however more difficult to perform than tensile tests. In order to prevent buckling of the test specimen the diameter of the specimen must not be to small compared with the length of the specimen. When yielding occurs, the cross-sectional area increases and the compressive force N necessary to further reduce the length of the specimen must increase considerably, and a compressive strength will be difficult to specify. The literature gives little information on compressive strengths for ductile materials. If we choose true stress and true strain as coordinates in the stress-strain diagrams for both uniaxial tension and compression, we will often see that the two diagrams have very similar forms. With nominal stress and nominal strain as coordinates, the compression diagram will show a higher degree of hardening.

Materials that do not show plastic strain before fracture, are called *brittle materials*. Cast iron, glass, concrete, and wood are examples of brittle materials.

When computing internal forces and deformations in continuous media, *constitutive models* are introduced. The models idealize the behavior and properties of the real materials. A model essentially based on the constitutive equation (1.2.7) for the relation between stress and strain, is called a *linearly elastic material*. The relationship (1.2.7) is called *Hooke's law* after Robert Hooke [1635–1703]. The *classical theory of elasticity*, presented in Chap. 7, presumes linear elasticity.

A mechanical model consisting of a *linear helical spring*, see Fig. 1.2.4, may illustrate the response of a *Hookean bar*. The response equation of the bar and the model is given by Hooke's law (1.2.7). The stress σ represents the force on the spring and η/L_o is the spring constant. A material without a definite yield point is often represented by the *Ramberg-Osgood model* [W. Ramberg and W. R. Osgood 1943], see Fig. 1.2.4. In a bar of this material the longitudinal strain ε and the cross-sectional normal stress σ are related through the constitutive equation:

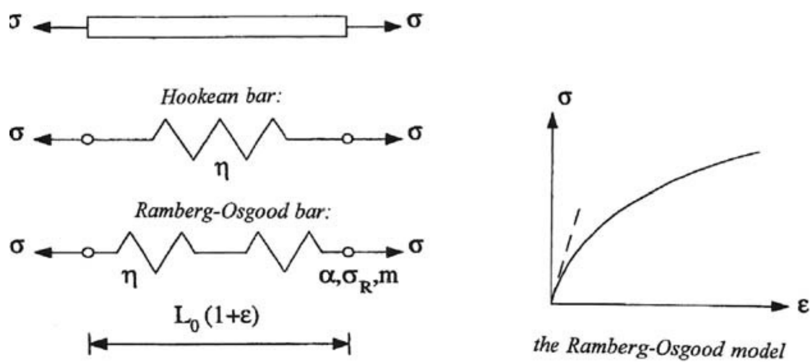


Fig. 1.2.4 Models for elastic material in uniaxial stress

$$\varepsilon = \frac{1}{\eta}\left[1 + \frac{\alpha}{\sigma_R}\left|\frac{\sigma}{\sigma_R}\right|^{m-1}\right]\sigma \tag{1.2.8}$$

where η, α, σ_R, and m are material parameters. An analogous mechanical model, *a Ramberg-Osgood bar*, may be series coupling of a linear ("Hookean") spring and a non-linear spring.

The simplest model for materials experiencing both elastic and plastic strains, called *elastoplastic materials*, is the *linearly elastic-perfectly plastic material*. The response for uniaxial stress may be represented by the mechanical model shown in Fig. 1.2.5a. The model consists of a series coupling of a linear spring and *a friction element*. The friction element can at most transfer a stress equal to the yield stress f_y. During loading the constitutive equation for uniaxial stress is:

$$\varepsilon = \frac{\sigma}{\eta} \text{ when } |\sigma| < f_y, \quad \varepsilon \text{ is indeterminate when } |\sigma| = f_y \tag{1.2.9}$$

During unloading the stress-strain curve is linear as shown in Fig. 1.2.5a.

A *linearly elastic-plastic material with hardening* may for uniaxial stress be represented by the mechanical model shown in Fig. 1.2.5b. During loading the response equation is:

$$\begin{aligned} &\varepsilon = \tfrac{\sigma}{\eta} \text{ for } |\sigma| < f_y, \quad \varepsilon = \left[\tfrac{1}{\eta_t} - \tfrac{f_y}{|\sigma|\eta_1}\right]\sigma \text{ for } |\sigma| > f_y \\ &\eta_t = \tfrac{\eta\eta_1}{\eta+\eta_1}, \quad \text{tangent modulus of elasticity} \end{aligned} \tag{1.2.10}$$

where η and η_1 are material parameters. For elastoplastic materials without a definite yield point the Ramberg-Osgood model may be used during increasing load, while unloading may be govern by a linear law. Elastoplastic materials will be discussed in Chap. 10 Theory of Plasticity.

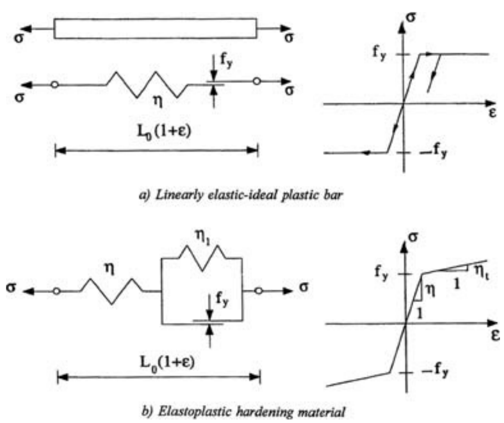


Fig. 1.2.5 Elastoplastic materials in uniaxial stress

1.3 Fluids

A common property of liquids and gases is that they at rest only can transmit a pressure normal to solid surfaces bounding the liquid or gas. Tangential forces on such surfaces will first occur when there is relative motion between the liquid or gas and the solid surface. Such forces are experienced as frictional forces on the surface of bodies moving through air or water. When we study the flow in a river we see that the flow velocity is greatest in the middle of the river and is reduced to zero at the riverbank. The phenomenon is explained by the notion of tangential forces, called *shear stresses* between the water layers that try to brake the flow. The volume of an element of flowing liquid is nearly constant. This means that the *density*: mass per unit volume, of a liquid is almost constant. Liquids are therefore usually considered to be incompressible. The *compressibility* of a liquid, that is change in volume and density, comes into play when convection and acoustic phenomena are considered.

Gases are easily compressible, but in many practical applications the compressibility of a gas may be neglected, and we may treat the gas as an incompressible medium. In elementary aerodynamics, for instance, it is customary to treat air as an incompressible matter. The condition for doing that is that the characteristic speed in the flow is less than 1/3 of the speed of sound in air.

Due to the fact that liquids and gases macroscopically behave similarly, the equations of motion and the energy equation have the same form, and the most common constitutive models applied are in principle the same for liquids and gases. A general name for these models is therefore of practical interest, and the models are called *fluids*. A *fluid* is thus a model for a liquid or a gas. *Fluid Mechanics* is the macromechanical theory for the mechanical behavior of liquids and gases. Solid materials may also show fluid behavior. Plastic deformation and creep, which is increasing deformation at constant stress, represent both fluid-like behavior. Steel experiences creep at high temperatures ($> 400\,°C$), but far below the melting temperature. Granite may obtain large deformations due to gravity during geological time intervals. All *thermo plastics* are, even in solid state, behaving like liquids, and therefore modelled as fluids. In continuum mechanics it seems natural to define a fluid on the basis of what is the most characteristic for a liquid or a gas. We choose the following definition:

> *A fluid is matter that deforms continuously when it is subjected to anisotropic states of stress.*

Figure 1.3.1 shows the difference of an *isotropic state of stress* and an *anisotropic state of stress*. At an isotropic state of stress in a material point all material surfaces through the point are subjected to the same normal stress, tension or compression, while tangential forces, i.e. *shear stresses*, on the surfaces are zero. At an anisotropic state of stress in a material point most material surfaces will experience shear stresses.

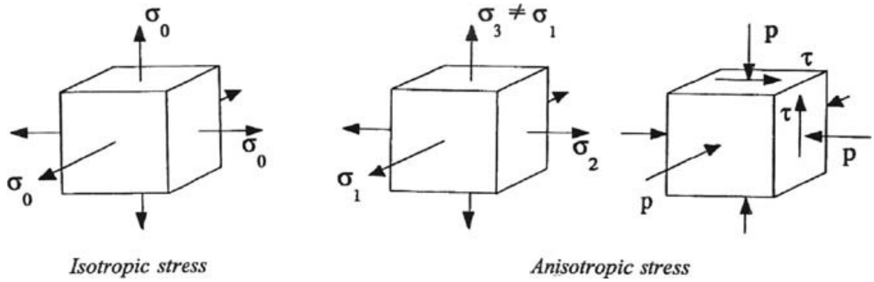


Fig. 1.3.1 Isotropic state of stress and anisotropic states of stress

As mentioned above, solid matter behaves in certain situations as fluids. Constitutive models that do not imply fluid-like behavior, will in this book be called *solids*. Continuum mechanics also introduces a third category of constitutive models called *liquid crystals*. Such materials are briefly mentioned in Sect. 11.6.5.

The ability of a fluid to transfer shear stresses may be measured by a viscometer. Figure 1.3.2 shows a cylinder viscometer. A cylinder can rotate in a cylindrical container about a vertical axis. The annular space between the two concentric cylindrical surfaces is filled with a liquid. The cylinder is subjected to a constant torque T and rotates with a constant angular velocity ω. The distance h between the two cylindrical surfaces is so small compared to the radius r of the cylinder that the motion of the liquid may be considered to be like the flow between two parallel plane surfaces, see Fig. 1.3.3. It may be shown that for moderate ω-values the velocity field is given by:

$$v_x = \frac{v}{h}y\ ,\ v_y = v_z = 0\ ,\ v = \omega r \tag{1.3.1}$$

where v_x, v_y, and v_z are velocity components in the directions of the axes in the Cartesian coordinate system $Oxyz$ shown in Fig. 1.3.3. The term $v = \omega r$ is the velocity of the fluid particle at the wall of the rotating cylinder.

A volume element having edges dx, dy, and dz, see Fig. 1.3.4, will during a short time interval dt change its form. The change in form is given by the *shear strain*:

$$d\gamma = \dot{\gamma} \cdot dt = \frac{dv_x \cdot dt}{dy} = \frac{v}{h}dt$$

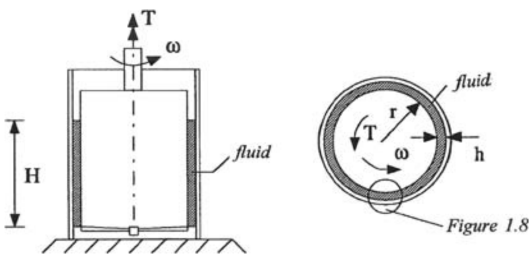


Fig. 1.3.2 Viscometer

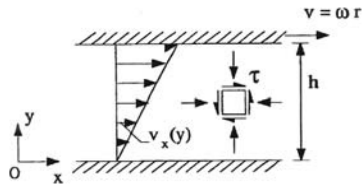


Fig. 1.3.3 Simple shear flow

The quantity:

$$\dot{\gamma} = \frac{v}{h} = \frac{r}{h}\omega \tag{1.3.2}$$

is called *the rate of shear strain*, or for short *the shear rate*. The flow described by the velocity field (1.3.1) and illustrated in Fig. 1.3.3, is called *simple shear flow*.

The fluid element in Fig. 1.3.4 is subjected to normal stresses on all six sides and shear stresses on four sides, see Fig. 1.3.3. The normal stresses on the three pairs of parallel sides may be different in a general case, but the shear stresses on the four sides are all equal. The shear stress τ may be determined from the law of balance of angular momentum applied to the rotating cylinder. For the case of steady flow at constant angular velocity ω the torque T is balanced by the shear stress τ on the cylindrical wall. Thus:

$$(\tau \cdot r) \cdot (2\pi r \cdot H) = T \;\Rightarrow \tau = \frac{T}{2\pi r^2 H} \tag{1.3.3}$$

The viscometer records the relationship between the torque T and the angular velocity ω. Using the formulas (1.3.2) and (1.3.3) we obtain a relationship between the shear stress τ and the shear rate $\dot{\gamma}$.

A fluid is said to be *purely viscous* if the shear stress τ is a function of only $\dot{\gamma}$:

$$\tau = \tau(\dot{\gamma}) \tag{1.3.4}$$

An incompressible *Newtonian fluid* is a purely viscous fluid with a linear constitutive equation:

$$\tau = \mu\dot{\gamma} \tag{1.3.5}$$

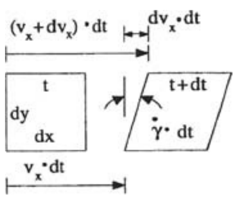


Fig. 1.3.4 Fluid element from Fig. 1.3.3

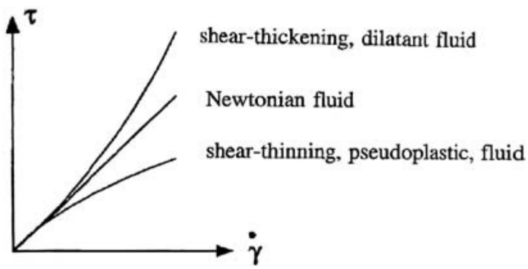


Fig. 1.3.5 Purely viscous fluids

The coefficient μ is called the *viscosity* of the fluid and has the unit $\mathrm{Ns/m^2} = \mathrm{Pa.s}$, *pascal-second*. An alternative unit is *poise* [P]: $10\mathrm{P} = 1$ Pa.s. The viscosity varies strongly with the temperature and to a certain extent also with the pressure in the fluid. For water at $0\,^\circ\mathrm{C}$ $\mu = 1.8 \cdot 10^{-3}\,\mathrm{Ns/m^2}$ and at $20\,^\circ\mathrm{C}$ $\mu = 1.0 \cdot 10^{-3}\,\mathrm{Ns/m^2}$. Usually a highly viscous fluid does not obey the linear law (1.3.5) and belongs to the *non-Newtonian fluids*. However, some highly viscous fluids are Newtonian. Mixing glycerin and water gives a Newtonian fluid with viscosity varying from $1.0 \cdot 10^{-3}$ to $1.5\,\mathrm{Ns/m^2}$ at $20\,^\circ\mathrm{C}$, depending upon the concentration of glycerin. This fluid is often used in tests comparing the behavior of a non-Newtonian fluid with that of a Newtonian fluid.

Figure 1.3.5 shows examples of the response of purely viscous non-Newtonian fluids. Most viscous non-Newtonian fluids are *shear thinning fluids*, sometime called *pseudoplastic fluids*. Examples are: nearly all polymer melts and polymer solutions, soap, biological fluids, and food products like mayonnaise. The expression "shear thinning" is reflecting that "the apparent viscosity" $\tau/\dot{\gamma}$ decreases with increasing shear rate. The expression "pseudoplastic" reflects the fact that the function $\tau(\dot{\gamma})$ has similar characteristics as for viscoplastic fluids, see Fig. 1.3.6.

For a relatively small group of real liquids *apparent viscosity* $\tau/\dot{\gamma}$ increases with increasing shear rate. These fluids are called *shear-thickening fluids* or *dilatant fluids* (expanding fluids). The last name reflects that these fluids often increase their volume when subjected to shear stresses.

Figure 1.3.6 shows the response of *viscoplastic fluids*. These material models are really solids without appreciable deformation until the shear stress has reached a

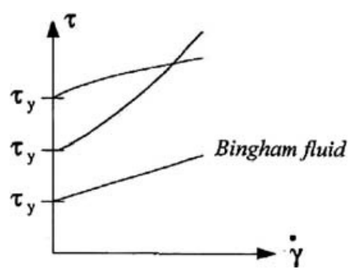


Fig. 1.3.6 Viscoplastic fluids

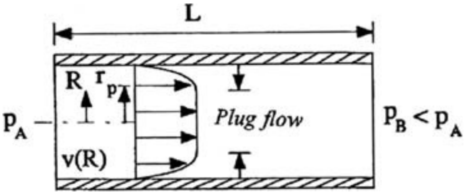


Fig. 1.3.7 Plug flow in a pipe

limit, called the *yield shear stress* τ_y. For shear stress $\tau > \tau_y$ the materials behave as fluids. A *Bingham fluid* is a linearly viscous fluid when the shear stress $\tau > \tau_y$ and the constitutive equation for simple shear flow is:

$$\tau = \left[\frac{\tau_y}{|\dot{\gamma}|} + \mu\right] \dot{\gamma} \text{ for } \dot{\gamma} \neq 0\,,\ |\tau| \leq \tau_y \text{ for } \dot{\gamma} = 0 \tag{1.3.6}$$

The Bingham fluid is discussed in Sect. 10.11.2. Fluids with a yield shear stress are: mud, clay, drilling fluids, sand in water, margarine, tooth paste, some paints, and some plastic melts.

A viscoplastic fluid in a pipe flow will obtain a velocity profile with a constant velocity over an inner core, see Fig. 1.3.7. The core flows like a plug, giving the name *plug flow* for this phenomenon. Tooth paste flows as a plug flow when it is squeezed from the tube.

Chapter 8 Fluid Mechanics gives a presentation of classical fluid mechanics for inviscid fluids and Newtonian fluids. Non-Newtonian fluids are discussed in Sects. 8.6 and 11.9.

1.4 Viscoelasticity

Propagation of sound in liquids and gases is an elastic response. Fluids are therefore in general both viscous and elastic, and the response is *viscoelastic*.

Figure 1.4.1 illustrates two phenomena that both are related to viscoelastic response in solid materials. Figure 1.4.1a shows the results of a *creep test*: A test specimen is subjected to a constant axial force resulting in a constant normal stress σ_o over the cross-section of the specimen. The longitudinal strain in the axial direction is recorded as a function of time. From the test diagram we see that the specimen momentarily gets an initial strain ε^i, which may be purely elastic or contain an elastic part $\varepsilon^{i,e}$ and a plastic part $\varepsilon^{i,p}$. Under constant stress the strain increases with time. This phenomenon is called *creep*. Creep may lead to fracture, partly because the material is weakening mechanically, and partly because the true stress over the cross-section of the specimen increases as the cross-sectional area decreases during the creep process. If the test specimen is unloaded abruptly at time t_1, the elastic part $\varepsilon^{i,e}$ of the initial strain disappears momentarily and is followed by a time dependent *restitution* to $\varepsilon = 0$, or to a strain $\varepsilon = \varepsilon^p$, which is a permanent or *plastic strain*.

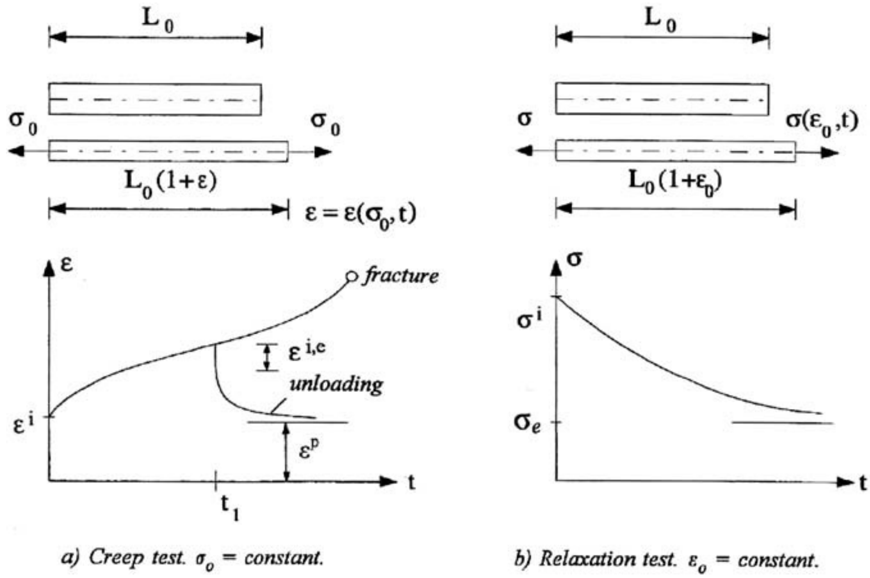


Fig. 1.4.1 Creep and stress relaxation

Figure 1.4.1b presents a *stress relaxation test*: A axial test specimen is suddenly subjected to a constant longitudinal strain. The axial force, and thereby the cross-sectional normal stress, is recorded as a function of time. The stress decreases asymptotically from the initial stress σ^i towards a lower limit, the *equilibrium stress* σ_e, which may be zero.

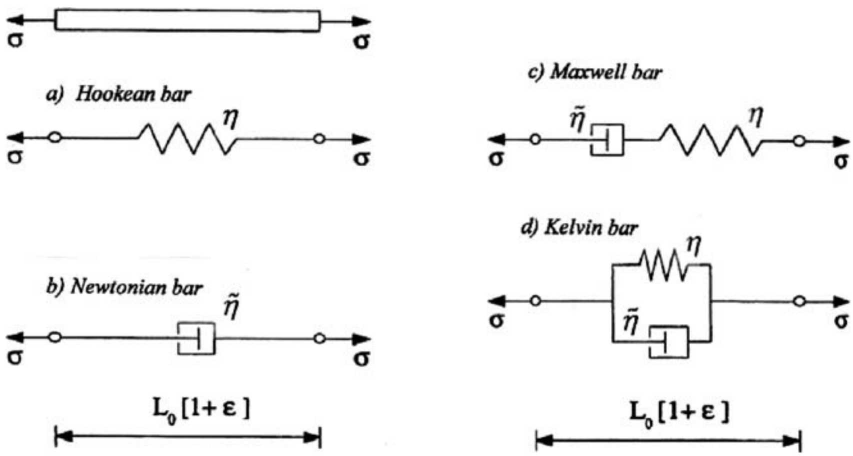


Fig. 1.4.2 Viscoelastic mechanical models

Materials, which under "normal" temperatures are responding purely elastically, may at higher, and sometimes lower, temperatures respond viscoelastically. For example, at temperatures above approximately 400 °C steel experiences creep and stress relaxation and thus has to be modelled as a viscoelastic material. Many plastics are viscoelastic at normal temperatures. Viscoelastic response is a dominating property of these materials. Concrete also shows apparent viscoelastic properties.

Viscoelastic response may be illustrated using simple mechanical models consisting of elastic springs and viscous dampers, see Fig. 1.4.2. The models a)–d) are all meant to represent an axially loaded bar with a cross-sectional normal stress σ and longitudinal strain ε in the axial direction.

A linearly elastic material, a *Hookean solid*, obeys Hooke's law: $\sigma = \eta\varepsilon$. A *linear helical spring*, see Fig. 1.4.2a, is a mechanical model for a Hookean bar. A linearly viscous material is called a *Newtonian material* or a *Newtonian fluid*. A *linear damper*, see Fig. 1.4.2b, is a mechanical model for a Newtonian bar and has the response equation:

$$\sigma = \tilde{\eta}\,\dot{\varepsilon} \tag{1.4.1}$$

where the *viscosity* $\tilde{\eta}$ is a material parameter, and $\dot{\varepsilon} \equiv d\varepsilon/dt$ is the *strain rate*.

For uniaxial stress the *Maxwell fluid* is defined by the response equation:

$$\frac{\sigma}{\tilde{\eta}} + \frac{\dot{\sigma}}{\eta} = \dot{\varepsilon} \tag{1.4.2}$$

and named after James Clerk Maxwell [1831–1879]. A linear damper and spring in series, see Fig. 1.4.2c, is a mechanical model for a *Maxwell bar*. Equation (1.4.2) is a result of adding the strain rates $\sigma/\tilde{\eta}$ for the damper and $\dot{\sigma}/\eta$ for the spring. Figure 1.4.3 shows how this material model responds in creep and relaxation tests.

For uniaxial stress the *Kelvin solid* is defined by the response equation:

$$\sigma = \eta\varepsilon + \tilde{\eta}\dot{\varepsilon} \tag{1.4.3}$$

and named after Lord Kelvin (William Thompson) [1824–1907]. A spring and damper in parallel, see Fig. 1.4.2d, is a mechanical model of a *Kelvin bar*. Equa-

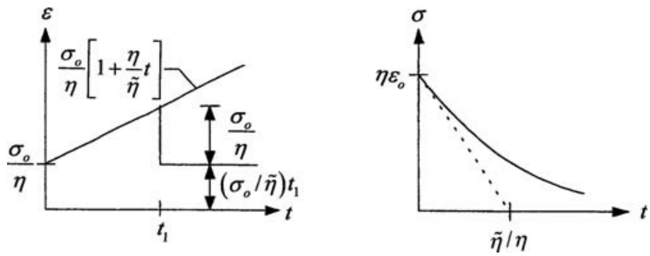


Fig. 1.4.3 Creep and relaxation tests of a Maxwell bar

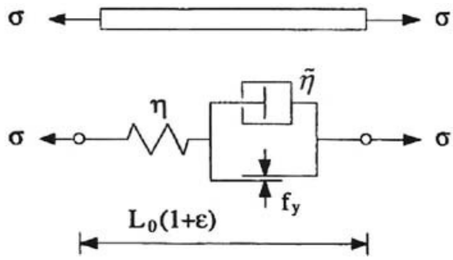


Fig. 1.4.4 Elasto-viscoplastic model

tion (1.4.3) is a result of adding the stress $\eta\varepsilon$ in the spring to the stress $\tilde{\eta}\dot{\varepsilon}$ in the damper.

The two material models represented by (1.4.2) and (1.4.3) respectively are *linearly viscoelastic materials* and are used to model plastics. Plastics are divided into two main groups: *thermo plastics and thermoset plastics*. The thermo plastics have their name because they may be melted, formed and cooled many times without changing their properties appreciably. Natural rubber is a thermo plastic. The molecules of the thermo plastics are long and interwoven without being chemical bonded with each other. The thermo plastics have a fluid-like response similar to the Maxwell fluid. The thermoset plastics are created by letting long molecules bind both in the longitudinal direction and in the cross direction such that they form a three-dimensional net. Usually a thermoset plastic is formed from a thermo plastic by adding a hardener and/or by adding heat, such that the chemical bonds between molecules are formed. After the hardening process the material cannot be remelted, and behaves like viscoelastic solid similar to the Kelvin material. Vulcanized rubber is a thermoset plastic.

The most commonly used constitutive model for the viscoelastic behavior of metals is the *Norton fluid*, named after F.H. Norton (1920). The response equation for uniaxial stress is:

$$\varepsilon = \frac{\dot{\sigma}}{\eta} + \frac{\dot{\varepsilon}_c}{\sigma_c}\left|\frac{\sigma}{\sigma_c}\right|^{n-1}\sigma \tag{1.4.4}$$

where $\dot{\varepsilon}_c, \sigma_c$, and $n > 1$ are temperature dependent material parameters. A mechanical model having the same response equation may consist of a series of a linear spring and a non-linear viscous damper. Viscoelastic materials are discussed in more details in Chap. 9 Viscoelasticity.

Figure 1.4.4 represents a mechanical model for an *elasto-viscoplastic material*. The response equation for uniaxial stress is:

$$\varepsilon = \frac{\sigma}{\eta} \text{ for } |\sigma| < f_y\,, \; \dot{\varepsilon} = \frac{\dot{\sigma}}{\eta} + \left[\frac{1 - f_y|\sigma|}{\mu}\right]\sigma \text{ for } |\sigma| > f_y \tag{1.4.5}$$

A discussion of this kind of behavior is presented in Sect. 10.10.

1.5 An Outline for the Present Book

The previous sections of this chapter present some of the basic concepts of macromechanical properties of materials when they are considered to be continuous matter. In the following chapters of the book the mathematical theory of Continuum mechanics will be introduced in a logical order. Detailed presentations of the most common and important constitutive models and their applications will be presented. The present chapter is now concluded by a short outline of the contents of the rest of the book.

Some basic mathematical concepts are needed first: matrix algebra, vector algebra and vector analysis. This will be the topic of Chap. 2 *Mathematical Foundation*. Vectors are presented using index notation, i.e. through vector components in a Cartesian coordinate system, by collecting the components in vector matrices, i.e. the index free notation, and finally by the coordinate invariant and bold face notation.

Chapter 3 *Dynamics* contains the basic Continuum Mechanics: Kinematics, Kinetics with the equations of motion for bodies and material points, and General Stress Analysis, common to all continuous materials.

When the concept of stress tensor has been introduced in Chap. 3, it becomes natural to discuss the general concept of *tensors* in Chap. 4. Tensors are defined as coordinate invariant quantities through *multilinear scalar-valued functions of vectors*, alternatively expressed by their components, primarily in Cartesian coordinates. The formal presentation of tensor analysis expressed by components in general curvilinear coordinates is postponed to Chap. 12 Tensors in Euclidean Space E_3.

Chapter 4 also contains sections on rigid body kinematics and kinetics. The rigid-body kinematics represents an important prerequisite for the understanding of the analysis of large deformation in Sect. 5.5. Rigid-body kinetics introduces the inertia tensor.

The analysis of small and large deformations and strains is the subject of Chap. 5 *Deformation Analysis*. The analysis of strain rates, rates of deformation, and rates of rotations, which is of major importance in Fluid Mechanics and in describing rate dependent materials, is also included in this chapter. A final section on the *Piola-Kirchhoff stress tensors*, which is important when dealing with materials undergoing large deformations, concludes Chap. 5.

The common foundation for all continuous media, solids, liquids, and gases, also includes Chap. 6 *Work and Energy*. Both the *mechanical energy balance equation* and the *thermal energy balance equation*, which is based on the *first law of thermodynamics*, are presented. The *second law of thermodynamics* is given a brief presentation. Some consequences of this law in constitutive modelling are given in Chap. 11 *Constitutive Equations*.

Chapter 7 *Theory of Elasticity* presents an extensive exposition of the classical theory for isotropic, linearly elastic materials. Many important applications in two-dimensional theory are treated, and the chapter gives a comprehensive discussion on elastic waves. A section on anisotropic elastic materials having different degrees of symmetries serves as an introduction to fiber composite materials. In a final section on large deformations Finite elasticity is briefly presented.

Fluid Mechanics is the subject of Chap. 8, which has sections on perfect fluids, i.e. inviscid fluid, linearly viscous fluids, i.e. Newtonian fluids, and potential flow of fluids. The chapter concludes with a presentation of non-Newtonian fluid models. Other non-Newtonian fluids will be presented in Chap. 9 Viscoelasticity and in Chap. 11 Constitutive Equations.

Chapter 9 Viscoelasticity presents the theory of viscoelastic models for both solids and fluids. Both linear models, most suitable for plastics, and non-linear models appropriate for metals, are presented.

Chapter 10 Theory of Plasticity starts with a presentation of criteria for yielding when a solid material is in a general state of stress. Constitutive equations relating stress and strain when the material yields are the *yield laws* or flow rules. These are presented for the most commonly used material models. Models for ideal plasticity and models for strain hardening materials are presented. A section discusses the important theorems for upper and lower bound solutions for elastoplastic materials. *Viscoplasticity*, which combines plastic behaviour with viscous response at yielding, is given a brief presentation. Finally the yield line theory is presented.

Principles and theories for the mathematical modelling of materials is the main theme in Chap. 11 Constitutive Equations. First it is necessary to include an extension of the deformation analysis from Chap. 5. The concepts of objective tensors and corotational time derivative are presented and discussed. The concepts of convected time derivatives are presented, but given a geometrical interpretation in Chap. 13. Non-linear materials with elastic, viscous, viscoelastic, plastic, viscoplastic and viscoelastic-plastic responses are discussed. Section 11.9 presents some advanced fluid models.

Chapter 12 Tensors in Euclidean Space E_3 presents the components of tensors in general curvilinear coordinate systems in the 3-dimensional physical Euclidean space E_3. The concept of tensors is the same as in Chap. 4 *Tensors*, but is now extended to more general coordinate systems.

Chapter 13 Continuum Mechanics in Curvilinear Coordinates presents the general field equations of Continuum Mechanics in curvilinear coordinates. Concepts like convected coordinates and the related deformation analysis are introduced. The concept of convected time derivatives is here given geometrically definitions.

Chapter 2
Mathematical Foundation

2.1 Matrices and Determinants

A *two-dimensional matrix* is a table or rectangular array of elements arranged in rows and columns. The matrix A with 2 rows and 3 columns is alternatively presented as:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix} \equiv (A_{\alpha i}) \equiv A \tag{2.1.1}$$

The elements may be numbers or functions. The symbol $A_{\alpha i}$ represents an arbitrary element for which the *row number* (α) may take the values 1 or 2, while the *column number (i)* may have any of the values 1, 2, or 3. Lower case Greek letter indices shall in general represent the numbers 1 and 2, while lower case Latin letter indices shall represent the numbers 1, 2 and 3. In the discussion of anisotropic elastic materials in Sect. 7.8 and composite materials in Sect. 7.9 we need to extend the number region of Greek indices to the numbers 1 to 6. In the present exposition matrices will in general be denoted by capital Latin letters. An exception to this notation applies to matrices with one row or one column and is presented below. Many textbooks on Matrix Analysis prefer to use capital boldface letters for matrices. But this book reserves boldface letters for the coordinate invariant notation of vectors and tensors.

It is convenient to let the symbol $A_{\alpha i}$ represent both an element in the matrix A, but also the complete matrix A. The table and the two other symbols in the identities (2.1.1), and the symbol $A_{\alpha i}$ now represent four different ways of presenting one and the same matrix. The matrix $A_{\alpha i}$, which has 2 rows and 3 columns, is called a 2×3 ("two by three") matrix. The matrices $B_{\alpha\beta}$ (2×2 matrix) and C_{ij} (3×3 matrix) have as many columns as rows and are called square matrices, due to the forms of their tables.

A *one-dimensional matrix* is an array of elements arranged in one column or one row, and is called a *vector matrix*. One-dimensional matrices will in general be denoted by lower case latin letters and will be considered to be a matrix with one column, *a column matrix*. For practical reasons it may be convenient to write a column matrix on a horizontal rather than a vertical line. An example is:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \equiv (a_i) \equiv a \equiv \left\{ a_1 \; a_2 \; a_3 \right\} \tag{2.1.2}$$

From a two-dimensional matrix $A = [A_{\alpha i}]$ we may construct a *transposed matrix* $A^T = [A_{i\alpha}]$ by interchanging rows and columns. For a square matrix C we have:

$$C^T_{ij} = C_{ji} \tag{2.1.3}$$

The transposed matrix a^T of a column matrix a is a *row matrix*:

$$a^T = \left(a_1 \; a_2 \; a_3 \right) \tag{2.1.4}$$

An n-dimensional matrix is represented by a set of elements with n indices. For example will the elements D_{ijk} represent a three-dimensional matrix. The matrix algebra, as presented below, is designed for one- and two-dimensional matrices.

Addition of matrices is defined only for matrices of the same size. The sum of two 3×3 matrices A and B is a 3×3 matrix C.

$$A + B = C \;\Leftrightarrow\; A_{ij} + B_{ij} = C_{ij} \tag{2.1.5}$$

The element C_{ij} in the matrix C is obtained by adding corresponding elements A_{ij} and B_{ij} in the matrices A and B. We easily see that the operation is both associative and commutative. Addition also applies to n-dimensional matrices.

The product of a matrix $A = (A_{ij})$ by a term α is a matrix $\alpha A = (\alpha A_{ij})$. The *matrix product* AB of two matrices A and B is a new matrix C, such that:

$$AB = C \;\Leftrightarrow\; \sum_{k=1}^{3} A_{ik} B_{kj} = C_{ij} \tag{2.1.6}$$

The element C_{ij} in the matrix C is obtained as the sum of the all product pairs $A_{ik}B_{kj}$ of the elements in row (i) of A and the elements in column (j) of B. It follows that the number of columns in A has to be equal to the number of rows in B. If all elements in C are zero, C is a *zero matrix 0*. Note that in general:

$$(AB)\,C = A\,(BC) = ABC \quad , \quad AB \neq BA \tag{2.1.7}$$

Thus multiplication of matrices is an associative, but not a commutative, operation. From the definition (2.1.6) it follows that:

$$(AB)^T = B^T A^T \tag{2.1.8}$$

We shall now introduce an important convention of great consequence when dealing with matrices in the index format.

The *Einstein's summation convention*: An index repeated once and only once in a term implies a summation over the number region of that index.

The convention is attributed to Albert Einstein [1879–1955]. By this convention we may write:

$$\sum_{k=1}^{3} A_{ik}B_{kj} \equiv A_{ik}B_{kj} \tag{2.1.9}$$

The summation convention does not apply if the *summation index* is repeated more than once. For example:

$$\sum_{k=1}^{3} A_{ik}B_{kj}a_k \neq A_{ik}B_{kj}a_k \tag{2.1.10}$$

An index that does not imply summation is called a *free index*. In the inequality (2.1.10) the index (k) on the left hand side is a summation index, while the index (k) on the right hand side is a free index. A summation index is also called a *"dummy" index* because it may be replaced by a different letter without changing the result of the summation. For example:

$$A_{ik}B_{kj} = A_{il}B_{lj} \tag{2.1.11}$$

The product of a square matrix and a vector matrix is a new vector matrix.

$$Aa = b \;\Leftrightarrow\; A_{ik}a_k = b_i \tag{2.1.12}$$

Note that $a_kA_{ki} \neq b_i$. The product a_kA_{ki} may alternatively be expressed as

$$a^TA = c^T \;\Leftrightarrow\; A^Ta = c \;\Leftrightarrow\; a_kA_{ki} = A_{ki}a_k = c_i \tag{2.1.13}$$

Two *unit matrices* or *identity matrices* are now defined:

$$1 = \left[\delta_{\alpha\beta}\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;,\; 1 = (\delta_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.1.14}$$

The element symbols $\delta_{\alpha\beta}$ and δ_{ij} are called *Kronecker deltas*, named after Leopold Kronecker [1823–1891]. The elements have the following values:

$$\delta_{ij} = 1 \text{ when } i = j \quad , \quad \delta_{ij} = 0 \text{ when } i \neq j \tag{2.1.15}$$

The effect of a unit matrix is shown in the following example.

$$A1 = 1A = A \;\Leftrightarrow\; A_{ik}\delta_{kj} = \delta_{ik}A_{kj} = A_{ij}$$

In the definition of the determinant of a matrix we need two special matrices:

$$\text{The } \textit{permutation symbol } e_{\alpha\beta} = 0,\ 1, \text{or} -1, \text{ such that: } \left(e_{\alpha\beta}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The *permutation symbol* e_{ijk}:

$$= \begin{cases} 0 \text{ when two or three indices are equal} \\ 1 \text{ when the indices form a cyclic permutation of the numbers 123} \\ \text{-1 when the indices form a cyclic permutation of the numbers 321} \end{cases}$$

For example: $e_{122} = e_{333} = 0, e_{123} = e_{231} = e_{312} = 1,\ e_{321} = e_{213} = e_{132} = -1$. It follows from the definitions that $e_{ijk} = e_{kij} = -e_{kji}$, and that $e_{\alpha\beta} = e_{\alpha\beta 3}$. We also find that:

$$e_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i) \tag{2.1.16}$$

The permutation symbol e_{ijk} and the Kronecker delta δ_{ij} are related through the identity:

$$e_{ijk}e_{rsk} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr} \tag{2.1.17}$$

The validity of the identity may be tested by selecting different sets of the free indices *i, j, r*, and *s* on both sides of the equation.

The *determinants*, det A, of 2×2 and 3×3 square matrices A are defined as follows.

$$\det A = \det\left(A_{\alpha\beta}\right) \equiv e_{\alpha\beta}A_{1\alpha}A_{2\beta} \equiv e_{\alpha\beta}A_{\alpha 1}A_{\beta 2} \tag{2.1.18}$$

$$\det A = \det\left(A_{ij}\right) \equiv e_{ijk}A_{1i}A_{2j}A_{3k} \equiv e_{ijk}A_{i1}A_{j2}A_{k3} \tag{2.1.19}$$

It follows that: $\det A^T = \det A$. From the definitions it follows that: $\det A^T = \det A$, and the determinant of a matrix is zero when two rows or two columns are identical. For example:

$$A_{1j} = A_{2j} \ \Rightarrow \det A = 0$$

By inspection we find that:

$$e_{\alpha\beta}A_{\alpha\gamma}A_{\beta\lambda} = (\det A)\, e_{\gamma\lambda} \quad , \quad e_{ijk}A_{ir}A_{js}A_{kt} = (\det A)\, e_{rst} \tag{2.1.20}$$

This result may be used to prove the *multiplication theorem for determinants*:

$$\det(AB) = (\det A)\,(\det B) \tag{2.1.21}$$

For 3×3 matrices we find, using formula (2.1.19) for B, then formula (2.1.20) for A, and finally formula (2.1.19) for the matrix product AB:

$$\begin{aligned}(\det A)\,(\det B) &= (\det A)\, e_{rst}B_{r1}B_{s2}B_{t3} = e_{ijk}A_{ir}A_{js}A_{kt}B_{r1}B_{s2}B_{t3} \\ &= e_{ijk}\left(A_{ir}B_{r1}\right)\left(A_{js}B_{s2}\right)\left(A_{kt}B_{t3}\right) = \det\,(AB)\end{aligned}$$

The determinant of a matrix A may be expressed as a linear function of the elements A_{ir} in any arbitrarily chosen row or column. The coefficients in the function are called the *cofactors* Co A_{ir}, or the *algebraic complements*, to the corresponding elements A_{ir}. For instance:

$$\det A = e_{ijk}A_{i1}A_{j2}A_{k3} = A_{i1}\left(e_{ijk}A_{j2}A_{k3}\right) = A_{i1}\mathrm{Co}A_{i1} \ \Rightarrow\ \mathrm{Co}A_{i1} = e_{ijk}A_{j2}A_{k3}$$

In general we find:

$$\mathrm{Co}A_{ir} = \frac{1}{2} e_{ijk} e_{rst} A_{js} A_{kt} = \frac{\partial (\det A)}{\partial A_{ir}} \tag{2.1.22}$$

$$\mathrm{Co}A_{\alpha\gamma} = e_{\alpha\beta} e_{\gamma\lambda} A_{\beta\lambda} = \frac{\partial (\det A)}{\partial A_{\alpha\gamma}} \tag{2.1.23}$$

$$\det A = \sum_i A_{ir} \mathrm{Co}A_{ir} = \sum_i A_{ri} \mathrm{Co}\,\mathrm{A}_{ri} \quad , \quad r = 1,\ 2,\ \text{or}\ 3 \tag{2.1.24}$$

The cofactors are elements in the matrix Co A, and we find that:

$$(\det A)1 = A^T \mathrm{Co}A = A\,\mathrm{Co}A^T = (\mathrm{Co}A)A^T \;\Leftrightarrow \tag{2.1.25}$$
$$(\det A)\,\delta_{ij} = A_{ki} \mathrm{Co}A_{kj} = A_{ik} \mathrm{Co}A_{jk} = (\mathrm{Co}A_{ik})A_{jk}$$

If the product of two square matrices A and B is equal to the unit matrix then we call one matrix the *inverse matrix* to the other. The inverse matrix to A is denoted A^{-1}. Thus:

$$AA^{-1} = A^{-1}A = 1 \tag{2.1.26}$$

A condition for the inverse matrix A^{-1} to exist is that the determinant det A is not equal to zero. The inverse matrix A^{-1} may be determined as follows. Using formula (2.1.25) we get:

$$A^{-1}(\det A) = A^{-1}(\det A)\,1 = A^{-1}A\,\mathrm{Co}A^T = \mathrm{Co}A^T \;\Rightarrow \tag{2.1.27}$$
$$A^{-1} = \frac{1}{\det A}\mathrm{Co}A^T \quad \Leftrightarrow \quad A^{-1}_{ij} = \frac{1}{\det A}\mathrm{Co}A_{ji}$$

Natural powers of A and A^{-1}, where n is a natural number, are defined by:

$$A^n = AA\cdots A \quad , \quad A^{-n} = A^{-1}A^{-1}\cdots A^{-1} \tag{2.1.28}$$

In addition to the determinant det A of a matrix A, we shall also need the *trace*, tr A, of A and the *norm* of A, denoted by norm A or $||A||$. These quantities are defined respectively by:

$$\mathrm{tr}A \equiv A_{kk} = A_{11} + A_{22} + A_{33} \tag{2.1.29}$$

$$normA \equiv \|A\| \equiv \sqrt{\mathrm{trAA^T}} = \sqrt{A_{ij}A_{ij}} \tag{2.1.30}$$

The norm of the matrix A is also called the *magnitude* of the matrix A.

2.2 Coordinate Systems and Vectors

In order to localize physical objects in space and to define motion in space, we need a *reference frame*, here for short called a *reference* and denoted by *Rf*. The reference may be the earth, a space laboratory, or the Milky Way. A quantity that is not defined relative to a reference, will be called a *reference invariant quantity*, an *objective quantity*, or a *reference invariant*. The distance between two space points and the temperature in a body are two examples of reference invariants. A quantity defined relative to a reference, will be called a *reference related quantity*. The velocity of a body gives an example of a reference related quantity.

Figure 2.2.1 shows three orthogonal axes x_i that intersect in a point O, the *origin*, and that are attached to the reference *Rf* we have chosen. The axes represent a *right-handed system* in the following sense: If the right hand is held about the x_3-axis such that the fingers point in the direction of a 90°-rotation of the positive x_1-axis toward the positive x_2-axis, the thumb then points to the direction of the positive x_3-axis. If the direction of one of the axes is reversed, or two of the axes are interchanged, the new system is a *left-handed system*.

The distances x_1, x_2, and x_3 from the three orthogonal planes Ox_2x_3, Ox_3x_1, and Ox_1x_2 to a point P in space are positive or negative according to which side of the planes the point lies. For instance, x_1 is positive when P is on the same side of the Ox_2x_3-plane as the positive x_1-axis. The three distances x_i are called the *coordinates* of P relative to an *orthogonal Cartesian right-handed coordinate system Ox*. The x_i-axes are called *coordinate axes*. The *coordinate planes* are defined by $x_1 =$ constant, $x_2 =$ constant, or $x_3 =$ constant. The lines of intersection between these planes are the *coordinate lines*. Three coordinate planes and three coordinate lines intersect in every point P in space.

In the general exposition we shall use orthogonal Cartesian right-handed coordinate systems until Chap. 12 where general curvilinear coordinate analysis is introduced. In special applications throughout the book we shall employ cylindrical coordinate and spherical coordinates. The quantities we call scalars, vectors, and tensors are invariant with respect to our choice of coordinate system, and they are therefore called *coordinate invariants*. Reference invariant quantities are coordinate invariant, but in general the opposite does not apply.

The following definitions of scalars and vectors are useful and important.

Scalar invariant or for short *scalar*: a coordinate invariant quantity uniquely expressed by a magnitude.

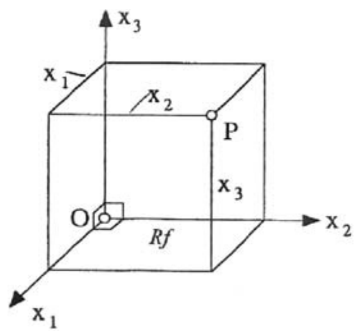


Fig. 2.2.1 Cartesian coordinate system

Examples of scalars: the distance between two points in space, the temperature θ, the pressure in a fluid p, and the time t. Scalars will in the present exposition preferably be denoted by Greek letters, the pressure p being the first exception.

Vector: a coordinate invariant quantity uniquely expressed by a magnitude and a direction in space, and that obeys the parallelogram law by addition.

Examples of vectors: force $\mathbf{f}$, velocity $\mathbf{v}$, acceleration $\mathbf{a}$, and angular velocity $\mathbf{w}$. Finite rotations may be expressed by magnitudes and directions. But since they cannot be added geometrically according to the parallelogram law, they are not proper vector quantities.

A vector is presented geometrically as an arrow giving the direction of the vector and with a length that represents the *magnitude* of the vector relative to the length of a chosen unit. Vectors will in the present exposition preferably be denoted by small Latin bold face letters, as indicated above in the examples. The magnitude of a vector $\mathbf{a}$ is denoted by:

$$\textit{magnitude} \text{ of } \mathbf{a} \equiv |\mathbf{a}| \equiv a$$

The magnitude is also called the *norm* of the vector.

We distinguish between free vectors, line vectors, and point vectors. A *point vector* is related to a point in space. For instance: The moment of a force about a point. The velocity of a mass particle at the time t is a point vector related to the position of the particle at the time t. A *line vector* is related to a straight line or axis in space. For instance: A force when we consider the dynamic action of the force on rigid body is a line vector. The force may be translated along the line of action of the force without altering its external action on the rigid body. A *free vector* is independent on any point or line in space. Examples are: The moment of a force couple, the angular velocity and the translational velocity of a rigid body.

The algebra of vectors, i.e. addition, subtraction, scalar and vector multiplication of vectors may be presented geometrically, i.e. without application of coordinate systems. This fact proves that vectors are indeed coordinate invariant quantities.

In the Ox-system, see Fig. 2.2.1, we define *base vectors* $\mathbf{e}_i$, which are unit vectors, i.e. dimensionless vectors of length 1, in the directions of the coordinate axes.

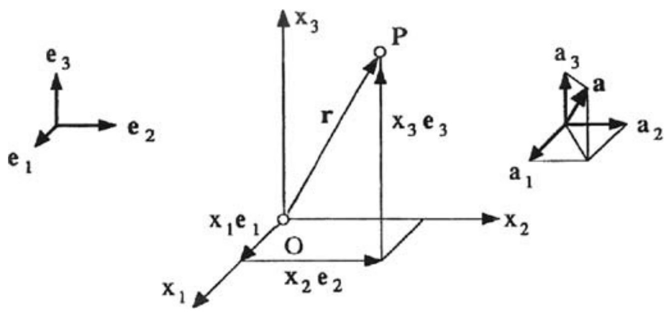


Fig. 2.2.2 Base vectors and vector components

A point P in space marks a *place* or *position*. The place P is given by the *place coordinates* x_i, or by the *place vector*:

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \equiv x_i\mathbf{e}_i \tag{2.2.1}$$

If the components x_i are presented in a vector matrix:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \equiv (x_i) \equiv x \equiv \{x_1 \ x_2 \ x_3\} \tag{2.2.2}$$

we have four equivalent ways of denoting a *place* in space: P, x_i, x, or $\mathbf{r}$.

A vector $\mathbf{a}$ may be decomposed into components a_i parallel to the coordinate axes, i.e. parallel to the base vectors, and then presented in three different ways:

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \equiv a_i\mathbf{e}_i \equiv [a_1, a_2, a_3] \tag{2.2.3}$$

a_i are called the *scalar components* of the vector $\mathbf{a}$, or for short the components of $\mathbf{a}$ in the coordinate system Ox. The vector may finally be represented by its vector matrix in the Ox-system:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \equiv (a_i) \equiv a \equiv \{a_1 \ a_2 \ a_3\} \tag{2.2.4}$$

Note that we use the symbol a both for the magnitude of the vector and for the vector matrix. However, we avoid using the two interpretations of a in the same connection.

Addition and *subtraction* of vectors are defined according to the parallelogram law as indicated in Fig. 2.2.3, but the geometrical form is transformed to the component form given by:

$$\mathbf{a} + \mathbf{b} = \mathbf{c} \ \Leftrightarrow \ a_i + b_i = c_i \ \Leftrightarrow \ a + b = c \tag{2.2.5}$$

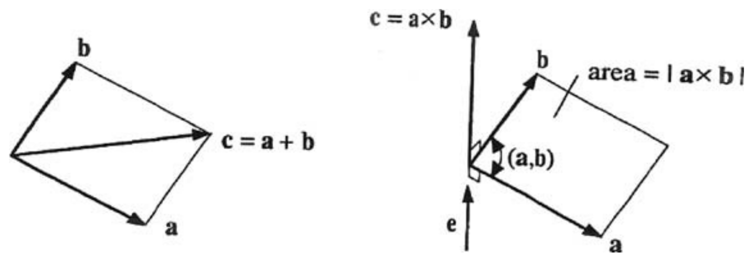


Fig. 2.2.3 Vector sum and vector product

The product of a scalar α and a vector $\mathbf{a}$ is a new vector $\mathbf{b} = \alpha\mathbf{a}$ with magnitude $b = \alpha a$ and the direction of $\mathbf{a}$ when α is positive and in the opposite direction when α is negative.

The *scalar product* of two vectors $\mathbf{a}$ and $\mathbf{b}$, also called the *dot product*, is defined by:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\mathbf{a}, \mathbf{b}) \tag{2.2.6}$$

(**a, b**) is the angle between the two vectors. The operation is commutative and distributive. For the base vectors of the coordinate system *Ox* we get:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \tag{2.2.7}$$

Now:

$$\begin{gathered}\mathbf{a} \cdot \mathbf{b} = (a_i\mathbf{e}_i) \cdot (b_j\mathbf{e}_j) = a_i b_j (\mathbf{e}_i \cdot \mathbf{e}_j) = a_i b_j \delta_{ij} \;\Rightarrow \\ \mathbf{a} \cdot \mathbf{b} = a_i b_i = a^T b = b^T a\end{gathered} \tag{2.2.8}$$

The magnitude of a vector may be computed from the formula:

$$a \equiv |\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_i a_i} \tag{2.2.9}$$

Note again the symbol a is used both for the components matrix of the vector $\mathbf{a}$, as in (2.2.4), and for the length of the vector $\mathbf{a}$.

The *vector product* of two vectors $\mathbf{a}$ and $\mathbf{b}$, also called the *cross product*, is defined by, see Fig. 2.2.3:

$$\mathbf{a} \times \mathbf{b} = \mathbf{c} \equiv |\mathbf{a}|\,|\mathbf{b}| \sin(\mathbf{a}, \mathbf{b})\,\mathbf{e} \tag{2.2.10}$$

The angle (**a, b**) shall be the smallest angle between the two vectors $\mathbf{a}$ and $\mathbf{b}$. The unit vector $\mathbf{e}$ is normal to the plane through $\mathbf{a}$ and $\mathbf{b}$, and point in the direction of the thumb if the right hand is held about $\mathbf{e}$ with the fingers pointing in the direction of a rotation from $\mathbf{a}$ to $\mathbf{b}$. The operation (2.2.10) is not commutative, but distributive.

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \quad , \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \tag{2.2.11}$$

From the definition of the vector product and the definition of a right-handed coordinate system it follows that the base vectors are related according to:

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3 \;,\; \mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{e}_3 \quad \text{etc.} \;\Rightarrow \mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k \tag{2.2.12}$$

This result may be used to derive the following result.

$$\begin{gathered}\mathbf{a} \times \mathbf{b} = (a_i\mathbf{e}_i) \times (b_j\mathbf{e}_j) = (\mathbf{e}_i \times \mathbf{e}_j)\, a_i b_j = e_{ijk} a_i b_j \mathbf{e}_k \Rightarrow \\ \mathbf{a} \times \mathbf{b} = \mathbf{c} \;\Leftrightarrow\; e_{ijk} a_i b_j = c_k\end{gathered} \tag{2.2.13}$$

It follows from this expression that the vector product also may be computed from a determinant of a matrix:

$$\mathbf{a} \times \mathbf{b} = e_{ijk} a_i b_j \mathbf{e}_k = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} \tag{2.2.14}$$

The *scalar triple product* of the three vectors **a, b**, and **c**, also called the *box product*, is defined by:

$$[\mathbf{abc}] = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) \tag{2.2.15}$$

Using formula (2.2.13) for the vector product and formula (2.2.8) for the scalar product, we obtain the result:

$$[\mathbf{abc}] = e_{ijk} a_i b_j c_k = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \tag{2.2.16}$$

The base vectors are related according to:

$$[\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3] = 1, \; [\mathbf{e}_3 \mathbf{e}_2 \mathbf{e}_1] = -1 \quad \text{etc.} \;\Leftrightarrow\; [\mathbf{e}_i \mathbf{e}_j \mathbf{e}_k] = e_{ijk} \tag{2.2.17}$$

This relation may be taken to define an *orthogonal right-handed system of unit vectors*.

2.3 Coordinate Transformations

Vectors and tensors, which will be defined in Chap. 4, are coordinate invariant quantities, which in any coordinate system we have chosen are represented by their components. It is important to determine the relations between sets of components relative to different coordinate systems. Let Ox and $\bar{O}\bar{x}$ denote two Cartesian coordinate systems, see Fig. 2.3.1. When the Ox-system has been given, the system $\bar{O}\bar{x}$ is determined by the position vector $-\mathbf{c}$ from the origin O to the origin $\bar{O}$ of the coordinate system, and the base vectors $\bar{\mathbf{e}}_i$.

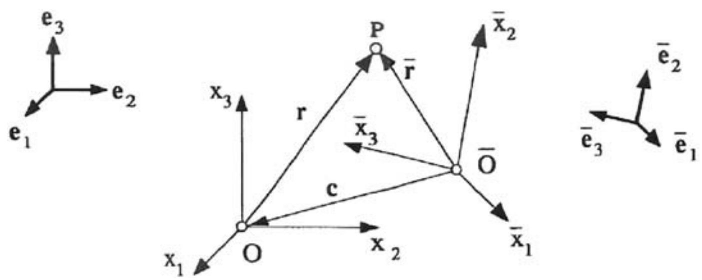


Fig. 2.3.1 Coordinate transformation

$$\mathbf{c} = c_k \mathbf{e}_k = \bar{c}_i \bar{\mathbf{e}}_i \tag{2.3.1}$$

$$\bar{\mathbf{e}}_i = Q_{ik}\mathbf{e}_k \quad , \quad \mathbf{e}_k = Q_{ik}\bar{\mathbf{e}}_i \tag{2.3.2}$$

where:

$$Q_{ik} = \cos(\bar{\mathbf{e}}_i, \mathbf{e}_k) = \bar{\mathbf{e}}_i \cdot \mathbf{e}_k \tag{2.3.3}$$

The *direction cosines* Q_{ik} are elements in a matrix that is called the *transformation matrix* Q for the coordinate transformation from Ox to $\bar{O}\bar{x}$. The 9 elements Q_{ik} are, as will be shown below, connected through 7 relations.

We require that the $\bar{O}\bar{x}$ -system is an orthogonal Cartesian right-handed coordinate system and that the base vectors $\bar{\mathbf{e}}_i$ are unit vectors. These requirements are satisfied by the conditions:

$$\bar{\mathbf{e}}_i \cdot \bar{\mathbf{e}}_j = \delta_{ij} \quad , \quad [\bar{\mathbf{e}}_1\bar{\mathbf{e}}_2\bar{\mathbf{e}}_3] = 1 \tag{2.3.4}$$

Confer the formulas (2.2.7) and (2.2.17). Now:

$$\bar{\mathbf{e}}_i \cdot \bar{\mathbf{e}}_j = (Q_{ik}\mathbf{e}_k) \cdot (Q_{jl}\mathbf{e}_l) = Q_{ik}Q_{jl}(\mathbf{e}_k \cdot \mathbf{e}_l) = Q_{ik}Q_{jl}\delta_{kl} = Q_{ik}Q_{jk}$$
$$[\bar{\mathbf{e}}_1\bar{\mathbf{e}}_2\bar{\mathbf{e}}_3] = [(Q_{1r}\mathbf{e}_r)(Q_{2s}\mathbf{e}_s)(Q_{3t}\mathbf{e}_t)] = Q_{1r}Q_{2s}Q_{3t}[\mathbf{e}_r\mathbf{e}_s\mathbf{e}_t] = Q_{1r}Q_{2s}Q_{3t}e_{rst} = \det Q$$

These results show, when compared with the conditions (2.3.4), that the transformation matrix Q has to satisfy the conditions:

$$Q_{ik}Q_{jk} = \delta_{ij} \Leftrightarrow QQ^T = 1 \Leftrightarrow Q^{-1} = Q^T \tag{2.3.5}$$

$$\det Q = 1 \tag{2.3.6}$$

A matrix with these properties is called an *orthogonal matrix*. From the formulas (2.3.2) we see that the rows in the matrix Q represent the components of the orthogonal vectors $\bar{\mathbf{e}}_i$ in the Ox-system, and that the columns in Q represent the components of the orthogonal vectors $\mathbf{e}_k$ in the $\bar{O}\bar{x}$-system.

In plane, two-dimensional analysis we introduce plane Cartesian coordinate systems Ox and $\bar{O}\bar{x}$ with the x_α-axes and the $\bar{x}_\alpha$ -axes in the plane. Let the $\bar{x}_1$ -axis make the angle θ with respect to the x_1 -axis. Then:

$$Q = (Q_{\alpha\beta}) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad Q = (Q_{ik}) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.3.7}$$

A position P in three-dimensional space may either be defined by the position vector $\mathbf{r}$ from the origin O or by the position vector $\bar{\mathbf{r}}$ from the origin $\bar{O}$. From Fig. 2.3.1 we obtain:

$$\bar{\mathbf{r}} = \mathbf{c} + \mathbf{r} \tag{2.3.8}$$

The coordinates of P are x_k in Ox and $\bar{x}_i$ in $\bar{O}\bar{x}$. Now:

$$\mathbf{r} = x_k\mathbf{e}_k = x_kQ_{ik}\bar{\mathbf{e}}_i \quad \text{and} \quad \bar{\mathbf{r}} = \bar{x}_i\bar{\mathbf{e}}_i$$

The representation of the relation (2.3.8) in the coordinate system $\bar{O}\bar{x}$ provides the *coordinate transformation formula*:

$$\bar{x}_i = \bar{c}_i + Q_{ik}x_k \quad \Leftrightarrow \quad \bar{x} = \bar{c} + Qx \tag{2.3.9}$$

The inverse transformation is easily found to be:

$$x_k = -c_k + Q_{ik}\bar{x}_i \;\Leftrightarrow\; x = -c + Q^T\bar{x} \tag{2.3.10}$$

For a vector **a** with the components a_k in Ox and $\bar{a}_i$ in $\bar{O}\bar{x}$, we get:

$$\begin{gathered}\mathbf{a} = \bar{a}_i\bar{\mathbf{e}}_i = \bar{a}_i Q_{ik}\mathbf{e}_k = a_k\mathbf{e}_k = a_k Q_{ik}\bar{\mathbf{e}}_i \Leftrightarrow \\ \bar{a}_i = Q_{ik}a_k \Leftrightarrow \; \bar{a} = Qa \;,\; a_k = Q_{ik}\bar{a}_i \Leftrightarrow \; a = Q\bar{a}\end{gathered} \tag{2.3.11}$$

2.4 Scalar Fields and Vector Fields

Scalars and vectors, representing physical or geometrical quantities related to places **r** in a defined region in space, are represented by fields.

$$\alpha = \alpha(\mathbf{r},t) \quad , \quad \mathbf{a} = \mathbf{a}(\mathbf{r},t) \tag{2.4.1}$$

are respectively a *scalar field* and a *vector field*. The symbol t is the *time*. The temperature θ in a body is an example of a scalar field, while the velocity **v** is a vector field. If the time t does not appear as an independent variable in a field function, the field is said to be a *steady field*. If the time t is the only argument in the field function, the field is called a *uniform field*. The fields are also denoted by:

$$\alpha = \alpha(x_1,x_2,x_3,t) \equiv \alpha(x,t) \quad , \quad \mathbf{a} = \mathbf{a}(x_1,x_2,x_3,t) \equiv \mathbf{a}(x,t) \tag{2.4.2}$$

The partial derivatives of a field function $f(x,t)$ with respect to the space coordinates x_i, $\bar{x}_i$, and the time t are expressed by the symbols:

$$\partial_t f \equiv \frac{\partial f}{\partial t} \tag{2.4.3}$$

$$f_{,i} \equiv \frac{\partial f}{\partial x_i} \quad , \quad \bar{f}_{,i} = \frac{\partial f}{\partial \bar{x}_i} \text{ comma notation} \tag{2.4.4}$$

$$f_{,i\,,j} \equiv f_{,ij} \equiv f_{,ji} = \frac{\partial^2 f}{\partial x_i x_j} \text{ comma notation} \tag{2.4.5}$$

For components a_i of a vector field we introduce the notations:

$$a_{i,j} = \frac{\partial a_i}{\partial x_j} \quad , \quad \bar{a}_{i,j} \equiv \frac{\partial \bar{a}_i}{\partial \bar{x}_j} \tag{2.4.6}$$

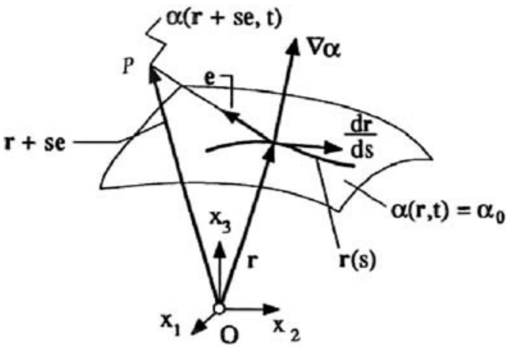


Fig. 2.4.1 The gradient of a scalar field

The equation $\alpha(\mathbf{r},t) = \alpha_o$, where α_o is a constant, represents at any time t a surface in space, see Fig. 2.4.1. The surface is called an *level surface* of the scalar field $\alpha(\mathbf{r},t)$. From the point $\mathbf{r}$ on the surface we introduce an axis in an arbitrary direction given by the unit vector $\mathbf{e} = e_i\mathbf{e}_i$. A coordinate s along the axis is introduced representing the distance from the point $\mathbf{r}$ to a point P on the axis. The point P is now given by the position vector $\mathbf{r} + s\mathbf{e}$. The α -value at the point P is equal to $\alpha(\mathbf{r} + s\mathbf{e},t)$. The rate of change of α at the point $\mathbf{r}$ for the direction $\mathbf{e}$ is defined by the *directional derivative* of α at the point $\mathbf{r}$ and in the direction $\mathbf{e}$, and is given by:

$$\left.\frac{d\alpha}{ds}\right|_{s=0} = \left[\frac{\partial\alpha}{\partial\left(x_i + se_i\right)} \cdot \frac{d\left(x_i + se_i\right)}{ds}\right]_{s=0} = \alpha_{,i}\, e_i \tag{2.4.7}$$

The *gradient of a scalar field* α $(\mathbf{r},t)$ is a vector field, which is denoted by three alternative symbols:

$$\text{grad}\alpha \equiv \frac{\partial\alpha}{\partial\mathbf{r}} \equiv \nabla\alpha \tag{2.4.8}$$

and defined by the formula:

$$\left.\frac{d\alpha}{ds}\right|_{s=0} \equiv \text{grad}\alpha \cdot \mathbf{e} \equiv \frac{\partial\alpha}{\partial\mathbf{r}} \cdot \mathbf{e} \equiv \nabla\alpha \cdot \mathbf{e} \tag{2.4.9}$$

It follows from the formulas (2.4.7) and (2.4.9) that the gradient has the components $\alpha_{,i}$. The symbol ∇ is called "del" and represents an operator, the *del-operator*, which in Cartesian coordinates is given by:

$$\nabla \equiv \mathbf{e}_i \frac{\partial}{\partial x_i} \tag{2.4.10}$$

Geometrically the gradient of a scalar field represents a normal vector to the level surface. This may be seen as follows. Let $\mathbf{r}(s) = x_i(s)\mathbf{e}_i$ be an arbitrary space curve in the surface α $(\mathbf{r}) = \alpha_o$, see Fig. 2.4.1. Now s is a parameter representing the length measured along the curve. The equation $\alpha(\mathbf{r}(s)) = \alpha_o$ is then satisfied for all values of s. Thus:

$$\frac{d\alpha}{ds} = \alpha_{,i}\frac{dx_i}{ds} = \nabla\alpha \cdot \frac{d\mathbf{r}}{ds} = 0 \tag{2.4.11}$$

The vector $d\mathbf{r}/ds$ is a tangent vector to the curve and therefore a tangent to the level surface. The result shows that the vector $\nabla\alpha$ is perpendicular to the tangent to an arbitrary curve on the surface. This implies that:

The gradient to a scalar field $\alpha(\mathbf{r})$ is at the place $\mathbf{r}$ represented by a normal vector to the level surface $\alpha(\mathbf{r}) = \alpha_o$.

From formula (2.4.9) it now follows that:

The maximum value of the directional derivative of a scalar field $\alpha(\mathbf{r})$ on the level surface $\alpha(\mathbf{r}) = \alpha_o$ is given by the magnitude of the gradient $\nabla\alpha$ of the scalar field. The vector $\nabla\alpha$ points in the direction of increasing α-value.

The coordinate invariant property of a vector defined by the components $\alpha_{,i}$ may also be shown in the following way. From formula (2.3.9) we get:

$$\frac{\partial\bar{x}_i}{\partial x_k} = \frac{\partial x_k}{\partial\bar{x}_i} = Q_{ik} \tag{2.4.12}$$

Using the chain rule, we may write:

$$\frac{\partial\alpha}{\partial\bar{x}_i} = \frac{\partial\alpha}{\partial x_k}\frac{\partial x_k}{\partial\bar{x}_i} \Rightarrow \frac{\partial\alpha}{\partial\bar{x}_i} \equiv \bar{\alpha}_{,i} = Q_{ik}\alpha_{,k}$$

which is the proper transformation relation for vector components, see formula $(2.3.11)_1$.

The *divergence of a vector field* $\mathbf{a}(\mathbf{r},t)$ is a scalar field defined by:

$$\operatorname{div}\mathbf{a} \equiv \nabla \cdot \mathbf{a} = a_{i,i} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} \tag{2.4.13}$$

The symbol $\nabla \cdot \mathbf{a}$ may be interpreted as "the scalar product" of the del-operator and the vector $\mathbf{a}$.

We shall show that div $\mathbf{a}$ is a scalar field, that is a coordinate invariant quantity, by showing that the value of div $\mathbf{a}$ is the same in all coordinate systems. Using successively formula $(2.3.11)_1$, the chain rule of differentiation, $(2.4.12)_2$, and finally (2.3.5), we obtain:

$$\bar{a}_{i,i} \equiv \frac{\partial\bar{a}_i}{\partial\bar{x}_i} = \frac{\partial(Q_{ik}a_k)}{\partial x_j}\frac{\partial x_j}{\partial\bar{x}_i} = Q_{ik}\frac{\partial a_k}{\partial x_j}Q_{ij} = Q_{ik}Q_{ij}a_{k,j} = \delta_{kj}a_{k,j} = a_{k,k} \text{ Q.E.D}$$

The *divergence of the gradient of a scalar field* $\alpha(\mathbf{r},t)$ is a new scalar field of special interest in applications, and is therefore given a special symbol:

$$\nabla^2\alpha \equiv \text{div}\,(\text{grad}\alpha) \equiv \nabla\cdot\nabla\alpha = \alpha_{,ii} \tag{2.4.14}$$

The symbol ∇^2 is called the *Laplace operator*, named after Pierre-Simon Laplace [1749–1827]. In a Cartesian coordinate system Ox the expression for the Laplace operator is:

$$\nabla^2 = \frac{\partial^2}{\partial x_i \partial x_i} \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \tag{2.4.15}$$

The *rotation of a vector field* $\mathbf{a}(\mathbf{r},t)$ is the vector:

$$\text{rot}\,\mathbf{a} \equiv \text{curl}\,\mathbf{a} \equiv \nabla\times\mathbf{a} = e_{ijk}\,a_{k,j}\,\mathbf{e}_i \tag{2.4.16}$$

The name and symbol *curl* is very often used in English literature. The symbol $\nabla\times\mathbf{a}$, read as "del cross **a**", may be interpreted as a vector product of the del-operator and the vector **a**. In a Cartesian coordinate system the rotation of a vector field may be computed from a determinant of a matrix.:

$$\nabla\times\mathbf{a} = \det\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{pmatrix} \tag{2.4.17}$$

In cylindrical coordinates (R, θ, z), see Fig. 2.4.2, the ∇-operator is given by:

$$\nabla = \mathbf{e}_R\frac{\partial}{\partial R} + \mathbf{e}_\theta\frac{1}{R}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z} \tag{2.4.18}$$

where the unit vectors in the directions of the coordinates are:

$$\mathbf{e}_R = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2\ ,\ \mathbf{e}_\theta = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2\ ,\ \mathbf{e}_z = \mathbf{e}_3 \tag{2.4.19}$$

In spherical coordinates (r,θ,ϕ), see Fig. 2.4.2, the ∇-operator is given by:

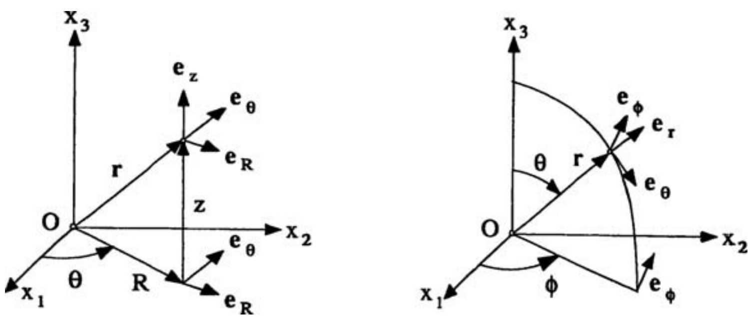


Fig. 2.4.2 Cylindrical coordinates (R,θ,z) and spherical coordinates (r,θ,φ)

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \tag{2.4.20}$$

where the unit vectors in the directions of the coordinates are:

$$\begin{aligned} \mathbf{e}_r &= \sin\theta \cos\phi \, \mathbf{e}_1 + \sin\theta \, \sin\phi \, \mathbf{e}_2 + \cos\theta \, \mathbf{e}_3 \\ \mathbf{e}_\theta &= \cos\theta \cos\phi \, \mathbf{e}_1 + \cos\theta \, \sin\phi \, \mathbf{e}_2 - \sin\theta \, \mathbf{e}_3 \\ \mathbf{e}_\phi &= -\sin\phi \, \mathbf{e}_1 + \cos\phi \, \mathbf{e}_2 \end{aligned} \tag{2.4.21}$$

Expressions for grad α, $\nabla^2\alpha$, div $\mathbf{a}$, and rot $\mathbf{a}$ in cylindrical coordinates and spherical coordinates are presented in Appendix A.

Problems

Problem 2.1. Compute: tr A, det A, norm A, Aa, $a^T b$, $b^T a$ for the matrices:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad , \quad a = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} , \; b = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

Problem 2.2. A and B are 3×3 matrices, and a and b are 3×1 matrices. Prove the implications:

a) $a^T Ab = 0$ for all a and $b \Rightarrow A = 0$
b) $A^T = -A \Leftrightarrow a^T Aa = 0$ for all a
c) $a^T Aa = a^T Ba$ for all $a \Rightarrow A + A^T = B + B^T$

Problem 2.3. Use formula (2.1.20) to show that for a 3×3 matrix A:

$$\det A = \frac{1}{6} e_{ijk} e_{rst} A_{ir} A_{js} A_{kt}$$

Problem 2.4. Validate the identity (2.1.17) through some test examples. Compute:

$$\delta_{ii} \quad , \quad \delta_{ij}\delta_{ij} \quad , \quad e_{ijk} e_{rjk} \,\{\text{use formula (2.1.17)}\} \quad , \quad e_{ijk} e_{ijk}$$

Problem 2.5. Determine the inverse matrix U^{-1} to the matrix:

$$U = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \frac{1}{\sqrt{2}}$$

Problem 2.6. The coordinate system $\bar{O}\bar{x}$ has the base vectors:

$$\bar{\mathbf{e}}_1 = [1,1,1] \left(1/\sqrt{3}\right) \quad , \quad \bar{\mathbf{e}}_2 = [1,0,-1] \left(1/\sqrt{2}\right)$$

referred to the coordinate system Ox. Determine:

$$\bar{\mathbf{e}}_3 \text{ and } Q = (\bar{\mathbf{e}}_i \cdot \mathbf{e}_j)$$

Problem 2.7. Three coordinate systems are denoted Ox, $\bar{O}\bar{x}$, and $\tilde{O}\tilde{x}$. The transformation matrices relating the base vectors of the three systems are:

$$Q_{ij} = \bar{\mathbf{e}}_i \cdot \mathbf{e}_j \quad , \quad \tilde{Q}_{ij} = \tilde{\mathbf{e}}_i \cdot \mathbf{e}_j \quad , \quad \bar{Q}_{ij} = \bar{\mathbf{e}}_i \cdot \tilde{\mathbf{e}}_j$$

a) Show that $Q = \bar{Q}\tilde{Q}$.
b) Compute Q when:

$$\bar{Q} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \sqrt{3} & 1 \\ 0 & -1 & \sqrt{3} \end{pmatrix} \frac{1}{2} \quad , \quad \tilde{Q} = \begin{pmatrix} 0 & \sqrt{3} & 1 \\ 2 & 0 & 0 \\ 0 & 1 & -\sqrt{3} \end{pmatrix} \frac{1}{2}$$

Check that $Q, \bar{Q}$, and $\tilde{Q}$ are orthogonal matrices.

Problem 2.8. The rotation rot $\mathbf{a}$ of a vector field $\mathbf{a}$ may be defined as the vector represented by the Cartesian components defined in formula (2.4.17). Show that rot $\mathbf{a}$ is a proper vector, i.e. the components obey the transformation rule (2.3.11).

Problem 2.9. Use identity (2.1.17) to prove the identities

a) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$
b) $\nabla^2 \mathbf{a} = \nabla(\nabla \cdot \mathbf{a}) - \nabla \times (\nabla \times \mathbf{a})$
c) $(\mathbf{a} \cdot \nabla)\mathbf{a} = (\nabla \times \mathbf{a}) \times \mathbf{a} + \nabla(\mathbf{a} \cdot \mathbf{a}/2)$
d) $\nabla \times \alpha\mathbf{a} = (\nabla\alpha) \times \mathbf{a} + \alpha(\nabla \times \mathbf{a})$

Problem 2.10. Use the formulas (2.4.18–2.4.19) to determine the expressions in cylindrical coordinates for the divergence and rotation of a vector field $\mathbf{a}(\mathbf{r},t)$, and the expression $\nabla^2\alpha$ of a scalar field $\alpha(\mathbf{r},t)$. The expressions are presented in Appendix A.

Chapter 3
Dynamics

This chapter presents the basic principles of dynamics related to Continuum Mechanics. *Dynamics*, which is the science of motion of bodies and the forces as causes of this motion, is often subdivided into *Kinematics* and *Kinetics*. Kinematics is the geometry of motion with velocity and acceleration as the most important concepts. The kinematics of continuous matter and material points is the subject matter in Sect. 3.1. Due to the complexity of the kinematics for rigid bodies this particular part of kinematics is postponed to Sect. 4.5. The kinematics of deformation is presented in Sect. 5.4. Kinetics treats the interrelationship between forces and the motion they cause. Section 3.2 introduces the types of forces generally considered in Continuum Mechanics, and the equations of motion that apply to all continuous materials. The equations of motion for rigid bodies are presented in Sect. 4.5. Section 3.3 Stress Analysis discusses the internal forces in a continuum.

3.1 Kinematics

3.1.1 Lagrangian Coordinates and Eulerian Coordinates

A portion of the continuum we are considering is called a *body*. A body has at any time t a volume V and a surface area A. A material point in the body is called a *particle*. The body consists at any time t of the same particles. The surface of the body is a closed material surface in the continuum and consists at any time of the same particles. In order to localize particles in a body and to describe their motion we introduce a *reference body*, for short a *reference* and denoted *Rf*, to which we may refer positions and change in positions of particles. Rigidly attached to the reference we introduce a coordinate system Ox, see Fig. 3.1.1.

A *place* or *position* in space may now be localized by a place vector $\mathbf{r}$ from the origin O of the coordinate system, or by three coordinate values x_1, x_2, and x_3. Alternatively we set:

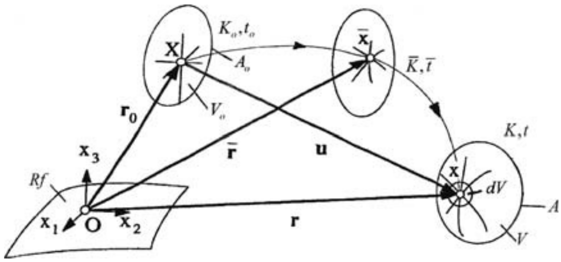


Fig. 3.1.1 Configurations of a body

$$\mathbf{r} = [x_1, x_2, x_3] \text{ and } (x_1, x_2, x_3) \equiv x_i \equiv x \equiv \{x_1 x_2 x_3\} \tag{3.1.1}$$

and we shall use the expressions:

$$\text{place } \mathbf{r} \equiv \text{place}\, [x_1, x_2, x_3] \equiv \text{place } x_i \equiv \text{place } x \tag{3.1.2}$$

The set of places that represents the body at any time is called the *configuration* of the body at that time. The time t at which we are investigating the continuum is called the *present time* and the corresponding configuration K is the *present configuration*, see Fig. 3.1.1.

In order to describe the motion of the body and its state of deformation at the present time t, we need to choose a *reference configuration* K_o, which may or may not represent a real reference state of the body. Normally the reference configuration will be chosen as an actual configuration. In this chapter K_o is taken to be an actual configuration of the body at a *reference time* t_o.

In the reference configuration K_o places are localized by position vectors $\mathbf{r}_{\text{o}}$ or by sets of coordinates X_i. Each particle in the body is at the reference time t_o given a position vector $\mathbf{r}_{\text{o}}$ and a set of coordinates X. Therefore we choose the expression the *particle* $\mathbf{r}_{\text{o}}$ or also the *particle* X. The notation $\mathbf{r}_{\text{o}}$ will be used when we want to emphasize independence of a particular coordinate system Ox.

At the present time t the particle X is at the position x and has moved on a space curve called a *particle path* or a *pathline*, see Fig. 3.1.1. We assume that there is a functional relationship between X and x, and between $\mathbf{r}_{\text{o}}$ and $\mathbf{r}$:

$$x_i = x_i(X_1, X_2, X_3, t) \equiv x_i(X, t) \;\Leftrightarrow\; x = x(X, t) \tag{3.1.3}$$

$$\mathbf{r} = \mathbf{r}(\mathbf{r}_o, t) \tag{3.1.4}$$

The relations (3.1.3) or (3.1.4) represent a one-to-one mapping between the sets of points x in K and X in K_o. The functions $x_i(X,t)$ or $\mathbf{r}(\mathbf{r}_{\text{o}},t)$ represent the *motion* of the body. Figure 3.1.1 defines the *displacement vector* $\mathbf{u}$:

$$\mathbf{u} = \mathbf{u}(X, t) = \mathbf{u}(\mathbf{r}_o, t) = \mathbf{r}(\mathbf{r}_o, t) - \mathbf{r}_o \tag{3.1.5}$$

which also represents the motion of the body.

The motion of the body from K_o to K will in general lead to deformation of the body. That means that material geometrical figures will change their shapes and sizes during the motion. Deformation is illustrated in Fig. 3.1.1 by *material lines*, which in K_o coincides with the coordinate lines. In K the deformed material lines represent a *curvilinear coordinate system*, and an analysis of this system would tell us about the state of deformation of the body relative to the chosen reference configuration. A coordinate system imbedded in a body, and which deforms with it, is called a *convected coordinate system*. Such systems will be analyzed and utilized in Sect. 13.3.6.

A *"moving" configuration* $\bar{K}$ at the time $\bar{t}$, where $\infty < \bar{t} \le t$, is used to describe the deformation process, or the deformation history, that the body has experienced prior to the present time t.

In fluids the deformations are normally so large that it is only practical and necessary to compare the present configuration K with configurations fairly close to K. In such cases K is chosen as a reference configuration, and is called a *relative reference configuration*. The deformation history is described by comparing the "moving" configuration $\bar{K}$ with the present configuration K.

Physical quantities are divided into extensive and intensive quantities. An *extensive quantity* is given for a body and may be a function of the volume V or the mass m of the body. Examples are the mass itself, the linear momentum of the body, and the kinetic energy of the body. The value of an extensive quantity is the sum of its values for the parts into which the body may be divided. An *intensive quantity* is related to the particles in a body and independent of the volume or the mass of the body. Examples are pressure, temperature, mass density, and particle velocity. An intensive quantity may represent the intensity of an extensive quantity. For instance, *mass density* is defined as mass per unit volume. In general an intensive quantity that is given per unit volume, is characterized as a *density*. Kinetic energy per unit volume is thus a *kinetic energy density*. For short it is customary to use the word *density* also when we actually mean mass density. An intensive quantity defined per unit mass, is characterized as a *specific quantity*. For instance, the velocity of a particle is also linear momentum per unit mass, and thus represents the intensity to the extensive quantity called the *linear momentum of the body*. Velocity is therefore *specific linear momentum.*

Intensive physical quantities are expressed by field functions either of the *particle coordinates* X, or *place vector* $\mathbf{r}_o$, and time t, or by the *place coordinates* x, *place vector* $\mathbf{r}$, and time t. We call the field function:

$$f(\mathbf{r},t) \equiv f(X,t) \equiv f(X_1,X_2,X_3,t) \tag{3.1.6}$$

a *particle function*, and we call the set (X and t) *Lagrangian coordinates*, named after Joseph Louis Lagrange [1736–1813], *material coordinates*, or *reference coordinates*. The application of these coordinates is called *Lagrangian description* or *reference description*. We call the field function:

$$f(\mathbf{r},t) \equiv f(x,t) \equiv f(x_1,x_2,x_3,t) \tag{3.1.7}$$

a *place function* or *position function*, and the set (x and t) is called *Eulerian coordinates*, named after Leonhard Euler [1707–1783], or *space coordinates*, and their application for *Eulerian description* or *spacial description*.

According to Truesdell and Toupin [54], the Lagrangian coordinates were introduced by Euler in 1762, while Jean le Rond D'Alembert [1717–1783] was the first to use the Eulerian coordinates i 1752. In general Continuum Mechanics Lagrangian coordinates and the reference description are the most common. The same holds true in solid Mechanics. However, in Fluid Mechanics, due to large displacements and complex deformations, it is usually necessary and most practical to use Eulerian coordinates and spacial description.

3.1.2 Material Derivative of an Intensive Quantity

Let the particle function $f(X,t)$, which may be a scalar or a vector, represent an arbitrary *intensive physical quantity*. For a particular choice of coordinate set X the function is connected to particle X at all times t. The change per unit time, i.e. the time rate of change, of f attached to the particle X, is called the *material derivative* of f and is denoted by a dot over f.

$$\dot{f} \equiv \left.\frac{df}{dt}\right|_{X=\text{constant}} = \frac{\partial f(X,t)}{\partial t} \equiv \partial_t f(X,t) \tag{3.1.8}$$

Other names for this quantity are the substantial derivative, particle derivative, and the individual derivative.

The *velocity* $\mathbf{v}$ and the *acceleration* $\mathbf{a}$ of the particle X is defined by:

$$\mathbf{v}(X,t) = \dot{\mathbf{r}} = \partial_t \mathbf{r}(X,t) \tag{3.1.9}$$

$$\mathbf{a}(X,t) = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \partial_t^2 \mathbf{r}(X,t) \tag{3.1.10}$$

Using the definition (3.1.5) of the displacement vector $\mathbf{u}$, we have as alternative expressions:

$$\mathbf{v} = \dot{\mathbf{u}} = \partial_t \mathbf{u}(X,t), \ \mathbf{a} = \ddot{\mathbf{u}} = \partial_t^2 \mathbf{u}(X,t) \tag{3.1.11}$$

The components of velocity and acceleration in the coordinate system Ox are:

$$v_i = \dot{x}_i = \partial_t x_i(X,t) = \dot{u}_i = \partial_t u_i(X,t) \tag{3.1.12}$$

$$a_i = \dot{v}_i = \ddot{x} = \partial_t^2 x_i(X,t) = \ddot{u} \tag{3.1.13}$$

Let the place function $f(x,t)$ represent an intensive quantity: scalar or vector. For any particular choice of coordinate set x the function is attached to the particular place x in space. The local change of f per unit time is:

$$\partial_t f(x,t) \equiv \frac{\partial f(x,t)}{\partial t} \tag{3.1.14}$$

In order to find the material derivative of the position function $f(x,t)$, we attach the function to the particle X that takes the position x at time t. Thus we write:

$$f = f(x,t) = f(x(X,t),t) \tag{3.1.15}$$

The definition (3.1.8) of the material derivative leads us to the result:

$$\dot{f} = \frac{\partial f(x,t)}{\partial t} + \frac{\partial f(x,t)}{\partial x_i}\frac{\partial x_i(X,t)}{\partial t} = \partial_t f + f_{,i}\,\partial_t x_i$$

The last factor in the last term represents the components v_i of the particle velocity. We then have an expression for the material derivative of a place function $f(x,t)$:

$$\dot{f} = \partial_t f + v_i f_{,i} \tag{3.1.16}$$

Using the *del-operator* from (2.4.10), the formula (3.1.16) may alternatively be written as:

$$\dot{f} = \partial_t f + (\mathbf{v} \cdot \nabla) f \tag{3.1.17}$$

where a new scalar operator has been introduced:

$$\mathbf{v} \cdot \nabla = v_i \frac{\partial}{\partial x_i} \tag{3.1.18}$$

The material derivative (3.1.17) contains two parts. The *local part* $\partial_t f$ registers the change of f at the place x. In Fluid Mechanics, for instance, recording of intensive quantities like pressure, temperatures and velocities will usually be performed with stationary instruments. A stationary instrument can register f at a definite place x and provides values of f for the particles passing through the place x. The recording of such an instrument can therefore only give $\partial_t f$. The *convective part*, $(\mathbf{v}\cdot\nabla)f$, of the material derivative represents the time rate of change of f due to the fact that the particle X, which at time t is at place x, moves to a new position in space.

The particle acceleration $\mathbf{a}$ may be computed from the velocity field $\mathbf{v}(x,t)$ by application of the formulas (3.1.16) and (3.1.17). We write:

$$\mathbf{a} = \dot{\mathbf{v}} = \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} \;\Leftrightarrow\; a_i = \dot{v}_i = \partial_t v_i + v_k v_{i,k} \tag{3.1.19}$$

The particle acceleration $\mathbf{a}$ consists of the *local acceleration* $\partial_t \mathbf{v}$ and the *convective acceleration* $(\mathbf{v}\cdot\nabla)\mathbf{v}$.

3.1.3 Material Derivative of an Extensive Quantity

A body of continuous matter and volume $V(t)$ has constant mass m. This statement is called the *principle of conservation of mass*. The mass is according to the continuum hypothesis distributed in the volume $V(t)$ of the body, such that it is possible to express the mass by a volume integral

$$m = \int_{V(t)} \rho\, dV \tag{3.1.20}$$

where the scalar field $\rho = \rho(\mathbf{r},t)$ is the *density* = mass per unit volume at the place $\mathbf{r}$, and dV is an element of volume in the reference configuration K, see Fig. 3.1.1.

Let $f(\mathbf{r},t)$ be a place function that represents an intensive quantity expressed per unit mass, i.e. a specific quantity, and let $F(t)$ be the corresponding extensive quantity for a body with volume $V(t)$:

$$F(t) = \int_{V(t)} f\rho dV \tag{3.1.21}$$

For instance, $f(\mathbf{r},t)$ may be the kinetic energy per unit mass and $F(t)$ the total kinetic energy for the body. The time rate of change of $F(t)$ attached to the body is called the *material derivative* of the extensive quantity $F(t)$. We will find that:

$$\dot{F} = \dot{F}(t) = \int_{V(t)} \dot{f}\rho dV \tag{3.1.22}$$

This result may be derived as follows. The body may be considered to consist of many small elements with volumes and mass symbolized by dV and ρdV respectively. The contribution to $F(t)$ from the volume element dV is $f\rho dV$. The integral in (3.1.22) represents the sum of contributions from all volume elements. The volume elements may change both in form and size with time, but the mass of any element ρdV is constant. The time rate of change of the quantity $f\rho dV$ is therefore $\dot{f}\rho dV$. The time rate of change of $F(t)$ is then the sum of the element contributions $\dot{f}\rho dV$, and thus equal to the integral in (3.1.22).

3.2 Equations of Motion

3.2.1 Euler's Axioms

Newton's laws for the motion of a mass particle cannot in the strict sense logically be transferred to apply to a body of continuously distributed matter. It has been customary in some texts to claim that this transformation is possible: From Newton's 2. law for the motion of a body, really a mass particle, and Newton's 3. law for the interaction between mass particles, the equations of motion for a system of mass particles are derived. The result, called the *law of forces* and the *law of moments*, is then taken to apply to a continuum. The consideration is either that the continuum is a system of a large number of elementary particles, or as a system of infinitely many infinitely small particles. The first method of consideration fails on the ground that elementary particles do not obey Newtonian mechanics. Quantum mechanical issues are involved here, and the transfer from the micro world to the macro world must include statistical mechanics. The other method contains logical difficulties of mathematical nature.

Continuum Mechanics postulates two axioms: *Euler's 1. axiom*, which corresponds to Newton's 2. law and the law of forces for a system of particles, and *Euler's 2. axiom*, which has its parallel in the law moments for a system of particles. The law of action and reaction of forces, i.e. Newton's 3. law, and of couples and counter couples for interaction between two bodies now follows as a consequence of the Eulerian axioms. This will be demonstrated in Sect. 3.2.2.

From a continuum at time t we now consider a body having mass m, volume V, and surface A. The body is assumed to be acted upon by two types of forces: *body force* **b** per unit mass, and *contact force* **t** per unit area over the surface of the body, see Fig. 3.2.1.

The body forces are external forces originating from sources outside of the body, and they represent actions at a distance from the surroundings. It is for practical reasons that all body forces are considered to be given per unit mass. Gravitation, centrifugal forces, electrostatic and magnetic influences are examples of body forces. The most typical body force is the *gravitational force* g in the parallel constant field of gravity, with its standard value $g = 9.81\,\mathrm{N/kg}$. Normally it is assumed that the body forces are independent of the state of the body, but in general the body forces are dependent upon the position in space of the particle on which they act. At the place **r** we express the body force by:

$$\mathbf{b} = \mathbf{b}(\mathbf{r},t) = \mathbf{b}(x,t) \tag{3.2.1}$$

If the body forces are given as force per unit volume, they are called *volume forces*.

In a solid material the contact force represents the internal forces between the physical particles, atoms and molecules, on both sides of the surface A separating the body from its surroundings. These forces decrease rapidly with the distance from their sources. Therefore the contribution to the body forces from the interparticular forces is neglected. *Surface force* and *traction* are other names of the contact force. The vector **t** will in the present exposition be called the *stress vector*. The stress vector varies with the place **r**, or the particle X, on the surface, with the time t, and with the normal **n** to the surface at the place **r**. The vector **n** is by definition a unit vector pointing out from the surface A at the place **r**. Alternatively we use the expressions:

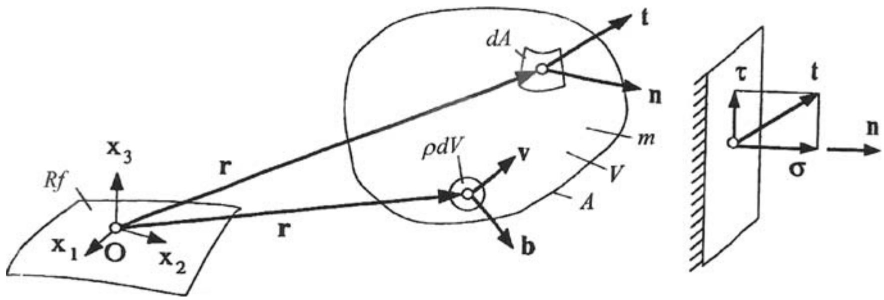


Fig. 3.2.1 Forces on a body of a continuum

$$\mathbf{t} = \mathbf{t}(\mathbf{r},t,\mathbf{n}) = \mathbf{t}(x,t,n) = \mathbf{t}(\mathbf{r}_o,t,\mathbf{n}) = \mathbf{t}(X,t,n) \tag{3.2.2}$$

The components of the stress vector **t** in the direction **n** and in the tangent plane to the surface A at the place **r** are the *normal stress* σ and the *shear stress* τ respectively, see Fig. 3.2.1.

When a gas is considered as a continuous medium, the surface of a gaseous body is a material surface. But if we take into consideration that the gas consists of molecules, moving about at great speeds, the surface of gaseous body is a mathematical surface through which molecules may pass, although the macromechanical flow through a surface element is assumed to be zero. The motion, or rather the momentum, of the molecules that pass through the boundary surface is in Continuum Mechanics represented by the contact forces. Intermolecular forces in a gas are negligible. When the boundary surface is an interface between a gas and a liquid or between a liquid and a solid material, the molecular motion is represented by a pressure and tangential forces, i.e. shear stresses, on the liquid surface or the surface of the solid material respectively. On an interface between two liquid bodies the contact forces represent both intermolecular forces and molecular motion.

In special situations it becomes necessary to add to the two types of forces introduced above, *body couples*, given as couple per unit mass, and surface couples or *couple stresses* on the surface of the body, given as couple per unit area. Body couples are for instance present as a result of an elastic strain wave passing through a body exposed to an electromagnetic field. Another example of the presence of body couples is given by the effect of the magnetic field of the earth on the needle of a compass. Couple stresses may appear when the molecular structure of a material is taken into consideration, and when dislocations in metals are considered. In this book these types of mechanical actions are not considered.

The body shown in Fig. 3.2.1 is subjected to a *resultant force* **f** given by:

$$\mathbf{f} = \int_A \mathbf{t}\,dA + \int_V \mathbf{b}\,dV \tag{3.2.3}$$

dA is an element of area on the surface A of the body, and dV is an element of volume of the body. The *resultant moment* $\mathbf{m}_O$ about the origin O of all forces acting of the body is:

$$\mathbf{m}_O = \int_A \mathbf{r}\times\mathbf{t}\,dA + \int_V \mathbf{r}\times\mathbf{b}\,dV \tag{3.2.4}$$

The velocity of a particle $\mathbf{r}_o$ at the place **r** in the body is given by the vector fields $\mathbf{v}(\mathbf{r},t)$ or by $\mathbf{v}(\mathbf{r}_o,t)$. The *linear momentum* **p** of the body and the *angular momentum* $\mathbf{l}_O$ of the body about the origin O are defined by:

$$\mathbf{p} = \int_V \mathbf{v}\rho\,dV\,, \quad \mathbf{l}_O = \int_V \mathbf{r}\times\mathbf{v}\rho\,dV \tag{3.2.5}$$

Note that the velocity **v** may be interpreted as the specific momentum, that is momentum per unit mass. Likewise, $\mathbf{r}\times\mathbf{v}$ is the specific angular momentum about O.

The general laws of motion for a body of mass m, which at time t has the volume V and the surface A, are postulated as *Euler's two axioms* or *laws* and given by:

$$\mathbf{f} = \dot{\mathbf{p}} \equiv \int_V \dot{\mathbf{v}} \rho \, dV \equiv \int_V \mathbf{a} \rho \, dV \quad \text{Euler's 1. axiom} \tag{3.2.6}$$

$$\mathbf{m}_O = \dot{\mathbf{l}}_O = \int_V \mathbf{r} \times \dot{\mathbf{v}} \rho \, dV \equiv \int_V \mathbf{r} \times \mathbf{a} \rho \, dV \quad \text{Euler's 2. axiom} \tag{3.2.7}$$

$\mathbf{a} = \dot{\mathbf{v}}$ is the particle acceleration. When computing the material derivative of the extensive quantities, we have used the formula (3.1.22). Furthermore we have used the result:

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \dot{\mathbf{r}} \times \mathbf{v} + \mathbf{r} \times \dot{\mathbf{v}} = \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a} = \mathbf{r} \times \mathbf{a}$$

Euler's 1. axiom is also called *the law of balance of linear momentum*. Euler's 2. axiom is also called *the law of balance of angular momentum*.

The *center of mass* of a body is defined by the point C that at time t is at the place given by the place vector:

$$\mathbf{r}_C = \frac{1}{m} \int_V \mathbf{r} \rho \, dV \tag{3.2.8}$$

The center of mass is a reference invariant point that moves with the body. The weight of a body, which represents the resultant of the constant and parallel gravitational field near the surface of the earth, always has its line of action through the mass center. The point C is therefore also called the *center of gravity* of the body. The velocity $\mathbf{v}_C$ and the acceleration $\mathbf{a}_C$ of the center of mass are found from the expressions:

$$\mathbf{v}_C = \dot{\mathbf{r}}_C = \frac{1}{m} \int_V \dot{\mathbf{r}} \rho \, dV = \frac{1}{m} \int_V \mathbf{v} \rho \, dV \tag{3.2.9}$$

$$\mathbf{a}_C = \dot{\mathbf{v}}_C = \ddot{\mathbf{r}}_C = \frac{1}{m} \int_V \dot{\mathbf{v}} \rho \, dV = \frac{1}{m} \int_V \mathbf{a} \rho \, dV \tag{3.2.10}$$

From the definition $(3.2.5)_1$ and the formula (3.2.9) it follow that the momentum $\mathbf{p}$ of the body may be expressed by:

$$\mathbf{p} = m\mathbf{v}_C \tag{3.2.11}$$

Then, according to Euler's 1. axiom (3.2.6) the motion of the center of mass is governed by the equation:

$$\mathbf{f} = m\mathbf{a}_C \tag{3.2.12}$$

Thus, the center of mass of a body moves as a physical particle with mass equal to the mass m of the body and subjected to the resultant force $\mathbf{f}$ on the body.

3.2.2 Newton's 3. Law

Newton formulated his 3. law about "action and reaction" when two bodies interacted, without taking into consideration the extent of the two bodies. Using the two fundamental axioms of Euler we can derive and extend the 3. law to two bodies of arbitrary shapes and extensions. Figure 3.2.2 shows a body, arbitrarily divided into two parts I and II by the interface A'. The axioms (3.2.6) and (3.2.7) are formulated for the body and for each of the parts I and II, using the point O' on in the interface A' as a moment point. Subtractions of corresponding equations for the parts from the equations for the total body give as results:

$$-\int_{A',I} \mathbf{t}dA - \int_{A',II} \mathbf{t}dA = 0, \quad -\int_{A',I} \mathbf{r}' \times \mathbf{t}dA - \int_{A',II} \mathbf{r}' \times \mathbf{t}dA = 0 \qquad (3.2.13)$$

The result shows that the resultant force $\mathbf{f}_{12}$ and the resultant moment $\mathbf{m}_{12}$ of the contact forces on part I from part II, are equal to the resultant force $\mathbf{f}_{21}$ and the resultant moment $\mathbf{m}_{21}$ of the contact forces acting on part II from part I, but with opposite signs:

$$\mathbf{f}_{12} = -\mathbf{f}_{21}, \quad \mathbf{m}_{12} = -\mathbf{m}_{21} \qquad (3.2.14)$$

The resultant forces act through the moment point O', and the resultant moments are couples. The result (3.2.14) may be interpreted as a generalization of *Newton's 3. law on "action and reaction"*.

Let $\mathbf{r}$ be the place of particle X on the interface A' and let A' shrink to zero. Then it follows from (3.2.13) that, see Fig. 3.2.3:

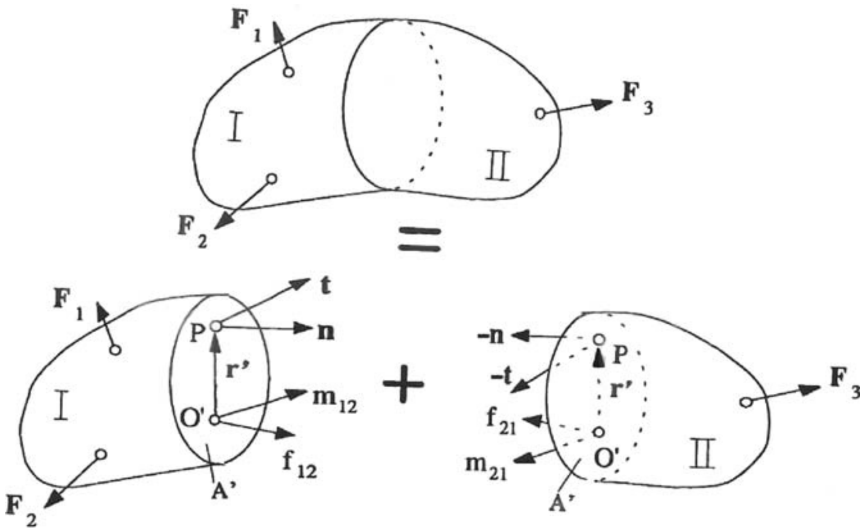


Fig. 3.2.2 Newton's 3. law

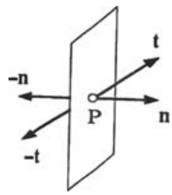


Fig. 3.2.3 Cauchy's lemma

$$\mathbf{t}(\mathbf{r},t,\mathbf{n}) = -\mathbf{t}(\mathbf{r},t,-\mathbf{n}) \tag{3.2.15}$$

This result is called *Cauchy's lemma*, named after Augustin Louis Cauchy [1789–1857]. The lemma shows that the stress vectors on the two sides of a material surface are equal but in opposite directions.

3.2.3 Coordinate Stresses

Figure 3.2.4 shows the stress vectors $\mathbf{t}_k$ on three surfaces through the particle P at the place $\mathbf{r}$ and perpendicular to the coordinate axes. The unit normals to these surfaces are thus the base vectors $\mathbf{e}_k$. The components of the stress vectors are denoted T_{ik}.

$$\mathbf{t}_k = T_{ik}\mathbf{e}_i \;\Leftrightarrow\; \mathbf{e}_i \cdot \mathbf{t}_k = T_{ik} \tag{3.2.16}$$

The components T_{ik} will be called the *coordinate stresses* in the particle P or at the place $\mathbf{r}$. The coordinate stresses are elements of the *stress matrix* T in P, or at $\mathbf{r}$, with respect to the coordinate system Ox, or with respect to the base vectors $\mathbf{e}_i$. The coordinate stresses T_{11}, T_{22}, and T_{33} are *normal stresses*, while the coordinate stresses T_{ik}, $i \neq k$, are *shear stresses*. The first index (i) of T_{ik} refers to the direction of the stress and the second index (k) refers to the normal vector $\mathbf{e}_k$ to the surface on which the stress acts. In the literature the meaning of the indices are often reversed.

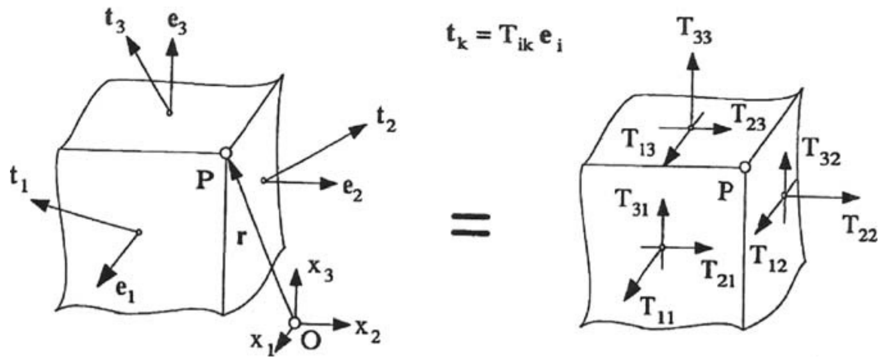


Fig. 3.2.4 Coordinate stresses

As will be shown in Sect. 3.2.5, the coordinate shear stresses T_{ik} and T_{ki} are equal: $T_{ik} = T_{ki}$, as long as we only consider the two types of forces: body forces **b** and contact forces **t**. Normally therefore the order of the indices is not really so important. An exception to the equality of shear stress pairs is provided when body couples and when stress couples are present. However, in some derivations it is advantageous to distinguish between T_{ik} and T_{ki}.

According to Cauchy's lemma the stress vectors are equal to $-\mathbf{t}_i$ on the sides of a coordinate surfaces for which the normals are $-\mathbf{e}_i$. We may also let the coordinate stresses T_{ik} represent the components of these stress vectors, see Fig. 3.2.5. Then we have the following sign rule for coordinate stresses.

A positive coordinate stress acts in direction of a positive coordinate axis on that side of a material coordinate surface facing the positive direction of a coordinate axis. On the side of the material surface facing the negative direction of a coordinate axis the positive coordinate stress acts in the direction of a negative coordinate axis.

Positive and negative normal stresses are *tensile stresses* and *compressive stresses*, respectively.

The literature uses a variety of symbols for coordinate stresses. When number indices are used the following symbols may be found.

$$T_{ik} = S_{ik} = \sigma_{ik} = \tau_{ik} = -p_{ik} \tag{3.2.17}$$

In texts using xyz-coordinates, normal stresses are denoted as σ_x or σ_{xx} etc. and shear stresses as τ_{xy} or σ_{xy} etc. Thus:

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \equiv \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} \equiv \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \tag{3.2.18}$$

In cylindrical coordinates $(R,\ \theta,\ z)$, Fig. 3.2.6a, and spherical coordinates (r, θ, ϕ), Fig. 3.2.6b, the stress matrix is respectively given by:

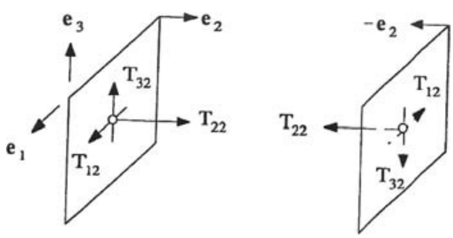
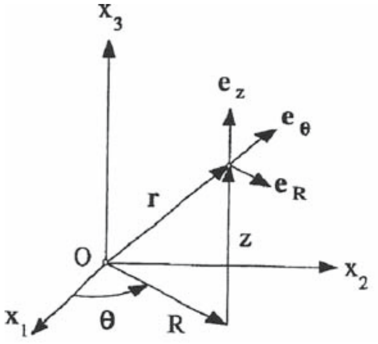


Fig. 3.2.5 Positive coordinate stresses



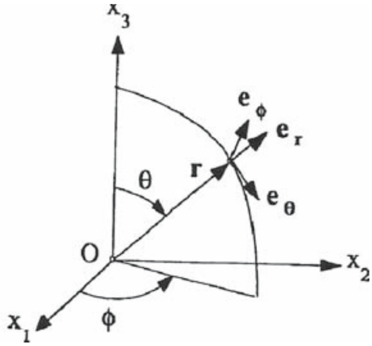


Fig. 3.2.6 **a**) Cylindrical coordinates. **b**) Spherical coordinates

$$T = \begin{pmatrix} \sigma_R & \tau_{R\theta} & \tau_{Rz} \\ \tau_{\theta R} & \sigma_\theta & \tau_{\theta z} \\ \tau_{zR} & \tau_{z\theta} & \sigma_z \end{pmatrix}, \; T = \begin{pmatrix} \sigma_r & \tau_{r\theta} & \tau_{r\phi} \\ \tau_{\theta r} & \sigma_\theta & \tau_{\theta\phi} \\ \tau_{\phi r} & \tau_{\phi\theta} & \sigma_\phi \end{pmatrix} \tag{3.2.19}$$

Example 3.1. Uniaxial Stress State

A rod with axis along the x_1-axis is subjected to an axial force N. The state of stress is given by the stress matrix:

$$T = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \sigma = \frac{N}{A}$$

σ is the normal stress on a cross-section A. The cross-section is free of shear stresses: $T_{12} = T_{31} = 0$. Sections parallel to the axis of the rod are stress free: $T_{i2} = T_{i3} = 0$. The term *uniaxial stress* is explained in Sect. 3.3.3.

Example 3.2. State of Pure Shear Stress

A thin-walled tube with a mean radius r and a wall thickness of $h(h << r)$ is subjected to a torsion moment, or *torque*, M, Fig. 3.2.7a. The resultant of the stresses over the cross-section of the tube must be equal to the torsion moment. Due to symmetry the cross-section will only carry a constant shear stress, which we find to be:

$$\tau = \tau_{\theta z} = \frac{M}{2\pi r^2 h}$$

Figure 3.2.7b shows the state of stress on an element of the tube wall. Moment equilibrium of the element requires that:

$$\tau_{z\theta} = \tau_{\theta z} = \tau$$

The state of stress in a particle P in the wall of the tube is now expressed by the stress matrix:

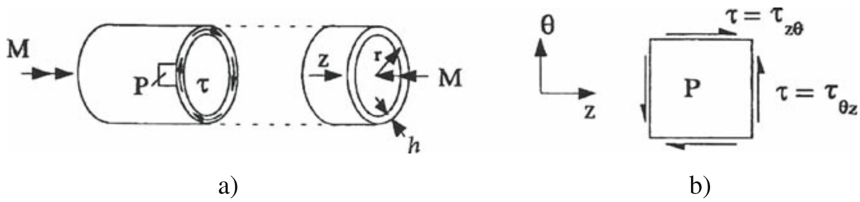


Fig. 3.2.7 Torsion of a thin-walled tube

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau \\ 0 & \tau & 0 \end{pmatrix}, \ \tau = \frac{M}{2\pi r^2 h}$$

The element shown in Fig. 3.2.7b is in a *state of pure shear stress*.

3.2.4 Cauchy's Stress Theorem and Cauchy's Stress Tensor

Figure 3.2.8 shows the stress vector **t** on a surface element dA with unit normal **n** through a particle P. The element may be a part of the boundary surface of a body or a material surface in a body. Let T_{ik} be the coordinate stresses in the particle and let:

$$\mathbf{t} = t_i \, \mathbf{e}_i, \ \mathbf{n} = n_k \, \mathbf{e}_k \tag{3.2.20}$$

Then the *Cauchy's stress theorem* states that:

$$t_i = T_{ik} \, n_k \ \Leftrightarrow \ t = T \, n \tag{3.2.21}$$

The proof of the theorem follows.

Figure 3.2.9 shows a small body in the form of a tetrahedron, a *Cauchy tetrahedron*, of volume V and a surface consisting of four triangles. Three of the triangles are parallel to coordinate planes and have the areas A_k through the particle P. The fourth triangular plane has the area A and a unit normal vector **n** at a distance h from particle P. The body is subjected to a body force **b**, the stress vectors $-\mathbf{t}_k$ on the coordinate planes and the stress vector **t** on the fourth triangle. Euler's 1. axiom

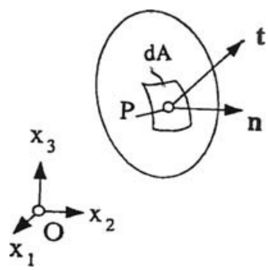


Fig. 3.2.8 The stress vector **t**

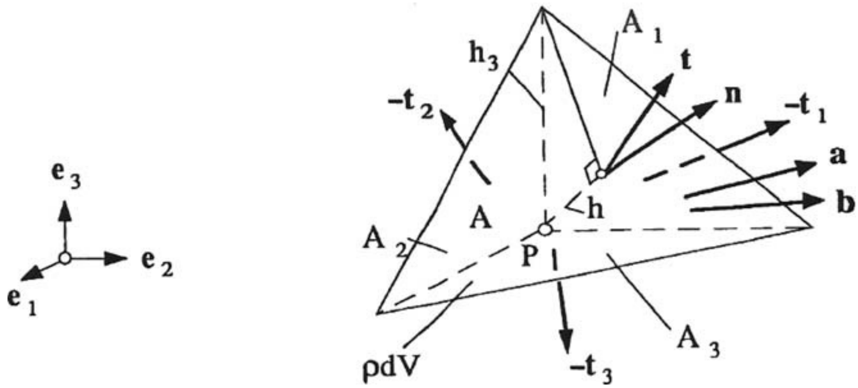


Fig. 3.2.9 Cauchy's tetrahedron

applied to the body results in the equations of motion:

$$\sum_{k=1}^{3} \int_{A_k} (-\mathbf{t}_k)\, dA + \int_{A} \mathbf{t}\, dA + \int_{V} \mathbf{b}\, dV = \int_{V} \mathbf{a}\rho\, dV$$

If we let $\mathbf{t}_k$, $\mathbf{t}$, $\mathbf{b}\rho$ and $\mathbf{a}\rho$ represent mean values on the respective surfaces and in the volume, the equation of motion may be presented as:

$$-\mathbf{t}_k A_k + \mathbf{t} A + \mathbf{b}\rho V = \mathbf{a}\rho V \tag{3.2.22}$$

The edges of the tetrahedron parallel to the base vectors $\mathbf{e}_k$ are denoted by h_k, and since $\mathbf{n}$ is a unit vector, we may write:

$$n_k = \frac{h}{h_k} \tag{3.2.23}$$

The volume V of the tetrahedron may be expressed in four different ways as:

$$V = \frac{1}{3} A \cdot h = \frac{1}{3} A_1 \cdot h_1 = \frac{1}{3} A_2 \cdot h_2 = \frac{1}{3} A_3 \cdot h_3$$

Using the result (3.2.23), we obtain the formulas:

$$V = \frac{1}{3} A \cdot h, \; A_k = A\, n_k \tag{3.2.24}$$

The results (3.2.24) are substituted into the equation of motion (3.2.22), and after division by A, we get:

$$-\mathbf{t}_k n_k + \mathbf{t} + \mathbf{b}\rho h/3 = \mathbf{a}\rho h/3$$

Now we let h approach zero. Then we are left with the following relation between the four stress vectors $\mathbf{t}$ and $\mathbf{t}_k$ on planes through the particle P.

$$\mathbf{t} = \mathbf{t}_k n_k \tag{3.2.25}$$

Using the relation (3.2.16) for the components of the vectors $\mathbf{t}_k$, we obtain:

$$\mathbf{t} = t_i \mathbf{e}_i = T_{ik} \mathbf{e}_i n_k \;\Rightarrow\; t_i = T_{ik} n_k$$

which completes the proof for the Cauchy's stress theorem (3.2.21). The theorem shows how the coordinate stresses T_{ik}, or the stress matrix $T = (T_{ik})$, related to a Cartesian coordinate system, completely determines the *state of stress* in a particle.

The unit normal $\mathbf{n}$ and the corresponding stress vector $\mathbf{t}$ are coordinate invariant properties. This implies that the relation (3.2.21) has a coordinate invariant character. The stress matrix T works as a vector operator, and we may state: The vector $\mathbf{t}$ is determined by the vector $\mathbf{n}$ through an operator $\mathbf{T}$, which in a Cartesian coordinate system Ox is represented by the matrix T. If a new coordinate system $\bar{O}\bar{x}$ with base vectors $\bar{\mathbf{e}}_i = Q_{ik}\mathbf{e}_k$ is introduced, we shall find the following relation between the components $\bar{t}_i$ of the vector $\mathbf{t}$, the components $\bar{n}_k$ of the vector $\mathbf{n}$, and the coordinate stresses $\bar{T}_{ik}$ in this coordinate system:

$$\bar{t}_i = \bar{T}_{ik} \bar{n}_k \;\Leftrightarrow\; \bar{t} = \bar{T}\,\bar{n} \tag{3.2.26}$$

Because the two relations (3.2.21) and (3.2.26) express the same connection between the vectors $\mathbf{n}$ and $\mathbf{t}$, it is reasonable to introduce a common symbol for the relation that is coordinate independent. We therefore say that the matrices T and $\bar{T}$ define the *stress tensor* $\mathbf{T}$ in the following sense:

The *stress tensor* $\mathbf{T}$ is a coordinate invariant intensive quantity that in any Cartesian coordinate system Ox is represented by the stress matrix T in that coordinate system. The coordinate stresses T_{ik} are called the components of the stress tensor in the Ox-system.

In the Sect. 5.6 and Sect. 13.5 other stress tensors, represented by other stress matrices and coordinate stresses are introduced, and the quantity $\mathbf{T}$, as defined above, will then be called *Cauchy's stress tensor*. The coordinate stresses T_{ik} defined by the relations (3.2.16) and in Fig. 3.2.4 are in this connection called *Cauchy stresses*.

The word tensor derives from the latin word "tensio" meaning tension and was originally the name of the stress matrix. Tensors are quantities that functions as linear operators in relations between vectors and scalars. Chapter 4 presents the general definition of tensors and their algebra.

Having defined the stress tensor, we may present the Cauchy stress theorem, represented by the relation (3.2.21) in a coordinate invariant form:

Cauchy's Stress Theorem: The stress vector $\mathbf{t}$ on a surface through a particle P is uniquely determined by the stress tensor $\mathbf{T}$ in the particle and the unit normal $\mathbf{n}$ to the surface through the relation:

$$\mathbf{t} = \mathbf{T}\mathbf{n} = \mathbf{T}\cdot\mathbf{n} \;\Leftrightarrow\; t_i = T_{ik} n_k \;\Leftrightarrow\; t = T\,n \tag{3.2.27}$$

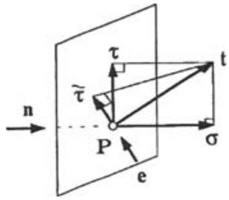


Fig. 3.2.10 Stress vector decomposed into a normal stress and shear stresses

The two coordinate invariant forms, $\mathbf{Tn}$ and $\mathbf{T}\cdot\mathbf{n}$, in the relation (3.2.27) are equivalent. In this book the latter form is preferred of reasons that will given later. The first form is sometimes chosen because it relates to the form of its matrix representation. Note that both the (3.2.21) and (3.2.26) represent the coordinate invariant relationship: $\mathbf{t} = \mathbf{T}\cdot\mathbf{n}$.

The normal stress σ on a surface with unit vector $\mathbf{n}$, Fig. 3.2.10, is given by the scalar product of $\mathbf{n}$ and the stress vector $\mathbf{t}$:

$$\sigma = \mathbf{n}\cdot\mathbf{t} = \mathbf{n}\cdot\mathbf{T}\cdot\mathbf{n} = n_i T_{ik} n_k = n^T T n \tag{3.2.28}$$

The shear stress τ on the surface may be computed from:

$$\tau = |\mathbf{n}\times\mathbf{t}| = \sqrt{\mathbf{t}\cdot\mathbf{t} - \sigma^2} \tag{3.2.29}$$

The projection $\tilde{\tau}$ of this shear stress in the direction given by the unit vector $\mathbf{e}$ in the surface may be expressed as the scalar product of the vectors $\mathbf{e}$ and $\mathbf{t}$:

$$\tilde{\tau} = \mathbf{e}\cdot\mathbf{t} = \mathbf{e}\cdot\mathbf{T}\cdot\mathbf{n} = e_i T_{ik} n_k = e^T T n \tag{3.2.30}$$

From (3.2.28) and (3.2.30) it follows that:

$$T_{ik} = \mathbf{e}_i\cdot\mathbf{T}\cdot\mathbf{e}_k \tag{3.2.31}$$

This result may be interpreted as a general relation between the stress tensor $\mathbf{T}$ and its components T_{ik} in a general Cartesian coordinate system Ox. The relation will now be used to find the relation between the coordinate stresses T and $\bar{T}$ in two coordinate systems Ox and $\bar{O}\bar{x}$. The base vectors of the two systems are related through: $\bar{\mathbf{e}}_i = Q_{ik}\mathbf{e}_k$, where $Q = (Q_{ik})$ is the transformation matrix for the transformation from Ox to $\bar{O}\bar{x}$. Using the general relation (3.2.31), we obtain:

$$\bar{T}_{ij} = \bar{\mathbf{e}}_i\cdot\mathbf{T}\cdot\bar{\mathbf{e}}_j = (Q_{ik}\mathbf{e}_k)\cdot\mathbf{T}\cdot(Q_{jl}\mathbf{e}_l) = Q_{ik}Q_{jl}\mathbf{e}_k\cdot\mathbf{T}\cdot\mathbf{e}_l = Q_{ik}Q_{jl}T_{kl} \Rightarrow$$
$$\bar{T}_{ij} = Q_{ik}Q_{jl}T_{kl} = Q_{ik}T_{kl}Q_{jl} \Leftrightarrow \bar{T} = QTQ^T \tag{3.2.32}$$

The two stress matrices, T and $\bar{T}$, represent the same state of stress, i.e. the same stress tensor $\mathbf{T}$. Each matrix is the representation of the tensor $\mathbf{T}$ in the respective coordinate system.

Example 3.3. Fluid at Rest. Isotropic State of Stress

In Sect. 1.3 a fluid is defined as a material that deforms continuously when subjected to anisotropic states of stress, see Fig. 1.3.1. In a fluid at rest over some time the state of stress is isotropic, i.e. all material surfaces through a fluid particle transmits the same normal stress, which is the pressure p, and the shear stress on the surfaces are zero. The stress matrix related to any Cartesian coordinate system Ox for a fluid at rest is therefore:

$$T = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} = -p\,1 \tag{3.2.33}$$

An alternative formulation of the definition of a fluid may be: *A fluid is a material that at rest only can transfer normal stresses on material surfaces*. We shall now show that this definition leads to the isotropic state of stress as presented by (3.2.33).

The definition implies that in any coordinate system Ox, the stress matrix in a fluid at rest has the form:

$$T = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

On a surface with unit normal **n** the stress vector **t** has only a normal stress component σ, i.e. the stress vector is parallel to the normal. The Cauchy's stress theorem (3.2.27) implies that:

$$\mathbf{t} = \mathbf{T}\cdot\mathbf{n} = \sigma\,\mathbf{n} \;\Rightarrow\; T_{ik}\,n_k = \sigma\,n_i \;\Rightarrow\; \sigma_i\,n_i = \sigma\,n_i \text{ for } i = 1, 2, \text{or } 3 \;\Rightarrow\; \sigma_1 = \sigma_2 = \sigma_3 = \sigma$$

The result shows that the stress matrix must have the form (3.2.33) with $p = -\sigma$. Thus: In a fluid at rest the pressure is the same in all directions.

3.2.5 Cauchy Equations of Motion

Euler's axioms, i.e. the laws of balance of linear momentum and angular momentum, are the equations of motion for a body of continuous material. From these two fundamental laws of motion field equations will be derived that represent the balance of linear and angular momentum of particles in a continuum. The derivation may be performed in two different ways. In the present section the axioms of Euler will be formulated for a differential element of volume about a particle. In the next section we start by writing the Euler's axioms for a finite volume of general shape. Then the integration theorem of Gauss is used, and the field equations are derived from the integrand of the integral equations.

Figure 3.2.11 shows a differential element of mass with volume $dV = dx_1dx_2dx_3$ about the particle X at the place x. The body is subjected to body forces and contact forces. The law of balance of linear momentum for the body, Euler's 1. axiom (3.2.6), yields:

$$\begin{aligned}(\mathbf{t}_{1,1}\,dx_1)\cdot(dx_2\,dx_3) + (\mathbf{t}_{2,2}\,dx_2)\cdot(dx_3\,dx_1) + (\mathbf{t}_{3,3}\,dx_3)\cdot(dx_1\,dx_2)& \\ + \mathbf{b}\,\rho\,dV = \mathbf{a}\,\rho\,dV \;\Rightarrow\; \mathbf{t}_{k,k} + \rho\,\mathbf{b} = \rho\,\mathbf{a}&\end{aligned} \tag{3.2.34}$$

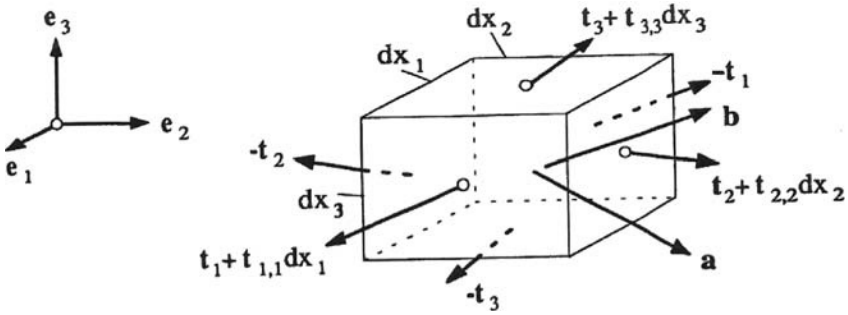


Fig. 3.2.11 Differential volume element

The terms in this equation represents mean values over the respective surfaces and in the volume. When the expression (3.2.16) for the stress vectors $\mathbf{t}_k$ is substituted into (3.2.34), the following component form for the balance of linear momentum is obtained.

$$T_{ik},_k + \rho\, b_i = \rho\, a_i \tag{3.2.35}$$

If we now let the volume of the element shrink to zero, (3.2.35) become field equations related to particle X at place x. The equations (3.2.35) are called *Cauchy's equations of motion, Cauchy's 1. law of motion*, or for short the *Cauchy equations*.

We now introduce the concept of the *divergence of a tensor*. The divergence of the stress tensor is a vector, div $\mathbf{T}$, with components $T_{ik,k}$ in the Ox-system.

$$\operatorname{div}\mathbf{T} = T_{ik},_k\, \mathbf{e}_i \tag{3.2.36}$$

The definition is coordinate invariant, a fact that will be demonstrated in Sect. 4.4 on Tensor Fields. Indirectly the coordinate invariance follows from the Cauchy equations (3.2.35): Since b_i and a_i are components of vectors, the terms $T_{ik,k}$ must also be components of a vector, that is a coordinate invariant quantity. The Cauchy equations may now be written in a index free form:

$$\operatorname{div}\mathbf{T} + \rho\,\mathbf{b} = \rho\,\mathbf{a} \tag{3.2.35a}$$

In a xyz-notation the Cauchy equations are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho\, b_x = \rho\, a_x \text{ etc.} \tag{3.2.37}$$

Under the assumption that the forces on the continuum only are contact forces and body forces, the second axiom of Euler, i.e. the law of balance of angular momentum, implies that the stress matrix is symmetric:

$$T^T = T \;\Leftrightarrow\; T_{ki} = T_{ik} \tag{3.2.38}$$

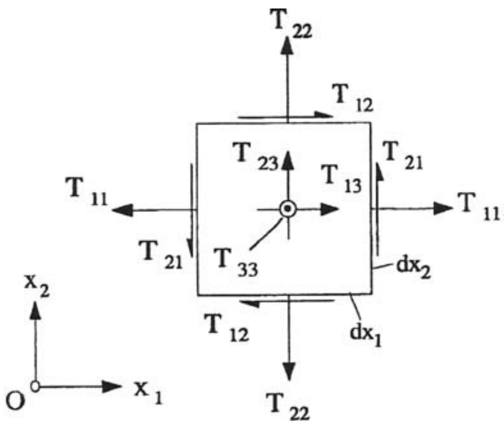


Fig. 3.2.12 Volume element subjected to the homogeneous stress field T

The symmetry of the stress matrix may be proved as follows. Let the state of stress in the particle X be given by the stress matrix T. The state of stress in the neighborhood of particle X may be represented by the stress matrix $T + \Delta T$. Since the matrix T now represents a homogeneous stress field and thus satisfies the Cauchy equations $T_{ik,k} = 0$, the additional stresses given by the matrix ΔT must satisfy the Cauchy equations (3.2.35) and thus balance the body forces **b** and the acceleration **a**.

Figure 3.2.12 shows an element of volume $dV = dx_1 dx_2 dx_3$ that contains the particle X. The element is subjected to the homogeneous stress field T. The law of balance of angular momentum applied to the element provides three component equations. The x_3-component equation is:

$$(T_{21} \cdot dx_2\, dx_3) \cdot dx_1 + (T_{12} \cdot dx_1\, dx_3) \cdot dx_2 = 0 \;\Rightarrow\; T_{21} = T_{12}$$

Similar results are obtained for the x_1- and the x_2-component equations. The results prove the statement (3.2.38). The symmetry of the stress matrix is a coordinate invariant property, and we therefore say that *the stress tensor is symmetric*. The result (3.2.38) may be interpreted as the law of balance of angular momentum for a particle and is also called *Cauchy's 2. law of motion*. Normally the symmetry of the stress matrix is assumed a priori such that the matrix is considered to contain only six independent element rather then nine. Thus the laws of motion for a particle are the three Cauchy equations of motion (3.2.35) or (3.2.35a).

From the symmetry (3.2.38) of the coordinate stresses we may extract the following statement, see Fig. 3.2.13:

On two orthogonal surfaces through a particle the shear stress components normal to the line intersecting the surfaces are equal.

The truth of the statement follows from the fact that according to the result (3.2.38) the statement is true for the shear stress components normal to the line intersecting any two coordinate planes through the particle.

The Cauchy equations in cylindrical coordinates (R, θ, z) are:

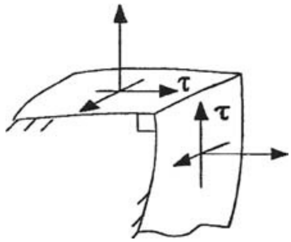


Fig. 3.2.13 Equality of shear stress components τ on orthogonal surfaces

$$\frac{\partial \sigma_R}{\partial R} + \frac{\sigma_R - \sigma_\theta}{R} + \frac{1}{R}\frac{\partial \tau_{R\theta}}{\partial \theta} + \frac{\partial \tau_{Rz}}{\partial z} + \rho\, b_R = \rho\, a_R \tag{3.2.39}$$

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\, \tau_{\theta R}\right) + \frac{1}{R}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \rho\, b_\theta = \rho\, a_\theta \tag{3.2.40}$$

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\, \tau_{zR}\right) + \frac{1}{R}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \rho\, b_z = \rho\, a_z \tag{3.2.41}$$

$\sigma_R,\ \sigma_\theta$, and σ_z are normal stresses and $\tau_{R\theta} = \tau_{\theta R},\ \tau_{\theta z} = \tau_{z\theta}$, and $\tau_{zR} = \tau_{Rz}$ are shear stresses.

The derivation of (3.2.39, 3.2.40, 3.2.41) from the general Cauchy equations (3.2.35a) is left as Problem 4.18 in Chap. 4.

The Cauchy equation in spherical coordinates (r, θ, ϕ) are:

$$\frac{\partial \sigma_r}{\partial r} + \frac{2\sigma_r - \sigma_\theta - \sigma_\phi}{r} + \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}\left(\sin\theta\, \tau_{r\theta}\right) + \frac{\partial \tau_{r\phi}}{\partial \phi}\right] + \rho\, b_r = \rho\, a_r \tag{3.2.42}$$

$$\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\, \tau_{\theta r}\right) + \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}\left(\sin\theta\, \sigma_\theta\right) + \frac{\partial \tau_{\theta\phi}}{\partial \phi}\right] - \frac{\sigma_\phi \cot\theta}{r} + \rho\, b_\theta = \rho\, a_\theta \tag{3.2.43}$$

$$\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\, \tau_{\phi r}\right) + \frac{1}{r\sin^2\theta}\frac{\partial}{\partial \theta}\left(\sin^2\theta\, \tau_{\phi\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial \sigma_\phi}{\partial \phi} + \rho\, b_\phi = \rho\, a_\phi \tag{3.2.44}$$

σ_r, σ_θ, and σ_ϕ are normal stresses and $\tau_{r\theta} = \tau_{\theta r}, \tau_{\theta\phi} = \tau_{\phi\theta}$, and $\tau_{\phi r} = \tau_{r\phi}$ are shear stresses.

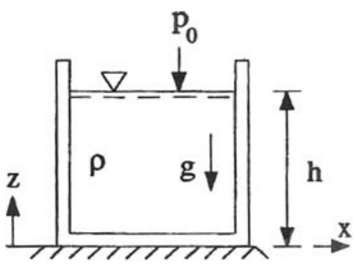


Fig. 3.2.14 Fluid at rest

Example 3.4. The Pressure in a Fluid at Rest

Figure 3.2.14 shows a vessel containing a homogeneous liquid of constant density ρ. The fluid is at rest and is subjected to the constant gravitational force g in the negative z-direction. The state of stress in the fluid is represented by the stress matrix (3.2.33) in Example 3.3. The pressure p is generally a function of the position coordinates x, y, and z. For a fluid at rest the Cauchy equations (3.2.35) are reduced to:

$$(-p\,\delta_{ik})_{,k} + \rho\, b_i = 0 \quad \Rightarrow$$

$$-p_{,i} + \rho\, b_i = 0 \quad \Leftrightarrow \quad -\nabla p + \rho\,\mathbf{b} = \mathbf{0} \tag{3.2.45}$$

This is called the *equilibrium equation of a fluid*.

In this example:

$$b_3 = b_z = -g\ ,\ b_1 = b_x = 0\ ,\ b_2 = b_y = 0,$$

and the equations (3.2.45) yield:

$$\frac{\partial p}{\partial x} = 0, \frac{\partial p}{\partial y} = 0, \frac{\partial p}{\partial z} = -\rho g$$

Partial integrations of these equations and use of the boundary condition: $p = p_o$, the atmospheric pressure, for $z = h$ on the free liquid surface, provide the following expression for the pressure in the liquid:

$$p(z) = p_o + \rho\, g\,(h - z)$$

3.2.6 Alternative Derivation of the Cauchy Equations

In this alternative derivation of the Cauchy equations of motion we shall apply *Gauss' integration theorem*, which is Theorem C.3 in Appendix C. The theorem provides a relation between a volume integral over a volume V and a surface integral over the surface A of the volume. If $f(\mathbf{r},t)$ is any field function, the Gauss' integration theorem states that:

$$\int_V f_{,i}\, dV = \int_A f\, n_i\, dA \tag{3.2.46}$$

n_i are the components of an outward unit normal to A.

The component form of Euler's 1. axiom (3.2.6) is:

$$\int_A t_i\, dA + \int_V b_i \rho\, dV = \int_V a_i \rho\, dV \tag{3.2.47}$$

The stress vector t_i is through Cauchy's stress theorem replaced by $T_{ik}n_k$, and by Gauss' theorem (3.2.46) we may write:

$$\int_A t_i \, dA = \int_A T_{ik} n_k \, dA = \int_V T_{ik},_k \, dV$$

Now the law of balance of linear momentum (3.2.47) may be written as:

$$\int_V [T_{ik},_k + b_i \rho - a_i \rho] \, dV = 0$$

These equations are valid for a body of arbitrary shape and volume. This means that the volume V may be chosen arbitrarily, and since the integral is zero always, the integrand must be zero:

$$T_{ik},_k + b_i \rho - a_i \rho = 0 \;\Rightarrow\; T_{ik},_k + \rho \, b_i = \rho \, a_i$$

Thus we are left with the Cauchy's equations of motion (3.2.35).

The component form of the law of balance of angular momentum (3.2.7) is:

$$e_{ijk} \left[\int_A x_i t_j \, dA + \int_V x_i (b_j - a_j) \rho \, dV \right] = 0$$

Again using the Cauchy's stress theorem and the Gauss' integration theorem (3.2.46), we find:

$$\int_A x_i t_j \, dA = \int_A x_i T_{jl} n_l \, dA = \int_V \left[x_{i},_l \, T_{jl} + x_i \, T_{jl},_l \right] dV = \int_V \left[T_{ji} + x_i \, T_{jl},_l \right] dV$$

where the results: $x_{i,l} = \delta_{il}$ and $x_{i,l} T_{jl} = \delta_{il} T_{jl} = T_{ji}$ have been utilized. Then the law of angular momentum balance may be rewritten to:

$$e_{ijk} \left[\int_V x_i (T_{jl},_l + \rho \, b_j - \rho \, a_j) \, dV \right] + \int_V e_{ijk} T_{ji} \, dV = 0$$

The Cauchy equations (3.2.35) imply that the first integral vanishes. Then, since the volume V may be chosen arbitrarily, the integrand in the second integral must be zero. Thus:

$$e_{ijk} T_{ji} = 0 \;\Leftrightarrow\; T_{ji} - T_{ij} = 0 \;\Rightarrow\; T_{ji} = T_{ij} \;\Leftrightarrow\; T^T = T$$

The stress matrix is symmetric. The result obviously holds for the stress matrices in all orthogonal coordinate systems, for instance the matrices T in (3.2.19) for cylindrical coordinates and spherical coordinates.

3.3 Stress Analysis

3.3.1 Principal Stresses

Let us assume that the stress tensor is known in a particle, and let the unit vector **n** be a normal vector to a plane through the particle. In Sect. 3.2.4 we have seen that using the Cauchy stress theorem we may compute the stress vector, the normal stress, and the shear stress on the plane. We shall now show that in general there are three orthogonal planes through the particle that are free of shear stress. In Sect. 3.3.3 we show that the normal stresses on these planes include the maximum and the minimum normal stress on planes through the particle. The result of the investigation may be formulated in the following theorem.

The Principal Stress Theorem: For any state of stress there exists, through a particle, three orthogonal planes free of shear stresses. The planes are called the *principal stress planes*, the three unit normals $\mathbf{n}_i$ to the planes are the *principal stress directions*, and the normal stresses σ_i on the planes are the *principal stresses* in the particle. The principal stresses include the maximum normal stress and minimum normal stress on planes through the particle.

First we will show that planes without shear stress through the particle do exist. We search for a plane, defined by its unit normal vector **n**, on which the stress vector is parallel to **n**. This unit normal vector has to satisfy the relation:

$$\mathbf{t} = \mathbf{T} \cdot \mathbf{n} = \sigma \mathbf{n} \tag{3.3.1}$$

The normal stress σ is the principal stress on the plane. The equation (3.3.1) may be rewritten in the matrix format:

$$(\sigma 1 - T)n = 0 \Leftrightarrow (\sigma \delta_{ik} - T_{ik}) n_k = 0 \tag{3.3.2}$$

This is a set of three linear, homogeneous equations for the three unknown components n_k.

The condition that the set of equations (3.3.2) has a solution is that the determinant of the coefficient matrix is equal to zero.

$$\det(\sigma 1 - T) = 0 \Rightarrow \det \begin{pmatrix} \sigma - T_{11} & -T_{12} & -T_{13} \\ -T_{21} & \sigma - T_{22} & -T_{23} \\ -T_{31} & -T_{32} & \sigma - T_{33} \end{pmatrix} = 0 \Rightarrow$$

$$\sigma^3 - I\sigma^2 + II\sigma - III = 0 \tag{3.3.3}$$

The three coefficients *I, II*, and *III* are called the *principal invariants of the stress tensor* and are expressed by:

$$
\begin{aligned}
I &= T_{kk} = \operatorname{tr} T \\
II &= \frac{1}{2}\left[T_{ii}T_{kk} - T_{ik}T_{ik}\right] = \frac{1}{2}\left[(\operatorname{tr} T)^2 - (\operatorname{norm} T)^2\right] \\
III &= \det T
\end{aligned} \tag{3.3.4}
$$

The fact that the three coefficients are coordinate invariant quantities, will formally be shown later. But from a physical point of view we may conclude that if the cubic equation (3.3.3) has a solution, this solution cannot depend upon the coordinate system applied to find it. So if the solution is coordinate invariant, then he coefficients in the equation from which the solution is found, must also be coordinate invariant. It will be shown that the cubic equation (3.3.3), called *the characteristic equation of the stress tensor*, has three real roots: σ_1, σ_2, and σ_3.

The mathematical problem related to (3.3.1, 3.3.2, 3.3.3) is known as an *eigenvalue problem*: σ_i are the *eigenvalues* and the corresponding $\mathbf{n}_i$ are the *eigenvectors* of the tensor T.

Any cubic equation has at least one real root, which we shall denote by σ_3. The corresponding principal direction is denoted $\mathbf{n}_3$. If we now choose a coordinate system Ox for which the base vector $\mathbf{e}_3$ is equal to $\mathbf{n}_3$, the stress vector $\mathbf{t}_3$ will be equal to $\sigma\mathbf{e}_3$ and $T_{\alpha 3} = 0$. The stress matrix in this coordinate system is then:

$$
T = \begin{pmatrix} T_{11} & T_{12} & 0 \\ T_{21} & T_{22} & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \tag{3.3.5}
$$

The problem to find the two other principal stresses σ_1 and σ_2, and the corresponding principal directions $\mathbf{n}_1$ and $\mathbf{n}_2$ is governed by the three equations (3.3.2), which we now rewrite to:

$$
\left(\sigma\,\delta_{\alpha\beta} - T_{\alpha\beta}\right) n_\beta = 0, \; (\sigma - \sigma_3)\, n_3 = 0 \tag{3.3.6}
$$

The principal stress σ is either σ_1 or σ_2, and $\mathbf{n} = n_i\mathbf{e}_i$ is the corresponding principal stress directions, either $\mathbf{n}_1$ or $\mathbf{n}_2$. Let us first assume that $\sigma \neq \sigma_3$. Then it follows from the last of (3.3.6) that $n_3 = 0$. Thus a new principal stress direction $\mathbf{n}$ is parallel to the $x_1\,x_2$-plane and therefore normal to $\mathbf{n}_3$. From Fig. 3.3.1 we find:

$$
\mathbf{n} = [\cos\phi,\ \sin\phi,\ 0]
$$

ϕ is the angle between $\mathbf{n}$ and the x_1-axis. The first two of (3.3.6) become:

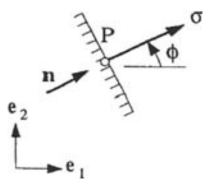


Fig. 3.3.1 Principal stress σ and principal stress direction $\mathbf{n}$ in a plane parallel to the x_1x_2 - plane

$$
\begin{aligned}
&(\sigma - T_{11})\cos\phi - T_{12}\sin\phi = 0 \\
&-T_{21}\cos\phi + (\sigma - T_{22})\sin\phi = 0
\end{aligned}
\tag{3.3.7}
$$

The determinant of the coefficient matrix for this set of linear equations has to be zero. Thus:

$$
\begin{aligned}
(\sigma - T_{11})(\sigma - T_{22}) - T_{12}T_{21} = 0 \quad \Rightarrow \\
\sigma^2 - (T_{11} + T_{22})\,\sigma + T_{11}T_{22} - T_{12}T_{21} = 0
\end{aligned}
\tag{3.3.8}
$$

The solution of this quadratic equation is, when the symmetry of T is utilized:

$$
\begin{matrix}\sigma_1 \\ \sigma_2\end{matrix} = \frac{1}{2}(T_{11} + T_{22}) \pm \sqrt{\left(\frac{T_{11} - T_{22}}{2}\right)^2 + (T_{12})^2}
\tag{3.3.9}
$$

The radicand can never be negative, which means that the two roots σ_1 and σ_2 are real. The principal stress directions are determined from (3.3.7), from which we find the results:

$$
\tan\phi_1 = \frac{\sigma_1 - T_{11}}{T_{12}}, \ \tan\phi_2 = \frac{\sigma_2 - T_{11}}{T_{12}}
\tag{3.3.10}
$$

If any two principal stresses are unequal, for instance $\sigma_1 \neq \sigma_2$, the corresponding principal directions $\mathbf{n}_1$ and $\mathbf{n}_2$ are orthogonal. This result may be demonstrated using (3.3.10) and (3.3.9). Another way of proving the orthogonality of the two principal directions $\mathbf{n}_1$ and $\mathbf{n}_2$ is as follows. Let:

$$
\mathbf{n}_1 = n_{1\alpha}\mathbf{e}_\alpha, \ \mathbf{n}_2 = n_{2\alpha}\mathbf{e}_\alpha
$$

Then from (3.3.6) we obtain:

$$
(\sigma_1\delta_{\alpha\beta} - T_{\alpha\beta})\,n_{1\beta}n_{2\alpha} = 0, \ (\sigma_2\delta_{\alpha\beta} - T_{\alpha\beta})\,n_{2\beta}n_{1\alpha} = 0
$$

The two sets of equations are subtracted and due to the symmetry of the stress matrix, we get:

$$
(\sigma_1 - \sigma_2)\,n_{1\alpha}n_{2\alpha} = (\sigma_1 - \sigma_2)\,\mathbf{n}_1 \cdot \mathbf{n}_2 = 0
$$

Since $\sigma_1 \neq \sigma_2$, it follows that $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$. This result proves that the principal stress directions $\mathbf{n}_1$ and $\mathbf{n}_2$ are orthogonal if $\sigma_1 \neq \sigma_2$.

We have now in fact proved that the three roots in the cubic equation (3.3.3) are real, and that if the three roots are distinct, the principal stress directions are orthogonal. Of physical reasons it is clear that the principal stresses and the principal stress directions are coordinate invariant properties of the stress tensor. A procedure for the computation of the principal stresses and directions is presented in Example 3.7 in Sect. 3.3.2.

The two roots in (3.3.9) coincide only if $T_{12} = 0$ and $T_{11} = T_{22}$. In that case:

$$
\sigma_1 = \sigma_2 = T_{11} = T_{22}
$$

From (3.3.7) it follows that the angle ϕ in that case becomes indeterminate. This means that any direction in a plane parallel to the x_1x_2-plane is a principal direction, and that the stress on any plane normal to $\mathbf{n}$, i.e. planes parallel to the x_3-axis, is a principal stress equal to $\sigma_1 = \sigma_2$. The situation is called a *state of plane-isotropic stress*, see Example 3.6.

We now return to (3.3.6) and consider the possibility that $\sigma = \sigma_3$. We may still choose $n_3 = 0$ and obtain the solution given by (3.3.9, 3.3.10), but if $\sigma = \sigma_1 = \sigma_3$, we obviously shall find that any direction $\mathbf{n}$ in the plane parallel to the $x_1\,x_3$-plane is a principal direction of stress with principal stress $\sigma = \sigma_1 = \sigma_3$.

If all three roots are coinciding: $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$, the unit normal vector $\mathbf{n}$ may be chosen freely, i.e. all directions are principal stress directions. The stress matrix will be the same in all coordinate systems and equal to the scalar σ multiplied by the unit matrix:

$$T_{ik} = \sigma\,\delta_{ik} \;\Leftrightarrow\; T = \sigma\,1 \tag{3.3.11}$$

In this case the stress tensor $\mathbf{T}$ is called an *isotropic tensor*, and we have an *isotropic state of stress*. Because the state of stress in a fluid at rest is isotropic, with the pressure p as the principal stress, see Example 3.3 and the stress matrix (3.2.33), and because water is the most typical fluid, and is called "hydro" in Latin (= hudor in Greek), the state of stress (3.3.11) is also called a *hydrostatic state of stress*.

The equations (3.3.2), which determine the principal directions $\mathbf{n}_i$, do not determine the direction of the arrow of the vectors $\mathbf{n}_i$. If $\mathbf{n}_i$ is a principal direction so is $-\mathbf{n}_i$. If we arrange the unit vectors $\mathbf{n}_i$ such that they form a right-handed system and choose a coordinate system Ox with base vectors $\mathbf{e}_i = \mathbf{n}_i$, the stress matrix in this coordinate system becomes the *diagonal matrix*:

$$T = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \tag{3.3.12}$$

The principal stress invariants (3.3.4) now take the simple forms:

$$\begin{aligned} I &= \sigma_1 + \sigma_2 + \sigma_3 \\ II &= \sigma_1\,\sigma_2 + \sigma_2\,\sigma_3 + \sigma_3\sigma_1 \\ III &= \sigma_1\sigma_2\sigma_3 \end{aligned} \tag{3.3.13}$$

The result show that *I, II*, and *III* are coordinate invariant properties, i.e. they are scalars. We shall call the unit vectors $\mathbf{n}_i$ the *principal axes of stress* in the particle.

The general case when all the principal stresses are different from zero is called *triaxial state of stress*. If only two of the principal stresses in a particle are different from zero, the particle is in a *biaxial state of stress*. This is also called *plane state of stress*, see Example 3.2 above and Examples 3.5 and 3.6 below. A *uniaxial state of stress* has only one non-zero principal stress, as in Example 3.1.

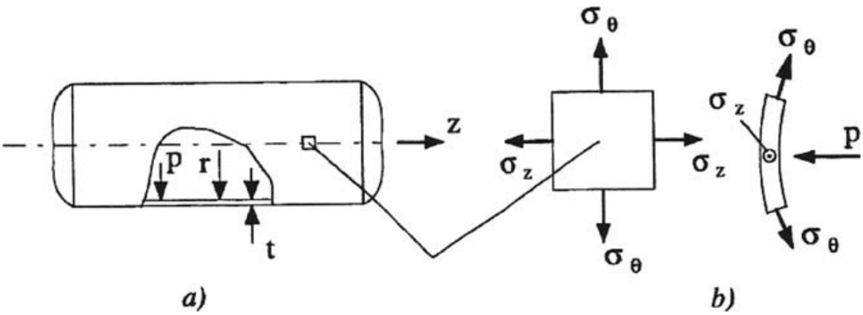


Fig. 3.3.2 **a**) Cylindrical container with internal pressure p. **b**) Element of container wall

Example 3.5. Biaxial State of Stress

A thin-walled circular cylindrical container is subjected to an internal pressure p. The middle radius of the cylindrical wall is r, and the wall thickness is $t(<< r)$.

From equilibrium considerations we shall find that an element of the container wall, as shown in Fig. 3.3.2 b, is subjected to the normal stresses:

$$\sigma_z = \frac{r}{2t}p, \; \sigma_\theta = \frac{r}{t}p \tag{3.3.14}$$

These equations are sometimes called the *Laplace equations of a cylindrical shell*. The normal stress σ_R on a surface normal to the R-direction has values in the interval $[-p, 0]$, given by the stresses at the inner and outer walls. The assumption $t << r$ guaranties that σ_R will be very small compared with σ_z and σ_θ from the formulas (3.3.14). For this reason σ_R may be neglected for thin-walled cylindrical containers. The state of stress in the container wall may therefore be represented by the following stress matrix related to cylindrical coordinates (R, θ, z).

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{r}{2t}p \tag{3.3.15}$$

The thin-walled cylinder with internal pressure is statically determinate. The state of stress (3.3.15) is directly determined from equilibrium considerations. A thick-walled cylindrical container is statically indeterminate, and we have to take into account the material properties of the container wall. The state of stress in an elastic thick-walled container is developed in Example 7.3 in Sect. 7.3.

Example 3.6. Plane-Isotropic State of Stress

A thin-walled spherical shell is subjected to an internal pressure p. The middle radius of the shell is r, and the wall thickness is $t(<< r)$. On a meridian plane through the shell the normal stress $\sigma \left(= \sigma_\theta = \sigma_\phi\right)$ in spherical coordinates (r, θ, ϕ) has the resultant force $\sigma \cdot (2\pi r \cdot t)$. This force has to balance the force $p \cdot \left(2\pi r^2\right)$ due to the pressure. The result of this equilibrium consideration is:

$$\sigma = \sigma_\theta = \sigma_\phi = \frac{r}{2t} p \tag{3.3.16}$$

The normal stress σ_r on a surface normal to the r-direction may be neglected because the value has to lie in the interval $[-p, \ 0]$, given by the stresses at the inner and outer walls. The assumption $t << r$ guaranties that σ_r will be very small compared with $\sigma_\theta = \sigma_\phi$ from the formulas (3.3.16). The state of stress in the container wall may therefore be represented by the stress matrix:

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{r}{2t} p \tag{3.3.17}$$

Since the principal stresses in the θ – and ϕ – directions are equal, any direction parallel to the shell surface is a principal stress direction. The state of stress is biaxial and plane-isotropic.

A thick-walled spherical shell subjected to an internal pressure is statically indeterminate, which means that the material properties must be taken into account when the state of stress is to be found. The solution to this problem when the shell is elastic, is given in Example 7.15 in Sect. 7.6.

3.3.2 Stress Deviator and Stress Isotrop

The stress matrix T may uniquely be decomposed into a trace-free matrix T' and an isotropic matrix T^o:

$$T = T' + T^o \tag{3.3.18}$$

$$T^o = \left(\frac{1}{3} \operatorname{tr} T\right) 1 \tag{3.3.19}$$

$$T' = T - T^o, \ \operatorname{tr} T' = 0 \tag{3.3.20}$$

The matrix T' represents a tensor $\mathbf{T}'$ which we shall call the *stress deviator*, while the matrix T^o represents a tensor $\mathbf{T}^o$ which we shall call the *stress isotrop*. Another name for this tensor is the *hydrostatic stress tensor*. Related to the principal axes of stress, all the matrices T, T', and T^o have diagonal form. The matrix for the deviator is:

$$T' = \begin{pmatrix} \sigma_1' & 0 & 0 \\ 0 & \sigma_2' & 0 \\ 0 & 0 & \sigma_3' \end{pmatrix}, \ \sigma_i' = \sigma_i - \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

This means that the tensors $\mathbf{T}$ and $\mathbf{T}'$ have coinciding principle directions. We express this fact by stating that the tensors $\mathbf{T}$ and $\mathbf{T}'$ are *coaxial tensors*.

The principal deviator stresses are determined from the characteristic equation (3.3.3) for the stress deviator $\mathbf{T}'$. Since the first principal invariant of $\mathbf{T}'$ by its definition (3.3.20) is equal to zero, $I' = \mathrm{tr}T' = 0$, the characteristic equation is reduced to:

$$\left(\sigma'\right)^3 - J_2\sigma' - J_3 = 0 \tag{3.3.21}$$

where $J_2\,(= II')$ is the negative second principal invariant, and $J_3\,(= III')$ is the third principal invariant of $\mathbf{T}'$:

$$\begin{aligned} J_2 &= -II' = \frac{1}{2}T'_{ik}\,T'_{ik} = \frac{1}{2}\left(\mathrm{norm}\;T'\right)^2 \geq 0 \\ J_3 &= III' = \det\;T' \end{aligned} \tag{3.3.22}$$

The reason why we choose the special J-symbols for the principal invariants for the stress deviator is that these symbols are commonly used in the literature on modelling of viscoelastic, viscoplastic, and plastic materials, as will be demonstrated in the Chaps. 9 and 10.

The general solution of (3.3.21) may be determined as follows. First the angle θ is computed from the formula:

$$\cos\theta = \frac{J_3}{2\sqrt{\left(J_2/3\right)^3}},\; 0 \leq \theta \leq 180^\circ \tag{3.3.23}$$

Then we compute the principal stresses of $\mathbf{T}'$ from:

$$\begin{aligned} &\sigma'_1 = 2\sqrt{\frac{J_2}{3}}\cos\frac{\theta}{3},\; \sigma'_2 = 2\sqrt{\frac{J_2}{3}}\cos\left(\frac{\theta}{3} + 60^\circ\right) \leq \sigma'_1 \\ &\sigma'_3 = 2\sqrt{\frac{J_2}{3}}\cos\left(\frac{\theta}{3} - 60^\circ\right) \leq \sigma'_2 \end{aligned} \tag{3.3.24}$$

The principal stresses σ_i of the stress tensor $\mathbf{T}$ are finally determined by:

$$\sigma_i = \sigma'_i + \frac{1}{3}\mathrm{tr}\,T \tag{3.3.25}$$

The principal directions of stress $\mathbf{n}_i$ are found from (3.3.2) with $\sigma = \sigma_i,\; i = 1,2,$ and 3.

Example 3.7. Principal Stresses and Directions from a Stress Matrix

The state of stress in a particle is defined by the stress matrix:

$$T = \begin{pmatrix} 90 & -30 & 0 \\ -30 & 120 & -30 \\ 0 & -30 & 90 \end{pmatrix} \mathrm{MPa}$$

We want to determine the principal stresses σ_i and the principal stress directions $\mathbf{n}_i$

In this example we choose to find the principal stresses by using the formulas developed above. The principal stresses may alternatively be found by a numerical solution of the characteristic equation (3.3.3) using a calculator that has a program for such a task.

The matrices for the stress isotrop and stress deviator, and the invariants of the stress deviator are computed.

$$T^o = \frac{1}{3}(\mathrm{tr}T)\,1 = \begin{pmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{pmatrix} \text{MPa}\,,\; T' = T - T^o = \begin{pmatrix} -10 & -30 & 0 \\ -30 & 20 & -30 \\ 0 & -30 & -10 \end{pmatrix} \text{MPa}$$

From the formulas (3.3.23) and (3.3.24) we compute:

$$J_2 = \frac{1}{2}T'_{ik} \quad T'_{ik} = 2100, \quad J_3 = def \quad T' = 20000$$

$$\cos\theta = \frac{J_3}{2\sqrt{(J_2/3)^3}} = 0.539949 \;\Rightarrow\; \theta = 57.31981^\circ$$

$$\sigma'_1 = 2\sqrt{\frac{J_2}{3}}\cos\frac{\theta}{3} = 50\,,\;\; \sigma'_2 = 2\sqrt{\frac{J_2}{3}}\cos\left(\frac{\theta}{3} + 60^\circ\right) = 10$$

$$\sigma'_3 = 2\sqrt{\frac{J_2}{3}}\cos\left(\frac{\theta}{3} - 60^\circ\right) = -\,40$$

The principal stresses for the state of stress are now:

$$\sigma_i = \sigma'_i + \frac{1}{3}\mathrm{tr\,T} \;\Rightarrow\; \sigma_1 = 150,\; \sigma_2 = 90,\; \sigma_3 = 60 \text{ MPa}$$

The principal stress directions $\mathbf{n}_i$ are determined from (3.3.2). For $\sigma = \sigma_1$ we have:

$$(\sigma_i\,\delta_{ik} - T_{ik})\,n_{1k} = 0 \;\Rightarrow\; \left\{ \begin{array}{llll} 60n_{11} & +30n_{12} & & = 0 \\ 30n_{11} & +30n_{12} & +\,30n_{13} & = 0 \\ & +30n_{12} & +\,60n_{13} & = 0 \end{array} \right\} \;\Rightarrow$$

$$n_{11} = 1/\sqrt{6},\; n_{12} = -2/\sqrt{6},\; n_{13} = 1/\sqrt{6}$$

The solutions for $\sigma = \sigma_2$ and for $\sigma = \sigma_3$ are found by the same procedure. The total solution is:

$$\sigma_1 = 150\,\text{MPa}\,,\; \mathbf{n}_1 = [1,-2,1]/\sqrt{6}$$
$$\sigma_2 = 90\,\text{MPa}\,,\; \mathbf{n}_2 = [1,0,-1]/\sqrt{2}$$
$$\sigma_3 = 60\,\text{MPa}\,,\; \mathbf{n}_3 = [1,1,1]/\sqrt{3}$$

Finally we check orthogonality of the principal directions.

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = n_{1i} n_{2i} = [1 \cdot 1 + (-2) \cdot 0 + 1 \cdot (-1)] / \sqrt{6 \cdot 2} = 0$$
$$\mathbf{n}_2 \cdot \mathbf{n}_3 = n_{2i} n_{3i} = [1 \cdot 1 + 0 \cdot 1 + (-1) \cdot 1] / \sqrt{2 \cdot 3} = 0$$
$$\mathbf{n}_3 \cdot \mathbf{n}_1 = n_{3i} n_{1i} = [1 \cdot 1 + 1 \cdot (-2) + 1 \cdot 1] / \sqrt{3 \cdot 6} = 0$$

3.3.3 Extremal Values for Normal Stress

The three principal stresses in a particle represent the extremal values for normal stress on planes through the particle. To show this we first choose a coordinate system Ox-system with base vectors $\mathbf{e}_i$ parallel to the principal stress directions $\mathbf{n}_i$. The stress matrix in this coordinate system is the diagonal matrix (3.3.12) having elements:

$$T_{ik} = \sigma_i \, \delta_{ik} \tag{3.3.26}$$

The principal stresses σ_i are now ordered such that:

$$\sigma_3 \leq \sigma_2 \leq \sigma_1 \tag{3.3.27}$$

The normal stress σ on a plane with unit normal $\mathbf{n}$ is given by:

$$\sigma = \mathbf{n} \cdot \mathbf{T} \cdot \mathbf{n} = \sum_{i,k} n_i \, \sigma_i \, \delta_{ik} \, n_k = \sigma_1 \, n_1^2 + \sigma_2 \, n_2^2 + \sigma_3 \, n_3^2 \tag{3.3.28}$$

Due to the arrangement (3.3.27) and because $\mathbf{n}$ is a unit vector, i.e.: $\mathbf{n} \cdot \mathbf{n} = n_1^2 + n_2^2 + n_3^2 = 1$, we find from the result (3.3.28) that:

$$\sigma_3 \left(n_1^2 + n_2^2 + n_3^2 \right) \leq \sigma \leq \sigma_1 \left(n_1^2 + n_2^2 + n_3^2 \right) \Rightarrow$$
$$\sigma_3 \leq \sigma \leq \sigma_1 \tag{3.3.29}$$

Thus we have shown that in a particle where the principal stresses are given by (3.3.27):

$$\sigma_{\max} = \sigma_1, \; \sigma_{\min} = \sigma_3 \tag{3.3.30}$$

The largest principal stress is therefore the maximum normal stress in the particle on planes through the particle, and the smallest principal stress is the minimum normal stress in the particle on planes through the particle. The third principal stress is called the intermediate principal stress $\sigma_{inf} = \sigma_2$.

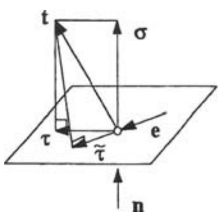


Fig. 3.3.3 Normal stress σ and maximum shear stress τ on a plane with unit normal $\mathbf{n}$

3.3.4 Maximum Shear Stress

The shear stress τ on a plane with unit normal $\mathbf{n}$ is given by formula (3.2.29). The normal projection $\tilde{\tau}$ of the stress vector $\mathbf{t}$ onto the direction $\mathbf{e}$ in the plane is determined by formula (3.2.30):

$$\tilde{\tau} = \mathbf{e} \cdot \mathbf{t} = \mathbf{e} \cdot \mathbf{T} \cdot \mathbf{n} = e_i T_{ik} n_k$$

From Fig. 3.3.3 it follows that $\tilde{\tau} \leq \tau$. The equality sign applies when $\mathbf{e}$ lies in the plane through $\mathbf{t}$ and $\mathbf{n}$. We shall determine $\mathbf{n}$ and $\mathbf{e}$ such that $\tilde{\tau}$ becomes a maximum. Once again we choose the representation (3.3.26) for the stress matrix and the principal stresses are ordered according to (3.3.27). Then:

$$\tilde{\tau} = \sum_{i,k} e_i \sigma_i \delta_{ik} n_k = \sigma_1 e_1 n_1 + \sigma_2 e_2 n_2 + \sigma_3 e_3 n_3$$

Now, since $\mathbf{e}$ and $\mathbf{n}$ are orthogonal vectors:

$$\mathbf{e} \cdot \mathbf{n} = e_1 n_1 + e_2 n_2 + e_3 n_3 = 0 \;\Rightarrow\; e_2 n_2 = -e_1 n_1 - e_3 n_3$$

Therefore we may write:

$$\tilde{\tau} = (\sigma_1 - \sigma_2)\, e_1 n_1 + (\sigma_2 - \sigma_3)\,(-e_3 n_3) \tag{3.3.31}$$

The terms $(\sigma_1 - \sigma_2)$ and $(\sigma_2 - \sigma_3)$ are both non-negative. In order to make $\tilde{\tau}$ as large as possible we must make the terms $e_1 n_1$ and $-e_3 n_3$ as large as possible. The vectors $\mathbf{e}$ and $\mathbf{n}$ are unit vectors, and the absolute values of e_1, e_3, n_1, and n_3 become largest if we set: $e_2 = n_2 = 0$. This means that we should choose $\mathbf{e}$ and $\mathbf{n}$ in a plane parallel to the principal directions $\mathbf{n}_1$ and $\mathbf{n}_3$, as shown in Fig. 3.3.4. Since the figure plane is a principal stress plane, the stress vector $\mathbf{t}$ has no component in the $\mathbf{n}_2$-direction. This means that $\tilde{\tau} = \tau$, i.e. the shear stress on the plane according to formula (3.2.29). From Fig. 3.3.4 we find that:

$$n_1 = -\,e_3 = \cos\phi,\; n_3 = e_1 = \sin\phi$$

and (3.3.31) gives:

$$\tilde{\tau} = \tau = (\sigma_1 - \sigma_2)\sin\phi\cos\phi - (\sigma_2 - \sigma_3)\,(-\cos\phi)\sin\phi \;\Rightarrow\; \tau = \frac{1}{2}(\sigma_1 - \sigma_3)\sin 2\phi$$

From this result we see that the maximum shear stress is:

$$\frac{\sigma_1 - \sigma_3}{2} \;\text{ for } 2\phi = \frac{\pi}{2} \;\Rightarrow\; \phi = 45^\circ.$$

Thus we have found that:

$$\tau_{\max} = \frac{1}{2}(\sigma_{\max} - \sigma_{\min}) \tag{3.3.32}$$

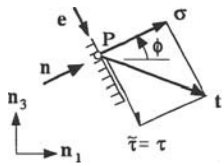


Fig. 3.3.4 Normal stress σ and maximum shear stress τ on a plane parallel to the principal stress direction $\mathbf{n}_2$

> *The maximum shear stress acts on planes that are inclined 45° with respect to the principal directions of the largest and the smallest principal stresses.*

There are four such planes. Figure 3.3.5 shows the orientation of one of these planes and how $\tau_{\max}$ acts on that plane.

3.3.5 Plane Stress

If a surface free of stress exists through a particle, the particle is in a *state of plane stress*, or a *state of biaxial stress*. This is often the case in engineering problems and especially where the stresses are at their extreme values. For instance, the surface of machine parts and structural elements may be free of loads, but the stresses in the surface may be very high. See Example 3.5 and Example 3.6. Even when a free surface is loaded, the contact forces representing the loads are often much smaller than the internal stresses they produce in the surface. In such cases the free surface may often be considered to be stress free. This is the situation for the insides of the thin-walled container in Example 3.5 and of the thin-walled spherical shell in

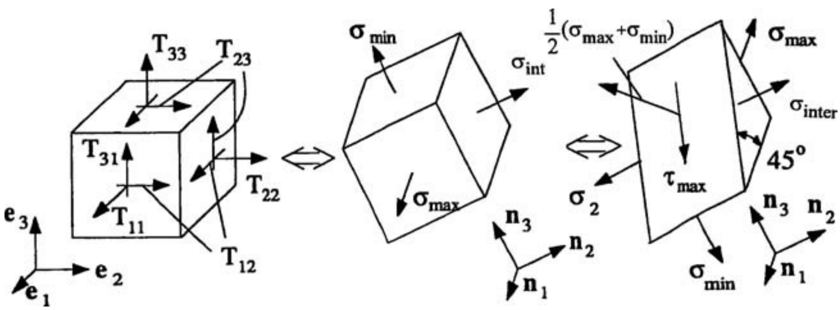


Fig. 3.3.5 Maximum shear stress. The left-hand figure shows a volume element about a particle P, with sides parallel to the initial coordinate surfaces, and subjected to the coordinate stresses T_{ik}. The center figure shows an element about P with sides parallel to the principal stress planes of P. The right-hand figure shows a triangular prism about P with $\tau_{\max}$ on the surface that is inclined 45° with respect to the principal planes of $\sigma_{\max}$ and $\sigma_{\min}$

Example 3.6. Plane stress is considerably simpler to analyze than a general, triaxial state of stress. The analysis of plane stress is therefore usually included in textbooks on Strength of Materials and Mechanics of Material.

Let the plane state of stress in a particle P be defined by the condition: $T_{i3} = 0$, which implies that the plane normal to the x_3-axis is stress free at the particle. The state of stress in the particle P is illustrated in Fig. 3.3.6a and is determined by the three coordinate stresses T_{11}, T_{22}, and T_{12}. The stress free plane through P normal to the x_3-direction is a principal stress plane with the principal stress $\sigma_3 = 0$. When discussing plane stress we do not use the ordering $\sigma_3 \leq \sigma_2 \leq \sigma_1$ for the three principal stresses.

The two other principal stresses, σ_1 and σ_2, are given by formula (3.3.9), repeated here.

$$\begin{matrix}\sigma_1 \\ \sigma_2\end{matrix} = \frac{1}{2}(T_{11} + T_{22}) \pm \sqrt{\left(\frac{T_{11} - T_{22}}{2}\right)^2 + (T_{12})^2} \tag{3.3.33}$$

The principal directions are represented by the angles ϕ_1 and ϕ_2, see Fig. 3.3.6b for ϕ_1 From (3.3.10) we obtain:

$$\tan\phi_1 = \frac{\sigma_1 - T_{11}}{T_{12}}, \ \tan\phi_2 = \frac{\sigma_2 - T_{11}}{T_{12}} \tag{3.3.34}$$

Since σ_1 and σ_2 may both be positive, both be negative, or have different signs, we must remember that the third principal stress, $\sigma_3 = 0$, may represent $\sigma_{\max}$ and $\sigma_{\min}$. Figure 3.3.7 shows $\tau_{\max}$ for the three different situations in plane stress, based on formula (3.3.32).

The principal stresses and the maximum shear stress in the wall of the cylindrical container in Example 3.5 are found to be:

$$\sigma_3 = \sigma_{\min} = \sigma_R \approx 0, \ \sigma_2 = \sigma_{int} = \sigma_z = \frac{r}{2t}p, \ \sigma_1 = \sigma_{\max} = \sigma_\theta = \frac{r}{t}p$$

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_\theta - \sigma_R}{2} = \frac{\sigma_\theta}{2} = \frac{r}{2t}p$$

For the spherical shell in Example 3.6 the principal stresses and the maximum shear stress in the wall are:

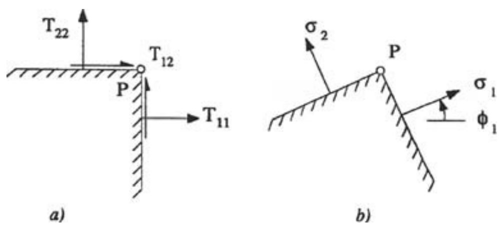


Fig. 3.3.6 Plane state of stress in particle P

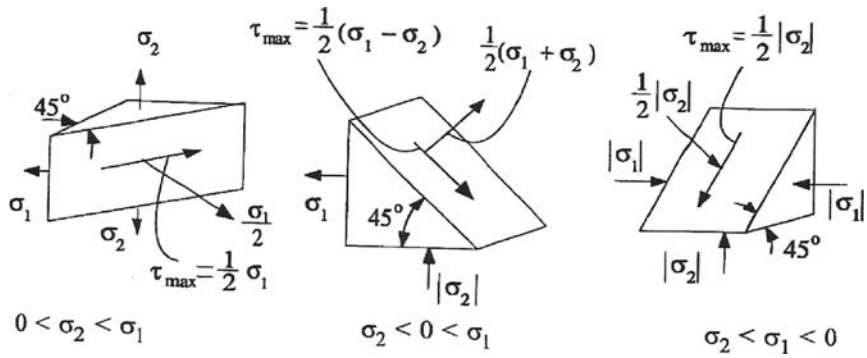


Fig. 3.3.7 Maximum shear stress for states of plane stress

$$\sigma_3 = \sigma_{\min} = \sigma_r \approx 0,\ \sigma_2 = \sigma_1 = \sigma_{\max} = \sigma_\theta = \sigma_\phi = \frac{r}{2t}p$$

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_\theta - \sigma_r}{2} = \frac{\sigma_\theta}{2} = \frac{r}{2t}p$$

Formulas for stresses on planes perpendicular to the stress free plane, i.e. planes parallel to the x_3-direction, will now be developed. Figure 3.3.8 presents the situation. The unit normal **n** to the plane and the unit vector **e** in the plane are given by:

$$\mathbf{n} = [\cos\phi, \sin\phi, 0],\ \mathbf{e} = [\sin\phi, -\cos\phi, 0] \tag{3.3.35}$$

ϕ is the angle between the direction of the normal **n** and the x_1-direction. The components of the stress vector **t** on the plane are obtained from Cauchy's stress theorem (3.2.27).

$$t_\alpha = T_{\alpha\beta}\, n_\beta = \begin{cases} t_1 = T_{11}\cos\phi + T_{12}\sin\phi \\ t_2 = T_{12}\cos\phi + T_{22}\sin\phi \end{cases}$$

The normal stress σ and the shear stress τ and the plane are then:

$$\sigma = \mathbf{n}\cdot\mathbf{t} = n_\alpha t_\alpha = T_{11}\cos^2\phi + T_{22}\sin^2\phi + 2T_{12}\sin\phi\cos\phi$$

$$\tau = \mathbf{e}\cdot\mathbf{t} = e_\alpha t_\alpha = (T_{11} - T_{22})\sin\phi\cos\phi - T_{12}\left(\cos^2\phi - \sin^2\phi\right)$$

Using the trigonometric formulas:

$$\sin 2\phi = 2\sin\phi\cos\phi,\ \cos 2\phi = \cos^2\phi - \sin^2\phi \tag{3.3.36}$$

we may transform the expressions for σ and τ to:

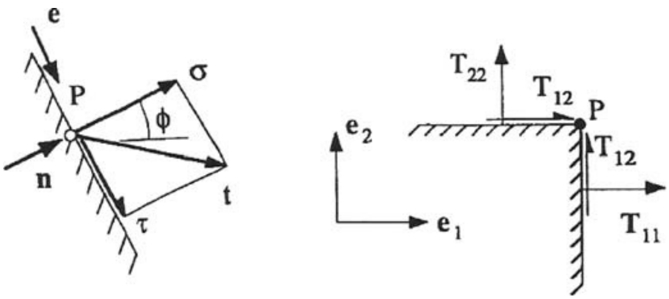


Fig. 3.3.8 Plane stress in particle *P*

$$\sigma = \frac{1}{2}(T_{11} + T_{22}) + \frac{1}{2}(T_{11} - T_{22})\cos 2\phi + T_{12}\sin 2\phi \tag{3.3.37}$$

$$\tau = \frac{1}{2}(T_{11} - T_{22})\sin 2\phi - T_{12}\cos 2\phi \tag{3.3.38}$$

The extremal values of σ and τ in the formulas (3.3.37) and (3.3.38) may be determined as follows. First we develop an expression for $[\sigma - (T_{11} + T_{22})/2]$ from formula (3.3.37). We then add the square of this expression and the square of τ obtained from formula (3.3.38). The result is:

$$\left[\sigma - \frac{T_{11} + T_{22}}{2}\right]^2 + \tau^2 = \left[\frac{T_{11} - T_{22}}{2}\right]^2 + [T_{12}]^2 \tag{3.3.39}$$

From this equation we see that σ obtains its extremal values when $\tau = 0$, and that the extremal values are given by the formulas (3.3.33). The extremal values of τ occur when $\sigma = (T_{11} + T_{22})/2$ and are:

$$\tau = \pm\sqrt{\left[\frac{T_{11} - T_{22}}{2}\right]^2 + [T_{12}]^2} = \pm(\sigma_1 - \sigma_2) \tag{3.3.40}$$

It is clear from the discussion above that the maximum shear stress in the particle *P* is in general not given by this result, but only when the two principal stresses σ_1 and σ_2 have opposite signs.

3.3.6 Mohr Diagram for Plane Stress

The stress analysis of plane stress may be illustrated graphically in a diagram. This graphic method has many important applications and gives a concentrated presentation of all aspects of the state of plane stress. The graphic method is also applicable in the analysis of any symmetric tensor of 2.order in two dimensions, for example the strain tensor for small strains in a surface, as will be demonstrated in Sect. 5.3.8, and the second moment of area tensor for plane areas.

We assume that the principal stresses and principal stress directions are known. In a coordinate system with axes parallel to the principal directions (3.3.37, 3.3.38, 3.3.39) become:

$$\sigma = \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\phi, \ \tau = \frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\phi \tag{3.3.41}$$

$$\left[\sigma - \frac{\sigma_1 + \sigma_2}{2}\right]^2 + \tau^2 = \left[\frac{\sigma_1 - \sigma_2}{2}\right]^2 \tag{3.3.42}$$

In a plane Cartesian coordinate system with σ and τ as coordinates, see Fig. 3.3.9, (3.3.42) describes a circle of radius $(\sigma_1 - \sigma_2)/2$ and with center C on the σ-axis at a distance $(\sigma_1 + \sigma_2)/2$ from the origin O. This circle is called *Mohr's circle* after Otto Mohr [1835–1918]. Figure 3.3.9 will be called a *Mohr diagram*. The points on the circle will be called stress points.

The stress point S having coordinates (σ, τ) represents the stresses on the physical plane that makes the angle ϕ with the principal direction for σ_1. The central angle between the σ-axis and the radius CS is equal to 2ϕ. This may be seen as follows. From the Mohr diagram we derive the formulas:

$$\sin 2\phi = \frac{2\tau}{\sigma_1 - \sigma_2}, \ \cos 2\phi = \frac{2\sigma - (\sigma_1 + \sigma_2)}{\sigma_1 - \sigma_2} \tag{3.3.43}$$

and these formulas are confirmed by the formulas (3.3.41).

The stresses, σ' and τ', on a plane with a normal making the angle $\phi + \pi/2$ with the σ_1-direction, i.e. a plane perpendicular to the plane defined by the angle ϕ, are by formula (3.3.41) given as:

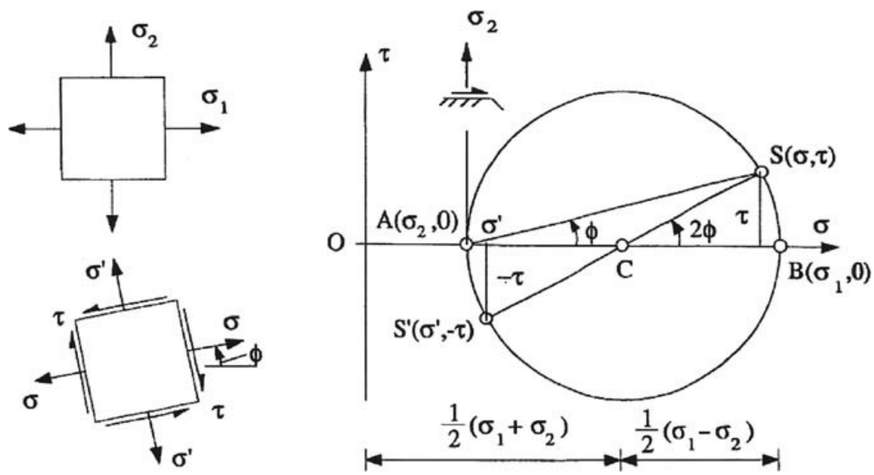


Fig. 3.3.9 Mohr diagram

$$\sigma' = \frac{1}{2}(\sigma_1+\sigma_2) + \frac{1}{2}(\sigma_1-\sigma_2)\cos\left[2\left(\phi+\frac{\pi}{2}\right)\right] = \frac{1}{2}(\sigma_1+\sigma_2) - \frac{1}{2}(\sigma_1-\sigma_2)\cos 2\phi$$

$$\tau' = \frac{1}{2}(\sigma_1-\sigma_2)\sin\left[2\left(\phi+\frac{\pi}{2}\right)\right] = -\frac{1}{2}(\sigma_1-\sigma_2)\sin 2\phi = -\tau$$

Note that $\tau' = -\tau$, in accordance with the general statement about shear stresses on orthogonal surfaces, see Fig. 3.2.13. The stress point S' with coordinate $(\sigma', -\tau)$ is located diametrically opposite to S on the circle. The central angle between the σ-axis and the radius CS' is equal to $2(\phi+\pi/2) = 2\phi+\pi$.

The stress point B represents the principal stress σ_1 and is the largest normal stress on planes parallel to the x_3-direction. The stress point A represents the principal stress σ_2 and is the smallest normal stress on planes parallel to the x_3-direction.

The point A in the Mohr diagram in Fig. 3.3.9 also represents a *pole of normals* to the planes the stresses act on, in the following sense. From the figure we see that the line from the pole A to the stress point S makes the angle ϕ with the x_1-direction: The periphery angle BAS is half of the central angle BCS. The line AS is therefore parallel to the normal to the plane that has the stresses σ and τ. Using this property of the pole we can include all information about the stresses on planes parallel to the x_3-direction in the Mohr diagram. Figure 3.3.10 shows how this may be presented.

The Mohr diagram may also be constructed in the general case of plane stress when the state of stress is given by the coordinate stresses T_{11}, T_{22}, and T_{12}. The equation (3.3.39) represents Mohr's circle in the Mohr diagram in Fig. 3.3.11. The radius of the circle is:

$$r = \sqrt{\left[\frac{T_{11}-T_{22}}{2}\right]^2 + [T_{12}]^2} \tag{3.3.44}$$

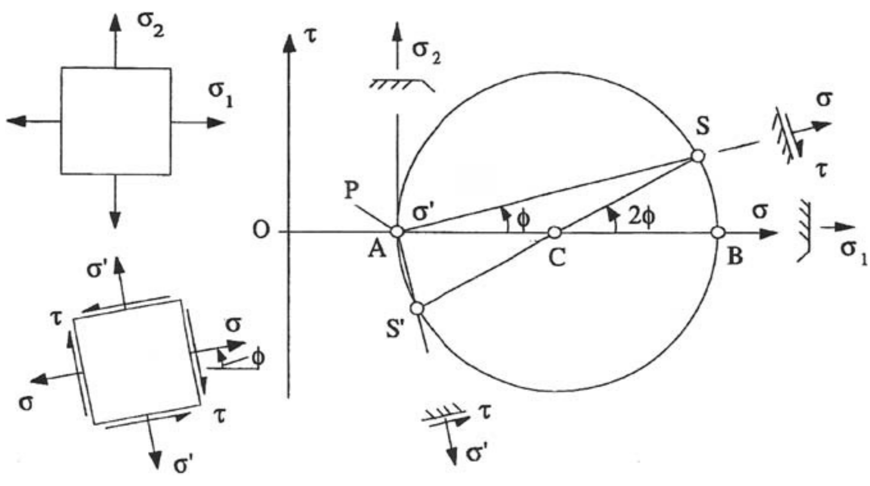


Fig. 3.3.10 Mohr diagram with the pole P of normals at A

The center of the circle is point C on the σ-axis a distance $(T_{11}+T_{22})/2$ from the origin O.

The stress point X with coordinates (T_{11}, T_{12}) represents the stresses on the plane normal to the x_1-direction, that is the plane for which $\phi = 0$. The stress point Y with coordinates (T_{22}, T_{12}) represents the stresses on the plane normal to the x_2-direction, that is the plane for which $\phi = \pi/2$. The two points X and Y lie on the same diameter of the circle. It follows from the discussion related to Fig. 3.3.10 that X and Y lie on the diameter that makes the angle $2\phi_1$ with the σ-axis, measured in the clockwise direction from the σ-axis. The angle ϕ_1 is given by formula (3.3.34). The Mohr's circle may be constructed by first marking off the points X and Y. The diameter between these points determines the center of the circle and the radius.

After the Mohr's circle has been constructed, all information about stresses on planes parallel to the x_3-direction may be determined graphically in the Mohr diagram. The pole of normals to the planes on which the stresses act are found as the point P of intersection between the lines from the stress points X and Y representing the normal directions related to those points. In order to see that point P really is the pole of normals, we draw the line PB and find that the periphery angle XPB is equal to ϕ_1, which is half of the central angle $2\phi_1$. The line PB is thus in the direction of the normal to the principal plane for σ_1, that is the direction of σ_1. The angle between the normal related to the stress point $S = (\sigma, \tau)$ and the x_1-direction is ϕ, and the angle between this normal and the σ_1-direction is $\phi - \phi_1$. Therefore the angle XCS equal to $2(\phi - \phi_1) + 2\phi_1 = 2\phi$. The periphery angle XPS is equal to half of the central angel 2ϕ, i.e. equal to ϕ.

From the Mohr diagram in Fig. 3.3.11 we may derive the formula (3.3.33) and the result:

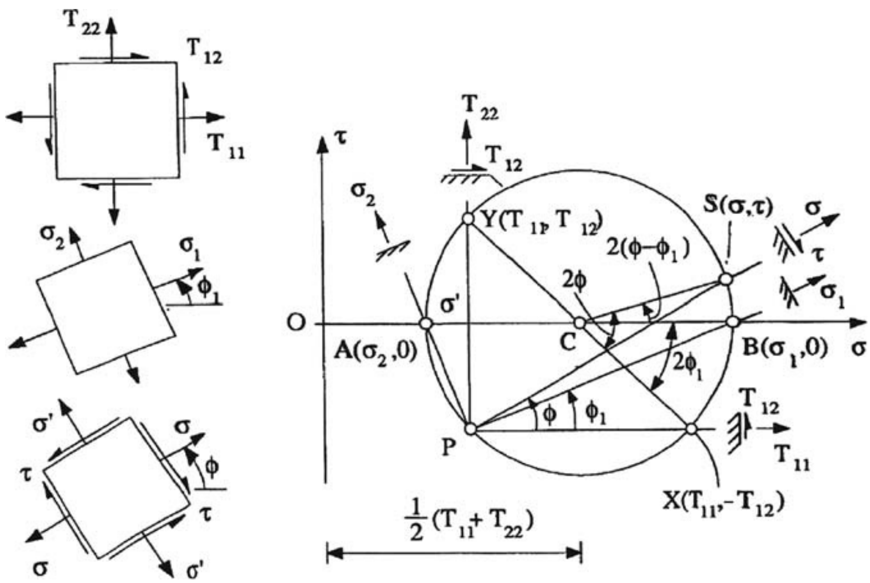


Fig. 3.3.11 A general Mohr diagram with the pole P of normals

$$\phi_1 = \frac{1}{2}\arctan\frac{2T_{12}}{T_{11}-T_{22}}, \ \phi_2 = \phi_1 + \pi/2 \tag{3.3.45}$$

These formulas for the principal directions do not distinguish between the two angles ϕ_1 and ϕ_2. Therefore, the formula (3.3.34) for ϕ_1 is to be preferred. The formula (3.3.34) may also be derived from the Mohr diagram.

The formulas (3.3.38) and $(3.3.41)_2$ for the shear stress τ implies a special sign convention illustrated in Fig. 3.3.12:

Shear stress pairs that tend to rotate a volume element in the clockwise direction are positive. While shear stress pairs that tend to rotate a volume element counter clockwise are negative.

This convention is followed in the Mohr diagram in the Figs. 3.3.9, 3.3.10, and 3.3.11.

3.3.7 *Mohr Diagram for General States of Stress*

A Mohr diagram may be used to illustrate general states of stress **T** in a particle *P*. For simplicity we choose a coordinate system *Ox* with base vectors $\mathbf{e}_i$ parallel to the principal direction of stress $\mathbf{n}_i$, and we order the principal stresses σ_i according to:

$$\sigma_{\min} = \sigma_3 \leq \sigma_2 \leq \sigma_1 = \sigma_{\max} \tag{3.3.46}$$

The normal stress σ and the shear stress τ on any plane parallel to the $\mathbf{n}_3$-direction must satisfy (3.3.42), which we may rewrite to:

$$(\sigma - \sigma_1)(\sigma - \sigma_2) + \tau^2 = 0 \tag{3.3.47}$$

Equation (3.3.42) was obtained for the case $\sigma_3 = 0$, but it is easily seen that the equation also applies when $\sigma_3 \neq 0$. Similarly we find for the stresses on planes parallel to the $\mathbf{n}_1$- and $\mathbf{n}_2$-directions:

$$\begin{aligned}(\sigma - \sigma_2)(\sigma - \sigma_3) + \tau^2 &= 0\\ (\sigma - \sigma_3)(\sigma - \sigma_1) + \tau^2 &= 0\end{aligned} \tag{3.3.48}$$

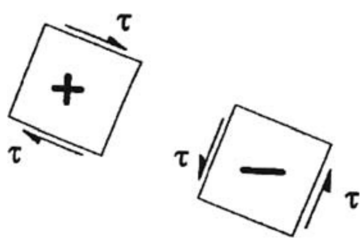


Fig. 3.3.12 Sign convention for shear stress related to the Mohr diagram and the formula (3.3.38)

The three equations (3.3.47, 3.3.48) may be illustrated by three stress circles in a Mohr diagram as shown in Fig. 3.3.13.

It will now be demonstrated that the normal stress σ and the shear stress τ on any plane through the particle P defined by the unit normal $\mathbf{n} = n_i \mathbf{e}_i \equiv n_i \mathbf{n}_i$ are represented by a stress point (σ, τ) in the Mohr diagram that lies outside of the two smaller circles and inside the largest circle. This means that we also obtain the result:

$$\sigma_{\min} \le \sigma \le \sigma_{\max}, \ \tau_{\max} = \frac{1}{2}(\sigma_{\max} - \sigma_{\min}) \tag{3.3.49}$$

which was demonstrated by different means in the Sects. 3.3.3 and 3.3.4.

The coordinate stresses T_{ij}, the stress vector $\mathbf{t}$ on the plane defined by the unit normal $\mathbf{n}$, and the normal stress σ and the shear stress τ on that plane are given by:

$$\begin{aligned} &T_{ij} = \sigma_i \delta_{ij}, \ t_i = T_{ij} n_j = \sigma_1 n_1 + \sigma_2 n_2 + \sigma_3 n_3 \\ &\sigma = \mathbf{n} \cdot \mathbf{t} = n_i t_i = n_1^2 \sigma_1 + n_2^2 \sigma_2 + n_3^2 \sigma_3 \\ &\tau^2 = \mathbf{t} \cdot \mathbf{t} - \sigma^2 = (\sigma_1 n_1)^2 + (\sigma_2 n_2)^2 + (\sigma_3 n_3)^2 - \sigma^2 \end{aligned} \tag{3.3.50}$$

Using these expressions for σ and τ and applying the condition: ${n_1}^2 + {n_2}^2 + {n_3}^2 = 1$, we obtain:

$$\begin{aligned} &(\sigma - \sigma_1)(\sigma - \sigma_2) + \tau^2 = \sigma^2 + \tau^2 - (\sigma_1 + \sigma_2)\sigma + \sigma_1 \sigma_2 \\ &= (n_1 \sigma_1)^2 + (n_2 \sigma_2)^2 + (n_3 \sigma_3)^2 - (\sigma_1 + \sigma_2)\left(n_1^2 \sigma_1 + n_2^2 \sigma_2 + n_3^2 \sigma_3\right) + \sigma_1 \sigma_2 \\ &= n_3^2 \left(\sigma_3^2 + \sigma_1 \sigma_2 - \sigma_1 \sigma_3 - \sigma_2 \sigma_3\right) = n_3^2 (\sigma_1 - \sigma_3)(\sigma_2 - \sigma_3) \end{aligned}$$

Two similar results are obtained and the three results are listed as:

$$\begin{aligned} (\sigma - \sigma_1)(\sigma - \sigma_2) + \tau^2 &= n_3^2 (\sigma_1 - \sigma_3)(\sigma_2 - \sigma_3) \\ (\sigma - \sigma_2)(\sigma - \sigma_3) + \tau^2 &= n_1^2 (\sigma_2 - \sigma_1)(\sigma_3 - \sigma_1) \\ (\sigma - \sigma_3)(\sigma - \sigma_1) + \tau^2 &= n_3^2 (\sigma_3 - \sigma_2)(\sigma_1 - \sigma_2) \end{aligned} \tag{3.3.51}$$

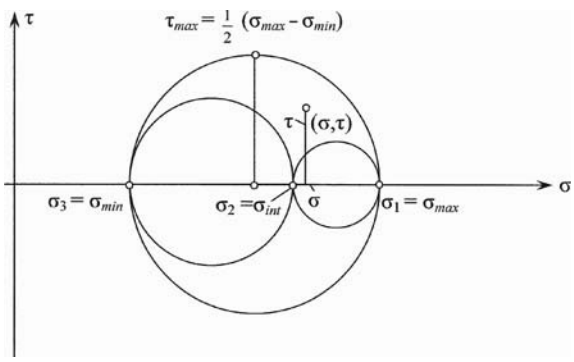


Fig. 3.3.13 Mohr diagram for general state of stress

Now, since:

$$(\sigma_1 - \sigma_3) \geq 0, \ (\sigma_2 - \sigma_3) \geq 0, \ (\sigma_1 - \sigma_2) \geq 0$$

it follows that:

$$\begin{aligned}(\sigma - \sigma_1)(\sigma - \sigma_2) + \tau^2 \geq 0 \\ (\sigma - \sigma_2)(\sigma - \sigma_3) + \tau^2 \geq 0 \\ (\sigma - \sigma_3)(\sigma - \sigma_1) + \tau^2 \leq 0\end{aligned} \tag{3.3.52}$$

By comparing these inequalities with the formulas (3.3.47, 3.3.48) for the three stress circles in the Mohr diagram in Fig. 3.3.13, we conclude that the normal stress σ and the shear stress τ on any plane through the particle are represented by a stress point (σ, τ) in the Mohr diagram that must lie outside of the two smaller stress circles and inside the largest stress circle. Furthermore, the results (3.3.49) follow directly from this observation.

Problems

Problem 3.1. A velocity field is given as:

$$v_1 = \frac{\alpha x_1}{t - t_o}, \ v_2 = -\frac{\alpha x_2}{t - t_o}, \ v_3 = 0, \ \alpha \text{ and } \mathrm{t}_o \text{ are constants}$$

a) Show that the flow is *isochoric* (= volume preserving), i.e. div $\mathbf{v} = 0$.
b) Determine the local acceleration, the convective acceleration, and the particle acceleration $\dot{\mathbf{v}}$.
c) Show that the flow is irrotational, i.e. rot $\mathbf{v} = \mathbf{0}$, and determine the velocity potential ϕ from the formula: $\mathbf{v} = \nabla\phi$.

Problem 3.2. The state of stress in a particle is represented by the stress matrix given in Example 3.7 with respect to the coordinate system Ox.

a) Determine the stress vector, the normal stress and the shear stress on a surface with unit normal: $\mathbf{n} = [1, 1, 0]/\sqrt{2}$.
b) Determine the stress matrix $\bar{T}$ with respect to a coordinate system $\bar{O}\bar{x}$ when the transformation matrix is:

$$Q = (\bar{\mathbf{e}}_i \cdot \mathbf{e}_k) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -1/2 & \sqrt{3}/2 \end{pmatrix}$$

Problem 3.3. A rectangular straight beam with horizontal axis has the length L, height h and width b. The x-axis is along the beam axis. The ends of the beam are defined by $x = 0$ and $x = L$. The beam end at $x = L$ is fixed. The beam is subjected to vertical load F in the positive y-direction at the end $x = 0$. According to the elementary beam theory the stress matrix at a particle in the beam is given by:

$$T = \begin{pmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \sigma_x = -\frac{12F}{h^3 b} xy, \ \tau = -\frac{3F}{2hb}\left[1 - \frac{4y^2}{h^2}\right]$$

Show that the stress matrix satisfies the Cauchy equations of motion.

Problem 3.4. A circular cylindrical vessel with a fluid rotates about its vertical axis at a constant angular velocity ω. The fluid moves as a rigid body with the vessel. The fluid has constant density ρ. On the free surface of the fluid the pressure is given by the atmospheric pressure p_o. Determine the shape of the free surface and find the expression for the pressure in the fluid.

Problem 3.5. A triangular plate is subjected to linearly varying pressure $p = p_o\, x/h$ on the surface $y = 0$. The body force is represented by the gravitational force g in the x-direction. The density of the plate is ρ. The following state of stress represents the solution from the theory of elasticity:

$$\sigma_x = \left[\frac{\rho g h}{b} - \frac{2p_o h^2}{b^3}\right] y + \left[\frac{p_o h}{b^2} - \rho g\right] x$$

$$\sigma_y = -\frac{p_o}{h} x, \ \tau_{xy} = -\frac{p_o h}{b^2} y$$

Show that the Cauchy equations are satisfied, and check that the free surface is stress free.

Problem 3.6. A thin-walled circular tube has middle radius $r = 150\,\text{mm}$ and wall thickness $h = 7\,\text{mm}$. The tube is closed in both ends, and is subjected to an internal pressure $p = 8\,\text{MPa}$, a torque $M = 60\,\text{kNm}$, and an axial force $N = 180\,\text{kN}$. The state of stress in the wall of the tube may be found by superposition of the states of stress given in the Examples 3.1, 3.2, and 3.5.

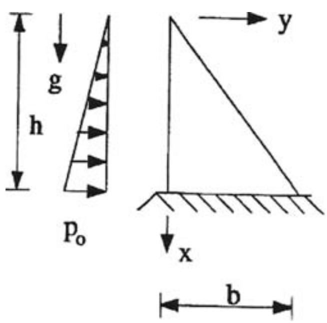


Fig. Problem 3.5 Triangular plate subjected to linearly varying pressure

a) Determine the coordinate stresses in cylinder coordinates for a particle in the wall of the tube.
b) Determine the principal stresses and the principal stress directions in the tube wall.
c) Determine the maximum shear stress in the tube wall.

Problem 3.7. The state of stress in a particle is represented by the stress matrix:

$$T = \begin{pmatrix} 200 & -100 & -100 \\ -100 & 100 & 0 \\ -100 & 0 & 100 \end{pmatrix}$$

a) Determine the matrices for the stress isotrop and the stress deviator.
b) Determine the principal stresses and the principal stress directions.
c) Compute the maximum shear stress, and determine the unit normal to one of the planes of maximum shear stress.
d) Compute the normal stress and the shear stress on the plane with unit normal:

$$\mathbf{n} = [1,1,0]/\sqrt{2}$$

Problem 3.8. The state of plane stress in a particle on the free surface of a solid is given by the stress matrix:

$$T = \left(T_{\alpha\beta}\right) = \begin{pmatrix} 180 & 80 \\ 80 & 60 \end{pmatrix} \text{MPa}\,,\; T_{i3} = 0$$

Draw Mohr's circle of stress for the state of stress. Use both formulas and the Mohr diagram to determine:

a) The principal stresses and principal stress directions.
b) The normal stress and the shear stress on the plane parallel with the x_3-axis and with unit normal 45° with the $x_1\ x_3$-plane.
c) The maximum shear stress in the particle. Present a figure showing a plane with maximum shear stress and how this plane relates to the principal stress directions. Present also the normal stress on this plane.

Problem 3.9. A continuum is in addition to contact forces and body forces also subjected to body couples, moment per unit mass, $\mathbf{c} = \mathbf{c}(\mathbf{r},t)$. Show that the Cauchy equations (3.2.35), i.e. Cauchy's 1. law, still apply while the Cauchy's 2. law (3.2.38) is replaced by:

$$e_{ijk}\,T_{kj} + \rho\,c_i = 0 \;\Leftrightarrow\; T_{kj} = T_{jk} + \rho\,c_i,\; k \neq j \neq i \neq k$$

The stress tensor is no longer symmetric.

Chapter 4
Tensors

4.1 Definition of Tensors

In Sect. 3.2.4 the stress tensor $\mathbf{T}$ was defined. The stress tensor is the "original" tensor as the word *tensor* means stress. We shall use the definition of the stress tensor as an introduction to the general concept of tensors. Let us consider a particle $\mathbf{r}_o$ at position $\mathbf{r}$ in a material body in the present configuration K. The contact force at the particle on a surface A through the particle, and with unit normal $\mathbf{n}$ at the particle, is represented by the stress vector $\mathbf{t}$. In a coordinate system Ox with base vectors $\mathbf{e}_i$ the two vectors, $\mathbf{n}$ and $\mathbf{t}$, are represented by their components n_k and t_i, such that:

$$\mathbf{n} = n_k \, \mathbf{e}_k \quad , \quad \mathbf{t} = t_i \, \mathbf{e}_i \tag{4.1.1}$$

Coordinate surfaces through the particle are acted upon by coordinate stresses T_{ik}. The Cauchy stress theorem by (3.2.27) states a linear relation between the components of the stress vector $\mathbf{t}$ and the components of the surface normal vector $\mathbf{n}$:

$$t_i = T_{ik} \, n_k \quad , \quad t = T \, n \tag{4.1.2}$$

Since such a relation holds in any Cartesian coordinate system and expresses a linear relationship between two coordinate invariant quantities: the vectors $\mathbf{n}$ and $\mathbf{t}$, we say that the coordinate stresses T_{ik}, or the stress matrix T, represent in Ox a coordinate invariant property $\mathbf{T}$, which we call the *stress tensor*. The coordinate invariant form of the relation (4.1.2) is written as:

$$\mathbf{t} = \mathbf{T} \cdot \mathbf{n} \tag{4.1.3}$$

We may interpret the symbol $\mathbf{T}$ in the relation (4.1.3) as a function with the vector $\mathbf{n}$ as an argument and the vector $\mathbf{t}$ as the value of the function: For any value of the argument $\mathbf{n}$ the relation (4.1.3), or (4.1.2), produces a vector $\mathbf{t}$. We may express this fact by stating that the tensor $\mathbf{T}$ represents a *vector-valued function of a vector*. The component version (4.1.2) of the relation (4.1.3) shows how the function $\mathbf{T}$ operates.

It follows that if we substitute the argument vector $\mathbf{n}$ in the relation (4.1.3) by an arbitrary vector $\mathbf{a}$, we will again get a vector $\mathbf{c} = \mathbf{T} \cdot \mathbf{a}$ as the value of the function. If we let $\mathbf{a}$ be equal to the sum of two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, we find that:

$$\mathbf{T} \cdot \mathbf{a} = \mathbf{T} \cdot (\mathbf{a}_1 + \mathbf{a}_2) = \mathbf{T} \cdot \mathbf{a}_1 + \mathbf{T} \cdot \mathbf{a}_2$$

This result follows directly from the component relation (4.1.2). The symbol $\mathbf{T}$ therefore represents a linear function of the vector argument. The relationship $\mathbf{c} = \mathbf{T} \cdot \mathbf{a}$ is called a *linear mapping of vectors* or a *linear transformation of vectors*. Both names are used in the literature. The tensor $\mathbf{T}$ represents in (4.1.3) a *linear vector-valued function of a vector*.

The normal stress σ on the surface A and the shear stress $\tilde{\tau}$ in an arbitrary direction $\mathbf{e}$ on the surface A are given by, see Fig. 3.2.10 and (3.2.28) and (3.2.30):

$$\sigma = \mathbf{n} \cdot \mathbf{t} = \mathbf{n} \cdot \mathbf{T} \cdot \mathbf{n} \quad , \quad \tilde{\tau} = \mathbf{e} \cdot \mathbf{t} = \mathbf{e} \cdot \mathbf{T} \cdot \mathbf{n} \tag{4.1.4}$$

In the Ox-system (4.1.4) have the representations:

$$\sigma = n_i T_{ik} n_k = n^T T n \quad , \quad \tilde{\tau} = e_i T_{ik} n_k = e^T T n \tag{4.1.5}$$

In the relations (4.1.4) $\mathbf{T}$ represents a function of two vector arguments $\mathbf{n}$ and $\mathbf{n}$, or $\mathbf{e}$ and $\mathbf{n}$, and with the scalars σ and $\tilde{\tau}$ as function values. The function $\mathbf{T}$ is linear with respect to both argument vectors. We may state that the tensor $\mathbf{T}$ represents a *bilinear scalar-valued function of two vectors*. The relations (4.1.5) show how the function operates. The coordinate stresses T_{ik} are called the *components of the stress tensor* $\mathbf{T}$ in the coordinate system Ox.

In Continuum Mechanics, and in field theory in general, the properties we call tensors appear primarily in relations between scalars and vectors, as shown in (4.1.3) and (4.1.4), and secondarily, as we shall see, in relation between already established tensors. Among the many possible definitions of the tensor concept, this book chooses the following general and completely coordinate invariant definition:

A tensor of order n is a multilinear scalar-valued function of n argument vectors.

The word multilinear means that the function is linear in every argument vector. The value of the function is a scalar. Based on this definition we see from (4.1.4) that the stress tensor $\mathbf{T}$ is a bilinear scalar-valued function of two argument vectors.

To see the implications of the general definition of tensors, we start to investigate the properties of a *tensor of 1. order*. Let $\alpha = \mathbf{a}[\mathbf{b}]$ be a linear function of a vector $\mathbf{b}$. Then $\mathbf{a}$ is a tensor of 1. order. In a Cartesian coordinate system Ox with base vectors $\mathbf{e}_i$ we may compute the function values:

$$a_i = \mathbf{a}[\mathbf{e}_i] \tag{4.1.6}$$

Due to the linear property or the tensor, the scalar value α for an arbitrarily chosen argument vector $\mathbf{b} = b_i \mathbf{e}_i$ may be computed thus:

$$\alpha = \mathbf{a}[\mathbf{b}] = \mathbf{a}[b_i \mathbf{e}_i] = b_i \, \mathbf{a}[\mathbf{e}_i] = b_i \, a_i \tag{4.1.7}$$

We see that the three function values a_i represent the tensor $\mathbf{a}$ in the Ox-system. For this reason the function values a_i are called the *components of the tensor* $\mathbf{a}$ *in the* Ox*-system*. The matrix $a = \{a_1 \; a_2 \; a_3\}$ is the *tensor matrix* of $\mathbf{a}$ in the Ox-system.

We shall now develop a relation between the components of a tensor $\mathbf{a}$ in two different coordinate systems Ox and $\bar{O}\bar{x}$, for which the base vectors $\mathbf{e}_j$ and $\bar{\mathbf{e}}_i$ are connected through the transformation matrix:

$$\begin{aligned} Q &= (Q_{ij}) \quad , \quad Q_{ij} = \cos(\bar{\mathbf{e}}_i \cdot \mathbf{e}_j) \\ \bar{\mathbf{e}}_i &= Q_{ij} \, \mathbf{e}_i \quad , \quad \mathbf{e}_j = Q_{ij} \, \bar{\mathbf{e}}_i \end{aligned} \tag{4.1.8}$$

In the two coordinate systems the tensor $\mathbf{a}$ is represented by the component sets:

$$a_j = \mathbf{a}[\mathbf{e}_j] \quad , \quad \bar{a}_i = \mathbf{a}[\bar{\mathbf{e}}_i] \tag{4.1.9}$$

We find that:

$$\begin{aligned} &\mathbf{a}[\bar{\mathbf{e}}_i] = \mathbf{a}[Q_{ij} \mathbf{e}_j] = Q_{ij} \, \mathbf{a}[\mathbf{e}_j] \quad \Rightarrow \\ &\bar{a}_i = Q_{ij} \, a_j \quad \Leftrightarrow \quad \bar{a} = Q \, a \quad \Leftrightarrow \quad a = Q^T \, \bar{a} \end{aligned} \tag{4.1.10}$$

This is also the transformation formula for the components of a vector $\mathbf{a}$. We may therefore make the statement that *vectors are tensors of 1. order*. From the expressions (4.1.7) it follows that the scalar-valued function $\mathbf{a}[\mathbf{b}]$ is equal to the scalar product of the two vectors $\mathbf{a}$ and $\mathbf{b}$.

$$\alpha = \mathbf{a}[\mathbf{b}] = \mathbf{a}[b_i \mathbf{e}_i] = b_i \, \mathbf{a}[\mathbf{e}_i] = b_i \, a_i = \mathbf{b} \cdot \mathbf{a} \tag{4.1.11}$$

According to the general definition of tensors given above, we may consider *scalars* as *tensors of zeroth order*.

Let $\mathbf{A}[\mathbf{b},\mathbf{c}]$ be *a bilinear scalar-valued function of two vectors* $\mathbf{b}$ and $\mathbf{c}$. The value of the function is a scalar α, and the function is linear in each vector argument.

$$\alpha = \mathbf{A}[\mathbf{b}, \mathbf{c}] \tag{4.1.12}$$

$$\mathbf{A}[\beta \, \mathbf{b}, \gamma \mathbf{c}] = \beta \, \gamma \mathbf{A}[\mathbf{b}, \mathbf{c}] \tag{4.1.13}$$

We say that $\mathbf{A}$ is *a tensor of 2. order*. The symbol for tensors will in general be capital bold face Latin letters. An exception is for tensors of zeroth order, i.e. scalars, for which we prefer small Greek letters, and for tensors of 1. order, for which we prefer lower case bold face Latin letters in accordance with what has been decided previously for vectors.

The *components of the tensor in an Ox-system* with base vectors $\mathbf{e}_i$ are defined as the following values of the tensor $\mathbf{A}$.

$$A_{ij} = \mathbf{A}[\mathbf{e}_i, \mathbf{e}_j] \tag{4.1.14}$$

The scalar value α for two arbitrarily chosen argument vectors:

$$\mathbf{b} = b_i \mathbf{e}_i \quad , \quad \mathbf{c} = c_j \mathbf{e}_j \tag{4.1.15}$$

may now be computed as follows. The tensor is a bilinear function. Thus:

$$\begin{aligned} \mathbf{A}[\mathbf{b}, \mathbf{c}] = \mathbf{A}[b_i \mathbf{e}_i, c_j \mathbf{e}_j] = b_i c_j \mathbf{A}[\mathbf{e}_i, \mathbf{e}_j] \quad \Rightarrow \\ \alpha = \mathbf{A}[\mathbf{b}, \mathbf{c}] = b_i c_j A_{ij} = b^T A c \end{aligned} \tag{4.1.16}$$

The $3^2 = 9$ tensor components A_{ij}, or the *tensor matrix* $A = (A_{ij})$, represent the tensor $\mathbf{A}$ in the Ox-system.

The expressions (4.1.4) for the normal stress σ on a surface with unit normal $\mathbf{n}$ and the shear stress $\tilde{\tau}$ in a direction $\mathbf{e}$ on the surface may now be written as:

$$\sigma = \mathbf{T}[\mathbf{n}, \mathbf{n}] = n_i n_j T_{ij} = n^T T n \tag{4.1.17}$$

$$\tilde{\tau} = \mathbf{T}[\mathbf{e}, \mathbf{n}] = e_i n_j T_{ij} = e^T T n \tag{4.1.18}$$

The coordinate stresses T_{ij} are components of the stress tensor $\mathbf{T}$ in the Ox-system.

The relation between the tensor components A_{ij} and $\bar{A}_{ij}$ in two coordinate systems Ox and $\bar{O}\bar{x}$ with base vectors related through the formulas (4.1.8) by the transformation matrix Q, are found as follows. By definition:

$$\bar{A}_{ij} = \mathbf{A}[\bar{\mathbf{e}}_i, \bar{\mathbf{e}}_j] \quad , \quad A_{kl} = \mathbf{A}[\mathbf{e}_k, \mathbf{e}_l] \tag{4.1.19}$$

Now:

$$\begin{aligned} \mathbf{A}[\bar{\mathbf{e}}_i, \bar{\mathbf{e}}_j] = \mathbf{A}[Q_{ik}\mathbf{e}_k, Q_{jl}\mathbf{e}_l] = Q_{ik} Q_{jl} \mathbf{A}[\mathbf{e}_k, \mathbf{e}_l] \quad \Rightarrow \\ \bar{A}_{ij} = Q_{ik} Q_{jl} A_{kl} \quad \Leftrightarrow \quad \bar{A} = Q A Q^T \end{aligned} \tag{4.1.20}$$

The inverse transformation is:

$$A_{kl} = Q_{ik} Q_{jl} \bar{A}_{ij} \quad \Leftrightarrow \quad A = Q^T A Q \tag{4.1.21}$$

In some presentations in the literature the relations (4.1.20) and (4.1.21) are used to define a tensor of 2. order:

A tensor of 2. order is an invariant quantity which in every Cartesian coordinate system Ox is represented by a two-dimensional matrix $A = (A_{ij})$, such that the tensor matrices in any two coordinate systems Ox and $\bar{O}\bar{x}$ are related by the formulas (4.1.20) and (4.1.21).

This definition is the most practical one in n-dimensional spaces in which the concept of vectors is abstract since it is not possible to use geometrical figures in the same fashion as in the three-dimensional *Euclidian space*. The definition chosen in the present exposition, that a tensor is a multilinear scalar-valued function of vectors, is preferred because the definition is obviously coordinate invariant, and because it is the most convenient definition when we want to introduce tensor components in general curvilinear coordinate systems, as we shall see in Chap. 12.

For convenience we shall let all the symbols $\mathbf{A}$, A, and A_{ij} represents one and the same tensor of 2. order. The bold face notation $\mathbf{A}$ is preferred when it is important to emphasize the coordinate invariance of the property described by the tensor. For a tensor of 1. order $\mathbf{a}$ we use alternatively the symbols $\mathbf{a}$, a, and a_i.

An *isotropic tensor* is a tensor represented by the same matrix in all Cartesian coordinate systems. Isotropic tensors of 2., 3., and 4. order will be presented below.

The scalar product of two vectors $\mathbf{b}$ and $\mathbf{c}$, i.e. $\mathbf{b}\cdot\mathbf{c} = b_i c_i$, may be given by the *unit tensor of 2. order* represented by the tensor symbol $\mathbf{1}$, such that:

$$\alpha = \mathbf{1}[\mathbf{b},\mathbf{c}] = \mathbf{b} \cdot \mathbf{c} \tag{4.1.22}$$

The components of the unit tensor in an Ox-system is a *Kronecker delta*:

$$\delta_{ij} = \mathbf{1}[\mathbf{e}_i,\mathbf{e}_j] = \mathbf{e}_i \cdot \mathbf{e}_j \quad \Leftrightarrow \quad 1 = (\delta_{ij}) \tag{4.1.23}$$

It follows from this result, and also from the transformation formula (4.1.20) for components of second order tensors, that $\bar{1} = Q1Q^T = 1$. The unit tensor is represented by the unit matrix in all Cartesian coordinate systems and is thus an isotropic tensor of 2. order.

A tensor of 3. order $\mathbf{C}$ is in an Ox-system represented by $3^3 = 27$ components:

$$C_{ijk} = \mathbf{C}[\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k] \tag{4.1.24}$$

which may be considered to be the elements in a three-dimensional matrix C. The relation between the tensor components C_{ijk} in the Ox-system and the tensor components $\bar{C}_{rst}$ in an $\bar{O}\bar{x}$-system is:

$$\bar{C}_{rst} = Q_{ri}\,Q_{sj}\,Q_{tk}\,C_{ijk} \tag{4.1.25}$$

The derivation of this result follows the development of the relation (4.1.20).

The box product of three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$: $[\mathbf{abc}] = e_{ijk}\;a_i\;b_j\;c_k$, may also be given by the *permutation tensor* represented by the symbol $\mathbf{P}$, such that:

$$\alpha = \mathbf{P}[\mathbf{a},\mathbf{b},\mathbf{c}] = [\mathbf{abc}] = a_i\,b_j\,c_j\,e_{ijk} \tag{4.1.26}$$

The components of the permutation tensor are given by the permutation symbol e_{ijk} in all Cartesian coordinate systems Ox.

$$P_{ijk} = [\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k] = e_{ijk} \quad , \quad \bar{P}_{rst} = [\bar{\mathbf{e}}_r,\bar{\mathbf{e}}_s,\bar{\mathbf{e}}_t] = e_{rst} \tag{4.1.27}$$

The permutation tensor is thus an isotropic tensor of 3. order.

The tensor properties of the components P_{ijk} and $\bar{P}_{rst}$ may also be demonstrated by applying the formula $(2.1.20)_2$ for the determinant of the transformation matrix Q for the transformation from an Ox-system to an $\bar{O}\bar{x}$-system. Since det $Q = 1$, we may use the formula $(2.1.20)_2$ to write:

$$Q_{ri}\,Q_{sj}\,Q_{tk}\,e_{ijk} = (\det Q)\,e_{rst} = e_{rst} \tag{4.1.28}$$

A tensor $\mathbf{N}$ of order n is in an Ox-system represented by 3^{n} components:

$$N_{ij\cdot\cdot} = \mathbf{N}[\mathbf{e}_i, \mathbf{e}_j, \cdot\cdot] \tag{4.1.29}$$

in an n-dimensional matrix N. The relations between the components $N_{ij..}$ in the Ox-system and the components $\bar{N}_{rs..}$ in an $\bar{O}\bar{x}$-system are:

$$\bar{N}_{rs..} = Q_{ri}\,Q_{sj}\cdot\cdot N_{ij\cdot\cdot} \quad \Leftrightarrow \quad N_{ij} = Q_{ri}Q_{sj}\cdot\cdot\bar{N}_{rs\cdot\cdot} \tag{4.1.30}$$

These relations may be used in an alternative definition of a tensor of n. order:

A tensor of n. order is a coordinate invariant quantity, which in every Cartesian coordinate system is represented by an n-dimensional matrix, such that the matrices in two coordinate systems are related by (4.1.30).

A tensor $\mathbf{N}$ of order $n(>1)$ is symmetric/antisymmetric with respect to two argument vectors, or two component indices, if:

$$\begin{gathered}\mathbf{N}[\cdot,\cdot\cdot,\mathbf{a},\cdot,\mathbf{b},\cdot] = +/-\mathbf{N}[\cdot,\cdot\cdot,\mathbf{b},\cdot,\mathbf{a},\cdot] \quad \Leftrightarrow \\ \mathbf{N}[\cdot,\cdot\cdot,\mathbf{e}_i,\cdot,\mathbf{e}_j,\cdot] = +/-\mathbf{N}[\cdot,\cdot\cdot,\mathbf{e}_j,\cdot,\mathbf{e}_i,\cdot] \quad \Leftrightarrow \quad N_{\dots i\cdot j\cdot} = +/-N_{\dots j\cdot i\cdot}\end{gathered} \tag{4.1.31}$$

The tensor is *completely symmetric/antisymmetric* if the symmetry/antisymmetry property applies to any two argument vectors, or any two component indices. The stress tensor $\mathbf{T}$ is an example of a symmetric tensor of 2. order. With reference to (4.1.18):

$$\mathbf{T}[\mathbf{e},\mathbf{n}] = \mathbf{T}[\mathbf{n},\mathbf{e}] \quad \Leftrightarrow \quad \mathbf{T}[\mathbf{e}_i,\mathbf{e}_j] = \mathbf{T}[\mathbf{e}_j,\mathbf{e}_i] \quad , \quad T_{ij} = T_{ji} \quad \Leftrightarrow \quad T = T^T$$

The unit tensor $\mathbf{1}$ is a symmetric tensor of 2. order: $\delta_{ij} = \delta_{ji}$. A completely symmetric tensor $\mathbf{S}$ of 3. order has components that satisfy the conditions:

$$S_{ijk} = S_{ikj} = S_{jki} = S_{jik} = S_{kij} = S_{kji} \tag{4.1.32}$$

The number of distinct components different from zero is reduced from 27 for a general 3. order tensor, to 10 for the symmetric $\mathbf{S}$. A completely antisymmetric tensor of 3. order has only one distinct component different from zero, such that its components in any Ox-system is given by αe_{ijk}, i.e. the product of a scalar α and the permutation symbol e_{ijk}. The proof of this statement is given as Problem 4.1.

4.2 Tensor Algebra

Tensors of the same order n may be added or subtracted, and the result are new tensors of order n. The *sum* of two 2. order tensors $\mathbf{A}$ and $\mathbf{B}$ is defined by:

$$\mathbf{A}+\mathbf{B}=\mathbf{C} \quad \Leftrightarrow \quad \mathbf{A}[\mathbf{a},\mathbf{b}]+\mathbf{B}[\mathbf{a},\mathbf{b}]=\mathbf{C}[\mathbf{a},\mathbf{b}] \tag{4.2.1}$$

This means that the scalar value of $\mathbf{C}$ for some argument vectors, $\mathbf{a}$ and $\mathbf{b}$, is obtained by adding the scalar values of $\mathbf{A}$ and $\mathbf{B}$ for the same argument vectors. It follows that:

$$\mathbf{A}+\mathbf{B}=\mathbf{C} \quad \Leftrightarrow \quad A_{ij}+B_{ij}=C_{ij} \quad \Leftrightarrow \quad A+B=C \tag{4.2.2}$$

The *difference* $\mathbf{A}-\mathbf{B}=\mathbf{D}$ is defined similarly. Addition and subtraction of tensor of other orders are defined analogously. The sum or difference of tensors of 1. order coincide with the result of these operations for vectors.

If the scalar value of tensor $\mathbf{A}$ is equal to the negative scalar value of a tensor $\mathbf{B}$ of the same order for all sets of the same argument vectors, we write:

$$\mathbf{A}=-\mathbf{B} \quad \Leftrightarrow \quad A=-B \tag{4.2.3}$$

The sum $\mathbf{A}+\mathbf{B}$ is then a *zero tensor* $\mathbf{O}$ with all components equal to zero in any coordinate system.

The *tensor product* of a tensor $\mathbf{A}$ of order m and a tensor $\mathbf{B}$ of order n is a tensor $\mathbf{C}$ of order $(m+n)$, and defined by:

$$\mathbf{A}\otimes\mathbf{B}=\mathbf{C} \quad \Leftrightarrow \quad \mathbf{A}[\mathbf{a},\cdot\cdot]\,\mathbf{B}[\mathbf{b},\cdot]=\mathbf{C}[\mathbf{a},\cdot\cdot,\mathbf{b},\cdot] \quad \Leftrightarrow \quad A_{i\cdot\cdot}B_{j\cdot}=C_{i\cdot\cdot j\cdot} \tag{4.2.4}$$

The tensor product is a multilinear scalar-valued function of the all the argument vectors of the two factor tensors such that the scalar value of the tensor product is the product of the scalar values of the factor tensors. In general $\mathbf{A}\otimes\mathbf{B}\neq\mathbf{B}\otimes\mathbf{A}$. The tensor products $\mathbf{c}\otimes\mathbf{d}$ and $\mathbf{d}\otimes\mathbf{c}$ of two vectors $\mathbf{c}$ and $\mathbf{d}$ are tensors of 2. order:

$$\begin{aligned}
&\mathbf{c}\otimes\mathbf{d}=\mathbf{E} \quad \Leftrightarrow \quad \mathbf{c}[\mathbf{a}]\,\mathbf{d}[\mathbf{b}]=\mathbf{E}[\mathbf{a},\mathbf{b}] \quad \Leftrightarrow \quad c_i d_j=E_{ij} \quad \Leftrightarrow \quad c^T d=E \\
&\mathbf{d}\otimes\mathbf{c}=\mathbf{F} \quad \Leftrightarrow \quad \mathbf{d}[\mathbf{a}]\,\mathbf{c}[\mathbf{b}]=\mathbf{F}[\mathbf{a},\mathbf{b}] \quad \Leftrightarrow \quad d_i c_j=F_{ij} \quad \Leftrightarrow \quad d^T c=F
\end{aligned} \tag{4.2.5}$$

We see that $E=F^T$. The two tensor products $\mathbf{c}\otimes\mathbf{d}$ and $\mathbf{d}\otimes\mathbf{c}$ are called the *dyadic products*, or for short *dyads* of the two vectors. Tensor products of many vectors are called *polyads*, for example the *triad*:

$$\mathbf{c}\otimes\mathbf{d}\otimes\mathbf{e}=(\mathbf{c}\otimes\mathbf{d})\otimes\mathbf{e}=\mathbf{c}\otimes(\mathbf{d}\otimes\mathbf{e})$$

In some presentations, see for instance Malvern [28], the multiplication symbol $\otimes$ in the tensor product is omitted. The tensor product in (4.2.4) is then denoted: $\mathbf{AB}=\mathbf{C}$. In the present book the product $\mathbf{AB}(\neq\mathbf{A}\otimes\mathbf{B})$ is defined only for 2. order tensors and is called the composition of the two second order tensors, and defined by the relation (4.3.13) below.

The tensor product of a scalar α, i.e. tensor of order zero, and a tensor **B** of order n is a tensor **C** of order n:

$$\alpha \mathbf{B} = \mathbf{C} \quad \Leftrightarrow \quad \alpha \mathbf{B}[\mathbf{a}, \cdot\cdot] = \mathbf{C}[\mathbf{a}, \cdot\cdot] \quad \Leftrightarrow \quad \alpha B_{i\cdot\cdot} = C_{i\cdot\cdot} \quad \Leftrightarrow \quad \alpha B = C \tag{4.2.6}$$

It may be shown that an isotropic tensor of 2. order, $\mathbf{I} \equiv \mathbf{I}_2$, always is the product of a scalar α and the unit tensor **1**.

$$\mathbf{I} \equiv \mathbf{I}_2 = \alpha \mathbf{1} \tag{4.2.7}$$

The proof of this is given as Problem 4.2. It may also be shown that the general isotropic tensor of order 3, denoted by $\mathbf{I}_3$, is a product of a scalar and the permutation tensor **P**.

$$\mathbf{I}_3 = \alpha \mathbf{P} \tag{4.2.8}$$

For a proof see Problem 4.3.

A *contraction* of a tensor of order n is an operation on the matrix of the tensor leading to a new tensor of order $(n-2)$. As an example, let **C** be a tensor of order 3. Then the component sets:

$$C_{ikk} \quad , \quad C_{kjk} \quad , \quad C_{iik} \tag{4.2.9}$$

represent three different tensors of order $(3-2) = 1$, in this case three different vectors. These operations are called contractions: two indices in the tensor matrix are set equal and a summation is implied over the region for this index. That the result of a contraction represents components of a new tensor of order two less then the original tensor will now be shown for the third of the contractions (4.2.9), C_{iik}. In the component relation (4.1.25):

$$\bar{C}_{rst} = Q_{ri} Q_{sj} Q_{tk} C_{ijk}$$

we set $s = r$ and perform the summation with respect to the index r:

$$\bar{C}_{rrt} = (Q_{ri} Q_{rj}) Q_{tk} C_{ijk} = \delta_{ij} Q_{tk} C_{ijk} \quad \Rightarrow \quad \bar{C}_{rrt} = Q_{tk} C_{iik} \quad Q.E.D$$

Let **B** be a tensor of 2. order with components B_{ij} and $\bar{B}_{kl}$ in two coordinate systems. The contraction $B_{ii} = \mathrm{tr}B$ gives a new tensor of order $(2-2) = 0$, a scalar. This scalar is called the *trace of* **B** and is denoted tr**B**. The value of the scalar is equal to the trace of the tensor matrix in any Cartesian coordinate system.

$$\mathrm{tr}\mathbf{B} = \mathrm{tr}B = B_{ii} = \mathrm{tr}\bar{B} = \bar{B}_{ii} \tag{4.2.10}$$

As an example: The trace of the stress tensor **T** is equal to the sum of the normal coordinate stresses on any set of three orthogonal planes:

$$\mathrm{tr}\mathbf{T} = \mathrm{tr}T = T_{ii} = T_{11} + T_{22} + T_{33} = \sigma_x + \sigma_y + \sigma_z = \sigma_1 + \sigma_2 + \sigma_3$$

The sum of normal stresses on three orthogonal planes is independent of the coordinate system. This fact is already shown in Sect. 3.3.1 by the formulas (3.3.4) and (3.3.13).

The *scalar product* of two tensors $\mathbf{A}$ and $\mathbf{B}$ of 2. order is defined as the scalar:

$$\alpha = \mathbf{A} : \mathbf{B} = A_{ij} B_{ij} \tag{4.2.11}$$

The following reasoning proves that the expression $\mathbf{A} : \mathbf{B}$ is coordinate invariant. The tensor product of $\mathbf{A}$ and $\mathbf{B}$ is a tensor of order $(2+2) = 4$ with components $A_{ij}B_{kl}$. Two contractions leading to $A_{ij}B_{ij}$ reduce the order of the tensor to $(4-2-2) = 0$, that is a tensor of order zero, or a scalar, and thus a coordinate invariant quantity.

The following results are easily proved.

$$\mathbf{A} : \mathbf{B} = \mathbf{B} : \mathbf{A} \quad , \quad \mathbf{A} : (\mathbf{B} + \mathbf{C}) = \mathbf{A} : \mathbf{B} + \mathbf{A} : \mathbf{C} \tag{4.2.12}$$

In the literature, e.g. Malvern [28], two different scalar products are introduced for tensors of 2. order:

$$\mathbf{A} : \mathbf{B} = A_{ij}\, B_{ij} \quad , \quad \mathbf{A} \cdot\cdot \mathbf{B} = A_{ij}\, B_{ji} \tag{4.2.13}$$

The latter scalar product is given an alternative presentation by formula (4.3.2) below.

A *linear mapping* of a vector $\mathbf{a}$ on another vector, $\mathbf{b}$ or $\mathbf{c}$, is given by a second order tensor $\mathbf{A}$:

$$\mathbf{A} \cdot \mathbf{a} = \mathbf{b} \quad \Leftrightarrow \quad A_{ij}a_j = b_i \quad \Leftrightarrow \quad Aa = b \tag{4.2.14}$$

$$\mathbf{a} \cdot \mathbf{A} = \mathbf{c} \quad \Leftrightarrow \quad a_i A_{ij} = c_j \quad \Leftrightarrow \quad a^T A = c^T \tag{4.2.15}$$

To see that the components b_i and c_j represent proper vectors, we argue as follows: The components $A_{ij}a_k$ represent the tensor product $\mathbf{A} \otimes \mathbf{a}$, a tensor of order 3. The contractions: $A_{ij}a_j$ og a_iA_{ij} lead to tensors of order $(3-2) = 1$, that is to the vectors b_i and c_j in the relations (4.2.14) and (4.2.15). The linear mapping in (4.2.14) is alternatively written as:

$$\mathbf{A}\mathbf{a} = \mathbf{b} \tag{4.2.16}$$

This notation may be attractive because it is analogous to notation for the related matrix product: $Aa = b$. In the present exposition the notation $\mathbf{A} \cdot \mathbf{a}$ is preferred because it fits in with the general definition of the dot product given in (4.2.21) below, but also because of the symmetry it provides in the following expression (4.2.17). Using the formulas (4.1.14) and (4.1.15) we find that:

$$\alpha = \mathbf{A}[\mathbf{b}, \mathbf{c}] = b_i c_j A_{ij} = b_i A_{ij} c_j = b^T A c = \mathbf{b} \cdot \mathbf{A} \cdot \mathbf{c} \tag{4.2.17}$$

This notation has already been used to express the normal stress and the shear stress on a surface by the stress tensor. See the formulas (4.1.4) and (4.1.5). A special application of the notation in (4.2.17) is:

$$A_{ij} = \mathbf{A}[\mathbf{e}_i, \mathbf{e}_j] = \mathbf{e}_i \cdot \mathbf{A} \cdot \mathbf{e}_j \tag{4.2.18}$$

A *linear mapping* of a tensor of 2. order **A** on another tensor of 2. order, **B** or **D**, is defined by:

$$\mathbf{C} : \mathbf{A} = \mathbf{B} \quad \Leftrightarrow \quad C_{ijkl} A_{kl} = B_{ij} \quad , \quad \mathbf{A} : \mathbf{C} = \mathbf{D} \quad \Leftrightarrow \quad A_{ij} C_{ijkl} = D_{kl} \tag{4.2.19}$$

where **C** is a 4. order tensor. The scalar product of **B**, or **D**, and a 2. order tensor **E** are the scalars:

$$\begin{aligned} \alpha &= \mathbf{B} : \mathbf{E} = \mathbf{E} : \mathbf{B} = \mathbf{E} : \mathbf{C} : \mathbf{A} = E_{ij} C_{ijkl} A_{kl} \\ \beta &= \mathbf{D} : \mathbf{E} = \mathbf{E} : \mathbf{D} = \mathbf{A} : \mathbf{C} : \mathbf{E} = A_{ij} C_{ijkl} E_{kl} \end{aligned} \tag{4.2.20}$$

The scalar product of two vectors, the scalar product of two tensors of 2. order, and the products in the linear mapping in (4.2.14), (4.2.15), and (4.2.19) are all called *inner products* or *dot products*. We now extend the dot product to apply to the following cases:

$$\begin{aligned} \mathbf{A} &= \mathbf{C} \cdot \mathbf{b} \quad \Leftrightarrow \quad A_{ij} = C_{ijk} b_k \\ \mathbf{A} &= \mathbf{B} \cdot \mathbf{D} = \mathbf{B}\mathbf{D} \quad \Leftrightarrow \quad A_{ij} = B_{ik} D_{kj} \\ \mathbf{a} &= \mathbf{C} : \mathbf{B} \quad \Leftrightarrow \quad a_i = C_{ijk} B_{jk} \end{aligned} \tag{4.2.21}$$

The second product is also called a *composition* of the tensors **B** and **D**. This operation is defined again below by (4.3.13).

The following operations for the tensor product, the scalar product and linear mapping of vectors are easily verified by their component versions.

$$\begin{aligned} (\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) \,, &\quad \mathbf{c} \cdot (\mathbf{a} \otimes \mathbf{b}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} \\ (\mathbf{a} \otimes \mathbf{b}) : (\mathbf{c} \otimes \mathbf{d}) &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) = a_i b_j c_i d_j \\ \mathbf{a} \cdot (\mathbf{b} \otimes \mathbf{c}) \cdot \mathbf{d} &= (\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{d}) = a_i b_i c_j d_j \end{aligned} \tag{4.2.22}$$

Above we have defined the stress tensor alternatively as a linear vector-valued function of a vector, by (4.1.3), and as a bilinear scalar-valued function of two vectors, by (4.1.4) and (4.1.5). In general tensors were defined as multilinear scalar-valued functions of vectors. The equations (4.2.19) express the 4. order tensor **C** as two linear tensor-valued functions of a 2. order tensor, while (4.2.20) may be interpreted as expressing the 4. order tensors **C** as a bilinear scalar-valued function of two 2. order tensors.

The Cauchy stress theorem, (3.2.27), states a linear relation between two tensors of 1. order **t** and **n** as expressed by (4.1.2) through their components in a Cartesian coordinate system Ox. The coefficients in these equations are the coordinate stresses T_{ik}. The component relation is valid in any other Cartesian coordinate system $\bar{O}\bar{x}$:

$$\bar{t}_i = \bar{T}_{ik} \bar{n}_k \tag{4.2.23}$$

The component relations (4.1.2) and (4.2.23) motivated the introduction of the concept of the stress tensor **T** in Sect. 3.2.4 as a coordinate invariant quantity represented by the coordinate stresses in any Cartesian coordinate system. The linear relations are given the coordinate invariant form:

$$\mathbf{t} = \mathbf{T} \cdot \mathbf{n} \tag{4.2.24}$$

In a sense the coordinate stresses may be characterized as the quotients between the tensor components of the tensors **t** and **n**. The fact that the matrices T and $\bar{T}$ represent a tensor is based on the observation that the linear relations (4.2.23) are valid in any coordinate system for any unit vector **n** and that the result **t** is a tensor 1.order.

Let us generalize to the following situation. Suppose that we have developed linear relations between the components of two tensors of 1. order **a** and **b** in two Cartesian coordinate systems Ox and $\bar{O}\bar{x}$:

$$a_i = C_{ij}\, b_j \quad , \quad \bar{a}_r = \bar{C}_{rs}\, \bar{b}_s \tag{4.2.25}$$

The two coordinate systems are related through the transformation matrix Q, such that:

$$\bar{a}_r = Q_{ri}\, a_i \quad , \quad b_j = Q_{sj}\, \bar{b}_s \tag{4.2.26}$$

For any choice of **b** the outcome of the relations (4.2.25) is the components of a tensor of 1. order **a**. We shall prove that the "quotients" C and $\bar{C}$ in the relations (4.2.25) represent the components of a 2. order tensor **C** in the respective coordinate systems, such that:

$$\bar{C}_{rs} = Q_{ri}\, Q_{sj}\, C_{ij} \tag{4.2.27}$$

Proof. We start with the relations $(4.2.26)_1$, apply the relations $(4.2.25)_1$, the relation $(4.2.26)_2$, and finally the relations $(4.2.25)_2$:

$$\bar{a}_r = Q_{ri}\, a_i = Q_{ri} C_{ij}\, b_j = Q_{ri} C_{ij}\, Q_{sj}\, \bar{b}_s = \bar{C}_{rs}\, \bar{b}_s \quad \Rightarrow \quad \left(\bar{C}_{rs} - Q_{ri}\, Q_{sj} C_{ij}\right) \bar{b}_s = 0 \tag{4.2.28}$$

This equation is to be valid for all choices of the argument tensor **b**, which means that the components $\bar{b}_s$ may be chosen freely. Then the expression in the parentheses must be zero, and the relation (4.2.27) is proved, i.e. the matrices C and $\bar{C}$ represent a 2. order tensor **C**.

The example above is further generalized to the *quotient theorem*:
The coefficient matrix C relating the matrix B to the matrix A, in any Cartesian coordinate system, in a linear tensor-valued function of a tensor **B** *and of value* **A**, *is representing a tensor* **C**. *The relation between the tensor matrices B and A through C must be valid for all argument tensors* **B**.

Another example of application of the quotient theorem may be as follows. Let **B** be a tensor of 2. order and **a** a tensor of 1. order. Suppose that we have developed the following linear relation between the matrices of the tensors:

$$a_i = C_{ijk}\, B_{jk} \tag{4.2.29}$$

valid in any Ox-system and for any argument $\mathbf{B}$. Then C_{ijk} are the components of a tensor of 3. order $\mathbf{C}$. The component relation (4.2.29) may be generalized to the invariant form:

$$\mathbf{a} = \mathbf{C} : \mathbf{B}$$

A *tensor equation* is a coordinate invariant equation of tensors. All terms in the equation have to be tensors of the same order. In the component format all terms must contain the same free indices. An example of a tensor equation is:

$$A_{ij} + C_{ijk} b_k = B_{ij} + a_i c_j \quad \Leftrightarrow \quad \mathbf{A} + \mathbf{C} \cdot \mathbf{b} = \mathbf{B} + \mathbf{a} \otimes \mathbf{c}$$

It often is convenient to develop physical or geometrical equations in a special Cartesian coordinate system. If such an equation can be identified as a tensor equation, it automatically is valid in any other coordinate system. In Chap. 12 we shall see how the components of tensors are defined in general curvilinear coordinates, and how a tensor equation is written in any general coordinate system. We should always try to formulate equations between physical or geometrical quantities that are represented by scalars, vectors, and tensors in a coordinate invariant format, which in fact means that the equations should be tensor equations.

4.2.1 Isotropic Tensors of 4. Order

In this section we present five special isotropic tensors of 4. order and a general 4. order isotropic tensor. First, we see that because δ_{ij} represent the components of an isotropic 2. order tensor, i.e. the 2. order unit tensor $\mathbf{1} \equiv \mathbf{1}_2$, the components $\delta_{ij}\ \delta_{kl}$ represent a 4. order isotrop tensor which is the tensor product of $\mathbf{1}$ by itself: $\mathbf{1} \otimes \mathbf{1}$. Two other isotropic tensors of 4. order are defined by their components $\delta_{ik}\ \delta_{jl}$ and $\delta_{il}\ \delta_{jk}$, where the order of the indices for the tensor components always shall be i, j, k, and l. The tensor with components $1_{ijkl} = \delta_{ik}\ \delta_{jl}$ is called the *4. order unit tensor* $\mathbf{1}_4$. The three isotropic tensors are:

$$\mathbf{1} \otimes \mathbf{1} \quad \Leftrightarrow \quad \delta_{ij}\ \delta_{kl} \quad , \quad \mathbf{1}_4 \equiv \mathbf{1} \bar{\otimes} \mathbf{1} \quad \Leftrightarrow \quad 1_{ijkl} = \delta_{ik}\ \delta_{jl} \quad , \quad \mathbf{1} \underline{\otimes} \mathbf{1} \quad \Leftrightarrow \quad \delta_{il}\ \delta_{jk} \tag{4.2.30}$$

The 4. order unit tensor has the symmetry properties:

$$1_{ijkl} = 1_{kjil} = 1_{ilkj} = 1_{klij} \tag{4.2.31}$$

The reason for the name unit tensor becomes apparent in the dot product of the tensor $\mathbf{1}_4$ and any 2. order tensor $\mathbf{B}$:

$$\begin{aligned} \mathbf{1}_4 : \mathbf{B} = \mathbf{B} \quad &\Leftrightarrow \quad 1_{ijkl} B_{kl} = \delta_{ik}\,\delta_{jl} B_{kl} = B_{ij} \\ \mathbf{B} : \mathbf{1_4} = \mathbf{B} \quad &\Leftrightarrow \quad B_{ij}\, 1_{ijkl} = B_{ij}\,\delta_{ik}\,\delta_{jl} = B_{kl} \end{aligned} \tag{4.2.32}$$

The unit tensor $\mathbf{1}_4$ may be decomposed into a symmetric part $\mathbf{1}_4^s$ and an antisymmetric part $\mathbf{1}_4^a$:

$$\mathbf{1}_4 = \mathbf{1}_4^s + \mathbf{1}_4^a \tag{4.2.33}$$

where:

$$\mathbf{1}_4^s = \frac{1}{2}\left(\mathbf{1}\bar{\otimes}\mathbf{1} + \mathbf{1}\underline{\otimes}\mathbf{1}\right) \quad \Leftrightarrow \quad 1_{ijkl}^s = \frac{1}{2}\left(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}\right) = 1_{jikl}^s = 1_{ijlk}^s = 1_{klij}^s \tag{4.2.34}$$

$$\mathbf{1}_4^a = \frac{1}{2}\left(\mathbf{1}\bar{\otimes}\mathbf{1} - \mathbf{1}\underline{\otimes}\mathbf{1}\right) \quad \Leftrightarrow \quad 1_{ijkl}^a = \frac{1}{2}\left(\delta_{ik}\,\delta_{jl} - \delta_{il}\,\delta_{jk}\right) = -1_{jikl}^a = -1_{ijlk}^a = 1_{klij}^a \tag{4.2.35}$$

The tensor product $\mathbf{1}\otimes\mathbf{1}$, has the following property in a dot product with a 2. order tensor $\mathbf{B}$:

$$(\mathbf{1}\otimes\mathbf{1}):\mathbf{B} = (\mathrm{tr}\,\mathbf{B})\,\mathbf{1} \quad \Leftrightarrow \quad \delta_{ij}\,\delta_{kl}\,B_{kl} = B_{kk}\,\delta_{ij} \tag{4.2.36}$$

It may be shown that the general isotropic tensor of 4. order is given by:

$$\mathbf{I}_4 = 2\mu\,\mathbf{1}_4^s + 2\theta\,\mathbf{1}_4^a + \lambda\,(\mathbf{1}\otimes\mathbf{1}) \quad \Leftrightarrow \tag{4.2.37}$$

$$I_{4ijkl} = \mu\left(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}\right) + \theta\left(\delta_{ik}\,\delta_{jl} - \delta_{il}\,\delta_{jk}\right) + \lambda\,\delta_{ij}\,\delta_{kl} \tag{4.2.38}$$

where μ, θ, and λ are three scalars. The dot product of the general isotropic tensor $\mathbf{I}_4$ and a symmetric 2. order tensor $\mathbf{E}$ is a symmetric 2. order tensor:

$$\mathbf{T} = \mathbf{I}_4:\mathbf{E} \quad \Leftrightarrow \quad T_{ij} = I_{4ijkl}\,E_{kl} \tag{4.2.39}$$

Because $\mathbf{E}$ is symmetric, the antisymmetric part $\mathbf{1}_4^a$ of $\mathbf{I}_4$ does not contribute in the dot product. Thus:

$$\mathbf{T} = \mathbf{I}_4:\mathbf{E} = \mathbf{I}_4^s:\mathbf{E} = (2\mu\,\mathbf{1}_4^s + \lambda\,\mathbf{1}\otimes\mathbf{1}):\mathbf{E} \quad , \quad \mathbf{I}_4^s = 2\mu\,\mathbf{1}_4^s + \lambda\,\mathbf{1}\otimes\mathbf{1} \tag{4.2.40}$$

The symmetric part $\mathbf{I}_4^s$ of $\mathbf{I}_4$ has the symmetry property:

$$I_{4ijkl}^s = I_{4jikl}^s = I_{4ijlk}^s = I_{4klij}^s \tag{4.2.41}$$

From (4.2.40) it follows that:

$$\mathbf{T} = 2\mu\,\mathbf{E} + \lambda\,(\mathrm{tr}\,\mathbf{E})\,\mathbf{1} \quad \Leftrightarrow \quad T_{ij} = 2\mu\,E_{ij} + \lambda\,E_{kk}\,\delta_{ij} \tag{4.2.42}$$

In Chap. 7 we identify $\mathbf{T}$ as the stress tensor, $\mathbf{E}$ as the *strain tensor for small strains*, defined in Sect. 5.3, and finally formula (4.2.42) as the generalized Hooke's law for an isotropic linearly elastic material. The scalars μ and λ are then called the *Lamé-constants*, after Gabriel Lamé [1795–1870], and are related to the modulus of elasticity and Poisson's ratio for the material. μ is the *shear modulus*, and $\lambda = \kappa - 2\mu/3$, where κ is the *bulk modulus*.

In Chap. 8 the formula (4.2.40) represents the stress contribution $\mathbf{T}$ due to the viscosity in a Newtonian fluid if $\mathbf{E}$ is replaced by the *rate of deformation tensor*, also called the *rate of strain tensor*, **D**. The tensor **D** is defined in Sect. 5.4. The parameter μ is now the *dynamic viscosity* and $\lambda = \kappa - 2\mu/3$, where κ is the *bulk viscosity*.

The tensor equation (4.2.42) has also other applications when we want to describe the state of stress in isotropic materials.

4.2.2 Tensors as Polyadics

The polyads $\mathbf{e}_i \otimes \mathbf{e}_j$ of the base vectors $\mathbf{e}_i$ in a coordinate system Ox and the polyads $\bar{\mathbf{e}}_k \otimes \bar{\mathbf{e}}_l$ of the base vectors $\bar{\mathbf{e}}_k$ in a coordinate system $\bar{O}\bar{x}$ may be interpreted as 2. order tensors. The relations between the two sets of base vectors are:

$$Q_{ki} = \cos(\bar{\mathbf{e}}_k, \mathbf{e}_i) \quad \Leftrightarrow \quad \bar{\mathbf{e}}_k = Q_{ki}\,\mathbf{e}_i \quad \Leftrightarrow \quad \mathbf{e}_i = Q_{ki}\,\bar{\mathbf{e}}_k$$

Then the components of $\mathbf{e}_i \otimes \mathbf{e}_j$ are:

$$\delta_{ki}\,\delta_{lj} \text{ in } Ox \text{ and } Q_{ki}\,Q_{lj} \text{ in } \bar{O}\bar{x}$$

while the components of $\bar{\mathbf{e}}_k \otimes \bar{\mathbf{e}}_l$ are:

$$Q_{ki}\,Q_{lj} \text{ in } Ox \text{ and } \delta_{ki}\,\delta_{lj} \text{ in } \bar{O}\bar{x}$$

Let **B** be any tensor of 2. order with components:

$$B_{ij} \text{ in } Ox \text{ and } \bar{B}_{kl} \text{ in } \bar{O}\bar{x} \quad \Leftrightarrow \quad \bar{B}_{kl} = Q_{ki}\,Q_{lj}\,B_{ij} \quad , \quad B_{ij} = Q_{ki}\,Q_{lj}\,\bar{B}_{kl}$$

Then the tensors $B_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ and $\bar{B}_{kl}\bar{\mathbf{e}}_k \otimes \bar{\mathbf{e}}_l$ are both identical to the tensor **B**. In order to see this, we evaluate the components of the two tensors in the two coordinate systems.

$$\begin{aligned}
&B_{ij}\,\mathbf{e}_i \otimes \mathbf{e}_j \quad \Rightarrow \quad B_{ij}\,\delta_{ki}\,\delta_{lj} = B_{kl} \text{ in } Ox \text{ and } B_{ij}\,Q_{ki}\,Q_{lj} = \bar{B}_{kl} \text{ in } \bar{O}\bar{x}\\
&\bar{B}_{kl}\,\bar{\mathbf{e}}_k \otimes \bar{\mathbf{e}}_l \quad \Rightarrow \quad \bar{B}_{kl}\,Q_{ki}\,Q_{lj} = B_{ij} \text{ in } Ox \text{ and } \bar{B}_{kl}\,\delta_{ki}\,\delta_{lj} = \bar{B}_{ij} \text{ in } \bar{O}\bar{x}
\end{aligned}$$

Thus we may write:

$$\mathbf{B} \equiv B_{ij}\,\mathbf{e}_i \otimes \mathbf{e}_j \equiv \bar{B}_{kl}\,\bar{\mathbf{e}}_k \otimes \bar{\mathbf{e}}_l \tag{4.2.43}$$

A linear combination of dyads of vectors is a called a *dyadic*. Formula (4.2.43) shows how a 2. order tensor may be expressed as a dyadic.

It is easy to see how this tensor representation may be extended to tensors of order n. order by using *polyadics*, i.e. linear combinations of polyads. For tensors of 1., 2., and 3. order the polyadic representation are:

$$\mathbf{a} \equiv a_i\,\mathbf{e}_i \quad , \quad \mathbf{B} \equiv B_{ij}\,\mathbf{e}_i \otimes \mathbf{e}_j \quad , \quad \mathbf{C} \equiv C_{ijk}\,\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \tag{4.2.44}$$

A contraction in a tensor may now be performed by replacing a tensor multiplication by a dot product, which is indicated by replacing the sign $(\otimes)$ by a dot $(\cdot)$

in the polyadic representation of the tensor and perform the dot-multiplication. For example:

$$C_{ijk}\,\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \quad \Rightarrow \quad C_{ijk}\,\mathbf{e}_i \cdot \mathbf{e}_j \otimes \mathbf{e}_k = C_{ijk}\,\delta_{ij}\,\mathbf{e}_k = C_{iik}\,\mathbf{e}_k \tag{4.2.45}$$

In *orthogonal coordinate systems* we introduce unit vectors $\mathbf{e}_i$ in the directions of the tangents to the coordinate lines. A vector $\mathbf{a}$ may be expressed by its *physical components* a_i, such that $\mathbf{a} = a_i\mathbf{e}_i$. The *physical components* of a tensor in an orthogonal coordinate system y are in general defined as the coefficients to polyads of the unit vectors $\mathbf{e}_i^y$ in the same way as in Cartesian coordinate systems, i.e. as shown by the formulas (4.2.44). As an example, the matrices (3.2.18, 3.2.19) represent the physical stress components in cylindrical and spherical coordinate systems respectively. While physical components and tensor components are identical in Cartesian coordinates, in curvilinear coordinates it becomes necessary to distinguish these two types of components of a tensor. Tensor components in curvilinear coordinates are not used in the first 11 chapters of this book. The formal definition of these components is therefore postponed until Chap. 12.

4.3 Tensors of 2. Order. Part One

Most of the important tensors relevant in continuum mechanics are of 2. order, the prominent example being the stress tensor $\mathbf{T}$. It is therefore natural to give 2. order tensors special attention and to investigate their properties thoroughly. For practical reasons the presentation of the properties of 2. order tensors is divided into two parts. Part one contains the material necessary for the applications in the chapters before Chap. 11. Section 4.6 "Tensors of 2.order. Part two" contains topics that give a more formal treatment of results developed geometrically or physically in the following chapters and also some results that are not used before Chap. 11 on general principles in constitutive modelling of materials.

Related to a tensor $\mathbf{A}$ of 2. order we define the *transposed tensor* $\mathbf{A}^T$ by:

$$\mathbf{A}^T \quad \Leftrightarrow \quad \mathbf{A}^T[\mathbf{b},\mathbf{c}] = \mathbf{A}[\mathbf{c},\mathbf{b}] \quad \Leftrightarrow \quad (A^T)_{ij} = A_{ji} \tag{4.3.1}$$

The matrix of $\mathbf{A}^T$ is the matrix A^T. The second scalar product in (4.2.13) may now be presented as:

$$\mathbf{A} \cdot\cdot \mathbf{B} = \mathbf{A} : \mathbf{B}^T \tag{4.3.2}$$

A tensor $\mathbf{B}$ of 2. order is *symmetric/antisymmetric* if it is symmetric/antisymmetric with respect to the argument vectors.

$$\mathbf{B}[\mathbf{b},\mathbf{c}] = (+/-)\,\mathbf{B}[\mathbf{c},\mathbf{b}] \quad \Leftrightarrow \quad B_{ij} = (+/-)\,B_{ji} \quad \Leftrightarrow \quad B = (+/-)\,B^T \tag{4.3.3}$$

The signs $(+/-)$ imply symmetry/antisymmetry respectively. We may alternatively indicate symmetry/antisymmetry this way:

$$\mathbf{B} = (+/-)\,\mathbf{B}^T \tag{4.3.4}$$

Any 2. order tensor **B** may be linearly decomposed uniquely into a symmetric tensor **S** and an antisymmetric tensor **A**:

$$\mathbf{S} = \mathbf{S}^T = \frac{1}{2}\left(\mathbf{B}+\mathbf{B}^T\right), \quad \mathbf{A} = -\mathbf{A}^T = \frac{1}{2}\left(\mathbf{B}-\mathbf{B}^T\right) \quad \Leftrightarrow \tag{4.3.5}$$

$$S = S^T = \frac{1}{2}\left(B+B^T\right), \quad S_{ij} = S_{ji} = \frac{1}{2}\left(B_{ij}+B_{ji}\right) \tag{4.3.6}$$

$$A = -A^T = \frac{1}{2}\left(B-B^T\right), \quad A_{ij} = -A_{ji} = \frac{1}{2}\left(B_{ij}-B_{ji}\right) \tag{4.3.7}$$

such that:

$$B = S+A \quad \Leftrightarrow \quad \mathbf{B} = \mathbf{S}+\mathbf{A} \tag{4.3.8}$$

Sometimes the following notation is convenient.

$$S_{ij} = B_{(ij)}, \quad A_{ij} = B_{[ij]} \tag{4.3.9}$$

The tensor **B** is specified when values for the 9 components B_{ij} in a Ox-system are given. To specify the symmetric part **S** of **B** we have to select values for 6 distinct components S_{ij}, while the antisymmetric part **A** of **B** is specified by the 3 distinct components A_{ij}. Together the two parts **S** and **A** contain the same information as the original 2. order tensor B.

To any vector **a** corresponds an antisymmetric tensor **A** of 2. order that contains the same information as the vector. The vector **a** and the tensor **A** are called *dual quantities*, and with the permutation tensor **P**, the tensor **A** is define by:

$$\mathbf{A} = -\mathbf{P}\cdot\mathbf{a} = -\mathbf{a}\cdot\mathbf{P} \Leftrightarrow A_{ij} = -e_{ijk}a_k = -a_k e_{kij} \Leftrightarrow A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \tag{4.3.10}$$

It follows that:

$$\begin{aligned} &\mathbf{a} = -\frac{1}{2}\mathbf{P}:\mathbf{A} = -\frac{1}{2}\mathbf{A}:\mathbf{P} \quad \Leftrightarrow \\ &a_i = -\frac{1}{2}e_{ijk}A_{jk} = -\frac{1}{2}A_{jk}e_{jki} \quad \Leftrightarrow \quad a = -\{A_{23}\,A_{31}\,A_{12}\} \end{aligned} \tag{4.3.11}$$

In some presentations in the literature **a** and $\mathbf{A}^T$ are defined as dual quantities.

We may use the relationship between the dual quantities **a** and **A** to express the vector product of **a** and any other vector **b**:

$$\mathbf{a}\times\mathbf{b} = \mathbf{A}\cdot\mathbf{b} \quad , \quad \mathbf{b}\times\mathbf{a} = \mathbf{b}\cdot\mathbf{A} \tag{4.3.12}$$

These relations are easily checked by writing their component versions. These alternative expressions for vector products are convenient in tensor equations and their matrix representations.

The *composition of two tensors* **A** and **B** of 2. order is a new tensor **K** of 2. order given by:

$$\mathbf{AB}\,(\equiv \mathbf{A}\cdot\mathbf{B}) = \mathbf{K} \quad \Leftrightarrow \quad A_{ik}B_{kj} = K_{ij} \quad \Leftrightarrow \quad AB = K \tag{4.3.13}$$

In general: $\mathbf{A}\,\mathbf{B} \neq \mathbf{B}\,\mathbf{A}$. It follows that:

$$\begin{aligned} &(\mathbf{A}\mathbf{B})\,\mathbf{C} = \mathbf{A}\,(\mathbf{B}\mathbf{C}), \quad \text{presented as } \mathbf{ABC} \\ &\mathbf{A}\,(\mathbf{B}+\mathbf{C}) = \mathbf{A}\mathbf{B}+\mathbf{A}\mathbf{C}, \quad (\mathbf{A}+\mathbf{B})\,\mathbf{C} = \mathbf{A}\mathbf{C}+\mathbf{B}\mathbf{C} \end{aligned} \tag{4.3.14}$$

Three important scalar invariants to a tensor **A** of 2. order are defined by:

$$\text{the } \textit{trace} \text{ of } \mathbf{A}: \ \operatorname{tr}\mathbf{A} = \operatorname{tr}A = A_{kk} \tag{4.3.15}$$

$$\text{the } \textit{determinant} \text{ of } \mathbf{A}: \ \det\mathbf{A} = \det A \tag{4.3.16}$$

$$\text{the } \textit{norm} \text{ of } \mathbf{A}: \ \operatorname{norm}\mathbf{A} \equiv \|\mathbf{A}\| = \sqrt{\mathbf{A}:\mathbf{A}} = \sqrt{\operatorname{tr}(\mathbf{A}\mathbf{A}^T)} = \sqrt{\operatorname{tr}(AA^T)} = \sqrt{A_{ij}A_{ij}} \tag{4.3.17}$$

It follows from their definitions that tr**A** and norm**A** are coordinate invariant quantities. Now we shall demonstrate that det**A** also is an invariant and thus a scalar. Let Q be the transformation matrix in a coordinate transformation from a system Ox to a system $\bar{O}\bar{x}$. Then since $\det Q = \det Q^T = 1$, we obtain from the multiplication theorem for determinants in (2.1.21):

$$\det\bar{A} = \det\left(QAQ^T\right) = (\det Q)\,(\det A)\left(\det Q^T\right) = \det A$$

If $\det\mathbf{A} \neq 0$, we may determined the *inverse tensor* $\mathbf{A}^{-1}$ of **A** from the tensor equation:

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{1} = \mathbf{A}\mathbf{A}^{-1} \quad , \quad A^{-1}A = 1 = AA^{1} \tag{4.3.18}$$

A tensor **A** is called an *orthogonal tensor* if:

$$\begin{aligned} &\mathbf{A}^T = \mathbf{A}^{-1} \text{ and } \det\mathbf{A} = \mathbf{1} \quad \Rightarrow \quad \mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{1} \quad \Leftrightarrow \\ &A^TA = AA^T = 1 \quad \Leftrightarrow \quad A_{ik}A_{jk} = \delta_{ij} \quad , \quad \text{and } \det A = 1 \end{aligned} \tag{4.3.19}$$

It follows from the definition (4.3.19) that the matrix A is an orthogonal matrix. This implies that the columns, or rows, of the matrix of an orthogonal tensor represent the components of an orthogonal set of unit vectors, which is called an *orthonormal set of vectors*. In the same order as they appear in the matrix they form a *right-handed system* in the same sense as the base vectors in a Cartesian right-handed coordinate system form a right-handed system. In the linear vector mapping $\mathbf{a} = \mathbf{A}\cdot\mathbf{b}$, where **A** is an orthogonal tensor, the tensor **A** represents a rotation of the argument vector **b**, such that $|\mathbf{a}| = |\mathbf{b}|$. This property will be further discussed in Sect. 4.5.1 on rigid-body kinematics.

The following rules may be directly transferred from matrix algebra. For any 2. order tensor **A** and **B**:

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \quad , \quad (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \tag{4.3.20}$$

The cofactor Co A of the matrix A of a 2. order tensor **A** represents a tensor Co **A**, the *cofactor tensor*. From formula (2.1.27) we get:

$$\mathrm{Co}\,\mathbf{A} = \mathbf{A}^{-T}\det\mathbf{A} \tag{4.3.21}$$

where $\mathbf{A}^{-T} = (\mathbf{A}^{-1})^T$, and from formula (2.1.22) we get the result:

$$\mathrm{Co}\,\mathbf{A} = \frac{\partial(\det\mathbf{A})}{\partial\mathbf{A}} \quad \Leftrightarrow \quad \mathrm{CoA}_{ij} = \frac{\partial(\det A)}{\partial A_{ij}} \tag{4.3.22}$$

4.3.1 Symmetric Tensors of 2. Order

Let **S** be a symmetric tensor of 2. order and **a** and **b** two orthogonal unit vectors, Fig. 4.3.1. We then define the vector **s** of the tensor **S** for the direction **a**:

$$\mathbf{s} = \mathbf{S}\cdot\mathbf{a} \quad \Leftrightarrow \quad s = Sa \tag{4.3.23}$$

The projection σ of this vector on **a** is called the *normal component of the tensor for the direction* **a**.

$$\sigma = \mathbf{a}\cdot\mathbf{s} = \mathbf{a}\cdot\mathbf{S}\cdot\mathbf{a} = a_i S_{ik} a_k = a^T S a \tag{4.3.24}$$

The projection τ of the vector **s** on **b** is called the *orthogonal shear component of the tensor for the directions* **a** *and* **b**.

$$\tau = \mathbf{b}\cdot\mathbf{s} = \mathbf{b}\cdot\mathbf{S}\cdot\mathbf{a} = b_i S_{ik} a_k = b^T S a \tag{4.3.25}$$

The names “normal component” and “shear component” are taken from the corresponding quantities for the stress tensor. Note that the three vectors **a, b**, and **s** do not necessarily lie in one and the same plane. We may also write:

$$\sigma = \mathbf{S}[\mathbf{a},\mathbf{a}] \quad , \quad \tau = \mathbf{S}[\mathbf{b},\mathbf{a}] = \mathbf{S}[\mathbf{a},\mathbf{b}] \tag{4.3.26}$$

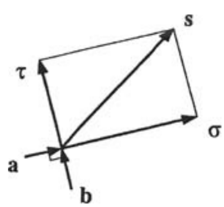


Fig. 4.3.1 Vector **s** of the tensor **S** for the direction **a**. Normal component σ for the direction **a**. Orthogonal shear component $\boldsymbol{\tau}$ for the directions **a** and **b**

If we choose $\mathbf{a} = \mathbf{e}_1$ and $\mathbf{b} = \mathbf{e}_2$, we get:

$$\sigma = \mathbf{S}[\mathbf{e}_1, \mathbf{e}_1] = S_{11} \quad , \quad \tau = \mathbf{S}[\mathbf{e}_2, \mathbf{e}_1] = S_{21} \tag{4.3.27}$$

Thus in the matrix S the elements on the diagonal represent normal components, while the off-diagonal elements are shear components.

Any symmetric 2. order tensor has mathematically the same properties as the stress tensor $\mathbf{T}$. In two dimensions we may analyze the properties in a Mohr diagram and otherwise use the formulas developed in Sects. 3.3.5 and 3.3.6. In the general three-dimensional case the three *principal values* $\sigma = \sigma_i$ and the three *principal directions* $\mathbf{a} = \mathbf{a}_i$ of $\mathbf{S}$ are determined from the condition:

$$\mathbf{s} = \mathbf{S} \cdot \mathbf{a} = \sigma \mathbf{a} \tag{4.3.28}$$

which is organized into the algebraic equations:

$$(\sigma \mathbf{1} - \mathbf{S}) \cdot \mathbf{a} = \mathbf{0} \quad \Leftrightarrow \quad (\sigma 1 - S)\, a = 0 \tag{4.3.29}$$

A solution of these equations requires that the determinant of the coefficient matrix $(\sigma 1 - S)$ is zero, i.e. $\det(\sigma 1 - S) = 0$, which results in the *characteristic equation* of the tensor:

$$\sigma^3 - I\sigma^2 + II\sigma - III = 0 \quad \text{the characteristic equation of } \mathbf{S} \tag{4.3.30}$$

The coefficients *I, II*, and *III* are the *principal invariants* of the tensor $\mathbf{S}$:

$$\begin{aligned} I &= \operatorname{tr}\mathbf{S} = \sigma_1 + \sigma_2 + \sigma_3 \\ II &= \frac{1}{2}\left[(\operatorname{tr}\mathbf{S})^2 - \operatorname{tr}\mathbf{S}^2\right] = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ III &= \det\mathbf{S} = \sigma_1\sigma_2\sigma_3 = \frac{1}{6}\left[(\operatorname{tr}\mathbf{S})^3 - 3\operatorname{tr}\mathbf{S}\operatorname{tr}\mathbf{S}^2 + 2\operatorname{tr}\mathbf{S}^3\right] \end{aligned} \tag{4.3.31}$$

The three principal values σ_i are determined from (4.3.30), and the three principal directions $\mathbf{a}_i$ are then determined from (4.3.29). The following properties and results, also demonstrated for the stress tensor $\mathbf{T}$ in Sect. 3.3.1, apply to all symmetric 2. order tensors $\mathbf{S}$. The principal values are real, and if they are all different, the principal directions are orthogonal. If two principal values are equal, but different from the third principal value, then all directions normal to the principal direction related to the third principal value, are principal directions. If all three principal values are equal, the tensor is isotropic, and any direction is a principal direction. The three principal directions $\mathbf{a}_i$ are said to represent the *principal axes* of the tensor $\mathbf{S}$. The principal values σ_i are also called the *eigenvalues* and the principal direction $\mathbf{a}_i$ the *eigenvectors* to the symmetric tensor $\mathbf{S}$. The mathematical problem related to (4.3.28, 4.3.29, 4.3.30, 4.3.31) is an *eigenvalue problem*.

In a coordinate system Ox with base vectors $\mathbf{e}_{\mathrm{i}} = \mathbf{a}_{\mathrm{i}}$ the tensor has the representation:

$$(\mathbf{S}[\mathbf{a}_i, \mathbf{a}_j]) = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} = (\sigma_i \, \delta_{ij}) \tag{4.3.32}$$

The principal values of the tensor represent extremal values for normal components. If the principal values are ordered such that:

$$\sigma_3 \leq \sigma_2 \leq \sigma_1$$

then:

$$\sigma_{\max} = \sigma_1 \quad , \quad \sigma_{\min} = \sigma_3 \tag{4.3.33}$$

The maximum value of the orthogonal shear component is:

$$\tau_{\max} = \frac{1}{2} (\sigma_{\max} - \sigma_{\min}) \text{ for } \mathbf{a} = (\mathbf{a}_1 + \mathbf{a}_3)/\sqrt{2}\,, \;\; \mathbf{b} = (\mathbf{a}_1 - \mathbf{a}_3)/\sqrt{2} \tag{4.3.34}$$

We now introduce the components a_{ki} in the coordinate system Ox of the principal directions $\mathbf{a}_k$:

$$\mathbf{a}_k = a_{ki} \mathbf{e}_i \quad \Leftrightarrow \quad a_{ki} = \mathbf{a}_k \cdot \mathbf{e}_i \quad \Rightarrow \quad \mathbf{e}_i = a_{ki} \mathbf{a}_k \tag{4.3.35}$$

Then we may write:

$$S_{ij} = \mathbf{S}[\mathbf{e}_i, \mathbf{e}_j] = \mathbf{S}[a_{ki} \mathbf{a}_k, a_{lj} \mathbf{a}_l] = \mathbf{S}[\mathbf{a}_k, \mathbf{a}_l] \; a_{ki} a_{lj} = \sum_{k,l} \sigma_k \, \delta_{kl} \, a_{ki} a_{lj} \quad \Rightarrow$$

$$S_{ij} = \sum_k \sigma_k \, a_{ki} a_{kj} \quad \Leftrightarrow \quad \mathbf{S} = \sum_k \sigma_k \, \mathbf{a}_k \otimes \mathbf{a}_k \tag{4.3.36}$$

This formula also follows directly from formula (4.2.44) when we choose $\mathbf{a}_k$ as base vectors.

Powers of a tensor $\mathbf{A}$ of 2. order is defined by:

$$\mathbf{A}^n = \mathbf{A}\mathbf{A} \cdots \mathbf{A} \; (n \text{ compositions}) \tag{4.3.37}$$

It may be shown that for a symmetric tensor of 2. order $\mathbf{S}$ given by (4.3.36), see Problem 4.5:

$$\mathbf{S}^n = \sum_k (\sigma_k)^n \, \mathbf{a}_k \otimes \mathbf{a}_k \quad , \quad n = 1, 2, 3, \cdots \tag{4.3.38}$$

The tensors $\mathbf{S}^n$ and $\mathbf{S}$ have the same principal directions and are therefore called *coaxial tensors*. Note that the exponent n is a natural number.

A symmetric 2. order tensor is called a *positive definite tensor* if:

$$\mathbf{c} \cdot \mathbf{S} \cdot \mathbf{c} > 0 \text{ for all vectors} \mathbf{c} \neq \mathbf{0} \tag{4.3.39}$$

From the definition (4.3.39) it follows that the principal values and the principal invariants of a positive definite symmetric tensor all are positive, see Problem 4.6. For any real number α we define real powers of a positive definite tensor by:

$$\mathbf{S}^{\alpha} = \sum_k (\sigma_k)^{\alpha}\, \mathbf{a}_k \otimes \mathbf{a}_k \quad , \quad \alpha \text{ is a real number} \tag{4.3.40}$$

$\mathbf{S}^{\alpha}$ and $\mathbf{S}$ are coaxial tensors. The inverse tensor $\mathbf{S}^{-1}$ is obtained from formula (4.3.40) when $\alpha = -1$. For the special case $\alpha = 1/2$, we write:

$$\mathbf{S}^{1/2} = \sqrt{\mathbf{S}}$$

All powers with exponents $n = 1, 2, 3, \ldots$ of a symmetric tensor $\mathbf{S}$ of 2. order may be expressed by $\mathbf{S}$, $\mathbf{S}^2$, and the principal invariants *I, II*, and *III*. In order to see this, we first prove the following theorem.

The **Cayley-Hamilton Theorem**. A symmetric tensor of 2. order satisfies its own characteristic equation (4.3.30), such that:

$$\mathbf{S}^3 - I\mathbf{S}^2 + II\mathbf{S} - III\mathbf{1} = \mathbf{0} \tag{4.3.41}$$

The theorem is named after Arthur Cayley [1821–1895] and William Rowan Hamilton [1805–1865]. The characteristic equation (4.3.30) of the tensor is satisfied by the principal values σ_k. When we multiply the equation by the tensor product $\mathbf{a}_k \otimes \mathbf{a}_k$ and sum with respect to the index k, we obtain:

$$\sum_k \left[(\sigma_k)^3\, \mathbf{a}_k \otimes \mathbf{a}_k - I\,(\sigma_k)^2\, \mathbf{a}_k \otimes \mathbf{a}_k + II\sigma_k\, \mathbf{a}_k \otimes \mathbf{a}_k - III\, \mathbf{a}_k \otimes \mathbf{a}_k \right] = 0$$

This result is directly transferred to (4.3.41) through application of formula (4.3.38), and the Cayley-Hamilton theorem is proved. By (4.3.41) all powers of $\mathbf{S}$ with natural number exponents may be expressed by the tensors $\mathbf{S}$ and $\mathbf{S}^2$ and the principal invariants *I, II*, and *III* of $\mathbf{S}$.

4.3.2 Alternative Invariants

The following sets of alternative invariants of a symmetric tensor of 2. order sometimes appear in the literature. The *moment invariants*:

$$\bar{I} = \operatorname{tr}\mathbf{S} = I \quad , \quad \overline{II} = \frac{1}{2}\operatorname{tr}\mathbf{S}^2 = \frac{1}{2}I^2 - II \quad , \quad \overline{III} = \frac{1}{3}\operatorname{tr}\mathbf{S}^3 \tag{4.3.42}$$

The *trace invariants*, see Problem 4.4:

$$\tilde{I} = \operatorname{tr}\mathbf{S} = \mathbf{1} : \mathbf{S} = I \quad , \quad \tilde{I}\tilde{I} = \operatorname{tr}\mathbf{S}^2 = \mathbf{S} : \mathbf{S} = I^2 - 2II$$
$$\tilde{I}\tilde{I}\tilde{I} = \operatorname{tr}\mathbf{S}^3 = \mathbf{S} : (\mathbf{SS}) = 3III + I^3 - 3I \cdot II \tag{4.3.43}$$

4.3.3 Deviator and Isotrop

A symmetric 2. order tensor $\mathbf{S}$ may be decomposed uniquely in a trace free *deviator* $\mathbf{S}'$ and an *isotrop* $\mathbf{S}^o$:

$$\mathbf{S} = \mathbf{S}' + \mathbf{S}^o \tag{4.3.44}$$

$$\mathbf{S}^o = \frac{1}{3}(\operatorname{tr}\mathbf{S})\,\mathbf{1} \quad \Leftrightarrow \quad S^o_{ij} = \frac{1}{3} S_{kk}\,\delta_{ij} \quad \text{isotrop} \tag{4.3.45}$$

$$\mathbf{S}' = \mathbf{S} - \mathbf{S}^o \quad , \quad \operatorname{tr}\mathbf{S}' = 0 \tag{4.3.46}$$

$\mathbf{S}'$ and $\mathbf{S}$ are *coaxial tensors*. The principal invariants and principal values of $\mathbf{S}'$ are:

$$I' = 0 \quad , \quad II' = -\frac{1}{2}\operatorname{tr}(\mathbf{S}')^2 = II - \frac{1}{3}I^2 \quad , \quad III' = \det\mathbf{S}' \tag{4.3.47}$$

$$\sigma'_i = \sigma_i - \frac{1}{3}I \quad \Leftrightarrow \quad \sigma'_1 = \frac{1}{3}(2\sigma_1 - \sigma_2 - \sigma_3) \quad \text{etc.} \tag{4.3.48}$$

The deviator may further be decomposed into five *states of shear*. First we write:

$$\operatorname{tr}\mathbf{S}' = S'_{11} + S'_{22} + S'_{33} = 0 \quad \Rightarrow \quad S'_{22} = -S'_{11} - S'_{33}$$

Then we get:

$$S' = \begin{pmatrix} S'_{11} & S'_{12} & S'_{13} \\ S'_{21} & S'_{22} & S'_{23} \\ S'_{31} & S'_{32} & S'_{33} \end{pmatrix} = S'_{11}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + S'_{33}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.3.49}$$
$$+ S'_{12}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + S'_{23}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + S'_{31}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Each of the last three matrices contains a shear component and is therefore said to represent a *state of shear*. One of the principal values of a state of shear is zero, while the other two have the same absolute value but have opposite signs. The first two matrices on the right-hand side of (4.3.49) are plane states with one principal value equal to zero and the other two of the same absolute value, equal to the absolute value of the shear component, but of opposite signs. It then follows that these two matrices also represent states of shear. Applications of the decompositions (4.3.44) and (4.3.49) will be demonstrated in connection with isotropic materials, i.e. materials with properties independent of directions.

4.4 Tensor Fields

Tensors in continuum mechanics represent often intensive quantities or properties related to particles or places in space at a given time t. This means that tensors really are *tensor fields*. A tensor $\mathbf{A}$ which is a tensor field is denoted by either $\mathbf{A}(\mathbf{r}_o,t)$, where $\mathbf{r}_o$ represents a particle, or by $\mathbf{A}(\mathbf{r},t)$, where $\mathbf{r}$ represent a place in space. Time or space derivatives of tensor fields represent new tensor fields. In this section we shall introduce some of the most important of these derived tensors.

The local time derivative of a tensor $\mathbf{A}(\mathbf{r},t)$ is a tensor $\partial_t\mathbf{A}$ with components $\partial_t A_{i\cdot\cdot}$ in a Ox-system:

$$\partial_t\mathbf{A} \quad \Leftrightarrow \quad \partial_t A_{i\cdot\cdot} = \frac{\partial A_{i\cdot\cdot}}{\partial t} \tag{4.4.1}$$

The definitions of the gradient, the divergence, and the rotation of a tensor field may vary somewhat in the literature, and this may lead to confusion. In the present exposition these quantities will first be defined in an analogous manner to how the corresponding quantities were defined for scalars and vectors in Sect. 2.4. Then the del-operator will be introduced, which together with the dyadic representation will indicate two possible definitions of the gradient, the divergence, and the rotation of a tensor field.

4.4.1 Gradient, Divergence, and Rotation

Let the unit vector $\mathbf{e}$ define an axis originating from a place $\mathbf{r}$. The distance from $\mathbf{r}$ to a position on the axis is given by the local coordinate s. The change of the value of the tensor field $\mathbf{A}(\mathbf{r},t)$ at place $\mathbf{r}$ and in the direction $\mathbf{e}$ is determined by the *directional derivative of the tensor* at the place $\mathbf{r}$ and in the direction $\mathbf{e}$, which is defined by:

$$\left.\frac{\partial\mathbf{A}\left(\mathbf{r}+s\mathbf{e},t\right)}{\partial s}\right|_{s=0} = \left[\frac{\partial\mathbf{A}}{\partial\left(x_i+se_i\right)}\frac{d\left(x_i+se_i\right)}{ds}\right]_{s=0} = \mathbf{A}_{,i}\, e_i \tag{4.4.2}$$

The gradient of a tensor field of n. order $\mathbf{A}(\mathbf{r},\mathrm{t})$ is a tensor field of order $(n+1)$ denoted by either of two symbols:

$$\operatorname{grad}\mathbf{A} \equiv \frac{\partial\mathbf{A}}{\partial\mathbf{r}} \tag{4.4.3}$$

and defined by the equation:

$$\left.\frac{\partial\mathbf{A}}{\partial s}\right|_{s=0} = \operatorname{grad}\mathbf{A}\cdot\mathbf{e} \tag{4.4.4}$$

for any given direction $\mathbf{e}$. For $\mathbf{e}=\mathbf{e}_i$, a base vector in the Ox-system, such that $s=x_i$, (4.4.4) gives:

$$\mathbf{A}_{,i} = \operatorname{grad}\mathbf{A} \cdot \mathbf{e}_i \tag{4.4.5}$$

The component form of this relation is:

$$A_{k\cdots,i} = (\operatorname{grad}\mathbf{A})_{k\cdots j}\, \delta_{ij} = (\operatorname{grad}\mathbf{A})_{k\cdots i} \tag{4.4.6}$$

which shows that the components to the tensor grad **A** is $A_{k...,i}$. The component form of the relation (4.4.4) is then:

$$\left.\frac{\partial A_{k\cdots}}{\partial s}\right|_{s=0} = A_{k\cdots,i}\, \mathrm{e}_i \tag{4.4.7}$$

The gradient of a tensor with components $A_k \ldots$ has therefore the components $A_{k...,i}$.

The gradient, grad **a**, of a vector **a**, i.e. a tensor of 1. order, is according to the general definition above, a tensor of 2. order and represented by the relations:

$$\left.\frac{\partial \mathbf{a}}{\partial s}\right|_{s=0} = \operatorname{grad}\mathbf{a} \cdot \mathbf{e} \quad \Leftrightarrow \quad \left.\frac{\partial a_i}{\partial s}\right|_{s=0} = a_{i,k}\, e_k \quad \Leftrightarrow \quad (\operatorname{grad}\mathbf{a})_{ik} = a_{i,k} \tag{4.4.8}$$

The divergence of a tensor field of n. order $\mathbf{A}(\mathbf{r},t)$, with the components $A_{i..j}$, is defined as the tensor field "div **A**" of order $(n-1)$ with components $A_{i..k,k}$.

$$\operatorname{div}\mathbf{A} \quad \Leftrightarrow \quad A_{i\cdot\cdot k,k} \tag{4.4.9}$$

The divergence of a vector **a** becomes a scalar $a_{i,i}$. The divergence of a second order tensor **A** is a vector **a**:

$$\operatorname{div}\mathbf{A} = \mathbf{a} \quad \Leftrightarrow \quad A_{ik,k} = a_i \tag{4.4.10}$$

This concept has already been used in connection with the development of the Cauchy equations of motion in Sect. 3.2.5.

The rotation of a tensor field of n. order $\mathbf{A}(\mathbf{r},t)$ is a tensor "rot **A**" of order n, also denoted "curl **A**". The tensor is defined by it components:

$$\operatorname{rot}\mathbf{A} \equiv \operatorname{curl}\mathbf{A} \quad \Leftrightarrow \quad e_{ijk} A_{r\cdot\cdot k,j} \tag{4.4.11}$$

The rotation of a vector **a** is the vector $e_{ijk} a_{k,j} \mathbf{e}_{\mathrm{i}}$.

The divergence of the gradient of a tensor $\mathbf{A}(\mathbf{r},t)$ of order n is a tensor of order n with the components $A_{i\cdot\cdot j,kk}$. Here we introduce the *Laplace operator* and write:

$$\nabla^2\mathbf{A} \equiv \operatorname{div}\operatorname{grad}\mathbf{A} \quad \Leftrightarrow \quad A_{i\cdot\cdot j,kk} \tag{4.4.12}$$

The gradient, the divergence, and the rotation of a tensor field $\mathbf{A}(\mathbf{r}_{\mathrm{o}},t)$ related to the position $\mathbf{r}_o$ in the reference configuration K_o of a continuum, are defined in similar manners as above but are denoted respectively as Grad **A**, Div **A**, and Rot **A**.

4.4.2 Del-Operator

The definitions of gradient, divergence, and rotation of tensors field are not universal. In the literature *right-gradient*, *left-gradient*, *right-divergence*, *left-gradient*, *right-divergence*, *left-divergence*, *right-rotation*, and *left-rotation* are defined. For instance, in the book by Malvern [28], the following vector operators are introduced:

$$\overleftarrow{\nabla} = \overleftarrow{\partial}_k \mathbf{e}_k \quad right - operator \quad , \quad \overrightarrow{\nabla} \equiv \nabla = \mathbf{e}_i \overrightarrow{\partial}_i \quad left - operator \tag{4.4.13}$$

The first operator is called a *right-operator* because it operates from the right. Note that the *left-operator* is identical to the del-operator defined by formula (2.4.10). Applying the operators in (4.4.13) to a 2. order tensor $\mathbf{A}$, we obtain:

$$\mathbf{A} \otimes \overleftarrow{\nabla} = \partial_k A_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k = A_{ij},_k \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \quad \text{right-gradient} \tag{4.4.14}$$

$$\overrightarrow{\nabla} \otimes \mathbf{A} = \partial_k A_{ij} \mathbf{e}_k \otimes \mathbf{e}_i \otimes \mathbf{e}_j = A_{ij},_k \mathbf{e}_k \otimes \mathbf{e}_i \otimes \mathbf{e}_j \quad \text{left-gradient} \tag{4.4.15}$$

In the literature the arrow over the ∇-symbol for the left-gradient is often left out, confer the definitions of the left operator in (4.4.13). The two tensors defined by the expressions (4.4.14) and (4.4.15) are represented by the same components, but the components are organized differently. What was defined in Sect. 4.4.1 as grad $\mathbf{A}$, is now seen to be the right-gradient of $\mathbf{A}$:

$$\operatorname{grad} \mathbf{A} = \mathbf{A}\overleftarrow{\nabla} \equiv \mathbf{A} \otimes \overleftarrow{\nabla} \tag{4.4.16}$$

The results in the expressions (4.4.14) and (4.4.15) for a tensor of 2. order may easily be generalized to tensors of any order. The right- and left-gradient of a vector $\mathbf{a}$ are 2. order tensors where one tensor is the transposed of the other:

$$\operatorname{grad} \mathbf{a} = \mathbf{a} \otimes \overleftarrow{\nabla} = a_{i,k} \, \mathbf{e}_i \otimes \mathbf{e}_k \quad \Rightarrow \quad (\mathbf{a}\overleftarrow{\nabla})_{ik} = \partial_k \, a_i \equiv a_{i,k} \tag{4.4.17}$$

$$\overrightarrow{\nabla} \otimes \mathbf{a} = (\operatorname{grad} \mathbf{a})^T = a_{k,i} \, \mathbf{e}_i \otimes e_k \quad \Rightarrow \quad (\overrightarrow{\nabla}\mathbf{a})_{ik} = \partial_i \, a_k \equiv a_{k,i} \tag{4.4.18}$$

The right-divergence and left-divergence of a tensor $\mathbf{A}$ are defined respectively by:

$$\mathbf{A} \cdot \overleftarrow{\nabla} = \partial_k A_{i \cdot\cdot j} \mathbf{e}_i \otimes \cdot\cdot \otimes \mathbf{e}_j \cdot \mathbf{e}_k = A_{i \cdot\cdot k},_k \mathbf{e}_i \otimes \cdot\cdot \quad right - divergence \tag{4.4.19}$$

$$\overrightarrow{\nabla} \cdot \mathbf{A} = \mathbf{e}_k \cdot \partial_k A_{i \cdot\cdot j} \mathbf{e}_i \otimes \cdot\cdot \otimes \mathbf{e}_j = A_{k \cdot\cdot j},_k \cdot\cdot \otimes \mathbf{e}_j \quad left - divergence \tag{4.4.20}$$

We see that the definition (4.4.10) of the divergence of a tensor is a right-divergence:

$$\operatorname{div} \mathbf{A} = \mathbf{A} \cdot \overleftarrow{\nabla} \tag{4.4.21}$$

For a vector $\mathbf{a}$ the right-divergence and the left-divergence are identical.

$$\operatorname{div}\mathbf{a} = \mathbf{a} \cdot \overleftarrow{\nabla} = \overrightarrow{\nabla} \cdot \mathbf{a} = \nabla \cdot \mathbf{a} = a_{i,i} \tag{4.4.22}$$

The divergence of the gradient of a tensor $\mathbf{A}$ of 2. order, defined by (4.4.12) may be given by:

$$\operatorname{div}\operatorname{grad}\mathbf{A} = \left(\mathbf{A} \otimes \overleftarrow{\nabla}\right) \cdot \overleftarrow{\nabla} = \overrightarrow{\nabla} \cdot \overrightarrow{\nabla} \otimes \mathbf{A} \quad \Leftrightarrow \quad A_{ij,kk} \tag{4.4.23}$$

The last version of div grad $\mathbf{A}$ in (4.4.23) indicates that it is natural to introduce the *Laplace operator*:

$$\nabla^2 = \operatorname{div}\operatorname{grad} = \nabla \cdot \nabla \equiv \overrightarrow{\nabla} \cdot \overrightarrow{\nabla} = \frac{\partial^2}{\partial x_i \partial x_i} \tag{4.4.24}$$

4.4.3 Orthogonal Coordinates

In the exposition of continuum mechanics in the Chaps. 1–11 Cartesian coordinates are used in the general presentation of the concepts and the basic equations. But in many applications we need to use *orthogonal coordinate systems*, i.e. systems where the coordinate lines are orthogonal, for instance *cylindrical coordinates* and *spherical coordinates*. The tensor analysis and the continuum mechanics in general curvilinear coordinates are presented in the Chaps. 12 and 13. The present section introduces some fundamental results for orthogonal coordinate systems.

In a general orthogonal coordinate system we denote the coordinates by y_i. The place vector $\mathbf{r}$ is then a function of y_i and we write $\mathbf{r} = \mathbf{r}(y_i, t)$. We assume a one-to-one correspondence between the y_i-coordinates and the Cartesian coordinates x_i. The tangent vectors to the coordinate lines of the y-coordinates are:

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial y_i} = \frac{\partial \mathbf{r}}{\partial x_k}\frac{\partial x_k}{\partial y_i} = \frac{\partial x_k}{\partial y_i}\mathbf{e}_k \quad \Leftrightarrow \quad \mathbf{e}_k = \frac{\partial y_i}{\partial x_k}\mathbf{g}_i \tag{4.4.25}$$

The vectors $\mathbf{g}_i$ are called the *base vectors* of the y-system. The magnitude of the base vectors are:

$$h_i = \sqrt{\mathbf{g}_i \cdot \mathbf{g}_i} = \sqrt{\frac{\partial x_k}{\partial y_i}\frac{\partial x_k}{\partial y_i}} \tag{4.4.26}$$

Because the coordinate lines are orthogonal, we have:

$$\mathbf{g}_i \cdot \mathbf{g}_j = 0 \quad \text{for } i \neq j \quad \Rightarrow \quad \frac{\partial x_k}{\partial y_i}\frac{\partial x_k}{\partial y_j} = 0 \text{ for } i \neq j \tag{4.4.27}$$

Thus:

$$\frac{\partial x_k}{\partial y_i}\frac{\partial x_k}{\partial y_j} = h_i^2 \delta_{ij} \tag{4.4.28}$$

We obtain:

$$\frac{\partial y_i}{\partial y_j} = \frac{\partial y_i}{\partial x_k}\frac{\partial x_k}{\partial y_j} = \delta_{ij} \quad \Rightarrow \quad \left[\frac{\partial y_i}{\partial x_k}\right] = \left[\frac{\partial x_k}{\partial y_i}\right]^{-1} \tag{4.4.29}$$

and it follows from the results (4.4.28) and (4.4.29) that:

$$\left(\frac{\partial y_i}{\partial x_k}\frac{\partial y_j}{\partial x_k}\right) = \left(\frac{\partial x_k}{\partial y_i}\frac{\partial x_k}{\partial y_j}\right)^{-1} = \left(\frac{1}{h_i^2}\,\delta_{ij}\right) \tag{4.4.30}$$

Now we introduce unit tangent vectors to the coordinate lines y_i:

$$\mathbf{e}_i^y = \frac{\mathbf{g}_i}{h_i} \tag{4.4.31}$$

The del-operator (4.4.13) in orthogonal coordinates becomes:

$$\nabla = \sum_i \frac{1}{h_i^2}\,\mathbf{g}_i \frac{\partial}{\partial y_i} = \sum_i \frac{1}{h_i}\,\mathbf{e}_i^y \frac{\partial}{\partial y_i} \tag{4.4.32}$$

To obtain this result we use (4.4.13), $(4.4.25)_2$, (4.4.30), and (4.4.31):

$$\nabla\,() = \mathbf{e}_k \frac{\partial\,()}{\partial x_k} = \left(\frac{\partial y_i}{\partial x_k}\,\mathbf{g}_i\right)\left(\frac{\partial\,()}{\partial y_j}\frac{\partial y_j}{\partial x_k}\right) = \sum_i \left(\frac{1}{h_i^2}\,\delta_{ij}\right)\mathbf{g}_i \frac{\partial\,()}{\partial y_j}$$

$$= \sum_i \frac{1}{h_i^2}\,\mathbf{g}_i \frac{\partial\,()}{\partial y_i} = \sum_i \frac{1}{h_i}\,\mathbf{e}_i^y \frac{\partial\,()}{\partial y_i} \quad \Rightarrow (4.4.32)$$

The expression for the del-operator in (4.4.32) may be used to obtain the expressions for the gradient, the divergence, and the rotation of tensors of first and second order. First we define the *physical components* $a(k)$ of a vector $\mathbf{a}$ and $A(kl)$ of a second order tensor $\mathbf{A}$ by:

$$\mathbf{a} = a(k)\,\mathbf{e}_k^y \quad , \quad \mathbf{A} = A(kl)\,\mathbf{e}_k^y \otimes \mathbf{e}_l^y \tag{4.4.33}$$

We obtain:

$$\nabla\alpha = \sum_i \frac{1}{h_i}\,\mathbf{e}_i^y \frac{\partial \alpha}{\partial y_i} \quad , \quad \nabla \cdot \mathbf{a} = \sum_i \frac{1}{h_i}\,\mathbf{e}_i^y \frac{\partial\left(a(k)\,\mathbf{e}_k^y\right)}{\partial y_i} \tag{4.4.34}$$

$$\operatorname{rot}\mathbf{a} \equiv \nabla \times \mathbf{a} = \sum_i \frac{1}{h_i}\,\mathbf{e}_i^y \times \frac{\partial\left(a(k)\,\mathbf{e}_k^y\right)}{\partial y_i} \quad , \quad \nabla\mathbf{a} \equiv \nabla \otimes \mathbf{a} = \sum_i \frac{1}{h_i}\,\mathbf{e}_i^y \otimes \frac{\partial\left(a(k)\,\mathbf{e}_k^y\right)}{\partial y_i} \tag{4.4.35}$$

$$\operatorname{grad}\mathbf{a} \equiv \mathbf{a}\overleftarrow{\nabla} \equiv \mathbf{a} \otimes \overleftarrow{\nabla} = \sum_i \frac{\partial\left(a(k)\,\mathbf{e}_k^y\right)}{\partial y_i} \otimes \frac{1}{h_i}\mathbf{e}_i^y \tag{4.4.36}$$

$$\nabla \mathbf{A} \equiv \nabla \otimes \mathbf{A} = \sum_i \frac{1}{h_i} \mathbf{e}_i^y \otimes \frac{\partial \left(A(kl)\, \mathbf{e}_k^y \otimes \mathbf{e}_l^y \right)}{\partial y_i} \tag{4.4.37}$$

$$\operatorname{grad} \mathbf{A} = \sum_i \frac{1}{h_i} \frac{\partial}{\partial y_i} \left(A(kl)\, \mathbf{e}_k^y \otimes \mathbf{e}_l^y \right) \otimes \mathbf{e}_i^y \quad , \quad \nabla \cdot \mathbf{A} = \sum_i \frac{1}{h_i} \mathbf{e}_i^y \cdot \frac{\partial}{\partial y_i} \left(A(kl)\, \mathbf{e}_k^y \otimes \mathbf{e}_l^y \right) \tag{4.4.38}$$

$$\operatorname{div} \mathbf{A} = \mathbf{A} \cdot \overleftarrow{\nabla} = \sum_i \frac{1}{h_i} \frac{\partial}{\partial y_i} \left(A(kl)\, \mathbf{e}_k^y \otimes \mathbf{e}_l^y \right) \cdot \mathbf{e}_i^y \tag{4.4.39}$$

The formula (4.4.39) is further expanded to give:

$$\operatorname{div} \mathbf{A} = \mathbf{A} \cdot \overleftarrow{\nabla} = \sum_i \left[\frac{1}{h_i} \frac{\partial A(kl)}{\partial y_i} \mathbf{e}_k^y + \frac{A(ki)}{h_i} \frac{\partial \mathbf{e}_k^y}{\partial y_i} + \frac{A(kl)}{h_i} \mathbf{e}_k^y \otimes \frac{\partial \mathbf{e}_l^y}{\partial y_i} \cdot \mathbf{e}_i^y \right] \tag{4.4.40}$$

The divergence of the gradient of a tensor $\mathbf{A}(\mathbf{r}, t)$ of any order may be presented as:

$$\operatorname{div} \operatorname{grad} \mathbf{A} = \nabla^2 \mathbf{A} = \sum_i \frac{1}{h_i} \frac{\partial}{\partial y_i} \left[\sum_j \frac{1}{h_j} \frac{\partial \mathbf{A}}{\partial y_i} \otimes \mathbf{e}_j^y \right] \cdot \mathbf{e}_i^y \tag{4.4.41}$$

In cylindrical coordinates (R, θ, z), Fig. 4.4.1, we have:

$$\mathbf{r} = R\, \mathbf{e}_R(\theta) + z\, \mathbf{e}_z \quad , \quad h_R = 1 \quad , \quad h_\theta = R \quad , \quad h_z = 1 \tag{4.4.42}$$

$$\frac{d\mathbf{e}_R}{d\theta} = \mathbf{e}_\theta \quad , \quad \frac{d\mathbf{e}_\theta}{d\theta} = -\mathbf{e}_R \tag{4.4.43}$$

In spherical coordinates (r, θ, ϕ), Fig. 4.4.1, we have:

$$\mathbf{r} = r\, \mathbf{e}_r(\theta, \phi) \quad , \quad h_r = 1 \quad , \quad h_\theta = r \quad , \quad h_\phi = r \sin\theta \tag{4.4.44}$$

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta \quad , \quad \frac{\partial \mathbf{e}_r}{\partial \phi} = \sin\theta\, \mathbf{e}_\phi \quad , \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\,\mathbf{e}_r \quad , \quad \frac{\partial \mathbf{e}_\theta}{\partial \phi} = \cos\theta\, \mathbf{e}_\phi$$

$$\frac{\partial \mathbf{e}_\phi}{\partial \theta} = \mathbf{0} \quad , \quad \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\cos\theta\, \mathbf{e}_\theta - \sin\theta\, \mathbf{e}_r \tag{4.4.45}$$

4.4.4 Material Derivative of a Tensor Field

The places of the particles in a continuum in the reference configuration K_o are identified by the $\mathbf{r}_\mathrm{o}$. The motion of the continuum is then given by the place vectors $\mathbf{r}(\mathbf{r}_\mathrm{o}, t)$ for the particle $\mathbf{r}_\mathrm{o}$ in the present configuration K.

The *material derivative* of a tensor field of order n is a new tensor field of order n. For the field $\mathbf{A}(\mathbf{r}_\mathrm{o}, \mathrm{t})$ the material-derivative is:

$$\dot{\mathbf{A}} = \partial_t \mathbf{A} \quad \Leftrightarrow \quad \dot{A}_{i\ldots} = \frac{\partial A_{i\ldots}}{\partial t} \equiv \partial_t A_{i\ldots} \tag{4.4.46}$$

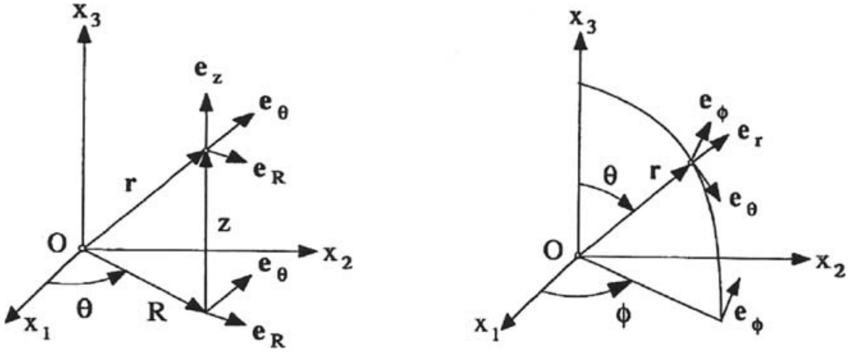


Fig. 4.4.1 Cylindrical coordinates (R, θ, z) and spherical coordinates (r, θ, ϕ)

For the tensor field $\mathbf{A}(\mathbf{r},t)$ we substitute the motion $\mathbf{r} = \mathbf{r}(\mathbf{r}_o,t)$, such that $\mathbf{A} = \mathbf{A}(\mathbf{r}(\mathbf{r}_o,t),t)$. Then:

$$\dot{\mathbf{A}} = \partial_t \mathbf{A} + \frac{\partial \mathbf{A}}{\partial \mathbf{r}} \cdot \frac{\partial \mathbf{r}}{\partial t} = \partial_t \mathbf{A} + \operatorname{grad} \mathbf{A} \cdot \frac{\partial \mathbf{r}}{\partial t} \quad \Leftrightarrow \quad \dot{A}_{ij..} = \partial_t A_{ij..} + A_{ij..,k}\, v_k \tag{4.4.47}$$

Alternatively we may write:

$$\dot{\mathbf{A}} = \partial_t \mathbf{A} + \mathbf{v} \cdot \vec{\nabla} \otimes \mathbf{A} \tag{4.4.48}$$

or we may introduce the operator:

$$(\mathbf{v} \cdot \nabla) = v_k \frac{\partial}{\partial x_k} \tag{4.4.49}$$

and write:

$$\dot{\mathbf{A}} = \partial_t \mathbf{A} + (\mathbf{v} \cdot \nabla)\mathbf{A} \tag{4.4.50}$$

In general orthogonal coordinate systems the operator (4.4.49) becomes:

$$(\mathbf{v} \cdot \nabla) = \sum_i \frac{v(i)}{h_i} \frac{\partial}{\partial y_i} \tag{4.4.51}$$

4.5 Rigid-Body Dynamics

All material bodies are deformable and will in general be deformed when subjected to forces. However, it is often convenient to divide the motion of a body into one contribution that is due to the deformation of the body and one contribution that neglects the deformation. The last contribution is called *rigid-body motion*. If the deformations are so small that they have little influence on the overall geometry of the body and the actions of the external forces that the body is subjected to, the motion of the body may be treated by what is called rigid-body dynamics. The subject matter of basic courses in Statics and Dynamics is essentially rigid body statics and dynamics. In the present section we shall include the fundamental aspects of rigid-body dynamics. The main purpose of placing this subject here is primarily to present rigid-body kinematics, to which we will referred in Sect. 5.5 about large deformations. An other reason why rigid-body dynamics is put into the present chapter, is that it provides us with an example of an important symmetric 2. order tensor, the *inertia tensor*.

4.5.1 Kinematics

Figure 4.5.1 shows a body in rigid-body motion from a reference configuration K_o at time t_o to the present configuration K at time t. The coordinate system Ox is fixed in the reference Rf we have chosen to describe the motion, while the coordinate system $\bar{O}\bar{x}$ moves rigidly with the body and coincide with the Ox-system at time t_o. This implies that the coordinates $\bar{x}$ of a particle P are equal to the reference coordinates X of that particle. A component version of the vector relation $\mathbf{r} = \mathbf{u}_\mathrm{o} + \bar{\mathbf{r}}_\mathrm{o}$, is provided by formula (2.3.10):

$$x = u_o + Q^T \bar{x} = u_o + Q^T X \tag{4.5.1}$$

$\mathbf{u}_\mathrm{o}(t)$ is the displacement vector of the origin $\bar{O}$ in the $\bar{O}\bar{x}$-system, and $Q(t)$ is the transformation matrix:

$$Q_{ik} = \cos\left(\bar{\mathbf{e}}_i, \mathbf{e}_k\right)$$

The relationship (4.5.1) may also be interpreted as a matrix representation of the vector equation:

$$\mathbf{r} = \mathbf{u}_o + \mathbf{R} \cdot \mathbf{r}_o \quad , \quad \mathbf{u}_o = \mathbf{u}_o(t) \quad , \quad \mathbf{R} = \mathbf{R}(t) \tag{4.5.2}$$

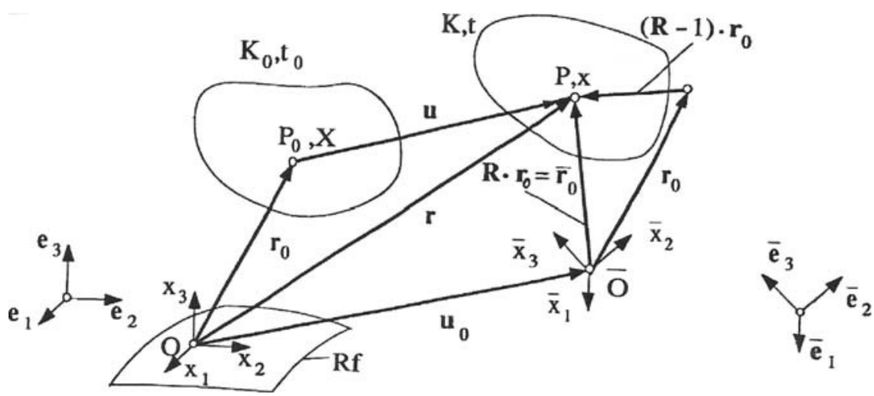


Fig. 4.5.1 Rigid-body motion

where the tensor $\mathbf{R}$ of 2. order has the components:

$$R = Q^T \quad \Leftrightarrow \quad R_{ik} = Q_{ki} \tag{4.5.3}$$

in the Ox-system. The place vector $\bar{\mathbf{r}}_o = \mathbf{R} \cdot \mathbf{r}_{\mathrm{o}}$ is a result of a rotation of the vector $\mathbf{r}_{\mathrm{o}}$, see Fig. 4.5.1. The tensor $\mathbf{R}$(t) is called the *rotation tensor* of the rigid-body motion. As shown in (4.5.3) the matrix R of the tensor $\mathbf{R}$ in the Ox-system is equal to the transposed of the transformation matrix Q for the transformation from the Ox-system to the $\bar{O}\bar{x}$-system, which coincides with the Ox-system at the reference time t_o. The rotation tensor is an orthogonal tensor:

$$\mathbf{R}\mathbf{R}^T = \mathbf{1} \quad , \quad \det \mathbf{R} = 1 \tag{4.5.4}$$

The formula (4.5.2) represents now a *rigid-body motion*.

A vector $\mathbf{c}(t)$ that rigidly follows the motion of the body, will, with respect to the reference *Rf*, transform during the motion according to:

$$\mathbf{c}(t) = \mathbf{R} \cdot \mathbf{c}_o \quad , \quad \mathbf{c}_o = \mathbf{c}(t_o) \tag{4.5.5}$$

The result follows from the expression (4.5.2) if we choose $\mathbf{c} = \mathbf{r}_A - \mathbf{r}_B$, where A and B are the head point and the tail point of the vector arrow $\mathbf{c}$. We write:

$$\mathbf{c} = \mathbf{r}_A - \mathbf{r}_B = (\mathbf{u}_o + \mathbf{R} \cdot \mathbf{r}_{oA}) - (\mathbf{u}_o + \mathbf{R} \cdot \mathbf{r}_{oB}) = \mathbf{R} \cdot (\mathbf{r}_{oA} - \mathbf{r}_{oB}) = \mathbf{R} \cdot \mathbf{c}_o \quad \Rightarrow \quad \mathbf{c} = \mathbf{R} \cdot \mathbf{c}_o$$

We say that $\mathbf{c}$ is the $\mathbf{R}$-*rotation* of the vector $\mathbf{c}_o$.

The rigid-body motion (4.5.2) may be considered as a combination of a *translation* $\mathbf{u}_{\mathrm{o}}(t)$, by which all particles of the body are given the same motion as the reference point $\bar{O}$ in the body, and a *rotation* $\mathbf{R}(t) \cdot \mathbf{r}_{\mathrm{o}}$ of the body about this reference point. The displacement of a particle P, from the reference configuration K_o to the present configuration K, is $\mathbf{u} = \mathbf{r} - \mathbf{r}_{\mathrm{o}}$, which by (4.5.2) is:

$$\mathbf{u} = \mathbf{u}_o + (\mathbf{R} - \mathbf{1}) \cdot \mathbf{r}_o \tag{4.5.6}$$

The displacement $\mathbf{u}$ is a sum of the translation $\mathbf{u}_{\mathrm{o}}(t)$ and the *rigid-body rotation*:

$$\bar{\mathbf{r}}_o - \mathbf{r}_o = (\mathbf{R} - \mathbf{1}) \cdot \mathbf{r}_o$$

We shall now continue to study a *pure rotation* about the origin, i.e. $O \equiv \bar{O}$, see Fig. 4.5.2. The motion is according to (4.5.2) given by the place vector:

$$\mathbf{r} = \mathbf{R} \cdot \mathbf{r}_o \tag{4.5.7}$$

or according to (4.5.6) by the displacement vector:

$$\mathbf{u} = (\mathbf{R} - \mathbf{1}) \cdot \mathbf{r}_o \tag{4.5.8}$$

It will now be shown that a material straight line through O exists that has the same position before and after the rotation $\mathbf{R}$(t). The line is called the *axis of rotation* related to the rotation $\mathbf{R}(t)$. $\mathbf{R}(t_o) = \mathbf{1}$ in the reference configuration K_o. The particles

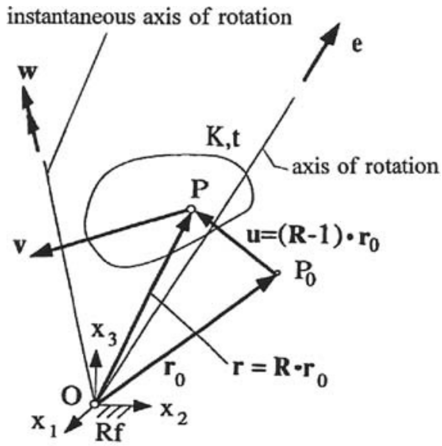


Fig. 4.5.2 Rigid-body rotation about a fixed point O

of the body on the axis of rotation may have moved but are in the same positions at time t, after the rotation, as they are before the rotation at time t_o. The particles of the body on the axis have zero displacement at time t. A unit vector **e** along the axis of rotation must satisfy the equation:

$$\mathbf{R}\cdot\mathbf{e}=\mathbf{e} \quad \Leftrightarrow \quad (\mathbf{R}-\mathbf{1})\cdot\mathbf{e}=\mathbf{0} \tag{4.5.9}$$

If this equation is compared with (4.3.28), we see that the unit vector **e** may be called a principal direction to the tensor **R**, and that the corresponding principal value is equal to 1. In order for (4.5.9) to have a solution, the following condition must be satisfied.

$$\det(\mathbf{R}-\mathbf{1})=0 \tag{4.5.10}$$

Using (4.5.4) and the multiplication theorem for determinants (2.1.21), we obtain:

$$\mathbf{R}-\mathbf{1}=\left(\mathbf{1}-\mathbf{R}^T\right)\mathbf{R}=-(\mathbf{R}-\mathbf{1})^T\mathbf{R} \Rightarrow \det(\mathbf{R}-\mathbf{1})=-\det(\mathbf{R}-\mathbf{1})^T\det\mathbf{R}=-\det(\mathbf{R}-\mathbf{1})$$

The result implies that (4.5.10) is satisfied. Thus we have shown that the unit vector **e**, which may be determined from (4.5.9), defines the axis of rotation of the body about the point O.

The motion of the body when it moves from configuration K_o to the configuration K in Fig. 4.5.2, is not in general a rotation about the axis of rotation defined by the vector **e**. However, it is possible to get the body from K_o to K by a pure rotation about this axis. The angle θ that the body must rotate about the axis, may be determined as follows. We choose the Ox-system such that the base vector $\mathbf{e}_3$ is parallel to the axis of rotation: $\mathbf{e}_3=\mathbf{e}$. Figure 4.5.3 shows the projection of the situation onto the x_1x_2-plane. The matrix R of the rotation tensor is determined from (4.5.3) and becomes:

$$R=\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix} \tag{4.5.11}$$

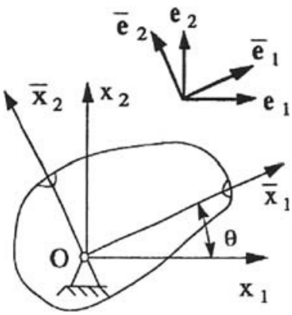


Fig. 4.5.3 Rotation about a fixed x_3-axis

It then follows that: $\mathrm{tr}\,\mathbf{R} = \mathrm{tr}\,R = 2\cos\theta + 1$, from which we derive the coordinate invariant formula:

$$\cos\theta = \frac{1}{2}\left(\mathrm{tr}\,\mathbf{R} - \mathbf{1}\right) \tag{4.5.12}$$

We now continue to discuss rotation of a rigid body about a fixed point O, Fig. 4.5.2. The *velocity* $\mathbf{v}$ of the particle P is determined from (4.5.7) as follows.

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\mathbf{R}} \cdot \mathbf{r}_o = \dot{\mathbf{R}} \cdot \left(\mathbf{R}^T \cdot \mathbf{r}\right) = \left(\dot{\mathbf{R}}\mathbf{R}^T\right) \cdot \mathbf{r}$$

The *rate of rotation tensor* $\mathbf{W}$(t) is defined by:

$$\mathbf{W} = \dot{\mathbf{R}}\mathbf{R}^T \tag{4.5.13}$$

such that $\mathbf{v} = \mathbf{W} \cdot \mathbf{r}$. The rate of rotation tensor is antisymmetric, as may be seen from the following development.

$$\mathbf{R}\mathbf{R}^T = \mathbf{1} \;\Rightarrow\; \dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\dot{\mathbf{R}}^T = \mathbf{0} \;\Rightarrow\; \dot{\mathbf{R}}\mathbf{R}^T = -\mathbf{R}\dot{\mathbf{R}}^T = -\left(\dot{\mathbf{R}}\mathbf{R}^T\right)^T \;\Rightarrow\; \mathbf{W} = -\mathbf{W}^T$$

The dual vector $\mathbf{w}$ of the antisymmetric tensor $\mathbf{W}$ is by definition:

$$\mathbf{w} = -\frac{1}{2}\mathbf{P} : \mathbf{W} = -\frac{1}{2}\mathbf{W} : \mathbf{P} \quad \Rightarrow \quad w_i = \frac{1}{2}e_{ijk}W_{kj} = \frac{1}{2}e_{ijk}v_{k,j} \quad \Rightarrow \quad \mathbf{w} = \frac{1}{2}\mathrm{rot}\,\mathbf{v} \tag{4.5.14}$$

where $\mathbf{P}$ is the permutation tensor. The vector $\mathbf{w}$ represents both mathematically and physically the same as the tensor $\mathbf{W}$ and is called the *angular velocity* or the *rotational velocity*. The first name is most common, although it is only when the body rotates about a fixed axis, or moves parallel to a fixed plane, that it is possible to associate the rotation of the body with a real angle of rotation. The inverse of the relation (4.5.14) is:

$$\mathbf{W} = -\mathbf{P} \cdot \mathbf{w} \quad \Leftrightarrow \quad W_{ij} = -W_{ji} = -e_{ijk}w_k \quad \Leftrightarrow \quad W = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} \tag{4.5.15}$$

Below we shall use the dual antisymmetric tensor $\mathbf{Z}$ to the position vector $\mathbf{r} = [x_1, x_2, x_3]$.

$$\mathbf{Z} = -\mathbf{P} \cdot \mathbf{r} \quad \Leftrightarrow \quad Z_{ij} = -Z_{ji} = -e_{ijk} x_k \quad \Leftrightarrow \quad Z = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \tag{4.5.16}$$

With these definitions of $\mathbf{W}$, $\mathbf{w}$, and $\mathbf{Z}$, and the formula $(4.3.12)_1$, the particle velocity $\mathbf{v}$ may be expressed alternatively by the following *velocity distribution formulas*.

$$\mathbf{v} = \dot{\mathbf{r}} = \mathbf{W} \cdot \mathbf{r} = \mathbf{w} \times \mathbf{r} \quad \Leftrightarrow \quad v_i = \dot{x}_i = W_{ik} x_k = e_{ijk}\, w_j x_k \tag{4.5.17}$$

$$\mathbf{v} = \dot{\mathbf{r}} = -\mathbf{Z} \cdot \mathbf{w} \quad \Leftrightarrow \quad v_i = \dot{x}_i = -Z_{ik} w_k = e_{ikj} x_j w_k = e_{ijk}\, w_j x_k \tag{4.5.18}$$

Particles on a straight line through the point of rotation O and parallel to the angular velocity vector $\mathbf{w}$, are at the present time t instantaneously at rest. This fact follows from the velocity distribution formula (4.5.17). The straight line is called the *instantaneous axis of rotation* and is show in Fig. 4.5.2. In the general case the instantaneous axis of rotation and the axis of rotation, defined by (4.5.9), do not coincide. The instantaneous axis of rotation will in general change its position both relative to the rigid body and to the reference *Rf*. Only when the rigid body is constrained to rotate about an axis fixed with respect to *Rf*, the instantaneous axis of rotation and the axis of rotation are one and the same axis at any time t.

The *acceleration* $\mathbf{a}$ of a particle P is developed from (4.5.17) as follows.

$$\mathbf{a} = \dot{\mathbf{v}} = \dot{\mathbf{W}} \cdot \mathbf{r} + \mathbf{W} \cdot \dot{\mathbf{r}} = \dot{\mathbf{w}} \times \mathbf{r} + \mathbf{w} \times \dot{\mathbf{r}} \quad \Rightarrow$$
$$\mathbf{a} = \left(\dot{\mathbf{W}} + \mathbf{W}^2\right) \cdot \mathbf{r} = \dot{\mathbf{w}} \times \mathbf{r} + \mathbf{w} \times (\mathbf{w} \times \mathbf{r}) \tag{4.5.19}$$

Alternatively, the acceleration may be expressed by, see Problem 4.9:

$$\mathbf{a} = -\mathbf{Z} \cdot \dot{\mathbf{w}} - \mathbf{W}\mathbf{Z} \cdot \mathbf{w} \tag{4.5.20}$$

$\dot{\mathbf{W}}(t)$ and $\dot{\mathbf{w}}(t)$ are called the *angular acceleration tensor* and the *angular acceleration*, respectively. In general the angular velocity and the angular acceleration are non-parallel vectors. The formulas (4.5.18) and (4.5.20) are convenient when rigid-body kinematics is presented or analyzed in a matrix format.

We now return to the general rigid body motion illustrated in Fig. 4.5.1. The motion of the particle P is given by the place vector $\mathbf{r}$ in (4.5.2) and repeated here.

$$\mathbf{r} = \mathbf{u}_o + \mathbf{R} \cdot \mathbf{r}_o \quad , \quad \mathbf{u}_o = \mathbf{u}_o(t) \quad , \quad \mathbf{R} = \mathbf{R}(t) \tag{4.5.21}$$

or by the displacement vector $\mathbf{u}$ in (4.5.6).

$$\mathbf{u} = \mathbf{u}_o + (\mathbf{R} - \mathbf{1}) \cdot \mathbf{r}_o \tag{4.5.22}$$

The velocity and the acceleration of the particle P are respectively:

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\mathbf{u}} = \mathbf{v}_o + \mathbf{W} \cdot \bar{\mathbf{r}}_o = \mathbf{v}_o + \mathbf{w} \times \bar{\mathbf{r}}_o \tag{4.5.23}$$

$$\mathbf{a} = \dot{\mathbf{v}} = \mathbf{a}_o + \left(\dot{\mathbf{W}} + \mathbf{W}^2\right) \cdot \bar{\mathbf{r}}_o = \mathbf{a}_o + \dot{\mathbf{w}} \times \bar{\mathbf{r}}_o + \mathbf{w} \times (\mathbf{w} \times \bar{\mathbf{r}}_o) \tag{4.5.24}$$

where $\mathbf{v}_o = \dot{\mathbf{u}}_o$ and $\mathbf{a}_o = \dot{\mathbf{v}}_o$ are the velocity and the acceleration of the reference point $\bar{O}$, which moves with the rigid body, and $\bar{\mathbf{r}}_o = \mathbf{R} \cdot \mathbf{r}_o$, see Fig. 4.5.1.

If the velocity $\mathbf{v}_P$ and the acceleration $\mathbf{a}_P$ of a point P in a rigid body are known, see Fig. 4.5.4, the velocity and the acceleration of any other point Q in the body may be determined from the formulas:

$$\mathbf{v}_Q = \mathbf{v}_P + \mathbf{W} \cdot \mathbf{r}_{Q/P} = \mathbf{v}_P + \mathbf{w} \times \mathbf{r}_{Q/P} \tag{4.5.25}$$

$$\mathbf{a}_Q = \mathbf{a}_P + \left(\dot{\mathbf{W}} + \mathbf{W}^2\right) \cdot \mathbf{r}_{Q/P} = \mathbf{a}_P + \dot{\mathbf{w}} \times \mathbf{r}_{Q/P} + \mathbf{w} \times (\mathbf{w} \times \mathbf{r}_{Q/P}) \tag{4.5.26}$$

where $\mathbf{r}_{Q/P}$ is the position vector from P to Q. The formulas may be developed directly from the formulas (4.5.23) and (4.5.24) and are respectively called the *velocity distribution formula* and the *acceleration distribution formula* of a rigid body.

The material derivative of a vector $\mathbf{c}(t)$ that rigidly follows the motion of the body, is:

$$\dot{\mathbf{c}} = \mathbf{W} \cdot \mathbf{c} = \mathbf{w} \times \mathbf{c} \tag{4.5.27}$$

This result is obtained from a differentiation of (4.5.5) followed by application of (4.5.13):

$$\dot{\mathbf{c}} = \dot{\mathbf{R}} \cdot \mathbf{c}_o = \dot{\mathbf{R}}\mathbf{R}^T \cdot \mathbf{c} = \mathbf{W} \cdot \mathbf{c}$$

4.5.2 Relative Motion

We let a rigid body represent a reference Rf^* that moves relative to the reference *Rf*. Two Cartesian coordinate systems are introduced: Ox fixed in *Rf* and O^*x^* fixed in Rf^*. The motion of Rf^* is given by the velocity $\mathbf{v}_\mathrm{o}$ of the reference point O^* in the body and the angular velocity $\mathbf{w}$ of the body, i.e. of Rf^*. The position of a particle in a continuum is at time t given by the place vector $\mathbf{r}$ from the origin O fixed in *Rf*, such that $\mathbf{r} = \mathbf{r}_o$ at a reference time t_o. The motion of the particle is thus given by $\mathbf{r} = \mathbf{r}(\mathbf{r}_o, t)$, such that $\mathbf{r} = \mathbf{r}(\mathbf{r}_o, t_o) = \mathbf{r}_o$. A vector field $\mathbf{c}(\mathbf{r}_\mathrm{o}, \mathrm{t})$ behaves differently with respect to time whether the field is observed from *Rf* or from Rf^*.

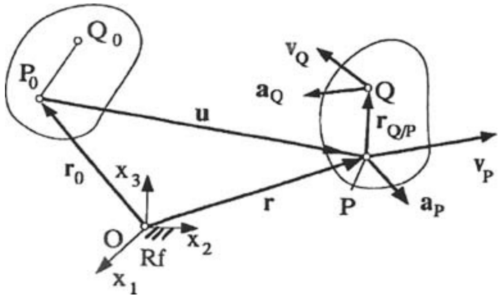


Fig. 4.5.4 General rigid-body motion

For instance, let $\mathbf{c}$ be a constant vector in *Rf*. Seen from Rf^* the vector represents a homogeneous vector field $\mathbf{c}(t)$. This implies that material differentiation of vectors, and of tensors, is a *reference related operation*. As before, we shall use a "superdot" to indicate material differentiation with respect to *Rf*, and introduce the symbol ˆ to mark material differentiation with respect to $\bar{R}\bar{f}$.

$$\dot{\mathbf{c}} = \left(\frac{\partial \mathbf{c}}{\partial t}\right)_{Rf} \text{ observed in} Rf \quad , \quad \hat{\mathbf{c}} = \left(\frac{\partial \mathbf{c}}{\partial t}\right)_{Rf*} \text{ observed in } Rf* \tag{4.5.28}$$

Material derivatives of scalars and vector and tensor components are denoted by the superdot. The components refer to a particular coordinate system and are thus not affected by any other reference.

For the vector field $\mathbf{c}(\mathbf{r}_\mathrm{o}, \mathrm{t})$ we write:

$$\mathbf{c} = c_i \mathbf{e}_i = c^*{}_k \mathbf{e}^*{}_k \quad \Rightarrow \quad \dot{\mathbf{c}} = \dot{c}_i \mathbf{e}_i \quad , \quad \hat{\mathbf{c}} = \dot{c}^*{}_k \mathbf{e}^*{}_k \tag{4.5.29}$$

Using formula (4.5.27), we obtain:

$$\dot{\mathbf{c}} = \dot{c}^*{}_k \mathbf{e}^*{}_k + c^*{}_k \dot{\mathbf{e}}^*{}_k = \hat{\mathbf{c}} + c^*{}_k \mathbf{W} \cdot \mathbf{e}^*{}_k \quad \Rightarrow$$
$$\dot{\mathbf{c}} = \hat{\mathbf{c}} + \mathbf{W} \cdot \mathbf{c} = \hat{\mathbf{c}} + \mathbf{w} \times \mathbf{c} \tag{4.5.30}$$

For a 2. order tensor field $\mathbf{T}(\mathbf{r}_\mathrm{o}, t)$ we shall obtain, see Problem 4.15:

$$\dot{\mathbf{T}} = \hat{\mathbf{T}} + \mathbf{W}\mathbf{T} - \mathbf{T}\mathbf{W} \tag{4.5.31}$$

Let the motion of the particle $\mathbf{r}_o$ relative to the reference Rf^* be given by the place vector $\mathbf{r}^*(\mathbf{r}^*{}_o, t)$ from the origin O^* fixed in Rf^*. The vector $\mathbf{r}^*{}_o$ represents the particle in Rf^*, such that $\mathbf{r}^*{}_o = \mathbf{r}^*(\mathbf{r}^*{}_o, t_o)$. Observed in *Rf* the particle has the velocity $\mathbf{v}$ and the acceleration $\mathbf{a}$. Observed in Rf^* the particle has the velocity $\mathbf{v}^*$ and the acceleration $\mathbf{a}^*$. These quantities are obtained from:

$$\mathbf{v} = \dot{\mathbf{r}} \quad , \quad \mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} \quad , \quad \mathbf{v}^* = \hat{\mathbf{r}}^* \quad , \quad \mathbf{a}^* = \hat{\mathbf{v}}^* = \hat{\hat{\mathbf{r}}}^* \tag{4.5.32}$$

Let the motion of O^* with respect to O be given by the displacement vector $\mathbf{u}_o(t)$. Then:

$$\mathbf{r} = \mathbf{u}_o + \mathbf{r}^* \tag{4.5.33}$$

Using formula (4.5.30), we obtain the following kinematical relations between the velocities $\mathbf{v}$ and $\mathbf{v}^*$, and between the accelerations $\mathbf{a}$ and $\mathbf{a}^*$:

$$\mathbf{v} = \mathbf{v}_o + \mathbf{w} \times \mathbf{r}^* + \mathbf{v}^* \quad , \quad \mathbf{v}_o = \dot{\mathbf{u}}_o \tag{4.5.34}$$
$$\mathbf{a} = \mathbf{a}_o + \dot{\mathbf{w}} \times \mathbf{r}^* + \mathbf{w} \times (\mathbf{w} \times \mathbf{r}^*) + 2\mathbf{w} \times \mathbf{v}^* + \mathbf{a}^* \quad , \quad \mathbf{a}_o = \dot{\mathbf{v}}_o \tag{4.5.35}$$

The first two terms in the formula (4.5.34) for $\mathbf{v}$ represent the velocity of a *place* in the reference Rf^*, i.e. a fixed point in Rf^*, relative to the reference *Rf*, and is therefore called the *place velocity*.

$$\mathbf{v}_{\text{place}} = \mathbf{v}_o + \mathbf{w} \times \mathbf{r}^* \tag{4.5.36}$$

The first three terms in the formula (4.5.35) for **a** represent the acceleration of a place in Rf^* relative to the reference *Rf*, and is therefore called the *place acceleration*.

$$\mathbf{a}_{\text{place}} = \mathbf{a}_o + \dot{\mathbf{w}} \times \mathbf{r}^* + \mathbf{w} \times (\mathbf{w} \times \mathbf{r}^*) \tag{4.5.37}$$

The second last term in the formula (4.5.35) for the acceleration **a** is called the *Coriolis acceleration*, named after Gustave Gaspard Coriolis [1792–1843].

4.5.3 Kinetics

The center of mass C of a body is defined by formula (3.2.8). For a rigid body the mass center is a point fixed with respect to the body, but does not necessarily coincide with a material point or particle in the body. The motion of the center of mass is governed by the equation of motion (3.2.12):

$$\mathbf{f} = m\mathbf{a}_C \tag{4.5.38}$$

This vector equation represents three component equations. A rigid body has six degrees of freedom. The additional three equations of motion are provided by the law of balance of angular momentum, i.e. Euler's 2. axiom (3.2.7). These additional equations motion will now be developed.

We shall start by considering a rigid body that rotates about a fixed point O as shown in Fig. 4.5.5. First we compute the angular momentum of the body about the point O from the definition formula (3.2.5).

$$\mathbf{l}_O = \int_V \mathbf{r} \times \mathbf{v}\, \rho\, dV \tag{4.5.39}$$

The velocity **v** of found in formula (4.5.17), and by using the property (4.3.12) of the two dual quantities **r** and **Z**, we obtain:

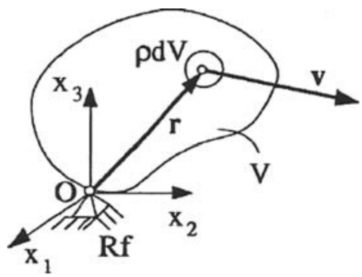


Fig. 4.5.5 Rotation about a fixed point O

$$\mathbf{r} \times \mathbf{v} = \mathbf{r} \times (\mathbf{w} \times \mathbf{r}) = \mathbf{Z} \cdot \mathbf{v} = -\mathbf{Z}\mathbf{Z} \cdot \mathbf{w} = -\mathbf{Z}^2 \cdot \mathbf{w}$$

It may be shown that:

$$\mathbf{Z}^2 = -(\mathbf{r} \cdot \mathbf{r})\,\mathbf{1} + \mathbf{r} \otimes \mathbf{r} \quad \Leftrightarrow$$
$$\left(\mathbf{Z}^2\right)_{ij} = -x_k x_k \,\delta_{ij} + x_i x_j \tag{4.5.40}$$

The angular momentum of the body about O then becomes:

$$\mathbf{l}_O = \int_V \mathbf{r} \times \mathbf{v}\,\rho\, dV = \int_V \mathbf{r} \times (\mathbf{w} \times \mathbf{r})\,\rho\, dV = \left[\int_V \left(-\mathbf{Z}^2\right)\rho\, dV\right] \cdot \mathbf{w} \quad \Rightarrow$$
$$\mathbf{l}_O = \mathbf{I} \cdot \mathbf{w} = I_{ij}\, w_j\, \mathbf{e}_i \tag{4.5.41}$$

where the *inertia tensor* $\mathbf{I}$ of the body with respect to the point O has been introduced.

$$\mathbf{I} = \int_V \left(-\mathbf{Z}^2\right)\rho\, dV = \int_V \left[(\mathbf{r} \cdot \mathbf{r})\,\mathbf{1} - \mathbf{r} \otimes \mathbf{r}\right]\rho\, dV$$
$$I_{ij} = \int_V \left[x_k x_k\, \delta_{ij} - x_i x_j\right]\rho\, dV \tag{4.5.42}$$

The inertia tensor is a symmetric second order tensor, and the six distinct components of the tensor in the Ox-system are:

$$I_{11}, \quad I_{22}, \quad I_{33} \qquad\qquad moments\ of\ inertia$$
$$I_{ij} = -\int_V x_i x_j\,\rho\, dV,\ i \neq j \quad products\ of\ inertia \tag{4.5.43}$$

In particular, I_{33} is the *moment of inertia about the* x_3*-axis*.

$$I_{33} = -\int_V \left(x_1^2 + x_2^2\right)\rho\, dV = \int_V R^2 \rho\, dV \tag{4.5.44}$$

where R is the distance from the x_3-axis to the element of mass ρdV, see Fig. 4.5.5. The quantity I_{12} is called the *product of inertia with respect to the* x_1 *and* x_2 *axes*.

The Euler's 2. axiom, (3.2.7), equates the resultant moment $\mathbf{m}_O$ about the point O of the forces on the body and the material derivative of the angular momentum of the body about the point O.

$$\mathbf{m}_O = \dot{\mathbf{l}}_O \tag{4.5.45}$$

The material derivative of the angular momentum about O, (4.5.41), becomes:

$$\dot{\mathbf{l}}_O = \dot{\mathbf{I}} \cdot \mathbf{w} + \mathbf{I} \cdot \dot{\mathbf{w}}$$

The inertia tensor is constant when referred to the rigid body. Therefore using the general formula (4.5.31), we get:

$$\dot{\mathbf{I}} = \mathbf{W}\mathbf{I} - \mathbf{I}\mathbf{W} \quad \Rightarrow \quad \dot{\mathbf{i}}_O = \mathbf{W}\mathbf{I} \cdot \mathbf{w} - \mathbf{I}\mathbf{W} \cdot \mathbf{w} + \mathbf{I} \cdot \dot{\mathbf{w}}$$

The product $\mathbf{IW} \cdot \mathbf{w}$ is a vector with components $I_{ij}W_{jk}w_k = I_{ij}e_{kjl}w_l w_k$, which are zero due to the properties of the permutation symbol e_{ijk}. The Euler's 2. axiom (4.5.45) therefore takes the form:

$$\begin{aligned} \mathbf{m}_O &= \mathbf{I} \cdot \dot{\mathbf{w}} + \mathbf{W}\mathbf{I} \cdot \mathbf{w} \quad \Leftrightarrow \\ m_{Oi} &= I_{ij}\,\dot{w}_j + W_{ik}\,I_{kl}\,w_l = I_{ij}\,\dot{w}_j + e_{ijk}\,w_j\,I_{kl}\,w_l \end{aligned} \tag{4.5.46}$$

The three component equations of (4.5.46) are called the *Euler equations for a rigid body*.

If the coordinate axes x_i are chosen to be parallel to the principal axes of the inertia tensor, called the *principal axes of inertia* of the body with respect to O, the Euler equations are:

$$\begin{aligned} m_{O1} &= I_1\,\dot{w}_1 + (I_3 - I_2)\,w_3\,w_2 \\ m_{O2} &= I_2\,\dot{w}_2 + (I_1 - I_3)\,w_1\,w_3 \\ m_{O3} &= I_3\,\dot{w}_3 + (I_2 - I_1)\,w_2\,w_1 \end{aligned} \tag{4.5.47}$$

I_1, I_2, and I_3 are the *principal moments of inertia*.

For general motion of a rigid body the expression of the angular momentum about the fixed point O is developed as follows, see Fig. 4.5.6. The place vector from O to any particle P in the body is expressed by the vector sum $\mathbf{r}_C + \mathbf{r}$, where $\mathbf{r}_C$ is the place vector from O to the mass center C of the body, and $\mathbf{r}$ is the place vector from C to the particle P. The velocity of P is given by formula (4.5.25), with reference to Fig. 4.5.6:

$$\mathbf{v} = \mathbf{v}_C + \mathbf{w} \times \mathbf{r} = \mathbf{v}_C - \mathbf{Z} \cdot \mathbf{w}$$

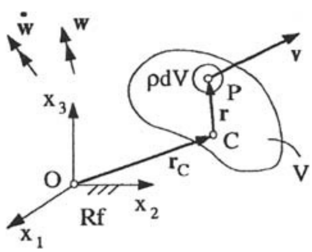


Fig. 4.5.6 General rigid body motion

In the last equation the alternative form (4.5.18) for $\mathbf{w} \times \mathbf{r}$ has been used. The angular momentum of the body about the fixed point O is:

$$\mathbf{l}_O = \int_V (\mathbf{r}_C + \mathbf{r}) \times \mathbf{v} \rho \, dV = \mathbf{r}_C \times \int_V \mathbf{v} \rho \, dV + \left[\int_V \mathbf{r} \rho \, dV \right] \times \mathbf{v}_C + \int_V \mathbf{r} \times (\mathbf{w} \times \mathbf{r}) \rho \, dV \tag{4.5.48}$$

The three integrals on the right hand side of the equation are: The first integral is equal to the linear momentum $m\mathbf{v}_C$ of the body. The second integral vanishes due the definition (3.2.8) of the center of mass. The third integral is denoted $\mathbf{l}_C$ and is called the *central angular momentum*. The central angular momentum may be developed similarly to the derivation of $\mathbf{l}_O$ in formula (4.5.41). Thus:

$$\mathbf{l}_C = \int_V \mathbf{r} \times (\mathbf{w} \times \mathbf{r}) \rho \, dV = \mathbf{I}_C \cdot \mathbf{w} \tag{4.5.49}$$

where $\mathbf{I}_C$ is the inertia tensor of the body with respect to mass center C. The components of this inertia tensor are given by the formulas (4.5.42) in a Cartesian coordinate fixed in the body, and with the origin in C. The angular momentum about O is then:

$$\mathbf{l}_O = \mathbf{r}_C \times m \mathbf{v}_C + \mathbf{l}_C \tag{4.5.50}$$

Note that the central angular momentum also represents the angular momentum of the body about the fixed point that at time t coincides with the center of mass C.

The law of balance of angular momentum (4.5.45) now takes the form:

$$\mathbf{m}_O = \mathbf{r}_C \times m \mathbf{a}_C + \dot{\mathbf{l}}_C \tag{4.5.51}$$

which may be transformed into:

$$\mathbf{m}_O = m\mathbf{Z}_C \cdot \mathbf{a}_C + \mathbf{I}_C \cdot \dot{\mathbf{w}} + \mathbf{W}\mathbf{I}_C \cdot \mathbf{w} \quad \Leftrightarrow \quad m_{Oi} = mZ_{Cik} a_{Ck} + I_{Cik} \dot{w}_k + W_{ij} I_{Cjk} w_k \tag{4.5.52}$$

$\mathbf{Z}_C$ is the dual antisymmetric tensor to the place vector $\mathbf{r}_C$, with components given by the formulas (4.5.16) when the coordinates x_i are replaced by the components x_{Ci} of the vector $\mathbf{r}_C$. The last two terms on the right-hand side of (4.5.52) are obtained analogously to the right-hand side of (4.5.46).

4.6 Tensors of 2. Order. Part Two

This section contains topics that are particularly relevant for mathematical modeling of non-linear materials, for which the non-linearity may be due to both large deformation and non-linear material behavior. Some of the results will also be presented and derived in later sections, when they are applied, and then using physical and geometrical arguments.

4.6.1 Rotation of Vectors and Tensors

Let **Q** be an orthogonal tensor of 2. order and **a** en vector. It follows from (4.5.5) that the vector:

$$\mathbf{b} = \mathbf{Q} \cdot \mathbf{a} \tag{4.6.1}$$

represents a rotation of the vector **a**, such that the two vectors have the same magnitude: $|\mathbf{b}| = |\mathbf{a}|$. We call the vector **b** the **Q**-*rotation* of the vector **a**. It follows from the discussion in Sect. 4.5.1 that there exists a direction given by the unit vector **e** such that:

$$\mathbf{Q} \cdot \mathbf{e} = \mathbf{e} \tag{4.6.2}$$

The unit vector **e** represents the *axis of rotation* related to the tensor **Q**. The vector mapping (4.6.1) represents a rotation of all vectors **a**, as if they were fixed in a rigid body rotating an angle θ about an axis of rotation parallel to the unit vector **e**. The *rotation* θ may according to (4.5.12) be determined from:

$$\cos\theta = \frac{1}{2}(\operatorname{tr}\mathbf{Q} - 1) \tag{4.6.3}$$

Let **A** be a tensor of 2. order. The tensor:

$$\mathbf{B} = \mathbf{Q}\mathbf{A}\mathbf{Q}^T \tag{4.6.4}$$

is called the **Q**-*rotation* of the tensor **A**. We shall investigate the properties of the tensor **B** if the tensor **A** is either an orthogonal tensor or a symmetric tensor.

1) **A** is an orthogonal tensor that represents a rotation ϕ with an axis of rotation parallel to a unit vector **a**.

$$\cos\phi = \frac{1}{2}(\operatorname{tr}\mathbf{A} - 1) \quad , \quad \mathbf{A} \cdot \mathbf{a} = \mathbf{a} \tag{4.6.5}$$

The **Q**-rotation of **A** is a new orthogonal tensor $\mathbf{B} = \mathbf{QAQ}^{\mathrm{T}}$ with the same rotation ϕ and with an axis of rotation parallel to the unit vector:

$$\mathbf{b} = \mathbf{Q} \cdot \mathbf{a} \tag{4.6.6}$$

The proof of these results is given as Problem 4.13.

2) **A** is a symmetric tensor with principal values α_k and principal directions $\mathbf{a}_k$. The **Q**-rotation of **A** is a new symmetric tensor $\mathbf{B} = \mathbf{QAQ}^{\mathrm{T}}$ with the same principal values α_k and with principal directions given by:

$$\mathbf{b}_k = \mathbf{Q} \cdot \mathbf{a}_k \tag{4.6.7}$$

To show this is given as Problem 4.14.

4.6.2 Polar Decomposition

A tensor $\mathbf{F}$ of 2. order is called *non-singular* if the determinant of $\mathbf{F}$ is non-zero: det $\mathbf{F} \neq 0$. A non-singular tensor $\mathbf{F}$ of 2. order having a positive determinant: det $\mathbf{F} > 0$, can always be expressed as a composition of an orthogonal tensor $\mathbf{R}$ and a positive definite symmetric tensor, $\mathbf{U}$, or as a composition of a positive definite symmetric tensor $\mathbf{V}$ and the same orthogonal tensor $\mathbf{R}$:

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R} \tag{4.6.8}$$

The tensors $\mathbf{R}$, $\mathbf{U}$, and $\mathbf{V}$ are uniquely determined by $\mathbf{F}$ through the relations:

$$\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}} \quad , \quad \mathbf{V} = \sqrt{\mathbf{F}\mathbf{F}^T} \quad , \quad \mathbf{R} = \mathbf{F}\mathbf{U}^{-1} \tag{4.6.9}$$

If det $\mathbf{F} < 0$, the tensor $\mathbf{R}$ is called an *improper orthogonal tensor*. The results (4.6.8, 4.6.9) represent the *polar decomposition theorem*. In Sect. 5.5 on large deformations a geometrical proof of the polar decomposition theorem is presented. An algebraic proof follows.

The composition $\mathbf{F}^T\mathbf{F}$ is symmetric since: $(\mathbf{F}^T\mathbf{F})^T = \mathbf{F}^T\mathbf{F}$, and positive definite. The latter property is demonstrated as follows. For any vector $\mathbf{c}$:

$$\mathbf{c} \cdot \left(\mathbf{F}^T\mathbf{F}\right) \cdot \mathbf{c} = c_i\,(F_{ki}\,F_{kj})\,c_j = (F_{ki}\,c_i)(F_{kj}\,c_j) = \text{sum of squares} > 0$$

Then according to the definition (4.3.39) $\mathbf{F}^T\mathbf{F}$ is a positive definite tensor. Now since $\mathbf{F}^T\mathbf{F}$ is a symmetric, positive definite tensor, we can through the relation (4.3.40) determine the symmetric and positive definite tensors:

$$\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}} = \left(\mathbf{F}^T\mathbf{F}\right)^{1/2} \quad , \quad \mathbf{U}^{-1} = \left(\mathbf{F}^T\mathbf{F}\right)^{-1/2} \tag{4.6.10}$$

Next we will show that the composition $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$ is an orthogonal tensor. First we find that, since $(\mathbf{U}^{-1})^T = \mathbf{U}^{-1}$:

$$\begin{aligned}\mathbf{R}\mathbf{R}^T &= \left(\mathbf{F}\mathbf{U}^{-1}\right)\left(\mathbf{U}^{-1}\mathbf{F}^T\right) = \mathbf{F}\left(\mathbf{U}^2\right)^{-1}\mathbf{F}^T = \mathbf{F}\left(\mathbf{F}^T\mathbf{F}\right)^{-1}\mathbf{F}^T = \left(\mathbf{F}\mathbf{F}^{-1}\right)\left(\mathbf{F}^{-T}\mathbf{F}^T\right)\\ &= \mathbf{1}\mathbf{1} = \mathbf{1}\end{aligned}$$

Thus:

$$\det\left(\mathbf{R}\mathbf{R}^T\right) = \det\left(\mathbf{R}\right)\det\left(\mathbf{R}^T\right) = (\det\mathbf{R})^2 = 1 \quad \Rightarrow \quad \det\mathbf{R} = \pm 1$$

Now, since det $\mathbf{F}$ by assumption is positive and the determinant of the positive definite tensor $\mathbf{U}^{-1}$ is positive, we get the result:

$$\det \mathbf{R} = (\det \mathbf{F}) \left(\det \mathbf{U}^{-1}\right) > 0$$

Thus we have shown that det $\mathbf{R} = +1$, which means that $\mathbf{R}$ is a proper orthogonal tensor. If det $\mathbf{F} < 0$, it follows that $\mathbf{R} = -1$, and $\mathbf{R}$ is an improper orthogonal tensor. This completes the proof of the decomposition $\mathbf{F} = \mathbf{RU}$.

We shall then prove that the decomposition $\mathbf{F} = \mathbf{RU}$ is unique. Let us assume that two decompositions are possible:

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{R}_1\mathbf{U}_1$$

Then:

$$\begin{aligned} \mathbf{F}^T\mathbf{F} &= \left(\mathbf{U}_1\mathbf{R}_1^T\right)(\mathbf{R}_1\mathbf{U}_1) = \mathbf{U}_1^2\,, \quad \mathbf{F}^T\mathbf{F} = \left(\mathbf{U}\mathbf{R}^T\right)(\mathbf{RU}) \\ &= \mathbf{U}^2 \quad \Rightarrow \quad \mathbf{U}_1 = \mathbf{U} \end{aligned}$$

The implication follows from the fact that the square root of the positive definite tensors $\mathbf{U}^2$ and $\mathbf{U}_1^2$ are unique tensors. Next we find: $\mathbf{R}_1 = \mathbf{F}\mathbf{U}_1{}^{-1} = \mathbf{F}\mathbf{U}^{-1} = \mathbf{R}$. Hence, the decomposition $\mathbf{F} = \mathbf{RU}$ is unique.

The decomposition $\mathbf{F} = \mathbf{VR}$ and its uniqueness may be shown similarly. The relation between $\mathbf{U}$ and $\mathbf{V}$ is found as follows.

$$\begin{aligned} &\mathbf{F}\mathbf{R}^T = (\mathbf{RU})\,\mathbf{R}^T = (\mathbf{V}\mathbf{R})\,\mathbf{R}^T = \mathbf{V}\left(\mathbf{R}\mathbf{R}^T\right) = \mathbf{V} \quad \Rightarrow \\ &\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T \quad \Leftrightarrow \quad \mathbf{U} = \mathbf{R}^T\mathbf{V}\mathbf{R} \end{aligned} \tag{4.6.11}$$

We see that $\mathbf{V}$ is the $\mathbf{R}$-rotation of $\mathbf{U}$. The symmetric tensors $\mathbf{U}$ and $\mathbf{V}$ have the same principal values, and the principal directions of $\mathbf{V}$ are $\mathbf{R}$-rotations of the principal direction of $\mathbf{U}$. An application of the polar decomposition theorem and a geometrical interpretation of the properties of $\mathbf{R}$, $\mathbf{U}$, and $\mathbf{V}$ are presented in Sect. 5.5 on large deformations.

4.6.3 Isotropic Functions of Tensors

Let $\gamma[\mathbf{A}]$ be scalar-valued function of a 2. order tensor $\mathbf{A}$. The function $\gamma[\mathbf{A}]$ is called an *isotropic scalar-valued function* of $\mathbf{A}$ if:

$$\gamma\left[\mathbf{Q}\mathbf{A}\mathbf{Q}^T\right] = \gamma\,[\mathbf{A}] \text{ for all orthogonal tensors } \mathbf{Q} \tag{4.6.12}$$

If $\mathbf{S}$ is a 2. order symmetric tensor with principal values σ_k and principal invariants *I, II*, and *III*, the isotropic scalar-valued function $\gamma[\mathbf{S}]$ has the representations:

$$\gamma[\mathbf{S}] = \gamma(\sigma_1, \sigma_2, \sigma_3) \quad , \quad \gamma[\mathbf{S}] = \gamma(I, II, III) \tag{4.6.13}$$

These results are found as follows. Let the orthogonal tensor $\mathbf{Q}$ be chosen such that the matrix of the tensor $\mathbf{QSQ}^{\mathrm{T}}$ is $[\sigma_i\,\delta_{ij}]$ (no sum). Equation (4.6.12) then implies the representation $(4.6.13)_1$. The representation $(4.6.13)_2$ then follows from the fact that the principal values σ_k according to (4.3.31) are unique functions of the principal invariants *I, II, III*.

Let $\mathbf{B}[\mathbf{A}]$ be a 2. order tensor-valued function of a 2. order tensor $\mathbf{A}$. If:

$$\mathbf{B}\left[\mathbf{Q}\,\mathbf{A}\,\mathbf{Q}^T\right] = \mathbf{Q}\,\mathbf{B}\,[\mathbf{A}]\,\mathbf{Q}^T \quad \text{for all orthogonal tensors } \mathbf{Q} \tag{4.6.14}$$

then $\mathbf{B}[\mathbf{A}]$ is called an *isotropic 2. order tensor-valued function* of $\mathbf{A}$.

We now assume that both the argument tensor $\mathbf{A}$ and the tensor value $\mathbf{B}$ of the function $\mathbf{B}[\mathbf{A}]$ are symmetric tensors with principal values and principal direction given respectively by $\alpha_k,\ \mathbf{a}_k$ and $\beta_k,\ \mathbf{b}_k$. The property of isotropy (4.6.14) results in a special representation of the function $\mathbf{B}[\mathbf{A}]$. First we shall prove the theorem:

Theorem 4.1. *a) An isotropic, symmetric 2. order tensor-valued function* $\mathbf{B}[\mathbf{A}]$ *of a symmetric 2. order tensor* $\mathbf{A}$ *is coaxial to the argument tensor* $\mathbf{A}$. *b) If* $\mathbf{A}$ *is a symmetric tensor of 2. order, and* $\mathbf{B}[\mathbf{A}]$ *is a symmetric 2. order tensor-valued function of* $\mathbf{A}$ *and coaxial with* $\mathbf{A}$*, then* $\mathbf{B}[\mathbf{A}]$ *is an isotropic, symmetric 2. order tensor-valued function of* $\mathbf{A}$.

Proof of part a): We need to show that any principal direction of $\mathbf{A}$, here denoted $\mathbf{a}_3$, also is a principal direction of $\mathbf{B}$. We choose a particular $\mathbf{Q}$-rotation that has an axis of rotation parallel to $\mathbf{a}_3$ and the angle of rotation $\theta = 180°$. The $\mathbf{Q}$-rotation of $\mathbf{A}$, $\bar{\mathbf{A}} = \mathbf{Q}\ \mathbf{A}\ \mathbf{Q}^T$, has the same principal values as $\mathbf{A}$, and has the principal directions:

$$\bar{\mathbf{a}}_3 = \mathbf{Q}\cdot\mathbf{a}_3 = \mathbf{a}_3 \quad , \quad \bar{\mathbf{a}}_\alpha = \mathbf{Q}\cdot\mathbf{a}_\alpha = -\mathbf{a}_\alpha \tag{4.6.15}$$

The general representation (4.3.36) of a symmetric tensor of 2. order:

$$\mathbf{A} = \sum_k \alpha_k\,\mathbf{a}_k \otimes \mathbf{a}_k \tag{4.6.16}$$

shows that $\bar{\mathbf{A}} = \mathbf{A}$. Then the property (4.6.14) and (4.6.15) implies that:

$$\mathbf{B} \equiv \mathbf{B}\,[\mathbf{A}] = \mathbf{B}\left[\bar{\mathbf{A}}\right] = \mathbf{Q}\,\mathbf{B}\,[\mathbf{A}]\,\mathbf{Q}^T \ \Rightarrow\ \mathbf{B}\mathbf{Q} = \mathbf{Q}\mathbf{B} \ \Rightarrow$$
$$\mathbf{B}\mathbf{Q}\cdot\mathbf{a}_3 = \mathbf{Q}\mathbf{B}\cdot\mathbf{a}_3 \ \Rightarrow \mathbf{B}\cdot\mathbf{a}_3 = \mathbf{Q}\,(\mathbf{B}\cdot\mathbf{a}_3)$$

This result shows that the vector $\mathbf{B}\cdot\mathbf{a}_3$ is not influenced by the special $\mathbf{Q}$-rotation we have chosen. Thus the two vector $\mathbf{B}\cdot\mathbf{a}_3$ and $\mathbf{a}_3$ are parallel. Any principal direction of the argument tensor $\mathbf{A}$, here represented by $\mathbf{a}_3$, is therefore also a principal direction of the tensorvalued function $\mathbf{B}[\mathbf{A}]$. This implies that $\mathbf{B}$ and $\mathbf{A}$ are coaxial tensors, and part a) of the Theorem 4.1 is thus proved.

Proof of part b): Since the two tensors $\mathbf{A}$ and $\mathbf{B}[\mathbf{A}]$ are coaxial, they have the same principal directions $\mathbf{a}_k$, and the principal values of $\mathbf{B}$ must be scalar-valued functions of the principal values of $\mathbf{A}$. Hence the tensor $\mathbf{B}$ may be expressed by:

$$\mathbf{B}\,[\mathbf{A}] = \sum_k \beta_k\,(\alpha)\,\mathbf{a}_k \otimes\ \mathbf{a}_k \quad , \quad \beta_k\,(\alpha) \equiv \beta_k\,(\alpha_1, \alpha_2, \alpha_3)$$

Let $\mathbf{Q}$ be an orthogonal tensor. Then since the tensor $\mathbf{QAQ}^T$ has the same principal values as the tensor $\mathbf{A}$ but principal directions that are $\mathbf{Q}$-rotations of the principal directions of $\mathbf{A}$, we obtain:

$$\begin{aligned}\mathbf{B}\left[\mathbf{Q}\,\mathbf{A}\,\mathbf{Q}^T\right] &= \sum_k \beta_k(\alpha)\,(\mathbf{Q}\cdot\,\mathbf{a}_k)\otimes\,(\mathbf{Q}\cdot\,\mathbf{a}_k) = \mathbf{Q}\left(\sum_K \beta_k(\alpha)\,\mathbf{a}_k\otimes\mathbf{a}_k\right)\mathbf{Q}^T\\ &= \mathbf{Q}\,\mathbf{B}\,[\mathbf{A}]\,\mathbf{Q}^T\end{aligned}$$

The result proves that $\mathbf{B}[\mathbf{A}]$ is an isotropic, symmetric 2. order tensor-valued function of $\mathbf{A}$.

Next we shall prove an important theorem giving the most general representation of a symmetric tensor-valued function of a 2. order symmetric tensor.

Theorem 4.2. *Let* $\mathbf{B}[\mathbf{A}]$ *be a 2. order symmetric tensor-valued function of a 2. order symmetric tensor* $\mathbf{A}$. *The function* $\mathbf{B}[\mathbf{A}]$ *is then isotropic if and only if the function has the representation:*

$$\mathbf{B}\,[\mathbf{A}] = \gamma_0\,\mathbf{1} + \gamma_1\,\mathbf{A} + \gamma_2\,\mathbf{A}^2 \tag{4.6.17}$$

where $\gamma_0, \gamma_1, \text{and}\gamma_2$ are isotropic scalar-valued function of $\mathbf{A}$.

Proof that: (4.6.17) $\Rightarrow$ (4.6.14). The functions γ_0, γ_1, and γ_2 are isotropic scalar-valued functions of $\mathbf{A}$:

$$\gamma_i\left[\mathbf{QAQ}^T\right] = \gamma_i\,[\mathbf{A}] \quad \text{for } i = 0,\ 1,\ 2$$

Then from (4.6.17):

$$\begin{aligned}\mathbf{B}\left[\mathbf{Q}\,\mathbf{A}\,\mathbf{Q}^T\right] &= \gamma_0\,[\mathbf{A}]\,\mathbf{1} + \gamma_1\,[\mathbf{A}]\,\mathbf{Q}\,\mathbf{A}\,\mathbf{Q}^T + \gamma_2\,[\mathbf{A}]\,\left(\mathbf{Q}\,\mathbf{A}\,\mathbf{Q}^T\right)\left(\mathbf{Q}\,\mathbf{A}\,\mathbf{Q}^T\right) \quad \Rightarrow\\ &= \mathbf{Q}\left(\gamma_0\,[\mathbf{A}]\,\mathbf{1} + \gamma_1\,[\mathbf{A}]\,\mathbf{A} + \gamma_2\,[\mathbf{A}]\,\mathbf{A}^2\right)\mathbf{Q}^T \quad \Rightarrow\\ \mathbf{B}\left[\mathbf{Q}\,\mathbf{A}\,\mathbf{Q}^T\right] &= \mathbf{Q}\,\mathbf{B}\,[\mathbf{A}]\,\mathbf{Q}^T \quad \Rightarrow (4.6.14)\end{aligned}$$

Proof that: (4.6.14) $\Rightarrow$ (4.6.17). Theorem 4.1 and (4.6.14) imply that the principal values β_k of $\mathbf{B}$ by the function $\mathbf{B}[\mathbf{A}]$ are functions of the principal values α_k of the argument tensor $\mathbf{A}$.

$$\beta_k = \beta_k\,(\alpha) \equiv \beta_k\,(\alpha_1, \alpha_2, \alpha_3) \tag{4.6.18}$$

Let us first assume that the principal values α_k are all unequal. The following three equations for the three unknown scalars γ_0, γ_1, and γ_2 will then have a unique solution.

$$\gamma_0 + \gamma_1\,\alpha_k + \gamma_2\,(\alpha_k)^2 = \beta_k \quad k = 1,2,3 \tag{4.6.19}$$

where β_k are given by (4.6.18). A unique solution of (4.6.19) is secured because the determinant of the coefficient matrix is different from zero:

$$\det\begin{pmatrix}1 & \alpha_1 & (\alpha_1)^2\\ 1 & \alpha_2 & (\alpha_2)^2\\ 1 & \alpha_3 & (\alpha_3)^2\end{pmatrix} = (\alpha_1 - \alpha_2)\,(\alpha_2 - \alpha_3)\,(\alpha_3 - \alpha_1) \neq 0 \tag{4.6.20}$$

The scalars γ_i are according to (4.6.19) functions of the principal values α_k, alternatively of the principal invariants *I, II*, and *III* of $\mathbf{A}$. From the general representation (4.6.16) of a 2. order tensor we now obtain:

$$\mathbf{B} = \sum_k \beta_k \mathbf{a}_k \otimes \mathbf{a}_k = \gamma_0 \sum_k \mathbf{a}_k \otimes \mathbf{a}_k + \gamma_1 \sum_k \alpha_k \mathbf{a}_k \otimes \mathbf{a}_k + \gamma_2 \sum_k (\alpha_k)^2 \mathbf{a}_k \otimes \mathbf{a}_k$$

which may be organized to:

$$\mathbf{B}[\mathbf{A}] = \gamma_0 \mathbf{1} + \gamma_1 \mathbf{A} + \gamma_2 \mathbf{A}^2 \tag{4.6.21}$$

If any two principal values are equal, say $\alpha_2 = \alpha_3$, any direction $\mathbf{a}$ normal to the principal direction $\mathbf{a}_1$ is then a principal direction of $\mathbf{A}$, confer Sect. 3.3.1, and according to Theorem 4.1 also of $\mathbf{B}$. This means that $\beta_2 = \beta_3$. The following two equations for the two unknown scalar-valued functions ϕ_0 and ϕ_1 of α_k have a unique solution:

$$\phi_0 + \phi_1 \alpha_\rho = \beta_\rho \quad \rho = 1,2 \tag{4.6.22}$$

The representation (4.6.16) now yields:

$$\mathbf{B}[\mathbf{A}] = \phi_0 \mathbf{1} + \phi_1 \mathbf{A} \tag{4.6.23}$$

If all three principal values α_k are equal: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, the argument tensor $\mathbf{A}$ and thus the function tensor $\mathbf{B}$ are both isotropic tensors, and we shall find:

$$\mathbf{B}[\mathbf{A}] = \psi_0 \mathbf{1} \quad \text{for } \mathbf{A} = \alpha \mathbf{1} \tag{4.6.24}$$

where ψ_0 is a function of α. The (4.6.21–4.6.24) show that (4.6.14) implies (4.6.17). This completes the proof of Theorem 4.2.

It may be shown that, see Problem 4.11:

$$\phi_0 = \gamma_0 - \alpha_1 \alpha_2 \gamma_2 \quad , \quad \phi_1 = \gamma_1 + (\alpha_1 + \alpha_2)\gamma_2 \quad \text{for } \alpha_2 = \alpha_3 \tag{4.6.25}$$

$$\psi_0 = \gamma_0 + \gamma_1 \alpha + \gamma_2 \alpha^2 \quad \text{for } \alpha_1 = \alpha_2 = \alpha_3 = \alpha \tag{4.6.26}$$

The proof of Theorem 4.2 is due to Serrin [41], who also shows that if the function $\mathbf{B}[\mathbf{A}]$ is three times differentiable with respect to $\mathbf{A}$, then the scalars γ_i are continuous functions of the principal invariants of $\mathbf{A}$.

An alternative form of the function (4.6.17) may be derived using the Cayley-Hamilton-theorem (4.3.41), which gives:

$$\mathbf{A}^2 = I\,\mathbf{A} - II\,\mathbf{1} + III\,\mathbf{A}^{-1}$$

When this expression for $\mathbf{A}^2$ is substituted into the function (4.6.17), we obtain:

$$\mathbf{B}[\mathbf{A}] = \lambda_0 \mathbf{1} + \lambda_1 \mathbf{A} + \lambda_{-1} \mathbf{A}^{-1} \tag{4.6.27}$$

where λ_i are isotropic scalar-valued functions of $\mathbf{A}$.

$$\lambda_0 = \gamma_0 - II\,\gamma_2 \quad , \quad \lambda_1 = \gamma_1 + I\,\gamma_2 \quad , \quad \lambda_{-1} = III\,\gamma_2 \tag{4.6.28}$$

If $\mathbf{B}[\mathbf{A}]$ is a linear function of $\mathbf{A}$, it follows from the general expression (4.6.17) that the function takes the form:

$$\mathbf{B}[\mathbf{A}] = (\gamma + \lambda\,\mathrm{tr}\mathbf{A})\,\mathbf{1} + 2\mu\,\mathbf{A} \tag{4.6.29}$$

where γ, λ, and μ are constants. An alternative form of the expression (4.6.29) is:

$$\mathbf{B}[\mathbf{A}] = \gamma\mathbf{1} + \mathbf{I}_4^s : \mathbf{A} \tag{4.6.30}$$

where $\mathbf{I}_4^s$ is the 4. order isotropic tensor presented in formula (4.2.40).

$$\mathbf{I}_4^s = 2\mu\,\mathbf{1}_4^s + \lambda\,\mathbf{1}\otimes\mathbf{1} \quad \Leftrightarrow \quad I_{4ijkl}^s = \mu\left(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}\right) + \lambda\,\delta_{ij}\,\delta_{kl} \tag{4.6.31}$$

If each of the two symmetric tensors $\mathbf{A}$ and $\mathbf{B}$ are decomposed into trace-free deviators: $\mathbf{A}'$ and $\mathbf{B}'$, and isotrops: $\mathbf{A}^{\mathrm{o}}$ and $\mathbf{B}^{\mathrm{o}}$, as shown by (4.3.44, 4.3.45, 4.3.46), the linear function (4.6.29) may be decomposed into:

$$\mathbf{B}' = 2\mu\,\mathbf{A}' \quad , \quad \mathbf{B}^o = 3\kappa\mathbf{A}^o + \gamma\mathbf{1} \tag{4.6.32}$$

where:

$$\kappa = \lambda + \frac{2}{3}\mu \tag{4.6.33}$$

The derivation of the result (4.6.32) is given as Problem 4.17. From the expressions (4.6.32) we get the alternative form of the function (4.6.29):

$$\mathbf{B}[\mathbf{A}] = 2\mu\mathbf{A} + \left(\kappa - \frac{2}{3}\mu\right)(\mathrm{tr}\mathbf{A})\,\mathbf{1} + \gamma\mathbf{1} \tag{4.6.34}$$

The linear functions (4.6.29, 4.6.30, 4.6.31, 4.6.32, 4.6.33, 4.6.34) are very important in constitutive modelling of linear, isotropic materials. An example will be the generalized Hooke's law for isotropic linearly elastic materials, presented in Sect. 7.2.

Problems

Problem 4.1. Show that a completely antisymmetric tensor of 3. order only has one distinct component different from zero, and that the tensor is represented by the product of a scalar and the permutation tensor.

Problem 4.2. Show that an isotropic tensor of 2. order always is a product of a scalar and the unit tensor as given by formula (4.2.7).

Hint: Transform the tensor matrix from the Ox-system to two new coordinate systems: $\bar{O}\bar{x}$, which results from a 90°-rotation of Ox about the x_3 - axis, and $\tilde{O}\tilde{x}$, resulting from a 90°-rotation about the x_1-axis. Equate the tensor matrices in the three coordinate systems. Then show that all elements outside of the main diagonal are zero, and that all the diagonal elements are equal.

Problem 4.3. The product of a scalar and the permutation tensor is an isotropic tensor of 3. order.

a) Show that this is correct.
b) Prove that this tensor represents the general isotropic of 3. order.

Hint: Transform the tensor matrix from the Ox-system to three new coordinate systems with base vectors:

$$\begin{aligned}
&1)\ \bar{\mathbf{e}}_1 = -\mathbf{e}_1 \quad , \quad \bar{\mathbf{e}}_2 = -\mathbf{e}_2 \quad , \quad \bar{\mathbf{e}}_3 = \mathbf{e}_3 \\
&2)\ \bar{\mathbf{e}}_1 = \mathbf{e}_2 \quad , \quad \bar{\mathbf{e}}_2 = \mathbf{e}_3 \quad , \quad \bar{\mathbf{e}}_3 = \mathbf{e}_1 \\
&3)\ \bar{\mathbf{e}}_1 = -\mathbf{e}_3 \quad , \quad \bar{\mathbf{e}}_2 = \mathbf{e}_2 \quad , \quad \bar{\mathbf{e}}_3 = \mathbf{e}_1
\end{aligned}$$

Equate the tensor matrices in the four coordinate systems. Then show that all elements outside of the main diagonal are zero, and that all the diagonal elements are equal.

Problem 4.4. Show that the third trace invariant $\tilde{I}\tilde{I}\tilde{I} = \mathrm{tr}\mathbf{S^3}$ may be expressed by

$$\tilde{I}\tilde{I}\tilde{I} = \mathrm{tr}\mathbf{S^3} = 3III + I^3 - 3I \cdot II$$

as presented in the formulas (4.3.43).

Problem 4.5. Show by induction that the result (4.3.38) follows from formula (4.3.36).

Problem 4.6. Show that the definition (4.3.39) of a positive definite and symmetric 2. order tensor implies that all the principal values are positive. From this result it also follows that the principal invariants of the tensor are positive.

Problem 4.7. Let $\mathbf{S}$ be a symmetric tensor of 2. order and $\mathbf{c}$ any vector. The normal component σ of the tensor $\mathbf{S}$ for the direction parallel to $\mathbf{c}$, may be calculated from the expression:

$$\sigma = \frac{\mathbf{c} \cdot \mathbf{S} \cdot \mathbf{c}}{\mathbf{c} \cdot \mathbf{c}}$$

The extremum values of σ, in other words the principal values of $\mathbf{S}$, may be determined from the three equations:

$$\frac{\partial \sigma}{\partial c_i} = 0$$

Show that this set of equations yields the set of (4.3.29).

Problem 4.8. Let $\mathbf{S}$ be the symmetric part and $\mathbf{A}$ be the antisymmetric part of a 2. order tensor $\mathbf{B}$. With $\mathbf{P}$ representing the permutation tensor, show that:

$$\mathbf{P} : \mathbf{S} = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{P} : \mathbf{B} = \mathbf{P} : \mathbf{A} \quad \Leftrightarrow \quad e_{ijk} B_{jk} \equiv e_{ijk} B_{[jk]} \equiv e_{ijk} A_{jk}$$

Problem 4.9. Derive the formula (4.5.20).

Problem 4.10. A rigid-body-motion about the origin O in an Ox-system is given by:

$$\mathbf{r} = \mathbf{R} \cdot \mathbf{r}_o \quad , \quad R = \begin{pmatrix} 0 & 2\sqrt{3} & 2 \\ 2\sqrt{3} & 1 & -\sqrt{3} \\ -2 & \sqrt{3} & -3 \end{pmatrix} \frac{1}{4}$$

Check that $\mathbf{R}$ is an orthogonal tensor. Determine the axis of rotation and the angle of rotation.

Problem 4.11. Derive the relations (4.6.25), (4.6.26), and (4.6.28).

Problem 4.12. Let $\mathbf{B}[\mathbf{A}]$ be the function (4.6.34). Determine the inverse function $\mathbf{A}[\mathbf{B}]$.

Problem 4.13. Let $\mathbf{A}$ be an orthogonal tensor with axis of rotation $\mathbf{a}$ and angle of rotation ϕ. Show that any $\mathbf{Q}$-rotation of $\mathbf{A}$ is an orthogonal tensor $\mathbf{B} = \mathbf{QAQ}^T$ with the same angle of rotation and with axis of rotation $\mathbf{b} = \mathbf{Q} \cdot \mathbf{a}$.

Problem 4.14. Let $\mathbf{A}$ be a symmetric tensor of 2. order with principal values α_k and principal directions $\mathbf{a}_k$. Show that any $\mathbf{Q}$-rotation of $\mathbf{A}$ is a symmetric 2. order tensor $\mathbf{B} = \mathbf{QAQ}^T$ with principal values α_k and principal directions $\mathbf{b}_k = \mathbf{Q} \cdot \mathbf{a}_k$.

Problem 4.15. Derive formula (4.5.31).

Problem 4.16. Use the Cayley-Hamilton-theorem (4.3.41) to show that:

$$III = \frac{1}{3}\left(\mathrm{tr}\,\mathbf{S}^3 - I\,\mathrm{tr}\,\mathbf{S}^2 + II\,\mathrm{tr}\,\mathbf{S}\right) = \frac{1}{3}\left(\mathrm{tr}\,\mathbf{S}^3 - \frac{2}{3}\,\mathrm{tr}\,\mathbf{S}^2\,\mathrm{tr}\,\mathbf{S} + \frac{1}{2}\left(\mathrm{tr}\,\mathbf{S}\right)^3\right)$$

Problem 4.17. Derive the relations (4.6.32) from the relation (4.6.29).

Problem 4.18. Derive the formula for the divergence of a 2. order tensor field $\mathbf{A}(\mathbf{r},t)$ in cylindrical coordinates. Then present the Cauchy-equations in cylindrical coordinates. The formulas (3.2.39–3.2.41) give the answers.

Chapter 5
Deformation Analysis

5.1 Strain Measures

The word strain is used about local deformation in a material, i.e. deformation in the neighborhood of a particle. Strain represents change in material lines, angles, and volume. Below we define three primary concepts of strain: *longitudinal strain* ε, *shear strain* γ, and *volumetric strain* ε_v. Strains are primarily due to mechanical stress and temperature changes in the material. But strain may also have contributions from other effects. For instance, changes in the water content in wood and in some plastics leads to swelling or shrinking, which may introduce both strains and stresses in the material.

Figure 5.1.1 shows a body in the reference configuration K_o at time t_o and in the present configuration K at time t. The body is assumed to be undeformed in the reference configuration. The motion of the body is given by the place vector $\mathbf{r}(\mathbf{r}_o, t)$, where $\mathbf{r}_o$ represent an arbitrarily chosen particle. Alternatively the motion may be given by the displacement vector $\mathbf{u}(\mathbf{r}_o, t)$:

$$\mathbf{u}(\mathbf{r}_o, t) = \mathbf{r}(\mathbf{r}_o, t) - \mathbf{r}_o \tag{5.1.1}$$

At the particle $\mathbf{r}_\text{o}$ we select a material line element, which in K_o is a straight line of length $s_o = P_oQ_o$, and with a direction given by the unit vector $\mathbf{e}$. In K the line element will in general have changed its length, which is denoted by $s = PQ$, and also have got a curved form. The *longitudinal strain* ε *in the direction* $\mathbf{e}$ *in a particle* $\mathbf{r}_o$ is defined by:

$$\varepsilon = \lim_{s_o \to 0} \frac{s - s_o}{s_o} = \frac{ds - ds_o}{ds_o} = \frac{ds}{ds_o} - 1 \tag{5.1.2}$$

The strain ε represents change of length per unit length of undeformed line element in the direction of $\mathbf{e}$ in particle $\mathbf{r}_o$. The longitudinal strain is also called the *normal strain*.

The *shear strain* γ *in a particle* $\mathbf{r}_\text{o}$ *with respect to two material elements*, which in K_o are orthogonal, is defined as the change in the right angle between the line

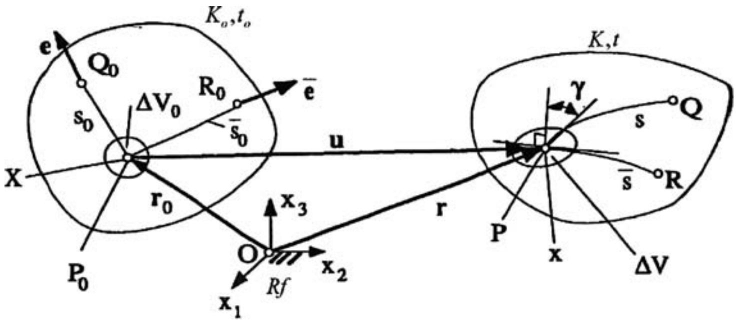


Fig. 5.1.1 General deformation of a body

elements, and measured in radians. By definition the shear strain is positive when the angle is reduced. Figure 5.1.1 illustrates the shear strain γ in $\mathbf{r}_o$ between material line elements that in K_o have the directions $\mathbf{e}$ and $\bar{\mathbf{e}}$.

The *volumetric strain* ε_v *in a particle* $\mathbf{r}_o$ is defined by:

$$\varepsilon_v = \lim_{\Delta V_o \to 0} \frac{\Delta V - \Delta V_o}{\Delta V_o} \tag{5.1.3}$$

see Fig. 5.1.1. ε_v represents change in volume per unit undeformed volume about the particle $\mathbf{r}_o$.

5.2 The Green Strain Tensor

In this section the primary measures of strain, introduced above, will be expressed in terms of the displacement vector $\mathbf{u}$. We look at the situation illustrated in Fig. 5.1.1 and here again presented in Fig. 5.2.1. The length s_o of the line element P_oQ_o is now chosen to be a curve parameter for the straight material line P_oQ_o and also for the deformed material line PQ in K. The length s of the deformed line PQ then becomes a function of s_o. The unit vector $\mathbf{e}$ in the direction P_oQ_o has the components e_i. The coordinates of the points Q_o and Q are $X_i + s_o\, e_i$ and $x_i(X + s_o e, t)$ respectively. We may now consider x_i as functions of X_i and s_o. With s_o as curve parameter, the arc length formula gives for the length s of the line PQ:

$$s(s_o) = \int_0^{s_o} \sqrt{\frac{\partial x_i}{\partial \bar{s}_o} \frac{\partial x_i}{\partial \bar{s}_o}}\, d\bar{s}_o \tag{5.2.1}$$

from which we obtain:

$$\frac{ds}{ds_o} = \sqrt{\frac{\partial x_i}{\partial s_o} \frac{\partial x_i}{\partial s_o}} \quad \Rightarrow \quad (ds)^2 = \left(\frac{\partial x_i}{\partial s_o} \frac{\partial x_i}{\partial s_o}\right) (ds_o)^2 \tag{5.2.2}$$

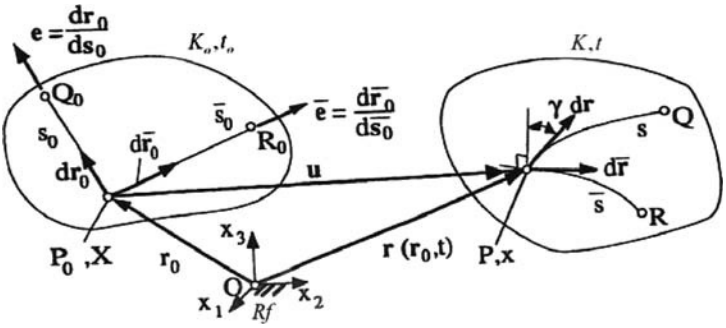


Fig. 5.2.1 General deformation of a body

Let ds_o be the length of the line element $d\mathbf{r}_o$ in P_o and along P_oQ_o, and let dX_k the components of $d\mathbf{r}_o$:

$$d\mathbf{r}_o = \mathbf{e}\,ds_o = dX_k\,\mathbf{e}_k \quad \Rightarrow \quad |d\mathbf{r}_o| = ds_o,\; dX_k = e_k\,ds_o \quad \Leftrightarrow \quad e_k = \frac{dX_k}{ds_o} \tag{5.2.3}$$

Note that e_k are components of the vector $\mathbf{e}$, while $\mathbf{e}_k$ are the base vector of the coordinate system: $\mathbf{e} = e_k\mathbf{e}_k$. The vector differential $d\mathbf{r}$, with components dx_i, represents a line element in P, which is a tangent vector to the curve PQ and defined by:

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial s_o} ds_o \quad \Rightarrow \quad dx_i = \frac{\partial x_i}{\partial s_o} ds_o \tag{5.2.4}$$

From (5.2.2) and (5.2.4) it follows that ds is the length of the differential line element $d\mathbf{r}$:

$$|d\mathbf{r}| = ds \tag{5.2.5}$$

The differential line elements $d\mathbf{r}_o$ and $d\mathbf{r}$ do not in general represent the same material line. That is only possible if the curve PQ is a straight line in the neighborhood of point P. However, it may be shown that PQ always is a straight line if the deformation of the body from K_o to K is a *homogeneous deformation*, see Problem 5.12.

The relation between the line elements $d\mathbf{r}_o$ in K_o and $d\mathbf{r}$ in K is determined as follows.

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \mathbf{r}_o} \cdot d\mathbf{r}_o \quad \Leftrightarrow \quad dx_i = \frac{\partial x_i}{\partial X_k} dX_k \tag{5.2.6}$$

The coordinate invariant form $\partial\mathbf{r}/\partial\mathbf{r}_o$ in the term to the left of the biimplication sign is a symbol for a tensor: Grad $\mathbf{r}$, with components $\partial x_i/\partial X_k$. This tensor is called the *deformation gradient* and denoted by $\mathbf{F}$.

$$\mathbf{F} = \operatorname{Grad}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \mathbf{r}_o} \quad \Leftrightarrow \quad F_{ik} = \frac{\partial x_i}{\partial X_k} \tag{5.2.7}$$

The relation (5.2.6) is now written as:

$$d\mathbf{r} = \mathbf{F} \cdot d\mathbf{r}_o \quad \Leftrightarrow \quad dx_i = F_{ik}\, dX_k \tag{5.2.8}$$

From (5.2.7) and (5.2.3) we get:

$$\left.\frac{\partial x_i\,(X + s_o e, t)}{\partial s_o}\right|_{s_o=0} = \frac{\partial x_i\,(X,t)}{\partial X_k}\left.\frac{d\,(X_k + s_o e_k)}{\partial s_o}\right|_{s_o=0} \Rightarrow \frac{\partial x_i}{\partial s_o} = \frac{\partial x_i}{\partial X_k}\frac{dX_k}{ds_o} = F_{ik}\, e_k$$

Thus we have from (5.2.2):

$$\left(\frac{ds}{ds_o}\right)^2 = \frac{\partial x_i}{\partial s_o} \cdot \frac{\partial x_i}{\partial s_o} = (F_{ik}\, e_k)\,(F_{il}\, e_l) = \mathbf{e} \cdot \left(\mathbf{F}^T\mathbf{F}\right) \cdot \mathbf{e} = \mathbf{e} \cdot \mathbf{C} \cdot \mathbf{e} \tag{5.2.9}$$

where $\mathbf{C}$ is a symmetric 2. order tensor called the *Green deformation tensor*, named after George Green [1793–1841].

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} \quad \Leftrightarrow \quad C_{kl} = F_{ik}F_{il} \tag{5.2.10}$$

From Fig. 5.2.1 we see that:

$$\mathbf{r} = \mathbf{r}\,(\mathbf{r}_o,t) = \mathbf{r}_o + \mathbf{u}\,(\mathbf{r}_o,t) \quad \Leftrightarrow \quad x_i\,(X,t) = X_i + u_i\,(X,t) \tag{5.2.11}$$

$\mathbf{u} = \mathbf{u}(\mathbf{r}_o,t)$ is the displacement of the particle P. We need to introduce the *displacement gradients* H_{ik}, which represent a tensor $\mathbf{H}$ called the *displacement gradient tensor*. The tensor is defined by:

$$\mathbf{H} = \frac{\partial \mathbf{u}}{\partial \mathbf{r}_o} \quad \Leftrightarrow \quad H_{ik} = \frac{\partial u_i}{\partial X_k} \tag{5.2.12}$$

From (5.2.11) it follows that:

$$\frac{\partial x_i}{\partial X_k} = \frac{\partial X_i}{\partial X_k} + \frac{\partial u_i}{\partial X_k} \quad \Leftrightarrow \quad F_{ik} = \delta_{ik} + H_{ik} \quad \Leftrightarrow \quad \mathbf{F} = \mathbf{1} + \mathbf{H} \tag{5.2.13}$$

Furthermore:

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \left(\mathbf{1} + \mathbf{H}^T\right)(\mathbf{1} + \mathbf{H}) = \mathbf{1} + \mathbf{H} + \mathbf{H}^T + \mathbf{H}^T\mathbf{H}$$

We define the *Green strain tensor* by:

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T\mathbf{H}\right) \quad \Leftrightarrow \quad E_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial X_l} + \frac{\partial u_l}{\partial X_k} + \frac{\partial u_i}{\partial X_k}\frac{\partial u_i}{\partial X_l}\right) \tag{5.2.14}$$

such that:

$$\mathbf{C} = \mathbf{1} + 2\mathbf{E} \tag{5.2.15}$$

The result (5.2.9) may rewritten to:

$$\left(\frac{ds}{ds_o}\right)^2 = \mathbf{1} + 2\mathbf{e} \cdot \mathbf{E} \cdot \mathbf{e} \tag{5.2.16}$$

We are now ready to express the primary strain measures ε, γ, and ε_v, presented in the previous section, in terms of the displacement vector or related quantities. According to (5.1.1) and (5.2.16) the longitudinal strain ε in direction $\mathbf{e}$ is given by:

$$\varepsilon = \frac{ds}{ds_o} - 1 = \sqrt{1 + 2\mathbf{e} \cdot \mathbf{E} \cdot \mathbf{e}} - 1 \tag{5.2.17}$$

In particular the longitudinal strain ε_{ii} (no summation) of a material line element that in K_o is parallel to the x_i-direction, is found from the formula (5.2.17) if $\mathbf{e}$ is chosen to be $\mathbf{e}_i$.

$$\varepsilon_{ii} = \sqrt{1 + 2E_{ii}} - 1 \quad \text{(no summation)} \tag{5.2.18}$$

In order to determine the shear strain γ with respect to two material line elements, which in K_o are orthogonal and have directions $\bar{\mathbf{e}}$ and $\mathbf{e}$, see Fig. 5.2.1, we compute the scalar product of the tangent vectors $d\bar{\mathbf{r}}$ of PR and $d\mathbf{r}$ of PQ. The vectors $d\mathbf{r}$ and $d\bar{\mathbf{r}}$ are given by (5.2.8), (5.2.13), and (5.2.3) as:

$$\begin{aligned} d\mathbf{r} &= \mathbf{F} \cdot d\mathbf{r}_o = (\mathbf{1} + \mathbf{H}) \cdot d\mathbf{r}_o = (\mathbf{1} + \mathbf{H}) \cdot \mathbf{e}\, ds_o \\ d\bar{\mathbf{r}} &= \mathbf{F} \cdot d\bar{\mathbf{r}}_o = (\mathbf{1} + \mathbf{H}) \cdot d\bar{\mathbf{r}}_o = (\mathbf{1} + \mathbf{H}) \cdot \bar{\mathbf{e}}\, d\bar{s}_o \end{aligned} \tag{5.2.19}$$

We now get:

$$d\bar{\mathbf{r}} \cdot d\mathbf{r} = d\bar{s}_o \bar{\mathbf{e}} \cdot \left(\mathbf{1} + \mathbf{H}^T\right)(\mathbf{1} + \mathbf{H}) \cdot \mathbf{e}\, ds_o = 2\bar{\mathbf{e}} \cdot \mathbf{E} \cdot \mathbf{e}\, d\bar{s}_o\, ds_o$$

By definition of the scalar product of two vectors this is also equal to $|d\bar{\mathbf{r}}| \cdot |d\mathbf{r}| \sin\gamma$. Equation (5.2.16) gives:

$$|d\mathbf{r}| = ds = \sqrt{1 + 2\mathbf{e} \cdot \mathbf{E} \cdot \mathbf{e}}\, ds_o, \; |d\bar{\mathbf{r}}| = d\bar{s} = \sqrt{1 + 2\bar{\mathbf{e}} \cdot \mathbf{E} \cdot \bar{\mathbf{e}}}\, d\bar{s}_o$$

The shear strain γ may now be expressed by:

$$\sin\gamma = \frac{2\bar{\mathbf{e}} \cdot \mathbf{E} \cdot \mathbf{e}}{\sqrt{(1 + 2\bar{\mathbf{e}} \cdot \mathbf{E} \cdot \bar{\mathbf{e}})(1 + 2\mathbf{e} \cdot \mathbf{E} \cdot \mathbf{e})}} = \frac{2\bar{\mathbf{e}} \cdot \mathbf{E} \cdot \mathbf{e}}{(1 + \bar{\varepsilon})(1 + \varepsilon)} \tag{5.2.20}$$

where $\bar{\varepsilon}$ is the longitudinal strain in the direction $\bar{P}\bar{R}$. The shear strain γ_{ij} for any two material line elements that in K_o have directions $\bar{\mathbf{e}} = \mathbf{e}_i$ and $\mathbf{e} = \mathbf{e}_j$, is:

$$\sin\gamma_{ij} = \frac{2E_{ij}}{\sqrt{(1 + 2E_{ii})(1 + 2E_{jj})}} = \frac{2E_{ij}}{(1 + \varepsilon_{ii})(1 + \varepsilon_{jj})} \quad i \neq j \tag{5.2.21}$$

The longitudinal strains ε_{ii}, $i = 1,\ 2$, or 3, from (5.2.18) and the shear strains γ_{ij} from (5.2.21) will be called *coordinate strains*.

The volumetric strain ε_v may be determined as follows. Figure 5.2.1 shows a material element about a particle P (or $\mathbf{r}_o$), undeformed in K_o and deformed in K. The volume dV_o of the undeformed element is:

$$dV_o = dX_1\, dX_2\, dX_3 \tag{5.2.22}$$

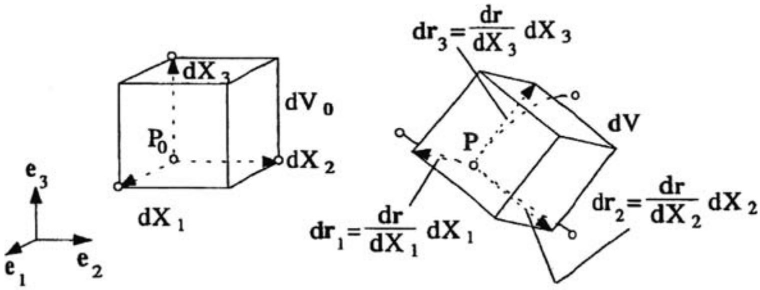


Fig. 5.2.2 Deformation of a material element of volume

The volume dV of the deformed element is given by the box product of the three vectors $d\mathbf{r}_i$ in Fig. 5.2.2. By (5.2.4):

$$d\mathbf{r}_1 = \frac{\partial \mathbf{r}}{\partial X_1} dX_1 = \frac{\partial x_i}{\partial X_1} \mathbf{e}_i dX_1 \quad \text{etc.}$$

Therefore:

$$dV = [d\mathbf{r}_1 \, d\mathbf{r}_2 \, d\mathbf{r}_3] = e_{ijk} \frac{\partial x_i}{\partial X_1} dX_1 \frac{\partial x_j}{\partial X_2} dX_2 \frac{\partial x_k}{\partial X_3} dX_3 = \det \mathbf{F} \, dV_o = \det(\mathbf{1} + \mathbf{H}) \, dV_o$$

The volumetric strain is then determined from the definition in (5.1.3):

$$\varepsilon_v = \frac{dV - dV_o}{dV_o} = \det \mathbf{F} - 1 = \det(\mathbf{1} + \mathbf{H}) - 1 = \sqrt{\det(\mathbf{1} + 2\mathbf{E})} - 1 \qquad (5.2.23)$$

It has now been demonstrated that the three strain measures ε, γ, and ε_v are all determined by the strain tensor **E**, which again is determined by the deformation gradient **F** or the displacement gradient **H**. As we shall see in the following chapters, most of the material models in Continuum Mechanics are defined by constitutive equations relating the stress tensor **T** and the strain tensor **E** and other measures of deformations derived from the deformation tensor **F** or the displacement gradient **H** through some functions or functionals.

A quantity related to longitudinal strain is the *stretch* λ defined by:

$$\lambda = \frac{ds}{ds_o} = 1 + \varepsilon \qquad (5.2.24)$$

The stretch is thus the ratio between the length of a deformed material line element to the length of the undeformed element. The stretch is always positive.

The stretch of a material line element that in K_o is parallel to the x_i-direction is according to the definition (5.2.24) and the equation (5.2.18):

$$\lambda_{ii} = \sqrt{1 + 2E_{ii}} \quad \text{(no summation)} \qquad (5.2.25)$$

and will be called a *coordinate stretch*.

All deformation effects and mechanical response related to the deformation tensor **F** or the displacement gradient tensor **H**, are most clearly exposed when we consider *homogeneous deformation*, i.e. the deformation gradient **F** and thus also the displacement gradient **H** are the same for all particles in the body. Homogeneous deformations are given by the special motion: $\mathbf{r}(\mathbf{r}_o, t)$ or $\mathbf{u}(\mathbf{r}_o, t)$:

$$\mathbf{r} = \mathbf{u}_o + \mathbf{F} \cdot \mathbf{r}_o \quad \Leftrightarrow \quad x_i = u_{oi} + F_{ik} X_k \tag{5.2.26}$$

$$\mathbf{u} = \mathbf{r} - \mathbf{r}_o = \mathbf{u}_o + \mathbf{H} \cdot \mathbf{r}_o \quad \Leftrightarrow \quad u_i = x_i - X_i = u_{oi} + H_{ik} X_k \tag{5.2.27}$$

where:

$$\mathbf{u}_o = \mathbf{u}_o(t), \ \mathbf{F} = \mathbf{F}(t), \ \mathbf{H} = \mathbf{H}(t) \tag{5.2.28}$$

The displacement vector $\mathbf{u}_\mathrm{o}$ represents a *translation* of the body. It may be shown, see Problem 5.12, that in a homogeneous deformation material planes and straight lines in K_o deform into planes and straight lines in K. In Sect. 5.5.1 we shall discuss special types of homogeneous deformations.

5.3 Small Strains and Small Deformations

In most applications of structural materials like steel, aluminum, concrete, and wood the strains are small. For instance, elastic strains in mild steel under uniaxial stress are less than 0.001, see Fig. 1.2.2. We will characterize a state in which the absolute values of all longitudinal strains are less than 0.01, as a state of *small strains*. According to (5.2.20) and (5.2.23) this implies that the absolute value of all shear strains is less than 0.02, and that the absolute value of the volumetric strain is less than 0.03. The analysis of small strains is significantly simpler than that for large strains. Small strains do not imply that the displacements are small or that rotations of line element will be small. The following example will illustrate this.

Figure 5.3.1a shows a thin steel bar AB fixed at one end and loaded on the free end B by a force F. Due to bending the bar experiences large elastic deformation, but as long as the bar remains elastic, the strain does not exceed 0.001. The displacements u_1 and u_2 of the end B may be of the magnitude as the length L of the bar. The angle of rotation ϕ_B of the tangent of the bar axis at B can be near 90°. Thus this is a case of *small strains and large deformations*.

In ordinary engineering structures the displacements of particles and the rotations of line elements will be small quantities. Measured in radians we may assume the angle of rotation to be much less than 1. The displacement will be considerably less than a characteristic length of the structure, for instance the length of a beam. This means that the change of the geometry of the structure, due to the deformation, need not to be taken into account when the action of the forces is considered. If both the strains and rotations are small, we use the expression *state of small deformation*.

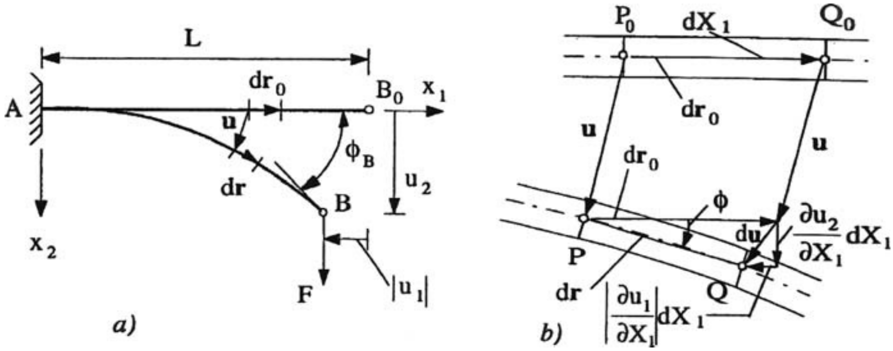


Fig. 5.3.1 Small strains and large deformations

5.3.1 Small Strains

We assume small strains, i.e. $\varepsilon < 0.01$. From formula (5.2.17) for the longitudinal strain ε in the direction $\mathbf{e}$ we obtain:

$$(\varepsilon+1)^2 = \left(\sqrt{1+2\mathbf{e}\cdot\mathbf{E}\cdot\mathbf{e}}\right)^2 \quad\Rightarrow\quad \varepsilon^2+2\varepsilon+1 = 1+2\mathbf{e}\cdot\mathbf{E}\cdot\mathbf{e} \quad\Rightarrow$$

$$\varepsilon = \mathbf{e}\cdot\mathbf{E}\cdot\mathbf{e} = e_i E_{ij} e_j \tag{5.3.1}$$

The term ε^2 has been neglected when compared with the term ε. In formula (5.2.20) for the shear strain γ with respect to the two orthogonal directions $\bar{\mathbf{e}}$ and $\mathbf{e}$ we set $\sin\gamma \approx \gamma$ and replace the denominator by 1. Then we get:

$$\gamma = 2\bar{\mathbf{e}}\cdot\mathbf{E}\cdot\mathbf{e} = 2\bar{e}_i E_{ij} e_j \tag{5.3.2}$$

For longitudinal strains and shear strains with respect to material line elements that in K_o are parallel to the coordinate axes, we find:

$$\varepsilon_{ii} = E_{ii} \quad \text{(no summation)}, \ \gamma_{ij} = 2E_{ij} \quad i \neq j \tag{5.3.3}$$

These strain components are *coordinate strains* for small strains. Sometimes it is convenient to also call the tensor components E_{ij} in the case of small strains for coordinate strains. Because all the coordinate strains E_{ij} are small quantities we get from (5.2.23):

$$(\varepsilon_v+1)^2 = \det(\mathbf{1}+2\mathbf{E}) \quad\Rightarrow$$

$$\varepsilon_v^2+2\varepsilon_v+1 = 1+2(E_{11}+E_{22}+E_{33}) + \text{products of the } E_{ij} \quad\Rightarrow$$

$$\varepsilon_v = E_{11}+E_{22}+E_{33} = E_{ii} = \operatorname{tr}\mathbf{E} \tag{5.3.4}$$

5.3.2 Small Deformations

In order to introduce a basis for definition of what we shall mean by small deformations, we shall consider the deformation of a line element PQ along the beam AB in Fig. 5.3.1a. The undeformed element P_oQ_o and the deformed element PQ are shown i Fig. 5.3.1b. It follows from the figure that for the angle of rotation ϕ and the strain of the element to be small, we have to require that:

$$\left|\frac{\partial u_2}{\partial X_1}\right| \equiv |H_{21}| \ll 1 \quad \text{and} \quad \left|\frac{\partial u_1}{\partial X_1}\right| \equiv |H_{11}| \ll 1$$

In the general case we define small deformations by the condition that all displacement gradients H_{ik} in absolute values must be small compared with 1:

$$\text{small deformations} \quad \Leftrightarrow \quad \left|H_{ij}\right| \equiv \left|\frac{\partial u_i}{\partial X_j}\right| \ll 1 \quad \Leftrightarrow \quad \text{norm}\,\mathbf{H} \ll 1 \tag{5.3.5}$$

Small deformations imply small strains and small rotations. In the literature the term *infinitesimal deformations* is sometimes used for what is here called small deformations.

Under the assumption of small deformations we get the following result for an arbitrary field $f(\mathbf{r},t) = f(\mathbf{r}(\mathbf{r}_o,t),t)$, using (5.2.11).

$$\frac{\partial f}{\partial X_i} = \frac{\partial f}{\partial x_k}\frac{\partial x_k}{\partial X_i} = \frac{\partial f}{\partial x_k}\left(\delta_{ki} + \frac{\partial u_k}{\partial X_i}\right) \approx \frac{\partial f}{\partial x_i} \equiv f_{,i} \tag{5.3.6}$$

Usually small deformations imply small displacements, and we may replace the particle reference $\mathbf{r}_o$ by the place vector $\mathbf{r}$, and use the place coordinates x_i as particle coordinates rather than X_i. For the same reason we will set:

$$\dot{f}(\mathbf{r},t) = \dot{f}(x,t) = \partial_t f(\mathbf{r},t) \equiv \frac{\partial f(\mathbf{r},t)}{\partial t} \tag{5.3.7}$$

For small deformations the displacement gradients are denoted by:

$$H_{ij} = u_{i,j} \quad \Leftrightarrow \quad \mathbf{H} = \text{grad}\,\mathbf{u} \tag{5.3.8}$$

In the expressions for strains we take into account that the displacement gradients are small quantities. Products of H_{ij} - components are neglected compared to the components H_{ij}, and the components H_{ij} are neglected when compared with 1. The Green strain tensor (5.2.14) is thus reduced to:

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right) \quad \Leftrightarrow \quad E_{ij} = \frac{1}{2}\left(H_{ij} + H_{ji}\right) = \frac{1}{2}\left(u_{i,j} + u_{i,j}\right) \tag{5.3.9}$$

This strain tensor is sometimes called the *small strain tensor*.

The expression for the volumetric strain for small strains is:

$$\varepsilon_v = \operatorname{tr}\mathbf{E} = E_{kk} = u_{k,k} = \operatorname{div}\mathbf{u} \tag{5.3.10}$$

The divergence of the displacement field is thus an expression for the change in volume per unit volume when the material has been deformed from the reference configuration K_o to the present configuration K. For the coordinate strains we get:

$$\varepsilon_{11} = E_{11} = u_{1,1}\,, \quad \gamma_{12} = 2E_{12} = u_{1,2} + u_{2,1} \quad \text{etc.} \tag{5.3.11}$$

The result (5.3.11) may be illustrated directly as shown in Fig. 5.3.2, which for clarity illustrates only the two-dimensional case. A material line element dx_1 in K_o is deformed to an approximate length $dx_1 + u_{1,1}\,dx_1$. The strain of the element is then $\varepsilon_{11} = u_{1,1} = E_{11}$. The right angle between the material line elements dx_1 and dx_2 is diminished by $\gamma_{12} = u_{1,2} + u_{2,1} = 2E_{12}$.

In the standard xyz-notation the formulas (5.3.11) are written as follows.

$$\varepsilon_x = \frac{\partial u_x}{\partial x},\ \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \quad etc. \tag{5.3.12}$$

5.3.3 *Coordinate Strains in Cylindrical Coordinates and Spherical Coordinates*

The two curvilinear coordinate systems that we make the most use of in applications are the ones applying cylindrical coordinates (R, θ, z) and spherical coordinates (r, θ, ϕ). Since both are orthogonal coordinate systems the coordinate strains may be presented in strain matrices in the same way as for the Cartesian coordinate systems. The formulas for the coordinate strains are presented below. The development of the formulas (5.3.13) for cylindrical coordinates is given as Problem 5.9.

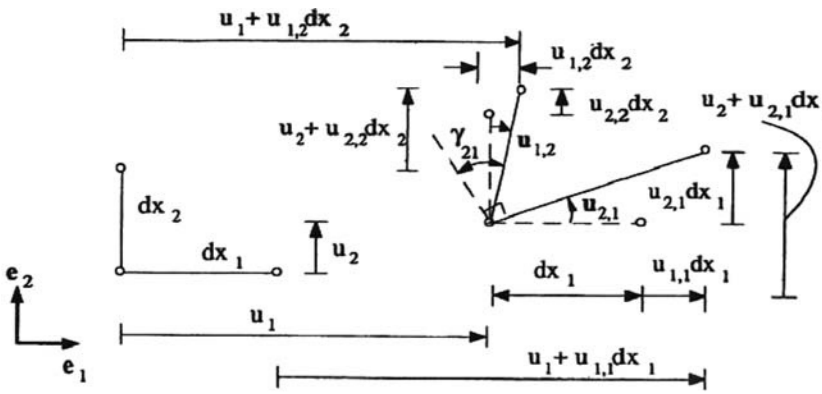
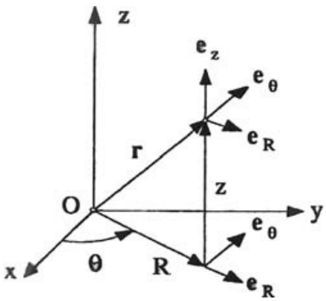


Fig. 5.3.2 Two-dimensional construction of coordinate strains

a) *Cylindrical coordinates* (R, θ, z)

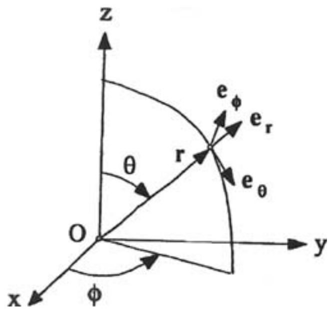
$$E = \begin{pmatrix} \varepsilon_R & \frac{1}{2}\gamma_{R\theta} & \frac{1}{2}\gamma_{Rz} \\ \frac{1}{2}\gamma_{\theta R} & \varepsilon_\theta & \frac{1}{2}\gamma_{\theta z} \\ \frac{1}{2}\gamma_{zR} & \frac{1}{2}\gamma_{z\theta} & \varepsilon_z \end{pmatrix} \tag{5.3.13}$$



$$\begin{aligned} \varepsilon_R &= \frac{\partial u_R}{\partial R},\ \varepsilon_\theta = \frac{1}{R}\frac{\partial u_\theta}{\partial \theta} + \frac{u_R}{R},\ \varepsilon_z = \frac{\partial u_z}{\partial z} \\ \gamma_{R\theta} &= \gamma_{\theta R} = \frac{1}{R}\frac{\partial u_R}{\partial \theta} + R\frac{\partial}{\partial R}\left(\frac{u_\theta}{R}\right) \\ \gamma_{\theta z} &= \gamma_{z\theta} = \frac{\partial u_\theta}{\partial z} + \frac{1}{R}\frac{\partial u_z}{\partial \theta},\ \gamma_{zR} = \gamma_{Rz} = \frac{\partial u_z}{\partial R} + \frac{\partial u_R}{\partial z} \end{aligned} \tag{5.3.14}$$

b) *Spherical coordinates* (r, θ, ϕ)

$$E = \begin{pmatrix} \varepsilon_r & \frac{1}{2}\gamma_{r\theta} & \frac{1}{2}\gamma_{r\phi} \\ \frac{1}{2}\gamma_{\theta r} & \varepsilon_\theta & \frac{1}{2}\gamma_{\theta\phi} \\ \frac{1}{2}\gamma_{\phi r} & \frac{1}{2}\gamma_{\phi\theta} & \varepsilon_\phi \end{pmatrix} \tag{5.3.15}$$



$$\begin{aligned} \varepsilon_r &= \frac{\partial u_r}{\partial r},\ \varepsilon_\theta = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r},\ \varepsilon_\phi = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{\cot\theta}{r}u_\theta \\ \gamma_{r\theta} &= \gamma_{\theta r} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right),\ \gamma_{\theta\phi} = \gamma_{\phi\theta} = \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{u_\phi}{\sin\theta}\right) \\ \gamma_{\phi r} &= \gamma_{r\phi} = r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} \end{aligned} \tag{5.3.16}$$

Note that the matrices E in the formulas (5.3.13) and (5.3.15) are not tensor matrices for the small strain tensor **E**. The coordinate strains (5.3.14) and (5.3.16) are *physical components* of the small strain tensor. The proper definition of tensor components in curvilinear coordinates will be presented in Chap. 12.

5.3.4 Principal Strains and Principal Directions of Strains

The analysis in this section is for *small strains*. This means that all results are also valid if the rotations are large. The condition small strains implies that all components of the strain tensor **E** are small in absolute values. The condition may be stated thus:

$$\left|E_{ij}\right| \ll 1 \quad \Leftrightarrow \quad \text{norm}\,\mathbf{E} \ll 1 \tag{5.3.17}$$

Because the strain tensor **E** is symmetric, we may determine three orthogonal principal directions of strain and corresponding principal values. We shall interpret these quantities geometrically, and we follow the general analysis for symmetric tensors of 2. order in Sect. 4.3.1. For any direction in K_o given by a unit vector **e** we define the vector $\mathbf{E}\cdot\mathbf{e}$. The longitudinal strain in the direction **e** is given by (5.3.1) and is the normal component of the strain tensor for the direction **e**:

$$\varepsilon = \mathbf{e}\cdot(\mathbf{E}\cdot\mathbf{e}) = \mathbf{e}\cdot\mathbf{E}\cdot\mathbf{e} = e_i E_{ij} e_j = e^T E e \tag{5.3.18}$$

The shear strain with respect to two orthogonal directions **e** and $\bar{\mathbf{e}}$ is given by (5.3.2) and is equal to twice the orthogonal shear component of the strain tensor for the directions **e** and $\bar{\mathbf{e}}$:

$$\gamma = 2\bar{\mathbf{e}}\cdot(\mathbf{E}\cdot\mathbf{e}) = 2\bar{\mathbf{e}}\cdot\mathbf{E}\cdot\mathbf{e} = 2\,\bar{e}_i E_{ij} e_j = 2\,\bar{e}^T E e \tag{5.3.19}$$

The *principal strains* ε_i and the *principal directions of strain* $\mathbf{a}_i$ are determined from the condition:

$$\mathbf{E}\cdot\mathbf{a} = \varepsilon\mathbf{a} \quad \Leftrightarrow \quad (\varepsilon\mathbf{1}-\mathbf{E})\cdot\mathbf{a} = \mathbf{0} \quad \Leftrightarrow \quad (\varepsilon\delta_{ij}-E_{ij})a_j = 0 \tag{5.3.20}$$

which gives the *characteristic equation for the small strain tensor* **E**:

$$\varepsilon^3 - I\varepsilon^2 + II\varepsilon - III = 0 \tag{5.3.21}$$

I, II, and *III* are the *principal invariants* of the strain tensor **E**.

$$I = \text{tr}\,\mathbf{E},\; II = \frac{1}{2}\left[(\text{tr}\,\mathbf{E})^2 - (\text{norm}\,\mathbf{E})^2\right],\; III = \det\mathbf{E} \tag{5.3.22}$$

The three principal strains ε_i are all real and are determined from (5.3.21). The principal directions $\mathbf{a}_i$, which are determined from (5.3.20), are orthogonal, under the condition that all the principal strains are different. In general three orthogonal directions may always be determined or specified. Shear strains related to a principal direction are zero. This may be expressed by the statement:

> *Through every material particle there exist three orthogonal material line elements before deformation that remain orthogonal after deformation.*

The principal strains in a particle represent extremal values of longitudinal strain in the particle.

$$\varepsilon_3 \le \varepsilon_2 \le \varepsilon_1 \quad \Rightarrow \quad \varepsilon_{\max} = \varepsilon_1,\ \varepsilon_{\min} = \varepsilon_3 \tag{5.3.23}$$

The *maximum shear strain* in a particle is given by:

$$\gamma_{\max} = \varepsilon_{\max} - \varepsilon_{\min} \tag{5.3.24}$$

and for the directions:

$$\mathbf{e} = (\mathbf{a}_1 + \mathbf{a}_3)/\sqrt{2}, \quad \bar{\mathbf{e}} = (\mathbf{a}_1 - \mathbf{a}_3)/\sqrt{2} \tag{5.3.25}$$

5.3.5 Strain Isotrop and Strain Deviator

In many applications, in particular in relation to isotropic materials, it is convenient to decompose the state of strain into a *form invariant part* and a *volume invariant part*. For small strains this is done by decomposing the strain tensor into a *strain isotrop* $\mathbf{E}^{\mathrm{o}}$ and a *strain deviator* $\mathbf{E}'$:

$$\begin{aligned}
&\mathbf{E} = \mathbf{E}^o + \mathbf{E}' \\
&\mathbf{E}^o = \frac{1}{3}(\operatorname{tr}\mathbf{E})\,\mathbf{1} = \frac{1}{3}\varepsilon_v\,\mathbf{1} \qquad \Leftrightarrow \quad \text{form invariant strain} \\
&\mathbf{E}' = \mathbf{E} - \mathbf{E}^o \quad \Rightarrow \quad \operatorname{tr}\mathbf{E}' = 0 \quad \Leftrightarrow \quad \text{volume invariant strain}
\end{aligned} \tag{5.3.26}$$

Because the strains are assumed to be small, the strain tensors $\mathbf{E}^{\mathrm{o}}$ and $\mathbf{E}'$ may be added commutatively.

The strain isotrop $\mathbf{E}^o$ represents a *state of form invariant strain* because the angle between any two material lines does not change due to this deformation. All shear strains are zero.

The strain deviator $\mathbf{E}'$ is trace free: $\operatorname{tr}\mathbf{E}' = \operatorname{tr}\mathbf{E} - \operatorname{tr}\mathbf{E}^o = \operatorname{tr}\mathbf{E} - (1/3)(\operatorname{tr}\mathbf{E})3 = 0$, and represents therefore a state of strain without change of volume. Hence the strain deviator represents a *state of volume invariant strain*. The strain deviator $\mathbf{E}'$ and the strain tensor $\mathbf{E}$ are coaxial tensors. The principal strains of the two tensors are related through:

$$\varepsilon_i' = \varepsilon_i - \frac{1}{3}\varepsilon_v \tag{5.3.27}$$

5.3.6 Rotation Tensor for Small Deformations

It is convenient to introduce a special *rotation tensor for small deformations*:

$$\tilde{\mathbf{R}} = \frac{1}{2}\left(\mathbf{H} - \mathbf{H}^T\right) \quad \Leftrightarrow \quad \tilde{R}_{ij} = \frac{1}{2}(H_{ij} - H_{ji}) = \frac{1}{2}(u_{i,j} - u_{j,i}) \tag{5.3.28}$$

Fig. 5.3.3 Rotation z_3 about an axis parallel to the x_3 - direction

This rotation tensor is the antisymmetrical part of the displacement gradient. The dual vector to the tensor $\tilde{\mathbf{R}}$ is called the *rotation vector* $\mathbf{z}$.

$$\mathbf{z} = -\frac{1}{2}\mathbf{P} : \tilde{\mathbf{R}} = \frac{1}{2}\mathrm{rot}\,\mathbf{u} \quad \Leftrightarrow \quad z_i = -\frac{1}{2}e_{ijk}\tilde{R}_{jk} = \frac{1}{2}e_{ijk}\,u_{k,j} \tag{5.3.29}$$

We shall now prove the following statement:

The rotation vector and the rotation tensor represent the rotation of three orthogonal material line elements through the particle in focus, and which are oriented in the principal directions to the strain tensor **E**.

A volume element with orthogonal edges parallel to the principal strain directions is rotated as determined by the rotation vector. Figure 5.3.3 illustrates this situation. In the figure the coordinate axes are for simplicity chosen to be parallel to the principal directions of strain, i.e.:

$$\gamma_{21} = 2E_{21} = 0 \quad \Rightarrow \quad u_{2,1} = -u_{1,2}$$

The rotation about an axis in the x_3-direction is represented by the angle:

$$z_3 = \frac{1}{2}e_{3jk}\,u_{k,j} = \frac{1}{2}e_{312}\,u_{2,1} + \frac{1}{2}e_{321}\,u_{1,2} = \frac{1}{2}\,(+1)\,u_{2,1} + \frac{1}{2}\,(-1)\,(-u_{2,1}) = u_{2,1}$$

For small deformations the deformation in the neighborhood of a particle $\mathbf{r}_o$ is represented by the displacement gradient tensor $\mathbf{H}$, which may be decomposed into two distinct contributions: a *pure strain* given by $\mathbf{E}(\mathbf{r},t)$, and a *pure rotation* about the particle $\mathbf{r}$ and given by $\tilde{\mathbf{R}}^T(\mathbf{r},t)$:

$$\mathbf{H} = \mathbf{E} + \tilde{\mathbf{R}} \quad \Leftrightarrow \quad H_{ij} \equiv u_{i,j} = E_{ij} + \tilde{R}_{ij} \tag{5.3.30}$$

A necessary and sufficient condition for a pure strain under small deformations, i.e. $\tilde{\mathbf{R}} = \mathbf{0}$, is that the displacement vector may be expressed by the gradient of a scalar field:

$$\mathbf{u} = \nabla\phi \quad \Leftrightarrow \quad \tilde{\mathbf{R}} = \mathbf{0} \quad \Leftrightarrow \quad \text{pure strain} \tag{5.3.31}$$

The scalar field $\phi(\mathbf{r},t)$ is called the *strain potential*. The proof of the biimplication (5.3.31) is given as Problem 5.10.

5.3.7 Small Strains in a Material Surface

Indirect measurements of stresses in the surface of an elastic body are performed by measuring the longitudinal strains in the surface. The strain measurements may be done by using electrical strain gages or strain rosettes. We shall now analyze the state of strain in a surface through a particle P and introduce a coordinate system Ox with the x_3 – axis normal to the tangent plane of the surface. The other two axes are then in the tangent plane. In Fig. 5.3.4 we have introduced two orthogonal unit directional vectors $\mathbf{e}$ and $\bar{\mathbf{e}}$ in the surface.

$$\mathbf{e} = [\cos\phi, \sin\phi, 0], \quad \bar{\mathbf{e}} = [\sin\phi, -\cos\phi, 0] \tag{5.3.32}$$

The longitudinal strain ε in the direction $\mathbf{e}$ in the material surface, and the shear strain γ with respect to the two orthogonal directions $\mathbf{e}$ and $\bar{\mathbf{e}}$ in the surface are given by:

$$\begin{aligned}
\varepsilon &= \varepsilon(\phi) = \mathbf{e}\cdot\mathbf{E}\cdot\mathbf{e} = e_\alpha E_{\alpha\beta} e_\beta = E_{11}\cos^2\phi + E_{22}\sin^2\phi + 2E_{12}\cos\phi\sin\phi \\
\gamma &= \gamma(\phi) = 2\bar{\mathbf{e}}\cdot\mathbf{E}\cdot\mathbf{e} = 2\bar{e}_\alpha E_{\alpha\beta} e_\beta \\
&= 2\left[E_{11}\sin\phi\cos\phi - E_{22}\cos\phi\sin\phi - E_{12}\left(\cos^2\phi - \sin^2\phi\right)\right]
\end{aligned} \tag{5.3.33}$$

We introduce the notation:

$$\varepsilon_x = E_{11}, \quad \varepsilon_y = E_{22}, \quad \gamma_{xy} = 2E_{12} \tag{5.3.34}$$

and the formulas (3.3.36) for $\sin 2\phi$ and $\cos 2\phi$. Then the formulas (5.3.33) may be transformed into:

$$\begin{aligned}
\varepsilon(\phi) &= \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\phi + \frac{1}{2}\gamma_{xy}\sin 2\phi \\
\gamma(\phi) &= (\varepsilon_x - \varepsilon_y)\sin 2\phi - \gamma_{xy}\cos 2\phi
\end{aligned} \tag{5.3.35}$$

These formulas are analogous to those developed in Sect. 3.3.5 for the state of plane stress.

The formulas for the principal strains ε_1 and ε_2 in the surface, and the angle ϕ_1 that the direction of the principal strain to ε_1 makes with the x_1 – axis, follow directly from the corresponding formula for plane stress:

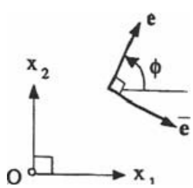


Fig. 5.3.4 Orthogonal unit vectors $\mathbf{e}$ and $\bar{\mathbf{e}}$ in a material surface

$$\begin{matrix}\varepsilon_1 \\ \varepsilon_2\end{matrix} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \tag{5.3.36}$$

$$\phi_1 = \arctan \frac{2(\varepsilon_1 - \varepsilon_x)}{\gamma_{xy}} \tag{5.3.37}$$

Note that the formulas (5.3.36) not necessarily represent the principal strains of the general state of strain. It is not assumed that the x_3 – direction, normal to the surface, is a principal direction of strain. However, very often this will be the case, especially when the material is isotropic and the surface is a free surface with electrical strain gages attached to it.

From the analogous analysis of plane stress we may also conclude that the *maximum shear strain in the surface* is given by:

$$\gamma_{\max} = |\varepsilon_1 - \varepsilon_2| = 2\sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \tag{5.3.38}$$

5.3.8 Mohr Diagram for Strain

We assume that the state of strain in a surface is given by the strain matrix:

$$E = \begin{pmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y \end{pmatrix} \tag{5.3.39}$$

The Mohr-diagram for strains in the surface is drawn in a Cartesian coordinate system with longitudinal strain ε and one half of shear strain $\gamma/2$ as coordinates. The Mohr strain circle is constructed by the same principles as the Mohr stress circle. First the points:

$$(\varepsilon_x, -\gamma_{xy}/2) \text{ and } (\varepsilon_y, \gamma_{xy}/2)$$

are marked in the diagram. These two points are on a diameter of the circle and thus determine the circle. The complete Mohr diagram and sign convention for ε and γ are shown in Fig. 5.3.5.

5.3.9 Equations of Compatibility

In this section the coordinate strains $E_{ij}(x,t)$ are assumed to be given as functions that are many times differentiable with respect to x_i. The 6 coordinate strains are according to (5.3.9) related through 3 components of the displacement vector. This fact imposes some conditions to the functions $E_{ij}(x,t)$. These conditions are expressed by the *compatibility equations*, which are necessary and sufficient conditions for

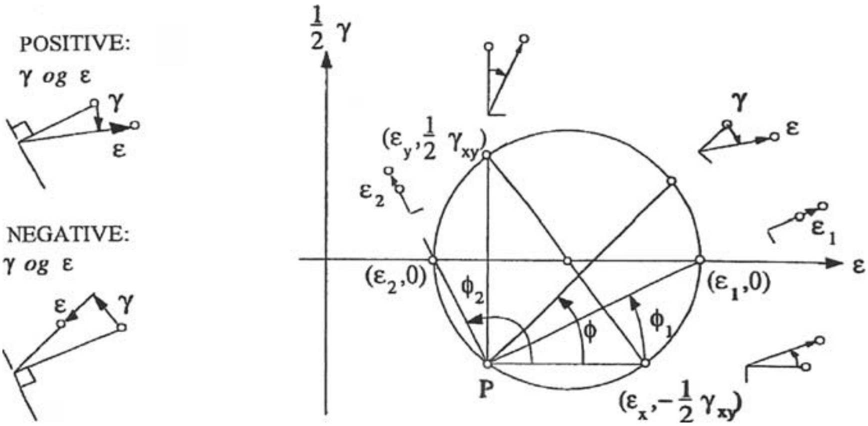


Fig. 5.3.5 Mohr diagram for strains

the strain tensor $\mathbf{E}(x,t)$ to correspond to a unique and continuous displacement field $\mathbf{u}(x,t)$. The compatibility equations are:

$$E_{ij,kl} + E_{kl,ij} - E_{il,jk} - E_{jk,il} = 0 \tag{5.3.40}$$

Due to the symmetry with respect to the pairs of indices (i and j) and (k and l), these $3^4 = 81$ equations are reduced to the following 6 independent equations:

$$\begin{aligned} &E_{11,23} + E_{23,11} - E_{12,13} - E_{13,12} = 0, \quad E_{11,22} + E_{22,11} - 2E_{12,12} = 0 \\ &E_{22,31} + E_{31,22} - E_{23,21} - E_{21,23} = 0, \quad E_{22,33} + E_{33,22} - 2E_{23,23} = 0 \\ &E_{33,12} + E_{12,33} - E_{31,32} - E_{32,31} = 0, \quad E_{33,11} + E_{11,33} - 2E_{31,31} = 0 \end{aligned} \tag{5.3.41}$$

For the state of *plane strain* defined by: $E_{\alpha\beta} = E_{\alpha\beta}(x_1, x_2, t)$ and $E_{3\alpha} = 0$, the 6 general compatibility equations are reduced to one non-trivial compatibility equation:

$$E_{11,22} + E_{22,11} - 2E_{12,12} = 0 \tag{5.3.42}$$

The compatibility equations, and especially (5.3.42), play an important role in the theory of elasticity.

Let us now convince ourselves that the compatibility equations (5.3.40) are necessary conditions for the strain field $\mathbf{E}(x,t)$. From the strain-displacement relations (5.3.9) we get:

$$\begin{aligned} 2E_{ij,kl} &= u_{i,jkl} + u_{j,ikl}, \quad 2E_{kl,ij} = u_{k,lij} + u_{l,kij} \\ -2E_{il,jk} &= -u_{i,ljk} - u_{l,ijk} \equiv -u_{i,jkl} - u_{l,kij}, \\ -2E_{jk,il} &= -u_{j,kil} - u_{k,jil} \equiv -u_{j,ikl} - u_{k,lij} \end{aligned}$$

The identities follow from the fact that the order of partial differentiation is immaterial. When the four equations are added, the sum of the left hand sides is equal to twice the left hand side of (5.3.40), while the sum of the right hand sides is zero. This implies that the compatibility equations (5.3.40) are necessary conditions for the strain field $\mathbf{E}(x,t)$.

5.3.10 Compatibility Equations as Sufficient Conditions

The strain tensor $\mathbf{E}(x,t)$ is assumed to be known in the region of interest of the material we are investigating. Let x^o be a particle for which we know the displacement $\mathbf{u}^o$ and the rotation tensor $\tilde{\mathbf{R}}^o$. The displacement $\mathbf{u}^1$ of an arbitrarily selected other particle x^1 may be determined from:

$$\mathbf{u}^1 = \mathbf{u}^o + \int_{x^o}^{x^1} d\mathbf{u} \tag{5.3.43}$$

The integration may be performed along any material curve between x^o and x^1. By (5.3.30) the components of the differential vector $d\mathbf{u}$ are:

$$du_i = u_{i,k}\, dx_k = H_{ik}\, dx_k = E_{ik}\, dx_k + \tilde{R}_{ik}\, dx_k \tag{5.3.44}$$

The integral in (5.3.43) is now split into two parts:

$$\int_{x^o}^{x^1} du_i = \int_{x^o}^{x^1} E_{ik}\, dx_k + \int_{x^o}^{x^1} \tilde{R}_{ik}\, dx_k \tag{5.3.45}$$

Partial integration of the second integral on the right hand yields:

$$\begin{aligned} \int_{x^o}^{x^1} \tilde{R}_{ik}\, dx_k &= \left[\tilde{R}_{ik}\left(x_k - x_k^1\right)\right]_{x^o}^{x^1} - \int_{x^o}^{x^1} \tilde{R}_{ik,j}\left(x_k - x_k^1\right) dx_j \\ &= -\tilde{R}_{ik}^o\left(x_k^o - x_k^1\right) - \int_{x^o}^{x^1} \tilde{R}_{ik,j}\left(x_k - x_k^1\right) dx_j \end{aligned} \tag{5.3.46}$$

From the expressions (5.3.9) for $\mathbf{E}$ and (5.3.28) for $\tilde{\mathbf{R}}$ we obtain the identity:

$$\tilde{R}_{ik,j} = E_{ij,k} - E_{kj,i} \tag{5.3.47}$$

Using the results (5.3.45, 5.3.46, 5.3.47), we may rewrite (5.3.43) to:

$$\mathbf{u}^1 = \mathbf{u}^o + \tilde{\mathbf{R}}^o \cdot \left(\mathbf{r}^1 - \mathbf{r}^o\right) + \int_{x^o}^{x^1} \mathbf{K} \cdot d\mathbf{r} \tag{5.3.48}$$

The tensor **K** is defined by its components:

$$K_{ij} = E_{ij} + \left(E_{ij},_k - E_{kj},_i\right)\left(x_k^1 - x_k\right) \tag{5.3.49}$$

If we require that the tensor **K** gives the same displacement $\mathbf{u}^1$ in (5.3.48), regardless of the material curve the integration is performed along, then according to Theorem C.1 in Appendix C, **K** must be the gradient of a vector **a**:

$$\mathbf{K} = \operatorname{grad}\mathbf{a} \quad \Leftrightarrow \quad K_{ij} = a_i,_j$$

From the identity $a_{i,jl} = a_{i,lj}$ it now follows that $K_{ij,l} - K_{il,j} = 0$, which by the definition (5.3.49) gives the conditions:

$$\left[\left(E_{ij},_{kl} - E_{kj},_{il}\right) - \left(E_{il},_{kj} - E_{kl},_{ij}\right)\right]\left(x_k^1 - x_k\right) = 0$$

Since these equations have to be valid for all choices of the terms $\left(x_k^1 - x_k\right)$, the terms in the square brackets have to be equal to zero. We now see from (5.3.40) that this condition is precisely the compatibility equations. We have in fact shown that the compatibility equations are sufficient conditions that the strain field $\mathbf{E}(x,t)$ must satisfy to give unique and continuous displacement field $\mathbf{u}(x,t)$.

The proof given above assumes that the material region we consider is a *simply-connected region*. This means that any closed material curve in the region can be made to shrink to a point. A material region with an open whole, for example an open thick-walled cylinder or pipe, is non-simply-connected. For such, so-called *multiply-connected regions*, we have to impose additional conditions, which we shall not discuss in this book.

5.4 Rates of Deformation, Strain, and Rotation

When describing the motion and deformation of fluids, and sometimes solids with fluidlike behavior we choose to work with Eulerian coordinates and the present configuration K as reference configuration. The particles are now denoted by the place vector $\mathbf{r}$, or the place coordinates x. A particle P at place $\mathbf{r}$ at the time t has the velocity $\mathbf{v}(\mathbf{r},t)$ and is during a short time increment dt given the displacement $d\mathbf{u} = \mathbf{v}\cdot dt$. This displacement field leads to small deformations with a displacement gradient tensor $d\mathbf{H}$:

$$dH_{ik} = \frac{\partial v_i}{\partial x_k}\,dt \equiv v_i,_k\,dt \tag{5.4.1}$$

and a small strain tensor $d\mathbf{E}$ and a rotation tensor for small deformations $d\tilde{\mathbf{R}}$:

$$d\mathbf{E} = \frac{1}{2}\left(d\mathbf{H} + d\mathbf{H}^T\right), \quad d\tilde{\mathbf{R}} = \frac{1}{2}\left(d\mathbf{H} - d\mathbf{H}^T\right) \tag{5.4.2}$$

We now define three new tensors: the *velocity gradient tensor* **L** at the time t, the symmetric *rate of deformation tensor* **D** at the time t, and the antisymmetric *rate of rotation tensor* **W** at time t:

$$\mathbf{L} = \operatorname{grad}\mathbf{v} \equiv \frac{\partial \mathbf{v}}{\partial \mathbf{r}} \quad \Leftrightarrow \quad L_{ik} = v_{i,k} \tag{5.4.3}$$

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{L}+\mathbf{L}^T\right) \quad \Leftrightarrow \quad D_{ik} = \frac{1}{2}\left(v_{i,k}+v_{k,i}\right) \tag{5.4.4}$$

$$\mathbf{W} = \frac{1}{2}\left(\mathbf{L}-\mathbf{L}^T\right) \quad \Leftrightarrow \quad W_{ik} = \frac{1}{2}\left(v_{i,k}-v_{k,i}\right) \tag{5.4.5}$$

$\mathbf{W}$ is also called the *spin tensor* and the *vorticity tensor*. It follows that:

$$d\mathbf{H} = \mathbf{L}\,dt, \quad d\mathbf{E} = \mathbf{D}\,dt, \quad d\tilde{\mathbf{R}} = \mathbf{W}\,dt, \quad \dot{\mathbf{E}} = \mathbf{D}, \quad \dot{\tilde{\mathbf{R}}} = \mathbf{W} \tag{5.4.6}$$

Note that $\dot{\mathbf{E}}$ is the material derivative of the small strain tensor (5.3.9) for small deformations. As will be demonstrated in Sect. 5.5 the relation between the material derivative of the general Green strain tensor $\mathbf{E}$ defined by (5.2.14) and the rate of deformation tensor $\mathbf{D}$ is more complex. The tensor $\mathbf{D}$ is sometimes called the *rate of strain tensor*, but only when we deal with small deformations is this appropriate.

The change length of a material line element per unit length and per unit time is called the *rate of longitudinal strain* or for short the *rate of strain*. From the analysis of small deformations in Sect. 5.3 it follows that the rate of strain in the direction $\mathbf{e}$ is, according to (5.3.1):

$$\dot{\varepsilon} = \mathbf{e} \cdot \mathbf{D} \cdot \mathbf{e} \tag{5.4.7}$$

In the coordinate direction $\mathbf{e}_i$ we get the *coordinate rates of strain*:

$$\dot{\varepsilon}_{ii} = D_{ii} = v_{i,i} \quad \text{(no summation)} \tag{5.4.8}$$

The change per unit time of the angle between two material line elements that in K are orthogonal, is called the *rate of shear strain* or for short the *shear rate*. The rate of shear strain with respect to orthogonal line elements in the directions $\mathbf{e}$ and $\bar{\mathbf{e}}$ is, in analogy to equation (5.3.2):

$$\dot{\gamma} = 2\bar{\mathbf{e}} \cdot \mathbf{D} \cdot \mathbf{e} \tag{5.4.9}$$

For the coordinate directions $\mathbf{e}_1$ and $\mathbf{e}_2$ we get the *coordinate shear rate*:

$$\dot{\gamma}_{21} = 2D_{21} = v_{2,1} + v_{2,1} \tag{5.4.10}$$

Figure 5.3.2 illustrates the coordinate rate of strain and shear rate when the displacement $\mathbf{u}$ is replaced by the increment in displacement $\mathbf{v} \cdot dt$.

The change in volume per unit of volume and per unit of time is called the *rate of volumetric strain*. By referring to the expression (5.3.4) for the volumetric strain, we may immediately write down the expression for the rate of volumetric strain.

$$\dot{\varepsilon}_v = D_{kk} = \operatorname{tr}\mathbf{D} = \operatorname{div}\mathbf{v} = v_{k,k} \tag{5.4.11}$$

Hence the divergence of the velocity field $\mathbf{v}(\mathbf{r},t)$ represents the rate of change of volume per unit volume about the particle under consideration.

The dual vector to the antisymmetric rate of rotation tensor **W** is the *angular velocity*:

$$\mathbf{w} = -\frac{1}{2}\mathbf{P} : \mathbf{W} = \frac{1}{2}\mathrm{rot}\,\mathbf{v} \quad \Leftrightarrow \quad w_i = \frac{1}{2}e_{ijk}W_{kj} = \frac{1}{2}e_{ijk}v_{k,j} \tag{5.4.12}$$

The matrix of **W** may now be written as:

$$W = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} \tag{5.4.13}$$

Note that angular velocity vector **w** is related to the rotation vector **z** for small deformation defined by (5.3.29):

$$\mathbf{w} = \dot{\mathbf{z}} \tag{5.4.14}$$

In Fluid Mechanics it is customary to introduce the concept of *vorticity* **c**:

$$\mathbf{c} = 2\mathbf{w} = \mathrm{rot}\,\mathbf{v} \equiv \mathrm{curl}\,\mathbf{v} \equiv \nabla \times \mathbf{v} \tag{5.4.15}$$

This concept and its importance will be discussed in Sect. 8.1 and Sect. 8.3.

Due to the relationship between the rate of deformation tensor and the rate of strain tensor for small deformation, and the corresponding relationship between the angular velocity vector and the rotation vector for small deformation, we may conclude based on the discussion in Sect. 5.3.6 that material line elements parallel to the principal directions of the rate of deformation tensor **D** rotate instantaneously with the angular velocity **w**. In other words:

The rate of rotation tensor **W** *represents the instantaneous angular velocity of the three orthogonal material line elements that are oriented in the principal directions of the rate of deformation tensor* **D**.

This fact is demonstrated in Fig. 5.4.1, in which the coordinate system is chosen to have axes parallel to the principal directions $\mathbf{a}_i$ of **D**. The matrix of **D** in this coordinate system has the elements:

$$D_{ik} = \dot{\varepsilon}_i\,\delta_{ik} \tag{5.4.16}$$

This implies that $D_{12} = v_{1,2} + v_{2,1} = 0$, and therefore $v_{1,2} = -v_{2,1}$. Thus:

$$w_3 = W_{21} = v_{2,1}$$

In Fig. 5.4.1 we now see that $w_3 dt = v_{2,1} dt$ represents, during the time interval dt, the angle of rotation about an axis parallel to the x_3-axis of material line elements in the principal directions $\mathbf{a}_i = \mathbf{e}_i$ of the rate of deformation tensor **D**. The quantity w_3 represents the angular velocity of those line elements about that axis.

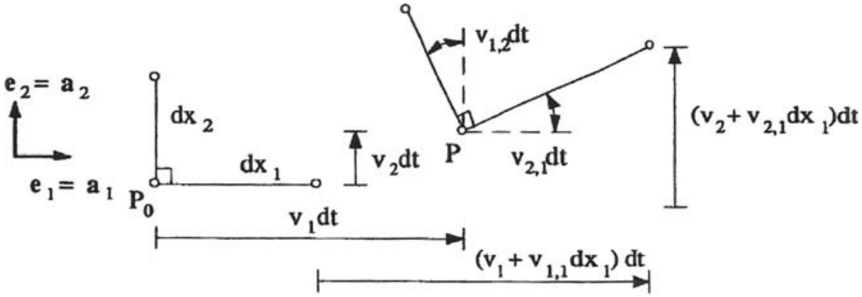


Fig. 5.4.1 Rotation about an axis parallel to the x_3-direction

To further illustrate the physical interpretation of the rate of rotation tensor we take a look at the special motion of rigid-body rotation about the x_3 – axis through the origin O of the coordinate system. The velocity field in this case is, see (4.5.17) and Fig. 5.4.2:

$$\mathbf{v} = \mathbf{W} \cdot \mathbf{r} = \mathbf{w} \times \mathbf{r}, \quad v = wR \quad \Rightarrow$$
$$v_1 = -v\sin\theta = -wx_2, \quad v_2 = v\cos\theta = wx_1, \quad v_3 = 0$$

The angular velocity may be a function of time, $\mathbf{w} = \mathbf{w}(t)$. We easily find that $\mathbf{D} = \mathbf{0}$ and:

$$W = \begin{pmatrix} 0 & -w & 0 \\ w & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In Fig. 5.4.2 a material volume element enclosing a particle P is shown to rotate as a rigid body about the x_3-axis. The general rigid-body rotation about a point is presented in Sect. 4.5.1. See in particular the formulas (4.5.15) and (4.5.17) for the relations between $\mathbf{W}$, $\mathbf{w}$, and $\mathbf{v}$.

If the rate of rotation tensor is zero during the motion, $\mathbf{W} = \mathbf{0}$, the motion is called *irrotational motion* or *irrotational flow*. According to Theorem C. 1 and Theorem C. 5 in Appendix C, in the case of irrotational flow the velocity field may be derived from a scalar field:

$$\mathbf{v} = \nabla\phi \tag{5.4.17}$$

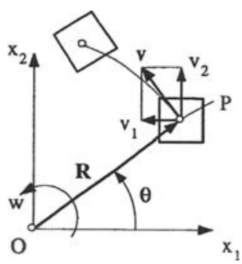


Fig. 5.4.2 Rigid-body rotation

The scalar field $\phi(\mathbf{r},t)$ is called the *velocity potential*. See (5.3.31) and Problem 5.10. Irrotational flow is for this reason also called *potential flow*. An important part of Fluid Mechanics is devoted to potential flow because we can in many practical flow cases show that the flow is in fact irrotational, or very close to be irrotational, and because potential flow introduces mathematical simplifications in Fluid Mechanics. Section 8.5 discusses potential flow.

In order to illustrate the concepts defined and discussed above in this section, two fundamental examples will now be presented. The two examples are given the provocative subtiles: "Rectilinear rotational flow" for a flow with rotation and "Circular irrotational flow" for a flow without rotation. The first example is of fundamental importance in the mechanics of Newtonian fluids, in Sect. 8.4 and viscous non-Newtonian fluids in Sect. 8.6. The second example illustrates an irrotational flow which contributes to many potential flows discussed in Sect. 8.5.

Example 5.1. Simple Shear Flow. Rectilinear Rotational Flow

A fluid flows between two parallel planes. One of the planes is at rest, while the other is moving with a constant velocity v parallel to the planes, as indicated in Fig. 5.4.3. We assume that the fluid particles move in straight parallel lines, and that the velocity field is:

$$v_x = \frac{v}{h}y, \quad v_x = v_x = 0$$

This velocity field implies that the fluid sticks to the rigid boundary planes, a common assumption in Fluid Mechanics. This particular flow is called *simple shear flow*. Only one rate of deformation component is different from zero, which gives the shear rate: $\dot{\gamma}_{xy}$, and only one rate of rotation component is different from zero: W_{xy}.

$$\dot{\gamma}_{xy} = \frac{v}{h}, \quad W_{xy} = \frac{v}{2h} \quad \Rightarrow \quad w_z \equiv \omega_z = -\frac{v}{2h}$$

Figure 5.4.3 shows the deformation of two fluid elements at the present time t and shortly thereafter at time $t+\Delta t$. Fluid element 1 experiences the shear rate $\dot{\gamma}_{xy}$. Fluid element 2, which is oriented with edges parallel to the principal directions to the rate of deformation tensor $\mathbf{D}$, does not directly show shear rates but show rotation with

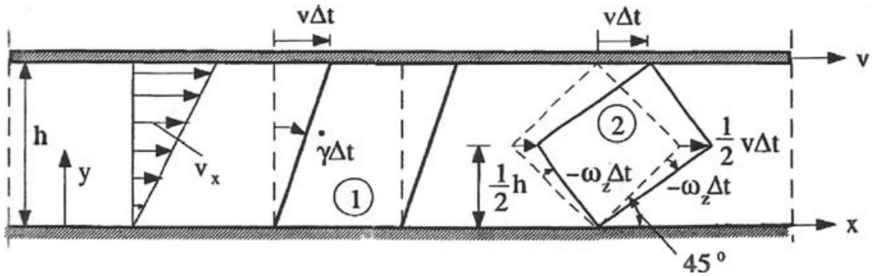


Fig. 5.4.3 Simple shear flow. Couette flow

instantaneous angular velocity w_z. Note that $\varepsilon_v = \text{div}\,\mathbf{v} = 0$. The flow is *isochoric* or volume preserving. The matrices of the velocity gradient tensor $\mathbf{L}$, the rate of deformation tensor $\mathbf{D}$, and the rate of rotation tensor $\mathbf{W}$ for this flow are:

$$L = \frac{v}{h}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D = \frac{v}{2h}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad W = \frac{v}{2h}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This flow occurs in many applications and is often called *Couette flow* after M. Couette (1890). The simple shear flow also plays an important part in the discussion of non-Newtonian fluids, as we shall see in Sect. 8.6.

Example 5.2. The Potential Vortex. Circular Irrotational Flow

A solid circular cylinder that rotates about its vertical axis with constant angular velocity ω in a large container filled with a Newtonian fluid, creates an irrotational flow in which the fluid particles move in concentric circles, see Fig. 5.4.4. It will be shown in Example 8.4 that this is indeed an irrotational flow, i.e. the vorticity is zero, and therefore a potential flow. Due to the rotational motion of the fluid particles the flow is called a *vortex*, and since the flow also is irrotational called a potential vortex.

The velocity field is:

$$v_\theta = \frac{\alpha}{R}, \quad v_R = v_z = 0, \quad \alpha = \omega a^2$$

where a is the radius of the cylinder. The container must be so large that it is reasonable to assume that the velocity approaches zero as $R \to \infty$. Referred to Cartesian coordinates the velocity components are:

$$v_x = -v_\theta \sin\theta = -\frac{\alpha y}{x^2 + y^2}, \quad v_y = v_\theta \cos\theta = \frac{\alpha x}{x^2 + y^2}, \quad v_z = 0, \quad \alpha = \omega a^2$$

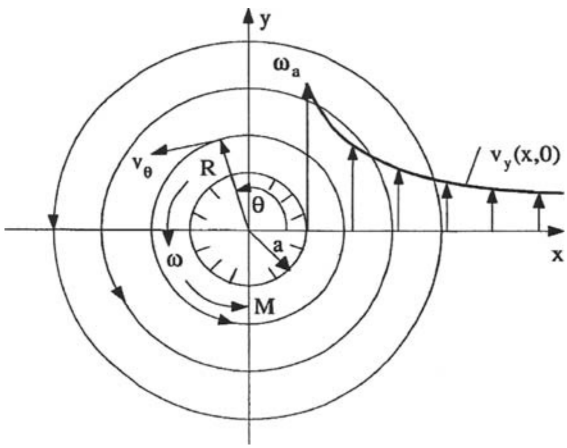


Fig. 5.4.4 The potential vortex. Vortex without vorticity

For the non-zero velocity gradients we find:

$$\frac{\partial v_x}{\partial x} = \frac{2\alpha yx}{(x^2+y^2)^2}, \quad \frac{\partial v_y}{\partial y} = \frac{-2\alpha xy}{(x^2+y^2)^2}$$

$$\frac{\partial v_x}{\partial y} = -\frac{\alpha}{x^2+y^2} + \frac{2\alpha y^2}{(x^2+y^2)^2} = \frac{\alpha\left(y^2-x^2\right)}{(x^2+y^2)^2}$$

$$\frac{\partial v_y}{\partial x} = \frac{\alpha}{x^2+y^2} - \frac{2\alpha x^2}{(x^2+y^2)^2} = \frac{\alpha\left(y^2-x^2\right)}{(x^2+y^2)^2}$$

This gives the result:

$$W_{xy} = -w_z = -\frac{1}{2}c_z = \frac{1}{2}\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}\right) = 0$$

Since this is a planar flow, all the other components of the rate of rotation tensor **W**, the angular velocity vector **w**, and the vorticity vector **c** are zero. Hence we have the results:

$$\mathbf{W} = \mathbf{0}, \quad \mathbf{c} = 2\mathbf{w} = \mathrm{rot}\,\mathbf{v} = \nabla \times \mathbf{v} = \mathbf{0}$$

For $y = 0$ the rates of strain are:

$$\dot{\varepsilon}_x = \dot{\varepsilon}_y = 0, \quad \dot{\gamma}_{xy} = -\frac{2\alpha}{x^2}$$

Since this is obviously an isochoric flow, i.e. a volume preserving flow, the volumetric strain rate is zero. Indeed we find:

$$\dot{\varepsilon}_v = \mathrm{div}\,\mathbf{v} = v_{k,k} = 0$$

The rates of deformation and rates of rotation in cylindrical coordinates are presented below in the formulas (5.4.18, 5.4.19, 5.4.20). For the present flow we get $W = 0$ and:

$$D = -\frac{\alpha}{R^2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It follows from this expression for D that two of the principal axes of **D** are rotated 45° with respect to the radial direction.

Figure 5.4.5 shows the deformation of two small fluid elements between the times t and $t+dt$, where dt is a short time increment. Fluid element 1, which experiences shear strain rate $\dot{\gamma}_{xy} = -2\alpha/R^2$, moves symmetrically with respect to the diagonal planes indicated in the figure. Fluid element 2, which is oriented according to the principal directions of the rate of deformation tensor **D**, retains its right angles and does not rotate during the short time interval.

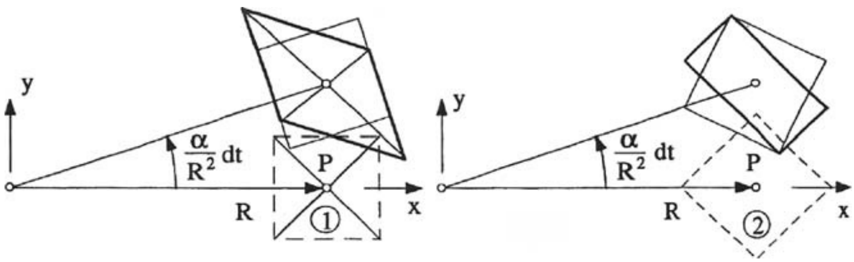


Fig. 5.4.5 Element 1 gets shear strain. Element 2, which is oriented according to the principal directions of **D**, is deformed without shear strain and without rotation

In constructing the deformed elements in Fig. 5.4.5 the velocity field is first represented by a series expansion about the particle P at position $x = R$ and $y = 0$.

$$v_\theta = \frac{\alpha}{R+\Delta R} = \frac{\alpha}{R}\,\frac{1}{1+\Delta R/R} \approx \frac{\alpha}{R}\left(1 - \frac{\Delta R}{R}\right) = \frac{\alpha}{R^2}(R+\Delta R) - \frac{2\alpha}{R^2}\Delta R$$

ΔR is a coordinate measured radially from particle P. We set:

$$v_\theta = v_{\theta 1} + v_{\theta 2} \quad \text{where} \quad v_{\theta 1} = \frac{\alpha}{R^2}(R+\Delta R), \quad v_{\theta 2} = -\frac{2\alpha}{R^2}\Delta R$$

The result shows that we may consider the velocity field to be decomposed into two parts: a rigid-body rotation about the origin O, represented by the velocity field $v_{\theta 1}$, and a motion with respect to P and represented by the velocity field $v_{\theta 2}$.

Because the flow is irrotational, the velocity field may be derived from a velocity potential: $\phi = \alpha\theta$, where α is a constant. Using the del-operator (2.4.18) we obtain:

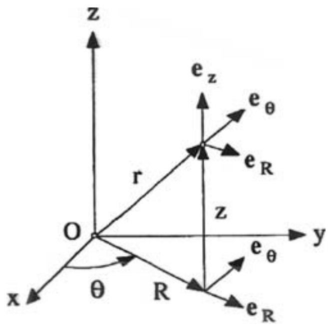
$$\mathbf{v} = \nabla\phi \quad \Leftrightarrow \quad v_R = \frac{\partial\phi}{\partial R} = 0, \quad v_\theta = \frac{1}{R}\frac{\partial\phi}{\partial\theta} = \frac{\alpha}{R}, \quad v_z = \frac{\partial\phi}{\partial z} = 0$$

We shall return to this example in Example 8.4 and Example 8.8.

5.4.1 Rate of Deformation Matrix and Rate of Rotation Matrix in Cylindrical and Spherical Coordinates

a) *Cylindrical coordinates* (R, θ, z). Rate of deformation matrix:

$$D = \begin{pmatrix} \dot\varepsilon_R & \frac{1}{2}\dot\gamma_{R\theta} & \frac{1}{2}\dot\gamma_{Rz} \\ \frac{1}{2}\dot\gamma_{\theta R} & \dot\varepsilon_\theta & \frac{1}{2}\dot\gamma_{\theta z} \\ \frac{1}{2}\dot\gamma_{zR} & \frac{1}{2}\dot\gamma_{z\theta} & \dot\varepsilon_z \end{pmatrix} \tag{5.4.18}$$



Rates of strain and rates of shear:

$$\dot{\varepsilon}_R = \frac{\partial v_R}{\partial R}, \quad \dot{\varepsilon}_\theta = \frac{1}{R}\frac{\partial v_\theta}{\partial \theta} + \frac{v_R}{R}, \quad \dot{\varepsilon}_z = \frac{\partial v_z}{\partial z}$$

$$\dot{\gamma}_{R\theta} = \frac{1}{R}\frac{\partial v_R}{\partial \theta} + R\frac{\partial}{\partial R}\left(\frac{v_\theta}{R}\right), \quad \dot{\gamma}_{\theta z} = \frac{\partial v_\theta}{\partial z} + \frac{1}{R}\frac{\partial v_z}{\partial \theta}, \quad \dot{\gamma}_{zR} = \frac{\partial v_z}{\partial R} + \frac{\partial v_R}{\partial z} \tag{5.4.19}$$

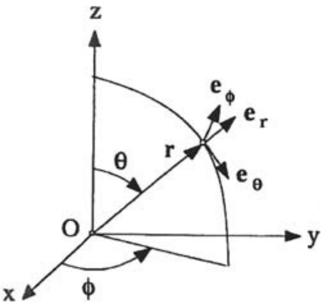
Rate of rotation components:

$$W_{R\theta} = \frac{1}{2R}\left[\frac{\partial v_R}{\partial \theta} - \frac{\partial}{\partial R}(R v_\theta)\right], \quad W_{\theta z} = \frac{1}{2}\left[\frac{\partial v_\theta}{\partial z} - \frac{1}{R}\frac{\partial v_z}{\partial \theta}\right]$$

$$W_{zR} = \frac{1}{2}\left[\frac{\partial v_z}{\partial R} - \frac{\partial v_R}{\partial z}\right] \tag{5.4.20}$$

b) *Spherical coordinates* (r, θ, ϕ). Rate of deformation matrix:

$$D = \begin{pmatrix} \dot{\varepsilon}_r & \frac{1}{2}\dot{\gamma}_{r\theta} & \frac{1}{2}\dot{\gamma}_{r\phi} \\ \frac{1}{2}\dot{\gamma}_{\theta r} & \dot{\varepsilon}_\theta & \frac{1}{2}\dot{\gamma}_{\theta\phi} \\ \frac{1}{2}\dot{\gamma}_{\phi r} & \frac{1}{2}\dot{\gamma}_{\phi\theta} & \dot{\varepsilon}_\phi \end{pmatrix} \tag{5.4.21}$$



Rates of strain and rates of shear:

$$\dot{\varepsilon}_r = \frac{\partial v_r}{\partial r}, \quad \dot{\varepsilon}_\theta = \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}, \quad \dot{\varepsilon}_\phi = \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{\cot\theta}{r}v_\theta$$

$$\dot{\gamma}_{r\theta} = \frac{1}{r}\frac{\partial v_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right), \quad \dot{\gamma}_{\theta\phi} = \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right)$$

$$\dot{\gamma}_{\phi r} = r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} \tag{5.4.22}$$

Rate of rotation components:

$$W_{r\theta} = \frac{1}{2r}\left[\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r}(r v_\theta)\right], \quad W_{\theta\phi} = \frac{1}{2r\sin\theta}\left[\frac{\partial v_\theta}{\partial \phi} - \frac{\partial}{\partial \theta}\left(\sin\theta\, v_\phi\right)\right]$$

$$W_{\phi r} = \frac{1}{2r}\left[\frac{\partial}{\partial r}\left(r v_\phi\right) - \frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi}\right] \tag{5.4.23}$$

Note that the matrices D in the formulas (5.4.18) and (5.4.21) are not tensor matrices for the rate of deformation tensor **D**. The coordinate strain rates in the formulas (5.4.19) and (5.4.22) are *physical components* of the rate of deformation tensor. The components (5.4.20) and (5.4.23) are physical components of the rate of rotation tensor **W**. The proper definition of tensor components in curvilinear coordinates will be presented in Chap. 12.

5.5 Large Deformations

In this section we shall analyze the general case of deformation in the neighborhood of a particle P that moves with a body from the reference configuration K_o at time t_o to the present configuration K at time t. The motion is given by, see Fig. 5.5.1:

$$\mathbf{r} = \mathbf{r}(\mathbf{r}_o,t) = \mathbf{r}_o + \mathbf{u}(\mathbf{r}_o,t) \quad \Leftrightarrow \quad x_i = x_i(X,t) = X_i + u_i(X,t) \tag{5.5.1}$$

$\mathbf{u}(\mathbf{r}_o,t)$ is the displacement vector. From the development in Sect. 5.2 we have seen that the three fundamental measures of strain ε, γ, and ε_v may be determined from the relation (5.2.8) between the material line element $d\mathbf{r}_o$ in K_o and the corresponding material line element $d\mathbf{r}$ in K:

$$d\mathbf{r} = \mathbf{F} \cdot d\mathbf{r}_o \quad \Leftrightarrow \quad dx_i = F_{ik}\, dX_k \tag{5.5.2}$$

The tensor **F** is the *deformation gradient* and defined by:

$$\mathbf{F} = \frac{\partial \mathbf{r}}{\partial \mathbf{r}_o} \quad \Leftrightarrow \quad F_{ik}(X,t) = \frac{\partial x_i(X,t)}{\partial X_k} \tag{5.5.3}$$

The deformation gradient **F** represents a linear transformation of a material line element $d\mathbf{r}_o$ in K_o to the corresponding line element $d\mathbf{r}$ in K. In a *homogeneous deformation*, in which the deformation is the same for all particles:

$$\mathbf{r} = \mathbf{u}_o + \mathbf{F} \cdot \mathbf{r}_o, \quad \mathbf{u}_o = \mathbf{u}_o(t), \quad \mathbf{F} = \mathbf{F}(t) \tag{5.5.4}$$

the differentials $d\mathbf{r}_o$ and $d\mathbf{r}$ are the same material line element in K_o and K respectively. Material planes and straight lines in K_o are deformed into planes and straight lines in K, see Problem 5.12.

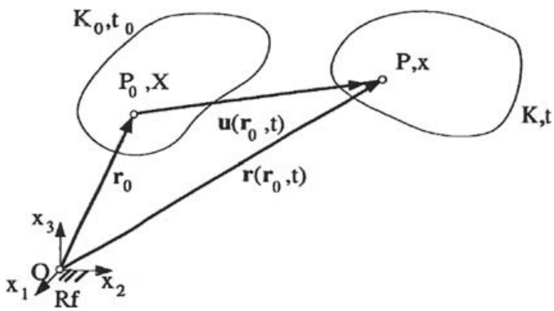


Fig. 5.5.1 General motion and deformation of a body

From Sect. 5.2 we import the general results expressing the longitudinal strain ε in the direction $\mathbf{e}$ in K_o, the shear strain γ with respect to two orthogonal directions $\bar{\mathbf{e}}$ and $\mathbf{e}$ in K_o, and the volumetric strain ε_v:

$$\varepsilon = \sqrt{1+2\mathbf{e}\cdot\mathbf{E}\cdot\mathbf{e}}-1 = \sqrt{\mathbf{e}\cdot\mathbf{C}\cdot\mathbf{e}}-1$$
$$\gamma = \arcsin\frac{2\bar{\mathbf{e}}\cdot\mathbf{E}\cdot\mathbf{e}}{\sqrt{(1+2\bar{\mathbf{e}}\cdot\mathbf{E}\cdot\bar{\mathbf{e}})(1+2\mathbf{e}\cdot\mathbf{E}\cdot\mathbf{e})}} = \arcsin\frac{\bar{\mathbf{e}}\cdot\mathbf{C}\cdot\mathbf{e}}{\sqrt{(\bar{\mathbf{e}}\cdot\mathbf{C}\cdot\bar{\mathbf{e}})(\mathbf{e}\cdot\mathbf{C}\cdot\mathbf{e})}}$$
$$\varepsilon_v = \det\mathbf{F}-1 = \det(\mathbf{1}+\mathbf{H})-1 = \sqrt{\det(\mathbf{1}+2\mathbf{E})}-1 = \sqrt{\det\mathbf{C}}-1 \tag{5.5.5}$$

$\mathbf{H}$ is the *displacement gradient tensor*:

$$\mathbf{H} = \frac{\partial\mathbf{u}}{\partial\mathbf{r}_o} = \mathbf{F}-\mathbf{1} \quad\Leftrightarrow\quad H_{ik} = \frac{\partial u_i}{\partial X_k} = \frac{\partial x_i}{\partial X_k} - \delta_{ik} \tag{5.5.6}$$

$\mathbf{E}$ is the *Green strain tensor*:

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H}+\mathbf{H}^T+\mathbf{H}^T\mathbf{H}\right) \quad\Leftrightarrow\quad E_{ik} = \frac{1}{2}\left(H_{ik}+H_{ki}+H_{ji}H_{jk}\right) \tag{5.5.7}$$

$\mathbf{C}$ is the *Green deformation tensor*:

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{1}+2\mathbf{E} \tag{5.5.8}$$

When dealing with large deformations, it is more convenient to use the deformation gradient tensor $\mathbf{F}$ and the deformation tensor $\mathbf{C}$ than the displacement gradient tensor $\mathbf{H}$ and the strain tensor $\mathbf{E}$. For practical reasons we shall continue the discussion under the assumption of homogeneous deformations, as expressed by (5.5.4). Before we consider the general case, we will study two special cases, which the general case may be decomposed into.

First we consider a *rigid-body motion*, as given by (4.5.2):

$$\mathbf{r} = \mathbf{u}_o+\mathbf{R}\cdot\mathbf{r}_o, \quad \mathbf{u}_o = \mathbf{u}_o(t), \quad \mathbf{R} = \mathbf{R}(t) \tag{5.5.9}$$

The displacement vector $\mathbf{u}_o(t)$ represents a translation, and the *rotation tensor* $\mathbf{R}$, which is an *orthogonal tensor*, represents a rigid-body rotation about the point $(\mathbf{r}-\mathbf{u}_o)$. The deformation gradient $\mathbf{F}$ is now equal to the orthogonal tensor $\mathbf{R}$:

$$\mathbf{F} = \mathbf{R}, \quad \mathbf{R}^{-1} = \mathbf{R}^T, \quad \det\mathbf{F} = \det\mathbf{R} = 1 \tag{5.5.10}$$

Next we consider a motion resulting in what is called *homogeneous pure strain*, in which the deformation gradient $\mathbf{F}$ is a *positive definite symmetric tensor* $\mathbf{U}$. For simplicity we set $\mathbf{u}_o(t) = 0$.

$$\mathbf{F} = \mathbf{U} \quad\Rightarrow\quad \mathbf{r} = \mathbf{U}\cdot\mathbf{r}_o, \quad \mathbf{U}^T = \mathbf{U} = \mathbf{U}(t), \quad \det\mathbf{U} > 0 \quad\Rightarrow\quad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2 \tag{5.5.11}$$

The symmetric tensors $\mathbf{C}$ and $\mathbf{U}$ are coaxial, i.e. both tensors have the same principal directions represented by the unit vectors $\mathbf{a}_i(t)$. If the principal values of $\mathbf{U}$ are $\lambda_i(t)$,

$\mathbf{C}$ has the principal values λ_i^2. A material line element $d\mathbf{r}_o$ in K_o that is parallel to one of the principal directions, such that $d\mathbf{r}_o = ds_o\,\mathbf{a}_i$, will in the deformed configuration in K be parallel to its original orientation. This follows from the motion (5.5.11) and the property (4.3.28) for symmetric second order tensors:

$$d\mathbf{r} = \mathbf{U}\cdot d\mathbf{r}_o = \mathbf{U}\cdot ds_o\,\mathbf{a}_i = \lambda_i\,ds_o\,\mathbf{a}_i \quad \Rightarrow \quad d\mathbf{r}||d\mathbf{r}_o||\mathbf{a}_i$$

Hence the three vectors $d\mathbf{r}$, $d\mathbf{r}_\text{o}$, and $\mathbf{a}_\text{i}$ are parallel. Furthermore we see that:

$$ds \equiv |d\mathbf{r}| = \lambda_i\,ds_o$$

The result shows that the λ_i are stretches, appropriately called *principal stretches* of the deformation. For geometrical reasons a stretch has to be positive. Therefore the tensor $\mathbf{U}$, which now is called a *stretch tensor*, has to be positive definite. The following example, shown in Fig. 5.5.2, will illustrate this property of the motion. We choose a coordinate system Ox with base vectors $\mathbf{e}_i$ that are parallel to the principal directions $\mathbf{a}_i$. The deformation gradient $\mathbf{F}=\mathbf{U}$, the deformation tensor $\mathbf{C}$, and the motion now have these representations:

$$F_{ik} = U_{ik} = \lambda_i\,\delta_{ik}, \quad C_{ik} = \lambda_i^2\,\delta_{ik}, \quad \mathbf{r} = \mathbf{U}\cdot\mathbf{r}_o \Leftrightarrow \quad x_i = \lambda_i X_i \qquad (5.5.12)$$

In Fig. 5.5.2 the principal stretches has arbitrarily been chosen to be: $\lambda_1 = 3, \lambda_2 = 2$, and $\lambda_3 = 1$. The same material volume element is shown in the two configurations K_o and K. In K_o the element is chosen to have orthogonal edges parallel to the principal directions $\mathbf{a}_i$. The figure shows that the edges of the same element in K are orthogonal, have been stretched but not rotated. Note that the material element may have rotated on its course from K_o to K.

The general homogeneous deformation, represented by the motion (5.5.4), may be decomposed into a deformation of pure strain and a rigid-body motion. We are going to demonstrate this in the following way, using the two-dimensional case in Fig. 5.5.3 as illustration. The deformation from the reference configuration K_o to the present configuration K at time t is assumed to be given by the deformation gradient $\mathbf{F}$. Then we can determine the deformation tensor $\mathbf{C}=\mathbf{F}^T\mathbf{F}$ and its principal directions $\mathbf{a}_i$. A material volume element with orthogonal edges parallel to $\mathbf{a}_i$ in K_o

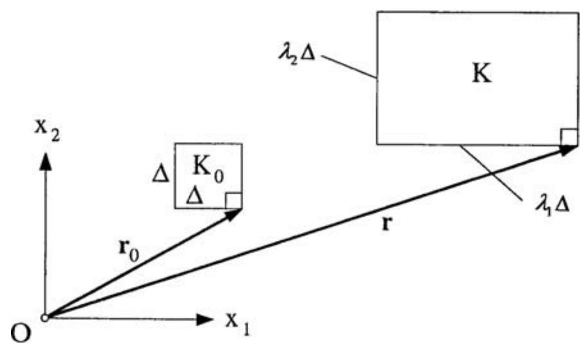


Fig. 5.5.2 Homogeneous pure strain

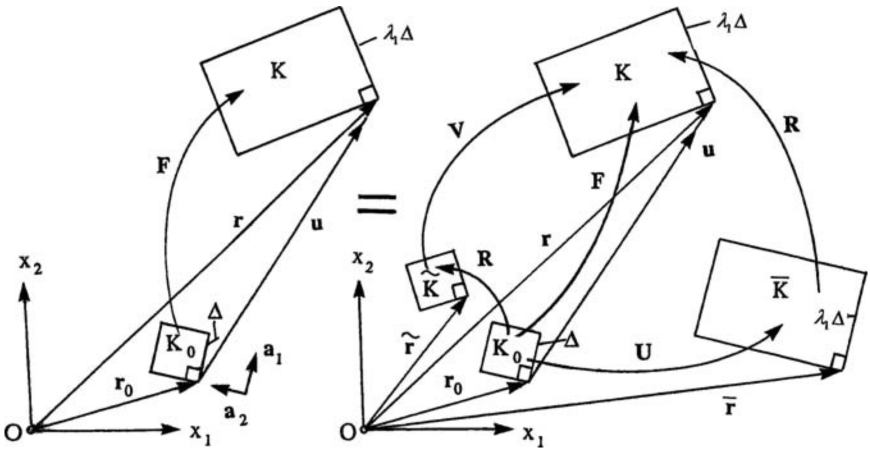


Fig. 5.5.3 Polar decomposition of a general homogeneous deformation

will also in K have orthogonal edges, as indicated in Fig. 5.5.3. Without regard to the actual motion and deformation history of the material element between K_o and K, we may imagine two different alternatives to accomplish the change of configuration from K_o to K, each alternative performed in two steps.

Alternative I: The material is first deformed from K_o to $\bar{K}$ by pure strain:

$$\bar{\mathbf{r}} = \mathbf{U} \cdot \mathbf{r}_o, \quad \mathbf{U}^T = \mathbf{U}$$

Then the material is given a rigid-body motion from $\bar{K}$ to K:

$$\mathbf{r} = \mathbf{u}_o + \mathbf{R} \cdot \bar{\mathbf{r}}, \quad \mathbf{R}^{-1} = \mathbf{R}^T, \quad \det \mathbf{R} = 1$$

The translation $\mathbf{u}_o$ is not shown in Fig. 5.5.3. The change from the reference configuration K_o to the present configuration K is then given by:

$$\mathbf{r} = \mathbf{u}_o + \mathbf{RU} \cdot \mathbf{r}_o \quad \Rightarrow \quad \mathbf{F} = \mathbf{RU}, \quad \mathbf{R}^{-1} = \mathbf{R}^T, \det \mathbf{R} = 1, \quad \mathbf{U}^T = \mathbf{U} \quad (5.5.13)$$

R is called the *rotation tensor* and **U** is called the *right stretch tensor*.

Alternative II: The material is first brought from K_o to $\tilde{K}$ by a rigid-body motion:

$$\tilde{\mathbf{r}} = \tilde{\mathbf{u}}_o + \mathbf{R} \cdot \bar{\mathbf{r}}, \quad \mathbf{R}^{-1} = \mathbf{R}^T, \quad \det \mathbf{R} = 1$$

The translation $\tilde{\mathbf{u}}_o$ is not shown in Fig. 5.5.3. Then the material is deformed from $\tilde{K}$ to K by pure strain:

$$\mathbf{r} = \mathbf{V} \cdot \tilde{\mathbf{r}}, \quad \mathbf{V}^T = \mathbf{V}$$

The change from the reference configuration K_o to the present configuration K is then given by:

$$\mathbf{r} = \mathbf{u}_o + \mathbf{VR} \cdot \mathbf{r}_o \quad \Rightarrow \quad \mathbf{u}_o = \mathbf{V} \cdot \tilde{\mathbf{u}}_o, \quad \mathbf{F} = \mathbf{VR} \quad (5.5.14)$$

R is again the *rotation tensor* and **V** is called the *left stretch tensor*.

Note that the real motion and deformation between K_o and K will in general be quite different from any of the two special configuration changes presented above.

The important results of the demonstrations related to Fig. 5.5.3 and the equations (5.5.13) and (5.5.14) are that the general homogeneous deformation (5.5.4) may be considered to consist of a pure strain and a rigid-body motion, and furthermore, that the deformation gradient **F** may be decomposed into a composition of an orthogonal tensor **R**, which represents a rigid-body rotation, and positive definite symmetric tensors **U** or **V**, which represent pure strain. The decomposition of the deformation gradient may be done in one of two alternatives ways:

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR} \tag{5.5.15}$$

It also follows from how the decompositions (5.5.15) were developed above, that these decompositions are unique.

An analytical proof for the result (5.5.15), which is called *polar decomposition* of the tensor **F**, is presented in Sect. 4.6.2. A mathematical condition for the tensor **F** in that proof is that **F** has to be a *non-singular tensor* with a positive determinant, $\det \mathbf{F} > O$. This condition also follows from the physical condition that the volumetric strain never can be equal to or less than -1. Material volume cannot vanish. From the formula for the volumetric strain in (5.5.5) this condition requires that the determinant of **F** is larger than zero.

The two tensors **U** and **V** represent the same state of strain. From Fig. 5.5.3 it is seen, and from (5.5.15) it follows, that the tensor **V** is the **R**-rotation of tensor **U**:

$$\mathbf{V} = \mathbf{RUR}^T \tag{5.5.16}$$

The principal directions of **V** are **R**-rotations of the principal directions $\mathbf{a}_i$ of **U**. The two tensors **V** and **U** have the same principal values $\lambda_i > 0$, called the *principal stretches*. The principal stretch λ_i represents the ratio between the lengths in K and K_o of a material line element that in K_o is parallel to the principal direction $\mathbf{a}_i$ of **U**, see Fig. 5.5.3. It is in reference to their positions relative to the rotation tensor **R** in (5.5.15) that **U** is called the *right stretch tensor* and **V** is called the *left stretch tensor*.

The right stretch tensor **U** is closely related to the Green deformation tensor **C**.

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2 \tag{5.5.17}$$

The two tensors **C** and **U** are coaxial. The principal values ζ_i of **C** and λ_i of **U** and **V** are related by:

$$\zeta_i = \lambda_i^2 \tag{5.5.18}$$

The principal strains are:

$$\varepsilon_i = \lambda_i - 1 \tag{5.5.19}$$

ε_i are different from the principal values E_i of the Green strain tensor **E**. Formula (5.2.18) gives:

$$\varepsilon_i = \sqrt{1+2E_i} - 1 \tag{5.5.20}$$

For the volumetric strain we get from (5.5.5):

$$\varepsilon_v = \det \mathbf{F} - 1 = \det \mathbf{U} - 1 = \det \mathbf{V} - 1 = \lambda_1 \lambda_2 \lambda_3 - 1 \tag{5.5.21}$$

The Green deformation tensor **C** is also called the *right deformation tensor* due to its relation to the right stretch tensor. A *left deformation tensor* **B** related to the left stretch tensor **V** plays an important role in the modeling of isotropic materials, e.g. in Sect. 7.10, and is defined by:

$$\mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^T = \mathbf{R}\mathbf{C}\mathbf{R}^T \tag{5.5.22}$$

We see that **B** is the **R**-rotation of **C**.

In a general, non-homogeneous deformation $\mathbf{r} = \mathbf{r}(\mathbf{r}_o, t)$ there is deformation gradient **F** for every particle $\mathbf{r}_o$, such that:

$$\begin{aligned} &\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R} \\ &\mathbf{F} = \mathbf{F}(\mathbf{r}_o, t), \quad \mathbf{R} = \mathbf{R}(\mathbf{r}_o, t), \quad \mathbf{U} = \mathbf{U}(\mathbf{r}_o, t), \quad \mathbf{V} = \mathbf{V}(\mathbf{r}_o, t) \end{aligned} \tag{5.5.23}$$

The deformation of the material in the neighborhood of the particle $\mathbf{r}_o$ is determined by a pure rotation **R** and a pure strain **U** or **V**. These tensors may be determined as follows. First we compute the deformation gradient **F** from its definition in (5.5.3). Then the Green deformation tensor **C**, the right stretch tensor **U**, and the rotation tensor **R** may be determined from:

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} = \mathbf{U}^2 \quad \Rightarrow \quad \mathbf{U} = \sqrt{\mathbf{C}}, \quad \mathbf{F} = \mathbf{R}\mathbf{U} \quad \Rightarrow \quad \mathbf{R} = \mathbf{F}\mathbf{U}^{-1} \tag{5.5.24}$$

If a deformation is an *isotropic deformation*, also called a *form invariant deformation* in a particle, any direction through the particle is a principal direction, and there is only one unique principal stretch λ:

$$\mathbf{U} = \mathbf{V} = \mu \mathbf{1} \quad \Leftrightarrow \quad \mathbf{F} = \lambda \mathbf{R} \quad \Leftrightarrow \quad \mathbf{C} = \mathbf{B} = \mathbf{U}^2 = \lambda^2 \mathbf{1} \tag{5.5.25}$$

All the tensors **U, V, C**, and **B** are isotropic tensors. The volumetric strain is:

$$\varepsilon_v = \det \mathbf{F} - 1 = \lambda^3 - 1 \tag{5.5.26}$$

If the deformation also is homogeneous the shape of any material volume element is unchanged by the deformation, see Fig. 5.5.4.

A deformation in which the volume is preserved is called an *isochoric deformation* or a *volume invariant deformation*, see Fig. 5.5.4. In this case:

$$\varepsilon_v = 0 \quad \Leftrightarrow \quad \det \mathbf{F} = \det \mathbf{U} = \det \mathbf{V} = 1 \tag{5.5.27}$$

From the definitions of **F** in (5.5.3) and **L** in (5.4.3) we develop the relationships:

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F} \quad \Leftrightarrow \quad \mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} \tag{5.5.28}$$

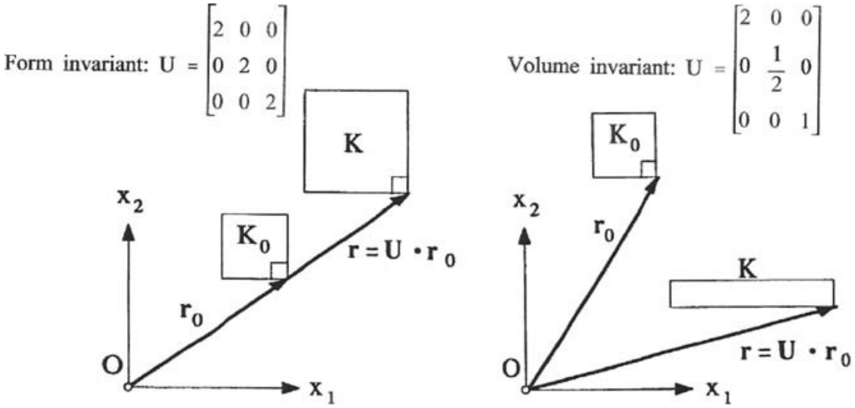


Fig. 5.5.4 Form invariant deformation and volume invariant deformation

The development of this result and the following relations is given as Problem 5.11:

$$\mathbf{D} = \frac{1}{2}\mathbf{R}\left(\dot{\mathbf{U}}\mathbf{U}^{-T} + \mathbf{U}^{-1}\dot{\mathbf{U}}\right)\mathbf{R}^T \tag{5.5.29}$$

$$\mathbf{W} = \dot{\mathbf{R}}\mathbf{R}^T + \frac{1}{2}\mathbf{R}\left(\dot{\mathbf{U}}\mathbf{U}^{-T} - \mathbf{U}^{-1}\dot{\mathbf{U}}\right)\mathbf{R}^T \tag{5.5.30}$$

$$\dot{\mathbf{E}} = \frac{1}{2}\mathbf{F}^T\left(\mathbf{L} + \mathbf{L}^T\right)\mathbf{F} = \mathbf{F}^T\mathbf{D}\mathbf{F} \tag{5.5.31}$$

For later applications we introduce the determinant J of the deformation gradient $\mathbf{F}$:

$$J = \det\mathbf{F} = \det(F_{ik}) = \det\left(\frac{\partial x_i}{\partial X_k}\right) \tag{5.5.32}$$

J is called the *Jacobi determinant*, or for short the *Jacobian*, of the mapping, or of the motion, $x(X,t)$. The name Jacobian is attributed to Carl Gustav Jacob Jacobi [1804–1851]. It may be shown that, see Problem 5.11:

$$\dot{J} = J\,\mathrm{div}\;\mathbf{v} \tag{5.5.33}$$

Note that the state of pure strain for large deformation, that is when $\mathbf{F}(\mathbf{r}_o,t) = \mathbf{U}(\mathbf{r}_o,t)$ and $\mathbf{R}(\mathbf{r}_o,t) = \mathbf{1}$, is not irrotational in the sense that $\mathbf{W} = \mathbf{0}$. Furthermore, we see that "irrotational motion", in the sense that $\dot{\mathbf{R}} = \mathbf{0}$, does not imply that $\mathbf{W} = \mathbf{0}$. For large deformations $\dot{\mathbf{E}} \neq \mathbf{D}$, i.e. the rate of strain tensor is not equal to the rate of deformation tensor. For rigid-body motion the right stretch tensor $\mathbf{U} = \mathbf{0}$, and formula (5.5.30) gives $\mathbf{W} = \dot{\mathbf{R}}\mathbf{R}^T$, which agrees with formula (4.5.13).

5.5.1 Special Types of Deformations and Flows

We shall discuss characteristic features of special types of deformations and flows that occur in experimental investigations of material behavior and material properties.

To simplify the illustrations of the deformation patterns we shall assume homogeneous deformation, which in general is given by the motion:

$$\mathbf{r} = \mathbf{u}_o + \mathbf{F} \cdot \mathbf{r}, \quad \mathbf{u}_o = \mathbf{u}_o(t), \quad \mathbf{F} = \mathbf{F}(t) \quad \Leftrightarrow \quad x_i = u_{oi} + F_{ik}X_k \tag{5.5.34}$$

It may be shown, see Problem 5.12, that in a homogeneous deformation material planes and straight lines in K_o deform into planes and straight lines in K. In the examples below we choose for simplicity, and without loss of generality, $\mathbf{u}_o = \mathbf{0}$.

Example 5.3. Pure Rotation About an Axis

The simplest homogeneous deformation is the rigid body motion presented in Sect. 4.5.1. Fig. 5.5.5a shows a rigid-body rotation about the x_3-axis of a material cube with edges of length d. The motion is given by:

$$x_1 = \cos\phi X_1 - \sin\phi X_2, \quad x_2 = \sin\phi X_1 + \cos\phi X_2, \quad x_3 = X_3 \tag{5.5.35}$$

The deformation gradient $\mathbf{F}$ is equal to the rotation tensor $\mathbf{R}$, and matrix of these tensors is:

$$F \equiv R = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.5.36}$$

This deformation obviously results in no strains.

Example 5.4. General Extension

The motion:

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3 \tag{5.5.37}$$

where the parameters λ_i may be functions of time only, results in a deformation gradient $\mathbf{F}$ equal to the right stretch tensor $\mathbf{U}$, and represented by the matrix:

$$F \equiv U = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{5.5.38}$$

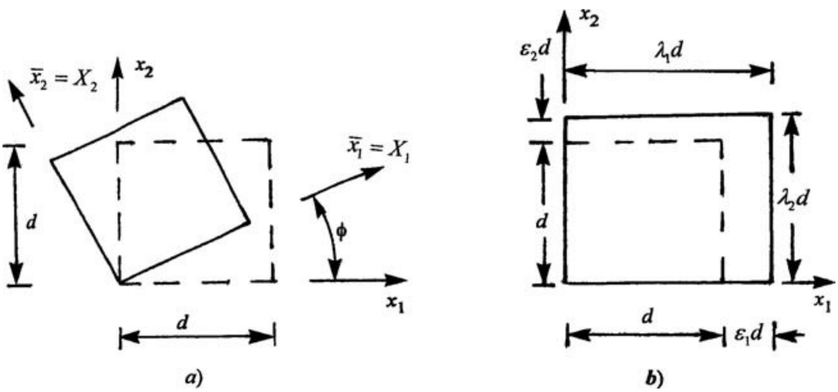


Fig. 5.5.5 Homogeneous deformation. **a)** Pure rotation about an axis. **b)** General extension

It follows that the parameters λ_i are the *principal stretches* and that the principal axes of strain are fixed in the directions of the coordinate axes and thus are represented by the same material lines at all times. Figure 5.5.5b shows the deformation of a material cube with edges of length d and volume d^3 in K_o. It follows that $\mathbf{R} = \mathbf{1}$. The deformation is a case of *pure strain* called *general extension*. The strain tensor $\mathbf{E}$ is obtained from (5.2.13, 5.2.14), and matrix of $\mathbf{E}$ is:

$$E = \frac{1}{2}\begin{pmatrix} \lambda_1^2 - 1 & 0 & 0 \\ 0 & \lambda_2^2 - 1 & 0 \\ 0 & 0 & \lambda_3^2 - 1 \end{pmatrix} \tag{5.5.39}$$

The longitudinal strains in the direction of the coordinate axes, (5.2.18), are the principal strains:

$$\varepsilon_{ii} = \varepsilon_i = \lambda_i - 1 \tag{5.5.40}$$

The volumetric strain may be found directly as:

$$\varepsilon_v = \frac{(\lambda_1 d)(\lambda_2 d)(\lambda_3 d) - d^3}{d^3} = \lambda_1 \lambda_2 \lambda_3 - 1 \tag{5.5.41}$$

But (5.2.23) gives the same result.

The particle velocity $\mathbf{v}$ is given by the components:

$$v_i = \frac{\partial x_i}{\partial t} = \dot{\lambda}_i X_i = \frac{\dot{\lambda}_i}{\lambda_i} x_i = \frac{\dot{\varepsilon}_i}{\lambda_i} x_i \tag{5.5.42}$$

The velocity gradient $\mathbf{L}$ and the deformation rate $\mathbf{D}$ may then be found directly:

$$L = D = \left(\frac{\partial v_i}{\partial x_k}\right) = \begin{pmatrix} \dot{\lambda}_1/\lambda_1 & 0 & 0 \\ 0 & \dot{\lambda}_2/\lambda_2 & 0 \\ 0 & 0 & \dot{\lambda}_3/\lambda_3 \end{pmatrix} = \begin{pmatrix} \dot{\varepsilon}_1/\lambda_1 & 0 & 0 \\ 0 & \dot{\varepsilon}_2/\lambda_2 & 0 \\ 0 & 0 & \dot{\varepsilon}_3/\lambda_3 \end{pmatrix} \tag{5.5.43}$$

The result may also be obtained from $(5.5.28)_2$. The rate of rotation tensor $\mathbf{W} = \mathbf{0}$, which means that the motion (5.5.37) is an *irrotational flow*. This type of irrotational flow is called *extensional flow* or *elongational flow*. Note that the strains ε_i in (5.5.40) are defined for large deformations. For small deformations the matrix elements in (5.5.43) reduce to $\dot{\varepsilon}_i$.

The special case $\lambda_i = \lambda$ is called *irrotational uniform dilatation* and is a *form invariant or shear free deformation*.

Example 5.5. Isochoric Uniaxial Extension

The word *isochoric* means equal volume. The expression uniaxial extension refers to one preferred direction. In (5.5.37, 5.5.38, 5.5.39, 5.5.40) we set:

$$\lambda_1 = c(t), \quad \lambda_2 = \lambda_3 = \frac{1}{\sqrt{c}} \tag{5.5.44}$$

Then according to formula (5.5.41):

$$\varepsilon_v = c\left(\frac{1}{\sqrt{c}}\right)^2 - 1 = 0 \tag{5.5.45}$$

A related isochoric deformation for which:

$$\lambda_1 = \lambda_2 = c(t), \quad \lambda_3 = \frac{1}{c^2} \tag{5.5.46}$$

is called *isochoric biaxial extension*.

The particle velocity **v** and the deformation rate **D** in isochoric uniaxial extension are obtained from (5.5.42, 5.5.43):

$$\mathbf{v} = \left[\frac{\dot{c}}{c}x_1, -\frac{\dot{c}}{2c}x_2, -\frac{\dot{c}}{2c}x_3\right], \quad D = \begin{pmatrix} \dot{c}/c & 0 & 0 \\ 0 & -\dot{c}/2c & 0 \\ 0 & 0 & -\dot{c}/2c \end{pmatrix} \tag{5.5.47}$$

It follows directly from (5.5.45) or from (5.4.11) and (5.5.47) that the volumetric strain rate is zero. This type of irrotational flow is called *isochoric uniaxial extensional flow* or *isochoric uniaxial elongational flow*.

Example 5.6. Isochoric Planar Extension. Pure Shear Flow

In (5.5.37) we set:

$$\lambda_1 = c(t), \quad \lambda_2 = c^{-1}, \quad \lambda_3 = 1 \tag{5.5.48}$$

for which (5.5.41) gives:

$$\varepsilon_v = \lambda_1\lambda_2\lambda_3 - 1 = cc^{-1}1 - 1 = 0 \tag{5.5.49}$$

The principal strains are:

$$\varepsilon_1 = c - 1, \quad \varepsilon_2 = c^{-1} - 1, \quad \varepsilon_3 = 0 \tag{5.5.50}$$

The motion is illustrated in Fig. 5.5.6a, from which we derive the following expressions for the strain along the material line *HG* and the shear strain with respect to the two lines *HE* and *HG*.

$$\varepsilon = \sqrt{\frac{1}{2}\left(c^2 + c^{-2}\right)} - 1, \quad \gamma = 2\arctan\frac{c^2 - 1}{c^2 + 1} \tag{5.5.51}$$

For small strains, i.e. $|\varepsilon_i| = |\lambda_i - 1| \ll 1$, we shall find that approximately:

$$\varepsilon_1 = -\varepsilon_2, \quad \varepsilon = 0, \quad \gamma = 2\varepsilon_1 \tag{5.5.52}$$

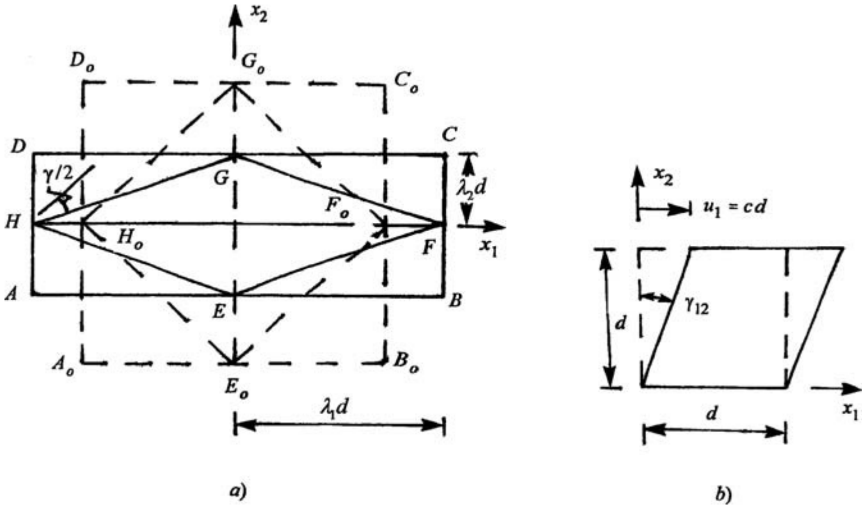


Fig. 5.5.6 **a)** Isochoric planar extension. Pure shear flow. **b)** Simple shear

The particle velocity **v** and the deformation rate **D** are obtained from (5.5.42, 5.5.43).

$$\mathbf{v} = \left[\frac{\dot{c}}{c}x_1, -\frac{\dot{c}}{c}x_2, 0\right], \quad D = \begin{pmatrix} \dot{c}/c & 0 & 0 \\ 0 & -\dot{c}/c & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.5.53}$$

The material lines H_oG_o and H_oE_o have directions given by the unit vectors:

$$\mathbf{e} = [1, 1, 0]\frac{1}{\sqrt{2}}, \quad \bar{\mathbf{e}} = [1, -1, 0]\frac{1}{\sqrt{2}} \tag{5.5.54}$$

The longitudinal strain rate along the material line H_oG_o and the shear strain with respect to the material lines H_oE_o and H_oG_o are:

$$\dot{\varepsilon} = \mathbf{e} \cdot \mathbf{D} \cdot \mathbf{e} = e_i D_{ik} e_k = 0, \quad \dot{\gamma} = 2\bar{\mathbf{e}} \cdot \mathbf{D} \cdot \mathbf{e} = 2\,\bar{e}_i D_{ik} e_k = 2\dot{c}/c \tag{5.5.55}$$

These results will be compared with similar results presented in the next two examples.

Example 5.7. Pure Shear Deformation

Response of solid materials to shear strain is sometimes performed using a shear panel. The quadratic element $E_oF_oG_oH_o$ in Fig. 5.5.6a may be used to illustrate the test specimen. Such a panel is fixed to rigid beams along the edges E_oF_o, F_oG_o, G_oH_o, and H_oE_o. The beams are attached to each other by pin joints at the ends E_o, F_o, G_o, and H_o. Deformation of the frame mechanism introduces a *pure shear deformation* of the specimen. We shall find that the deformation in the plane of the panel is described by:

$$\lambda_1 = c(t), \quad \lambda_2 = \sqrt{2 - c^2}, \quad \lambda_3 = 1 \tag{5.5.56}$$

The edges $E_oF_o,\ F_oG_o,\ G_oH_o$, and H_oE_o of the panel are not strained:

$$(G_oF_o)^2 = d^2 + d^2 = 2d^2, \quad GF = (\lambda_1 d)^2 + (\lambda_2 d)^2 = c^2 d^2 + \left(2 - c^2\right) d^2 = 2d^2 \quad \text{etc.}$$

The shear strain γ with respect to the material lines and H_oE_o and H_oG_o is found from the trigonometric formula:

$$\tan\frac{\gamma}{2} = \tan(\pi - \angle FHG) = \frac{\tan(\pi/4) - \tan(\angle FHG)}{1 + \tan(\pi/4)\tan(\angle FHG)},$$

$$\tan(\angle FHG) = \frac{\lambda_2 d}{\lambda_1 d} = \frac{\sqrt{2-c^2}}{c} \Rightarrow$$

$$\gamma = 2\arctan\frac{c - \sqrt{2-c^2}}{c + \sqrt{2-c^2}} \tag{5.5.57}$$

For small strains, i.e. $|\varepsilon_i| = |\lambda_i - 1| \ll 1$, we shall find that approximately:

$$\varepsilon_1 = \lambda_1 - 1 = c - 1 \quad \Rightarrow \quad \varepsilon_2 = \lambda_2 - 1 = \sqrt{2-c^2} - 1 = \sqrt{2 - (1+\varepsilon_1)^2} - 1 \approx -\varepsilon_1 \tag{5.5.58}$$

Example 5.8. Simple Shear Deformation and Simple Shear Flow

The deformation resulting from the motion given by:

$$x_1 = X_1 + c(t)X_2, \quad x_2 = X_2, \quad x_3 = X_3 \tag{5.5.59}$$

is called *simple shear deformation*. Figure 5.5.6b shows the deformation of a material element which in K_o is a cube with edges of length d. The deformation gradient $\mathbf{F}$ and the strain tensor $\mathbf{E}$, obtained from (5.2.13, 5.2.14), are represented by the matrices:

$$F = \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E = \frac{1}{2}\begin{pmatrix} 0 & c & 0 \\ c & c^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.5.60}$$

The principal strains, the principal stretches, and the shear strain with respect to the x_α-directions are:

$$\varepsilon_1 = \varepsilon_3 = 0, \quad \varepsilon_2 = \sqrt{1+c^2} - 1, \quad \lambda_1 = \lambda_3 = 1, \quad \lambda_2 = \sqrt{1+c^2}$$
$$\gamma_{12} = \arctan c \tag{5.5.61}$$

The volumetric strain $\varepsilon_v = \det\mathbf{F} - 1 = 0$, so this is an isochoric deformation.

The velocity $\mathbf{v}$, the rate of deformation tensor $\mathbf{D}$, and the rate of rotation tensor $\mathbf{W}$ are represented by:

$$\mathbf{v} = [\dot{c}x_2, 0, 0], \quad D = \frac{1}{2}\begin{pmatrix} 0 & \dot{c} & 0 \\ \dot{c} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad W = \frac{1}{2}\begin{pmatrix} 0 & c & 0 \\ -\dot{c} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.5.62}$$

These equations define *simple shear flow*. Simple shear deformation and simple shear flow are important in testing the shear response of solids and fluids.

5.5.2 The Continuity Equation in a Particle

The principle of conservation of mass for a body of volume V_o and density ρ_o in the reference configuration K_o and volume V and density ρ in the present configuration K, may be stated by the following expressions for the mass m of the body.

$$m = \int_{V_o} \rho_o \, dV_o = \int_{V} \rho \, dV \quad \text{at all times } t \tag{5.5.63}$$

We apply formula (5.5.32) and rewrite the expression (5.2.23) for the volumetric strain:

$$\varepsilon_v = \frac{dV}{dV_o} - 1 = \det \mathbf{F} - 1 = J - 1 \quad \Rightarrow$$

$$J = \det \mathbf{F} = \frac{dV}{dV_o} \tag{5.5.64}$$

The Jacobian J thus represents the ratio between the volume dV in the present configuration K and the volume dV_{o} in the reference configuration K_o of the same material element. The result $dV = J dV_{\mathrm{o}}$ is introduced into the integral on the right hand side of (5.5.63):

$$m = \int_{V_o} \rho_o \, dV_o = \int_{V_o} \rho J \, dV_o \quad \text{at all times } t$$

The two integral are equal for any choice of volume V_o. This implies that the integrands must be equal. Therefore we have obtained the result:

$$\rho J = \rho_o \quad \Leftrightarrow \quad \rho \det \mathbf{F} = \rho_o \tag{5.5.65}$$

This result is called the *continuity equation in a particle*.

5.5.3 Reduction to Small Deformations

In Sect. 5.3.2 the concept of *small deformations* was defined by the condition:

$$\mathrm{norm}\mathbf{H} \ll 1 \quad \Rightarrow \quad \mathrm{norm}\mathbf{E} \ll 1 \tag{5.5.66}$$

$\mathbf{H}$ is the displacement gradient from (5.5.6). Due to the condition (5.5.66) the Green strain tensor $\mathbf{E}$ from (5.57) is reduced to:

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right) \quad \Leftrightarrow \quad E_{ik} = \frac{1}{2}\left(u_{i,k} + u_{k,i}\right) \tag{5.5.67}$$

In Sect. 5.3.6 a special *rotation tensor for small deformations* was defined through (5.3.28):

$$\tilde{\mathbf{R}} = \frac{1}{2}\left(\mathbf{H} - \mathbf{H}^T\right) \quad \Leftrightarrow \quad \tilde{R}_{ik} = \frac{1}{2}\left(u_{i,k} - u_{k,i}\right) \tag{5.5.68}$$

Since the absolute values of the components to $\mathbf{E}$ are small numbers $(\ll 1)$, we may make the following approximations:

$$(\mathbf{1} + \mathbf{E})^2 = (\mathbf{1} + \mathbf{E})(\mathbf{1} + \mathbf{E}) \approx \mathbf{1} + 2\mathbf{E} = \mathbf{C} = \mathbf{U}^2 \quad \text{and} \quad (\mathbf{1} + \mathbf{E})(\mathbf{1} - \mathbf{E}) \approx \mathbf{1}$$

Hence:

$$\mathbf{U} \approx \mathbf{1} + \mathbf{E}, \quad \mathbf{U}^{-1} \approx \mathbf{1} - \mathbf{E}$$

Furthermore:

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1} \approx (\mathbf{1} + \mathbf{H})(\mathbf{1} - \mathbf{E}) \approx \mathbf{1} + \tilde{\mathbf{R}} \quad \text{and} \quad \mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T \approx \mathbf{U}$$

We conclude that for small deformations the following *linear decomposition* applies.

$$\mathbf{F} = \mathbf{1} + \mathbf{H} = \mathbf{1} + \tilde{\mathbf{R}} + \mathbf{E} \quad \Leftrightarrow \quad \mathbf{F} = \tilde{\mathbf{R}} + \mathbf{U} \tag{5.5.69}$$

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1} = \mathbf{1} + \tilde{\mathbf{R}}, \quad \mathbf{R}^T = \mathbf{1} - \tilde{\mathbf{R}}, \quad \mathbf{U} = \mathbf{V} = \mathbf{1} + \mathbf{E} \tag{5.5.70}$$

5.5.4 Deformation with Respect to the Present Configuration

As mention in Sect. 5.4, in order to describe the motion and deformation of fluids, and sometimes solids with fluidlike behavior, we choose to work with Eulerian coordinates and with the present configuration K as reference configuration. The particles are now denoted by their place vectors $\mathbf{r}$ in K, or their place coordinates x. In order to present the motion of a particle P, which at time t is in place $\mathbf{r}$, we introduce a *"moving" configuration* of the material $\bar{K}$ at the time $\bar{t}$ in the interval $-\infty < \bar{t} \leq t$. The motion of the particle is then given by the place vector $\bar{\mathbf{r}}$. in Fig. 3.1.1, which for convenience now is denoted by $\mathbf{r}_t$.

$$\bar{\mathbf{r}} \equiv \mathbf{r}_t = \mathbf{r}_t(\mathbf{r}, \bar{t}) = \mathbf{r}(\mathbf{r}_o, \bar{t}), \quad -\infty < \bar{t} \leq t \tag{5.5.71}$$

The deformation of the material in $\bar{K}$ relative to K is defined by the *relative deformation gradient* $\mathbf{F}_t$:

$$\mathbf{F}_t = \mathbf{F}_t(\mathbf{r}, \bar{t}) = \frac{\partial \mathbf{r}_t}{\partial \mathbf{r}} = \operatorname{grad} \mathbf{r}_t \tag{5.5.72}$$

The relative deformation gradient $\mathbf{F}_t$ is polar decomposed into the *relative rotation tensor* $\mathbf{R}_t$ and the *relative right stretch tensor* $\mathbf{U}_t$, or into the *relative left stretch tensor* $\mathbf{V}_t$ and the *relative rotation tensor* $\mathbf{R}_t$:

$$\mathbf{F}_t = \mathbf{R}_t \mathbf{U}_t = \mathbf{V}_t \mathbf{R}_t \tag{5.5.73}$$

From $\mathbf{F}_t$ we define the *relative right deformation tensors* $\mathbf{C}_t$ and the *relative left deformation tensor* $\mathbf{B}_t$, and can then determine the *relative right stretch tensor* $\mathbf{U}_t$, the *relative left stretch tensor* $\mathbf{V}_t$, and the *relative rotation tensor* $\mathbf{R}_t$:

$$\mathbf{C}_t = \mathbf{F}_t^T \mathbf{F}_t, \quad \mathbf{B}_t = \mathbf{F}_t \mathbf{F}_t^T, \quad \mathbf{U}_t = \sqrt{\mathbf{C}_t}, \quad \mathbf{V}_t = \sqrt{\mathbf{B}_t}, \quad \mathbf{R}_t = \mathbf{F}_t \mathbf{U}_t^{-1} \tag{5.5.74}$$

It is convenient to introduce the following notation. For any field $f_t(\mathbf{r},\bar{t})$:

$$\dot{f}_t = \dot{f}_t(\mathbf{r},t) = \frac{\partial}{\partial \bar{t}} f_t(\mathbf{r},\bar{t})\bigg|_{\bar{t}=t} \tag{5.5.75}$$

Then we get:

$$\dot{\mathbf{F}}_t = \frac{\partial}{\partial \bar{t}} \frac{\partial \mathbf{r}_t(\mathbf{r},\bar{t})}{\partial \mathbf{r}}\bigg|_{\bar{t}=t} = \frac{\partial}{\partial \mathbf{r}} \frac{\partial \mathbf{r}_t(\mathbf{r},\bar{t})}{\partial \bar{t}}\bigg|_{\bar{t}=t} = \frac{\partial \mathbf{v}(\mathbf{r},t)}{\partial \mathbf{r}} = \mathbf{L} \tag{5.5.76}$$

$\mathbf{L}$ is the *velocity gradient* at the time t.

At the time $\bar{t} = t$ we find that $\mathbf{F}_t = \mathbf{R}_t = \mathbf{U}_t = \mathbf{V}_t = \mathbf{1}$. Therefore:

$$\dot{\mathbf{F}}_t = \dot{\mathbf{R}}_t + \dot{\mathbf{U}}_t, \quad \dot{\mathbf{V}}_t = \dot{\mathbf{U}}_t = \frac{1}{2}\dot{\mathbf{C}}_t = \frac{1}{2}\dot{\mathbf{B}}_t \tag{5.5.77}$$

The relation $\mathbf{R}_t \mathbf{R}_t^T = \mathbf{1}$ implies that:

$$\dot{\mathbf{R}}_t \mathbf{R}_t^T + \mathbf{R}_t \dot{\mathbf{R}}_t^T = \dot{\mathbf{R}}_t + \dot{\mathbf{R}}_t^T = \mathbf{0} \quad \Rightarrow \quad \dot{\mathbf{R}}_t^T = -\dot{\mathbf{R}}_t \tag{5.5.78}$$

The result shows that $\dot{\mathbf{R}}_t$ is an antisymmetric tensor. By its definition it follows that $\dot{\mathbf{U}}_t$ is a symmetric tensor, and from $(5.5.77)_1$ we see that $\dot{\mathbf{U}}_t$ is the symmetric part of $\dot{\mathbf{F}}_t$. The rate of deformation tensor $\mathbf{D}$ defined by (5.4.4) and the rate of rotation tensor $\mathbf{W}$ defined by (5.4.5) may now be expressed by:

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right) = \dot{\mathbf{U}}_t = \frac{1}{2}\dot{\mathbf{C}}_t \tag{5.5.79}$$

$$\mathbf{W} = \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^T\right) = \dot{\mathbf{R}}_t \tag{5.5.80}$$

The *n. acceleration gradient* $\mathbf{L}_n$, the *n. deformation acceleration* $\mathbf{D}_n$, and the *n. rotation acceleration tensor* $\mathbf{W}_n$ are defined by:

$$\mathbf{L}_n = \frac{\partial^n}{\partial \bar{t}^n} \mathbf{F}_t(\mathbf{r},\bar{t})\bigg|_{\bar{t}=t} \equiv \mathbf{F}_t^{(n)}(\mathbf{r},t) \tag{5.5.81}$$

$$\mathbf{D}_n = \mathbf{U}_t^{(n)}(\mathbf{r},t), \quad \mathbf{W}_n = \mathbf{R}_t^{(n)}(\mathbf{r},t) \tag{5.5.82}$$

From these definitions we see that:

$$\mathbf{D}_0 = \mathbf{1}, \quad \mathbf{D}_1 = \mathbf{D}, \quad \mathbf{W}_0 = \mathbf{1}, \quad \mathbf{W}_1 = \mathbf{W} \tag{5.5.83}$$

The *Rivlin-Ericksen tensors*, named after R. S. Rivlin and J. L. Ericksen (1955), are defined by:

$$\mathbf{A}_n = \mathbf{C}_t^{(n)}(\mathbf{r},t) \tag{5.5.84}$$

It may be shown that, see Problem 5.16:

$$\mathbf{A}_n = \sum_{i=1}^{n} \binom{n}{i} \mathbf{D}_i \mathbf{D}_{n-i} = 2\mathbf{D}_n + \sum_{i=1}^{n-1} \binom{n}{i} \mathbf{D}_i \mathbf{D}_{n-i} \tag{5.5.85}$$

Note that $\mathbf{A}_1 = \mathbf{D}$. The Rivlin-Ericksen tensors appear in the constitutive equations of some complex fluid models, as for instance in the *Rivlin-Ericksen fluids* discussed in Sect. 11.5.

5.6 The Piola-Kirchhoff Stress Tensors

In problems involving large deformations it is often advantageous to apply Lagrangian description. All equations are then formulated in the reference configuration K_o, and Lagrangian coordinates (X,t) are used to describe the field functions. We shall now see what we then have to do with the equations of motion and the consequences this will have for the stress concept.

Figure 5.6.1 shows a body in the reference configuration K_o and in the present configuration K. In K the body is subjected to contact forces on its surface A, represented by the stress vector $\mathbf{t}$, and body forces, represented by the vector $\mathbf{b}$. The first axiom of Euler from Sect. 3.2.1, equation (3.2.6), balances the total force on the body and the rate of change of the linear momentum of the body:

$$\int_A \mathbf{t}\, dA + \int_V \mathbf{b}\rho\, dV = \int_V \mathbf{a}\rho\, dV \tag{5.6.1}$$

dA is the area of a surface element on A, V is the volume of the body, ρ is the mass density, dV the volume of an element of volume, and $\mathbf{a}$ is the particle acceleration. The fields $\mathbf{t}$, $\mathbf{b}$, ρ, and $\mathbf{a}$ are all given in the present configuration K. But through the motion $\mathbf{r}(\mathbf{r}_o,t)$ the fields may be transferred to the reference configuration K_o. The material surfaces A and dA are in K_o represented by A_o and dA_o respectively. The volume of the body in K_o is V_o, and the material volume element is given by dV_o.

We define a stress vector $\mathbf{t}_o$ in K_o by the equalities:

$$\mathbf{t}_o\, dA_o = \mathbf{t}\, dA \quad \Leftrightarrow \quad \int_{A_o} \mathbf{t}_o\, dA_o = \int_A \mathbf{t}\, dA \tag{5.6.2}$$

Hence:

$$\mathbf{t}_o = \frac{dA}{dA_o}\mathbf{t} \tag{5.6.3}$$

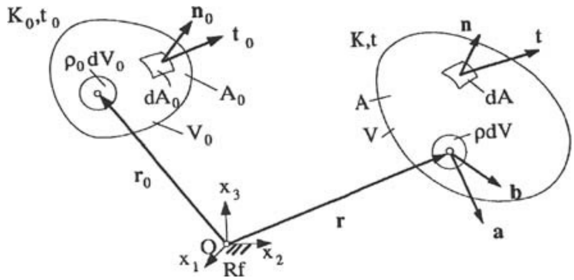


Fig. 5.6.1 Lagrangian description

The vector $\mathbf{t}_o$ thus represents contact forces in K and has the direction as $\mathbf{t}$ but is scaled such that (5.6.2) holds: The contact force $\mathbf{t}_o dA_o$ on the material element of area dA_o in K_o is equal to the contact force $\mathbf{t}\, dA$ on the same material element of area dA in K.

The mass of the body is conserved:

$$\rho\, dV = \rho_o\, dV_o \tag{5.6.4}$$

ρ_o is the mass density in K_o. The body force acting on an element of volume dV in K is $\mathbf{b}\rho dV$. The rate of change of momentum of the volume element is $\mathbf{a}\rho dV$. Due to the equality (5.6.4), we may write:

$$\int_V \mathbf{b}\rho\, dV = \int_{V_o} \mathbf{b}\rho_o\, dV_o, \quad \int_V \mathbf{a}\rho\, dV = \int_{V_o} \mathbf{a}\rho_o\, dV_o \tag{5.6.5}$$

The results in (5.6.2) and (5.6.4) make it possible to rewrite the expression (5.6.1) for Euler's 1. axiom to:

$$\int_{A_o} \mathbf{t}\, dA_o + \int_{V_o} \mathbf{b}\rho_o\, dV_o = \int_{V_o} \mathbf{a}\rho_o\, dV_o \tag{5.6.6}$$

By the Cauchy stress theorem in Sect. 3.2.4 the stress vector $\mathbf{t}$ on a surface through a particle P is uniquely defined by the Cauchy stress tensor $\mathbf{T}$ in the particle and the unit normal vector $\mathbf{n}$ to the surface through the relation:

$$\mathbf{t} = \mathbf{T}\mathbf{n} \equiv \mathbf{T} \cdot \mathbf{n} \quad \Leftrightarrow \quad t_i = T_{ik}\, n_k \quad \Leftrightarrow \quad t = T\, n \tag{5.6.7}$$

On the surface element dA_o in K_o we now introduce a unit normal vector $\mathbf{n}_o$, see Fig. 5.6.1. Analogous to the way the Cauchy stress tensor is defined above, we may then define the *first Piola-Kirchhoff stress tensor* $\mathbf{T}_o$ through the invariant linear relation:

$$\mathbf{t}_o = \mathbf{T}_o \cdot \mathbf{n}_o \tag{5.6.8}$$

This stress tensor is named after G. Piola (1833) and Kirchhoff (1852). The normal stress σ_o (1.2.6) in Sect. 1.2 was called a nominal stress or engineering stress. We see now that this is an example of a first Piola-Kirchhoff stress.

In order to determine the relationship between the Piola-Kirchhoff stress tensor and the Cauchy stress tensor, we first need to find a relation between the two unit normal vectors $\mathbf{n}$ and $\mathbf{n}_o$. Figure 5.6.2 shows a surface A_o in K_o with two material line elements $d\mathbf{s}_1$ and $d\mathbf{s}_2$ and the plane surface element dA_o they represent. In K the two corresponding line elements, represented by the vectors $d\mathbf{r}_1$ and $d\mathbf{r}_2$, form a plane surface element dA, which represents the deformed dA_o material element. The line elements in K_o and K are related through:

$$d\mathbf{r}_\alpha = \mathbf{F} \cdot d\mathbf{s}_\alpha = F_{ir}\, ds_{\alpha r}\, \mathbf{e}_i \tag{5.6.9}$$

Now:

$$dA_o\, \mathbf{n}_o = d\mathbf{s}_1 \times d\mathbf{s}_2, \quad dA\, \mathbf{n} = d\mathbf{r}_1 \times d\mathbf{r}_2 \tag{5.6.10}$$

We obtain from (5.6.10) and (5.6.9), and the expression (5.5.32) for the Jacobian J, the following result:

$$dA\, \mathbf{n} \cdot \mathbf{F} = \left(e_{ijk}\, F_{ir}\, ds_{1r}\, F_{js}\, ds_{2s}\right) F_{kt}\, \mathbf{e}_t = e_{rst} \det \mathbf{F}\, ds_{1r} ds_{2s}\, \mathbf{e}_t = J\, d\mathbf{s}_1 \times d\mathbf{s}_2 \quad \Rightarrow$$
$$dA\, \mathbf{n} \cdot \mathbf{F} = J\, dA_o\, \mathbf{n}_o \tag{5.6.11}$$

Hence:

$$\mathbf{n} = J \frac{dA_o}{dA} \mathbf{n}_o \cdot \mathbf{F}^{-1} \quad \Leftrightarrow \quad \frac{dA}{dA_o} = \mathbf{n}_o \cdot \mathbf{F}^{-1} \cdot \mathbf{n} \tag{5.6.12}$$

The first formula, to the left of the biimplication sign, is called the Nanson formula (1878). From the relations (5.6.8), (5.6.3), and (5.6.12) we obtain the result:

$$\mathbf{t}_o = \mathbf{T}_o \cdot \mathbf{n}_o = \frac{dA}{dA_o} \mathbf{T} \cdot \mathbf{n} = \frac{dA}{dA_o} \mathbf{T} \cdot \left(\frac{dA_o}{dA} J \mathbf{n}_o \cdot \mathbf{F}^{-1} \right) = J\, \mathbf{T}\, \mathbf{F}^{-T} \cdot \mathbf{n}_o \quad \Rightarrow$$

$$\left[\mathbf{T}_o - J\, \mathbf{T}\, \mathbf{F}^{-T}\right] \cdot \mathbf{n}_o = \mathbf{0} \tag{5.6.13}$$

Because this result is valid for any choice of $\mathbf{n}_o$, and since J, $\mathbf{T}$ and $\mathbf{F}^{-T}$ are independent of $\mathbf{n}_o$, the factor in the brackets [] in (5.5.13) has to be a zero tensor. Hence:

$$\mathbf{T}_o = J\, \mathbf{T}\, \mathbf{F}^{-T} \quad \Leftrightarrow \quad T_{oij} = J\, T_{ik} \frac{\partial X_j}{\partial x_k} \tag{5.6.14}$$

The *second Piola-Kirchhoff stress tensor* $\mathbf{S}$ is defined by the relation:

$$\mathbf{S} = \mathbf{F}^{-1}\, \mathbf{T}_o = J\, \mathbf{F}^{-1}\, \mathbf{T}\, \mathbf{F}^{-T} \tag{5.6.15}$$

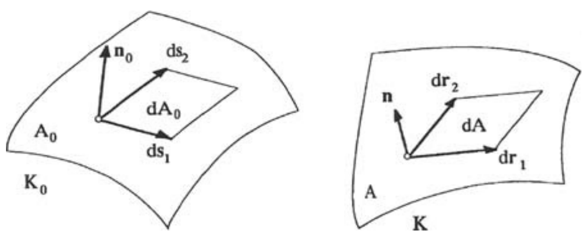


Fig. 5.6.2 Material surface elements in K_o and K

In Sect. 3.2.5 the Cauchy equations of motion (3.2.35) were derived from Euler's 1. axiom (5.6.1). Similarly, from the alternative form (5.6.6) of Euler's 1. axiom we may derive the following *alternative Cauchy equations of motion*:

$$\operatorname{Div}\mathbf{T}_o + \rho_o\,\mathbf{b} = \rho_o\,\mathbf{a} \quad \Leftrightarrow \quad \frac{\partial T_{oik}}{\partial X_k} + \rho_o\,b_i = \rho_o\,a_i \tag{5.6.16}$$

$$\operatorname{Div}\left(\mathbf{F}\,\mathbf{S}\right) + \rho_o\,\mathbf{b} = \rho_o\,\mathbf{a} \quad \Leftrightarrow \quad \frac{\partial\left(F_{ij}\,S_{jk}\right)}{\partial X_k} + \rho_o\,b_i = \rho_o\,a_i \tag{5.6.17}$$

In these equations we have introduced the divergence "Div" of a 2. order tensor related to the reference coordinates X in K_o, as opposed to the divergence "div" related to the coordinates x in the present configuration K. The equations (5.6.16) and (5.6.17) are formulated with respect to the reference configuration K_o, but we should remember that they are in fact representing the situation in the present configuration K.

From the symmetry of the Cauchy stress tensor it follows that:

$$\mathbf{T}_o^T = \mathbf{F}^{-1}\,\mathbf{T}_o\,\mathbf{F}^T, \quad \mathbf{S}^T = \mathbf{S} \tag{5.6.18}$$

Hence, the first Piola-Kirchhoff stress tensor is not symmetric, while the second Piola-Kirchhoff stress tensor is symmetric.

Problems

Problem 5.1. A steel rod with cross-sectional area $1000\,\mathrm{mm}^2$ is subjected to a tensile force of 160 kN. The modulus of elasticity is $\mathrm{E} = 210\,\mathrm{GPa}$, and Poisson's ratio is $\nu = 0.3$. Assume that the material is linearly elastic and determine the longitudinal strains in the axial direction and in the cross-section of the rod. Determine the maximum shear strain and the volumetric strain.

Problem 5.2. A 5 m long tube with outer diameter 150 mm is subjected to torsion. The angle of twist is measured to be 5.0°. The longitudinal strains in the axial and circumferential directions are zero. What is the maximum shear strain in the surface of the tube? Determine the principal strains in the surface of the tube.

Problem 5.3. A stain gage is glued on the surface of a tube. The strain gage records the longitudinal strain ε in a direction 45° with a line parallel to the axis of the tube. The length of the tube is 4 m, and the outer diameter is 160 mm. The tube is subjected to torsion. The longitudinal strains in the axial and circumferential directions are zero. Compute the angle of twist when the strain gage records a strain of $\varepsilon = 9.6 \cdot 10^{-4}$.

Problem 5.4. Assume small deformations. Show that the displacement field:

a) $u_i = \alpha x_i (x_k x_k)^{3/2}$, $\alpha =$ constant, results in a deformation with constant volume,
b) $\mathbf{u} = \nabla\phi \Leftrightarrow u_i = \phi_{,i}\,, \phi = \phi(\mathbf{r})$, results in a pure (rotation free) strain,
c) $\mathbf{u} = \mathbf{u}_o(t) + \mathbf{z}(t) \times \mathbf{r}$ is free of strains.

Problem 5.5. The strains in the surface of a structural element are measured by a strain rosette. The rosette records the longitudinal strains in three directions: the x-direction and the y-direction, and a direction 45° to the x-direction, see Fig. Problems 5.5 and 5.6. During a particular loading of the structure the following strains are recorded:

$$\varepsilon_x = 420 \cdot 10^{-6}, \quad \varepsilon_y = 250 \cdot 10^{-6}, \quad \varepsilon_{45} = 600 \cdot 10^{-6}$$

a) Determine the shear strain γ_{xy}. Compute the principal strains and the principal strain directions in the surface, and the maximum shear strain in the surface.
b) Draw the Mohr-diagram for strains, and read from the diagram the principal strains in the surface, the maximum shear strain in the surface, and the corresponding directions.

Problem 5.6. The strain rosette shown in Fig. Problems 5.5 and 5.6 is attached to the surface of a structural part such that the directions of the three longitudinal strains $\varepsilon_I, \varepsilon_{II}$, and ε_{III} make the following angles with the x-axis in a local xy-system on the surface:

$$\phi_I = \phi, \quad \phi_{II} = \phi + 45^o, \quad \phi_{III} = \phi + 90^o$$

a) Show that the coordinate strains in the xy-system on the surface are expressed by:

$$\varepsilon_x = \frac{\varepsilon_I + \varepsilon_{III}}{2}[1 + \sin 2\phi] + \frac{\varepsilon_I - \varepsilon_{III}}{2}\cos 2\phi - \varepsilon_{II}\sin 2\phi$$
$$\varepsilon_y = \frac{\varepsilon_I + \varepsilon_{III}}{2}[1 - \sin 2\phi] - \frac{\varepsilon_I - \varepsilon_{III}}{2}\cos 2\phi + \varepsilon_{II}\sin 2\phi$$
$$\gamma_{xy} = [\varepsilon_I - \varepsilon_{III}]\sin 2\phi - [\varepsilon_I + \varepsilon_{III}]\cos 2\phi + 2\varepsilon_{II}\cos 2\phi$$

b) Derive the following formulas for the principal strains ε_1 and ε_2 in the surface and the angle ϕ_1 between the direction of ε_1 and the x-direction:

$$\begin{matrix}\varepsilon_1 \\ \varepsilon_2\end{matrix} = \varepsilon_I + \varepsilon_{III} \pm \sqrt{\frac{1}{2}\left(\varepsilon_I^2 + \varepsilon_{III}^2\right) + \varepsilon_{II}\left(\varepsilon_{II} - \varepsilon_I - \varepsilon_{III}\right)}$$
$$\phi_1 = \phi + \arctan\frac{2\left(\varepsilon_1 - \varepsilon_I\right)}{2\varepsilon_{II} - \varepsilon_I - \varepsilon_{III}}$$

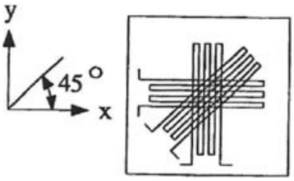


Fig. Problem 5.5 and 5.6 Electric 45°–90°– strain rosette recording longitudinal strains in three directions on a surface: $\varepsilon_x, \varepsilon_y$, and ε_{45}

Problem 5.7. A 60°–120°- strain rosette is attached to the surface of a structural part. The rosette records longitudinal strains in three directions: ε_I, ε_{II}, ε_{III}.

a) Develop for the strain rosette general formulas for the coordinate strains with respect to the local *xy*-system shown in Fig. Problem 5.7.
b) Develop for the strain rosette general formulas for the principal strains and the principal strain directions in the surface, and maximum shear strain in the surface.
c) For a particular loading of the structure the following strains are recorded by the strain rosette:

$$\varepsilon_I = 390 \cdot 10^{-6}, \quad \varepsilon_{II} = 190 \cdot 10^{-6}, \quad \varepsilon_{III} = -120 \cdot 10^{-6}$$

Compute the coordinate strains with respect to the local *xy*-system, the principal strains and the principal strain directions in the surface, and the maximum shear strain in the surface of the structural part.

Problem 5.8. Show that if the state of strain under small deformations is form invariant, then the volume strain ε_v must be a linear function of the x_i-coordinates. Use the compatibility equations.

Problem 5.9. Use the relation (5.3.9) between the strain tensor **E** for small deformations and the displacement vector **u** and formula (4.4.36) for the gradient of a vector to develop the expressions (5.3.14) for the coordinate strains in cylindrical coordinates.

Problem 5.10. Prove the bi-implication (5.3.31) by the Theorems C.1 and C.5.

Problem 5.11. Develop the relations (5.5.28, 5.5.29, 5.5.30, 5.5.31, 5.5.32 and 5.5.33).

Problem 5.12. Show that in the case of homogeneous deformation material planes and straight lines in the reference configuration K_o deform into planes and straight lines in the present configuration K.

Problem 5.13. A state of simple shear is given by:

$$x_1 = X_1 + \alpha X_2, \quad x_2 = X_2, \quad x_3 = X_3, \quad \alpha = \alpha(t)$$

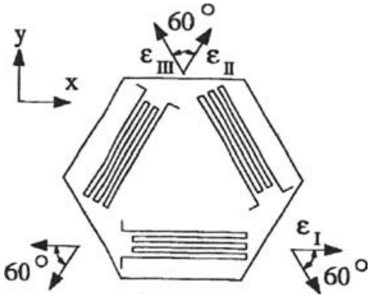


Fig. Problem 5.7 Electric 60°–120°– strain rosette recording longitudinal strains in three directions on a surface

a) Determine the deformation gradient **F** and the deformation tensor **C**.
b) Determine the displacement gradient **H** and the strain tensor **E**.
c) Assume small deformations and determine the strain tensor **E** and the rotation tensor for small deformations $\tilde{\mathbf{R}}$.
d) Set $\alpha = 2$ and compute the right stretch tensor **U** and the rotation tensor **R**.

Problem 5.14. A homogeneous deformation is given by

$$x_1 = X_1 - 2\alpha X_2 + 2\alpha X_3, \quad x_2 = 2\alpha X_1 + X_2, \quad x_3 = -2\alpha X_1 + X_3, \quad \alpha = \alpha(t)$$

a) Determine the deformation gradient **F** and the deformation tensor **C**.
b) Compute the principal values and principal directions of **C**.

Problem 5.15. A of homogeneous deformation is given by:

$$x_1 = \frac{2}{3}X_1 - 2X_2 + 2X_3, \quad x_2 = \frac{4}{3}X_1 + X_2, \quad x_3 = -\frac{4}{3}X_1 + X_3$$

Determine:

a) The deformation gradient **F** and the deformation tensor **C**,
b) The principal values λ_k^2 and the principal directions $\mathbf{a}_k$ of **C**.
c) The stretch tensor **U** and its inverse $\mathbf{U}^{-1}$ from the formulas:

$$\mathbf{U} = \sqrt{\mathbf{C}} = \sum_{k=1}^{3} \lambda_k \, \mathbf{a}_k \otimes \mathbf{a}_k, \quad \mathbf{U}^{-1} = \sum_{k=1}^{3} \frac{1}{\lambda_k} \mathbf{a}_k \otimes \mathbf{a}_k$$

d) The rotation tensor **R** from the relation $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$, the axis of rotation and the angle of rotation.
e) The left stretch tensor **V** from the relation $\mathbf{V} = \mathbf{F}\mathbf{R}^T$, and from the relation:

$$\mathbf{V} = \sum_{k=1}^{3} \lambda_k \, \mathbf{b}_k \otimes \mathbf{b}_k, \quad \text{where } \mathbf{b}_k = \mathbf{R} \cdot \mathbf{a}_k.$$

f) The shear strain γ_{23}, the volumetric strain ε_v, and the extremal values of longitudinal strain.

Problem 5.16. Derive the formulas (5.5.85) from the definition (5.5.84).

Chapter 6
Work and Energy

6.1 Mechanical Energy Balance

For a system of mass particles and of rigid bodies it may be shown that the work W done by the external forces on the system is equal to the change ΔK of the kinetic energy K of the system:

$$W = \Delta K \tag{6.1.1}$$

This *work-energy equation for rigid bodies* is a special result of the general mechanical energy balance equation for a body of continuous material derived below.

When a body in motion, Fig. 6.1.1, is subjected to body forces $\mathbf{b}$ and contact forces $\mathbf{t}$, work is done on the body. Per unit time this work is expressed by:

$$P = \int_V \mathbf{b} \cdot \mathbf{v} \rho \, dV + \int_A \mathbf{t} \cdot \mathbf{v} \, dA \tag{6.1.2}$$

V is the volume of the body, A is the area of the surface of the body, and $\mathbf{v}$ is the particle velocity at time t. Work per unit time is called *mechanical power P* supplied to the body.

The expression (6.1.2) for the mechanical power will be transformed through the use of the Cauchy equations of motion (3.2.35a), the Cauchy stress theorem (3.2.27), and the Gauss theorem C.3 in Appendix C. The Cauchy equations of motion:

$$\operatorname{div} \mathbf{T} + \rho \, \mathbf{b} = \rho \, \dot{\mathbf{v}} \tag{6.1.3}$$

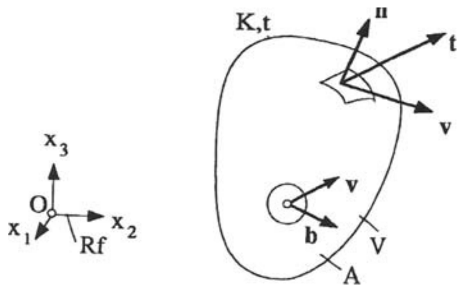
are used to write:

$$\int_V \mathbf{b} \cdot \mathbf{v} \rho \, dV = \int_V \dot{\mathbf{v}} \cdot \mathbf{v} \rho \, dV - \int_V (\operatorname{div} \mathbf{T}) \cdot \mathbf{v} \, dV \tag{6.1.4}$$

The integrand in the last integral is further developed.

$$(\operatorname{div} \mathbf{T}) \cdot \mathbf{v} = T_{ik,k} \, v_i = (v_i \, T_{ik})_{,k} - T_{ik} \, v_{i,k} = \operatorname{div} (\mathbf{v} \cdot \mathbf{T}) - \mathbf{T} : \mathbf{D} \tag{6.1.5}$$

Fig. 6.1.1 Body and forces



D is the rate of deformation tensor:

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right) \quad \Leftrightarrow \quad D_{ik} = \frac{1}{2}\left(v_{i,k} + v_{k,i}\right)$$

and we have used the result:

$$T_{ik}\, v_{i,k} = \mathbf{T} : \mathbf{L} = \frac{1}{2} T_{ik}\, v_{i,k} + \frac{1}{2} T_{ki}\, v_{k,i} = T_{ik} \frac{1}{2}\left(v_{i,k} + v_{k,i}\right) = T_{ik} D_{ik} = \mathbf{T} : \mathbf{D} \quad (6.1.6)$$

By the Gauss integration theorem C.3 and Cauchy's stress theorem $\mathbf{t} = \mathbf{T} \cdot \mathbf{n}$, we obtain:

$$\int_V \operatorname{div}(\mathbf{v} \cdot \mathbf{T})\, dV = \int_A (\mathbf{v} \cdot \mathbf{T}) \cdot \mathbf{n}\, dA = \int_A \mathbf{v} \cdot \mathbf{T} \cdot \mathbf{n}\, dA = \int_A \mathbf{v} \cdot \mathbf{t}\, dA \quad (6.1.7)$$

The results (6.1.4), (6.1.5), and (6.1.7) give:

$$\int_V \mathbf{b} \cdot \mathbf{v} \rho\, dV = \int_V \dot{\mathbf{v}} \cdot \mathbf{v} \rho\, dV - \int_A \mathbf{v} \cdot \mathbf{t}\, dA + \int_V \mathbf{T} : \mathbf{D}\, dV$$

When this result is substituted into the expression (6.1.2) for the mechanical power, we obtain:

$$P = \int_V \mathbf{b} \cdot \mathbf{v} \rho\, dV + \int_A \mathbf{t} \cdot \mathbf{v}\, dA = \int_V \dot{\mathbf{v}} \cdot \mathbf{v} \rho\, dV + \int_V \mathbf{T} : \mathbf{D}\, dV \quad (6.1.8)$$

Now we introduce the *kinetic energy* of the body:

$$K = \int_V \frac{v^2}{2} \rho\, dV \equiv \int_V \frac{\mathbf{v} \cdot \mathbf{v}}{2} \rho\, dV \quad (6.1.9)$$

and the *stress power* supplied to the body:

$$P^d = \int_V \mathbf{T} : \mathbf{D}\, dV \quad (6.1.10)$$

Below we shall show that P^d represents the deformation work done in the body per unit time. The stress power results in a change in the internal energy of the body, which partly may be a recoverable elastic energy, and which partly is represented by an increase in the temperature in the particles of the body, and in heat conducted to the surroundings of the body. These issues will be discussed further in Sect. 6.3.

By the differentiation formula (3.1.22) we get:

$$\dot{K} = \int_V \dot{\mathbf{v}} \cdot \mathbf{v} \rho \, dV \tag{6.1.11}$$

The result (6.1.8) may now be presented as:

$$P = \dot{K} + P^d \tag{6.1.12}$$

This result is called the *power theorem* or the *mechanical energy balance equation*:

> *The work done per unit time by the external forces on a body is equal to the time rate of change of the kinetic energy plus the stress power supplied to the body.*

From the formula (6.1.10) it follows that the stress power per unit volume is:

$$\omega = \mathbf{T} : \mathbf{D} = T_{ik} D_{ik} \tag{6.1.13}$$

A physically interpretation of the individual terms in the sum $T_{ik}\ D_{ik}$ will now be presented. The sum contains two kinds of terms represented by:

$$T_{11} D_{11} \quad \text{and} \quad T_{12} D_{12} + T_{21} D_{21} = 2 T_{12} D_{12}$$

Figure 6.1.2 shows an element of volume $dV = dx_1 dx_2 dx_3$ subjected to the coordinate stresses T_{ik} and the strains $D_{11}dt$ and $2D_{12}dt$. The work done on the element by the stresses due to the two coordinate strains is the sum of two terms:

$$\Delta W_{11} = (T_{11}\, dx_2\, dx_3)\,((D_{11}\, dt)\, dx_1) \quad , \quad \Delta W_{12} = (T_{12}\, dx_1\, dx_3)\,((2D_{12}\, dt)\, dx_2)$$

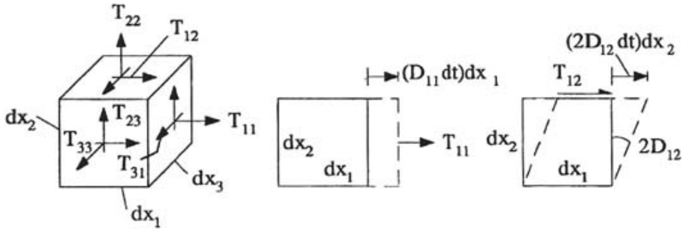


Fig. 6.1.2 Volume element with coordinate stresses and coordinate deformation rates D_{11} and D_{12}

The total work by the stresses T_{ik} on the element when the element is subjected to the deformation $D_{ik}dt$, is:

$$\Delta W = (T_{11}D_{11} + 2T_{12}D_{12} + \cdots)\,dx_1\,dx_2\,dx_3\,dt = T_{ik}D_{ik}\,dV\,dt \tag{6.1.14}$$

The stress work per unit volume and per unit time is then $\omega = T_{ik}D_{ik}$, as given by formula (6.1.13). It follows that P^d in (6.1.10) represents the work done per unit time by the stresses due to deformation.

When the tensors **T** and **D** are decomposed into isotrops and deviators:

$$\mathbf{T} = \mathbf{T}^o + \mathbf{T}' \quad , \quad \mathbf{D} = \mathbf{D}^o + \mathbf{D}' \tag{6.1.15}$$

the stress power per unit volume, ω, is split into the sum of two distinct components:

$$\omega = \omega^o + \omega' \tag{6.1.16}$$

$$\omega^o = \mathbf{T}^o\mathbf{D}^o = \frac{1}{3}(\mathrm{tr}\,\mathbf{T})\,\mathrm{div}\,\mathbf{v} = \sigma^o\,\dot{\varepsilon}_v \quad , \quad \omega' = \mathbf{T}' : \mathbf{D}' \tag{6.1.17}$$

$\sigma^o \equiv (1/3)\mathrm{tr}\mathbf{T}$ is the mean normal stress. The details which leads to the result (6.1.16), is given as Problem 6.1. The component $\omega^o$is called the *volumetric stress power* and the component ω' is called the *distortion stress power*. For an incompressible material $\omega^o = 0$, and the total stress power is a distortion stress power.

6.1.1 The Work-Energy Equation for Rigid Bodies

For a rigid body the rate of deformation tensor is zero, $\mathbf{D} = \mathbf{0}$, and the stress power is zero. The mechanical energy balance equation (6.1.12) for a rigid body is therefore:

$$P = \dot{K} \tag{6.1.18}$$

Integration of (6.1.18) with respect to time and between the two times t_1 and t_2 leads to an expression for the work W done on the body by the forces **b** and **t**.

$$W = \int_{t_1}^{t_2} P\,dt = \int_{t_1}^{t_2} \dot{K}(t)\,dt = K(t_2) - K(t_1) \equiv \Delta K \tag{6.1.19}$$

This equation also applies to a system of rigid bodies when the work W also includes the work done by the internal forces between the bodies. The result (6.1.19) is identical to the work- energy equation (6.1.1). Based on (6.1.19) we may state that:

Kinetic energy represents a work resource through motion.

The work done by the forces on a rigid body is stored as kinetic energy in the body. A negative change in the kinetic energy of a rigid body implies that the external forces on the body have done negative work on the body, or that the body has performed positive work on the surroundings, i.e. the bodies from which the external forces originate.

The motion of a rigid body may be represented by the velocity $\mathbf{v}_C$ of the center of mass C of the body and the angular velocity $\mathbf{w}$. The velocity of a particle in the body is given by the velocity distribution formula (4.5.25):

$$\mathbf{v} = \mathbf{v}_C + \mathbf{w} \times \mathbf{r} \tag{6.1.20}$$

$\mathbf{r}$ is the place vector from the center of mass C to the particle. The kinetic energy of the body is:

$$K = \int_V \frac{1}{2}\mathbf{v} \cdot \mathbf{v}\,\rho\,dV = \int_V \frac{1}{2}(\mathbf{v}_C + \mathbf{w} \times \mathbf{r}) \cdot (\mathbf{v}_C + \mathbf{w} \times \mathbf{r})\,\rho\,dV$$

$$= \frac{1}{2}\mathbf{v}_C \cdot \mathbf{v}_C \int_V \rho\,dV + \mathbf{v}_C \cdot \left(\mathbf{w} \times \int_V \mathbf{r}\rho\,dV\right) + \frac{1}{2}\mathbf{w} \cdot \int_V \mathbf{r} \times (\mathbf{w} \times \mathbf{r})\,\rho\,dV$$

In the last term on the right-hand side of the formula we have used the property of the scalar triple product: a dot and a cross may exchange positions. The integral in that term is according to formula (4.5.49) equal the *central angular momentum* $\mathbf{l}_C = \mathbf{I}_C \cdot \mathbf{w}$, in which $\mathbf{I}_C$ is the inertia tensor of the body with respect to the center of mass. The second integral on the right-hand side is zero according to the definition (3.2.8) of the center of mass of the body. The first integral on the right-hand side is the mass m of the body. Hence the kinetic energy of a rigid body may be expressed by:

$$K = \frac{1}{2}m v_C{}^2 + \frac{1}{2}\mathbf{w} \cdot \mathbf{I}_C \cdot \mathbf{w} \quad , \quad K = \frac{1}{2}\mathbf{p} \cdot \mathbf{v}_C + \frac{1}{2}\mathbf{l}_C \cdot \mathbf{w} \tag{6.1.21}$$

$\mathbf{p} = m\mathbf{v}_C$ is the linear momentum of the body. The first term in the expression for the kinetic K is called the *translational energy* K^t of the body, while the second term is called the *rotational energy* K^r.

$$K^t = \frac{1}{2}m v_C{}^2 \quad \text{translational energy} \tag{6.1.22}$$

$$K^r = \frac{1}{2}\mathbf{w} \cdot \mathbf{I}_C \cdot \mathbf{w} = \frac{1}{2}w_i I_{ik} w_k = \frac{1}{2}\mathbf{l}_C \cdot \mathbf{w} \quad \text{rotational energy} \tag{6.1.23}$$

If we choose a coordinate system with axes parallel to the principal axes of inertia of the body with respect to the center of mass C, we find that:

$$K^r = \frac{1}{2}\left(I_1 w_1{}^2 + I_2 w_2{}^2 + I_3 w_3{}^2\right) \tag{6.1.24}$$

I_1, I_2, and I_3 are the principal moments of inertia with respect to C.

Let $\mathbf{f}(\mathbf{r})$ be a force applied to the point in a body at the place $\mathbf{r}$, and such that the force may be derived from a scalar valued function of position $\mathrm{B}(\mathbf{r})$:

$$\mathbf{f} = -\nabla \mathrm{B} \tag{6.1.25}$$

The work done by that force as the point of application is moved from place $\mathbf{r}_1$ to place $\mathbf{r}_2$ during the motion of the body, may be computed as the negative change in the value of the function $\mathrm{B}(\mathbf{r})$:

$$\begin{aligned} W_f &= \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{f} \cdot d\mathbf{r} = -\int_{\mathbf{r}_1}^{\mathbf{r}_2} \nabla \mathrm{B} \cdot d\mathbf{r} = -\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathrm{B}_{,i}\, dx_i = \int_{\mathbf{r}_1}^{\mathbf{r}_2} d\mathrm{B} \\ &= -\left[\mathrm{B}(\mathbf{r}_2) - \mathrm{B}(\mathbf{r}_1)\right] = -\left(\mathrm{B}_2 - \mathrm{B}_1\right) \equiv -\Delta \mathrm{B} \quad \Rightarrow \\ W_f &= \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{f} \cdot d\mathbf{r} = -\left[\mathrm{B}(\mathbf{r}_2) - \mathrm{B}(\mathbf{r}_1)\right] = -\left(\mathrm{B}_2 - \mathrm{B}_1\right) \equiv -\Delta \mathrm{B} \end{aligned} \tag{6.1.26}$$

The scalar function $\mathrm{B}(\mathbf{r})$ is called the *force potential* of the force $\mathbf{f}$, or a *potential energy* of the body due to the force $\mathbf{f}$. We may state that:

A potential energy represents a work resource through position.

The work of the force $\mathbf{f} = -\nabla \mathrm{B}$ is conserved in the sense that when the work on the body is positive there is an equal decrease in the potential energy $\mathrm{B}(\mathbf{r})$. The force $\mathbf{f} = -\nabla \mathrm{B}$ is therefore called a *conservative force*.

If all the forces acting on a body and producing work are conservative, the total work done on the body may be expressed by the sum of the work done by the individual forces. We write this as:

$$W = \sum W_f = -\sum \Delta \mathrm{B} = -\Delta U \tag{6.1.27}$$

The quantity U is called the *potential energy of the body*.

$$U = \sum \mathrm{B} \tag{6.1.28}$$

The work-energy equation (6.1.19) may now be rewritten to:

$$\begin{aligned} -\Delta U = \Delta K \quad &\Rightarrow \quad \Delta(K + U) = 0 \quad \Rightarrow \\ K + U &= \text{ constant} \end{aligned} \tag{6.1.29}$$

$K+U$ is called the *total mechanical energy of the rigid* body. The result (6.1.29) is *the law of conservation of total mechanical energy* $K+U$. The law also applies to systems of rigid bodies and may be formulated thus:

> *If all work performing forces acting on a rigid body or a system of rigid bodies are conservative, the total mechanical energy of the body or bodies is constant.*

6.1.2 Conjugate Stress Tensors and Deformation Tensors

The Cauchy stress tensor $\mathbf{T}$ is defined in Sect. 3.2, while the first Piola-Kirchhoff stress tensor $\mathbf{T}_\mathrm{o}$ and the second Piola-Kirchhoff stress tensor $\mathbf{S}$ are defined in Sect. 5.6. We shall now demonstrate how the two last stress tensors naturally appear in connection with expressions for the stress power when we change measures of the rates of deformation. The results may be used in connection with applications of non-linear finite element methods and large deformations. According to (5.6.14) and (5.6.15) the three stress tensors are related through:

$$\mathbf{T} = \frac{1}{J}\mathbf{T}_o\mathbf{F}^T = \frac{1}{J}\mathbf{F}\mathbf{S}\mathbf{F}^T \tag{6.1.30}$$

We shall show that the stress power supplied to the body alternatively may be given by the following three expressions:

$$P^d = \int_V \mathbf{T} : \mathbf{D}dV = \int_{V_o} \mathbf{T}_o : \dot{\mathbf{F}}\,dV_o = \int_{V_o} \mathbf{S} : \dot{\mathbf{E}}\,dV_o \tag{6.1.31}$$

V and V_o are the volumes of the body in the present configuration K and the reference configuration K_o respectively. Due to the relations (6.1.31) the stress tensors: $\mathbf{T}, \mathbf{T}_o$, and $\mathbf{S}$ are called conjugated variables to the rates of deformations: $\mathbf{D}, \dot{\mathbf{F}}$, and $\dot{\mathbf{E}}$ respectively.

Using Theorem C.8 in Appendix C on the change of variable in a volume integral, we obtain:

$$P^d = \int_V \omega\,dV = \int_{V_o} \omega J\,dV_o = \int_{V_o} \omega_o\,dV_o \tag{6.1.32}$$

ω_o represents the stress power supplied to the body in K, but given per unit volume in K_o. Now we obtain from (6.1.13), (6.1.6), (6.1.30), (5.5.28), and (5.5.31) that:

$$\omega_o = J\,\omega = J\,\mathbf{T}:\mathbf{D} = J\,\mathbf{T}:\mathbf{L} = J\left(\frac{1}{J}\mathbf{T}_o\mathbf{F}^T\right):\left(\dot{\mathbf{F}}\mathbf{F}^{-1}\right)$$

$$= \mathbf{T}_o:\dot{\mathbf{F}} \; = J\left(\frac{1}{J}\mathbf{F}\mathbf{S}\mathbf{F}^T\right):\left(\mathbf{F}^{-T}\dot{\mathbf{E}}\mathbf{F}^{-1}\right) = \mathbf{S}:\dot{\mathbf{E}} \quad \Rightarrow$$

$$\omega_o = J\,\omega = J\,\mathbf{T}:\mathbf{D} = \mathbf{T}_o:\dot{\mathbf{F}} \; = \mathbf{S}:\dot{\mathbf{E}} \tag{6.1.33}$$

The details in the derivations are given as Problem 6.3. When the results (6.1.33) are substituted into the right hand side of (6.1.32), the two last integral expressions in (6.1.31) are obtained.

6.2 The Principle of Virtual Power

We consider a body subjected to body forces $\mathbf{b}$. On one part A_σ of the total surface A of the body the contact forces are prescribed by the stress vector $\mathbf{t}^*$. On the rest of the surface, $A_u = A - A_\sigma$, the displacements $\mathbf{u}^*$ are prescribed. The boundary conditions are therefore:

$$\mathbf{t} = \mathbf{T}\cdot\mathbf{n} = \mathbf{t}^* \text{ on } A_\sigma \tag{6.2.1}$$

$$\mathbf{u} = \mathbf{u}^* \text{ on } A_u \tag{6.2.2}$$

$\mathbf{t}^*$ and $\mathbf{u}^*$ are prescribed functions of position.

The stress field $\mathbf{T}^d$ is called *a dynamically permissible stress field* if the stress field obeys the Cauchy equations of motion (6.1.3) and the boundary conditions (6.2.1). It is not required of the stress field that it through some constitutive equations produces compatible strains, or that the displacements resulting from these strains satisfy the boundary conditions (6.2.2). If it happens that these requirements also are satisfied, we have a real solution to the stress problem for the body under consideration.

In the Cauchy equations (6.1.3) we assume that the body force $\mathbf{b}$ and the acceleration $\mathbf{a}$ are prescribed vector fields. The equations are written as the equilibrium equations:

$$\operatorname{div}\mathbf{T} + \rho\,(\mathbf{b} - \mathbf{a}) = \mathbf{0} \tag{6.2.3}$$

The quantity -$\mathbf{a}$ may be treated as an *extraordinary body force* called a *specific inertia force*. Extraordinary body forces are discussed in Sect. 11.2. The equation (6.2.3) is called a *dynamic equilibrium equation*.

A velocity field $\mathbf{v}^k$ is called *a kinematically permissible velocity field* if it satisfies the boundary conditions (6.2.2) and represents displacements to unique positions of the particles of the body. However, the velocity field may introduce fractures in the body. From the velocity field $\mathbf{v}^k$ a deformation rate tensor $\mathbf{D}^k$ is computed. The boundary condition (6.2.2) provides the condition:

$$\mathbf{v}^k = \mathbf{0} \text{ on } A_u \tag{6.2.4}$$

Due to the velocity field $\mathbf{v}^k$ the prescribed forces $(\mathbf{b}-\mathbf{a})$ and $\mathbf{t}^*$ perform work per unit time on the body equal to:

$$P^e = \int_V \rho\,(\mathbf{b}-\mathbf{a}) \cdot \mathbf{v}^k dV + \int_{A_\sigma} \mathbf{t}^* \cdot \mathbf{v}^k dA \tag{6.2.5}$$

We call P^e the *external virtual mechanical power*. Using the Cauchy equations (6.2.3), we get:

$$\rho\,(\mathbf{b}-\mathbf{a}) \cdot \mathbf{v}^k = -T^d{}_{ij,j}\, v^k{}_i = -\left(T^d{}_{ij} v^k_i\right)_{,j} + T^d{}_{ij} v^k{}_{i,j} \tag{6.2.6}$$

The Gauss' theorem C.3 and the boundary condition (6.2.4) give:

$$-\int_V \left(T^d{}_{ij} v^k{}_i\right)_{,j} dV = -\int_A T^d{}_{ij} v^k{}_i n_j\, dA = -\int_{A_\sigma} \mathbf{t}^* \cdot \mathbf{v}^k\, dA \tag{6.2.7}$$

Furthermore:

$$T^d{}_{ij} v^k{}_{i,j} = T^d{}_{ij} L^k{}_{ij} = T^d{}_{ij} D^k{}_{ij} = \mathbf{T}^d : \mathbf{D}^k \tag{6.2.8}$$

Using the results (6.2.6, 6.2.7, 6.2.8), we may write:

$$\int_V \rho\,(\mathbf{b}-\mathbf{a}) \cdot \mathbf{v}^k dV = -\int_{A_\sigma} \mathbf{t}^* \cdot \mathbf{v}^k dA + \int_V \mathbf{T}^d : \mathbf{D}^k dV \tag{6.2.9}$$

The second integral on the right hand side is a *virtual stress power*, which we shall call the *internal virtual mechanical power*:

$$P^i = \int_V \mathbf{T}^d : \mathbf{D}^k dV \tag{6.2.10}$$

By substituting the result (6.2.9) into (6.2.5), we obtain:

$$P^e = P^i \tag{6.2.11}$$

The result expresses the *theorem of virtual power*:

> *The external virtual mechanical power supplied by the forces acting on a body when the body is subjected to a kinematically permissible velocity field, is equal to the internal virtual mechanical power due to a dynamically permissible stress field.*

The terms virtual mechanical power and virtual stress power are used to emphasize the fact that no connection between the kinematically permissible velocity field and the dynamically permissible stress field is intended. The kinematically permissible velocity field may be replaced by a kinematically permissible displacement field, under the assumption of small displacements. The external virtual mechanical power is then replaced by *external virtual mechanical work*, and the internal virtual mechanical power is replaced by *internal virual mechanical work*. The above theorem then becomes the *theorem of virtual work* or the *principle of virtual work*.

6.3 Thermal Energy Balance

6.3.1 Thermodynamic Introduction

The temperature in a body expresses how hot the body is. Most materials expand when heat is supplied to them or when mechanical work is performed on them. Changes in volume of certain test materials are therefore used to measure the temperature. Experience shows that it is always possible to choose a temperature scale such that the temperature θ is always positive.

$$\theta > 0 \tag{6.3.1}$$

θ is then called *absolute temperature*. The *state of a body* in thermodynamical equilibrium is defined by a set of values of $\kappa + 1$ *state variables*:

$$\theta, \theta_1, \theta_2, \cdots \theta_\kappa \tag{6.3.2}$$

The state variables $\theta_1, \theta_2, \cdots \theta_\kappa$ may be the volume of the body, the density, measures of strain, or other so-called *internal variables*, which will not be defined here. We assume that the state of the body is homogeneous, i.e. all state variables that relate to particles are the same for all particles in the body.

A change in the state of a body that is due only through the mechanical work performed on the body is called an *adiabatic change of state* of the body. The body must then be thermally insulated from the surrounding such that heat is not supplied from or transmitted to the surroundings. A transport of heat from or to the surroundings may be registered by a change in the temperature in the surroundings.

Experiments show that if a body is subjected to an adiabatic change of state, from state 1 to state 2, and then back to state 1, the total work done on the body will be zero. If we denote the work done on the body when the state is changed from state 1 to state 2 by W_{12}, and the work done on the body when it is taken from state 2 and back to state 1 by W_{21}, we may write:

$$W_{12} + W_{21} = 0 \tag{6.3.3}$$

This result may be interpreted thus: The work done on a body W_{12} during an adiabatic change of state is independent of how the change of state comes about. The work W_{12} may therefore be expressed by a change in a function of state. We introduce as this function of state the property:

$$E = E\left(\theta, \theta_1, \theta_2, \cdots, \theta_\kappa\right) \tag{6.3.4}$$

called the *internal energy of the body*, such that:

$$W_{12} = E_2 - E_1 \equiv \Delta E \tag{6.3.5}$$

ΔE in (6.3.5) is the change in the internal energy of the body due to a change of state from state 1 to state 2.

For non-adiabatic changes of states experiments show that:

$$W_{12} \neq \Delta E$$

Based on this result the concept of *supplied heat* Q_{12} to a body is introduced as the difference between the change in the internal energy ΔE of the body and the work W_{12} done on the body:

$$W_{12} + Q_{12} = \Delta E \tag{6.3.6}$$

This is the *first law of thermodynamics*.

6.3.2 Thermal Energy Balance

The thermodynamics presented above is related to homogeneous states of thermal equilibrium. We now assume that a body is subjected to *thermomechanical processes*. A thermomechanical process is defined by a set of particle functions, of which we have defined already

the motion	$\mathbf{r} = \mathbf{r}(\mathbf{r}_o, t)$
the Cauchy stress tensor	$\mathbf{T} = \mathbf{T}(\mathbf{r}, t)$
the density	$\rho = \rho(\mathbf{r}, t)$
the temperature	$\theta = \theta(\mathbf{r}, t)$

In addition other particle functions will be defined in the following.

Specific internal energy $\varepsilon = \varepsilon(\mathbf{r}, t)$ is defined as the internal energy per unit mass. The internal energy of a body of volume V and with density ρ is then expressed by:

$$E = \int_V \varepsilon \rho \, dV \tag{6.3.7}$$

Internal energy per unit volume $\varepsilon\rho$ is called *energy density*.

The heat supplied to a body per unit time may be expressed by the *heat power*, which in general has two contributions, see Fig. 6.3.1:

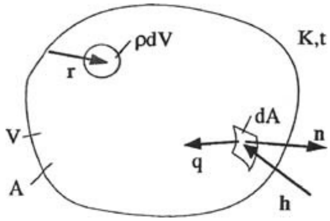


Fig. 6.3.1 Heat supplied to a body

1) *Heat flux* through the surface A of the body:

$$\int_A q\,dA \tag{6.3.8}$$

where $q = q(\mathbf{n}, \mathbf{r}, t)$ is heat flux per unit area. The vector $\mathbf{n}$ is an outward unit normal from A.

2) Heat generated from heat sources in the body or heat radiated into the body from outer external distant radiating sources:

$$\int_V r\rho\,dA \tag{6.3.9}$$

$r = r(\mathbf{r}, t)$ is the *specific heat source* or heat radiation. In the present exposition we shall not consider contributions from the heat supply (6.3.9).

The work done on a body per unit time is expressed by the *mechanical power P* and is given by (6.1.2). Some of this work is used to change the kinetic energy K of the body. The first law of thermodynamics (6.3.6) is now generalized to:

$$P + \int_A q\,dA = \dot{E} + \dot{K} \tag{6.3.10}$$

This is the *total energy balance equation*, which may be expressed by the statement:

> *The sum of the mechanical power and the heat power supplied to a body is equal to the sum of the time rate of change of the internal energy and the time rate of change of the kinetic energy of the body.*

By subtracting the mechanical energy balance equation (6.1.12) from the total energy balance equation (6.3.10), we obtain the *thermal energy balance equation*:

$$\dot{E} = \int_A q\,dA + P^d \tag{6.3.11}$$

This result is also called the *first law of themomechanics*.

We shall now develop the local form of the thermal energy balance equation (6.3.11), which must be satisfied at every place in the body and for every particle of the body. The expressions (6.3.7) and (6.1.10) are substituted into (6.3.11) and the result is:

$$\int_V [\dot{\varepsilon}\rho - \mathbf{T} : \mathbf{D}]\, dV = \int_A q\, dA \tag{6.3.12}$$

This equation applies for any volume V. Then according to Theorem C.9 in Appendix C the heat flux per unit area may be expressed by:

$$q(\mathbf{n}, \mathbf{r}, t) = -\mathbf{h}(\mathbf{r}, t) \cdot \mathbf{n} \tag{6.3.13}$$

The vector field $\mathbf{h}(\mathbf{r}, t)$ is called the *heat flux vector*. The minus sign on the left-hand side is introduced so that the vector $\mathbf{h}$ gives the direction of the heat flow, see Fig. 6.3.1. The relationship (6.3.13) represents *Fourier's heat flux principle*, named after Jean Baptiste Joseph Fourier [1768–1830]. By Theorem C.9 the equations (6.3.12, 6.3.13) imply that:

$$\rho\, \dot{\varepsilon} = -\operatorname{div} \mathbf{h} + \mathbf{T} : \mathbf{D} \tag{6.3.14}$$

This is the *thermal energy balance equation at a place or for a particle*.

We may interpret Cauchy's equation of motion (6.1.3) as representing the local form of the mechanical energy balance equation (6.1.12). The Cauchy equation of motion may therefore by called the mechanical energy balance equation for a place and for a particle.

6.4 The Second Law of Thermodynamics

The second law of thermodynamics introduces the concept of entropy. It is particularly in relation to the formulation of the 2. law of thermodynamics and the definition of the concept of entropy for thermodynamic processes that the thermodynamics of continuous media still has some fundamental problems. However, the following exposition seems to be generally accepted in modern literature.

First we define *the time rate of increase in entropy supplied to the body due to the heat power*:

$$\dot{S}_q = -\int_A \frac{\mathbf{h} \cdot \mathbf{n}}{\theta}\, dA \tag{6.4.1}$$

By application of the Gauss integration theorem and thermal energy balance equation (6.3.14) the expression (6.4.1) may be transformed into:

$$\dot{S}_q = \int_V \frac{1}{\theta} \left[\rho\, \dot{\varepsilon} - \mathbf{T} : \mathbf{D} + \frac{\mathbf{h} \cdot \operatorname{grad} \theta}{\theta}\right] dV \tag{6.4.2}$$

The *ideal gas* is defined by the constitutive equation:

$$\mathbf{T} = -p\mathbf{1} \quad , \quad p = R\rho\,\theta \tag{6.4.3}$$

where R is the gas constant of the gas. The ideal gas is often taken as an "introductory" material in classical thermodynamics. It is often reasonable to assume that the specific energy of an ideal gas is a function of temperature alone:

$$\varepsilon = \varepsilon(\theta) \tag{6.4.4}$$

If a body of an ideal gas is subjected to a process slowly enough for the temperature gradient grad θ in (6.4.2) to be neglected, the time rate of entropy increase $\dot{S}_q$ may be expressed by the material derivative of an extensive scalar property S called the *entropy of the body*.

$$S = \int_V s\rho\,dV \tag{6.4.5}$$

The intensive scalar $s = s(\rho,\theta)$ is the *specific entropy*. Hence, for sufficiently slow processes in an ideal gas we may write:

$$\dot{S}_q = \dot{S} \tag{6.4.6}$$

Classical thermodynamics defines the entropy S of a body by arguing that in a so-called *reversible change of state*, the change in the entropy ΔS_q due to heat supplied to the body, and obtained from (6.4.1), may be expressed by a change in a extensive function of state $S = S(\theta,\theta_1,\theta_2,\cdots,\theta_\kappa)$, called the entropy of the body:

$$\Delta S_q = \Delta S \tag{6.4.7}$$

A reversible change of state is defined thus:

A change of state is a reversible change of state if it may be reversed such that both the body and the surroundings are brought back to their original states.

Experiments then show that for any other changes of states, i.e. a *irreversible change of state*:

$$\Delta S_q < \Delta S \tag{6.4.8}$$

This inequality represents the *second law of thermodynamics*.

In modern thermomechanics the existence of entropy S of a body and specific entropy s are postulated. The rate of change of the entropy of a body $\dot{S}$ is an upper bound for the rate of change of the entropy $\dot{S}_q$ due to the heat power supplied to the body. This postulate constitutes the *second law of thermomechanics* and is expressed by:

$$\dot{S}_q \leq \dot{S} \tag{6.4.9}$$

The *entropy of the body S* is given by (6.4.5) and $\dot{S}_q$ is given by (6.4.1).

The local, intensive form of the second law of thermomechanics (6.4.9) is found as follows. Using (6.4.2) and the expression (6.4.5), we may write the second law (6.4.9) in the form:

$$\int_V \frac{1}{\theta}\left[\rho\,\dot{\varepsilon} - \mathbf{T}:\mathbf{D} + \frac{\mathbf{h}\cdot\operatorname{grad}\theta}{\theta} - \rho\,\theta\,\dot{s}\right] dV \leq 0 \tag{6.4.10}$$

Because this integral relation must be valid for any body, i.e. for any choice of the volume V, the integrand must be zero. Hence:

$$\rho\,\dot{\varepsilon} - \mathbf{T}:\mathbf{D} + \frac{\mathbf{h}\cdot\operatorname{grad}\theta}{\theta} - \rho\,\theta\,\dot{s} \leq 0 \tag{6.4.11}$$

This particular form of the second law of thermomechanics is called the *Clausius-Duhem inequality* and has its name after Rudolph Clausius [1822–1888] and Pierre Duhem [1861–1916].

The time rate of entropy production in a body of volume V is defined by the extensive quantity:

$$\Gamma = \int_V \gamma\rho\, dV = \dot{S} - \dot{S}_q \tag{6.4.12}$$

The intensive quantity γ is the *specific entropy production*. It now follows from (6.4.1), (6.4.5), and (6.4.12) that:

$$\gamma = \gamma_{\mathrm{loc}} + \gamma_{\mathrm{con}} \tag{6.4.13}$$

$$\gamma_{\mathrm{loc}} = \dot{s} + \frac{1}{\rho\,\theta}\operatorname{div}\mathbf{h} \quad \text{local specific entropy production} \tag{6.4.14}$$

$$\gamma\mathrm{con} = -\frac{1}{\rho\,\theta^2}\,\mathbf{h}\cdot\operatorname{grad}\theta \quad \text{specific entropy production by heat conduction‘} \tag{6.4.15}$$

The details in deriving (6.4.13, 6.4.14, 6.4.15) are given as Problem 6.4.

The Clausius-Duhem inequality may now alternatively be expressed by:

$$\gamma \geq 0 \tag{6.4.16}$$

This result may be stated thus: *The specific entropy production cannot be negative*.

We define the *internal specific dissipation* δ by;

$$\delta = \mathbf{T}:\mathbf{D} - \rho\,(\dot{\varepsilon} - \theta\,\dot{\mathbf{s}}) \tag{6.4.17}$$

Now the Clausius-Duham inequality (6.4.11) may be given the alternative form:

$$\delta - \frac{\mathbf{h} \cdot \text{grad}\,\theta}{\theta} \geq 0 \tag{6.4.18}$$

This inequality is satisfied if:

$$\delta \geq 0 \quad \text{and} \quad \frac{\mathbf{h} \cdot \text{grad}\,\theta}{\theta} \leq 0 \tag{6.4.19}$$

The last inequality implies that heat flows in the direction opposite to the temperature gradient. In Fluid Mechanics, see Sect. 8.4, the dissipation function δ of a linearly viscous fluid is defined, and it follows that this dissipation function is identical to the internal specific dissipation.

Specific heat c is heat supplied to the material per unit mass and per unit temperature. The thermal energy balance equation (6.3.14) provides the general formula for c:

$$c = \frac{\dot{\varepsilon}}{\dot{\theta}} - \frac{\mathbf{T} : \mathbf{D}}{\rho\,\dot{\theta}} \tag{6.4.20}$$

By application of the definition (6.4.17) the formula (6.4.20) may alternatively be written as:

$$c = \frac{\theta\,\dot{s}}{\dot{\theta}} - \frac{\delta}{\rho\,\dot{\theta}} \tag{6.4.21}$$

Finally, the thermal energy balance equation (6.3.14) may be given the form:

$$\rho\,c\,\dot{\theta} = -\text{div}\,\mathbf{h} \tag{6.4.22}$$

The second law of thermomechanics will be applied in Sect. 11.7 on thermoelastic materials.

Problems

Problem 6.1. Perform the development of the expressions (6.1.17) for the volumetric stress power and the distortion stress power.

Problem 6.2. Let the body forces $\mathbf{b}(\mathbf{r})$ be conservative such that:

$$\mathbf{b} = -\nabla\beta$$

$\beta(\mathbf{r})$ is a scalar field called the force potential of the force field $\mathbf{b}(\mathbf{r})$ or the potential energy of the body related to the force field $\mathbf{b}(\mathbf{r})$.

Problem 6.3. Perform the development of (6.1.33).

Problem 6.4. Derive the expressions (6.4.13, 6.4.14, 6.4.15) and show that Clausius-Duhem inequality may be expressed by the inequality (6.4.18).

Chapter 7
Theory of Elasticity

7.1 Introduction

The classical theory of elasticity is primary a theory for isotropic, linearly elastic materials subjected to small deformations. All governing equations in this theory are linear partial differential equations, which means that the *principle of superposition* may be applied: The sum of individual solutions to the set of equations is also a solution to the equations. The classical theory of elasticity has a *theorem of uniqueness of solution* and a *theorem of existence of solution*. The theorem of uniqueness insures that if a solution of the pertinent equations and the proper boundary conditions for a particular problem is found, then this solution is the only solution to the problem. The theorem is presented and proven in Sect. 7.6.3. The theorem of existence of a solution is fairly difficult and complicated to prove and perhaps not so important, as from a practical view point we understand that a physical solution must exist. Although this chapter is primarily devoted to the classical theory of elasticity, Sect. 7.10 includes some fundamental aspects of the general theory of elasticity.

Temperature changes in a material result in strains and normally also in stresses, so-called *thermal stresses*. Section 7.5 presents the basis for determining the thermal stresses in elastic material.

Two-dimensional theory of elasticity in Sect. 7.3 presents analytical solutions to many relatively simple but important problems. Examples are thick-walled circular cylinders subjected to internal and external pressure and a plate with a hole. Analytical solutions are, apart from being of importance as solutions to practical problems, also serving as test examples for numerical solution procedures like finite difference methods and finite element methods.

An important technical application of the theory of elasticity is the theory of torsion of rods. The elementary torsion theory applies only to circular cylindrical bars. The *Saint-Venant theory of torsion* for cylindrical rods of arbitrary cross-section is presented in Sect. 7.4.

The theory of stress waves in elastic materials is treated in Sect. 7.7. The introductory part of the theory of elastic waves is mathematically relatively simple, and some of the most important aspects of elastic wave propagation are revealed, using simple one-dimensional considerations. The general theory of elastic waves is fairly complex and will only be given in an introductory exposition.

Anisotropic linearly elastic materials are presented in Sect. 7.8. Materials having different kinds of symmetry are discussed. This basis is then applied in the theory of fiber-reinforced composite materials in Sect. 7.9.

An introduction to non-linearly elastic materials and elastic materials that are subjected to large deformations, is presented in Sect. 7.10. We shall return to these topics in Sect. 11.7.

Energy methods have great practical importance, both for analytical solution and in relation to the finite element method. The basic concepts of elastic energy are presented in Sect. 7.2 and 7.10. However, the energy methods are not included in this book.

7.2 The Hookean Solid

We shall now assume *small deformations and small displacements* such that the strains may be given by the *strain tensor for small deformations* **E**, which in a Cartesian coordinate system Ox has the components:

$$E_{ik} = (u_{i,k} + u_{k,i}) \tag{7.2.1}$$

where u_i are the components of the displacement vector $\mathbf{u}$, and $u_{i,j}$ are the displacement gradients. E_{ii} (not summed) are longitudinal strains in the directions of the coordinate axes, and E_{ij} $(i \neq j)$ are half of the shear strains for the directions $\mathbf{e}_i$ and $\mathbf{e}_j$.

A material is called *elastic*, also called *Cauchy-elastic*, if the stresses in a particle $\mathbf{r} = x_i \mathbf{e}_\mathrm{i}$ are functions only of the strains in the particle.

$$T_{ik} = T_{ik}(E, x) \Leftrightarrow \quad \mathbf{T} = \mathbf{T}[\mathbf{E}, \mathbf{r}] \tag{7.2.2}$$

These equations are the basic *constitutive equations* for *Cauchy-elastic materials*. In Sect. 7.6 another definition of elasticity is introduced, *hyperelasticity*, also called *Green-elasticity*, which reflects that a material may store deformation work as *elastic energy*, also called *strain energy*, a concept to be defined already in Sect. 7.2.2.

If the elastic properties are the same in every particles in a material, the material is *elastically homogeneous*. If the elastic properties are the same in all directions through one and the same particle, the material is *elastically isotropic*. Metals, rocks, and concrete are in general considered to be both homogeneous and isotropic materials. The crystals in polycrystalline materials are assumed to be small and their orientations so random that the crystalline structure may be neglected. Each individual crystal is normally anisotropic. By milling or other forms of

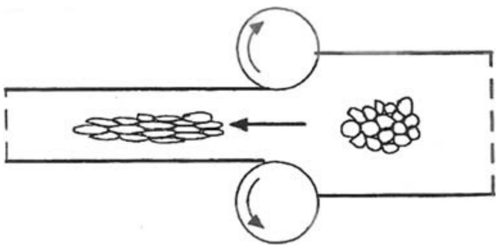


Fig. 7.2.1 Anisotropy due to milling

macro-mechanical forming of polycrystalline metals, an originally isotropic material may be anisotropic, as indicated in Fig. 7.2.1. Inhomogeneities in concrete due to the presence of large particles of gravel may for practical reasons be overlooked when the concrete is treated as a continuum. Materials with fiber structure and well defined fiber directions have anisotropic elastic response. Wood and fiber reinforced plastic are typical examples. The elastic properties of wood are very different in the directions of the fibers and in the cross-fiber direction. Anisotropic elastic materials are discussed in Sect. 7.8 and 7.9.

Isotropic elasticity implies that the principal directions of stress and strain coincide: *The stress tensor and the strain tensor are coaxial.* This may be demonstrated by the following arguments. Figure 7.2.2 shows a material element with orthogonal edges in the undeformed configuration.The element is subjected to a state of stress with principal directions parallel to the undeformed edges of the element. The deformed element is also shown. For the sake of illustration the deformation of the element is exaggerated considerably. Due to the symmetry of the configuration of stress and the isotropy of the elastic properties the diagonal planes marked p_1 and p_2 are equally deformed. This means that the element retains the right angles between its edges through the deformation. Thus the principal directions of strains coincide with the principal directions of stress.

Homogeneous elasticity implies that the stress tensor is independent of the particle coordinates. Therefore the constitutive equation of a homogeneous Cauchy-elastic material should be of the form:

$$T_{ik} = T_{ik}(E) \Leftrightarrow \quad \mathbf{T} = \mathbf{T}[\mathbf{E}] \tag{7.2.3}$$

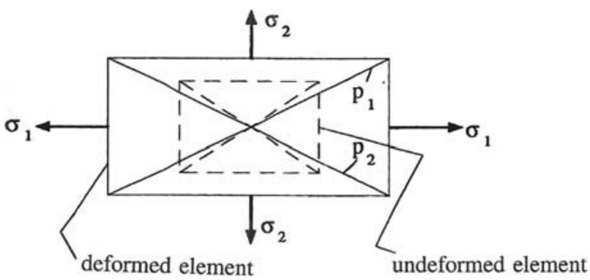


Fig. 7.2.2 Coaxial stresses and strains

If the relation (7.2.2) is linear in **E**, the material is said to be *linearly elastic*. The six coordinate stresses T_{ij} with respect to a coordinate system Ox are now linear functions of the six coordinate strains E_{ij}. For a generally anisotropic material these linear relations contain $6 \times 6 = 36$ coefficients or material parameters, which are called *elasticities* or *stiffnesses*. For a homogeneously elastic material the stiffnesses are constant material parameters. We shall now prove that for an isotropic, linearly elastic material the number of independent stiffnesses is two. Different types of anisotropy are presented and discussed in Sects. 7.8 and 7.9.

In tension or compression tests of isotropic materials a test specimen is subjected to uniaxial stress σ and experiences the strains ε in direction of the stress and ε_t in any transverse direction, i.e. normal to the stress. For a linearly elastic material the following relations may be stated:

$$\varepsilon = \frac{\sigma}{\eta}\,, \quad \varepsilon_t = -\nu\,\varepsilon = \nu\,\frac{\sigma}{\eta} \tag{7.2.4}$$

where η is the *modulus of elasticity* and ν is the *Poisson's ratio*. Values for η and ν for some characteristic materials are given in Table 7.2.1. The symbol η for the modulus of elasticity rather then the more common symbol E is used to prevent confusion between the modulus of elasticity and the strain matrix E. Throughout the book the symbol η for the modulus of elasticity will be used in constitutive equations were the bold face tensor notation or the index notation are used. However, when the xyz – notation is used and in cylindrical coordinates and spherical coordinates the more common symbol E will be used for the modulus of elasticity.

A linearly elastic material in a state of uniaxial stress: $\sigma_1 \neq 0, \sigma_2 = \sigma_3 = 0$, obtains the strains:

$$\varepsilon_1 = \frac{\sigma_1}{\eta}\,, \quad \varepsilon_2 = \varepsilon_3 = \varepsilon_t = -\nu\,\frac{\sigma_1}{\eta}$$

In a general state of triaxial stress, with principal stresses σ_1, σ_2, and σ_3, the principal strains are:

Table 7.2.1 Density ρ, modulus of elasticity $\eta \equiv E$, shear modulus $\mu \equiv G$, poisson's ratio ν, and thermal expansion coefficient α for some characteristic materials

	ρ [10^3 kg/m^3]	η, E [GPa]	μ, G [GPa]	ν	α [$10^{-6}\,^\circ\mathrm{C}^{-1}$]
Steel	7.83	210	80	0.3	12
Aluminium	2.68	70	26	0.25	23
Concrete	2.35	20–40		0.15	10
Copper	8.86	118	41	0.33	17
Glass	2.5	80	24	0.23	3–9
Wood	0.5	4–11	fiber dir.	anisotropic	3–8
Cork				0	
Rubber	1.5		0.007	0.49	
Bronze	8.30	97	39		
Brass	8.30	97	39		19
Magnesium	1.77	40	16		25
Cast iron	7.75	103	41	0.25	11

$$\varepsilon_1 = \frac{\sigma_1}{\eta} - \frac{\nu}{\eta}(\sigma_2 + \sigma_3) = \frac{1+\nu}{\eta}\sigma_1 - \frac{\nu}{\eta}(\sigma_1 + \sigma_2 + \sigma_3) \quad \text{etc. for } \varepsilon_2 \text{ and } \varepsilon_3$$

The result follows from the fact that the relations between stresses and strains are linear such that the principal of superposition applies. The result may be rewritten to:

$$\varepsilon_i = \frac{1+\nu}{\eta}\sigma_i - \frac{\nu}{\eta}\,\mathrm{tr}T\,, \quad \mathrm{tr}T = \sigma_1 + \sigma_2 + \sigma_3 = \mathrm{tr}\ \mathbf{T}$$

The expression may also be presented in the matrix representation:

$$\varepsilon_i \delta_{ik} = \frac{1+\nu}{\eta}\sigma_i\,\delta_{ik} - \frac{\nu}{\eta}(\mathrm{tr}T)\,\delta_{ik} \tag{7.2.5}$$

This is furthermore the matrix representation of a tensor equation between the tensors **E** and **T** in a coordinate system with base vectors parallel to the principal directions of stress. The matrix equation (7.2.5) now represents the tensor equation:

$$\mathbf{E} = \frac{1+\nu}{\eta}\mathbf{T} - \frac{\nu}{\eta}(\mathrm{tr}\ \mathbf{T})\,\mathbf{1} \tag{7.2.6}$$

In any Cartesian coordinate system Ox (7.2.6) has the representation:

$$E_{ik} = \frac{1+\nu}{\eta}T_{ik} - \frac{\nu}{\eta}T_{jj}\,\delta_{ik} \tag{7.2.7}$$

From the (7.2.6) and (7.2.7) the inverse relations between the stress tensor and the strain tensor is obtained:

$$\mathbf{T} = \frac{\eta}{1+\nu}\left[\mathbf{E} + \frac{\nu}{1-2\nu}(\mathrm{tr}\ \mathbf{E})\,\mathbf{1}\right] \quad \Leftrightarrow \quad T_{ik} = \frac{\eta}{1+\nu}\left[E_{ik} + \frac{\nu}{1-2\nu}E_{jj}\,\delta_{ik}\right] \tag{7.2.8}$$

The inversion procedure is given as Problem 7.1. The relations (7.2.6, 7.2.7, 7.2.8) represent the *generalized Hooke's law* and are the constitutive equations of the *Hookean material* or *Hookean solid*, which are names for a *isotropic, linearly elastic material*. Note that normal stresses T_{ii} only result in longitudinal strains E_{ii}, and visa versa, and that shear stresses T_{ij} only result in shear strains $\gamma_{ik} = 2E_{ik}$, and visa versa. This property is in general not the case for anisotropic materials, see Sect. 7.8.

For the relations between the coordinate shear stresses and coordinate shear strains the (7.2.8) give:

$$T_{ik} = 2\mu\,E_{ik} = \mu\,\gamma_{ik}\,, \quad i \neq k \tag{7.2.9}$$

The material parameter μ ($\equiv G$ in common notation) is called the *shear modulus* and is given by:

$$\mu = \frac{\eta}{2(1+\nu)} \quad \Leftrightarrow \quad G = \frac{E}{2(1+\nu)} \tag{7.2.10}$$

Table 7.2.1 presents the elasticities $\eta\,(\equiv E), \mu\,(\equiv G)$, and ν for some characteristic materials. The values vary some with the quality of the materials listed and with temperature. The values given in the table are for ordinary, or room, temperature of 20°C. We shall find that the relationship for the shear modulus in formula (7.2.10) is not exactly satisfied. This is due to the fact that the values given in Table 7.2.1 are standard values found from different sources.

The relationship between the elastic volumetric strain ε_v and the stresses is found by computing the trace of the strain matrix from (7.2.7).

$$\varepsilon_v = E_{ii} = \frac{1+\nu}{\eta} T_{ii} - \frac{\nu}{\eta} T_{jj}\,\delta_{ii} = \frac{1-2\nu}{\eta} T_{ii} \tag{7.2.11}$$

The *mean normal stress* σ^o and the *bulk modulus* or the *compression modulus* of elasticity κ are introduced:

$$\sigma^o = \frac{1}{3} T_{ii} = \frac{1}{2} \text{tr } \mathbf{T} \quad \text{the mean normal stress} \tag{7.2.12}$$

$$\kappa = \frac{\eta}{3(1-2\nu)} \quad \text{the bulk modulus} \tag{7.2.13}$$

Then the result (7.2.11) may now be presented as:

$$\varepsilon_v = \frac{1}{\kappa} \sigma^o \tag{7.2.14}$$

A more appropriate name for κ than the compression modulus had perhaps been the *expansion modulus*. For an isotropic state of stress the mean normal stress is equal to the normal stress, i.e. $\mathbf{T} = \sigma^o \mathbf{1}$.

Fluids are considered as linearly elastic materials when sound waves are analyzed. The only elasticity relevant for fluids is the bulk modulus κ. For water $\kappa = 2.1$GPa, for mercury $\kappa = 27$GPa, and for alcohol $\kappa = 0.91$GPa.

It follows from (7.2.13) that a Poisson ratio $\nu > 0.5$ would have given $\kappa < 0$, which according to (7.2.14) would lead to the physically unacceptable result that the material increases its volume when subjected to isotropic pressure. Furthermore we may expect to find that $\nu \geq 0$ because a Poisson ratio $\nu < 0$ would, according to $(7.2.4)_2$, give an expansion in the transverse direction when the material is subjected to uniaxial stress. Thus we may expect that:

$$0 \leq \nu \leq 0.5 \tag{7.2.15}$$

The upper limit for the Poisson ratio, $\nu = 0.5$, which according to (7.2.13) gives $\kappa = \infty$, characterizes an *incompressible material*. Among the real materials rubber, having $\nu = 0.49$, is considered to be (nearly) incompressible, while the other extreme, $\nu = 0$, is represented by cork, which is an advantageous property when corking bottles. As a curiosity and a historical note it may be mentioned that a theory developed by Poisson and based on an atomic model of materials, led to a universal value of ν equal to 0.25. From Table 7.2.1 it is seen that this "universal" value is not universal, although close to the values found in experiments for some important materials.

A very simple form for Hooke's law (7.2.8) is obtained when we decompose the stress tensor **T** and the strain tensor **E** into isotrops and deviators:

$$\mathbf{T} = \mathbf{T}^o + \mathbf{T}' \,, \quad \mathbf{E} = \mathbf{E}^o + \mathbf{E}' \tag{7.2.16}$$

$$\mathbf{T}^o = \frac{1}{3}\,(\text{tr }\mathbf{T})\,\mathbf{1} = \sigma^o\,\mathbf{1} \,, \quad \mathbf{E}^o = \frac{1}{3}\,(\text{tr }\mathbf{E})\,\mathbf{1} = \frac{1}{3}\,\varepsilon_v\mathbf{1} \tag{7.2.17}$$

From (7.2.8) or (7.2.6) we find that:

$$\mathbf{T}^o = 3\kappa\mathbf{E}^o, \ \mathbf{T}' = 2\mu\,\mathbf{E}' \tag{7.2.18}$$

The development of these results is given as Problem 7.2. Alternative forms for Hooke's law are:

$$\mathbf{T} = 2\mu\,\mathbf{E} + \left(\kappa - \frac{2}{3}\mu\right)(\text{tr }\mathbf{E})\,\mathbf{1} \tag{7.2.19}$$

$$\mathbf{E} = \frac{1}{2\mu}\mathbf{T} - \frac{3\kappa - 2\mu}{18\mu\,\kappa}\,(\text{tr }\mathbf{T})\,\mathbf{1} \tag{7.2.20}$$

The parameters:

$$\mu \text{ and } \lambda \equiv \kappa - \frac{2}{3}\mu \tag{7.2.21}$$

are called the *Lamé constants*. The parameter λ does not have any independent physical interpretation.

For an *incompressible material*: $\varepsilon_v \equiv 0$, the mean stress: $\sigma^o = (1/3)\text{tr}\mathbf{T}$, cannot be determined from Hooke's law. For these materials it is customary to replace (7.2.8) or (7.2.19) by:

$$\mathbf{T} = -p\,\mathbf{1} + 2\mu\,\mathbf{E} \tag{7.2.22}$$

$p = p(\mathbf{r},t)$ is an unknown pressure, which is an unknown tension if p is negative. The pressure p can only be determined from the equations of motion and the corresponding boundary conditions.

7.2.1 An Alternative Development of the Generalized Hooke's Law

The constitutive equations for a Hookean solid, or what we have called the generalized Hooke's law for general states of stress and strain, represented by (7.2.6, 7.2.7, 7.2.8), may also be found on the basis of mathematical results in Sect. 4.6.3 on isotropic tensor functions. Since the relationship between the stress tensor **T** and the strain tensor **E** is linear, we may write:

$$\mathbf{T} = \mathbf{S} : \mathbf{E} \quad \Leftrightarrow \quad T_{ij} = S_{ijkl}\,E_{kl} \tag{7.2.23}$$

$$\mathbf{E} = \mathbf{K} : \mathbf{T} \quad \Leftrightarrow \quad E_{ij} = K_{ijkl}\,T_{kl} \tag{7.2.24}$$

The fourth order tensor $\mathbf{S}$ is called the *elasticity tensor* or the *stiffness tensor*. The fourth order tensor $\mathbf{K}$ is called the *compliance tensor* or the *flexibility tensor*. Since we assume that the material model defined by the constitutive equations (7.2.23) and (7.2.24) is isotropic, each of the tensors $\mathbf{S}$ and $\mathbf{K}$ must be represented by the same matrix in all coordinate systems Ox. This implies that $\mathbf{S}$ and $\mathbf{K}$ are isotropic fourth order tensors. With reference to (4.6.30) we see that $\mathbf{S} = \mathbf{I}_4^s$, and with the symmetric 4.order isotropic tensor $\mathbf{I}_4^s$ from (4.6.31) we may write:

$$\mathbf{S} = 2\mu\,\mathbf{1}_4^s + \left(\kappa - \frac{2}{3}\mu\right)\mathbf{1}\otimes\mathbf{1} \quad \Leftrightarrow \quad S_{ijkl} = \mu\left(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}\right) + \left(\kappa - \frac{2}{3}\mu\right)\delta_{ij}\,\delta_{kl} \tag{7.2.25}$$

$$\mathbf{K} = \frac{1}{2\mu}\,\mathbf{1}_4^s - \frac{3\kappa - 2\mu}{18\mu\,\kappa}\mathbf{1}\otimes\mathbf{1} \quad \Leftrightarrow \quad K_{ijkl} = \frac{1}{4\mu}\left(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}\right) - \frac{3\kappa - 2\mu}{18\mu\,\kappa}\,\delta_{ij}\,\delta_{kl} \tag{7.2.26}$$

The material parameters have been chosen such that the result coincide with what is found above in (7.2.19) and (7.2.20). Compare (7.2.25) with (4.6.34).

7.2.2 Strain Energy

In Chap. 6 the concept of stress power was defined. The stress power per unit volume is given by the expression $\omega = \mathbf{T} : \mathbf{D}$, where $\mathbf{D}$ is the rate of deformation tensor. When we assume small deformations, we may set $\mathbf{D} = \dot{\mathbf{E}}$, and thus:

$$\omega = \mathbf{T} : \dot{\mathbf{E}} \tag{7.2.27}$$

Since the stresses are linear functions of the strains through Hooke's law (7.2.8), the work done on the material per unit volume when the state of stress is increased from zero stress to the state given by $\mathbf{T}$, is equal to:

$$W = \frac{1}{2}\mathbf{T} : \mathbf{E} \tag{7.2.28}$$

This work is recoverable in the sense that the material may perform an equal amount of work on the environment when the stresses are relieved, and W may thus be considered to be stored in the material in the form of *elastic energy per unit volume* or *strain energy per unit volume*:

$$\phi = \frac{1}{2}\mathbf{T} : \mathbf{E} = \mu\,\mathbf{E} : \mathbf{E} + \frac{1}{2}\left(\kappa - \frac{2}{3}\mu\right)(\text{tr }\mathbf{E})^2 \tag{7.2.29}$$

By linear decompositions of the stress tensor $\mathbf{T}$ and the strain tensor $\mathbf{E}$ according to (7.2.16) it may be shown, see Problem 7.7, that the strain energy consists of a *volumetric strain energy* ϕ^o and a *deviatoric strain energy* ϕ':

$$\phi = \phi^o + \phi'$$
$$\phi^o = \frac{1}{2}\mathbf{T}^o : \mathbf{E}^o = \frac{\kappa}{2}\varepsilon_v{}^2 = \frac{1}{2\kappa}(\sigma^o)^2 = \frac{1}{18\kappa}(\text{tr } \mathbf{T})^2$$
$$\phi' = \frac{1}{2}\mathbf{T}' : \mathbf{E}' = \mu\mathbf{E}' : \mathbf{E}' = \frac{1}{4\mu}\mathbf{T}' : \mathbf{T}' \tag{7.2.30}$$

For uniaxial stress:

$$\phi = \frac{1}{2}\sigma\,\varepsilon = \frac{1}{2}\eta\,\varepsilon^2 = \frac{1}{2\eta}\sigma^2 \tag{7.2.31}$$

7.3 Two-Dimensional Theory of Elasticity

The general equations of the theory of elasticity can only be solved by elementary analytical methods in a few special and simple cases. In many problems of practical interest we may however introduce simplifications with respect to the state of stress or the state of displacements, such that a useful solution may be found by relatively simple means.

7.3.1 Plane Stress

Thin plates or slabs that are loaded parallel to the middle plane, see Fig. 7.3.1, by body forces **b** and contact forces **t** on the boundary surface *A*, such that:

$$b_\alpha = b_\alpha(x_1, x_2, t)\ ,\quad b_3 = 0 \quad \text{in the volume } V \tag{7.3.1}$$
$$t_\alpha = t_\alpha(x_1, x_2, t)\ ,\quad t_3 = 0 \quad \text{on the surface } A \tag{7.3.2}$$

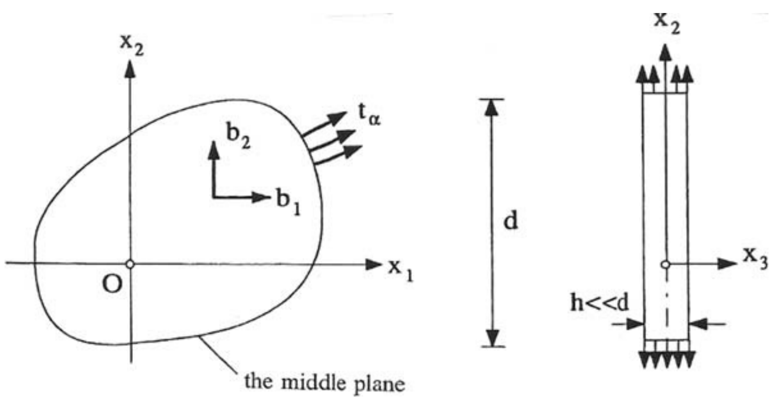


Fig. 7.3.1 Thin plate in plane stress

obtain approximately a state of plane stress:

$$T_{i3} = 0\,,\quad T_{\alpha\beta} = T_{\alpha\beta}\left(x_1, x_2, t\right) \tag{7.3.3}$$

A stricter analysis of the problem presented in Fig. 7.3.1 and with the loading conditions given by (7.3.1) and (7.3.2), will show that the conditions (7.3.3) for plane stress are not completely satisfied. However, the approximation (7.3.3) is acceptable for a thin plate if the stresses $T_{\alpha\beta}$ and the displacements u_α are considered to represent mean values over the thickness h of the plate. The thickness h is assumed to be much smaller than a characteristic diameter d of the plate, see Fig. 7.3.1.

The fundamental equations for a thin plate in plane stress are:

1) The Cauchy equations of motion:

$$\text{div}\;\mathbf{T} + \rho\,\mathbf{b} = \rho\,\ddot{\mathbf{u}} \quad\Leftrightarrow\quad T_{\alpha\beta,\beta} + \rho\, b_\alpha = \rho\,\ddot{u}_\alpha \tag{7.3.4}$$

The acceleration **a** has been substituted by the second material derivative of the displacement vector **u**, which in turn will be represented by the second partial derivative of **u** with respect to time:

$$\ddot{\mathbf{u}} = \frac{\partial^2 \mathbf{u}}{\partial t^2} \tag{7.3.5}$$

2) *Hooke's law for plane stress*, see Problem 7.3:

$$T_{\alpha\beta} = 2\mu\left[E_{\alpha\beta} + \frac{\nu}{1-\nu}E_{\rho\rho}\,\delta_{\alpha\beta}\right],\; 2\mu = \frac{\eta}{1+\nu} \tag{7.3.6}$$

$$\sigma_x = \frac{2G}{1-\nu}\left[\varepsilon_x + \nu\,\varepsilon_y\right]\,,\;\; \sigma_y = \frac{2G}{1-\nu}\left[\varepsilon_y + \nu\,\varepsilon_x\right]\,,\;\; \tau_{xy} = G\gamma_{xy}\,,\;\; 2G = \frac{E}{1+\nu} \tag{7.3.7}$$

$$E_{\alpha\beta} = \frac{1}{2\mu}\left[T_{\alpha\beta} - \frac{\nu}{1+\nu}T_{\rho\rho}\,\delta_{\alpha\beta}\right],\; E_{33} = -\frac{\nu}{\eta}T_{\rho\rho} \tag{7.3.8}$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\,\sigma_y\right],\; \varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\,\sigma_x\right],\; \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\varepsilon_z = -\frac{\nu}{E}\left[\sigma_x + \sigma_y\right],\; E = \frac{G}{2\left(1+\nu\right)} \tag{7.3.9}$$

Note that for a convenient notation two different symbols have been used for the modulus of elasticity: $\eta \equiv E$ and for the shear modulus: $\mu = G$.

3) Strain-displacement relations:

$$E_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right). \tag{7.3.10}$$

Equations (7.3.4), (7.3.6), and (7.3.10) represent all together 8 equations for the 8 unknown functions $T_{\alpha\beta}$, $E_{\alpha\beta}$, and u_α.

It is assumed that the contact forces, or stresses, are given on a part A_σ of the surface A of the plate, while the displacements are given over the remaining part: $A_u = A - A_\sigma$, such that:

$$t_\alpha = T_{\alpha\beta}\, n_\beta = t_\alpha^* \quad \text{on } A_\sigma \tag{7.3.11}$$

$$u_\alpha = u_\alpha^* \quad \text{on } A_u \tag{7.3.12}$$

t_α^* and u_α^* are prescribed functions of position. Equations (7.3.11) and (7.3.12) are thus boundary conditions in the problem governed by the (7.3.4), (7.3.6), and (7.3.10).

In an analytical solution to a problem in the theory of elasticity it is customary to choose either displacements or stresses as the primary unknown functions. In the present section we consider the first alternative. In Sect. 7.3.3 the stresses are chosen as the primary unknown functions.

When the displacements are selected as the primary unknowns, the fundamental equations are transformed as follows. The relations (7.3.10) are substituted into Hooke's law (7.3.6), and the result is:

$$T_{\alpha\beta} = \mu \left[u_{\alpha,\beta} + u_{\beta,\alpha} + \frac{2\nu}{1-\nu}\, u_{\rho,\rho}\, \delta_{\alpha\beta} \right] \tag{7.3.13}$$

These expressions for the stresses are then substituted into the Cauchy equations of motion (7.3.4) to give:

$$u_{\alpha,\beta\beta} + \frac{1+\nu}{1-\nu} u_{\beta,\beta\alpha} + \frac{1}{\mu}\,\rho\,(b_\alpha - \ddot{u}_\alpha) = 0 \tag{7.3.14}$$

These equations of motion are called the *Navier equations for plane stress*, named after Claude L. M. H. Navier [1785–1836]. The two displacement components u_α are to be found from the two Navier equations. The stresses may then be determined from the expressions (7.3.13), and the strains are determined from the expressions (7.3.10). Finally the boundary conditions (7.3.11) and (7.3.12) complete the solution to the problem.

Below we shall consider two problems for which the state of displacements is axisymmetrical. The result of this assumption is that the state of deformation is irrotational, or in other words the result is a state of pure strain:

$$\tilde{R}_{\alpha\beta} = 0 \Leftrightarrow u_{\alpha,\beta} = u_{\beta,\alpha} \quad \Rightarrow \quad u_{\alpha,\beta\beta} = u_{\beta,\alpha\beta} = u_{\beta,\beta\alpha}$$

The two Navier equations (7.3.14) may now be transformed into:

$$\varepsilon_{A,\alpha} + \frac{1-\nu}{2\mu}\,\rho\,(b_\alpha - \ddot{u}_\alpha) = 0 \tag{7.3.15}$$

ε_A is the invariant:

$$\varepsilon_A \equiv u_{\beta,\beta} = E_{\beta\beta} = E_{11} + E_{22} \tag{7.3.16}$$

The invariant represents the change in area per unit area in the plane of the plate. We shall call this quantity the *area strain*. We now introduce the radial displacement u as the only unknown displacement function, see Fig. 7.3.2 which shows a circular slab. Polar coordinates (R, θ) are applied and due to axisymmetry the radial displacement is a function of R alone: $u = u(R)$. The relevant strains are the longitudinal strains:

$$\varepsilon_R = \frac{du}{dR}, \quad \varepsilon_\theta = \frac{l - l_o}{l_o} = \frac{2\pi(R+u) - 2\pi R}{2\pi R} = \frac{u}{R} \tag{7.3.17}$$

The area strain is now:

$$\varepsilon_A = \varepsilon_R + \varepsilon_\theta = \frac{du}{dR} + \frac{u}{R} = \frac{1}{R}\frac{d(Ru)}{dR} \tag{7.3.18}$$

The Navier equation for the R-direction is supplied by (7.3.15) when x_α is replaced by R. The result is:

$$\frac{d}{dR}\left[\frac{1}{R}\frac{d(Ru)}{dR}\right] + \frac{1-\nu}{2G}\rho\left(b_R - \ddot{u}_R\right) = 0 \tag{7.3.19}$$

The coordinate stresses in polar coordinates are $\sigma_R, \tau_{R\theta}$, and σ_θ, see Fig. 7.3.2. Due to axisymmetry the shear stress $\tau_{R\theta}$ is zero. When the displacement u has been determined from (7.3.19), the stresses may be determined from Hooke's law (7.3.7), now written as:

$$\sigma_R(R) = \frac{2G}{1-\nu}\left[\frac{du}{dR} + \nu\frac{u}{R}\right], \quad \sigma_\theta(R) = \frac{2G}{1-\nu}\left[\frac{u}{R} + \nu\frac{du}{dR}\right] \tag{7.3.20}$$

Example 7.1. Circular Plate with a Hole

A circular plate of radius b has a concentric hole of radius a, as shown in Fig. 7.3.3. The edge of the hole is subjected to a pressure p, and the outer edge of the slab is subjected to a pressure q. We shall determine the state of stress and the radial displacement of the plate. The Navier equation (7.3.19) is in this case reduced to the equilibrium equation:

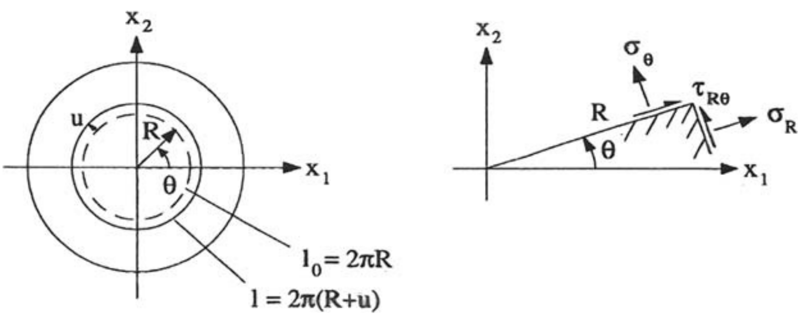


Fig. 7.3.2 Circular plate with axissymmetrical displacement

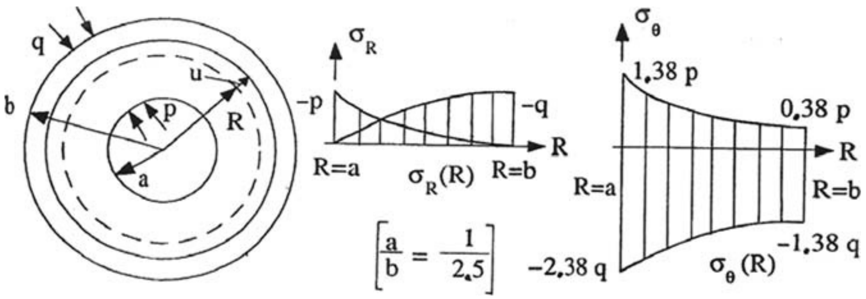


Fig. 7.3.3 Circular plate with a concentric hole. Examples 7.1 and 7.2. Radial stress σ_R and tangential stress σ_θ as functions of the radial distance R

$$\frac{d}{dR}\left[\frac{1}{R}\frac{d\,(Ru)}{dR}\right] = 0$$

Two integrations result in:

$$u = AR + \frac{B}{R}\,,\quad \frac{du}{dR} = A - \frac{B}{R^2}\,,\quad A \text{ and } B \text{ are constants of integration}$$

An expression for the radial stress is now obtained from (7.3.20):

$$\sigma_R(R) = 2G\left[\frac{1+\nu}{1-\nu}A - B\frac{1}{R^2}\right]$$

The boundary conditions give:

$$\sigma_R(a) = -p \;\Rightarrow\; 2G\left[\frac{1+\nu}{1-\nu}A - B\frac{1}{a^2}\right] = -p\,,$$

$$\sigma_R(b) = -q \;\Rightarrow\; 2G\left[\frac{1+\nu}{1-\nu}A - B\frac{1}{b^2}\right] = -q \quad\Rightarrow$$

$$2G\frac{1+\nu}{1-\nu}A = \frac{p\,(a/b)^2 - q}{1-(a/b)^2}\,,\quad 2GB = \frac{(p-q)\,a^2}{1-(a/b)^2}$$

The final solution to the problem is then:

$$\sigma_R(R) = \frac{1}{1-(a/b)^2}\left\{-\left[\left(\frac{a}{R}\right)^2 - \left(\frac{a}{b}\right)^2\right]p - \left[1-\left(\frac{a}{R}\right)^2\right]q\right\} \tag{7.3.21}$$

$$\sigma_\theta(R) = \frac{1}{1-(a/b)^2}\left\{\left[\left(\frac{a}{R}\right)^2 + \left(\frac{a}{b}\right)^2\right]p - \left[1+\left(\frac{a}{R}\right)^2\right]q\right\} \tag{7.3.22}$$

$$u(R) = \frac{1}{2G}\frac{a}{1-(a/b)^2}\left\{\left[\frac{1-\nu}{1+\nu}\left(\frac{a}{b}\right)^2\frac{R}{a} + \frac{a}{R}\right]p - \left[\frac{1-\nu}{1+\nu}\frac{R}{a} + \frac{a}{R}\right]q\right\} \tag{7.3.23}$$

The stress distribution in the radial direction and in the circumferential direction are shown in Fig. 7.3.3. For the particular case when $p = q$, we get the reasonable result:

$$\sigma_R = \sigma_\theta = -p \,, \quad u = -\frac{1-\nu}{2G(1+\nu)} pR$$

The strain in the direction normal to the plane of the plate is found from $(7.3.9)_4$:

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_R + \sigma_\theta) = \frac{\nu}{G(1+\nu)} \frac{q-(a/b)^2 p}{1-(a/b)^2} = \text{constant} \tag{7.3.24}$$

Example 7.2. Rotating Circular Plate

The plate in Fig. 7.3.3 rotates with a constant angular velocity ω about its axis. The pressures $p = q = 0$. The state of stress and the radial displacement are to be determined when the boundary conditions alternatively are given by:

Case I. Plate without a hole, $a = 0$. Outer edge, $R = b$, of the plate is stress free.

Case II. Plate with a hole, $a > 0$. Stress free inner and outer edges.

The rotation of the plate results in a normal acceleration a_n towards the axis of rotation. Hence:

$$\ddot{u} = \ddot{u}_R = -a_n = -\omega^2 R$$

The Navier equation (7.3.19) is reduced to:

$$\frac{d}{dR}\left[\frac{1}{R}\frac{d(Ru)}{dR}\right] = -\frac{1-\nu}{2G}\rho\,\omega^2 R$$

Two integrations provide the result:

$$u(R) = AR + \frac{B}{R} - \frac{1-\nu}{16G}\rho\,\omega^2 R^3 \,, \quad A \text{ and } B \text{ are constants of integration} \tag{7.3.25}$$

The stresses are given by the (7.3.20):

$$\begin{aligned} \sigma_R(R) &= 2G\left[\frac{1+\nu}{1-\nu}A - B\frac{1}{R^2}\right] - \frac{3+\nu}{8}\rho\,\omega^2 R^2 \\ \sigma_\theta(R) &= 2G\left[\frac{1+\nu}{1-\nu}A + B\frac{1}{R^2}\right] - \frac{1+3\nu}{8}\rho\,\omega^2 R^2 \end{aligned} \tag{7.3.26}$$

The constants A and B will be found from the boundary conditions in Case I and Case II respectively.

Case I: Plate without a hole, $a = 0$. Outer edge, $R = b$, is stress free. The boundary conditions are:

$$u(0) = 0 \,, \quad \sigma_R(b) = 0$$

Using the expressions (7.3.25) and (7.3.26) in these conditions, we get two equations from which A and B can be solve. The solution of the equations is:

$$B = 0\,, \quad 2G\frac{1+\nu}{1-\nu}A = \frac{3+\nu}{8}\rho\,\omega^2 b^2\,,$$

The (7.3.24, 7.3.25, and 7.3.20) provide the complete solution to the problem:

$$u(R) = \frac{1-\nu}{16G(1+\nu)}\rho\,\omega^2 b^3\left[(3+\nu)\frac{R}{b} - (1+\nu)\left(\frac{R}{b}\right)^3\right]$$

$$\sigma_R(R) = \frac{3+\nu}{8}\rho\,\omega^2 b^2\left[1 - \left(\frac{R}{b}\right)^2\right], \sigma_\theta(R) = \frac{1}{8}\rho\,\omega^2 b^2\left[(3+\nu) - (1+3\nu)\left(\frac{R}{b}\right)^2\right]$$

$$\sigma_{\max} = \sigma_R(0) = \sigma_\theta(0) = \frac{3+\nu}{8}\rho\,\omega^2 b^2$$

Case II: Plate with a hole. Stress free inner and outer edges, $R = a$ and $R = b$. The boundary conditions are:

$$\sigma_R(a) = 0\,, \quad \sigma_R(b) = 0$$

Using the expression (7.3.26) in these conditions, we get two equations from which the constants of integration A and B can be solve. The solution of the equations is:

$$2G\frac{1+\nu}{1-\nu}A = \frac{3+\nu}{8}\left(a^2+b^2\right)\rho\,\omega^2\,, \quad 2GB = \frac{3+\nu}{8}a^2 b^2 \rho\,\omega^2$$

The (7.3.25, 7.3.26, and 7.3.20) provide the complete solution to the problem:

$$u(R) = \frac{(3+\nu)\rho\,\omega^2 b^3}{16G}\left\{\frac{1-\nu}{1+\nu}\left[1+\left(\frac{a}{b}\right)^2\right]\frac{R}{b} + \left(\frac{a}{b}\right)^2\frac{b}{R} - \frac{1-\nu}{3+\nu}\left(\frac{R}{b}\right)^3\right\}$$

$$\sigma_R(R) = \frac{(3+\nu)\rho\,\omega^2 b^2}{8}\left\{1+\left(\frac{a}{b}\right)^2 - \left(\frac{a}{R}\right)^2 - \left(\frac{R}{b}\right)^2\right\}$$

$$\sigma_\theta(R) = \frac{(3+\nu)\rho\,\omega^2 b^2}{8}\left\{1+\left(\frac{a}{b}\right)^2 + \left(\frac{a}{R}\right)^2 - \frac{1+3\nu}{3+\nu}\left(\frac{R}{b}\right)^2\right\}$$

$$\sigma_{\max} = \sigma_\theta(a) = \frac{(3+\nu)\rho\,\omega^2 b^2}{4}\left\{1+\frac{1-\nu}{3+\nu}\left(\frac{a}{b}\right)^2\right\}$$

If we let the hole radius approach zero, $a \to 0$, we get the result that $\sigma_{\max} = (3+\nu)\rho b^2\omega^2/4$, which is twice the value we found above for $\sigma_{\max}$ in a plate without a hole.

7.3.2 Plane Displacements

A body is in the state of *plane displacements* parallel to the x_1x_2 – plane when:

$$u_\alpha = u_\alpha(x_1, x_2, t)\,, \; u_3 = 0 \tag{7.3.27}$$

The strains are expressed by:

$$E_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha},_{\beta} + u_{\alpha},_{\beta}\right), \ E_{i3} = 0 \tag{7.3.28}$$

This is also called a state of *plane deformations*, or a state of *plane strains*.

States of plane displacements occur in cylindrical bodies that are held between two rigid and parallel planes, as shown in Fig. 7.3.4, and that are subjected to forces of the type (7.3.1) and (7.3.2). The rigid planes can only transfer normal stresses T_{33}. If the cylindrical body is not held between two rigid planes, but has an appreciable length, we may find the stresses and the displacements in the body by first assuming the body is held between two rigid planes, compute the stress T_{33} from Hooke's law, and then superimpose a solution for the problem where the body is subjected to negative T_{33} stress. This type of problem will be demonstrated in Example 7.3.

The fundamental equations for plane displacements are provided by *Hooke's law for plane displacements*, see Problem 7.4:

$$\begin{aligned}
T_{\alpha\beta} &= 2\mu\left[E_{\alpha\beta} + \frac{\nu}{1-2\nu}E_{\rho\rho}\,\delta_{\alpha\beta}\right], \ T_{33} = \frac{2\nu\mu}{(1-2\nu)}E_{\alpha\alpha} = \nu T_{\alpha\alpha} \\
\sigma_x &= \frac{2G}{1-2\nu}\left[(1-\nu)\,\varepsilon_x + \nu\,\varepsilon_y\right], \ \tau_{xy} = G\gamma_{xy} \qquad (7.3.29)\\
\sigma_y &= \frac{2G}{1-2\nu}\left[(1-\nu)\,\varepsilon_y + \nu\,\varepsilon_x\right], \ \sigma_z = \frac{2\nu G}{1-2\nu}\left(\varepsilon_x + \varepsilon_y\right)
\end{aligned}$$

$$\begin{aligned}
E_{\alpha\beta} &= \frac{1}{2\mu}\left[T_{\alpha\beta} - \nu\,T_{\rho\rho}\,\delta_{\alpha\beta}\right] \qquad (7.3.30)\\
\varepsilon_x &= \frac{1-\nu}{2G}\left[\sigma_x - \frac{\nu}{1-\nu}\sigma_y\right], \ \varepsilon_y = \frac{1-\nu}{2G}\left[\sigma_y - \frac{\nu}{1-\nu}\sigma_x\right], \ \gamma_{xy} = \frac{1}{G}\tau_{xy}
\end{aligned}$$

The stress tensor, the displacement vector, and the strain tensor must satisfy the Cauchy equations (7.3.4), the strain-displacement relations (7.3.10), and the boundary conditions (7.3.11) and (7.3.12). When the stresses (7.3.29) are substituted into

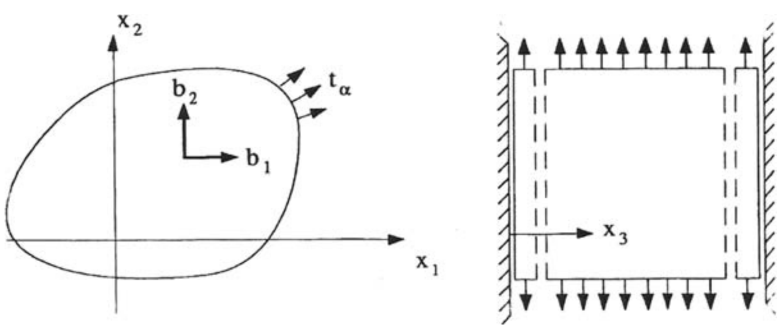


Fig. 7.3.4 Elastic body in plane displacements

the Cauchy equations, the result is the *Navier equations for plane displacements*:

$$u_{\alpha,\beta\beta} + \frac{1}{1-2\nu} u_{\beta,\beta\alpha} + \frac{1}{\mu} \rho \left(b_\alpha - \ddot{u}_\alpha \right) = 0 \tag{7.3.31}$$

We shall find that the fundamental equations (7.3.29, 7.3.30, 7.3.31) for plane displacements mathematically are identical to the similar (7.3.6), (7.3.8), and (7.3.14) for the case of plane stress if we in (7.3.29, 7.3.30, 7.3.31) keep the shear modulus $\mu = \eta/(2(1+\nu))$ unchanged but otherwise replace ν by $\nu/(1+\nu)$. Alternatively we may in the fundamental equations for plane stress keep $\mu = \eta/(2(1+\nu))$ unchanged but otherwise replace ν by $\nu/(1-\nu)$, and the result is the fundamental (7.3.29, 7.3.30, 7.3.31) for plane displacements. Due to the analogy between the two sets of fundamental equations it becomes easy to transfer solutions of problems in plane stress to analogous problems in plane displacements. We may state the rules of transformation as follows.

In the plane displacement equations expressed with the elastic parameters

μ and ν, replace ν by $\dfrac{\nu}{1+\nu}$ $\quad\Rightarrow\quad$ plane stress equations (7.3.32)

In the plane stress equations expressed with the elastic parameters μ and ν

replace ν by $\dfrac{\nu}{1-\nu}$ $\quad\Rightarrow\quad$ plane displacement equations (7.3.33)

The Navier equation for *axisymmetrical displacements u(R)* may be developed along similar lines to the (7.3.19) for plane stress. The result is:

$$\frac{d}{dR}\left[\frac{1}{R}\frac{d(Ru)}{dR}\right] + \frac{(1-2\nu)}{2G(1-\nu)} \rho \left(b_R - \ddot{u}_R \right) = 0 \tag{7.3.34}$$

The relevant stresses are:

$$\sigma_R(R) = \frac{2G(1-\nu)}{(1-2\nu)}\left[\frac{du}{dR} + \frac{\nu}{1-\nu}\frac{u}{R}\right], \; \sigma_\theta(R) = \frac{2G(1-\nu)}{(1-2\nu)}\left[\frac{u}{R} + \frac{\nu}{1-\nu}\frac{du}{dR}\right] \tag{7.3.35}$$

Example 7.3. Thick-Walled Cylinder with Internal and External Pressure

A circular thick-walled cylinder with inner radius a and outer radius b is subjected to an internal pressure p and an external pressure q, see Fig. 7.3.5. The boundary conditions for the radial stress $\sigma_R(R)$ are:

$$\sigma_R(a) = -p\,, \quad \sigma_R(b) = -q \tag{7.3.36}$$

We shall consider three different situations for the plane end surfaces of the cylinder:

1) The end surfaces are fixed as shown in Fig. 7.3.5.
2) The end surfaces are free without stresses.
3) The cylinder is closed with rigid end plates, as indicated to right in Fig. 7.3.5. A force F is necessary to hold the rigid plates in place without axial motion.

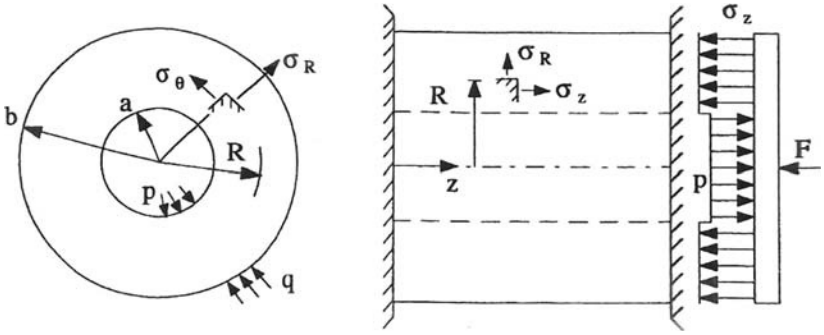


Fig. 7.3.5 Thick walled cylinder with internal pressure p and external pressure q

Let us start by assuming that the end surfaces of the cylinder are prevented from moving in the axial direction. The cylinder will then be in a state of plane displacements. Since we are not considering body forces or accelerations in this problem, the Navier equation (7.3.34) is identical to (7.3.19) for plane stress. The solution to the Navier equation with the boundary conditions (7.3.36) and the general stress formulas (7.3.35), results in the same formulas for the stresses $\sigma_R(R)$ and $\sigma_\theta(R)$ as in Example 7.1. For the radial displacement $u(R)$ however we find:

$$u(R) = \frac{1}{2G}\frac{a}{1-(a/b)^2}\left\{\left[\frac{a}{R}+(1-2\nu)\left(\frac{a}{b}\right)^2\frac{R}{a}\right]p - \left[\frac{a}{R}+(1-2\nu)\frac{R}{a}\right]q\right\} \tag{7.3.37}$$

This expression may be also be obtained directly from formula (7.3.23) by using the rule (7.3.33). The stress on a plane normal to the z-axis is determined from the second formula in the set (7.3.29):

$$\sigma_z \equiv T_{33} = \nu\, T_{\alpha\alpha} = \nu\,(\sigma_R + \sigma_\theta) = -2\nu\,\frac{q-(a/b)^2 p}{1-(a/b)^2} = \text{ constant} \tag{7.3.38}$$

We shall now assume that the end surfaces of the cylinder are free and without stresses. To the solution above we need only to add a normal stress in the z-direction that is equal to the constant σ_z in formula (7.3.38) for plane displacements, but with opposite sign. This addition does not influence the stresses σ_R and σ_θ. The cylinder will now in fact be in a state of plane stress and the radial displacement $u(R)$ is given by (7.3.23) in Example 7.1. The strain in the z-direction is:

$$\varepsilon_z = -\frac{\sigma_z}{E} = \frac{\nu}{G(1+\nu)}\frac{q-(a/b)^2 p}{1-(a/b)^2} = \text{ constant} \tag{7.3.39}$$

The result is identical to the result (7.3.24) in Example 7.1. The radial and tangential strains get a constant addition equal to:

$$\varepsilon_R = \varepsilon_\theta = -\nu\,\varepsilon_z = -\frac{\nu^2}{G(1+\nu)}\frac{q-(a/b)^2 p}{1-(a/b)^2} = \text{ constant} \tag{7.3.40}$$

Using formula $(7.3.17)_2$ we obtain the additional radial displacement due to this tangential strain:

$$\Delta u(R) = R\varepsilon_\theta = -\frac{\nu^2}{G(1+\nu)}\frac{q-(a/b)^2 p}{1-(a/b)^2}R \tag{7.3.41}$$

When this displacement is added to the radial displacement given by (7.3.37), we obtain the displacement (7.3.23) in Example 7.1.

If the cylinder is closed by rigid end plates, as indicated to the right in Fig. 7.3.5, these plates will, under the assumption of plane displacements for the cylinder, be subjected to an axial tension $\sigma_z \cdot \pi(b^2 - a^2)$, an axial compression $p \cdot \pi a^2$, and an additional external force F, see Fig. 7.3.5. For the case when $q = 0$, i.e. when the cylinder is subjected to internal pressure p only, the extra force is a compressive force equal to:

$$F = p \cdot \pi a^2 - \sigma_z \cdot \pi\left(b^2 - a^2\right) = \pi a^2 (1-2\nu)\, p$$

This extra force may be eliminated by superposition of a constant tensile stress in the z-direction equal to:

$$\sigma_z = \frac{F}{\pi\left(b^2 - a^2\right)} = \frac{1-2\nu}{1-(a/b)^2}\left(\frac{a}{b}\right)^2 p = \text{constant}$$

The constant tensile stress results in constant strains in the z-direction and in the radial and tangential directions. Using formula $(7.3.17)_2$ we obtain the additional radial displacement due to the tangential strain:

$$\Delta u(R) = R\varepsilon_\theta = R\cdot(-\nu\varepsilon_z) = R\cdot\left(-\nu\frac{\sigma_z}{E}\right) = \frac{\nu(1-2\nu)}{2G(1+\nu)}\frac{1}{1-(a/b)^2}\left(\frac{a}{b}\right)^2 pR \tag{7.3.42}$$

7.3.3 Airy's Stress Function

The choice of stresses as primary unknown functions is only natural in static problems, i.e. when the acceleration $\ddot{\mathbf{u}} = \mathbf{0}$, or in problems where the acceleration is known a priori. In the latter case we introduce an extraordinary body force $(-\ddot{\mathbf{u}})$ and a "corrected body force", $\mathbf{b} - \ddot{\mathbf{u}}$, and the problem is again a static one. The Cauchy equations (7.3.4) are now *equations of equilibrium*:

$$\operatorname{div}\mathbf{T} + \rho\,\mathbf{b} = \mathbf{0} \quad \Leftrightarrow \quad T_{\alpha\beta},_\beta + \rho\, b_\alpha = 0 \tag{7.3.43}$$

Let us assume that we have found three stress components $T_{\alpha\beta}$ that satisfy the two equations of equilibrium (7.3.43). The strain components $E_{\alpha\beta}$ may then be determined from Hooke's law (7.3.8) in the case of plane stress, or from Hooke's law (7.3.30) in the case of plane displacements. Then the displacements should follow from (7.3.10). But we cannot be sure that "any" three components of strain will give two displacement components u_α. The relation (7.3.10) represents three equations for the two unknown functions u_α. From the strain-displacement (7.3.10) we may develop the equation:

$$E_{11,22} + E_{22,11} - 2E_{12,12} = 0 \quad \Leftrightarrow \quad \varepsilon_{x,yy} + \varepsilon_{y,xx} - \gamma_{xy,xy} = 0 \tag{7.3.44}$$

This equation is called the *compatibility equation* and represents a necessary and sufficient condition for the three strain functions $E_{\alpha\beta}$ to give two displacement functions u_α.

Sections 5.3.9 and 5.3.10 present the compatibility equations for a general state of small deformations, and furthermore give the proof for their necessity and sufficiency. The proof assumes that the material region considered is *simply-connected*. This implies that any closed curve in the region may be shrunk to a point. A region containing a piercing hole does not represent a simply connected region. For such regions, in general called *multply-connected regions*, additional conditions have to be imposed. Example 7.9 provides a case where the region is doubly-connected and an extra condition is introduced for the unknown displacement function.

Because we will use stress components as primary unknown functions, we write the compatibility (7.3.44) in terms of the stress components. In the case of plane stress Hooke's law (7.3.8) and the equations of equilibrium (7.3.43) are used to transform the compatibility (7.3.44) into:

$$\nabla^2 T_{\alpha\alpha} + (1+\nu)\,\rho\, b_{\alpha,\alpha} = 0 \quad \text{plane stress} \tag{7.3.45}$$

In the case of plane displacements we apply Hooke's law (7.3.30) and the equations of equilibrium (7.3.43) to express the equation of compatibility (7.3.44) as:

$$\nabla^2 T_{\alpha\alpha} + \frac{1}{1-\nu}\,\rho\, b_{\alpha,\alpha} = 0 \quad \text{plane displacements} \tag{7.3.46}$$

Note that this also may be derived from (7.3.45) by use of the transformation (7.3.33).

In cases where the body forces may be neglected, $b_\alpha = 0$, the equations of equilibrium (7.3.43) and the compatibility equation (7.3.45) or (7.3.46) are reduced to the following set of equations:

$$T_{\alpha\beta,\beta} = 0\,, \quad \nabla^2 T_{\alpha\alpha} = 0 \tag{7.3.47}$$

For any scalar field $\Psi(\mathbf{r})$ coordinate stresses defined by the expressions:

$$T_{11} = \Psi_{,22}\,, \quad T_{22} = \Psi_{,11}\,, \quad T_{12} = -\Psi_{,12} \tag{7.3.48}$$

satisfy the equations of equilibrium, $T_{\alpha\beta,\beta} = 0$, identical. The compatibility equation, $\nabla^2 T_{\alpha\alpha} = 0$, now becomes:

$$\nabla^2\nabla^2\Psi = 0 \quad \Leftrightarrow \quad \nabla^4\Psi = 0 \tag{7.3.49}$$

This equation is a *biharmonic partial differential equation*. The operator ∇^4 is called the *biharmonic operator* and is in Cartesian coordinates:

$$\nabla^4 \equiv \nabla^2 \nabla^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 y^2} + \frac{\partial^4}{\partial y^4} \tag{7.3.50}$$

The scalar field $\Psi(\mathbf{r})$ is called *Airy's stress function*, named after George Biddell Airy [1801–1892]. Any scalar field $\Psi(\mathbf{r})$ that is the solution to the biharmonic (7.3.49), i.e. the compatibility equation, gives stresses from (7.3.48) that satisfy the equations of equilibrium in $(7.3.47)_1$ and that provide compatible strains through Hooke's law.

In cases where the body forces b_α may not be neglected, we proceed as follows. First we try to find any particular solution to the equations of equilibrium (7.3.43) and the compatibility equation (7.3.45) or (7.3.46), but without necessarily satisfying any boundary conditions. The complete solution to the problem in question is then given by the sum of the particular solution and a homogeneous solution determined from an Airy's stress function. This total solution must satisfy the boundary conditions of the problem.

A series of simple states of stress may be derived from the stress function:

$$\Psi = Ax^2 + Bxy + Cy^2 + Dx^3 + Ex^2y + Fxy^2 + Gy^3 + Hx^3y + Kxy^3 \tag{7.3.51}$$

$A, B, \ldots, K$ are constants. Each term in this stress function satisfies the compatibility equation (7.3.49).

Example 7.4. Cantilever Beam with Rectangular Cross-Section

The stress function:

$$\Psi = Bxy + Kxy^3$$

provides a satisfactory solution to the beam problem illustrated in Fig. 7.3.6. The height h of the beam is assumed to be greater than the width b of the beam, which leads us to assume a state of plane stress. The beam is loaded by shear stresses on the free end surface at $x = 0$. The resultant of the shear stresses is a known force F, while the distribution of F is not given, and we shall accept the distribution of the shear stresses at the end surface that the solution requires. For convenience we shall use a mixture of x and y and numbers for indices in this example.

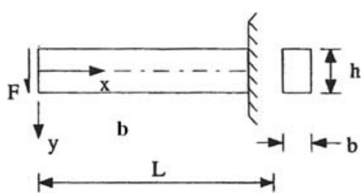


Fig. 7.3.6 Cantilever beam

The stress components become:

$$\sigma_x(x,y) = T_{11} = \frac{\partial^2\Psi}{\partial y^2} = 6Kxy\,, \quad \sigma_y = T_{22} = \frac{\partial^2\Psi}{\partial x^2} = 0\,,$$
$$\tau_{xy}(y) = T_{12} = -\frac{\partial^2\Psi}{\partial x\partial y} = -B - 3Ky^2$$

The constants B and K will now be determined from the boundary conditions:

$$\sigma_x(0,y) = 0 \text{ is satified}\,,\ \tau_{xy}(\pm h/2) = -B - 3K\,(\pm h/2)^2 = 0 \Rightarrow\ B = -3Kh^2/4$$
$$\int_{-h/2}^{h/2} T_{12}\,b\,dy = -Bbh - 3Kb\left[\frac{y^3}{3}\right]_{-h/2}^{h/2} = -F \Rightarrow \quad -Bbh - Kbh^3/4 = -F$$

From these two equations we find the constants B and K.

$$B = \frac{3F}{2bh}\,,\ \ K = -\frac{2F}{bh^3}$$

The stress components are then:

$$\sigma_x(x,y) = -\frac{12F}{bh^3}xy\,,\ \ \sigma_y = 0\,,\ \ \tau_{xy}(y) = -\frac{3F}{2bh}\left[1 - \left(\frac{2y}{h}\right)^2\right]$$

These are the same stresses that are given by the elementary beam theory.

If the real distribution of shear stresses on the free end surface is known and deviates from the result obtained from the stress function, we may assume that the state of stress in the beam is everywhere approximately the one found above from the stress function, except in a small region near the free end. Such an assumption, which we very often have to make in the theory of elasticity, is called the application of the *Saint-Venant's principle*, named after Barré de Saint-Venant [1797–1886].

We now turn to displacements: $u_1(x,y) \equiv u_x(x,y)$ and $u_2(x,y) \equiv u_y(x,y)$. From Hooke's law for plane stress (7.3.9) we get:

$$\varepsilon_x = E_{11} = u_{1,1} \equiv \frac{\partial u_x}{\partial x} = \frac{\sigma_x}{E} = -\frac{12F}{Ebh^3}xy\,,\ \ \varepsilon_y = E_{22} = u_{2,2} \equiv \frac{\partial u_y}{\partial y} = -\nu\frac{\sigma_x}{E}$$
$$= -\frac{12F\nu}{Ebh^3}xy$$
$$\gamma_{xy} = 2E_{12} = u_{1,2} + u_{2,1} \equiv \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = \frac{1}{G}\tau_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$
$$= -\frac{3(1+\nu)F}{Ebh^2}\left[1 - \left(\frac{2y}{h}\right)^2\right]$$

Integrations of $u_{1,1}$ and $u_{2,2}$ result in:

$$u_1(x,y) = -\frac{6F}{Ebh^3}x^2y + f_1(y)\,, \quad u_2(x,y) = \frac{6\nu F}{Ebh^3}xy^2 + f_2(x)$$

$f_1(y)$ and $f_2(x)$ are unknown functions. The expressions for u_1 and u_2 are substituted into the expression for $u_{1,2} + u_{2,1}$, and we get:

$$-\frac{6F}{Ebh^3}x^2 + \frac{df_1}{dy} + \frac{6\nu F}{Ebh^3}y^2 + \frac{df_2}{dx} = -\frac{3(1+\nu)F}{Ebh}\left[1 - \left(\frac{2y}{h}\right)^2\right]$$

From this equation we deduce the results:

$$f_1(y) = \frac{(4+2\nu)F}{Ebh^3}y^3 - \left[A + \frac{3(1+\nu)F}{Ebh}\right]y + C\,, \quad f_2(x) = \frac{2F}{Ebh^3}x^3 + Ax + B$$

A, B, and C are constants of integration to be determined from the boundary conditions at the fixed support, where $x = L$. It appears from the expressions for the displacements u_α that it is not possible to demand that $u_\alpha = 0$ over the entire cross section at $x = L$. Such a requirement would indeed not be realistic anyway since we must expect that the material in the support itself will be somewhat deformed due to the stresses transmitted from the beam. We therefore choose first to require: $u_\alpha = 0$ at $x = L, y = 0$. These two conditions give:

$$u_1|_{x=L,y=o} = 0 \quad \Rightarrow \quad C = 0$$
$$u_2|_{x=L,y=o} = 0 \quad \Rightarrow \quad \frac{2FL^3}{Ebh^3} + AL + B = 0 \quad \Rightarrow \quad B = -\frac{2FL^3}{Ebh^3} - AL \qquad (7.3.52)$$

A third boundary condition is obtained by either requiring: 1) $u_{2,1} = 0$ at $x = L, y = 0$, which means that the axis of the beam is parallel to the x-axis, as in the elementary beam theory, or: 2) $u_{1,2} = 0$ at $x = L, y = 0$, which implies that the cross section at the support is parallel to the y-axis where the cross section intersects the beam axis. Figure 7.3.7 shows both alternative boundary conditions at the fixed support. For the two alternative boundary conditions we get, when also the result (7.3.52) is applied:

$$1)\; u_{2,1}|_{x=L,y=o} = 0 \quad \Rightarrow \quad \frac{6FL^2}{Ebh^3} + A = 0 \quad \Rightarrow \quad A = -\frac{6FL^2}{Ebh^3}\,, \quad B = \frac{4FL^3}{Ebh^3}$$
$$2)\; u_{1,2}|_{x=L,y=o} = 0 \quad \Rightarrow \quad -\frac{6FL^2}{Ebh^3} - \left[A + \frac{3(1+\nu)F}{Ebh}\right] = 0 \quad \Rightarrow$$
$$A = -\frac{6FL^2}{Ebh^3} - \frac{3(1+\nu)F}{Ebh}\,, \quad B = \frac{4FL^3}{Ebh^3} + \frac{3(1+\nu)FL}{Ebh}$$

The displacements according to alternative 1) become:

$$u_1(x,y) = \frac{FL^3}{Ebh^3}\left\{6\left[1 - \left(\frac{x}{L}\right)^2\right]\frac{y}{L} + 2(2+\nu)\left(\frac{y}{L}\right)^3 - 3(1+\nu)\left(\frac{h}{L}\right)^2\frac{y}{L}\right\}$$

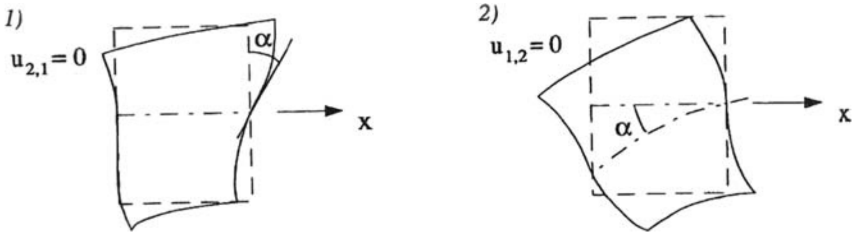


Fig. 7.3.7 Two alternative boundary conditions at the fixed support

$$u_2(x,y) = \frac{FL^3}{Ebh^3}\left\{4 + 2\left(\frac{x}{L}\right)^3 - 6\left[1 - \nu\left(\frac{y}{L}\right)^2\right]\frac{x}{L}\right\}\ , \ u_{2,\max} = u_2\,(0,y) = \frac{4FL^3}{Ebh^3}$$

The displacements according to alternative 2) become:

$$u_1(x,y) = \frac{FL^3}{Ebh^3}\left\{6\left[1 - \left(\frac{x}{L}\right)^2\right]\frac{y}{L} + 2\,(2+\nu)\left(\frac{y}{L}\right)^3\right\}$$

$$u_2(x,y) = \frac{FL^3}{Ebh^3}\left\{4 + 3\,(1+\nu)\left(\frac{h}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 - 6\left[1 - \nu\left(\frac{y}{L}\right)^2 + \frac{1}{2}\,(1+\nu)\left(\frac{h}{L}\right)^2\right]\frac{x}{L}\right\}$$

$$u_{2,\max} = u_2\,(0,y) = \frac{FL^3}{Ebh^3}\left[4 + 3\,(1+\nu)\left(\frac{h}{L}\right)^2\right]$$

Elementary beam theory gives:

$$u_2(x,0) = \frac{FL^3}{Ebh^3}\left[4 + 2\left(\frac{x}{L}\right)^3 - 6\frac{x}{L}\right],\ u_{2,\max} = u_2\,(0,0) = \frac{4FL^3}{Ebh^3}$$

The displacement $u_2(x,y)$ according to the alternative boundary condition 2) may also be determined by adding to the displacement $u_2(x,y)$ for the alternative boundary condition 1) found above a rigid-body counter-clockwise rotation given by the angle, see Problem 7.8:

$$\alpha = -\,u_{1,2},|_{x=L,y=o} = \frac{3\,(1+\nu)\,F}{E\,b\,h}$$

In this example we have found a solution that gives stresses and displacement in the cantilever beam in Fig. 7.3.6 when we accept the approximation regarding the distribution of shear stresses on the free end surface and the uncertainty about the proper displacement conditions at the fixed end surface.

7.3.4 Airy's Stress Function in Polar Coordinates

The general Cauchy equations of motion in polar coordinates are given by (3.2.39) and (3.2.40). By setting the accelerations and body forces equal to zero, we obtain the proper equations of equilibrium. The following formulas for the coordinate stresses in polar coordinates, Fig. 7.3.8, will satisfy these equations of equilibrium:

$$\sigma_R = \frac{1}{R^2}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{1}{R}\frac{\partial\Psi}{\partial R}, \quad \sigma_\theta = \frac{\partial^2\Psi}{\partial R^2}$$
$$\tau_{R\theta} = -\frac{1}{R}\frac{\partial^2\Psi}{\partial R\partial\theta} + \frac{1}{R^2}\frac{\partial\Psi}{\partial\theta} = -\frac{\partial}{\partial R}\left[\frac{1}{R}\frac{\partial\Psi}{\partial\theta}\right] \tag{7.3.53}$$

The Laplace operator and the biharmonic operator in polar coordinates are:

$$\nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} \tag{7.3.54}$$

$$\nabla^4 = \nabla^2\nabla^2 = \left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}\right) \tag{7.3.55}$$

Example 7.5. Edge Load on a Semi-Infinite Elastic Plate

The plate is presented in Fig. 7.3.9 and defined by the region $x > 0$. The material is subjected to an edge load q. The load intensity q is given as a force per unit length in the z-direction. We shall find the stresses on planes parallel to the z-axis, which is normal to plane of the figure.

This stress function will provide us with the solution:

$$\Psi = -\frac{q}{\pi}R\,\theta\sin\theta \tag{7.3.56}$$

The polar angle θ is measured from the direction line of the load, as shown in Fig. 7.3.9. The load direction line forms the angle α with respect to the x–axis. By applying the biharmonic operator (7.3.53) to this stress function we shall find that the equation of compatibility (7.3.49 is satisfied. The coordinate stresses become:

$$\sigma_R = \frac{1}{R^2}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{1}{R}\frac{\partial\Psi}{\partial R} = -\frac{2q}{\pi}\frac{1}{R}\cos\theta$$

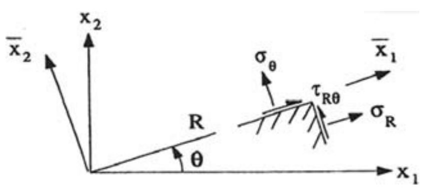


Fig. 7.3.8 Coordinate stresses in polar coordinates

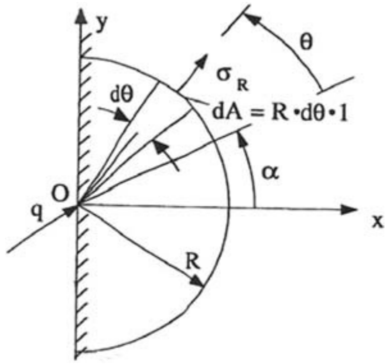


Fig. 7.3.9 Semi-infinite elastic plate subjected to an edge load q

$$\sigma_\theta = \frac{\partial^2 \Psi}{\partial R^2} = 0\,, \quad \tau_{R\theta} = -\frac{\partial}{\partial R}\left[\frac{1}{R}\frac{\partial \Psi}{\partial \theta}\right] = 0$$

The boundary condition: $\mathbf{t} = \mathbf{0}$ on the free surface at $x = 0$, except at the origin O, which is a singular point, is clearly satisfied. In order to show that the state of stress is in equilibrium with the externally applied edge load q, we check the equilibrium of a body formed as a semi cylinder with radius R, see Fig. 7.3.9. Equating to zero the forces in the direction of the edge load q and in the direction normal to q and referring to Fig. 7.3.9, we obtain:

$$q = -\int_A (\sigma_R \cdot \cos\theta)\, dA = \frac{2q}{\pi} \int_{-\pi/2-\alpha}^{\pi/2-\alpha} \cos^2\theta\, d\theta = \frac{2q}{\pi}\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{-\pi/2-\alpha}^{\pi/2-\alpha} = q$$

$$0 = \int_A (\sigma_R \cdot \sin\theta)\, dA = -\frac{2q}{\pi} \int_{-\pi/2-\alpha}^{\pi/2-\alpha} \cos\theta \sin\theta\, d\theta = -\frac{2q}{\pi}\left[\frac{1}{2}\sin^2\theta\right]_{-\pi/2-\alpha}^{\pi/2-\alpha} = 0$$

We have thus shown that all the necessary requirements to the stress function (7.3.56) are satisfied.

This problem was first investigated and solved by Boussinesq in 1885 and Flamant in 1892, and is therefore often referred to as *the Boussinesq-Flamant problem*.

Example 7.6. Edge Load on a Wedge

The stress function in the previous example also gives the solution to the two problems shown in Fig. 7.3.10 if we only adjust the constant $(-q/\pi)$. Figure 7.3.10 shows two wedges, each having a wedge angle of 2α. With C as an initially unknown constant we first set:

$$\Psi = -CR\theta\sin\theta \quad \Rightarrow \quad \sigma_R = -\frac{2C}{R}\cos\theta, \sigma_\theta = \tau_{R\theta} = 0$$

Equilibrium of the cylinder segments bounded by the radius R requires that:

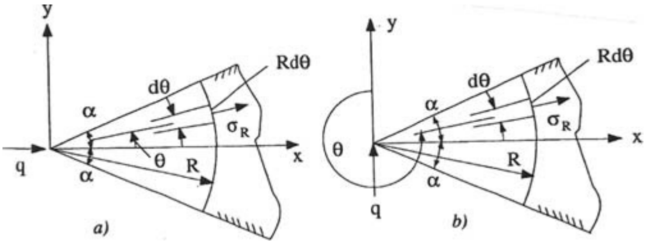


Fig. 7.3.10 Wedges subjected to edge load q

Case a in Fig. 7.3.10a:

$$q = -\int_A (\sigma_R \cdot \cos\theta)\, dA = 2C \int_{-\alpha}^{+\alpha} \cos^2\theta\, d\theta = 2C \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{-\alpha}^{+\alpha}$$

$$= 2C\left[\alpha + \frac{1}{2}\sin 2\alpha\right] \quad \Rightarrow \quad C = \frac{q}{2\alpha + \sin 2\alpha}$$

Case b in Fig. 7.3.10b:

$$q = -\int_A (\sigma_R \cdot \cos\theta)\, dA = 2C \int_{3\pi/2-\alpha}^{3\pi/2+\alpha} \cos^2\theta\, d\theta = 2C \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{3\pi/2-\alpha}^{3\pi/2+\alpha}$$

$$= 2C\left[\alpha - \frac{1}{2}\sin 2\alpha\right] \quad \Rightarrow \quad C = \frac{q}{2\alpha - \sin 2\alpha}$$

We now have the following non-zero coordinate stresses in the two cases:

$$\text{Case a: } \sigma_R = -\frac{2q}{2\alpha + \sin 2\alpha}\frac{\cos\theta}{R}, \quad \text{Case b: } \sigma_R = -\frac{2q}{2\alpha - \sin 2\alpha}\frac{\cos\theta}{R}$$

Example 7.7. Circular Cylinder with Edge Loads

The state of stress in a circular cylinder or a thin cylindrical plate of diameter d and which is subjected to two diametrically opposite edge loads q, see Fig. 7.3.11d, is obtained by a superposition of the following three states of stress:

a) Edge load q on a semi-infinite space, Fig. 7.3.11a.
b) Edge load q on a semi-infinite space, Fig. 7.3.11b.
c) Plane isotropic tensile stress $\sigma_o = 2q/\pi d$, Fig. 7.3.11c.

We start by considering the material between two parallel planes a distance d apart, see Fig. 7.3.11 a and b. An edge force q on the upper plane, Fig. 7.3.11a, is balanced by normal and shear stresses on the lower plane in such a manner that the state of stress is given by Example 7.5:

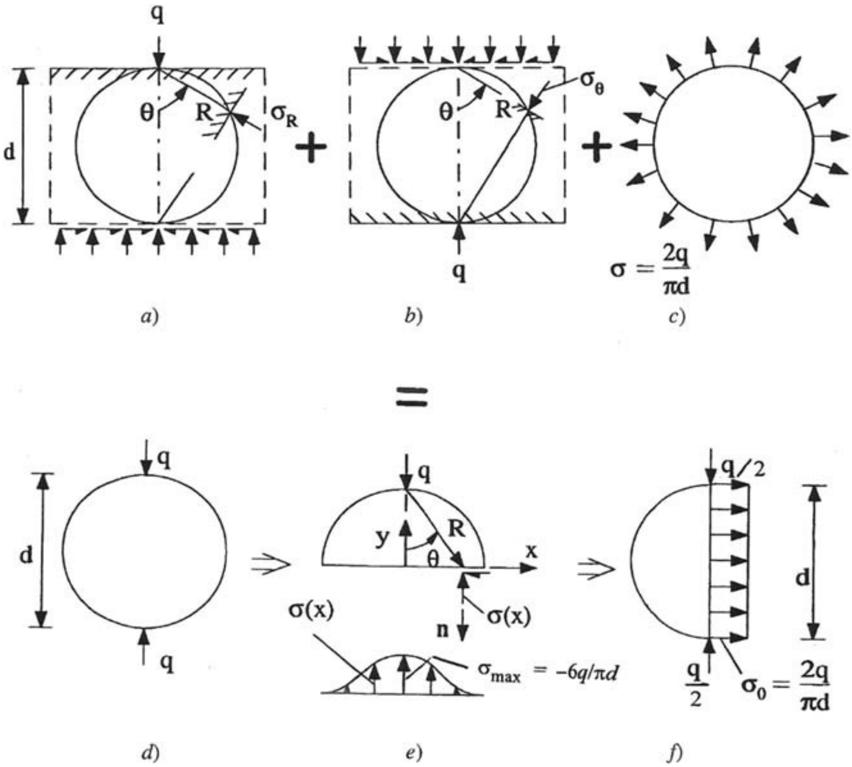


Fig. 7.3.11 Superposition of edge loads

$$\sigma_R = -\frac{2q}{\pi}\frac{1}{R}\cos\theta\ ,\quad \sigma_\theta = \tau_{R\theta} = 0$$

In the circular cylindrical surface of diameter $d = R/\cos\theta$ shown in Fig. 7.3.11, the radial stress is constant and equal to $\sigma_R = -2q/\pi d$. An edge force q on the lower plane, Fig. 7.3.11b, is similarly balanced by normal and shear stresses on the upper plane. In the circular cylindrical surface the only non-zero coordinate stress is $\sigma_\theta = -2q/\pi d$. This implies that when the effects of the two edge loads, Fig. 7.3.11 a and b, are superimposed, the state of stress in the cylindrical surface is plane-isotropic. The stress on the cylindrical surface is therefore a constant pressure equal to $2q/\pi d$. If we now add an isotropic state of tensile stress $\sigma_o = 2q/\pi d$, as shown in Fig. 7.3.11c, the cylindrical surface becomes stress free. The superposition of the three load cases in Fig. 7.3.11a, b, and c results in the situation shown in Fig. 7.3.11d: A circular cylinder of diameter d is loaded by two opposite edge forces q.

The computational work to find the stress matrix in any particle in the cylinder shown in Fig. 7.3.11d, becomes very extensive. Therefore we shall confine the computation to the stresses on two characteristic diametrical sections as shown in Fig. 7.3.11e and f.

For the section in Fig. 7.3.11e the unit normal is $\mathbf{n} = \cos\theta\mathbf{e}_R - \sin\theta\mathbf{e}_\theta$. The contribution to the normal stress σ from the upper knife load is then:

$$\sigma = \mathbf{n} \cdot \mathbf{T} \cdot \mathbf{n} = \cos\theta \cdot \sigma_R \cdot \cos\theta = -\frac{2q}{\pi}\frac{\cos^3\theta}{R}$$

We set:

$$\cos\theta = \frac{d/2}{R}\,, \quad R^2 = \left(\frac{d}{2}\right)^2 + x^2 = \left(\frac{d}{2}\right)^2\left[1+\left(\frac{2x}{d}\right)^2\right]$$

Then:

$$-\frac{2q}{\pi}\frac{\cos^3\theta}{R} = -\frac{2q}{\pi}\frac{(d/2R)^3}{R} = \frac{2q}{\pi}\frac{d^3}{8R^4} = -\frac{q}{\pi d}\left[\frac{2}{1+(2x/d)^2}\right]^2$$

Totally from the three load cases in Figure a, b, and c we obtain:

$$\sigma(x) = \left[-\frac{2q}{\pi}\frac{\cos^3\theta}{R}\right]\cdot 2 + \frac{2q}{\pi d} = \frac{2q}{\pi d}\left\{1-\left[\frac{2}{1+4(x/d)^2}\right]^2\right\}$$

Figure 7.3.11e shows the normal stress distribution. The maximum (compressive) normal stress $\sigma_{\max} = -6q/\pi d$. Due to symmetry the shear stress on the section is zero.

On the section shown in Fig. 7.3.11f only the load case in Fig. 7.3.11c contributes. Thus the normal stress is a constant tension $\sigma_o = 2q/\pi d$, while the shear is zero. This particular result is used to find the ultimate stress of brittle materials having high compressive strength and low tensile strength, as for instance rocks and soils. The result of the load case in Fig. 7.3.11d is also used in the determination of the stress optical constant in the method of *photoelasticity*.

Example 7.8. Rectangular Plate with a Hole. Kirsch's Problem (1898)

A rectangular plate of width b and height h and with a small hole of radius $a << h$ and b, Fig. 7.3.12, is subjected to a normal stress $\sigma_x = \sigma$ on the sides $x = \pm b/2$. The sides $y = \pm h/2$ are stress free. The stresses in the neighborhood of the hole are to be determined.

Sufficiently far from the hole we may assume that the stresses are as for a plate without the hole. Thus:

$$\sigma_x = \sigma\,, \quad \sigma_y = \tau_{xy} = 0 \text{ for } |x| \gg a\,,\ |y| \gg a$$

Because we choose to use polar coordinates (R,θ), this state of stress may be expressed through the formulas (3.3.37) and (3.3.38) as:

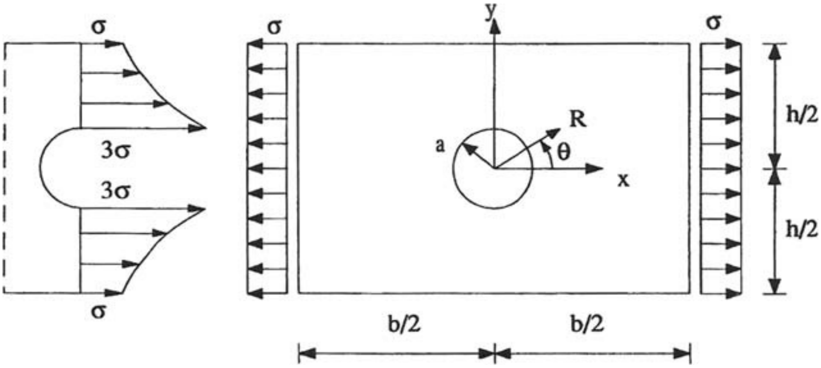


Fig. 7.3.12 Rectangular plate with a hole

$$\begin{aligned} \sigma_R = \tfrac{\sigma}{2}(1+\cos 2\theta)\ , \quad \sigma_\theta = \tfrac{\sigma}{2}(1-\cos 2\theta) \\ \tau_{R\theta} = -\tfrac{\sigma}{2}\sin 2\theta \end{aligned} \qquad \text{for } |x| \gg a\ ,\ |y| \gg a \tag{7.3.57}$$

The following stress function provides approximately the correct stresses in the plate.

$$\Psi = -\frac{\sigma a^2}{2}\ln R + \frac{\sigma}{4}R^2 + \frac{\sigma a^2}{4}\left[2 - \left(\frac{a}{R}\right)^2 + \left(\frac{R}{a}\right)^2\right]\cos 2\theta \tag{7.3.58}$$

The compatibility (7.3.49) is satisfied, and the state of stress is:

$$\sigma_R(R,\theta) = \frac{1}{R}\frac{\partial\Psi}{\partial R} + \frac{1}{R^2}\frac{\partial^2\Psi}{\partial\theta^2} = -\frac{\sigma}{2}\left[1 - \left(\frac{a}{R}\right)^2\right] + \frac{\sigma}{2}\left[1 - 4\left(\frac{a}{R}\right)^2 + 3\left(\frac{a}{R}\right)^4\right]\cos 2\theta$$

$$\sigma_\theta(R,\theta) = \frac{\partial^2\Psi}{\partial R^2} = \frac{\sigma}{2}\left[1 + \left(\frac{a}{R}\right)^2\right] - \frac{\sigma}{2}\left[1 + 3\left(\frac{a}{R}\right)^4\right]\cos 2\theta$$

$$\tau_{R\theta}(R,\theta) = -\frac{\partial}{\partial R}\left[\frac{1}{R}\frac{\partial\Psi}{\partial\theta}\right] = -\frac{\sigma}{2}\left[1 + 2\left(\frac{a}{R}\right)^2 - 3\left(\frac{a}{R}\right)^4\right]\sin 2\theta$$

At the edge of the hole, $R = a$, the stresses are:

$$\sigma_R(a,\theta) = 0\ , \quad \tau_{R\theta}(a,\theta) = 0\ , \quad \sigma_\theta(a,\theta) = (1 - 2\cos 2\theta)\,\sigma$$

$$\sigma_{\theta,\max} = \sigma_\theta(a,\pm\pi/2) = 3\sigma\ , \quad \sigma_{\theta,\min} = \sigma_\theta(a,0) = \sigma_\theta(a,\pi) = -\sigma$$

The state of stress is in accordance with the condition of a stress free surface in the hole. On the sides of the plate, $x = \pm b/2$ and $y = \pm h/2$, we may assume that:

$$\frac{a}{R} \le \frac{2a}{b} \ll 1 \text{ for } x = \pm\frac{b}{2} \quad \text{and} \quad \frac{a}{R} \le \frac{2a}{h} \ll 1 \text{ for y } = \pm\frac{h}{2}$$

The stress formulas reduce to the expressions (7.3.57) for the boundary conditions on the edges. On the section $x = 0 \Leftrightarrow \theta = \pm\pi/2$, the shear stress is zero and the normal stress becomes, see Fig. 7.3.12:

$$\sigma_\theta(R, \pi/2) = \left[1 + \frac{1}{2}\left(\frac{a}{R}\right)^2 + \frac{3}{2}\left(\frac{a}{R}\right)^4\right]\sigma$$

At the hole, $R = a$, the plate has a *stress concentration* $\sigma_\theta = 3\sigma$, independent of the radius a, as long as the condition $a << h$ and b is satisfied. For $h = 4a$, R. C. J. Howland, see [48], has found the solution that: $\sigma_\theta(a, \pi/2) = 4.3\sigma$ and $\sigma_\theta(h/2, \pi/2) = 0.75\sigma$. The stress concentration increases as the hole approaches the edges $y = \pm h/2$.

7.3.5 Axial Symmetry

For problems with a plane state of stress that is symmetrical with respect to a z-axis, we may assume that the Airy stress function is a function only of the radial coordinate R:

$$\Psi = \Psi(R) \tag{7.3.59}$$

The general solution of the equation of compatibility (7.3.49) is:

$$\Psi(R) = A\ln R + BR^2\ln R + CR^2 + D \tag{7.3.60}$$

A, B, C, and *D* are constants of integration. The coordinate stresses become:

$$\begin{aligned}
\sigma_R(R) &= \frac{1}{R}\frac{d\Psi}{dR} = \frac{A}{R^2} + B(1 + 2\ln R) + 2C \\
\sigma_\theta(R) &= \frac{d^2\Psi}{dR^2} = -\frac{A}{R^2} + B(3 + 2\ln R) + 2C
\end{aligned} \tag{7.3.61}$$

The shear stress $\tau_{R\theta}$ is zero, which is in accordance with the symmetry condition.

Example 7.9. Circular Plate with a Hole

The problem in Example 7.1, see Fig. 7.3.3, is now revisited. The general solution (7.3.61) contains three unknown constants of integration: A, B, and C. To determine these three constants we have only two boundary conditions:

$$\sigma_R(a) = -p\,, \quad \sigma_R(b) = -q$$

We know that the general solution satisfies the equations of equilibrium and the compatibility (7.3.49). As mentioned in connection with the development of the compatibility (7.3.49), for multi-connected region we have to introduce additional conditions that secure a unique and continuous displacement field. The hole introduces a double-connected region: A closed material curve surrounding the hole, may not be shrunk to zero. For this reason we have to investigate closer the displacements that result in the general expressions (7.3.61) for the stresses. Using the expressions (7.3.17) for the strains, Hooke's law (7.3.9), and the stress expressions (7.3.61), we obtain:

$$\varepsilon_R = \frac{du}{dR} = \frac{1}{E}\left\{\frac{A}{R^2} + B(1+2\ln R) + 2C - \nu\left[-\frac{A}{R^2} + B(3+2\ln R) + 2C\right]\right\} \tag{7.3.62}$$

$$\varepsilon_\theta = \frac{u}{R} = \frac{1}{E}\left\{-\frac{A}{R^2} + B(3+2\ln R) + 2C - \nu\left[\frac{A}{R^2} + B(1+2\ln R) + 2C\right]\right\} \quad \Rightarrow$$

$$u = \frac{1}{E}\left\{-\frac{A}{R} + B(3+2\ln R)R + 2CR - \nu\left[\frac{A}{R} + B(1+2\ln R)R + 2CR\right]\right\} \quad \Rightarrow$$

$$\frac{du}{dR} = \frac{1}{E}\left\{\frac{A}{R^2} + B(3+2\ln R+2) + 2C - \nu\left[-\frac{A}{R^2} + B(1+2\ln R+2) + 2C\right]\right\} \tag{7.3.63}$$

If we compare the two expressions (7.3.62) and (7.3.63) for *du/dR*, we see that the constant B must be zero. Thus we have the following general expressions for the stresses and for the radial displacement:

$$\sigma_R(R) = \frac{A}{R^2} + 2C\,, \quad \sigma_\theta(R) = -\frac{A}{R^2} + 2C$$

$$u(R) = \frac{1}{E}\left[-(1+\nu)\frac{A}{R} + 2(1-\nu)CR\right] \tag{7.3.64}$$

These expressions have the same structures as those found in Example 7.1. The boundary conditions in the present example and in Example 7.1 are the same, and the final results must therefore be the same as given in Example 7.1.

Example 7.10. Pure Bending of Curved Beam

Figure 7.3.13 shows a curved beam with a circular axis and constant rectangular cross-section of height $h = b - a$ and width t. The beam is subjected to a bending

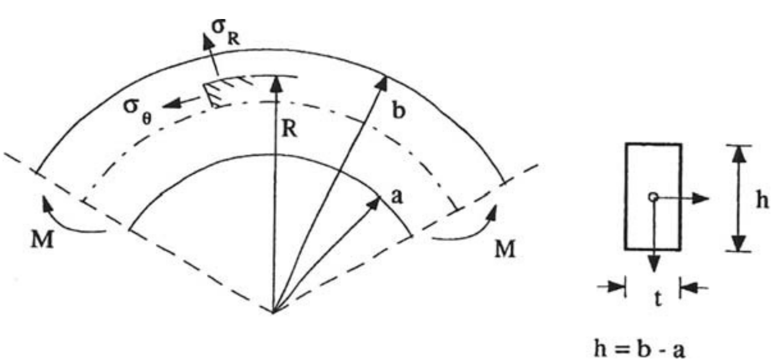


Fig. 7.3.13 Bending of a curved beam

moment M at each end. The stress distribution in the beam is to be determined based on the condition that it is uniform along the axis of the beam.

As a starting point we use the general solution (7.3.61). The boundary conditions are formulated as:

$$\sigma_R(a)=0\,,\ \ \sigma_R(b)=0\,,\ \int_a^b \sigma_\theta(R)\,t\,dR=0\,,\ \int_a^b \sigma_\theta(R)\,R\,t\,dR=-M \tag{7.3.65}$$

Substituting the general expressions (7.3.61) for the stresses $\sigma_R(R)$ and $\sigma_\theta(R)$ into the first, the second, and the fourth of these boundary conditions, we obtain three equations for the three unknown constants A, B, and C:

$$\frac{A}{a^2}+B(1+2\ln a)+2C=0\,,\ \ \frac{A}{b^2}+B(1+2\ln b)+2C=0$$
$$\left[A\ln\left(\frac{b}{a}\right)+B\left(b^2\ln b-a^2\ln a\right)+C\left(b^2-a^2\right)\right]t=-M \tag{7.3.66}$$

It may be shown that the third of the boundary conditions (7.3.65) is automatically satisfied when the first two are satisfied. The solution of the three linear equations (7.3.66) for the constants A, B, and C is:

$$A=-\frac{4M}{K}a^2b^2\ln\left(\frac{b}{a}\right),\ B=-\frac{2M}{K}\left(b^2-a^2\right),\ C=\frac{M}{K}\left[b^2-a^2+2\left(b^2\ln b-a^2\ln a\right)\right]$$
$$K=\left\{\left(b^2-a^2\right)^2-4a^2b^2\left[\ln\left(\frac{b}{a}\right)\right]^2\right\}t$$

The stresses are then:

$$\sigma_R=-\frac{4Mb^2}{K}\left[\left(\frac{a}{R}\right)^2\ln\left(\frac{b}{a}\right)-\ln\left(\frac{b}{R}\right)-\left(\frac{a}{b}\right)^2\ln\left(\frac{R}{a}\right)\right] \tag{7.3.67}$$
$$\sigma_\theta=-\frac{4Mb^2}{K}\left[\left(\frac{a}{R}\right)^2\ln\left(\frac{b}{a}\right)-\ln\left(\frac{b}{R}\right)-\left(\frac{a}{b}\right)^2\ln\left(\frac{R}{a}\right)+1-\left(\frac{a}{b}\right)^2\right] \tag{7.3.68}$$

The problem does not specify how the stresses, represented by the bending moment M, are distributed over the end section of the beam. A deviation from the stress distribution provided by the solution for $\sigma_\theta(R)$ given above, will according to the Saint-Venant's principle have little influence on the stress distribution at a short distance from the end sections of the beam.

The elementary beam theory, which really presupposes that the axis of the beam is straight before deformation, gives a linear distribution of $\sigma_\theta(R)$ over the cross-section of the beam. The extremal values for $\sigma_\theta(R)$ according to this theory are:

$$\sigma_{\theta,\max}=\sigma_\theta(a)=+\frac{6M}{th^2}\,,\ \ \sigma_{\theta,\min}=\sigma_\theta(b)=-\frac{6M}{th^2}$$

The exact theory represented by formula (7.3.68) gives:

$$\text{For } b = 2a: \quad \sigma_{\theta,\max} = \sigma_\theta(a) = +7.73\frac{M}{th^2}\,, \quad \sigma_{\theta,\min} = \sigma_\theta(b) = -4.86\frac{M}{th^2}$$
$$\text{For } b = 3a: \quad \sigma_{\theta,\max} = \sigma_\theta(a) = +9.14\frac{M}{th^2}\,, \quad \sigma_{\theta,\min} = \sigma_\theta(b) = -4.38\frac{M}{th^2}$$

7.4 Torsion of Cylindrical Bars

A bar is defined by an axis and a cross-section. A *cylindrical bar* has a straight axis and constant cross-section. When the bar is subjected to loads perpendicular to its axis, the bar is called a *beam*.

The bending of beams having symmetrical and unsymmetrical cross-sections is discussed in books on Strength of Materials. The elementary beam theory provides sufficiently accurate results for the distribution of normal stresses on cross-sections and of the shear stress distribution on simple thin-walled cross-sections. Normally the shear stresses on the cross sections of beams that have a ratio of beam height to beam length less than 1/10, are negligible when compared with the normal stresses on the cross sections. The elementary beam theory also gives satisfactory results for computing beam deflections.

A bar twisted by torques M at the ends, Fig. 7.4.1, is said to be in *pure torsion*. The elementary theory of torsion found in textbooks on Strength of Materials applies only to circular cylindrical bars. This theory is also called the *Coulomb theory of torsion*, after Charles Augustin Coulomb [1789–1857]. In the present section we shall primarily discuss the “non-elementary” theory of torsion of cylindrical bars with arbitrary cross-sections. This theory is called the *Saint-Venant’s theory of torsion*. However, it is convenient to start this section by a short presentation of the Coulomb theory.

7.4.1 The Coulomb Theory of Torsion

The circular cylindrical bar in Fig. 7.4.1 is subjected to *torques*, or *torsional moments, M* at the ends. It is most convenient to use the *xyz* notation for the Cartesian

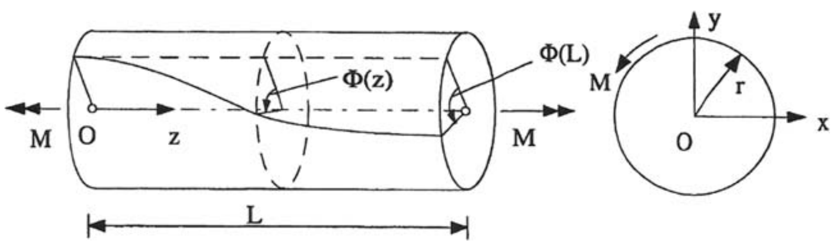


Fig. 7.4.1 Torsion of a circular cylindrical bar

coordinates in this section, but also apply the index-notation. The torques result in a rotation of the end cross-section at $z = L$ relative to the end cross section at $z = 0$. The angle of rotation is called the *angle of twist* and is denoted by Φ. If we on the cylindrical surface of the bar draw contours of cross-sections, we shall see that when the bar is twisted the plane contours remain plane and that the lengths of diameters do not change. This observation leads us to formulate the following *deformation hypothesis*:

During torsion of a circular cylindrical bar plane cross-sections are rotated as rigid planes.

The deformation hypothesis is the basis of the *Coulomb theory of torsion* for circular cylindrical bars. The theory will now be developed.

Since the stress resultant on any cross section must be the same and equal to the torque M, it follows that the angle of twist $\Phi(z)$ is proportional to the distance z from the end of the bar, where $z = 0$. The constant angle of twist per unit length of the rod is denoted by ϕ. Thus:

$$\Phi(z) = \phi z \tag{7.4.1}$$

In order to find the state of strain in the bar we consider an element of the bar between two cross-sections, a distance dz apart, and within a cylindrical surface of radius R, as shown in Fig. 7.4.2. From the figure we derive the only non-zero strain component:

$$\gamma = \phi R \tag{7.4.2}$$

The material is assumed to be linearly elastic, and from Hooke's law it follows that the only non-zero stress component is:

$$\tau \equiv \tau_{\theta z} = G\gamma = G\phi R$$

Since M represents the resultant of the shear stress distribution over the cross-section, it follows that:

$$M = \int_A (\tau \cdot R)\, dA = G\phi \int_A R^2\, dA \tag{7.4.3}$$

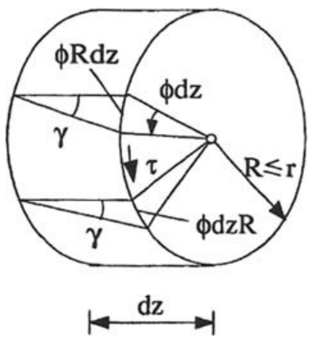


Fig. 7.4.2 Element of bar

The integral on the right hand side is the *polar moment of area* I_p. For a massive circular bar of radius r and a thick-walled pipe of inner radius a and outer radius r the polar moments of area are:

$$\text{Massive bar: } I_p = \frac{\pi r^4}{2}\,, \quad \text{Thick-walled pipe:} I_p = \frac{\pi}{2}\left(r^4 - a^4\right) \tag{7.4.4}$$

From (7.4.2, 7.4.3) we obtain the results:

$$\phi = \frac{M}{GI_p}\,, \quad \tau(R) = \frac{M}{I_p}R\,, \quad \tau_{\max} = \frac{M}{I_p}r \tag{7.4.5}$$

These formulas also apply when the torque varies along the bar: $M = M(z)$. It is not necessary to assume that the angle of twist Φ is small. The theory only requires that ϕ should be small, see (7.4.2). Material lines on the cylindrical surface that are generatrices before deformation, i.e. straight lines parallel to the axis of the bar, are deformed into helices, see Fig. 7.4.1.

7.4.2 The Saint-Venant Theory of Torsion

The analysis of torsion of elastic cylindrical bars of non-circular cross-sections is more complex than for bars having circular cross-section. Figure 7.4.3 shows the result of torsion of a bar with rectangular cross section. Large strains are allowed in the figure to clearly demonstrate the mode of deformation. The deformation in Fig. 7.4.3 may be demonstrated with a rubber shaft, or simply by subjecting a rectangular everyday eraser to torsion. We may draw generatrices and contours of cross-sections on the surface of the bar. When the bar is deformed the generatrices become helices, while the deformed cross-sectional contours indicate that cross-sections are deformed to curved surfaces. The cylindrical outer surface of the bar is stress free and the resultant of the normal stresses on a cross-section is zero. We shall use Cartesian coordinates in the analysis, and with the $z(\equiv x_3)$-axis along the axis of the bar. It is furthermore convenient to mix the xy- and the x_1x_2-notation. Based on the observation about stresses presented above it seems reasonable to assume a state of stress represented by a stress matrix without normal stress components, i.e.:

$$T_{11} = T_{22} = T_{33} = 0 \tag{7.4.6}$$

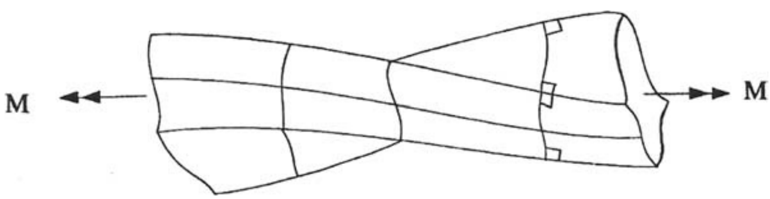


Fig. 7.4.3 Torsion of non-circular cylindrical bar

This stress assumption is realized by assuming the following *deformation hypothesis*:

When a bar is subjected to torsion, plane material cross-sections deform without strains in their surfaces and in such a way that all cross-sections obtain the same warped form. The longitudinal strain in the axial direction is zero.

The deformation hypothesis is the starting point of the *Saint-Venant theory of torsion*. As a consequence of the deformation hypothesis the projections of cross-sections onto the xy-plane rotates as rigid planes, and we may introduce an angle of twist Φ, see Fig. 7.4.4, which is proportional to the distance z from the end surface of the bar:

$$\Phi = \phi z \,, \quad \phi = \text{constant} \tag{7.4.7}$$

Since the longitudinal strain in the axial direction is zero, it follows that the displacement u_3 in the z-direction is only proportional to the angle of twist ϕ per unit length and independent of z. Thus we may set:

$$u_3 = \phi\, \psi(x,y) \tag{7.4.8}$$

The unknown function $\psi(x,y)$ is called the *warping function*. We assume small displacements, which imply that ϕ must be a small angle. The displacements of a particle P in the x- and y-directions may therefore be given as:

$$u_1 = -\Phi y = -\phi z y \,, \quad u_2 = \Phi x = \phi z x \tag{7.4.9}$$

The formulas (7.4.7, 7.4.8) are mathematical expressions of the deformation hypothesis. From Hooke's law and the relations between strains and displacements we find that the state of stress is given by:

$$\begin{aligned} T_{13} &= G\gamma_{13} = G\left(u_{1,3} + u_{3,1}\right) = G\phi\left(-y + \psi_{,1}\right) \\ T_{23} &= G\gamma_{23} = G\left(u_{2,3} + u_{3,2}\right) = G\phi\left(x + \psi_{,2}\right) \end{aligned} \tag{7.4.10}$$

All other coordinate stresses are zero. For a circular cylindrical bar the warping function $\psi = 0$, and the stresses T_{13} and T_{23} are components of the shear stress $\tau(R)$ given by (7.4.5).

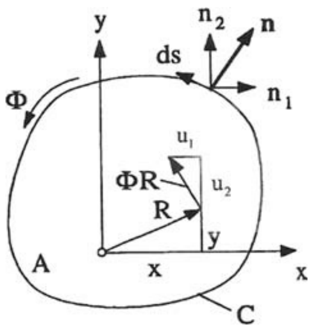


Fig. 7.4.4 Cross-section of the bar

The state of stress, given by the formulas (7.4.10) must satisfy the equilibrium conditions given by the Cauchy equations of motion, of which only one is not trivially satisfied:

$$\operatorname{div}\mathbf{T}=\mathbf{0} \quad \Leftrightarrow \quad T_{ik,k}=0 \quad \Rightarrow \quad T_{31,1}+T_{32,2}=0 \quad \Rightarrow \quad T_{13,1}+T_{23,2}=0 \quad \text{on } A \tag{7.4.11}$$

A is the projected cross-section as shown in Fig. 7.4.4. On the outer, cylindrical surface of the bar, i.e. along the contour C of the surface A, the stress boundary condition is:

$$\mathbf{t}=\mathbf{T}\cdot\mathbf{n}=\mathbf{0} \quad \text{along } C \;\Rightarrow\; T_{31}n_1+T_{32}n_2=0 \;\Rightarrow\; T_{13}n_1+T_{23}n_2=0 \quad \text{along } C \tag{7.4.12}$$

On the end surfaces of the bar, we can only require that the resultant of the stress distribution is represented by the torque M alone. Thus the resultant forces in the x- and y-directions must be zero, and the resultant moment about the z-axis is equal to the torque M:

$$\int_A T_{13}\,dA=\int_A T_{23}\,dA=0\,, \quad \int_A (T_{23}\,x-T_{13}\,y)\,dA=M \tag{7.4.13}$$

The real distribution of the stresses over the end surfaces is dependent upon how the torque M is supplied to the bar. But regardless of how this is done, we shall accept a solution that gives another distribution of stresses over the end surfaces. We only demand that the stress distribution is equivalent to the real one in the sense that the conditions (7.4.13) are satisfied. We may assume that the effect of this discrepancy between the real and computed stress distribution over the end surfaces is negligible except in small regions near the end surfaces. This assumption is supported by the *Saint-Venant's principle*, also referred to in Example 7.4.

The stresses (7.4.10) are now substituted into the equilibrium equation (7.4.11) and the stress boundary condition (7.4.12), and we obtain the following conditions for the warping function ψ:

$$\nabla^2\psi=0 \quad \text{on } A \tag{7.4.14}$$

$$\frac{d\psi}{dn}\equiv\psi_{,1}\frac{dx}{dn}+\psi_{,2}\frac{dy}{dn}\equiv\psi_{,\alpha}\,n_\alpha=yn_1-xn_2 \quad \text{along } C \tag{7.4.15}$$

The solution of (7.4.14, 7.4.15) is a standard problem in *potential theory* called the *Neumann's problem*, after Franz Ernst Neumann [1798–1895]. Equation (7.3.14) is called the *Laplace equation*.

When the warping function has been found from the (7.4.14, 7.4.15), it will be shown that the first and the second of the conditions (7.4.13) are satisfied. First we find, using (7.4.10) and (7.4.14), that:

$$\int_A T_{13}\,dA=G\phi\int_A(-y+\psi_{,1})\,dA=G\phi\int_A\{[x(-y+\psi_{,1})]_{,1}+[x(x+\psi_{,2})]_{,2}\}\,dA$$

Then, using the Gauss integration theorem in a plane, Theorem C.2, followed by application of the condition (7.3.15), we obtain:

$$\int_A T_{13}\, dA = G\phi \oint_C \{[x(-y+\psi_{,1})]\, n_1 + [x(x+\psi_{,2})]\, n_2\}\, ds$$
$$= G\phi \oint_C \{x[(\psi_{,1} n_1 + \psi_{,2} n_2) - (yn_1 - xn_2)]\}\, ds = 0$$

The fulfillment of the second condition in (7.4.13) is shown similarly.

The third of the conditions (7.4.13) provides the following relation between the torque M and the angle of twist ϕ.

$$M = GJ\phi\ , \quad J = \int_A \left[x^2 + y^2 + x\Psi_{,2} - y\Psi,1\right] dA \tag{7.4.16}$$

The parameter J is called the *torsion constant* of the cross section, and the combination GJ is called the *torsional stiffness* of the bar.

Example 7.11. Elliptical Cross-Section

The simplest non-trivial solution of the Laplace equation (7.4.14) is given by the warping function:

$$\psi = kxy\ , \quad k = \text{constant} \tag{7.4.17}$$

We will now determine the contour curve C that together with this warping function satisfies the boundary condition (7.4.15). The condition (7.4.15) gives:

$$kyn_1 + kxn_2 = yn_1 - xn_2 \quad \text{along } C \tag{7.4.18}$$

From Fig. 7.4.4 we derive the relations:

$$n_1 = \frac{dy}{ds}\ , \quad n_2 = -\frac{dx}{ds} \tag{7.4.19}$$

Equation (7.4.18) is now be reorganized to give:

$$\frac{d}{ds}\left[x^2 + \frac{1-k}{1+k}y^2\right] = 0 \quad \text{along } C \quad \Rightarrow \quad x^2 + \frac{1-k}{1+k}y^2 = \text{constant along } C$$

The contour C is thus an ellipse. We write the equation of the ellipse C on the standard form:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1\ , \quad k = -\frac{a^2 - b^2}{a^2 + b^2}$$

a and b are the semiaxes, see Fig. 7.4.5. The function $\psi = kxy$ is thus the warping function for the torsion of a massive bar with elliptical cross-section. The torsion constant J is determined from the integral in equation (7.4.16), and the stress distribution is found from the expressions (7.4.10) when ϕ is supplied from (7.4.16). The results are:

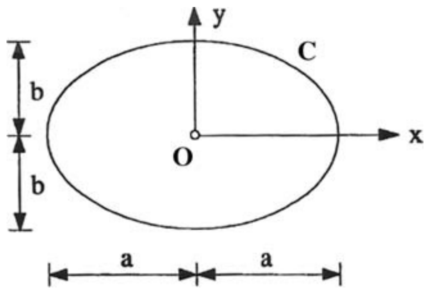


Fig. 7.4.5 Elliptical cross-section

$$J = \frac{\pi a^3 b^3}{a^2 + b^2}\ , \quad \tau_{xz} \equiv T_{13} = -\frac{2M}{\pi a b^3} y\ , \quad \tau_{yz} \equiv T_{23} = \frac{2M}{\pi a^3 b} x$$

The combined shear stress on the cross-section is:

$$\tau = \sqrt{T_{13}^2 + T_{23}^2} = \frac{2M}{\pi a b^2} \sqrt{\left(\frac{y}{b}\right)^2 + \left(\frac{b}{a}\right)^2 \left(\frac{x}{a}\right)^2}$$

At the contour C the cross-sectional shear is:

$$\tau = \frac{2M}{\pi a b^2} \sqrt{\left(\frac{y}{b}\right)^2 + \left(\frac{b}{a}\right)^2 \left[1 - \left(\frac{y}{b}\right)^2\right]} = \frac{2M}{\pi a b^2} \sqrt{\left(\frac{b}{a}\right)^2 + \left[1 - \left(\frac{b}{a}\right)^2\right] \left(\frac{y}{b}\right)^2}$$

When $b < a$ the maximum shear stress acts at the particles $(x, y) = (0, \pm b)$:

$$\tau_{\max} = \frac{2M}{\pi a b^2} \quad \text{on } A \text{ for } x = 0 \text{ and } y = \pm b$$

For the case $a = b = r$ we get:

$$J = \pi r^4 = I_p$$

and the results (7.4.5) for the circular cross-section.

7.4.3 Prandtl's Stress Function

An alternative mathematical formulation of the torsion problem is obtained by the introduction of *Prandtl's stress function* $\Omega(x, y)$ named after Ludwig Prandtl [1875–1953]. This stress function has the property that the equilibrium equation (7.4.11) is satisfied identically. This is achieved by setting:

$$T_{13} = \Omega_{,2}\ , \quad T_{23} = -\Omega_{,1} \tag{7.4.20}$$

Because this indirectly implies that we use the stresses as primary unknown functions, we must ensure that the resulting strains are compatible. That is, the strains γ_{13} and γ_{23} calculated from the stresses T_{13} and T_{23} by Hooke's law, must be compatible and give unique displacement function $u_3(x,y)$. From the strain-displacement relations (7.2.1) and the assumptions (7.4.9) concerning the displacements, we obtain:

$$\gamma_{13} = u_{1,3} + u_{3,1} = -\phi\, y + u_{3,1} \ , \quad \gamma_{23} = u_{2,3} + u_{3,2} = \phi\, x + u_{3,2}$$

From these equations we derive the following *compatibility condition* for the strains.

$$\begin{aligned} u_{3,12} - u_{3,21} = (\gamma_{13,2} + \phi) - (\gamma_{23,1} - \phi) \equiv 0 \quad \Rightarrow \\ \gamma_{13,2} - \gamma_{23,1} = -2\phi \quad \text{compatibility equation for strains} \end{aligned} \tag{7.4.21}$$

By Hooke's law:

$$T_{13} = G\gamma_{13} \ , \quad T_{23} = G\gamma_{23}$$

and (7.4.20), the compatibility condition (7.4.21) is rewritten to:

$$\nabla^2\Omega = -2G\phi \text{ on } A \tag{7.4.22}$$

The stress function $\Omega(x,y)$ must also satisfy a boundary condition on the contour curve C of the cross-section A. This condition will be derived from the stress condition (7.4.12) using the relations (7.4.19).

Substituting the stresses (7.4.20) into the boundary condition (7.4.12), and then using the relations (7.4.19), we obtain:

$$\Omega_{,2}\frac{dy}{ds} + (-\Omega_{,1})\left(-\frac{dx}{ds}\right) = \frac{d\Omega}{ds} = 0 \text{ along } C$$

This implies that $\Omega(x,y)$ must be a constant on the contour C. For the sake of simplicity we set the constant equal to zero. Thus we have obtained the boundary condition:

$$\Omega = 0 \text{ along } C \tag{7.4.23}$$

We must now show that the first two boundary conditions (7.4.13) are satisfied by the stress function $\Omega(x,y)$ and find what the third of the conditions (7.4.13) leads to. Using the expressions (7.4.20), Gauss theorem in a plane, Theorem C.2, and finally (7.4.23), we obtain:

$$\int_A T_{13}\, dA = \int_A \Omega_{,2}\, dA = \oint_C \Omega n_2\, ds = 0 \ , \quad \int_A T_{23}\, dA = -\int_A \Omega_{,1}\, dA = -\oint_C \Omega n_1\, ds = 0$$

These results prove that the first two boundary conditions (7.4.13) are satisfied

From the third of the conditions (7.4.13) it follows that:

$$M = \int_A (T_{23}x - T_{13}y)\,dA = -\int_A (\Omega_{,1}\,x + \Omega_{,2}\,y)\,dA = -\int_A (\Omega x_\alpha)_{,\alpha}\,dA + 2\int_A \Omega\,dA$$
$$= -\oint_C \Omega x_\alpha n_\alpha\,ds + 2\int_A \Omega\,dA$$

Due to (7.4.23) the result is that:

$$M = 2\int_A \Omega\,dA \tag{7.4.24}$$

We may now conclude that the Prandtl stress function $\Omega(x,y)$ has to satisfy the differential equation (7.4.22) and the boundary condition (7.4.23):

$$\nabla^2\Omega = -2G\phi \text{ on } A\,,\quad \Omega = 0 \text{ along } C \tag{7.4.25}$$

The mathematical problem represented by (7.4.25) is called the *Poisson problem* in Potential Theory.

The warping function $\psi(x,y)$ and the Prandtl stress function $\Omega(x,y)$ are related through:

$$\Omega_{,2} = G\phi\,(-y + \psi_{,1})\,,\ \Omega_{,1} = G\phi\,(-x - \psi_{,2}) \tag{7.4.26}$$

These relations follow by comparing the stress expressions (7.4.10) and (7.4.20). The compatibility condition (7.4.22) may also be derived directly from the relations (7.4.26) and the compatibility equation (7.4.14).

In the next section we will present a practical further application of the use of the Prandtl stress function.

Example 7.12. Elliptical Cross-Section

For torsion of a bar having an elliptical cross-section with semiaxes a and b, Fig. 7.4.5, the Prandtl stress function is:

$$\Omega = -\frac{a^2b^2}{a^2+b^2}G\phi\left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1\right] \tag{7.4.27}$$

The boundary condition (7.4.23) is obviously satisfied, and it is easily shown that the differential equation (7.4.22) also is satisfied. From (7.4.24) we obtain:

$$M = 2\int_A \Omega\,dA = -\frac{2a^2b^2}{a^2+b^2}G\phi\left[\frac{1}{a^2}\int_A x^2\,dA + \frac{1}{b^2}\int_A y^2\,dA - \int_A dA\right]$$

The integrals are found to be:

$$\int_A x^2\,dA = I_y = \frac{\pi a^3 b}{4}\,,\quad \int_A y^2\,dA = I_x = \frac{\pi a b^3}{4}\,,\quad \int_A dA = A = \pi a b$$

I_y and I_x are the second moments of area of the cross-section A about the y-axis and the x-axis respectively. Finally, the torque M becomes:

$$M = \frac{\pi a^3 b^3}{a^2 + b^2} G\phi$$

The stress function $\Omega(x,y)$, the torsion constant J in (7.4.16), and the shear stresses on the cross section, according to the formulas (7.4.20), are now:

$$\Omega = -\frac{M}{\pi ab}\left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1\right], \; J = \frac{\pi a^3 b^3}{a^2 + b^2}$$

$$T_{13} = \Omega_{,2} = -\frac{2M}{\pi a b^3} y\,, \;\; T_{23} = -\Omega_{,1} = \frac{2M}{\pi a^3 b} x$$

Compare with the results in Example 7.11.

7.4.4 The Membrane Analogy

Figure 7.4.6 shows a flexible membrane attached to a plane stiff border C of a hole with a plane area A. The membrane is subjected to a constant pressure p and is

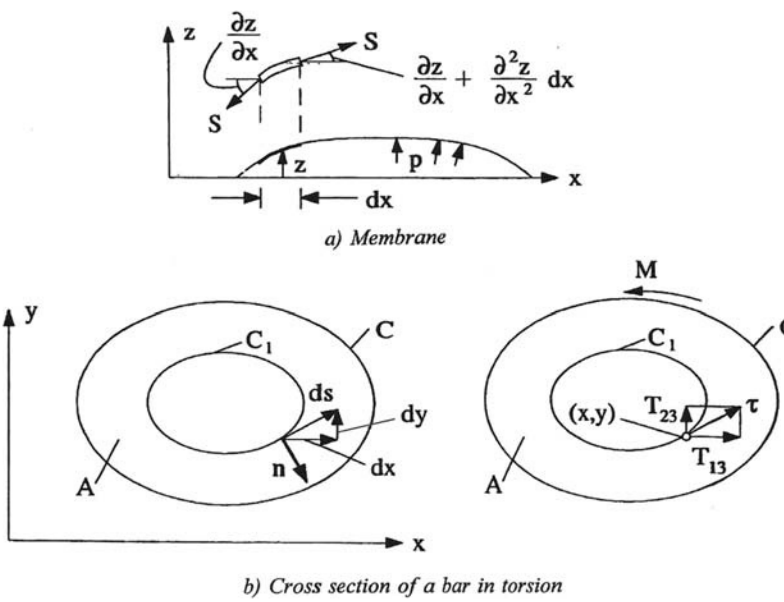


Fig. 7.4.6 **a**) Membrane subjected to a pressure p. **b**) Cross-section of a bar in torsion

stretched to a curved form given by a function $z = z(x,y)$. We assume the membrane force S, given as a force per unit length, is a constant and equal in all directions. The condition of a constant membrane force S is satisfied by a soap film membrane.

We assume that z is small compared to the smallest diameter of the area A. An element of the membrane with the projection $dx \cdot dy$ on the xy-plane is in equilibrium under the action of the pressure p and the membrane force S. The equilibrium equation in the z-direction is:

$$p \cdot (dx \cdot dy) - (S \cdot dy) \cdot \frac{\partial z}{\partial x} + (S \cdot dy) \cdot \left(\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial x^2} dx \right) - (S \cdot dx) \cdot \frac{\partial z}{\partial y} +$$
$$(S \cdot dx) \cdot \left(\frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial y^2} dy \right) = 0 \quad \Rightarrow$$
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \equiv \nabla^2 z = -\frac{p}{S} \tag{7.4.28}$$

The boundary condition for the z-function along the border curve C is:

$$z = 0 \quad \text{along } C \tag{7.4.29}$$

The mathematical problem consisting of finding the function $z(x,y)$ from the differential equation (7.4.28) and the boundary condition (7.4.29), is a Poisson problem and thus mathematically analogous to the problem (7.4.25) of finding the Prandtl stress function $\Omega(x,y)$. When we choose for the plane A of the hole a cross-section A of a cylindrical bar subjected to torsion, and the constant p/S equal to the constant $2G\phi\alpha$, where:

$$\alpha = \frac{p}{2G\phi S} \tag{7.4.30}$$

then the membrane function $z(x,y)$ becomes identical to:

$$z(x,y) = \alpha\, \Omega(x,y) \tag{7.4.31}$$

Let C_1 be the contour line of the membrane, such that $z \equiv \alpha\Omega = \text{constant}$ along C_1. In direction of the tangent to the line C_1, Fig. 7.4.6:

$$\frac{\partial z}{\partial s} = 0 \quad \Rightarrow \quad \frac{\partial \Omega}{\partial s} = \frac{\partial \Omega}{\partial x}\frac{dx}{ds} + \frac{\partial \Omega}{\partial y}\frac{dy}{ds} = 0 \quad \Rightarrow \quad -T_{23}\frac{dx}{ds} + T_{13}\frac{dy}{ds} = 0$$

This result and Fig. 7.4.6 show that the resultant of the shear stresses on the cross section does not have a component normal to the contour line C_1 in the cross-section. This means that the contour lines of the membrane are vector lines of the shear stresses on the cross-section of the bar. In other words, in every particle on the cross-section the resultant shear stress τ is directed along the contour line $\Omega(x,y) =$ constant through the particle. We may call the contour lines *shear stress lines* in the cross-section. Since the resultant shear stress τ is tangential to the contour line C_1

and thus perpendicular to the unit normal **n** to the contour line, pointing outward, we find from Fig. 7.4.6:

$$\tau = T_{13}\frac{dx}{ds} + T_{23}\frac{dy}{ds} = \frac{\partial \Omega}{\partial y}\left(-\frac{dy}{dn}\right) + \left(-\frac{\partial \Omega}{\partial x}\right)\frac{dx}{dn} \quad \Rightarrow \quad \tau = -\frac{\partial \Omega}{\partial n} = -\frac{1}{\alpha}\frac{\partial z}{\partial n} \tag{7.4.32}$$

The n-coordinate is measured along the normal **n**. The result (7.4.32) shows that the maximum cross-sectional shear stress occurs in particles where the membrane has the steepest slope. If a net of contour lines have been drawn, these points of maximum shear stress will be where the contour lines are closest together.

The formula (7.4.24) shows that the torque M is proportional to the volume V between the membrane and the surface A.

$$M = 2\int_A \Omega\, dA = \frac{2}{\alpha}\int_A z\, dA = \frac{2V}{\alpha} \tag{7.4.33}$$

The membrane analogy may be used in an experimental determination of the stress distribution on a cross section A of a bar in torsion. A membrane of soap is often used for this purpose. The constant p/S may be determined by introducing a calibrating membrane over a circular hole near the hole representing the cross-sectional area A. The circular hole, representing a circular cross section, and the hole with cross-sectional area A are covered by a soap film having the same membrane force S, and are subjected to the same pressure p. The known solution of the torsion problem for a circular cylindrical bar is then used to compute the constant p/S.

The membrane analogy may be used for bars with open and closed thin-walled cross sections and for bars with cell like cross sections. We shall not elaborate on the subject, only refer to the literature, e.g. [48].

Example 7.13. Narrow Rectangular Cross-Section

Figure 7.4.7 shows a membrane over a rectangular hole for which the width b is much smaller the height h. We may discard the small regions near the short sides and thus consider the membrane as a cylindrical surface, implying that $\partial z/\partial x \equiv 0$. Then from the formulas (7.4.28) and (7.4.29) we obtain:

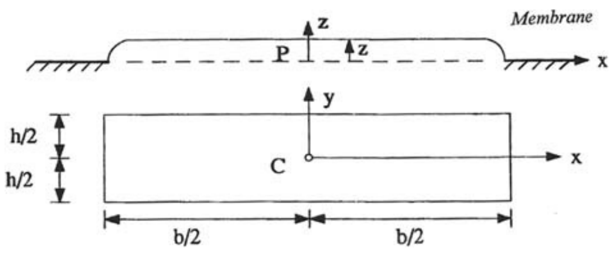


Fig. 7.4.7 Membrane for a narrow rectangular cross-section

$$\frac{d^2 z}{dy^2} = -\frac{p}{S}, \ z = 0 \quad \text{for } y = \pm\frac{h}{2} \quad \Rightarrow \quad z(y) = \frac{ph^2}{8S}\left[1 - \left(\frac{2y}{h}\right)^2\right]$$

This membrane provides the solution to the torsion problem for a cylindrical bar with a narrow rectangular cross-section with width b and height h. The Prandtl stress function is:

$$\Omega(y) = \frac{z(y)}{\alpha} = \frac{G\phi h^2}{4}\left[1 - \left(\frac{2y}{h}\right)^2\right]$$

This function is substituted into formula (7.4.24) for the torque, which then gives:

$$M = 2\int_A \Omega \, dA = \frac{G\phi h^2}{2}\left[b\int_{-h/2}^{h/2}\left[1 - \left(\frac{2y}{h}\right)^2\right]dy\right] = \frac{G\phi h^3 b}{3}$$

The angle of twist ϕ per unit length of the bar, the torsion constant J, and the maximum shear stress on the cross-section are then, formulas (7.4.16, 7.4.20):

$$\phi = \frac{3M}{Gh^3 b}, \ J = \frac{M}{G\phi} = \frac{h^3 b}{3}, \ \tau_{\max} = -\left.\frac{d\Omega}{dy}\right|_{y=h/2} = G\phi h = \frac{3M}{h^2 b}$$

7.5 Thermoelasticity

7.5.1 Constitutive Equations

When the strains in an isotropic, linearly elastic material, i.e. a Hookean material, do not originate only from the stresses in the material, Hooke's law given by (7.2.6, 7.2.7, 7.2.8), has to be modified. We shall in this section discuss the important case of a Hookean material subjected to a temperature field $\theta(\mathbf{r},t)$, which may imply expansions and contractions in the material. It is assumed that the changes in the temperature, from a homogeneous temperature θ_o in the reference configuration K_o, are small enough not to influence the elastic properties of the material. Furthermore, we assume that the thermal properties are homogeneous and isotropic. Thermal isotropy implies that when the material experiences *free thermal deformation*, i.e. stress-free deformation, the longitudinal strain is the same in all directions in a particle of the material and given by $\varepsilon^t = \alpha \cdot (\theta - \theta_o)$. The parameter α is the *linear coefficient of thermal expansion* of the material. The state of strain in free thermal deformation is therefore isotropic and form invariant. The strain tensor is:

$$E^t_{ij} = \alpha \cdot (\theta - \theta_o)\,\delta_{ij} \quad \Leftrightarrow \quad \mathbf{E}^t = \alpha \cdot (\theta - \theta_o)\,\mathbf{1} \tag{7.5.1}$$

A volume element of the material subjected to a constant change in temperature $(\theta - \theta_o)$ preserves its shape and the volumetric strain due to the thermal deformation is $\varepsilon_v = 3\alpha(\theta - \theta_o)$. The strains (7.5.1) are in general not compatible, which implies that elastic strains due to stresses are developed to make the total strains, thermal plus elastic, compatible. It may be shown, see Problem 5.8 that the condition for the strains in (7.5.1) to be compatible is that the change in temperature $(\theta - \theta_o)$ is a linear function of the place vector $\mathbf{r}$, or the x-coordinates. The total strains in a body of a Hookean material, represented by the strain tensor $\mathbf{E}$, may be considered to comprise of the sum of three contributions:

1) *thermal strains* according to (7.5.1),
2) *elastic strains* to create compatible strains when the material resists free thermal deformation,
3) *elastic strains* produced by the stresses due to the external forces on the body and the motion of the body.

The two contributions of elastic strains are represented by the elastic strain tensor:

$$\mathbf{E}^e = \mathbf{E} - \mathbf{E}^t = \mathbf{E} - \alpha \cdot (\theta - \theta_o)\mathbf{1} \quad \Leftrightarrow \quad E^e_{ij} = E_{ij} - E^t_{ij} = E_{ij} - \alpha \cdot (\theta - \theta_o)\,\delta_{ij} \tag{7.5.2}$$

Substitution of this strain tensor into Hooke's law, (7.2.6) and (7.2.7), yields:

$$\mathbf{E} = \frac{1+\nu}{\eta}\mathbf{T} - \frac{\nu}{\eta}(\mathrm{tr}\,\mathbf{T})\,\mathbf{1} + \alpha \cdot (\theta - \theta_o)\,\mathbf{1} \quad \Leftrightarrow$$
$$E_{ij} = \frac{1+\nu}{\eta}T_{ij} - \frac{\nu}{\eta}T_{kk}\,\delta_{ij} + \alpha \cdot (\theta - \theta_o)\,\delta_{ij} \tag{7.5.3}$$

The inverse form becomes:

$$\mathbf{T} = \frac{\eta}{1+\nu}\left[\mathbf{E} + \frac{\nu}{1-2\nu}(\mathrm{tr}\,\mathbf{E})\,\mathbf{1}\right] - 3\kappa\alpha \cdot (\theta - \theta_o)\,\mathbf{1} \quad \Leftrightarrow$$
$$T_{ij} = \frac{\eta}{1+\nu}\left[E_{ij} + \frac{\nu}{1-2\nu}E_{kk}\,\delta_{ij}\right] - 3\kappa\alpha \cdot (\theta - \theta_o)\,\delta_{ij} \tag{7.5.4}$$

The constitutive equations (7.5.3) and (7.5.4) represent the *Duhamel-Neumann law*, named after Duhamel [1838] and Neumann [1841].

7.5.2 Plane Stress

In the case of plane stress, $T_{i3} = 0$, the constitutive equations (7.5.3) and (7.5.4) are replaced by:

$$E_{\alpha\beta} = \frac{1+\nu}{\eta}\left[T_{\alpha\beta} - \frac{\nu}{1+\nu}T_{\rho\rho}\,\delta_{\alpha\beta}\right] + \alpha\cdot(\theta-\theta_o)\,\delta_{\alpha\beta}$$
$$E_{33} = -\frac{\nu}{\eta}T_{\rho\rho} + \alpha\cdot(\theta-\theta_o) \tag{7.5.5}$$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) + \alpha\cdot(\theta-\theta_o)$$
$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) + \alpha\cdot(\theta-\theta_o)\,, \quad \gamma_{xy} = \frac{1}{G}\tau_{xy} \tag{7.5.6}$$
$$T_{\alpha\beta} = \frac{\eta}{1+\nu}\left[E_{\alpha\beta} + \frac{\nu}{1-\nu}E_{\rho\rho}\,\delta_{\alpha\beta}\right] - \frac{\eta}{1-\nu}\alpha\cdot(\theta-\theta_o)\,\delta_{\alpha\beta} \tag{7.5.7}$$

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) - \frac{E}{1-\nu}\alpha\cdot(\theta-\theta_o)$$
$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) - \frac{E}{1-\nu}\alpha\cdot(\theta-\theta_o)\,, \quad \tau_{xy} = G\gamma_{xy} \tag{7.5.8}$$

The equilibrium equations (7.3.43) and the constitutive equations (7.5.5) may be used to express the compatibility equations (7.3.44) as:

$$\nabla^2 T_{\alpha\alpha} + (1+\nu)\,\rho\, b_{\alpha},_{\alpha} + \eta\,\alpha\,(\theta-\theta_o),_{\alpha\alpha} = 0 \tag{7.5.9}$$

From these equations it readily follows that a linear temperature field $(\theta-\theta_o)$ does not contribute to the stresses if the surface of the body is not subjected to displacement conditions.

The Navier equations (7.3.14) represent the Cauchy equations of motion expressed in displacements. When the stresses are expressed by the Duhamel-Neumann equations (7.5.7) rather than Hooke's law, the following modified Navier equations are obtained:

$$u_{\alpha},_{\beta\beta} + \frac{1+\nu}{1-\nu}\left[u_{\beta},_{\beta\alpha} - 2\alpha\,(\theta-\theta_0),_{\alpha}\right] + \frac{1}{\mu}\rho\,(b_{\alpha} - \ddot{u}_{\alpha}) = 0 \tag{7.5.10}$$

In an axisymmetrical case these equations are reduced to one equation for the radial displacement $u \equiv u_R(R)$, confer equation (7.3.19):

$$\frac{d}{dR}\left[\frac{1}{R}\frac{d\,(Ru)}{dR}\right] - (1+\nu)\,\alpha\frac{d\,(\theta-\theta_o)}{dR} + \frac{1-\nu}{2G}\rho\,(b_R - \ddot{u}_R) = 0 \tag{7.5.11}$$

Example 7.14. Circular Plate Mounted on a Rigid Rod

A circular plate of radius b has a concentric hole of radius a. The plate is mounted on a rigid rod of radius a, see Fig. 7.5.1. The plate is cooled from the temperature θ_o to the temperature $\theta_1 < \theta_o$. We shall determine the state of stress in the plate and the radial displacement.

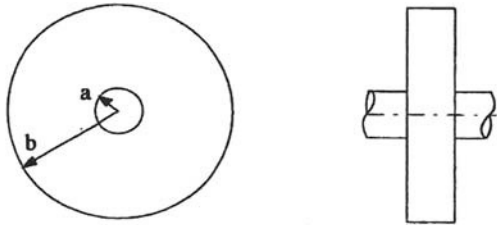


Fig. 7.5.1 Circular plate mounted on a rigid rod

Since body forces are not considered in this problem, and since the term involving the temperature in the compatibility equation (7.5.9) in this case becomes zero, we may choose between the general solution from Example 7.9 using Airy's stress function, or we may solve the Navier equation (7.5.11) as was done in Example 7.1. In the case of a more complex but axisymmetric temperature field $\theta(R)$, it is most convenient to use the Navier equation. The solution procedure is then as follows. The general solution of the Navier equation (7.5.11), with $b_R = \ddot{u} = 0$, is:

$$u(R) = AR + \frac{B}{R} + (1+\nu)\,\alpha \frac{1}{R}\int [\theta\,(R) - \theta_o]\,R\,dR \tag{7.5.12}$$

A and B are constants of integration. In the present example: $\theta(R) - \theta_o = \theta_1 - \theta_o < 0$. Therefore:

$$u(R) = AR + \frac{B}{R} - (1+\nu)\,\alpha\,(\theta_o - \theta_1)\,\frac{R}{2} \tag{7.5.13}$$

The strains are given by the formulas (7.3.17) as:

$$\varepsilon_R = \frac{du}{dR}, \quad \varepsilon_\theta = \frac{u}{R} \tag{7.5.14}$$

The stresses are determined from the constitutive equations (7.5.8), and become:

$$\begin{aligned}\sigma_R(R) &= 2G\left[\frac{1+\nu}{1-\nu}A - B\frac{1}{R^2}\right] + G(1+\nu)\,\alpha\,(\theta_o - \theta_1)\\ \sigma_\theta(R) &= 2G\left[\frac{1+\nu}{1-\nu}A + B\frac{1}{R^2}\right] + G(1+\nu)\,\alpha\,(\theta_o - \theta_1)\end{aligned} \tag{7.5.15}$$

The boundary conditions are: 1) $u(a) = 0$ and 2) $\sigma_R(b) = 0$. The final solution to the problem is then:

$$u(R) = -C\left[R - \frac{a^2}{R}\right], \quad C = \alpha\cdot(\theta_o - \theta_1)\left[1 + \frac{1-\nu}{1+\nu}\left(\frac{a}{b}\right)^2\right]^{-1} \tag{7.5.16}$$

$$\sigma_R(R) = 2GC\left(\frac{a}{b}\right)^2\left[1 - \left(\frac{b}{R}\right)^2\right], \quad \sigma_\theta(R) = 2GC\left(\frac{a}{b}\right)^2\left[1 + \left(\frac{b}{R}\right)^2\right] \tag{7.5.17}$$

If we let $a \to 0$, we see that the stresses disappear, in accordance with the fact that the rigid core has vanished and the plate therefore has free thermal deformation, with:

$$u(R) = -\alpha \cdot (\theta_o - \theta_1)\, R \tag{7.5.18}$$

7.5.3 Plane Displacements

In the case of plane displacements, $u_3 = 0 \Rightarrow E_{33} = 0$, the constitutive equations (7.5.3) and (7.5.4) are reduced to:

$$E_{\alpha\beta} = \frac{1}{2\mu}\left[T_{\alpha\beta} - \nu T_{\rho\rho}\,\delta_{\alpha\beta}\right] + (1+\nu)\,\alpha\cdot(\theta-\theta_o)\,\delta_{\alpha\beta} \tag{7.5.19}$$

$$\varepsilon_x = \frac{1-\nu}{2G}\left(\sigma_x - \frac{\nu}{1-\nu}\sigma_y\right) + (1+\nu)\,\alpha\cdot(\theta-\theta_o)$$

$$\varepsilon_y = \frac{1-\nu}{2G}\left(\sigma_y - \frac{\nu}{1-\nu}\sigma_x\right) + (1+\nu)\,\alpha\cdot(\theta-\theta_o)\,, \quad \gamma_{xy} = \frac{1}{G}\tau_{xy} \tag{7.5.20}$$

$$T_{\alpha\beta} = 2\mu\left[E_{\alpha\beta} + \frac{\nu}{1-2\nu}E_{\rho\rho}\,\delta_{\alpha\beta}\right] - 3\kappa\,\alpha\cdot(\theta-\theta_o)\,\delta_{\alpha\beta}, \quad \kappa = \frac{2\mu\,(1+\nu)}{3\,(1-2\nu)} \tag{7.5.21}$$

$$T_{33} = \frac{2\nu\mu}{1-2\nu}E_{\rho\rho} - 3\kappa\alpha\cdot(\theta-\theta_o) \tag{7.5.22}$$

$$\sigma_x = \frac{2G}{1-2\nu}\left[(1-\nu)\,\varepsilon_x + \nu\,\varepsilon_y\right] - 3\kappa\alpha\cdot(\theta-\theta_o)$$

$$\sigma_y = \frac{2G}{1-2\nu}\left[(1-\nu)\,\varepsilon_y + \nu\,\varepsilon_x\right] - 3\kappa\alpha\cdot(\theta-\theta_o)\,, \quad \tau_{xy} = G\gamma_{xy} \tag{7.5.23}$$

The equilibrium equations (7.3.43) and the constitutive equations (7.5.20) may be used to express the compatibility equations (7.3.44) as:

$$\nabla^2 T_{\alpha\alpha} + \frac{1}{1-\nu}\left[\rho\, b_{\alpha,\alpha} + \eta\,\alpha\,(\theta-\theta_o)_{,\alpha\alpha}\right] = 0 \tag{7.5.24}$$

As in the case of plane stress we see that a linear temperature field $(\theta-\theta_o)$ does not contribute to the stresses $T_{\alpha\beta}$ if the surface of the body is not subjected to displacement conditions for u_α.

Modified *Navier equations*, which represent the Cauchy equations of motion expressed in displacements, now become:

$$u_{\alpha,\beta\beta} + \frac{1}{1-2\nu}\left[u_{\beta,\beta\alpha} - 2\,(1+\nu)\,\alpha\cdot\ (\theta-\theta_0)_{,\alpha}\right] + \frac{1}{\mu}\rho\,(b_\alpha - \ddot{u}_\alpha) = 0 \tag{7.5.25}$$

7.6 Hyperelasticity

7.6.1 Elastic Energy

In the case of a uniaxial stress history $\sigma(t)$ with the strain $\varepsilon(t)$ in the direction of the stress σ, the stress power per unit volume is $\omega = \sigma\dot{\varepsilon}$. The stress work w per unit volume when the material is deformed from the reference configuration K_o at time t_o to the present configuration K at time t will be:

$$w = \int_{t_o}^{t} \omega\, dt = \int_{t_o}^{t} \sigma\,\dot{\varepsilon}\, dt = \int_{\varepsilon_o}^{\varepsilon} \sigma\, d\varepsilon \tag{7.6.1}$$

ε and ε_o are the strains in K and K_o espectively. Figure 7.6.1 shows that the stress work is represented by the area under the stress curve $\sigma(\varepsilon)$ in the $\sigma\varepsilon$-diagram.

Under the assumption of small deformations the stress power per unit volume for a general state of stress $\mathbf{T}$ and the corresponding state of strain $\mathbf{E}$ is given by $\omega = \mathbf{T} : \dot{\mathbf{E}}$. The stress work done between the configurations K and K_o is given by:

$$w = \int_{t_o}^{t} \omega\, dt = \int_{t_o}^{t} \mathbf{T} : \dot{\mathbf{E}}\, dt = \int_{\mathbf{E}_o}^{\mathbf{E}} \mathbf{T} : d\mathbf{E} \tag{7.6.2}$$

A material is called *hyperelastic, Green elastic*, or *conservative* if the response of the material is such that the stress power and the stress work may be derived from a scalar valued potential $\phi = \phi\,[\mathbf{E}]$ such that:

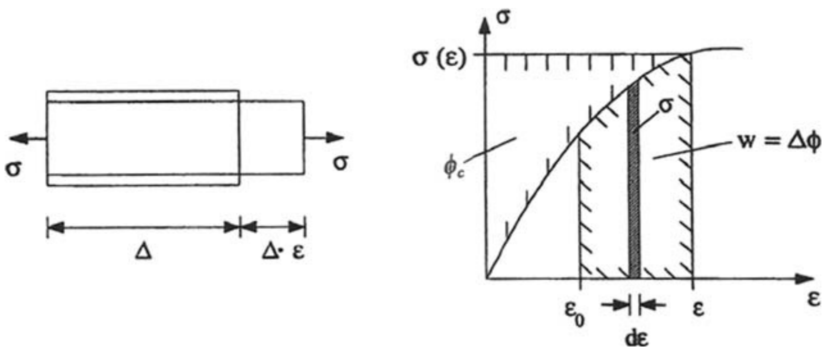


Fig. 7.6.1 Stress work w, elastic energy ϕ, and complementary energy ϕ_c

$$\omega = \dot{\phi} = \frac{\partial \phi}{\partial \mathbf{E}} : \dot{\mathbf{E}} = \frac{\partial \phi}{\partial E_{ij}} \dot{E}_{ij} \tag{7.6.3}$$

$$w = \int_{t_o}^{t} \omega \, dt = [\phi]_{t_o}^{t} = \phi\,[\mathbf{E}] - \phi\,[\mathbf{E}_o] \equiv \Delta\phi \tag{7.6.4}$$

The potential $\phi\,[\mathbf{E}]$ is called *elastic energy* or *strain energy per unit volume*. We choose to set $\phi\,[\mathbf{0}] = 0$, such that the elastic energy is zero when the material is free of strain. A body with volume V has a total elastic energy:

$$\Phi = \int_V \phi \, dV \tag{7.6.5}$$

The stress power supplied to the body is:

$$P^d = \int_V \omega \, dV = \int_V \dot{\phi} \, dV = \dot{\Phi} \tag{7.6.6}$$

The result for the material derivative of the integral in (7.6.5) is obtained under the assumption of small volumetric strains, which means that we may neglect the fact that the volumes V and dV change with time. The result (7.6.6) shows that the stress power P^d supplied to the body is equal to the time rate of change in the elastic energy of the body. The balance equation of mechanical energy (6.1.12) may now be written as:

$$P = \dot{K} + \dot{\Phi} \tag{7.6.7}$$

Integration with respect to time yields the *work and energy equation*:

$$\int_{t_o}^{t} P \, dV = [K + \Phi]_{t_o}^{t} \quad \Rightarrow \quad W = \Delta(K + \Phi) \tag{7.6.8}$$

W represents the work of the external forces on the body. For a body of a hyperelastic material the work of the external forces is converted to two kinds of energy: *kinetic energy* and *elastic energy*. Positive work on the body is conserved in the body as kinetic energy and elastic energy. When the work is negative, the body delivers energy to the surroundings.

For a hyperelastic material the stress power per unit volume is represented by two expressions:

$$\omega = T_{ij} \dot{E}_{ij} = \frac{\partial \phi}{\partial E_{ij}} \dot{E}_{ij} \tag{7.6.9}$$

Note that in the differentiation of ϕ in (7.6.9) we must treat E_{ij} and E_{ji} for $i \neq j$ as independent variables. In other words, ϕ must be treated as a function of 9 independent variables E_{ij}. Since $\partial\phi/\partial\mathbf{E}$ is not dependent upon the rate of strain $\dot{\mathbf{E}}$, the stress power ω is a linear function of $\dot{\mathbf{E}}$. This implies that the stresses T_{ij} are also independent of $\dot{\mathbf{E}}$. Equation (7.6.9) will therefore imply that:

$$\mathbf{T} = \frac{\partial \phi}{\partial \mathbf{E}} \quad \Leftrightarrow \quad T_{ij} = \frac{\partial \phi}{\partial E_{ij}} \tag{7.6.10}$$

The result may be obtained as follows. First we choose the case:

$$\dot{E}_{11} \neq 0 \text{ all other } \dot{E}_{ij} = 0$$

which results in the stress power:

$$\omega = T_{11}\dot{E}_{11} = \frac{\partial \phi}{\partial E_{11}}\dot{E}_{11} \quad \Rightarrow \quad T_{11} = \frac{\partial \phi}{\partial E_{11}} \tag{7.6.11}$$

Then we choose:

$$\dot{E}_{12} = \dot{E}_{21} \neq 0 \text{ all other } \dot{E}_{ij} = 0$$

which implies the stress power:

$$\omega = T_{12}\dot{E}_{12} + T_{21}\dot{E}_{22} = (T_{12} + T_{21})\,\dot{E}_{12} = \left[\frac{\partial \phi}{\partial E_{12}} + \frac{\partial \phi}{\partial E_{21}}\right]\dot{E}_{12} \quad \Rightarrow$$

$$T_{12} = \frac{1}{2}\left[\frac{\partial \phi}{\partial E_{12}} + \frac{\partial \phi}{\partial E_{21}}\right] \quad \Rightarrow \quad T_{12} = \frac{\partial \phi}{\partial E_{12}} \tag{7.6.12}$$

From the results (7.6.11, 7.6.12) and similar results for other (*ij*)-index pairs the general result (7.6.10) follows. Remember that in the differentiation of ϕ in (7.6.10) we must treat E_{ij} and E_{ji} for $i \neq j$ as independent variables, i.e. ϕ must be treated as a function of 9 independent variables E_{ij}.

According to the constitutive equations (7.6.10) for hyperelastic materials the stresses are functions of the strains: $\mathbf{T} = \mathbf{T}[\mathbf{E}]$. Thus the following implication:

$$Hyperelasticity \Rightarrow Elasticity$$

All material models that have been proposed in the literature for real elastic materials are hyperelastic. In principle however, an elastic material model may be dissipative, i.e. some of the work done on such a material may turn into heat.

Complementary energy per unit volume ϕ_c is defined by the expression:

$$\phi_c = \mathbf{T} : \mathbf{E} - \phi\,[\mathbf{E}] \tag{7.6.13}$$

For uniaxial stress the complimentary energy is:

$$\phi_c = \sigma\varepsilon - \phi\,[\varepsilon] \tag{7.6.14}$$

In the $\sigma\varepsilon$ – diagram ϕ_c is represented by the vertically hatched area above the $\sigma(\varepsilon)$ – curve in Fig. 7.6.1. From the definitions (7.6.13) and equation (7.6.10) we obtain the result:

$$\frac{\partial \phi_c}{\partial \mathbf{T}} = \frac{\partial \mathbf{T}}{\partial \mathbf{T}} : \mathbf{E} + \mathbf{T} : \frac{\partial \mathbf{E}}{\partial \mathbf{T}} - \frac{\partial \phi}{\partial \mathbf{E}} : \frac{\partial \mathbf{E}}{\partial \mathbf{T}} = \mathbf{E} + \mathbf{T} : \frac{\partial \mathbf{E}}{\partial \mathbf{T}} - \mathbf{T} : \frac{\partial \mathbf{E}}{\partial \mathbf{T}} = \mathbf{E} \quad \Rightarrow$$

$$\mathbf{E} = \frac{\partial \phi_c}{\partial \mathbf{T}} \quad \Leftrightarrow \quad E_{ij} = \frac{\partial \phi_c}{\partial T_{ij}} \tag{7.6.15}$$

It will be shown in Sect. 7.8 that for linearly hyperelastic materials, for which the stresses are linear functions of the strains or the strains are linear functions of the stresses, i.e.:

$$T_{ij} = S_{ijkl} E_{kl} \quad \Leftrightarrow \quad E_{ij} = K_{ijkl} T_{kl} \tag{7.6.16}$$

the number of independent stiffnesses S_{ijkl} or compliances K_{ijkl} is at most 21.

In Sect. 7.2.2 the following expression was developed for the elastic energy per unit volume for an isotropic, linearly elastic material, i.e. a *Hookean material*:

$$\phi = \frac{1}{2}\mathbf{T} : \mathbf{E} = \mu\, \mathbf{E} : \mathbf{E} + \frac{1}{2}\left(\kappa - \frac{2}{3}\mu\right)(\mathrm{tr}\,\mathbf{E})^2 \tag{7.6.17}$$

Expressed in terms of stresses we find:

$$\phi = \frac{1}{2}\mathbf{T} : \mathbf{E} = \frac{1}{4\mu}\left[\mathbf{T} : \mathbf{T} - \frac{\nu}{1+\nu}(\mathrm{tr}\,\mathbf{T})^2\right] \tag{7.6.18}$$

The complementary energy per unit volume for a Hookean material becomes:

$$\phi_c = \mathbf{T} : \mathbf{E} - \phi = \phi \tag{7.6.19}$$

The Hookean material model is thus hyperelastic, and elastic energy and complementary energy are equal. In Sect. 7.8 it will be shown that this is true for all linearly hyperelastic materials.

It is reasonable to require that the elastic energy always is positive when $\mathbf{E} \neq 0$, i.e. the elastic energy per unit volume ϕ must be a *positive definite scalar-valued function* of the strain tensor $\mathbf{E}$. It is given as Problem 7.21 to show that this requirement implies the following conditions for the elasticities of Hookean materials:

$$E \equiv \eta > 0\,, \quad G \equiv \mu > 0\,, \quad \kappa > 0 \quad \Rightarrow \quad -1 < \nu \leq 0.5 \tag{7.6.20}$$

In the expression (7.2.15) we suggested that the Poisson's ratio ν should be expected to be a number between 0 and 0.5. Negative values of ν appear to be very unrealistic, and no real materials have been found with negative Poisson's ratio.

7.6.2 The Basic Equations of Hyperelasticity

The primary objective of the theory of elasticity is to provide methods for calculating stresses, strains, and displacements in elastic bodies subjected to body forces and prescribed boundary conditions for contact forces and/or displacements on the surface of the bodies. The basic equations of the theory are:

The Cauchy equations of motion:

$$\operatorname{div}\mathbf{T}+\rho\,\mathbf{b}=\rho\,\ddot{\mathbf{u}} \quad \Leftrightarrow \quad T_{ik,k}+\rho\,b_i=\rho\,\ddot{u}_i \tag{7.6.21}$$

Hooke's law for isotropic, linearly elastic materials:

$$\mathbf{T}=2\mu\left[\mathbf{E}+\frac{\nu}{1-2\nu}(\operatorname{tr}\mathbf{E})\,\mathbf{1}\right] \quad \Leftrightarrow \quad T_{ik}=2\mu\left[E_{ik}+\frac{\nu}{1-2\nu}E_{jj}\,\delta_{ik}\right] \tag{7.6.22}$$

$$\mathbf{E}=\frac{1}{2\mu}\left[\mathbf{T}-\frac{\nu}{1+\nu}(\operatorname{tr}\mathbf{T})\,\mathbf{1}\right] \quad \Leftrightarrow \quad E_{ik}=\frac{1}{2\mu}\left[T_{ik}-\frac{\nu}{1+\nu}T_{jj}\,\delta_{ik}\right] \tag{7.6.23}$$

The strain-displacement relations:

$$\mathbf{E}=\frac{1}{2}\left(\mathbf{H}+\mathbf{H}^T\right) \quad \Leftrightarrow \quad E_{ik}=\frac{1}{2}\left(u_{i,k}+u_{k,i}\right) \tag{7.6.24}$$

The component form of these equations applies only to Cartesian coordinate systems.

If temperature effects have to be included, Hooke's law (7.6.22) should be replaced by the Duhamel-Neumann law (7.5.4). Equations (7.6.21, 7.6.22, 7.6.23, 7.6.24) represent 15 equations for the 15 unknown functions T_{ik}, E_{ik}, and u_i. The boundary conditions are expressed by the contact forces **t** and the displacements **u** on the surface A of the body. The part of the surface A on which **t** is prescribed, is denoted A_σ. On the rest of the surface, denoted A_u, we assume that the displacement **u** is prescribed. For static problems ($\ddot{\mathbf{u}}=\mathbf{0}$) the boundary conditions are:

$$\mathbf{t}=\mathbf{T}\cdot\mathbf{n}=\mathbf{t}^* \text{ on } A_\sigma, \quad \mathbf{u}=\mathbf{u}^* \text{ on } A_u \tag{7.6.25}$$

$\mathbf{t}^*$ and $\mathbf{u}^*$ are the prescribed functions, and **n** is the unit normal vector on A_σ. We may have cases where the boundary conditions on parts of A are given as combinations of prescribed components of the contact force **t** and displacement **u**. For dynamic problems conditions with respect to time must be added, for instance as initial conditions on the displacement field $\mathbf{u}(\mathbf{r},t)$:

$$\mathbf{u}=\mathbf{u}^{\#}(\mathbf{r},0) \text{ and } \dot{\mathbf{u}}=\dot{\mathbf{u}}^{\#}(\mathbf{r},0) \text{ in } V \tag{7.6.26}$$

V denotes the volume of the body. $\mathbf{u}^{\#}\,(\mathbf{r},0)$ and $\dot{\mathbf{u}}^{\#}\,(\mathbf{r},0)$ are prescribed functions.

In analytical solutions of problems in the theory of elasticity we may use *Saint-Venant's semi-inverse method*. In this method the stresses, displacements and strains are to a certain extent assumed and then completely determined by the basic equations and the boundary conditions. The assumptions are based on reasonable physical considerations, usually related to the state of deformations. Examples on such assumptions are provided by the deformation hypotheses in the elementary beam theory and in the theory of torsion of cylindrical bars. The semi-inverse method may be employed for most problems in continuum mechanics.

It is convenient to transform the basic equations in accordance to which unknown functions we like to choose as primary unknown variables in a problem.

The Displacement Vector u as Primary Unknown Variable

The strain-displacement relation (7.6.24) is introduced in Hooke's law (7.6.22), and the result is:

$$T_{ij} = \mu \left[u_{i,j} + u_{j,i} + \frac{2\nu}{1-2\nu} u_{k,k}\, \delta_{ij} \right] \tag{7.6.27}$$

When these expressions for the stresses are substituted into the Cauchy equations of motion (7.6.21), we obtain the equations of motion in terms of displacements.

$$\nabla^2 \mathbf{u} + \frac{1}{1-2\nu} \nabla (\nabla \cdot \mathbf{u}) + \frac{\rho}{\mu} (\mathbf{b} - \ddot{\mathbf{u}}) = \mathbf{0} \quad \Leftrightarrow$$
$$u_{i,kk} + \frac{1}{1-2\nu} u_{k,ki} + \frac{\rho}{\mu} (b_i - \ddot{u}_i) = 0 \tag{7.6.28}$$

These three equations are the *Navier equations* in the general three-dimensional case. When the three u_i – functions are found from the Navier equations, the stresses and the strains may be computed from (7.6.27) and (7.6.24) respectively. The boundary conditions (7.6.25) and the initial conditions (7.6.26) complete the solution to the problem.

In the case of symmetry with respect to a point O the displacement vector $\mathbf{u}$ may be expressed by:

$$\mathbf{u} = u(r)\, \mathbf{e}_r = u_i \mathbf{e}_i \,, \quad u_i = u(r) \frac{x_i}{r} \tag{7.6.29}$$

$u(r)$ is the displacement radially from the symmetry point $O, \mathbf{e}_r$ is the unit vector in the radial direction, and u_i are the displacement components in a Cartesian coordinate system Ox. The acceleration vector is now:

$$\ddot{\mathbf{u}} = \ddot{u}(r)\, \mathbf{e}_r = \ddot{u}_i \mathbf{e}_i \,, \quad \ddot{u}_i = \ddot{u}(r) \frac{x_i}{r} \tag{7.6.30}$$

The body force will be expressed by a radial component related to the three Cartesian components:

$$\mathbf{b} = b(r)\, \mathbf{e}_r = b_i \mathbf{e}_i \,, \quad b_i = b(r) \frac{x_i}{r} \tag{7.6.31}$$

Referring to spherical coordinates, see Sect. 5.3.3 and the formulas (7.3.17), we find the longitudinal strains:

$$\varepsilon_r = \frac{du}{dr} \,, \quad \varepsilon_\theta = \varepsilon_\phi = \frac{u}{r} \tag{7.6.32}$$

and the volumetric strain:

$$\varepsilon_v = \mathrm{tr}\, \mathbf{E} = u_{i,i} = \varepsilon_r + \varepsilon_\theta + \varepsilon_\phi = \frac{du}{dr} + 2\frac{u}{r} \tag{7.6.33}$$

The displacement field (7.6.29) implies that the radial direction and any direction normal to the radial direction are principal directions of strain. Material line elements in these directions do not rotate due to the deformation, which means that the rotation tensor for small deformations is zero:

$$\tilde{R}_{ik} = \frac{1}{2}\left(u_{i,k} - u_{k,i}\right) = 0 \quad \Rightarrow \quad u_{i,k} = u_{k,i} \quad \Rightarrow$$
$$u_{i,kk} = u_{k,ik} = u_{k,ki} = (\varepsilon_v)_{,i} \tag{7.6.34}$$

Using the result:

$$r^2 = x_i x_i \quad \Rightarrow \quad \frac{\partial r}{\partial x_i} = \frac{x_i}{r} \tag{7.6.35}$$

we can write:

$$(\varepsilon_v)_{,i} = \frac{d\varepsilon_v}{dr}\frac{x_i}{r} \tag{7.6.36}$$

The results (7.6.30, 31, 34, 36) are now used to reduce the Navier equations (7.6.28) to one Navier equation for point symmetric deformation:

$$\frac{d}{dr}\left[\frac{du}{dr} + 2\frac{u}{r}\right] + \frac{1-2\nu}{1-\nu}\frac{\rho}{2G}(b - \ddot{u}) = 0 \tag{7.6.37}$$

Symmetry implies that the state of stress in any particle is expressed by the radial stress σ_r and the stress σ_ϕ on meridian planes. Hooke's law (7.6.22) then gives:

$$\sigma_r = \frac{2G}{1-2\nu}\left[(1-\nu)\frac{du}{dr} + 2\nu\frac{u}{r}\right], \quad \sigma_\phi = \frac{2G}{1-2\nu}\left[\nu\frac{du}{dr} + \frac{u}{r}\right] \tag{7.6.38}$$

Example 7.15. Thick-Walled Spherical Shell with Internal Pressure

A spherical shell with internal radius a and external radius b is subjected to an internal pressure p as shown in Fig. 7.6.2. We want to determine the state of stress $\sigma_r(r)$ and $\sigma_\phi(r)$ and the radial displacement $u(r)$.

The boundary conditions are:

$$\sigma_r(a) = -p\,, \quad \sigma_r(b) = 0 \tag{7.6.39}$$

The Navier equation (7.6.37) in this case becomes:

$$\frac{d}{dr}\left[\frac{du}{dr} + 2\frac{u}{r}\right] = 0 \tag{7.6.40}$$

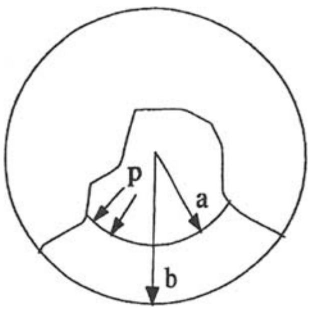


Fig. 7.6.2 Spherical thick-walled shell with internal pressure p

The general solution of this equation is:

$$u(r) = C_1\, r + \frac{C_2}{r^2} \tag{7.6.41}$$

The constants of integration C_1 and C_2 will be determined from the boundary conditions (7.6.39), but first we have to obtain expressions for the stresses using the formulas (7.6.38):

$$\begin{aligned}\sigma_r(r) &= \frac{2G}{1-2\nu}\left[(1+\nu)\,C_1 - 2\,(1-2\nu)\,\frac{C_2}{r^3}\right] \\ \sigma_\phi(r) &= \frac{2G}{1-2\nu}\left[(1+\nu)\,C_1 + (1-2\nu)\,\frac{C_2}{r^3}\right]\end{aligned} \tag{7.6.42}$$

When these stresses are substituted into the boundary conditions (7.6.39), we obtain two equations for the unknown constants of integration C_1 and C_2. The solution of the two equations is:

$$C_1 = \frac{1-2\nu}{1+\nu}\,\frac{(a/b)^3}{1-(a/b)^3}\,\frac{p}{2G}\,, \quad C_2 = \frac{a^3}{1-(a/b)^3}\,\frac{p}{4G}$$

The complete solution to the problem is then:

$$\begin{aligned}u(r) &= \frac{b\,(a/b)^3}{1-(a/b)^3}\left[\frac{1-2\nu}{1+\nu}\,\frac{r}{b} + \frac{1}{2}\left(\frac{b}{r}\right)^2\right]\frac{p}{2G} \\ \sigma_r(r) &= -\frac{1}{1-(a/b)^3}\left[\left(\frac{a}{r}\right)^3 - \left(\frac{a}{b}\right)^3\right]p \quad \leq 0\,, \\ \sigma_\phi(r) &= \frac{1}{1-(a/b)^3}\left[\frac{1}{2}\left(\frac{a}{r}\right)^3 + \left(\frac{a}{b}\right)^3\right]p \quad \geq 0\end{aligned}$$

The maximum compressive and tensile stresses occur at the inner surface of the shell:

$$\sigma_{\min} = \sigma_r(a) = -p\,, \quad \sigma_{\max} = \sigma_\phi(a) = \frac{\frac{1}{2} + \left(\frac{a}{b}\right)^3}{1-(a/b)^3}\,p$$

If the thickness of the shell $t = b - a$ is very small, such that $a \approx b$, we find that:

$$\sigma_\phi = \frac{a}{2t}\,p \gg \sigma_r\,, \quad t = b - a$$

Confer equation (3.3.16) for a thin–walled shell in Example 3.6.

An interesting special case is obtained if we let the external radius b become very large. Then we get at the inner surface of the shell:

$$\sigma_r(a) = -p\,, \quad \sigma_\phi(a) = \frac{p}{2}$$

which are the stresses in the spherical boundary surface to a spherical cavity in a large elastic body.

In this example it would have been natural to apply the basic equations expressed in spherical coordinate. Due to symmetry we could also have derived the Navier equation (7.6.37) indirectly from the equation of motion of an infinitesimal shell element. But the main purpose of the example has been to demonstrate the application of the basic equations as they have been presented above, and furthermore to illustrate the semi-inverse method, which in this case consists of starting with the displacement (7.6.29).

The Stresses T_{ij} as Primary Unknown Variables

This choice is only natural for static problems: $\ddot{\mathbf{u}} = \mathbf{0}$, or for problems where the acceleration $\ddot{\mathbf{u}}$ is prescribed. In the latter cases we may replace the body force $\mathbf{b}$ by a corrected body force $(\mathbf{b} - \ddot{\mathbf{u}})$, where - $\ddot{\mathbf{u}}$ represents an *extraordinary body force*, also called an *inertia force*. In what follows we assume that $\ddot{\mathbf{u}} = \mathbf{0}$. The Cauchy equations of motion (7.6.21) are three equations for six unknown stresses. Let us assume that we have found a solution for the stresses T_{ij} from of these equations. The strains E_{ij} may then be determined from the Hooke's law (7.6.23) and the displacements u_i should follow from (7.6.24). But it is not certain that the six functions for the strains are compatible and therefore provide unique displacement functions. The six (7.6.24) represent six equations for the three unknown displacement functions u_i. The compatibility equations (5.3.41) must also be satisfied. Since we have chosen the stresses as the primary unknowns, it is natural to express the compatibility equations in terms of stresses. Hooke's law (7.6.23) is used to express the coordinate strains E_{ij} as functions of the coordinate stresses T_{ij}. These relations are substituted into the compatibility equations (5.3.40) and the Cauchy equations (7.6.21) are used to simplify in the process. The result is the *compatibility equations expressed in stresses*, also called the *Beltrami-Michell equations*:

$$\begin{aligned}
&\nabla^2\mathbf{T} + \frac{1}{1+\nu}\nabla\left(\nabla \mathrm{tr}\,\mathbf{T}\right) + \frac{\nu}{1-\nu}\left[\nabla\cdot\ (\rho\,\mathbf{b})\right]\mathbf{1} + \nabla(\rho\,\mathbf{b}) + \left(\nabla(\rho\,\mathbf{b})\right)^T = \mathbf{0} \quad \Leftrightarrow \\
&T_{ij,kk} + \frac{1}{1+\nu}T_{kk,ij} + \frac{\nu}{1-\nu}(\rho\, b_k)_{,k}\delta_{ij} + (\rho\, b_i)_{,j} + (\rho\, b_j)_{,i} = 0
\end{aligned} \tag{7.6.43}$$

In principle the problem may now be solved from the three (7.6.21) and the three (7.6.43) with the boundary conditions (7.6.25). A method to simplify the problem is to introduce three stress functions, which satisfy identically the Cauchy equations (7.6.21), and which reduce the compatibility equations (7.6.43) to three equations. Very few three-dimensional problems have been solved by this solution procedure. For two-dimensional problems the (7.6.43) are replaced by (7.3.45) or (7.3.46), and Airy's stress function is introduced to simplify the solution procedure.

7.6.3 The Uniqueness Theorem

The uniqueness theorem. The basic equations (7.6.21, 7.6.22, 7.6.23, 7.6.24) with the boundary conditions (7.6.25) and the initial conditions (7.6.26) have only one solution in the general dynamic case. In a static case the solution is unique if the displacement boundary $A_u \neq 0$, and unique except from an undetermined rigid-body displacement if $A_u = 0$.

The theorem insures that an obtained solution to an elasticity problem is *the only solution* of that problem. The proof of the theorem depends on the elastic energy ϕ per unit volume being a positive definite quantity.

Proof. Let us assume that the following two solutions satisfy the basic equations (7.6.21, 7.6.22, 7.6.23, 7.6.24):

$$\mathbf{u}^{(1)}(\mathbf{r},t) \text{ and } \mathbf{T}^{(1)}(\mathbf{r},t)\,, \quad \mathbf{u}^{(2)}(\mathbf{r},t) \text{ and } \mathbf{T}^{(2)}(\mathbf{r},t) \tag{7.6.44}$$

Because the basic equations are linear, the displacement state and the stress state:

$$\mathbf{u} = \mathbf{u}^{(1)} - \mathbf{u}^{(2)}\,, \quad \mathbf{T} = \mathbf{T}^{(1)} - \mathbf{T}^{(2)} \tag{7.6.45}$$

will also satisfy equation (7.6.21), now with $\mathbf{b} = \mathbf{0}$, i.e. such that: $\mathrm{div}\mathbf{T} = \rho\ddot{\mathbf{u}}$. The displacement $\mathbf{u}$ will also satisfy the boundary conditions and the initial conditions:

$$\mathbf{u}(\mathbf{r},t) = \mathbf{0} \text{ on } A_u \quad \Rightarrow \quad \dot{\mathbf{u}}(\mathbf{r},t) = \mathbf{0} \text{ on } A_u \tag{7.6.46}$$

$$\mathbf{u}(\mathbf{r},0) = \mathbf{0} \text{ and } \dot{\mathbf{u}}(\mathbf{r},0) = \mathbf{0} \text{ in } V \tag{7.6.47}$$

The stress tensor $\mathbf{T}$ will satisfy:

$$\mathbf{t}(\mathbf{r},t) = \mathbf{T} \cdot \mathbf{n} = \mathbf{0} \text{ on } A_\sigma \tag{7.6.48}$$

From the conditions (7.6.46) and (7.6.48) it follows that:

$$\mathbf{t} \cdot \dot{\mathbf{u}} = 0 \text{ on } A \tag{7.6.49}$$

The equation of balance of mechanical energy (6.1.12) gives for the solution (7.6.45), with no body force $\mathbf{b} = \mathbf{0}$ and the boundary condition (7.6.49):

$$\dot{K} + \dot{\Phi} = 0 \tag{7.6.50}$$

where the stress power has been expressed by the total elastic energy Φ, i.e. $P^d = \dot{\Phi}$. The integration of equation (7.6.50) leads to:

$$K + \Phi = C$$

C is a constant of integration, which due to the condition (7.6.47) is equal to zero. In general $K \geq 0$ and $\Phi \geq 0$. Hence we may conclude that:

$$K(t) = \Phi(t) = 0 \tag{7.6.51}$$

The scalars K and Φ are expressed as integrals in which the integrands are positive definite forms with respect to the variables $\mathbf{v} \equiv \dot{\mathbf{u}}$ and $\mathbf{E}$, respectively. The result (7.6.51) therefore implies that:

$$\dot{\mathbf{u}}(\mathbf{r},t) = \mathbf{0} \text{ and } \mathbf{E}(\mathbf{r},t) = \mathbf{0} \quad \text{in } V \tag{7.6.52}$$

The integration of $\dot{\mathbf{u}}$ in (7.6.52) with respect to time and with equation $(7.6.47)_1$ as initial condition yields:

$$\mathbf{u}(\mathbf{r},t) = \mathbf{0} \quad \text{in } V \tag{7.6.53}$$

It now follows from equation (7.6.51) that elastic energy per unit volume is zero, and then from the equations (7.6.52) and (7.6.22) that:

$$\mathbf{T}(\mathbf{r},t) = \mathbf{0} \quad \text{in } V \tag{7.6.54}$$

The results (7.6.53, 7.6.54) show that the two solutions assumed in equation (7.6.44) are identical. This proves the uniqueness theorem.

In a static case time does not enter any of the relevant equations. The displacement $\mathbf{u}$ in equation (7.6.45) must satisfy the boundary condition:

$$\mathbf{u}(\mathbf{r}) = \mathbf{0} \quad \text{on } A_u \tag{7.6.55}$$

and result in stresses satisfying the equilibrium equation and the boundary condition:

$$\operatorname{div}\mathbf{T} = \mathbf{0} \text{ in } V, \quad \mathbf{t}(\mathbf{r}) = \mathbf{T} \cdot \mathbf{n} = \mathbf{0} \text{ on } A_\sigma \tag{7.6.56}$$

We may now write:

$$\mathbf{t} \cdot \mathbf{u} = \mathbf{0} \text{ on } A \quad \Rightarrow \quad \int_A \mathbf{t} \cdot \mathbf{u}\, dA = \int_A \mathbf{u} \cdot \mathbf{T} \cdot \mathbf{n}\, dA = 0$$

which by Gauss' theorem C.3 yields:

$$\int_A \mathbf{u} \cdot \mathbf{T} \cdot \mathbf{n}\, dA = \int_A u_i T_{ij} n_j\, dA = \int_V (u_i T_{ij})_{,j}\, dV = 0 \quad \Rightarrow$$

$$\int_V u_i T_{ij,j}\, dV + \int_V u_{i,j} T_{ij}\, dV = 0 \tag{7.6.57}$$

The first integral vanishes due to the equilibrium equation (7.6.56). The integrand in the second integral may according to equation (7.6.17) be changed to:

$$u_{i,j}T_{ij} = \frac{1}{2}(u_{i,j}T_{ij} + u_{j,i}T_{ji}) = T_{ij}E_{ij} = \mathbf{T}:\mathbf{E} = 2\phi$$

Then equation (7.6.57) has been reduced to:

$$2\int_V \phi\, dV = 0 \tag{7.6.58}$$

In general $\phi \geq 0$, which by equation (7.6.58) implies that $\phi = 0$ in V, which again shows that:

$$\mathbf{E} = \mathbf{0} \ \text{ in } V \quad \Rightarrow \quad \mathbf{T} = \mathbf{0} \text{ in } V \tag{7.6.59}$$

Thus the two displacement fields $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ result in the same strains and stresses. The difference between the two displacement fields may only be a small rigid-body displacement, which does not result in strains and changes in the stresses. If the displacement boundary condition applies, i.e. $A_u \neq 0$, the rigid-body displacement must vanish. These arguments prove the uniqueness theorem in a static case.

7.7 Stress Waves in Elastic Materials

In this section we discuss some relatively simple but fundamental aspects of propagation of *stress pulses* or *stress waves* in isotropic, linearly elastic materials. We may also consider these pulses or waves as displacement, deformation, or strain pulses or waves, according to which of these quantities we are interested in. In the general presentation we do not distinguish between pulses and waves, and since the materials are assumed to be elastic, we use the common name *elastic waves*.

Sound in gasses and liquids is elastic waves and will be discussed in Sect. 8.3.3 in the chapter on Fluid Mechanics.

Our analysis starts with longitudinal elastic waves in a bar. Then we treat plane waves in an infinite elastic body. A general discussion of elastic waves in bodies of infinite extent concludes the chapter.

7.7.1 Longitudinal Waves in Cylindrical Bars

Figure 7.7.1 shows an elastic bar of length L and cross-sectional area A. The material has density ρ and modulus of elasticity E. The right end of the bar, at $x = L$, is subjected to an axial force $F(t)$. The force may be given as an impact representing an impulsive force $\hat{F}$ or may be oscillatory, for instance of the form:

$$F(t) = F_o \sin \omega t \tag{7.7.1}$$

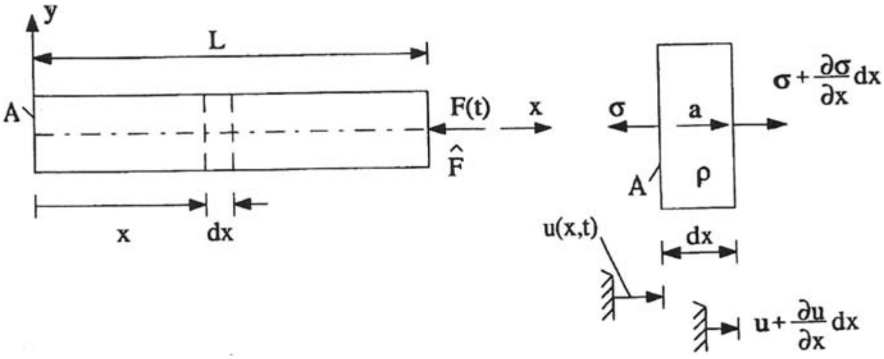


Fig. 7.7.1 Cylindrical bar subjected to a force F(t) or an impulsive force $\hat{F}$ at the end $x = L$

The normal stresses over the cross-section at the end of the bar, $x = L$, will propagate through the rod as what we may call a wave $\sigma(x,t)$ towards the other end of a bar, at $x = 0$. At the end $x = 0$ the stress wave is reflected as a new, reflected wave $\sigma_r(x,t)$. The reflected wave propagates in the positive x-direction towards the end at $x = L$. How the incoming wave $\sigma(x,t)$ is reflected at the end $x = 0$ depends on the boundary condition, i.e. whether the end is free, as indicated in Fig. 7.7.1, or the bar is attached to another body.

As a basis for a simplified analysis of the stress wave propagation problem we make the following assumptions:

1. The cross sections of the bar moves as planes. The displacement in the axial direction is given by:
$$u = u(x,t) \tag{7.7.2}$$
2. The state of stress is uniaxial and given by the normal stress over the cross section of the bar:
$$\sigma = \sigma(x,t) \tag{7.7.3}$$

In addition we assume that the material is linearly elastic with a modulus of elasticity E, and that the deformations are small. Thus we may state:

$$\sigma = E\varepsilon, \quad \varepsilon = \frac{\partial u}{\partial x} \quad \Rightarrow \quad \sigma = E\,\frac{\partial u}{\partial x} \tag{7.7.4}$$

Due to the *Poisson effect*, i.e. the transverse strain due to the Poisson ratio $\nu > 0$, the axial motion will also create motion in the directions normal to the axis of the bar. This motion will result in shear and normal stresses on surfaces parallel to the axis of the bar, and also result in warped cross-sections. We shall discard such secondary effects in the following development of the theory and comment on them at the end of the presentation.

Now we consider a short element of the bar of length dx and mass $dm = \rho \cdot (A \cdot dx)$. The acceleration of the element is $\ddot{u}$. The element is subjected to normal stresses on the two cross-sections. The1. axiom of Euler is applied to the element:

$$\mathbf{f} = m\ddot{\mathbf{u}} \quad \Rightarrow \quad \left[\frac{\partial \sigma}{\partial x} dx\right] \cdot A = [\rho \cdot (A\,dx)] \cdot \ddot{u} \quad \Rightarrow \quad \frac{\partial \sigma}{\partial x} = \rho\,\ddot{u}$$

which by using equation (7.7.4) is transformed to:

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad c = \sqrt{\frac{E}{\rho}} \tag{7.7.5}$$

The partial differential equation (7.7.5) is called a *one-dimensional wave equation*, and the parameter c is called *wave velocity*. The reason for this will be demonstrated below.

The general solution of the wave equation (7.7.5) is given as:

$$u(x,t) = f(ct + x) + g(ct - x) \tag{7.7.6}$$

$f(\alpha)$ and $g(\alpha)$ are two arbitrary functions of one variable α. The solution $u(x,t) = f(ct+x)$ may be interpreted as shown in Fig. 7.7.2. The graph of u in a xu-diagram has constant shape independent of time. The figure shows the graphs at the times t_1 and $t_2 > t_1$. The distance $x_1 - x_2$ between two corresponding points on the graphs is determined by the condition:

$$f(ct_1 + x_1) = f(ct_2 + x_2) \quad \Rightarrow \quad ct_1 + x_1 = ct_2 + x_2 \quad \Rightarrow \quad x_1 - x_2 = c\,(t_2 - t_1)$$

From this result it follows that the displacement $u(x,t) = f(ct + x)$ propagates in the negative x-direction with the velocity c. We therefore call $u(x,t) = f(ct + x)$ a *plane longitudinal displacement wave with the wave velocity* c. We introduce the notation:

$$f'(\alpha) \equiv \frac{df}{d\alpha} \tag{7.7.7}$$

The corresponding strain wave and stress wave are now:

$$\varepsilon(x,t) = \frac{\partial u}{\partial x} = f'(ct + x), \quad \sigma(x,t) = E\varepsilon = E f'(ct + x) \tag{7.7.8}$$

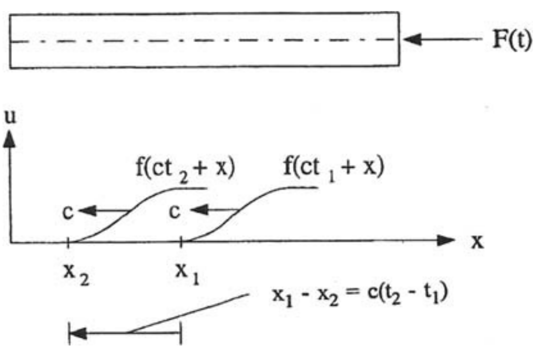


Fig. 7.7.2 Displacement wave $u = f(ct + x)$ in the negative x-direction

The function $f(\alpha)$ is determined by the force $F(t)$ applied at the end $x = L$ of the bar.

$$F(t) = A\sigma(L,t) = AE\varepsilon(L,t) = AEf'(ct+L) \quad \Rightarrow \quad f'(\alpha) = \frac{1}{AE}F([\alpha - L]/c)$$

$$\Rightarrow \quad \sigma(x,t) = E\varepsilon(x,t) = Ef'(ct+x) = \frac{1}{A}F(t+[x-L]/c) \tag{7.7.9}$$

For the harmonic force function (7.7.1), we obtain:

$$\sigma(x,t) = Ef'(ct+x) = \frac{F_o}{A}\sin\left(\left[\frac{2\pi c}{\lambda}\right][ct+x-L]\right) \tag{7.7.10}$$

λ is the *wave length* and defined by:

$$\lambda = \frac{2\pi c}{\omega} \tag{7.7.11}$$

If we choose the initial condition $u(L,0) = 0$, the displacement wave becomes:

$$u(x,t) = f(ct+x) = \frac{F_o}{EA}\frac{\lambda}{2\pi}\left[1 - \cos\left(\left[\frac{2\pi}{\lambda}\right][ct+x-L]\right)\right] \tag{7.7.12}$$

The particle velocity at the cross section x is:

$$v(x,t) = \frac{\partial u}{\partial t} = cf'(ct+x) = \frac{c}{E}\,\sigma(x,t) \tag{7.7.13}$$

The part $u(x,t) = g(ct-x)$ of the general solution (7.7.6) of the wave equation (7.7.5) represents a plane longitudinal displacement wave propagating in the positive x-direction with the wave velocity c given by the formula in (7.7.5). This wave may be established by a reflection at the end $x = 0$ of the incoming displacement wave $u(x,t) = f(ct+x)$. We shall consider two different types of end conditions at $x = 0$:

1) Reflection from a free end at $x = 0$, and
2) Reflexion from a fixed end at $x = 0$.

Reflection from a Free End at $x = 0$

When a displacement wave $u(x,t) = f(ct+x)$ moving in the negative x-direction reaches the free end at $x = 0$, the reflection $u(x,t) = g(ct-x)$ is a displacement wave in the positive x-direction. The total displacement will be the sum of the two waves:

$$u(x,t) = f(ct+x) + g(ct-x) \tag{7.7.14}$$

At the free end, $x = 0$, the resulting stress must vanish at all times:

$$\sigma(0,t) = 0 \quad \Rightarrow \quad \varepsilon(0,t) = \frac{\partial u}{\partial x} = 0 \quad \Rightarrow \quad f'(ct) - g'(ct) = 0 \text{ at all times } t \tag{7.7.15}$$

This result implies that the functions $f(\alpha)$ and $g(\alpha)$ must be equal except for a constant:

$$g(\alpha) = f(\alpha) + C$$

In order to have the natural initial condition that $u(0,0) = 0$, we subtract a constant $f(0)$ from both $f(\alpha)$ and $g(\alpha)$ in equation (7.7.14), and set:

$$u(x,t) = f(ct + x) + f(ct - x) - 2f(0) \tag{7.7.16}$$

Figure 7.7.3 shows the incoming wave $f(ct + x)$, the reflected wave $f(ct - x)$, and the total displacement $u(x,t)$ from equation (7.7.16). In the xu-plane in Fig. 7.7.3 the graphs of the functions $f(ct + x)$ and $f(ct - x)$ are mirror images of each other with respect to the u-axis. The displacement (7.7.16) gives the *axial strain* and *axial stress*:

$$\varepsilon(x,t) = \frac{\partial u}{\partial x} = f'(ct + x) - f'(ct - x) \tag{7.7.17}$$

$$\sigma(x,t) = E\varepsilon = E f'(ct + x) - E f'(ct - x) \tag{7.7.18}$$

The development of the stress function $\sigma(x,t)$ is illustrated in Fig. 7.7.3, where the stress wave $E\ f(ct + x)$ is assumed to represent tension, i.e. a *tension wave*. The reflected stress wave $E\ f(ct - x)$ will then represent compressive stress, that is a *compression wave*. In general:

A tension/compression wave is reflected from a free end as a compression/tension wave.

A heavy impulsive blow to one end of a bar or a detonation of an explosive at the end will initiate a wave of high compressive stress. The reflection of this wave

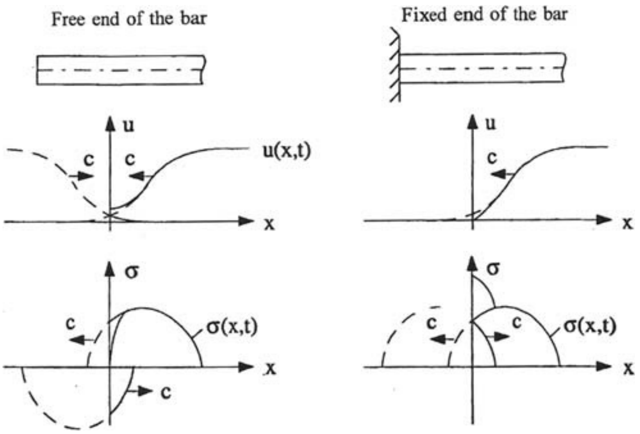


Fig. 7.7.3 Reflection of displacement wave and stress wave from a free and from a fixed end

is a tension wave. Interference of the two waves may result in a total stress of high tension values a short distance from the free end. A bar of a material with a tensile strength that is lower than the compressive strength may therefore experience a tensile fracture a short distance from the free end. We shall return to this phenomenon in Sect. 7.7.7.

Reflection from a Fixed End at $x = 0$

When a displacement wave $f(ct + x)$ reaches a fixed end at $x = 0$, the reflected displacement wave $g(ct - x)$ will be determined by the condition that the combined displacement $u(x,t)$ must be zero at the end:

$$u(0,t) = f(ct) + g(ct) = 0 \text{ at any time } t \tag{7.7.19}$$

This implies that $g(\alpha) = -f(\alpha)$, and hence the total displacement is:

$$u(x,t) = f(ct + x) - f(ct - x) \tag{7.7.20}$$

Figure 7.7.3 illustrates the incoming wave, the reflected wave, and the total displacement. The resulting *axial strain* and *axial stress* are:

$$\varepsilon(x,t) = \frac{\partial u}{\partial x} = f'(ct + x) + f'(ct - x) \tag{7.7.21}$$

$$\sigma(x,t) = E\varepsilon = E f'(ct + x) + E f'(ct - x) \tag{7.7.22}$$

From the expression (7.7.21), and as illustrated in Fig. 7.7.3, it is seen that a stress wave representing tension/compression is reflected as a tensile/compressive stress wave. In general:

A tension/compression wave is reflected from a fixed end as a tension/compression wave.

An interesting consequence of this result will be demonstrated in Sect. 7.7.2.

Critical Comments to the Simplified Theory Presented Above

It was mentioned in the introduction to the theory developed above that the Poisson effect will influence the wave propagation. The assumptions (7.7.2) and (7.7.3), on which the simplified theory is based, neglect the *Poisson effect*. H. Kolsky discusses in "Stress waves in solids" [23] the exact theory for displacement waves in a circular cylindrical rod. The displacement wave is represented by a trigonometric function:

$$u(x,t) = u_o \sin\left(\frac{2\pi}{\lambda}[ct - x]\right) \tag{7.7.23}$$

λ is the wave length. The diameter of the rod is denoted by $2r$. If the ratio $\lambda/r > 10$, the simplified theory presented above is sufficiently accurate, and the wave velocity c is given by the formula $(7.7.5)_2$ and thus independent of the wave length. For $1 < \lambda/r < 10$ the exact theory has to be considered, according to which the wave velocity decreases with the wave length. For wave lengths less than the radius of the rod, $\lambda/r < 1$, the wave velocity again becomes practically constant, but now approximately equal to c_r, which is the wave velocity of the so-called *Rayleigh waves*. The Rayleigh waves are surface waves and are discussed in Sect. 7.7.8. These waves represent motion in the vicinity of free surfaces, and with one displacement component in the direction of the wave propagation and one component in the direction normal to the surface. For steel with $\nu = 0.3,\ E = 210\,\mathrm{GPa}$, and $\rho = 7.83 \cdot 10^3\,\mathrm{kg/m^3}$ we will find:

$$c = \sqrt{\frac{E}{\rho}} = 5179\ \mathrm{m/s},\ c_r = 2952\ \mathrm{m/s}$$

In order to have a numerical example with some relevant numbers we shall consider sound waves with a frequency f between 10 and 10 000 Hz propagating down a steel bar. For steel the velocity of propagation is $c = 5179\,\mathrm{m/sec}$. From the relationship $f = c/\lambda$ we find that the wave length λ lies between 0.52 m and 520m: $0.52\,\mathrm{m} < \lambda < 520\,\mathrm{m}$. Based on the condition that $\lambda_{\min}/r > 10$, we obtain the result $r_{\min} = 52\,\mathrm{mm}$. We may thus conclude that the simplified wave propagation theory presented above, may be used for sound waves of frequencies less than 10 000 Hz in steel bars when the bar diameter is less than 104 mm.

Due to the induced motion perpendicular to the direction of propagation and internal damping, the wave length of displacement waves will increase while the intensity decreases along the direction of propagation. The effect of internal damping will be further discussed in Sect. 9.5 on stress waves in viscoelastic materials.

7.7.2 The Hopkinson Experiment

J. Hopkinson published in 1872 the results of an experiment that showed how the interference of stress waves may lead to fracture. The experiment is discussed by G.I. Taylor in the paper: The Testing of materials at high rates of loading [47]. Figure 7.7.4 shall illustrate the experiment. A steel wire with cross sectional area A and modulus of elasticity E is fixed at one end. The axis of the wire is vertical. The lower end of the wire is attached to a plate with negligible mass, but sufficiently heavy to keep the wire straight. A body of mass m can slide freely on the wire. The body is released from rest from a height h above the plate. The wire used by Hopkinson had static fracture strength of 500 N. The weight of the body could be varied between 31 N and 182 N. The experiment showed that the minimum height $h_{\min}$ resulting in fracture, was independent of the mass of the falling body and that the location of the fracture for this height occurred near the upper fixed end.

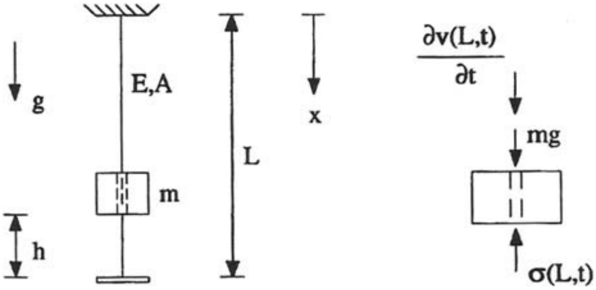


Fig. 7.7.4 The Hopkinson experiment

The Hopkinson theory is as follows. For any given height h the falling body hits the plate at a velocity v_o, which may be determined from the work and energy equation for a rigid body:

$$W = \Delta K \quad \Rightarrow \quad mg \cdot h = \frac{1}{2} m v_o^2 \quad \Rightarrow \quad v_o = \sqrt{2gh} \tag{7.7.24}$$

The velocity v_o is independent of the mass of the body. When the body hits the plate, say at time $t = 0$, the lower end of the wire is suddenly given the velocity v_o. If we denote the particle velocity of the wire at a distance x from the upper end by $v(x,t)$, then:

$$v(L,0) = v_o \tag{7.7.25}$$

From (7.7.13) and the formula for the wave velocity c in (7.7.5) it follows that the velocity wave $v(x,t)$ introduces a tensile stress wave in the wire:

$$\sigma(x,t) = \rho\, c\, v(x,t) \tag{7.7.26}$$

The velocity field $v(x,t)$ and thus the stress field $\sigma(x,t)$ may be determined from the law of motion for the body on the plate, see Fig. 7.7.4:

$$f = m\,a \quad \Rightarrow \quad m\,g - \sigma(L,t) \cdot A = m \frac{\partial v(L,t)}{\partial t} \tag{7.7.27}$$

Using equation (7.7.26), we get:

$$\frac{\partial v(L,t)}{\partial t} + \frac{\rho A c}{m} v(L,t) = g \tag{7.7.28}$$

The solution of equation (7.7.28) that satisfies the condition (7.7.25), is:

$$v(L,t) = \{ v_o \exp(-[\rho A/m]\, ct) + (mg/\rho A c)\,[1 - \exp(-[\rho A/m]\, ct)] \}\, H(t) \tag{7.7.29}$$

The function $H(t)$ is the *Heaviside unit step function*, Oliver Heaviside [1850–1925]:

$$H(t) = \begin{cases} 0 \text{ for } t \leq 0 \\ 1 \text{ for } t > 0 \end{cases} \tag{7.7.30}$$

From the results (7.7.29) and (7.7.26) we get the expression for the stress wave in the wire:

$$\begin{aligned} \sigma(x,t) = &\{\rho c v_o \exp(-[\rho A/m][ct + x - L])\} H(t + (x - L)/c) \\ &+ \{(mg/A)[1 - \exp(-[\rho A/m][ct + x - L])]\} H(t + (x - L)/c) \end{aligned} \tag{7.7.31}$$

This is a tensile wave with a front stress $\rho c v_o$. The wave reflected from the fixed upper end of the bar represents also tension. A maximum tensile stress $2\rho c v_o$ will occur at the upper fixed end. Successive reflections from both ends will superimpose and a very complex stress picture develops in the wire. Due to internal material damping and other effects discussed at the conclusion of Sect. 7.7.1, the system wire/rigid body comes to rest after a relatively short time. In the analysis of the experiments performed by J. Hopkinson a theoretical value of maximum tensile stress $4.33\rho c v_o$ occurs after the 3. reflection from the upper fixed end. B. Hopkinson (1905) repeated his father's experiments, with better equipment and a somewhat different objective. In these experiments the theoretical value of the maximum tensile stress is $2.15\rho c v_o$ and occurs after the 2. reflection from the upper end. These results are based only on the first term on the right hand side of equation (7.7.31) and reflections of this stress wave.

7.7.3 Plane Elastic Waves

We assume that a relatively small region in an infinite body of isotropic elastic material is subjected to a mechanical disturbance of some sort. The disturbed region may be considered to be a point source of a displacement field propagating into the undisturbed material as a displacement wave. Sufficiently far away from the source the displacement propagates as a plane wave. Figure 7.7.5 illustrates the situation. In the neighborhood of the x_3-axis and a distance from the source we assume the displacement field:

$$\mathbf{u} = \mathbf{u}(x_3, t) \quad \Leftrightarrow \quad u_i = u_i(x_3, t) \tag{7.7.32}$$

This field really represents three plane waves: a *longitudinal wave* $u_3(x_3, t)$ and two *transverse waves* $u_1(x_3, t)$ and $u_2(x_3, t)$. We shall see that the wave velocities are different for the two types of waves.

The equations of motion in this case are the Navier equations (7.6.28). If we neglect the body forces, the equations are reduced to:

$$u_{i,kk} + \frac{1}{1 - 2\nu} u_{k,ki} = \frac{\rho}{\mu} \ddot{u}_i \tag{7.7.33}$$

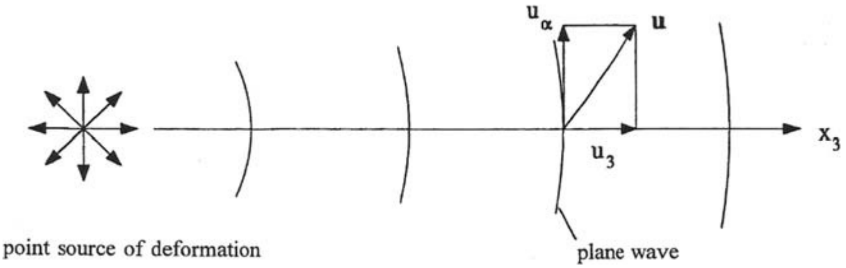


Fig. 7.7.5 Plane displacement wave from a distant point source

The displacement field (7.7.32) is substituted into the Navier equations (7.7.33), and the result is three one-dimensional wave equations:

$$c_l \frac{\partial^2 u_3}{\partial x_3^2} = \frac{\partial^2 u_3}{\partial t^2}, \quad c_l = \sqrt{\frac{2(1-\nu)}{1-2\nu}\frac{\mu}{\rho}} = \sqrt{\frac{1-\nu}{(1+\nu)(1-2\nu)}\frac{\eta}{\rho}} = \sqrt{\frac{\kappa + 4\mu/3}{\rho}} \tag{7.7.34}$$

$$c_t \frac{\partial^2 u_\alpha}{\partial {x_3}^2} = \frac{\partial^2 u_\alpha}{\partial t^2}, \quad c_t = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{1}{2(1+\nu)}\frac{\eta}{\rho}} \tag{7.7.35}$$

$\mu (= G)$ is the shear modulus, $\eta (= E)$ is the modulus of elasticity, and κ is the bulk modulus. The wave equations (7.7.34) and (7.7.35) have general solutions of the form (7.7.6).

The displacement $u_3(x_3,t)$ represents a motion in the direction of the propagation x_3, and c_l is the *wave velocity of longitudinal waves*. The displacements $u_\alpha(x_3,t)$ represent motions in the directions normal to the direction of propagation, and c_t is the *wave velocity of transverse waves*. Because $0 \leq \nu \leq 0.5$ we will find that in general:

$$c_t < c_l \tag{7.7.36}$$

For steel with $\nu = 0.3$, $E = 210\,\text{GPa}$, and $\rho = 7.83 \cdot 10^3\,\text{kg/m}^3$, we find $c_t = 3212\,\text{m/s}$, $c_l = 6009\,\text{m/s}$, and $c = 5179\,\text{m/s}$ (from 7.7.5).

The longitudinal wave implies volume changes: $\varepsilon_v = u_{3,3} = E_{33}$, and the wave is therefore called a *dilatational wave* or *volumetric wave*. Because:

$$\text{rot}\,(u_3\,\mathbf{e_3}) \equiv \text{curl}\,(u_3\,\mathbf{e_3}) \equiv \nabla \times (u_3\,\mathbf{e_3}) = \mathbf{0}$$

the displacement $u_3(x_3,t)$ is also called an *irrotational wave*. A physical consequence of an irrotational wave is that the principal directions of strain do not rotate. For the displacement field $u_3(x_3,t)$ the x_i-directions are the principal strain directions. According to Hooke's law expressed by the equations (7.6.27) the stresses are determined by:

$$T_{33} = \frac{2(1-\nu)\mu}{1-2\nu} u_{3,3}, \quad T_{11} = T_{22} = \frac{\nu}{1-\nu} T_{33} \tag{7.7.37}$$

The transverse waves $u_1(x_3,t)$ and $u_2(x_3,t)$ are isochoric, i.e. $\varepsilon_v = 0$, and are therefore also called *dilatation free waves* or *equivoluminal waves*. The stresses are according to Hooke's law (7.6.27):

$$T_{13} = \mu u_{1,3}, \quad T_{23} = \mu u_{2,3} \tag{7.7.38}$$

Because these waves also represent shear stresses on planes normal to the direction of propagation, they are also called *shear waves*. Other names are *distortional waves* and *rotational waves*.

7.7.4 Elastic Waves in an Infinite Medium

We consider a body of an isotropic, linearly elastic material. The extension of the body is so large that the influence from boundary surfaces to other media may be neglected. It is assumed that the material region bounded by a closed surface A is in motion, while the material outside A is undisturbed and at rest. The surface A will spread out into the previously undisturbed region. The motion of the surface is now called a *wave*, and the surface A is called the *wave front*. We will find that the mathematical points on A have a constant velocity in the direction normal to A. This *wave velocity c* is independent of the shape of the wave front A. A further investigation will show that A represents two surfaces: 1) a longitudinal wave front for particle motions normal to the wave front, and which represents volume changes and has the wave velocity c_l, and 2) a transverse wave front for motions parallel to the wave front, and which represents shear stresses on the wave front and rotation, and has the wave velocity c_t.

7.7.5 Seismic Waves

An earth quake initiates three elastic displacement waves. The fastest wave is a longitudinal wave, a volumetric wave, called the *primary wave*, the *P-wave*. The second fastest wave is a transverse wave, a shear wave, called the *secondary wave*, the *S-wave*. Both waves propagate from the earth quake region in all directions, and their intensities, or energy per unit area, decrease with the square of the distance from the earth quake. These two waves therefore are registered by relatively weak signal on a seismograph far away from the epicenter of the earth quake. The third and generally the strongest wave propagates along the surface of the earth and represents a combination of a Rayleigh wave and a *Love wave*.

Rayleigh waves, presented below in Sect. 7.7.8, are surface waves with displacement components in the direction of propagation and in the direction normal to the surface, the latter component being the strongest. The intensity of the Rayleigh waves will in principle decrease with the distance from the surface. The propagation velocity c_r is somewhat smaller than c_t, approximately 10% smaller, depending upon the Poisson ratio of the medium. In the neighborhood of the free surface of a homogeneous material the Rayleigh wave is the only possible surface wave. The wave is named after Lord Rayleigh, 3rd baron (John William Strutt) [1842–1919].

Love waves are surface waves, which near the surface of the earth are equally important as the Rayleigh waves, and represent displacements in the surface and in the direction normal to the direction of propagation. Love waves were introduced by A. E. H. Love [1863–1940]. This type of surface waves is also called *SH-waves* (surface - horizontal). The Love waves may be explained by considering a region of the upper surface of the earth to have different material properties than the rest of the earth surface. The velocity of the Love waves is somewhat smaller than c_t for the earth surface and lies between the transverse wave velocity c'_t for the earth crust and c_t. A condition for the existence of Love waves is that $c'_t < c_t$. For a harmonic wave the wave velocity depends on the wavelength. This means that non-harmonic waves are distorted as they travel through the material, a phenomenon called dispersion.

The surface waves must travel a longer distance than the *P*-waves and the *S*-waves, and are therefore registered somewhat later. The energy per unit area of the surface waves decreases proportionally to the travelled distance. This fact explains that the surface waves give a relatively stronger seismic signal than the primary and secondary waves.

7.7.6 Reflection of Elastic Waves

When a plane volumetric wave meets a plane free surface, it is reflected as a plane volume wave and a plane shear wave, see Fig. 7.7.6. It may be shown that angle α_2 between the surface normal and the direction of propagation of the reflected dilatational wave is equal to the angle α_1 between the surface normal and the direction of the incoming volumetric wave. For the angle β_2 of the reflected shear wave we will find:

$$\frac{\sin\beta_2}{\sin\alpha_1} = \frac{c_t}{c_l} < 1 \tag{7.7.39}$$

The intensity of the shear wave approaches zero as the angle α_1 approaches $0°$. The intensity of the reflected volumetric wave is equal to the intensity of the incoming wave when the angle $\alpha_1 = 0$, otherwise it is less. Similarly as for reflection of waves from a free end of a bar, we find that a tensile/compressive wave is reflected as a compressive/tensile wave.

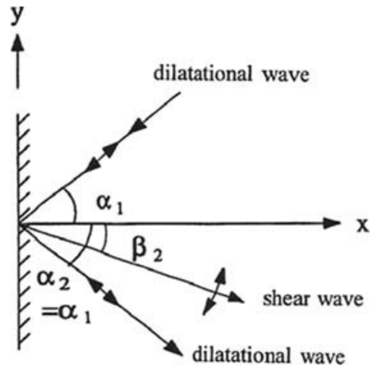


Fig. 7.7.6 Reflection of a dilatation wave from a free surface

When a plane shear wave meets a plane free surface, the displacement component parallel to the surface is reflected as a shear wave and such that the angle β_2 of the reflected wave is equal to the angle β_1 of the incoming wave. The wave intensity is unchanged but the phase, that is the direction of shear strain/stress, is opposite. In Fig. 7.7.7 this displacement component is in the in z-direction that is normal to the figure plane. The displacement component parallel to the xy – plane in Fig. 7.7.7 is reflected as two waves: a shear wave and a dilatational wave. The angle of the reflected shear wave is equal to the angle of the incoming wave: $\beta_2 = \beta_1$. The angle α_2 of the reflected dilatational wave is given by:

$$\frac{\sin \alpha_2}{\sin \beta_1} = \frac{c_l}{c_t} > 1 \tag{7.7.40}$$

7.7.7 Tensile Fracture Due to Compression Wave

Figure 7.7.8 shows a thick steel plate. At the position P on the top face of the plate an explosive charge is detonated, resulting in a compressive stress wave of very high

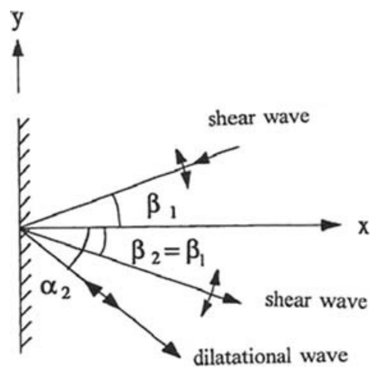


Fig. 7.7.7 Reflection of shear waves from a free surface

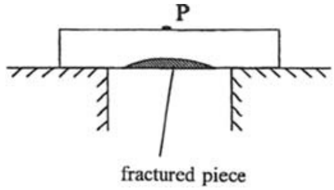


Fig. 7.7.8 Fracture due to reflected tensile wave

intensity. From the free bottom face of the plate the compression wave is reflected as a tension wave. The two waves combine, and a short distance inside the plate from the bottom face the tensile stresses may become larger than the tensile strength of the material, resulting in a fracture surface, with the result that a piece of the plate, indicated by the hatched area in Fig. 7.7.8 is detached from the plate and falls off

7.7.8 Surface Waves. Rayleigh Waves

In the vicinity of a free surface of an elastic body elastic waves are propagated in a special way. Figure 7.7.9 shows a free surface of a body at $y = 0$. We assume that the body, for $y \geq 0$, has plane displacements represented by the displacement field:

$$u_3 = 0, \quad u_\alpha = u_\alpha(x, y, t) \tag{7.7.41}$$

It is further assumed that the deformation is concentrated near the free surface, and we take as a condition for the displacement field that:

$$u_\alpha \to 0 \text{ as } y \to \infty \tag{7.7.42}$$

Using the formulas (7.7.34, 7.7.35) for the wave velocities, we may write the Navier equations (7.7.33) as:

$$c_t{}^2 u_{\alpha,\beta\beta} + \left(c_l{}^2 - c_t{}^2\right) u_{\beta,\beta\alpha} = \ddot{u}_\alpha \tag{7.7.43}$$

The condition of a stress free surface at $y = 0$ provides the following boundary conditions for the displacement field u_α, where Hooke's law for plane displacements (7.3.29) has been used:

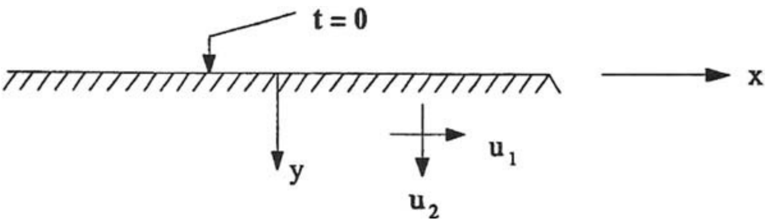


Fig. 7.7.9 Elastic body with free surface

$$T_{22}|_{y=0} = 0 \quad \Rightarrow \quad (1-\nu)\, u_{2,2} + \nu u_{1,1} = 0 \text{ at } y = 0 \tag{7.7.44}$$

$$T_{12}|_{y=0} = 0 \quad \Rightarrow \quad u_{1,2} + u_{2,1} = 0 \text{ at } y = 0 \tag{7.7.45}$$

The solution of the differential equation (7.7.43) satisfying the boundary conditions (7.7.42, 44, 45), is obtained as the following displacement waves, called *Rayleigh waves*:

$$u_1(x,y,t) = A\left[\exp(-ay) - \frac{2ab}{k^2+b^2}\exp(-by)\right]\sin\left[k\left(c_r t - x\right)\right] \tag{7.7.46}$$

$$u_2(x,y,t) = A\frac{a}{k}\left[\exp(-ay) + \frac{2k^2}{k^2+b^2}\exp(-by)\right]\cos\left[k\left(c_r t - x\right)\right] \tag{7.7.47}$$

A and k are undetermined constants, and a and b are given by the formulas:

$$a = k\sqrt{1-\left(\frac{c_r}{c_l}\right)^2}, \quad b = k\sqrt{1-\left(\frac{c_r}{c_t}\right)^2} \tag{7.7.48}$$

The wave speed c_r is determined from the cubic equation:

$$\left(\frac{c_r}{c_l}\right)^6 - 8\left(\frac{c_r}{c_t}\right)^4 + 24\left(\frac{c_r}{c_t}\right)^2 - 16\left(\frac{c_r}{c_l}\right)^2 - 16\left[1-\left(\frac{c_t}{c_l}\right)^2\right] = 0 \tag{7.7.49}$$

For the Poisson's ratio $\nu = 0$, we obtain $c_l = \sqrt{2}\, c_t$ and the solution of (7.7.49) only gives one acceptable value for the wave speed: $c_r = 0.874\ c_t$. *The* other two roots of equation (7.7.49) give complex values for a and b. For incompressible materials, $\nu = 0.5$, we find $c_r = 0.955\ c_t$.

7.8 Anisotropic Materials

In this section we shall discuss anisotropic, linearly elastic materials having simple symmetries. The presentation will cover some types of elastic crystals, wood materials, biological materials like bones, and fiber reinforced materials.

It is convenient in the presentation to introduce a special notation for coordinate stresses and coordinate strains referred to a Cartesian coordinate system Ox:

$$\begin{aligned} &T_1 \equiv T_{11} \equiv \sigma_x, \quad T_2 \equiv T_{22} \equiv \sigma_y, \quad T_3 \equiv T_{33} \equiv \sigma_z \\ &T_4 \equiv T_{23} \equiv \tau_{yz}, \quad T_5 \equiv T_{31} \equiv \tau_{zx}, \quad T_6 \equiv T_{12} \equiv \tau_{xy} \end{aligned} \tag{7.8.1}$$

$$\begin{aligned} &E_1 \equiv E_{11} \equiv \varepsilon_x, \quad E_2 \equiv E_{22} \equiv \varepsilon_y, \quad E_3 \equiv E_{33} \equiv \varepsilon_z \\ &E_4 \equiv 2E_{23} \equiv \gamma_{yz}, \quad E_5 \equiv 2E_{31} \equiv \gamma_{zx}, \quad E_6 \equiv 2E_{12} \equiv \gamma_{xy} \end{aligned} \tag{7.8.2}$$

The material response of a fully anisotropic, linearly elastic material is defined by the constitutive equations:

$$T_\alpha = S_{\alpha\beta} E_\beta \quad \Leftrightarrow \quad T = SE \tag{7.8.3}$$

when Greek indices represent the numbers 1 to 6. T and E are (6×1) vector matrices, and S is a (6×6) *elasticity matrix* or *stiffness matrix*.

$$S \equiv \left(S_{\alpha\beta}\right) \equiv \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{pmatrix} \tag{7.8.4}$$

The 36 elements of S are called *elasticities* or *stiffnesses*. It will be shown that if the material is hyperelastic, the stiffness matrix S is symmetric: $S^T = S$, such that only 21 stiffnesses are independent for full anisotropy.

The constitutive equation of a linear elastic material may also be presented as:

$$E_\alpha = K_{\alpha\beta} T_\beta \quad \Leftrightarrow \quad E = KT \tag{7.8.5}$$

$K = [K_{\alpha\beta}]$ is a (6×6) *compliance matrix* or *flexibility matrix*. The elements $K_{\alpha\beta}$ are called *compliances* or *flexibilities*. It follows from (7.8.3) and (7.8.5) that S and K are inverse matrices:

$$K = S^{-1} \tag{7.8.6}$$

If the material is hyperelastic only 21 compliances are independent for full anisotropy.

The stiffness matrix and the compliance matrix of an isotropic, linearly elastic material, i.e. a Hookean material, are:

$$S = \frac{\mu}{1-2\nu} \begin{pmatrix} 2(1-\nu) & 2\nu & 2\nu & 0 & 0 & 0 \\ 2\nu & 2(1-\nu) & 2\nu & 0 & 0 & 0 \\ 2\nu & 2\nu & 2(1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{pmatrix} \tag{7.8.7}$$

$$K = \frac{1}{2\mu(1+\nu)} \begin{pmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{pmatrix} \tag{7.8.8}$$

Note that S and K are symmetric matrices. We shall show in the following section that this a general property of anisotropic, hyperelastic materials.

7.8.1 Hyperelasticity

For a hyperelastic material the stress tensor may, according to (7.6.10), be derived from the elastic energy ϕ per unit volume:

$$\mathbf{T} = \frac{\partial \phi}{\partial \mathbf{E}} \quad \Leftrightarrow \quad T_{ij} = \frac{\partial \phi}{\partial E_{ij}} \quad \Leftrightarrow \quad T_\alpha = \frac{\partial \phi}{\partial E_\alpha} \tag{7.8.9}$$

Note that if we differentiate ϕ with respect to the coordinate strains E_{ij} we must treat E_{ij} as 9 independent variables; confer the development of (7.6.10). For a linearly hyperelastic material (7.8.3) and (7.8.9) imply:

$$T_\alpha = S_{\alpha\beta} E_\beta = \frac{\partial \phi}{\partial E_\alpha} \tag{7.8.10}$$

Because:

$$\frac{\partial^2 \phi}{\partial E_\beta \, \partial E_\alpha} = \frac{\partial^2 \phi}{\partial E_\alpha \, \partial E_\beta}$$

it follows from (7.8.10) that:

$$S_{\alpha\beta} = S_{\beta\alpha} \quad \Leftrightarrow \quad S^T = S \tag{7.8.11}$$

Thus, the stiffness matrix S is symmetric. This property reduces the number of independent stiffnesses from 36 for a fully anisotropic, linearly elastic material to 21 for a fully anisotropic, linearly hyperelastic material.

For a hyperelastic material the strain tensor may, according to (7.6.15), be derived from the complementary energy ϕ_c per unit volume.

$$\mathbf{E} = \frac{\partial \phi_c}{\partial \mathbf{T}} \quad \Leftrightarrow \quad E_{ij} = \frac{\partial \phi_c}{\partial T_{ij}} \quad \Leftrightarrow \quad E_\alpha = \frac{\partial \phi_c}{\partial T_\alpha} \tag{7.8.12}$$

For a linearly hyperelastic material (7.8.5) and (7.8.12) imply:

$$K_{\alpha\beta} = K_{\beta\alpha} \quad \Leftrightarrow \quad K^T = K \tag{7.8.13}$$

The compliance matrix is symmetric. Since the stiffness matrix S is symmetric, this result also follows from the relation (7.8.6).

Partial integrations of equations (7.8.10) and (7.8.12), with the boundary conditions:

$$\phi = \phi_c = 0 \quad \text{for} \quad E_\alpha = T_\alpha = 0 \tag{7.8.14}$$

result in the following expressions for the elastic energy and the complementary energy per unit volume for *linearly hyperelastic materials*, defined by the (7.8.4) and (7.8.8):

$$\phi = \phi_c = \frac{1}{2}E_\alpha S_{\alpha\beta} E_\beta = \frac{1}{2}E^T SE = \frac{1}{2}E^T T = \frac{1}{2}T_\alpha K_{\alpha\beta} T_\beta = \frac{1}{2}T^T K T = \frac{1}{2}T^T E \tag{7.8.15}$$

The stiffnesses $S_{\alpha\beta}$ and the compliances $K_{\alpha\beta}$ are determined by testing under uniaxial stress, biaxial stress, and pure shear stress. In these tests so-called *engineering parameters* are introduced: moduli of elasticity, Poisson's ratios, normal stress couplings, and shear moduli. Section 7.8.3, in an example with biomaterial, and Sect. 7.9 on composite materials, present some engineering parameters and relate them to the stiffnesses $S_{\alpha\beta}$ and the compliances $K_{\alpha\beta}$.

7.8.2 *Materials with one Plane of Symmetry*

If the structure of the material in a particle is symmetric with respect to a plane through the particle, such that the mirror image of the structure with respect to the plane is identical to the structure itself, the number of stiffnesses, and compliances, is reduced from 21 to 13.

Crystals with one symmetry plane are called *monoclinic crystals*. Materials having fibrous structure and with one characteristic fiber direction are symmetric about the plane normal to the fibers.

Figure 7.8.1a shows a volume element of a material with one plane of symmetry normal to the $\mathbf{e}_3$-direction. I Figure 7.8.1b the element is rotated 180° about the $\mathbf{e}_3$-direction. Two material lines symmetrically placed with respect to the plane of symmetry are drawn in the element. A state of strain $\mathbf{E}$ will give the same state of stress $\mathbf{T}$ whether it is introduced to the element before or after the rotation. In other words: the material is insensitive to a rotation 180° about the normal to the plane

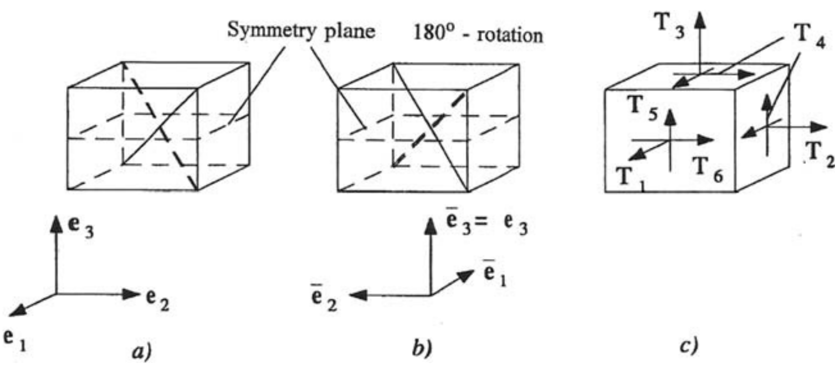


Fig. 7.8.1 Material with one plane of symmetry

of symmetry before the material is subjected to the strain $\mathbf{E}$. With respect to the coordinate directions $\mathbf{e}_i$ the coordinate stresses, see Fig. 7.8.1c, and the coordinate strains are:

$$T = \{T_1, T_2, T_3, T_4, T_5, T_6\}, \quad E = \{E_1, E_2, E_3, E_4, E_5, E_6\} \tag{7.8.16}$$

With respect to the coordinate directions $\bar{\mathbf{e}}_i$ the same state of strain $\mathbf{E}$ and stress $\mathbf{T}$ applied to the element in Fig. 7.8.1a is represented by the coordinate stresses and the coordinate strains:

$$\bar{T} = \{T_1, T_2, T_3, -T_4, -T_5, -T_6\}, \quad \bar{E} = \{E_1, E_2, E_3, -E_4, -E_5, -E_6\} \tag{7.8.17}$$

The constitutive equations expressed with respect to the directions $\mathbf{e}_i$ and $\bar{\mathbf{e}}_i$ are:

$$T_\alpha = S_{\alpha\beta} E_\beta, \quad \bar{T}_\alpha = \bar{S}_{\alpha\beta} \bar{E}_\beta \tag{7.8.18}$$

Since the configurations in Fig. 7.8.1a and b are equivalent as far as elastic properties are concerned, the stiffness matrices related to the directions $\mathbf{e}_i$ and $\bar{\mathbf{e}}_i$ must be identical:

$$\bar{S} = S \quad \Leftrightarrow \quad \bar{S}_{\alpha\beta} = S_{\alpha\beta} \tag{7.8.19}$$

We now choose special values for special Greek indices:

$$\lambda = 4 \text{ and } 5, \ \rho, \gamma = 1, 2, 3, 6$$

The constitutive equations (7.8.18) are expressed as:

$$\begin{aligned} &T_\alpha = S_{\alpha\beta} E_\beta \quad \Rightarrow \\ &T_\lambda = S_{\lambda\rho} E_\rho + S_{\lambda 4} E_4 + S_{\lambda 5} E_5, \quad T_\gamma = S_{\gamma\rho} E_\rho + S_{\gamma 4} E_4 + S_{\gamma 5} E_5 \end{aligned} \tag{7.8.20}$$

$$\begin{aligned} &\bar{T}_\alpha = \bar{S}_{\alpha\beta} \bar{E}_\beta \quad \Rightarrow \\ &-T_\lambda = S_{\lambda\rho} E_\rho + S_{\lambda 4}(-E_4) + S_{\lambda 5}(-E_5), \ T_\gamma = S_{\gamma\rho} E_\rho + S_{\gamma 4}(-E_4) + S_{\gamma 5}(-E_5) \end{aligned} \tag{7.8.21}$$

When (7.8.20) is compared with (7.8.21), we find that:

$$S_{\lambda\rho} = S_{\gamma 4} = S_{\gamma 5} = 0 \quad \Rightarrow \quad S_{4\rho} = S_{5\rho} = S_{\rho 4} = S_{\rho 5} = 0 \quad \text{for } \rho = 1, 2, 3, 6 \tag{7.8.22}$$

Thus the stiffness matrix for materials having one symmetry plane is:

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ & S_{22} & S_{23} & 0 & 0 & S_{26} \\ & & S_{33} & 0 & 0 & S_{36} \\ & symmetry & & S_{44} & S_{45} & 0 \\ & & & & S_{55} & 0 \\ & & & & & S_{66} \end{pmatrix} \tag{7.8.23}$$

The stiffness matrix contains 13 *independent stiffnesses*. The corresponding compliance matrix likewise contains 13 *independent compliances*.

7.8.3 Three Orthogonal Symmetry Planes. Orthotropy

If an elastic material has a structure that is mirror symmetrical about two orthogonal planes, the number of independent elasticities is reduced from 13, for one plane of symmetry, to 9. We shall see that two orthogonal symmetry planes imply three orthogonal planes of symmetry.

Let the two planes of symmetry be normal to the $\mathbf{e}_3$- and $\mathbf{e}_2$-directions in Fig. 7.8.1. We then find, using the same arguments that gave the results in equations (7.8.22), that:

$$\begin{aligned} &S_{4\rho} = S_{5\rho} = S_{6\rho} = S_{\rho 4} = S_{\rho 5} = S_{\rho 6} = 0 \quad \text{for } \rho = 1,2,3 \\ &S_{45} = S_{54} = S_{46} = S_{64} = S_{65} = S_{56} = 0 \end{aligned} \tag{7.8.24}$$

The stiffness matrix is now:

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ & S_{22} & S_{23} & 0 & 0 & 0 \\ & & S_{33} & 0 & 0 & 0 \\ & symmetry & & S_{44} & 0 & 0 \\ & & & & S_{55} & 0 \\ & & & & & S_{66} \end{pmatrix} \tag{7.8.25}$$

The matrix contains 9 *independent stiffnesses*. The corresponding compliance matrix has 9 *independent compliances*.

It follows from the result (7.8.25) that the material is symmetric also with respect to a plane normal to the $\mathbf{e}_1$-direction. Hence, two orthogonal planes of symmetry imply three orthogonal planes of symmetry. This type of symmetry is called *orthotropy*.

Materials having fiber structures and one distinct fiber direction may be orthotropic. If the fibers are directed in three orthogonal directions, the material is orthotropic. Wood gives an example of an approximate orthotropic material. Many crystals are orthotropic.

The elements of the stiffness matrix S in (7.8.25) and the elements $K_{\alpha\beta} = K_{\beta\alpha}$ in the compliance matrix:

$$K = \begin{pmatrix} K_{11} & K_{12} & K_{13} & 0 & 0 & 0 \\ & K_{22} & K_{23} & 0 & 0 & 0 \\ & & K_{33} & 0 & 0 & 0 \\ & symmetry & & K_{44} & 0 & 0 \\ & & & & K_{55} & 0 \\ & & & & & K_{66} \end{pmatrix} \tag{7.8.26}$$

may be determined experimentally from uniaxial tests and pure shear tests. For uniaxial stress T_1 in the direction $\mathbf{e}_1$ normal to one of the symmetry planes the relation $E = KT$ results in three longitudinal strains:

$$\begin{aligned} &E_1 = K_{11} T_1 = \frac{1}{\eta_1} T_1 \\ &E_2 = K_{21} T_1 = -\nu_{21} E_1 = -\frac{\nu_{21}}{\eta_1} T_1, \quad E_3 = K_{31} T_1 = -\nu_{31} E_1 = -\frac{\nu_{31}}{\eta_1} T \end{aligned} \tag{7.8.27}$$

η_1 is a *modulus of elasticity*, and ν_{21} and ν_{31} are *Poisson's ratios*. Similar expressions are formed for a uniaxial stress T_2 in the direction $\mathbf{e}_2$ and a uniaxial stress T_3 in the direction $\mathbf{e}_3$. These expressions introduces the *moduli of elasticity* η_2 and η_3 and the *Poisson's ratios* $\nu_{12}, \nu_{32}, \nu_{13}$, and ν_{23} Because K must be a symmetric matrix, the following relations must be satisfied:

$$\frac{\nu_{21}}{\eta_1} = \frac{\nu_{12}}{\eta_2}, \quad \frac{\nu_{31}}{\eta_1} = \frac{\nu_{13}}{\eta_3}, \quad \frac{\nu_{32}}{\eta_2} = \frac{\nu_{23}}{\eta_3} \tag{7.8.28}$$

Pure states of shear T_4, T_5, or T_6 with respect to two orthogonal directions $\mathbf{e}_i$ and $\mathbf{e}_j$ will only result in shear strains:

$$E_4 = K_{44} T_4 = \frac{1}{\mu_4} T_4, \quad E_5 = K_{55} T_5 = \frac{1}{\mu_5} T_5, \quad E_6 = K_{66} T_6 = \frac{1}{\mu_6} T_6 \tag{7.8.29}$$

where μ_4, μ_5, and μ_6 are *shear moduli*. The compliance matrix in (7.8.26) may now be represented by:

$$K = \begin{pmatrix} 1/\eta_1 & -\nu_{12}/\eta_2 & -\nu_{13}/\eta_3 & 0 & 0 & 0 \\ & 1/\eta_2 & -\nu_{23}/\eta_3 & 0 & 0 & 0 \\ & & 1/\eta_3 & 0 & 0 & 0 \\ & symmetry & & 1/\mu_4 & 0 & 0 \\ & & & & 1/\mu_5 & 0 \\ & & & & & 1/\mu_6 \end{pmatrix} \tag{7.8.30}$$

The 9 independent compliances in the expression (7.8.26) are now replaced by 9 *engineering parameters*: 3 moduli of elasticity, 3 shear moduli, and 3 Poisson's ratios. The 3 remaining Poisson's ratios are found from the three equations (7.8.28). When the compliance matrix K has been found, the stiffness matrix S may be found by inversion. An example of values of elastic parameters for fiber reinforced composites is presented in Sect. 7.9.

Example 7.16. Elastic Parameters of Human Femur and Bovine Femur

Bone as an elastic material is considered to be orthotropic. The table below shows elastic parameters for human femur (thighbone) found from standard material testing (Reilly, D. T. and Burstein, A. H., 1975), and elastic parameters from bovine (ox) femur found from ultra sound testing (Burris, 1983). The $\mathbf{e}_1$-direction represents the long axis of the bone, $\mathbf{e}_2$ is in the radial direction, and $\mathbf{e}_3$ is in the circumferential direction.

Elastic parameters of human femur and bovine femur

	η_1 [GPa]	η_2 [GPa]	η_3 [GPa]	μ_4 [GPa]	μ_5 [GPa]	μ_6 [GPa]	ν_{12}	ν_{13}	ν_{23}	ν_{21}	ν_{31}	ν_{32}
Human	11.5	11.5	17.0	3.3	3.3	3.6	0.58	0.46	0.46	0.58.	0.31	0.31
Bovine	10.79	12.24	18.90	5.96	4.47	3.38	0.51	0.42	0.33	0.45	0.24	0.22

The formulas (7.8.28) have been used to obtain some of the parameters in the table.

7.8.4 Transverse Isotropy

A material is transverse isotropic if an *axis of symmetry* exists with respect to the elastic properties through every particle. A symmetry axis implies that every plane through the axis is a plane of symmetry. Materials with fiber structure may exhibit such properties, For example is this approximately true for wood, for which the direction of the grains may be considered an axis of symmetry. Transverse isotropy implies orthotropy, but the reverse is not true.

The number of independent stiffnesses or compliances is 5 for transverse isotropy. To show this, we start with the stiffness matrix (7.8.25) for orthotropy. The axis of symmetry must be parallel with one of the $\mathbf{e}_i$ - directions. A 90°-rotation of the material about the symmetry axis does not change the apparent elastic structure of the material with respect to a fixed reference, represented for instance by the directions of the coordinate axes $\mathbf{e}_i$. With $\mathbf{e}_3$ parallel to the symmetry axis, we may argue that:

$$\begin{gathered} S_{11} = S_{22}\,, \quad S_{13} = S_{23}\,, \quad S_{44} = S_{55} \\ K_{11} = K_{22}\,, \quad K_{13} = K_{23}\,, \quad K_{44} = K_{55} \\ \eta_1 = \eta_2, \quad \nu_{13} = \nu_{23}, \quad \mu_4 = \mu_5, \quad \nu_{21} = \nu_{12} \end{gathered} \tag{7.8.31}$$

The last result follows from formula $(7.8.28)_1$.

For plane strain or plane stress with respect to a plane normal to the symmetry axis, the material responds isotropically. This means that principal axes of stress and of strain are coinciding. A state of pure shear stress T_6, see Fig. 7.8.2, results according to (7.8.29) to a state of pure shear strain:

$$E_6 = K_{66}T_6 = \frac{1}{\mu_6}T_6 \tag{7.8.32}$$

Principal stresses and principal strains are obtained from the formulas (3.3.9) and (5.3.36), see Fig. 7.8.2:

$$\sigma_1 = -\sigma_2 = T_6, \quad \varepsilon_1 = -\varepsilon_2 = \frac{1}{2}E_6 \tag{7.8.33}$$

The principal directions $\bar{\mathbf{e}}_1$ and $\bar{\mathbf{e}}_2$ are rotated 45° with respect to the directions $\mathbf{e}_1$ and $\mathbf{e}_2$. The compliance matrix (7.8.30) also applies for the principal directions $\bar{\mathbf{e}}_1$, $\bar{\mathbf{e}}_2$, and $\mathbf{e}_3$. We may therefore use the matrix (7.8.30) and the formulas (7.8.31, 7.8.32 ,7.8.33) to obtain:

$$\left.\begin{aligned} \varepsilon_1 = \tfrac{1}{\eta_1}\sigma_1 - \tfrac{\nu_{12}}{\eta_2}\sigma_2 = \tfrac{1+\nu_{12}}{\eta_1}T_6 \equiv \tfrac{1}{2}E_6 = K_{11}T_6 + K_{12}(-T_6) = (K_{11}-K_{12})T_6 \\ E_6 = \tfrac{1}{\mu_6}T_6 \equiv K_{66}T_6 \end{aligned}\right\} \Rightarrow$$

$$\mu_6 = \frac{\eta_1}{2(1+\nu_{12})} \quad \Leftrightarrow \quad K_{66} = 2(K_{11}-K_{12}) \tag{7.8.34}$$

Similarly we find:

$$S_{66} = \frac{1}{2}(S_{11}-S_{12}) \tag{7.8.35}$$

The four conditions for the elements of the matrices K and S provided by the formulas (7.8.31, 34, 35) result in the following forms for the compliance matrix and the stiffness matrix in the case of transverse isotropy:

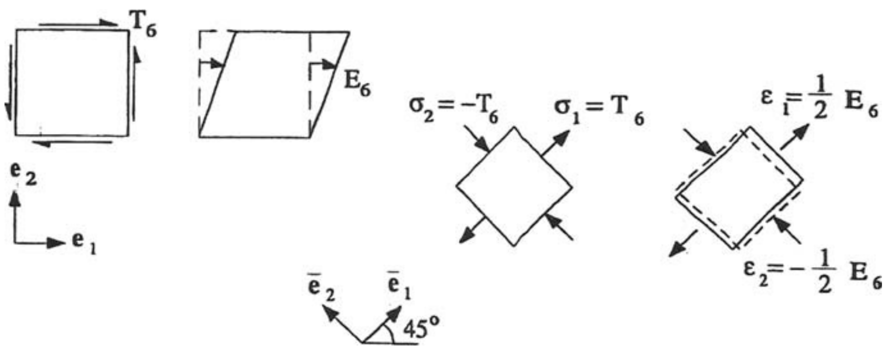


Fig. 7.8.2 Pure shear stress and pure shear strain

$$K = \begin{pmatrix} 1/\eta_1 & -\nu_{12}/\eta_1 & -\nu_{13}/\eta_3 & 0 & 0 & 0 \\ & 1/\eta_1 & -\nu_{13}/\eta_3 & 0 & 0 & 0 \\ & & 1/\eta_3 & 0 & 0 & 0 \\ & symmetry & & 1/\mu_4 & 0 & 0 \\ & & & & 1/\mu_4 & 0 \\ & & & & & 2\left(1+\nu_{12}\right)/\eta_1 \end{pmatrix} \tag{7.8.36}$$

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ & S_{11} & S_{13} & 0 & 0 & 0 \\ & & S_{33} & 0 & 0 & 0 \\ & symmetry & & S_{44} & 0 & 0 \\ & & & & S_{44} & 0 \\ & & & & & \frac{1}{2}\left(S_{11}-S_{12}\right) \end{pmatrix} \tag{7.8.37}$$

The compliance matrix K has 5 *independent compliances*, represented by 5 independent engineering parameters, and the stiffness matrix S has 5 *independent stiffnesses*. Note that the engineering parameters given in Example 7.16 show that the human femur is a transverse isotropic elastic material.

7.8.5 Isotropy

For an isotropic elastic material all axes are symmetry axes. Starting from the matrices (7.8.36) and (7.8.37) we may set:

$$\eta_3 = \eta_1 = \eta_2 = \eta, \quad \nu_{12} = \nu_{13} = \nu, \quad \mu_4 = \mu = \frac{\eta}{2\left(1+\nu\right)}$$
$$S_{33} = S_{11}, \quad S_{44} = \frac{1}{2}\left(S_{11}-S_{12}\right), \quad S_{13} = S_{12} \tag{7.8.38}$$

These conditions reduce the number of independent compliances to two and independent stiffnesses to two. The stiffness matrix S has the symmetry shown by the matrix (7.8.7). Inverting the compliance matrix (7.8.36), we find:

$$S_{11} = \frac{1-\nu}{\left(1+\nu\right)\left(1-2\nu\right)}\eta = \frac{2\left(1-\nu\right)}{\left(1-2\nu\right)}\mu, \quad S_{12} = \frac{\nu}{\left(1+\nu\right)\left(1-2\nu\right)}\eta = \frac{2\nu}{\left(1-2\nu\right)}\mu$$
$$S_{44} = \frac{1}{2}\left(S_{11}-S_{12}\right) = \mu = \frac{\eta}{2\left(1+\nu\right)} \tag{7.8.39}$$

When the results (7.8.39) and (7.8.38) are substituted into (7.8.37), we obtain the stiffness matrix (7.8.7) for a Hookean material.

7.9 Composite Materials

A composite material, a *composite* for short, is a macroscopic composition of two or more materials. The material properties of a composite may to some extent be calculated from the knowledge of the material properties of its components. Plywood and reinforced concrete are two typical examples of composites.

It is customary to distinguish between three main types of composites:

1. *Fiber composites*
2. *Laminates*
3. *Particular composites*.

Materials of type 1 consist of *fibers* of one material baked into another base material called a *matrix*. Reinforced concrete belongs to this type. A laminate is made of layers, which may have different properties in different directions. A particular composite is mixture of particles of one material in a base material. In the present section we shall concentrate the presentation to fiber reinforced two-dimensional layers, called *laminas*, see Fig. 7.9.1, and a combination of these laminas into a *plate laminate*, see Fig. 7.9.2.

Fibers of a material are much stronger than the bulk material. Glass, for example, may have 300 times higher fiber strength than the base material. A fiber consists of crystals of the material arranged parallel to the fiber axis. The fiber diameter is of the same order of magnitude as the crystal diameter. The fiber therefore has fewer internal defects than the material. Very short fibers are called *whiskers*, and they are even stronger then ordinary fibers.

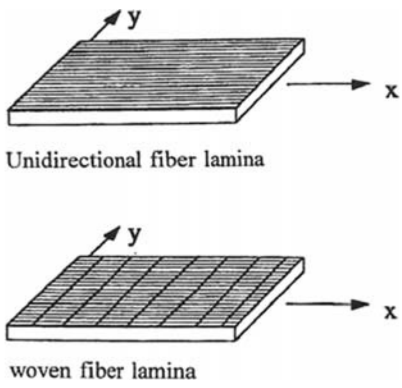
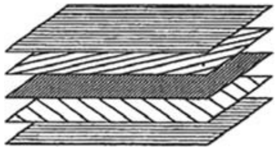


Fig. 7.9.1 Lamina

Fig. 7.9.2 Plate laminate



7.9.1 Lamina

Figure 7.9.1 shows two typical laminas: a *unidirectional fiber lamina* and a *woven fiber lamina*. Both types have one direction of dominating strength, represented by the x-direction. Fibers are normally linearly elastic, while the matrix material often shows a viscoelastic or visco-elasto-plastic response. In the present exposition we shall assume that the lamina as a composite material is a linearly elastic material. Furthermore, we shall concentrate the attention to laminas with unidirectional fiber reinforcement, called *unidirectional laminas*. However, all the general results we obtain also apply to laminas with woven fiber reinforcement.

Figure 7.9.3 shows an element of a unidirectional lamina oriented after the *lamina axes* x and y, with the x-axis in the fiber direction. The coordinate stresses with respect to the lamina axes, and for the state of plane stress, are denoted $T_x,\ T_y$, and $T_s(= T_{xy})$. The corresponding coordinate strains are $E_x,\ E_y$, and $E_s(= \gamma_{xy})$.

A unidirectional lamina represents an orthotropic, linearly elastic material. In the present case of plane stress only 4 independent stiffnesses or 4 independent compliances are relevant. With respect to the *lamina axes* x and y, see Fig. 7.9.3, we introduce the constitutive equations:

$$\bar{T} = \bar{S}\bar{E} \quad \Leftrightarrow \quad \begin{pmatrix} T_x \\ T_y \\ T_s \end{pmatrix} = \begin{pmatrix} S_{xx} & S_{xy} & 0 \\ S_{xy} & S_{yy} & 0 \\ 0 & 0 & S_{ss} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_s \end{pmatrix} \tag{7.9.1}$$

$$\bar{E} = \bar{K}\bar{T} \quad \Leftrightarrow \quad \begin{pmatrix} E_x \\ E_y \\ E_s \end{pmatrix} = \begin{pmatrix} K_{xx} & K_{xy} & 0 \\ K_{xy} & K_{yy} & 0 \\ 0 & 0 & K_{ss} \end{pmatrix} \begin{pmatrix} T_x \\ T_y \\ T_s \end{pmatrix} \tag{7.9.2}$$

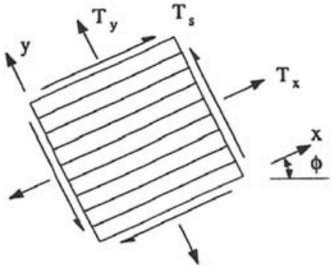


Fig. 7.9.3 Lamina axes

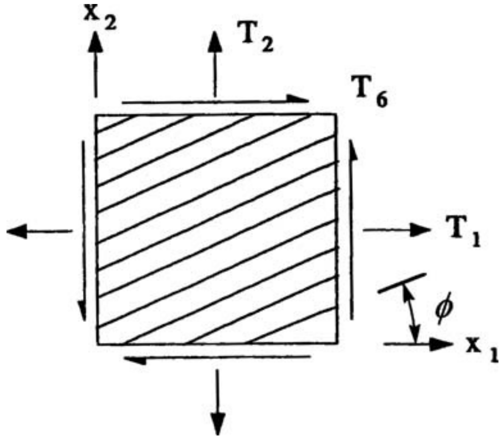


Fig. 7.9.4 Laminate axes

The compliance matrix and the stiffness matrix expressed in *engineering parameters* are according to the form (7.8.30):

$$\bar{K}=\begin{pmatrix}\frac{1}{\eta_x} & -\frac{\nu_x}{\eta_x} & 0\\ -\frac{\nu_x}{\eta_x} & \frac{1}{\eta_y} & 0\\ 0 & 0 & \frac{1}{\mu}\end{pmatrix},\quad \bar{S}=\begin{pmatrix}\alpha\,\eta_x & \alpha\,\nu_x\,\eta_y & 0\\ \alpha\,\nu_x\,\eta_y & \alpha\,\eta_y & 0\\ 0 & 0 & \mu\end{pmatrix},\quad \alpha=\frac{1}{1-\nu_x\,\nu_y} \tag{7.9.3}$$

In the formulas for $\bar{K}$ and $\bar{\mathrm{S}}$ the symmetry of the matrices has been invoked. The new symbols in the formulas are:

$$\begin{aligned}\eta_x &= \frac{1}{K_{xx}} && \text{longitudinal modulus of elasticity}\\ \eta_y &= \frac{1}{K_{yy}} && \text{transverse modulus of elasticity}\\ \nu_x &= -\frac{K_{yx}}{K_{xx}} && \text{longitudinal Poisson's ratio}\end{aligned} \tag{7.9.4}$$

We shall demonstrate how the engineering parameters may be estimated when the elastic properties of the matrix and the fibers are known. The matrix is assumed to be an isotropic elastic material having the modulus of elasticity η_m and the Poisson's ratio ν_m. The fibers are also isotropic elastic with the modulus of elasticity η_f and Poisson's ratio ν_f.

Figure 7.9.5 shows a cubic element of the composite. The length of the edges is L. The volume of the element is then $V = L^3$ and each side has the area $A = L^2$. The volume fraction of matrix and of fibers are c_m and c_f respectively, such that:

$$c_m + c_f = 1$$

Uniaxial stress in the fiber direction, T_x, results in longitudinal strains E_x and E_y. Matrix and fibers get the same strains E_x but have unequal stresses T_{xm} and T_{xf}. The

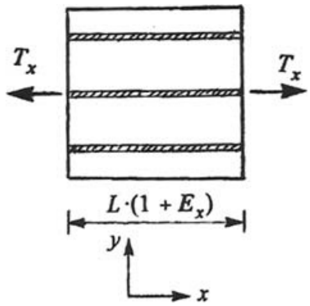


Fig. 7.9.5 Unidirectional lamina

area A of the sides subjected to the stresses T_x, T_{xm}, and T_{xf} is the sum of the area A_m of matrix and the area A_f of fibers. It follows that:

$$A = L^2\,,\ A_m = c_m L^2 = c_m A,\ \text{and}\ A_f = c_f L^2 = c_f A$$

The modulus of elasticity η_x may be determined as follows. The normal force N_x on the area A is expressed by:

$$N_x = T_x A = T_{xf} A_f + T_{xm} A_m = \quad \Rightarrow \quad (\eta_x E_x) A = \left(\eta_f E_x\right) A_f + (\eta_m E_x)\, A_m \quad \Rightarrow$$
$$\eta_x = \eta_f c_f + \eta_m c_m \tag{7.9.5}$$

The other engineering parameters can only be estimated. Their determination depends on the way the fibers are distributed in the direction normal to the fibers. Figure 7.9.6 shows two extreme cases. In the case in Fig. 7.9.6a we can derive the results:

$$\nu_x = \nu_f c_f + \nu_m c_m, \quad \eta_y = \frac{\eta_f \eta_m}{\eta_f c_m + \eta_m c_f}, \quad \mu = \frac{\mu_f \mu_m}{\mu_f c_m + \mu_m c_f} \tag{7.9.6}$$

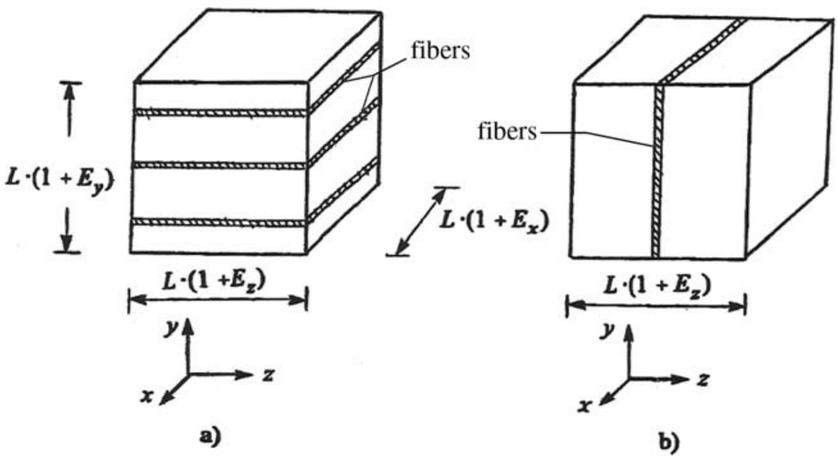


Fig. 7.9.6 Extreme distribution of fibers in the x-direction

where:

$$\mu_f = \frac{\eta_f}{2\left(1+\nu_f\right)}, \quad \mu_m = \frac{\eta_m}{2\left(1+\nu_m\right)} \tag{7.9.7}$$

It may be shown that the formula for η_y represents a lower limit for this modulus of elasticity.

Example 7.17. Elastic Parameters for Fiber Reinforced Epoxy

The table below shows the elastic parameters for three composite materials with epoxy matrix and 0.6 volume fraction of unidirectional fibers of three different fiber materials. The values are obtained from the book: "Mechanical behavior of materials" by Norman E. Dowling [11]. The numbers in parenthesis indicate the elastic properties of the fiber material. Epoxy is isotropic with the modulus of elasticity $\eta = 3.5\,\text{GPa}$ and Poisson's ratio $\nu = 0.33$. Using the formulas (7.9.5) and (7.9.6), we find that the η_x - and ν_x-values are in accordance with the table, while the other parameters do not correspond to the table-values.

parameter	E-glass (GPa)	Kevlar 49 (GPa)	Graphite, T-300 (GPa)
η_x	45 (72.3)	76 (124)	132 (218)
η_y	12	5.5	10.3
μ	4.4	2.1	6.5
ν_y	0.067 (0.22)	0.025 (0.35)	0.020 (0.20)
ν_x	0.25 (0.22)	0.34 (0.35)	0.25 (0.20)

7.9.2 From Lamina Axes to Laminate Axes

When we want to construct a laminate using lamina with different fiber orientations, we need to transform lamina stresses and strains to the stresses T_1, T_2, and $T_6(= T_{12})$, and the strains E_1, E_2, and $E_6(= 2E_{12})$ related to the *laminate axes* x_i, see Fig. 7.9.4. The strain matrix E is the same for all laminas, while the stress matrix T varies with the orientation angle ϕ. For each lamina we define the stiffness matrix S and the compliance matrix K through the relations:

$$T = SE \quad \Leftrightarrow \quad \begin{pmatrix} T_1 \\ T_2 \\ T_6 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_6 \end{pmatrix} \tag{7.9.8}$$

$$E = KT \quad \Leftrightarrow \quad \begin{pmatrix} E_1 \\ E_2 \\ E_6 \end{pmatrix} = \begin{pmatrix} K_{11} & K_{12} & K_{16} \\ K_{12} & K_{22} & K_{26} \\ K_{16} & K_{26} & K_{66} \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_6 \end{pmatrix} \tag{7.9.9}$$

The coordinate transformation from the lamina axes x and y to the laminate axes x_1 and x_2 in Fig. 7.9.3 and Fig. 7.9.4, is given by:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \Leftrightarrow \quad x = Q\bar{x} \tag{7.9.10}$$

The transformation formulas for coordinate stresses and coordinate strains are:

$$T_{\alpha\beta} = Q_{\alpha\gamma} Q_{\beta\lambda} \bar{T}_{\gamma\lambda} \quad \Leftrightarrow \quad T = Q_\sigma \bar{T}, \quad E_{\alpha\beta} = Q_{\alpha\gamma} Q_{\beta\lambda} \bar{E}_{\gamma\lambda} \quad \Leftrightarrow \quad E = Q_\varepsilon \bar{E} \tag{7.9.11}$$

T, $\bar{T}$, E, and $\bar{E}$ are vector matrices for the stresses and the strains in (7.9.1) and (7.9.8), and:

$$Q_\sigma = \begin{pmatrix} \cos^2\phi & \sin^2\phi & -\sin 2\phi \\ \sin^2\phi & \cos^2\phi & \sin 2\phi \\ \frac{1}{2}\sin 2\phi & -\frac{1}{2}\sin 2\phi & \cos 2\phi \end{pmatrix}, \quad Q_\varepsilon = \begin{pmatrix} \cos^2\phi & \sin^2\phi & -\frac{1}{2}\sin 2\phi \\ \sin^2\phi & \cos^2\phi & \frac{1}{2}\sin 2\phi \\ \sin 2\phi & -\sin 2\phi & \cos 2\phi \end{pmatrix} \tag{7.9.12}$$

The difference between the matrices Q_ε and Q_σ is due to the fact that $T_{12} = T_6$ and $E_{12} = (1/2)E_6$. The two matrices Q_ε and Q_σ are related through:

$$Q_\varepsilon = Q_\sigma^{-T} \quad \Leftrightarrow \quad Q_\varepsilon^T = Q_\sigma^{-1} \quad \Leftrightarrow \quad Q_\varepsilon^{-1} = Q_\sigma^T \tag{7.9.13}$$

This may be shown as follows. The elastic energy per unit volume may be expressed alternatively by:

$$\frac{1}{2} T^T E = \frac{1}{2} \bar{T}^T \bar{E} \tag{7.9.14}$$

Then using the formulas (7.9.14) and (7.9.11), we obtain:

$$\bar{T}^T \bar{E} = T^T E = \bar{T}^T Q_\sigma^T Q_\varepsilon \bar{E} \quad \Rightarrow \quad Q_\sigma^T Q_\varepsilon = 1 \quad \Rightarrow \quad (7.9.13)$$

Now we are ready to develop relations between the stiffness matrices and the compliance matrices related to the lamina axes and laminate axes. Using the formulas (7.9.8), (7.9.11), (7.9.1), and (7.9.13), we find:

$$SE = T = Q_\sigma \bar{T} = Q_\sigma \bar{S} \bar{E} = Q_\sigma \bar{S} Q_\varepsilon^{-1} E = Q_\sigma \bar{S} Q_\sigma^T E \tag{7.9.15}$$

Since S and $Q_\sigma \bar{S} Q_\sigma^T$ both are independent of E, it follows from the result (7.9.15) that:

$$S = Q_\sigma \bar{S} Q_\sigma^T \tag{7.9.16}$$

Using similar arguments, we get:

$$K = Q_\varepsilon \bar{K} Q_\varepsilon^T \tag{7.9.17}$$

7.9.3 Engineering Parameters Related to Laminate Axes

The engineering parameters for a lamina related to the laminate axes are defined similarly to the engineering parameters related to the lamina axes. The lamina is subjected in turn to uniaxial stress in the x_1- and the x_2-directions, and to the shear stress T_6. Then:

$$\begin{aligned}
&T_1 \neq 0, \quad T_2 = T_6 = 0 \quad \Rightarrow \\
&\eta_1 = T_1/E_1 = 1/K_{11} \quad \text{modulus of elasticity} \\
&\nu_{21} = -E_2/E_1 = -K_{21}/K_{11} \quad \text{Poisson's ratio} \\
&\nu_{61} = E_6/E_1 = K_{61}/K_{11} \quad \text{shear coupling coefficient}
\end{aligned} \tag{7.9.18}$$

$$\begin{aligned}
&T_2 \neq 0, \quad T_1 = T_6 = 0 \quad \Rightarrow \\
&\eta_2 = T_2/E_2 = 1/K_{22} \quad \text{modulus of elasticity} \\
&\nu_{12} = -E_1/E_2 = -K_{12}/K_{22} \quad \text{Poisson's ratio} \\
&\nu_{62} = E_6/E_2 = K_{62}/K_{22} \quad \text{shear coupling coefficient}
\end{aligned} \tag{7.9.19}$$

$$\begin{aligned}
&T_6 \neq 0, \quad T_1 = T_2 = 0 \quad \Rightarrow \\
&\mu_6 = T_6/E_6 = 1/K_{66} \quad \text{shear modulus} \\
&\nu_{16} = E_1/E_6 = K_{16}/K_{66} \quad \text{normal stress coupling coefficient} \\
&\nu_{26} = E_2/E_6 = K_{26}/K_{66} \quad \text{normal stress coupling coefficient}
\end{aligned} \tag{7.9.20}$$

We use the fact that K is symmetric. Per definitions (7.9.18 ,7.9.19, 7.9.20):

$$K_{12} = -\frac{\nu_{12}}{\eta_2},\ K_{21} = -\frac{\nu_{21}}{\eta_1},\ K_{16} = \frac{\nu_{16}}{\mu_6},\ K_{61} = \frac{\nu_{61}}{\eta_1},\ K_{26} = \frac{\nu_{26}}{\mu_6},\ K_{62} = \frac{\nu_{62}}{\eta_2}$$

Hence:

$$\frac{\nu_{12}}{\eta_2} = \frac{\nu_{21}}{\eta_1},\ \frac{\nu_{16}}{\mu_6} = \frac{\nu_{61}}{\eta_1},\ \frac{\nu_{26}}{\mu_6} = \frac{\nu_{62}}{\eta_2} \tag{7.9.21}$$

From the above it follows that:

$$\bar{K} = \begin{pmatrix} \frac{1}{\eta_1} & -\frac{\nu_{21}}{\eta_1} & \frac{\nu_{61}}{\eta_1} \\ -\frac{\nu_{21}}{\eta_1} & \frac{1}{\eta_2} & \frac{\nu_{62}}{\eta_2} \\ \frac{\nu_{61}}{\eta_1} & \frac{\nu_{62}}{\eta_2} & \frac{1}{\mu_6} \end{pmatrix} \tag{7.9.22}$$

7.9.4 Plate Laminate of Unidirectional Laminas

We shall consider a laminate made of many laminas bound together by the same material as in the matrix of the laminas. The plate is symmetric with respect to its middle surface, see Fig. 7.9.7, and is loaded by forces in the middle surface.

The forces acting on an element of the plate are called *stress resultants* and are given as forces N_1, N_2, and N_6 per unit length along the edges of the element.

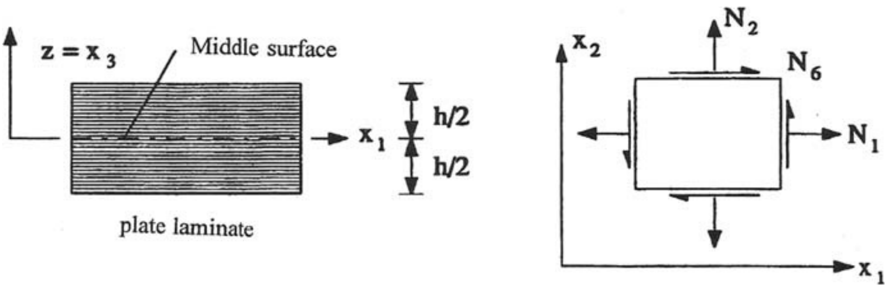


Fig. 7.9.7 Laminate and stress resultants

$$N_1 = \int_{-h/2}^{h/2} T_1\,dz, \quad N_2 = \int_{-h/2}^{h/2} T_2\,dz, \quad N_6 = \int_{-h/2}^{h/2} T_6\,dz \tag{7.9.23}$$

If the middle surface is plane, the laminate is a plate. A laminate with a curved middle surface for which the curvature is small relative to the inverse thickness $1/h$, and with stress resultants given by the formulas (7.9.23), is a *membrane shell*. Problem 7.25 presents an example of a cylindrical laminated membrane shell.

We assume that the strains E_1, E_2, and E_6 are constant over the thickness of the laminate. But due to the different orientations of the laminas of the laminate, the stresses T_1, T_2, and T_6 may vary through the thickness. Let the strains E_1, E_2, and E_6 and the stress resultants N_1, N_2, and N_6 be related through the matrix equations:

$$N \equiv \begin{pmatrix} N_1 \\ N_2 \\ N_6 \end{pmatrix} = AE \quad \Leftrightarrow \quad E = BN, B = A^{-1} \tag{7.9.24}$$

The 3×3-matrices A and B are called the *plate stiffness matrix* and the *plate flexibility matrix*, respectively. Introducing the relation $T = SE$ into (7.9.23), we obtain:

$$A = \int_{-h/2}^{h/2} S\,dz \tag{7.9.25}$$

Since S is symmetric it follows that A and B are symmetric matrices. The integral in formula (7.9.25) may be replaced by a sum. Let the laminate have n different orientations of laminas, specified by n angles $\phi_i, i = 1,2,\ldots,n$. The laminas with orientation angle ϕ_i have the stiffness matrix S_i. The total height is h_i of the laminas with orientation angle ϕ_i. Then from the formula (7.9.25) we get the result:

$$A = \sum_{i=1}^{n} S_i h_i \tag{7.9.26}$$

7.10 Large Deformations

In general an *elastic material*, or more precisely a *Cauchy-elastic material*, is defined by the constitutive equation of the form:

$$\mathbf{T} = \mathbf{T}[\mathbf{F}, \mathbf{r}_o] \tag{7.10.1}$$

The Cauchy stress tensor $\mathbf{T}$ is given by a tensor-valued function of the deformation gradient tensor $\mathbf{F}$ and the position vector $\mathbf{r}_o$ of the particle in the reference configuration K_o. This definition does not assume small deformations or small strains. The deformation gradient $\mathbf{F}$ may be polar decomposed into a stretch tensor $\mathbf{U}$ and the rotation tensor $\mathbf{R}$, i.e. $\mathbf{F} = \mathbf{RU}$. Due to the geometrical interpretation of polar decomposition in Fig. 5.5.3 it is reasonable to require of the constitutive equation (7.10.1) that the two states of deformation represented by $\mathbf{F} = \mathbf{RU}$ and $\mathbf{F} = \mathbf{U}$ respectively result in states of stress that are equal apart from the a rotation of the principal axes of stress. We may therefore expect of the tensor-valued function in (7.10.1) that:

$$\mathbf{T} = \mathbf{T}[\mathbf{F}, \mathbf{r}_o] = \mathbf{R}\,\mathbf{T}[\mathbf{U}, \mathbf{r}_o]\,\mathbf{R}^T \tag{7.10.2}$$

The response function $\mathbf{T}[\mathbf{F}, \mathbf{r}_o]$ may thus be determined from experiments with pure strain, $\mathbf{F} = \mathbf{U} = \mathbf{U}^T$. The argument used to obtain the result (7.10.2) is really an application of a fundamental principle in the general theory of constitutive equations, which is presented in Sect. 11.5 as: *the principle of material objectivity*.

In the case of small deformations between the reference configuration K_o and the present configuration K we have from the formulas (5.5.69) that:

$$\mathbf{R} = \mathbf{1} + \tilde{\mathbf{R}}, \quad \mathbf{R}^T = \mathbf{1} - \tilde{\mathbf{R}}, \quad \mathbf{U} = \mathbf{1} + \mathbf{E} \tag{7.10.3}$$

We assume that the state of stress in K_o is given by $\mathbf{T}_o = \mathbf{T}[\mathbf{1}, \mathbf{r}_o]$. The two first terms in a Taylor series expansion of the response function in (7.10.2) provide the following approximation to the response function, valid for small deformations.

$$\mathbf{T}[\mathbf{U}, \mathbf{r}_o] = \mathbf{T}_o + \mathbf{S}[\mathbf{r}_o] : \mathbf{E} \tag{7.10.4}$$

The *stiffness tensor* $\mathbf{S}(\mathbf{r}_o)$ is defined by:

$$\mathbf{S}[\mathbf{r}_o] = \left.\frac{\partial \mathbf{T}}{\partial \mathbf{U}}\right|_{\mathbf{U}=\mathbf{1}} = \left.\frac{\partial \mathbf{T}}{\partial \mathbf{E}}\right|_{\mathbf{E}=\mathbf{0}}, \quad S_{ijkl}[X] = \left.\frac{\partial T_{ij}}{\partial U_{kl}}\right|_{U=1} = \left.\frac{\partial T_{ij}}{\partial E_{kl}}\right|_{E=0} \tag{7.10.5}$$

When $\mathbf{R}$ from the formulas (7.10.3) and $\mathbf{T}[\mathbf{U}, \mathbf{r}_o]$ from (7.10.4) are substituted into the result (7.10.2), this result is obtained:

$$\mathbf{T} = \mathbf{T}_o + \tilde{\mathbf{R}}\,\mathbf{T}_o - \mathbf{T}_o\,\tilde{\mathbf{R}} + \mathbf{S} : \mathbf{E} \quad \Leftrightarrow \quad T_{ij} = T_{oij} + \tilde{R}_{ik}\,T_{okj} - T_{oik}\,\tilde{R}_{kj} + S_{ijkl}\,E_{kl} \tag{7.10.6}$$

The formula is applicable in incremental solutions in non-linear problems where the non-linearity is due to non-linear elastic material properties and/or to large deformations. The response equation (7.10.6) may also be used in stability investigations.

7.10.1 Isotropic Elasticity

For an isotropic material with a stress free reference configuration the response function (7.10.1) may be replaced by a isotropic tensor-valued function of the *left deformation tensor* $\mathbf{B} = \mathbf{F}\mathbf{F}^T$:

$$\mathbf{T} = \mathbf{T}[\mathbf{B}, \mathbf{r}_o] \tag{7.10.7}$$

In order to understand this we use the polar decomposition theorem to write: $\mathbf{F} = \mathbf{VR}$, where $\mathbf{V}$ is the left stretch tensor. Now, to deform the material from the reference configuration K_o to the present configuration K we may first let the material in the neighborhood of the particle under consideration be rotated according to the rotation tensor $\mathbf{R}$, and then subject the material to pure strain through the stretch tensor $\mathbf{V}$. Because the material is isotropic, the rotation $\mathbf{R}$ does not influence the stresses resulting from the deformation gradient $\mathbf{F}$. This implies that we may replace $\mathbf{F}$ by $\mathbf{V}$ as the argument tensor in the tensor-valued function (7.10.1).

$$\mathbf{T} = \mathbf{T}[\mathbf{F}, \mathbf{r}_o] = \mathbf{T}[\mathbf{V}, \mathbf{r}_o] \tag{7.10.8}$$

Since $\mathbf{B} = \mathbf{V}^2$, we may now consider the stress to be a tensor-valued function of $\mathbf{B}$ and the result is the function (7.10.7), although the mathematical functions (7.10.8) and (7.10.7) are not the same. Because the material is isotropic, the function $\mathbf{T}[\mathbf{B}, \mathbf{r}_o]$ must be isotropic with respect to the argument tensor $\mathbf{B}$, i.e. if the deformation $\mathbf{B}$ result in the stress tensor $\mathbf{T}$, a $\mathbf{Q}$ – rotated deformation $\mathbf{QBQ}^T$ will result in the stress tensor $\mathbf{QTQ}^T$. According to the results (4.6.17) and (4.6.27) the constitutive equation (7.10.7) may be represented by the two alternative forms:

$$\mathbf{T} = \gamma_o \mathbf{1} + \gamma_1 \mathbf{B} + \gamma_2 \mathbf{B}^2 \tag{7.10.9}$$

$$\mathbf{T} = \phi_o \mathbf{1} + \phi_1 \mathbf{B} + \phi_{-1} \mathbf{B}^{-1} \tag{7.10.10}$$

γ_i and ϕ_i are scalar-valued functions of the principal invariants I_B, II_B, and III_B, or of the principal values of the deformation tensor $\mathbf{B}$.

From the definitions of the displacement gradient $\mathbf{H} = \mathbf{F} - \mathbf{1}$ and the strain tensor:

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T\mathbf{H}\right) \tag{7.10.11}$$

we obtain:

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = (\mathbf{1} + \mathbf{H})\left(\mathbf{1} + \mathbf{H}^T\right) \quad \Rightarrow \quad \mathbf{B} = \mathbf{1} + 2\mathbf{E} + \mathbf{H}\mathbf{H}^T - \mathbf{H}^T\mathbf{H} \tag{7.10.12}$$

Because we assume that the material is stress free in the reference configuration K_o, a linearization of the general constitutive equation of an isotropic elastic material (7.10.9) may be presented as the following form of Hooke's law:

$$\mathbf{T} = \lambda\,(\mathrm{tr}\,\mathbf{E})\,\mathbf{1} + 2\mu\,\mathbf{E} \tag{7.10.13}$$

Confer equation (7.2.19). The parameters λ and μ are the *Lamé-constants*. λ is presented by formula (7.2.21) and μ is identical to the shear modulus.

7.10.2 Hyperelasticity

The stress power of a body with volume V is by definition:

$$P^d = \int_V \omega \, dV = \int_V \mathbf{T} : \mathbf{D} \, dV \tag{7.10.14}$$

The stress work done on the body when it moves from the reference configuration K_o at the time t_o to the present configuration K at the time t is:

$$W = \int_{t_o}^{t} P^d \, dt \tag{7.10.15}$$

A material is called *hyperelastic*, or *Green-elastic*, if the stress work may be derived from a scalar potential $\Phi(t)$, called the *elastic energy* in the body:

$$\Phi(t) = \int_V \psi \rho \, dV \tag{7.10.16}$$

The potential ψ, which is the *specific elastic energy*, i.e. is *elastic energy per unit mass*, is a scalar-valued function of the deformation tensor $\mathbf{C} = \mathbf{F}^T \mathbf{F}$:

$$\psi = \psi [\mathbf{C}, \mathbf{r}_o] \tag{7.10.17}$$

The rotation part $\mathbf{R}$ of $\mathbf{F}$ does obviously not contribute to the elastic energy ψ.

For a hyperelastic material the stress work on the body may now be expressed by:

$$W = \Phi(t) - \Phi(t_o) \tag{7.10.18}$$

such that:

$$P^d = \int_V \omega \, dV = \int_V \mathbf{T} : \mathbf{D} \, dV = \dot{\Phi} = \int_V \dot{\psi} \rho \, dV \tag{7.10.19}$$

It follows that the stress power per unit volume ω is equal to $\dot{\psi}\rho$:

$$\omega = \mathbf{T} : \mathbf{D} = \dot{\psi} \rho \tag{7.10.20}$$

From (7.10.17) it follows that:

$$\dot{\psi} = \frac{\partial \psi}{\partial \mathbf{C}} : \dot{\mathbf{C}} = \frac{\partial \psi}{\partial C_{ij}} \dot{C}_{ij} \tag{7.10.21}$$

An expression for $\dot{\mathbf{C}}$ is found using (5.5.28): $\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$, where $\mathbf{L}$ is the velocity gradient tensor:

$$\dot{\mathbf{C}} = \dot{\mathbf{F}}^T \mathbf{F} + \mathbf{F}^T \dot{\mathbf{F}} = \mathbf{F}^T \mathbf{L}^T \mathbf{F} + \mathbf{F}^T \mathbf{L} \mathbf{F} = 2\mathbf{F}^T \mathbf{D} \mathbf{F}$$

Then (7.10.21) may be further developed:

$$\dot{\psi} = \frac{\partial \psi}{\partial C_{ij}} \dot{C}_{ij} = \frac{\partial \psi}{\partial C_{ij}} \left(2F_{ki} D_{kl} F_{lj}\right) \quad \Rightarrow \quad \dot{\psi} = \left(2\mathbf{F} \frac{\partial \psi}{\partial \mathbf{C}} \mathbf{F}^T\right) : \mathbf{D} \qquad (7.10.22)$$

This expression is substituted into (7.10.20):

$$\omega = \mathbf{T} : \mathbf{D} = \left(2\rho \mathbf{F} \frac{\partial \psi}{\partial \mathbf{C}} \mathbf{F}^T\right) : \mathbf{D} \qquad (7.10.23)$$

Because the tensor in the parenthesis is symmetric and independent of the rate of deformation tensor $\mathbf{D}$, and $\mathbf{D}$ may be chosen arbitrarily, we may conclude from the equation (7.10.23) that:

$$\mathbf{T} = 2\rho \mathbf{F} \frac{\partial \psi}{\partial \mathbf{C}} \mathbf{F}^T \qquad (7.10.24)$$

Note: When ψ is differentiated with respect to the symmetric tensor $\mathbf{C}$, the components C_{ij} must be treated as 9 independent quantities. Confer the commentaries to (7.6.10) and to (7.8.9). The result (7.10.24) represents the *general constitutive equation of a hyperelastic material*. The equation may be given an alternative form by introducing the strain tensor $\mathbf{E}$:

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}) \qquad (7.10.25)$$

the *second Piola-Kirchhoff stress tensor* $\mathbf{S}$ from (5.6.15):

$$\mathbf{S} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T} \qquad (7.10.26)$$

and the *continuity equation in a particle* (5.5.65):

$$\rho J = \rho_o \qquad (7.10.27)$$

From the constitutive equation (7.10.24) we then obtain the alternative form:

$$\mathbf{S} = \rho_o \frac{\partial \psi}{\partial \mathbf{E}} \qquad (7.10.28)$$

In the case of small deformation, we may set $\mathbf{S} = \mathbf{T}$. By introducing the elastic energy per unit volume:

$$\phi = \phi[\mathbf{E}, \mathbf{r}_o] = \rho_o \psi \qquad (7.10.29)$$

equation (7.10.28) may be expressed as:

$$\mathbf{T} = \frac{\partial \phi}{\partial \mathbf{E}} \tag{7.10.30}$$

Confer equation (7.6.10).

For isotropic materials we may express the elastic energy of the body as an isotropic scalar-valued function of the left deformation tensor $\mathbf{B}$:

$$\psi = \psi[\mathbf{B}, \mathbf{r}_o] = \psi(I_B, II_B, III_B) \tag{7.10.31}$$

I_B, II_B, and III_B are the principal invariants of $\mathbf{B}$. Now:

$$\dot{\psi} = \frac{\partial \psi}{\partial \mathbf{B}} : \dot{\mathbf{B}} \tag{7.10.32}$$

The expression for $\dot{\mathbf{B}}$ is found using (5.5.28), i.e. $\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$:

$$\dot{\mathbf{B}} = \dot{\mathbf{F}}\,\mathbf{F}^T + \mathbf{F}\,\dot{\mathbf{F}}^T = \mathbf{L}\,\mathbf{F}\,\mathbf{F}^T + \mathbf{F}\mathbf{F}^T\,\mathbf{L}^T = \mathbf{L}\,\mathbf{B} + \mathbf{B}\,\mathbf{L}^T$$

Hence:

$$\dot{\psi} = \frac{\partial \psi}{\partial \mathbf{B}} : \dot{\mathbf{B}} = \frac{\partial \psi}{\partial B_{ij}} \dot{B}_{ij} = \frac{\partial \psi}{\partial B_{ij}} \left(L_{ik} B_{kj} + B_{ik} L_{jk} \right) = \left(2 \frac{\partial \psi}{\partial \mathbf{B}} \mathbf{B} \right) : \mathbf{L} \tag{7.10.33}$$

The latter equality is due to the symmetric property of the tensor $\mathbf{B}$. Now, from (7.10.20) and (7.10.33) we obtain:

$$\omega = \mathbf{T} : \mathbf{D} = \mathbf{T} : \mathbf{L} = \dot{\psi} \rho = \left(2\rho \frac{\partial \psi}{\partial \mathbf{B}} \mathbf{B} \right) : \mathbf{L} \tag{7.10.34}$$

Because $\mathbf{L}$ may be chosen arbitrarily, we conclude from the expressions above that:

$$\mathbf{T} = 2\rho \frac{\partial \psi}{\partial \mathbf{B}} \mathbf{B} \tag{7.10.35}$$

Alternative forms of this result are represented by (7.10.9) and (7.10.10).

The *Mooney-Rivlin material* is an example of an isotrop, incompressible, non-linear hyperelastic material used as a material model for rubber. The model is defined by the specific elastic energy:

$$\psi = \frac{1}{2} \frac{\mu}{\rho} \left(\frac{1}{2} + \alpha \right) (I_B - 3) + \frac{1}{2} \frac{\mu}{\rho} \left(\frac{1}{2} - \alpha \right) (II_B - 3) \tag{7.10.36}$$

The material parameters μ and α are elasticities. From (7.10.35) we obtain the stress tensor:

$$\mathbf{T} = \mu \left(\frac{1}{2} + \alpha \right) \mathbf{B} - \mu \left(\frac{1}{2} - \alpha \right) \mathbf{B}^{-1} - p\,\mathbf{1} \tag{7.10.37}$$

Because the material is incompressible, an isotropic stress $-p\mathbf{1}$, representing an indeterminate pressure p, has to be added to the stress tensor. The special case with $\alpha = 1/2$ (7.10.37) defines the model called the *neo-Hookean material*.

Problems

Problem 7.1. Develop the form (7.2.8) of Hooke's law from the form (7.2.7).

Problem 7.2. Develop the decomposition (7.2.18) from Hooke's law, (7.2.8).

Problem 7.3. Develop Hooke's law for plane stress (7.3.6, 7.3.7, 7.3.8, 7.3.9) from the general law (7.2.6, 7.2.7, 7.2.8).

Problem 7.4. Develop Hooke's law for plane displacement (7.3.29, 7.3.30) from the general law (7.2.6, 7.2.7, 7.2.8).

Problem 7.5. A 45°–90° strain rosette, see Problem 5.5, is fixes to the surface of a machine part of steel. For a certain load the following strains are recorded:

$$\varepsilon_x = 560 \cdot 10^{-6},\ \varepsilon_y = 120 \cdot 10^{-6},\ \varepsilon_{45} = 200 \cdot 10^{-6}$$

Determine the principle stresses and the principle stress directions.

Problem 7.6. A thin-walled steel pipe, closed in both ends, is a part of a larger structure. The outer diameter of the pipe is 500 mm, and the wall thickness is 20 mm. The pipe is subjected to an axial tensile force N, a torque M, and an internal pressure p. A 45°–90° strain rosette, see Problem 5.5, is fixed to the surface of the pipe. The x-direction of the rosette is parallel to axis of the pipe. For a certain load on the structure the following strains are recorded:

$$\varepsilon_x = 620 \cdot 10^{-6}, \varepsilon_y = 320 \cdot 10^{-6}, \varepsilon_{45} = 230 \cdot 10^{-6}$$

Assume homogeneous state of strain in the pipe wall. Determine N, M and p for this load.

Problem 7.7. Show how the elastic energy per unit volume may be decomposed into volumetric energy and distortion energy as shown by the formulas (7.2.30).

Problem 7.8. Determin the displacement $u_2(x,y)$ according to the alternative boundary condition 2) by adding to the displacement $u_2(x,y)$ for the alternative boundary condition 1) found in Example 7.4 a rigid-body counterclockwise rotation given by the angle:

$$\alpha = -\left. u_{1,2}\right|_{x=L,y=0} = \frac{3\,(1+\nu)\,F}{E\,b\,h}$$

found from the displacement $u_2(x,y)$ for the alternative boundary condition 1).

Problem 7.9. Derive the Navier equations for plane displacements and for plane stress.

Problem 7.10. Derive the basic equations of thermoelasticity under the condition of plane stress and plane displacements. Check with Sect. 7.5.2 and Sect. 7.5.3.

Problem 7.11. Show that the state of stress given in Problem 3.3 satisfies the compatibility equation (7.3.45) for plane stress.

Problem 7.12. Show that the state of stress given in Problem 3.5 satisfies the compatibility equation (7.3.45) for plane stress.

Problem 7.13. A copper plate is restrained from expansion in the plane of the plate. $E = 118\,\text{GPa}, \nu = 0.33$, and $\alpha = 17 \cdot 10^{-6}\,{}^\circ\text{C}^{-1}$. Compute the stresses in the plate due to a temperature increase of $15\,{}^\circ$C.

Answer: –45 MPa.

Problem 7.14. A circular plate with a concentric hole is mounted on an undeformable shaft. The radius of the plate is b. The radius of the hole and of the shaft is a. The plate has the modulus of elasticity E and the Poisson's ratio ν. The plate is subjected to an external pressure p. Assume plane stress and determine the pressure against the shaft. Answer:

$$\frac{2}{(1-\nu)\,(a/b)^2+1+\nu}p$$

Problem 7.15. A simply supported horizontal beam has the length L and a rectangular cross section with horizontal width b and vertical height h. A coordinate system has the origin O on the beam axis a distance $L/2$ from left support, the x-axis along the beam axis, and a y-axis vertically downward. The beam is subjected to uniform pressure p on the surface $y = -h/2$.

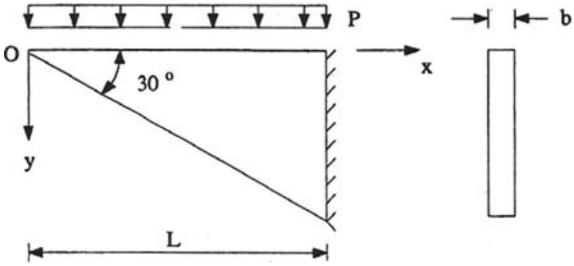
a) Show that the Airy's stress function:

$$\Psi = -\frac{p}{h^3}x^2y^3 + \frac{p}{5h^3}y^5 + \frac{3p}{4h}x^2y - \frac{p}{4}x^2 - \left[\frac{3}{2}\left(\frac{L}{h}\right)^2 - \frac{3}{5}\right]\frac{p}{6h}y^3$$

satisfies the compatibility equation (7.3.49).

b) Compute the stresses and show that they satisfy the boundary conditions on the surfaces $y = \pm h/2$. Show also the stress resultants over the cross section at $x = \pm L/2$ satisfy the support conditions.
c) Show that the stresses computed in b) satisfy the Cauchy equations for the beam.
d) Show that the stresses computed in b) gives the correct axial force, shear force and bending moment over the cross-section of the beam. Compare the stresses with those obtained from elementary (engineering) beam theory.

Fig. Problem 7.16



Problem 7.16. A triangular plate with constant thickness b is rigidly fixed along the side $x = L$. The plate is subjected to a uniform pressure p on the surface $y = 0$. The surface $y = x \tan 30°$ is free of stresses. The following state of plane stress is suggested:

$$\sigma_x = a\left[\frac{\pi}{6} - \arctan\frac{y}{x} - \frac{xy}{x^2+y^2}\right]p, \; a = \frac{6}{2\sqrt{3}-\pi}$$

$$\sigma_y = -a\left[\frac{1}{a} + \arctan\frac{y}{x} - \frac{xy}{x^2+y^2}\right]p, \; \tau_{xy} = -a\,\frac{y^2}{x^2+y^2}p$$

a) Show that the Cauchy equations are satisfied.
b) Show that the equation of compatibility (7.3.45) is satisfied.
c) Check that the boundary conditions on the surfaces $y = 0$ and $y = x\tan 30°$ are satisfied.
d) Determined the principal stresses, the principal stress directions, and the maximum shear stress at the surfaces $y = 0$ and $y = x\tan 30°$.
e) Consider the plate as a beam and compare the stresses σ_x and τ_{xy} with those obtained from the elementary (engineering) beam theory.

Problem 7.17. The rectangular plate in Fig. 7.3.12 is subjected to stresses $\sigma_x = \sigma$ on the sides $x = \pm b/2$ and $\sigma_y = -\sigma$ on the sides $y = \pm h/2$. Use the solution provided by Example 7.8 to determine the stresses σ_R, σ_θ, and $\tau_{R\theta}$. Determine the extremal principal stresses at the hole.

Problem 7.18. The rectangular plate in Fig. 7.3.12 is subjected to shear stresses $\tau_{xy} = \tau$ on the surfaces $x = \pm b/2$ and $y = \pm h/2$. Use the solution in Example 7.8, or Problem 7.17, to determine the stresses σ_R, σ_θ, and $\tau_{R\theta}$. Determine the extreme principal stresses at the hole.

Problem 7.19. A thin-walled pipe with external diameter $d = 210\,\mathrm{mm}$ and wall thickness $h = 10\,\mathrm{mm}$ is subjected to a torque $M = 24\,\mathrm{kNm}$ and an axial force $S = 280\,\mathrm{kN}$. A circular hole is cut through the wall of the pipe. Assume that the diameter of the hole is very small compared to the diameter d of the pipe. Compute the maximum normal stress in the wall. Hint: Superimpose the states of stress in Example 7.8 and the Problems 7.17 and 7.18. Answer: 221 MPa.

Problem 7.20. Consider the Prandtl stress function of torsion:

$$\Omega = k\,(x-a)\left(x-y\sqrt{3}+2a\right)\left(x+y\sqrt{3}+2a\right), \quad k = \text{ constant}$$

a) Show that Ω is the stress function of torsion of a cylindrical bar with an equilateral triangular cross section with side a.
b) Determine the distribution of stress on the cross section.
c) Show that relation between the torque M and the torsion angle ϕ per unit length of the bar is:

$$M = \frac{9\sqrt{3}}{5}a^4\mu\,\phi$$

Problem 7.21. Show that the condition $\phi \geq 0$ for elastic energy per unit volume implies the conditions (7.6.20) for the elastic parameters: $E \equiv \eta, G \equiv \mu$, and κ.

Problem 7.22. Derive the differential equation (7.4.14) for the warping function ψ directly from the Navier equations (7.6.28).

Problem 7.23. Show that the compatibility equations (7.6.43) implies for the Prandl's stress function Ω that:

$$\frac{\partial}{\partial x_\alpha}\left(\nabla^2\Omega\right) = 0 \quad \Rightarrow \quad \nabla^2\Omega = \text{constant}$$

Compare the result with the differential equations (7.4.22).

Problem 7.24. A unidirectional lamina denoted T300/5208 consists of graphite fibers imbedded in epoxy. The lamina has the following engineering parameters with respect to the lamina axes x and y:

$$\eta_x = 181 \text{ GPa}, \ \eta_y = 10.3 \text{ GPa}, \quad \mu = 7.17 \text{ GPa}, \ \nu_x = 0.28$$

Compute the following quantities:

a) ν_y.
b) The stiffness and compliance matrices $\bar{S}$ and $\bar{K}$.
c) The stiffness and compliance matrices S and K for $\phi = (\mathbf{e}_1, \mathbf{e}_x) = 45°$.
d) The engineering parameters with respect to the laminate axes x_1 and x_2.

Problem 7.25. A laminate is made of layers of the lamina T300/5208 described in Problem 7.24. The laminate is symmetric with respect to the x_1x_2-plane. The direction of the fibers makes an angle with respect to the x_1-axis of $\phi = 90°$ for 30 laminas and of $\phi = 0°$ for 20 laminas. The thickness of each lamina is 0.125 mm.

a) Determine the plate stiffness matrix A and the plate compliance matrix B.
b) A circular cylindrical container with internal diameter $d = 1200$ mm is made of the laminate. The x_1-direction is parallel to the axis of the cylinder. The container is subjected to an internal pressure $p = 8$ MPa. Determine the stresses T_1, T_2, and T_6.

Problem 7.26. Derive the following formulas for the modulus of elasticity η_1 and the shear modulus μ_6 with respect to the laminate axes for a unidirectional lamina with fibers making a 45° angle with the laminate axes.

$$\frac{1}{\eta_1} = \frac{1}{4}\left[\frac{1-2\nu_x}{\eta_x} + \frac{1}{\eta_y} + \frac{1}{\mu}\right], \frac{1}{\eta_6} = \frac{1+2\nu_x}{\eta_x} + \frac{1}{\eta_y}$$

Check if the formulas give the correct values for an isotropic laminate.

Chapter 8
Fluid Mechanics

8.1 Introduction

In continuum mechanics it is natural to define a fluid on the basis of what seems to be the most characteristic macromechanical aspects of liquids and gases as opposed to solid materials. Therefore we chose the following definition in Sect. 1.3:

A fluid is a material that deforms continuously when subjected to anisotropic states of stress.

An anisotropic state of stress in a particle implies surfaces through the particle subjected to shear stress. A fluid will therefore deform continuously when subjected to shear stresses. The fluid may be at rest without further deformation only when the state of stress is isotropic. This implies that the constitutive equations of any fluid at rest relative to any reference must reduce to:

$$\mathbf{T} = -p\,\mathbf{1}, \quad p = p(\rho, \theta) \tag{8.1.1}$$

p is the *thermodynamic pressure*, which is a function of the *density* ρ and the *temperature* θ. The relationship for p in (8.1.1) is called an *equation of state*.

An *ideal gas* is defined by the equation of state:

$$p = R\rho\,\theta \tag{8.1.2}$$

R is the *gas constant* for the gas, and θ is the absolute temperature, given in degrees Kelvin. The model ideal gas may be used with good results for many real gases, for example air.

Due to the large displacements and the chaotic motions of the fluid particles it is in general impossible to follow the motion of the individual particles. Therefore the physical properties of the particles or quantities related to particles are observed or described at fixed positions in space. In other words we employ *spatial description* and *Euler coordinates*. The primary kinematic quantity in Fluid Mechanics is the velocity vector $\mathbf{v}(\mathbf{r},t)$.

The concept of *streamlines* is introduced to illustrate fluid flow. The streamlines are *vector lines to the velocity field*, i.e. lines that have the velocity vector as a tangent in every point in the space of the fluid. The stream line pattern of a *non-steady flow*: $\mathbf{v} = \mathbf{v}(\mathbf{r},t)$, will in general change with time, see Problem 8.1. In a *steady flow*: $\mathbf{v} = \mathbf{v}(\mathbf{r})$, the streamlines coincide with the particle trajectories, called the *pathlines*. For a given velocity field $\mathbf{v}(\mathbf{r},t)$ the streamlines are determined from the differential equations:

$$d\mathbf{r} \times \mathbf{v}(\mathbf{r},t) = \mathbf{0} \quad \Leftrightarrow \quad \frac{dx_1}{v_1} = \frac{dx_2}{v_2} = \frac{dx_3}{v_3} \text{ at constant time } t \tag{8.1.3}$$

The pathlines are determined by the differential equation:

$$\dot{\mathbf{r}} = \mathbf{v}(\mathbf{r},t) \tag{8.1.4}$$

The streamlines through a closed curve in space form a *streamtube*.

The vector lines of the *vorticity field*:

$$\mathbf{c} = \operatorname{rot}\mathbf{v} \equiv \operatorname{curl}\mathbf{v} \equiv \nabla \times \mathbf{v} \tag{8.1.5}$$

are called *vortex lines*. The vortex lines through a closed curve in space form a *vortex tube*. If the velocity field is *irrotational*, which is the same as *vorticity free*, i.e. $\mathbf{c} = \mathbf{0}$, the velocity field may be developed from a velocity potential, see Theorem C.10:

$$\mathbf{v} = \nabla\phi, \quad \phi = \phi(\mathbf{r},t) \tag{8.1.6}$$

This kind of flow is called *potential flow* and will be discussed in Sect. 8.5. The scalar field ϕ is called the *velocity potential*.

The fundamental field equations of fluid mechanics are the Cauchy equations in the form:

$$\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = \frac{1}{\rho}\operatorname{div}\mathbf{T} + \mathbf{b}, \quad \partial_t v_i + v_k\, v_{i,k} = \frac{1}{\rho}\mathrm{T}_{ik,k} + \mathrm{b}_i \tag{8.1.7}$$

the continuity equation, to be presented in Sect. 8.2.3, constitutive equations that relate the stress tensor to the velocity field, and finally an energy equation. The energy equation of a linearly viscous fluid is discussed in Sect. 8.4.4. The model *perfect fluid*, also called the *Eulerian fluid*, and which does not transfer shear stresses even when the fluid is deforming, is treated in Sect. 8.3. The most important fluid model: the *linearly viscous fluid*, also called the *Newtonian fluid*, is presented in Sect. 8.4. Non-Newtonian fluids are presented in Sect. 8.6.

The governing equations of Fluid Mechanics do not provide unique solutions, in contrast to the equations of the classical theory of elasticity. It is often necessary to check whether an obtained solution is stable. For instance, in pipe flow steady state conditions give steady state flow provided the velocities are small enough. Increasing the level of velocities the flow may change into a chaotic non-steady flow. Osborne Reynolds [1842–1912] performed in 1883 an experiment illustrating

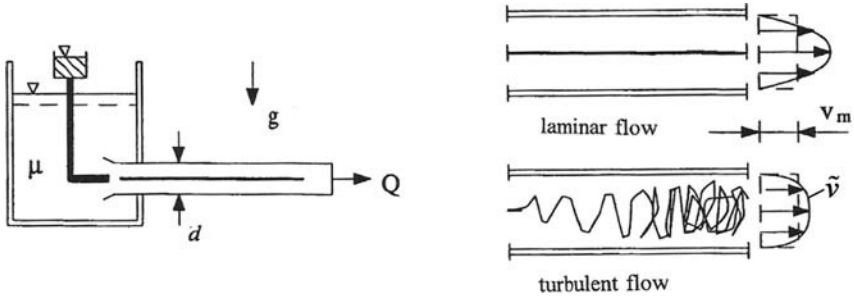


Fig. 8.1.1 The Reynolds experiment. Laminar and turbulent pipe flow

this phenomenon, see Fig. 8.1.1. A fluid of density ρ and viscosity μ flows through a pipe under steady state conditions. The quality of the flow is shown by injecting a thin colored fluid into the main flow. At low velocities the colored fluid is seen as a nearly straight line parallel to the axis of the pipe. This type of flow is called *laminar flow*: the fluid flows in cylindrical layers which move relative to each other. If the velocities are increased the colored fluid line becomes unstable and eventually disintegrates into a complex flow which gives the fluid in the pipe a general colored look. This type of flow is characterized as *turbulent flow*. The particle velocity at a specific place will vary strongly with time, but the time averaged velocity, or *time mean velocity* $\tilde{v}$, at the place over a certain time interval is constant. When a flow has become turbulent, it is customary to express the flow through the time mean velocity $\tilde{v}$. The equipments that record velocities may measure automatically the time average velocity at a place. Figure 8.1.1 shows the velocity distributions in the pipe for the two types of flow. Reynolds found in the experiment that the transition from laminar flow to turbulent flow is primarily dependent upon four factors: the *volumetric flow* Q, i.e. the fluid volume that per unit time flows through a cross-section of the pipe, the diameter of the pipe d, and the viscosity μ and the density ρ of the fluid. The result of the Reynolds' experiment may then be expressed thus: The *Reynolds number Re* defined by:

$$\mathrm{Re} = \rho \frac{v \cdot d}{\mu}, \; v \equiv v_m = \frac{4Q}{\pi d^2} \tag{8.1.8}$$

must be less than approx. 2000 for the flow to be laminar. In the expression for the Reynolds number v is the mean velocity over the cross-section of the pipe, $v = v_{\mathrm{m}} = Q/A$, where A is the cross-sectional area. For $Re > 2000$ the flow becomes turbulent. A Reynolds number may be defined for most flows, and we shall return to more general definition of the Reynolds number in Sect. 8.4 on linearly viscous fluids.

Figure 8.1.2 shows a rigid body in a *uniform flow*. Far away upstream from the body the velocity field is constant, independent of place and time. This situation occurs when a body moves with a constant velocity through a fluid at rest and when the reference for the motion is chosen fixed in the body. Apart from a thin region

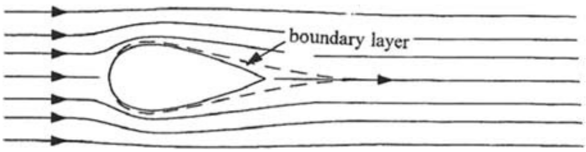


Fig. 8.1.2 Rigid Body in uniform flow. Free irrotational flow and boundary layer

near the surface of the body, we may neglect the viscosity of the fluid and assume irrotational flow. In the thin region near the surface of the body the viscosity has to be taken into account. This region is called the *boundary layer*.

Downstream of a rigid body in a flowing fluid a *wake* is created in which the flow is very chaotic and therefore is characterized as turbulent. The wake is due to the flow in the boundary layer. The fluid in the wake is rotational with a high content of vorticity. Bodies that create a narrow wake are often called *streamlined bodies* because the streamlines in a steady flow form a stable pattern surrounding the body, see Fig. 8.1.3. A rigid body creating a broad wake are called *blunt body*, see Fig. 8.1.3.

8.2 Control Volume. Reynolds' Transport Theorem

The fundamental laws of thermomechanics are: the principle of conservation of mass, the first and second axiom of Euler, the equation of mechanical energy balance, and the 1. law of thermomechanics. These laws are first expressed through equations for material bodies in motion, and the equations will be presented below in the form they are used when spacial or Eulerian description is chosen. This description is the natural one in Fluid Mechanics.

A fluid body of volume $V(t)$ and surface area $A(t)$ contains by definition the same mass m at any time t. The mass per unit volume at the place $\mathbf{r}$ and the present time t is represented by the *mass density*, or *density* for short, $\rho(\mathbf{r},t)$.

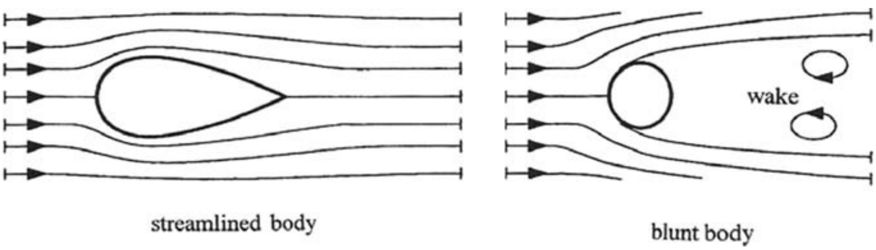


Fig. 8.1.3 Streamlined body with a negligible wake and blunt body with a broad wake

$$m = \int_{V(t)} \rho(\mathbf{r},t)\,dV = \text{constant} \tag{8.2.1}$$

Equation (8.2.1) expresses the *principle of conservation of mass*.

For a body of continuous matter we have in the Sect. 3.2.1, 6.1, and 6.3.2 defined a series of time-dependent *extensive quantities*, of which the most important are:

$$\mathbf{p}(t) = \int_{V(t)} \mathbf{v}\rho\,dV, \text{ linear momentum} \tag{8.2.2}$$

$$\mathbf{l}_O(t) = \int_{V(t)} \mathbf{r}\times\mathbf{v}\rho\,dV, \text{ angular momentum about a point } O \tag{8.2.3}$$

$$K(t) = \int_{V(t)} \frac{1}{2}v^2\rho\,dV, \text{ kinetic energy} \tag{8.2.4}$$

$$E(t) = \int_{V(t)} \varepsilon\,\rho\,dV, \text{ internal energy} \tag{8.2.5}$$

All these extensive quantities are expressed in the general form:

$$B(t) = \int_{V(t)} \beta\,\rho\,dV, \text{ extensive quantity} \tag{8.2.6}$$

$\beta = \beta(\mathbf{r},t)$ is a *specific intensive quantity*, representing the quantity per unit mass. The fundamental laws of thermomechanics contain material time derivatives of both intensive and extensive quantities. In Sect. 3.1.3 the following expression for the material derivative of an extensive quantity was developed. For the extensive quantity $B(t)$ we have:

$$\dot{B}(t) = \frac{d}{dt}\int_{V(t)} \beta\,\rho\,dV = \int_{V(t)} \dot{\beta}\,\rho\,dV \tag{8.2.7}$$

In Fluid Mechanics it is often more convenient to transform the equations of the fundamental laws to apply to a region fixed in space, or a region moving in a prescribed fashion. Such a region is called a *control volume* and is assumed to coincide with a fluid body with the volume $V(t)$ and the surface area $A(t)$ at the present time t, see Fig. 8.2.1. We shall first assume that the control volume V is fixed relative to the reference *Rf* chosen to describe the motion of the fluid. The surface A of the control volume V is called a *control surface*. To obtain the transformations of the equations of the laws of thermomechanics, which basically is meant to apply to a material body, such that they apply for a control volume, we shall derive an alternative expression for the material derivative of an extensive quantity B, which is related to the body.

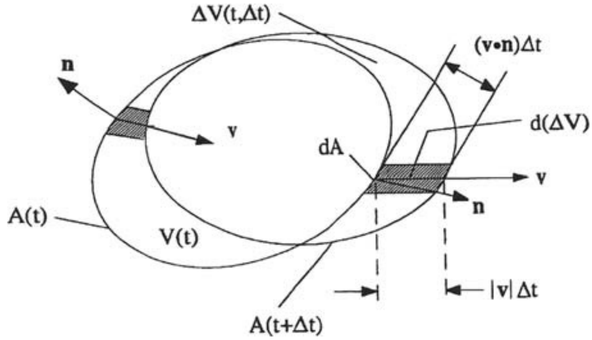


Fig. 8.2.1 Control volume V and control surface A

First let:

$$B(t) = \int_{V(t)} b\,dV,\; b = b(\mathbf{r},t) = \beta(\mathbf{r},t)\,\rho(\mathbf{r},t) \tag{8.2.8}$$

$b = \beta\rho$ is called *the density of the quantity* and expresses the quantity per unit volume. We may write:

$$\dot{B} = \lim_{\Delta t \to \infty} \frac{B(t+\Delta t) - B(t)}{\Delta t} \tag{8.2.9}$$

in which, see Fig. 8.2.1:

$$B(t+\Delta t) = \int_{V(t+\Delta t)} b(\mathbf{r},t+\Delta t)\,dV = \int_{V(t)} b(\mathbf{r},t+\Delta t)\,dV + \int_{\Delta V(t,\Delta t)} b(\mathbf{r},t+\Delta t)\,d(\Delta V)$$

$$\Delta V(t,\Delta t) = V(t+\Delta t) - V(t),\; d(\Delta V) = dA \cdot [(\mathbf{v}\cdot\mathbf{n})\,\Delta t] \tag{8.2.10}$$

The unit vector $\mathbf{n}$ is a normal to the control surface A directed out from the surface of the control volume. The third volume integral in (8.2.10) is transformed to a surface integral:

$$\int_{\Delta V(t,\Delta t)} b(\mathbf{r},t+\Delta t)\,d(\Delta V) = \int_{A(t)} b(\mathbf{r},t+\Delta t)\cdot[(\mathbf{v}\cdot\mathbf{n})\,\Delta t]\,dA \tag{8.2.11}$$

The contribution to the surface integral in (8.2.11) is positive whenever $\mathbf{v}\cdot\mathbf{n} > 0$ and mass is flowing out of the control volume V through the control surface A, and the contribution is negative whenever $\mathbf{v}\cdot\mathbf{n} < 0$ and mass is flowing into the control volume through the control surface A. We may now write:

$$\frac{B(t+\Delta t) - B(t)}{\Delta t} = \int_{V(t)} \frac{b(\mathbf{r},t+\Delta t) - b(\mathbf{r},t)}{\Delta t}\,dV + \int_{A(t)} b(\mathbf{r},t+\Delta t)\,(\mathbf{v}\cdot\mathbf{n})\,dA \tag{8.2.12}$$

Substitution of the result (8.2.12) into (8.2.9) yields:

$$\dot{B} = \int_V \frac{\partial b}{\partial t}\, dV + \int_A b\,(\mathbf{v}\cdot\mathbf{n})\, dA \tag{8.2.13}$$

This result expresses the *Reynolds' transport theorem*. The terms on the right side are:

term 1: The time rate of change of the quantity B inside the fixed control volume V.
term 2: The net flow out of the fixed control surface A.

Because the control volume V is fixed in space, the transport theorem may alternatively be expressed by:

$$\dot{B} = \frac{d}{dt}\int_V b\, dV + \int_A b\,(\mathbf{v}\cdot\mathbf{n})\, dA \tag{8.2.14}$$

In some application we choose a control volume V that moves and deforms. Let the velocity of points on the moving control surface be denoted $\bar{\mathbf{v}}(\mathbf{r},t)$. Equation (8.2.13) expressing the transport theorem is still valid, but (8.2.14) has to be replaced by:

$$\dot{B} = \frac{d}{dt}\int_{V(t)} b\, dV + \int_A b\,[(\mathbf{v}-\bar{\mathbf{v}})\cdot\mathbf{n}]\, dA \tag{8.2.15}$$

8.2.1 Alternative Derivation of the Reynolds' Transport Theorem

The result (8.2.13) may be derived directly using mathematics. The Theorem C.8 is first used to transform the integral in (8.2.8):

$$B(t) = \int_{V(t)} b\, dV = \int_{V_o} b\, J\, dV_o \tag{8.2.16}$$

V_o is the volume of the body in the reference configuration K_o, and J is the Jacobian to the deformation gradient:

$$J = \det \mathbf{F} = \det\ F_{ij} \equiv \det\left(\frac{\partial x_i}{\partial X_j}\right) \tag{8.2.17}$$

The transformation from the integral in (8.2.8) to the integral in (8.2.16) may also be obtained directly by the use of the results (5.5.64) and (5.5.65), from which:

$$dV = \frac{\rho_o}{\rho}\, dV_o = J\, dV_o \tag{8.2.18}$$

Because the volume V_o is independent of the time t, the material derivative of the integral in (8.2.8) may be obtained by performing the differentiation under the integral sign. Using formula (5.5.33):

$$\dot{J} = J \operatorname{div} \mathbf{v} \tag{8.2.19}$$

the following two formulas and Gauss' integral theorem C.3:

$$\dot{b} = \frac{\partial b}{\partial t} + \mathbf{v} \cdot \nabla b, \ \mathbf{v} \cdot \nabla b + b \operatorname{div} \mathbf{v} = \operatorname{div}(b\mathbf{v}), \ \int\limits_{V(t)} \operatorname{div}(b\mathbf{v})\, dV = \int\limits_{A} b\,(\mathbf{v} \cdot \mathbf{n})\, dA$$

we obtain:

$$\dot{B} = \int\limits_{V_o} \left(\dot{b}J + b\dot{J}\right) dV_o = = \int\limits_{V_o} \left(\dot{b} + b \operatorname{div} \mathbf{v}\right) J\, dV_o = \int\limits_{V(t)} \left(\frac{\partial b}{\partial t} + \mathbf{v} \cdot \nabla b + b \operatorname{div} \mathbf{v}\right) dV$$

$$= \int\limits_{V(t)} \left(\frac{\partial b}{\partial t} + \operatorname{div}(b\mathbf{v})\right) dV = \int\limits_{V(t)} \frac{\partial b}{\partial t}\, dV + \int\limits_{A} b\,(\mathbf{v} \cdot \mathbf{n})\, dA \quad \Rightarrow \tag{8.2.13}$$

8.2.2 Control Volume Equations

The Reynolds' transport theorem, (8.2.13), will now be used to transform the fundamental laws of thermomechanics for a body to a fixed control volume V with a control surface A.

The *principle of conservation of mass*, (8.2.1), implies:

$$\dot{m} = \frac{d}{dt} \int\limits_{V(t)} \rho\, dV = 0 \quad \Rightarrow \quad \int\limits_{V} \frac{\partial \rho}{\partial t}\, dV + \int\limits_{A} \rho\,(\mathbf{v} \cdot \mathbf{n})\, dA = 0 \tag{8.2.20}$$

The result (8.2.20) is called the *continuity equation for a control volume*.

The *law of balance of linear momentum*, Euler's 1. axiom (3.2.6):

$$\mathbf{f} = \dot{\mathbf{p}} = \frac{d}{dt} \int\limits_{V(t)} \mathbf{v}\rho\, dV \quad \Rightarrow$$

$$\int\limits_{V} \frac{\partial(\mathbf{v}\rho)}{\partial t}\, dV + \int\limits_{A} \mathbf{v}\rho\,(\mathbf{v} \cdot \mathbf{n})\, dA = \int\limits_{A} \mathbf{t}\, dA + \int\limits_{V} \mathbf{b}\rho\, dV \tag{8.2.21}$$

The *law of balance of angular momentum*, Euler's 2. axiom (3.2.7):

$$\mathbf{m}_O = \dot{\mathbf{l}}_O \equiv \frac{d}{dt} \int\limits_{V(t)} \mathbf{r} \times \mathbf{v}\rho\, dV \quad \Rightarrow$$

$$\int\limits_{V} \frac{\partial(\mathbf{r} \times \mathbf{v}\rho)}{\partial t}\, dV + \int\limits_{A} \mathbf{r} \times \mathbf{v}\rho\,(\mathbf{v} \cdot \mathbf{n})\, dA = \int\limits_{A} \mathbf{r} \times \mathbf{t}\, dA + \int\limits_{V} \mathbf{r} \times \mathbf{b}\rho\, dV \tag{8.2.22}$$

The *mechanical energy balance equation* (6.1.12):

$$\dot{K} = \frac{d}{dt}\int_{V(t)} \frac{v^2}{2}\,\rho\, dV = P - P^d \quad \Rightarrow$$

$$\int_V \frac{\partial}{\partial t}\left(\frac{v^2}{2}\rho\right) dV + \int_A \frac{v^2}{2}\rho\,(\mathbf{v}\cdot\mathbf{n})\, dA = \int_A \mathbf{t}\cdot\mathbf{v}\, dA + \int_V \mathbf{b}\cdot\mathbf{v}\,\rho\, dV - \int_V \mathbf{T}:\mathbf{D}\, dV \tag{8.2.23}$$

The *first law of thermomechanics*, (6.3.11):

$$\dot{E} = \frac{d}{dt}\int_{V(t)} \varepsilon\,\rho\, dV = \int_A q\, dA + P^d \quad \Rightarrow$$

$$\int_V \frac{\partial}{\partial t}(\varepsilon\,\rho)\, dV + \int_A \varepsilon\,\rho\,(\mathbf{v}\cdot\mathbf{n})\, dA = \int_A q\, dA + \int_V \mathbf{T}:\mathbf{D}\, dV \tag{8.2.24}$$

Example 8.1. Forces on a Turbine Vane

Figure 8.2.2a shows a vane hit by a water jet of cross-section A. The velocity of the water hitting the vane is $v_{in} = v$. The jet leaves the vane with the velocity $v_{out} = v$ and in a direction that makes an angle θ with respect to the direction of the incoming yet. Referring to Fig. 8.2.2b, we want to determine the forces K_x and K_y and the couple moment M at the point O where the vane is attached to a foundation, and which are due to the action of the water jet.

A control volume V is selected as shown by the dashed line in Fig. 8.2.2b. The law of balance of linear momentum (8.2.21) applied to this control volume gives:

$$\int_A \mathbf{v}\rho\,(\mathbf{v}\cdot\mathbf{n})\, dA = \int_A \mathbf{t}\, dA \quad \Rightarrow$$

$$v_{out}\cos\theta\cdot\rho\cdot(v_{out})A + v_{in}\cdot\rho\cdot(-v_{in})A = -K_x,\; v_{out}\sin\theta\cdot\rho\cdot(v_{out})A = K_y$$

The law of balance of angular momentum (8.2.22) with O as moment point and applied to the control volume V, reads:

$$\int_A \mathbf{r}\times\mathbf{v}\rho\,(\mathbf{v}\cdot\mathbf{n})\, dA = \int_A \mathbf{r}\times\mathbf{t}\, dA \quad \Rightarrow$$

$$-h\cdot(v_{out}\cos\theta)\cdot\rho\cdot(v_{out})A + c\cdot(v_{out}\sin\theta)\cdot\rho\cdot(v_{out})A - h\cdot v_{in}\cdot\rho\cdot(-v_{in})A = M$$

From these three equations we get the result:

$$K_x = \rho A v^2(1-\cos\theta),\; K_y = \rho A v^2\sin\theta,\; M = \rho A v^2\left[c\sin\theta + h(1-\cos\theta)\right]$$

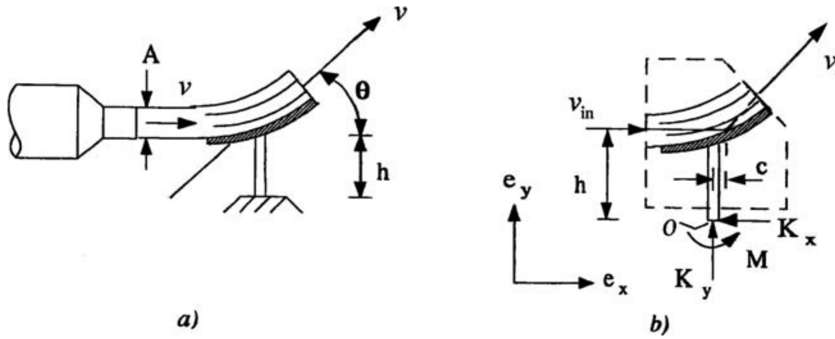


Fig. 8.2.2 Vane in a water jet

8.2.3 Continuity Equation

The *continuity equation at a place*, commonly just called the *continuity equation* in Fluid Mechanics, shall now be derived from the continuity (8.2.20) for a control volume. Using the Gauss' theorem C.3, (8.2.20) is rewritten to:

$$\int_V \frac{\partial \rho}{\partial t} dV + \int_A \rho \, (\mathbf{v} \cdot \mathbf{n}) \, dA = \int_V \left[\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \, \mathbf{v}) \right] dV = 0$$

Because the equation must apply to an arbitrary control volume V, the integrand must be zero. Hence:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \, \mathbf{v}) = 0 \quad \Leftrightarrow \quad \frac{\partial \rho}{\partial t} + (\rho \, v_i)_{,i} = 0 \tag{8.2.25}$$

Alternatively the result may be written as:

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0 \quad \Leftrightarrow \quad \dot{\rho} + \rho \, v_{i,i} = 0 \tag{8.2.26}$$

Equations (8.2.25, 8.2.26) are alternative expressions for the *equation of continuity at a place*. For an incompressible fluid $\dot{\varepsilon}_v = \operatorname{div} \mathbf{v} = 0$, and (8.2.26) implies that $\dot{\rho} = 0$, i.e. the density in a fluid particle is constant. Note that $\partial_t \rho$ not necessarily has to be zero. The equality $\partial_t \rho = 0$ does apply only in the case the fluid is homogeneous, which means that $\rho(\mathbf{r}, t) =$ constant. For incompressible fluids the equation of continuity, (8.2.25) or (8.2.26) may be replaced by the *condition of incompressibility*:

$$\operatorname{div} \mathbf{v} = 0 \quad \Leftrightarrow \quad v_{i,i} = 0 \tag{8.2.27}$$

The equation of continuity (8.2.25) or (8.2.26) can alternatively be derived as follows. Let the element of volume dV represent a small body of mass ρdV, where ρ is the mean value of the density in dV. The time rate of change of the volume dV may be expressed by the volumetric strain:

$$\dot{\varepsilon}_v = \operatorname{div}\mathbf{v} \quad \Rightarrow \quad d\dot{V} = (\operatorname{div}\mathbf{v})\, dV \tag{8.2.28}$$

Since the mass of a body is constant, we obtain:

$$\frac{d}{dt}(\rho\, dV) = \dot{\rho}\, dV + \rho\, d\dot{V} = (\dot{\rho} + \rho \operatorname{div}\mathbf{v})\, dV = 0 \quad \Rightarrow \quad \dot{\rho} + \rho \operatorname{div}\mathbf{v} = 0 \Rightarrow (8.2.26)$$

In a third alternative derivation of the equation of continuity at a place (8.2.26), we start with the *continuity equation in a particle* (5.5.65):

$$\rho\, J = \rho_o \tag{8.2.29}$$

Using (8.2.19), we obtain by material differentiation of (8.2.29):

$$\frac{d}{dt}(\rho\, J) = \dot{\rho}\, J + \rho\, \dot{J} = (\dot{\rho} + \rho \operatorname{div}\mathbf{v})\, J = 0 \quad \Rightarrow \quad \dot{\rho} + \rho \operatorname{div}\mathbf{v} = 0 \quad \Rightarrow \quad (8.2.26)$$

8.3 Perfect Fluid ≡ Eulerian Fluid

In many practical applications we may neglect shear stresses in fluid flows. For instance, when analyzing the flow of a liquid or a gas surrounding rigid bodies, it may often be sufficient to take into consideration the viscosity of the fluid only in a relatively thin layer, the *boundary layer*, near the solid surfaces. Outside of the boundary layer the shear stresses may be neglected, and the liquid or gas may be modelled as an *inviscid fluid*, i.e. a fluid without viscosity. The fluid model is called a *perfect fluid* or a *Eulerian fluid*. The constitutive equation of a perfect fluid is:

$$\mathbf{T} = -p\mathbf{1} \quad \Leftrightarrow \quad T_{ij} = -p\,\delta_{ij} \tag{8.3.1}$$

$$p = p(\rho, \theta) \tag{8.3.2}$$

$\rho(\mathbf{r},t)$ is the density of the fluid and $\theta(\mathbf{r},t)$ is the temperature in the fluid. The perfect fluid gives the simplest example of a *thermoelastic material*. The elasticity is expressed by the fact that the pressure $p(\rho,\theta)$ is a function of the density, which again is a function of the volumetric strain, as implied by (5.5.64) and (5.2.23).

The compressibility of a fluid may often be disregarded. Liquids are only rarely considered to be compressible. Gases, which are relatively easily compressible, may also in many practical cases be modelled as incompressible media. In elementary aerodynamics the compressibility of air may be neglected when the velocity of the flying body is less than approx. 1/3 of the speed of sound in air. For incompressible perfect fluids it is customary to replace formula (8.3.2) by:

$$p = p(\mathbf{r}, \theta) \tag{8.3.3}$$

because the pressure p no longer can be considered to be a state variable.

The motion of a perfect fluid is governed by the following four equations:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{b} \qquad \text{the Euler equations} \tag{8.3.4}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\,\mathbf{v}) = 0 \qquad \text{the continuity equation} \tag{8.3.5}$$

The *Euler equations* follow from the Cauchy equations (8.1.7) and the constitutive equations (8.3.1). The 4 equations (8.3.4) and (8.3.5) contain 5 unknown functions v_i, ρ, and p. In general the set of equations is supplemented by an energy equation and by an equation of state, for example (8.3.2), which then introduces the temperature θ as an additional 6. unknown field function.

It is often reasonable to assume as a boundary condition for the velocity field of a real fluid that the velocity relative to a rigid surface is zero, i.e. the fluid sticks to the rigid surface. For a perfect fluid only the relative velocity component normal to the boundary surface may be assumed to be zero.

An *ideal gas* is a perfect fluid having the equation of state:

$$p = R\rho\,\theta \tag{8.3.6}$$

R is the gas constant of the gas, and θ is the absolute temperature in degrees Kelvin [$^{\mathrm{o}}$K]. This fluid model may be applied with success for many real gases, for instance air for which $R = 287\,\mathrm{Nm/kg^oK} = 287\,\mathrm{m^2/s^{2\,o}K}$.

We call a deformation process polytropic if the equation of state for the pressure may be presented as:

$$p = p_o\left(\frac{\rho}{\rho_o}\right)^{\alpha} \tag{8.3.7}$$

α is a constant, and p_o and ρ_o are reference values for pressure and density. The following known processes are represented by (8.3.7):

$$\begin{aligned} &a)\ \text{Isobaric process} &&\Leftrightarrow\quad \text{constant pressure field: } \alpha = 0 \\ &b)\ \text{Isothermal process} &&\Leftrightarrow\quad \text{constant temperature field: } \alpha = 1 \\ &c)\ \text{Isentropic process} &&\Leftrightarrow\quad \text{constant entropy field: } \alpha = \kappa = c_p/c_v \end{aligned} \tag{8.3.8}$$

In the case c), where the specific entropy is constant, c_p and c_v are the *specific heats* at constant pressure and constant volume respectively.

The motion of a perfect fluid is called *barotropic* if a one-to one relation exists between pressure and density:

$$p = p(\rho) \quad \Leftrightarrow \quad \rho = \rho(p) \tag{8.3.9}$$

If we can assume that a barotropic relation exists in a particular problem, we may consider (8.3.9) as a property of the fluid. A fluid with (8.3.1) and (8.3.9) as constitutive equations is called an *elastic fluid*. The literature also applies the names *barotropic fluid*, autobarotropic fluid, and piezotropic fluid in this case. An incompressible fluid is also called a barotropic fluid, but this fluid can obviously not be considered to be elastic.

An elastic fluid, defined by the constitutive equations (8.3.1) and (8.3.9), is hyperelastic, as defined in the Sect. 7.6.1 and 7.10.2. To see this, we start by forming the stress power supplied to a fluid body per unit volume, and then using the continuity equation (8.2.26):

$$\omega = \mathbf{T} : \mathbf{D} = -p\delta_{ij}D_{ij} = -pD_{ii} = -p \operatorname{div} \mathbf{v} = p\,\frac{\dot{\rho}}{\rho} \quad \Rightarrow$$

$$\omega = \mathbf{T} : \mathbf{D} = p\,\frac{\dot{\rho}}{\rho} \tag{8.3.10}$$

We introduce the potential:

$$\psi = \int_{p_o}^{p} \frac{d\bar{p}}{\rho} - \frac{p}{\rho} = \int_{\rho_o}^{\rho} \frac{p}{\bar{\rho}^2}\,d\bar{\rho} + \text{ constant} \tag{8.3.11}$$

The task to show that the two expressions (8.3.11) for the potential ψ are equivalent, is left as an exercise in Problem 8.6. We now find that:

$$\omega = \rho\,\dot{\psi} \tag{8.3.12}$$

The stress power supplied to a body of volume V is:

$$P^d = \int_V \omega\,dV = \int_V \dot{\psi}\rho\,dV = \frac{d}{dt}\int_V \psi\rho\,dV = \dot{\Psi} \tag{8.3.13}$$

where:

$$\Psi = \int_V \psi\rho\,dV \tag{8.3.14}$$

The result (8.3.13) shows that the stress power P^d may be derived from a potential Ψ, and this proves that the elastic fluid is hyperelastic. The field $\psi = \psi(\mathbf{r},t)$ is the *elastic energy per unit mass*, i.e. the *specific elastic energy*, and Ψ is the elastic energy of the body.

8.3.1 Bernoulli's Equation

From the definition (8.3.11) we find:

$$\nabla\psi = \frac{d\psi}{d\rho}\nabla\rho = \frac{p}{\rho^2}\nabla\rho$$

Then the first term on the right-hand side of the Euler equation (8.3.4) may be transformed:

$$-\frac{1}{\rho}\nabla p = -\nabla\left(\frac{p}{\rho}\right) - \frac{p}{\rho^2}\nabla\rho = -\nabla\left(\frac{p}{\rho} + \psi\right) \tag{8.3.15}$$

For barotropic fluids moving in a conservative body force field **b(r)**, such that:

$$\mathbf{b}(\mathbf{r}) = -\nabla\beta \tag{8.3.16}$$

where $\beta = \beta(\mathbf{r})$ is the force potential, the Euler equations (8.3.4) may be presented as:

$$\dot{\mathbf{v}} \equiv \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\left(\frac{p}{\rho} + \psi + \beta\right) \tag{8.3.17}$$

$\beta = \beta(\mathbf{r})$ may also interpreted as the potential energy due to the conservative body force **b(r)**. The result (8.3.17) shows that the acceleration of a barotropic fluid in a conservative force field may be found from a scalar potential. From this result we may derive a series of important theorems.

Bernoulli's theorem for steady flow of a barotropic fluid: There exist surfaces, called *Bernoulli surfaces*, covered by stream lines and vortex lines, and defined by:

$$\frac{1}{2}v^2 + \pi + \beta = \text{ constant} \tag{8.3.18}$$

The theorem is named after Daniel Bernoulli [1700–1782].

Proof. Using the identity c) in Problem 2.9 we can rewrite (8.3.17) to:

$$\partial_t\mathbf{v} + \mathbf{c}\times\mathbf{v} = -\nabla\left(\frac{1}{2}v^2 + \frac{p}{\rho} + \psi + \beta\right) \tag{8.3.19}$$

c is the vorticity. In steady flows $\partial_t\mathbf{v} = 0$, and since the vector $\mathbf{c}\times\mathbf{v}$ at a place is normal to the stream line and the vortex line through the place, the left-hand side of (8.3.19) is zero. This fact proves the theorem.

From (8.3.18) follows the *Bernoulli equation*:

$$\frac{1}{2}v^2 + \frac{p}{\rho} + \psi + \beta = \text{ constant along a stream line} \tag{8.3.20}$$

For incompressible fluids the specific elastic energy ψ is constant and will for convenience be set equal to zero. It is now necessary to require a steady pressure because the pressure in an incompressible medium may be changed uniformly without influencing the motion. The reason for this is that velocity of pressure propagation, i.e. the velocity of sound, is infinite for incompressible media.

The Bernoulli equation may be interpreted as a statement of conservation of mechanical energy. In order to see that we apply (8.2.23) for mechanical energy balance on a control volume V of a stream tube between to surfaces A_1 and A_2 normal to the stream line, as shown in Fig. 8.3.1. The control surface A consists of the surfaces A_1, A_2, and A_3.

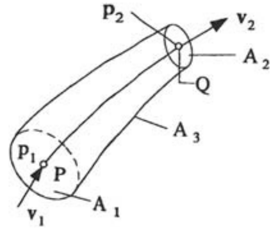


Fig. 8.3.1 Control volume V of a streamtube between the surfaces A_1 and A_2 normal to the stream lines. The control surface A consists of the surfaces A_1, A_2, and A_3

First we perform some initial manipulations. For steady flow the continuity (8.2.25) yields: $\nabla \cdot (\mathbf{v}\rho) = 0$. Equation (8.3.12) and formula (3.1.16) for the material derivative of the steady field $\psi(\mathbf{r})$ imply that:

$$\mathbf{T} : \mathbf{D} \equiv \omega = \rho\,\dot{\psi} = \rho\, v_i\, \psi_{,i} = \nabla \cdot (\psi \rho\, \mathbf{v})$$

Furthermore, with $\mathbf{t} = -p\mathbf{n}$ as the stress vector on any material surface and $\mathbf{b}$ as the body force defined by (8.3.16), we get:

$$\mathbf{t} \cdot \mathbf{v} = (-p\,\mathbf{n}) \cdot \mathbf{v} = -\frac{p}{\rho}(\rho\,\mathbf{v} \cdot \mathbf{n}),\; \mathbf{b} \cdot \mathbf{v}\rho = -\nabla\beta \cdot \mathbf{v}\rho = -\nabla \cdot (\beta\,\mathbf{v}\rho)$$

Using Gauss integration theorem C.3, we then obtain:

$$\int_V [\mathbf{b} \cdot \mathbf{v}\rho - \mathbf{T} : \mathbf{D}]\, dV = \int_V [-\nabla \cdot (\beta\,\mathbf{v}\rho + \psi\rho\,\mathbf{v})]\, dV = -\int_A [\beta + \psi\,]\,\rho\,\mathbf{v} \cdot \mathbf{n}\, dA$$

The energy equation (8.2.23) applied to the control volume V now becomes:

$$\int_A \left[\frac{1}{2}v^2 + \frac{p}{\rho} + \psi + \beta\right] \rho\,\mathbf{v} \cdot \mathbf{n}\, dA = 0 \tag{8.3.21}$$

On the surface A_3 of the stream tube in Fig. 8.3.1 $\mathbf{v} \cdot \mathbf{n} = 0$. The mean value theorem C.6 then gives:

$$\left[\frac{1}{2}v^2 + \frac{p}{\rho} + \psi + \beta\right]_2 \int_{A_2} \rho\,\mathbf{v} \cdot \mathbf{n}\, dA + \left[\frac{1}{2}v^2 + \frac{p}{\rho} + \psi + \beta\right]_1 \int_{A_1} \rho\,\mathbf{v} \cdot \mathbf{n}\, dA = 0$$

The terms $[\,]_1$ and $[\,]_2$ are calculated at places on A_1 and A_2, respectively. According to the continuity equation (8.2.20) the two integrals must, be equal but of opposite signs. If we let A_1 approach zero about P_1 and let A_2 approach zero about P_2, where P_1 and P_2 are two places on the same streamline, we obtain the result (8.3.20). All terms in (8.3.20) represent specific energies, i.e. energies per unit mass: $v^2/2$ is kinetic energy, $(p/\rho + \beta)$ is potential energy, and ψ is elastic energy.

A *Bernoulli equation for non-steady flow* is obtained by integration of (8.3.19) along a streamline between two points P_1 and P_2. The result is:

$$\int_{P_1}^{P_2} \frac{\partial \mathbf{v}}{\partial t} \cdot d\mathbf{r} + \left[\frac{1}{2}v^2 + \frac{p}{\rho} + \psi + \beta\right]_{P_2} = \left[\frac{1}{2}v^2 + \frac{p}{\rho} + \psi + \beta\right]_{P_1} \tag{8.3.22}$$

An important family of flows, presented in Sect. 8.5, are called *potential flows* or *irrotational flows*. These flows are characterized by zero vorticity: $\mathbf{c} = \nabla \times \mathbf{v} = \mathbf{0}$, which implies that the velocity may be expressed by a velocity potential ϕ, such that $\mathbf{v} = \nabla\phi$. From (8.3.19) we then get:

$$\nabla\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}v^2 + \frac{p}{\rho} + \psi + \beta\right] = 0$$

This result implies:
Bernoulli's theorem for irrotational flow of a barotropic fluid:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}v^2 + \frac{p}{\rho} + \psi + \beta = f(t) \quad \text{in the fluid} \tag{8.3.23}$$

where $f(t)$ is a function of time.

The result is also called the *Euler pressure equation*. For steady state flows (8.3.23) is reduced to:

$$\frac{1}{2}v^2 + \frac{p}{\rho} + \psi + \beta = \text{constant in the fluid} \tag{8.3.24}$$

For incompressible fluids the specific elastic energy ψ is set equal to zero.

Example 8.2. The Torricelli Law

We want to determine the exit velocity v through the orifice in an open vessel containing a fluid. The vessel is assumed to have a large free surface as compared to the area of the orifice. The fluid is subjected to the constant specific gravitational body force g for which the specific potential energy is:

$$\beta = gz$$

We consider the streamline indicated in Fig. 8.3.2, from point A at the free surface to point B at the orifice. The fluid velocity at A may be neglected. The Bernoulli equation (8.3.24) then gives:

$$\frac{p_o}{\rho} + g(h_1 + h) = \frac{v^2}{2} + \frac{p_o}{\rho} + gh_1 \quad \Rightarrow \quad v = \sqrt{2gh}$$

The result is called *Torricelli's law*, named after Evangelista Torricelli [1608–1647].

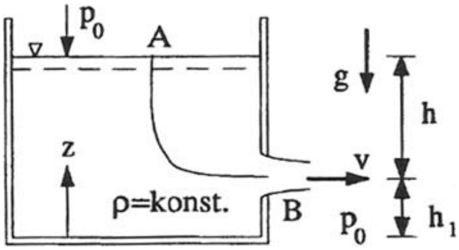


Fig. 8.3.2 Efflux from an orifice in an open vessel

8.3.2 Circulation and Vorticity

We shall discuss some properties of the vorticity field $\mathbf{c}(\mathbf{r},t)$ of barotropic fluids in conservative force fields. From (8.3.19) it follows that:

$$\nabla \times \partial_t \mathbf{v} + \nabla \times (\mathbf{c} \times \mathbf{v}) = 0 \tag{8.3.25}$$

The two terms will now in turn be transformed. The definition of the vorticity $\mathbf{c}$ yields:

$$\nabla \times \partial_t \mathbf{v} = \partial_t \mathbf{c} \tag{8.3.26}$$

The following identity is to be derived as Problem 8.7:

$$\nabla \times (\mathbf{c} \times \mathbf{v}) = \mathbf{c}\,\mathrm{div}\,\mathbf{v} + (\mathbf{v} \cdot \nabla)\;\mathbf{c} - (\mathbf{c} \cdot \nabla)\;\mathbf{v} \tag{8.3.27}$$

The formulas (8.3.26–8.3.27) are substituted into (8.3.25), and the result is:

$$\dot{\mathbf{c}} \equiv \partial_t \mathbf{c} + (\mathbf{v} \cdot \nabla)\;\mathbf{c} = (\mathbf{c} \cdot \nabla)\;\mathbf{v} - \mathbf{c}\,\mathrm{div}\,\mathbf{v} \tag{8.3.28}$$

Using the continuity (8.2.26), we can rewrite (8.3.28) to obtain *Nanson's formula* [E. Nanson 1874]:

$$\frac{d}{dt}\left(\frac{\mathbf{c}}{\rho}\right) = \mathbf{L} \cdot \frac{\mathbf{c}}{\rho} \tag{8.3.29}$$

$\mathbf{L} =$ grad $\mathbf{v}$ is the velocity gradient tensor, and the time derivative is the material derivative. The solution of this differential equation is:

$$\frac{\mathbf{c}}{\rho} = \frac{\mathbf{F} \cdot \mathbf{c}_o}{\rho_o} \tag{8.3.30}$$

$\mathbf{F}$ is the deformation gradient tensor, and $\mathbf{c}_o$ and ρ_o are respectively the vorticity and the density in the reference configuration K_o. Application of $(5.5.28)_1$ in the material derivative of (8.3.30) will show that (8.3.30) is the solution to (8.3.29).

From (8.3.30) we may draw two important conclusions:

a) *The vortex lines are material lines*: Let $d\mathbf{r}_o$ be tangent at the particle X to a material line that also is a vortex line in the reference configuration K_o, i.e $d\mathbf{r}_o$ is parallel to the vorticity $\mathbf{c}_o$. Then we may set $d\mathbf{r}_o = \mathbf{c}_o d\tau$, where $d\tau$ is a constant. In the present configuration K the material line has a tangent at the particle X that is parallel to $d\mathbf{r} = \mathbf{F} \cdot d\mathbf{r}_o$. Thus from (8.3.30):

$$d\mathbf{r} = \mathbf{F} \cdot d\mathbf{r}_o = \mathbf{F} \cdot \mathbf{c}_o d\tau = \frac{\rho_o}{\rho}\mathbf{c}\, d\tau \tag{8.3.31}$$

The result shows that the tangent vector $d\mathbf{r}$ is parallel to the vorticity $\mathbf{c}$. This means that the material line is identical to the vortex line at all times. Using (8.3.31) and the relations $d\mathbf{r}_o = \mathbf{c}_o d\tau$ and $ds_\mathrm{o} = |d\mathbf{r}_\mathrm{o}|$, we get this formula for the line differential $ds = |d\mathbf{r}|$:

$$ds = \frac{\rho_o}{\rho}\frac{c}{c_o}\, ds_o \tag{8.3.32}$$

b) *The Lagrange-Cauchy theorem*: If a barotropic fluid subjected to a conservative body force field has irrotational motion, $\mathbf{c} = \mathbf{0}$, at a certain time, the motion will be irrotational at all times. This theorem may be used to identify potential flows.

Let C be a closed material line in a flowing fluid. The *circulation* around C is defined by the integral:

$$\Gamma(t) = \oint_C \mathbf{v} \cdot d\mathbf{r} \tag{8.3.33}$$

The *Kelvin circulation theorem*, named after William Thomson, Lord Kelvin [1824–1907] states that: For a barotropic fluid in a conservative force field the circulation around a closed material line C is time independent:

$$\Gamma(t) = \oint_C \mathbf{v} \cdot d\mathbf{r} = \text{constant} \tag{8.3.34}$$

Proof. Let the closed curve C be represented by $\mathbf{r} = \mathbf{r}(s_o, t)$, where s_o is the arc length parameter in the reference configuration K_o, and $0 \le s_o \le L_o$, where L_o is the length of the curve in K_o. Then the circulation around the curve C at time t is:

$$\Gamma(t) = \int_0^{L_o} \mathbf{v} \cdot \frac{\partial \mathbf{r}}{\partial s_o} ds_o$$

Because:

$$\mathbf{v} \cdot \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{r}}{\partial s_o}\right) = \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial s_o} = \frac{\partial}{\partial s_o}\left(\frac{\mathbf{v} \cdot \mathbf{v}}{2}\right) = \frac{\partial}{\partial s_o}\left(\frac{v^2}{2}\right)$$

and the curve C is closed, we obtain:

$$\dot{\Gamma}(t) = \int_0^{L_o} \dot{\mathbf{v}} \cdot \frac{\partial \mathbf{r}}{\partial s_o} ds_o = \oint_C \dot{\mathbf{v}} \cdot d\mathbf{r} \tag{8.3.35}$$

For a barotropic fluid in a conservative force field (8.3.17) applies. Thus we get:

$$\dot{\Gamma}(t) = \oint_C \dot{\mathbf{v}} \cdot d\mathbf{r} = -\oint_C \nabla\left(\frac{p}{\rho} + \psi + \beta\right) \cdot d\mathbf{r} = -\oint_C d\left(\frac{p}{\rho} + \psi + \beta\right) = 0 \quad (8.3.36)$$

This result proves the theorem.

The circulation around a closed curve C may also be computed from the vorticity $\mathbf{c}$. Let A be any surface bounded by the curve C. Then, using Stokes's theorem C.5, we get the result:

$$\Gamma(t) = \oint_C \mathbf{v} \cdot d\mathbf{r} = \int_A \mathbf{c} \cdot \mathbf{n}\, dA \quad (8.3.37)$$

$\mathbf{n}$ is the unit normal to the surface A. This result together with Kelvin's theorem gives an alternative proof of the Lagrange-Cauchy theorem.

Kelvin's theorem implies *the three vortex theorems of Helmholtz*, named after Herman Ludwig Ferdinand von Helmholtz [1821–1894]. Kelvin presented his theorem in 1869 to prove the three vortex theorems.

Theorem 1. The circulation is the same about any closed curve surrounding a vortex tube.

Theorem 2. The vortex lines are material lines.

Theorem 3. The *strength of a vortex tube*, defined by the surface integral in (8.3.37), is constant.

Proof. Figure 8.3.3 shows two curves C_1 and C_2 surrounding a vortex tube. Using Stokes' theorem C.5, we get:

$$\oint_{C_1} \mathbf{v} \cdot d\mathbf{r} - \oint_{C_2} \mathbf{v} \cdot d\mathbf{r} = \oint_C \mathbf{v} \cdot d\mathbf{r} = \int_A \mathbf{c} \cdot \mathbf{n}\, dA$$

A is a surface bounded by the curve C, marked in Fig. 8.3.3 by a broken line, and $\mathbf{n}$ is the unit normal to A. Since the vectors $\mathbf{n}$ and $\mathbf{c}$ are orthogonal, $\mathbf{c} \cdot \mathbf{n} = 0$, and we may conclude that:

$$\oint_{C_1} \mathbf{v} \cdot d\mathbf{r} = \oint_{C_2} \mathbf{v} \cdot d\mathbf{r} \quad (8.3.38)$$

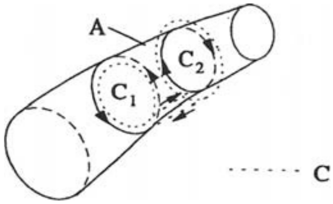


Fig. 8.3.3 Vortex tube

This result proves Theorem 1. Theorem 2 is already proved by (8.3.31). The strength of the vortex tube may through Theorem 1 be expressed by the circulation around any curve surrounding the vortex tube. Kelvin's theorem then completes the proof of Theorem 3.

8.3.3 Sound Waves

Sound propagates as elastic waves. In fluids the elastic waves represent small variations in the pressure. The loudest sound the human ear can receive without pain corresponds to an amplitude of the pressure variation of 28 Pa. The sound pressure $p_t = 28\,\text{Pa}$ is therefore called the *threshold of pain*. The weakest sound the human ear can hear is called the *threshold of hearing* and corresponds to a sound pressure of about $2 \cdot 10^{-5}\,\text{Pa}$. For comparison the normal atmospheric pressure is $p_o = 101.32\,\text{kPa}$.

Propagation of sound is an isentropic process governed by the Euler equations (8.3.4), the equation of continuity (8.3.5), and a constitutive equation on the form:

$$p = p(\rho) \tag{8.3.39}$$

A reference pressure p_o and the corresponding reference density ρ_o represent an equilibrium state governed by the equilibrium equation:

$$\mathbf{0} = -\frac{1}{\rho_o}\nabla p_o + \mathbf{b} \tag{8.3.40}$$

Sound waves are small variations $\tilde{p}$ in the pressure and $\tilde{\rho}$ in the density, such that:

$$\rho = \rho_o + \tilde{\rho},\; p = p_o + \tilde{p} = p_o + \left.\frac{dp}{d\rho}\right|_{\rho=\rho_o}\tilde{\rho} \quad \Rightarrow \quad \frac{\tilde{p}}{\tilde{\rho}} = \left.\frac{dp}{d\rho}\right|_{\rho=\rho_o} \tag{8.3.41}$$

A linearization of the Euler equations and the continuity equation yields, after the equilibrium (8.3.40) has been applied, the set of equations:

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho_o}\nabla\tilde{p},\; \frac{\partial\tilde{\rho}}{\partial t} + \rho_o\nabla\cdot\mathbf{v} = 0 \tag{8.3.42}$$

We introduce the constant parameter:

$$c = \sqrt{\frac{\tilde{p}}{\tilde{\rho}}} = \sqrt{\left.\frac{dp}{d\rho}\right|_{\rho=\rho_o}} \tag{8.3.43}$$

which will be shown to be the *velocity of sound*. Formula (8.3.43) and the linearized basic equations (8.3.42) now yield:

$$\nabla \cdot \frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho_o}\nabla^2 \tilde{p}, \ \frac{\partial^2 \tilde{p}}{\partial t^2} + \rho_o \nabla \cdot \frac{\partial \mathbf{v}}{\partial t} = 0 \quad \Rightarrow$$
$$\frac{\partial^2 \tilde{p}}{\partial t^2} = c^2 \nabla^2 \tilde{p} \tag{8.3.44}$$

This is a linear wave equation, as presented in Sect. 7.7, e.g. (7.7.5), and c may be identified as the velocity of sound in the fluid.

For gasses we may use the constitutive (8.3.7), and for air we choose:

$$p_o = 101.32\,\text{kPa} \quad , \quad \rho_o = 1.225\,\text{kg/m}^3, \ \alpha = 1.4 \quad \text{at } 15^{\text{o}}C$$

The velocity of sound in air is then:

$$c = \sqrt{\left.\frac{dp}{d\rho}\right|_{\rho=\rho_o}} = \sqrt{p_o\,\alpha\,\frac{1}{\rho_o}} = 340\text{ m/s at } 15^{\text{o}}C$$

For liquids we may use the constitutive equation:

$$p = C_1 \left(\frac{\rho}{\rho_o}\right)^{\gamma} - C_2 \tag{8.3.45}$$

C_1, C_2, and γ are constant parameters. For water we may set:

$$\rho_o = 1000\,\text{kg/m}^3, \ C_1 = 304.06\,\text{MPa} \quad , \quad C_2 = 303.96\,\text{MPa} \quad , \quad \gamma = 7$$

The velocity of sound in water then becomes:

$$c = \sqrt{\left.\frac{dp}{d\rho}\right|_{\rho=\rho_o}} = \sqrt{C_1\,\gamma\frac{1}{\rho_o}} = 1460\text{ m/s}$$

8.4 Linearly Viscous Fluid = Newtonian Fluid

8.4.1 Constitutive Equations

The presence of shear stresses in a flowing fluid is realized when we observe the velocity field near rigid boundary surfaces, see Fig. 8.4.1. The fluid particles are slowed down in the neighborhood of the rigid surface, and very close to the surface the relative velocity is practically zero. Shear stresses are present everywhere in the flow, but their influence on the velocity field is normally very slight, except in the boundary layer near the rigid boundary surface. In the analysis of fluid flow around rigid bodies it is customary to first model the fluid as perfect fluid, and then use the solution as an asymptotic external flow to a viscous boundary layer solution near the

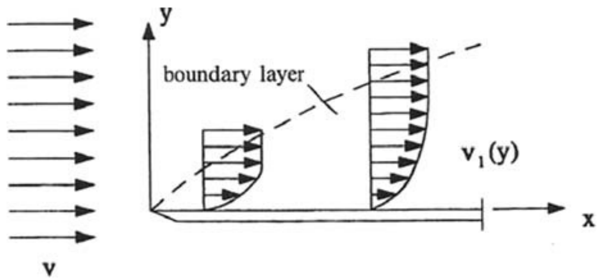


Fig. 8.4.1 Uniform flow and viscous boundary layer near a rigid surface

rigid surface. In the analysis of flows in pipes and through other narrow passages the viscous boundary layer fills the entire flow regime.

As a starting point for the development of a constitutive equation for a linearly viscous fluid we again look at the experiment with the viscometer in Fig. 1.3.2 in Sect. 1.3. The flow between the two cylindrical surfaces is considered to be a steady flow between two parallel planes, and called *simple shear flow*, see Fig. 8.4.2.

The velocity field of simple shear flow is:

$$v_1 = \frac{v}{h} x_2, \; v_2 = v_3 = 0 \tag{8.4.1}$$

The rate of deformation matrix becomes:

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\dot{\gamma}}{2}, \; \dot{\gamma} = 2D_{12} = \frac{dv_1}{dx_2} = \frac{v}{h} \tag{8.4.2}$$

We assume that the result of the viscometer test is a shear stress proportional to the shear strain rate $\dot{\gamma}$:

$$\tau = \mu \dot{\gamma} \tag{8.4.3}$$

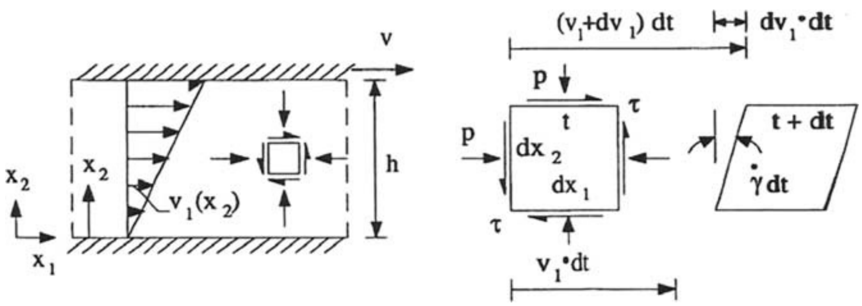


Fig. 8.4.2 Simple shear flow between two parallel planes

The *dynamic viscosity* μ is a temperature dependent material parameter. A first generalization of the constitutive (8.4.3) to apply to a general flow is called *Newton's law of fluid friction* and may be presented as:

$$T_{ij} = 2\mu D_{ij} = \mu \left(v_{i,j} + v_{j,i}\right) \text{ for } i \neq j \tag{8.4.4}$$

A further generalization of (8.4.4) leads to the *constitutive equation of a linearly viscous fluid*, or what is called a *Newtonian fluid*, and is presented below.

George Gabriel Stokes [1819–1903] presented the following four criteria for the relationships between the stresses and the velocity field in a viscous fluid:

1. The stress tensor $\mathbf{T}$ is a continuous function of the rate of deformation tensor $\mathbf{D}$.
2. The stress tensor $\mathbf{T}$ is explicitly independent of the particle coordinates, which implies that the fluid is homogeneous.
3. When the fluid is not deforming, i.e. $\mathbf{D} = \mathbf{0}$, the stress tensor is: $\mathbf{T} = -p(\rho,\theta)\mathbf{1}$.
4. Viscosity is an isotropic property.

The first three criteria imply the following general form of the constitutive equation of a viscous fluid:

$$\mathbf{T} = \mathbf{T}\left[\mathbf{D},\rho,\theta\right],\ \mathbf{T}\left[\mathbf{0},\rho,\theta\right] = -p(\rho,\theta)\,\mathbf{1} \tag{8.4.5}$$

The pressure $p(\rho,\theta)$ is the *thermodynamic pressure*. The requirement of isotropy is really superfluous because the (8.4.5) implies viscous isotropy. This fact will be demonstrated in Sect. 11.9.2 on *Stokesian fluids*. Equation (8.4.5) represents the constitutive equation of a Stokesian fluid.

We now restrict (8.4.5) to be linear with respect to $\mathbf{D}$. Using arguments along the lines used for isotropic, linearly elastic materials in Sect. 7.2, we may conclude that viscous isotropy implies that the tensors $\mathbf{T}$ and $\mathbf{D}$ are coaxial and furthermore that the linear version of (8.4.5) has the form:

$$\mathbf{T} = -p(\rho,\theta)\,\mathbf{1} + 2\mu\mathbf{D} + \left(\kappa - \frac{2\mu}{3}\right)(\mathrm{tr}\,\mathbf{D})\,\mathbf{1} \quad \Leftrightarrow$$
$$T_{ij} = -p(\rho,\theta)\delta_{ij} + 2\mu D_{ij} + \left(\kappa - \frac{2\mu}{3}\right) D_{kk}\,\delta_{ij} \tag{8.4.6}$$

The *viscosities* μ and κ are functions of the temperature and in some cases also of the pressure. The *dynamic viscosity* μ is relatively easy to determine experimentally, for instance using a viscometer of the type described in Sect. 1.3. The viscosity μ of water is $1.8 \cdot 10^{-3}\,\mathrm{Ns/m^2}$ at $0\,^\circ\mathrm{C}$ and $1.0 \cdot 10^{-3}\,\mathrm{Ns/m^2}$ at $20\,^\circ\mathrm{C}$. For air the viscosity μ is $1.7 \cdot 10^{-5}\,\mathrm{Ns/m^2}$ at $0\,^\circ\mathrm{C}$ and $1.8 \cdot 10^{-5}\,\mathrm{Ns/m^2}$ at $20\,^\circ\mathrm{C}$. The parameter κ is called the *bulk viscosity*, and is far more difficult to measure. Its physical implication will be discussed below. In the literature the (8.4.6) is often presented with λ in stead of $(\kappa - 2\mu/3)$, but the parameter λ has no direct physical interpretation, contrary to what is true for both μ and κ. With reference to (4.2.42) in Sect. 4.2.1 on isotropic tensors of 4. order, the parameters λ and μ may be identified as the Lamé-constants for viscous fluids.

In modern literature (8.4.6) is sometimes called the *Cauchy-Poisson law*. A material model defined by the constitutive (8.4.6) is called a *Newtonian fluid*. For an incompressible fluid, for which $\mathrm{tr}\mathbf{D} = 0$, (8.4.6) has to be replaced by:

$$\mathbf{T} = -p(\mathbf{r},t)\,\mathbf{1} + 2\mu\mathbf{D} \quad \Leftrightarrow \quad T_{ij} = -p(\mathbf{r},t)\delta_{ij} + 2\mu D_{ij} \tag{8.4.7}$$

The pressure $p(\mathbf{r},t)$ is a function of position $\mathbf{r}$ and time t, and can only be determined from the equations of motion and the boundary conditions. An equation of state, $p = p(\rho,\theta)$, loses its meaning when incompressibility is assumed.

If the symmetric tensors $\mathbf{T}$ and $\mathbf{D}$ are decomposed into isotrops and deviators, the constitutive equations (8.4.6) takes the alternative form:

$$\mathbf{T}^o = -p(\rho,\theta)\,\mathbf{1} + 3\kappa\mathbf{D}^o \quad \Leftrightarrow \quad T_{ij}^o = -p(\rho,\theta)\delta_{ij} + 3\kappa D_{ij}^o \tag{8.4.8}$$

$$\mathbf{T}' = 2\mu\mathbf{D}' \quad \Leftrightarrow \quad T_{ij}' = 2\mu D_{ij}' \tag{8.4.9}$$

For isotropic states of stress, we may replace (8.4.8) by:

$$\mathbf{T} = -\tilde{p}\mathbf{1},\ \tilde{p} = p(\rho,\theta) - \kappa\,\dot{\varepsilon}_v,\ \dot{\varepsilon}_v = \mathrm{tr}\mathbf{D} = \mathrm{div}\,\mathbf{v} = -\frac{\dot{\rho}}{\rho} \tag{8.4.10}$$

Note that the total pressure $\tilde{p}$ is not the same as the thermodynamic pressure $p(\rho,\theta)$.

The bulk viscosity κ expresses the resistance of the fluid toward rapid volume changes. Due to the fact that it is difficult to measure κ, values are hard to find in the literature. Kinetic theory of gasses shows that $\kappa = 0$ for monatomic gasses. But as shown by Truesdell [49] this result is implied in the stress assumption that is the basis for the kinetic theory. Experiments show that for monatomic gasses it is reasonable to set $\kappa = 0$, while for other gasses and for all liquids the bulk viscosity κ, and values of $\lambda = \kappa - 2\mu/3$, are larger than, and often much larger than μ. The assumption $\kappa = 0$, which is sometimes taken for granted in older literature on Fluid Mechanics, is called the *Stokes relation*, since it was introduced by him. However, Stokes did not really believe the relation to be relevant. Usually the deviator $\mathbf{D}'$ dominates over $\mathbf{D}^o$ such that the effects of the bulk viscosity are small. The bulk viscosity κ has dominating importance for the dissipation and absorption of sound energy.

A Newtonian fluid provides an example of a *visco-thermoelastic material*. If the thermodynamic pressure p is a function only of the density, the Newtonian fluid represents a *viscoelastic fluid*.

Example 8.3. Flow Between Parallel Planes

In this example we shall use the *Saint-Venant's semi-inverse method*. By this method the unknown functions in a problem are partly assumed known. The governing equations and the boundary conditions in the problem are then used to determine these functions completely.

The flow of an incompressible Newtonian fluid between two rigid plates, as shown in Fig. 8.4.3, is driven by a constant pressure gradient $c = -\partial p/\partial x$ in the direction of the flow and by constant velocity v of one of the plates. Gravity

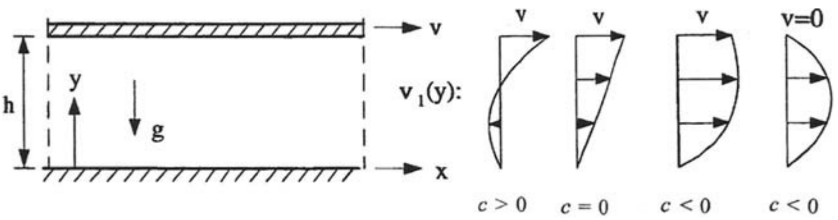


Fig. 8.4.3 Flow between parallel planes. Velocity profiles $v_x(y)$ for different combinations of the parameters c and v

represents a body force -g in the y-direction. We assume steady state motion and the velocity field:

$$v_x = v_x(y),\ v_y = v_z = 0 \tag{8.4.11}$$

which satisfies the incompressibility condition, div $\mathbf{v} = 0$. From the constitutive (8.4.7) we obtain the following expression for the stresses in the fluid:

$$\sigma_x = \sigma_y = \sigma_z = -p(x,y,z),\ \tau_{xy} = \mu\,\frac{dv_x}{dy},\ \tau_{yz} = \tau_{zx} = 0$$

When these stresses are substituted into the Cauchy equations (8.1.7), we find:

$$0 = -\frac{\partial p}{\partial x} + \mu\frac{d^2 v_x}{dy^2},\ 0 = -\frac{\partial p}{\partial y} - \rho g,\ 0 = -\frac{\partial p}{\partial z} \tag{8.4.12}$$

It follows from these equations that the pressure is independent of z and that the pressure gradient in the x-direction, $\partial p/\partial x = -c$, is constant as assumed. Integrations of the equations yield:

$$p(x,y) = -\rho g y - cx + A,\ v_x = -\frac{c}{2\mu}y^2 + By + C \tag{8.4.13}$$

A, B, and C are constants of integration. We assume that the fluid sticks to the rigid surfaces of the plates. The boundary conditions for the velocity and their implications are therefore:

$$v_x(0) = 0,\ v_x(h) = v \quad \Rightarrow \quad C = 0,\ B = \frac{v}{h} + \frac{ch}{2\mu}$$

The velocity field has then been determined.

$$v_x(y) = \frac{ch^2}{2\mu}\left[\frac{y}{h} - \left(\frac{y}{h}\right)^2\right] + v\frac{y}{h}$$

Figure 8.4.3 illustrates the *velocity profile* $v_x(y)$ for different combinations of the constant parameters c and v.

A boundary condition for the pressure may be $p(0,0) = p_o$, which implies that $A = p_o$ and:

$$p(x,y) = p_o - \rho g y - c x$$

Example 8.4. Flow Around a Rotating Cylinder

Figure 8.4.4 shows a cylindrical container with inner radius b and a rigid cylinder with radius a. The container and the cylinder have a common vertical axis, and their length is L. The annular space between the two concentric cylindrical surfaces of the container and the cylinder contains a Newtonian fluid. The rigid cylinder rotates with a constant angular velocity ω due to a constant external couple moment M, which is counteracted by the shear stress from the fluid. The motion of the cylinder results in a steady flow of the fluid, and we assume the flow to be two-dimensional with the velocity field:

$$v_\theta = v_\theta(R), \; v_R = v_z = 0 \tag{8.4.14}$$

Using (5.4.19) for deformation rates in cylindrical coordinates, we find only one deformation rate different from zero:

$$\dot{\gamma} \equiv \dot{\gamma}_{\theta R} = \frac{dv_\theta}{dR} - \frac{v_\theta}{R} = R\frac{d}{dR}\left(\frac{v_\theta}{R}\right)$$

The state of stress in the fluid is thus given by a pressure p and a shear stress:

$$\tau \equiv \tau_{\theta R} = \mu \, \dot{\gamma}_{\theta R} = \mu \, R\frac{d}{dR}\left(\frac{v_\theta}{R}\right) \tag{8.4.15}$$

The law of balance of angular momentum, (8.2.22), for a cylindrical body of radius R containing the rigid cylinder and fluid, provides the following equilibrium equation:

$$0 = R \cdot \tau \cdot (2\pi R L) + M \quad \Rightarrow \quad \tau = -\frac{M}{2\pi L}\frac{1}{R^2} \tag{8.4.16}$$

A combination of (8.4.15) and (8.4.16) results in a differential equation for the unknown velocity field v_θ:

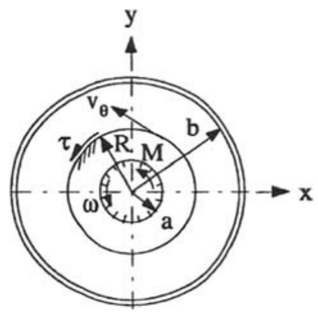


Fig. 8.4.4 Steady flow in a cylindrical container around a rotating cylinder

$$\frac{d}{dR}\left(\frac{v_\theta}{R}\right) = -\frac{M}{2\pi\mu L}\frac{1}{R^3} \tag{8.4.17}$$

We assume that the fluid sticks to the rigid cylindrical surfaces and obtain the boundary conditions:

$$v_\theta(a) = \omega a, \; v_\theta(b) = 0$$

The solution of the differential (8.4.17) is then:

$$v_\theta(R) = \frac{\omega a}{\left[1-(a/b)^2\right]}\left[\frac{a}{R} - \frac{aR}{b^2}\right], \; M = \frac{4\pi\mu L\omega a^2}{\left[1-(a/b)^2\right]}$$

An interesting special case, which will be referred to in Sect. 8.5, is obtained if we let $b \to \infty$. The result is the potential flow:

$$v_\theta(R) = \frac{\omega a^2}{R} = \nabla\phi, \; \phi = \omega a^2\theta \tag{8.4.18}$$

This flow is also discussed in Example 5.2 in Sect. 5.4, and called the *potential vortex*.

If the fluid is incompressible with constant density ρ, the pressure $p(R,\theta,z)$ in the fluid in a potential vortex may be determined as follows. In cylindrical coordinates the state of stress is expressed by:

$$\sigma_R = \sigma_\theta = \sigma_z = -p(R,\theta,z), \; \tau_{\theta R} \equiv \tau = -\frac{M}{2\pi LR^2}, \; \tau_{\theta z} = \tau_{zR} = 0$$

The body force is given by the gravitational force $-g\mathbf{e}_z$. The particle acceleration is:

$$\mathbf{a} = \dot{\mathbf{v}} = -\frac{v_\theta^2}{R}\mathbf{e}_R = -\frac{\omega^2 a^4}{R^3}\mathbf{e}_R$$

The Cauchy equations (3.2.39–41) yield:

$$-\frac{\partial p}{\partial R} = -\frac{\rho\omega^2 a^4}{R^3}, \; -\frac{\partial p}{\partial \theta} = 0, \; -\frac{\partial p}{\partial z} - \rho g = 0$$

The result of integrations of these equations is:

$$p(R,z) = -\frac{\rho\omega^2 a^4}{2R^2} - \rho g z + C$$

The constant of integration C may be determined from a pressure boundary condition. For comparison we may note that the pressure in a fluid in rigid-body rotation, see Problem 3.4, is:

$$p(R,z) = \frac{\rho\,\omega^2 R^2}{2} - \rho g z + C$$

8.4.2 The Navier-Stokes Equations

The general equations of motion of a linearly viscous fluid are called the *Navier-Stokes equations*. These equations are obtained by the substitution of the constitutive equations (8.4.6) into the Cauchy equations of motion (8.1.7). If it is assumed that the viscosities μ and κ may be considered to be constant parameters, the resulting equations are:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \mathbf{v} + \frac{1}{\rho}\left(\kappa + \frac{\mu}{3}\right)\nabla(\nabla \cdot \mathbf{v}) + \mathbf{b} \tag{8.4.19}$$

In a Cartesian coordinate system the Navier-Stokes equations are:

$$\partial_t v_i + v_k\, v_{i,k} = -\frac{1}{\rho} p_i + \frac{\mu}{\rho} v_{i,kk} + \frac{1}{\rho}\left(\kappa + \frac{\mu}{3}\right) v_{k,ki} + b_i \tag{8.4.20}$$

For incompressible fluids $\nabla \cdot \mathbf{v} = 0$, and (8.4.19) is reduced to:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \mathbf{v} + \mathbf{b} \tag{8.4.21}$$

The Navier-Stokes equations (8.4.19, 8.4.20, 8.4.21) are the most important equations in the study of viscous fluids. The complexity of the equations indicates that analytical solutions in most cases require major simplifications and approximations. Modern computer codes make it possible to use the Navier-Stokes equations in numerical solutions of very complex fluid flow problems.

For incompressible fluids it is often convenient to combine the pressure gradient ∇p and the body force term $\rho \mathbf{b}$ in the Navier-Stokes equations (8.4.21) by introducing the *modified pressure P*. First we compute the static pressure p_s that would exist in the fluid at rest only subjected to the body force $\mathbf{b}$. The static pressure p_s is thus determined from the equilibrium equation:

$$\mathbf{0} = -\frac{1}{\rho}\nabla p_s + \mathbf{b} \tag{8.4.22}$$

If the body force is conservative such that $\mathbf{b} = -\nabla\beta$, as presented in (8.3.16), we write for (8.4.22):

$$\mathbf{0} = -\nabla\left(\frac{p_s}{\rho} + \beta\right) \quad \Rightarrow \quad p_s = p_o - \rho\beta$$

p_o is constant reference pressure.

For example, let $\mathbf{b}$ be the constant gravitational force g and z the vertical height above a chosen reference level at which the pressure is p_o. Then we find $\beta = gz$ and:

$$p_s(z) = p_o - \rho g z$$

The modified pressure P is defined by:

$$P = p - p_s \quad \text{or} \quad P = p + \rho\beta - p_o \tag{8.4.23}$$

Then the Navier-Stokes equations for an incompressible fluid may be reduced to:

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\frac{1}{\rho}\nabla P + \frac{\mu}{\rho}\nabla^2\mathbf{v} \tag{8.4.24}$$

In Example 8.4 we may set $p_s = C - \rho g z$ and the modified pressure becomes:

$$P(R) = -\frac{\rho\omega^2 a^4}{2R^2}$$

Example 8.5. Film Flow

Figure 8.4.5 illustrates the transportation of an incompressible fluid on a wide conveyer belt as a film with constant thickness h. The belt has the width w and is inclined an angle α with respect to the horizontal plane. The belt moves with a constant velocity v_o. The free fluid surface is only subjected to the atmospheric pressure p_a. The fluid sticks to the belt. The body force is given by the gravitational force g and is expressed by:

$$\mathbf{b} = -g\sin\alpha\,\mathbf{e}_x - g\cos\alpha\,\mathbf{e}_y$$

We assume steady two-dimensional flow with the velocity field and the pressure field:

$$v_x = v_x(y),\ v_y = 0,\ p = p(x,y)$$

The special situation at the edges of the belt needs not be considered if the width of the belt is sufficiently large. The rate of deformation field $\mathbf{D}$ will naturally be functions of the y-coordinate only. The stress field $\mathbf{T}$ will be dependent on both x and y. The boundary conditions for the flow are:

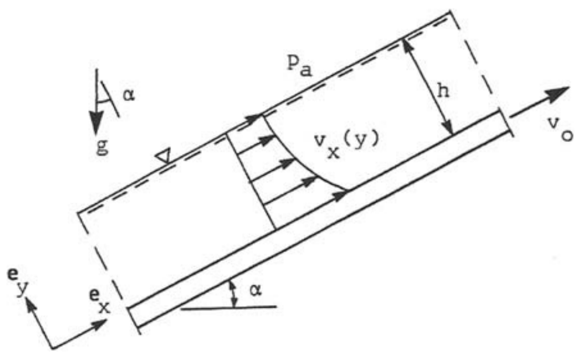


Fig. 8.4.5 Film flow on a conveyer belt

$$v_x(0) = v_o,\ \sigma_y(h,z) = -p(h,x) = -p_a,\ \tau_{xy}(h,z) = 0$$

The Navier-Stokes equations (8.4.21) are reduced to:

$$\mathbf{0} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 v_x \mathbf{e}_x + \mathbf{b} \Rightarrow 0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{d^2 v_x}{dy^2} - g\cos\alpha,\ 0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g\sin\alpha$$

The solution to the two partial differential equations that also satisfies the boundary conditions is:

$$v_x(y) = v_o - \frac{\rho g \sin\alpha\, h^2}{2\mu}\left[1 - \left(1 - \frac{y}{h}\right)^2\right]$$
$$\sigma_x = \sigma_y = \sigma_z = -p = -p_a - \rho g \cos\alpha\,(h - y)$$
$$\tau_{zy} = -\rho g \sin\alpha\,(h - y),\ \tau_{xy} = \tau_{zx} = 0$$

The volumetric flow Q of fluid volume transported by the conveyer belt per unit time is calculated from:

$$Q = w\int_0^h v_x(y)\,dy = w\int_0^h \left\{v_o - \frac{\rho g \sin\alpha\, h^2}{2\mu}\left[1 - \left(1 - \frac{y}{h}\right)^2\right]\right\} dy \quad \Rightarrow$$
$$Q = v_o w h - \frac{\rho g \sin\alpha\, w h^3}{3\mu}$$

Example 8.6. Laminar Flow in Pipes

An incompressible Newtonian fluid flows through a circular cylindrical pipe of internal diameter d, see Fig. 8.4.6. The flow is driven by a constant modified pressure gradient in the axial direction z. The flow is assumed to be laminar and steady, with streamlines parallel to the axis of the pipe and with the velocity field:

$$v_z = v(R),\ v_R = v_\theta = 0 \tag{8.4.25}$$

In order to find the velocity function $v(R)$ we shall use the Navier-Stokes equations (8.4.24) in cylindrical coordinates, as given in Appendix B. First we note the

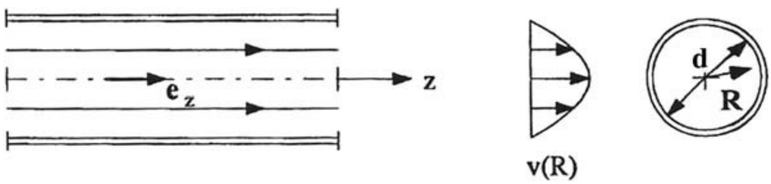


Fig. 8.4.6 Laminar pipe flow

particle acceleration is zero. Then the Navier-Stokes equations expressed in cylindrical coordinates are reduced to:

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial R},\; 0 = -\frac{1}{\rho R}\frac{\partial P}{\partial \theta},\; 0 = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{\mu}{\rho R}\frac{\partial}{\partial R}\left(R\frac{\partial v}{\partial R}\right)$$

When these equations are integrated we end up with two integration constants. These are found by requiring a finite velocity at the center of the pipe and by assuming that the fluid sticks to the pipe wall:

$$v(0) \neq \infty,\; v(d/2) = 0$$

The solution of the partial differential equations is then:

$$P = P(z) = p_o - cz,\; v = \frac{d^2}{16\mu}c\left[1 - \left(\frac{2R}{d}\right)^2\right]$$

The parameter c is the constant negative modified pressure gradient in the direction of the flow. The velocity function may alternatively be expressed in terms of the maximum v_o:

$$v_z(R) \equiv v(R) = v_o\left[1 - \left(\frac{2R}{d}\right)^2\right],\; v_o = \frac{d^2}{16\mu}c \tag{8.4.26}$$

The velocity profile is shown in Fig. 8.4.6.

According to the general expressions (5.4.19) for the rates of strain and rates of shear in cylindrical coordinates the assumed velocity field (8.4.25) provides only one non-zero value:

$$\dot{\gamma}_{zR} = \frac{dv}{dz}$$

The constitutive equations (8.4.7) then give the stresses:

$$\sigma_R = \sigma_\theta = \sigma_z = -p,\; \tau_{zR} = \mu\frac{dv_z}{dR} = -\frac{c}{2}R,\; \tau_{R\theta} = \tau_{\theta z} = 0 \tag{8.4.27}$$

8.4.3 Dissipation

The viscosity results in dissipation of mechanical energy in a flowing fluid, i.e. mechanical energy is converted to heat and internal energy. We now compute the stress power per unit volume for a Newtonian fluid. Using the constitutive (8.4.6), we get:

$$\omega = \mathbf{T} : \mathbf{D} = -p\,\mathrm{div}\,\mathbf{v} + 2\mu\,\mathbf{D} : \mathbf{D} + \left(\kappa - \frac{2\mu}{3}\right)(\mathrm{tr}\,\mathbf{D})^2 \tag{8.4.28}$$

A decomposition of the stress tensor **T** and the rate of deformation tensor **D** into isotrops and deviators, using the expressions (8.4.8) and (8.4.9) yields:

$$\omega = \mathbf{T} : \mathbf{D} = -p \,\mathrm{div}\,\mathbf{v} + 2\mu\,\mathbf{D}' : \mathbf{D}' + \kappa\,(\mathrm{div}\,\mathbf{v})^2 \tag{8.4.29}$$

For a fluid with barotropic pressure: $p = p(\rho)$, we introduce the specific elastic energy ψ defined by (8.3.11). Using the continuity (8.2.26), we obtain from (8.3.11):

$$\rho\,\dot{\psi} = p\,\frac{\dot{\rho}}{\rho} = -p\,\mathrm{div}\mathbf{v} \tag{8.4.30}$$

The expression (8.4.28) may then be rewritten to:

$$\omega = \mathbf{T} : \mathbf{D} = \rho\dot{\psi} + 2\mu\mathbf{D} : \mathbf{D} + \left(\kappa - \frac{2}{3}\mu\right)(\mathrm{tr}\,\mathbf{D})^2 \tag{8.4.31}$$

The first term on the right-hand side of this expression for the stress power per unit volume represents the recoverable part of the mechanical energy, while the two last terms, which are seen always to be positive, represent a loss or dissipation of mechanical energy.

The *viscous-dissipation function* is a positive semidefinite scalar-valued function and for any fluid defined by:

$$\delta = \omega - (-p\,\mathrm{div}\,\mathbf{v}) = \omega - p\frac{\dot{\rho}}{\rho} \tag{8.4.32}$$

For a fluid with barotropic pressure we get:

$$\delta = \omega - (-p\,\mathrm{div}\,\mathbf{v}) = \omega - p\frac{\dot{\rho}}{\rho} = \omega - \rho\,\dot{\psi} \tag{8.4.33}$$

For a Newtonian fluid the dissipation function becomes:

$$\delta = 2\mu\,\mathbf{D} : \mathbf{D} + \left(\kappa - \frac{2\mu}{3}\right)(\mathrm{tr}\,\mathbf{D})^2 = 2\mu\,\mathbf{D}' : \mathbf{D}' + \kappa\,(\mathrm{div}\,\mathbf{v})^2 \tag{8.4.34}$$

For a fluid with barotropic pressure the mechanical energy balance (6.1.12) for a fluid body with volume V may be presented as:

$$P = \dot{K} + \dot{\Psi} + \Delta \tag{8.4.35}$$

Ψ is the elastic energy of the body, confer (8.3.14), and Δ is the dissipation in the fluid body and given by:

$$\Delta = \int_V \delta\,dV \tag{8.4.36}$$

8.4.4 The Energy Equation

The general balance of thermal energy for a place or a particle is given (6.3.14):

$$\rho\dot{\varepsilon} = -\mathrm{div}\,\mathbf{h} + \mathbf{T} : \mathbf{D} \tag{8.4.37}$$

The equation will now be developed further for a Newtonian fluid. The specific internal energy may be replaced by the *specific enthalpy h* through thee relationship:

$$h = \varepsilon + \frac{p}{\rho} \tag{8.4.38}$$

p is the thermodynamic pressure. The heat flux vector $\mathbf{h}$ is expressed by the *Fourier heat conduction equation*, named after Jean Baptiste Joseph Fourier [1768–1830]:

$$\mathbf{h} = -k\nabla\theta \tag{8.4.39}$$

The parameter k is a temperature dependent *heat conduction coefficient*. In many cases k is taken to be a constant parameter. From the definition (8.4.38) we obtain:

$$\dot{\varepsilon} = \dot{h} - \frac{\dot{p}}{\rho} + p\frac{\dot{\rho}}{\rho^2} \tag{8.4.40}$$

For the last term in the energy (8.4.37) we use the definition (8.4.32) to write:

$$\mathbf{T} : \mathbf{D} = \omega = \delta + p\frac{\dot{\rho}}{\rho} \tag{8.4.41}$$

The results (8.4.39, 8.4.40, 8.4.41) are now substituted into the energy (8.4.37), and we obtain the alternative form of the thermal energy equation:

$$\rho\dot{h} = \dot{p} + \nabla\cdot(k\nabla\theta) + \delta \tag{8.4.42}$$

For an incompressible fluid we introduce the *specific heat at constant pressure*:

$$c_p = \left.\frac{\partial h}{\partial\theta}\right|_{p=\text{constant}} \tag{8.4.43}$$

It may be shown that for a gas the term $\dot{p}$ in (8.4.42) may be neglected if incompressibility is assumed. The energy equation for an incompressible gas then takes the form:

$$\rho\, c_p\,\dot{\theta} = \nabla\cdot(k\nabla\theta) + \delta \tag{8.4.44}$$

For a Newtonian fluid the dissipation function δ is given by (8.4.34).

If we had started with the energy equation in the form (8.4.37) and then assumed that the fluid was incompressible, we would have obtained the alternative energy (8.4.44) with c_p replaced by the *specific heat at constant volume, or constant density*:

$$c_v = \left.\frac{\partial \varepsilon}{\partial \theta}\right|_{\rho=\text{constant}} \tag{8.4.45}$$

For a liquid, which is nearly an incompressible material, $c_v = c_p$. It is customary for an incompressible liquid to replace both c_p and c_v by a common *specific heat c*.

A commentary to the definition of the specific heat may be of interest at this point. The specific heat c represents the heat that must be supplied per unit mass and per unit of temperature. From the thermal energy (8.4.37) we obtain:

$$c = \frac{\operatorname{div}\mathbf{h}}{\rho\dot{\theta}} = \frac{\dot{\varepsilon}}{\dot{\theta}} - \frac{\mathbf{T}:\mathbf{D}}{\rho\dot{\theta}} \tag{8.4.46}$$

For a perfect fluid we find, using (8.3.10) and (8.4.40), that:

$$c = \frac{\dot{\varepsilon}}{\dot{\theta}} - \frac{p\dot{\rho}}{\rho^2\dot{\theta}} = \frac{\dot{h}}{\dot{\theta}} - \frac{\dot{p}}{\rho\,\dot{\theta}} \tag{8.4.47}$$

At constant volume, or constant density, i.e. $\dot{\rho} = 0$:

$$c = c_v = \left.\frac{\dot{\varepsilon}}{\dot{\theta}}\right|_{\rho=\text{constant}} = \left.\frac{\partial \varepsilon}{\partial \theta}\right|_{\rho=\text{constant}} \tag{8.4.48}$$

At constant pressure:

$$c = c_p = \left.\frac{\dot{h}}{\dot{\theta}}\right|_{p=\text{constant}} = \left.\frac{\partial h}{\partial \theta}\right|_{p=\text{constant}} \tag{8.4.49}$$

8.4.5 The Bernoulli Equation for Pipe Flow

An incompressible Newtonian fluid flows through a pipe. It is assumed that the flow is laminar and steady, and that the fluid sticks to the pipe wall. As shown in Fig. 8.4.7 the pipe is cylindrical at the two positions where the cross-sections are A_1 and A_2. It then follows from Example 8.6 that we may assume at these positions that the stream lines are parallel to the axis of the pipe. We shall derive a Bernoulli equation from the equation of balance of mechanical energy (8.2.23) for a control volume V

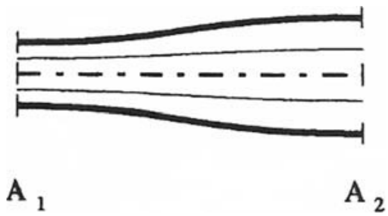


Fig. 8.4.7 Pipe flow

defined by the fluid body in the pipe and between the cross-sections A_1 and A_2. The control surface A consists then of the pipe wall A_w and the cross-sectionsA_1 and A_2.

For the body force we set $\mathbf{b} = -\nabla\beta$, and due to the condition of incompressibility: $\nabla \cdot \mathbf{v} = 0$, we can write: $\mathbf{b} \cdot \mathbf{v}\rho = -\nabla \cdot (\beta \mathbf{v}\rho)$. By the divergence theorem C.4 we get:

$$\int_V \nabla \cdot (\beta\, \mathbf{v}\rho)\, dV = \int_A \beta\, \rho\, \mathbf{v} \cdot \mathbf{n}\, dA \tag{8.4.50}$$

At the pipe wall A_w the fluid velocity $\mathbf{v}$ is zero, which implies that $\mathbf{t} \cdot \mathbf{v} = 0$ on A_w. Based on the result (8.4.27) in Example 8.6 we assume that on the cross-sections A_1 and A_2: $\mathbf{t} \cdot \mathbf{v} = -p\mathbf{v} \cdot \mathbf{n}$. Equation (8.4.32) yields: $\mathbf{T} : \mathbf{D} \equiv \omega = \delta$. It now follows that the energy (8.2.23) for the control volume V with the control surface $A = A_w + A_1 + A_2$ can be presented as:

$$\left[\int_A \left(\frac{v^2}{2} + \frac{p}{\rho} + \beta \right) \rho\, \mathbf{v} \cdot \mathbf{n}\, dA \right]_{A_2}^{A_1} = \Delta \equiv \int_V \delta\, dV = \int_V \mathbf{T} : \mathbf{D}\, dV = \int_V 2\mu\, \mathbf{D} : \mathbf{D}\, dV \tag{8.4.51}$$

Confer (8.3.21). The Navier-Stokes equation (8.4.20) may be presented as:

$$\dot{\mathbf{v}} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \mathbf{v} - \nabla\beta = -\nabla\left(\frac{p}{\rho} + \beta\right) + \frac{\mu}{\rho}\nabla^2 \mathbf{v}$$

At the cross-sections A_1 and A_2 we assume zero particle acceleration and zero velocity gradient. The Navier-Stokes equation then implies:

$$\nabla\left(\frac{p}{\rho} + \beta\right) = 0 \quad \Rightarrow \quad \frac{p}{\rho} + \beta = \text{constant over the cross-sections } A_1 \text{ and } A_2 \tag{8.4.52}$$

Hence:

$$\int_A \left(\frac{p}{\rho} + \beta\right) \rho\, \mathbf{v} \cdot \mathbf{n}\, dA = \left(\frac{p}{\rho} + \beta\right) \rho\, Q, \ Q = \int_A \mathbf{v} \cdot \mathbf{n}\, dA \text{ on } A = A_1 \text{ or } A_2$$

Q is the *volumetric flow*. We introduce the dimensionless parameter α such that:

$$\int_A \frac{v^2}{2} \rho\, \mathbf{v} \cdot \mathbf{n}\, dA = \left(\alpha \frac{v_m^2}{2}\right) \rho\, Q, \ v_m = \frac{Q}{A} = \text{ mean velocity over cross-section } A$$

Equation (8.4.51) may now be transformed into the *Bernoulli equation*:

$$\left[\alpha \frac{v_m^2}{2} + \frac{p}{\rho} + \beta\right]_{A_1} - \left[\alpha \frac{v_m^2}{2} + \frac{p}{\rho} + \beta\right]_{A_2} = \frac{\Delta}{\rho Q} \tag{8.4.53}$$

The right-hand side of this equation is called the loss term.

We shall compute the parameter α for a cylindrical pipe with circular cross section of diameter d. The velocity distribution over the cross section is given by (8.4.26). Thus:

$$\int_A \frac{v^2}{2}\rho\,\mathbf{v}\cdot\mathbf{n}\,dA = \int_0^{d/2} \frac{\rho v^3(R)}{2}\,2\pi R\,dR = \left(2\frac{v_m^2}{2}\right)\rho Q,\; v_m = \frac{v_o}{2}$$

Thus $\alpha = 2$ for circular pipes with laminar flow. For the same case we shall compute the loss term along a pipe of length L and with constant diameter d. The deformation rate matrix D in cylindrical coordinates is given by formulas (5.4.18–19). The velocity field (8.4.26) provides this rate of deformation matrix:

$$D = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \frac{1}{2}\frac{dv}{dR},\quad \frac{dv}{dR} = -\frac{16\,v_m}{d^2}R$$

Then from the definition of Δ in (8.4.51) we get:

$$\Delta = \int_V 2\mu\,\mathbf{D}:\mathbf{D}\,dV = L\int_0^{d/2} 2\mu\left[\left(-\frac{1}{2}\frac{16v_m}{d^2}R\right)^2 2\right](2\pi R\,dR) = 32\frac{\mu\,v_m L}{d^2}Q$$

The loss term becomes:

$$\frac{\Delta}{\rho Q} = 32\frac{\mu\,v_m L}{d^2} \tag{8.4.54}$$

For steady turbulent flow in a pipe it is experimentally found that the time-averaged velocity is practically constant over the cross section, which means that $\alpha \approx 1$. The loss term for turbulent flow in a pipe of length L and with constant diameter d, is presented as:

$$\frac{\Delta}{\rho Q} = \lambda\,\frac{L}{d}\frac{v_m^2}{2} \tag{8.4.55}$$

The parameter λ has to be determined experimentally. It is found that the parameter λ depends on the *Reynolds number* $Re = v_m d/(\mu/\rho)$ and the roughness of the inside surface of the pipe. The last effect dominates, and it is customary to consider λ independent of Re in the case of turbulent flow.

For the special case of laminar flow discussed above, the loss term may also be computed from the Bernoulli (8.4.53). From (8.4.26) in Example 8.6 we obtained for the modified pressure gradient in the flow direction: $c = 16\mu v_o/d^2$. Using (8.4.23) we obtain:

$$cL = \frac{16\mu\,v_o L}{d^2} = 32\frac{\mu\,v_m L}{d^2} = P_{A_1} - P_{A_2} = [p+\rho\beta]_{A_1} - [p+\rho\beta]_{A_2} \quad\Rightarrow$$

$$[p+\rho\beta]_{A_1} - [p+\rho\beta]_{A_2} = 32\frac{\mu\,v_m L}{d^2} \tag{8.4.56}$$

Since the mean velocity over the cross-sections A_1 and A_2 are the same, the result (8.4.54) follows from (8.4.56) and the Bernoulli equation (8.4.53).

8.5 Potential Flow

It was shown in Sect. 8.3 that under the assumptions: 1) barotropic fluid and 2) conservative body forces, a fluid will remain in irrotational flow if the fluid has at any time been in irrotational flow. These conditions exist when for instance fluid flows from a reservoir as shown in Fig. 8.3.2. The same conditions are approximately satisfied in a flow created by a rigid body moving in fluid originally at rest. A similar case occurs in a fluid flow around a rigid body at rest if the fluid before it meets the body and far away from the body, has parallel stream lines and constant velocity, i.e. the fluid is in *uniform flow*. As illustrated in Fig. 8.5.1, the two last types of flows are essentially the same: The flow in the second case is obtained from the first if the motion is referred to the rigid body. Obviously the vorticity is zero in the uniform flow. The flow in the vicinity of rigid surfaces must be corrected by a boundary layer analysis, and the wake must be excluded from the irrotational flow analysis.

Irrotational flow is also called *potential flow* because the velocity field in the flow may be derived from a scalar valued function of position $\phi(\mathbf{r},t)$, called the *velocity potential*, such that:

$$\mathbf{v} = \nabla\phi \tag{8.5.1}$$

For potential flow the continuity (8.2.26) may be rewritten to:

$$\dot{\rho} + \rho\nabla^2\phi = 0 \tag{8.5.2}$$

In this section we shall assume that the fluid is incompressible. The continuity (8.5.2) is then reduced to the Laplace equation:

$$\nabla^2\phi = 0 \tag{8.5.3}$$

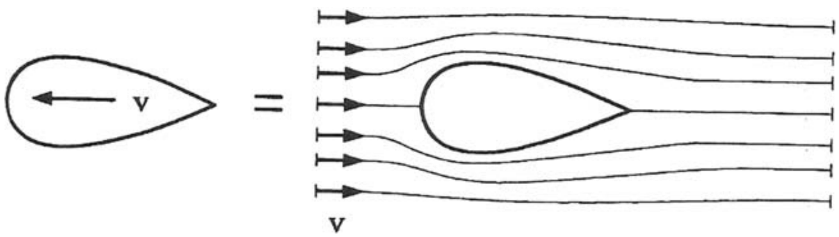


Fig. 8.5.1 Potential flow. Flow created by a rigid body moving in a fluid originally at rest is equivalent to the fluid flow around a rigid body at rest approached by a uniform flow

Solutions of this equation provide possible potential flows of incompressible fluids. The boundary conditions for ϕ are determined from the fact that the velocity component normal to rigid surfaces must be zero. Let $\mathbf{n}$ be the unit normal to a rigid surface. Then:

$$\mathbf{n} \cdot \nabla\phi = \frac{d\phi}{dn} = 0 \text{ on rigid surfaces} \tag{8.5.4}$$

The differential equation (8.5.3) and the boundary condition (8.5.4) are linear, which means that we may superimpose known solutions to obtain new solutions. This is demonstrated in Example 8.10 below. The theory of potential flows is highly developed and mathematically extensive. In two-dimensional flows conform mapping may be applied by which the real flow is transformed mathematically to a flow around a rigid cylinder. The reader is referred to the Fluid Mechanics literature for further presentation of this application.

Example 8.7. Uniform Flow

In a uniform flow: v_1 =constant and $v_2 = v_3 = 0$, referred to a Cartesian coordinate system Ox, the velocity potential is: $\phi = v_1 x_1$.

Example 8.8. The Potential Vortex

In Example 8.4 we found that the fluid flow created by a vertical cylinder of radius a rotating with a constant angular velocity ω in a cylindrical container of inner radius b, becomes a potential flow if the radius of the container is very large, i.e. $b \to \infty$, see (8.4.18). Now we introduce a constant parameter Γ such that the velocity potential and the corresponding velocity field are:

$$\phi = \frac{\Gamma\theta}{2\pi}, \; v_\theta = \frac{\Gamma}{2\pi R}, \; v_R = v_z = 0$$

The parameter Γ represents the circulation along any closed curve around the z-axis. To see this we compute the circulation along a circle C with center on the z-axis and of radius R:

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \int_0^{2\pi} \frac{\Gamma}{2\pi R} R\, d\theta = \Gamma = \text{constant} \tag{8.5.5}$$

This result seams to be in contradiction with what is implied by (8.3.37): Since the vorticity is zero in a potential flow, $\mathbf{c} = \nabla \times \mathbf{v} = \mathbf{0}$, (8.3.37) implies that the circulation Γ is zero. However, the result (8.3.37) is based on the application of Stokes' theorem C.5, which requires that the velocity $\mathbf{v}$ is regular on the integration surface A. This condition is not satisfied for the present flow because any surface bounded by C will intersect the z-axis, at which $\mathbf{v} = \infty$.

By proper choice of the integration path the result (8.5.5) may easily be generalized to apply to any closed curve around the z-axis.

Example 8.9. Rigid Cylinder in Uniform Flow

Potential flow around a rigid cylinder of radius a, as shown in Fig. 8.5.2, is represented by the velocity potential:

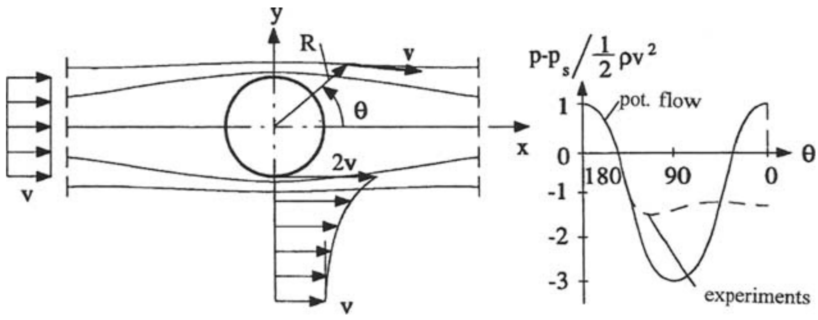


Fig. 8.5.2 Rigid cylinder in uniform flow

$$\phi = v\left[x + a^2\frac{\cos\theta}{R}\right] \equiv v\left[R + \frac{a^2}{R}\right]\cos\theta$$

v is the velocity of a uniform flow far away upstream from the cylinder.

The velocity field is found by applying (8.5.1) and (2.4.18) for the del-operator in cylindrical coordinates:

$$v_R = \frac{\partial\phi}{\partial R} = v\left[1 - \left(\frac{a}{R}\right)^2\right]\cos\theta,\ v_\theta = \frac{1}{R}\frac{\partial\phi}{\partial\theta} = -v\left[1 + \left(\frac{a}{R}\right)^2\right]\sin\theta$$

The Euler pressure equation (8.3.24) provides the pressure against the cylinder wall. First we introduce the modified pressure P from (8.4.23):

$$P = p - p_s = p + \rho\beta - p_o$$

p_s is the static pressure and p_o is a reference pressure. When the modified pressure is substituted into (8.3.24), we obtain an alternative form of the *Euler pressure equation*:

$$\frac{v^2}{2} + \frac{P}{\rho} = \text{ constant in the fluid} \tag{8.5.6}$$

In this general equation v is the fluid velocity at the chosen place in the fluid. In the present case the pressure p far away from the rigid cylinder is equal to the static pressure p_s and the modified pressure P is zero. Hence, when the pressure equation (8.5.6) is applied to the present flow example, we get for the modified pressure P on the rigid cylinder:

$$\frac{v^2}{2} + \frac{0}{\rho} = \frac{v_R^2 + v_\theta^2}{2} + \frac{P}{\rho} \quad \Rightarrow \quad P = p - p_s = \frac{\rho\, v^2}{2}\left(1\text{–}4\sin^2\theta\right)$$

Figure 8.5.2 shows how this theoretical pressure deviates from the pressure obtained experimentally. The theoretical velocity field and the corresponding pressure are only realistic near the front of the cylinder. The reason for this is the creation of the wake downstream, which is highly rotational. Confer Fig. 8.1.3.

Example 8.10. Rotating Cylinder in Uniform Flow

A long cylinder is lying on a horizontal table, as illustrated in Fig. 8.5.3. Around the cylinder is wound a tape. We pull the tape with a force T and thereby give the cylinder a horizontal velocity v and an angular velocity ω. Referred to the cylinder the air approaches the cylinder from a uniform flow with constant velocity v. The rotation of the cylinder creates a potential vortex as described in Example 5.2, Example 8.4, and Example 8.8. We will experience that the cylinder is subjected to a lifting force counteracting the gravitational force. In fact the cylinder may lift itself and rise higher then the level of the table it has left. The lift is obviously due to the rotation of the cylinder and thereby the circulation created by this rotation. The lifting effect is called the *Magnus effect*, named after Gustav Heinrich Magnus [1802–1870].

The Magnus effect is also present when balls in sports, like tennis, golf, and soccer, are thrown, kicked, or batted with a rotation. The ball may be given a higher or flatter vertical path, or a path curving to the left or to the right. The present example will give a theoretical explanation to the Magnus effect.

We shall compute the lifting force on a rigid cylinder in a flow described by the velocity potential:

$$\phi = v\left[R + \frac{a^2}{R}\right]\cos\theta - \frac{\Gamma\theta}{2\pi}, \; \Gamma = 2\pi\omega a^2$$

This potential is obtained by superposition of the velocity potentials in Example 8.9 and Example 8.8. Note however that the sense of ω is opposite in Example 8.8 and in the present example. The velocity field is obtained directly by addition of the velocities in the two examples:

$$v_R = v\left[1 - \left(\frac{a}{R}\right)^2\right]\cos\theta, \; v_\theta = -v\left[1 + \left(\frac{a}{R}\right)^2\right]\sin\theta - \frac{\Gamma}{2\pi R}$$

The modified pressure P on the rigid cylinder is obtained from the Euler pressure equation (8.5.6):

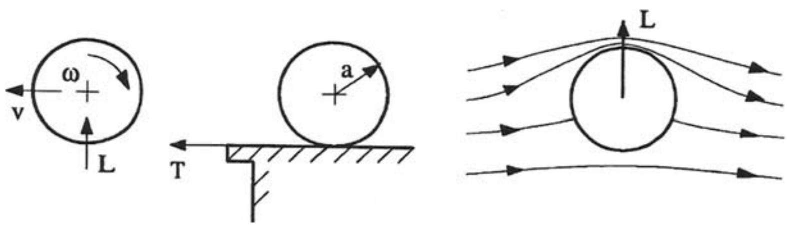


Fig. 8.5.3 The Magnus effect. Lifting force L due to the combined rotation and translation of the cylinder

$$\frac{v^2}{2} + \frac{0}{\rho} = \frac{v_R^2 + v_\theta^2}{2} + \frac{P}{\rho} \quad \Rightarrow \quad P = p - p_s = \frac{\rho v^2}{2}\left[1 - \left(2\sin\theta + \frac{\Gamma}{2\pi a v}\right)^2\right]$$

This pressure gives the resulting vertical lift force:

$$L = -\int_0^{2\pi} P \sin\theta \, a \, d\theta = \Gamma \rho \, v$$

8.5.1 The D'alembert Paradox

It may be shown theoretically that any body moving with constant velocity through a barotropic fluid, originally at rest, will not be subjected to a resulting force from the fluid. This phenomenon is called *the d'Alembert paradox* after Jean Le Rond d'Alembert [1717–1783]. Two-dimensional potential theory gives another result, as we just have seen in Example 8.10 above. In general we shall find that a rigid body in an originally uniform flow is subjected to a lift force L in the direction normal to the uniform flow which is given by the expression:

$$L = \Gamma \rho \, v_o \tag{8.5.7}$$

v_o is the velocity of the uniform flow and Γ is the circulation around any closed curve surrounding the two-dimensional body. This fascinating result is called *the Kutta-Joukowsky theorem* after Wilhelm Kutta [1867–1944] and Nikolai Egorovich Joukowsky [1847–1921].

8.6 Non-Newtonian Fluids

8.6.1 Introduction

Viscous fluids that do not follow Newton's law of fluid friction, (8.4.4) are called non-Newtonian fluids. These fluids are usually highly viscous fluids and their elastic properties are also of importance. The viscoelastic fluids discussed in Chap. 9 are also characterized as non-Newtonian. Typical real non-Newtonian fluids are polymer solutions, thermo plastics, drilling fluids, paints, fresh concrete and biological fluids. The theory of non-Newtonian fluids is called rheology.

The term rheology was invented in 1920 by Professor Eugene C. Bingham at Lafayette College in Indiana, USA. Bingham who was a professor of Chemistry, studied new materials with strange flow behavior, in particular paints. The syllable Rheo is from the Greek word "rhein", meaning flow, so the name rheology was taken to mean *the theory of deformation and flow of matter*. Rheology has also come to include the constitutive theory of highly viscous fluids and solids exhibiting viscoelastic and viscoplastic properties.

Materials in the solid state can behave fluid-like under special conditions. Plastic deformation of solids at yield and creep may be considered to be fluid-like behavior. At high temperatures ($> 400\,^\circ$C) common structural steel shows creep and stress relaxation. In many simulations of forming processes with metals and polymers the material is modelled as a fluid although the temperature is below the melting temperature of the material.

Fluid models may be classified into three main groups:

A. Time independent fluids for which the fluid properties are explicitly independent of time.

   A1. Viscoplastic fluids. Two examples are presented in Sect. 8.6.2, and more examples are presented in Sect. 10.11.
   A2. Purely viscous fluids. Some examples are presented below, and some advanced models are discussed in the Sect. 11.9.

B. Time dependent fluids. The constitutive modeling of these fluids is very complex in will not be treated in this book.

   B1. Thixotropic fluids. For constant deformation rates the stresses in a thixotropic fluid decrease monotonically.
   B2. Rheopectic fluids or antithixotropic fluids. For constant deformation rates the stresses in a rheopectic fluid increase monotonically.

C. Viscoelastic fluids. Linear and non-linear models are discussed in Chap. 9.

8.6.2 Generalized Newtonian Fluids

The most commonly used models for incompressible non-Newtonian fluids are called *generalized Newtonian fluids*. The constitutive equation defining this fluid model is:

$$\mathbf{T} = -p\mathbf{1} + 2\eta\mathbf{D} \tag{8.6.1}$$

p is the pressure p ($\mathbf{r}$,t) and η, called the *viscosity function*, is a function of the *magnitude of shear rate* or the *shear rate measure* $\dot{\gamma}$:

$$\eta = \eta(\dot{\gamma}),\ \dot{\gamma} = \sqrt{2\mathbf{D}:\mathbf{D}} \equiv 2\sqrt{-II_D} \tag{8.6.2}$$

II_D is the 2. principal invariant of the rate of deformation tensor $\mathbf{D}$. This model is called a *purely viscous fluid* because the stress tensor depends solely on the rate of deformation tensor. The viscosity function, which is temperature dependent, is determined in experiments with simple shear flow, for instance as described in Sect. 1.3. In a simple shear flow the shear measure reduces to the absolute value of the rate of shear strain:

$$v_x = c(t)y,\ v_y = v_z = 0 \quad \Rightarrow \quad \dot{\gamma} = \left|\frac{dv_x}{dy}\right| = |c(t)| \tag{8.6.3}$$

Figure 8.6.1 shows a characteristic experimental curve of the viscosity function of real fluids. If η is constant and independent of the shear measure, (8.6.1) represents an incompressible Newtonian fluid, with $\eta \equiv \mu$ the shear viscosity.

Different analytical functions for the viscosity functions represent different fluid models, all of which are generalized Newtonian fluids. Some of the most common of these are presented in the following.

a) POWER-LAW FLUID (W. Ostwald 1925, A. de Waele 1923):

$$\eta = K\dot{\gamma}^{n-1} \tag{8.6.4}$$

K and n are temperature dependent material parameters. K is called the *consistency parameter* and n is the *power-law index*. Table 8.6.1 gives examples of values of K and n for some real fluids. It is often practical to set:

$$K = K_o \exp\left[-A\left(\theta - \theta_o\right)\right],\ n = \text{constant} \tag{8.6.5}$$

θ is the temperature and K_o, A, and the temperature θ_o are reference values. The power-law fluid has the weakness that it cannot fit the experimental curve of the viscosity function for very small and very large values of the shear measure. However, the model is relatively easy to work with in analytical solutions. For most real non-Newtonian fluids $n < 1$, see Fig. 8.6.1. The viscosity of the fluid decreases with increasing shear measure and the fluid is therefore called *shear-thinning*. For $n > 1$ the viscosity increases with increasing shear measure $\dot{\gamma}$, and the fluid is called *shear-thickening* or *dilatant*. The latter name is due to the fact that such a fluid normally also expands when deformed.

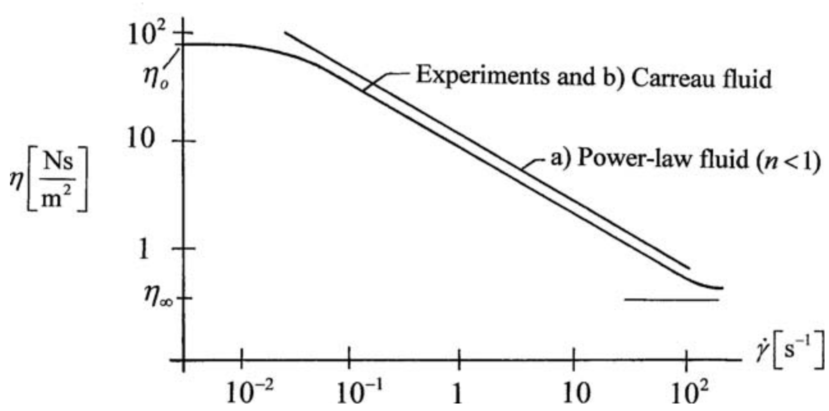


Fig. 8.6.1 The viscosity function

Table 8.6.1 Consistency K and power-law index n for some fluids

Fluid	Region of $\dot{\gamma}$ [s^{-1}]	K[Ns^n/m^2]	n
54.3% cement rock in water, 300 °K	10–200	2.51	0.153
23.3% Illinois clay in water, 300 °K	1800–6000	5.55	0.229
Polystyrene, 422 °K	0.03–3	$1.6 \cdot 10^5$	0.4
Tomato Concentrate, 90 °F 30% solids		18.7	0.4
Applesauce, 80 °F 11.6% solids		12.7	0.28
Banana puree, 68 °F		6.89	0.46

b) CARREAU FLUID (P.J. Carreau 1968):

$$\eta = \eta_\infty + (\eta_o - \eta_\infty)\left[1 + (\lambda\dot{\gamma})^2\right]^{(n-1)/2} \tag{8.6.6}$$

$\eta_\infty = \eta(\infty)$, called the *infinite-shear-rate viscosity*, $\eta_o = \eta(0)$, called the *zero-shear-rate- viscosity*, and λ is a time constant. This model adjusts to the experimental curve in Fig. 8.6.1 very well for all $\dot{\gamma}$-values.

c) ZENER-HOLLOMON FLUID (Zener, C. and Hollomon, J.H. 1944):

$$\eta(\dot{\gamma},\theta) = \frac{1}{\sqrt{3}\,\alpha\,\dot{\gamma}}\text{arcsinh}\left[\left(\frac{Z}{A}\right)^{1/n}\right], \; Z = \dot{\gamma}\exp\left[\frac{Q}{R\theta}\right] \tag{8.6.7}$$

α, A, and n are material parameters, and θ is the absolute temperature. The material parameter Q is called the activation energy, and R is the universal gas constant. The parameter Z, called the *Zener-Hollomon parameter*, is a temperature compensated shear measure. The model has been applied in simulations of forming processes, for instance in extrusion of light metals.

The next two models are not really purely viscous fluids but rather *viscoplastic fluids*. Section 10.11 presents a further discussion of viscoplastic fluids.

d) BINGHAM FLUID (E.C. Bingham 1922):

$$\eta = \infty \text{ when} |\tau_{\max}| < \tau_y, \; \eta = \mu + \frac{\tau_y}{\dot{\gamma}}\text{when} |\tau_{\max}| \geq \tau_y \tag{8.6.8}$$

$\tau_{\max}$ is the maximum shear stress in the fluid particle. τ_y is called the *yield shear stress*. μ is a constant viscosity parameter. This fluid model will be discussed further in Sect. 10.11.2.

e) CASSON FLUID (N. Casson 1959):

$$\eta = \infty \text{ when} |\tau_{\max}| < \tau_y, \; \eta = \mu + \frac{\tau_y}{\dot{\gamma}} + 2\sqrt{\frac{\mu\tau_y}{\dot{\gamma}}} \text{ when} |\tau_{\max}| \geq \tau_y \tag{8.6.9}$$

This model was originally introduced to describe flow of mixtures of pigments and oil. The model is now often used to describe flow of blood for low values of the shear measure. For high values of $\dot{\gamma}$ blood behaves as a Newtonian fluid.

8.6.3 Viscometric Flows. Kinematics

Steady flow between to parallel plates and without modified pressure gradient, Fig. 8.6.2, is called *steady simple shear flow*. As this type of flow has some characteristic features common for many more complex flows important in applications, we shall take a closer look at the characteristic aspects of steady simple shear flow. The velocity field is:

$$v_1 = \dot{\gamma} x_2,\ v_2 = v_3 = 0,\ \dot{\gamma} = \frac{v}{h} = \text{ constant} > 0 \tag{8.6.10}$$

resulting in the deformation rate matrix and the magnitude of shear rate:

$$D = \frac{1}{2}\dot{\gamma}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \sqrt{2\mathbf{D}:\mathbf{D}} = \sqrt{2tr D^2 = j} \tag{8.6.11}$$

The flow has the following characteristic features:

a) The flow is isochoric: $\nabla \cdot \mathbf{v} = \mathrm{tr} D = 0$.
b) Material planes parallel to the x_1x_3-plane moves in the x_1-direction without in-plane strains. We say that these planes represent a one-parameter family of *isometric surfaces*. The coordinate x_2 is the parameter defining each plane in the family. The word "isometric" is used to indicate that the distances between particles in each surface, measured in the surface, do not change during the flow. The isometric surfaces are called *shearing surfaces*.
c) The deformation rate matrix D is given by (8.6.11).
d) The magnitude of shear rate $\dot{\gamma}$ in (8.6.11) is constant.

The traces of two shearing surfaces are shown in Fig. 8.6.2. The particles in the upper surface have the velocity $v_{1,2} \cdot dx_2$ relative to the lower surface. The stream-

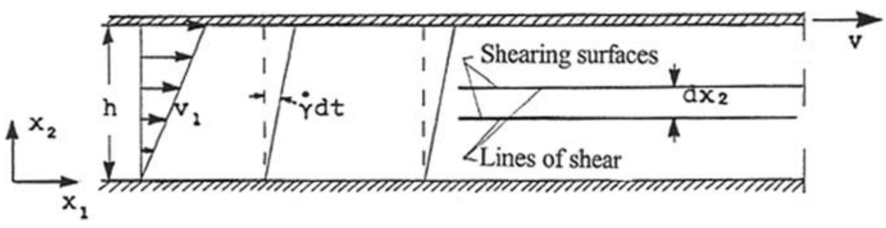


Fig. 8.6.2 Steady simple shear flow between a fixed plane and a moving parallel plate

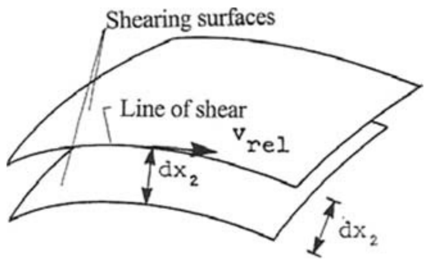


Fig. 8.6.3 Shear surfaces and line of shear

lines related to the velocity field $v_{1,2} \cdot dx_2$, when $dx_2 \to 0$, are called *lines of shear*. In the simple shear flow the shearing surfaces are planes and the lines of shear are straight lines parallel to the x_1-axis. Because the fluid particles are fixed to the same line of shear at all times, the lines of shear are material lines.

The general shear flow has features parallel to those of the simple steady shear flow. A flow is a *shear flow* if the following conditions are fulfilled:

a) The flow is isochoric: $\nabla \cdot \mathbf{v} = \mathrm{tr}D = 0$.

b) A one-parameter family of material surfaces exists that move isometrically, i.e. is without in-surface strains. These surfaces are called *shearing surfaces*, Fig. 8.6.3.

   The streamlines related to the velocity field v_{rel} of one shearing surface relative to a neighbor shearing surface are called *lines of shear*. The particles on one line of shear at the time t will not in general stay on the same line of shear at a later time. In other words, the lines of shear are not necessarily material lines. The condition a) implies zero strain rate normal to the shearing surfaces.

   A shear flow that in addition to the conditions a) and b) of a general shear flow, also satisfies the condition:

c) The lines of shear are material lines, is called a *unidirectional shear flow*. The material lines coinciding with the lines of shear at a particular time, will continue to be lines of shear as time passes. We may imaged that the lines of shear are "drawn" on the shearing surfaces and these material lines would than represent the lines of shear at later times. Unidirectional shear flow is the most common shear flow in applications and in particular in experiments designed to investigate the properties of non-Newtonian fluids.

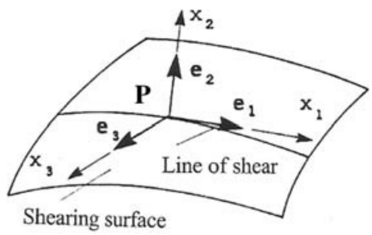


Fig. 8.6.4 Shear axes

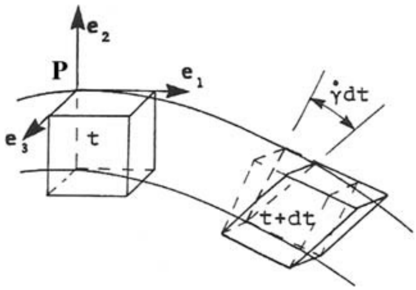


Fig. 8.6.5 Deformation of a fluid element

The analysis of the deformation kinematics of shear flows in the neighborhood of a particle P is simplified by introducing a local Cartesian coordinate system Px at the particle, as shown in Fig. 8.6.4. The coordinate axes are chosen such that the base vector $\mathbf{e}_1$ and $\mathbf{e}_3$ are tangents to the shearing surface, with $\mathbf{e}_1$ in the direction of the relative velocity $\mathrm{v}_{\mathrm{rel}}$ of the shearing surface relative to the neighbor shearing surface. The base vector $\mathbf{e}_1$ is thus tangent to the line of shear through the particle. The base vector $\mathbf{e}_2$ is normal to the shear surface. The three vectors $\mathbf{e}_\mathrm{i}$ are called the *shear axes*, and the vector $\mathbf{e}_1$ is the *shear direction*.

A fluid element $dV = dx_1 dx_2 dx_3$ is during a short time interval dt deformed as indicated in Fig. 8.6.5. The deformation is govern by the shear strain rate $\dot{\gamma}_{12} = v_{1,2}$. The deformation rate matrix D in the Px-system is therefore equal to the deformation rate matrix (8.6.11) of a simple shear flow, and the magnitude of shear rate is $\dot{\gamma} = |\dot{\gamma}_{12}|$.

A unidirectional flow that also satisfies the condition:

d) For every particle the magnitude of shear rate $\dot{\gamma}$ is independent of time, is called a *viscometric flow*. Another name of this kind of flow is *rheological steady flow*. The flow is not necessarily a steady flow as defined in fluid mechanics. Rheological steady means that the magnitude of shear rate of the fluid is not changing with time. Viscometric flows play an important role in investigating the properties of non-Newtonian fluids. We shall now present a series of important viscometric flows and identify shearing surfaces, lines of shear, and shear axes for each flow.

Example 8.11. Steady Axial Annular Flow. Steady Pipe Flow

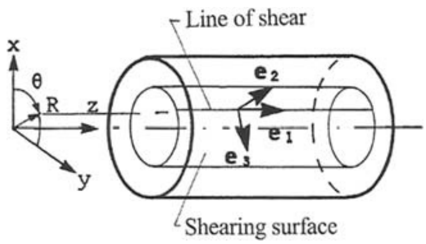


Fig. 8.6.6 Axial annular flow. Steady pipe flow

The fluid flows in the *annular space* between two solid, concentric cylindrical surfaces, or, as shown in Fig. 8.6.6, the fluid flows in a cylindrical pipe. The flow is steady and the velocity is parallel to the axis of the cylindrical surfaces:

$$v_z = v_z(R), \; v_R = v_\theta = 0 \tag{8.6.12}$$

The *shearing surfaces are concentric cylindrical surfaces*. The *lines of shear are straight lines* parallel to the axis of the cylindrical surfaces, and they coincide with the streamlines of the flow and with the pathlines of the fluid particles. The *shear axes* are:

$$\mathbf{e}_1 = \mathbf{e}_z, \; \mathbf{e}_2 = \mathbf{e}_R, \; \mathbf{e}_3 = \mathbf{e}_\theta \tag{8.6.13}$$

The magnitude of shear rate is:

$$\dot{\gamma} = |\dot{\gamma}_{zR}| = \left| \frac{dv_z}{dR} \right| \tag{8.6.14}$$

Example 8.12. Steady Tangential Annular Flow

The fluid flows in the annular space between two concentric solid cylindrical surfaces. One of the solid surfaces rotates with a constant angular velocity ω. Figure 8.6.7 shows the case where the inner cylindrical surface rotates. The velocity field is:

$$v_\theta = v_\theta(R), \; v_z = v_R = 0 \tag{8.6.15}$$

The *shearing surfaces are concentric cylindrical surfaces*. The *lines of shear are circles* with constant R and z, and they coincide with streamlines of the flow and the pathlines of the particles. The *shear axes* are:

$$\mathbf{e}_1 = \mathbf{e}_\theta, \; \mathbf{e}_2 = \mathbf{e}_R, \; \mathbf{e}_3 = -\mathbf{e}_z \tag{8.6.16}$$

The magnitude of shear rate is found from the formulas (5.4.19):

$$\dot{\gamma} = |\dot{\gamma}_{R\theta}| = \left| R \frac{d}{dR} \left(\frac{v_\theta}{R} \right) \right| \tag{8.6.17}$$

It is found that for a Newtonian fluid the velocity field presented by (8.6.15) is unstable when the angular velocity ω is increased above a certain limit. The instability introduces a secondary flow with velocities both in the z- and the R-directions and is described as *Taylor vortexes*. Instability and Taylor vortexes occur when:

$$T_a \equiv \left(\frac{\rho}{\mu} \omega \right)^2 r_i \left(r_o - r_i \right)^3 > 1700 \tag{8.6.18}$$

ρ is the density, μ is the viscosity, and r_i and r_o are the radii of the inner and outer solid boundary surfaces. T_a is called the Taylor number. At $T_a > 160 \cdot 10^3$ the flow becomes turbulent. Similar instabilities can occur for non-Newtonian fluids.

Example 8.13. Steady Torsion Flow

The fluid is set in motion between two plane concentric circular disks. One disk is at rest while the other disk rotates about its axis at a constant angular velocity ω. Figure 8.6.8 illustrates the situation. The dashed curved line indicates a free surface. In the case of a thick fluid this is really a free surface, while in the case of a thin fluid, the disks are submerged in a fluid bath. The rotating disk is touching the free surface of the bath and the dashed line marks an artificial free surface. Only the fluid between the disks is considered in the analysis.

The velocity field is assumed to be:

$$v_\theta = \frac{\omega}{h} R z, \; v_z = v_R = 0 \tag{8.6.19}$$

Based on the assumption that the fluid sticks to the solid disks, the velocity $v_\theta(R,z)$ satisfies the boundary conditions:

$$v_\theta\left(R,h\right) = \omega R, \; v_\theta\left(R,0\right) = 0 \tag{8.6.20}$$

The *shearing surfaces are planes normal to the axis of rotation*. The *lines of shear are concentric circles*, see Fig. 8.6.8b, and they coincide with the streamlines of the flow and the pathlines of the particles. Figure 8.6.8c shows an unfolded part of the cylinder surface $R \cdot dz$ between two shearing surfaces a distance dz apart. From the deformation of the fluid element shown in the figure we conclude that the *shear axes* are:

$$\mathbf{e}_1 = \mathbf{e}_\theta, \; \mathbf{e}_2 = \mathbf{e}_z, \; \mathbf{e}_3 = \mathbf{e}_R \tag{8.6.21}$$

and that the magnitude of shear rate becomes, as seen from Fig. 8.6.8c:

$$\dot{\gamma} = |\dot{\gamma}_{\theta z}| = \left|\frac{\partial v_\theta}{\partial z}\right| = \frac{\omega}{h} R \tag{8.6.22}$$

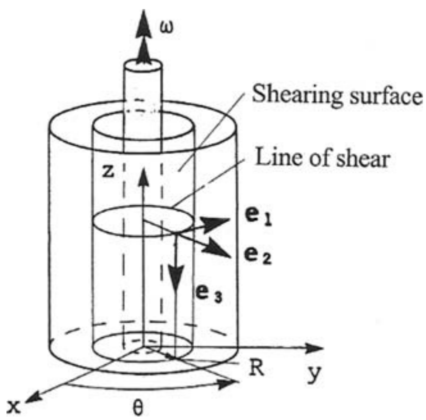


Fig. 8.6.7 Tangential annular flow

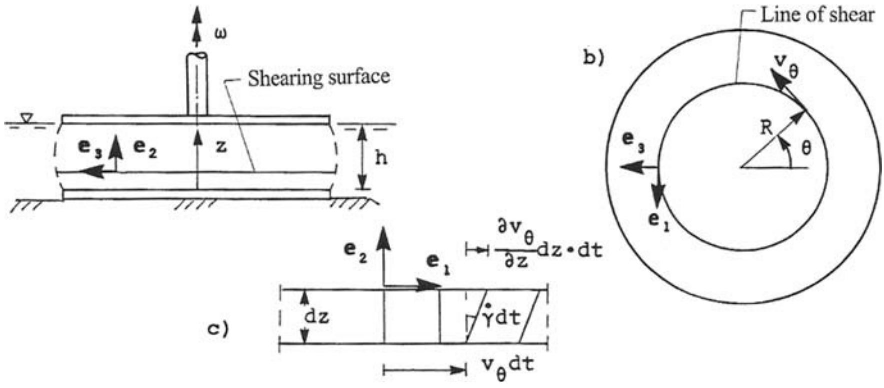


Fig. 8.6.8 Torsion flow

This result can is also be obtain from the formulas (5.4.19).

Example 8.14. Steady Helix Flow

The flow of the fluid in the annular space between to solid cylindrical surfaces is driven by the rotation and the axial translation of the inner cylindrical surface, see Fig. 8.6.9. The angular velocity ω and the axial velocity v are constants.

The velocity field is assumed as:

$$v_\theta = v_\theta(R), \ v_z = v_z(R), \ v_R = 0 \tag{8.6.23}$$

This kind of flow may also be obtained by a combination of a rotation of the inner cylinder and a constant modified pressure gradient $\partial P/dz$. The *shearing surfaces are concentric cylindrical surfaces*, which rotate and move in the axial direction. A fluid particle moves in a helix. Thus pathlines and streamlines are helices. A fluid particle on a shearing surface moves relative to a neighbor shearing surface also in a helix. Hence the *lines of shear are helices*, but they do not coincide with the

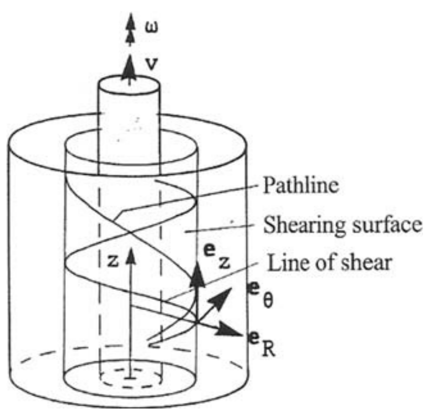


Fig. 8.6.9 Helix flow

streamlines or the pathlines. This is shown in Fig. 8.6.9. The rate of deformation matrix in cylindrical coordinates obtained from (5.4.18) and (5.4.19), contains only two independent elements for the flow (8.6.23):

$$\dot{\gamma}_{R\theta} = R\frac{d}{dR}\left(\frac{v_\theta}{R}\right), \ \dot{\gamma}_{zR} = \frac{dv_z}{dR} \tag{8.6.24}$$

The magnitude of shear rate becomes, see Problem 8.9:

$$\dot{\gamma} = \sqrt{2\mathbf{D}:\mathbf{D}} = \sqrt{2\left[\left(\frac{1}{2}\dot{\gamma}_{R\theta}\right)^2 2 + \left(\frac{1}{2}\dot{\gamma}_{zR}\right)^2 2\right]} \quad \Rightarrow$$

$$\dot{\gamma} = \sqrt{(\dot{\gamma}_{R\theta})^2 + (\dot{\gamma}_{zR})^2} \tag{8.6.25}$$

The shear axis normal to the shearing surface is $\mathbf{e}_2 = \mathbf{e}_R$. The *shear direction* and the third shear axis are found to be, Problem 8.9:

$$\mathbf{e}_1 = \frac{\dot{\gamma}_{R\theta}}{\dot{\gamma}}\mathbf{e}_\theta + \frac{\dot{\gamma}_{zR}}{\dot{\gamma}}\mathbf{e}_z, \ \mathbf{e}_3 = \frac{\dot{\gamma}_{zR}}{\dot{\gamma}}\mathbf{e}_\theta - \frac{\dot{\gamma}_{R\theta}}{\dot{\gamma}}\mathbf{e}_z \tag{8.6.26}$$

8.6.4 Material Functions for Viscometric Flows

Relations between stress components and deformation components, like strains and strain rates, in characteristic and simple flows are expressed by *material functions*. The *viscosity function* $\eta(\dot{\gamma})$ presented in Sect. 8.6.2 is an example of a material function for *unidirectional shear flows*. The characteristic flows for which the material functions are defined occur in standard experiments designed to investigate the properties of non-Newtonian fluids. In general the material functions may be functions of stresses, stress rates, strains, strain rates, temperature, time, and other parameters. The material functions are determined experimentally and are represented by data or mathematical functions representing these data.

In analyses of general flows fluid models are introduced. These models are defined by *constitutive equations*. A constitutive equation is a relationship between stresses and different measures of deformations, as strains, strain rates, and vorticities. A general constitutive equation is intended to represent a fluid in any flow, although it is experienced that most constitutive equations have limited application to only a few types of flows. The material functions may enter the constitutive equations or are used to determine material parameters in the constitutive equations. It might be a goal when constructing a fluid model that the constitutive equations of the model contain the material functions that is relevant for the special test flows that most resemblance the actual flow the fluid model is intended for.

We shall consider an *isotropic, incompressible, and purely viscous fluid* in a general viscometric flow, as described in Sect. 8.6.3. Figure 8.6.10a shows a particle

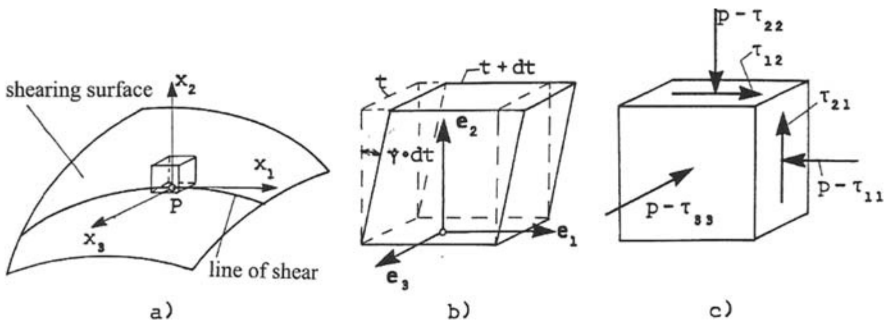


Fig. 8.6.10 Stresses in the viscometric flow

P and the shearing surface and the line of shear going through the particle. A local Cartesian coordinate system Px is introduced such that the base vectors $\mathbf{e}_i$ are the shear axes, see Fig. 8.6.10b. The rate of deformation matrix is given by (8.6.11), where $\dot{\gamma}$ is the magnitude of strain rate. The stress tensor $\mathbf{T}$ is decomposed into an isotropic part containing the pressure p, and a deviatoric extra stress $\mathbf{T}'$ due to the shear flow and assumed to be only a function of the magnitude of shear rate $\dot{\gamma}$ and the temperature θ. The temperature dependence will not be reflected implicitly in the following. Thus we set:

$$\mathbf{T} = -p\mathbf{1} + \mathbf{T}', \; T'_{ik} \equiv \tau_{ik}(\dot{\gamma}) \tag{8.6.27}$$

The condition of material isotropy implies that the state of stress must have the same symmetry as the state of deformation rate. With reference to Fig. 8.6.10 the x_1x_2-plane is a symmetry plane. This implies that the shear stresses $\tau_{13} = \tau_{31}$ and $\tau_{23} = \tau_{32}$ must be zero because these stresses act antisymmetrically with respect to the symmetry plane. The state of stress in the fluid is therefore given by the stress matrix:

$$T = (-p\delta_{ik} + \tau_{ik}) = \begin{pmatrix} -p+\tau_{11} & \tau_{12} & 0 \\ \tau_{12} & -p+\tau_{22} & 0 \\ 0 & 0 & -p+\tau_{33} \end{pmatrix} \tag{8.6.28}$$

Incompressibility implies that the pressure p cannot be given by a constitutive equation but has to be determined from the equations of motion and the boundary conditions for the flow. For an incompressible fluid the pressure level cannot influence the flow. Only pressure gradients are of importance.

In measuring directly or indirectly the normal stresses, it is not possible to distinguish between the pressure p and the contribution from the extra stresses due to the deformation of the fluid. The implication of this is that only normal stress differences may be expressed by material functions. In a viscometric flow we seek material functions for the following three stresses:

The shear stress: τ_{12}

The primary normal stress difference: $N_1 = T_{11} - T_{22} = \tau_{11} - \tau_{22}$

The secondary normal stress difference: $N_2 = T_{22} - T_{33} = \tau_{22} - \tau_{33}$ (8.6.29)

The third normal stress difference, $T_{11} - T_{33}$, is determined by the two others:

$$T_{11} - T_{33} = (T_{11} - T_{22}) + (T_{22} - T_{33}) = N_1 + N_2 \tag{8.6.30}$$

Three material functions, called *viscometric functions*, are introduced in a viscometric flow:

$$\begin{aligned}
\eta(\dot{\gamma}) &= \frac{|\tau_{12}(\dot{\gamma})|}{\dot{\gamma}} \quad \text{the viscosity function} \\
\psi_1(\dot{\gamma}) &= \frac{N_1(\dot{\gamma})}{(\dot{\gamma})^2} \quad \text{the primary normal stress coefficient} \\
\psi_2(\dot{\gamma}) &= \frac{N_2(\dot{\gamma})}{(\dot{\gamma})^2} \quad \text{the secondary normal stress coefficient}
\end{aligned} \tag{8.6.31}$$

$\dot{\gamma}$ is the magnitude of shear rate. The viscosity function is also called the *apparent viscosity*.

Figure 8.6.11 shows characteristic behavior of the viscometric functions for shear-thinning fluids. For low values of the magnitude of shear rate $\dot{\gamma}$ the viscosity function $\eta(\dot{\gamma})$ is nearly constant and equal to $\eta_o = \eta(0)$, called the *zero-shear-rate-viscosity*. For high values of the magnitude of shear rate $\dot{\gamma}$ the viscosity function $\eta(\dot{\gamma})$ may approach asymptotically a *infinite-shear-rate viscosity* η_∞.

For some fluids, for example highly concentrated polymer solutions and polymer melts, it may be impossible to measure η_∞. For the fluids mentioned the reason is that the polymer chains may be destroyed at very high shear rates.

The primary normal stress coefficient $\psi_1(\dot{\gamma})$ is positive, and is almost constant and equal to $\psi_{1,o} = \psi_1(0)$ for low magnitude of shear rate, and then decreases more rapidly with increasing magnitude of shear rate than the viscosity function $\eta(\dot{\gamma})$. A lower bound for ψ_1 when $\dot{\gamma} \to \infty$, is not registered.

The secondary normal stress coefficient $\psi_2(\dot{\gamma})$ is usually negative and is found for polymeric fluids to be approximately 10% of $\psi_1(\dot{\gamma})$ for the same fluid.

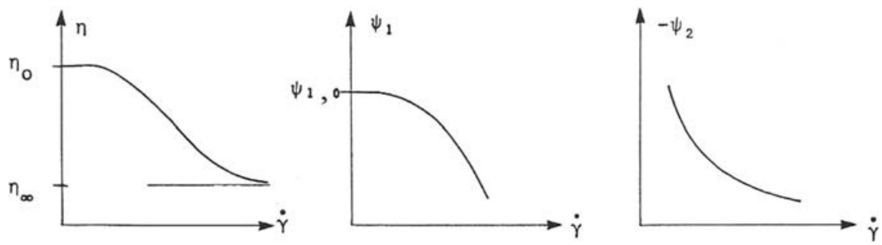


Fig. 8.6.11 Characteristic behavior of viscometric functions

8.6.5 Extensional Flows

As mentioned in Sect. 5.4, in any flow there exist through each particle at any time t three orthogonal material line elements that do not show shear strain rates: The lines remain orthogonal after a short time increment dt. Confer the elements 2 in the Examples 5.1 and 5.2. The three material line elements represent the principal directions of strain rates at the time t. We assume that the fluid is incompressible and introduce a local Cartesian coordinate system Px in the particle P, and with base vectors $\mathbf{e}_i$ coinciding with the *principal directions PD of strain rates* at the time t. Then the rate of deformation matrix takes the form:

$$D = \begin{pmatrix} \dot{\varepsilon}_1 & 0 & 0 \\ 0 & \dot{\varepsilon}_2 & 0 \\ 0 & 0 & \dot{\varepsilon}_3 \end{pmatrix}, \quad \dot{\varepsilon}_1 + \dot{\varepsilon}_2 + \dot{\varepsilon}_3 = 0 \tag{8.6.32}$$

A flow is called an *extensional flow if the same material line elements ML through each particle represent the principal directions PD of strain rates at all times*. This also implies the principal directions of strain rates are identical to the principal direction of strains. The literature also uses the names *elongational flow* and *shear free flow* for this type of flow. See Example 5.4.

A simple extensional flow is given by the velocity field:

$$v_x = \dot{\varepsilon}_x(t)\, x, \; v_y = \dot{\varepsilon}_y(t)\, y, \; v_z = \dot{\varepsilon}_z(t)\, z \tag{8.6.33}$$

The deformation of a volume element in this flow is illustrated in Fig. 8.6.12. Material lines parallel to the coordinate axes represent the principal directions PD of rates of strain at all times. The principal directions are fixed in space for this simple extensional flow.

It follows from Fig. 5.4.3 that in a simple shear flow the material line elements representing the principal directions of rates of strain at a time t do not represent the principal directions at a later time $t + dt$. The principal directions are fixed in space but the material lines coinciding with the principal directions at one time are not fixed in space. This difference between shear flows and extensional flows is very important in modelling of non-Newtonian fluids.

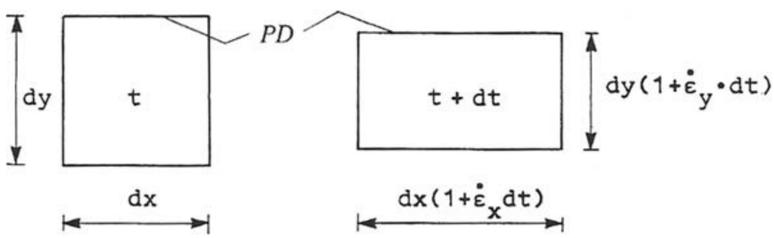


Fig. 8.6.12 Extensional flow. PD = principal directions of strain rates

Extensional flows are important in experimental investigations of the properties of non-Newtonian fluids. These flows are also relevant in connection with forming processes for plastics, as for example in vacuum forming, blow molding, foaming operations, and spinning. In metal forming extensional flows are important in milling and extrusion.

We continue to discuss incompressible and isotropic fluids for which the deviatoric stresses τ_{ik} only depend on the rate of deformation matrix (8.6.32). Isotropy implies that the principal axes of stress coincide with the principal directions of strain and strain rates. Based on the structure of the rate of deformation matrix (8.6.32) the stress matrix is presented as:

$$T = (-p\delta_{ik} + \tau_{ik}) = \begin{pmatrix} -p+\tau_{11} & 0 & 0 \\ 0 & -p+\tau_{22} & 0 \\ 0 & 0 & -p+\tau_{33} \end{pmatrix} \tag{8.6.34}$$

Because the pressure is constitutively indeterminate, only normal stress differences may be modelled:

$$T_{11} - T_{22} = \tau_{11} - \tau_{33}, \; T_{22} - T_{33} = \tau_{22} - \tau_{33} \tag{8.6.35}$$

Three special cases of steady extensional flow will now be presented.
UNIAXIAL EXTENSIONAL FLOW. See also Example 5.5. The rate of deformation matrix for this flow is given by (8.6.32) with:

$$\dot{\varepsilon}_1 = \dot{\varepsilon} = \text{ constant}, \; \dot{\varepsilon}_2 = \dot{\varepsilon}_3 = -\frac{\dot{\varepsilon}}{2} \tag{8.6.36}$$

This type of flow is relevant when the fluid is stretched axisymmetrically in one direction. Material isotropy and the strain rates (8.6.36) imply that the normal stresses τ_{22} and τ_{33} are equal. Thus only one normal stress difference need be modelled, and the relevant material function is:

$$\eta_E(|\dot{\varepsilon}|) = \frac{\tau_{11} - \tau_{22}}{\dot{\varepsilon}} \tag{8.6.37}$$

called the *extensional viscosity* or the *Trouton viscosity*, F. T. Trouton(1906). For some fluids the extensional viscosity is decreasing with increasing strain rate. This is called *tension-thinning*. If the extensional viscosity is increasing with increasing strain rate the fluid is said to exhibit *tension-thickening*.

For Newtonian fluids with shear viscosity μ:

$$\tau_{11} = 2\mu\,\dot{\varepsilon}, \; \tau_{22} = \tau_{33} = -\mu\,\dot{\varepsilon} \quad \Rightarrow \quad \eta_E = \frac{\tau_{11} - \tau_{22}}{\dot{\varepsilon}} = 3\mu \tag{8.6.38}$$

The relationship between the extensional viscosity and the shear viscosity is in classical Newtonian fluid mechanics associated with the name Trouton.

The behavior of the extensional viscosity is often qualitatively different from that of the shear viscosity. It is found that highly elastic polymer solutions that show shear-thinning often exhibit a dramatic tension-thickening. Experiments and further analysis in continuum mechanics show that in the limit, as the strain rate approaches zero the extensional viscosity approaches a value three times the zero-shear rate viscosity:

$$\eta_E\left(|\dot{\varepsilon}|\right)|_{\dot{\varepsilon}\to 0} = 3\,\eta\left(\dot{\gamma}\right)|_{\dot{\gamma}\to 0} \quad \Rightarrow \quad \eta_E\left(0\right) = 3\eta\left(0\right) \quad \Leftrightarrow \quad \eta_{Eo} = 3\eta_o \tag{8.6.39}$$

BIAXIAL EXTENSIONAL FLOW. When a fluid is stretched or compressed equally in two orthogonal directions the flow is called a biaxial extension. The rate of deformation matrix for this flow is given by (8.6.32) with:

$$\dot{\varepsilon}_1 = \dot{\varepsilon}_2 = \dot{\varepsilon} = \text{ constant}, \quad \dot{\varepsilon}_3 = -2\dot{\varepsilon} \tag{8.6.40}$$

The material function relevant for this type of flow is called *biaxial extensional viscosity*:

$$\eta_{EB}\left(|\dot{\varepsilon}|\right) = \frac{\tau_{11} - \tau_{33}}{\dot{\varepsilon}} \tag{8.6.41}$$

A comparison of the biaxial extensional flow and uniaxial extensional flow shows that constitutive modelling of a fluid in either flow should be the same. In fact it follows that:

$$\eta_{EB}\left(|\dot{\varepsilon}|\right) = 2\eta_E\left(2\,|\dot{\varepsilon}|\right) \tag{8.6.42}$$

PLANAR EXTENSIONAL FLOW. See also Example 5.6. The rate of deformation matrix for this flow is given by (8.6.32) with:

$$\dot{\varepsilon}_1 = -\dot{\varepsilon}_2 = \dot{\varepsilon} = \text{ constant}, \; \dot{\varepsilon}_3 = 0 \tag{8.6.43}$$

The material function relevant for this type of flow is called *planar extensional viscosity*:

$$\eta_{EP}\left(|\dot{\varepsilon}|\right) = \frac{\tau_{11}}{\dot{\varepsilon}} \tag{8.6.44}$$

Problems

Problem 8.1. A closed vessel filled with a fluid is given a translatoric motion defined by the velocity field:

$$v_1 = -v_o \sin\omega t, \; v_2 = v_o \cos\omega t$$

v_o and ω are constants. The fluid moves with the vessel as a rigid body. Show the streamlines at time t are straight lines, and that the path lines are circles.

Problem 8.2. Show that the streamlines and the path lines coincide for the following type of non-steady two-dimensional flow:

$$v_1 = f(t)\,g(x,y), \; v_2 = f(t)\,h(x,y), \; v_3 = 0$$

$f(t)$, $g(x,y)$, and $h(x,y)$ are arbitrary functions of the variables: time t and Cartesian coordinates x and y.

Problem 8.3. Let $\alpha(\mathbf{r}) = 0$ represent a fixed rigid boundary surface A in a flow of a perfect fluid. Show that the velocity field $\mathbf{v}(\mathbf{r},t)$ must satisfy the condition:

$$\mathbf{v} \cdot \nabla\alpha = 0 \text{ on } A$$

Problem 8.4. Let $\alpha(\mathbf{r},t) = 0$ represent a moving rigid boundary surface A in a flow of a perfect fluid. Show that the velocity field $\mathbf{v}(\mathbf{r},t)$ must satisfy the condition:

$$\partial_t\alpha + \mathbf{v} \cdot \nabla\alpha = 0 \text{ on } A$$

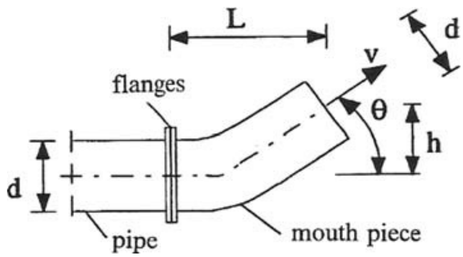


Fig. Problem 8.5

Problem 8.5. A mouth piece is attached to a pipe by flanges and bolts. The cross-sections of the piece and the pipe are the same with the area A. Water of constant velocity v flows through the pipe and out through the mouth piece. The pressure in the fluid is equal to the atmospheric pressure p_o. Determine the shear force, the axial force and the bending moment at the flanges.

Problem 8.6. Show that the two expressions for the elastic energy per unit mass in (8.3.11) are equivalent.

Problem 8.7. Use the identity (2.1.17) to prove the identity (8.3.27).

Problem 8.8. Use a differentiation test and (5.5.28) that (8.3.30) is the solution of the differential equation (8.3.29).

Problem 8.9. Derive the results presented as (8.6.25) and (8.6.26).

Chapter 9
Viscoelasticity

9.1 Introduction

Viscoelastic response results in two important quasi-static phenomena: *creep* and *stress relaxation*, and in damping and energy dissipation when the material is subjected to dynamic loading. Many materials that under "normal" temperatures may be considered purely elastic, will at higher temperatures respond viscoelastically. It is customary to introduce a *critical temperature* θ_c for these materials such that the material is considered to be viscoelastic at temperatures $\theta > \theta_c$. For example for common structural steel the critical temperature θ_c is approximately 400 °C. For plastics a *glass transition temperature* θ_g is introduced. At temperatures below the glass transition temperature the materials behave elastically, more or less like brittle glass. Established plastic materials have θ_g-values from $-120\,°\mathrm{C}$ to $+120\,°\mathrm{C}$. Some plastic materials behave viscoelastically within a certain temperature interval: $\theta_g < \theta < \theta'$. For temperatures $\theta < \theta_g$ and $\theta > \theta'$ these materials are purely elastic. Vulcanized rubber is an example of such a material.

In order to expose the most characteristic properties of real viscoelastic materials, we shall now present typical results from tests in which a specimen is subjected to uniaxial tension. Figure 9.1.1 shows creep at constant stress σ_o, while Fig. 9.1.2 show stress relaxation at constant strain ε_o. In Sect. 9.4 we will analyze results from dynamic testing in which the test specimen is subjected to harmonic varying stress $\sigma_o \sin \omega t$ or strain $\varepsilon_o \sin \omega t$.

In a creep test the test specimen is subjected to uniaxial stress σ_o during a time interval $< 0, t_1 >$. Thereafter the stress is removed. During the test the axial strain in the specimen is registered as a function of time and the stress level: $\varepsilon(\sigma_o, t)$. The creep diagram shown in Fig. 9.1.1 may be divided into the following regions:

I. *Initial strain* $\varepsilon^i = \varepsilon^{i,e} + \varepsilon^{i,p}$. Almost instantaneously the specimen gets an initial strain, which may be purely elastic, or contain an elastic part $\varepsilon^{i,e}$ and a plastic part $\varepsilon^{i,p}$.

P. *Primary creep*. The time rate of strain $\dot{\varepsilon} = d\varepsilon/dt$ is at first relatively high, but decreases towards a stationary value.

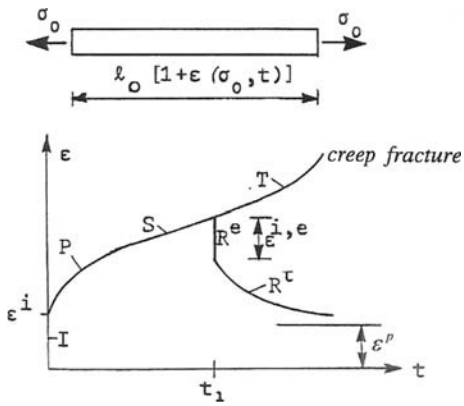


Fig. 9.1.1 Creep test

S. *Secondary creep*. The rate of strain $\dot{\varepsilon}$ is constant.
T. *Tertiary creep*. If the specimen is under constant stress for a long period of time, the rate of strain $\dot{\varepsilon}$ starts to increase until eventually *fracture* occurs. Two completely different effects may explain the increasing strain rate: 1) The material weakens mechanically and 2) The true stress over the cross-section of the specimen increases because the cross-sectional area is reduced during the creep process. The constant stress σ_o in a creep test is a nominal stress based on a constant tensile force on the specimen. The latter effect is the dominating one, and *creep fracture analysis* is normally based on this.
R^e. *Elastic restitution*. At unloading the initial elastic strain $\varepsilon^{i,e}$ disappears momentarily.
R^t. *Time dependent restitution*, also called elastic after-effect.

Note that the creep test diagrams in Fig. 9.1.1 and in the figures to follow show both the actual creep test and the restitution process after the stress has been removed.

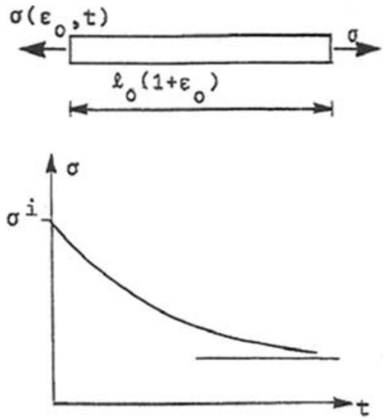


Fig. 9.1.2 Relaxation test

After the restitution is completed, in principle it may take infinitely long time, the test specimen has got a permanent or plastic strain ε^p. The different regions described above are more or less prominent for different materials. Tests show that an increase in the stress level or of the temperature will lead to increasing strain rates in all the creep regions.

In a stress relaxation test the test specimen is subjected to an initial uniaxial stress σ^i, which produce a axial strain ε_o. This strain is kept constant equal to ε_o, and the development of the axial stress history $\sigma(\varepsilon_o,t)$ is registered. Tests show that the stress σ decreases asymptotically towards a value, which may be zero. We shall see that there is a close correspondence between the response of a material in a creep test and in a relaxation test.

A viscoelastic material may be classified as a solid or a fluid, see Fig. 9.1.3. The creep diagram for a *viscoelastic solid* will exhibit elastic initial strain, primary creep, and complete restitution without plastic strain. The primary creep will after sufficiently long time reach an "elastic ceiling", which is given by the *equilibrium strain* $\varepsilon_e = \varepsilon_e(\sigma_o)$. In a relaxation test of a viscoelastic solid the stress decreases towards an *equilibrium stress* $\sigma_e = \sigma_e(\varepsilon_o)$. The creep diagram for a *viscoelastic fluid* may exhibit all the regions mentioned in connection with Fig. 9.1.1. For a solid material that responds as a viscoelastic fluid, for instance steel with temperatures over the critical value θ_c, tertiary creep will eventually result in a *creep fracture*. The relaxation graph of a viscoelastic fluid approaches the zero stress level asymptotically. Figure 9.1.3 also presents the response curves for an elastic material, for example a Hookean solid, and a viscous material, for example a Newtonian fluid.

In the mathematical description of the creep test and the relaxation test we need to apply two special functions: the Heaviside unit step function and the Dirac delta function. The *Heaviside function* $H(t)$ is defined in (7.7.30) as:

$$H(t) = \begin{cases} 0 \text{ for } t \leq 0 \\ 1 \text{ for } t > 0 \end{cases} \tag{9.1.1}$$

Figure 9.1.4 illustrates the graph of the function.

The *Dirac delta function* $\delta(t)$, named after Paul A.M. Dirac [1902–1984], has the following properties:

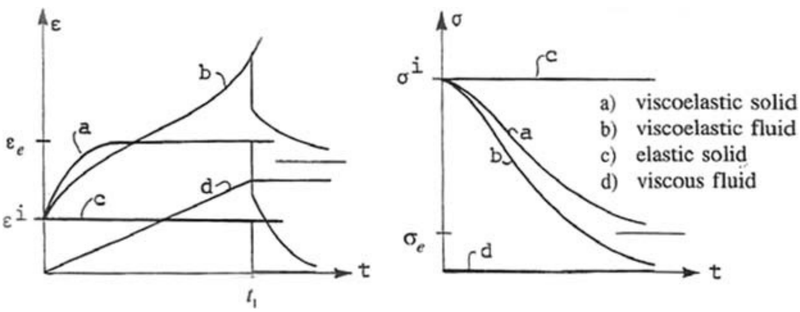


Fig. 9.1.3 Solid and fluid response in creep and relaxation

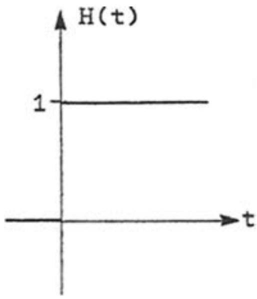


Fig. 9.1.4 Heaviside function $H(t)$

$$\delta(t) = \dot{H}(t) = \begin{cases} 0 \text{ for } t \neq 0 \\ \infty \text{ for } t = 0 \end{cases} \tag{9.1.2}$$

$$\int_0^\infty \delta(t)\, dt = 1\,, \quad \int_b^c f(t)\, \delta(t)\, dt = f(0)\,[H(c) - H(b)] \tag{9.1.3}$$

The last integral formula applies to any continuous function $f(t)$.

To get a better understanding of the delta function, we may first introduce the functions:

$$H_a(t) = \begin{cases} 0 & \text{for} \quad t \leq 0 \\ t/a & \text{for } 0 < t \leq a \\ 1 & \text{for} \quad t > a \end{cases} \qquad \text{see Fig. 9.1.5} \tag{9.1.4}$$

$$\delta_a(t) = \begin{cases} 0 & \text{for } t \leq 0 \text{ and } t > a \\ 1/a & \text{for} \quad 0 < t \leq a \end{cases} \qquad \text{see Fig. 9.1.6} \tag{9.1.5}$$

The function $H_a(t)$ is continuous everywhere and differentiable everywhere, except for $t = 0$ and $t = a$, for which the function is unilaterally differentiable:

$$\dot{H}_a(t) = \delta_a(t) \text{ for } t \neq 0,\ a\,, \quad \lim_{a \to 0} H_a(t) = H(t) \tag{9.1.6}$$

Furthermore:

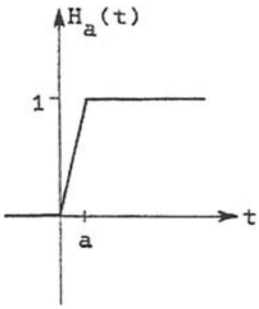


Fig. 9.1.5 Graph of the function $H_a(t)$ in (9.1.4)

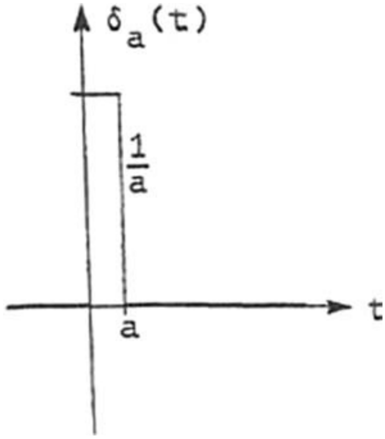


Fig. 9.1.6 Graph of the function $\delta_a(t)$ in (9.1.5)

$$\int_0^\infty \delta_a(t)\,dt = \int_0^\infty \dot{H}_a(t)\,dt = 1 \tag{9.1.7}$$

Let $f(t)$ be a continuous function and b and c two time points outside the interval $[0,a]$. Then:

$$\int_b^c f(t)\,\delta_a(t)\,dt = f(e)\,[H_a(c) - H_a(b)] = f(e)\,[H(c) - H(b)] \tag{9.1.8}$$

e is a time point in the interval $[0,a]$, such that $f(e)$ is the mean value of $f(t)$ within the interval.

We may define the delta function $\delta(t)$ such that:

$$\delta(t) = \lim_{a\to 0} \delta_a(t) \tag{9.1.9}$$

Now, let $a \to 0$, then (9.1.8) $\to$ (9.1.3)$_2$, (9.1.7) $\to$ (9.1.3)$_1$, and (9.1.6) $\to$ (9.1.2)$_1$.

The definition of $H(t)$ and $\delta(t)$ may vary somewhat in the literature. The value $H(0)$ is often not defined or defined as $1/2$. The delta function may be defined symmetrically about $t = 0$, for instance as the limit of a function $\delta_{\mathrm{a}}(t)$ that is equal to $1/(2a)$ for t in the interval $-a \leq t \leq a$, and equal to 0 for $t \leq -a$ and for $t > a$.

In a creep test the axial stress in the test specimen may be described by the function:

$$\sigma(t) = \sigma_o\,[H(t) - H(t - t_1)] \tag{9.1.10}$$

The result of the creep phase of the test, that is when $\sigma = \sigma_o H(t)$, may be described by a *creep function* $\alpha(\sigma_{\mathrm{o}}, t)$, such that the axial strain becomes:

$$\varepsilon(\sigma_o, t) = \alpha(\sigma_o, t)\,\sigma_o\,H(t) \tag{9.1.11}$$

In a relaxation test the specimen is subjected to an axial stress $\sigma(\varepsilon_o,t)$ which produces the axial strain:

$$\varepsilon(t) = \varepsilon_o H(t) \tag{9.1.12}$$

The test result may be described by a *relaxation function* $\beta(\varepsilon_o,t)$ such that:

$$\sigma(\varepsilon_o,t) = \beta(\varepsilon_o,t)\,\varepsilon_o H(t) \tag{9.1.13}$$

If we obtain the result that the creep function and the relaxation function are functions only of time:

$$\alpha = \alpha(t),\ \beta = \beta(t) \tag{9.1.14}$$

we say that the material shows *linearly viscoelastic response*. A linearly viscoelastic material model may be used as a first approximation in many cases. The creep function and the relaxation function are examples of what are called *material functions* in Sect. 8.6.

Viscoelastic response is a dominating property for plastics. For small strains these materials are linearly viscoelastic. We shall now present some special features of plastics in relation to the two quasi-static tests discussed above and use the creep function $\alpha(t)$ and the relaxation function $\beta(t)$ in the description. Plastics are divided into two main categories, which may appear in amorphous or crystalline forms:

a) *Thermo plastics* also called linear plastics
b) *Thermoset plastics* also called crosslinked plastics

The molecules in a thermo plastic are long and interwoven without being chemical bound together. Figure 9.1.7a and c show respectively a creep curve and a relaxation curve for this kind of plastics. Secondary creep, permanent strain after unloading, and complete stress relaxation are characteristic effects. The instantaneous response is given by the *glass compliance* $\alpha_g = \alpha(0)$ and the *glass modulus*, also called the *short time modulus*, $\beta_g = \beta(0)$. Both these parameters are temperature dependent. The material is called a thermoplastic because it may be melted, formed and cooled a large number of times without losing essential properties. Natural rubber represents a thermo plastic. With reference to Fig. 9.1.3 we see that the thermo plastics are viscoelastic fluids.

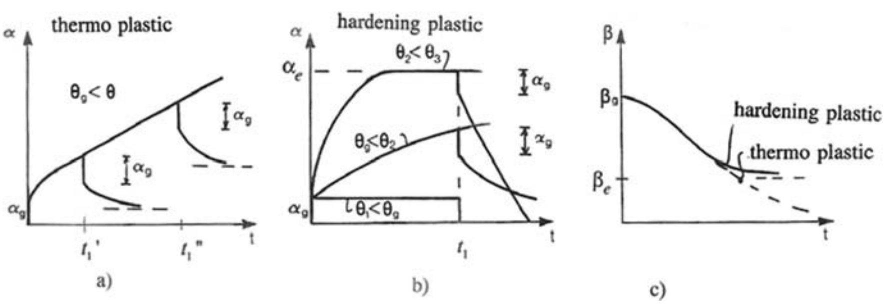


Fig. 9.1.7 Characteristic creep and relaxation functions for plastics

The thermoset plastics are created when long chain molecules are bound together chemically, both in the directions of the chains and across the chains. Thus a three-dimensional net is formed. These plastics do not show permanent strains after a loading-unloading procedure, see Fig. 9.1.7b. Primary creep is predominant. At high temperatures in the viscoelastic temperature interval, $\theta_g < \theta < \theta'$, the material behaves "rubber-like": the creep strain approaches an "elastic ceiling", in Fig. 9.1.7b marked by the *equilibrium compliance* $\alpha_e = \alpha(\infty)$. In a relaxation test the stress decreases toward a limit value, in Fig. 9.1.7c marked by the *equilibrium modulus*, or the *long time modulus*, $\beta_e = \beta(\infty)$. A comparison with Fig. 9.1.3 indicates that the hardening plastics are viscoelastic solids.

Usually a hardening plastic is made from a thermoplastic by adding a hardening component and/or by heating such that chemical bounds are established between the chain molecules. This process gives the name hardening plastics to these materials. The material cannot be remelted after the first hardening process. Vulcanized rubber represents a hardening plastic.

All parameters $\alpha_g, \alpha_e, \beta_g$, and β_e are temperature dependent. Because it is the same whether we set $\sigma(0^+) = \beta_g \varepsilon_o$ or $\varepsilon(0^+) = \alpha_g \sigma_o$, and $\sigma(\infty) = \beta_e \varepsilon_o$ or $\varepsilon(\infty) = \alpha_e \sigma_o$, we have the result:

$$\alpha_g \beta_g = 1 \,, \quad \alpha_e \beta_e = 1 \tag{9.1.15}$$

Tests with multiaxial states of stress show that viscoelastic response is primary a shear stress-shear strain effect. Very often materials subjected to isotropic stress will deform elastically. This fact fits well with the common notion that there is a close micro-mechanical correspondence between viscous deformation and plastic deformation, and that plastic deformation is approximately isochoric, i.e. without volumetric strain. Therefore it is natural to investigate separately viscoelastic properties in shear stress tests and in tests with isotropic states of stress. The shear tests may be performed with torsion of thin-walled tubes, or by subjecting rectangular panels to shear forces. Tests with isotropic states of stress to determine creep and relaxation response are difficult to perform. For isotropic, linearly viscoelastic materials we may get sufficient information describing the material through a model by performing shear tests and uniaxial normal stress tests.

The present chapter is divided into sections as follows. Section 9.2 presents the constitutive equations for isotropic, linearly viscoelastic materials, with emphasis on isotropic materials. The *principle of correspondence* is introduced in Sect. 9.3, by which boundary value problems for linearly viscoelastic materials may be transformed mathematically to similar problems from the theory of linearly elastic materials. We present applications of the principle in some quasi-static problems. Section 9.4 discusses materials subjected to periodical loading or deformation, and discusses damping effects in viscoelastic materials. Wave propagation is the subject matter in Sect. 9.5. Non-linear material response is given a brief presentation in Sect. 9.6. The Norton fluid, which is the most widely used material model to describe viscoelasticity for metals, will be applied in a few examples.

9.2 Linearly Viscoelastic Materials

The classical theory of viscoelastic materials assumes small deformations and concentrates the attention to linear constitutive equations. Constitutive equations for uniaxial stress may be constructed from mechanical models consisting of springs and dashpot with analogous response. This method of developing constitutive equations is not a necessity, but it provides a good physical understanding and feeling for viscoelastic behavior. Constitutive equations for general states of stress may then be constructed through a generalization of the equations for the uniaxial stress. We shall now start by a presentation of the most common mechanical models. More models may be found in the book by Flügge [16].

9.2.1 Mechanical Models

The mechanical models shall represent a test specimen under axial loading and thus uniaxial stress, see Fig. 9.2.1. We assume that the material behaves equally in tension and compression. The length of the specimen and thus of the mechanical model is l_o, the cross-sectional area is A_o, and the axial force is N. The axial stress in the specimen is then: $\sigma = N/A_o$, and the axial strain is: $\varepsilon = \Delta l/l_o$, where $\Delta l = \varepsilon l_o$ is the elongation of the specimen.

A linearly elastic material under uniaxial stress has the constitutive equation: $\sigma = \eta\varepsilon$, where η is the *modulus of elasticity*. The response of a test specimen of this material is the same as that of a *linear spring*, Fig. 9.2.2, with a length l_o and spring stiffness $k = \eta A_o/l_o$, and subjected to the force:

$$N = \sigma A_o = k\Delta l = \left(\frac{\eta A_o}{l_o}\right)(\varepsilon l_o) = \eta A_o \varepsilon$$

The response equation of the spring is: $N = k\Delta l$ and may be simplified to:

$$\sigma = \eta\,\varepsilon \tag{9.2.1}$$

The linear spring shown in Fig. 9.2.2, will be called a *Hookean model*. The creep function $\alpha(t) = \varepsilon(t)/\sigma_o$, and the relaxation function $\beta(t) = \sigma(t)/\varepsilon_o$ of the model is found by introducing in turns:

$$\sigma(t) = \sigma_o H(t) \quad \text{and} \quad \varepsilon(t) = \varepsilon_o H(t)$$

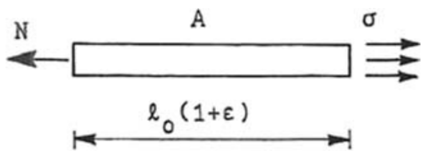
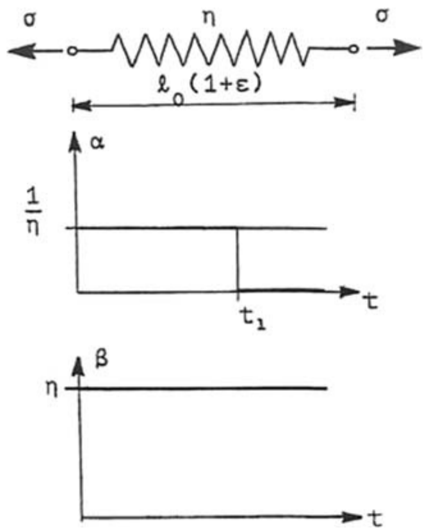


Fig. 9.2.1 Uniaxial stress

Fig. 9.2.2 Hookean model



into (9.2.1). The result is:

$$\alpha(t) = \frac{1}{\eta} = \alpha_g = \alpha_e \,, \quad \beta(t) = \eta = \beta_g = \beta_e \tag{9.2.2}$$

A linearly viscous material in the state of uniaxial stress has the constitutive equation: $\sigma = \tilde{\eta}\dot{\varepsilon}$, where $\tilde{\eta}$ is a *viscosity*. The response of a test specimen of this material is the same as of a *linear dashpot*, or a linear damper, shown in Fig. 9.2.3, with length l_o and coefficient of damping: $c = \tilde{\eta}A_o/l_o$, subjected to the force:

$$N = \sigma A_o = c\Delta\dot{l} = c\,(\dot{\varepsilon}\,l_o) = \left(\frac{\tilde{\eta}\,A_o}{l_o}\right)(\dot{\varepsilon}\,l_o) = \tilde{\eta}\,A_o\,\dot{\varepsilon}$$

The response equation of the dashpot is: $N = c\,(\dot{\varepsilon}l_o)$ and may be simplified to:

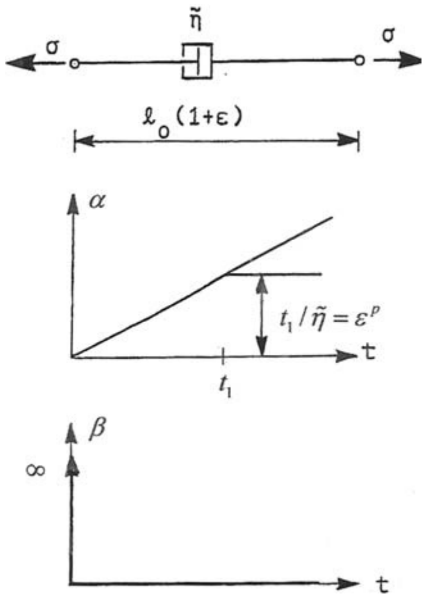
$$\sigma = \tilde{\eta}\,\dot{\varepsilon} \tag{9.2.3}$$

The linear dashpot shown in Fig. 9.2.3 will be called a *Newtonian model*. For the creep test $\sigma(t) = \sigma_o H(t)$, and for the relaxation test $\varepsilon(t) = \varepsilon_o H(t)$, (9.2.3) gives:

$$\alpha(t) = \frac{t}{\tilde{\eta}} \,, \quad \alpha_g = 0 \,, \quad \alpha_e = \infty \,, \quad \beta(t) = \tilde{\eta}\,\delta(t) \,, \quad \beta_g = \infty \,, \quad \beta_e = 0 \tag{9.2.4}$$

The *Maxwell model*, Fig. 9.2.4, is a series of a linear spring and a linear dashpot. The rate of strain $\dot{\varepsilon}$ of the model is equal to the sum of the rates of strain $\sigma/\tilde{\eta}$ of the damper and $\dot{\sigma}/\eta$ of the spring. The model therefore has the response equation:

Fig. 9.2.3 Newtonian model



$$\frac{\sigma}{\tilde{\eta}} + \frac{\dot{\sigma}}{\eta} = \dot{\varepsilon} \tag{9.2.5}$$

The relaxation function $\beta(t)$ is found from the solution of the differential equation (9.2.5) with $\varepsilon(t) = \varepsilon_o H(t)$ and the initial condition $\sigma(0^-) = 0$. First we write the differential equation as:

$$\frac{\sigma}{\tilde{\eta}} + \frac{\dot{\sigma}}{\eta} = \varepsilon_o\,\delta(t) \tag{9.2.6}$$

The homogeneous solution, with $\varepsilon_o = 0$, is $C\exp(-\eta t/\tilde{\eta})$, where C is a constant. For the inhomogeneous equation (9.2.6) we try the solution $\sigma(t) = f(t)\exp(-\eta t/\tilde{\eta})$ and find that the unknown function $f(t)$ has to satisfy the equation:

$$\dot{f}(t) = \varepsilon_o\,\eta\exp\left(\frac{t}{\lambda}\right)\,\delta(t),\ \lambda = \frac{\tilde{\eta}}{\eta} \tag{9.2.7}$$

This equation is integrated, and with the initial condition $\sigma(0^-) = 0$ we obtain:

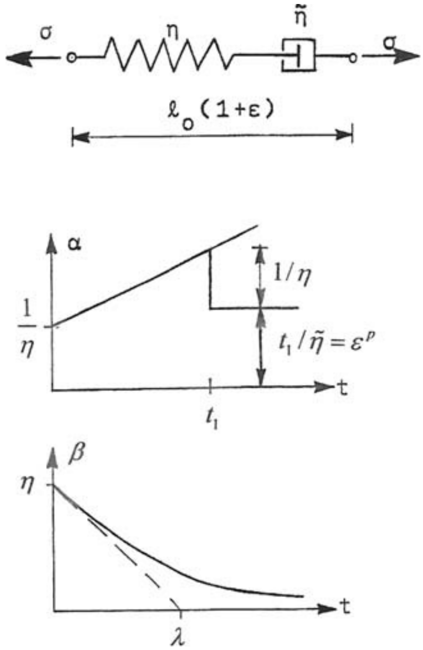
$$f(t) = \int_{0^-}^{t} \varepsilon_o\,\eta\exp\left(\frac{\bar{t}}{\lambda}\right)\,\delta(\bar{t})\,d\bar{t} = \varepsilon_o\,\eta\,H(t)$$

Hence:

$$\sigma(t) = \eta\exp\left(-\frac{t}{\lambda}\right)\,\varepsilon_o H(t) = \beta(t)\,\varepsilon_o H(t)$$

and we have found the relaxation function:

Fig. 9.2.4 Maxwell model



$$\beta(t) = \eta \exp\left(-\frac{t}{\lambda}\right), \; \lambda = \frac{\tilde{\eta}}{\eta} \text{ the relaxation time, } \beta_g = \eta, \; \beta_e = 0 \qquad (9.2.8)$$

The significance of the *relaxation time* is shown in the diagram for $\beta(t)$ in Fig. 9.2.4.

The creep function $\alpha(t)$ is found from the response equation (9.2.5) with $\sigma(t) = \sigma_o H(t)$. That is, we have to integrate the differential equation:

$$\dot{\varepsilon} = \frac{\sigma_o}{\eta}\left[\frac{1}{\lambda}H(t) + \delta(t)\right]$$

Since $\varepsilon(t)$ is equal to zero for $t < 0$, the integration yields:

$$\varepsilon = \frac{\sigma_o}{\eta}\int_{0^-}^{t}\left[\frac{1}{\lambda}H(\bar{t}) + \delta(\bar{t})\right]d\bar{t} = \frac{\sigma_o}{\eta}\left[\frac{t}{\lambda} + 1\right]H(t) = \alpha(t)\,\sigma_o H(t)$$

Hence the creep function is:

$$\alpha(t) = \frac{1}{\eta}\left[1 + \frac{t}{\lambda}\right], \; \alpha_g = \frac{1}{\eta}, \; \alpha_e = \infty \qquad (9.2.9)$$

When we compare the response curves in Fig. 9.2.4 with the general curves in Fig. 9.1.1 and Fig. 9.1.2, we see that the Maxwell model results in: elastic initial strain, no primary and tertiary creep, elastic restitution, no time dependent restitution, permanent strain upon unloading, and complete stress relaxation to $\sigma = 0$. The Maxwell model represents a viscoelastic fluid.

The *Kelvin model* in Fig. 9.2.5 consists of a linear spring and linear dashpot in parallel. Another name for this model is the Kelvin-Voigt-model. The stress σ in the model is equal to the sum of the stress $\eta\varepsilon$ in the spring and the stress $\tilde{\eta}\dot{\varepsilon}$ in the dashpot. The model therefore has the response equation:

$$\sigma = \eta\,\varepsilon + \tilde{\eta}\,\dot{\varepsilon} \tag{9.2.10}$$

The response of the model to a stress $\sigma(t) = \sigma_o H(t)$ is given by the solution to the differential equation:

$$\eta\,\varepsilon + \tilde{\eta}\,\dot{\varepsilon} = \sigma_o H(t)$$

with the initial condition $\varepsilon(0^-) = 0$. The solution is:

$$\varepsilon(t) = \frac{\sigma_o}{\eta}\left[1 - \exp\left(-\frac{t}{\lambda}\right)\right] H(t) = \alpha(t)\,\sigma_o H(t),\ \lambda = \frac{\tilde{\eta}}{\eta}$$

The creep function is therefore:

$$\alpha(t) = \frac{1}{\eta}\left[1 - \exp\left(-\frac{t}{\lambda}\right)\right] \quad , \quad \lambda = \frac{\tilde{\eta}}{\eta} \quad \text{retardation time} \quad , \quad \alpha_g = 0 \quad , \quad \alpha_e = \frac{1}{\eta} \tag{9.2.11}$$

The diagram of $\alpha(t)$ in Fig. 9.2.5 illustrates the meaning of the *retardation time* λ.

The response of the model to a constant strain $\varepsilon(t) = \varepsilon_o H(t)$ is according to (9.2.10):

$$\sigma(t) = \eta\,\varepsilon_o\,[H(t) + \lambda\,\delta(t)] = \beta(t)\,\varepsilon_o H(t)$$

Hence the relaxation function is:

$$\beta(t) = \eta\,[1 + \lambda\,\delta(t)],\ \beta_g = \infty,\ \beta_e = \eta \tag{9.2.12}$$

From Fig. 9.2.5 it is apparent that the Kelvin model results in: no initial strain, no secondary and tertiary creep, no elastic restitution, complete time dependent restitution after unloading, instantaneous relaxation, and no time dependent relaxation. The Kelvin model represents a viscoelastic solid.

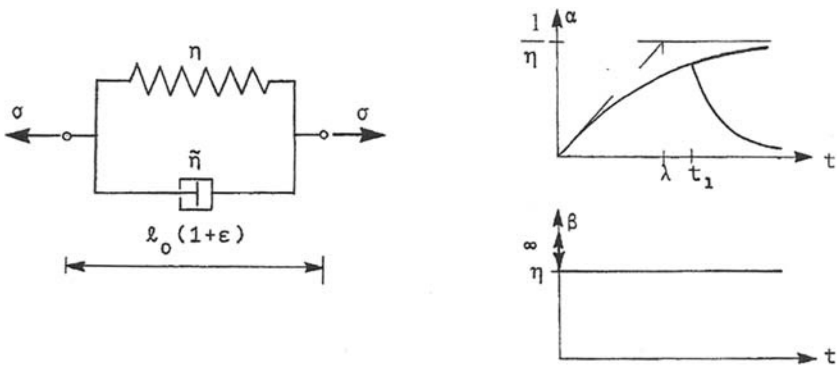


Fig. 9.2.5 The Kelvin model

The *Burgers model*, Fig. 9.2.6, consists of a series of a Maxwell element and a Kelvin element. The response equation is constructed from (9.2.5) and (9.2.10), and the conditions: a) The two elements have the same stress σ, and b) The strain of the elements are added to give ε. We then find:

$$\sigma + p_1\dot{\sigma} + p_2\ddot{\sigma} = q_1\dot{\varepsilon} + q_2\ddot{\varepsilon} \tag{9.2.13}$$

where:

$$p_1 = \lambda_1\left[1 + \frac{\eta_1}{\eta_2}\right] + \lambda_2\,, \quad p_2 = \lambda_1\lambda_2$$
$$q_1 = \lambda_1\eta_1\,, \quad q_2 = \lambda_2\tilde{\eta}_1\,, \quad \lambda_1 = \frac{\tilde{\eta}_1}{\eta_1}\,, \quad \lambda_2 = \frac{\tilde{\eta}_2}{\eta_2} \tag{9.2.14}$$

The creep function $\alpha(t)$ is directly equal to the sum of the creep function of the Maxwell element and the creep function of the Kelvin element:

$$\alpha(t) = \frac{1}{\eta_1}\left[1 + \frac{t}{\lambda_1}\right] + \frac{1}{\eta_2}\left[1 - \exp\left(-\frac{t}{\lambda_2}\right)\right],\ \alpha_g = \frac{1}{\eta_1},\ \alpha_e = \infty \tag{9.2.15}$$

The relaxation function $\beta(t)$ is determined by solving the differential equation (9.2.13) for the strain $\varepsilon(t) = \varepsilon_o H(t)$ and the initial condition $\sigma(0^-) = 0$. We then find:

$$\beta(t) = \frac{1}{\sqrt{p_1^2 - 4p_2}}\left[(q_1 - \rho_2 q_2)\exp(-\rho_2 t) - (q_1 - \rho_1 q_2)\exp(-\rho_1 t)\right] \tag{9.2.16}$$

$$\begin{matrix}\rho_1\\ \rho_2\end{matrix} = \frac{1}{2p_2}\left[p_1 \pm \sqrt{p_1^2 - 4p_2}\right]\,, \quad \beta_g = \eta_1\,, \quad \beta_e = 0 \tag{9.2.17}$$

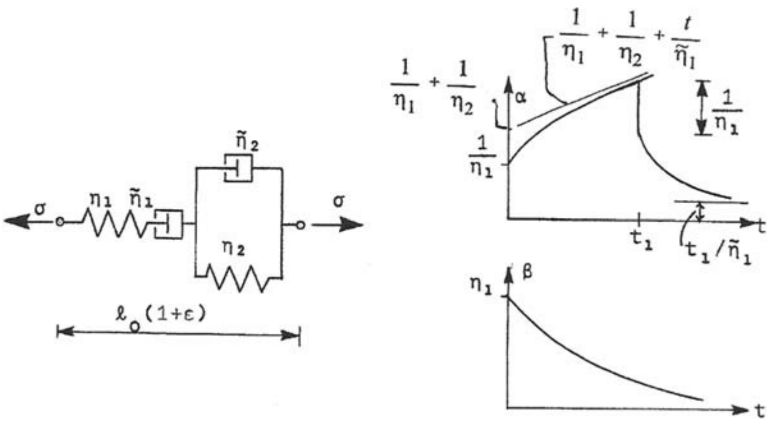


Fig. 9.2.6 Burgers model

From Fig. 9.2.6 we conclude that the Burgers model results in all the time dependent $\sigma\varepsilon$-characteristics, except tertiary creep, that is presented by the general curves in Fig. 9.1.1 and Fig. 9.1.2. The Burgers model represents a viscoelastic fluid.

The *Jeffreys model*, Fig. 9.2.7, is essentially a Burgers model having an infinitely stiff spring in the Maxwell element, and represents a viscoelastic fluid. The response equation, the creep function, and the relaxation function are respectively:

$$\sigma + \lambda_1 \dot{\sigma} = \tilde{\eta}_1 \dot{\varepsilon} + \tilde{\eta}_1 \lambda_2 \ddot{\varepsilon}, \; \lambda_1 = \frac{\tilde{\eta}_1 + \tilde{\eta}_2}{\eta}, \; \lambda_2 = \frac{\tilde{\eta}_2}{\eta} \tag{9.2.18}$$

$$\alpha(t) = \frac{1}{\eta}\left[1 + \frac{t}{\lambda_1 - \lambda_2} - \exp\left(-\frac{t}{\lambda_2}\right)\right], \; \alpha_g = 0, \; \alpha_e = \infty \tag{9.2.19}$$

$$\beta(t) = \eta\left[\lambda_2\left(1 - \frac{\lambda_2}{\lambda_1}\right)\delta(t) + \left(1 - \frac{\lambda_2}{\lambda_1}\right)^2 \exp\left(-\frac{t}{\lambda_1}\right)\right], \; \beta_g = \infty, \; \beta_e = 0 \tag{9.2.20}$$

The *standard linear model*, Fig. 9.2.8, is essentially a Burgers model having an infinite damping coefficient in the dashpot of the Maxwell element. An analogous model that has the same response consists of a Maxwell element and a Hookean element in parallel. These models represent viscoelastic solids. The response equation, creep function, and relaxation function for the standard linear model are respectively:

$$\sigma + \lambda_1 \dot{\sigma} = \frac{\eta_1 \lambda_1}{\lambda_2}\varepsilon + \eta_1 \lambda_1 \dot{\varepsilon}, \; \lambda_1 = \frac{\tilde{\eta}}{\eta_1 + \eta_2}, \; \lambda_2 = \frac{\tilde{\eta}}{\eta_2} \tag{9.2.21}$$

$$\alpha(t) = \frac{1}{\eta_1}\left\{1 + \frac{\eta_1}{\eta_2}\left[1 - \exp\left(-\frac{t}{\lambda_2}\right)\right]\right\}, \; \alpha_g = \frac{1}{\eta_1}, \; \alpha_e = \frac{\eta_1 + \eta_2}{\eta_1 \eta_2} \tag{9.2.22}$$

$$\beta(t) = \frac{\eta_1 \eta_2}{\eta_1 + \eta_2}\left[1 + \frac{\eta_1}{\eta_2}\exp\left(-\frac{t}{\lambda_1}\right)\right], \; \beta_g = \eta_1, \; \beta_e = \frac{\eta_1 \eta_2}{\eta_1 + \eta_2} \tag{9.2.23}$$

The *generalized Kelvin model* consists of a series of one Maxwell element and (m) Kelvin elements. The model is applied in describing results from creep tests.

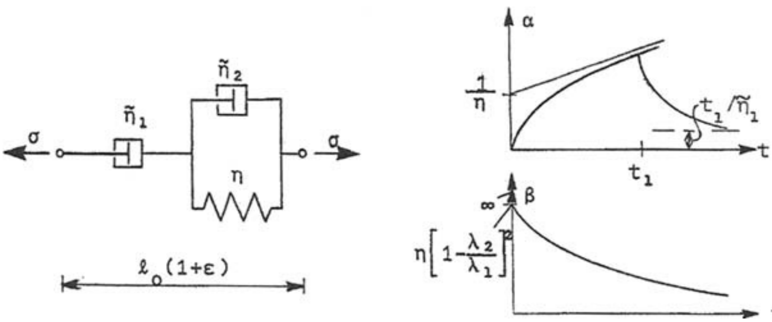


Fig. 9.2.7 Jeffreys model

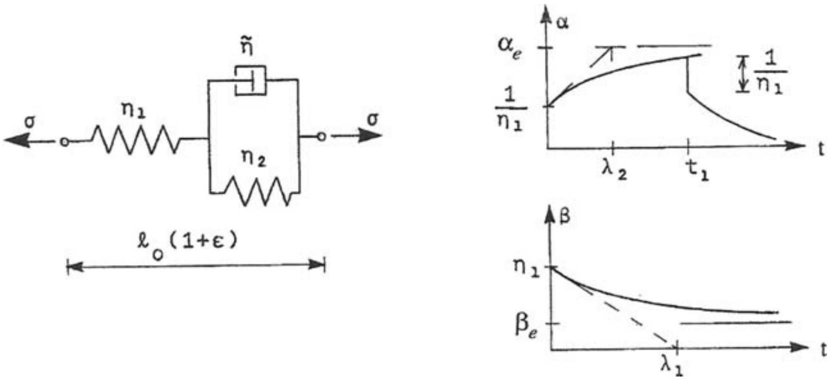


Fig. 9.2.8 Standard linear model

The creep function may be developed directly as a sum of functions like (9.2.9) and (9.2.11):

$$\alpha(t) = \frac{1}{\eta}\left[1+\frac{t}{\lambda_o}\right] + \sum_{n=1}^{m}\frac{1}{\eta_n}\left[1-\exp\left(-\frac{t}{\lambda_n}\right)\right],\ \alpha_g = \frac{1}{\eta},\ \alpha_e = \infty \tag{9.2.24}$$

In the case that $\lambda_o = \infty$, which means that the Maxwell element has degenerated to a Hookean element, we have:

$$\alpha_g = \frac{1}{\eta},\ \alpha_e = \frac{1}{\eta} + \sum_{n=1}^{m}\frac{1}{\eta_n} = \frac{1}{\eta} + \frac{1}{\eta^*},\ \frac{1}{\eta^*} \equiv \sum_{n=1}^{m}\frac{1}{\eta_n} \tag{9.2.25}$$

We may generalize the model further by replacing the sum in (9.2.24) by an integral. The creep function is now written as:

$$\alpha(t) = \frac{1}{\eta}\left[1+\frac{t}{\lambda_o}\right] + \frac{1}{\eta_1}\int_0^{\infty} f(\lambda)\left[1-\exp\left(-\frac{t}{\lambda}\right)\right]d\lambda\,, \quad \alpha_g = \frac{1}{\eta}\,, \quad \alpha_e = \infty \tag{9.2.26}$$

The term $f(\lambda)$ is a continuous function of the parameter λ and is called the *retardation spectrum*. The function is scaled such that:

$$\int_0^{\infty} f(\lambda)\,d\lambda = 1 \tag{9.2.27}$$

The *generalized Maxwell model* consists of a Hookean element and (m) Maxwell elements in parallel. The model is used to model results from relaxation tests. The relaxation function is found as the sum of functions of the type given by (9.2.2) and (9.2.8). The result is:

$$\beta(t) = \eta + \sum_{n=1}^{m} \eta_n \exp\left(-\frac{t}{\lambda_n}\right), \quad \beta_g = \eta + \sum_{n=1}^{m} \eta_n\,, \quad \beta_e = \eta \quad (9.2.28)$$

As for the generalized Kelvin model, we also here may generalize further by setting:

$$\beta(t) = \eta + \eta_1 \int_0^\infty f(\lambda) \exp\left(-\frac{t}{\lambda}\right) d\lambda\,, \quad \beta_g = \eta + \eta_1\,, \quad \beta_e = \eta \quad (9.2.29)$$

The function $f(\lambda)$, which is now called the *relaxation spectrum*, shall satisfy the condition (9.2.27).

9.2.2 General Response Equation

All the mechanical models we may construct by adding together linear springs and linear dashpots in series and in parallel, have *response equations* of the form:

$$\sum_{n=0}^{m} p_n \frac{d^n \sigma}{dt^n} = \sum_{n=0}^{m+1} q_n \frac{d^n \varepsilon}{dt^n} \quad (9.2.30)$$

p_n and q_n are model parameters. Because this equation is developed from the basic response equations: $\sigma = \eta\varepsilon$ for the Hookean element and $\sigma = \tilde{\eta}\dot{\varepsilon}$ for the Newtonian element, the number of terms on the right hand side of (9.2.30) may exceed the number of terms on the left hand side by at most one. All models will have $p_o > 0$.

If $q_o > 0$, (9.2.30) has a solution with a constant stress σ and a constant strain $\varepsilon = (p_o/q_o)\sigma$. This is the asymptotical solutions for a creep test and a relaxation test, which implies that:

$$\alpha_e = \frac{1}{\beta_e} = \frac{p_o}{q_o} \quad (9.2.31)$$

A mechanical model with $q_o > 0$ thus behaves as a viscoelastic solid. If $q_o = 0$, then $\beta_e = 0$ and $\alpha_e = \infty$. In that case the mechanical model represents a viscoelastic fluid.

If $q_{m+1} = 0$, the model has a glass compliance and a glass modulus:

$$\alpha_g = \frac{1}{\beta_g} = \frac{p_m}{q_m} \quad (9.2.32)$$

This result may be obtained as follows. First we integrate the response equation (9.2.30) (m) times from $t = 0^-$ to t. Then we set $t = 0^+$. The stress σ and strain ε may in the creep test or the relaxation test have a discontinuity of the type $f(t)H(t)$, where $f(t)$ is any continuous function of t. But the integral of σ and ε from 0^- to t will be continuous functions, which have the value 0 for $t = 0^+$. This means that the result of the (m) integrations of (9.2.30) from $t = 0^-$ to $t = 0^+$ is:

$$p_m \sigma(0^+) = q_m \varepsilon(0^+) \quad \Rightarrow \quad \frac{\varepsilon(0^+)}{\sigma(0^+)} = \frac{p_m}{q_m}$$

This result proves the formulas (9.2.32). For $q_{m+1} > 0$, we will find that $\alpha_g = 0$ and $\beta_g = \infty$.

9.2.3 The Boltzmann Superposition Principle

For a material in a state of uniaxial stress and a given stress or strain history we may compute the resulting strain or stress by solving the response equation of the type (9.2.30). We have seen above examples of solutions for the special stress history: $\sigma(t) = \sigma_o H(t)$, and the special strain history: $\varepsilon(t) = \varepsilon_o H(t)$. We shall now develop a method of solutions of the response equation (9.2.30) for a general strain history in the case of a problem under strain control, and for a general stress history in the case of a problem under stress control. In these solutions the relaxation function $\beta(t)$ or the creep function $\alpha(t)$ will represent the material response. The method is based on the principle of superposition and was introduced by Ludwig Boltzmann [1844–1906] in 1874.

Let t be the present time and $\bar{t}$ a "moving" time, such that: $\bar{t} \leq t$. A given strain history $\varepsilon(\bar{t})$, which satisfy the condition $\varepsilon(\bar{t}) = 0$ for $\bar{t} < t_o$, is replaced by the step function $\tilde{\varepsilon}(\bar{t})$ as shown in Fig. 9.2.9a. The function $\tilde{\varepsilon}(\bar{t})$ is constructed in the following way. The time interval $[t_o, t]$ is divided into (m) equal subintervals $[t_n, t_{n-1}]$ or increments: $\Delta t = t_n - t_{n-1} = (t - t_o)/m$. For the present time t we have $t = t_m$. Within each time subinterval $[t_n, t_{n-1}]$ we will find a $\bar{t}_n$ such that: $t_{n-1} \leq \bar{t}_n \leq t_n$ and:

$$\Delta\varepsilon_n \equiv \varepsilon(t_n) - \varepsilon(t_{n-1}) = \dot{\varepsilon}(\bar{t}_n)\,\Delta t$$

See Fig. 9.2.9b. The step function $\tilde{\varepsilon}(\bar{t})$ is now defined by:

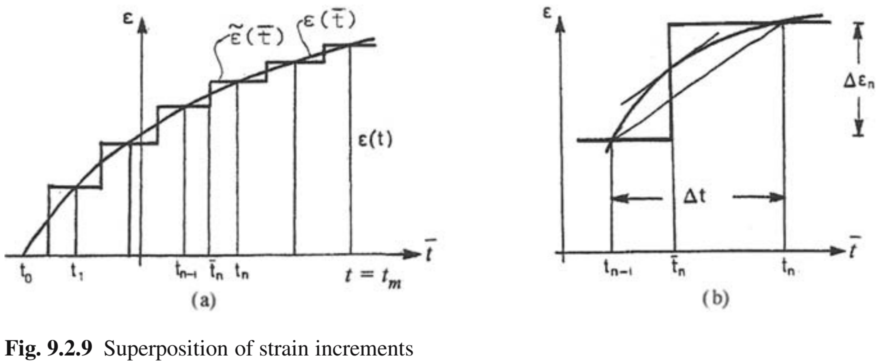


Fig. 9.2.9 Superposition of strain increments

$$\tilde{\varepsilon}(\bar{t}) = \sum_{n=1}^{m} \Delta\varepsilon_n H(\bar{t} - \bar{t}_n) \tag{9.2.33}$$

The stress at the "moving" time $\bar{t}$ due to the strain increment $\Delta\varepsilon_n$ is:

$$\Delta\sigma(\bar{t}) = \beta(\bar{t} - \bar{t}_n)\,\Delta\varepsilon_n H(\bar{t} - \bar{t}_n) = \beta(\bar{t} - \bar{t}_n)\,\dot{\varepsilon}(\bar{t}_n)\,\Delta t\, H(\bar{t} - \bar{t}_n)$$

Since the response equation (9.2.30) is linear we may superimpose the stress contributions $\Delta\sigma_n(\bar{t})$. Hence the strain history $\tilde{\varepsilon}(\bar{t})$ results in the following stress at the present time t:

$$\tilde{\sigma}(t) = \sum_{n=1}^{m} \Delta\sigma_n(t) = \sum_{n=1}^{m} \beta(t - \bar{t}_n)\,\dot{\varepsilon}(\bar{t}_n)\,\Delta t \tag{9.2.34}$$

If we now let $m \to \infty$, the approximate strain history $\tilde{\varepsilon}(\bar{t})$ approaches the actual strain history $\varepsilon(t)$, and the approximate stress history $\tilde{\sigma}(t)$ converges towards the actual stress history $\sigma(t)$. Thus we have obtained the result:

$$\sigma(t) = \int_{-\infty}^{t} \beta(t - \bar{t})\,\dot{\varepsilon}(\bar{t})\,d\bar{t} \tag{9.2.35}$$

The lower integration value is really t_o, but because $\varepsilon(t) = 0$ for $t < t_o$, the contribution to the integral is zero in the interval $-\infty < t < t_o$. The integral equation (9.2.35) is a completely general *response functional* for a linearly viscoelastic material under uniaxial stress. The relaxation function $\beta(t)$ must be determined by tests, either completely or by determining the relevant parameters in a suitable analogous mechanical model.

If the material has a glass modulus β_g, we may transform (9.2.35) by performing integration by parts. First we get:

$$\sigma(t) = [\beta(t - \bar{t})\,\varepsilon(\bar{t})]_{-\infty}^{t} - \int_{-\infty}^{t} \frac{d\beta(t - \bar{t})}{d\bar{t}}\,\varepsilon(\bar{t})\,d\bar{t} \tag{9.2.36}$$

Since $\varepsilon(-\infty) = 0$, the first term on the right hand side becomes $\beta(0)\varepsilon(t) = \beta_g\varepsilon(t)$. In the integral we change variable from $\bar{t}$ to the *past time* defined by:

$$s = t - \bar{t} \quad \Rightarrow \quad d\bar{t} = -ds \tag{9.2.37}$$

The upper integration limit $\bar{t} = t$ corresponds to $s = 0$, while the lower integration limit $\bar{t} = -\infty$ gives $s = \infty$. Furthermore:

$$\frac{d\beta(t - \bar{t})}{d\bar{t}} = -\frac{d\beta(s)}{ds}$$

Finally we reverse the direction of integration in (9.2.36), which is then transformed to the response functional:

$$\sigma(t) = \beta_g\, \varepsilon(t) + \int_0^{\infty} \frac{d\beta\,(s)}{ds}\, \varepsilon(t-s)\, ds \tag{9.2.38}$$

When a stress history $\sigma(t)$ is given, we may in a similar fashion develop response functionals for the strain history $\varepsilon(t)$:

$$\varepsilon(t) = \int_{-\infty}^{t} \alpha\,(t-\bar{t})\, \dot{\sigma}(\bar{t})\, d\bar{t},\ \varepsilon(t) = \alpha_g\, \sigma(t) + \int_0^{\infty} \frac{d\alpha\,(s)}{ds}\, \sigma(t-s)\, ds \tag{9.2.39}$$

The glass compliance α_g always exists, but may be zero.

For one and the same material a relation must exist between the creep function $\alpha(t)$ and the relaxation function $\beta(t)$. By (9.1.15) we have already seen that:

$$\alpha_g\, \beta_g = 1,\ \alpha_e\, \beta_e = 1 \tag{9.2.40}$$

Per definition the strain history $\alpha(t)\sigma_o H(t)$ corresponds to the stress history $\sigma_o H(t)$. From (9.2.35) we therefore find:

$$\sigma_o\, H(t) = \int_{-\infty}^{t} \beta\,(t-\bar{t})\, \left[\dot{\alpha}(\bar{t})\, H(\bar{t}) + \alpha(\bar{t})\, \dot{H}(\bar{t})\right] \sigma_o\, d\bar{t}$$

which because $\dot{H}(\bar{t}) = \delta(\bar{t})$ gives:

$$H(t) = \beta(t)\, \alpha_g + \int_0^{t} \beta\,(t-\bar{t})\, \dot{\alpha}(\bar{t})\, d\bar{t} \tag{9.2.41}$$

By similar arguments we find from the first of the response functionals (9.2.39), on the condition that β_g exists:

$$H(t) = \alpha(t)\, \beta_g + \int_0^{t} \alpha\,(t-\bar{t})\, \dot{\beta}(\bar{t})\, d\bar{t} \tag{9.2.42}$$

Example 9.1. Maxwell Bar in Tension

An axial bar is subjected to the stress history, see Fig. 9.2.10a:

$$\sigma(t) = \begin{cases} 0 & \text{for } t \leq 0 \text{ and } t > 2t_1 \\ \sigma_o t/t_1 & \text{for } 0 < t \leq t_1 \\ \sigma_o(2 - t/t_1) & \text{for } t_1 < t \leq 2t_1 \end{cases}$$

σ_o and t_1 are constants. We shall determine the axial strain $\varepsilon(t)$. We assume that the Maxwell model applies, with the creep function (9.2.9).

The given stress history $\sigma(t)$ may be expressed as:

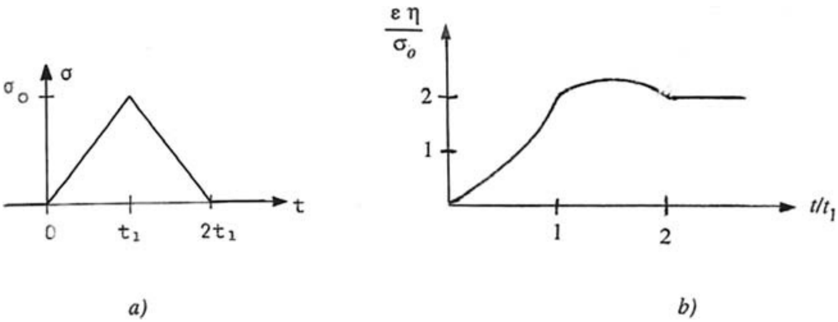


Fig. 9.2.10 **a**) Stress history and **b**) strain history

$$\sigma(t) = \frac{\sigma_o}{t_1}\left[tH(t) - 2(t-t_1)H(t-t_1) + (t-2t_1)H(t-2t_1)\right]$$

The first of the response functional (9.2.39) gives:

$$\varepsilon(t) = \int_{-\infty}^{t} \frac{1}{\eta}\left[1+\frac{t-\bar{t}}{\lambda}\right]\frac{\sigma_o}{t_1}\left[H(\bar{t}) + \bar{t}\,\delta(\bar{t}) - 2H(\bar{t}-t_1)\right.$$
$$\left. -2(\bar{t}-t_1)\,\delta(\bar{t}-t_1) + H(\bar{t}-2t_1) + (\bar{t}-2t_1)\,\delta(\bar{t}-2t_1)\right] d\bar{t} \quad \Rightarrow$$
$$\varepsilon(t) = \frac{\sigma_o}{\eta t_1}\left[t\left(1+\frac{t}{2\lambda}\right)H(t) - (t-t_1)\left(2+\frac{t-t_1}{\lambda}\right)\right.$$
$$\left.\times\, H(t-t_1) + (t-2t_1)\left(1+\frac{t-2t_1}{2\lambda}\right)H(t-2t_1)\right]$$

Figure 9.2.10b shows the graph of the $\varepsilon(t)$. For the graph t_1 is chosen to be equal to 2λ. The permanent plastic strain in the bar after unloading is found to be $2\sigma_o/\eta$. This result also follows if the stress had been constant and equal to the mean value $\sigma_o/2$ of the original stress during the time interval $[0, 2t_1]$. With reference to the creep curve in Fig. 9.2.4 for the Maxwell model the plastic strain in this case will be:

$$\frac{2t_1}{\tilde{\eta}}\frac{\sigma_o}{2} = \frac{2t_1}{\eta\lambda}\frac{\sigma_o}{2} = \frac{2\sigma_o}{\eta}$$

9.2.4 Linearly Viscoelastic Material Models

In a relaxation test with an isotropic, linearly viscoelastic material and an arbitrary constant state of strain $\mathbf{E}_o H(t)$ the state of stress $\mathbf{T}[\mathbf{E}_o, t]$ will be coaxial with the state of strain $\mathbf{E}_o$. The arguments for this are as for an isotropic elastic material, see Sect. 7.2. We may therefore, by referring to Hooke's law for an isotropic elastic material, i.e. $\mathbf{T} = 2\mu\mathbf{E} + (\kappa - 2\mu/3)(\mathrm{tr}\mathbf{E})\mathbf{1}$, presume that the stress in an isotropic viscoelastic material under a constant state of strain $\mathbf{E}_o H(t)$ will be given by:

$$\mathbf{T}(t) = 2\beta'(t)\,\mathbf{E}_o H(t) + \left[\beta^o(t) - \frac{2\beta'(t)}{3}\right](\mathrm{tr}\,\mathbf{E}_o H(t))\,\mathbf{1} \tag{9.2.43}$$

For a state of pure shear strain γ_o the response equation (9.2.43) reduces to an expression for the shear stress:

$$\tau(t) = \beta'(t)\,\gamma_o H(t) \tag{9.2.44}$$

The function $\beta'(t)$ is therefore the *relaxation function for shear strain*. For an isotropic state of strain: $\mathbf{E}_o = (\varepsilon_{vo}/3)\mathbf{1}H(t)$, where ε_{vo} is a constant volumetric strain, (9.2.43) gives an isotropic stress:

$$\mathbf{T}^o(t) = \sigma^o(t)\,\mathbf{1},\ \sigma^o(t) = \beta^o(t)\,\varepsilon_{vo} H(t) \tag{9.2.45}$$

The function $\beta^o(t)$ is therefore called the *relaxation function for isotropic strain*.

The response of an isotropic, linearly viscoelastic material to a constant state of stress $\mathbf{T}_o\,H(t)$ will be given by a formula analogous to (9.2.43):

$$\mathbf{E}(t) = \frac{1}{2}\alpha'(t)\,\mathbf{T}_o H(t) + \left[\frac{1}{9}\alpha^o(t) - \frac{1}{6}\alpha'(t)\right](\mathrm{tr}\,\mathbf{T}_o H(t))\,\mathbf{1} \tag{9.2.46}$$

For a pure shear stress τ_o formula (9.2.46) reduces to:

$$\gamma(t) = \alpha'(t)\,\tau_o H(t) \tag{9.2.47}$$

The function $\alpha'(t)$ is called the *creep function for shear stress*. For an isotropic state of stress: $\mathbf{T}_o = \sigma_o\mathbf{1}$, formula (9.2.46) results in an isotropic strain $\mathbf{E}^o = (1/3)\varepsilon_v\mathbf{1}$, where the volumetric strain is:

$$\varepsilon_v(t) = \alpha^o(t)\,\sigma_o H(t) \tag{9.2.48}$$

The function α^o is called the *creep function for isotropic stress*.

Shear tests to determine the creep function $\alpha'(t)$ and the relaxation function $\beta'(t)$ are relatively easy to perform. They may be executed as torsion tests with thin-walled tubes of the material or by subjecting rectangular panels of the material to shear forces. A test with isotropic states of stress to determine the functions $\alpha^o(t)$ and $\beta^o(t)$ is more difficult to accomplish. However the two functions may be found indirectly from a combination of results from shear tests and uniaxial stress tests. For a constant uniaxial stress $\sigma_o H(t)$ formula (9.2.46) gives the axial strain:

$$\varepsilon(t) = \left[\frac{1}{9}\alpha^o(t) + \frac{1}{3}\alpha'(t)\right]\sigma_o H(t) \tag{9.2.49}$$

The *creep function for uniaxial stress* is therefore:

$$\alpha(t) = \frac{1}{9}\alpha^o(t) + \frac{1}{3}\alpha'(t) \tag{9.2.50}$$

For uniaxial stress under constant axial strain $\varepsilon_o\, H(t)$ the formula (9.2.43) gives:

$$\sigma(t) = 2\beta'(t)\,\varepsilon_o H(t) + \left[\beta^o(t) - \frac{2\beta'(t)}{3}\right](\varepsilon_o + 2\varepsilon_t)\,H(t) \tag{9.2.51}$$

The quantity ε_t is the strain in the direction normal to the direction of the stress. The stresses on planes parallel to the axial direction are zero. Therefore we find from formula (9.2.43) that:

$$\mathrm{tr}\,\mathbf{T} = \sigma(t) = 3\beta^o(t)\,(\varepsilon_o + 2\varepsilon_c)\,H(t) \quad \Rightarrow \quad (\varepsilon_o + 2\varepsilon_c)\,H(t) = \frac{\sigma(t)}{3\beta^o(t)} \tag{9.2.52}$$

From (9.2.51) and (9.2.52) we obtain:

$$\sigma(t) = \frac{9\beta^o(t)\,\beta'(t)}{3\beta^o(t) + \beta'(t)}\,\varepsilon_o H(t) \tag{9.2.53}$$

The *relaxation function for uniaxial stress* is thus:

$$\beta(t) = \frac{9\beta^o(t)\,\beta'(t)}{3\beta^o(t) + \beta'(t)} \tag{9.2.54}$$

A general state of stress $\mathbf{T}$ may be decomposed into a *stress isotrop* and a *stress deviator*:

$$\mathbf{T}^o = \sigma^o\,\mathbf{1} = \left(\frac{1}{3}\mathrm{tr}\,\mathbf{T}\right)\mathbf{1}\,, \quad \mathbf{T}' = \mathbf{T} - \mathbf{T}^o \tag{9.2.55}$$

Similarly a general state of strain $\mathbf{E}$ may be decomposed into a *strain isotrop* and a *strain deviator*:

$$\mathbf{E}^o = \frac{1}{3}\varepsilon_v\,\mathbf{1} = \left(\frac{1}{3}\mathrm{tr}\,\mathbf{E}\right)\mathbf{1}\,, \quad \mathbf{E}' = \mathbf{E} - \mathbf{E}^o \tag{9.2.56}$$

It now follows from (9.2.43) and (9.2.46) that:

$$\mathbf{T}^o = 3\beta^o(t)\,\mathbf{E}^o_o H(t)\,, \quad \mathbf{T}' = 2\beta'(t)\,\mathbf{E}'_o H(t) \tag{9.2.57}$$

$$\mathbf{E}^o = \frac{1}{3}\alpha^o(t)\,\mathbf{T}^o_o H(t)\,, \quad \mathbf{E}' = \frac{1}{2}\alpha'(t)\,\mathbf{T}'_o H(t) \tag{9.2.58}$$

The principle of superposition, which gave the results (9.2.35) and (9.2.38) for uniaxial stress, may be applied to write the general constitutive equations for an isotropic, linearly viscoelastic material in the following alternative forms:

$$\mathbf{T}(t) = \int_{-\infty}^{t} \left\{2\beta'(t-\bar{t})\,\dot{\mathbf{E}}(\bar{t}) + \left[\beta^o(t-\bar{t}) - \frac{2\beta'(t-\bar{t})}{3}\right]\left(\mathrm{tr}\,\dot{\mathbf{E}}(\bar{t})\right)\mathbf{1}\right\} d\bar{t} \tag{9.2.59}$$

$$\mathbf{T}(t) = 2\beta_g' \mathbf{E}(t) + \left[\beta_g^o - \frac{2\beta_g'}{3}\right] (\mathrm{tr}\,\mathbf{E}(t))\,\mathbf{1}$$
$$+ \int_0^\infty \left\{ 2\frac{d\beta'(s)}{ds}\mathbf{E}(t-s) + \left[\frac{d\beta^o(s)}{ds} - \frac{2}{3}\frac{d\beta'(s)}{ds}\right][\mathrm{tr}\,\mathbf{E}(t-s)]\,\mathbf{1}\right\} ds \tag{9.2.60}$$

$$\mathbf{T}^o(t) = 3\int_{-\infty}^{t} \beta^o(t-\bar{t})\dot{\mathbf{E}}^o(\bar{t})\,d\bar{t}\,, \quad \mathbf{T}'(t) = 2\int_{-\infty}^{t} \beta'(t-\bar{t})\dot{\mathbf{E}}'(\bar{t})\,d\bar{t} \tag{9.2.61}$$

$$\sigma^o(t) = \int_{-\infty}^{t} \beta^o(t-\bar{t})\,\dot{\varepsilon}_v(\bar{t})\,d\bar{t}\,, \quad \tau(t) = \int_{-\infty}^{t} \beta'(t-\bar{t})\,\dot{\gamma}(\bar{t})\,d\bar{t} \tag{9.2.62}$$

Similarly we find:

$$\mathbf{E}(t) = \int_{-\infty}^{t} \left\{\frac{1}{2}\alpha'(t-\bar{t})\,\dot{\mathbf{T}}(\bar{t}) + \left[\frac{1}{9}\alpha^o(t-\bar{t}) - \frac{1}{6}\alpha'(t-\bar{t})\right]\left(\mathrm{tr}\,\dot{\mathbf{T}}(\bar{t})\right)\mathbf{1}\right\} d\bar{t} \tag{9.2.63}$$

$$\mathbf{E}(t) = \frac{1}{3}\alpha'_g\,\mathbf{T}(t) + \left[\frac{1}{9}\alpha_g^o - \frac{1}{6}\alpha'_g\right][\mathrm{tr}\,\mathbf{T}(t)]\,\mathbf{1}$$
$$+ \int_0^\infty \left\{\frac{1}{2}\frac{d\alpha'(s)}{ds}\mathbf{T}(t-s) + \left[\frac{1}{9}\frac{d\alpha^o(s)}{ds} - \frac{1}{6}\frac{d\alpha'(s)}{ds}\right][\mathrm{tr}\,\mathbf{T}(t-s)]\,\mathbf{1}\right\} ds \tag{9.2.64}$$

$$\mathbf{E}^o(t) = \frac{1}{3}\int_{-\infty}^{t} \alpha^o(t-\bar{t})\,\dot{\mathbf{T}}^o(\bar{t})\,d\bar{t}\,, \quad \mathbf{E}'(t) = \frac{1}{2}\int_{-\infty}^{t} \alpha'(t-\bar{t})\,\dot{\mathbf{T}}'(\bar{t})\,d\bar{t} \tag{9.2.65}$$

$$\varepsilon_v(t) = \int_{-\infty}^{t} \alpha^o(t-\bar{t})\,\dot{\sigma}^o(\bar{t})\,d\bar{t}\,, \quad \gamma(t) = \int_{-\infty}^{t} \alpha'(t-\bar{t})\,\dot{\tau}(\bar{t})\,d\bar{t} \tag{9.2.66}$$

Viscoelastic behavior is mainly a shear stress-strain effect, i.e. relationship between the deviators $\mathbf{T}'$ and $\mathbf{E}'$. Isotropic stress-strain response is often nearly elastic. It is therefore customary to assume that:

$$\beta^o(t) = \kappa = \text{constant} \quad , \quad \alpha^o(t) = \frac{1}{\kappa} \tag{9.2.67}$$

κ is the *bulk modulus of elasticity* of the material. The response functionals $(9.2.61)_1$ and $(9.2.65)_1$, are now reduced to the response equations:

$$\mathbf{T}^o(t) = 3\kappa\,\mathbf{E}^o(t)\,, \quad \mathbf{E}^o(t) = \frac{1}{3\kappa}\mathbf{T}^o(t) \tag{9.2.68}$$

The classical constitutive models for isotropic, linearly viscoelastic materials are based on the constitutive equations (9.2.68), $(9.2.61)_2$, and $(9.2.65)_2$. As an alternative to the response functionals $(9.2.61)_2$ and $(9.2.65)_2$, we may use the response equation:

$$\sum_{n=0}^{m} p'_n \frac{\partial^n \mathbf{T}'}{\partial t^n} = \sum_{n=0}^{m+1} q'_n \frac{\partial^n \mathbf{E}'}{\partial t^n} \tag{9.2.69}$$

p'_n and q'_n are material parameters. The creep function $\alpha(t)$ and the relaxation function $\beta(t)$ for uniaxial stress are according to (9.2.50), (9.2.54), and (9.2.67):

$$\alpha(t) = \frac{1}{9\kappa} + \frac{1}{3}\alpha'(t),\ \beta(t) = \frac{9\kappa\beta'(t)}{3\kappa + \beta'(t)} \tag{9.2.70}$$

The response functional for uniaxial stress due to a history $\varepsilon(t)$ of the axial strain $\varepsilon(t)$ is given by (9.2.35) or (9.2.38). The response functional for axial strain due to uniaxial stress history $\sigma(t)$ is given by the $(9.2.39)_1$ or by:

$$\varepsilon(t) = \frac{1}{9\kappa}\sigma(t) + \frac{1}{3}\int_{-\infty}^{t} \alpha'(t-\bar{t})\,\dot{\sigma}(\bar{t})\,d\bar{t} \tag{9.2.71}$$

This result is obtained by substitution of the expression for $\alpha(t)$ from $(9.2.70)_1$ into $(9.2.39)_1$.

We shall now present a set of constitutive models for isotrop, linearly viscoelastic materials. With reference to the corresponding mechanical models in Sect. 9.2.1 we define the models by their relaxation function and creep function for shear, and the response equation relating the stress deviator and strain deviator. For all the models the response for isotropic stress and strain is given by the (9.2.68).
Maxwell fluid. The material model has 2 elasticities: κ and μ, and 1 viscosity $\tilde{\mu}$.

$$\beta'(t) = \mu \exp\left(-\frac{t}{\lambda}\right),\ \beta'_g = \mu,\ \beta'_e = 0,\ \lambda = \frac{\tilde{\mu}}{\mu} \tag{9.2.72}$$

$$\alpha'(t) = \frac{1}{\mu}\left[1 + \frac{t}{\lambda}\right],\ \alpha'_g = \frac{1}{\mu},\ \alpha'_e = \infty \tag{9.2.73}$$

$$\mathbf{T}' + \lambda\,\dot{\mathbf{T}}' = 2\tilde{\mu}\,\dot{\mathbf{E}}' \tag{9.2.74}$$

Kelvin solid. The material model has 2 elasticities: κ and μ, and 1 viscosity $\tilde{\mu}$.

$$\beta'(t) = \mu + \tilde{\mu}\,\delta(t),\ \beta'_g = \infty,\ \beta'_e = \mu \tag{9.2.75}$$

$$\alpha'(t) = \frac{1}{\mu}\left[1 - \exp\left(-\frac{t}{\lambda}\right)\right],\ \alpha'_g = 0,\ \alpha'_e = \frac{1}{\mu},\ \lambda = \frac{\tilde{\mu}}{\mu} \tag{9.2.76}$$

$$\mathbf{T}' = 2\mu\,\mathbf{E}' + 2\tilde{\mu}\,\dot{\mathbf{E}}' \tag{9.2.77}$$

Jeffreys fluid. The material model has 2 elasticities: κ and μ, and 2 viscosities: $\tilde{\mu}_1$ and $\tilde{\mu}_2$.

$$\beta'(t) = \mu \left\{ \lambda_2 \left[1 - \frac{\lambda_2}{\lambda_1} \right] \delta(t) + \left[1 - \frac{\lambda_2}{\lambda_1} \right]^2 \exp\left(-\frac{t}{\lambda_1} \right) \right\}, \ \beta'_g = \infty, \ \beta'_e = 0 \tag{9.2.78}$$

Two time parameters are defined:

$$\lambda_1 = \frac{\tilde{\mu}_1}{\mu}, \ \lambda_2 = \frac{\tilde{\mu}_2}{\mu} \tag{9.2.79}$$

$$\alpha'(t) = \frac{1}{\mu} \left[1 + \frac{t}{\lambda_1 - \lambda_2} - \exp\left(-\frac{t}{\lambda_2} \right) \right], \ \alpha'_g = 0, \ \alpha'_e = \infty \tag{9.2.80}$$

$$\mathbf{T}' + \lambda_1 \dot{\mathbf{T}}' = 2\tilde{\mu}_1 \dot{\mathbf{E}}' + 2\tilde{\mu}_1 \lambda_2 \ddot{\mathbf{E}}' \tag{9.2.81}$$

Standard linear solid. The material model has 3 elasticities: κ, μ_1 and μ_2, and 1 viscosity: $\tilde{\mu}$.

$$\beta'(t) = \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} \left[1 + \frac{\mu_1}{\mu_2} \exp\left(-\frac{t}{\lambda_1} \right) \right], \ \beta'_g = \mu_1, \ \beta'_e = \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} \tag{9.2.82}$$

Two time parameters are defined:

$$\lambda_1 = \frac{\tilde{\mu}}{\mu_1 + \mu_2}, \ \lambda_2 = \frac{\tilde{\mu}}{\mu_2}, \ \frac{\lambda_1}{\lambda_2} = \frac{\mu_2}{\mu_1 + \mu_2} \tag{9.2.83}$$

$$\alpha'(t) = \frac{1}{\mu_1} \left\{ 1 + \frac{\mu_1}{\mu_2} \left[1 - \exp\left(-\frac{t}{\lambda_2} \right) \right] \right\}, \ \alpha'_g = \frac{1}{\mu_1}, \ \alpha'_e = \frac{\mu_1 + \mu_2}{\mu_1 \mu_2} \tag{9.2.84}$$

$$\mathbf{T}' + \lambda_1 \dot{\mathbf{T}}' = \frac{2\mu_1 \mu_2}{\mu_1 + \mu_2} \mathbf{E}' + 2\mu_1 \lambda_1 \dot{\mathbf{E}}' \tag{9.2.85}$$

Burgers fluid. This fluid model is based on the Burgers model, has 3 elasticities and 2 viscosities.

9.2.5 Beam Theory

Figure 9.2.11 shows a straight beam subjected to a distributed load $q(z,t)$ normal to the axis of the beam. The z-axis represents the undeformed axis of the beam. The beam axis intersects the cross-sections A through the centroids of the cross-sections. The xz-plane is a symmetry plane for the beam. The displacement $u(z,t)$ of the axis of the beam takes place in this symmetry plane.

The elementary beam theory for elastic beams is founded on four basic hypotheses:

1. *Bernoulli's deformation hypothesis*: Plane cross-sections remain plane cross-sections during bending.
2. *Stress hypothesis*: Normal stresses on plane parallel to the axis of the beam are in general small and may be neglected.

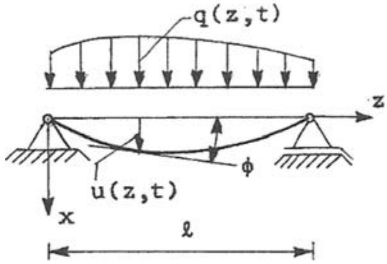


Fig. 9.2.11 Simply supported beam

3. *Material hypothesis*: The material is linearly elastic.
4. *Displacement hypothesis*: The displacement of the axis of the beam is small enough to let us assume that the slope of the deformed beam axis with respect to the undeformed beam axis is very small. This hypothesis implies that:

$$\phi \ll 1 \quad \Rightarrow \quad \phi \approx \tan\phi = \frac{\partial u}{\partial z}$$

The theory of viscoelastic beams is based on the hypotheses 1, 2, and 4. The material hypothesis is replaced by the response functional:

$$\varepsilon(x,z,t) = \int_{-\infty}^{t} \alpha(t-\bar{t})\,\dot{\sigma}(x,z,\bar{t})\,d\bar{t} \tag{9.2.86}$$

$\sigma(x,z,t)$ is the normal stress over the cross-section A of the beam, and ε is the corresponding strain. The deformation hypothesis and the displacement hypothesis lead to the same result as in the elementary beam theory:

$$\varepsilon(x,z,t) = -x\frac{\partial^2 u(z,t)}{\partial z^2} \tag{9.2.87}$$

From the (9.2.86) and (9.2.87) it follows that the stress is a linear function of x:

$$\sigma(x,z,t) = x\,f(z,t) \tag{9.2.88}$$

f(z,t) is a function to be determined by the bending moment $M(z,t)$. The relation between the stress (9.2.88) and the bending moment is:

$$M(z,t) = \int_A \sigma(x,z,t)\cdot x\,dA = f(z,t)\int_A x^2\,dA = f(z,t)\,I \tag{9.2.89}$$

I is the second moment of area of the beam cross-section with respect to the y-axis. Equations (9.2.88) and (9.2.89) show that the *bending stress formula* is identical to the one for an elastic beam:

$$\sigma(x,z,t) = \frac{M(z,t)}{I}x \tag{9.2.90}$$

From the (9.2.86), (9.2.87), and (9.2.90) we develop the following differential equation for the displacement function *u(z,t)*:

$$\frac{\partial^2 u(z,t)}{\partial z^2} = -\frac{1}{I}\int_{-\infty}^{t} \alpha(t-\bar{t})\,\dot{M}(z,\bar{t})\,d\bar{t} \tag{9.2.91}$$

For a quasi-static case, that is when inertia effects are neglected, the bending moment due to the loading on the beam, is found from an equilibrium equation. We limit the further discussion to statically determinate beams. The right-hand side of (9.2.91) is then independent of *u*(z,t), and the equation may be integrated. If the load on the beam is given on the form:

$$q(z,t) = \bar{q}(z)\,g(t) \tag{9.2.92}$$

where $g(t)$ is a dimensionless time function, (9.2.91) gives:

$$\frac{\partial^2 u(z,t)}{\partial z^2} = -\frac{\bar{M}(z)}{I}\int_{-\infty}^{t} \alpha(t-\bar{t})\,\dot{g}(\bar{t})\,d\bar{t} \tag{9.2.93}$$

$\bar{M}(z)$ is the bending moment computed from a load $\bar{q}(z)$. In particular we obtain for a load that is applied instantaneously at time $t = 0$, i.e. $q(z,t) = \bar{q}(z)\,H(t)$:

$$\frac{\partial^2 u(z,t)}{\partial z^2} = -\frac{\bar{M}(z)}{I}\,\alpha(t)\,H(t) \tag{9.2.94}$$

For an elastic beam with modulus of elasticity E the differential equation for the displacement function $u(z,t)$ is given by:

$$\frac{\partial^2 u(z,t)}{\partial z^2} = -\frac{M(z,t)}{EI} \tag{9.2.95}$$

Example 9.2. Uniform Load on a Simply Supported Beam

A simply supported beam is subjected to a uniform load $q(z,t) = q_o H(t)$. The length of the beam is l and the cross-section is uniform with second moment of area I. The displacement of an equivalent elastic simply supported beam is:

$$u^e(z,t) = \frac{l^4}{24EI}\left[\frac{z}{l} - 2\left(\frac{z}{l}\right)^3 + \left(\frac{z}{l}\right)^4\right] q_o H(t) \tag{9.2.96}$$

The displacement for a viscoelastic beam is then according to (9.2.94) and (9.2.95):

$$u(z,t) = u^e(z,t)\,E\,\alpha(t)\,H(t) \tag{9.2.97}$$

9.2.6 Torsion

The Saint-Venant theory of torsion of elastic cylindrical bars, see Sect. 7.4, is based on the two fundamental hypotheses:

1. *Deformation hypothesis*: All plane material cross-sections deform to the same curved shape and without strains in the cross-sections.
2. *Material hypothesis*: the material is a Hookean solid.

The torsion theory of linearly viscoelastic materials is based on the same deformation hypothesis, but the material hypothesis is replaced by the constitutive equation of one of the linearly viscoelastic models. Let J be the torsion constant, ψ the warping function, and x and y Cartesian coordinates in the cross-section of the bar, see Sect. 7.4. Following an analogous line of reasoning that we used for the beam theory in Sect. 9.2.5, we may now develop the following expressions for the torque $M(t)$, the shear stresses over the cross-section, and the angle of torsion $\phi(t)$ per unit length:

$$M(t) = J\int_{-\infty}^{t} \beta'(t-\bar{t})\,\dot{\phi}(\bar{t})\,d\bar{t}\,, \quad \phi(t) = \frac{1}{J}\int_{-\infty}^{t} \alpha'(t-\bar{t})\,\dot{M}(\bar{t})\,d\bar{t} \tag{9.2.98}$$

$$\tau_{xz} = \left[-y + \frac{\partial\psi}{\partial x}\right]\frac{M(t)}{J}\,, \quad \tau_{yz} = \left[x + \frac{\partial\psi}{\partial y}\right]\frac{M(t)}{J} \tag{9.2.99}$$

For the special case of a rod of circular cross-section: $\psi = 0$ and $J = I_p$, the *polar moment of area*, and the cross-sectional shear stress in the circumferential direction becomes: $\tau(R,t) = M(t)\cdot R/I_p$.

9.3 The Correspondence Principle

If the field equations in the theory of linear elasticity and the theory of linear viscoelasticity are Laplace transformed, we shall find a correspondence between related equations. This fact makes it possible to transform known solutions of problems from elasticity theory to solutions to the same problems but with viscoelastic materials. This method is called application of the *principle of correspondence*.

The *Laplace transform* of a function of time $f(t)$ is defined by:

$$\bar{f}(\zeta) \equiv L\{f(t)\} = \int_{0^-}^{\infty} f(t)\exp(-\zeta t)\,dt \tag{9.3.1}$$

Note that the Laplace transform $\bar{f}(\zeta)$ is not influenced by the function $f(t)$ for $t < 0$. The following properties of Laplace transforms are recorded for later usage:

$$L\left\{\frac{d^n f(t)}{dt^n}\right\} = \zeta^n \bar{f}(\zeta) - \sum_{r=1}^{n}\zeta^{n-r}\left[\frac{d^{r-1}f(t)}{dt^{r-1}}\right]_{t=0} \quad n = 1,2,.. \tag{9.3.2}$$

$$L\left\{\int_{0-}^{t} f(\bar{t})\,d\bar{t}\right\} = \frac{1}{\zeta}\bar{f}(\zeta)\,, \quad L\left\{\int_{0-}^{t} f(t-\bar{t})\,g(\bar{t})\,d\bar{t}\right\} = \bar{f}(\zeta)\,\bar{g}(\zeta) \tag{9.3.3}$$

In Laplace transforming equations derivatives and integrals with respect to time are replaced by algebraic operations with respect to ζ. The solutions of the transformed equations are thus functions of ζ, and must be transformed back to functions of time t. Inversion, i.e. the process to determine the function $f(t)$ from its known Laplace transform $\bar{f}(\zeta)$, may be a very complicated task, and we need not discuss that in this book. In the few examples we are going to present, it is sufficient to apply the results presented in Table 9.3.1 below.

The fundamental equations in linear elasticity are the *Cauchy equations of motion*:

$$\operatorname{div}\mathbf{T} + \rho\mathbf{b} = \rho\ddot{\mathbf{u}} \tag{9.3.4}$$

the *strain-displacement relations*:

$$\mathbf{E} = \frac{1}{2}\left(\operatorname{grad}\mathbf{u} + \operatorname{grad}\mathbf{u}^T\right) \tag{9.3.5}$$

and the *constitutive equations* (Hooke's law):

$$\mathbf{T}^o = 3\kappa\,\mathbf{E}^o \tag{9.3.6}$$

$$\mathbf{T}' = 2\mu\,\mathbf{E}' \tag{9.3.7}$$

These equations are supplemented with boundary conditions and initial conditions. We assume in the following that the material is undisturbed for times $t \le 0$, which implies that stresses, strains, and displacements are zero for $t \le 0$. The body under consideration has volume V and surface area A. The surface area is divided into an area A_σ for which the contact force is given as $\mathbf{t}^*(\mathbf{r},t)$, and an area A_u for which the displacement is specified as $\mathbf{u}^*(\mathbf{r},t)$. The subdivision of A into A_σ and A_u is assumed to be time independent. The *initial conditions* are given as:

$$\mathbf{u} = \mathbf{0} \text{ and } \dot{\mathbf{u}} = \mathbf{0} \text{ in } V \text{ for } t = 0 \tag{9.3.8}$$

and the *boundary conditions* are expressed by:

$$\mathbf{T}\cdot\mathbf{n} = \mathbf{t}^*(\mathbf{r},t) \text{ on } A_\sigma\,, \quad \mathbf{u} = \mathbf{u}^*(\mathbf{r},t) = \mathbf{0} \text{ on } A_u \tag{9.3.9}$$

Table 9.3.1 Laplace transforms

$f(t)$	$\bar{f}(\zeta)$	$f(t)$	$\bar{f}(\zeta)$
$H(t-a)$	$\frac{e^{-a\zeta}}{\zeta}$	t^n	$\frac{n!}{\zeta^{n+1}} n = 0,1,2,..$
$\delta(t-a)$	$e^{-a\zeta}$	$1-\exp(-at)$	$\frac{a}{\zeta(\zeta+a)}$
$\exp(-at)$	$\frac{1}{\zeta+a}$	$at-1+\exp(-at)$	$\frac{a^2}{\zeta^2(\zeta+a)}$

For an isotropic, linearly viscoelastic material all (9.3.4, 9.3.5, 9.3.6, 9.3.7, 9.3.8, 9.3.9), with the exception of the constitutive equations (9.3.7, 9.3.8), still apply. If we assume that the material behaves linearly elastic for isotropic stress, we keep (9.3.6) while (9.3.7) is replaced by either of the following equations, obtained from $(9.2.61)_2$, $(9.2.65)_2$, and (9.2.69):

$$\mathbf{T}'(t) = 2\int_0^t \beta'(t-\bar{t})\,\dot{\mathbf{E}}'(\bar{t})\,d\bar{t}\,, \quad \mathbf{E}'(t) = \frac{1}{2}\int_0^t \alpha'(t-\bar{t})\,\dot{\mathbf{T}}'(\bar{t})\,d\bar{t} \tag{9.3.10}$$

$$\sum_{n=0}^{m} p'_n \frac{\partial^n \mathbf{T}'}{\partial t^n} = \sum_{n=0}^{m+1} q'_n \frac{\partial^n \mathbf{E}'}{\partial t^n} \tag{9.3.11}$$

Because the material is assumed to be undisturbed for $t \le 0$, the lower integration limits in (9.3.10) have been changed from $-\infty$ to 0.

Equations (9.3.4, 9.3.5, 9.3.6, 9.3.7, 9.3.8, 9.3.9, 9.3.10, 9.3.11), with the exception of (9.3.8), are now Laplace transformed. For the transformation $L\{\dot{\mathbf{u}}\}$ we use formula (9.3.2) and the condition (9.3.8). For the transformation of (9.3.10) the formulas $(9.3.3)_2$ and (9.3.2) are applied, and (9.3.11) is transformed using formula (9.3.2) and the condition (9.3.8). The result of the transformations is:

$$\operatorname{div}\bar{\mathbf{T}} + \rho\,\bar{\mathbf{b}} = \rho\,\zeta^2\,\bar{\mathbf{u}} \tag{9.3.4)*}$$

$$\bar{\mathbf{E}} = \frac{1}{2}\left(\operatorname{grad}\bar{\mathbf{u}} + \operatorname{grad}\bar{\mathbf{u}}^T\right) \tag{9.3.5)*}$$

$$\bar{\mathbf{T}}^o = 3\kappa\,\bar{\mathbf{E}}^o \tag{9.3.6)*}$$

$$\bar{\mathbf{T}}' = 2\mu\,\bar{\mathbf{E}}' \tag{9.3.7)*}$$

$$\bar{\mathbf{T}}\cdot\mathbf{n} = \bar{\mathbf{t}}^*(\mathbf{r},\zeta) \text{ on } A_\sigma\,, \quad \bar{\mathbf{u}} = \bar{\mathbf{u}}^*(\mathbf{r},\zeta) = \mathbf{0} \text{ on } A_u \tag{9.3.9)*}$$

$$\bar{\mathbf{T}}'(\zeta) = 2\zeta\,\bar{\beta}'(\zeta)\,\bar{\mathbf{E}}'(\zeta)\,, \quad \bar{\mathbf{E}}'(\zeta) = \frac{1}{2}\zeta\,\alpha'(\zeta)\,\bar{\mathbf{T}}'(\zeta) \tag{9.3.10)*}$$

$$\bar{\mathbf{T}}'(\zeta) = \frac{\bar{Q}(\zeta)}{\bar{P}(\zeta)}\,\bar{\mathbf{E}}'(\zeta)\,, \quad \bar{Q}(\zeta) = \sum_{n=0}^{m+1} q'_n\,\zeta^n\,, \quad \bar{P}(\zeta) = \sum_{n=0}^{m} p'_n\,\zeta^n \tag{9.3.11)*}$$

A comparison of the relationships in $(9.3.10)^*$ shows that the Laplace transforms of the creep function and the relaxation function are related by:

$$\bar{\alpha}(\zeta)\,\bar{\beta}(\zeta) = \frac{1}{\zeta^2} \tag{9.3.12}$$

From $(9.3.10)^*$ and $(9.3.11)^*$ we obtain the relation:

$$\frac{\bar{Q}(\zeta)}{\bar{P}(\zeta)} = 2\zeta\,\bar{\beta}'(\zeta) \tag{9.3.13}$$

By comparing (9.3.4, 9.3.5, 9.3.6, 9.3.7, 9.3.8, 9.3.9, 9.3.10, 9.3.11) with (9.3.4, 9.3.5, 9.3.6, 9.3.7, 9.3.8, 9.3.9, 9.3.10)*, we can state the following theorem.

The correspondence theorem. The solution of the Laplace transformed basic equations for a boundary value problem involving an isotropic, linearly viscoelastic material is identical to the solution of the corresponding problem with an isotropic, linearly elastic material if the coefficient $2\zeta\bar{\beta}'(\zeta)$ is replaced by 2μ.

The correspondence theorem may be applied as follows. Stresses and/or strains in a linearly viscoelastic body are wanted. The solution to the corresponding problem when the material is assume to be linearly elastic is known and is Laplace transformed. By replacing the shear modulus μ by the coefficient $\zeta\bar{\beta}'(\zeta)$ we obtain the Laplace transforms of the solution to the problem when the material is viscoelastic. The inverse Laplace transformation provides the solution to the original problem. The method will be demonstrated in the next section.

Note that the division of the area A into A_σ and A_u is time independent. This implies that A must be time independent. This is for instance not the case when parts of the material is melting or burning. Furthermore, the areas A_σ and A_u must be the same material parts of the total area A at all times $t > 0$. If the contact force $\mathbf{t}^*$ is prescribed on a part of the surface area at certain time intervals and the displacement $\mathbf{u}^*$ is given on the same area at the rest of the time $t > 0$, then it is not possible to Laplace transform the boundary conditions.

9.3.1 Quasi-Static Problems

A problem is called *quasi-static* if the acceleration term in the equations of motion may be neglected, or is independent of time. The application of the correspondence principle becomes particularly simple for quasi-static cases.

In many of the standard solutions in the theory of elasticity the formulas are expressed through the modulus of elasticity η and the Poisson ratio ν. These two parameters will now be given in terms of μ and κ. From the relationships between the elastic parameters η, ν, μ, and κ established in Sect. 7.2 we may obtain:

$$\eta = \frac{18\kappa\mu}{6\kappa + 2\mu}\,, \quad \nu = \frac{3\kappa - 2\mu}{6\kappa + 2\mu} \tag{9.3.14}$$

The following three formulas become useful:

$$1 + \nu = \frac{9\kappa}{6\kappa + 2\mu}\,, \quad 1 - \nu = \frac{3\kappa + 4\mu}{6\kappa + 2\mu}\,, \quad 1 - 2\nu = \frac{6\mu}{6\kappa + 2\mu} \tag{9.3.15}$$

To get from a solution based on the theory of elasticity to the corresponding solution for a viscoelastic material, the shear modulus μ should be replaced by the coefficient $\zeta\bar{\beta}'(\zeta)$ in the Laplace transformed elastic solution. This process may be accomplished by retaining the compression modulus κ, while replacing the elasticities μ, η, and ν by $\bar{\mu}, \bar{\eta}$, and $\bar{\nu}$ defined by:

$$\bar{\mu}(\zeta) = \zeta\bar{\beta}'(\varsigma) \quad , \quad \bar{\eta}(\zeta) = \frac{18\kappa\bar{\mu}}{6\kappa + 2\bar{\mu}} = \frac{18\kappa\zeta\bar{\beta}'(\zeta)}{6\kappa + 2\zeta\bar{\beta}'(\zeta)}$$
$$\bar{\nu}(\zeta) = \frac{3\kappa - \bar{\mu}}{6\kappa + 2\bar{\mu}} = \frac{3\kappa - \zeta\bar{\beta}'(\zeta)}{6\kappa + 2\zeta\bar{\beta}'(\zeta)} \tag{9.3.16}$$

In the example below and in the Problems 9.6, 9.7, 9.8, 9.9, 9.10 and 9.12, which are related to the present section, the viscoelastic material is modelled either as a Maxwell fluid or as a Kelvin solid. Using Table 9.3.1 we shall find for the coefficient $\bar{\mu}$ for these two material models:

Maxwell fluid:

$$\beta'(t) = \mu\exp\left(-\frac{t}{\lambda}\right) \quad \Rightarrow \quad \bar{\beta}'(\zeta) = \frac{\mu\lambda}{\lambda\zeta + 1} \quad \Rightarrow \quad \bar{\mu}(\zeta) = \zeta\bar{\beta}'(\zeta) = \frac{\mu\lambda\zeta}{\lambda\zeta + 1} \tag{9.3.17}$$

Kelvin fluid:

$$\beta'(t) = \mu + \tilde{\mu}\,\delta(t) \quad \Rightarrow \quad \bar{\beta}'(\zeta) = \frac{\mu}{\zeta} + \tilde{\mu} \quad \Rightarrow \quad \bar{\mu}(\zeta) = \zeta\bar{\beta}'(\zeta) = \mu + \zeta\tilde{\mu} \tag{9.3.18}$$

Example 9.3. Thick-Walled Cylinder with Internal Pressure

A circular thick-walled cylinder with inner radius a and outer radius b is subjected to an internal pressure $p_o H(t)$. We want to determine the stresses and the radial displacement u. In this example we shall model the material as a Maxwell fluid. Furthermore, we assume plane stress and shall therefore also determine the axial strain ε_z. Other assumptions are presented as Problems 9.6, 9.7, 9.8, 9.9.

The solution to the corresponding problem with an isotropic, linearly elastic material is given in Example 7.1 and Example 7.3. Under the assumption of plane stress or plane displacements we have from the formulas (7.3.21, 7.3.22):

$$\sigma_R(R,t) = -\frac{1}{1-(a/b)^2}\left[\left(\frac{a}{R}\right)^2 - \left(\frac{a}{b}\right)^2\right] p_o H(t)$$
$$\sigma_\theta(R,t) = \frac{1}{1-(a/b)^2}\left[\left(\frac{a}{R}\right)^2 + \left(\frac{a}{b}\right)^2\right] p_o H(t) \tag{9.3.19}$$

For plane stress: $\sigma_z = 0$ the formulas (7.3.23, 7.3.24) give:

$$u(R,t) = \frac{1}{2\mu}\frac{a}{1-(a/b)^2}\left[\frac{a}{R} + \frac{3\kappa + 4\mu}{9\kappa}\left(\frac{a}{b}\right)^2\frac{R}{a}\right] p_o H(t) \tag{9.3.20}$$

$$\varepsilon_z(t) = -\frac{3\kappa - 2\mu}{9\mu\kappa}\frac{(a/b)^2}{1-(a/b)^2}\,p_o H(t)$$

As reference for Problems 9.6 and 9.7 the following results is presented for plane displacements: $\varepsilon_z = 0$, obtained from the formulas (7.3.37, 7.3.38):

$$u(R,t) = \frac{1}{2\mu}\,\frac{a}{1-(a/b)^2}\left[\frac{a}{R} + (1-2\nu)\left(\frac{a}{b}\right)^2\frac{R}{a}\right] p_o H(t)$$
$$\sigma_z(t) = \frac{2\nu\,(a/b)^2}{1-(a/b)^2}\,p_o H(t) \tag{9.3.21}$$

The development of the stress formulas is based on the assumption that the elastic material is isotropic and linear. But because the stresses σ_R and σ_θ are independent of the material parameters, the formulas for the stresses are the same for an isotropic and linearly viscoelastic material.

To obtain the solution for the radial displacement and axial strain for the cylinder modelled as a Maxwell fluid, the solution (9.3.20) for the elastic cylinder is first Laplace transformed. The result is:

$$\bar{u}(R,\zeta) = \frac{a}{1-(a/b)^2}\left[\frac{1}{2\mu}\bar{H}(\zeta)\frac{a}{R} + \frac{3\kappa+4\mu}{18\mu\kappa}\bar{H}(\zeta)\left(\frac{a}{b}\right)^2\frac{R}{a}\right] p_o$$
$$\bar{\varepsilon}_z(\zeta) = -\frac{3\kappa-2\mu}{9\mu\kappa}\bar{H}(\zeta)\frac{(a/b)^2}{1-(a/b)^2}\,p_o$$

For a viscoelastic cylinder the corresponding transformed solution is:

$$\bar{u}(R,\zeta) = \frac{a}{1-(a/b)^2}\left[\frac{1}{2\bar{\mu}(\zeta)}\bar{H}(\zeta)\frac{a}{R} + \frac{3\kappa+4\bar{\mu}(\zeta)}{18\bar{\mu}(\zeta)\kappa}\bar{H}(\zeta)\left(\frac{a}{b}\right)^2\frac{R}{a}\right] p_o$$
$$\bar{\varepsilon}_z(\zeta) = -\frac{3\kappa-2\bar{\mu}(\zeta)}{9\bar{\mu}(\zeta)\kappa}\bar{H}(\zeta)\frac{(a/b)^2}{1-(a/b)^2}\,p_o \tag{9.3.22}$$

Table 9.3.1 is used to express the Laplace transform $\bar{H}(\zeta)$ and formula (9.3.17) for the Maxwell fluid is used to express the Laplace transformed modulus $\bar{\mu}(\zeta)$. Then the Table 9.3.1 is again used to invert the three terms that contain Laplace transforms. The procedure is as follows:

$$\frac{1}{2\bar{\mu}(\zeta)}\bar{H}(\zeta) = \frac{\lambda\zeta+1}{2\mu\lambda\zeta}\frac{1}{\zeta} = \frac{1}{2\mu\zeta} + \frac{1}{2\mu\lambda\zeta^2} \quad\Rightarrow\quad \frac{1}{2\mu}\left[1+\frac{t}{\lambda}\right]H(t) \tag{9.3.23}$$

$$\frac{3\kappa+4\bar{\mu}(\zeta)}{18\bar{\mu}(\zeta)\kappa}\bar{H}(\zeta) = \frac{3}{18}\frac{1}{\bar{\mu}(\zeta)}\bar{H}(\zeta) + \frac{4}{18\kappa}\bar{H}(\zeta) = \frac{3}{18}\frac{\lambda\zeta+1}{\mu\lambda\zeta}\frac{1}{\zeta} + \frac{4}{18\kappa}\frac{1}{\zeta}$$
$$= \frac{3\kappa+4\mu}{18\mu\kappa}\frac{1}{\zeta} + \frac{3}{18}\frac{1}{\mu\lambda\zeta^2} \quad\Rightarrow$$

$$\frac{3\kappa+4\bar{\mu}(\zeta)}{18\bar{\mu}(\zeta)\kappa}\bar{H}(\zeta) \quad \Rightarrow \quad \frac{1}{2\mu}\left[\frac{3\kappa+4\mu}{9\kappa}+\frac{t}{3\lambda}\right]H(t) \tag{9.3.24}$$

$$\begin{aligned}\frac{3\kappa-2\bar{\mu}(\zeta)}{9\bar{\mu}(\zeta)\kappa}\bar{H}(\zeta) &= \frac{3}{9}\frac{1}{\bar{\mu}(\zeta)}\bar{H}(\zeta)-\frac{2}{9\kappa}\bar{H}(\zeta)\\ &= \frac{3}{9}\frac{\lambda\zeta+1}{\mu\lambda\zeta}\frac{1}{\zeta}-\frac{2}{9\kappa}\frac{1}{\zeta}=\left[\frac{3}{9}\frac{1}{\mu}-\frac{2}{9\kappa}\right]\frac{1}{\zeta}+\frac{3}{9}\frac{1}{\mu\lambda}\frac{1}{\zeta^2} \quad \Rightarrow\end{aligned}$$

$$\frac{3\kappa-2\bar{\mu}(\zeta)}{9\bar{\mu}(\zeta)\kappa}\bar{H}(\zeta) \quad \Rightarrow \quad \frac{1}{\mu}\left[\frac{3\kappa-2\mu}{9\kappa}+\frac{t}{3\lambda}\right]H(t) \tag{9.3.25}$$

The solution for a cylinder of a material modelled as a Maxwell fluid then follows by the inverse Laplace transform of the expressions (9.3.22), using the inverse transforms (9.3.23, 9.3.24, 9.3.25). We obtain:

$$\begin{aligned}u(R) &= \frac{1}{2\mu}\frac{a}{1-(a/b)^2}\left\{\left[1+\frac{t}{\lambda}\right]\frac{a}{R}+\left[\frac{3\kappa+4\mu}{9\kappa}+\frac{t}{3\lambda}\right]\left(\frac{a}{b}\right)^2\frac{R}{a}\right\}p_oH(t)\\ \varepsilon_z &= -\frac{(a/b)^2}{1-(a/b)^2}\frac{1}{\mu}\left[\frac{3\kappa-2\mu}{9\kappa}+\frac{t}{3\lambda}\right]p_oH(t)\end{aligned} \tag{9.3.26}$$

9.4 Dynamic Response

A state of strain in a linearly viscoelastic material that varies harmonically with time, produces a harmonically varying state of stress. There may be three reasons why we are interested in investigating harmonic response:

1) In many practical problems the loading is a harmonic time function and at the same time the motions developed by the loads are small enough for the accelerations to be neglected. The result is then harmonically varying stresses and strains.
2) Propagation of waves in viscoelastic media may be described by superposition of harmonic waves.
3) Tests with harmonically varying loads provide a means to find creep functions and relaxation functions for a particular material.

Figure 9.4.1 shows a bar of linearly viscoelastic material. The bar is fixed at one end, while the other end is given a harmonic displacement $u = u_o \sin \omega t$. The *period of oscillation* $2\pi/\omega$ is large compared to the time it takes for deformation waves to travel through the bar from one end to the other. We may then assume homogeneous strain and stress states in the bar. The strain in the axial direction is given by $\varepsilon = u/l_o$, or if we introduce the *amplitude* $\varepsilon_o = u_o/l_o$, by:

$$\varepsilon(t) = \varepsilon_o \sin \omega t \tag{9.4.1}$$

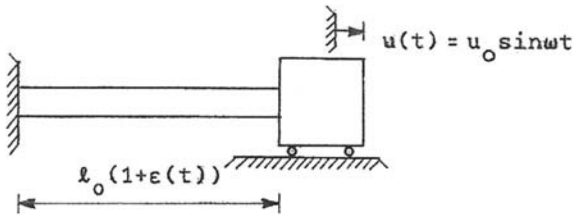


Fig. 9.4.1 Linearly viscoelastic bar subjected to a dynamic load

We shall assume uniaxial stress and use the general form of the response equation for linearly viscoelasticity:

$$\sum_{n=0}^{m} p_n \frac{d^n \sigma}{dt^n} = \sum_{n=0}^{m+1} q_n \frac{d^n \varepsilon}{dt^n} \tag{9.4.2}$$

Substitution of the strain (9.4.1) into the right-hand side of (9.4.2), gives:

$$\sum_{n=0}^{m} p_n \frac{d^n \sigma}{dt^n} = \varepsilon_o \left[f(q) \sin \omega t - g(q) \cos \omega t \right] \tag{9.4.3}$$

where:

$$f(q) = \sum_{n=0} q_{2n}\, \omega^{2n} \left(-1\right)^n,\ g(q) = \sum_{n=1} q_{2n-1}\, \omega^{2n-1} \left(-1\right)^n \tag{9.4.4}$$

The upper limit in the sums is not given because it depends on whether (m) in (9.4.3) is an even or odd number. The steady state solution of (9.4.3) is a phase shifted harmonic stress, as shown in Fig. 9.4.2:

$$\sigma(t) = \sigma_o \sin\left(\omega t + \delta\right) \tag{9.4.5}$$

The phase angle δ is, of reasons given below, called the *loss angle*. The expression (9.4.5) is rewritten to:

$$\sigma(t) = \left[\beta_1 \sin \omega t + \beta_2 \cos \omega t\right] \varepsilon_o \tag{9.4.6}$$

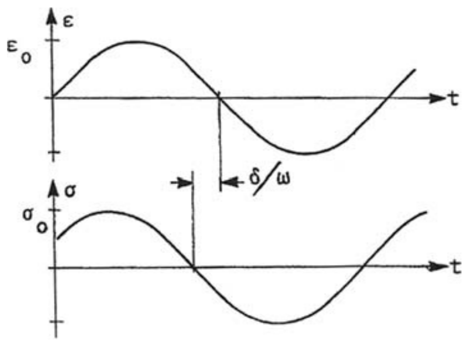


Fig. 9.4.2 Harmonic strain and stress

where:
$$\beta_1 = \frac{\sigma_o}{\varepsilon_o}\cos\delta,\ \beta_2 = \frac{\sigma_o}{\varepsilon_o}\sin\delta \tag{9.4.7}$$

The parameters β_1 and β_2 are called the *storage modulus* and the *loss modulus* respectively. These names will be explained below.

If we substitute the expression (9.4.6) into equation (9.4.3), we obtain:

$$\beta_1(\omega) = \frac{f(p)\,f(q)+g(p)\,g(q)}{f^2(p)+g^2(p)},\ \beta_2(\omega) = \frac{f(q)\,g(p)-f(p)\,g(q)}{f^2(p)+g^2(p)} \tag{9.4.8}$$

The functions $f(p)$ and $g(p)$ are given by the formulas (9.4.4) when the parameters q_n are replaced by p_n.

The relation between $\sigma(t)$ and $\varepsilon(t)$, based on formulas (9.4.6) and (9.4.1), is represented by a closed ellipse in a $\sigma\varepsilon$-diagram, see Fig. 9.4.3. The equation of this ellipse is found as follows. Formulas (9.4.1) and (9.4.6) are changed to:

$$\beta_2\,\varepsilon = \beta_2\,\varepsilon_o\,\sin\omega t\ ,\ \ (\sigma-\beta_1\,\varepsilon) = \beta_2\,\varepsilon_o\,\cos\omega t \tag{9.4.9}$$

The two equations are squared and summed. The result is the ellipse equation:

$$(\sigma-\beta_1\,\varepsilon)^2 + (\beta_2\,\varepsilon)^2 = (\beta_2\,\varepsilon_o)^2 \tag{9.4.10}$$

A stress-strain curve like the one in Fig. 9.4.3 is called a *hysteresis loop*. Hysteresis loops for real materials have similar appearances.

The area inside the hysteresis loop represents *dissipation*, or loss, of mechanical energy per unit volume and during one period $2\pi/\omega$. The dissipation is equal to the deformation work w per unit volume in one period. We find:

$$w = \int_0^{2\pi/\omega} \sigma\,\dot{\varepsilon}\,dt = \varepsilon_o^2\,\omega \int_0^{2\pi/\omega} \left[\beta_1 \sin\omega t\cos\omega t + \beta_2\cos^2\omega t\right]\,dt = \pi\,\varepsilon_o^2\,\beta_2 \tag{9.4.11}$$

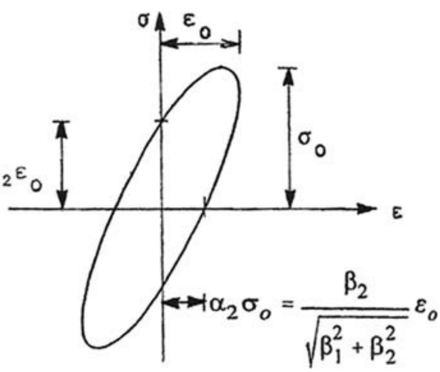


Fig. 9.4.3 Hysteresis loop

The loss of mechanical energy is thus proportional to the loss modulus. On the other hand, the part of the deformation work that is at any time stored as *strain energy*, is proportional to the storage modulus.

If we instead of introducing the harmonic strain (9.4.1) start with the harmonic stress:

$$\sigma(t) = \sigma_o \sin \omega t$$

and determine the axial strain $\varepsilon(t)$ from the response equation (9.4.2), we obtain:

$$\varepsilon(t) = \varepsilon_o \sin(\omega t - \delta) = [\alpha_1 \sin \omega t - \alpha_2 \cos \omega t]\, \sigma_o \tag{9.4.12}$$

where:

$$\begin{aligned} \alpha_1(\omega) &= \frac{\varepsilon_o}{\sigma_o} \cos \delta = \frac{f(p)\, f(q) + g(p)\, g(q)}{f^2(q) + g^2(q)} \\ \alpha_2(\omega) &= \frac{\varepsilon_o}{\sigma_o} \sin \delta = \frac{f(q)\, g(p) - f(p)\, g(q)}{f^2(q) + g^2(q)} \end{aligned} \tag{9.4.13}$$

The parameters α_1 and α_2 are called the *storage compliance* and the *loss compliance* respectively. We will find that the deformation work per unit volume in one period is: $w = \pi \sigma_o^2 \alpha_2$ and that the strain energy stored in the material is proportional to α_1.

It follows from (9.4.7) and (9.4.13) that:

$$\tan \delta = \frac{\beta_2}{\beta_1} = \frac{\alpha_2}{\alpha_1} = \frac{f(q)\, g(p) - f(p)\, g(q)}{f(p)\, f(q) + g(p)\, g(q)} \tag{9.4.14}$$

It is actually not the same whether the specimen initially is subjected to a harmonic strain or a harmonic stress. A linear material with fluid-like response will when subjected to a harmonic stress $\sigma_o \sin \omega t$, respond with a harmonic strain $\varepsilon_o \sin(\omega t - \delta)$ superimposed on a mean strain ε_m. The strain ε_m is a "transient" response that does not vanish with time, see Example 9.5 below.

For the mechanical models presented in Sect. 9.2.1 we find the following dynamic material parameters.

The Maxwell model:

$$\begin{aligned} &\beta_1 = \frac{(\lambda \omega)^2}{1 + (\lambda \omega)^2}\, \eta\ , \quad \beta_2 = \frac{\lambda \omega}{1 + (\lambda \omega)^2}\, \eta \\ &\alpha_1 = \frac{1}{\eta}, \ \alpha_2 = \frac{1}{\lambda\, \omega\, \eta}\ , \quad \tan \delta = \frac{1}{\lambda\, \omega} \end{aligned} \tag{9.4.15}$$

The Kelvin model:

$$\begin{aligned} &\beta_1 = \eta\ , \quad \beta_2 = \lambda\, \omega\, \eta\ , \quad \tan \delta = \lambda\, \omega \\ &\alpha_1 = \frac{1}{1 + (\lambda \omega)^2} \frac{1}{\eta}\ , \quad \alpha_2 = \frac{\lambda \omega}{1 + (\lambda \omega)^2} \frac{1}{\eta} \end{aligned} \tag{9.4.16}$$

The Jeffreys model:

$$\beta_1 = \frac{(\lambda_1 - \lambda_2)^2 \omega^2}{1 + (\lambda\omega)^2}\eta\ ,\ \ \beta_2 = \frac{\left(1 + \lambda_1 \lambda_2 \omega^2\right)(\lambda_1 - \lambda_2)\,\omega}{1 + (\lambda\omega)^2}\eta$$

$$\alpha_1 = \frac{1}{1 + (\lambda\omega)^2}\frac{1}{\eta}\ ,\ \ \alpha_2 = \frac{1 + \lambda_1 \lambda_2 \omega^2}{\left(1 + \lambda_2^2 \omega^2\right)(\lambda_1 - \lambda_2)\ \omega}\frac{1}{\eta}\ ,\ \ \tan\delta = \frac{1 + \lambda_1 \lambda_2 \omega^2}{(\lambda_1 - \lambda_2)\,\omega} \tag{9.4.17}$$

The Standard linear model:

$$\beta_1 = \frac{\lambda_1/\lambda_2 + (\lambda_1\omega)^2}{1 + (\lambda_1\omega)^2}\eta_1\ ,\ \ \beta_2 = \frac{\lambda_1\omega\,(1 - \lambda_1/\lambda_2)}{1 + (\lambda_1\omega)^2}\eta_1$$

$$\alpha_1 = \frac{\lambda_2/\lambda_1 + (\lambda_2\omega)^2}{1 + (\lambda_2\omega)^2}\frac{1}{\eta_1}\ ,\ \ \alpha_2 = \frac{(\lambda_2/\lambda_1 - 1)\,\lambda_2\,\omega}{1 + (\lambda_2\omega)^2}\frac{1}{\eta_1}$$

$$\tan\delta = \frac{(\lambda_2/\lambda_1 - 1)\,\lambda_2\,\omega}{\lambda_2/\lambda_1 + (\lambda_2\omega)^2} \tag{9.4.18}$$

The development above is for uniaxial stress. The results may easily be transformed to a general stress-strain state by replacing the uniaxial creep and relaxation functions by the creep and relaxation functions for shear stress, and then applying the formulas in relations between the deviatoric stress and strain tensors. The stress and strain isotrops are related through the linear equations (9.2.68).

9.4.1 Complex Notation

In applications of harmonic analysis it is often convenient to use complex notation. Harmonic varying stress and strain are then denoted by:

$$\sigma(t) = \sigma_o \exp(i\omega t)\ ,\ \ \varepsilon(t) = \varepsilon_o \exp(i\omega t) \tag{9.4.19}$$

σ_o and ε_o are now complex amplitudes. When the functions (9.4.19) are substituted into the response equation (9.4.2), we get:

$$\sigma(t) = \beta^*(\omega)\,\varepsilon(t),\ \varepsilon(t) = \alpha^*(\omega)\,\sigma(t) \tag{9.4.20}$$

where:

$$\beta^*(\omega) = \frac{1}{\alpha^*(\omega)} = \frac{\sum\limits_{n=0}^{m+1} q_n\,(i\omega)^n}{\sum\limits_{n=0}^{m} p_n\,(i\omega)^n} \tag{9.4.21}$$

$\beta^*(\omega)$ is called the *complex modulus* and $\alpha^*(\omega)$ is called the *complex compliance*. They may be expressed by their real and imaginary parts as follows:

$$\beta^*(\omega) = \beta_1(\omega) + i\beta_2(\omega)\,,\quad \alpha^*(\omega) = \alpha_1(\omega) - i\alpha_2(\omega) \tag{9.4.22}$$

When the response equation (9.4.2) and the response functional (9.2.35):

$$\sigma(t) = \int_{-\infty}^{t} \beta\,(t-\bar{t})\,\dot{\varepsilon}\,(\bar{t})\,d\bar{t} \tag{9.4.23}$$

are Laplace transformed, we shall find, using the formulas (9.3.2) and $(9.3.3)_2$:

$$\bar{\sigma}\,(\zeta) = \frac{\bar{Q}\,(\zeta)}{\bar{P}\,(\zeta)}\,\bar{\varepsilon}\,(\zeta) = \zeta\,\bar{\beta}\,(\zeta)\,\bar{\varepsilon}\,(\zeta) \tag{9.4.24}$$

where $\bar{\beta}\,(\zeta)$ is the Laplace transform of the relaxation function $\beta(t)$, and:

$$\bar{Q}\,(\zeta) = \sum_{n=0}^{m+1} q_n\,\zeta^n,\ \bar{P}\,(\zeta) = \sum_{n=0}^{m} p_n\,\zeta^n \tag{9.4.25}$$

The expressions (9.4.24) and (9.4.25) are analogous to the expressions $(9.3.10)^*$ and $(9.3.11)^*$. It follows from (9.4.21), (9.4.24), and (9.4.25) that:

$$\beta^*(i\omega) = \frac{1}{\alpha^*(i\omega)} = \frac{\bar{Q}\,(i\,\omega)}{\bar{P}\,(i\,\omega)} = i\,\omega\,\bar{\beta}\,(i\,\omega) \tag{9.4.26}$$

If the material behaves as a fluid, which means that the equilibrium modulus $\beta_e = 0$, we may establish from the last of the results (9.4.26) a relationship between the complex modulus $\beta^*(\omega)$ and the relaxation function $\beta(t)$. We have to assume the existence of the integral:

$$\int_0^{\infty} \beta(t)\,dt$$

Then we may write:

$$\bar{\beta}\,(i\,\omega) = \int_0^{\infty} \beta(t)\exp(-i\,\omega\,t)\,dt = \int_0^{\infty} \beta(t)\cos\omega t\,dt - i\int_0^{\infty} \beta(t)\sin\omega t\,dt \tag{9.4.27}$$

which by (9.4.26) and (9.4.22) gives:

$$\beta_1(\omega) = \omega\int_0^{\infty} \beta(t)\sin\omega t\,dt\,,\quad \beta_2(\omega) = \omega\int_0^{\infty} \beta(t)\cos\omega t\,dt \tag{9.4.28}$$

From the theory of Fourier transforms it follows that $\beta_1(\omega)$ and $\beta_2(\omega)$ are respectively the sine transform and the cosine transform of the relaxation function $\beta(t)$.

The inversion theorem for Fourier transforms gives:

$$\beta(t) = \frac{2}{\pi}\int_0^\infty \frac{\beta_1(\omega)}{\omega}\sin\omega t\, d\omega = \frac{2}{\pi}\int_0^\infty \frac{\beta_2(\omega)}{\omega}\cos\omega t\, d\omega \tag{9.4.29}$$

The results (9.4.28) may also be developed directly from the response functional (9.4.23). First we introduce the variable $s = t - \bar{t}$, which we have called the *past time*. Then:

$$\sigma(t) = \int_{-\infty}^{\infty} \beta(s)\, \frac{d\varepsilon(t-s)}{d(t-s)}\, ds \tag{9.4.30}$$

which we transform into:

$$\sigma(t) = \beta(t)\,\varepsilon(t) + \int_{-\infty}^{\infty} [\beta(s) - \beta(t)]\, \frac{d\varepsilon(t-s)}{d(t-s)}\, ds$$

The strain history $\varepsilon(t) = \varepsilon_o \exp(i\omega t)$ results in the uniaxial stress:

$$\sigma(t) = \left\{ \beta(t) + i\omega \int_0^t [\beta(s) - \beta(t)] \exp(-i\omega s)\, ds \right\} \varepsilon(t)\,, \quad \varepsilon(t) = \varepsilon_o \exp(i\omega t)$$

We assume now that the relaxation function $\beta(t)$ approaches the equilibrium modulus β_e fast enough for the integral:

$$\int_0^\infty [\beta(s) - \beta_e]\, ds$$

to exist. Then, for sufficiently long times, $t >> 0$, the stress becomes:

$$\sigma(t) = \beta^*(\omega)\,\varepsilon(t) = \beta^*(\omega)\,\varepsilon_o \exp(i\,\omega\, t)$$

where:

$$\beta^*(\omega) = \beta_e + i\omega \int_0^\infty [\beta(s) - \beta_e] \exp(-i\omega s)\, ds \tag{9.4.31}$$

from which we obtain:

$$\beta_1(\omega) = \beta_e + \omega \int_0^\infty [\beta(s) - \beta_e] \sin\omega s\, ds\,, \quad \beta_2(\omega) = \omega \int_0^\infty [\beta(s) - \beta_e] \cos\omega s\, ds \tag{9.4.32}$$

For a viscoelastic material behaving fluid-like, $\beta_e = 0$, the formulas (9.4.28, 9.4.29) are again obtained.

For viscoelastic models with an existing glass modulus β_g, (9.4.32) may be transformed by partial integration to:

$$\beta_1(\omega) = \beta_g + \int_0^\infty \frac{d\beta(s)}{ds} \cos\omega s\, ds, \ \beta_2(\omega) = -\int_0^\infty \frac{d\beta(s)}{ds} \sin\omega s\, ds$$

$$\beta^*(i\omega) = \beta_1(i\omega) + i\beta_2(i\omega) = \beta_g + \int_0^\infty \frac{d\beta(s)}{ds} \exp(-i\omega s)\, ds \tag{9.4.33}$$

The result may also be derived directly from the response functional on the form (9.2.38).

From the theory of Fourier transforms it follows that the integrals in (9.4.32) are respectively the sine transform and the cosine transform of the function $[\beta(t) - \beta_e]$. The inversion theorem of Fourier transformation gives:

$$\beta(t) = \beta_e + \frac{2}{\pi}\int_0^\infty \frac{\beta_1(\omega) - \beta_e}{\omega} \sin\omega t\, d\omega = \frac{2}{\pi}\int_0^\infty \frac{\beta_1(\omega)}{\omega} \sin\omega t\, d\omega \tag{9.4.34}$$

or, alternatively:

$$\beta(t) = \frac{2}{\pi}\int_0^\infty \frac{\beta_2(\omega)}{\omega} \cos\omega t\, d\omega \tag{9.4.35}$$

As we shall see in Example 9.4 that these two inverse formulas do not always give the same result, and in such a case do not represent a proper solution. To obtain the latter form in formula (9.4.34) a well-known integral formula has been applied.

For viscoelastic solids: $\alpha_e > 0$, we may, following an analogous development used above from (9.4.30) to (9.4.33), derive the formulas:

$$\alpha_1(\omega) = \alpha_e + \omega\int_0^\infty [\alpha(s) - \alpha_e] \sin\omega s\, ds\,, \ \ \alpha_2(\omega) = \omega\int_0^\infty [\alpha(s) - \alpha_e] \cos\omega s\, ds \tag{9.4.36}$$

$$\alpha_1(\omega) = \alpha_g + \int_0^\infty \frac{d\alpha(s)}{ds} \cos\omega s\, ds\,, \ \ \alpha_2(\omega) = -\int_0^\infty \frac{d\alpha(s)}{ds} \sin\omega s\, ds$$

$$\alpha^*(i\omega) = \alpha_1(i\omega) - i\alpha_2(i\omega) = \alpha_g + \int_0^\infty \frac{d\alpha(s)}{ds} \exp(-i\omega s)\, ds \tag{9.4.37}$$

Example 9.4. In this example we shall check out some of the formulas given above for a Maxwell model and for a Kelvin model.

The Maxwell Model

The relaxation function is: $\beta(t) = \eta \exp(-t/\lambda)$, and the equilibrium modulus is $\beta_e = 0$. With reference to the formulas (9.3.17) we write:

$$\bar{\beta}(\zeta) = \frac{\eta\lambda}{1+\zeta\lambda} \tag{9.4.38}$$

From (9.4.26) we obtain:

$$\beta^*(i\omega) = i\omega\frac{\eta\lambda}{1+i\lambda\omega} = \frac{(\lambda\omega)^2\eta}{1+(\lambda\omega)^2} + \frac{i\lambda\omega\eta}{1+(\lambda\omega)^2} \quad \Rightarrow$$

$$\beta_1(\omega) = \frac{(\lambda\omega)^2\eta}{1+(\lambda\omega)^2},\ \beta_2(\omega) = \frac{\lambda\omega\eta}{1+(\lambda\omega)^2} \tag{9.4.39}$$

which is in agreement with the formulas in (9.4.15). Using standard integral tables we obtain from (9.4.28, 9.4.29):

$$\begin{aligned}
\beta_1(\omega) &= \omega\int_0^\infty \eta\exp(-t/\lambda)\sin\omega t\,dt \\
&= \omega\left[\eta\exp(-t/\lambda)\,\frac{(-1/\lambda)\sin\omega t - \omega\cos\omega t}{(-1/\lambda)^2+\omega^2}\right]_0^\infty = \frac{(\lambda\omega)^2\eta}{1+(\lambda\omega)^2} \\
\beta(t) &= \frac{2}{\pi}\int_0^\infty \frac{(\lambda\omega)^2\eta}{1+(\lambda\omega)^2}\sin\omega t\,d\omega = \frac{2}{\pi}\frac{\pi}{2}\eta\exp(-t/\lambda) = \eta\exp(-t/\lambda)
\end{aligned}$$

The Kelvin Model

The relaxation function is $\beta(t) = \eta + \lambda\eta\delta(t)$ and the equilibrium modulus is $\beta_e = \eta$. With reference to formula (9.3.18) we write:

$$\bar{\beta}(\zeta) = \frac{\eta}{\zeta} + \tilde{\eta} \equiv \frac{\eta}{\zeta} + \lambda\eta \tag{9.4.40}$$

From $(9.4.26)_3$ we obtain:

$$\beta^*(i\omega) = i\omega\left[\frac{\eta}{i\omega} + \lambda\eta\right] = \eta + i\omega\lambda\eta \quad \Rightarrow \quad \beta_1 = \eta\ ,\ \ \beta_2(\omega) = \omega\lambda\eta \tag{9.4.41}$$

which is in agreement with the formulas in (9.4.16). From (9.4.32) we obtain:

$$\beta_1(\omega) = \eta + \omega\int_0^\infty \lambda\eta\delta(s)\sin\omega s\,ds = \eta\ ,\ \ \beta_2(\omega) = \omega\int_0^\infty \lambda\eta\delta(s)\cos\omega s\,ds = \omega\lambda\eta$$

In this case the inversion formulas (9.4.34, 9.4.35) do not work. The first formula gives $\beta(t) = \eta$, while the second formula does not give a result at all. We must conclude that the inversion formulas do not work for all cases.

Example 9.5. It was mentioned above that a linearly viscoelastic model with fluid-like behavior subjected to a harmonic uniaxial stress $\sigma_o \sin \omega t$ responds with an axial strain oscillating around a mean value ε_m:

$$\varepsilon(t) = \varepsilon_m + \varepsilon_o \sin(\omega t - \delta)$$

This result will now be demonstrated for the Maxwell model.

The stress is applied at time $t = 0$, such that:

$$\sigma(t) = [\sigma_o \sin \omega t]\, H(t)$$

The creep function and the glass compliance are:

$$\alpha(t) = \frac{1}{\eta}\left[1 + \frac{t}{\lambda}\right], \; \alpha_g = \frac{1}{\eta}$$

Using the response functional $(9.2.39)_2$:

$$\varepsilon(t) = \alpha_g\, \sigma(t) + \int_0^\infty \frac{d\alpha(s)}{ds}\, \sigma(t-s)\, ds$$

we get:

$$\begin{aligned}
\varepsilon(t) &= \left\{ \frac{1}{\eta} \sigma_o \sin \omega t + \int_0^t \frac{1}{\eta \lambda}\, \sigma_o \sin[\omega(t-s)]\, ds \right\} H(t) \\
&= \left\{ \frac{1}{\eta} \sin \omega t + \frac{1}{\eta\, \lambda\, \omega} (1 - \cos \omega t) \right\} \sigma_o H(t)
\end{aligned}$$

Using the formulas (9.4.15), we may rewrite this result and obtain:

$$\varepsilon(t) = [\alpha_1 \sin \omega t - \alpha_2 \cos \omega t]\, \sigma_o H(t) + \alpha_2\, \sigma_o H(t)$$

When this result is compared with (9.4.12), we see that because it is the stress that is initiated at the time $t = 0$ and not the strain, the material gets a mean strain:

$$\varepsilon_m = \alpha_2\, \sigma_o$$

The mean strain represents a "transient" response, which for the Maxwell model does not depend upon time. For other viscoelastic models, as in Problem 9.11, the transient response may represent a time dependent strain that approaches a mean strain asymptotically.

9.4.2 Viscoelastic Foundation

Viscoelastic materials are often used in supports for heavy machinery in order to reduce the transfer of large forces and displacements due to vibrations of the machines. In this section we shall discuss some aspects and properties of viscoelastic foundations.

Figure 9.4.4 illustrates a machine of mass m supported on viscoelastic blocks. The supports have heights h and a total supporting area A. In the vertical direction the machine is subjected to its weight mg, a harmonically varying force:

$$F(t) = F_o \exp(i\omega t) \tag{9.4.42}$$

which may originate from unbalanced rotating machine parts, and reactions from the supports. The vertical motion of the machine is expressed by the displacement $u(t)$, measured from the equilibrium position of the machine, i.e. when $u = 0$ and $F_o = 0$. The initial conditions are taken to be:

$$u(0^+) = u_o\,, \quad \dot{u}(0^+) = 0 \tag{9.4.43}$$

First we shall investigate free vibrations of the machine, i.e. for case $F_o = 0$. Secondly we consider forced vibrations due to the harmonic force (9.4.42).

The stress in the viscoelastic blocks contains a mean value $-\sigma_m$ that supports the weight mg. The equation of motion for the machine is then:

$$F(t) - mg - [\sigma(t) - \sigma_m]A = m\,\ddot{u} \quad \Rightarrow \quad m\,\ddot{u} + A\,\sigma(t) = F_o \exp(\omega t) \tag{9.4.44}$$

As constitutive equation we choose the response functional (9.2.38):

$$\sigma(t) = \beta_g\,\varepsilon(t) + \int_0^\infty \frac{d\beta(s)}{ds}\,\varepsilon(t-s)\,ds\,, \quad \varepsilon(t) = \frac{u(t)}{h} \tag{9.4.45}$$

Free vibrations: $F_o = 0$. The solution for the vertical displacement is suggested to have the complex form:

$$u(t) = u^* \exp(ibt)\,H(t) \tag{9.4.46}$$

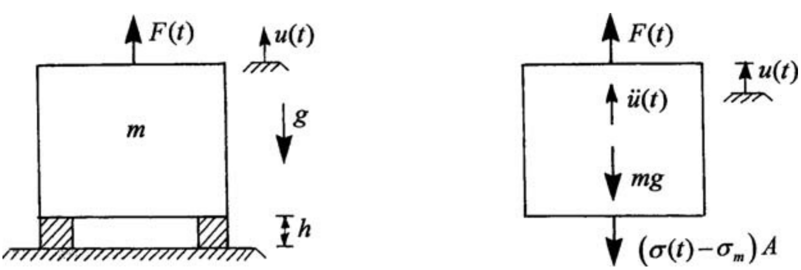


Fig. 9.4.4 Machine on viscoelastic foundation. Free-body diagram

where u^* and b are complex constants. The resulting stress in the viscoelastic foundations is:

$$\sigma(t) = \left[\beta_g \exp(ibt) + \int_0^\infty \frac{d\beta(s)}{ds} \exp(i\,b\,t - ibs)\,ds\right] \frac{u^*}{h} H(t) \quad \Rightarrow$$

$$\sigma(t) = \left[\beta_g + \int_0^\infty \frac{d\beta(s)}{ds} \exp(-ibs)\,ds\right] \frac{u(t)}{h} \tag{9.4.47}$$

The expression in the square brackets is according to (9.4.33) equal to $\beta^*(b)$. Thus the suggested solution (9.4.46) gives a stress:

$$\sigma(t) = \beta^*(b)\,\frac{u(t)}{h} = \frac{\beta^*(b)\,u^*}{h} \exp(ibs)\,H(t) \tag{9.4.48}$$

This constitutive equation is substituted into the equation of motion (9.4.44), with $F_o = 0$, and the result is:

$$-\,m u^* b^2 \exp(ibt)\,H(t) + \frac{A\,\beta^*(b)\,u^*}{h} \exp(ibt)\,H(t) = 0 \quad \Rightarrow \quad b^2 = \frac{A}{mh}\beta^*(b) \tag{9.4.49}$$

We may conclude directly from this result that for any viscoelastic material b must be a complex quantity: If b had been real, then $\beta^*(b)$ is complex by definition, and that does not agree with (9.4.44). Only if the material had been purely elastic would b be a real quantity. We anticipate a damped vibration or a damped aperiodic motion and set:

$$ib = -\rho + i\omega^d \quad \Leftrightarrow \quad b = \omega^d + i\rho \tag{9.4.50}$$

ρ and ω^d are real quantities. If $\omega^d \neq 0$, the real general solution of (9.4.44) can be presented as:

$$u(t) = a_o \exp(-\rho\,t) \cos\left(\omega^d\,t - \phi\right) H(t) \tag{9.4.51}$$

a_o is a real amplitude of the displacement function and ϕ is real phase angle. The initial conditions (9.4.43) imply that:

$$a_o = u_o \sqrt{1 + \left(\frac{\rho}{\omega^d}\right)^2}, \; \phi = \arctan \frac{\rho}{\omega^d} \tag{9.4.52}$$

As an example we assume that the foundation blocks respond as a *Kelvin model*. The complex modulus $\beta^*(b)$ is given by (9.4.22) and (9.4.16). Equation (9.4.49) for the parameter b then gives:

$$b^2 = \frac{A}{mh}(\eta + i\lambda b\eta) \quad \Rightarrow \quad ib = -\frac{A}{2mh}\lambda\eta \pm i\sqrt{\frac{A}{mh}\eta - \left[\frac{A}{2mh}\lambda\eta\right]^2} \tag{9.4.53}$$

The condition that the solution (9.4.51) represents a damped free vibration is that the radicand in (9.4.53) is positive. If the radicand is negative, the motion $u(t)$ is aperiodic.

Forced vibrations: $F_o > 0$. The solution $u(t)$ of the equation of motion (9.4.44) will have a transient part, for example as given by the function (9.4.51), and a steady-state of the form:

$$u(t) = u^* \exp(i\,\omega t) \tag{9.4.54}$$

u^* is a complex displacement amplitude. We shall focus on the steady-state solution. Substitution of the solution (9.4.54) into the equation of motion (9.4.44) results in a stress:

$$\sigma(t) = \left[\frac{F_o}{A} + \frac{m\,\omega^2}{A\,u^*}\right] \exp(i\,\omega t) \tag{9.4.55}$$

According to the response equation (9.4.20) the stress due to the strain obtained from the displacement (9.4.54) is:

$$\sigma(t) = \beta^*(\omega)\,\frac{u^*}{h} \exp(i\omega t) \tag{9.4.56}$$

When the two expressions (9.4.55) and (9.4.56) for the stress are set equal, we obtain an expression for the complex displacement amplitude:

$$u^* = \frac{F_o}{m}\,\frac{1}{\frac{A}{m\,h}\beta^*(\omega) - \omega^2} \tag{9.4.57}$$

For a Kelvin model the complex modulus is given by (9.4.41), and we obtain:

$$u^* = \frac{F_o}{m}\,\frac{1}{\left[\frac{A}{m\,h}\eta - \omega^2\right] + i\frac{A}{m\,h}\,\tilde{\eta}\,\omega} = \frac{F_o}{m}\,\frac{\left[\frac{A}{m\,h}\eta - \omega^2\right] - i\frac{A}{m\,h}\,\tilde{\eta}\,\omega}{\left[\frac{A}{m\,h}\eta - \omega^2\right]^2 + \left[\frac{A}{m\,h}\,\tilde{\eta}\,\omega\right]^2} \tag{9.4.58}$$

To be in accordance with standard theory for a simple mass/spring/damper system we introduce the parameters: the *spring stiffness* k, the *eigenfrequency* ω_e, the *damping coefficient* c, the *critical damping coefficient* c_{cr}, and the *damping ratio* ζ:

$$k = \frac{\eta A}{h}\,, \quad \omega_e = \sqrt{\frac{k}{m}}\,, \quad c = \frac{\tilde{\eta} A}{h}\,, \quad c_{cr} = 2\sqrt{km}\,, \quad \varsigma = \frac{c}{c_{cr}} \tag{9.4.59}$$

The formula (9.4.58) can now be transformed into:

$$u^* = \frac{F_o}{k}\,\frac{1 - \left(\frac{\omega}{\omega_e}\right)^2 - 2i\,\varsigma\,\frac{\omega}{\omega_e}}{\left[1 - \left(\frac{\omega}{\omega_e}\right)^2\right]^2 + \left[2\,\varsigma\,\frac{\omega}{\omega_e}\right]^2} \tag{9.4.60}$$

To obtain the real part of the solution $u(t)$ we set:

$$F(t) = F_o \cos\omega t \tag{9.4.61}$$

F_o is now a real force amplitude. Then:

$$u(t) = \text{Real}\left\{u^* \exp(i\omega t)\right\} = \frac{F_o}{kS}\left[\frac{1-\left(\frac{\omega}{\omega_e}\right)^2}{S}\cos\omega t + \frac{2\varsigma\frac{\omega}{\omega_e}}{S}\sin\omega t\right] \tag{9.4.62}$$

where:

$$S = \sqrt{\left[1-\left(\frac{\omega}{\omega_e}\right)^2\right]^2 + \left[2\varsigma\frac{\omega}{\omega_e}\right]^2} \tag{9.4.63}$$

Alternatively we may write:

$$u(t) = u_o\cos(\omega t - \phi_o)\ , \quad u_o = \frac{F_o}{kS}\ , \quad \tan\phi_o = \frac{2\varsigma\frac{\omega}{\omega_e}}{1-\left(\frac{\omega}{\omega_e}\right)^2} \tag{9.4.64}$$

This last alternative is the one most common in standard presentations of the theory of vibrations of a mass/spring/damper system.

9.5 Viscoelastic Waves

Fundamental aspects related to propagation of displacement waves in elastic materials were presented and discussed in Sect. 7.7. Parts of the analysis are purely kinematical and may be transferred directly to the following presentation of viscoelastic waves. However, we shall anyway present and develop all the necessary equations needed. It was shown in Sect. 7.7 that uniaxial stress waves in elastic rods and plane waves in an elastic material may be analyzed from one-dimensional wave equations. The same is not possible for viscoelastic materials.

9.5.1 Acceleration Waves in a Cylindrical Bar

A cylindrical viscoelastic bar is at time $t = 0$ given an axial impact $\hat{F}$ or a time dependent axial force $F(t)$ at one end. The result is a deformation pulse, or strain pulse, that travels in the axial direction. A Cartesian coordinate x is introduced along the axis of the bar, and $x = 0$ at the end where the impact or force is applied. Uniaxial stress $\sigma(x,t)$ is assumed, constant over the cross-section of the bar. The axial displacement is $u(x,t)$ and the axial strain becomes $\varepsilon(x,t) = \partial_x u$. Figure 9.5.1 illustrates the situation.

At time t the strain pulse $\varepsilon(x,t)$ has reached a distance $y = y(t)$ along the rod. We call the pulse a *wave*, more specifically a *displacement wave u(x,t)*, a *strain wave* $\varepsilon(x,t)$, or a *stress wave* $\sigma(x,t)$. The cross-section of the bar at the distance $x = y(t)$

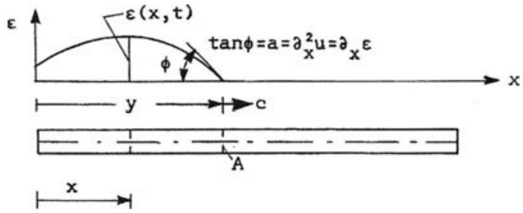


Fig. 9.5.1 Acceleration wave in a viscoelastic rod

marks the *wave front A*. The velocity of propagation of the wave front, which we call the *wave velocity*, is given by:

$$c = \frac{dy}{dt} \tag{9.5.1}$$

It will be shown below that the wave velocity c is constant and depends upon the characteristic material parameters of the constitutive model used to represent the viscoelastic material of the bar.

For any function $f(x,t)$, related to the wave front $x = y(t)$, we write:

$$\frac{df}{dt} \equiv \frac{df\,(y(t),t)}{dt} = \partial_t f\,(y,t) + \partial_y f\,(y,t)\,\frac{dy}{dt} \text{ on } A \quad \Rightarrow$$

$$\frac{df}{dt} = \partial_t f + c\,\partial_y f \text{ on } A \tag{9.5.2}$$

If the function $f(x,t)$ is constant on A at all times t, (9.5.2) provides the implication:

$$f(x,t) = \text{constant on } A \text{ at all times} \quad \Rightarrow \quad \frac{df}{dt} = \partial_t f + c\,\partial_x f = 0 \text{ at } x = y \tag{9.5.3}$$

The displacement field $u(x,t)$ and the particle velocity field $v \equiv \partial_t u(x,t)$ are assume to be continuous functions and satisfy the conditions:

$$u(x,t) = 0 \text{ for } x \geq y\,, \quad v(x,t) \equiv \partial_t\, u(x,t) = 0 \text{ for } x \geq y \tag{9.5.4}$$

When the implication (9.5.3) is applied to the displacement field and the velocity field vi obtain the conditions:

$$\partial_t u + c\,\partial_x u = 0 \text{ at } x = y^-\,,\; \partial_t^2 u + c\,\partial_x \partial_t u = 0 \text{ at } x = y^- \tag{9.5.5}$$

The notation $x = y^-$ is used to indicate that the conditions apply for the displacement wave, i.e. for the function $u(x,t)$ for $x \leq y$. From the conditions (9.5.4) and (9.5.5), and because the velocity field has to be continuous, it follows that:

$$\partial_x u = 0 \text{ at } x = y^- \tag{9.5.6}$$

This result implies that the strain $\varepsilon = \partial_x u$ vanishes at the wave front. We apply the implication (9.5.3) to the strain field and obtain:

$$\partial_t \partial_x u + c\,\partial_x^2\, u = 0 \text{ at } x = y^- \tag{9.5.7}$$

The *amplitude at the wave front* is defined by:

$$a(t) \equiv \partial_x^2 u \quad \text{at} \quad x = y^- \tag{9.5.8}$$

and represents the gradient of the strain $\varepsilon(x,t)$ at the wave front. Figure 9.5.1 shows that $a = a(t) = \tan\phi$, where ϕ is the slope of the graph of the strain field $\varepsilon(x,t)$ at the wave front. From the (9.5.5), (9.5.7), and (9.5.8) we derive two *kinematical conditions* at the wave front:

$$\partial_t \partial_x u = -c\,a \text{ at } x = y^-\,,\; \partial_t^2 u = c^2\,a \text{ at } x = y^- \tag{9.5.9}$$

A *dynamic condition* at the wave front is given directly by a Cauchy equation of motion:

$$\partial_x \sigma = \rho\, \partial_t^2 u \tag{9.5.10}$$

The three conditions (9.5.9) and (9.5.10) are independent of the material properties of the bar and presume only small deformations and the state of uniaxial stress. In order to obtain an expression for the wave velocity c and the amplitude $a(t)$, we must introduce a constitutive equation. For a linearly viscoelastic material we introduce the following constitutive equation, obtained from (9.2.38):

$$\sigma(x,t) = \beta_g\, \partial_x u(x,t) + \int_0^{\Delta t} \frac{d\beta(s)}{ds}\, \partial_x u(x,t-s)\, ds \tag{9.5.11}$$

Δt is the time interval for which the material at the cross section at x is subjected to strain. This time interval represents then the time it takes from when the wave front passes the cross-section at x to the present time t. Because the wave velocity c is constant, a fact that will be shown below, and we set $y(0) = 0$, we may write:

$$\Delta t = \frac{y - x}{c} = t - \frac{x}{c} \tag{9.5.12}$$

In (9.5.11) both the integrand and the upper limit Δt are functions of x and t. Therefore we get:

$$\partial_x \sigma(x,t) = \beta_g\, \partial_x^2 u(x,t) + \frac{d\beta(\Delta t)}{d(\Delta t)} \partial_x u(x,t-\Delta t) \frac{\partial(\Delta t)}{\partial x} + \int_0^{\Delta t} \frac{d\beta(s)}{ds}\, \partial_x^2 u(x,t-s)\, ds \tag{9.5.13}$$

The following implications are needed:

$$\begin{gathered} \Delta t \;\to\; 0 \quad \Rightarrow \quad x \to y^- \text{ and } \partial_x^2 u(x,t) \;\to\; a \quad \Rightarrow \\ \frac{d\beta(\Delta t)}{d(\Delta t)} \;\to\; \dot{\beta}(0)\,,\;\; \partial_x u(x,t-\Delta t) \;\to\; 0\,,\;\; \frac{\partial(\Delta t)}{\partial x} \;\to\; -\frac{1}{c} \end{gathered} \tag{9.5.14}$$

By these implications we obtain from (9.5.13) the following *material condition* at the wave front:

$$\partial_x \sigma(x,t) = \beta_g \, a \text{ at } x = y^- \tag{9.5.15}$$

The wave velocity c can now be determined. The expression for $\partial_x \sigma$ from (9.5.15) and the expression for $\partial_t^2 u$ from (9.5.9) are substituted into (9.5.10). The result is:

$$\beta_g \, a = \rho \, c^2 \mathrm{a} \quad \Rightarrow \quad c = \sqrt{\frac{\beta_g}{\rho}} \tag{9.5.16}$$

The wave velocity is constant and only exists for material models exhibiting a glass modulus. The constitutive equation (9.5.11) actually assumes that the glass modulus exists and that the relaxation function is continuous with a continuous first derivative. The Kelvin model and the Jeffreys model have infinite glass modulus and thus do not admit strain pulses with a distinct wave front. The wave velocity is infinite for these models. A disturbance at one end of the bar is instantaneously registered at the other end of the bar. For a Hookean model the glass modulus is $\beta_g = \eta \equiv E$ and the wave speed becomes $c = \sqrt{E/\rho}$, as found in Sect. 7.7.1.

Next, we shall determine an expression for the amplitude $a(t)$ at the wave front. The two kinematic conditions (9.5.9) provide two expressions for the time derivative of $a(t)$, obtained from (9.5.2). The time derivatives of $a(t)$ from both these expressions yield:

$$\frac{da}{dt} = -\frac{1}{c}\left[\partial_t^2 \partial_x u + c \partial_t \partial_x^2 u\right] = \frac{1}{c^2}\left[\partial_t^3 u + c \partial_x \partial_t^2 u\right] \text{ at } x = y^- \quad \Rightarrow$$

$$\frac{da}{dt} = \frac{1}{2}\left[\frac{1}{c^2}\partial_t^3 u - \partial_t \partial_x^2 u\right] \text{ at } x = y^- \tag{9.5.17}$$

The expression will now be transformed. First the equation of motion (9.5.10) yields:

$$\partial_t \partial_x \sigma = \rho \, \partial_t^3 u \tag{9.5.18}$$

Then from (9.5.13) and relation (9.5.12):

$$\partial_t \, \partial_x \, \sigma(x,t) = \beta_g \, \partial_t \, \partial_x^2 u(x,t) + \frac{d^2 \beta(\Delta t)}{(d(\Delta t))^2} \partial_x u(x, x/c) \left(-\frac{1}{c}\right)$$
$$+ \frac{d\beta(\Delta t)}{d(\Delta t)} \partial_x^2 u(x, t - \Delta t) \left(-\frac{1}{c}\right) + \int_0^{\Delta t} \frac{d\beta(s)}{ds} \, \partial_t \, \partial_x^2 u(x, t - s) \, ds$$

By the implications (9.5.14) and the relation (9.5.12) this result is reduced to the condition:

$$\partial_t \, \partial_x \, \sigma = \beta_g \, \partial_t \, \partial_x^2 u + \dot{\beta}(0) \, a \quad \text{at} \quad x = y^- \tag{9.5.19}$$

Combination of (9.5.18) and (9.5.19) gives:

$$\rho \, \partial_t^3 u - \beta_g \, \partial_t \partial_x^2 u = \dot{\beta}(0) \, a \quad \text{at} \quad x = y^- \tag{9.5.20}$$

Finally, (9.5.17) and (9.5.20) provide the differential equation:

$$\frac{da}{dt} = \frac{\dot{\beta}(0)}{2\beta_g}\, a \tag{9.5.21}$$

The general solution of this equation is:

$$a(t) = a_o \exp(-\theta t), \; \theta = -\frac{\dot{\beta}(0)}{2\beta_g} \tag{9.5.22}$$

a_o is a constant of integration. The parameter θ is called the *damping coefficient at the wave front*. The parameter $\dot{\beta}(0)$ is always negative, and the constant θ is therefore always positive. For the Maxwell model, for example:

$$\theta = \frac{\eta}{2\tilde{\eta}} = \frac{1}{2\lambda}$$

The result (9.5.22) indicates that the graph of the strain $\varepsilon = \partial_x u$ as function of x has a slope ϕ at the wave front that decreases with time. The graph flattens, and the extreme values of the strain at any particular cross section are reduced as the wave propagates through the rod.

It has implicitly been assumed so far that the amplitude a at the wave front is non-zero, i.e. $a \neq 0$. According to the second of the kinematical conditions (9.5.9) this implies a discontinuity or *jump* in the particle acceleration at the wave front. The motion $u(x,t)$ is therefore called an *acceleration wave*. If we assume that the acceleration is continuous across the wave front, that is $a = 0$, we may continue the analysis and assume jumps in higher order derivatives of $u(x,t)$ with respect to time and place at the wave front $x = y$. The result is again formula (9.5.16) for the wave velocity and the formula $(9.5.22)_2$ for the damping coefficient for the relevant jump function.

9.5.2 Progressive Harmonic Wave in a Cylindrical Bar

For the bar shown in Fig. 9.5.1 we shall investigate the following motion:

$$u(x,t) = u_o \exp\left[i\omega\left(t - \frac{x}{c}\right) - \theta\frac{x}{c}\right] \tag{9.5.23}$$

u_o is a complex amplitude. The angular frequency ω, the wave velocity c, and the damping coefficient θ are real and constant quantities. As will be shown below, c and θ are dependent upon the frequency ω. The motion (9.5.23) represents a progressive damped harmonic displacement wave moving in the positive x-direction, and may exist in an infinitely long bar subjected to the motion:

$$u(0,t) = u_o \exp(i\omega t)$$

at the end $x = 0$ of the bar. It will be shown that the parameters θ and c may be determined such that the motion (9.5.23) satisfies the equation of motion (9.5.10).

The strain $\varepsilon = \partial_x u$ has according to (9.5.23) the form $\varepsilon = \varepsilon_o \exp(i\omega t)$. Thus we may use the constitutive equation (9.4.20):

$$\sigma = \beta^* \varepsilon = \beta^* \partial_x u \tag{9.5.24}$$

When this expression for stress is substituted into the equation of motion (9.5.10), we obtain the *one-dimensional wave equation*:

$$\beta^* \partial_x^2 u = \rho \partial_t^2 u \tag{9.5.25}$$

A condition for the wave velocity c and the damping coefficient θ is obtained from this equation when the displacement function (9.5.23) is introduced. We get:

$$\beta^* \left(\frac{i\omega}{c} + \frac{\theta}{c}\right)^2 = \rho\,(i\omega)^2 \tag{9.5.26}$$

The real and imaginary parts of this equation are respectively:

$$\left(\theta^2 - \omega^2\right)\beta_1 - 2\omega\,\theta\,\beta_2 = -\rho\,c^2\omega^2,\ 2\omega\,\theta\,\beta_1 + \left(\theta^2 - \omega^2\right)\beta_2 = 0$$

The solution so this set of equation becomes:

$$c^2(\omega) = \frac{2\beta_1}{\rho}\left[1 + \left(\frac{\beta_1}{\beta_2}\right)^2\right]\left[\sqrt{1 + \left(\frac{\beta_2}{\beta_1}\right)^2} - 1\right] = \frac{\sqrt{\beta_1^2 + \beta_2^2}}{\rho\cos^2(\delta/2)} \tag{9.5.27}$$

$$\theta(\omega) = \omega\frac{\beta_1}{\beta_2}\left[\sqrt{1 + \left(\frac{\beta_2}{\beta_1}\right)^2} - 1\right] = \omega\tan\frac{\delta(\omega)}{2}\,,\quad \delta(\omega) = \arctan\frac{\beta_2}{\beta_1} \tag{9.5.28}$$

$\delta(\omega)$ is the *loss angle*. The parameters β_1 and β_2 are frequency dependent, which therefore also applies to the wave velocity c and the damping coefficient θ. For at Hookean solid with $\beta_1 = \eta$ and $\beta_2 = 0$, the formulas (9.5.27, 9.5.28) yield $c = \sqrt{\eta/\rho}$ and $\theta = 0$, i.e. the wave velocity c and the damping coefficient θ are independent of frequency ω.

Figure 9.5.2 presents the wave velocity $c(\omega)$ and the damping coefficient $\theta(\omega)$ for a viscoelastic fluid, e.g. the Maxwell model, and for a viscoelastic solid, e.g. the Standard linear solid model. The formulas (9.4.32) show that $\beta_1(0) = \beta_e$ and $\beta_2(0) = 0$, such that $\delta(0) = 0$. Thus, we obtain from (9.5.27, 9.5.28):

$$c_o \equiv c(0) = \sqrt{\frac{\beta_e}{\rho}},\ \theta_o = \theta(0) = 0 \tag{9.5.29}$$

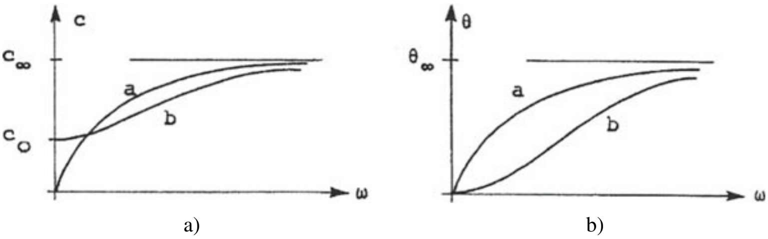


Fig. 9.5.2 Wave velocity c and damping coefficient θ as functions of the angular frequency ω. **a**) Viscoelastic fluid. **b**) Viscoelastic solid

In order to obtain expressions for the asymptotes $c(\infty) \equiv c_\infty$ and $\theta(\infty) \equiv \theta_\infty$ we have to do some calculations. From (9.4.8) and (9.2.32) we get that:

$$\lim_{\omega\to\infty} \beta_1(\omega) = \beta_g\,, \quad \lim_{\omega\to\infty} \beta_2(\omega) = 0 \tag{9.5.30}$$

$$\lim_{\omega\to\infty} \tan\delta(\omega) = \lim_{\omega\to\infty} \frac{\beta_2(\omega)}{\beta_1(\omega)} = 0\,, \quad \lim_{\omega\to\infty} \delta(\omega) = 0 \tag{9.5.31}$$

Equations (9.5.27), (9.5.30), and (9.5.31) provide the result:

$$c_\infty \equiv c(\infty) = \lim_{\omega\to\infty} c(\omega) = \sqrt{\frac{\beta_g}{\rho}} \tag{9.5.32}$$

Partial integration of the second of the equations (9.4.33) yields:

$$\omega\beta_2 = -\int_0^\infty \frac{d\beta(s)}{ds}\omega\sin\omega s\,ds = \left[\frac{d\beta(s)}{ds}\cos\omega s\right]_0^\infty - \int_0^\infty \frac{d^2\beta(s)}{ds^2}\cos\omega s\,ds \quad \Rightarrow$$

$$\lim_{\omega\to\infty} \omega\beta_2 = -\dot{\beta}(0) \tag{9.5.33}$$

The integral vanishes according to a theorem from Fourier analysis. Using a trigonometric formula for $\tan(\delta/2)$ and that:

$$\tan\delta = \frac{\sin\delta}{\cos\delta} = \frac{\beta_2}{\beta_1}$$

we obtain from the formula (9.5.28):

$$\theta(\omega) = \omega\tan\frac{\delta}{2} = \omega\frac{\sin\delta}{1+\cos\delta} = \frac{\omega\beta_2}{\beta_1}\frac{\cos\delta}{1+\cos\delta}$$

which with the results (9.5.30), (9.5.31), and (9.5.33) yields:

$$\theta_\infty \equiv \theta(\infty) = \lim_{\omega\to\infty} \theta(\omega) = -\frac{\dot{\beta}(0)}{2\beta_g} \tag{9.5.34}$$

The parameters c_∞ and θ_∞ are called the *ultrasonic wave velocity* and the *ultrasonic damping coefficient*. By comparing with the expressions (9.5.16) and (9.5.22), we see that these parameters are equal to the corresponding quantities for acceleration waves.

9.5.3 Waves in Infinite Viscoelastic Medium

We now consider a viscoelastic medium large enough for any influence from boundaries to other media to be of no importance. A region R^- of the medium bounded by a closed surface A is in motion, while the medium in the region R^+ outside of A is undisturbed. The situation is illustrated in Fig. 9.5.3a. The motion inside of A may for instance be initiated by an explosion. The surface A is a mathematical surface, which moves and expands with time in the medium. The surface is therefore called a *wave front A(t)*. We assume small displacements and small deformations and that the displacement field $\mathbf{u}(\mathbf{r},t)$ and the velocity field $\mathbf{v}(\mathbf{r},t)$ are continuous functions and that they satisfy the boundary conditions:

$$\mathbf{u}(\mathbf{r},t) = \mathbf{0} \ \text{ in } \ R^+ \text{ and on } \ A^- \tag{9.5.35}$$

$$\mathbf{v}(\mathbf{r},t) \equiv \dot{\mathbf{u}}(\mathbf{r},t) = \partial_t \mathbf{u}(\mathbf{r},t) = \mathbf{0} \text{ in } R^+ \text{ and on } A^- \tag{9.5.36}$$

The symbol A^- marks that side of the boundary surface A that faces the region R^-.

To describe the motion of the wave front through the medium we introduce a Cartesian coordinate system Ox. Let the place vector $\mathbf{y}(q,t)$ with the components $y_i(q,t)$ represent a curve on the wave front A, with q as curve parameter. The velocity of the point on the curve given by q is $\partial\mathbf{y}/\partial t$. The component of this velocity normal to the wave front A is called the *wave velocity c* and is determined by:

$$c = \mathbf{n} \cdot \frac{\partial \mathbf{y}}{\partial t} = n_i \frac{\partial y_i}{\partial t} \tag{9.5.37}$$

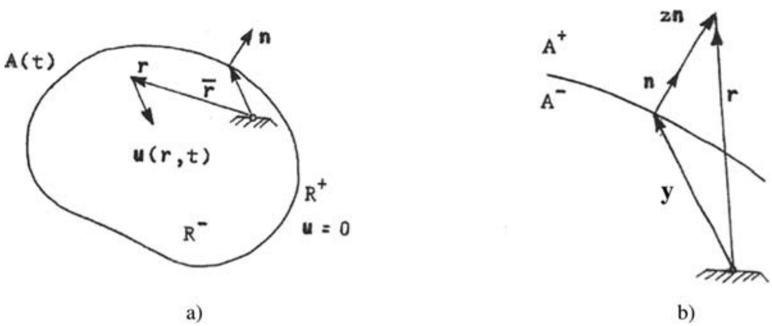


Fig. 9.5.3 Acceleration wave. Deformed region R^-. Undeformed region R^+. Wave front A(t)

$\mathbf{n}(q,t)$ is the unit normal to the wave front and pointing out from the region R^- towards the region R^+. The vector $\mathbf{n}$ is called the *direction of propagation*. It will be shown that the wave velocity c is a constant and dependent on the material properties, but independent of the shape of the wave front A and of the displacement field $\mathbf{u}(\mathbf{r},t)$.

The following analysis, which will provide an expression for the wave velocity c, is parallel to the analysis for one-dimensional development in Sect. 9.5.1. Let z denote a coordinate along the direction of propagation $\mathbf{n}$. Then, with reference to Fig. 9.5.3b:

$$\mathbf{r} = \mathbf{y} + z\mathbf{n} \quad \Leftrightarrow \quad x_i(q,t,z) = y_i(q,t) + zn_i(q,t) \tag{9.5.38}$$

It follows from this that:

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial z} \quad \Leftrightarrow \quad n_i = \frac{\partial x_i}{\partial z} \text{ on } A \tag{9.5.39}$$

Any field function $f(\mathbf{r},t) = f(\mathbf{r}(q,t,z),t)$ satisfies the relations:

$$\partial_z f \equiv \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x_i} n_i \equiv f_{,i}\, n_i,\ \partial_z^2 f \equiv \frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial x_k \partial x_i} n_k n_i \equiv f_{,ki}\, n_k n_i \tag{9.5.40}$$

$$\partial_t f|_{q,z} = \partial_t f(\mathbf{r},t) + f_{,i}\, \frac{\partial x_i}{\partial t} \tag{9.5.41}$$

If $f(\mathbf{r},t)$ is constant on the wave front A at all times t, which means that $f(\mathbf{r},t)$ is a level function on the surface A, the partial derivatives $f_{,i}$ represent a normal vector to the wave front, see Sect. 2.4:

$$f(\mathbf{r},t) = \text{constant on } A \text{ at all times } t \quad \Rightarrow \quad f_{,i} = \partial_z f\, n_i \text{ on } A \tag{9.5.42}$$

From (9.5.42) and (9.5.37) we obtain:

$$f_{,i}\, \frac{\partial y_i}{\partial t} = \partial_z f\, n_i \frac{\partial y_i}{\partial t} = \partial_z f\, c = c f_{,i}\, n_i \text{ on } A \tag{9.5.43}$$

The results (9.5.41) and (9.5.43) are now used to express the implication:

$$f(\mathbf{r},t) = \text{constant on } A \text{ at all times } t \quad \Rightarrow \quad \partial_t f + c\partial_z f = 0 \text{ on } A \tag{9.5.44}$$

The result is a generalization of the result (9.5.3) for the one-dimensional case.

The implications (9.5.42) and (9.5.44) are now applied to the displacement field $\mathbf{u}(\mathbf{r},t)$ and the velocity field $\dot{\mathbf{u}}(\mathbf{r},t) = \partial_t \mathbf{u}(\mathbf{r},t)$. Both these fields are according to the boundary conditions (9.5.35, 9.5.36) equal to zero on the wave front. We thus have these conditions on the wave front:

$$\mathbf{u}(\mathbf{r},t) = \mathbf{0} \text{ on } A^- \quad \Rightarrow \quad \mathbf{u},_i = \partial_z \mathbf{u} n_i, \; \dot{\mathbf{u}} + c\,\mathbf{u},_i n_i = \mathbf{0} \text{ on } A^- \tag{9.5.45}$$

$$\dot{\mathbf{u}}(\mathbf{r},t) = \mathbf{0} \text{ on } A^- \quad \Rightarrow \quad \dot{\mathbf{u}},_i = \partial_z \dot{\mathbf{u}} n_i, \; \ddot{\mathbf{u}} + c\,\dot{\mathbf{u}},_i n_i = \mathbf{0} \text{ on } A^- \tag{9.5.46}$$

The conditions (9.5.36) and (9.5.45) imply that:

$$\partial_z \mathbf{u} = \mathbf{0} \text{ on } A^- \quad \Rightarrow \quad \partial_z \dot{\mathbf{u}} + c\,\partial_z^2 \mathbf{u} = \mathbf{0} \text{ on } A^- \tag{9.5.47}$$

which by (9.5.46) yields the results:

$$\ddot{\mathbf{u}} = c^2\,\partial_z^2 \mathbf{u} \text{ on } A^-, \; \dot{\mathbf{u}},_i = -c\,\partial_z^2 \mathbf{u} n_i \text{ on } A^- \tag{9.5.48}$$

From (9.5.42), (9.5.47), and (9.5.44) we obtain the results:

$$\mathbf{u},_i = \mathbf{0} \text{ on } A^- \quad \Rightarrow \quad \mathbf{u},_{ik} = \partial_z \mathbf{u},_i n_k, \; \dot{\mathbf{u}},_i + c\partial_z \mathbf{u},_i = \mathbf{0} \text{ on } A^- \tag{9.5.49}$$

which by $(9.5.48)_2$ yield:

$$\mathbf{u},_{ik} = \partial_z^2 \mathbf{u},_i n_i n_k \text{ on } A^- \tag{9.5.50}$$

The *amplitude at the wave front* is defined as the vector:

$$\mathbf{a} \equiv \partial_z^2 \mathbf{u} = \mathbf{u},_{ik} n_i n_k \text{ on } A^- \tag{9.5.51}$$

We summarize the results (9.5.48) in the following two *kinematic conditions at the wave front* A:

$$\ddot{\mathbf{u}} \equiv c^2 \mathbf{a} \text{ on } A^-, \; \dot{\mathbf{u}},_i \equiv -c\,\mathbf{a} n_i \text{ on } A^- \tag{9.5.52}$$

These two conditions are analogous to the conditions (9.5.9) for the one-dimensional case. The result (9.5.50) may now be expressed as the condition:

$$\mathbf{u},_{ik} = \mathbf{a} n_i n_k \text{ on } A^- \tag{9.5.53}$$

Let us now investigate the kinematic implications of the kinematic conditions (9.5.52). While the strain tensor $\mathbf{E}$ is continuous across the wave front A and zero there, the strain rate tensor is discontinuous on A:

$$\mathbf{E} = \mathbf{0} \text{ on } A^-, \; \dot{E}_{ik} = \frac{1}{2}\left(\dot{u}_{i},_k + \dot{u}_{k},_i\right) = -\frac{c}{2}\left(a_i n_k + a_k n_i\right) \text{ on } A^- \tag{9.5.54}$$

The rate of longitudinal strain in the direction of propagation $\mathbf{n}$ is given by:

$$\dot{\varepsilon}_z = \mathbf{n} \cdot \dot{\mathbf{E}} \cdot \mathbf{n} = n_i \dot{E}_{ik} n_k = n_i \left[-\frac{c}{2}\left(a_i n_k + a_k n_i\right)\right] n_k \quad \Rightarrow$$
$$\dot{\varepsilon}_z = -c\,a_k n_k = -c\,\mathbf{a} \cdot \mathbf{n} \text{ on } A^- \tag{9.5.55}$$

The volumetric strain rate becomes:

$$\dot{\varepsilon}_v = \dot{E}_{kk} = -c\,a_k n_k = -c\,\mathbf{a} \cdot \mathbf{n} \text{ on } A^- \tag{9.5.56}$$

Let $\mathbf{e}$ be any unit tangent vector to the wave front at the point $\mathbf{y}(q,t)$ and $\mathbf{d}$ any unit vector perpendicular to $\mathbf{e}$. Then the shear strain rate with respect to the material directions $\mathbf{e}$ and $\mathbf{d}$ is:

$$\dot{\gamma} = 2\mathbf{e}\cdot\dot{\mathbf{E}}\cdot\mathbf{d} = 2e_i\dot{E}_{ik}d_k = e_i\left[-\frac{c}{2}\left(a_i n_k + a_k n_i\right)\right]d_k \quad\Rightarrow$$

$$\dot{\gamma} = -c\left[(\mathbf{e}\cdot\mathbf{a})(\mathbf{n}\cdot\mathbf{d}) + (\mathbf{d}\cdot\mathbf{a})(\mathbf{n}\cdot\mathbf{e})\right] \text{ on } A^- \tag{9.5.57}$$

From the results (9.5.55, 9.5.56, 9.5.57) it follows that there are no strain rates in the wave front. The rate of the rotation vector for small rotation $\dot{\mathbf{z}}$, which is equal to the angular velocity vector $\mathbf{w}$, is by (5.4.12) and $(9.5.52)_2$ equal to:

$$w_i \equiv \dot{z}_i = \frac{1}{2}e_{ijk}\dot{u}_{k,j} = -\frac{c}{2}e_{ijk}a_k n_j,\ \mathbf{w}\equiv\dot{\mathbf{z}} = \frac{c}{2}\mathbf{a}\times\mathbf{n} \text{ on } A^- \tag{9.5.58}$$

In addition to the two kinematic conditions (9.5.52) we need the *kinetic condition* provided by the Cauchy equation of motion:

$$\operatorname{div}\mathbf{T} = \rho\,\ddot{\mathbf{u}} \text{ on } A^- \tag{9.5.59}$$

and a constitutive equation specifying the material response.

The response of an isotropic, linearly viscoelastic material may be represented by the constitutive equation (9.2.60):

$$\mathbf{T}(\mathbf{r},t) = 2\beta'_g\mathbf{E}(\mathbf{r},t) + \left[\beta^o_g - \frac{2}{3}\beta'_g\right][\operatorname{tr}\mathbf{E}(\mathbf{r},t)]\,\mathbf{1}$$
$$+\int_0^{\Delta t}\left\{2\frac{d\beta'(s)}{ds}\mathbf{E}(\mathbf{r},t-s) + \left[\frac{d\beta^o(s)}{ds} - \frac{2}{3}\frac{d\beta'(s)}{ds}\right][\operatorname{tr}\mathbf{E}(\mathbf{r},t-s)]\,\mathbf{1}\right\}ds \tag{9.5.60}$$

Δt is the time interval during which the material at the particle $\mathbf{r}$ is subjected to deformation. From (9.5.60) we compute div $\mathbf{T}$ and then let $\mathbf{r}\to\mathbf{y}$ in the region R^-. Using the same arguments that led to (9.5.11) from (9.5.15), and then replacing the strain components by displacement gradients and finally applying the result (9.5.53), we obtain the result:

$$T_{ik,k} = 2\beta'_g E_{k,ki} + \left[\beta^o_g - \frac{2}{3}\beta'_g\right]E_{kk,i} = \beta'_g u_{i,kk} + \left[\beta^o_g + \frac{1}{3}\beta'_g\right]u_{k,ki} \text{ on } A^- \quad\Rightarrow$$

$$\operatorname{div}\mathbf{T} = \beta'_g\,\mathbf{a} + \left[\beta^o_g + \frac{1}{3}\beta'_g\right](\mathbf{a}\cdot\mathbf{n})\,\mathbf{n} \text{ on } A^- \tag{9.5.61}$$

The wave speed c is now determined by combining the kinematic conditions (9.5.52), the kinetic condition (9.5.59), and the material condition (9.5.61). First we derive from these conditions the result:

$$\beta'_g \mathbf{a} + \left[\beta_g^o + \frac{1}{3}\beta'_g\right](\mathbf{a}\cdot\mathbf{n})\,\mathbf{n} = \rho c^2 \mathbf{a} \text{ on } A^- \tag{9.5.62}$$

The scalar product and the vector product of this equation by the propagation direction $\mathbf{n}$ yield:

$$\left[\beta_g^o + \frac{4}{3}\beta'_g - \rho c^2\right](\mathbf{a}\cdot\mathbf{n}) = 0 \text{ on } A^-, \ \left[\beta'_g - \rho c^2\right](\mathbf{a}\times\mathbf{n}) = \mathbf{0} \text{ on } A^- \tag{9.5.63}$$

If we assume that $\mathbf{a}\cdot\mathbf{n} \neq 0$, the (9.5.63) tell us that the wave velocity is:

$$c = c_l = \sqrt{\frac{\beta_g^o + \frac{4}{3}\beta'_g}{\rho}} \tag{9.5.64}$$

and that in general $\mathbf{a}\times\mathbf{n} = \mathbf{0}$. If we assume that $\mathbf{a}\times\mathbf{n} \neq 0$, the (9.5.63) tell us that the wave velocity is:

$$c = c_t = \sqrt{\frac{\beta'_g}{\rho}} \tag{9.5.65}$$

and that in general $\mathbf{a}\cdot\mathbf{n} = 0$. It may be concluded from the results above that the wave amplitude $\mathbf{a}$ represents two wave fronts A_l and A_t:

1) The wave front A_l, for $\mathbf{a}\cdot\mathbf{n} \neq 0$ and $\mathbf{a}\times\mathbf{n} = \mathbf{0}$, moves with the velocity c_l and represents a propagation of motion in the direction of propagation $\mathbf{n}$. It follows from (9.5.55, 9.5.56) that the longitudinal strain rate in the direction of propagation and the volumetric strain rate are non-zero. Equation (9.5.57) shows that the shear strain rates with respect to the material directions: $\mathbf{e}$ in the tangent plane to the wave front and any $\mathbf{d}$ perpendicular to $\mathbf{e}$, are zero. Equation (9.5.58) shows that the rate of the rotation vector is zero. For these reasons the motion is alternatively called a *longitudinal wave*, a *volumetric wave*, and an *irrotational wave*.
2) The wave front A_t, for $\mathbf{a}\times\mathbf{n} \neq \mathbf{0}$ and $\mathbf{a}\cdot\mathbf{n} = 0$, moves with the velocity c_t and represents a propagation of motion in the directions perpendicular to the direction of propagation $\mathbf{n}$. Equation (9.5.57) shows that the shear strain rates with respect to the material directions: $\mathbf{e}$ in the tangent plane to the wave front and any $\mathbf{d}$ perpendicular to $\mathbf{e}$, are non-zero. Equation (9.5.58) shows that the rate of the rotation vector is non-zero. It follows from (9.5.55, 9.5.56) that longitudinal strain rate in the direction of propagation and the volumetric strain rate are zero in this case. For these reasons the motions are called *transverse waves, isochoric waves*, and *shear waves*.

According to (9.5.52) and (9.5.53) it follows that if the wave amplitude $\mathbf{a}$ is non-zero, the partial derivatives of second order with respect to time and place of the displacement $\mathbf{u}(\mathbf{r},t)$ are discontinuous functions on the wave front A. The discontinuities are called *jumps*, and (9.5.52, 9.5.53) are *jump conditions* on the displacement field $\mathbf{u}(\mathbf{r},t)$. Waves of the type discussed above, with jump in the acceleration $\ddot{\mathbf{u}}(\mathbf{r},t)$, are called *acceleration waves*.

For the case when $\mathbf{a} = \mathbf{0}$ at all times, the analysis can continue and jumps in the partial derivative of higher order may be found. Regardless of how far we have to seek for jumps, we shall find that A represents two wave fronts A_l and A_t propagating with the wave speeds c_l and c_t respectively.

For a Hookean solid $\beta_g{}' = \mu$ and $\beta_g{}^o = \kappa$, and we get the wave velocities:

$$c_l = \sqrt{\frac{\kappa + \frac{4}{3}\mu}{\rho}}, \; c_t = \sqrt{\frac{\mu}{\rho}} \tag{9.5.66}$$

which agree with what we found as the velocities (7.7.34–7.7.35) for plane elastic waves in Sect. 7.7.3.

9.6 Non-Linear Viscoelasticity

In a uniaxial creep test with constant stress σ_o the axial strain is a function of the stress σ_o and time t: $\varepsilon = \varepsilon(\sigma_o, t)$. If ε is a non-linear function of σ_o, the material is *non-linearly viscoelastic*. All *metals* show non-linearly viscoelastic response, except for very low levels of stress, whenever the temperature is higher than the critical value θ_c. Also *plastics* must often be treated as non-linear materials.

Another kind of non-linearity, which we will not consider in this chapter, is due to large deformations. In the case of large deformations all constitutive equations are non-linear. Viscoelastic liquids must therefore usually be modelled by non-linear fluid models. Section 11.9 presents models that are relevant when large deformations have to be considered. In the present section we will limit the presentation to the simplest and therefore often the most applicable part of the theory.

The experimental result of a uniaxial creep test, see Fig. 9.6.1, shows that the specimen experiences initial strain ε^i, primary creep P, and secondary creep S. The initial strain may be given by Hooke's law:

$$\varepsilon^i = \frac{\sigma_o}{\eta} \tag{9.6.1}$$

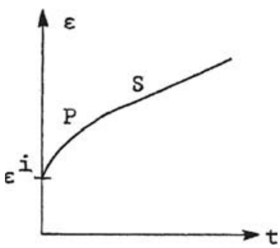


Fig. 9.6.1 Creep strain for uniaxial stress. Initial strain ε^i, primary creep P, secondary creep S

η is the *modulus of elasticity*. If the stress level is high enough for plastic strain to occur, or if the instantaneous response of the material is non-linearly elastic, we may, for instance, use the *Ramberg-Osgood law*:

$$\varepsilon^i = \frac{1}{\eta}\left[1 + \frac{\alpha}{\sigma_R}\left|\frac{\sigma_o}{\sigma_R}\right|^{m-1}\right]\sigma_o \tag{9.6.2}$$

The quantities η, α, σ_R, and m are temperature dependent material parameters. Equation (9.6.2) is written such that it applies both for compressive and for tensile stress.

The primary creep may be expressed by the function:

$$\varepsilon_{pc}(\sigma_o, t) = k\,|\sigma_o|^{p-1}\,\sigma_o\,t^q \quad \Leftrightarrow \quad \dot{\varepsilon}_{pc}(\sigma_o, t) = k\,q\,|\sigma_o|^{p-1}\,\sigma_o\,t^{q-1} \tag{9.6.3}$$

k, p, and q are temperature dependent material parameters. It is commonly found that $q < 0.5$. The graph of formula (9.6.3) appear to be nearly rectilinear for high values of t, and that means that formula (9.6.3) also may be used to describe secondary creep, which is in particular relevant for metals. Primary creep is the dominating creep region for plastics.

Secondary creep may be described by the functions:

$$\dot{\varepsilon}_{sc}(\sigma_o) = \dot{\varepsilon}_c\left|\frac{\sigma_o}{\sigma_c}\right|^{n-1}\frac{\sigma_o}{\sigma_c}\ \text{Norton's law} \tag{9.6.4}$$

$$\dot{\varepsilon}_{sc}(\sigma_o) = \dot{\varepsilon}_c\sinh\left(\frac{\sigma_o}{\sigma_c}\right)\ \text{Prandtl's law} \tag{9.6.5}$$

$\dot{\varepsilon}_c, \sigma_c$, and $n(> 1)$ are temperature dependent material parameters. Formula (9.6.4) is called *Norton's law* or the *Norton-Bailey* law, after F.H. Norton (1920) and R.W. Bailey (1929). The formula (9.6.5) is *Prandtl's law*, after L. Prandtl (1928). Because Norton's law (9.6.4) is the easiest to use in analytical computations, it is the preferred one of the two. But Norton's law (9.6.4) has a weakness not present in Prandtl's law (9.6.5): For low levels of stress σ_o any creep law should approach a linear relation of the form: $\dot{\varepsilon} = \sigma_o/\tilde{\eta}$, see Fig. 9.6.2. From the two suggested laws for secondary creep rates we get:

$$\begin{aligned} &\text{Norton's law (9.6.4)} \quad \Rightarrow \quad \lim_{\sigma_o \to 0}\frac{\partial \dot{\varepsilon}}{\partial \sigma_o} = 0 \\ &\text{Prandtl's law(9.6.5)} \quad \Rightarrow \quad \lim_{\sigma_o \to 0}\frac{\partial \dot{\varepsilon}}{\partial \sigma_o} = \frac{\dot{\varepsilon}_c}{\sigma_c} \end{aligned} \tag{9.6.6}$$

A further development of Prandtl's law leads to the *Zener-Hollomon law*:

$$\dot{\varepsilon}_{sc}(\sigma_o) = \dot{\varepsilon}_c\exp\left(-\frac{Q}{R\theta}\right)\left[\sinh\left(\frac{\sigma_o}{\sigma_c}\right)\right]^n\ \text{Zener-Hollomon law} \tag{9.6.7}$$

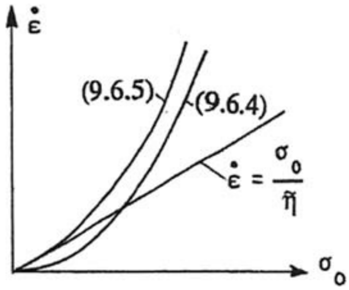


Fig. 9.6.2 Secondary creep according to Norton's law (9.6.4) and Prandtl's law (9.6.5)

$\dot{\varepsilon}_c$, Q, R, σ_c, and n are temperature dependent material parameters. Confer the description in Sect. 8.6.2 of the generalized Newtonian fluid called the Zener-Hollomon fluid.

Creep tests with variable stress show that the strain rate is a function of stress, strain, and time. We distinguish between two types of creep laws:

$$\dot{\varepsilon} = f(\sigma, \varepsilon) \text{ strain hardening law} \tag{9.6.8}$$

$$\dot{\varepsilon} = f(\sigma, t) \text{ time hardening law} \tag{9.6.9}$$

Most tests show that the stain hardening law gives the best description. The mostly used form of the law (9.6.8) is Nadai's law, according to A. Nadai (1963):

$$\dot{\varepsilon} = K\sigma^r \varepsilon^{-s} \text{ Nadai's law} \tag{9.6.10}$$

Little information is found in the literature about the material parameters K, r, and s.

For materials showing a dominant secondary creep region, like metals, and that is subjected over long time to weakly varying stress, Norton's law (9.6.4) may represent the creep rate adequately. For simplicity we shall assume that the material is linearly elastic for temperatures $\theta < \theta_c$ and has linearly elastic instantaneous response for $\theta > \theta_c$. A *non-linear Maxwell model*, which consists of a linear spring in series with a non-linear damper with the response given by formula (9.6.4), has the response equation:

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{\eta} + \frac{\dot{\varepsilon}_c}{\sigma_c} \left| \frac{\sigma}{\sigma_c} \right|^{n-1} \sigma \tag{9.6.11}$$

The model will be called the *Norton model*. η is the modulus of elasticity. It describes the $\sigma\varepsilon$-relationship for uniaxial stress and is suitable for metals. The data in Table 9.6.1 are taken from the book by Odqvist [36]. The parameters n and σ_{c7} are found from creep tests. The stress σ_{c7} corresponds to the strain $\varepsilon = 0.01$ after a time of 10^5 h $\approx$ 12 years. This means that:

$$\sigma_c = \sigma_{c7} \text{ for } \dot{\varepsilon}_c = 10^{-7}\text{h}^{-1} \tag{9.6.12}$$

Table 9.6.1 Material parameters in Norton's law, adapted from Odqvist's book [36]

	θ[°C]	n	σ_{c7}[MPa]
Rolled carbon steel	450	5	70
	500	3,3	35
Aluminium (24S-T4)	150	10	190
	190	5,3	71

9.6.1 The Norton Fluid

The following constitutive equation for general states of stress:

$$\dot{\mathbf{E}} = \frac{1+\nu}{\eta}\dot{\mathbf{T}} - \frac{\nu}{\eta}\left[\mathrm{tr}\,\dot{\mathbf{T}}\right]\mathbf{1} + \frac{3}{2}\frac{\dot{\varepsilon}_c}{\sigma_c}\left(\frac{\sigma_e}{\sigma_c}\right)^{n-1}\mathbf{T}' \tag{9.6.13}$$

defines a *Norton fluid*. ν is the Poisson's ratio. The quantity:

$$\sigma_e = \sqrt{\frac{3}{2}\mathbf{T}' : \mathbf{T}'} = \sqrt{\frac{3}{2}T'_{ij}T'_{ij}} \tag{9.6.14}$$

is called the *effective stress*, the *equivalent stress*, or the *Mises stress* and is a stress invariant. The name Mises stress is taken from the theory of plasticity, see Sect. 10.2.1. The Norton fluid model presumes small strains and has the following properties: a) Isotropic stress results in elastic strains, b) The material is isotropic, and c) The constitutive equations for uniaxial stress are:

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{\eta} + \frac{\dot{\varepsilon}_c}{\sigma_c}\left|\frac{\sigma}{\sigma_c}\right|^{n-1}\sigma, \; \dot{\varepsilon}_t = -\frac{\nu}{\eta}\dot{\sigma} - \frac{1}{2}\frac{\dot{\varepsilon}_c}{\sigma_c}\left|\frac{\sigma}{\sigma_c}\right|^{n-1}\sigma \tag{9.6.15}$$

ε_t is the strain in the direction perpendicular to the axial direction. Note that response equation (9.6.11) is included here.

We shall now demonstrate that the constitutive equation (9.6.13) in fact implies these properties.

a) For an isotropic state of stress: $\mathbf{T} = \sigma\mathbf{1}$, the last term on the right-hand side of (9.6.13) disappears, and we are left with the time-differentiated version and Hooke's law for isotropic states of stress.
b) Material isotropy implies that for a stress history $\mathbf{T}(t)$ with fixed principal directions $\mathbf{a}_i$, the strain history $\dot{\mathbf{E}}(t)$ and the strain rate history $\dot{\mathbf{E}}(t)$ are coaxial with $\mathbf{T}$, confer the arguments in Sect. 7.2. Isotropy also implies that the contribution to $\dot{\mathbf{E}}(t)$ from $\dot{\mathbf{T}}$ must be coaxial with $\dot{\mathbf{T}}$. The constitutive equation (9.6.13) satisfies both these conditions. The material has no preferred directions.
c) For uniaxial stress $T_{11} = \sigma$, we find that:

$$T^o = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{\sigma}{2}, \; T' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{\sigma}{3}$$

$$\sigma_e = \sqrt{\frac{3}{2} T'_{ij} T'_{ij}} = \sqrt{\frac{3}{2} \left(\frac{\sigma}{3}\right)^2 \left[2^2 + (-1)^2 + (-1)^2\right]} = |\sigma|, \; \mathrm{tr}\,\dot{\mathbf{T}} = \dot{\sigma}$$

Equation (9.6.13) now gives:

$$\dot{\varepsilon} = \dot{E}_{11} = \frac{1+\nu}{\eta} \dot{\sigma} - \frac{\nu}{\eta} \dot{\sigma} + \frac{3}{2} \frac{\dot{\varepsilon}_c}{\sigma_c} \left(\frac{|\sigma|}{\sigma_c}\right)^{n-1} \left(\frac{2\sigma}{3}\right) \quad \Rightarrow$$

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{\eta} + \frac{\dot{\varepsilon}_c}{\sigma_c} \left|\frac{\sigma}{\sigma_c}\right|^{n-1} \sigma \quad \Leftrightarrow \quad (9.6.11)$$

Example 9.6. Uniaxial Stress Relaxation

A steel bar is modelled as a Norton fluid. The bar is suddenly given a constant positive axial strain ε_o. We shall determine the stress history $\sigma(t)$.

The initial stress becomes: $\sigma^i = \eta \varepsilon_o$. For $t > 0$ the constitutive equation $(9.6.15)_1$ yields:

$$\frac{\dot{\sigma}}{\eta} + \dot{\varepsilon}_c \left(\frac{\sigma}{\sigma_c}\right)^n = 0$$

This differential equation is integrated with the result:

$$\int_{\sigma^i}^{\sigma} \bar{\sigma}^{-n} \, d\bar{\sigma} = -\eta \, \dot{\varepsilon}_c \, \sigma_c^{-n} \int_0^t d\bar{t} \quad \Rightarrow \quad \frac{\sigma^{1-n} - \left(\sigma^i\right)^{1-n}}{1-n} = -\eta \, \dot{\varepsilon}_c \, \sigma_c^{-n} \, t \quad \Rightarrow$$

$$\sigma(t) = \left[\left(\sigma^i\right)^{1-n} + (n-1) \, \eta \, \dot{\varepsilon}_c \, \sigma_c^{-n} \, t\right]^{\frac{1}{1-n}}$$

9.6.2 Steady Bending of Non-Linearly Viscoelastic Beams

The beam theory applied to viscoelastic beams is based on the hypotheses 1, 2, and 4 from the elementary beam theory, see Sect. 9.2.5. The deformation hypothesis and the displacement hypothesis imply that the strain parallel with the beam axis is given by:

$$\varepsilon(x,z,t) = -\frac{x}{R}, \; \frac{1}{R} = -\frac{\partial^2 u(z,t)}{\partial z^2} \qquad (9.6.16)$$

R is the radius of curvature of the deflected beam axis, and $u(z,t)$ is the displacement of the beam axis. The stress hypothesis and the constitutive hypothesis, which is $(9.6.15)_1$ for a Norton fluid, give for the axial strain rate:

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{\eta} + \frac{\dot{\varepsilon}_c}{\sigma_c} \left|\frac{\sigma}{\sigma_c}\right|^{n-1} \sigma \qquad (9.6.17)$$

Figure 9.6.3 shows a simply supported beam subjected to a distributed load $q(z)H(t)$ normal to the axis of the beam. The instantaneous response is elastic:

$$\sigma^i = \eta\,\varepsilon^i \tag{9.6.18}$$

This means that initially the stress distribution over the cross-section of the beam is linear, see Fig. 9.6.3d. According to the elementary beam theory we then have:

$$\sigma^i(x,z) = \frac{M(z)}{I}x, \ \frac{1}{R^i(z)} = \frac{M(z)}{\eta I} \tag{9.6.19}$$

$M(z)$ is the bending moment due to the load $q(z), I$ is the second moment of area of the beam cross-section and with respect to the y-axis through the center of area C, and $R^i(z)$ is the initial radius of curvature of the deflected beam axis. The state of stress approaches a steady state distribution $\sigma^s(x,z)$, presented in Fig. 9.6.3e. The change from the initial stress $\sigma^i(x,z)$ to the steady state stress $\sigma^s(x,z)$ occurs asymptotically as creep during variable stress. To find the stress as a function of time requires numerical integration. This transient creep, which represents a redistribution of stresses, is called *static primary creep* as a contrast to *physical primary creep* in a creep test under constant stress.

We shall develop a stress distribution formula and a differential equation for the deflection $u(z,t)$ of the beam axis for the steady state. The constitutive equation (9.6.17) will for the steady state reduce to:

$$\dot{\varepsilon} = \frac{\dot{\varepsilon}_c}{\sigma_c}\left|\frac{\sigma}{\sigma_c}\right|^{n-1}\sigma \tag{9.6.20}$$

If we combine (9.6.16) and (9.6.20), we obtain:

$$\frac{\partial}{\partial t}\frac{\partial^2 u(z,t)}{\partial z^2} = -\frac{\dot{\varepsilon}_c}{h_u}\left(\frac{\sigma_u(z)}{\sigma_c}\right)^n, \ \sigma(x,z) = \frac{\sigma_u(z)}{h_u}\left|\frac{x}{h_u}\right|^{\frac{1}{n}-1} x \tag{9.6.21}$$

σ_u is the stress at $x = h_u$, see Fig. 9.6.3b. The resultant of the stress distribution over the cross-section of the beam is the bending moment $M(z)$:

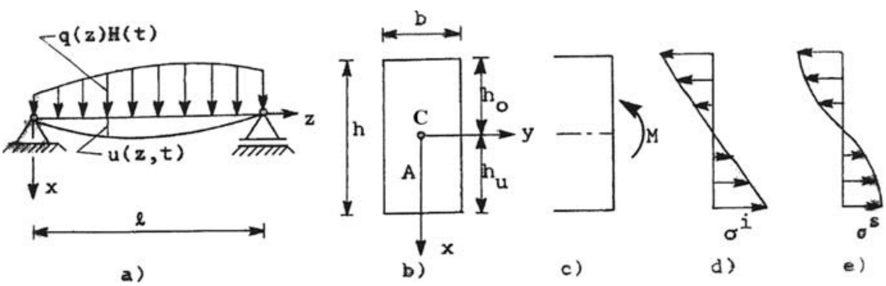


Fig. 9.6.3 Bending of a viscoelastic beam

$$\int_A \sigma(x,z)\,dA = 0, \int_A \sigma(x,z)\,x\,dA = M(z) \tag{9.6.22}$$

The first equation determines the neutral axis in the cross-section. For a symmetrical cross-section, as in Fig. 9.6.3b, the neutral axis coincides with the geometrical y-axis. From $(9.6.22)_2$ and $(9.6.21)_2$ we obtain:

$$\sigma(x,z) = \frac{\sigma_u(z)}{h_u}\left|\frac{x}{h_u}\right|^{\frac{1}{n}-1} x,\ \sigma_u(z) = \frac{M(z)}{I_n}h_u,\ I_n = \int_A \left|\frac{x}{h_u}\right|^{\frac{1}{n}-1} x^2\,dA \tag{9.6.23}$$

The quantity I_n is called the *second moment of area for creep*. For a rectangular cross-section with height h and width b we find:

$$I_n = \frac{3n}{2n+1}\frac{bh^3}{12} \tag{9.6.24}$$

From $(9.6.21)_1$, (9.6.23), and $(9.6.19)_2$ the curvature of the beam at steady state is expressed by:

$$\frac{\partial^2 u(z,t)}{\partial z^2} = -\dot{\varepsilon}_c\left(\frac{h_u M(z)}{\sigma_c I_n}\right)^n t - \frac{M(z)}{\eta I} \tag{9.6.25}$$

from which we can compute the deflection $u(z,t)$ of the beam axis when the bending moment $M(z)$ has been established.

Problems

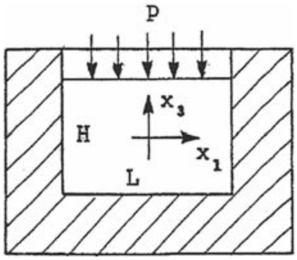
Problem 9.1. Solve the problem in Example 9.1 by using the second of the response functions (9.2.39)

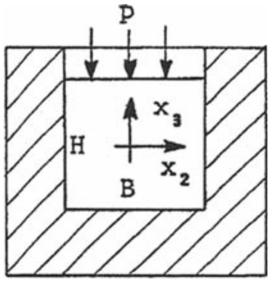
Problem 9.2. The figure shows a viscoelastic material which is placed in a undeformable container. The material is initially free of stress and strain. At time $t = 0$ the material is subjected to a constant pressure on the free surface. Neglect the friction between the material and the container wall and assume homogeneous state of stress. Determine the strain $\varepsilon = E_{33}$ and the stresses T_{ik} as functions of time when the material is modelled as a) a Maxwell fluid and b) a Kelvin solid.

Problem 9.3. A thin-walled tube of radius r and wall thickness h is subjected to torsion with a torque:

$$M(t) = \begin{cases} 0 & \text{for } t \le 0 \\ M_o \frac{t}{t_1} & \text{for } 0 < t \le t_1 \quad M_o \text{ and } t_1 \text{ are constants} \\ M_o & \text{for } t_1 < t \end{cases}$$

Fig. Problem 9.2





The material in the tube is assumed to be a Kelvin solid. Determine the torsion angle per unit length of the tube as a function of time. Use the constitutive equation both on a) rate form (response equation) and b) integral form (functional equation).

Problem 9.4. A viscoelastic material is bounded by two parallel rigid plane plates. The distance between the plates is h. The material sticks to the plates. The relaxation function for shear stress is $\beta'(t)$. The shear stress from the material on the plates is $\tau(t)$ and the related shear strain is $\gamma(t)$.

a) For $t < 0$ the plates are at rest. At time $t = 0$ one of the plates is set in motion with a constant velocity v_o. Develop the following expression for the shear stress $\tau(t)$, and determine $\tau(t)$ for a Maxwell fluid:

$$\tau(t) = \frac{v_o}{h} H(t) \int_0^t \beta'(s)\, ds$$

b) For $t < 0$ one plate is moving with a constant velocity v_o. For $t > 0$ both plates are at rest. Develop the following expression for the shear stress $\tau(t)$, and determine $\tau(t)$ for a Maxwell fluid:

$$\tau(t) = \frac{v_o}{h} \left[\int_0^\infty \beta'(s)\, ds - H(t) \int_0^t \beta'(s)\, ds \right]$$

c) For $t < 0$ one plate is moving with a constant velocity v_o. At time $t = 0$ the force that moves the plate is removed, such that the shear stress $\tau = 0$ for $t > 0$. Show that the restitution may be expressed by the following formula, and determine the restitution for a Maxwell fluid:

$$\gamma_\infty \equiv \int_0^\infty \dot{\gamma}(\bar{t})\,d\bar{t} = -\frac{v_o}{h}\,\frac{\int_0^\infty s\,\beta'(s)\,ds}{\int_0^\infty \beta'(s)\,ds}$$

Problem 9.5. Develop the response equation (9.2.13) for a Burgers-model. Determine also the relaxation function $\beta(t)$ in formula (9.2.16).

Problem 9.6. Assume the state of plane displacement for the cylinder in Example 9.3 and compute the stress $\sigma_z(t)$ and the radial displacement $u(R,t)$.

Problem 9.7. Solve Problem 9.6 if the cylinder is modelled as a Kelvin solid.

Problem 9.8. The cylinder in Example 9.3 is modelled as a Kelvin solid. Show that the solution to the problem is:

$$\begin{aligned} u(R,t) &= \frac{1}{2\mu}\frac{(a/b)^2}{1-(a/b)^2} \\ &\times \left\{\frac{3\kappa+4\mu}{9\kappa}\left[1-\frac{3\kappa}{3\kappa+4\mu}\exp\left(-\frac{t}{\lambda}\right)\right]R+\left[1-\exp\left(-\frac{t}{\lambda}\right)\right]\right\}p_o H(t) \\ \varepsilon_z(t) &= -\frac{(a/b)^2}{1-(a/b)^2}\,\frac{3\kappa-2\mu}{9\kappa\mu}\left[1-\frac{3\kappa}{3\kappa-2\mu}\exp\left(-\frac{t}{\lambda}\right)\right]p_o H(t) \end{aligned}$$

Problem 9.9. The outer wall $R = b$ of the cylinder in Example 9.3 is prevented from displacement in the radial direction: $u(b,t) = 0$. The material is modelled as a Maxwell fluid. Determine the stresses and the radial displacement $u(R,t)$ under the assumption of plane displacements. Comment on the special situations at $t = 0^+$ and $t = \infty$.

The theory of elasticity provides the following quasi-static solution for stresses and the radial displacement:

$$\begin{aligned} \begin{matrix}\sigma_R(R,t)\\ \sigma_\theta(R,t)\end{matrix} &= -\frac{1}{1+(1-2\nu)(b/a)^2}\left[1\pm(1-2\nu)\left(\frac{b}{R}\right)^2\right]p(t) \\ \sigma_z(t) &= -\frac{2\nu}{1+(1-2\nu)(b/a)^2}p(t), \\ u(R,t) &= \frac{b}{1+(1-2\nu)(b/a)^2}\,\frac{1-2\nu}{2\mu}\left[\frac{b}{R}-\frac{R}{b}\right]p(t) \end{aligned}$$

Problem 9.10. Compute the stresses and the radial displacement in a thick-walled spherical shell subjected to an internal pressure $p(t) = p_o H(t)$. The inner radius and outer radius of the shell are a and b respectively. The material is modelled as a Kelvin solid.

The quasi-static solution for an elastic shell is presented in Example 7.15 as:

$$\sigma_r(r,t) = -\frac{1}{1-(a/b)^3}\left[\left(\frac{a}{r}\right)^3 - \left(\frac{a}{b}\right)^3\right] p(t),$$

$$\sigma_\phi(r,t) = \frac{1}{1-(a/b)^3}\left[\frac{1}{2}\left(\frac{a}{r}\right)^3 + \left(\frac{a}{b}\right)^3\right] p(t)$$

$$u(r,t) = \frac{b\,(a/b)^3}{1-(a/b)^3}\left[\frac{1-2\nu}{1+\nu}\frac{r}{b} + \frac{1}{2}\left(\frac{b}{r}\right)^2\right]\frac{1}{2G}p(t)$$

Problem 9.11. A bar is subjected to a axial stress $\sigma_o \sin \omega t$ from time $t = 0$. The bar has the same viscoelastic response as the Jeffreys model. Compute the axial strain $\varepsilon(t)$. Show that the mean strain ε_m about which $\varepsilon(t)$ oscillates after sufficiently long time, is:

$$\varepsilon_m = \frac{\sigma_o}{\eta\,(\lambda_1 - \lambda_2)\,\omega}$$

Problem 9.12. A viscoelastic material modelled as a Kelvin solid fills the semi-infinite space defined by: $z \geq 0$, related to cylindrical coordinates (R, θ, z). The surface at $z = 0$ is free of stresses except at the point: $R = 0$, where the surface is subjected to a point load $F(t) = F_o H(t)$ perpendicular to the surface. Determine the stresses $\sigma_R, \sigma_\theta, \sigma_z$, and τ_{Rz}, and the displacement u_z. Investigate in particular the situation at $t = 0^+$ and the asymptotic solution when $t \to \infty$. The solution for a Hookean solid is:

$$\sigma_R(R,t) = \frac{1}{2\pi}\left\{(1-2\nu)\left[1 - \frac{z}{(R^2+z^2)^{1/2}}\right]\frac{1}{R^2} - \frac{3R^2 z}{(R^2+z^2)^{5/2}}\right\}F(t)$$

$$\sigma_\theta(R,t) = -\frac{1}{2\pi}\left\{(1-2\nu)\left[1 - \frac{z}{(R^2+z^2)^{1/2}}\right]\frac{1}{R^2} - \frac{z}{(R^2+z^2)^{3/2}}\right\}F(t)$$

$$\sigma_z(R,t) = -\frac{3z^3 F(t)}{2\pi(R^2+z^2)^{5/2}}\,,\ \tau_{Rz}(R,t) = \frac{R}{z}\sigma_z(R,t)\,,\ u_z(R,t) = \frac{1-\nu}{2\pi G R}F(t)$$

Problem 9.13. Determine the free damped vibration $u(t)$ in the problem described in Fig. 9.4.4 when the response of the foundation blocks is that of the Maxwell model. Use $1/\alpha^*$ rather than β^* in this case.

Problem 9.14. A bolt of stainless steel is subjected to an axial strain $\varepsilon_o = 6.0 \cdot 10^{-4}$ from time $t = 0$. The axial stress $\sigma(t)$ in the bolt at the constant temperature 600°C is to be determined.

a) Use the Norton fluid as material model with:

$$\eta = 210\,\text{GPa},\ \nu = 0.3,\ n = 5,\ \sigma_{c7} = 50\,\text{MPa}\ at\ \theta = 600\,^{o}C$$

b) Use the Maxwell fluid as material model. Determine the necessary material parameters from the data given in a).

Problem 9.15. A thin-walled tube with mean radius r and length L is subjected to torsion with the torque:

$$M(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ M_o \frac{t}{t_1} & \text{for } t > 0 \end{cases} \qquad M_o \text{ and } t_1 \text{ are constant parameters}$$

The tube is modelled as a Maxwell fluid. Determine the torsion angle as a function of time.

Problem 9.16. A bar with cross-sectional area A is subjected to an axial force:

$$N(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ N_o \frac{t}{t_1} & \text{for } t > 0 \end{cases} \qquad \text{N}_o \text{ and } t_1 \text{ are constant parameters}$$

The bar is modelled as a Maxwell fluid. Determine the axial strain as a function of time.

Problem 9.17. A thin-walled tube of length L, mean radius r, and wall thickness $h << r$ is subjected to an axial tensile force N, a torque $M = Nr$, and an internal pressure $p = N/2\pi r^2$. The pressure does not introduce stresses on the cross section of the tube. The tensile force is given as a function of time:

$$N(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ N_o \left(\frac{t}{t_1}\right)^2 & \text{for } t > 0 \end{cases} \qquad \text{N}_o \text{ and } t_1 \text{ are constant parameters}$$

The tube is modelled as a Norton fluid. Determine the torsion angle $\phi(t)$.

Problem 9.18. The response to uniaxial stress in a viscoelastic material is simulated by a mechanical model consisting of a parallel coupling of a Maxwell model and a linearly elastic spring. Derive the response equation for this model, and determine the creep function and the relaxation function.

Problem 9.19. A bar is subjected to an axial force N which is given as a function of time:

$$N(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ N_o \frac{t}{t_1} & \text{for } 0 < t \leq t_1 \\ N_o & \text{for } t_1 < t \end{cases} \qquad M_o \text{ and } t_1 \text{ are constants}$$

The bar is modelled as a Maxwell fluid. Determine the axial strain in the bar.

Problem 9.20. A thin-walled tube is subjected to an internal pressure p_o from the time $t \geq 0$. For the time $t \leq 0$ the dimensions of the tube are: length L_o, mean radius r_o, and wall thickness $h_o << r_o$. The tube is modelled as a Maxwell fluid. Determine the mean tube diameter as a function of time.

Problem 9.21. The surface properties of paper may be improved by a thermomechanical process called *calendaring*. In principle the paper is compressed between two rollers where the paper is subjected to heat and pressure with the purpose to compact the outer surface layer of the paper. The rolling machine, which usually consists of several pairs of rollers, is called a *calendar*. The calendaring process provides smoother surfaces of the paper, which is advantageous when the paper is used in a printing process. The present problem and the two following problems discuss a simplified procedure to calculate the strain and stress histories in the calendaring process.

Figure *a* illustrates half of the geometry of one pair of rollers in a calendar. The undeformed paper enters from the left with a constant velocity v. The paper is then compressed between the two rollers of which only the upper roller is shown in the figure. The rollers are pressed together with a load F, given as force per unit width of the paper. Due to the visco-elastic-plastic response of the paper the paper thickness is reduced from the initial thickness $2z_i$ to the minimum thickness $2z_m$ between the rollers, and then due to partial restitution the paper thickness increases again to the final thickness $2z_p$. It is assumed that the strains are small and that the state of stress

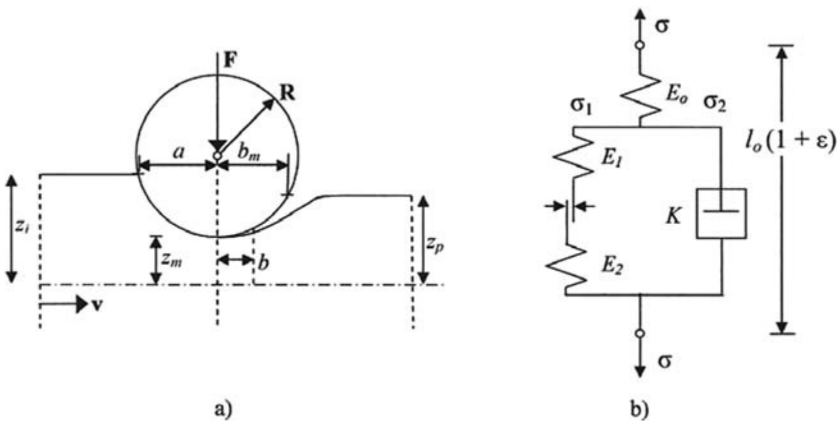


Fig. Problem 9.21 a) *Calender for improving paper surface quality.* **b**) *Mechanical model for uniaxial stress in the material model suggested for paper in the calendaring process* through the thickness of the paper is uniaxial and homogeneous. Assume also that the paper comes in contact with the rollers when the thickness is $2z_i$ as indicated in the figure. The following geometrical relations exist:

$$a = \sqrt{-2R\, z_i\, \varepsilon_m},\; b_{\mathrm{m}} = \sqrt{2R\, z_i\, (\varepsilon^p - \varepsilon_m)} \tag{9.6.1}$$

ε_m is the maximum (negative) strain, and ε^p is the final (negative) plastic strain in the thickness direction of the paper. As long as the paper is in contact with the rollers, the deformation process is strain controlled by the following strain history:

$$\varepsilon(t) = \varepsilon_m \frac{vt}{a}\left(2 - \frac{vt}{a}\right) \tag{9.6.2}$$

When the paper has lost contact with the rollers, the process is stress controlled with the stress being constant equal to zero.

Figure *b* illustrates a mechanical model for uniaxial stress in the material model suggested to be used in calendaring process. The mechanical model consists of three springs and a damper.

a) Derive the formulas (1) and (2).
b) Assume that all the elements of the mechanical model are linear and develop the response equation for the model.
c) Solve the response equation analytically separately for the strain controlled part and the stress governed part of the process.

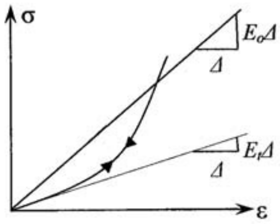
Problem 9.22.

a) Present a numerical solution procedure for the process described in Problem 9.21 that evaluates the strains $\varepsilon_1(t), \varepsilon_2(t)$, and $\varepsilon(t)$, and the stresses $\sigma_1(t), \sigma_2(t)$, and $\sigma(t)$.
b) Let the calendaring process be specified by: the radius of the roller $R = 150\,\mathrm{mm}$, the initial paper thickness $h = 2z_i = 100\,\mu\mathrm{m}$, the paper velocity $v = 15\,\mathrm{m/s}$, and the load $F = 100\,\mathrm{kN/m}$. The material parameters are chosen to be: $E_o = 215\,\mathrm{MPa}$, $E_1 = 420\,\mathrm{MPa}, E_2 = 215\,\mathrm{MPa}$, and $K = 12.4\,\mathrm{kPa\cdot s}$. Follow the numerical procedure in a) and compute the strains $\varepsilon_1(t)$, $\varepsilon_2(t)$, and $\varepsilon(t)$, and the stresses $\sigma_1(t)$, $\sigma_2(t)$, and $\sigma(t)$. Present graphs of these functions. Determine also the distance b, which indicate where the paper loses contact with the rollers.

Problem 9.23. In the mechanical model illustrated in Fig. Problem 9.21b the series spring marked E_o is now assumed to be non-linear. The parameter E_o is the secant modulus, see Fig. problem 9.23a:

$$E_o = E_t + n\,|\sigma|\,,\; E_t \text{ and } n \text{ are constant material parameters} \tag{1}$$

The spring marked E_2 is replaced by a plastic element having a response illustrated by Fig. Problem 9.23b.



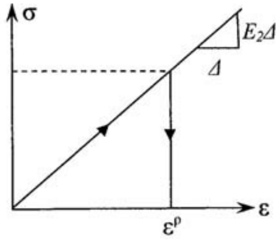


Fig. Problem 9.23 a) Response of non-linear spring. **b**) Response of plastic element

a) Derive the following response equation for the mechanical model under the above conditions:

$$\left(\frac{1}{E_o}+\frac{1}{E_1}+\frac{1}{E_2}\right)\sigma+K\left(\frac{1}{E_1}+\frac{1}{E_2}\right)\frac{d}{dt}\left(\frac{\sigma}{E_o}\right)=\varepsilon+\left(\frac{1}{E_1}+\frac{1}{E_2}\right)\frac{d\varepsilon}{dt} \quad (2)$$

b) Show that the plastic strain may be expressed by the formula:

$$\varepsilon^p=\frac{\varepsilon-\sigma/E_o}{1+E_2/E_1} \quad (3)$$

Problem 9.24.

a) Present a numerical solution procedure that evaluates the strains $\varepsilon(t)$ and $\varepsilon_p(t)$, and the stress $\sigma(t)$ for the process described in Problem 9.21 and with the material model presented in Problem 9.23.
b) Let the calendaring process be specified by: the radius of the roller $R = 150\,\text{mm}$, the initial paper thickness $h = 2z_i = 100\,\mu\text{m}$, the paper velocity $v = 15\,\text{m/s}$, and the load $F = 100\,\text{kN/m}$. The material parameters are chosen to be: $E_i = 13.0\,\text{MPa}$, $n = 2.6$, $E_1 = 420\,\text{MPa}$, $E_2 = 215\,\text{MPa}$, and $K = 12.4\,\text{kPa}\cdot\text{s}$. Follow the numerical procedure to compute the strains $\varepsilon(t)$ and ε^p, and the stress $\sigma(t)$. Present graphs of these functions. Determine also the distance b, which indicate where the paper loses contact with the rollers.

Chapter 10
Theory of Plasticity

10.1 Introduction

Permanent deformations in a solid resulting from a load cycle: loading and unloading, are called *plastic deformations*. Real solid materials that exhibit elastic response up to a certain level of strains or stresses, are considered either to be *brittle materials* or *ductile materials*. When the level of elastic response is exceeded in a brittle material, fracture occurs. A ductile material shows elastic-plastic response before fracture.

In order to illustrate the elastic-plastic response of a ductile material, the stress-strain curve of mild steel in uniaxial tension is shown in Fig. 10.1.1. The experimental curves for uniaxial compression and for a shear test exhibit the same characteristics as the tension curve in the figure. Experiments often show that the curves for tension and compression will coincide if the stress σ is the absolute value of the "*true normal stress*" $=$ force per unit deformed area, and the strain is the absolute value of the *logarithmic strain* ε_l. In Fig. 10.1.1 the stress is the engineering stress and the strain is the conventional, engineering strain. Mild steel has a well defined *yield stress* or *yield strength* f_y, at which the test material gets an increasing strain without an increase in stress. The real test curve shows an upper yield stress f_y^u. Both the actual value of f_y^u and the curve between the points A and B in Fig. 10.1.1 are however sensitive to the loading condition, especially the loading time. Until the stress has reached the yield stress the stress-strain curve is a straight line and expressed by Hooke's law: $\sigma = E\varepsilon$. Sometimes two other stress parameters are introduced before yielding: the *proportional limit* σ_P and the *elastic limit* σ_E. In the region $0 < \sigma < \sigma_P$ Hooke's law applies. In the region $0 < \sigma < \sigma_E$ the material is elastic and an unloading curve from a stress in this region follows the loading curve. For mild steel we may assume that $\sigma_P \approx \sigma_E \approx f_y$.

From the point B on the $\sigma\varepsilon$-curve in Fig. 10.1.1 the material hardens and the stress must be increased to obtain further deformation. Fracture occurs at the *tensile strength* f_u.

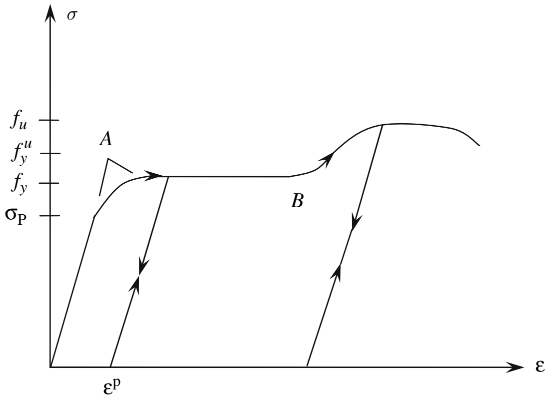


Fig. 10.1.1 Stress-strain diagram for mild steel in uniaxial tension

Unloading the material from a maximum stress above the yield stress, $\sigma > f_y$, to zero stress leaves the material with a permanent or *plastic strain* ε^p. The unloading curves in Fig. 10.1.1 are practically straight lines parallel with the loading curve from $\sigma = 0$. A new loading gives approximately the same straight line up to the stress level at which the unloading started. It is seen that hardening leads to an increase in the elastic limit σ_E.

Other ductile materials than mild steel may have $\sigma\varepsilon$-curves for uniaxial stress similar to the one shown in Fig. 10.1.2. Real test curves for some metals are shown in Fig. 1.2.3 in Sect. 1.2. For materials without a clearly marked yield stress, it is common to define a yield stress f_y equal to the stress $\sigma_{0.2}$ resulting in a plastic strain $\varepsilon^p = 0.002 (= 0.2\text{‰})$, see Fig. 10.1.2.

It is customary to introduce two main types of elastic-plastic material models:

linearly elastic-perfectly plastic materials,
linearly elastic-plastic hardening materials.

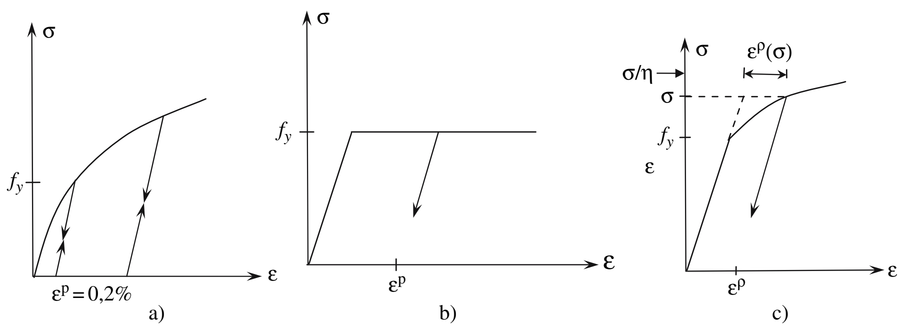


Fig. 10.1.2 Elastic-plastic material response. **a)** General response. **b)** Linearly elastic-perfectly plastic material. **c)** Linearly elastic-plastic hardening material

The uniaxial response of the first type of models is illustrated by Fig. 10.1.2. The yield stress f_y is assumed to be the same in tension and compression. The other type of models has the stress-strain curve in uniaxial stress as shown by Fig. 10.1.2 and described by:

$$\varepsilon = \begin{cases} \frac{\sigma}{\eta} & \text{for } \sigma \leq f_y \\ \frac{\sigma}{\eta} + \varepsilon^p(\sigma) & \text{for } \sigma > f_y \end{cases} \tag{10.1.1}$$

The function $\varepsilon^p(\sigma)$ has to be determined experimentally.

Figure 10.1.3 illustrates the result of a reversal of the stress from tension to compression. Case a) may apply for moderate strains. The yield stress decreases when the material has been plastically predeformed by a stress of opposite sign: $f_y'' < f_y$. The phenomena is called the *Bauschinger-effect*, named after Johan Bauschinger (1897). If: $f_y'' + f_y' = 2f_y$, the material is said to exhibit *kinematic hardening*. Case b) may apply when the strains are large, for instance in forming processes. The yield stress increases when the material has been plastically predeformed by a stress of opposite sign: $f_y'' > f_y$. If $f_y'' = f_y'$, see Fig. 10.1.3b, we say that the material shows *isotrop hardening*.

For general states of stress experiments show that isotropic stress superimposed on a stress state that represents elastic response, often does not lead to plastic deformations. For example in 1923 Percy Williams Bridgman [1882–1961] demonstrated that an isotropic pressure of the same order of magnitude as the yield stress f_y does not influence the criterion describing when the material gets plastic deformations. He also showed that high isotropic pressures will increase the ductility, i.e. the yielding capacity, of the material. It is also a fact that plastic strains result in minor volumetric strain, i.e. plastic deformations may be considered to be isochoric. A test bar in uniaxial tension that gets an axial strain of ε^p experiences a cross-sectional strain of about $-\varepsilon^p/2$, which gives a volumetric strain of $\varepsilon_v^p = 0$, assuming small strains.

The theory of plasticity has two primary objectives:

1) To develop a yield criterion that determines the states of stress at which the material response changes from elastic to elastic-plastic. This is the topic of Sect. 10.2.
2) To develop a flow rule that determines the relationship between stresses and the plastic strains. This objective is discussed in Sect. 10.3.

The theory of plasticity is still not fully developed, and the present exposition only tries to give an introduction to a part of the theory that seems to be accepted and applied.

10.2 Yield Criteria

We want to formulate a *yield criterion* that defines when a material behaves purely elastically and when the material starts to yield such that plastic strains result. Let $f[\mathbf{T}]$ be a scalar-valued function of the stress tensor $\mathbf{T}$. Then the yield criterion for

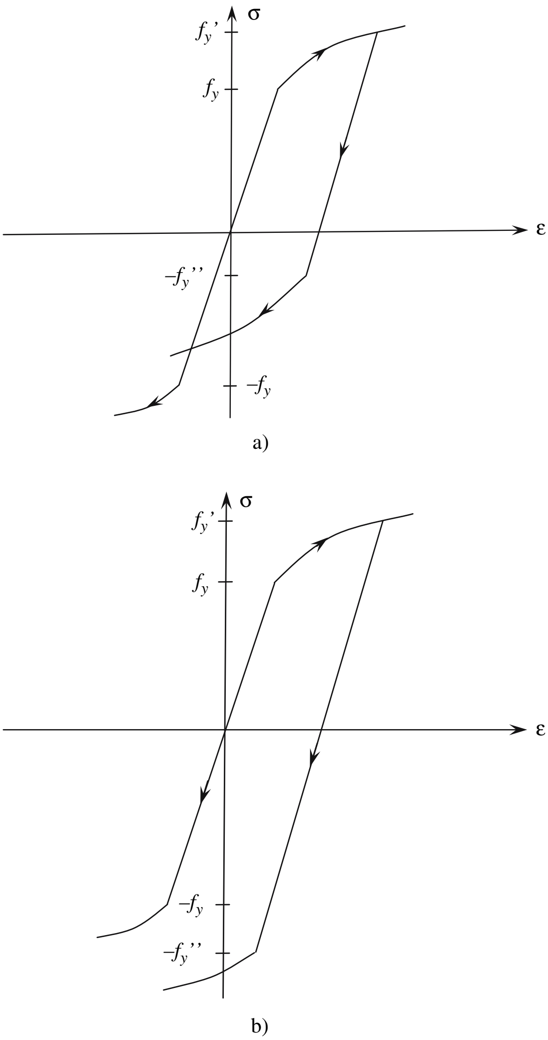


Fig. 10.1.3 Linearly elastic-plastic hardening material. **a)** Material showing the Bauschinger-effect $f_y'' < f_y$. **b)** Isotropic hardening $f_y'' = f_y'$

elastic-perfectly plastic materials shall be:

$$
\begin{aligned}
&f[\mathbf{T}] = 0 \Rightarrow \text{yielding may start}\\
&f[\mathbf{T}] < 0 \Rightarrow \text{elastic behavior}\\
&f[\mathbf{T}] > 0 \Rightarrow \text{not acceptable}
\end{aligned}
\qquad (10.2.1)
$$

The function $f[\mathbf{T}]$ is called the *yield function* for the material. For elastic-plastic hardening materials a yield function will also be influenced by the plastic deformation, and this will be reflected by introducing some hardening parameters in the yield function. The yield criteria for hardening materials will be discussed in Sect. 10.2.3.

In order to obtain useful yield functions, some fundamental hypotheses are usually formulated. These hypotheses are based on experience and experiments. For simplicity we shall use the same symbol f for the different yield functions defined below. The hypotheses are:

1. In uniaxial tension and compression the yield condition is:

$$|\sigma| = f_y \text{ at yielding} \tag{10.2.2}$$

2. The yield function $f[\mathbf{T}]$ is symmetric with respect to reversing of the stresses:

$$f[-\mathbf{T}] = f[\mathbf{T}] \tag{10.2.3}$$

Hypothesis 1 supports this statement.

3. Isotropic states of stress or superposition of isotropic stress on states of stress corresponding to elastic response, do not lead to yielding. The consequence of this is that we may assume that only the deviatoric part $\mathbf{T}'$ of the stress tensor $\mathbf{T}$ appears in the yield function $f[\mathbf{T}]$:

$$f[\mathbf{T}] = f[\mathbf{T}'] \tag{10.2.4}$$

4. The material is isotropic. The representation (4.3.36) of a tensor of second order shows that the stress tensor is uniquely given by the principal stresses σ_i and the principal directions $\mathbf{a}_i$. For an isotropic material the principal directions cannot be significant for when the material yields. Furthermore, the principal stresses are uniquely determined by the principal stress invariants I, II, and III, as shown by the characteristic equation of stress (3.3.3). Thus the hypothesis of material isotropy implies that:

$$f[\mathbf{T}] = f(\sigma_1, \sigma_2, \sigma_3) = f(I, II, III) \tag{10.2.5}$$

Using the terminology from Sect. 4.6.3 we may state that material isotropy implies that the yield function $f[\mathbf{T}]$ is an isotropic scalar-valued function of the stress tensor $\mathbf{T}$.

It is customary in the theory of plasticity to introduce the alternative deviatoric stress invariants J_2 and J_3, introduced in Sect. 3.3.2:

$$\begin{aligned} J_2 &\equiv -II' = \frac{1}{2}\left(\text{norm}\mathbf{T}'\right)^2 = \frac{1}{2}\mathbf{T}' : \mathbf{T}' = \frac{1}{2}T'_{ij}T'_{ij} = \frac{1}{2}\left[\mathbf{T} : \mathbf{T} - \frac{1}{3}(\text{tr}\mathbf{T})^2\right] \\ J_3 &= III' = \det \mathbf{T}' \end{aligned} \tag{10.2.6}$$

The first principal deviatoric stress invariant $I' = \text{tr}\mathbf{T}'$ is zero by the definition of $\mathbf{T}'$. The hypotheses 3 and 4 together therefore imply that:

$$f[\mathbf{T}] = f(J_2, J_3) \tag{10.2.7}$$

The yield condition in (10.2.1) using the yield function from (10.2.7) may be illustrated by a *yield surface in the principal stress space*, in which the principal stresses σ_i are coordinates in a Cartesian coordinate system, see Fig. 10.2.1. Points inside the yield surface correspond to purely elastic states. Yielding starts when the stress point is moved to the yield surface. Points outside the yield surface are not accessible. In case of hardening materials, yielding will move and/or extend the yield surface.

The yield surface must be a cylindrical surface, not necessarily with a circular cross-section, but with axis parallel to the unit vector:

$$\mathbf{e} = [1,1,1]\,\frac{1}{\sqrt{3}} \tag{10.2.8}$$

This fact may be understood from the following reasoning: If an isotropic state of stress is added to a stress state represented by a point inside the yield surface, the stress point is moved along a straight line parallel to the directional vector **e**. According to hypothesis 3 this line will never reach the yield surface.

The part of the cylindrical surface that represents the yield condition is bounded by a *fracture surface*, see Fig. 10.2.1. We expect that fracture may occur when a sufficiently large isotropic tensile stress is superimposed on a state of stress not alone leading to fracture. But superposition of a compressive isotropic stress is assumed not to lead to fracture. This explains the shape of the fracture surface in Fig. 10.2.1. For ductile materials the fracture surface intersects the yield surface. At lower temperatures the yield stress f_y increases while the tensile strength f_u does not change appreciably. The yield surface will then grow and include most of the fracture surface and for most states of stress the material behaves as a brittle material.

The further discussion of the cylindrical yield surface may be limited to a discussion of the form of its cross-sectional curve, which is called a *yield curve*. Figure 10.2.2 represents a plane through the origin in the principal stress space and normal to the directional vector **e**. This plane is called the π*-plane* and is represented by the equation:

$$\sigma_1 + \sigma_2 + \sigma_3 = 0 \tag{10.2.9}$$

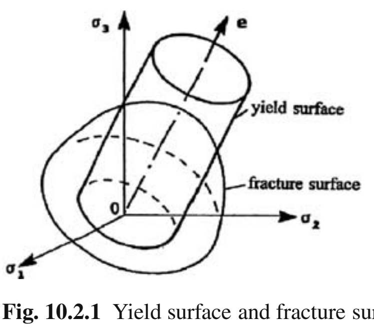


Fig. 10.2.1 Yield surface and fracture surface in the principal stress space

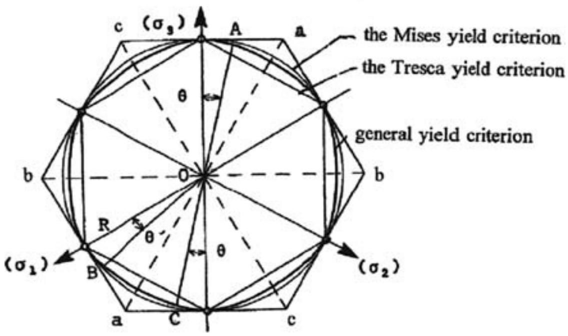


Fig. 10.2.2 Yield curves in the π-plane

The stress deviator $\mathbf{T}'$ of an arbitrary state of stress is represented by a point in this plane because:

$$\mathrm{tr}\mathbf{T}' = \sigma_1' + \sigma_2' + \sigma_3' = 0$$

Points on the yield curve in the π-plane indicate yielding.

The six points marked by rings on the yield curve, represent the stress deviators for uniaxial tension and compression. Because we have assumed that the yield stress is the same in tension and compression, hypothesis 1, the six points must lie on a circle. The radius R of this circle may be determined as follows. Uniaxial stress with $\sigma_1 = f_y$ has the stress deviator:

$$T' = T - T^o = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} f_y - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{f_y}{3} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{f_y}{3} \tag{10.2.10}$$

Then:

$$R = \sqrt{(\sigma_1')^2 + (\sigma_2')^2 + (\sigma_3')^2} = \sqrt{\left(\frac{2f_y}{3}\right)^2 + \left(-\frac{f_y}{3}\right)^2 + \left(-\frac{f_y}{3}\right)^2} = \sqrt{\frac{2}{3}}\, f_y \tag{10.2.11}$$

Due to material isotropy, hypothesis 4, the yield curve is symmetric with respect to the 3 axes marked (σ_i) in Fig. 10.2.2. Hypothesis 2 implies that the yield curve also is symmetric with respect to the axes marked with dashed lines in Fig. 10.2.2, and which are normal to the (σ_i)-axes. This may be explained from the following arguments.

Let the stress points A, B, and C in Fig. 10.2.2 represent yielding. The point A is chosen arbitrarily on the yield curve. Material isotropy implies that the points A and B are analogous states of stress. This means that the distance OB is equal to the distance OA. The stress point C represents yielding after a reversion of the stress state from the state corresponding to point A. According to hypothesis 2 the distance OC is equal to the distance OA. From the Fig. 10.2.2 it then follows that the stress points B and C lies symmetrically with respect to the dashed axis marked a-a. Since the starting point A is arbitrary, it follows that the axis a-a, and the other two dashed axes b-b and c-c, are symmetry axes for the yield curve.

Figure 10.2.2 shows a general yield curve that is symmetric with respect to the six axes. In the figure the curve is drawn convex outward. This latter property is implied by the *Drucker postulate*, which will be presented in Sect. 10.6. Due to the symmetry properties the yield curve consists of 12 parts that are identical or mirror images of each other. This implies that in principle only a 1/12-part of the yield curve in the π-plane must be determined experimentally.

Figure 10.2.2 indicates that the yield curve must be a convex curve between the two regular hexagons. Experiments with metals give stress points usually between the inner hexagon and the circle of radius R as given by (10.2.11). The circle and the inner hexagon provide yield curves for the two most important elastic-plastic material models: the *Mises material* and the *Tresca material*. The yield criteria of these two material models will now be discussed separately.

10.2.1 The Mises Yield Criterion

The yield surface that has the simplest mathematical representation and which agrees with the four hypotheses presented above, is a circular cylinder of radius R, and shown in Fig. 10.2.3. A stress point on the yield surface has the position vector $\mathbf{r} = [\sigma_1,\ \sigma_2,\ \sigma_3]$, which has to satisfy the equation:

$$\mathbf{r} \cdot \mathbf{r} - (\mathbf{e} \cdot \mathbf{r})^2 = R^2 \tag{10.2.12}$$

From this equation and formula (10.2.11) we derive, see Problem 10.1, the *Mises yield criterion*:

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 = f_y^2 \quad \Leftrightarrow \quad \text{yielding may start} \tag{10.2.13}$$

This yield criterion represents the yield condition of a *Mises material*. Richard von Mises [1883–1953] presented the criterion in 1913. For *plane stress*, $\sigma_3 = 0$, the yield criterion reduces to:

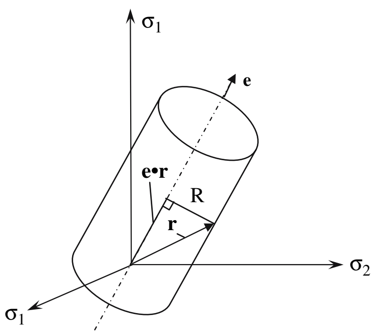


Fig. 10.2.3 The yield surface for the Mises criterion

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = f_y^2 \quad \Leftrightarrow \quad \text{yielding} \tag{10.2.14}$$

The yield surface intersects the $\sigma_1\sigma_2$-plane in an ellipse represented by the equation (10.2.14), see Fig. 10.2.4. It may be shown, see Problem 10.2, that the left side of the equation (10.2.13) can be transformed to:

$$\frac{3}{2}\mathbf{T}:\mathbf{T} - \frac{1}{2}(\text{tr}\mathbf{T})^2 = \frac{3}{2}\mathbf{T}':\mathbf{T}' = I^2 - 3II = -3II' = 3J_2 \tag{10.2.15}$$

Using the formulas (10.2.15) the yield criterion (10.2.13) can be given the presentation:

The Mises Yield Criterion. Yielding may start in a particle when the state of stress $\mathbf{T}$ in the particle satisfies the equation:

$$f[\mathbf{T}] = 0 \tag{10.2.16}$$

The yield function $f[\mathbf{T}]$ has these alternative forms:

$$f[\mathbf{T}] = \frac{3}{2}\mathbf{T}:\mathbf{T} - \frac{1}{2}(\text{tr}\mathbf{T})^2 - f_y^2 \equiv \frac{3}{2}\mathbf{T}':\mathbf{T}' - f_y^2 \tag{10.2.17}$$

$$f[\mathbf{T}] = I^2 - 3II - f_y^2 \equiv -3II' - f_y^2 \equiv 3J_2 - f_y^2 \tag{10.2.18}$$

$$f[\mathbf{T}] = T_{11}^2 + T_{22}^2 + T_{33}^2 - T_{11}T_{22} - T_{22}T_{33} - T_{33}T_{11} + 3T_{12}^2 + 3T_{23}^2 + 3T_{31}^2 - f_y^2 \tag{10.2.19}$$

For plane stress, $T_{i3} = 0$:

$$f[\mathbf{T}] = \sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2 - f_y^2 \tag{10.2.20}$$

The yield criterion has also been proposed by M.T. Huber in 1904 and H. Hencky in 1924, but then from a different motivation based on strain energy. The material is now assumed to be linearly and isotropic elastic until yielding occurs. The strain

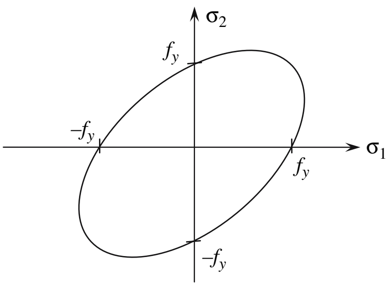


Fig. 10.2.4 The plane stress yield curve for the Mises criterion

energy ϕ before yielding is given in Sect. 7.2.2 as:

$$\phi = \frac{1}{2}\mathbf{T}:\mathbf{E} = \phi^o + \phi' = \frac{1}{18\kappa}(\mathrm{tr}\,\mathbf{T})^2 + \frac{1}{4\mu}\mathbf{T}':\mathbf{T}'$$

$$\phi^o = \frac{1}{2}\mathbf{T}^o:\mathbf{E}^o = \frac{1}{18\kappa}(\mathrm{tr}\,\mathbf{T})^2 \quad \text{volumetric strain energy per unit volume}$$

$$\phi' = \frac{1}{2}\mathbf{T}':\mathbf{E}' = \frac{1}{4\mu}\mathbf{T}':\mathbf{T}' = \frac{1}{2\mu}J_2 \text{ distortion energy per unit volume} \quad (10.2.21)$$

For uniaxial stress σ the stress deviator $\mathbf{T}'$ is given by the matrix T' expressed by (10.2.10), and the distortion energy is:

$$\phi' = \frac{1}{4\mu}\mathbf{T}':\mathbf{T}' = \frac{1}{4\mu}T'_{ij}T'_{ij} = \frac{\sigma^2}{6\mu} \quad (10.2.22)$$

Huber and Hencky based the yield criterion on the hypothesis that yielding occurs when the distortion energy reaches a limit ϕ'_y determined at uniaxial stress. Thus we have:

$$\phi' = \frac{1}{4\mu}\mathbf{T}':\mathbf{T}' = \frac{1}{4\mu}T'_{ij}T'_{ij} = \phi'_y,\ \phi'_y = \frac{f_y^2}{6\mu} \quad \Leftrightarrow \quad \text{yielding may start} \quad \Rightarrow$$

$$\frac{3}{2}\mathbf{T}':\mathbf{T}' - f_y^2 = 0 \quad \Leftrightarrow \quad \text{yielding may start} \quad (10.2.23)$$

This result is identical to the yield condition given by (10.2.16) and (10.2.17).

An alternative formulation of the Mises yield criterion is based on the concept of a *Mises stress* σ_M or *equivalent stress* σ_e:

$$\sigma_M \equiv \sigma_e = \sqrt{3J_2} = \sqrt{\frac{3}{2}\mathbf{T}':\mathbf{T}'} = \sqrt{\frac{3}{2}\mathbf{T}:\mathbf{T} - \frac{1}{2}(\mathrm{tr}\mathbf{T})^2} \quad (10.2.24)$$

A third name used in the literature for this stress is the *effective stress*. The yield criterion may now be formulated as:

$$\sigma_M \equiv \sigma_e = f_y \quad \Leftrightarrow \quad \text{yielding may start} \quad (10.2.25)$$

The Mises stress σ_M is sometimes replaced by *octahedral shear stress*:

$$\tau_{oct} = \sqrt{\frac{2}{3}J_2} = \frac{\sqrt{2}}{3}\sigma_M \quad (10.2.26)$$

which has the following interpretation. Let the principal stress directions and the principal stresses in the particle be $\mathbf{a}_i$ and σ_i. Then each of eight planes through the particle and defined by their normals:

$$\mathbf{n}_1 = \frac{1}{\sqrt{3}}(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3)\ ,\ \mathbf{n}_2 = \frac{1}{\sqrt{3}}(\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3)\ ,\ \mathbf{n}_3 = \frac{1}{\sqrt{3}}(\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3)$$
$$\mathbf{n}_4 = \frac{1}{\sqrt{3}}(-\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3)\ ,\ \mathbf{n}_5 = \frac{1}{\sqrt{3}}(\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3)\ ,\ \mathbf{n}_6 = \frac{1}{\sqrt{3}}(-\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3)$$
$$\mathbf{n}_7 = \frac{1}{\sqrt{3}}(-\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3),\ \mathbf{n}_8 = \frac{1}{\sqrt{3}}(-\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3) \tag{10.2.27}$$

are subjected to the stress vectors:

$$\mathbf{t}_\alpha = \mathbf{T} \cdot \mathbf{n}_\alpha \quad \alpha = 1,2,3,\ldots,8 \tag{10.2.28}$$

These eight planes are called the *octahedral planes* for the particle under the state of stress $\mathbf{T}$. Eight planes parallel to each of the octahedral planes may be used to form an octahedron about the particle. The normal stress is the same on all the octahedral planes and is called the *octahedral normal stress* σ_{oct}. Because $\mathbf{a}_i \cdot \mathbf{T} \cdot \mathbf{a}_i = \sigma_i$ and $\mathbf{a}_i \cdot \mathbf{T} \cdot \mathbf{a}_j = 0$ for $i \neq j$, it follows that:

$$\sigma_{\text{oct}} = \mathbf{n}_\alpha \cdot \mathbf{t}_\alpha = \mathbf{n}_\alpha \cdot \mathbf{T} \cdot \mathbf{n}_\alpha = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \sigma_m \tag{10.2.29}$$

σ_m is the *mean normal stress*. The shear stress is also the same on all the octahedral planes and is called the *octahedral shear stress*:

$$\tau_{\text{oct}} = \sqrt{\mathbf{t}_\alpha \cdot \mathbf{t}_\alpha - (\sigma_{\text{oct}})^2} = \sqrt{\frac{1}{3}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - \frac{1}{9}(\sigma_1 + \sigma_2 + \sigma_3)^2} \quad \Rightarrow$$
$$\tau_{\text{oct}} = \sqrt{\frac{1}{9}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1\right)} = \sqrt{\frac{2}{3}J_2} = \frac{\sqrt{2}}{3}\sigma_M \tag{10.2.30}$$

The Mises yield criterion may now alternatively be expressed as the *maximum octahedral shear stress criterion*:

$$\tau_{\text{oct}} = \frac{\sqrt{2}}{3} f_y \quad \Leftrightarrow \quad \text{yielding may start} \tag{10.2.31}$$

Example 10.1. Torsion of a Thin-Walled Pipe

A thin-walled pipe with radius r and wall thickness h is subjected to a torque M. We want to determine M when yielding starts. The material is assumed to behave as a Mises material.

The state of stress is that of *pure shear stress*. From Example 3.2 in Sect. 3.2.3 we get the stress matrix:

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau \\ 0 & \tau & 0 \end{pmatrix},\ \tau = \frac{M}{2\pi r^2 h}$$

The Mises yield criterion (10.2.16), with the expression (10.2.19) for the yield function, gives:

$$3T_{23}^2 - f_y^2 = 0 \quad \Rightarrow \quad \tau = \frac{1}{\sqrt{3}} f_y \quad \Rightarrow \quad M = \frac{2\pi r^2 h}{\sqrt{3}} f_y = 1.15\,\pi r^2 h f_y$$

The shear stress at yielding according to the Mises yield criterion in a state of pure shear is called the *Mises yield shear stress* τ_{yM}. It follows from this example that:

$$\tau_{yM} = \frac{f_y}{\sqrt{3}} \tag{10.2.32}$$

Example 10.2. Circular Cylindrical Container with Internal Pressure

A thin-walled cylindrical container with radius r and wall thickness h is subjected to internal pressure p. We want to determine the pressure at which yielding starts. The material is assumed to behave as a Mises material.

The state of stress is presented in Example 3.5 in Sect. 3.3.1. The stress matrix is:

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{pr}{2h}$$

The Mises yield criterion (10.2.16), with the expression (10.2.19) for the yield function, gives:

$$T_{22}^2 + T_{33}^2 - T_{22}T_{33} - f_y^2 = 0 \quad \Rightarrow \quad \left(2^2 + 1^2 - 2\cdot 1\right)\left(\frac{pr}{2h}\right)^2 - f_y^2 = 0 \quad \Rightarrow$$
$$p = \frac{2h}{\sqrt{3}\,r} f_y = 1.15\frac{h}{r} f_y$$

10.2.2 The Tresca Yield Criterion

The second yield criterion to be presented has a very direct physical description.

The Tresca Yield Criterion. Yielding starts in a particle when the maximum shear stress $\tau_{\max}$ reaches the *Tresca yield shear stress* τ_{yT} determined at uniaxial stress.

Another name of the criterion is *the maximum shear stress criterion*. The criterion represents the yield criterion for the *Tresca material model*. Henri Edouard Tresca presented the criterion in 1864. From Sect. 3.3, formula (3.3.32), we have that:

$$\tau_{\max} = \frac{1}{2}\left(\sigma_{\max} - \sigma_{\min}\right) \tag{10.2.33}$$

$\sigma_{\max}$ and $\sigma_{\min}$ are respectively the maximum and minimum principal stress in the particle. For uniaxial stress $\sigma, \sigma_{\max} = \sigma$ and $\sigma_{\min} = 0$. The yield shear stress is therefore determined from:

$$\tau_{yT} = \frac{f_y}{2} \tag{10.2.34}$$

The mathematical formulation of the Tresca yield criterion is:

$$\tau_{\max} = \tau_{yT} = \frac{f_y}{2} \quad \Leftrightarrow \quad \sigma_{\max} - \sigma_{\min} = f_y \quad \Leftrightarrow \quad \text{yielding may start} \tag{10.2.35}$$

The general form of the *Tresca yield function* may be defined as:

$$f\,[\mathbf{T}] = \sigma_{\max} - \sigma_{\min} - f_y \tag{10.2.36}$$

In the principal stress space, see Fig. 10.2.1, the criterion (10.2.35) is represented by a regular hexagonal cylindrical yield surface with a regular hexagonal yield curve in the π-plane, as shown in Fig. 10.2.5. The yield surface consists of the six planes:

$$\begin{aligned} f_1 &= \sigma_1 - \sigma_3 - f_y = 0, \; f_2 = \sigma_2 - \sigma_3 - f_y = 0 \\ f_3 &= \sigma_2 - \sigma_1 - f_y = 0, \; f_4 = \sigma_3 - \sigma_1 - f_y = 0 \\ f_5 &= \sigma_3 - \sigma_2 - f_y = 0, \; f_6 = \sigma_1 - \sigma_2 - f_y = 0 \end{aligned} \tag{10.2.37}$$

A comparison of (10.2.35) and (10.2.37) confirms that the hexagonal cylindrical surface really is the yield surface of the Tresca material. The intersection between the yield surface and the $\sigma_1\sigma_2$-plane in the principal stress space is shown in Fig. 10.2.5. This curve represents the *yield curve for plane stress*.

The equivalent stress for a Tresca material may be defined as:

$$\sigma_T \equiv \sigma_{eT} = 2\tau_{\max} = \sigma_{\max} - \sigma_{\min} \tag{10.2.38}$$

The Tresca yield criterion is then presented by the statement:

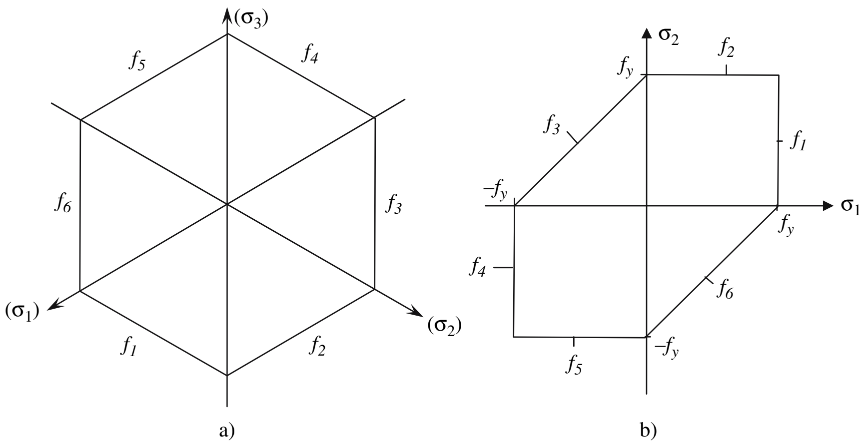


Fig. 10.2.5 Tresca yield curves: **a)** In the π-plane, **b)** Plane stress

$$\sigma_T \equiv \sigma_{eT} = f_y \quad \Leftrightarrow \quad \text{yielding may start} \tag{10.2.39}$$

Figure 10.2.6 shows the yield curves for plane stress in both the Mises material and the Tresca material. The figure will be used in the discussion in the following two examples. In applications the two material models are used interchangeably. As mentioned above the Mises criterion is usually better in agreement with experimental results.

Example 10.3. Torsion of a Thin-Walled Pipe

The Tresca material is used as a model for the pipe in Example 10.1. The Tresca yield criterion (10.2.35) gives:

$$\tau = \frac{M}{2\pi r^2 h} = \tau_{\max} = \frac{f_y}{2} \quad \Rightarrow \quad M = \pi r^2 h f_y$$

In Fig. 10.2.6 the points on the "loading curve" OP_1P_2 represent the principal stresses $\sigma_1 = \tau$ and $\sigma_2 = -\tau$ when the torque M is increased from 0 to when yielding starts. It is seen from the figure that the maximum deviation of results from the two criteria is precisely in the case of pure shear stress. The torque M at yielding is 15% greater according to the Mises criterion than the result given by the Tresca criterion. This number corresponds to the ratio between the yield stresses for the two material models:

$$\frac{\tau_{yM}}{\tau_{yT}} = \frac{f_y/\sqrt{3}}{f_y/2} = \frac{2}{\sqrt{3}} = 1.15 \tag{10.2.40}$$

Example 10.4. Circular Cylindrical Container with Internal Pressure

The container in Example 10.2 is now modelled as a Tresca material. The Tresca criterion (10.2.35) gives:

$$\sigma_{\max} - \sigma_{\min} = \left(\frac{p\,r}{h}\right) - 0 = f_y \quad \Rightarrow \quad p = \frac{h}{r} f_y$$

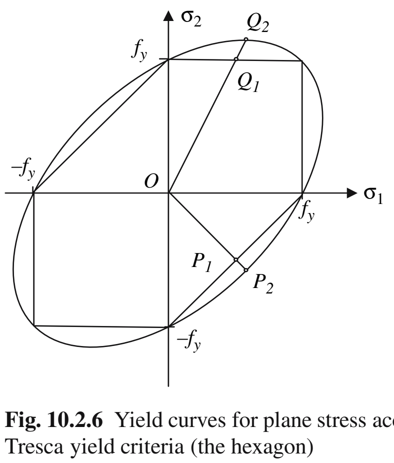


Fig. 10.2.6 Yield curves for plane stress according to the Mises yield criterion (the ellipse) and the Tresca yield criteria (the hexagon)

Figure 10.2.6 shows the "loading line" OQ_1Q_2 representing stress states when the pressure is increased from zero to its final yield value given above.

A comparison with the result from Example 10.2 shows that the Mises criterion gives a 15% higher value for the yield pressure, just like when comparing the results in Example 10.1 and Example 10.3. The reason for this may easily be understood by constructing the deviatoric stress matrix in the two problems.

$$\text{For the pipe in the Examples 10.1 and 10.3: } T' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tau, \ \tau = \frac{M}{2\pi r^2 h}$$

$$\text{For the container in the Examples 10.2 and 10.4: } T' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{pr}{2h}$$

The principal stresses for the deviatoric stress state for the pipe is clearly τ and $-\tau$. Therefore the deviatoric stress states in the pipe and in the container represent the same type of stress, namely pure shear. In the case of the container the state of stress may also be considered to be a superposition of a pure shear and an isotropic stress $\sigma = pr/2h$. The isotropic stress does not influence the yielding condition. The projections of the loading lines OP_1P_2 and OQ_1Q_2 onto the π-plane in Fig. 10.2.2 are represented by the axes marked b-b and c-c respectively. The symmetry of the yield curve in the π-plane explains why the differences in results from the two criteria are the same in per cent in the two cases and why the difference between results from the two yield criteria is at a maximum for the two cases.

Example 10.5. Thread Pulling

A tread is pulled through a rigid conical die with the purpose to reduce the diameter from $2r_o$ to $2r_1$, as shown in Fig. 10.2.7. The force between the thread and die is represented by a pressure $p(z)$ and a frictional shear stress μp, where μ is a coefficient of friction. We want to find an expression for the pulling force F. The material is assumed to be elastic-perfectly plastic, i.e. hardening is neglected, and to yield either according to the Tresca yield criterion or the Mises yield criterion.

We assume that the coefficient of friction μ and the aperture angle 2α are small quantities. This assumption makes it reasonable to treat σ_R, σ_θ, and σ_z as

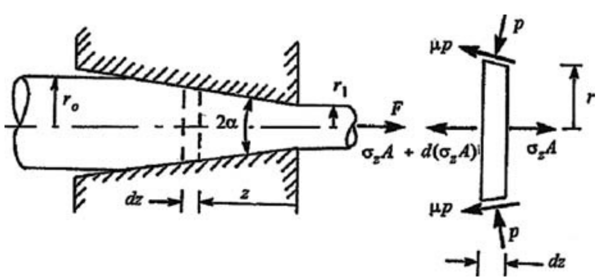


Fig. 10.2.7 Thread pulling

principal stresses, all three functions of the axial coordinate z. It then follows from Example 7.1 that: $\sigma_R(z) = \sigma_\theta(z) = -p(z)$ independent of the polar coordinates R and θ.

Because $\sigma_z(z) > 0$, it follows that $\sigma_{\max} = \sigma_z(z)$ and $\sigma_{\min} = -p(z)$. The Tresca yield criterion (10.2.35) then gives:

$$\sigma_{\max} - \sigma_{\min} = \sigma_z + p = f_y \quad \Rightarrow \quad p = f_y - \sigma_z \tag{10.2.41}$$

The Mises yield criterion (10.2.16, 19) gives:

$$\sigma_R^2 + \sigma_\theta^2 + \sigma_z^2 - \sigma_R\,\sigma_\theta - \sigma_\theta\,\sigma_z - \sigma_z\,\sigma_R = f_y^2 \quad \Rightarrow$$
$$(-p)^2 + (-p)^2 + \sigma_z^2 - (-p)(-p) - (-p)\,\sigma_z - \sigma_z\,(-p) = (p + \sigma_z)^2 = f_y^2 \quad \Rightarrow$$
$$p = f_y - \sigma_z \quad \Rightarrow \quad (10.2.41)$$

Thus the two yield criteria give the same result namely equation (10.2.41). Next we form the equilibrium equation in the z-direction of a thin slab of thickness dz, see Fig. 10.2.7:

$$\sigma_z A + d(\sigma_z A) - \sigma_z A + (p\sin\alpha + \mu\,p\cos\alpha)\left[\frac{dz}{\cos\alpha}2\pi r\right] = 0 \quad \Rightarrow$$
$$\frac{d}{dz}(\sigma_z A) + 2\pi r p\,(1+k)\tan\alpha = 0,\; k = \frac{\mu}{\tan\alpha} \tag{10.2.42}$$

The radius of the slab and the cross-sectional area of the slab are respectively:

$$r = r_1 + z\tan\alpha,\; A = \pi r^2 = \pi(r_1 + z\tan\alpha)^2$$

We observe that $dz = dr/\tan\alpha$. The expression for A is substituted into (10.2.42) and the result is:

$$\frac{d\sigma_z}{dr} + \frac{2\sigma_z}{r} + \frac{2p}{r}(1+k) = 0 \tag{10.2.43}$$

Eliminating the pressure p by the use of (10.2.41), we obtain:

$$\frac{d\sigma_z}{dr} - \frac{2k}{r}\sigma_z = -\frac{2f_y}{r}(1+k) \tag{10.2.44}$$

This equation is integrated, and with the boundary condition: $\sigma_z = 0$ for $r = r_o$, the result is:

$$\sigma_z(r) = f_y\left[1 + \frac{1}{k}\right]\left[1 - \left(\frac{r}{r_o}\right)^{2k}\right]$$

The pulling force is determined by setting:

$$F = \sigma_z(r_1)\cdot A_1,\; A_1 = \pi r_1^2$$

Thus the expression for the pulling force becomes:

$$F = f_y A_1 \left[1 + \frac{1}{k}\right] \left[1 - \left(\frac{A_1}{A_o}\right)^k\right], \; A_1 = \pi r_1^2, \; A_o = \pi r_o^2 \tag{10.2.45}$$

If the friction is neglected, i.e. is $\mu = 0$, then $k = 0$. The pulling force F may then be found from formula (10.2.45) by a limiting process. However, the simplest way of obtaining the formula for the pulling force in this case is to solve the differential equation (10.2.44) directly with $k = 0$. The result is:

$$F|_{\mu=0} = f_y A_1 \ln \left(\frac{A_o}{A_1}\right)$$

10.2.3 Yield Criteria for Hardening Materials

For hardening materials some *hardening parameters* $\kappa_1, \; \kappa_2, \; \ldots$ are included as parameters in the yield function, which is now denoted $f[\mathbf{T}, \kappa]$, where $\kappa = \{\kappa_1, \kappa_2, \ldots\}$. Two examples of hardening parameters are presented below. The yield criterion may be formulated as:

$$f[\mathbf{T}, \kappa] = f_1[\mathbf{T} - \mathbf{T}_o(\kappa)] - f_y(\kappa) \begin{cases} = 0 & \Leftrightarrow \quad \text{yielding may start} \\ < 0 & \Leftrightarrow \quad \text{elastic response} \end{cases} \tag{10.2.46}$$

$\mathbf{T}_o(\kappa)$ is a stress tensor only dependent upon the hardening parameters, and the scalar $f_y(\kappa)$ represents a varying, normally increasing, *yield stress*. The yield function $f[\mathbf{T}, \kappa]$ is still assumed to satisfy the four hypotheses represented by (10.2.2–10.2.5).

Plastic work per unit volume may be used as a hardening parameter:

$$\kappa = \int_{\mathbf{0}}^{\mathbf{E}^p} \mathbf{T} : d\mathbf{E}^p \tag{10.2.47}$$

The tensor $d\mathbf{E}^p$ represents the *plastic strain increment*, which as we shall see in Sect. 10.9 is dependent upon the present state of stress $\mathbf{T}$ and the stress increment $d\mathbf{T}$ during the yielding process.

Another example of a hardening parameter is *accumulated plastic strain*:

$$\kappa = \int_0^{\varepsilon_e^p} \sqrt{\frac{2}{3} d\mathbf{E}^p : d\mathbf{E}^p} \equiv \int_0^{\varepsilon_e^p} d\varepsilon_e^p = \varepsilon_e^p \tag{10.2.48}$$

The scalars $d\varepsilon_e^p$ and ε_e^p are called the *equivalent plastic strain increment* and the *equivalent plastic strain*. Small strains are implied here. For uniaxial stress σ_1 material isotropy and incompressibility of plastic strains imply that the principal plastic strain increments must satisfy the condition:

$$d\varepsilon_2^p = d\varepsilon_3^p = -\frac{1}{2}d\varepsilon_1^p$$

In this case the equivalent plastic strain increment becomes:

$$d\varepsilon_e^p = \sqrt{\frac{2}{3}d\mathbf{E}^p : d\mathbf{E}^p} = \sqrt{\frac{2}{3}\left[\left(d\varepsilon_1^p\right)^2 + \left(d\varepsilon_2^p\right)^2 + \left(d\varepsilon_3^p\right)^2\right]} = d\varepsilon_1^p$$

This result explains why the scalar $d\varepsilon_e^p$ is called the equivalent plastic strain increment.

For isotropic materials a yield function may be illustrated by a yield surface in the 3-dimensional principal stress space, as in Fig. 10.2.1. For anisotropic materials the yield function, being a function of 6 independent stresses, may mathematically be imagined illustrated in 6-dimensional stress space, with the 6 stresses T_{ij} as coordinates.

Three special cases of the yield criterion (10.2.46) may be relevant:

$$\begin{aligned} &\mathbf{T}_o(\kappa) = 0,\ \frac{\partial f_y(\kappa)}{\partial \kappa} = 0 \quad \Leftrightarrow \quad \text{perfect plasticity} \\ &\mathbf{T}_o(\kappa) = 0,\ \frac{\partial f_y(\kappa)}{\partial \kappa} > 0 \quad \Leftrightarrow \quad \text{isotropic hardening} \\ &\mathbf{T}_o(\kappa) \neq 0,\ \frac{\partial f_y(\kappa)}{\partial \kappa} = 0 \quad \Leftrightarrow \quad \text{kinematic hardening} \end{aligned} \qquad (10.2.49)$$

Isotropic hardening implies that the yield surface expands during yielding, and in a succession of stress increments the yield surface is represented by conforming and concentric surfaces in the stress space. The hypothesis of isotropic hardening provides acceptable results when the stresses increase proportionally, or for relatively large strains. An isotropic material with isotropic hardening may be considered to retain its isotropic properties when the material has been subjected to a load cycle. In the case of plane stress the yield curve in the $\sigma_1\sigma_2$-plane expands as a conform and concentric curve during yielding.

Kinematic hardening implies that the yield surface retains its form and size during a yielding process, but moves in the stress space. In the case of plane stress the yield curve in the $\sigma_1\sigma_2$-plane retains its form and size but moves in the direction of a vector with components proportional to the strain increments $d\varepsilon_1$ and $d\varepsilon_2$.

For the Mises material with isotropic hardening, presented in Sect. 10.9, the accumulated equivalent plastic strain ε_e^p is the hardening parameter and the monotonically increasing yield stress $f_y(\varepsilon_e^p)$ is defined by:

$$f_y\left(\varepsilon_e^p\right) = \left[\frac{\eta\varepsilon_e^p}{\alpha}\right]^{1/m} \qquad (10.2.50)$$

η is the modulus of elasticity, and α and σ_R are material parameters. The yield stress (10.2.50) is related to the non-linear part of the Ramberg-Osgood law, see Sect. 1.2:

$$\varepsilon = \frac{1}{\eta}\left[1 + \frac{\alpha}{\sigma_R}\left|\frac{\sigma}{\sigma_R}\right|^{m-1}\right]\sigma \tag{10.2.51}$$

The yield criterion for the isotropic hardening Mises material is:

$$f\left[\mathbf{T}, \varepsilon_e^p\right] \equiv \sigma_M^2 - f_y^2\left(\varepsilon_e^p\right) = 0 \tag{10.2.52}$$

σ_M is the Mises stress defined by formula (10.2.24).

10.3 Flow Rules

10.3.1 The General Flow Rule

A flow rule is the relationship between the plastic strains or strain increments and the stresses and stress increments. The strains in a particle in an elastic-plastic material depend on the stress history the particle has been subjected to. This was illustrated for uniaxial stress in Sect. 10.1. For perfectly plastic material flow rules may be formulated that provide relations between the components of *plastic strain tensor* $\mathbf{E}^p$ when the stress tensor $\mathbf{T}$ satisfies a yield criterion. If one plastic strain component is known, for instance from given displacement conditions, the other strain components may be determined from this relation, and perhaps some other constraints on the displacements. For a hardening material the flow rule provides a relation between the *plastic strain increments tensor* $d\mathbf{E}^p$, the stress tensor $\mathbf{T}$, and the *stress increment tensor* $d\mathbf{T}$. In general we chose to let a flow rule express the plastic strain increments expressed by the tensor $d\mathbf{E}^p$. For an elastic-perfectly plastic material we let $d\mathbf{E}^p$ represent the total plastic strain tensor.

A general flow rule of an isotropic, elastic-plastic material is based on the following hypotheses:

1. The strain increment tensor $d\mathbf{E}$ is a sum of an elastic part $d\mathbf{E}^e$ and a plastic part $d\mathbf{E}^p$:

$$d\mathbf{E} = d\mathbf{E}^e + d\mathbf{E}^p \tag{10.3.1}$$

   The plastic strain increment $d\mathbf{E}^p$ applies only when the stress tensor satisfies a yield criterion of the form, see equation (10.2.46) in Sect. 10.2.3:

$$f[\mathbf{T}, \kappa] = f_1\left[\mathbf{T} - \mathbf{T}_o(\kappa)\right] - f_y(\kappa) \begin{cases} = 0 & \Leftrightarrow \quad \text{yielding may start} \\ < 0 & \Leftrightarrow \quad \text{elastic response} \end{cases} \tag{10.3.2}$$

2. The plastic strain increment tensor $d\mathbf{E}^p$ and the stress tensor are coaxial, and is a function of the stress tensor $\mathbf{T}$, the hardening parameters κ, and the stress increment tensor $d\mathbf{T}$.
3. The stress tensor $\mathbf{T}$ may be derived from an elastic potential $\phi[\mathbf{E}^e]$, representing elastic strain energy per unit volume, and where $\mathbf{E}^e$ is accumulated elastic strain:

$$\mathbf{T} = \frac{\partial \phi}{\partial \mathbf{E}^e} \quad \Leftrightarrow \quad T_{ij} = \frac{\partial \phi}{\partial E^e_{ij}} \quad \Leftrightarrow \quad T_\alpha = \frac{\partial \phi}{\partial E^e_\alpha} \tag{10.3.3}$$

In the last version the notation from (7.8.1) and (7.8.2) is used. In differentiating with respect to the tensorial components E_{ik}, we must keep in mind that the coordinate strains E_{ik} must be regarded as 9 independent variables, confer the derivation of the relation (7.6.9).

4. Plastic deformation is isochoric, i.e. volume preserving, which implies that plastic strains are deviatoric:

$$dE^p_{11} + dE^p_{22} + dE^p_{33} = 0 \quad \Leftrightarrow \quad d\mathbf{E}^p = d\mathbf{E}'^p \tag{10.3.4}$$

This hypothesis originates from the conception that plastic deformation is related to sliding on atomic planes in the crystals of the material.

5. The material is stable according to the Drucker postulate, presented in Sect. 10.6. The postulate is not employed in the presentation of the two most applied flow rules, presented in the two following sections, but as will be demonstrated in Sect. 10.6, the consequences of the postulate is satisfied by both flow rules.

10.3.2 Elastic-Perfectly Plastic Tresca Material

Yielding is governed by the *Tresca yield criterion*. The flow rule is formulated as the statement:

> The Tresca Flow Rule. Yielding is an isochoric deformation by relative sliding on planes of maximum shear stress and in the direction of the shear stress.

Figure 10.3.1 shows two material elements surrounding the same particle in a yielding state. The first element shows the principal stresses $\sigma_1 = \sigma_{\max}$ and $\sigma_3 = \sigma_{\min}$, while the second element shows τ_{max}. The element with maximum shear stress is according to the Tresca flow rule deformed with a plastic shear strain increment $d\gamma^p$ with a direction given by the direction of τ_{max}. A Mohr diagram of strain increments, Fig. 10.3.1, shows that the principal strain increments are:

$$d\varepsilon^p_1 = -d\varepsilon^p_3 = \frac{d\gamma^p}{2}$$

The condition of incompressibility (10.3.4) gives in the present case $d\varepsilon^p_2 = 0$. Introducing the parameter $d\lambda = d\varepsilon^p_1$, we may characterize the plastic strain increments by:

$$d\varepsilon^p_1 = d\varepsilon^p_{\max} = d\lambda, \; d\varepsilon^p_3 = d\varepsilon^p_{\min} = -d\lambda, \; d\varepsilon^p_2 = 0 \tag{10.3.5}$$

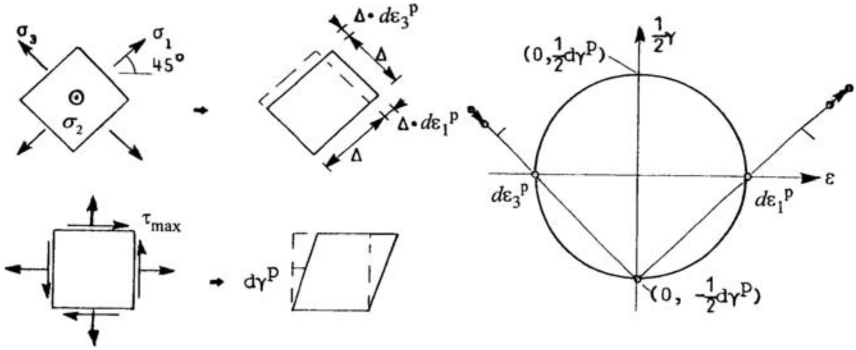


Fig. 10.3.1 Plastic strain increments in a Tresca material: Element with principal stresses and resulting plastic strain increments, element with maximum shear stress and resulting plastic shear strain increment, and Mohr-diagram for plastic strain increments

In the principal stress space we define the principal strain increment vector:

$$d\mathbf{e}_1^p = [1, 0, -1]\, d\lambda \tag{10.3.6}$$

using the principal strain increments $d\varepsilon_i^p$ as additional coordinates along the axes of the principal stresses σ_i. The yield surface for the case $\sigma_{\max} = \sigma_1$ and $\sigma_{\min} = \sigma_3$ is represented by the yield function:

$$f_1 = \sigma_1 - \sigma_3 - f_y = 0$$

A normal to this surface in the principal stress space is given by the gradient of f_1:

$$\nabla f_1 = [1, 0, -1] \tag{10.3.7}$$

This vector is parallel to the principal strain vector (10.3.6). Thus we may write:

$$d\mathbf{e}_1^p = d\lambda\, \nabla f_1$$

The result is generalized to:

$$d\mathbf{e}^p = d\lambda\, \nabla f \quad \Leftrightarrow \quad d\varepsilon_i^p = d\lambda\, \frac{\partial f}{\partial \sigma_i} \tag{10.3.8}$$

The vector $d\mathbf{e}^p$ contains the components:

$$\begin{aligned} d\varepsilon_{\max}^p &= d\lambda \quad \text{in the direction of } \sigma_{\max} \\ d\varepsilon_{\min}^p &= -d\lambda \quad \text{in the direction of } \sigma_{\min} \\ d\varepsilon_{\text{int}}^p &= 0 \quad \text{in the direction of } \sigma_{\text{int}} \end{aligned} \tag{10.3.9}$$

The unknown parameter $d\lambda$ has to be determined by some given displacement condition. This will be demonstrated in the example below. The result (10.3.8) may be stated thus:

The principal plastic strain incremental vector $d\mathbf{e}^p$ is normal to the yield surface: $f[\mathbf{T}] = 0$ in the principal stress space.

On the edges of the yield surface in the principal stress space, i.e. where two yield planes meet, the vector $d\mathbf{e}^p$ is undetermined. Figure 10.3.2 shows the situation for the plane stress case, $T_{i3} = 0$.

The result (10.3.8) may be further generalized to the flow rule:

$$d\mathbf{E}^p = d\lambda \frac{\partial f}{\partial \mathbf{T}} \quad \Leftrightarrow \quad dE_{ij}^p = d\lambda \frac{\partial f}{\partial T_{ij}} \quad \Leftrightarrow \quad dE_\alpha^p = d\lambda \frac{\partial f}{\partial T_\alpha} \tag{10.3.10}$$

In the last version of this flow rule we have used the symbols defined by the formulas (7.8.1) and (7.8.2). In order to see that the flow rule (10.3.10) follows from the result (10.3.8), we first show that the quantity $\partial f/\partial \mathbf{T}$ is a tensor that is coaxial with the stress tensor $\mathbf{T}$ and with principal values $\partial f/\partial \sigma_i$, where σ_i are the principal stresses. Let $\mathbf{n}_i = n_{ik}\mathbf{e}_k$ be the principal stress directions. Then:

$$\sigma_i = \mathbf{T}[\mathbf{n}_i, \mathbf{n}_i] = n_{ik} n_{il} \mathbf{T}[\mathbf{e}_k, \mathbf{e}_l] = n_{ik} n_{il} T_{kl}$$

and we obtain:

$$\frac{\partial f}{\partial T_{kl}} = \frac{\partial f}{\partial \sigma_i} \frac{\partial \sigma_i}{\partial T_{kl}} \quad \Rightarrow \quad \frac{\partial f}{\partial T_{kl}} = \sum_i \frac{\partial f}{\partial \sigma_i} n_{ik} n_{il}$$

By formula (4.3.36) this is the component form of the second order tensor:

$$\frac{\partial f}{\partial \mathbf{T}} = \sum_i \frac{\partial f}{\partial \sigma_i} \mathbf{n}_i \otimes \mathbf{n}_i \tag{10.3.11}$$

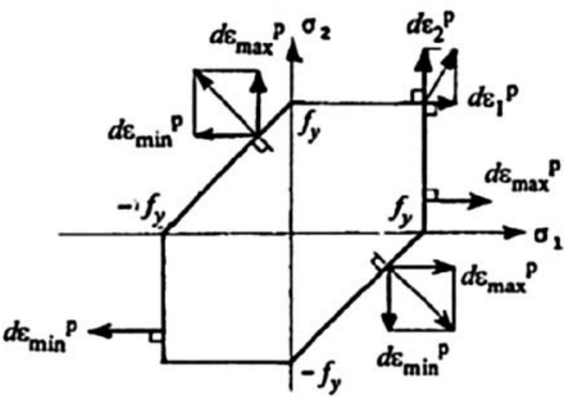


Fig. 10.3.2 Plastic strain increments in a Tresca material. Plane stress

Because the material is assumed to be isotropic, the plastic strain increment tensor $d\mathbf{E}^p$ is also coaxial with the stress tensor $\mathbf{T}$. From this fact and from the result (10.3.8) it then follows that:

$$d\mathbf{E}^p = \sum_i d\varepsilon_i^p \, \mathbf{n}_i \otimes \mathbf{n}_i = d\lambda \sum_i \frac{\partial f}{\partial \sigma_i} \mathbf{n}_i \otimes \mathbf{n}_i = d\lambda \, \frac{\partial f}{\partial \mathbf{T}}$$

which proves the flow rule (10.3.10).

For plane stress, $T_{i3} = 0$, the general form (10.3.10) of the flow rule may be specialized as follows. Here we need the formula for the principal stresses, which we rewrite slightly in order to simplify its use in connection with the flow rule (10.3.10):

$$\begin{matrix} \sigma_1 \\ \sigma_2 \end{matrix} = \frac{1}{2}\left[\sigma_x + \sigma_y \pm \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right] \tag{10.3.12}$$

We may distinguish between three cases.
Case 1. The principal stresses σ_1 and σ_2 have opposite signs:

$$\sigma_2 < 0 < \sigma_1 \quad \Rightarrow \quad \sigma_{\max} = \sigma_1 \text{ and } \sigma_{\min} = \sigma_2$$

The yield function (10.2.36) becomes:

$$f[\mathbf{T}] = \sigma_{\max} - \sigma_{\min} - f_y = \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} - f_y$$

The flow rule $(10.3.10)_3$ gives:

$$d\varepsilon_x^p = d\lambda \, \frac{\partial f}{\partial \sigma_x} = \frac{d\lambda}{\sigma_{\max} - \sigma_{\min}} (\sigma_x - \sigma_y) = -d\varepsilon_y^p, \; d\varepsilon_z^p = 0$$

$$d\gamma_{xy}^p = d\lambda \, \frac{\partial f}{\partial \tau_{xy}} = \frac{d\lambda}{\sigma_{\max} - \sigma_{\min}} 4\tau_{xy} \tag{10.3.13}$$

$$\tag{10.3.14}$$

Case 2. The principal stresses σ_1 and σ_2 are both positive:

$$0 < \sigma_2 < \sigma_1 \quad \Rightarrow \quad \sigma_{\max} = \sigma_1 \text{ and } \sigma_{\min} = 0$$

The yield function (10.2.36) becomes:

$$f[\mathbf{T}] = \sigma_{\max} - \sigma_{\min} - f_y = \frac{1}{2}\left[\sigma_x + \sigma_y + \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right] - f_y$$

The flow rule $(10.3.10)_3$ gives:

$$d\varepsilon_x^p = d\lambda\,\frac{\partial f}{\partial \sigma_x} = d\lambda\frac{1}{2}\left[1+\frac{\sigma_x-\sigma_y}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}\right],\ d\varepsilon_y^p = d\lambda - d\varepsilon_x^p$$

$$d\gamma_{xy}^p = d\lambda\,\frac{\partial f}{\partial \tau_{xy}} = d\lambda\,\frac{2\tau_{xy}}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}},\ d\varepsilon_z^p = -d\lambda \tag{10.3.15}$$

Case 3. The principal stresses σ_1 and σ_2 are both negative:

$$\sigma_2 < \sigma_1 < 0 \quad \Rightarrow \quad \sigma_{\max} = 0 \text{ and } \sigma_{\min} = \sigma_2$$

The yield function (10.2.36) becomes:

$$f[\mathbf{T}] = \sigma_{\max} - \sigma_{\min} - f_y = \frac{1}{2}\left[-\sigma_x - \sigma_y + \sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}\right] - f_y$$

The flow rule $(10.3.10)_3$ gives:

$$d\varepsilon_x^p = d\lambda\,\frac{\partial f}{\partial \sigma_x} = d\lambda\frac{1}{2}\left[-1+\frac{\sigma_x-\sigma_y}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}\right],\ d\varepsilon_y^p = -d\lambda - d\varepsilon_x^p$$

$$d\gamma_{xy}^p = d\lambda\,\frac{\partial f}{\partial \tau_{xy}} = d\lambda\,\frac{2\tau_{xy}}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}},\ d\varepsilon_z^p = d\lambda \tag{10.3.16}$$

Example 10.6. Thin-Walled Pipe

A thin-walled pipe with radius r, wall thickness h, and length L is subjected to a torque M and an axial force $N = M/r$. The torque is increased until yielding occurs and the pipe is given a permanent plastic torsion angle $\Delta\phi$. We want to determine the corresponding plastic elongation ΔL of the tube when the Tresca material is used as material model.

The state of stress is plane and given by the stress matrix, see Fig 10.3.3:

$$T = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{M}{2\pi r^2 h} \tag{10.3.17}$$

Since $\sigma_y = 0$, the principal stresses σ_1 and σ_2 have opposite signs. We therefore use the flow rule (10.3.13):

$$d\varepsilon_x^p = d\lambda\,\frac{\partial f}{\partial \sigma_x} = \frac{d\lambda}{\sigma_{\max}-\sigma_{\min}}\,(\sigma_x-\sigma_y) = \frac{d\lambda}{\sigma_{\max}-\sigma_{\min}}\,\frac{M}{2\pi r^2 h} = -d\varepsilon_y^p,\ d\varepsilon_z^p = 0$$
$$d\gamma_{xy}^p = d\lambda\,\frac{\partial f}{\partial \tau_{xy}} = \frac{d\lambda}{\sigma_{\max}-\sigma_{\min}}\,4\tau_{xy} = \frac{d\lambda}{\sigma_{\max}-\sigma_{\min}}\,4\frac{M}{2\pi r^2 h}$$
$$\Rightarrow \quad \frac{d\varepsilon_x^p}{d\gamma_{xy}^p} = \frac{1}{4}$$

By definition: $d\varepsilon_x^p = \Delta L/L$, and from Fig. 10.3.3 we geometrically derive the result:

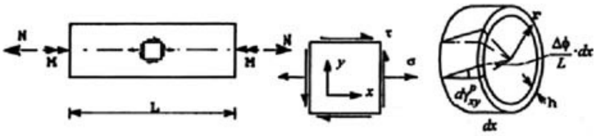


Fig. 10.3.3 Thin-walled tube under torsion and axial force

$$d\gamma_{xy}^{p} = \left(\frac{\Delta\phi}{L}\cdot dx\right)\cdot r\frac{1}{dx} = \frac{\Delta\phi}{L}\cdot r$$

Hence we obtain:

$$\frac{d\varepsilon_x^p}{d\gamma_{xy}^p} = \frac{\Delta L/L}{\Delta\phi\cdot r/L} = \frac{1}{4} \quad\Rightarrow\quad \Delta L = \frac{1}{4}\Delta\phi\cdot r$$

10.3.3 Elastic-Perfectly Plastic Mises Material

The elastic-perfectly plastic *Mises material* follows the Mises yield criterion (10.2.16, 10.2.17, 10.2.18, 10.2.19, 10.2.20). The flow rule is based on the assumption that the principal plastic strain increments $d\varepsilon^p{}_i$ represents a vector $d\mathbf{e}^p$ in the principal stress space that is normal to the yield surface, such that, as for the Tresca material:

$$d\mathbf{e}^p = d\lambda\,\nabla f \quad\Leftrightarrow\quad d\varepsilon_i^p = d\lambda\,\frac{\partial f}{\partial\sigma_i} \tag{10.3.18}$$

The parameter $d\lambda$ is unknown and has to be determined by some given displacement condition.

> The vector $d\mathbf{e}^p$ is normal to the yield surface: $f[\mathbf{T}] = 0$ in the principal stress space.

As for the Tresca material the flow rule (10.3.18) is generalized to the flow rule:

$$d\mathbf{E}^p = d\lambda\,\frac{\partial f}{\partial\mathbf{T}} \quad\Leftrightarrow\quad dE_{ij}^p = d\lambda\,\frac{\partial f}{\partial T_{ij}} \quad\Leftrightarrow\quad dE_{\alpha}^p = d\lambda\,\frac{\partial f}{\partial T_{\alpha}} \tag{10.3.19}$$

Using the expressions (10.2.17) for the Mises yield function, we obtain these specific and alternative forms for the flow rule for the perfectly plastic Mises material:

$$
\begin{aligned}
d\mathbf{E}^p &= d\lambda\,[3\mathbf{T} - (\mathrm{tr}\,\mathbf{T})\,\mathbf{1}] = 3\,d\lambda\,\mathbf{T}' \quad \Leftrightarrow \\
dE_{ij}^p &= d\lambda\,[3T_{ij} - T_{kk}\,\delta_{ij}] = 3\,d\lambda\,T_{ij}' \quad \Leftrightarrow \\
d\varepsilon_i^p &= d\lambda\,[3\,\sigma_i - (\sigma_1 + \sigma_2 + \sigma_3)] = 3\,d\lambda\,\sigma_i'
\end{aligned} \tag{10.3.20}
$$

Equations (10.3.20) express what is called the *Lévy Flow Rule*, proposed by M. Lévy (1871), which is alternatively written as:

$$\dot{E}_{ij}^p = 3\,\dot{\lambda}\,T_{ij}' \tag{10.3.21}$$

$\dot{\lambda}$ is an unknown scalar parameter. The Lévy flow rule was originally proposed for the total strain increments, neglecting any elastic strains.

The result of adding elastic strains to the plastic strains in equations (10.3.20) is called the *Prandtl-Reuss equations*, after L. Prandtl (1924) and A. Reuss (1930):

$$d\mathbf{E} = d\mathbf{E}^p + d\mathbf{E}^e \quad \Leftrightarrow \quad d\mathbf{E}' = 3\,d\lambda\,\mathbf{T}' + \frac{1}{2\mu}d\mathbf{T}',\ d\mathbf{E}^o = \frac{1}{3\kappa}d\mathbf{T}^o \tag{10.3.22}$$

Example 10.7. Thin-Walled Pipe

The tube in Example 10.6 is now modelled as a Mises-material. We want to determine the plastic elongation ΔL of the tube.

The stress state is given by the matrix (10.3.17). The flow rule (10.3.20) implies:

$$dE^p \equiv \begin{pmatrix} d\varepsilon_x^p & \frac{1}{2}d\gamma_{xy}^p & \frac{1}{2}d\gamma_{xz}^p \\ \frac{1}{2}d\gamma_{yx}^p & d\varepsilon_y^p & \frac{1}{2}d\gamma_{yz}^p \\ \frac{1}{2}d\gamma_{zx}^p & \frac{1}{2}d\gamma_{zy}^p & d\varepsilon_z^p \end{pmatrix} = d\lambda \begin{pmatrix} 2 & 3 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{M}{2\pi r^2 h} \quad \Rightarrow \quad \frac{d\varepsilon_x^p}{\frac{1}{2}d\gamma_{xy}^p} = \frac{2}{3}$$

In Example 10.6 we found geometrical expressions for the plastic strains:

$$d\varepsilon_x^p = \frac{\Delta L}{L},\ d\gamma_{xy}^p = \frac{\Delta\phi}{L}\cdot r \quad \Rightarrow \quad \frac{d\varepsilon_x^p}{d\gamma_{xy}^p/2} = \frac{2\Delta L/L}{\Delta\phi\cdot r/L}$$

Hence we obtain:

$$\frac{d\varepsilon_x^p}{d\gamma_{xy}^p/2} = \frac{2\Delta L/L}{\Delta\phi\cdot r/L} = \frac{2}{3} \quad \Rightarrow \quad \Delta L = \frac{1}{3}\Delta\phi\cdot r$$

10.4 Elastic-Plastic Analysis

In this section we shall work through three examples in which we combine results from the theory of elasticity in Chap. 7 with results based on the constitutive models: the linear elastic-perfectly plastic Tresca and Mises materials. For convenience we shall in this section use plastic strains rather than plastic strain increments.

10.4.1 Plane Stress Problems

Example 10.8. Circular Plate with a Hole. Plane Stress

A circular plate with radius b has a hole of radius a, Fig. 10.4.1. The outer edge of the plate is subjected to a pressure q. Under the assumption of plane stress we want to determine: a) the pressure q_o when yielding starts, b) the limit pressure q_l, that is the maximum pressure when perfect plasticity is assumed, and finally the radius c of the interface between a plastic zone and an elastic zone of the slab at partially developed yielding.

a) The Pressure q_o at Initial Yielding

The whole plate is now elastic and the stresses according to the theory of elasticity are given by the formulas (7.3.21, 7.3.22) in Example 7.1, from which we get:

$$\sigma_R(R) = -\frac{1-(a/R)^2}{1-(a/b)^2}q\ ,\ \sigma_\theta(R) = -\frac{1+(a/R)^2}{1-(a/b)^2}q,\ \sigma_z = 0 \qquad (10.4.1)$$

It follows from this solution that: $\sigma_\theta < \sigma_R \le 0$. Hence: $\sigma_{\max} = 0$ everywhere, and:

$$\sigma_{\min} = \sigma_\theta(a) = -\frac{2q}{1-(a/b)^2}$$

The *Tresca yield criterion* implies:

$$\sigma_{\max} - \sigma_{\min} - f_y = 0 \Rightarrow 0 - \left[-\frac{2q_o}{1-(a/b)^2}\right] - f_y = 0 \Rightarrow q_o = \frac{f_y}{2}\left[1-\left(\frac{a}{b}\right)^2\right] [6pt] \qquad (10.4.2)$$

The *Mises yield criterion* (10.2.16–20) implies:

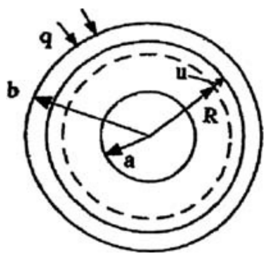


Fig. 10.4.1 Circular plate with hole

$$\sigma_R^2 + \sigma_\theta^2 - \sigma_R\sigma_\theta - f_y^2 = 0 \quad \Rightarrow$$
$$\left[\frac{q}{1-(a/b)^2}\right]^2 \left[\left\{-\left[1-(a/R)^2\right]\right\}^2 + \left\{1+(a/R)^2\right\}^2\right.$$
$$\left. - \left\{-\left[1-(a/R)^2\right]\right\}\left\{1+(a/R)^2\right\}\right] - f_y^2 = 0 \quad \Rightarrow$$
$$\left[\frac{q}{1-(a/b)^2}\right]^2 \left[1+3\left(\frac{a}{R}\right)^4\right] = f_y^2$$

Yielding starts at $R = a$, and we get:

$$q_o = \frac{f_y}{2}\left[1-\left(\frac{a}{b}\right)^2\right] \tag{10.4.3}$$

The two yield criteria give the same result.

b) The Limit Pressure q_l

When the pressure is increased from the value q_o, a plastic zone emanates from the inner edge at $R = a$ towards the outer edge at $R = b$. At the limit pressure q_l, the complete plate is plasticized. For a perfectly plastic material, i.e. for which the yield stress f_y is constant, the pressure cannot be any higher than q_l. For simplicity we shall only use the Tresca material in this case, as the Mises criterion gives a considerably more complicated development.

On the basis of the elastic solution given by (10.4.1) we assume that:

$$\sigma_\theta(R) < \sigma_R(R) \leq 0 = \sigma_z \tag{10.4.4}$$

This assumption will be verified below. The Tresca yield criterion implies:

$$\sigma_{\max} - \sigma_{\min} - f_y = 0 \quad \Rightarrow \quad 0 - \sigma_\theta = f_y \quad \Rightarrow \quad \sigma_\theta = -f_y \tag{10.4.5}$$

The Cauchy equation (3.2.39) in the R-direction, provides the equilibrium equation:

$$\frac{d\sigma_R}{dR} + \frac{\sigma_R - \sigma_\theta}{R} = 0 \quad \Leftrightarrow \quad \frac{d\left(R\sigma_R\right)}{dR} - \sigma_\theta = 0 \tag{10.4.6}$$

The result (10.4.5) is substituted into (10.4.6) and the equation is integrated:

$$\frac{d\left(R\sigma_R\right)}{dR} = -f_y \quad \Rightarrow \quad R\sigma_R = -f_y R + C \tag{10.4.7}$$

The constant of integration C is determined by the boundary condition at $R = a$:

$$\sigma_R(a) = 0 \quad \Rightarrow \quad C = f_y a$$

Thus we have the stresses:

$$\sigma_R(R) = -f_y\left[1 - \frac{a}{R}\right], \ \sigma_\theta = -f_y \tag{10.4.8}$$

The result shows that the assumption (10.4.4) holds. The limit pressure is found from the boundary condition at $R = b$:

$$\sigma_R(b) = -q_l \quad \Rightarrow \quad q_l = f_y\left[1 - \frac{a}{b}\right] \tag{10.4.9}$$

It is interesting to take a look at the plastic deformation of the plate. We assume that the elastic strains are much smaller than the plastic strains and can be neglected. Due to the inequalities (10.4.4) the flow rule for the Tresca material implies the principal plastic strains:

$$\varepsilon_\theta(R) = -\lambda(R), \ \varepsilon_z(R) = \lambda(R), \ \varepsilon_R = 0 \tag{10.4.10}$$

Note that the unknown parameter λ will be function of the R-coordinate. The strain-displacement relations in cylindrical coordinates (5.3.14) will in the present case reduce to:

$$\varepsilon_R = \frac{\partial u_R}{\partial R}, \ \varepsilon_\theta = \frac{u_R}{R}, \ \varepsilon_z = \frac{\partial u_z}{\partial z}, \tag{10.4.11}$$

From (10.4.10) and (10.4.11) we derive the results:

$$\frac{\partial u_R}{\partial R} = 0 \quad \Rightarrow \quad u_R = -u(= \text{ constant}), \ u_z(R,z) = u\frac{z}{R} \tag{10.4.12}$$

which gives the plastic strains:

$$\varepsilon_R = 0, \ \varepsilon_\theta(R) = -\frac{u}{R}, \ \varepsilon_z = \frac{u}{R} \tag{10.4.13}$$

Figure 10.4.2 shows the deformation of an axial section through the plate.

c) Partially Developed Yielding

In the case where the pressure q is larger than q_o at which yielding starts, and smaller than the limit pressure q_l, i.e. $q_o < q < q_l$, the material inside a cylindrical interface of radius $R = c$, is plasticized, while the rest of the plate is in an elastic state:

$$\text{Plastic region}: \ a \le R \le c \le b. \ \text{Elastic region}: \ a \le c \le R \le b.$$

We shall determine the radius c of the interface between the elastic and the plastic regions as a function of the pressure q. Again we will use the Tresca material as a model in the plastic region, and the stresses are given by the formulas (10.4.8). On the interface, $R = c$, the radial stress is an unknown pressure p obtained from:

$$p = -\sigma_R(c) = f_y\left[1 - \frac{a}{c}\right] \tag{10.4.14}$$

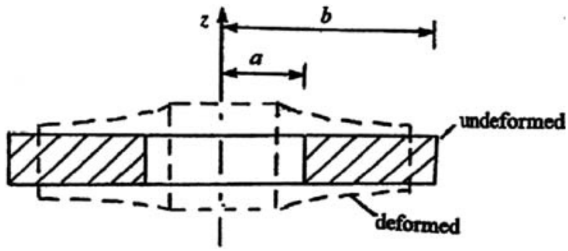


Fig. 10.4.2 Axial section through the undeformed and deformed plate in plane stress

The state of stress in the elastic region is governed by the formulas (7.3.21, 7.3.22) which give:

$$\sigma_R(R) = \frac{1}{1-(c/b)^2}\left\{-\left[\left(\frac{c}{R}\right)^2 - \left(\frac{c}{b}\right)^2\right]p - \left[1-\left(\frac{c}{R}\right)^2\right]q\right\}$$

$$\sigma_\theta(R) = \frac{1}{1-(c/b)^2}\left\{\left[\left(\frac{c}{R}\right)^2 + \left(\frac{c}{b}\right)^2\right]p - \left[1+\left(\frac{c}{R}\right)^2\right]q\right\} \quad (10.4.15)$$

(10.4.16)

On the interface, $R = c$, the material is in the state of initial yielding and the stress σ_θ from (10.4.15) for the elastic region is equal to $-f_y$ given by the solution for the plastic region. Using the expression (10.4.14) for the interface pressure, we obtain:

$$\sigma_\theta(c) = \frac{1}{1-(c/b)^2}\left\{\left[1+\left(\frac{c}{b}\right)^2\right]f_y\left[1-\frac{a}{c}\right] - 2q\right\} = -f_y$$

$$\Rightarrow \quad \frac{q}{f_y} = 1 - \frac{a}{2c}\left[1+\left(\frac{c}{b}\right)^2\right] \quad (10.4.17)$$

As a check we set $c = a$ and recover the pressure q_o in formula (10.4.2) for initial yielding, while $c = b$ provides the result (10.4.9) for the limit pressure q_l. When equation (10.4.17) is solved with respect to the interface radius c, we get:

$$\frac{c}{a} = \left[1-\frac{q}{f_y}\right]\left(\frac{b}{a}\right)^2 - \sqrt{\left[1-\frac{q}{f_y}\right]^2\left(\frac{b}{a}\right)^4 - \left(\frac{b}{a}\right)^2} \quad (10.4.18)$$

The stresses in the elastic region may now be expressed by:

$$\sigma_R(R) = -\left[1-\frac{a}{2c}-\frac{a}{2c}\left(\frac{c}{R}\right)^2\right]f_y, \; \sigma_\theta(R) = -\left[1-\frac{a}{2c}+\frac{a}{2c}\left(\frac{c}{R}\right)^2\right]f_y \quad (10.4.19)$$

10.4.2 Plane Strain Problems

Many problems in elastic-plastic analysis may be treated as plane strain problems, which will simplify the analysis considerably. With respect to Cartesian coordinates *xyz* let plane strain be defined by the conditions:

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0 \tag{10.4.20}$$

We assume that the elastic strains are small enough compared to the plastic strains that they may be neglected. Then the condition $\varepsilon_z = 0$ implies that both the plastic strain and the elastic strain in the *z*-direction are assumed to be zero:

$$\varepsilon_z^p = \varepsilon_z^e = 0 \tag{10.4.21}$$

The *z*-direction is a principal direction of stress and strain. The two other principal directions are in the *xy*-plane. From the condition (10.4.21) and the incompressibility condition for plastic strains:

$$\varepsilon_v^p = \varepsilon_{\min}^p + \varepsilon_{\text{int}}^p(\text{i.e. the intermediate principal strain}) + \varepsilon_{\max}^p = 0$$

it follows that:

$$\varepsilon_{\text{int}}^p = \varepsilon_z^p = 0 \text{ and } \varepsilon_{\max}^p \text{and } \varepsilon_{\min}^p \text{ are in the } xy - \text{plane} \tag{10.4.22}$$

This implies that the principal strains and the principal stresses in the *xy*-plane:

$$\varepsilon_1^p, \varepsilon_2^p \text{ and } \sigma_1, \sigma_2 \tag{10.4.23}$$

represent the extremal values of longitudinal strains and normal stresses.

The yield criterion for the Tresca material is then:

$$|\sigma_1 - \sigma_2| = f_y \tag{10.4.24}$$

The flow rule (10.3.20) for the Mises material implies:

$$\varepsilon_z^p = \lambda\,[3\sigma_z - (\sigma_1 + \sigma_2 + \sigma_z)] = 0 \quad \Rightarrow \quad \sigma_z = \frac{1}{2}(\sigma_1 + \sigma_2) \tag{10.4.25}$$

The yield condition (10.2.16, 19) for the Mises material then gives:

$$\sigma_1^2 + \sigma_2^2 + \left[\frac{1}{2}(\sigma_1 + \sigma_2)\right]^2 - \sigma_1\,\sigma_2 - \sigma_2\left[\frac{1}{2}(\sigma_1 + \sigma_2)\right] - \left[\frac{1}{2}(\sigma_1 + \sigma_2)\right]\sigma_1 - f_y^2 = 0$$

$$\Rightarrow |\sigma_1 - \sigma_2| = \frac{2}{\sqrt{3}} f_y \tag{10.4.26}$$

If we introduce the yield shear stress for the two material models:

$$\tau_y = \begin{cases} \tau_{yT} = f_y/2 \text{ Tresca material} \\ \tau_{yM} = f_y/\sqrt{3} \text{ Mises material} \end{cases} \tag{10.4.27}$$

we may combine the yield criteria for plane strain in the expression:

$$|\sigma_1 - \sigma_2| = 2\tau_y \quad \Leftrightarrow \quad \sigma_{\max} - \sigma_{\min} = 2\tau_y \quad \Leftrightarrow \quad \text{yielding may start} \tag{10.4.28}$$

Example 10.9. Circular Plate with a Hole. Plane Strain

This example is the same as the previous one, except that we now assume the condition (10.4.20) of plane strain, rather then $\boldsymbol{\sigma} = 0$.

a) The Pressure q_o at Initial Yielding

According to the theory of elasticity the stresses σ_R and σ_θ are given by the formulas (10.4.1). The stress σ_z follows from formula (7.3.38), which gives:

$$\sigma_z = \nu\left(\sigma_R + \sigma_\theta\right) = -\frac{2\nu}{1-(a/b)^2}\, q \tag{10.4.29}$$

The formulas (10.4.1) and (10.4.29) imply that:

$$\sigma_{\min} = \sigma_\theta < \sigma_z < \sigma_R = \sigma_{\max}$$

and the formulas (10.4.1) show that yielding starts at $R = a$, where:

$$\sigma_{\min} = \sigma_\theta(a) = -\frac{2}{1-(a/b)^2}\, q, \ \sigma_{\max} = \sigma_R(a) = 0$$

The *Tresca yield criterion* implies:

$$\sigma_{\max} - \sigma_{\min} = f_y \quad \Rightarrow \quad q_{o,T} = \frac{f_y}{2}\left[1 - \left(\frac{a}{b}\right)^2\right] \tag{10.4.30}$$

The *Mises criterion* (10.2.16, 19) implies, with stresses from the equations (10.4.1) and (10.4.29):

$$\sigma_R^2 + \sigma_\theta^2 + \sigma_z^2 - \sigma_R\sigma_\theta - \sigma_\theta\sigma_z - \sigma_z\sigma_R - f_y^2 = 0 \Leftrightarrow \quad \text{yielding may start} \quad \Rightarrow$$

$$q_{o,M} = \frac{f_y}{2}\left[1 - \left(\frac{a}{b}\right)^2\right] \frac{1}{\sqrt{1-\nu+\nu^2}} \tag{10.4.31}$$

A comparison of the results (10.4.30) and (10.4.31) obtained from the two different yield criteria shows:

$$q_{o,T} \leq q_{o,M}, \ \nu = 0 \ \Rightarrow \ q_{o,T} = q_{o,M}, \ \nu = 0.5 \ \Rightarrow \ q_{o,T} = \frac{\sqrt{3}}{2} q_{o,M}$$

b) The Limit Pressure q_l
The flow rule (10.3.9) for the *Tresca material* implies that:

$$\varepsilon_z = 0 = \varepsilon_{\text{int}} \quad \Rightarrow \quad \sigma_{\text{int}} = \sigma_z$$

which makes it reasonable to assume for the stresses that:

$$\sigma_{\min} = \sigma_\theta < \sigma_z < \sigma_R = \sigma_{\max} \tag{10.4.32}$$

The assumption is checked below. The yield criterion for both material models, given by (10.4.28), implies then that:

$$\sigma_R - \sigma_\theta = 2\tau_y \tag{10.4.33}$$

When this equation is substituted into the equation of equilibrium (10.4.6) in the R-direction, we get:

$$\frac{d\sigma_R}{dR} = -\frac{2\tau_y}{R} \tag{10.4.34}$$

Integration of this equation, followed by application of the boundary condition: $\sigma_R(a) = 0$, and by (10.4.33), gives:

$$\sigma_R(R) = -2\tau_y \ln\frac{R}{a}, \ \sigma_\theta(R) = \sigma_R - 2\tau_y = -2\tau_y \left[1 + \ln\frac{R}{a}\right] < \sigma_R \tag{10.4.35}$$

The result shows that the assumption (10.4.32) was correct. The limit pressure q_l is found from the boundary condition at $R = b$:

$$\sigma_R(b) = -q_l \quad \Rightarrow \quad q_l = 2\tau_y \ln\frac{b}{a} \tag{10.4.36}$$

Plastic deformation.
For both material models:

$$\varepsilon_z^p = 0 \quad \Rightarrow \quad \varepsilon_\theta^p = -\varepsilon_R^p \tag{10.4.37}$$

Using this result together with the formulas (10.4.11) for the strain-displacement relations, we obtain:

$$\frac{u_R}{R} = -\frac{\partial u_R}{\partial R} \quad \Rightarrow \quad \frac{1}{R}\frac{\partial (Ru_R)}{\partial R} = 0 \tag{10.4.38}$$

Integration provides us with the radial displacement $u_R(R)$. We introduce as an unknown parameter the radial displacement u at $R = b$, i.e. $u_R(b) = -u$, and obtain the displacement fields and strain fields as:

$$u_R(R) = u_R(b)\frac{b}{R} \equiv -u\frac{b}{R}, \ u_z = u_\theta = 0, \ \varepsilon_R^p = -\varepsilon_\theta^p = u\frac{b}{R^2}, \ \varepsilon_z^p = 0 \tag{10.4.39}$$

10.4.3 General Two-Dimensional Problem

Example 10.10. Circular Cylinder with Internal Pressure

A circular cylinder with internal radius a and external radius b is subjected to an internal pressure p. We want to determine: a) the pressure p_o when yielding starts, and b) the state of stress when yielding has developed such that the cylinder has a plastic region and an elastic region separated by an interface at radius $R = c$. The material is modelled as a linear, isotropic elastic-perfectly plastic Tresca or Mises material. The state of stress in an elastic cylinder will be obtained from Example 7.1 and Example 7.3. Three alternative conditions will be assumed at the ends of the cylinder:

Alt. 1: The end surfaces are not allowed to move in the axial direction, resulting in plane strain: $\varepsilon_z = 0$.

Alt. 2: The cylinder is open in both ends, which implies that the axial force over a cylindrical cross-section is zero. If the cylinder is in an elastic state, then according to the solution in Example 7.3, the state of stress is plane: $\sigma_z = 0$.

Alt. 3: The cylinder is closed with rigid end plates.

a) The Pressure p_o at Initial Yielding

The cylinder is in the elastic state and (7.3.21–7.3.22) provide the stresses. The stress formulas are also given by the formulas (10.4.15) by setting $q = 0$ and replacing c by a:

$$\sigma_R(R) = -\frac{1}{1-(a/b)^2}\left[\left(\frac{a}{R}\right)^2 - \left(\frac{a}{b}\right)^2\right]p$$

$$\sigma_\theta(R) = \frac{1}{1-(a/b)^2}\left[\left(\frac{a}{R}\right)^2 + \left(\frac{a}{b}\right)^2\right]p \qquad (10.4.40)$$

In all three alternative conditions at the ends of the cylinder the axial strain ε_z is constant and given by Hooke's law (7.2.7), from which we obtain, using the stresses from the formulas (10.4.40):

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_R + \sigma_\theta\right)\right] \quad \Rightarrow \quad \varepsilon_z = \frac{1}{E}\left[\sigma_z - \frac{2\nu\,(a/b)^2}{1-(a/b)^2}\,p\right] \qquad (10.4.41)$$

Because the strain ε_z for all three alternative end conditions is constant over the cross-section of the cylinder, the equation above shows that the stress σ_z also is constant. The axial force over a cross-section of the cylinder is therefore:

$$F = \pi\left(b^2 - a^2\right)\sigma_z \qquad (10.4.42)$$

For a cylinder closed in both ends by rigid plates the axial force F must be balanced by the pressure p on the end surface πa^2. The stress σ_z in alternative 3 is thus:

$$\sigma_z = \frac{p \cdot \pi a^2}{\pi (b^2 - a^2)} = \frac{(a/b)^2}{1-(a/b)^2} p \tag{10.4.43}$$

We shall now introduce formulas for the axial strain and axial stress that apply for all three alternative end conditions and set:

$$\varepsilon_z = \left(\frac{\alpha}{E}\right) \frac{(a/b)^2}{1-(a/b)^2} p, \; \sigma_z = (2\nu + \alpha) \frac{(a/b)^2}{1-(a/b)^2} p \tag{10.4.44}$$

The parameter α is given as follows for the three alternative end conditions:

Alt. 1: The end surfaces are not allowed to move in the axial direction, resulting in *plane strain*: $\varepsilon_z = 0$. Then (10.4.40 and 43) imply: $\alpha = 0$.
Alt. 2: The cylinder is open in both ends, resulting in *plane stress*: $\sigma_z = 0$. Then (10.4.40 and 43) imply: $\alpha = -2\nu$.
Alt. 3: The cylinder is closed with rigid end plates. Then (10.4.42 and 43) imply: $\alpha = 1 - 2\nu$.

The *Tresca yield criterion*. Under the conditions for Poisson's ratio ν that: $0 \leq \nu \leq 0.5$, it follows from the expressions (10.4.40) and (10.4.44) for the principal stresses that they always satisfy the conditions:

$$\sigma_R \leq 0 \leq \sigma_z \leq \sigma_\theta \tag{10.4.45}$$

The Tresca yield criterion (10.2.35) then implies:

$$\sigma_{\max} - \sigma_{\min} = f_y \quad \Rightarrow \quad \sigma_\theta - \sigma_R = f_y \quad \Rightarrow \quad \frac{2}{1-(a/b)^2}\left(\frac{a}{R}\right)^2 p = f_y$$

from which we see that yielding starts at $R = a$, and at the pressure:

$$p_{oT} = \frac{f_y}{2}\left[1 - \left(\frac{a}{b}\right)^2\right] \tag{10.4.46}$$

The *Mises yield criterion* (10.2.16 and 19) gives:

$$\sigma_R^2 + \sigma_\theta^2 + \sigma_z^2 - \sigma_R\sigma_\theta - \sigma_\theta\sigma_z - \sigma_z\sigma_R - f_y^2 = 0 \quad \Leftrightarrow \quad \text{yielding may start} \quad \Rightarrow$$

yielding starts at $R = a$ and when the pressure is:

$$p_{oM} = \frac{f_y}{2}\left[1 - \left(\frac{a}{b}\right)^2\right] \frac{2}{\sqrt{3 + (1 - 2\nu - \alpha)^2 \left(\frac{a}{b}\right)^4}} \tag{10.4.47}$$

b) Partially Developed Yielding for $p > p_o$
We assume that a plastic region has developed from the internal cylinder wall at $R = a$, where yielding starts, to a cylindrical interface at $R = c$. The material in the rest of the cylinder is still in an elastic state. Thus:

Plastic region: a $\leq$ R $\leq$ c. Elastic region: c $\leq$ R $\leq$ b

The radius c is unknown, but as we shall see below, c may be found when the pressure p is given.

In the elastic region the stresses are given by the solution for an elastic cylinder with internal radius c. The stresses in the elastic region are therefore given by the formulas (10.4.40) and (10.4.44) when the radius a is replaced by the unknown radius c:

$$\sigma_R(R) = -\frac{1}{1-(c/b)^2}\left[\left(\frac{c}{R}\right)^2 - \left(\frac{c}{b}\right)^2\right] p_c$$

$$\sigma_\theta(R) = \frac{1}{1-(c/b)^2}\left[\left(\frac{c}{R}\right)^2 + \left(\frac{c}{b}\right)^2\right] p_c \qquad c \leq R \leq b$$

$$\sigma_z = (2\nu+\alpha)\frac{(c/b)^2}{1-(c/b)^2} p_c, \; p_c = -\sigma_R(c) \tag{10.4.48}$$

In the plastic region we have to choose a material model. The Tresca material will be favoured because the Mises material results in far mores complex analysis in this case. We assume that the relations between the principal stresses are as given by the inequalities (10.4.45). This assumption will be discussed after we have found the complete state of stress. The Tresca criterion (10.2.35) implies:

$$\sigma_\theta - \sigma_R = f_y \quad \Leftrightarrow \quad \text{yielding may start} \tag{10.4.49}$$

Substituting the stresses from the (10.4.48) at the interface, $R = c$, between the elastic and plastic regions, into the yielding condition (10.4.49), we obtain:

$$p_c = \frac{f_y}{2}\left[1 - \left(\frac{c}{b}\right)^2\right] \tag{10.4.50}$$

The inequalities (10.4.45) indicates that σ_z is the intermediate principal stress and the flow rule for the Tresca material implies then that $\varepsilon_z^p = 0$, which means that the strain in the axial direction ε_z is purely elastic as determined by (10.4.41) from Hooke's law. The axial stress σ_z over the cross-section of the cylinder, both in the elastic and the plastic regions, may therefore be computed from the formula:

$$\sigma_z = E\,\varepsilon_z + \nu\,(\sigma_R + \sigma_\theta) \tag{10.4.51}$$

In the plastic region the yield criterion (10.4.49) is substituted into the equilibrium equation (10.4.6) in the radial direction. The result is:

$$\frac{d\sigma_R}{dR} = \frac{f_y}{R}, \; a \leq R \leq c$$

Integration followed by use of the boundary condition: $\sigma_R(a) = -p$, results in:

$$\sigma_R(R) = -p + f_y \ln \frac{R}{a}, \; a \leq R \leq c \tag{10.4.52}$$

which by the condition: $\sigma_R(c) = -p_c$ at the interface between the elastic region and the plastic region, implies that:

$$p = \frac{f_y}{2}\left[1 - \left(\frac{c}{b}\right)^2 + 2\ln\left(\frac{c}{a}\right)\right] \tag{10.4.53}$$

This equation may be used to determine the radius c of the interface for any given pressure p:

$$p_o < p < p_l \tag{10.4.54}$$

p_l is the limit pressure corresponding to fully plasticized cylinder. The limit pressure p_l is determined by setting $c = b$ in formula (10.4.53). Hence:

$$p_l = f_y \ln\left(\frac{b}{a}\right) = 2\tau_{yT} \ln\left(\frac{b}{a}\right) \tag{10.4.55}$$

This result is also correct for Mises material if the yield shear stress $\tau_{yT} = f_y/2$ is replaced by $\tau_{yM} = 2f_y/\sqrt{3}$. See Problem 10.9.

In order to find an expression for the stress σ_z we first calculate the axial force F on a cross-section of the cylinder. Equation (10.4.51) is used to express the axial stress σ_z.

$$F = \int_a^b \sigma_z \, 2\pi R \, dR = \int_a^b \left[E\varepsilon_z + \nu\left(\sigma_R + \sigma_\theta\right)\right] 2\pi R \, dR$$

Before the integration can be performed the equilibrium equation (10.4.6) is used to obtain the result:

$$(\sigma_R + \sigma_\theta)R = [2\sigma_R - (\sigma_R - \sigma_\theta)]R = 2\sigma_R R + R^2 \frac{d\sigma_R}{dR} = \frac{d}{dR}\left[R^2 \sigma_R\right]$$

Then, with the boundary conditions: $\sigma_R(a) = -p$ and $\sigma_R(b) = 0$, we get:

$$F = \int_a^b \left[E\varepsilon_z + \nu\left(\sigma_R + \sigma_\theta\right)\right] 2\pi R \, dR = 2\pi \int_a^b \left[E\varepsilon_z R + \nu \frac{d}{dR}\left(R^2 \sigma_R\right)\right] dR \quad \Rightarrow$$

$$F = E\varepsilon_z \pi \left(b^2 - a^2\right) + 2\nu \pi a^2 p \tag{10.4.56}$$

For the three alternative conditions at the ends of the cylinder we obtain:

Alt. 1: The end surfaces are not allowed to move in the axial direction:

$$\text{Plane strain: } \varepsilon_z = 0,\ \alpha = 0 \quad \Rightarrow \quad F = 2\nu\pi a^2 p$$

Alt. 2: The cylinder is open in both ends:

$$\text{Plane stress: } \sigma_z = 0,\ \alpha = -2\nu \quad \Rightarrow \quad F = 0$$

Alt. 3: The cylinder is closed with rigid end plates and the force F must balance the internal pressure:

$$F = \pi a^2 p,\ \alpha = 1 - 2\nu$$

A general expression for the axial force F, covering all three alternative end conditions is:

$$F = (2\nu + \alpha)\,\pi a^2 p \tag{10.4.57}$$

From the formulas (10.4.55, 56, and 50) we obtain the results:

$$\varepsilon_z = \left(\frac{\alpha}{E}\right)\frac{(a/b)^2}{1-(a/b)^2}p,\ a \le R \le b$$
$$\sigma_z = \frac{\alpha\,(a/b)^2}{1-(a/b)^2}p + \nu\,(\sigma_R + \sigma_\theta),\ a \le R \le b \tag{10.4.58}$$

The complete state of stress in the cylinder may now be presented:
Elastic region: $c \le R \le b$.

$$\sigma_R(R) = -\frac{f_y}{2}\left[\left(\frac{c}{R}\right)^2 - \left(\frac{c}{b}\right)^2\right],\ \sigma_\theta(R) = \frac{f_y}{2}\left[\left(\frac{c}{R}\right)^2 + \left(\frac{c}{b}\right)^2\right]$$
$$\sigma_z = \frac{\alpha\,(a/b)^2}{1-(a/b)^2}p + \nu\left(\frac{c}{b}\right)^2 f_y \qquad c \le R \le b \tag{10.4.59}$$

Plastic region: $a \le R \le c$.

$$\sigma_R(R) = -\frac{f_y}{2}\left[1 - \left(\frac{c}{b}\right)^2 + 2\ln\left(\frac{c}{R}\right)\right]$$
$$\sigma_\theta(R) = \frac{f_y}{2}\left[1 + \left(\frac{c}{b}\right)^2 - 2\ln\left(\frac{c}{R}\right)\right] \qquad a \le R \le c$$
$$\sigma_z(R) = \frac{\alpha\,(a/b)^2}{1-(a/b)^2}p + \nu\left[\left(\frac{c}{b}\right)^2 - \ln\left(\frac{c}{R}\right)\right]f_y \tag{10.4.60}$$

From (10.4.59) and (10.4.60) we see that the solutions in the elastic region and the plastic region match at the interface $R = c$.
It may be shown, by fairly complex argumentation, that the assumption (10.4.45) is fulfilled if, for $\nu = 0.3$, the following condition for the ratio b/a is

satisfied under the alternative conditions at the ends of the cylinder:

$$\text{Alt. 1:}b/a < 5.75,\ \text{Alt. 2:}b/a < 6.19,\ \text{Alt. 3:}b/a < 5.43$$

When the ratio b/a exceeds these values, the assumption of small plastic strains now longer is sufficient. A detailed discussion of this problem is presented in the book: Chakrabarty: Theory of Plasticity, [6].

10.5 Limit Load Analysis for Plane Beams and Frames

10.5.1 Introduction

Section 10.4 presents examples of elastic-plastic analysis of a few special problems. A complete elastic-plastic analysis of stresses and strains is in most cases a very complex task, and will require the use of numerical methods and computer programs. In the present section, and in the Sect. 10.7 and 10.8, we shall see how it is possible by simple means to estimate a *limit load* for a structure or a body when it is reasonable to use an elastic-perfectly plastic material model. It is assumed that the actual load on the body is increased until the material in sufficiently large parts of the body has reached a plastic state, such that the body may be treated as a *collapse mechanism*. The load is now called the *limit load*. Any further increase of the load beyond the limit load results in large, theoretically speaking uncontrollable, deformations, and the situation is characterized as collapse or failure. When hardening of the material is taken into account, the load may be increased beyond the limit load, but the deformations will in general be too large to be acceptable.

For plane beams and frames this limit load analysis becomes fairly simple, and we shall in the present section show how the limit load may be found or at least estimated for such systems.

10.5.2 Plastic Collapse

For beam and frame systems yielding usually occurs first over cross sections with high values of the bending moment. Influence of an axial force N on a cross-section may be of importance in some cases. However, it is often reasonable to neglect the influence of a shear force V in limit load analysis.

In the case of *pure bending*, i.e. $N = V = 0$, the bending moment capacity over a beam cross-section is given by the *plastic moment* M_p, representing the yield stress distribution shown in Fig. 10.5.1c for the plane beam in Fig. 10.5.1a. Figure 10.5.1b shows the bending stress distribution when the cross-section is in the elastic state, i.e. the bending stress $|\sigma| \leq f_y$. The plastic moment is presented as:

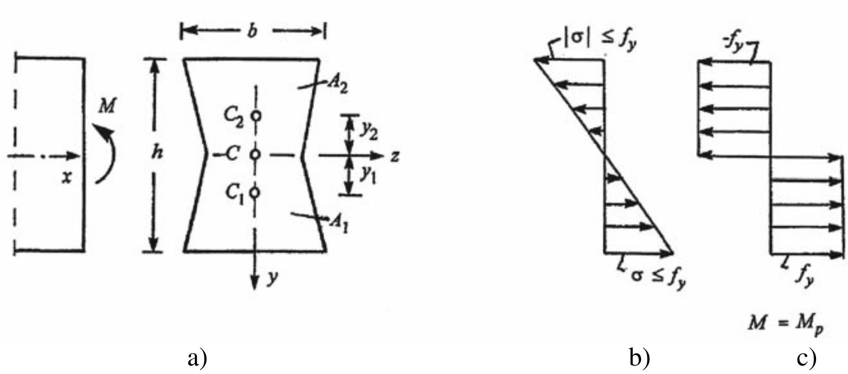


Fig. 10.5.1 Beam in pure bending. **a)** Beam geometry and cross-section. **b)** Stress distribution for elastic material. **c)** Stress distribution for fully developed plastic zone

$$M_p = W_p f_y \tag{10.5.1}$$

where W_p is the *plastic section modulus*. Since the axial force is zero, the area A_1 of the cross-section in tension is equal to the area A_2 of the cross-section in compression, both equal to half of the total area A of the cross-section: $A_1 = A_2 = A/2$. Let C_1 and C_2 be the center of area of the tensile part and the compressive part of the cross-section. Referring to Fig. 10.5.1 and c we then obtain for the plastic section modulus:

$$W_p = \frac{A}{2}(y_1 + y_2) \tag{10.5.2}$$

For a rectangular cross section of height h and width b: $A = bh, y_1 = y_2 = h/4$, and the sectional modulus becomes: $W_p = bh^2/4$.

The presence of an axial force N on a cross-section will reduce the plastic moment. Figure 10.5.2 shows the cross-section of an I-beam subjected to a reduced plastic moment $M_{p,r}$ and to an axial force N. The stress distribution over the cross-section may be split into two contributions: one corresponding to the bending moment and one representing the axial force. If the plastic moment for the cross-section in pure bending is M_p, it follows that the *reduced plastic moment* will be:

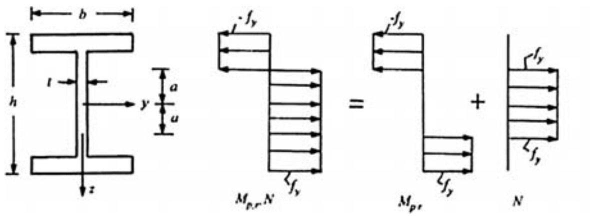


Fig. 10.5.2 Plastic cross-section with normal stress distribution due to a reduced plastic moment $M_{p,r}$ and an axial force N

$$M_{p,r} = M_p - \frac{ta^2}{4} f_y = W_{p,r} f_y, \; a = \frac{N}{2t f_y}, \; W_{p,r} = W_p - \frac{ta^2}{4} \qquad (10.5.3)$$

$W_{p,r}$ is called the reduced plastic section modulus. For simplicity we shall in the remainder of Sect. 10.5 not distinguish between the reduced plastic moment and the plastic moment and use the symbol M_p for both.

Figure 10.5.3 illustrates a fully developed plastic zone at a beam cross-section with a bending moment capacity equal to the plasticity moment M_p. The plastic zone represents a *plastic hinge* in the beam, which transmits the constant moment M_p.

Figure 10.5.4 shows a simply supported horizontal beam with a vertical load F at the middle of the beam. The limit load $F = F_l$ is determined by equating the maximum bending moment $F_l L/4$ to the plasticity moment M_p. Hence, the limit load is:

$$F_l = \frac{4M_p}{L} \qquad (10.5.4)$$

Under the limit load the beam may be considered as a mechanism of rigid beam parts joined by a plastic hinge, see Fig. 10.5.4b. The mechanism will be called a *collapse mechanism*. Applying the theorem of virtual power from Sect. 6.2, we can use the mechanism to determine the limit load in an alternative way. The load point is given a velocity v in the direction of the limit load F_l, and the beam parts rotates with angular velocities $\omega = v/(L/2)$, as shown in Fig. 10.5.4. Thus we have a *kinematically permissible velocity field* satisfying the displacement boundary conditions at the supports. The beam is in equilibrium and the theorem of virtual power states that the external virtual power, $P^e = F_l v$, is equal to the internal virtual power P^i, which is obtained at the plastic hinge. Note that the support reactions will not participate in the calculation of the external virtual power. An expression for the internal virtual power P^i at a plastic hinge is obtained by applying the virtual power theorem to the beam element shown in Fig. 10.5.3. The element consists of rigid parts connected by a plastic hinge and subjected to end moments equal to the plastic moment M_p. For this beam element $P^e = M_p \Omega$, where Ω is the relative angular velocity between the rigid beam parts. The power equation $P^e = P^i$ gives then the internal virtual power for a plastic hinge with plasticity moment M_p:

$$P^i = M_p \Omega \qquad (10.5.5)$$

For the collapse mechanism in Fig. 10.5.4 we obtain from the power equation:

$$P^e = P^i \quad \Rightarrow \quad F_l v = (M_p \omega) 2 \quad \Rightarrow \quad F_l = \frac{4M_p}{L}$$

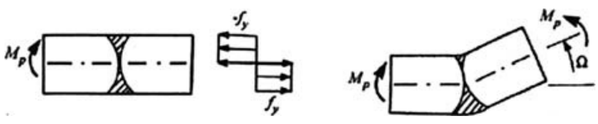


Fig. 10.5.3 Plastic zone and plastic hinge

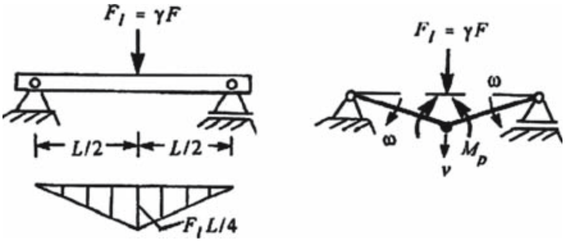


Fig. 10.5.4 **a)** Statically determinate beam and bending moment diagram under the limit load, **b)** Collapse mechanism for the beam

which is exactly the same result as obtained above in (10.5.4).

To obtain a plastic mechanism for a statically determinate beam one hinge will suffice, and the position of the yield hinge is decided by the position of the maximum bending moment. For statically indeterminate beams and frames of degree (n) a plastic mechanism requires $(n+1)$ plastic hinges. An example will illustrate this.

Figure 10.5.5a shows a statically indeterminate beam of degree $n = 1$, with constant bending stiffness EI when the beam is in the elastic state, and with constant plasticity moment M_p. The bending moment diagram for the elastic state is shown in Fig. 10.5.5b. The load F is increased towards the limit load F_l. Yielding of the material starts where the bending moment is largest. According to the moment diagram in Fig. 10.5.5b this is at the fixed support. As the load is increased a plastic zone develops towards a plastic hinge. Because the beam is statically indeterminate of degree $n = 1$, the number of plastic hinges necessary to create a collapse mechanism

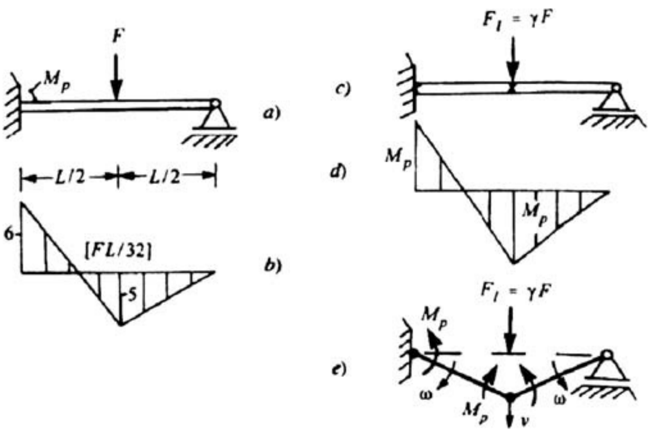


Fig. 10.5.5 **a)** Statically indeterminate beam. **b)** Bending moment diagram for the elastic state. **c)** Collapse mechanism for the beam. d) Bending moment diagram at the limit load. **d)** Kinematically permissible velocity field

is: $(n+1) = 2$. The presence of the first plastic hinge makes the beam statically determinate and the load may be increased further until the second plastic hinge is developed at the load, see Figure 10.5.5c and the moment diagram in Fig. 10.5.5d. Figure 10.5.5c shows the collapse mechanism for the beam, with plastic hinges at the fixed support and under the load. The limit load F_l may be determined by an equilibrium condition. Alternatively the limit load may be determined directly from the moment diagram in Fig. 10.5.5d. However, because we shall introduce a general method of obtaining the limit load of beam and frame systems through the application of the following limit load theorem, we shall prefer to use the theorem of virtual power to obtain the limit load.

A kinematically permissible velocity field is shown in Fig. 10.5.5e. The limit load is given a velocity v and the beam parts get the angular velocity $\omega = v/(L/2)$ as shown. The virtual power equation: $P^e = P^i$, gives then:

$$F_l\, v = (M_p\,\omega)\,3 = \left(\frac{6M_p}{L}\right) v \quad \Rightarrow \quad F_l = \frac{6M_p}{L}$$

For a statically indeterminate structure it is not always evident where the plastic hinges are positioned to give an actual collapse mechanism for the given load situation. Figure 10.5.6a shows a statically indeterminate frame of degree $n = 1$. A collapse mechanism requires $(n+1) = 2$ plastic hinges. Figures 10.5.6b, c, and d show three possible collapse mechanisms and corresponding kinematically permissible velocity fields. The load point of the load F is given a velocity v in the direction of

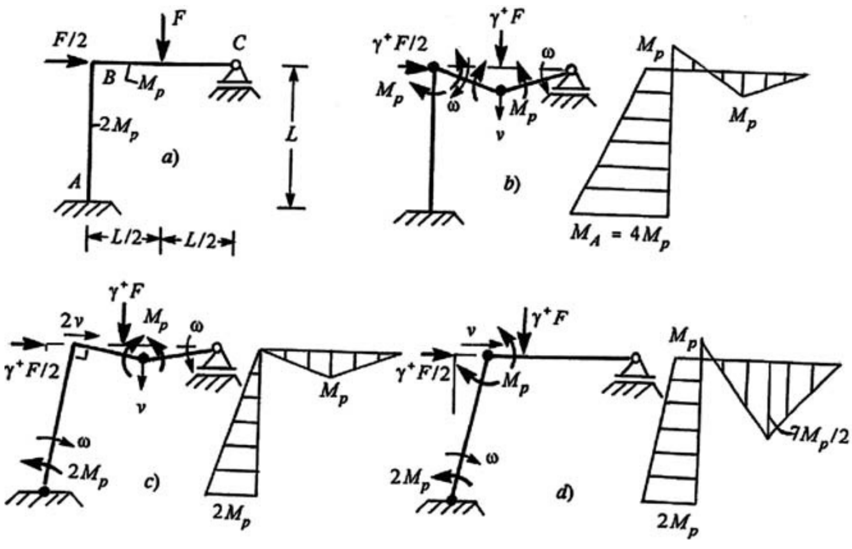


Fig. 10.5.6 **a)** Statically indeterminate frame. **b)**- **d)** Possible collapse mechanisms, corresponding kinematically permissible velocity field and bending moment diagram

the load and the beam parts rotates with angular velocities $\omega = v/(L/2)$, as shown. We shall see that two of the mechanisms give too high values for the limit load. At this stage it is convenient to introduce the concept of a *limit load coefficient* γ defined by:

$$\gamma = \frac{F_l}{F} \quad \Leftrightarrow \quad F_l = \gamma F \tag{10.5.6}$$

The limit load coefficient is the factor the actual load has to be increased by to obtain the limit load. Because we do not know which of the three possible collapse mechanisms that is a collapse mechanism, and thus gives the limit load and the limit load coefficient, we denote the loads corresponding to the three the real mechanism by $\gamma^+ F$. The smallest γ^+ - value is the limit load coefficient, and the corresponding mechanism is an actual collapse mechanism. The load coefficients γ^+ are thus upper bounds for the limit load coefficient:

$$\gamma^+ \geq \gamma \tag{10.5.7}$$

The virtual power equation: $P^e = P^i$, gives for the three possible collapse mechanisms:

Fig. 10.5.6b: $\gamma^+ F \cdot v = (M_p \cdot \omega)\, 3 \quad \Rightarrow \quad \gamma^+ = \dfrac{6M_p}{FL}$

Fig. 10.5.6c: $\gamma^+ F \cdot v + \dfrac{\gamma^+ F}{2} \cdot 2v = 2M_p \cdot \omega + (M_p \cdot \omega)\, 2 \quad \Rightarrow \quad \gamma^+ = \dfrac{4M_p}{FL}$

Fig. 10.5.6d: $\dfrac{\gamma^+ F}{2} \cdot v = 2M_p \cdot \omega + M_p \cdot \omega \quad \Rightarrow \quad \gamma^+ = \dfrac{6M_p}{FL}$

Hence we may conclude that the mechanism in Fig. 10.5.6c is the actual collapse mechanism for the frame subjected to the loads in Fig. 10.5.6a, and the limit load coefficient and the limit load are:

$$\gamma = \frac{4M_p}{FL} \quad \Leftrightarrow \quad F_l = \frac{4M_p}{L}$$

From the moment diagrams corresponding to the three possible collapse mechanisms in Fig. 10.5.6 we observe that for the mechanisms b and d the bending moment is larger than the relevant plastic moment in parts of the frame. We shall call the moment diagram in Fig. 10.5.6c for the actual collapse mechanism a *safe statically permissible moment diagram*. The adjective "statically permissible" indicates that the moment diagram agrees with the external load, i.e. γF. The word "safe" is used to mark that the bending moment nowhere exceeds the relevant plastic moment.

10.5.3 Limit Load Theorem for Plane Beams and Frames

For complex beam and frame systems the number of possible collapse mechanisms can be large, and the calculation of the load coefficients for all the mechanisms can be extensive work. The limit load theorem, to be presented below, will reduce the amount of calculations to obtain an estimate of the limit load. A more general form of the theorem applying to any elastic-plastic body will be presented in Sect. 10.7. A limit load theorem for beams and frames may be stated as follow.

Limit Load Theorem for Plane Beam and Frame Systems.

A load coefficient γ^+ calculated from any possible collapse mechanism for the system, represents an upper bound for the limit load coefficient γ:

$$\gamma \leq \gamma^+ \tag{10.5.8}$$

A lower bound γ^- for the load coefficient γ may be obtained as follows. Let M^+ represent the bending moment for the possible collapse mechanism, and let α be the largest value of the ratio $|M^+/M_p|$. M_p is the plastic moment of the system where the bending moment is M^+:

$$\alpha = \left|\frac{M^+}{M_p}\right|_{\max} \tag{10.5.9}$$

Then:

$$\gamma^- = \frac{\gamma^+}{\alpha} \leq \gamma \tag{10.5.10}$$

The theorem may also be called the *Upper and lower bound theorem for the limit load on plane beam and frame systems*.

A proof of the theorem: A possible and reasonable collapse mechanism for the system is chosen. The mechanism, consisting of rigid beam parts and plastic hinges, is subjected to a kinematically admissible velocity field satisfying the support conditions given for the system. The relative angular velocities of the beam tangents at the plastic hinges are denoted ω^+. Let P^{e+} be the external virtual power of the external load F due to the velocity field. Here the symbol F stands for a general load situation: single loads and distributed loads on the system. The mechanism is in equilibrium both for the load $\gamma^+ F$ and the limit load γF. For the load $\gamma^+ F$ the bending moment is assumed to be equal to the plastic moment M_p in the plastic hinges. For the limit load γF the bending moment M is less then or equal to the plastic moment, i.e. $M \leq M_p$ everywhere. The virtual power equations for the mechanism for the two load situations are:

$$\gamma P^{e+} = \sum M\,\omega^+,\; M \leq M_p,\; \gamma^+ P^{e+} = \sum M_p\,\omega^+$$

from which we get the result:

$$\gamma P^{e+} \leq \gamma^{+} P^{e+} \quad \Rightarrow \quad \gamma \leq \gamma^{+} \tag{10.5.11}$$

To find a lower bound γ^{-} for the limit load coefficient γ we proceed as follows. The moment diagram M^{+} for the mechanism subjected to the load $\gamma^{+}F$ and with plastic moments M_p in the hinges is determined by equilibrium equations. If the mechanism does not qualify as a collapse mechanism, the bending moment will exceed the plastic moment in parts of the system. The parameter α in formula (10.5.9) is determined. If the load $\gamma^{+}F$ and the corresponding bending moment M^{+} everywhere is reduced by the factor α, the assumed collapse mechanism with the load $\gamma^{+}F/\alpha$ is a system in equilibrium and with the bending moment, $M^{-} = M^{+}/\alpha$, which everywhere satisfies the condition $M^{-} \leq M_p$. We then say that the system is in a *safe statically permissible state*, and M^{-} is called a *safe statically permissible moment diagram*. The load $\gamma^{+}F/\alpha$ gives the load coefficient γ^{-} in the formula (10.5.10). It will then be shown that this is a lower bound for the limit load coefficient γ.

An actual collapse mechanism, which we do not need to know, is subjected to a kinematically admissible velocity field, having a sufficient number of plastic hinges and satisfying the condition of rigid beam parts. The relative angular velocity of the beam tangents at the hinges are denoted by ω. Let P^e be the external virtual power of the load F due to the velocity field. The collapse mechanism is in equilibrium both for the load $\gamma^{-}F$ and for the limit load γF. For the load $\gamma^{-}F$ the bending moment M^{-} satisfies the condition: $M^{-} \leq M_p$ everywhere. For the limit load the bending moment is equal to the plastic moment M_p in the hinges. The virtual power equations for the collapse mechanism due to the two loads $\gamma^{-}F$ and γF are:

$$\gamma^{-} P^{e} = \sum M^{-}\,\omega,\; M^{-} \leq M_p,\; \gamma P^{e} = \sum M_p\,\omega$$

from which we obtain:

$$\gamma^{-} P^{e} \leq \gamma P^{e} \quad \Rightarrow \quad \gamma^{-} \leq \gamma \tag{10.5.12}$$

We conclude from the results (10.5.11) and (10.5.12) that:

$$\gamma^{-} \leq \gamma \leq \gamma^{+} \tag{10.5.13}$$

The result proves the limit load theorem for plane beam and frame systems.

As an example we apply the limit load theorem to the frame in Fig. 10.5.6. Let us assume that an upper bound for limit load coefficient has been found from the possible collapse mechanism in Fig. 10.5.6b: $\gamma^{+} = 6M_p/FL$. From the moment diagram in Fig. 10.5.6b we obtain:

$$\alpha = \left|\frac{M^{+}}{M_p}\right|_{\max} = \frac{4M_p}{2M_p} = 2$$

A lower bound for the limit load coefficient is therefore:

$$\gamma^- = \frac{\gamma^+}{\alpha} = \frac{6M_p}{FL}\frac{1}{2} = \frac{3M_p}{FL}$$

For the limit load coefficient γ we now have:

$$\frac{3M_p}{FL} \le \gamma \le \frac{6M_p}{FL}$$

10.6 The Drucker Postulate

D. C. Drucker [12–13] has proposed a postulate for elastic-plastic hardening materials. A consequence of the postulate is that the yield surface represented by the yield criterion (10.2.46):

$$\begin{aligned} f[\mathbf{T},\kappa] = 0 \quad &\Leftrightarrow \quad \text{yielding may start} \\ f[\mathbf{T},\kappa] < 0 \quad &\Leftrightarrow \quad \text{elastic response} \end{aligned} \tag{10.6.1}$$

is convex, and that the flow rule is represented by the general equation:

$$d\mathbf{E}^p = d\lambda\,\frac{df}{d\mathbf{T}} \tag{10.6.2}$$

$d\lambda$ is scalar-valued function of the stress tensor $\mathbf{T}$ and of the stress increment tensor $d\mathbf{T}$. The postulate may be stated thus:

The Drucker Postulate. When a particle of an elastic-plastic hardening material is subjected to a stress history from one state of stress $\mathbf{T}^{\mathrm{a}}$ to another state of stress $\mathbf{T}^{\mathrm{b}}$, the deformation work done by the "additional stresses", $(\mathbf{T}-\mathbf{T}^{\mathrm{a}})$, is always non-negative:

$$\int_a^b (\mathbf{T}-\mathbf{T}^a) : d\mathbf{E} \ge 0 \tag{10.6.3}$$

The Drucker postulate generalizes what we experience for uniaxial stress histories, as illustrated in Fig. 10.6.1a and b. For a uniaxial stress history from the stress σ^a to the stress σ^b the deformation work per unit volume is:

$$W = \int_a^b \sigma\,d\varepsilon$$

The work per unit volume of the additional stress $(\sigma - \sigma^a)$ is:

$$\bar{W} = \int_a^b (\sigma - \sigma^a)\, d\varepsilon = W - \sigma^a\left(\varepsilon^b - \varepsilon^a\right)$$

In Fig. 10.6.1 the work W is represented by areas under $\sigma\varepsilon$-graph, and the work $\bar{W}$. is represented by the hatched areas. We may conclude from the figures that the work $\bar{W}$ is always positive if the yield curve $\sigma(\varepsilon)$ is part of the area and $d\sigma/d\varepsilon > 0$. If the stress σ throughout the stress history is less than the yield stress, the work $\bar{W} > 0$ whenever $\sigma^b \neq \sigma^a$ and $\bar{W} = 0$ for a loading cycle, $\sigma^b = \sigma^a$. Figure 10.6.1c shows a yield curve with a softening portion, i.e. $d\sigma/d\varepsilon < 0$, and a stress history that includes the portion. The additional work $\bar{W}$ is negative.

The Drucker postulate may be interpreted as a hypothesis on stability of an elastic- plastic hardening material: A change in the state of stress in a particle requires a non-negative work of the additional stresses.

We shall investigate the consequences of the Drucker postulate for the yield function $f[\mathbf{T},\kappa]$ and the yield surface $f[\mathbf{T},\kappa] = 0$, and for the general flow rule. In order to simplify the presentation we shall use the notation T_α from the formulas (7.8.1) for coordinate stresses, and the notation E_α from the formulas (7.8.2) for the coordinate strains. Greek letters have values from 1 to 6.

$$T = \{T_1,\, T_2,\, T_3, T_4, T_5, T_6\},\; E = \{E_1,\, E_2,\, E_3, E_4, E_5, E_6\} \tag{10.6.4}$$

The equality/inequality (10.6.3) in the postulate can now be restated as:

$$\int_a^b (T_\alpha - T_\alpha^a)\, dE_\alpha \geq 0 \tag{10.6.5}$$

Let a particle be subjected to a stress cycle from an elastic state at the stress $\mathbf{T}^a$ and back to the same stress: $\mathbf{T}^b = \mathbf{T}^a$. The cycle may consist of an elastic loading from $\mathbf{T}^a$ to initial yielding at the stress state $\mathbf{T}^c$, followed by a plastic stress history

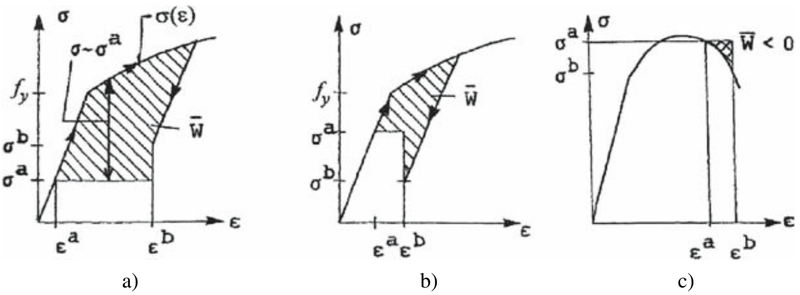


Fig. 10.6.1 Uniaxial stress histories in an elastic-plastic hardening material **a)** and **b)**, and **c)** a softening material

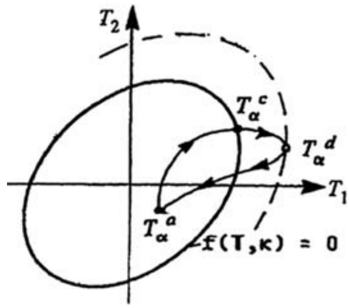


Fig. 10.6.2 Stress cycle: Elastic loading from T^a for inital yielding at $\mathbf{T}^c$, then plastic stress history to $\mathbf{T}^d$, and finally elastic unloading to $\mathbf{T}^b = \mathbf{T}^a$

from $\mathbf{T}^c$ to a stress state $\mathbf{T}^d$, and finally an elastic unloading history from the stress state $\mathbf{T}^d$ back to the starting state of stress $\mathbf{T}^b = \mathbf{T}^a$. The situation is illustrated in Fig. 10.6.2, which for simplicity only shows two coordinate stresses. During the plastic part of the history the yield surface (curve) is changed and/or moved such that the stress state is always represented by a point on the yield surface. Thus:

$$f[\mathbf{T}, \kappa] = 0 \quad \text{for states of stress } \mathbf{T} \text{ between } \mathbf{T}^c \text{ and } \mathbf{T}^d$$

According to hypothesis 1 for a general flow rule, (10.3.1), the strain increment $d\mathbf{E}$ is assumed to be a sum of an elastic contribution $d\mathbf{E}^e$ and a plastic contribution $d\mathbf{E}^p$:

$$dE_\alpha = dE_\alpha^e + dE_\alpha^p$$

The plastic strain increments appear only from the stress $\mathbf{T}^c$ to the stress $\mathbf{T}^d$. Therefore the statement (10.6.5) is changed to:

$$\int_a^b (T_\alpha - T_\alpha^a)\, dE_\alpha = \int_a^b (T_\alpha - T_\alpha^a)\, dE_\alpha^e + \int_c^d (T_\alpha - T_\alpha^a)\, dE_\alpha^p \geq 0$$

Because the 3. hypothesis, (10.3.3), states that:

$$T_\alpha = \frac{\partial \phi}{\partial E_\alpha^e}$$

the first integral on the right-hand side can be shown to be zero:

$$\int_a^{b=a} (T_\alpha - T_\alpha^a)\, dE_\alpha^e = \int_a^{b=a} \left(\frac{\partial \phi}{\partial E_\alpha^e} dE_\alpha^e - T_\alpha^a dE_\alpha^e \right) = [\phi - T_\alpha^a E_\alpha^e]_a^{b=a} = 0$$

This result reduces the statement (10.6.5) to:

$$\int_c^d (T_\alpha - T_\alpha^a)\, dE_\alpha^p \geq 0 \tag{10.6.6}$$

Because this equality/inequality has to be satisfied for any stress cycle, we may state that:

$$(T_\alpha^c - T_\alpha^a)\, dE_\alpha^p \geq 0 \tag{10.6.7}$$

where dE_α^p is the plastic strain increment when the stress $\mathbf{T}^c$ satisfies the yield criterion (10.6.1). If in particular we choose:

$$T_\beta^a = T_\beta^c \text{ for } \beta = 3{,}4{,}5{,}6$$

the statement (10.6.7) is reduced to:

$$(T_\alpha^c - T_\alpha^a)\, dE_\alpha^p \geq 0 \text{ for } \alpha = 1,2 \tag{10.6.8}$$

In a Cartesian coordinate system with the stresses T_α and strains dE_α^p as coordinates, and with the same axes for stress and strain, the yield criterion (10.6.1) is represented by a yield curve, which in the case of hardening changes form and/or moves in the coordinate plane, Fig. 10.6.2. The statement (10.6.8) tells us that the scalar product of the vectors $(T_\alpha^c - T_\alpha^a)$ and dE_α^p is non-negative. From this it follows the angle θ between the vectors must be less than or equal to 90°. This implies first that the yield curve must be convex, and secondly that the vector dE_α^p is perpendicular to the yield curve at the stress point T_α^c and points outward from the yield curve. This is illustrated in Fig. 10.6.3, while Fig. 10.6.4 shows the consequence if the yield curve is not convex. Figure 10.6.5 illustrates a situation where the plastic strain increment vector is not normal to the yield curve.

We know that the stress points inside the yield curve represent elastic states, i.e. $f[\mathbf{T},\kappa] < 0$. This means that the vector $\partial f/\partial T_\alpha$, i.e. the gradient to the yield function $f[\mathbf{T},\kappa]$ is pointing outward from the yield curve. The consequence of these observations is that we may state:

$$dE_\alpha^p = d\lambda \frac{\partial f}{\partial T_\alpha} \quad \alpha = 1,2 \tag{10.6.9}$$

The parameter $d\lambda$ is a scalar, which in the case of perfectly plastic material is an unknown constant, and in the general case of hardening materials is a scalar-valued function of the state of stress $\mathbf{T}$, the hardening parameters κ_i, and of the stress increment tensor $d\mathbf{T}$. The case of a hardening material will be discussed in Sect. 10.9.

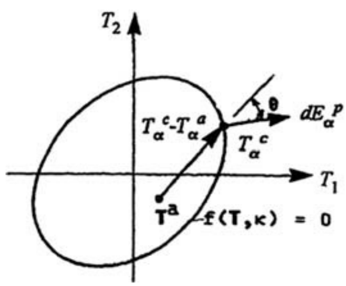


Fig. 10.6.3 The scalar product $(T_\alpha^c - T_\alpha^a)\, dE_d^p \geq 0$ implying that the vector dE_α^p points outward from the yield curve (surface)

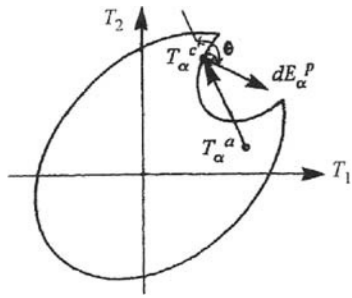


Fig. 10.6.4 The consequence of a non-convex of the yield curve (surface): Formula (10.68) is not satisfied everywhere

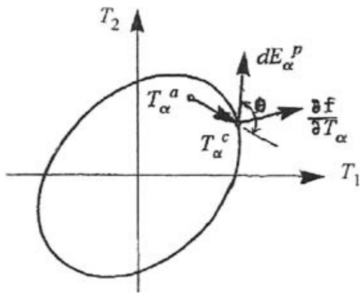


Fig. 10.6.5 A plastic strain increment vector not normal to the yield curve (surface)

Because the result (10.6.9) also must apply for arbitrary choices of two α-values, we may conclude, based on the Drucker postulate, that the general flow rule must have the form:

$$dE_\alpha^p = d\lambda \frac{\partial f}{\partial T_\alpha} \quad = 1,2,3,4,5,6 \quad \Leftrightarrow \quad d\mathbf{E}^p = d\lambda \frac{\partial f}{\partial \mathbf{T}} \tag{10.6.10}$$

When the flow rule (10.6.10) is used with the tensor components T_{ik}, we must to treat the coordinate stresses as nine independent variables when the yield function is differentiated.

The final conclusion based on the discussion above is the following statement:

The yield surface $f[\mathbf{T}, \kappa] = 0$ is a convex surface in any stress space.

10.7 Limit Load Analysis

The objective of this section is to present the general limit load theorems for a body of elastic-perfectly plastic material. It is assumed that the body is subjected to loads: body forces and/or contact forces, which are increased proportionally until the body behaves as a mechanism. The load is now called the *limit load*. Further increase of the load leads to large, theoretically uncontrollable deformations of the body. The situation is called *collapse*, and the mechanism is called a *collapse mechanism*. As the elastic strains are small compared with the plastic strains in the mechanism, we shall neglect the elastic strains, i.e. we treat the material as *rigid-perfectly plastic*. If hardening of the material is taken into consideration the load may be increased beyond the limit load. However, the deformations will be unacceptable in most cases. The limit load is therefore often characterized as the *ultimate load*. The limit load will also represent the load necessary to initiate a forming process.

Figure 10.7.1 shows a body with volume V loaded with body forces $\mathbf{b}$ and contact forces $\mathbf{t}^*$ on a part A_σ of the surface A of the body. We set:

$$\mathbf{t} = \mathbf{T} \cdot \mathbf{n} = \mathbf{t}^* \text{ prescribed contact force on } A_\sigma \tag{10.7.1}$$

$\mathbf{T}$ is the stress tensor in the body and $\mathbf{n}$ is a unit normal to the surface A_σ. On the remainder of the surface $A: \ A_u = A - A_\sigma$, the displacements $\mathbf{u}$ are specified:

$$\mathbf{u} = \mathbf{u}^* \text{ prescribed displacement on } A_u \tag{10.7.2}$$

It is assumed that the load situation $\{\mathbf{b}, \mathbf{t}^*\}$ does not lead to a collapse of the body.

The *limit load* is denoted by:

$$\{\mathbf{b}_l, \mathbf{t}_l^*\} \equiv \{\gamma\mathbf{b}, \gamma\mathbf{t}^*\} = \gamma\{\mathbf{b}, \mathbf{t}^*\} \tag{10.7.3}$$

γ is the *limit load coefficient*. The limit load coefficient is in a certain sense a safety factor for the situation in question. The state of stress at the limit load (10.7.3) will be represented by the tensor field $\mathbf{T}^l$.

To actually find the limit load coefficient can often be very complicated and sometimes impossible. But using a relatively simple analysis, we are able to find a *lower bound* and an *upper bound* for the limit load and thereby obtain a reasonable estimate for the real limit load. If it should happen that the two bounds coincide, then we have found a collapse mechanism and the limit load.

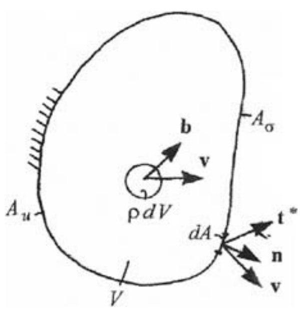


Fig. 10.7.1 Elastic-perfectly plastic body with loads and velocities

10.7.1 Lower Bound Limit Load Theorem

A stress field $\mathbf{T}^s$ is called a *statically permissible stress field* if the stress field everywhere in the body satisfies the Cauchy equations of equilibrium:

$$T^s_{ik,k} + \rho\, b_i = 0 \tag{10.7.4}$$

and the boundary conditions (10.7.1). The stress field does not have to result in compatible strains. Nor do the resulting displacements have to satisfy the boundary condition (10.7.2). However, if the displacement conditions (10.7.2) also are satisfied, then we have obtained a real static solution for the situation at hand. A statically permissible stress field $\mathbf{T}^s$ that also satisfies the yield condition:

$$f\,[\mathbf{T}^s] \leq 0 \quad \text{everywhere in the body} \tag{10.7.5}$$

where $f\,[\mathbf{T}]$ is the yield function related to a chosen yield criterion, is called a *safe statically permissible stress field*.

Let a safe statically permissible stress field $\mathbf{T}^s$ correspond, via the Cauchy equations (10.7.4) and the boundary condition (10.7.1), to a load state:

$$\left\{\gamma^- \mathbf{b}, \gamma^- \mathbf{t}^*\right\} = \gamma^- \left\{\mathbf{b}, \mathbf{t}^*\right\} \tag{10.7.6}$$

The *lower bound limit load theorem* implies that the load coefficient γ^- is a lower bound for the load coefficient γ. The theorem is stated thus:

Lower Bound Limit Load Theorem. To any safe statically permissible stress field $\mathbf{T}^s$ corresponds a lower bound γ^- for the limit load coefficient γ:

$$\gamma^- \leq \gamma \tag{10.7.7}$$

Proof. The theorem of virtual power in Sect. 6.2 will be applied. At the limit load the body is a collapse mechanism. As a *kinematically permissible velocity field* $\mathbf{v}$, with a corresponding rate of plastic strain field $\dot{\mathbf{E}}$, we choose one that agrees with this mechanism and that satisfies the displacement boundary condition (10.7.2), which implies that the velocities are zero on the surface A_u. Furthermore, the mechanical power due to the loads shall be positive. It follows that:

$$f\left[\mathbf{T}^l\right] = 0 \quad \text{wherever } \dot{\mathbf{E}} \neq \mathbf{0} \tag{10.7.8}$$

The theorem of virtual work applied on the two load situations (10.7.3) and (10.7.6) yields:

$$\int_V \gamma \mathbf{b} \cdot \mathbf{v} \rho \, dV + \int_{A_\sigma} \gamma \mathbf{t}^* \cdot \mathbf{v} \, dA = \int_V \mathbf{T}^l : \dot{\mathbf{E}} \, dV$$
$$\int_V \gamma^- \mathbf{b} \cdot \mathbf{v} \rho \, dV + \int_{A_\sigma} \gamma^- \mathbf{t}^* \cdot \mathbf{v} \, dA = \int_V \mathbf{T}^s : \dot{\mathbf{E}} \, dV \tag{10.7.9}$$

These two equations are subtracted from each other and the result is:

$$\left(\gamma - \gamma^-\right) \left[\int_V \gamma \mathbf{b} \cdot \mathbf{v} \rho \, dV + \int_{A_\sigma} \gamma \mathbf{t}^* \cdot \mathbf{v} \, dA \right] = \int_V \left(\mathbf{T}^l - \mathbf{T}^s \right) : \dot{\mathbf{E}} \, dV \tag{10.7.10}$$

Due to the conditions (10.7.5) and (10.7.8) and to Drucker's postulate (10.6.3), the right-hand side in (10.7.10) is non-negative. By our assumption of positive mechanical power of the loads, the term in the square brackets on the left-hand side of equation (10.7.10) is positive. From these two observations we conclude that the factor $\left(\gamma - \gamma^-\right)$ must be non-negative. Thus we have proved the equality/inequality (10.7.7), and by that the lower bound limit load theorem. The equality sign in the statement (10.7.7) applies if the proposed permissible stress field is identical to the stress field in the collapse mechanism. In order to determine the load coefficient γ^- we use the power equation $(10.7.9)_2$. The stress field $\mathbf{T}^s$ will be given by the yield criterion where the strain field $\dot{\mathbf{E}}^k \neq \mathbf{0}$. It can be shown that more than one collapse mechanism may exist, which have different stress field and velocity fields, but the same limit load coefficient.

10.7.2 Upper Bound Limit Load Theorem

For any possible collapse mechanism we choose a *kinematically permissible velocity field* $\mathbf{v}^k$, with a corresponding rate of plastic strain $\dot{\mathbf{E}}^k$. The velocity field must agree with the mechanism and satisfy the displacement boundary condition (10.7.2), which implies that the velocities are zero on the surface A_u. Furthermore, the mechanical power due to the loads shall be positive. A statically permissible stress field $\mathbf{T}^s$ corresponding to the load state:

$$\left\{\gamma^+ \mathbf{b}, \gamma^+ \mathbf{t}^*\right\} = \gamma^+ \left\{\mathbf{b}, \mathbf{t}^*\right\} \tag{10.7.11}$$

is introduced, i.e. $\mathbf{T}^s$ satisfies the Cauchy equations (10.7.4) and the stress boundary condition (10.7.1). We only need to know the stress field $\mathbf{T}^s$ in region with plastic strain rates and where the yield criterion is satisfied:

$$f\left[\mathbf{T}^s\right] = 0 \quad \text{wherever } \dot{\mathbf{E}}^k \neq \mathbf{0} \tag{10.7.12}$$

The stress field $\mathbf{T}^s$ need not be a safe stress field, i.e. we do not require that the condition (10.7.5) is satisfied. However, it follows that the stress field $\mathbf{T}^l$ satisfies the condition:

$$f\left[\mathbf{T}^l\right] \leq 0 \quad \text{wherever } \dot{\mathbf{E}}^k \neq \mathbf{0} \tag{10.7.13}$$

The *upper bound limit load theorem* then states that the load coefficient γ^+ is an upper bound for the limit load coefficient γ. The theorem is formulated as follows.

Upper Bound Limit Load Theorem. A kinematically permissible velocity field $\mathbf{v}^k$, with corresponding rate of strain field $\dot{\mathbf{E}}^k$, that satisfies the boundary condition (10.7.2), i.e. $\mathbf{v}^k = \mathbf{0}$ on A_u, and a statically permissible stress field $\mathbf{T}^s$ that satisfies the yield criterion (10.7.12), provides an upper bound γ^+ for the limit load coefficient γ:

$$\gamma \leq \gamma^+ \tag{10.7.14}$$

Proof. The theorem of virtual power used with the load states (10.7.3) and (10.7.11) provides the following power equations:

$$\int_V \gamma \mathbf{b} \cdot \mathbf{v}^k \rho \, dV + \int_{A_\sigma} \gamma \mathbf{t}^* \cdot \mathbf{v}^k \, dA = \int_V \mathbf{T}^l : \dot{\mathbf{E}}^k \, dV$$

$$\int_V \gamma^+ \mathbf{b} \cdot \mathbf{v}^k \rho \, dV + \int_{A_\sigma} \gamma^+ \mathbf{t}^* \cdot \mathbf{v}^k \, dA = \int_V \mathbf{T}^s : \dot{\mathbf{E}}^k \, dV \tag{10.7.15}$$

The two equations are subtracted and the result is:

$$\left(\gamma^+ - \gamma\right) \left[\int_V \gamma \mathbf{b} \cdot \mathbf{v}^k \rho \, dV + \int_{A_\sigma} \gamma \mathbf{t}^* \cdot \mathbf{v}^k \, dA \right] = \int_V \left(\mathbf{T}^s - \mathbf{T}^l\right) : \dot{\mathbf{E}}^k \, dV \tag{10.7.16}$$

Due to the conditions (10.7.12) and (10.7.13) and to Drucker's postulate (10.6.3) the right-hand side in (10.7.16) is non-negative. By our assumption of positive mechanical power of the loads, the term in the square brackets on the left-hand side of (10.7.16) is positive. From these two observations we conclude that the factor $(\gamma^+ - \gamma)$ must be non-negative. This proves the equality/inequality (10.7.14) and thus the upper bound limit load theorem. The equality sign in the statement (10.7.14) applies if the proposed collapse mechanism is a real collapse mechanism. In order to determine the load coefficient γ^+ we use the power equation $(10.7.15)_2$. The stress field $\mathbf{T}^s$ will be given by the yield criterion where the strain field $\dot{\mathbf{E}}^k \neq \mathbf{0}$.

A simple application of the upper bound limit load theorem will be shown below. Both limit load theorems will be applied in the sections to follow. The limit load theorem for beams and frames combines the lower bound and the upper bound theorems.

Example 10.11. Circular Cylinder with Internal Pressure

A circular cylinder with internal radius a, external radius b, and length L is subjected to internal pressure p. The cylinder is not allowed to move in the axial

z-direction, i.e. plane strain is assumed with $\dot{\varepsilon}_z = 0$. The material is modelled as a rigid-perfectly plastic Tresca or Mises material with yield stress f_y. We want to determine an upper bound $\gamma^+ p$ for the limit pressure $p_l = \gamma p$. The limit pressure has already been found in Example 10.10, see (10.4.55):

$$p_l = 2\tau_y \ln\left(\frac{b}{a}\right) \tag{10.7.17}$$

We assume a kinematically permissible velocity field:

$$v_R^k = \frac{C}{R},\ v_\theta^k = v_z^k = 0 \tag{10.7.18}$$

C is an unknown constant. The corresponding strain rates are according to (5.4.19):

$$\dot{\varepsilon}_R^k = \frac{d}{dR} v_R^k = -\frac{C}{R^2},\ \dot{\varepsilon}_\theta^k = \frac{v_R^k}{R} = \frac{C}{R^2} \tag{10.7.19}$$

Since the strain rate field is present throughout the cylinder a statically permissible stress field has to satisfy the yield criterion of the material everywhere. Based on what we found in Example 10.9, we can make the general statement that for *plane strain*, $\varepsilon_z = 0$, the yield criterions for the two material models are covered by the equation:

$$|\sigma_R^s - \sigma_\theta^s| = 2\tau_y \quad \Leftrightarrow \quad \text{yielding may start} \tag{10.7.20}$$

The virtual power equation $(10.7.15)_2$ implies:

$$\int_{A_\sigma} \gamma^+ \mathbf{t}^* \cdot \mathbf{v}^k \, dA = \int_V \mathbf{T}^s : \dot{\mathbf{E}}^k \, dV \quad \Rightarrow$$

$$\gamma^+ p \cdot \frac{C}{a} \cdot 2\pi a L = \int_a^b \left[\sigma_R^s \cdot \left(-\frac{C}{R^2}\right) + \sigma_\theta^s \cdot \frac{C}{R^2}\right] 2\pi R L \, dR = C\left(-\sigma_R^s + \sigma_\theta^s\right) 2\pi L \ln\left(\frac{b}{a}\right)$$

$$\Rightarrow \gamma^+ p = \left(\sigma_\theta^s - \sigma_R^s\right) \ln\left(\frac{b}{a}\right) \tag{10.7.21}$$

From the equations (10.7.20) and (10.7.21) we obtain:

$$p^+ \equiv \gamma^+ p = 2\tau_y \ln\left(\frac{b}{a}\right) \quad \Rightarrow \quad \gamma^+ = \frac{2\tau_y}{p} \ln\left(\frac{b}{a}\right)$$

Thus we have found an upper bound for the limit load coefficient, and for the limit pressure. By comparing the upper bound p^+ with the limit load given in (10.7.17) we see that the upper bound for the limit pressure is identical to the limit load. This means that the possible collapse mechanism represented by equation (10.7.19) is a real collapse mechanism for the cylinder. This should not come as a surprise since the chosen statically permissible stress field satisfies the yield criterion everywhere in the cylinder and is thus a safe statically permissible stress field, which implies that the solution γ^+ also is a lower bound for the limit load coefficient, or in fact the solution γ^+ is the limit load coefficient γ. Note also that the rate of strain field obeys the requirement:

$$\dot{\varepsilon}_R^k = -\dot{\varepsilon}_\theta^k$$

in agreement with the flow rules for both material models under plane strain.

10.7.3 Discontinuity in Stress and Velocity

Discontinuities in the stress field may be accepted in a statically permissible stress field as long as the stresses on both sides of every internal material surface are the same. Figure 10.7.2 demonstrates the situation in the case of plane stress. The shear stresses and the normal stresses on both sides of the interface must be the same:

$$\tau_a = \tau_b, \ \sigma_{na} = \sigma_{nb} \tag{10.7.22}$$

On the other hand we can accept that $\sigma_{ta} \neq \sigma_{tb}$.

A kinematically permissible velocity field must be continuous, but we may introduce artificial discontinuities by representing narrow *shear zones* by discontinuity surfaces. Figure 10.7.3 illustrates a shear zone of thickness h, length L, and width b. A shear zone has plane strain and the state of strain rate is represented by a shear rate $\dot{\gamma}$, as shown in Fig. 10.7.3, and the corresponding shear stress is the yield shear stress τ_y. The stress power in the shear zone becomes:

$$P_s^i = \int_V \mathbf{T}^s : \mathbf{E}^k \, dV = (\tau_y \cdot \dot{\gamma})\, hLb = \tau_y \, vLb \tag{10.7.23}$$

which is independent of the thickness h. It is convenient to replace this shear zone by a shear zone with "infinitesimal small" thickness, and then consider the zone as a surface of discontinuous displacements, as illustrated in Fig. 10.7.4. The stress power in a shear zone, or discontinuity surface is, according to (10.7.23) equal to the product of the yield shear stress τ_y, the relative velocity v in the shear zone, and the area $A_s = Lb$ of the discontinuity:

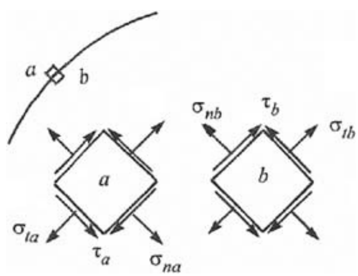


Fig. 10.7.2 Discontinuity surface ab for stresses

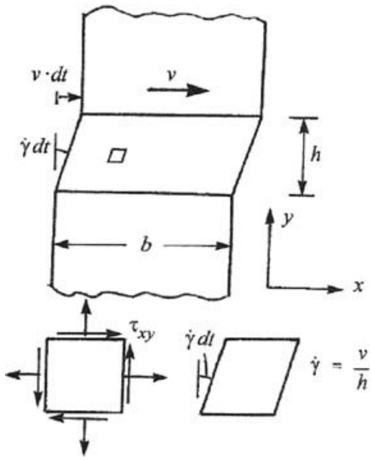


Fig. 10.7.3 Shear zone of thickness h

$$P_s^i = \tau_y v A_s \tag{10.7.24}$$

Example 10.12. Bar with a Notch

Figure 10.7.5a shows a bar with cross sectional area $w \cdot h$ and with a notch that reduces the area of the bar to $w \cdot d$. The bar is subjected to tensile forces F with the line of action through the centroid of the reduced cross section. We want to find a lower and an upper bound for the limit load coefficient γ.

To determine a lower bound γ^- for the limit load coefficient we introduce a safe statically permissible stress field that is homogeneous within each of the zones a and b, see Fig. 10.7.5b:

$$\text{Zone}\,a\text{:}\,\mathbf{T} = \mathbf{0},\ \text{Zone}\ b\text{: Uniaxial stress}\ \sigma = f_y \tag{10.7.25}$$

On the interface ab between the two zones the continuity condition (10.7.22) is satisfied. From the condition of equilibrium we obtain:

$$\left(\gamma^- F\right) = f_y \cdot (w \cdot d) \quad \Rightarrow \quad \gamma^- = \frac{f_y w d}{F} \tag{10.7.26}$$

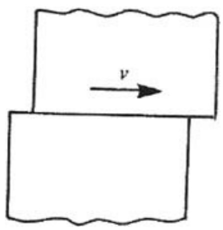


Fig. 10.7.4 Discontinuity surface

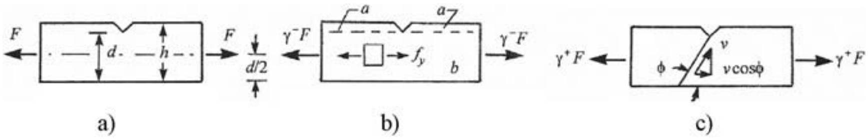


Fig. 10.7.5 Bar with a notch: **a)** The given tensile forces. **b)** Safe statically permissible stress field to provide the lower bound for the load coefficient. **c)** Kinematically permissible velocity field to provide the upper bound for the load coefficient

To find an upper bound γ^+ for the limit load coefficient we introduce a kinematically permissible velocity field with a plane shear zone as shown in Fig. 10.7.5c. The shear zone is defined by an unknown angle ϕ with respect to the axis of the bar. The right-hand part of the beam is given a virtual velocity v relative to the left-hand part. The virtual power equation gives:

$$P^e = P^i \quad \Rightarrow \quad \left(\gamma^+ F\right) \cdot v\cos\phi = (\tau_y \cdot v)\left(w \cdot \frac{d}{\sin\phi}\right) \quad \Rightarrow \quad \gamma^+ = \frac{2\tau_y wd}{F\sin 2\phi} \tag{10.7.27}$$

The angle $\tau = 45^\circ$ gives the lowest value of the upper bound coefficient.

$$\gamma^+_{\min} = \frac{2\tau_y wd}{F} \tag{10.7.28}$$

For the Tresca material $\tau_y = f_y/2$, and the lower bound and the upper bound for the limit load coefficient are the same. Thus we have found the limit load coefficient:

$$\gamma = \gamma^+_{\min} = \gamma^- = \frac{f_y wd}{F} \tag{10.7.29}$$

For the Mises material $\tau_y = f_y/\sqrt{3}$, and the upper bound for the limit load coefficient becomes:

$$\gamma^+_{\min} = \frac{2f_y wd}{\sqrt{3}F} = 1.15\frac{f_y wd}{F} \tag{10.7.30}$$

We conclude that for a bar of a Mises material:

$$\frac{f_y wd}{F} < \gamma < 1.15\frac{f_y wd}{F} \tag{10.7.31}$$

10.7.4 Indentation

We shall study the following type of problems: What force F is necessary to press a rigid piston into an elastic-perfectly plastic material such that the result is a permanent indentation. In other words, we want to find the limit load for the indentation force F for a rigid piston on a surface of an elastic-perfectly plastic material.

This type of problems has applications within forming of materials, design of foundations of buildings, and hardness tests of materials.

In hardness tests a pointed diamond or a hardened steel ball is pressed into the surface of the material to be examined by a sufficiently large force F. The projected area A of the indent is measured and the hardness of the material is defined by the parameter $H = F/A$. It may then be shown that $f_y = H/3$.

The problem type will be illustrated by a relatively simple example.

Example 10.13. Indentation of a Rigid Piston

Figure 10.7.6 illustrates an indentation problem. The rigid piston has the width w. The extension in direction normal the plane of the figure is large and the indentation force $F = F_l$ is force per unit length in this direction. The piston is pressed against a surface of a semi-infinite space of elastic-perfectly plastic material.

The condition of plane strain, with $\varepsilon_z = 0$, is assumed. The yield condition for both a Tresca material and a Mises material may be expressed by (10.4.28):

$$|\sigma_1 - \sigma_2| = 2\tau_y \quad \Leftrightarrow \quad \text{yielding may start} \tag{10.7.32}$$

σ_1 and σ_2 are the principal stresses in the xy-plane, and τ_y is the yield shear stress appropriate for the material model and given by the formulas (10.4.27).

To find a lower bound F^- for the necessary indentation force F_l, we introduce the safe statically permissible stress field shown in Fig. 10.7.7. The stress field is discontinuous, but homogeneous in each of the zones a and b:

$$\text{Zone } a : \mathbf{T} = \mathbf{0}, \text{ Zone } b : \sigma_{xb} = \tau_{xyb} = 0, \ \sigma_{yb} = -2\tau_y \tag{10.7.33}$$

The zones a are stress free, and the zone b has uniaxial stress satisfying the yield criterion (10.7.32). On the interfaces ab between two zones the necessary continuity conditions are satisfied:

$$\sigma_n = \sigma_{xa} = \sigma_{xb} = 0, \ \tau = \tau_{xya} = \tau_{xyb} = 0 \tag{10.7.34}$$

The stress circle in the Mohr-diagram in Fig. 10.7.7 represents the state of stress in Zone b, while point $a =$ point ab represents the state of stress in zone a and the zero stresses on the interfaces ab between zone a and zone b. Point bp is the pole in the Mohr -diagram and represents also the stress on the contact surface bp between

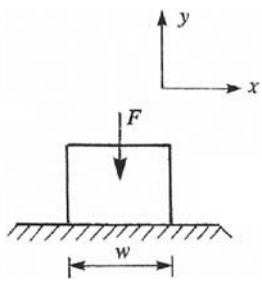


Fig. 10.7.6 Indentation of a rigid piston

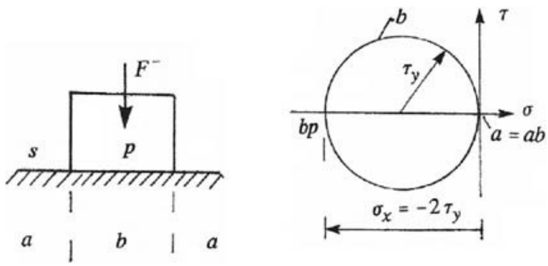


Fig. 10.7.7 Safe statically permissible stress field (10.7.33) with Mohr-diagram

the piston and the material. A lower bound for the indentation force is now directly determined by the boundary condition on the surface *bp*:

$$F^{-} = \left(-\sigma_{yb}\right) \cdot w = 2\tau_y w \tag{10.7.35}$$

A better proposition for a safe statically permissible stress field and with same zones as in Fig. 10.7.7 is:

$$\begin{aligned} &\text{Zone } a: \sigma_{xa} = -2\tau_y,\ \tau_{xya} = \sigma_{ya} = 0 \\ &\text{Zone } b: \sigma_{xb} = -2\tau_y,\ \tau_{xyb} = 0,\ \sigma_{yb} = -\,4\tau_y \end{aligned} \tag{10.7.36}$$

The stress fields in the zones *a* and *b* are homogeneous and satisfy the yield criterion (10.7.32), which means that the material is in a fully developed yielding modus in all three zones. In addition the stress continuity conditions are satisfied on the interfaces between the zones and on the free surface. Figure 10.7.8 shows the Mohr-diagram for the states of stress in the zones. The boundary condition on the contact surface *bp* between the piston and the semi-infinite material gives:

$$F^{-} = \left(-\sigma_{yb}\right) \cdot w = 4\tau_y w \tag{10.7.37}$$

A third proposition for a safe statically permissible stress field is the following, related to the zones defined in Fig. 10.7.9:

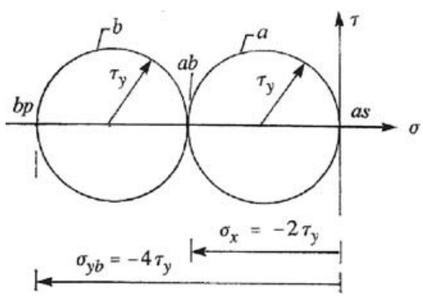


Fig. 10.7.8 Mohr-diagram for the state of stress (10.7.36)

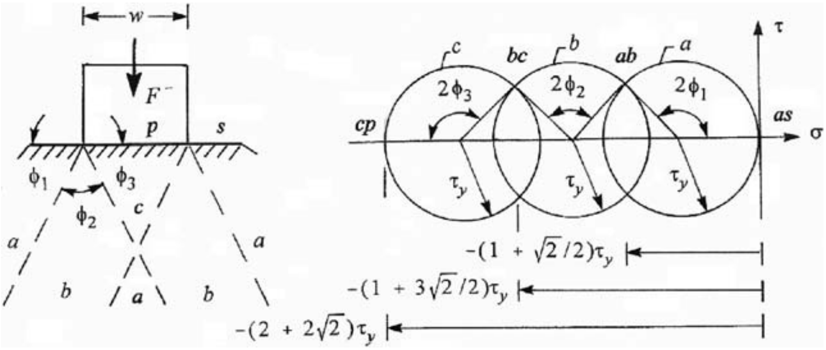


Fig. 10.7.9 Safe statically permissible stress field (10.7.38) with Mohr-diagram

$$\begin{aligned}
&\text{Zone } a: \sigma_{xa} = -2\tau_y,\ \tau_{xya} = \sigma_{ya} = 0 \\
&\text{Zone } b: \sigma_{xb} = \sigma_{yb} = -\left(1+\sqrt{2}\right)\tau_y,\ \tau_{xyb} = -\,\tau_y \\
&\text{Zone } c: \sigma_{xc} = -2\sqrt{2}\,\tau_y,\ \tau_{xyc} = 0,\ \sigma_{yc} = -\left(2+2\sqrt{2}\right)\tau_y
\end{aligned} \tag{10.7.38}$$

The angles defining the zones are: $\phi_1 = \phi_3 = 67.7°$, and $\phi_2 = 45°$. The Mohr stress circles for the three states of stress are shown in the Mohr-diagram in Fig. 10.7.9. Point *as* represents the stress free surface of zone *a*. Point *ab* represents the stresses on the interface between zone *a* and zone *b*:

$$\sigma = -\left(1+\frac{\sqrt{2}}{2}\right)\tau_y,\ \tau = \frac{\sqrt{2}}{2}\tau_y \tag{10.7.39}$$

Point *bc* gives the stresses on the interface between zone *b* and zone *c*:

$$\sigma = -\left(1+\frac{3\sqrt{2}}{2}\right)\tau_y,\ \tau = \frac{\sqrt{2}}{2}\tau_y \tag{10.7.40}$$

Point *cp* represents the stresses on the contact surface to the piston:

$$\sigma = \sigma_{yc} = -\left(2+2\sqrt{2}\right)\tau_y,\ \tau = 0 \tag{10.7.41}$$

The boundary condition on the contact surface gives the lower bound for the limit load:

$$F^- = (-\sigma_{yc})\cdot w = \left(2+2\sqrt{2}\right)\tau_y\,w = 4.83\,\tau_y\,w \tag{10.7.42}$$

To determine an upper bound F^+ for the indentation load, we use the kinematically permissible velocity field indicated in Fig. 10.7.10. We assume that the piston is allowed to rotate. A semi-cylindrical part of the material with radius w rotates with

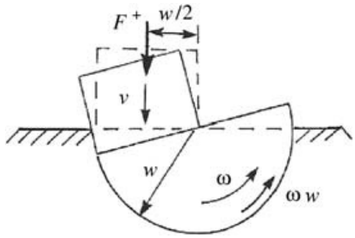


Fig. 10.7.10 Yield mechanism with kinematically permissible velocity field

the angular velocity ω. The velocity in the shear zone becomes ωw, and the velocity for the load F^+ is $v = \omega w/2$. The virtual power theorem gives:

$$P^e = P^i \quad \Rightarrow \quad F^+ \cdot \left(\omega \frac{w}{2}\right) = (\tau_y \cdot \omega\, w) \cdot (\pi w) \quad \Rightarrow$$
$$F^+ = 2\pi\, \tau_y\, w = 6.28\, \tau_y\, w \tag{10.7.43}$$

With a lower bound and an upper bound for the indentation force F_l we have the result:

$$4.83\, \tau_y\, w < F_l < 6.28\, \tau_y\, w \tag{10.7.44}$$

We shall return to this problem in Example 10.14.

10.8 Yield Line Theory

In many forming processes the state of plane strain may be assumed. Such cases are: milling, pressing, extrusion or pressing of plates in which the change in the width is prohibited or not essential, cutting with a wide edge, and indentation by a long narrow tool.

The yield line theory describes the limit state when a *rigid-perfectly plastic Tresca or Mises material* is subjected to plane strain. The adjective "rigid" indicates that the elastic strains are neglected, i. e. $\mathbf{E} = \mathbf{E}^p$, when the loading has reach the limit load. For convenience we shall in this section express plastic strains by plastic strain rates rather than strain increments. We assume that all plastic deformation takes place in the xy-plane, such that:

$$\tau_{zx} = \tau_{zy} = 0, \; \dot{\varepsilon}_z = 0 \tag{10.8.1}$$

Let σ_1 and σ_2 be the principal stresses and $\dot{\varepsilon}_1$ and $\dot{\varepsilon}_2$ be the principal plastic strain rates in the xy-plane. The condition of incompressibility: $\dot{\varepsilon}_v = 0$, and the plane strain condition $\dot{\varepsilon}_z = 0$ imply the flow rule: $\dot{\varepsilon}_1 = -\dot{\varepsilon}_2$. The yield condition for both the Tresca material and the Mises material in plane strain is be expressed by (10.4.28). Hence:

$$\dot{\varepsilon}_1 = -\dot{\varepsilon}_2 \text{ flow rule}$$
$$\sigma_1 - \sigma_2 = 2\tau_y \text{ yield condition} \tag{10.8.2}$$

The yield shear stress τ_y for each of the two types of material models is given by formula (10.4.27). The assumption that both material models are isotropic implies that the principal directions of stresses and plastic strain rates coincide.

Figure 10.8.1 shows the Mohr-diagram for stresses in the *xy*-plane. The maximum shear stress, $\tau_{max} = \tau_y$, acts on planes that makes a 45° angle with the directions of the principal stresses σ_1 and σ_2.

The fact that the principal directions of stresses and plastic strains coincide helps us to draw the Mohr-diagram for plastic strain rates as shown in Fig. 10.8.2. However, as the level of plastic strain rates is unknown, the radius of the strain rate circle is unknown, but by use of the flow rule in (10.8.2) and the known angle ϕ_1 from the Mohr-diagram for stresses in Fig. 10.8.1, the strain rate circle can be drawn. Yielding occurs by sliding on planes with maximum shear stress equal to the yield shear stress, i.e. $\tau_{max} = \tau_y$. In the *xy*-plane the sliding takes place along two sets of orthogonal lines called *yield lines*: α*-lines* and β*-lines*. The direction of the α-line is defined by the angle θ, measured from the horizontal *x*-direction, positive *clockwise*, as shown in Fig. 10.8.3. An α-line and a β-line are drawn in the Mohr-diagram for strains. Note that in the direction *counter-clockwise* the α-line is passed before and the β-line is passed after the σ_1-direction. The general rule is:

In the counter-clockwise direction the α*-line is passed before and the* β*-line after the direction of the largest principal stress.*

Figure 10.8.3 shows a material element with coordinate stresses and an α-line and a β-line through a particle *P* in the element. We let the yield lines represent an

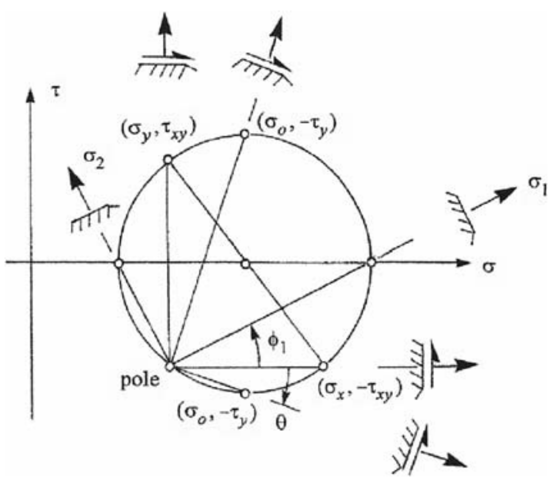


Fig. 10.8.1 Mohr-diagram for stress at plane strain

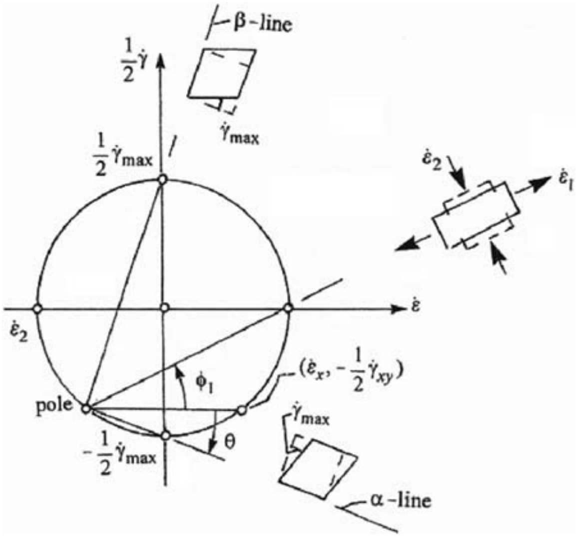


Fig. 10.8.2 Mohr-diagram for plastic strain rates

orthogonal curvilinear coordinate system with metric coordinates s_α and s_β, which measure lengths along the yield lines.

From the Mohr-diagram for stress we derive the stress relations:

$$\tau_y = \frac{1}{2}(\sigma_1 - \sigma_2), \ \sigma_o = \frac{1}{2}(\sigma_1 + \sigma_2)$$
$$\sigma_x = \sigma_o + \tau_y \sin 2\theta, \ \sigma_y = \sigma_o - \tau_y \sin 2\theta, \ \tau_{xy} = \tau_y \cos 2\theta \tag{10.8.3}$$

The equations of equilibrium in the xy-plane are obtained from the Cauchy equations (3.2.37):

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

The equations are transformed by use of the formulas (10.8.3):

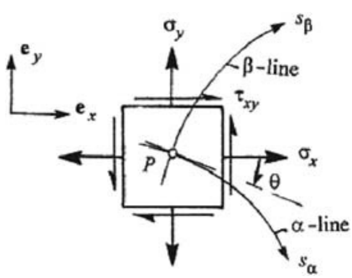


Fig. 10.8.3 Material element with coordinate stresses and yield lines: α – line and β – line

$$\left[\frac{\partial \sigma_o}{\partial x} + 2\tau_y \cos 2\theta \frac{\partial \theta}{\partial x}\right] - 2\tau_y \sin 2\theta \frac{\partial \theta}{\partial y} = 0$$
$$-2\tau_y \sin 2\theta \frac{\partial \theta}{\partial x} + \left[\frac{\partial \sigma_o}{\partial y} - 2\tau_y \cos 2\theta \frac{\partial \theta}{\partial y}\right] = 0$$

If we temporarily choose the *xy-system* such that the angle $\theta = 0$, and with the axes along the tangents to the system $s_\alpha s_\beta$, the above equilibrium equations are reduced to:

$$\frac{\partial \sigma_o}{\partial s_\alpha} + 2\tau_y \frac{\partial \theta}{\partial s_\alpha} = 0, \; \frac{\partial \sigma_o}{\partial s_\beta} - 2\tau_y \frac{\partial \theta}{\partial s_\beta} = 0 \tag{10.8.4}$$

These two equations are integrated along the respective coordinate lines. The resulting equations are:

$$\begin{aligned} &\sigma_o + 2\tau_y\theta = C_\alpha \text{ (constant) along an } \alpha\text{-line} \\ &\sigma_o - 2\tau_y\theta = C_\beta \text{ (constant) along a } \beta\text{-line} \end{aligned} \tag{10.8.5}$$

and are called the *Hencky equations*. They represent equilibrium equations in the material when the yield criterion is satisfied.

Figure 10.8.4 shows a section of a *yield line field*. Using the Hencky equations, we can show that:

$$\theta_B - \theta_A = \theta_D - \theta_C \quad \Leftrightarrow \quad \theta_C - \theta_A = \theta_D - \theta_B \tag{10.8.6}$$

This result is called the *Hencky's first theorem*. From the theorem we may draw the following conclusion:

> The angle is constant between the tangents to any two yield lines from the same family (e.g. two α- lines) at the intersections between the yield lines and any yield line from the other family (points *A* and *C* or points *B* and *D* in Fig. 10.8.4).

With reference to Fig. 10.8.5, it follows that:

> If a segment (*AC*) of a β-line between two α-lines is a straight line, then all line segments (for instance *BD*) of the β-lines between the two α-lines are also straight lines. Furthermore, all the straight elements have the same length.

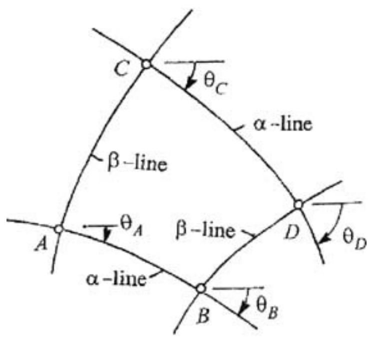
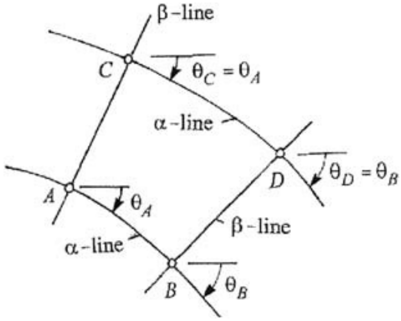


Fig. 10.8.4 Sectiont of a yield line field

Fig. 10.8.5 Straight β – lines



A complete solution of a problem according to the yield line theory requires:

a. A safe statically permissible stress field $\mathbf{T}^s$, i.e. a stress field that satisfies the equilibrium equations and a yield criterion: $f[\mathbf{T}^s] \leq 0$, everywhere.
b. A kinematically permissible velocity field $\mathbf{v}^k$, i.e. a velocity field that satisfies the given displacement boundary conditions.
c. The kinematically permissible velocity field $\mathbf{v}^k$ must give plastic strain increments along the α-lines and the β-lines that are in accordance with the directions of yield shear stress in the yield lines.

If the requirements *b.* and *c.* are satisfied, and $\mathbf{T}^s$ is a statically permissible stress field that satisfies the yield condition $f[\mathbf{T}^s] = 0$ where the velocity field $\mathbf{v}^k$ results in plastic strain rates, then according to the upper bound theorem in Sect. 10.7.2, the solution represents an upper bound for the collapse load.

Example 10.14. Indentation of a Rigid Piston

We seek an exact solution to the problem described in Example 10.12, i.e. we shall find the *limit load* $F = F_l$, as a force per unit length in the direction perpendicular to the figure plane, which is necessary to press a piston into a rigid-perfectly plastic material, such that the process leaves a permanent indentation. Figure 10.8.6 shows the situation and a proposed yield line field. The construction of the yield line

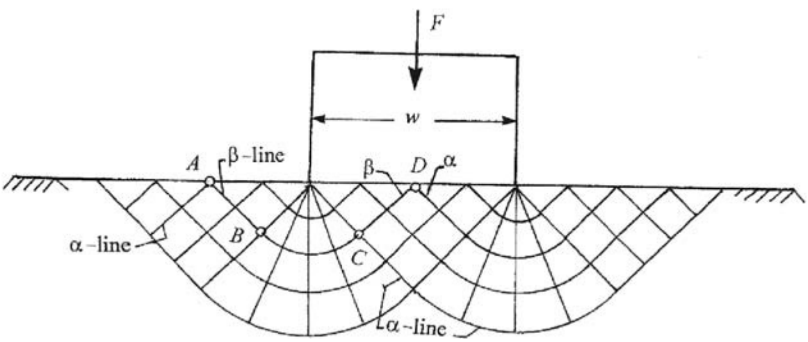


Fig. 10.8.6 Indentation of a piston. Yield line field

field starts at the particle A where we know the principal stress directions and one principal stress, and where the other principal stress obviously is negative:

$$\sigma_1 = 0, \ \sigma_2 < 0 \tag{10.8.7}$$

The state of stress is the same in the particles A and B. Figure 10.8.7 illustrates the situation and the Mohr-diagram of stress. Because it is assumed that the yield condition is satisfied, the radius of the stress circle is equal to the yield shear stress τ_y. The α-line and the β-line are identified from the general rule that: in the counter-clockwise direction the α-line is passed before and the β-line after the vertical direction of the largest principal stress σ_1. From the Mohr-diagram we read:

$$\sigma_o = -\tau_y, \ \sigma_2 = -2\tau_y, \ \theta = \frac{3\pi}{4} \tag{10.8.8}$$

Applying the Hencky-equation (10.8.5) for a β-*line* from particle A to particle B we get:

$$\sigma_o - 2\tau_y\,\theta = C_\beta \Rightarrow \quad C_\beta = -\tau_y - 2\tau_y \cdot \frac{3\pi}{4} = -\tau_y\left(1 + \frac{3\pi}{2}\right) \tag{10.8.9}$$

The β-line is followed from B to the particles C and D. Between C and D the angle θ is equal to $\pi/4$, and the Hencky-equation for the β-line gives:

$$\sigma_o - 2\tau_y\,\theta = C_\beta \quad \Rightarrow \quad \sigma_o = -\tau_y\left(1 + \frac{3\pi}{2}\right) + 2\,\tau_y\frac{\pi}{4} = -\tau_y\,(1+\pi) \tag{10.8.10}$$

Figure 10.8.8 shows the situation at the particles C and D and the Mohr-diagram for the state of stress. We know the directions for the α-line and the β-line. It follows that the direction for the largest principal stress σ_1 is horizontal, and we can localize the pole in the Mohr-diagram. We obtain:

$$\sigma_2 = \sigma_o - \tau_y = -\tau_y\,(2+\pi) \tag{10.8.11}$$

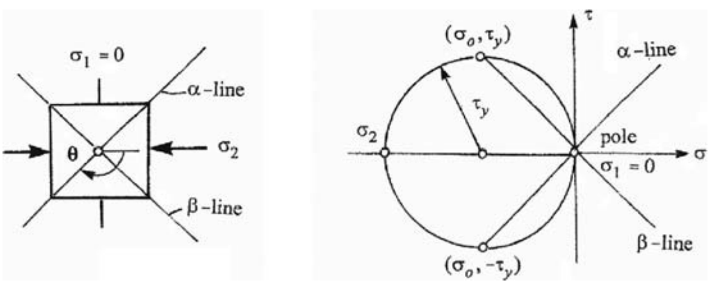


Fig. 10.8.7 The state of stress in the particles A and B

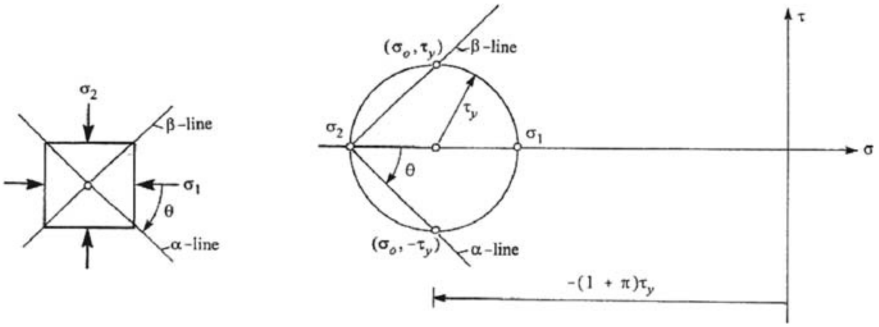


Fig. 10.8.8 The state of stress in the particles C and D

This is the compressive stress that carries the piston. Hence we obtain for the indentation load:

$$F = -\sigma_2 \cdot w = \tau_y (2 + \pi) w = 5.14 \, \tau_y w \tag{10.8.12}$$

It may be shown that this is an exact solution, which means that the *limit load* is: $F_l = 5.14 \ \tau_y w$. The strain rate field is kinematically permissible and the yield condition: $f[\mathbf{T}] \le 0$, is satisfied everywhere. In Example 10.13 we found that:

$$4.83 \, \tau_y w < F_l < 5.66 \, \tau_y w$$

Example 10.15. Plate Extrusion

A rigid-perfectly plastic plate of thickness h is pressed through a die such that the thickness is reduced to $h/2$, see Fig. 10.8.9. We shall determine the necessary press force F per unit width of the plate.

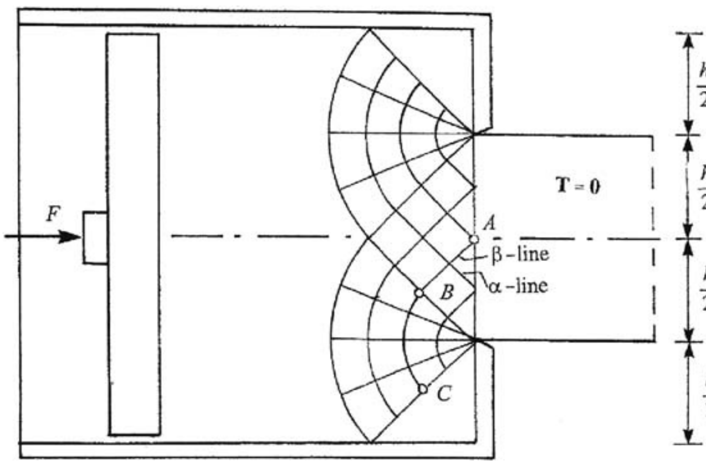


Fig. 10.8.9 Plate extrusion. Yield line field

In Fig. 10.8.9 a suitable yield line field is indicated. Outside of the field we assume that the plate material is rigid. In the extruded plate the material is assume to be stress free, i.e. $\mathbf{T} = \mathbf{0}$. In the particle A we know the principal stress directions and the largest principal stress $\sigma_1 = 0$. The other principal stress is necessarily negative:

$$\sigma_1 = 0, \; \sigma_2 < 0 \tag{10.8.13}$$

Figure 10.8.10 shows the situation at the particles A and B, which lie on the same β-line, and the corresponding Mohr-diagram for stress. Since we assume that the yield condition is satisfied, the radius of the stress circle is equal to the yield shear stress. The α-line and the β-line are identified based on the rule that in counter-clockwise direction the α-line is passed before and the β-line after the horizontal principal direction of the largest principal stress σ_1. From the Mohr-diagram we obtain:

$$\sigma_o = -\tau_y, \; \sigma_2 = -2\tau_y, \; \theta = \frac{\pi}{4} \tag{10.8.14}$$

Applying the Hencky-equation for the β-line, we find:

$$\sigma_o - 2\tau_y\,\theta = C_\beta \quad \Rightarrow \quad C_\beta = -\tau_y - 2\tau_y \cdot \frac{\pi}{4} = -\tau_y\left(1 + \frac{\pi}{2}\right) \tag{10.8.15}$$

The β-line is followed from particle B to particle C where the angle θ is equal to $-\pi/4$, corresponding to a counter-clockwise rotation of the yield line field by the angle $\pi/2$. The Hencky equation for the β-line implies:

$$\sigma_o - 2\tau_y\,\theta = C_\beta \quad \Rightarrow \quad \sigma_o = -\tau_y\left(1 + \frac{\pi}{2}\right) + 2\tau_y\left(-\frac{\pi}{4}\right) = -\tau_y\,(1 + \pi) \tag{10.8.16}$$

Fig. 10.8.11 shows the situation at the particle C and the corresponding Mohr-diagram. Because we know the directions to the α-line and the β-line, we see that the direction of the largest principal stress σ_1 is vertical. This fact determines the pole in the Mohr-diagram. We obtain the result:

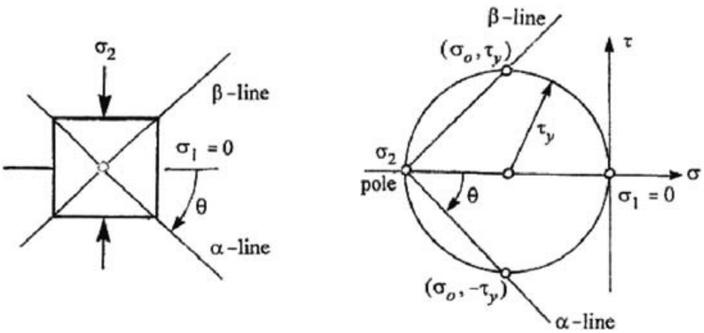


Fig. 10.8.10 The state of stress in the particles A and B

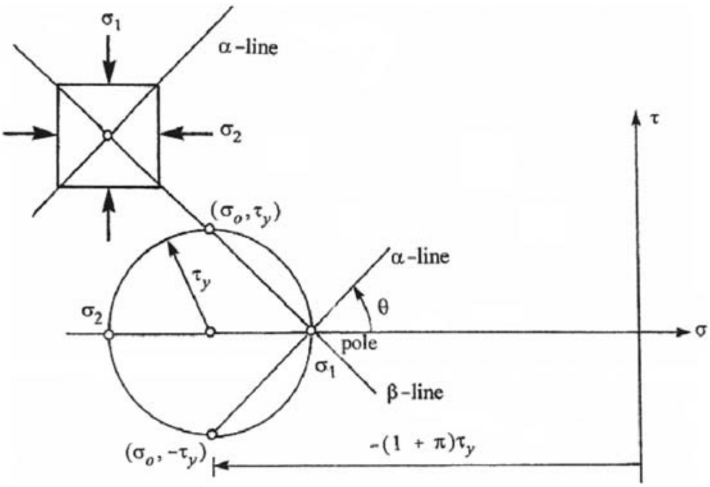


Fig. 10.8.11 The state of stress in particle C

$$\sigma_2 = \sigma_o - \tau_y = -\tau_y (2 + \pi) \tag{10.8.17}$$

It follows from the Hencky-equation for the straight α-line through particle C that all the particle on this α-line have the same state of stress. The horizontal compressive stress σ_2 from the expression (10.8.17) acts on a vertical plane surface with area $(h/2) \cdot 1$ and shall due to symmetry balance half the press force F. Thus we obtain:

$$F = 2\left[-\sigma_2 \cdot \left(\frac{h}{2} \cdot 1\right)\right] = \tau_y (2 + \pi) h = 5.14\, \tau_y h \tag{10.8.18}$$

It may be shown that this is the exact solution. The deformation field is kinematically permissible and the yield criterion is satisfied everywhere.

10.9 Mises Material with Isotropic Hardening

This material model is defined by the yield criterion presented in (10.2.52):

$$f\,[\mathbf{T}, \varepsilon_e^p] = \sigma_M^2 - f_y^2\,(\varepsilon_e^p) = 0 \tag{10.9.1}$$

where:

$$\sigma_M \equiv \sigma_e = \sqrt{3J_2} = \sqrt{\frac{3}{2}\mathbf{T}' : \mathbf{T}'} = \sqrt{\frac{3}{2}\mathbf{T} : \mathbf{T} - \frac{1}{2}\,(\mathrm{tr}\,\mathbf{T})^2} \tag{10.9.2}$$

is the *Mises stress* or *equivalent stress*. The scalar $f_y(\varepsilon_e^p)$ is a *yield stress*, monotonically increasing with the *equivalent plastic strain* ε_e^p, defined by (10.2.48). For uniaxial stress σ the yield stress is given by the function $f_y(\varepsilon^p)$, where ε^p is the plastic strain in the direction of the stress σ and a function of the stress σ.

The flow rule for the Mises material with isotropic hardening will be developed using the general flow rule (10.6.10) for an isotropic material:

$$dE^p_\alpha = d\lambda \frac{\partial f}{\partial T_\alpha}, \ \alpha = 1,2,3,4,5,6 \quad \Leftrightarrow \quad d\mathbf{E}^p = d\lambda \frac{\partial f}{\partial \mathbf{T}} \tag{10.9.3}$$

and the yield criterion (10.9.1). From the expression (10.9.2) for the Mises stress we derive:

$$\frac{\partial \sigma_M}{\partial \mathbf{T}} = \frac{1}{2} \frac{1}{\sqrt{\frac{3}{2}\mathbf{T}:\mathbf{T} - \frac{1}{2}(\mathrm{tr}\,\mathbf{T})^2}} \left[\frac{3}{2} 2\mathbf{T} - \frac{1}{2} 2(\mathrm{tr}\,\mathbf{T})\,\mathbf{1}\right] \quad \Rightarrow$$

$$\frac{\partial \sigma_M}{\partial \mathbf{T}} = \frac{3}{2\sigma_M}\mathbf{T}' \quad \Leftrightarrow \quad \frac{\partial \sigma_M}{\partial T_\alpha} = \frac{3}{2\sigma_M} T'_\alpha \tag{10.9.4}$$

From the general flow rule (10.6.10) for an isotropic material, the yield criterion (10.9.1), and the expression (10.9.4), we obtain:

$$dE^p_\alpha = d\lambda \frac{\partial f}{\partial T_\alpha} = d\lambda \frac{\partial f}{\partial \sigma_M} \frac{\partial \sigma_M}{\partial T_\alpha} = d\lambda\ (2\sigma_M) \frac{3}{2\sigma_M} T'_\alpha \quad \Rightarrow$$

$$dE^p_\alpha = 3 d\lambda\, T'_\alpha \quad \Leftrightarrow \quad d\mathbf{E}^p = 3 d\lambda\, \mathbf{T}' = d\lambda\,[3\,\mathbf{T} - (\mathrm{tr}\,\mathbf{T})\,\mathbf{1}] \tag{10.9.5}$$

In terms of the principal values of stress and plastic strains the flow rule is:

$$d\varepsilon^p_i = d\lambda\,[3\sigma_i - (\sigma_1 + \sigma_2 + \sigma_3)] \tag{10.9.6}$$

The expressions (10.9.5, 10.9.6) for the flow rule are naturally the same as the expressions (10.3.19) for the flow rule for a perfectly plastic Mises material. But while the parameter $d\lambda$ is not determined by the material properties for the perfectly plastic material, we shall see below how a constitutive equation for the parameter $d\lambda$ may be obtained.

In the case of uniaxial stress σ we introduce the function:

$$\alpha(\sigma) = \frac{d\varepsilon^p}{d\sigma} \tag{10.9.7}$$

and write for the plastic strain increment in the direction of the uniaxial stress:

$$d\varepsilon^p = \alpha(\sigma)\, d\sigma = \left[\frac{\alpha(\sigma)}{\sigma} d\sigma\right] \sigma \tag{10.9.8}$$

The flow rule (10.9.6) implies that:

$$d\varepsilon^p = d\lambda\,[3\sigma - \sigma] = 2 d\lambda\, \sigma \tag{10.9.9}$$

Comparing the expressions (10.9.8) and (10.9.9), we obtain the result:

$$d\lambda = \frac{\alpha(\sigma)}{2\sigma} d\sigma \quad \text{for uniaxial stress} \tag{10.9.10}$$

For a general state of stress we set:

$$d\lambda = \frac{\alpha(\sigma_M)}{2\sigma_M} d\sigma_M \quad \text{for general states of stress} \tag{10.9.11}$$

and obtain from the expressions (10.9.5) the *flow rule for the Mises material with isotropic hardening*:

$$d\mathbf{E}^p = \left[\frac{3\alpha(\sigma_M)}{2\sigma_M} d\sigma_M\right] \mathbf{T}' = \left[\frac{\alpha(\sigma_M)}{2\sigma_M} d\sigma_M\right] [3\mathbf{T} - (\text{tr}\,\mathbf{T})\,\mathbf{1}] \tag{10.9.12}$$

The flow rule (10.9.12) will now be transformed such that the plastic strain increment tensor $d\mathbf{E}^p$ is expressed as a function of the stress increment tensor $d\mathbf{T}$. It follows from equation (10.9.4) that:

$$d\sigma_M = \frac{\partial \sigma_M}{\partial \mathbf{T}} : d\mathbf{T} = \frac{3}{2\sigma_M} \mathbf{T}' : d\mathbf{T} \tag{10.9.13}$$

The flow rule (10.9.12) may now be rewritten to:

$$d\mathbf{E}^p = \beta\,\mathbf{T}' \left(\mathbf{T}' : d\mathbf{T}\right) \equiv \beta\,\mathbf{T}' \otimes \mathbf{T}' : d\mathbf{T} \quad \Leftrightarrow \quad dE^p_{ij} = \beta\, T'_{ij} T'_{kl} d T_{kl} \tag{10.9.14}$$

where:

$$\beta = \beta(\sigma_M) = \frac{9\alpha(\sigma_M)}{4\sigma_M^2} \tag{10.9.15}$$

For an isotropic, linearly elastic-plastic material the elastic strain tensor $\mathbf{E}^e$ and the stress tensor $\mathbf{T}$ are related through Hooke's law (7.2.23, 7.2.24):

$$\mathbf{T} = \mathbf{S} : \mathbf{E}^e \quad \Leftrightarrow \quad \mathbf{E}^e = \mathbf{K} : \mathbf{T} \tag{10.9.16}$$

$\mathbf{S}$ is the stiffness tensor (7.2.25) and $\mathbf{K}$ the compliance tensor (7.2.26). It follows from the relations (10.9.16) that:

$$d\mathbf{T} = \mathbf{S} : d\mathbf{E}^e \quad \Leftrightarrow \quad d\mathbf{E}^e = \mathbf{K} : d\mathbf{T} \tag{10.9.17}$$

Alternatives to the formulas (10.9.16) and (10.9.17) for $\mathbf{T}$ and $d\mathbf{T}$ are obtained by adding the relations (7.2.18), which gives:

$$\mathbf{T} = \mathbf{S} : \mathbf{E}^e = 2\mu \mathbf{E}^{e\prime} + 3\kappa \mathbf{E}^{eo}, \; d\mathbf{T} = \mathbf{S} : d\mathbf{E}^e = 2\mu\, d\mathbf{E}^{e\prime} + 3\kappa\, d\mathbf{E}^{eo} \tag{10.9.18}$$

The parameter μ is the *shear modulus* and the parameter κ is the *bulk modulus* of the material. The total strain increment tensor $d\mathbf{E}$ is the sum of the elastic strain increment tensor $d\mathbf{E}^e$ from $(10.9.17)_2$ and the plastic increment tensor $d\mathbf{E}^p$ from (10.9.14):

$$d\mathbf{E} = d\mathbf{E}^e + d\mathbf{E}^p = \left(\mathbf{K} + \beta\,\mathbf{T}' \otimes \mathbf{T}'\right) : d\mathbf{T} \tag{10.9.19}$$

It will be shown below that this relation may be inverted to:

$$d\mathbf{T} = \left(\mathbf{S} - \gamma \mathbf{T}' \otimes \mathbf{T}'\right) : d\mathbf{E} \tag{10.9.20}$$

where:

$$\gamma = \frac{3\mu}{\sigma_M^2}\left[1 + \frac{1}{3\mu\,\alpha(\sigma_M)}\right]^{-1} \tag{10.9.21}$$

The expressions (10.9.19) and (10.9.20) represent the *constitutive equations for a Mises material with isotropic hardening*. The equations are only valid under the condition that if the Mises stress σ_M gives $f\left[\mathbf{T}, \varepsilon_e^p\right] < 0$, i.e. elastic response, the parameters β and γ are set equal to zero.

The inversion of (10.9.19) to obtain (10.9.20) may be performed as follows. The relation between the isotrops and deviators of the strain and stress increment tensors are:

$$d\mathbf{E}^{eo} = \frac{1}{3\kappa} d\mathbf{T}^o,\; d\mathbf{E}^{e\prime} = \frac{1}{2\mu} d\mathbf{T}',\; d\mathbf{E}^{p\prime} = d\mathbf{E}^p = 3\,d\lambda\,\mathbf{T}' \tag{10.9.22}$$

from which:

$$d\mathbf{E}' = \frac{1}{2\mu} d\mathbf{T}' + 3d\lambda\,\mathbf{T}' \tag{10.9.23}$$

We then get:

$$\mathbf{T}' : d\mathbf{E} = \mathbf{T}' : \left(d\mathbf{E}^{e\prime} + d\mathbf{E}^{eo} + d\mathbf{E}^{p\prime}\right) = \frac{1}{2\mu}\mathbf{T}' : d\mathbf{T}' + 3d\lambda\,\mathbf{T}' : \mathbf{T}' \tag{10.9.24}$$

From the definition (10.9.2) of the Mises stress σ_M it follows that:

$$\mathbf{T}' : \mathbf{T}' = \frac{2}{3}\sigma_M^2 \tag{10.9.25}$$

which by (10.9.11) gives:

$$\mathbf{T}' : d\mathbf{T}' = \frac{4\sigma_M^2}{3\alpha(\sigma_M)} d\lambda \tag{10.9.26}$$

From the equations (10.9.24, 10.9.25, 10.9.26) we get:

$$\mathbf{T}' : d\mathbf{E} = \left[\frac{2\sigma_M^2}{3\mu\,\alpha(\sigma_M)} + 2\sigma_M^2\right] d\lambda \quad \Rightarrow$$

$$d\lambda = \frac{1}{2\sigma_M^2}\left[1 + \frac{1}{3\mu\,\alpha(\sigma_M)}\right]^{-1} \mathbf{T}' : d\mathbf{E} = \frac{\gamma}{6\mu}\mathbf{T}' : d\mathbf{E} \tag{10.9.27}$$

Then (10.9.23) and (10.9.27) give:

$$d\mathbf{T}' = 2\mu\,d\mathbf{E}' - \gamma\mathbf{T}'\left(\mathbf{T}' : d\mathbf{E}\right) \tag{10.9.28}$$

The equation: $d\mathbf{T}^{\mathbf{o}} = 3\kappa d\mathbf{E}^{\mathrm{o}}$, relating the isotrops of stress and strain increments and derived from (7.2.18), is added to the deviatoric stress increment from (10.9.28), followed by application of (10.9.18). The result is:

$$d\mathbf{T} = 2\mu d\mathbf{E}' + 3\kappa d\mathbf{E}^{o} - \gamma\mathbf{T}'\left(\mathbf{T}' : d\mathbf{E}\right) = \left(\mathbf{S} - \gamma\mathbf{T}' \otimes \mathbf{T}'\right) : d\mathbf{E}$$

which proves (10.9.20).

10.10 Yield Criteria Dependent on the Mean Stress

When plasticity theory is applied to *geomaterials* as soils, rocks, and concrete, the yield criterion has to include the effect of the mean normal stress:

$$\sigma_m = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}T_{kk} = \frac{1}{3}I \tag{10.10.1}$$

I is the first principal stress invariant.

In this section two such yield criteria will be presented. The reader is referred to the book [7] by Chen and Saleeb for a more detailed discussion of these criteria and extension of them.

10.10.1 The Mohr-Coulomb Criterion

The *Mohr theory* states that yielding starts on a material plane when the shear stress and the normal stress on that plane achieve a critical combination. The simplest combination of the shear stress τ and the normal stress σ on a material plane is given by the *Mohr-Coulomb criterion*:

$$\tau = c - \mu\sigma \quad \Leftrightarrow \quad \text{yielding may start} \tag{10.10.2}$$

The *cohesion c* and the *coefficient of internal friction* μ are two material parameters. The friction coefficient is alternatively given by the *friction angle* ϕ:

$$\tan\phi = \mu \tag{10.10.3}$$

Figure 10.10.1 shows three stress circles in a Mohr-diagram based on the minimum, the intermediate, and the maximum principal stresses: $\sigma_{\min}, \sigma_{\mathrm{int}}$, and $\sigma_{\max}$. It is shown in Sect. 3.3.7 that the state of stress (σ, τ) on any plane through a particle may be represented by a point within the area outside the two smallest stress circles and inside the largest stress circle. The formula in the yield criterion (10.10.2) is represented by a straight line in the Mohr-diagram, and the criterion is satisfied when

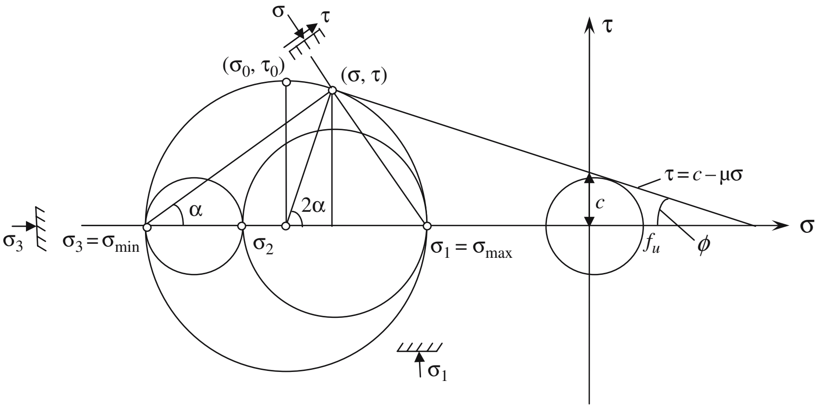


Fig. 10.10.1 Mohr diagram for general state of stress

the greatest stress circle touches the straight line. This circle may be expressed by:

$$\sigma = \sigma_o + \tau_o \cos 2\alpha, \ \tau = \tau_o \sin 2\alpha \tag{10.10.4}$$

where:

$$\sigma_o = \frac{1}{2}\left(\sigma_{\max} + \sigma_{\min}\right), \ \tau_o = \tau_{\max} = \frac{1}{2}\left(\sigma_{\max} - \sigma_{\min}\right) \tag{10.10.5}$$

τ_o is the radius of the stress circle, and α is the angle between the yield plane and the principal direction of the minimum normal stress $\sigma_{\min}$. It follows from Fig. 10.10.1 that:

$$\cot 2\alpha = \tan\phi = \mu \quad \Rightarrow \quad 2\alpha = \frac{\pi}{2} - \phi \ , \ \sin 2\alpha = \cos\phi \ , \ \cos 2\alpha = \sin\phi \tag{10.10.6}$$

By application of (10.10.3–10.10.6) the Mohr-Coulomb criterion (10.10.2) may be rewritten to:

$$\begin{aligned} \tau_o \cos\phi = c - \tan\phi \ (\sigma_o + \tau_o \sin\phi) \quad &\Rightarrow \\ \tau_o + \sigma_o \sin\phi = c\cos\phi \quad \Leftrightarrow \quad &\text{yielding may start} \end{aligned} \tag{10.10.7}$$

In terms of the principal stresses the criterion is:

$$\left(\sigma_{\max} - \sigma_{\min}\right) + \left(\sigma_{\max} + \sigma_{\min}\right)\sin\phi = 2\, c\cos\phi \quad \Leftrightarrow \quad \text{yielding may start} \tag{10.10.8}$$

The yield stress in uniaxial tension and compression are respectively:

$$f_y^+ = \frac{2c\cos\phi}{1+\sin\phi}, \ f_y^- = \frac{2c\cos\phi}{1-\sin\phi} \tag{10.10.9}$$

However, for materials weak in tension it is necessary to introduce a limit to the maximum tensile stress:

$$\sigma_{\max} \leq f_u \equiv \text{ the tensile strength} \tag{10.10.10}$$

This is called the *tension cutoff condition*. In Fig. 10.10.1 a stress circle with maximum normal stress f_u illustrates the extreme position of the largest of the three stress circles for which the yield criterion may be used for materials weak in tension. Also, to account for the limited compressive ductility of some materials, e.g. concrete, a maximum principal compressive strain criterion may be introduced: When the maximum compressive strain reaches a critical value, crushing of the material is assumed to occur and the material loses its strength.

The yield shear stress in pure shear, e.g. $\sigma_1 = -\sigma_3 = \tau_y$ and $\sigma_2 = 0$, is:

$$\tau_y = c\cos\phi \tag{10.10.11}$$

For the special case of zero internal friction: $\phi = 0$, the yield criterion (10.10.2) is reduced to the Tresca yield criterion with:

$$f_y^+ = f_y^- = f_y = 2c, \;\; \tau_y = \tau_{yT} = c = \frac{f_y}{2} \tag{10.10.12}$$

For plane stress, with $\sigma_3 = 0$, the Mohr-Coulomb criterion on the form (10.10.8) may be illustrated in a $\sigma_1\sigma_2$ – plane as shown in Figure 10.10.2a. Yielding occurs when the stress point (σ_1, σ_2) reaches any of the six lines of the hexagon. In the principal stress space, i.e. the $\sigma_1\sigma_2\sigma_3$ – space, the yield surface is a hexagonal pyramid about the axis $\sigma_1 = \sigma_2 = \sigma_3$.

It is shown in Sect. 3.3 that the principal stresses may be expressed in terms of the mean stress $\sigma_m = \mathrm{tr}\mathbf{T}/3$ and the principal invariants J_2 and J_3 of the stress deviator. Thus the Mohr-Coulomb yield criterion (10.10.8) may be described by an equation of the form:

$$f(J_2, J_3, \sigma_m) = 0 \tag{10.10.13}$$

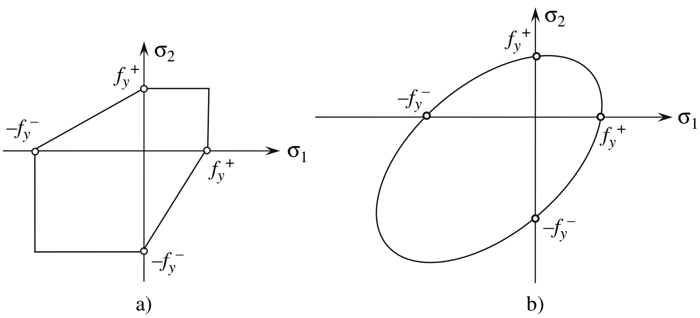


Fig. 10.10.2 **a)** Mohr-Coulomb criterion and **b**) Drucker-Prager criterion in plane stress

10.10.2 The Drucker-Prager Criterion

The Mohr-Coulomb criterion may be considered to be an extension of the Tresca yield criterion. A similar extension of the Mises yield criterion has been proposed by Drucker and Prager in 1952. The *Drucker-Prager yield criterion* is expressed by the condition:

$$\tau_{\text{oct}} = \sqrt{\frac{2}{3}}\,\tau_y - \mu\,\sigma_m \quad \Leftrightarrow \quad \text{yielding may start} \tag{10.10.14}$$

The octahedral shear stress τ_{oct} is defined by (10.2.26):

$$\tau_{\text{oct}} = \sqrt{\frac{2}{3}J_2} \tag{10.10.15}$$

σ_m is the mean normal stress and μ is a friction coefficient. The parameter τ_y is the *yield shear stress* obtained for yielding in pure shear:

$$\sigma_{\max} = -\sigma_{\min} = \tau_y,\ \sigma_{\text{int}} = 0,\ \sigma_m = 0$$

To see this we compute J_2 and the octahedral shear stress for pure shear:

$$J_2 = \frac{1}{2}T'_{ij}T'_{ij} = \tau_y^2,\ \tau_{\text{oct}} = \sqrt{\frac{2}{3}J_2} = \sqrt{\frac{2}{3}}\,\tau_y$$

When the expression (10.10.15) for the octahedral shear stress is introduced into the yield criterion (10.10.14), we obtain the associated *yield function* $f[\mathbf{T}]$ and the alternative expression for the Drucker-Prager yield criterion:

$$\begin{aligned} &f[\mathbf{T}] = f(I, J_2) = \sqrt{J_2} + \sqrt{\frac{3}{2}}\,\mu\,\sigma_m - \tau_y \\ &f[\mathbf{T}] = 0 \quad \Leftrightarrow \quad \text{yielding may start} \end{aligned} \tag{10.10.16}$$

The yield surface in the principal stress space, described by: $f[\mathbf{T}] = 0$, is a circular cone about the axis $\sigma_1 = \sigma_2 = \sigma_3$. For plane stress, with $\sigma_3 = 0$, the Drucker-Prager criterion is illustrated in the $\sigma_1\sigma_2$ – plane in Fig. 10.10.2b by the yield curve:

$$\sqrt{\frac{1}{3}\left(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2\right)} + \sqrt{\frac{1}{6}}\,\mu\,(\sigma_1 + \sigma_2) - \tau_y = 0 \quad \Leftrightarrow \quad \text{yielding may start} \tag{10.10.17}$$

The yield stress in uniaxial tension and compression are respectively:

$$f_y^+ = \frac{\sqrt{6}}{\sqrt{2} + \mu}\tau_y,\ f_y^- = \frac{\sqrt{6}}{\sqrt{2} - \mu}\tau_y \tag{10.10.18}$$

Again the tension cutoff condition (10.10.10) may be needed.

10.11 Viscoplasticity

10.11.1 Introduction

The classical theory of viscoplasticity describes deformation rate dependent behavior of a material after a well-defined yield criterion has been satisfied by the stress state. A yield function:

$$f[\mathbf{T}, \theta, \kappa], \ \kappa = \{\kappa_1, \kappa_2, \ldots, \kappa_n\} \tag{10.11.1}$$

is defined. The parameter θ is the temperature and the scalars κ are some internal variables or hardening parameters defined such that:

$$\begin{aligned} f[\mathbf{T}, \theta, \kappa] \geq 0 \quad &\Leftrightarrow \quad \text{elasto-viscoplastic response} \\ f[\mathbf{T}, \theta, \kappa] < 0 \quad &\Leftrightarrow \quad \text{elastic response} \end{aligned} \tag{10.11.2}$$

Thus a yield criterion has been introduced.

The term viscoplasticity is also used by some authors for models of highly viscoelastic behavior without any purely elastic region, i.e. without a yield criterion. All viscoelastic models with fluidlike behavior may be said to comply with this interpretation of viscoplasticity.

Although any yield criterion may be selected freely in a presentation of viscoplastic material models, we shall choose the Mises criterion for perfectly plastic behavior. Thus the yield function is:

$$f[\mathbf{T}] = \sigma_M^2 - f_y^2, \ \sigma_M = \sqrt{3J_2}, \ J_2 = \frac{1}{2}\mathbf{T}' : \mathbf{T}' \tag{10.11.3}$$

σ_M is the *Mises stress*. For simplicity we have suppressed the temperature dependency and left out any internal variables.

10.11.2 The Bingham Elasto-Viscoplastic Models

In the Chaps. 1 and 9 elastic, elastoplastic, viscoelastic, and elasto-viscoplastic behavior of materials in uniaxial stress was illustrated by means of mechanical models consisting of *springs, viscous dampers*, and *friction elements*. See the Figs. 1.4.4, and 9.2.2–9.2.8. Elasto-viscoplastic response to uniaxial stress may be illustrated by the two typical mechanical models in Fig. 10.11.1.

The model in Fig. 10.11.1 a may be called a *Bingham-Maxwell model* and is represented by the response equations:

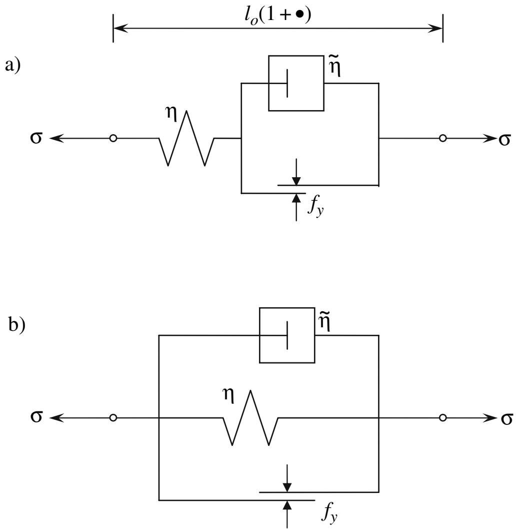


Fig. 10.11.1 Elasto-viscoplastic models: **a)** Bingham-Maxwell model, **b)** Bingham-Kelvin model

$$\begin{aligned} &\sigma = \eta\varepsilon \quad \text{when}\,|\sigma| < f_y \\ &\frac{1}{\tilde{\eta}}\left[1 - \frac{f_y}{|\sigma|}\right]\sigma + \frac{\dot{\sigma}}{\eta} = \dot{\varepsilon} \quad \text{when}\,|\sigma| > f_y \end{aligned} \tag{10.11.4}$$

The model has one elasticity η, one viscosity $\tilde{\eta}$, and a yield stress f_y. For zero yield stress the model is reduced to a Maxwell model illustrated in Fig. 9.2.4. If the viscous element is non-linear and represented by the strain rate:

$$\dot{\varepsilon} = \frac{\dot{\varepsilon}_c}{\sigma_c}\left|\frac{\sigma}{\sigma_c}\right|^{n-1}\sigma \tag{10.11.5}$$

defined in (9.6.11), the model in Fig. 10.11.1a may be called a *Bingham-Norton model*, and the response equation for $|\sigma| > f_y$ is:

$$\frac{\dot{\varepsilon}_c}{\sigma_c}\left[1 - \frac{f_y}{|\sigma|}\right]^n \left|\frac{\sigma}{\sigma_c}\right|^{n-1}\sigma + \frac{\dot{\sigma}}{\eta} = \dot{\varepsilon} \quad \text{when}\,|\sigma| > f_y \tag{10.11.6}$$

The model in Fig. 10.11.1b, with the similar parameters as the one in Fig. 10.11.1a, may be called a *Bingham-Kelvin model* and is represented by the response equations:

$$\begin{aligned} &\varepsilon = 0 \quad \text{when}\,|\sigma| < f_y \\ &\sigma = \eta\,\varepsilon + \left[\tilde{\eta} + \frac{f_y}{|\dot{\varepsilon}|}\right]\dot{\varepsilon} \quad \text{when}\,|\sigma| > f_y \end{aligned} \tag{10.11.7}$$

For $|\sigma| < f_y$ the model behaves a rigid body. For zero yield stress the model is reduced to a Kelvin model illustrated in Fig. 9.2.5.

When generalizing from uniaxial stress response to material response for general states of stress, we shall use the response equation (10.11.4) based on the mechanical model in Figure 10.11.1a as reference. We assume that the material is isotropic and that the deformations are small. The *Bingham-Maxwell elasto-viscoplastic fluid* is defined by the response equations:

$$\begin{aligned} &\mathbf{T}^o = 3\kappa\mathbf{E}^o \\ &\mathbf{T}' = 2\mu\,\mathbf{E}' \quad \text{when } f[\mathbf{T}] < 0 \\ &\frac{1}{2\tilde{\mu}}\left[1 - \frac{f_y}{\sigma_M}\right]\mathbf{T}' + \frac{1}{2\mu}\dot{\mathbf{T}}' = \dot{\mathbf{E}}' \quad \text{when } f[\mathbf{T}] > 0 \end{aligned} \tag{10.11.8}$$

This material model has two elasticities: a bulk modulus κ and a shear modulus μ, one viscosity $\tilde{\mu}$, and a yield stress f_y. As in Chap. 9 on viscoelasticity and the previous sections in Chap. 10 for elastoplastic materials, it is assumed that isotropic stress-strain response is purely elastic.

For the case of a simple shear rate history:

$$\dot{E}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\dot{\gamma}(t)}{2} \quad \Leftrightarrow \quad T(t) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tau(t),\ \sigma_M = \sqrt{3}\,\tau(t) \tag{10.11.9}$$

the response equations (10.11.8) are reduced to:

$$\begin{aligned} &\tau = \mu\,\gamma \quad \text{when } \tau < \tau_y = \frac{f_y}{\sqrt{3}} \\ &\frac{1}{\tilde{\mu}}\left[1 - \frac{f_y}{\sqrt{3}\,|\tau|}\right]\tau + \frac{1}{\mu}\dot{\tau} = \dot{\gamma} \quad \text{when } \tau > \tau_y = \frac{f_y}{\sqrt{3}} \end{aligned} \tag{10.11.10}$$

Equation (10.11.10) is an analogous form of (10.11.4) for uniaxial stress. The first terms in $(10.11.8)_3$ and $(10.11.10)_2$ are viscoplastic contributions to the strain rates:

$$\dot{\mathbf{E}}^p = \frac{1}{2\tilde{\mu}}\left[1 - \frac{f_y}{\sigma_M}\right]\mathbf{T}',\ \dot{\gamma}^p = \frac{1}{\tilde{\mu}}\left[1 - \frac{\tau_y}{|\tau|}\right]\tau \tag{10.11.11}$$

Note that alternatively the term f_y/σ_M may be replaced by $\tau_y/\sqrt{J_2}$. The two equation (10.10.11) may be inverted to give the results:

$$\mathbf{T}' = 2\tilde{\mu}\left[1 + \frac{\tau_y}{2\tilde{\mu}\sqrt{-II}}\right]\dot{\mathbf{E}}^p,\ \tau = \tilde{\mu}\left[1 + \frac{\tau_y}{\tilde{\mu}\,|\dot{\gamma}^p|}\right]\dot{\gamma}^p \tag{10.11.12}$$

II is the second principal invariant of the plastic strain rate tensor:

$$II = -\frac{1}{2}\dot{\mathbf{E}}^p : \dot{\mathbf{E}}^p \tag{10.11.13}$$

To obtain (10.11.12) from (10.11.11) we first use (10.11.11) to compute the scalar product:

$$\dot{\mathbf{E}}^p : \dot{\mathbf{E}}^p = \left(\frac{1}{2\tilde{\mu}}\left[1 - \frac{f_y}{\sigma_M}\right]\right)^2 \mathbf{T}' : \mathbf{T}' \Rightarrow \sqrt{-II} = \sqrt{\frac{1}{2}\dot{\mathbf{E}}^p : \dot{\mathbf{E}}^p} = \frac{1}{2\tilde{\mu}}\left[1 - \frac{f_y}{\sigma_M}\right]\sqrt{J_2}$$

After some algebra we obtain:

$$\frac{1}{1 - f_y/\sigma_M} = 1 + \frac{\tau_y}{2\tilde{\mu}\sqrt{-II}} \tag{10.11.14}$$

Using this result we easily see how (10.10.12) are obtained from (10.10.11) by inversion.

When large deformations are accepted, we may disregard the elastic strains, and the rate of strain tensor $\dot{\mathbf{E}}^p$ must be replaced by the deformation rate tensor $\mathbf{D}$. The response equations (10.11.8) and $(10.11.12)_1$ are now replaced by:

$$\mathbf{D} = \frac{1}{2\tilde{\mu}}\left[1 - \frac{f_y}{\sigma_M}\right]\mathbf{T}',\ \mathbf{T} = -p\mathbf{1} + 2\tilde{\mu}\left[1 + \frac{\tau_y}{\tilde{\mu}\dot{\gamma}}\right]\mathbf{D} \quad \text{when } \tau_{\max} \geq \tau_y$$

$$\mathbf{T} \text{ is undetermined and } \mathbf{D} = \mathbf{0} \quad \text{when } \tau_{\max} \leq \tau \tag{10.11.15}$$

p is the indeterminate pressure for incompressible fluids and $\dot{\gamma}$ is the *magnitude of shear rate*:

$$\dot{\gamma} = \sqrt{2\mathbf{D} : \mathbf{D}} \tag{10.11.16}$$

The material defined by the constitutive equations (10.11.15) is the *Bingham fluid* presented in Sect. 8.6. In simple shear flow defined by (10.11.9) when the strain matrix E is replaced by the deformation rate matrix D, $(10.11.15)_2$ yields the shear stress:

$$\tau = \tilde{\mu}\left[1 + \frac{\tau_y}{\tilde{\mu}\,|\dot{\gamma}|}\right]\dot{\gamma} \quad \text{when } \dot{\gamma} \neq 0 \tag{10.11.17}$$

which is called the *Bingham law*. In Sect. 8.6 the Bingham law was presented as:

$$\tau = \eta\,(\dot{\gamma})\,\dot{\gamma},\ \eta\,(\dot{\gamma}) = \tilde{\mu} + \frac{\tau_y}{\dot{\gamma}},\ \tilde{\mu} \text{ is a constant viscosity} \tag{10.11.18}$$

$\eta\,(\dot{\gamma})$ is the *viscosity function*.

The *Bingham-Norton fluid* is defined by the constitutive equation:

$$\frac{\dot{\varepsilon}_c}{\sigma_c}\left[1 - \frac{f_y}{\sigma_M}\right]^n \left|\frac{\sigma_M}{\sigma_c}\right|^{n-1} \mathbf{T}' + \frac{1+\nu}{\eta}\dot{\mathbf{T}} - \frac{\nu}{\eta}\left(\mathrm{tr}\,\dot{\mathbf{T}}\right)\mathbf{1} = \dot{\mathbf{E}} \tag{10.11.19}$$

which is a generalization of the uniaxial equation (10.11.6). For large deformations we may disregard elastic strains, and the constitutive equation is:

$$\frac{\dot{\varepsilon}_c}{\sigma_c}\left[1-\frac{f_y}{\sigma_M}\right]^n\left|\frac{\sigma_M}{\sigma_c}\right|^{n-1}\mathbf{T}' = \mathbf{D} \quad \text{when } \tau_{\max} \geq \tau_y$$
$$\mathbf{T} \text{ is undetermined and } \mathbf{D} = \mathbf{0} \quad \text{when } \tau_{\max} \geq \tau_y \qquad (10.11.20)$$

Examples of applications of the model may be found in Prager's book [39]: "Introduction to Mechanics of Continua" [1961]. In that book it is pointed out that there is a similar correspondence between the constitutive equations of the viscoplastic model and the perfectly plastic Mises material as there is between the constitutive equations of the Newtonian fluid and the perfect fluid, i.e. the inviscid or Euler fluid. Therefore, if the viscosity is small, it is reasonable to expect that the effects of viscosity on the behavior of the Bingham elasto-viscoplastic models defined above are restricted to boundary layers, just as in a viscous fluid of small viscosity.

Problems

Problem 10.1. The yield surface for a Mises material in the principal stress space, see Fig. 10.2.3, is a circular cylinder with radius R. Points on the yield surface represent states of stress at yielding. These points are also given by the position vector $\mathbf{r}$ and the axis of the cylinder is given by the unit vector $\mathbf{e}$:

$$\mathbf{r} = [\sigma_1, \sigma_2, \sigma_3], \; \mathbf{e} = [1, 1, 1]/\sqrt{3}$$

Equations (10.2.10, 10.2.11) related to uniaxial stress show that the radius of the cylinder is:

$$R = \sqrt{2/3}\, f_y$$

Derive the version of the Mises yield criterion (10.2.13) from the geometrical observation (10.2.12).

Problem 10.2. Derive the expressions (10.2.18) for the Mises yield function.

Problem 10.3. A thick-walled circular cylindrical container with internal radius $a = 50\,\text{mm}$ and external radius $b = 100\,\text{mm}$ is closed in both ends and subjected to an internal pressure p. The yield stress is $f_y = 250\,\text{MPa}$. The material is isotropic and is modelled as a linearly elastic-perfectly plastic material. Determine the pressure that results in yielding according to a) The Mises criterion, and b) The Tresca criterion.

Answers: a) 108 MPa, b) 94 MPa.

Problem 10.4. A thin-walled pipe of length L, radius r, and wall thickness h is subjected to a tensile force N and a torque $M = Nr/2$. The pipe material is modelled as an elastic-perfectly plastic Mises-material. The yield stress is f_y.

a) Determine the value of N for which yielding starts.
b) Due to plastic strains the pipe elongates ΔL. Determine the resulting angle of twist $\Delta\phi$.

Answers: *a*) $4\pi r h f_y/\sqrt{7}$, *b*) $3\Delta L/2r$

Problem 10.5. A thin-walled pipe of length L, radius r, and wall thickness $h << r$ is closed in both ends. The pipe is subjected to an internal pressure p and a torque M, which is proportional with the pressure: $T = 2\pi r^3 p$. The pressure is increased until yielding occurs. Due to the plastic deformation the length of the pipe is increased by ΔL and the radius by Δr. In addition the pipe is twisted by the angle $\Delta\ \phi$. The pipe is modelled as a perfectly plastic Mises material with yield stress f_y.

a) Find the pressure at yielding.
b) Determine the ratios: $\Delta L/\Delta r/r\Delta\phi$.

Problem 10.6. A thin-walled pipe of length L, radius r, and wall thickness $h << r$ is subjected to an axial force N, a torque $M = Nr$ and an internal pressure $p = N/2\Pi r^2$. The pressure does not contribute to stresses on the cross-section of the pipe. The material is modelled as a perfectly plastic Mises material with yield stress f_y. The axial force N is increased until yielding takes place. After unloading the pipe has obtained a permanent angle of twist $\Delta\phi$.

a) Find an expression for the axial force N at yielding.
b) Determine the permanent elongation ΔL of the pipe.

Problem 10.7. Solve Problem 10.6 when the pipe is modelled as a Tresca material.

Problem 10.8. A plate is subjected to a load that results in a plane state of stress given by the stress matrix:

$$T = \begin{pmatrix} 21 & 12 & 0 \\ 12 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sigma$$

in a Cartesian coordinate system Ox. The stress parameter σ at yielding is to be determined. The yield stress of the material is $f_y = 350\,\text{MPa}$. The plate is then unloaded such that it becomes stress free. The plastic strain increment in the x_1-direction is measured to be ε.

a) The plate is modelled as an elastic-perfectly plastic Mises material. Compute the stress parameter σ at yielding. Determine the strain increment matrix in the Ox-system.
b) The plate is modelled as an elastic-perfectly plastic Tresca material. Draw the Mohr-diagram for stresses. Compute the stress parameter σ at yielding. Draw the Mohr-diagram for plastic strain increments and determine the plastic strain increment matrix in the Ox-system.

Problem 10.9. The elastic-perfectly plastic Mises material is chosen as the model for the cylinder in Example 10.10. Show that the limit pressure is given by:

$$p_l = \frac{2}{\sqrt{3}} \ln\left(\frac{b}{a}\right) f_y \equiv 2\ln\left(\frac{b}{a}\right) \tau_{yM}$$

Chapter 11
Constitutive Equations

11.1 Introduction

Macroscopic thermomechanical properties of materials are in general described by constitutive equations. The preceding chapters have treated the classical theory of linearly elastic materials, the mechanics of inviscid and linearly viscous fluids, linearly viscoelastic material, plastic and viscoplastic materials. Some aspects of non-linear constitutive theory have been mentioned. It is the purpose of the present chapter to give a broader and more general exposition of the theory of modelling solid and fluid materials.

A body of continuous matter moves with respect to a reference *Rf*. At a reference time t_o a particle of the body is at the place $\mathbf{r}_\mathrm{o}$ given by the particle coordinates X_i in a Cartesian coordinate system Ox, see Fig. 11.1.1. The vector $\mathbf{r}_\mathrm{o}$ or the coordinates X_i identify the particle. The set of places representing the body at any one time is called a configuration of the body at that time. Figure 11.1.1 shows the *reference configuration* K_o at the time t_o, the *present configuration* K at the present time t, and the "moving" *configuration* $\bar{K}$ at the time $\bar{t} \leq t$. At the present time the particle $\mathbf{r}_o$ is at the place $\mathbf{r}$ with coordinates x_i. The motion of the body is described alternatively by the functions:

$$\mathbf{r} = \mathbf{r}(\mathbf{r}_o, t) \quad \Leftrightarrow \quad x_i = x_i(X, t) \tag{11.1.1}$$

$$\bar{\mathbf{r}} = \bar{\mathbf{r}}(\mathbf{r}_o, \bar{t}) \quad \Leftrightarrow \quad \bar{x}_i = \bar{x}_i(X, \bar{t}),\ \bar{t} \leq t \tag{11.1.2}$$

A *dynamic process* is defined by a given motion $\mathbf{r}(\mathbf{r}_o, t)$ and a stress field $\mathbf{T}(\mathbf{r}_\mathrm{o}, t)$ that together with the proper body forces $\mathbf{b}(\mathbf{r}_o, t)$ satisfy the Cauchy equations of motion:

$$\mathrm{div}\mathbf{T} + \rho\mathbf{b} = \rho\ddot{\mathbf{r}} \tag{11.1.3}$$

In order to find the relevant dynamic process that satisfies the given initial conditions and the boundary conditions for contact forces $\mathbf{t}(\mathbf{r}_o, t)$, displacements $\mathbf{u}(\mathbf{r}_o, t)$, and velocities $\mathbf{v}(\mathbf{r}_o, t)$, the material properties have to be taken into account. This is done through the selection of a proper *material model* defined by *constitutive equations*

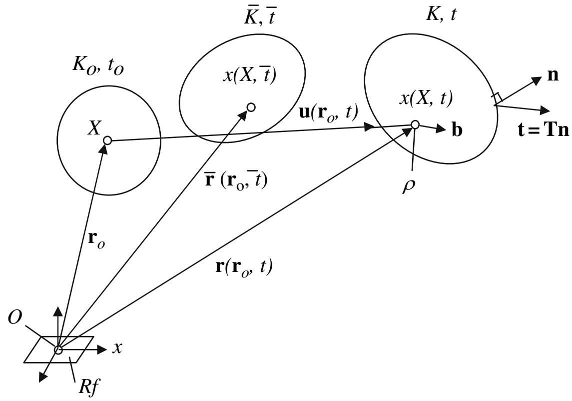


Fig. 11.1.1 Body in the reference configuration K_o, the "moving" configuration $\bar{k}$, and the present configuration K

relating the state of stress and the state of deformation. In Chap. 7 and Chap. 8 the three most common material models have been presented: the *Hookean solid* defined by the *generalized Hooke's law*, the *Eulerian fluid* defined by an isotropic pressure, and the *Newtonian fluid* defined by the *Cauchy-Poisson law*. Other more complex material models were also presented in Chap. 7, 8, 9, and 10. In a purely dynamic process a *purely mechanical material model* is used. The constitutive equations are relations between the stress tensor $\mathbf{T}[\mathbf{r}_o, t]$ at the present time t and measures of deformation derived from the motion $\mathbf{r}(\mathbf{r}_o, \bar{t})$ for $\bar{t} \leq t$. Hooke's law gives an example of a purely mechanical constitutive equation. The elastic - ideally plastic materials, presented in Chap. 10, are examples of material models without complete sets of constitutive equations. In the general three-dimensional case the constitutive equations are 6 relations for the 6 coordinate stresses T_{ij}, related to a Cartesian coordinate system Ox, which together with the 3 component equations in the Cauchy equations of motion (11.1.3) provide 9 equations for the 9 unknown functions $T_{ij}(X,t)$ and $x_i(X,t)$.

A *thermomechanical process* is defined in Sect. 6.3.2. The material model will be defined by *thermomechanical constitutive equations* defined through relations between the stress tensor $\mathbf{T}$, the temperature $\theta(\mathbf{r}_o, t)$, or other thermodynamic quantities, and deformation quantities derived from the motion $\mathbf{r}(\mathbf{r}_o, \bar{t})$ for $\bar{t} \leq t$. The Cauchy-Poisson law (8.4.6), in which the pressure is the thermodynamic pressure $p = p(\rho, \theta)$ and the viscosity is temperature dependent: $\mu = \mu(\theta)$, provides an example of a set of *thermomechanical constitutive equations*.

The present chapter concentrates on general principles for constructing constitutive equations of purely mechanical material models. Section 11.2 introduces the concepts of *reference invariant* or *objective tensor quantities* and *reference related tensor quantities*. An objective quantity is independent of the reference chosen to describe the motion of the body. Examples of objective quantities are: the pressure p, the temperature θ, contact forces $\mathbf{t}$, the stress tensor $\mathbf{T}$, and the deformation rate tensor $\mathbf{D}$. The material derivative of a reference invariant quantity does not lead to a new reference invariant tensor field. This fact motivates the introduction of the concept the *corotational derivative* of a tensor field presented in Sect. 11.3, and of the *convected derivative* of a tensor field in Sect. 11.4. The general principles of constitutive modelling are the main topics of Sect. 11.5. Material symmetry and the classification of the symmetry properties of materials are presented in Sect. 11.6. Short presentations of thermoelastic materials and thermoviscous materials are given in Sect. 11.7 and Sect. 11.8. Section 11.9 presents some advanced fluid models.

11.2 Objective Tensor Fields

Let *Rf* and Rf^* be two references that move relative to each other. In each reference we introduce Cartesian coordinate systems: *Ox* in *Rf* and O^*x^* in Rf^*. In Fig. 11.2.1 the reference *Rf* is at rest relative to the plane of the figure, and the figure shows the situation at the present time t.

A *place in space* at the time t is given by the position vectors $\mathbf{r}$ and $\mathbf{r}^*$: To give the concept of a place in space an objective meaning, we must define a place relative to a reference. A *place in the reference Rf* is thus defined as the point given by a constant, time independent position vector $\mathbf{r}$, or by the time dependent position vector:

$$\mathbf{r}^*(t) = \mathbf{c}(t) + \mathbf{r} \tag{11.2.1}$$

Similarly, a *place in the reference* Rf^* is defined as the point given by a constant, time independent position vector $\mathbf{r}^*$ or by the position vector:

$$\mathbf{r}(t) = -\mathbf{c}(t) + \mathbf{r}^* \tag{11.2.2}$$

This means that the concept *a place in space* has only an objective meaning at a particular time t.

It is convenient to introduce the concept of events, important in the theory of relativity. An *event* is defined by a place or position vector $\mathbf{r}$ and a time t, and given by the pair $(\mathbf{r}, t)$. The set of all events is called *space-time*. The two pairs $(\mathbf{r}, t)$ and $(\mathbf{r}^*, t)$, where $\mathbf{r}$ and $\mathbf{r}^*$ are related through (11.2.1, 11.2.2) represent the same event. We may therefore state that a *change of reference*, from *Rf* to Rf^*, is a one-to-one mapping of space-time onto itself, and in such a way that distances in space and time are preserved. This kind of change of reference is called an *Eulerian transformation*. If the reference Rf^* has translatoric motion with constant velocity, i.e. an unaccelerated motion, relative to the reference *Rf*, the transformation is called

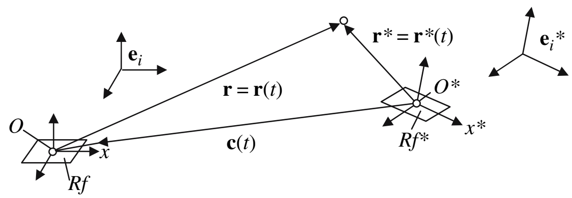


Fig. 11.2.1 References Rf and Rf^*. Coordinate systems: Ox in Rf and O^*x^* in Rf^*

a *Galilean transformation*. It is then said that the two references *Rf* and Rf^* belong to the same *Galilean class* of reference systems.

The primary reference in classical mechanics is *the Milkey Way RfM*. In this reference all body forces and contact forces are mutual forces acting on pairs of bodies according to Newton's 3. law: *Action equals reaction*. Body forces that do obey Newton's 3. law will be called *ordinary body forces*. In all references belonging to the same Galilean class as the Milkey Way, *RfM*, the body forces and the contact forces are the same. These references are called *inertial references* or inertial frames.

In a non-inertial reference *extraordinary body forces* appear. These are also called *inertia forces* and do not in the ordinary sense follow Newton's 3. law of action and reaction. The extraordinary body forces are reference dependent quantities. It is assumed that the ordinary body forces are objective quantities in the sense that they do not depend upon the choice of reference.

In order to understand why the extraordinary body forces are present in a non-inertial reference, we consider Euler's 1. axiom (3.2.6) for a body with volume V, first in a inerial reference *Rf*:

$$\mathbf{f} = \int_V \mathbf{a}\rho\,\mathbf{dV} \tag{11.2.3}$$

f is the result of body forces and contact forces acting on the body. In a non-inertial reference Rf^* we write for Euler's 1. axiom for the same body:

$$\mathbf{f}^* = \int_V \mathbf{a}^*\rho\,\mathbf{dV} \tag{11.2.4}$$

The acceleration **a** with respect to *Rf* and $\mathbf{a}^*$ with respect to Rf^* are related through formula (4.5.35) from which:

$$\mathbf{a}^* = \mathbf{a} - \mathbf{a}_{\text{place}} - 2\mathbf{w}\times\mathbf{v}^* \tag{11.2.5}$$

$$\mathbf{a}_{\text{place}} = \mathbf{a}_o + \dot{\mathbf{w}}\times\mathbf{r}^* + \mathbf{w}\times(\mathbf{w}\times\mathbf{r}^*) \tag{11.2.6}$$

$\mathbf{v}^*$ is the particle velocity relative to Rf^*, $\mathbf{a}_o$ is the acceleration of the reference point O^* fixed in Rf^*, and **w** and $\dot{\mathbf{w}}$ are respectively the angular velocity and the angular acceleration of Rf^* relative to *Rf*. The place acceleration is the acceleration of a place $\mathbf{r}^*$ in Rf* relative to Rf. The third term in (11.2.5) is the *Coriolis acceleration*.

When formula (11.2.5) is substituted into (11.2.4), and (11.2.3) is applied, we obtain:

$$\mathbf{f}^* = \mathbf{f} + \int_V \mathbf{b}^*{}_e\rho\,dV \tag{11.2.7}$$

$$\mathbf{b}^*{}_e = -\mathbf{a}_{\text{place}} - 2\mathbf{w}\times\mathbf{v}^* \quad \text{extraordinary body forces in } Rf^* \tag{11.2.8}$$

From this result we conclude that the fundamental laws of motion apply in any reference Rf^* if extraordinary body forces $\mathbf{b}^*{}_e$ are added to the forces already included in the reference *Rf*. The first term on the right-hand side of (11.2.8) is called the specific *place force*, while the second term is the specific *Coriolis force*.

The Earth is not an inertial reference. Due to the motion of the Earth around the Sun, which we may for the present arguments consider fixed in *RfM*, and the rotation of the Earth about its north-south axis and with respect to *RfM*, extraordinary body forces are present on the earth. The first motion is partly responsible for the tides in the oceans, to which however the action of the moon gives the major contribution. The rotation of the Earth about its axis gives rise to a *centrifugal force*, which normally is incorporated in the effective gravitational force on the surface of the Earth, and the Coriolis force, which is responsible for the turning of the equatorial winds, clockwise on the northern hemisphere and counter-clockwise on the southern hemisphere. The centrifugal force originates from the place acceleration on the Earth.

11.2.1 Tensor Components in Two References

The base vectors $\mathbf{e}_{\mathrm{i}}$ in the Cartesian coordinate system *Ox* fixed in *Rf* and the base vectors $\mathbf{e}^*{}_{\mathrm{i}}$ in the Cartesian coordinate system O^*x^* fixed in the reference Rf^*, see Fig. 11.2.1, are related by:

$$\mathbf{e}^*{}_i = Q_{ik}\,\mathbf{e}_k \quad \Leftrightarrow \quad \mathbf{e}_k = Q_{ik}\,\mathbf{e}^*{}_i \tag{11.2.9}$$

The transformation matrix Q is a function of time:

$$Q_{ik} = Q_{ik}(t) = \cos\left(\mathbf{e}^*{}_i, \mathbf{e}_k\right) = \mathbf{e}^*{}_i \cdot \mathbf{e}_k \tag{11.2.10}$$

The base vectors $\mathbf{e}_k$ are constants relative to *Rf* and functions of time relative to Rf^*, while $\mathbf{e}^*{}_i$ are functions of time relative to *Rf* and constants relative to Rf^*.

The motion of a particle $\mathbf{r}_o$, or X, is given by (11.1.1) or by:

$$\mathbf{r} = \mathbf{r}\left(\mathbf{r}_o, t\right) = x_i\left(X, t\right)\mathbf{e}_i \quad \Rightarrow \quad \mathbf{v} = \mathbf{v}\left(\mathbf{r}_o, t\right) = \dot{\mathbf{r}} = \dot{x}_i\left(X, t\right)\mathbf{e}_i \tag{11.2.11}$$

$\mathbf{v}$ is the particle velocity.

Let $\mathbf{a}$ be an objective vector and $\mathbf{A}$ an objective 2. order tensor. In *Rf* the quantities are represented by the components:

$$a_i = \mathbf{a}\left[\mathbf{e}_i\right] = \mathbf{a}\cdot\mathbf{e}_i,\; A_{ik} = \mathbf{A}\left[\mathbf{e}_i, \mathbf{e}_k\right] = \mathbf{e}_i \cdot \mathbf{A} \cdot \mathbf{e}_k \tag{11.2.12}$$

In Rf^* the same quantities are represented by the components:

$$a^*{}_i = \mathbf{a}\left[\mathbf{e}^*{}_i\right] = \mathbf{a}\cdot\mathbf{e}^*{}_i,\; A^*{}_{ik} = \mathbf{A}\left[\mathbf{e}^*{}_i, \mathbf{e}^*{}_k\right] = \mathbf{e}^*{}_i \cdot \mathbf{A} \cdot \mathbf{e}^*{}_k \tag{11.2.13}$$

It follows from (11.2.10) that:

$$a^* = Q\,a \quad \Leftrightarrow \quad a = Q^T a^* \quad , \quad A^* = Q A\, Q^T \quad \Leftrightarrow \quad A = Q^T A^*\, Q \tag{11.2.14}$$

The place vector $\mathbf{r}$ has the components x and Qx in the two coordinate systems Ox and O^*x^* in Fig. 11.2.1, and the place vector $\mathbf{r}^*$ has the components x^* and $Q^T x^*$ in the same two coordinate systems. The transformations of reference from Rf to Rf^* and from Rf^* to Rf may therefore be expressed by:

$$\begin{aligned} Rf \quad &\rightarrow \quad Rf^* : \quad \mathbf{r}^* = \mathbf{c} + \mathbf{r} \quad \Leftrightarrow \quad x^* = c^* + Q\,x \\ Rf^* \quad &\rightarrow \quad Rf \;: \quad \mathbf{r} = -\mathbf{c} + \mathbf{r}^* \quad \Leftrightarrow \quad x = -c + Q^T x^* \end{aligned} \tag{11.2.15}$$

The rotation of the reference Rf with respect to the reference Rf^* is given by the orthogonal tensor $\mathbf{Q}$, which in both coordinate systems Ox and O^*x^* is represented by the transformation matrix Q. To see this we argue as follows. First we note that it follows from the transformation formula $(11.2.14)_3$ for 2. order tensors that $Q^* = Q$. Let $\mathbf{a}$ be a vector fixed in Rf^*, i.e.:

$$\mathbf{a} = a^*{}_i \mathbf{e}^*{}_i, \; a^*{}_i = \text{ constants} \tag{11.2.16}$$

The $\mathbf{Q}$-rotation of $\mathbf{a}$, see Sect. 4.6.1, is a vector:

$$\mathbf{b} = \mathbf{Q}\,\mathbf{a} \quad \Leftrightarrow \quad b = Qa = QQ^T a^* = a^* = \text{ constants} \tag{11.2.17}$$

The result shows that the vector $\mathbf{b}$ is a vector that does not change with respect to the reference Rf. Equation (11.2.17) is thus a general relation between a vector $\mathbf{a}$ fixed in Rf^* and a vector $\mathbf{b}$ fixed in Rf. This means that the tensor $\mathbf{Q}$ really represents the rotational motion of the reference Rf with respect to the reference Rf^*. Similarly we understand that the rotation of Rf^* relative to Rf is given by the tensor $\mathbf{Q}^{\mathrm{T}}$. The relationship:

$$\mathbf{a} = \mathbf{Q}^T \mathbf{b} \tag{11.2.18}$$

which follows from (11.2.17), expresses this rotational motion. If we let the base vectors of the x-system and the x^*-system coincide at the reference time t_o, such that $\mathbf{e}_i(t_o) = \mathbf{e}^*{}_i(t_o)$ and $\mathbf{Q}(t_o) = \mathbf{1}$, the two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel at the time $t = t_o$.

A tensor representing the rotation of the reference Rf with respect to the reference Rf^* is not unique. Let $\mathbf{P}$ and $\mathbf{P}^*$ be any two orthogonal tensors, such that $\mathbf{P}$ is time independent relative to Rf, and $\mathbf{P}^*$ time independent relative to Rf^*. If $\mathbf{a}$ is a vector fixed in Rf^* such that $\mathbf{b} = \mathbf{Q}\mathbf{a}$ is a vector fixed in Rf, then the vector $\mathbf{PQa} = \mathbf{Pb}$ is also fixed in Rf, while the vector $\mathbf{P}^*\mathbf{a} = \mathbf{P}^*\mathbf{Q}^T\mathbf{b}$ is fixed in Rf^*. It also follows that the vector $\mathbf{QP}^*\mathbf{a} = \mathbf{Q}(\mathbf{P}^*\mathbf{a})$ is a vector fixed in Rf. These results show that both tensors $\mathbf{PQ}$ and $\mathbf{QP}^*$ represent the rotation of Rf relative to Rf^*.

11.2.2 Material Derivative of Objective Tensors

Let $\mathbf{a} = \mathbf{a}(\mathbf{r}_o, t)$ be an objective vector field observed both in Rf and in Rf^* and connected to a particle $\mathbf{r}_o$ or a material body. In the latter case we write $\mathbf{a} = \mathbf{a}(t)$.

The vector field will be different functions of time whether the field is observed in *Rf* or in Rf^*. The material derivative of **a** observed in *Rf* is marked with a "superdot", while the material derivative of **a** observed in Rf^* is marked with the symbol $^\wedge$:

$$\dot{\mathbf{a}} = \left(\frac{\partial \mathbf{a}}{\partial t}\right)_{Rf} \quad \text{observed in } Rf, \ \hat{\mathbf{a}} = \left(\frac{\partial \mathbf{a}}{\partial t}\right)_{Rf^*} \quad \text{observed in } Rf^* \tag{11.2.19}$$

This notation has already been introduced in Sect. 4.5.2. The relationship between the two material derivatives is according to (4.5.30) given by:

$$\hat{\mathbf{a}} = \dot{\mathbf{a}} + \mathbf{S} \cdot \mathbf{a} \tag{11.2.20}$$

where $\mathbf{S}(t)$ is the *rate of rotation tensor* related to the rotation of *Rf* relative to Rf^*:

$$\mathbf{S} = \dot{\mathbf{Q}}\mathbf{Q}^T = \mathbf{Q}^T\dot{\mathbf{Q}} \tag{11.2.21}$$

Confer with equation (4.5.13). To see that (11.2.20) and (11.2.21) follow directly from (4.5.30) and (4.5.13) respectively, remember that **R** and **W** in the latter equations are in the former equations represented by $\mathbf{Q}^T$ and $\mathbf{S}^T = -\mathbf{S}$, respectively.

An alternative expression for the formula (11.2.20) is:

$$\hat{\mathbf{a}} = \dot{\mathbf{a}} + \mathbf{s} \times \mathbf{a} \tag{11.2.22}$$

s is the dual vector to the antisymmetric tensor **S** and is called the *angular velocity of Rf relative to* Rf^*:

$$\mathbf{s} = -\frac{1}{2}\mathbf{P} : \mathbf{S} \quad \Leftrightarrow \quad s_i = \frac{1}{2}e_{ijk}S_{kj} \tag{11.2.23}$$

In the special case of an objective vector **a** fixed in Rf^*:

$$\hat{\mathbf{a}} = \mathbf{0} \quad \Leftrightarrow \quad \dot{\mathbf{a}} = -\mathbf{S} \cdot \mathbf{a} = -\mathbf{s} \times \mathbf{a} \tag{11.2.24}$$

For an objective vector **b** fixed in *Rf*:

$$\dot{\mathbf{b}} = \mathbf{0} \quad \Leftrightarrow \quad \hat{\mathbf{b}} = \mathbf{S} \cdot \mathbf{b} = \mathbf{s} \times \mathbf{b} \tag{11.2.25}$$

Let **A** be 2. order objective tensor field observed both in *Rf* and in Rf^* and connected to a particle or a material body. The material derivative of **A** observed in *Rf* is marked with a "superdot", while the material derivative of **A** observed in Rf^* is marked with the symbol $\hat{}$:

$$\begin{aligned} \dot{\mathbf{A}} &= \left(\frac{\partial \mathbf{A}}{\partial t}\right)_{Rf} \ \text{observed in } Rf \text{ and represented by the matrix } \dot{A} \text{ in } Ox \\ \hat{\mathbf{A}} &= \left(\frac{\partial \mathbf{A}}{\partial t}\right)_{Rf^*} \ \text{observed in } Rf^* \text{ and represented by the matrix } \dot{A}^* \text{ in } O^*x^* \end{aligned} \tag{11.2.26}$$

The relationship between the two material derivatives is according to (4.5.31) given by:

$$\hat{\mathbf{A}} = \dot{\mathbf{A}} + \mathbf{SA} - \mathbf{AS} \tag{11.2.27}$$

Special results are:

$$\hat{\mathbf{S}} = \dot{\mathbf{S}} \quad \Leftrightarrow \quad \hat{\mathbf{s}} = \dot{\mathbf{s}} \tag{11.2.28}$$

The rate of rotation tensor **S** and angular velocity vector **s** are unique, even though the rotation tensor **Q** is not. If **Q** is replaced by $\mathbf{QP}^*$, where $\mathbf{P}^*$ is an orthogonal tensor, time independent in Rf^*, or by **PQ**, where **P** is an orthogonal tensor, time independent in *Rf*, we obtain:

$$\left(\hat{\mathbf{Q}}\,\mathbf{P}^*\right)(\mathbf{Q}\,\mathbf{P}^*)^T = \hat{\mathbf{Q}}\,\mathbf{P}^*\mathbf{P}^{T*}\,\mathbf{Q}^T = \hat{\mathbf{Q}}\,\mathbf{Q}^T,\ (\,\mathbf{PQ})^T\left(\mathbf{P}\dot{\mathbf{Q}}\right) = \mathbf{Q}^T\,\mathbf{P}^T\mathbf{P}\dot{\mathbf{Q}} = \ \mathbf{Q}^T\dot{\mathbf{Q}}$$

which by (11.2.21) show that the rotations **Q**, $\mathbf{QP}^*$, and **PQ** all result in the same rate of rotation tensor **S**.

11.2.3 Deformations with Respect to Fixed Reference Configuration

Figure 11.2.2 shows a body in a *reference configuration* K_o at time t_o, the *reference time point*, and in the *present configuration* K at the *present time* t. The two configurations are shown as they are observed from the reference *Rf*, which is at rest (fixed) relative to the figure. The reference Rf^* moves with respect to the "fixed" reference *Rf*. In each of the references a coordinate system is introduced. The system Ox is fixed in *Rf*, while the system O^*x^* moves rigidly with Rf^* and is chosen such that at the reference time t_o the two systems coincide. The vector $\mathbf{c}(t)$ connecting the two origins O and O^* is zero at time t_o, i.e. $\mathbf{c}(t_o) = \mathbf{0}$. An observer in Rf^* will see K at

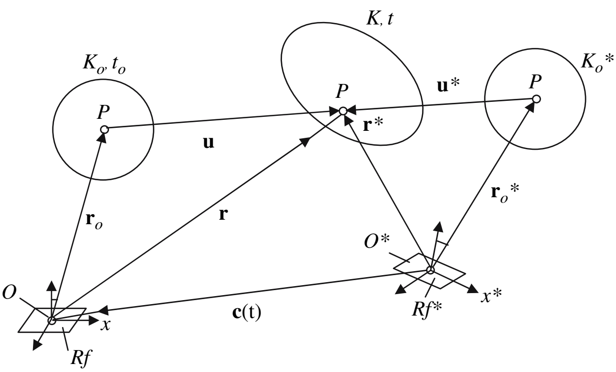


Fig. 11.2.2 Configurations of a body. References Rf and Rf^* moving relative to each other

the time t and compares it with the configuration K_o^* at time t_o. Figure 11.2.2 shows K_o^* as it is seen from the observer in *Rf*.

For an observer in *Rf* the particle P is at the place $\mathbf{r}_o$ at the time t_o and at the place $\mathbf{r}$ at the time t. The particle is also identified by the *particle coordinates* X in K_o. The place $\mathbf{r}$ is also identified by the *place coordinates* x. The observer describes the motion and deformation of the body either by the *place vector field*:

$$\mathbf{r} = \mathbf{r}(X,t) \quad \Leftrightarrow \quad x_i = x_i(X,t) \tag{11.2.29}$$

or by the *displacement vector*:

$$\mathbf{u} = \mathbf{u}(X,t) = \mathbf{r}(X,t) - \mathbf{r}_o(X) \tag{11.2.30}$$

and vectors and tensors derived from either of these vector fields.

For the observer in Rf^* the particle P is at the place $\mathbf{r}^*$ at the time t. The place $\mathbf{r}^*$ is also identified by the *place coordinates* x^*. As seen from the observer in *Rf* the reference configuration used by the observer in Rf^* has between the times t_o and t moved with Rf^* to its position $K_o{}^*$ and the reference position of the particle P is the place vector $\mathbf{r}_o{}^*$. The particle coordinates in Rf^* are $X^* = X$. The observer in Rf^* describes the motion and deformation of the body either by the *place vector field*:

$$\mathbf{r}^* = \mathbf{r}^*(X,t) \quad \Leftrightarrow \quad x^*{}_i = x^*{}_i(X,t) \tag{11.2.31}$$

or by the *displacement vector*:

$$\mathbf{u}^* = \mathbf{u}^*(X,t) = \mathbf{r}^*(X,t) - \mathbf{r}^*{}_o(X) \tag{11.2.32}$$

It follows from (11.2.15) and (11.2.17) that the place vector $\mathbf{r}_o$ is the $\mathbf{Q}$-rotation of the place vector $\mathbf{r}_o{}^*$, and that the vector $\mathbf{r}_o{}^*$ is the $\mathbf{Q}^T$-rotation of the vector $\mathbf{r}_o$:

$$\mathbf{r}_o = \mathbf{Q}\mathbf{r}^*{}_o \quad \Leftrightarrow \quad \mathbf{r}^*{}_o = \mathbf{Q}^T\mathbf{r}_o,\ \mathbf{Q} = \mathbf{Q}(t),\ \mathbf{Q}(t_o) = \mathbf{1} \tag{11.2.33}$$

The *deformation gradients* $\mathbf{F}$ related to *Rf* and $\mathbf{F}^*$ related to Rf^* are defined by:

$$\mathbf{F} = \frac{\partial \mathbf{r}}{\partial \mathbf{r}_o} \quad \Leftrightarrow \quad F_{ik} = \frac{\partial x_i}{\partial X_k} \quad , \quad \mathbf{F}^* = \frac{\partial \mathbf{r}^*}{\partial \mathbf{r}^*{}_o} \quad \Leftrightarrow \quad F^*{}_{ik} = \frac{\partial x^*{}_i}{\partial X^*{}_k} = \frac{\partial x^*{}_i}{\partial X_k} \tag{11.2.34}$$

It follows from (11.2.1) and (11.2.33) that:

$$\frac{\partial \mathbf{r}^*}{\partial \mathbf{r}} = \mathbf{1},\ \frac{\partial \mathbf{r}_o}{\partial \mathbf{r}^*{}_o} = \mathbf{Q} \tag{11.2.35}$$

Hence:

$$\mathbf{F}^* = \frac{\partial \mathbf{r}^*}{\partial \mathbf{r}^*{}_o} = \frac{\partial \mathbf{r}^*}{\partial \mathbf{r}}\frac{\partial \mathbf{r}}{\partial \mathbf{r}_o}\frac{\partial \mathbf{r}_o}{\partial \mathbf{r}^*{}_o} = \mathbf{1FQ} \quad \Rightarrow$$
$$\mathbf{F}^* = \mathbf{FQ} \tag{11.2.36}$$

The result shows that the deformation gradient is not represented by an objective tensor.

The result (11.2.36) may also be derived from the following physical reasoning. The difference in the apparent deformations of a body as seen from the reference *Rf* and the reference Rf^* is a rigid body rotation given by the tensor $\mathbf{Q}$, which represents the rotation of the *Rf* with respect to Rf^*. In the polar decomposition of the deformation gradients:

$$\mathbf{F} = \mathbf{V}\mathbf{R}, \ \mathbf{F}^* = \mathbf{V}^*\mathbf{R}^* \tag{11.2.37}$$

the left stretch tensors $\mathbf{V}$ and $\mathbf{V}^*$ represent pure strain after rotation, and the two tensors must therefore be identical: $\mathbf{V}^* \equiv \mathbf{V}$. Since the tensor $\mathbf{Q}$ represents the rotation of Rf with respect to Rf^*, the rotation tensors $\mathbf{R}^*$ and $\mathbf{R}$ are related through:

$$\mathbf{R}^* = \mathbf{R}\mathbf{Q} \tag{11.2.38}$$

Hence:

$$\mathbf{F}^* = \mathbf{V}^*\mathbf{R}^* = \mathbf{V}\mathbf{R}\mathbf{Q} = \mathbf{F}\mathbf{Q} \quad \Rightarrow \quad (11.2.36)$$

The components matrix of $\mathbf{F}^*$ in the Ox-system is according to formula (11.2.36) equal to FQ, where F is the component matrix of $\mathbf{F}$ in the same coordinate system. Therefore the component matrix F^* of $\mathbf{F}^*$ in the O^*x^*-system becomes:

$$F^* = Q(FQ)Q^T = QF \tag{11.2.39}$$

From the deformation gradient $\mathbf{F}$ a series of tensors related to deformations have been defined in Chap. 5. The relations between the corresponding tensors related to the two references *Rf* and Rf^* will now be derived.

The *displacement gradients* are:

$$\mathbf{H} = \frac{\partial \mathbf{u}}{\partial \mathbf{r}_o} = \mathbf{F} - \mathbf{1} \quad \Leftrightarrow \quad H_{ik} = \frac{\partial u_i}{\partial X_k} = F_{ik} - \delta_{ik} \tag{11.2.40}$$

$$\mathbf{H}^* = \frac{\partial \mathbf{u}^*}{\partial \mathbf{r}^*{}_o} = \mathbf{F}^* - \mathbf{1} \quad \Leftrightarrow \quad H^*{}_{ik} = \frac{\partial u^*{}_i}{\partial X_k} = F^*{}_{ik} - \delta_{ik} \tag{11.2.41}$$

$$\mathbf{H}^* = \mathbf{F}\mathbf{Q} - \mathbf{1} = (\mathbf{H} + \mathbf{1})\,\mathbf{Q} - \mathbf{1} = \mathbf{H}\mathbf{Q} + \mathbf{Q} - \mathbf{1} \tag{11.2.42}$$

The *Green deformation tensors* also called *the right deformation tensors* are:

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{C}^T \quad \text{symmetric} \tag{11.2.43}$$

$$\mathbf{C}^* = \mathbf{F}^{*T}\mathbf{F}^* = (\mathbf{F}\mathbf{Q})^T\,(\mathbf{F}\mathbf{Q}) = \mathbf{Q}^T\mathbf{F}^T\mathbf{F}\mathbf{Q} \quad \Rightarrow$$

$$\mathbf{C}^* == \mathbf{Q}^T\mathbf{C}\mathbf{Q} \quad \Leftrightarrow \quad C^* = QC^*Q^T = C \quad \text{reference related} \tag{11.2.44}$$

The *left deformation tensors* are:

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{B}^T \quad \text{symmetric} \tag{11.2.45}$$

$$\mathbf{B}^* = \mathbf{F}^*\mathbf{F}^{*T} = (\mathbf{F}\mathbf{Q})(\mathbf{F}\mathbf{Q})^T = \mathbf{F}\mathbf{Q}\mathbf{Q}^T\mathbf{F}^T = \mathbf{F}\mathbf{F}^T = \mathbf{B} \quad \Rightarrow$$

$$\mathbf{B}^* = \mathbf{B} \quad \Leftrightarrow \quad B^* = QBQ^T \quad \text{objective} \tag{11.2.46}$$

The *right stretch tensors* are:

$$\mathbf{U} = \sqrt{\mathbf{C}} = \mathbf{U}^T \quad \text{symmetric} \tag{11.2.47}$$

$$\mathbf{U}^* = \sqrt{\mathbf{C}^*} = \mathbf{Q}^T\sqrt{\mathbf{C}}\mathbf{Q} \quad \Rightarrow$$

$$\mathbf{U}^* = \mathbf{Q}^T\mathbf{U}\mathbf{Q} \quad \Leftrightarrow \quad U^* = U \quad \text{reference related} \tag{11.2.48}$$

The *left stretch tensors* are:

$$\mathbf{V} = \sqrt{\mathbf{B}} = \mathbf{V}^T \quad \text{symmetric} \tag{11.2.49}$$

$$\mathbf{V}^* = \sqrt{\mathbf{B}^*} = \mathbf{V} \quad \Leftrightarrow \quad V^* = QVQ^T \quad \text{objective} \tag{11.2.50}$$

The *Green strain tensors*, also called the *Lagrange strain tensors*, are:

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}) = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T\mathbf{H}\right) \quad \text{symmetric} \tag{11.2.51}$$

$$\mathbf{E}^* = \frac{1}{2}(\mathbf{C}^* - \mathbf{1}) = \mathbf{Q}^T\mathbf{E}\mathbf{Q} \quad \Leftrightarrow \quad E^* = E \quad \text{reference related} \tag{11.2.52}$$

11.2.4 Deformation with Respect to the Present Configuration

For materials that are modelled as fluids it is natural to describe the response partially through the deformation relative to the present configuration K at the present time t. Figure 11.2.3 shows a body in a *reference configuration* K_o at time t_o, the *reference time point*, in a *"moving" configuration* $\bar{K}$ at the time $\bar{t}, \infty < \bar{t} \leq t$, and in the *present configuration* K at the *present time* t. The three configurations are shown as they are observed from the reference *Rf*, which is at rest (fixed) relative to the figure. The reference Rf^* moves with respect to the "fixed" reference *Rf*. In each of the references a coordinate system is introduced. The system Ox is fixed in *Rf*, while the system O^*x^* moves rigidly with Rf^* and is chosen such that at the reference time t_o the two systems coincide. An observer in Rf^* will see $\bar{K}$ at the time $\bar{t}$ and compares it with the configuration K^* at the time t. The configuration K^* in Fig. 11.2.3 is an image of K which must be used when the observer in Rf^* compares $\bar{K}$ and K.

The motion of a particle P relative to the reference *Rf* and the deformation of the material surrounding the particle may be described by the place vector to the place for the particle in the configuration $\bar{K}$:

$$\mathbf{r}_t(x,\bar{t}) = \mathbf{r}(X,\bar{t}), \; X = X(x,t) \tag{11.2.53}$$

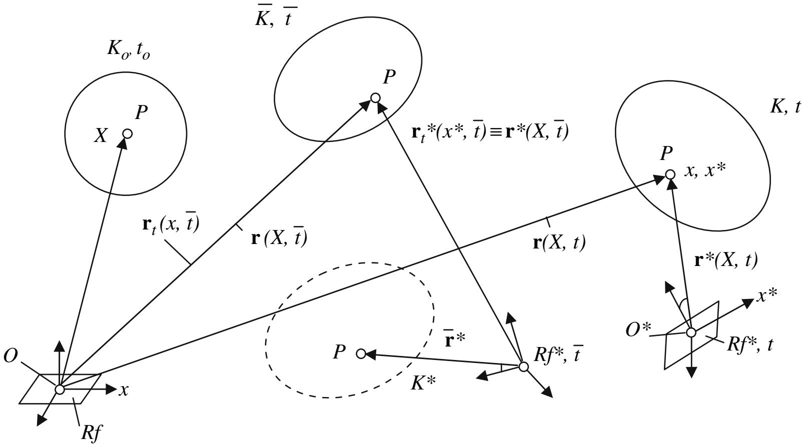


Fig. 11.2.3 Configurations of a body. References Rf and Rf^* moving relative to each other

The motion of a particle P relative to the reference Rf^* and the deformation of the material surrounding the particle may be described by the place vector to the place of the particle in the configuration $\bar{K}$:

$$\mathbf{r}_t{}^*(x^*, \bar{t}) = \mathbf{r}^*(X, \bar{t}), \; X = X(x^*, t) \tag{11.2.54}$$

The rotation of Rf with respect to Rf^* is given by the orthogonal tensor $\mathbf{Q}_t(\bar{t})$, such that:

$$\bar{\mathbf{r}}^* = \bar{\mathbf{r}}^*(X, \bar{t}) = \mathbf{Q}_t(\bar{t})\mathbf{r}^*(X, t), \; \mathbf{Q}_t(t) = \mathbf{1} \tag{11.2.55}$$

$\bar{\mathbf{r}}^* = \bar{\mathbf{r}}^*(X, \bar{t})$ is the place vector of the particle P in Rf^* at the time $\bar{t}$ as observed in Rf, see Fig. 11.2.3. The relation (11.2.55) is analogous to the relation (11.2.33). The rate of rotation tensor $\mathbf{S}$ for the rotation of Rf with respect to Rf^* at the present time t may according to (11.2.21) and (11.2.55) be expressed by:

$$\mathbf{S} = \hat{\mathbf{Q}}_t(t) \equiv \hat{\mathbf{Q}}_t = \dot{\mathbf{Q}}_t \tag{11.2.56}$$

The deformation gradient based on the present configuration as reference configuration and related to the reference Rf is called the relative deformation gradient and is in (5.5.72) defined as:

$$\mathbf{F}_t(x, \bar{t}) = \frac{\partial \mathbf{r}_t}{\partial \mathbf{r}} = \operatorname{grad} \mathbf{r}_t \quad \Leftrightarrow \quad (F_t)_{ik} = \frac{\partial x_i(X, \bar{t})}{\partial x_k(X, t)} \tag{11.2.57}$$

The corresponding relative deformation gradient related to the reference Rf^* is then given by:

$$\mathbf{F}_t{}^*(x^*, \bar{t}) = \frac{\partial \mathbf{r}_t{}^*}{\partial \mathbf{r}^*} = \operatorname{grad}^* \mathbf{r}_t{}^* \quad \Leftrightarrow \quad (\mathrm{F}_t^*)_{ik} = \frac{\partial x_i{}^*(X, \bar{t})}{\partial x_k(X, t)} \tag{11.2.58}$$

A series of deformation tensors derived from $\mathbf{F}_t$ related to reference *Rf* were defined in Sect. 5.5.4. The corresponding tensors related to reference Rf^* will now be presented and their relations to the similar tensors in *Rf* will be given.

The relationship between $\mathbf{F}^*$ and $\mathbf{F}_t{}^*$ is obtained from (11.2.36) as:

$$\mathbf{F}_t{}^* = \mathbf{F}_t\,\mathbf{Q}_t \tag{11.2.59}$$

The relative right and left deformation tensors, relative right and left stretch tensors, and the relative rotation tensor with respect to reference *Rf* are all defined in Sect. 5.5.4. The corresponding tensors with respect to reference Rf^* and the relations to their counterpart in *Rf* are listed below:

$$\mathbf{C}_t{}^* = \mathbf{Q}_t^T\mathbf{C}_t\mathbf{Q}_t,\ \mathbf{U}_t{}^* = \mathbf{Q}_t^T\mathbf{U}_t\mathbf{Q}_t \quad \text{reference related} \tag{11.2.60}$$

$$\mathbf{B}_t{}^* = \mathbf{B}_t,\ \mathbf{V}_t{}^* = \mathbf{V}_t \quad \text{objective} \tag{11.2.61}$$

$$\mathbf{R}_t{}^* = \mathbf{R}_t\,\mathbf{Q}_t \tag{11.2.62}$$

The particle velocities $\mathbf{v}$ relative to reference *Rf* and $\mathbf{v}^*$ relative to reference Rf^* are:

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\mathbf{u}} \quad \text{particle velocity relative to } Rf \tag{11.2.63}$$

$$\mathbf{v}^* = \hat{\mathbf{r}} = \hat{\mathbf{u}} \quad \text{particle velocity relative to } Rf^* \tag{11.2.64}$$

From (11.2.1), (11.2.20) and (11.2.22) we obtain:

$$\begin{gathered}\mathbf{v}^* = \hat{\mathbf{r}}^* = \hat{\mathbf{c}} + \hat{\mathbf{r}} = \hat{\mathbf{c}} + \dot{\mathbf{r}} + \mathbf{S}\cdot\mathbf{r} \quad \Rightarrow \\ \mathbf{v}^* = \hat{\mathbf{c}} + \mathbf{S}\cdot\mathbf{r} + \mathbf{v} = \hat{\mathbf{c}} + \mathbf{s}\times\mathbf{r} + \mathbf{v}\end{gathered} \tag{11.2.65}$$

In Sect. 5.5.4 we derived the following expressions for the velocity gradient $\mathbf{L}$, the rate of deformation tensor $\mathbf{D}$, and the rate of rotation tensor $\mathbf{W}$, all with respect to the reference *Rf*:

$$\mathbf{L} = \operatorname{grad}\mathbf{v} \equiv \frac{\partial\mathbf{v}}{\partial\mathbf{r}} = \dot{\mathbf{F}}_t\,(\mathbf{r},t) \tag{11.2.66}$$

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{L}+\mathbf{L}^T\right) = \dot{\mathbf{U}}_t\,(\mathbf{r},t) = \frac{1}{2}\dot{\mathbf{C}}_t\,(\mathbf{r},t) \tag{11.2.67}$$

$$\mathbf{W} = \frac{1}{2}\left(\mathbf{L}-\mathbf{L}^T\right) = \dot{\mathbf{R}}_t\,(\mathbf{r},t) \tag{11.2.68}$$

The velocity gradient with respect to reference Rf^* is:

$$\mathbf{L}^* = \operatorname{grad}^*\mathbf{v}^* \equiv \frac{\partial\mathbf{v}^*}{\partial\mathbf{r}^*} = \hat{\mathbf{F}}^*{}_t\,(\mathbf{r}^*,t) \tag{11.2.69}$$

From (11.2.59) it follows that:

$$\mathbf{L}^* = \hat{\mathbf{F}}^*{}_t\,(\mathbf{r}^*,t) = \hat{\mathbf{F}}_t\,(\mathbf{r},t)\,\mathbf{Q}_t\,(t) + \mathbf{F}_t\,(\mathbf{r},t)\,\hat{\mathbf{Q}}_t\,(t) \tag{11.2.70}$$

Now since $\mathbf{F}_t(\mathbf{r},t) = \mathbf{1}$ and $\mathbf{Q}_t(t) = \mathbf{1}$, it follows from (11.2.27) that:

$$\hat{\mathbf{F}}_t(\mathbf{r},t) = \dot{\mathbf{F}}_t(\mathbf{r},t) = \mathbf{L}(\mathbf{r},t) \tag{11.2.71}$$

This result and (11.2.56) reduces (11.2.70) to:

$$\mathbf{L}^* = \mathbf{L} + \mathbf{S} \tag{11.2.72}$$

Since the rate of rotation tensor $\mathbf{S}$ is antisymmetric, we obtain for the relations between the rate of deformation tensors $\mathbf{D}$ and $\mathbf{D}^*$, and for the rate of rotation tensors $\mathbf{W}$ and $\mathbf{W}^*$:

$$\mathbf{D}^* = \mathbf{D} \quad \text{objective} \tag{11.2.73}$$

$$\mathbf{W}^* = \mathbf{W} + \mathbf{S} \quad \text{reference related} \tag{11.2.74}$$

11.3 Corotational Derivative

The material derivative of a quantity is the time rate of change of that quantity with respect to a particular reference Rf when the quantity represents a particle or a body. It may therefore be expected that the material derivative of an objective tensor does not result in a new objective tensor. The formulas (11.2.20) and (11.2.27) show the results of time differentiations of objective vectors and second order tensors. We may however define other forms of time differentiations of objective tensors that by definition lead to objective tensor quantities. In the presentations of constitutive equations either of two major forms of time derivatives is usually chosen: the *corotational derivative* and the *convective derivatives*.

We shall introduce a special reference Rf^r that rotates with the particle P in a material body, such that the rotation tensor $\mathbf{Q}_t(\bar{t})$ of Rf relative to Rf^r is equal to the transposed of the relative rotation tensor $\mathbf{R}_t(\mathbf{r},\bar{t})$ for the particle:

$$\mathbf{Q}_t(\mathbf{r},\bar{t}) = \mathbf{R}_t^T(\mathbf{r},\bar{t}), \; -\infty < \bar{t} \le t \tag{11.3.1}$$

From (11.2.62) it follows that the relative rotation tensor $\mathbf{R}_t^r(\mathbf{r},\bar{t})$ with respect to Rf^r is a unit tensor:

$$\mathbf{R}_t^r = \mathbf{R}_t\,\mathbf{Q}_t = \mathbf{R}_t\,\mathbf{R}_t^T = \mathbf{1} \tag{11.3.2}$$

The rate of rotation tensor $\mathbf{S}$ for Rf relative to Rf^r becomes according to (11.2.56) and (11.2.68):

$$\mathbf{S} = \dot{\mathbf{Q}}_t = \dot{\mathbf{R}}_t^T = \mathbf{W}^T = -\mathbf{W} \tag{11.3.3}$$

It follows from (11.2.74) that the material spin tensor with respect to Rf^r is a zero tensor and from (11.3.2) and (5.5.82) that all the n rotation acceleration tensors $\mathbf{W}_n^r$ with respect to the corrotational reference Rf^r are zero tensors:

$$\mathbf{W}^r = \mathbf{0}, \; \mathbf{W}_n^r = \mathbf{0}, \; n = 1,2,3,\ldots \tag{11.3.4}$$

Due to these properties of the reference Rf^r it is called the *corotational reference to the particle P*.

The material derivative of a tensor field **A** relative to the corotational reference Rf^r for a particle P is called the *corotational material derivative*, or for short the *corotational derivative*, $\partial_r\mathbf{A}$ of the tensor in the particle. The definition implies that *the corotational derivative of an objective tensor is also an objective tensor*. The corotational derivative is also called the *Jaumann derivative*, G. Jaumann 1906, and was apparently first introduced by S. Zaremba in 1903.

For an objective scalar field α we obviously have the result:

$$\partial_r\alpha = \dot{\alpha} \tag{11.3.5}$$

For a vector field **a** the corotational derivative is found by using (11.2.20) and replacing **S** by - **W**. Then:

$$\partial_r\mathbf{a} = \dot{\mathbf{a}} - \mathbf{W}\cdot\mathbf{a} \quad \Leftrightarrow \quad \partial_r a_i = \dot{a}_i - W_{ik}a_k = \frac{\partial a_i}{\partial t} + a_{i,k}v_k - \frac{1}{2}v_{i,k}a_k + \frac{1}{2}v_{k,i}a_k \tag{11.3.6}$$

The corotational derivative $\partial_r\mathbf{A}$ of a 2. order tensor **A** may be determined from formulas (11.2.27) and (11.3.3):

$$\partial_r\mathbf{A} = \dot{\mathbf{A}} - \mathbf{W}\,\mathbf{A} + \mathbf{A}\,\mathbf{W} \quad \Leftrightarrow \quad \partial_r A_{ij} = \dot{A}_{ij} - W_{ik}A_{kj} + A_{ik}W_{kj} \tag{11.3.7}$$

11.4 Convected Derivative

Convected time differentiation of tensors was presumably first introduced by Oldroyd [37]. The results are called *convected derivatives* or *Oldroyd derivatives of the tensor*. The general definition of convected differentiation of tensors leads to more than one convected derivative from each tensor. There are at least two kinds of convected derivatives for a tensor of order $n \geq 1$. These are called *lower-convected derivatives* and *upper-convected derivatives*. The convected derivatives of an objective tensor lead to objective tensors. The kinematical interpretation of the convected derivatives is based on convected coordinates, i.e. material coordinates moving and deforming with the material. These coordinates are not in general Cartesian, and the application of convected coordinate systems is therefore postponed to Chap. 13. The proper geometrical presentation of convected derivatives is given in Sect. 13.4.

In the present section we define the convected derivatives of a vector **a**, a second order tensor **B**, and a *scalar* α *by* the following expressions:

$$\mathbf{a}^{\Delta} = \dot{\mathbf{a}} + \mathbf{L}^T\mathbf{a} \;\Leftrightarrow\; a_i^{\Delta} = \dot{a}_i + v_{k,i}a_k \quad \text{lower-convected derivative of } \mathbf{a} \tag{11.4.1}$$

$$\mathbf{a}^{\nabla} = \dot{\mathbf{a}} - \mathbf{L}\mathbf{a} \;\Leftrightarrow\; a_i^{\nabla} = \dot{a}_i - v_{i,k}a_k \quad \text{upper-convected derivative of } \mathbf{a} \tag{11.4.2}$$

$$\begin{array}{l}\mathbf{B}^{\Delta} = \dot{\mathbf{B}} + \mathbf{L}^T\mathbf{B} + \mathbf{B}\,\mathbf{L} \\ B_{ij}^{\Delta} = \dot{B}_{ij} + v_{k,i}B_{kj} + B_{ik}v_{k,j}\end{array} \;\Leftrightarrow\; \text{lower-convected derivative of } \mathbf{B} \tag{11.4.3}$$

$$\begin{aligned}&\mathbf{B}^{\nabla} = \dot{\mathbf{B}} - \mathbf{L}\,\mathbf{B} - \mathbf{B}\,\mathbf{L}^{T} \\ &B_{ij}^{\nabla} = \dot{B}_{ij} - v_{i,k}\,B_{kj} - B_{ik}\,v_{j,k}\end{aligned} \quad \Leftrightarrow \quad \text{upper-convected derivative of } \mathbf{B} \tag{11.4.4}$$

$$\alpha^{\Delta} \equiv \alpha^{\nabla} \equiv \dot{\alpha} \quad \Leftrightarrow \quad \text{convected derivative of} \alpha \tag{11.4.5}$$

It follows that:

$$\partial_r \mathbf{B} = \frac{1}{2}\left(\mathbf{B}^{\Delta} + \mathbf{B}^{\nabla}\right) \tag{11.4.6}$$

11.5 General Principles of Constitutive Theory

We consider a continuous medium that subjected to body forces and contact forces moves and deforms. Our task is to formulate a relationship between the stress tensor $\mathbf{T}(X,t)$ in a particle X at time t and the motion and the deformation. Causality is the bases for constitutive modelling. The constitutive equations shall be deterministic. For that purpose we need the concept of the *history* f^t *of a field* $f(X,t)$:

$$f^t \equiv f^t(X,s) = f(X,t-s), \; s \geq 0 \tag{11.5.1}$$

The time parameter s will be called the *past time*. The history $\mathbf{r}^t$ of the motion $\mathbf{r}(X,t)$ represents the continuous set of configurations of a body at all times before and including the present configuration K.

The principle of determinism: The stress $\mathbf{T}(X,t)$ in a particle is determined by the history $\mathbf{r}^t$ of the motion $\mathbf{r}(X',t)$ of all particles X' in the reference configuration K_o.

The intermolecular forces have limited action radius and are in continuum mechanics represented by contact forces, see Sect. 3.2. On the basis of this it is natural to presume:

The principle of locality: The motions of particles X' that have finite distance from the particle X have no direct influence on the state of stress in the particle X.

We now assume that the history $\mathbf{r}^t$ of the motion $\mathbf{r}(X,t)$ only enters the constitutive equations through the deformation gradient:

$$\mathbf{F} = \frac{\partial \mathbf{r}}{\partial \mathbf{r}_o} \quad \Leftrightarrow \quad F_{ik} = \frac{\partial x_i}{\partial X_k} \tag{11.5.2}$$

According to the two general principles above, a constitutive equation must be a form where the stress tensor is given by a *response functional*:

$$\mathbf{T} = \mathop{\Gamma}_{s=0}^{s=\infty}\left[\mathbf{F}^t(X,s);X,t\right] \tag{11.5.3}$$

The response functional is a time operator on the history $\mathbf{F}^t = \mathbf{F}(X,t-s)$ and a function of the arguments X and t. If the particle coordinates X do not enter the response functional explicitly as an argument, the material is said to be *mechanically homogeneous*. The time t is included explicitly to cover a possible time effect not due to the motion of the material, i.e. the material properties may change with time due to structural changes or chemical processes. For simplicity we shall in the following presentation leave out both the integration limits, i.e. $s = 0$ and $s = \infty$, and the explicit arguments X and t from the response functional. Thus we set:

$$\mathbf{T} = \Gamma\left[\mathbf{F}^t\right] \quad \Leftrightarrow \quad T = \Gamma\left[F^t\right] \tag{11.5.4}$$

Equations (11.5.3) and (11.5.4) are now supposed to be the same, and they are called *the constitutive equation of a simple material*. All "commonly known" materials are represented by simple material models. We shall present some examples.

An elastic material may be defined by:

$$\mathbf{T} = \mathbf{T}[\mathbf{F}] \tag{11.5.5}$$

The response functional is here reduced to a function of the deformation gradient $\mathbf{F}(X,t)$ at the present time t.

A Newtonian fluid is defined by:

$$\mathbf{T} = -p(\rho,\theta)\,\mathbf{1} + 2\mu\mathbf{D} + \left(\kappa - \frac{2}{3}\mu\right)(\mathrm{tr}\,\mathbf{D})\,\mathbf{1} \tag{11.5.6}$$

See Sect. 8.4. If we may assume barotropic pressure: $p = p(\rho)$, and introduce the formulas (5.5.65), (5.4.4), and (5.5.28):

$$\rho = \frac{\rho_o}{\det\mathbf{F}}, \ \mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right) = \frac{1}{2}\left[\dot{\mathbf{F}}\mathbf{F}^{-1} + \left(\dot{\mathbf{F}}\mathbf{F}^{-1}\right)^T\right] \tag{11.5.7}$$

the constitutive equation (11.5.6) may be transformed to:

$$\mathbf{T} = -p(\det\mathbf{F})\,\mathbf{1} + \mu\left[\dot{\mathbf{F}}\mathbf{F}^{-1} + \left(\dot{\mathbf{F}}\mathbf{F}^{-1}\right)^T\right] + \left(\kappa - \frac{2}{3}\mu\right)\left(\mathrm{tr}\,\dot{\mathbf{F}}\mathbf{F}^{-1}\right)\mathbf{1} \tag{11.5.8}$$

An isotropic and linearly viscoelastic material is defined by the constitutive equation (9.2.60):

$$\begin{aligned}\mathbf{T} =& 2\beta'_g\mathbf{E}(t) + \left(\beta^o_g - \frac{2}{3}\beta'_g\right)(\mathrm{tr}\,\mathbf{E}(t))\,\mathbf{1} \\ &+ \int_0^\infty \left\{2\frac{d\beta'(s)}{ds}\mathbf{E}(t-s) + \left[\frac{d\beta^o(s)}{ds} - \frac{2}{3}\frac{d\beta'(s)}{ds}\right](\mathrm{tr}\,\mathbf{E}(t-s))\,\mathbf{1}\right\}ds\end{aligned} \tag{11.5.9}$$

$\mathbf{E}$ is the strain tensor for small deformations, which according to the formulas (5.5.67) and (5.5.69) may be presented as:

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F} + \mathbf{F}^T\right) - \mathbf{1} \tag{11.5.10}$$

It seems reasonable to assume that the material properties are not influenced by rigid body motion of the material. For example, the modulus of elasticity of a test specimen subjected to uniaxial stress will be taken to be the same regardless of the position of the specimen in space and or of motion of the specimen as a rigid body. Translation and rotation of a particle may always be eliminated by choosing a reference that moves with the particle. Thus we expect that material properties are invariant with respect to a) a change of position and orientation of the material in space, and b) a change of reference. Item a) implies that the space is homogeneous and isotropic with respect to the material properties, while item b) implies that the properties are reference invariant. These statements may be formulated in the following principle.

The Principle of Material Objectivity

Constitutive equations must be reference invariant. This implies that constitutive equations that in reference *Rf* result in a stress tensor $\mathbf{T}$, shall in any other reference Rf^* result in a stress tensor $\mathbf{T}^* = \mathbf{T}$.

In the literature this principle has many different names and formulations. Oldroyd [37] was probably the first to explicitly formulate the principle in 1950. He satisfied the principle by using convected coordinates. Noll [32] introduced in 1955 the *principle of isotropy of space*. In a paper from 1958 Noll [33] called it the *principle of objectivity of material properties* and gave a formulation near the one given above. Truesdell [49, 50] seemed to prefer the name *the principle of material reference invariance*.

In the present book the principle will for short be called the *objectivity principle*. The implication of the principle is that the general response functional in the constitutive equation (11.5.4) of a simple material has to satisfy the condition:

$$\Gamma\left[\mathbf{Q}^t\mathbf{F}^t\right] = \mathbf{Q}\Gamma\left[\mathbf{F}^t\right]\mathbf{Q}^T \tag{11.5.11}$$

$\mathbf{Q}^t = \mathbf{Q}^t(s) = \mathbf{Q}(t-s)$ is any history of an orthogonal tensor and $\mathbf{Q} = \mathbf{Q}(t)$.

To demonstrate that (11.5.11) follows from the objectivity principle, we may argue as follows. Let $\mathbf{Q}^t$ represent the rotation of the reference *Rf* with respect to Rf^*. The deformation history $\mathbf{F}^t$ is represented by the matrix F^t in the coordinate system Ox fixed in *Rf*. The same deformation history is with respect to the reference Rf^* is according to (11.2.36) given by the tensor:

$$\mathbf{F}^{*t} = \mathbf{F}^t\mathbf{Q}^t \tag{11.5.12}$$

In a coordinate system O^*x^* fixed in Rf^* and chosen such that the two coordinate systems Ox and O^*x^* coincide at the reference time t_o the deformation history $\mathbf{F}^{*t}$ is according to equation (11.2.39) represented by the matrix Q^tF^t.

$$F^{*t} = Q^tF^t \tag{11.5.13}$$

Since the reference configurations K_o and $K_o{}^*$ in Fig. 11.2.2 are the same relatively to the coordinate systems Ox and O^*x^* respectively, we use the same response functional in Rf and Rf^*. The objectivity principle then states that $\mathbf{T}^* = \mathbf{T}$, which implies that $T^* = QTQ^t$. Hence we may write that:

$$\Gamma\left[Q^t F^t\right] = Q\Gamma\left[F^t\right] Q^T \qquad (11.5.14)$$

This is in fact the matrix representation of equation (11.5.11).

The condition (11.5.11) may also be derived from the equivalent principle of isotropy of space (Noll 1955). This principle states that two deformation histories $\bar{\mathbf{F}}^t$ and $\mathbf{F}^t$ that only differ by a rigid body motion, such that:

$$\bar{\mathbf{F}}^t = \mathbf{Q}^t \mathbf{F}^t \qquad (11.5.15)$$

result in states of stress $\bar{\mathbf{T}}^t$ and $\mathbf{T}^t$ that also only differ by a rotation of the principal stress axes. Hence:

$$\bar{\mathbf{T}} = \mathbf{Q}\mathbf{T}\mathbf{Q}^T \qquad (11.5.16)$$

According to (11.5.15) and (11.5.16) the response functional (11.5.4) must now satisfy the condition (11.5.11).

The mathematical formulation (11.5.11) of the objectivity principle implies that the general constitutive equation (11.5.14) of a simple material must take the form:

$$\mathbf{T} = \mathbf{R}\,\Gamma\left[\mathbf{U}^t\right] \mathbf{R}^T \qquad (11.5.17)$$

$\mathbf{R} = \mathbf{R}(X,t)$ is the rotation tensor for the particle X, and $\mathbf{U}^t = \mathbf{U}^t(X, t-s)$ is the history of the right stretch tensor for the particle X.

Proof of the implication: (11.5.11) $\Rightarrow$ (11.5.17). By polar decomposition: $\mathbf{Q}^t\mathbf{F}^t = \mathbf{Q}^t\mathbf{R}^t\mathbf{U}^t$. If we chose $\mathbf{Q}^t = \mathbf{R}^{tT}$, we get from (11.5.11):

$$\Gamma\left[\mathbf{U}^t\right] = \mathbf{R}^T\Gamma\left[\mathbf{F}^t\right]\ \mathbf{R} \quad \Rightarrow \quad (11.5.17)$$

Proof of the implication: (11.5.17) $\Rightarrow$ (11.5.11). For a deformation history: $\bar{\mathbf{F}}^t = \mathbf{Q}^t\mathbf{F}^t$, where $\mathbf{Q}^t$ is any orthogonal tensor history, the rotation tensor becomes $\bar{\mathbf{R}}^t = \mathbf{Q}^t\mathbf{R}^t$, and the right stretch tensor is unchanged $\bar{\mathbf{U}}^t = \mathbf{U}^t$. Thus from (11.5.4) and (11.5.17) we get:

$$\bar{\mathbf{T}} = \Gamma\left[\mathbf{Q}^t\mathbf{F}^t\right] = \mathbf{Q}\mathbf{R}\Gamma\left[\mathbf{U}^t\right]\mathbf{R}^T\mathbf{Q}^T \quad \Rightarrow \quad \Gamma\left[\mathbf{Q}^t\mathbf{F}^t\right] = \mathbf{Q}\Gamma\left[\mathbf{F}^t\right]\mathbf{Q}^T \quad \Rightarrow \quad (11.5.12)$$

The general form (11.5.17) of a constitutive equation for a simple material shows that the mathematical expression for the functional for any particular material model may be discussed on the bases of or be determined from experiments using pure stretch histories: $\mathbf{F}^t = \mathbf{U}^t$. It is important to note that the response functional Γ of any particular material is closely related to the choice of reference configuration K_o. If we choose a new reference configuration, the response functional of the material model will change. We shall discuss this further in Sect. 11.6.

The result (11.5.17) has already been used in Sect. 7.10 in connection with the discussion of elastic materials. Hooke's law of linearly elastic materials does not satisfy the objectivity principle. But as long as this constitutive equation is only used for small deformations: $\mathbf{R} \approx \mathbf{1} + \tilde{\mathbf{R}} \approx \mathbf{1}$, this objection does not interfere with the application of Hooke's law.

11.5.1 Present Configuration as Reference Configuration

The deformation gradients:

$$\mathbf{F}(X,\bar{t}) = \frac{\partial \mathbf{r}(X,\bar{t})}{\partial \mathbf{r}_o} = \frac{\partial \mathbf{r}_t}{\partial \mathbf{r}_o}, \ \mathbf{F}(X,t) = \frac{\partial \mathbf{r}}{\partial \mathbf{r}_o}, \ \mathbf{F}_t(x,t) = \frac{\partial \mathbf{r}_t}{\partial \mathbf{r}} \tag{11.5.18}$$

are related through:

$$\mathbf{F}(X,\bar{t}) = \frac{\partial \mathbf{r}_t}{\partial \mathbf{r}} \frac{\partial \mathbf{r}}{\partial \mathbf{r}_o} = \mathbf{F}_t(x,\bar{t})\,\mathbf{F}(X,t) \tag{11.5.19}$$

We simplify the notation and write:

$$\mathbf{F}(\bar{t}) = \mathbf{F}_t(t-s)\,\mathbf{F}(t) \tag{11.5.20}$$

or, using the notation from (11.5.1):

$$\mathbf{F}^t(s) = \mathbf{F}_t^t(s)\,\mathbf{F}(t) \tag{11.5.21}$$

Polar decomposition of the three tensors in this equation leads to the relation:

$$\mathbf{F}^t = \mathbf{R}^t\,\mathbf{U}^t = \mathbf{R}_t^t\,\mathbf{U}_t^t\,\mathbf{R}\mathbf{U} \tag{11.5.22}$$

The principle of objectivity requires that (11.5.11) is satisfied for any orthogonal tensor history $Q^t \equiv Q(t-s)$. If we in particular choose $\mathbf{Q}(t-s) = \mathbf{R}(t)^T \mathbf{R}_t(t-s)$, such that $\mathbf{Q}(t) = \mathbf{R}^T(t)$, we get:

$$\mathbf{Q}^t\,\mathbf{F}^t = \mathbf{R}^T\,\mathbf{R}_t^{tT}\,\mathbf{R}_t^t\,\mathbf{U}_t^t\,\mathbf{R}\mathbf{U} = \mathbf{R}^T\,\mathbf{U}_t^t\,\mathbf{R}\,\mathbf{U} \tag{11.5.23}$$

and (11.5.11) requires that the response functional $\Gamma[\mathbf{F}^t]$ satisfies the condition:

$$\Gamma\left[\mathbf{R}^T\,\mathbf{U}_t^t\,\mathbf{R}\,\mathbf{U}\right] = \mathbf{R}^T\,\Gamma\left[\mathbf{F}^t\right]\mathbf{R} \tag{11.5.24}$$

Thus constitutive equation of a simple material has to be on the form:

$$\mathbf{T} = \Gamma\left[\mathbf{F}^t\right] = \mathbf{R}\,\Gamma\left[\mathbf{R}^T\,\mathbf{U}_t^t\,\mathbf{R}\,\mathbf{U}\right]\,\mathbf{R}^T \tag{11.5.25}$$

The response functional is now a functional of the history $\mathbf{U}_t^t(s)$ and a function of the stretch tensor $\mathbf{U}(t)$. It turns out to be convenient to replace the stretch tensor $\mathbf{U}$ by the deformation tensor $\mathbf{C} = \mathbf{U}^2$, and we introduce a new functional:

$$\Lambda\left[\mathbf{R}^T\,\mathbf{C}_t^t\,\mathbf{R};\,\mathbf{C}\right] = \Gamma\left[\mathbf{R}^T\,\mathbf{U}_t^t\,\mathbf{R}\,\mathbf{U}\right] \tag{11.5.26}$$

The constitutive equation (11.5.25) becomes:

$$\mathbf{T} = \mathbf{R}\Lambda\left[\mathbf{R}^T\,\mathbf{C}_t^t\,\mathbf{R};\,\mathbf{C}\right]\mathbf{R}^T \tag{11.5.27}$$

This equation is called *Noll's reduced constitutive equation*.

It may be advantageous to replace the functional Λ with the sum of a *rest stress* given by the tensor-valued function $\Pi[\mathbf{C}]$ of $\mathbf{C}$ and a functional $\tilde{\Lambda}$ that give zero-response for a *rest history*:

$$\mathbf{C}_t^t(s) = \mathbf{C}_t(t-s) = \mathbf{1} \text{ for } s \geq 0 \tag{11.5.28}$$

Thus we set:

$$\mathbf{T} = \mathbf{R}\,\Pi[\mathbf{C}]\,\mathbf{R}^T + \mathbf{R}\tilde{\Lambda}\left[\mathbf{R}^T\,\mathbf{C}_t^t\,\mathbf{R};\,\mathbf{C}\right]\mathbf{R}^T \quad , \quad \tilde{\Lambda}[\mathbf{1};\,\mathbf{C}] = \mathbf{0} \tag{11.5.29}$$

The constitutive equation (11.5.29) for an isotropic, linearly viscoelastic material provides an example on the form (11.5.29).

If the stress tensor $\mathbf{T}$ only depends on the right deformation tensor $\mathbf{C}$(t) and the deformation history $\mathbf{C}_t(t-s)$ is very close to the present configuration, we may try a *constitutive equation of the differential type*:

$$\begin{aligned}\mathbf{T} =& \mathbf{R}\,\Pi[\mathbf{C}]\,\mathbf{R}^T + \mathbf{R}\Sigma\left[\mathbf{R}^T\,\mathbf{A}_1\,\mathbf{R},\mathbf{R}^T\,\mathbf{A}_2\,\mathbf{R},..,\mathbf{R}^T\,\mathbf{A}_n\,\mathbf{R},\,\mathbf{C}\right]\mathbf{R}^T\\ &\Sigma[\mathbf{1},\mathbf{1},..,\mathbf{1},\,\mathbf{C}] = \mathbf{0}\end{aligned} \tag{11.5.30}$$

Σ is a tensor-valued function of **C, R**, and the *n Rivlin-Ericksen tensors* $\mathbf{A}_1,\ \mathbf{A}_1,..,\mathbf{A}_n$ see (5.5.84). The transition from (11.5.29) to (11.5.30) is based on the assumption that the history $\mathbf{C}_t(t-s)$ may be replaced by a sufficient part of the Taylor series at $s = 0$:

$$\mathbf{C}_t(t-s) = \mathbf{C}_t(t) + s\,\dot{\mathbf{C}}_t(t) + \frac{s^2}{2}\,\dot{\mathbf{C}}_t(t) + \ldots. \tag{11.5.31}$$

The constitutive equation (11.5.30) defines a *Rivlin-Ericksen material of complexity n*. A *Rivlin-Ericksen fluid of complexity n* is defined by the constitutive equation:

$$\mathbf{T} = -p(\rho)\,\mathbf{1} + \Theta[\mathbf{A}_1,\,\mathbf{A}_2,..,\,\mathbf{A}_n,\,\rho] \quad , \quad \Theta[\mathbf{1},\mathbf{1},..,\mathbf{1},\,\rho] = \mathbf{0} \tag{11.5.32}$$

Θ is an isotropic tensor-valued function of all the argument tensors. Because $= 2\mathbf{D}$, the *Stokesian fluid*, mentioned in Sect. 8.4.1 and further discussed in Sect. 11.9.2, is a Rivlin-Ericksen fluid of complexity 1.

Equation (11.5.29) is called a *constitutive equation of the integral type* if the functional in the equation has the form:

$$\begin{aligned}&\Lambda\left[\mathbf{R}^T\,\mathbf{C}_t^t\,\mathbf{R};\,\mathbf{U}\right]\\ &= \sum_{n=1}^{m}\int_0^\infty ..\int_0^\infty \mathbf{G}(s_1,s_2,..,s_n) : \left[\mathbf{R}^T\mathbf{E}_t(t-s_1)\,\mathbf{R}\otimes\cdot\cdot\otimes\mathbf{R}^T\mathbf{E}_t(t-s_n)\,\mathbf{R}\right]ds_1\cdot\cdot ds_n\end{aligned} \tag{11.5.33}$$

$\mathbf{G}$ is a tensor of order $(2n+2)$ and $\mathbf{E}_t$ is a strain tensor:

$$\mathbf{E}_t = \frac{1}{2}(\mathbf{C}_t - \mathbf{1}) \tag{11.5.34}$$

Equation (11.5.9) provides an example of a constitutive equation of the integral type. This constitutive equation only applies for small deformation, i.e. $\mathbf{R} \approx \mathbf{1}$, and:

$$\mathbf{E}_t(t-s) = \mathbf{E}(t-s) - \mathbf{E}(t) \tag{11.5.35}$$

The tensor $\Pi[\mathbf{C}]$ is represented by the tensor:

$$2\beta'_e \mathbf{E}(t) + \left[\beta^o{}_e - \frac{2}{3}\beta'_e\right](\mathrm{tr}\,\mathbf{E}(t))\,\mathbf{1} \tag{11.5.36}$$

$\mathbf{G}$ is the 4. order tensor with components:

$$G_{ijkl} = 2\frac{d\beta'(s)}{ds}\delta_{ik}\delta_{jl} + \left[\frac{d\beta^o(s)}{ds} - \frac{2}{3}\frac{d\beta'(s)}{ds}\right]\delta_{ij}\delta_{kl} \tag{11.5.37}$$

A *constitutive equation of the rate type* has the form:

$$\overset{(n)}{\mathbf{T}} = \mathbf{K}\left[\mathbf{T}, \dot{\mathbf{T}}, \cdot\cdot \overset{(n-1)}{\mathbf{T}}, \mathbf{F}, \dot{\mathbf{F}}, \cdot\cdot \overset{(n)}{\mathbf{F}}\right] \tag{11.5.38}$$

$\overset{(n)}{\mathbf{T}}$ is the n. material derivative of the tensor $\mathbf{T}$. It turns out that (11.5.38) defines a class of material models, confer Truesdell and Noll [52].

The constitutive equation (11.5.38) does not satisfy the objectivity principle and must be transformed for this principle to be satisfied. On the basis of the Noll's reduced form (11.5.27) three alternative kinds rate-type equations may be constructed:

$$\partial_r^n \mathbf{T}^* = \mathbf{K}\left[\mathbf{T}^*, \partial_r \mathbf{T}^*, \cdot\cdot, \partial_r^{n-1}\mathbf{T}^*, \mathbf{D}^*, \mathbf{D}_2{}^*, \cdot\cdot, \mathbf{C}\right] \tag{11.5.39}$$

$$\mathbf{T}^{\Delta n*} = \mathbf{K}\left[\mathbf{T}^*, \mathbf{T}^{\Delta *}, \cdot\cdot, \mathbf{T}^{\Delta(n-1)*}, \mathbf{A}_1{}^*, \mathbf{A}_2{}^*, \cdot\cdot, \mathbf{B}\right] \tag{11.5.40}$$

$$\mathbf{T}^{\nabla n*} = \mathbf{K}\left[\mathbf{T}^*, \mathbf{T}^{\nabla *}, \cdot\cdot, \mathbf{T}^{\nabla(n-1)*}, \mathbf{A}_1{}^*, \mathbf{A}_2{}^*, \cdot\cdot, \mathbf{B}\right] \tag{11.5.41}$$

where:

$$\partial_r^n \mathbf{T}^* = \mathbf{R}^T\,\partial_r \mathbf{T}\,\mathbf{R},\ \partial_r \mathbf{T} = \text{corotational time derivative of } \mathbf{T}$$

$$\mathbf{T}^{\Delta *} = \mathbf{R}^T\,\mathbf{T}^{\Delta}\,\mathbf{R},\ \mathbf{T}^{\Delta} = \text{lower convected time derivative}$$

$$\mathbf{T}^{\nabla *} = \mathbf{R}^T\,\mathbf{T}^{\nabla}\,\mathbf{R},\ \mathbf{T}^{\nabla} = \text{upper convected time derivative} \tag{11.5.42}$$

The Sect. 11.9.4 and 11.9.5 present some fluid models with these types of constitutive equations.

The development of the theory of constitutive equations for continuum materials has been focused on simple materials. In a more general form of constitutive equations the stress tensor is assumed to be a functional of the n first gradients to the motion $\mathbf{r}(\mathbf{r}_o,t)$:

$$\mathbf{T} = \mathbb{F}\left[\mathbf{F}^t, \mathbf{F}_2^t, \cdot\cdot, {}_n\mathbf{F}_n^t\right],\ \mathbf{F}_n = \frac{\partial^n \mathbf{r}}{\partial \mathbf{r}_o^n} \quad \Leftrightarrow \quad F_{nik_1k_2\cdot\cdot k_n} = \frac{\partial^n x_i}{\partial X_{k_1}\partial X_{k_2}\cdot\cdot\partial X_{k_n}} \tag{11.5.43}$$

The constitutive equation (11.5.42) characterizes a material of degree n. This implies that simple materials are materials of degree 1.

11.6 Material Symmetry

In this section we need to understand what is meant by a *group*. A group is a set of elements. In the present application of the group concept the elements are 2. order tensors.

Definition 11.1. A group g is a set of elements that satisfy the following four axioms:

1. A binary operation * exists that from any two elements $A,B \in g$, i.e. elements A and B belonging to the group, creates another element in the group:

$$A^*B \in g$$

2. The binary operation is associative:

$$A,B,C \in g \quad \Rightarrow \quad \left(A^*B\right)^* C = A^*\left(B^*C\right)$$

3. A unit element 1 belongs to the group, $1 \in g$, such that for any element $A \in g$:

$$A^*1 = 1^*A$$

4. For every element $A \in g$, an inverse element A^{-1} exists, such that:

$$A^*A^{-1} = A^{-1}A = 1$$

If the elements $A,B,\cdot\cdot$ are the elements of the group g, we identify the group by:

$$g = \{A,B,\cdot\cdot\}$$

In the application below of the group concept the elements are 2. order tensors and the binary operation is the composition of any two 2. order tensors.

11.6.1 Symmetry Groups

A homogeneous elastic material, assumed to be stress free in a reference configuration K_o, is subjected to a homogeneous deformation given by the deformation gradient $\mathbf{F}$. The state of stress in the material in the deformed configuration K, see Fig. 11.6.1b, is given by the stress tensor $\mathbf{T}[\mathbf{F}]$. If the material is deformed from K_o by the homogeneous deformation gradient $\mathbf{P}$ to the configuration $\tilde{K}_o$, Fig. 11.6.1a, and this configuration is used as a new reference configuration, from which the material is given the homogeneous deformation $\mathbf{F}$, the state of stress will in general not be $\mathbf{T}[\mathbf{F}]$. If however the elastic material is isotropic, a predeformation deformation gradient $\mathbf{P}$ equal to an orthogonal tensor $\mathbf{Q}$, will not influence the response of the material when subjected to the deformation $\mathbf{F}$ from the configuration $\bar{K}_o$ to the configuration $\bar{K}$, see Fig. 11.6.1c. The state of stress in the two configurations K and $\bar{K}$ are the same.

A characteristic property of the materials we call fluids is that for such materials the response to a deformation history is not influenced by a isochoric predeformation followed by a sufficient time at rest. Volumetric changes due to a predeformation will change the pressure in the new reference configuration and thus may change the response of the fluid to following volumetric deformation. The viscous properties of the fluid are also to some extend dependent upon the pressure level.

The considerations above will now be generalized. Figure 11.6.2 is meant to illustrate a simple material subjected to the deformation history $\mathbf{F}(t-s)$ determined from the reference configuration K_o. The state of stress in the present configuration K at time t is determined from the response functional Γ:

$$\mathbf{T} = \Gamma\left[\mathbf{F}^t\right], \; \mathbf{F}(t) = \frac{\partial \mathbf{r}}{\partial \mathbf{r}_o} \tag{11.6.1}$$

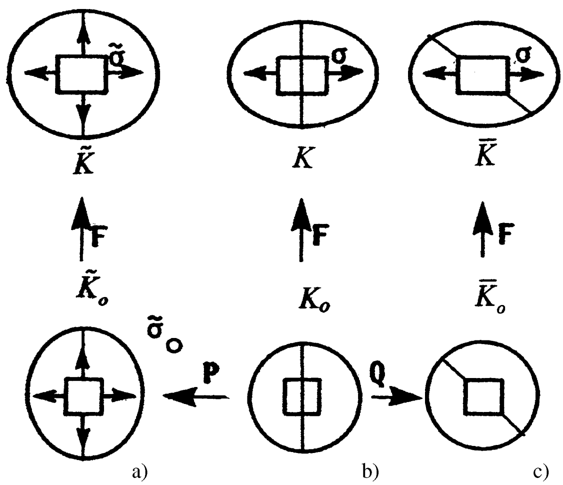


Fig. 11.6.1 Deformation of an isotropic elastic material. **a**) Predeformation from the reference configuration K_o by the homogeneous deformation gradient $\mathbf{P}$ to the configuration $\tilde{K}_o$, which is used as a new reference configuration. The material is then given the homogeneous deformation $\mathbf{F}$. **b**) Homogeneous deformation $\mathbf{F}$ from the reference configuration K_o. The state of stress in $\tilde{K}$ and K will in general be different. **c**) Predeformation gradient $\mathbf{P}$ equal to an orthogonal tensor $\mathbf{Q}$. The state of stress in $\bar{K}$ is the same as in K

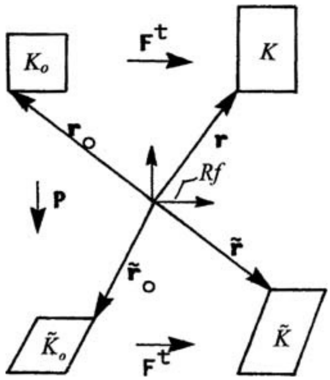


Fig. 11.6.2 A simple material subjected to the deformation history $\mathbf{F}(t-s)$ determined from the reference configuration K_o and from the reference configuration $\tilde{K}_o$ after a predeformation $\mathbf{P}$

Alternatively let the same material be predeformed from K_o to the configuration $\tilde{K}_o$ by the deformation gradient $\mathbf{P}$:

$$\mathbf{P} = \frac{\partial \tilde{\mathbf{r}}_o}{\partial \mathbf{r}_o} \tag{11.6.2}$$

and then subjected to the deformation history $\mathbf{F}^t(s) = \mathbf{F}(t-s)$, as indicated in Fig. 11.6.2. The response functional $\tilde{\Gamma}$ relative to $\tilde{K}_o$ will in general be different from Γ, and the state of stress in the present configuration $\tilde{K}$ at the time t is:

$$\tilde{\mathbf{T}} = \tilde{\Gamma}\left[\mathbf{F}^t\right] = \Gamma\left[\tilde{\mathbf{F}}^t\right] = \Gamma\left[\mathbf{F}^t\,\mathbf{P}\right], \; \tilde{\mathbf{F}} = \frac{\partial \tilde{\mathbf{r}}}{\partial \tilde{\mathbf{r}}_o}\frac{\partial \tilde{\mathbf{r}}_o}{\partial \mathbf{r}_o} = \mathbf{F}\,\mathbf{P} \tag{11.6.3}$$

The set of predeformation gradients $\mathbf{P}$ that results in the same response from the reference configuration K as from the reference configuration $\tilde{K}_o$ for any deformation histories $\mathbf{F}^t$, defines the *symmetry group g_o of the material with respect to the configuration K_o*. The elements of the symmetry group are denoted by the symbol $\mathbf{S}$. A deformation gradient $\mathbf{S}$, which belongs to the symmetry group, must satisfy the condition that for any deformation history $\mathbf{F}^t = \mathbf{F}(t-s)$ the state of stress at time t may be obtain from either of the response functionals Γ or $\tilde{\Gamma}$, such that:

$$\mathbf{T} = \tilde{\Gamma}\left[\mathbf{F}^t\right] = \Gamma\left[\mathbf{F}^t\right] \tag{11.6.4}$$

By the result (11.6.2) this condition may be restated as:

$$\mathbf{T} = \Gamma\left[\mathbf{F}^t\right] = \Gamma\left[\mathbf{F}^t\mathbf{S}\right] \tag{11.6.5}$$

When two predeformations $\mathbf{S}_1$ and $\mathbf{S}_2$ are known, it may be shown that the composition $\mathbf{S}_3 = \mathbf{S}_1\mathbf{S}_2$ also satisfies the condition (11.6.5) and thus belongs to the symmetry group of the material:

$$\mathbf{T} = \Gamma\left[\mathbf{F}^t\right] = \Gamma\left[\mathbf{F}^t\mathbf{S}_1\right] = \Gamma\left[\mathbf{F}^t\mathbf{S}_2\right] = \Gamma\left[\mathbf{F}^t\mathbf{S}_1\mathbf{S}_2\right] = \Gamma\left[\mathbf{F}^t\mathbf{S}_3\right] \quad , \quad \mathbf{S}_3 = \mathbf{S}_1\mathbf{S}_2 \tag{11.6.6}$$

The unit tensor **1** is a trivial member of any symmetry group, and it follows that if the deformation gradient **S** belongs to the symmetry group of a material so does the inverse $\mathbf{S}^{-1}$:

$$\mathbf{T} = \Gamma\left[\mathbf{F}^t\right] = \Gamma\left[\mathbf{F}^t\mathbf{1}\right] = \Gamma\left[\mathbf{F}^t\mathbf{S}^{-1}\mathbf{S}\right] = \Gamma\left[\mathbf{F}^t\mathbf{S}^{-1}\right] \tag{11.6.7}$$

According to the definition of the group concept the results (11.6.6, 11.6.7) show that the set of tensors **S** that satisfy the condition (11.6.5) represents a group under the operation composition.

Experiments with real materials and thermodynamic arguments indicate that the predeformation gradient **S** must be isochoric. Thus we assume a priory that:

$$\det \mathbf{S} = 1 \tag{11.6.8}$$

A tensor with this property is called a *unimodular tensor*. All unimodular tensors constitute a group under the operation composition. This group is called the *unimodular group u*. The elements in a symmetry group g_o of a material are contained in u and g_o is thus a subgroup of u. This property is denoted by:

$$g_o \subset u \tag{11.6.9}$$

An especially important symmetry group is the *orthogonal group o* consisting of all orthogonal tensors **Q**. It has been shown by Noll [34] that if all orthogonal tensors are contained in a symmetry group g_o of a material, such that $o \subset g_o$, then $g_o = o$ or $g_o = u$:

$$g_o \supset o \quad \Rightarrow \quad g_o = o \text{ or } u \tag{11.6.10}$$

11.6.2 Isotropy

Definition 11.2. A material is said to be isotropic if a reference configuration K_o can be found such that $g_o = o$ or $g_o = u$. The reference configuration K_o is then called an undeformed state for the isotropic material.

An isotropic material has in the undeformed state no preferred directions that can reveal whether the material has been rotated before it is subjected to a deformation history. If $g_o = o$, the isotropy will in general be destroyed by a change of reference configuration.

We shall see what the consequences of isotropy are to the Noll reduced constitutive equation (11.5.29). As a consequence of the geometrical interpretation of the polar decomposition $\mathbf{F} = \mathbf{VR}$ shown in Fig. 5.5.3, it follows that two deformation histories $\mathbf{F}(t-s)$ and $\bar{\mathbf{F}}(t-s) = \mathbf{F}(t-s)\ \mathbf{R}(t)^T$ result in the same state of stress $\mathbf{T}(t)$. Now:

$$\begin{aligned}&\bar{\mathbf{F}} = \bar{\mathbf{V}}\bar{\mathbf{R}} = \mathbf{F}\mathbf{R}^T = \mathbf{V}\mathbf{R}\mathbf{R}^T = \mathbf{V} \quad \Rightarrow \quad \bar{\mathbf{R}} = \mathbf{1},\ \bar{\mathbf{V}} = \mathbf{V} \quad \Rightarrow \\ &\bar{\mathbf{U}} = \bar{\mathbf{R}}^T\bar{\mathbf{V}}\bar{\mathbf{R}} = \mathbf{V},\ \bar{\mathbf{C}} = \bar{\mathbf{U}}^2 = \bar{\mathbf{V}}^2 = \bar{\mathbf{B}}\end{aligned} \tag{11.6.11}$$

Furthermore, from (11.5.21):

$$\bar{\mathbf{F}}_t^t = \bar{\mathbf{F}}^t \bar{\mathbf{F}}^{-1} = \mathbf{F}^t \mathbf{R}^T \mathbf{R} \mathbf{F}^{-1} = \mathbf{F}_t^t \tag{11.6.12}$$

The equations (11.6.11, 12) yield:

$$\bar{\mathbf{R}}^T \bar{\mathbf{C}}_t^t \bar{\mathbf{R}} = \mathbf{C}_t^t \text{ and } \bar{\mathbf{C}}_t^t = \left(\bar{\mathbf{F}}_t^t\right)^T \bar{\mathbf{F}}_t^t = \left(\mathbf{F}_t^t\right)^T \mathbf{F}_t^t = \mathbf{C}_t^t \tag{11.6.13}$$

The results (11.6.11, 12, 13) imply that the Noll reduced constitutive equation (11.5.29) may be presented as:

$$\mathbf{T} = \Pi[\mathbf{B}] + \tilde{\Lambda}\left[\,\mathbf{C}_t^t\,;\,\mathbf{B}\right],\ \tilde{\Lambda}[\mathbf{1};\,\mathbf{B}] = \mathbf{0} \tag{11.6.14}$$

We shall now show that $\Pi(\mathbf{B})$ is an isotropic tensor-valued function of the left deformation tensor $\mathbf{B}$ and that $\tilde{\Lambda}[\mathbf{C}_t^t;\mathbf{B}]$ is an isotropic tensor-valued functional of the right deformation history $\mathbf{C}_t^t$ and a function of $\mathbf{B}$, such that:

$$\Pi[\mathbf{Q}\mathbf{B}\mathbf{Q}^T] = \mathbf{Q}\Pi[\mathbf{B}]\,\mathbf{Q}^T,\ \tilde{\Lambda}\left[\,\mathbf{Q}\mathbf{C}_t^t\mathbf{Q}^T\,;\,\mathbf{Q}\mathbf{B}\mathbf{Q}^T\right] = \mathbf{Q}\tilde{\Lambda}\left[\,\mathbf{C}_t^t\,;\,\mathbf{B}\right]\mathbf{Q}^T \tag{11.6.15}$$

for all orthogonal tensors $\mathbf{Q}$.

The equation (11.5.17) implies that two deformation histories $\mathbf{F}(t-s)$ and $\tilde{\mathbf{F}}(t-\ s) = \mathbf{Q}\mathbf{F}(t-\ s)$ result in states of stress that are rotated with respect to each other:

$$\tilde{\mathbf{T}} = \mathbf{Q}\mathbf{T}\mathbf{Q}^T \tag{11.6.16}$$

It follows that:

$$\begin{gathered}\tilde{\mathbf{F}} = \tilde{\mathbf{R}}\tilde{\mathbf{U}} = \mathbf{Q}\mathbf{F} = \mathbf{Q}\mathbf{R}\mathbf{U} \quad \Rightarrow \quad \tilde{\mathbf{U}} = \mathbf{U} \quad , \quad \tilde{\mathbf{R}} = \mathbf{Q}\mathbf{R} \quad \Rightarrow \\ \tilde{\mathbf{C}} = \tilde{\mathbf{U}}^2 = \mathbf{U}^2 = \mathbf{C},\ \tilde{\mathbf{B}} = \tilde{\mathbf{R}}\tilde{\mathbf{C}}\bar{\mathbf{R}}^T = \mathbf{Q}\mathbf{B}\mathbf{Q}^T\end{gathered} \tag{11.6.17}$$

Furthermore we get (11.5.21):

$$\begin{aligned}&\tilde{\mathbf{F}}_t^t = \tilde{\mathbf{F}}^t\,\tilde{\mathbf{F}}^{-1} = \mathbf{Q}\mathbf{F}^t\mathbf{F}^{-1}\mathbf{Q}^T = \mathbf{Q}\mathbf{F}_t^t\mathbf{Q}^T \quad \Rightarrow \\ &\tilde{\mathbf{C}}_t^t = \left(\tilde{\mathbf{F}}_t^t\right)^T\tilde{\mathbf{F}}_t^t = \mathbf{Q}\left(\mathbf{F}_t^t\right)^T\mathbf{Q}^T\mathbf{Q}\mathbf{F}_t^t\mathbf{Q}^T = \mathbf{Q}\left(\mathbf{F}_t^t\right)^T\mathbf{F}_t^t\mathbf{Q}^T = \mathbf{Q}\mathbf{C}_t^t\mathbf{Q}^T \quad \Rightarrow \\ &\tilde{\mathbf{C}}_t^t = \mathbf{Q}\mathbf{C}_t^t\mathbf{Q}^T\end{aligned} \tag{11.6.18}$$

The equations (11.6.16, 17, 18) now prove that the tensors on the right-hand side of (11.5.14) satisfy the conditions (11.6.15).

11.6.3 Change of Reference Configuration

Let g_o and $\tilde{g}_o$ be the symmetry groups for a material with respect to the references K_o and $\tilde{K}_o$ in Fig. 11.6.2. For an element $\mathbf{S} \in g_o$ the constitutive equations (11.6.4) and (11.4.7) imply:

$$\tilde{\Gamma}\left[\mathbf{F}^t\right] = \Gamma\left[\tilde{\mathbf{F}}^t\right] = \Gamma\left[\mathbf{F}^t\mathbf{P}\right] = \Gamma\left[\mathbf{F}^t\mathbf{P}\mathbf{S}\right] = \Gamma\left[\mathbf{F}^t\mathbf{P}\mathbf{S}\mathbf{P}^{-1}\mathbf{P}\right] = \tilde{\Gamma}\left[\mathbf{F}^t\mathbf{P}\mathbf{S}\mathbf{P}^{-1}\right] \quad \Rightarrow$$

$$\tilde{\Gamma}\left[\mathbf{F}^t\right] = \tilde{\Gamma}\left[\mathbf{F}^t\mathbf{P}\mathbf{S}\mathbf{P}^{-1}\right] \quad \Rightarrow$$

$$\mathbf{S} \in g_o \quad \Leftrightarrow \quad \mathbf{P}\mathbf{S}\mathbf{P}^{-1} \in \tilde{g}_o \tag{11.6.19}$$

This result may also be stated as follows, which is called *Noll's rule*:

$$\tilde{g}_o = \mathbf{P}\, g_o \mathbf{P}^{-1} \tag{11.6.20}$$

It follows from Noll's rule that the symmetry group of a material is invariant to dilatations:

$$\mathbf{P} = \alpha\mathbf{1} \quad \Rightarrow \quad \tilde{g}_o = g_o \tag{11.6.21}$$

It may be shown that the only symmetry groups that are invariant for all tensors $\mathbf{P}$, is the two trivial groups $\{\mathbf{1}\}$, i.e. the group containing only the unit tensor, and u, the unimodular group. Because according to the result (11.6.20) any unimodular tensor $\mathbf{S}$ in g_o is a mapping of the tensor $\mathbf{P}^{-1}\mathbf{S}\mathbf{P}$ in $\tilde{g}_o$ for any deformation gradient $\mathbf{P}$, the unimodular group u certainly satisfies Noll's rule:

$$\mathbf{P}\, u\, \mathbf{P}^{-1} = u \tag{11.6.22}$$

11.6.4 Classification of Simple Materials

A material is considered to be a *solid* if the response of the material is influenced by a non-orthogonal deformation $\mathbf{P}$ from K_o to $\tilde{K}_o$. We may formulate this in the following statement:

Definition 11.3. A simple material with the symmetry group g_o is a solid if a reference configuration K_o exists such that:

$$g_o \subset o \subset u \tag{11.6.23}$$

The reference configuration K_o characterizes an undeformed state.

If a solid does not have any real symmetry as far as the response is concerned, the symmetry group is the trivial group $g_o = \{\mathbf{1}\}$ with respect to any reference configuration. The material is called *triclinic* and is represented by two of the crystal classes.

Crystals are represented in 32 crystal classes. Considered as simple materials these classes are arranged in 11 types, each type given a symmetry group.

Solids with a plane of symmetry, let for the sake of argument the plane be normal to a x_3-axis, has a symmetry group containing two elements:

$$g_o = \{\mathbf{1}, \mathbf{Q}_3^{\pi}\} \tag{11.6.24}$$

The tensor $\mathbf{Q}_i^\phi$ represents a rotation by the angle ϕ about an x_i-axis.

Three orthogonal planes of symmetry normal to the axes in a Cartesian coordinate system result in the symmetry group:

$$g_o = \{\mathbf{1}, \mathbf{Q}_1^\pi, \mathbf{Q}_2^\pi, \mathbf{Q}_3^\pi\} \tag{11.6.25}$$

Solids with the symmetry group (11.6.25) are called orthotropic solids. Note that since $\mathbf{Q}_1^\pi = \mathbf{Q}_3^\pi \mathbf{Q}_2^\pi$, it suffices to identify two orthogonal symmetry planes and the third plane orthogonal to the other two follows. Confer Sect. 7.8.3 on orthotropic elastic materials.

The symmetry groups (11.6.24 and 11.6.25) are member of the 11 groups relevant for crystals. All the 11 groups are finite groups, i.e. groups with a finite number of elements. Transverse isotropic solids, confer Sect. 7.8.4, belong to a 12. type of anisotropic materials characterized by the symmetry group:

$$g_o = \{\mathbf{1}, \mathbf{Q}_3^\phi \text{ for all angles } \phi\text{: } 0 \le \phi < 2\pi\} \tag{11.6.26}$$

Let K_o and $\tilde{K}_o$ be two undeformed configurations according to the definition of a solid, and $\mathbf{P}$ the deformation gradient that brings the material from K_o to $\tilde{K}_o$. The symmetry groups related to the two reference configurations are g_o and $\tilde{g}_o$ respectively. The deformation gradient $\mathbf{P}$ is polar decomposed into a rotation tensor $\mathbf{R}$ and a stretch tensor $\mathbf{U}$, such that $\mathbf{P} = \mathbf{RU}$. We shall now show that according to Noll's rule:

$$\tilde{g}_o = \mathbf{R}\, g_o \, \mathbf{R}^T \tag{11.6.27}$$

Let $\mathbf{Q}$ be an orthogonal tensor belonging to the group g_o, i.e. $\mathbf{Q} \in g_o$. Then according to Noll's rule the symmetry group $\tilde{g}_o$ has an element:

$$\begin{gathered}
\tilde{\mathbf{Q}} = \mathbf{PQP}^{-1} = (\mathbf{RU})\,\mathbf{Q}\,(\mathbf{RU})^{-1} \quad \Rightarrow \\
\tilde{\mathbf{Q}}\mathbf{RU} = (\mathbf{RU})\,\mathbf{Q} = \mathbf{R}\left(\mathbf{QQ}^T\right)\mathbf{UQ} = \mathbf{RQ}\left(\mathbf{Q}^T\mathbf{UQ}\right) \quad \Rightarrow \\
\left(\tilde{\mathbf{Q}}\mathbf{R}\right)\mathbf{U} = (\mathbf{RQ})\left(\mathbf{Q}^T\mathbf{UQ}\right)
\end{gathered} \tag{11.6.28}$$

The tensors $\tilde{\mathbf{Q}}\mathbf{R}$ and $\mathbf{RQ}$ are orthogonal tensors, and $\mathbf{Q}^T\mathbf{UQ}$ is a symmetric tensor. Then because polar decomposition is a unique operation, (11.6.28) implies the results:

$$\tilde{\mathbf{Q}}\mathbf{R} = \mathbf{RQ} \Rightarrow \tilde{\mathbf{Q}} = \mathbf{RQR}^T \tag{11.6.29}$$

$$\mathbf{U} = \mathbf{Q}^T\mathbf{UQ} \quad \text{for all orthogonal tensors } \mathbf{Q} \in g_o \tag{11.6.30}$$

The result (11.6.29) proves the statement (11.6.27). The condition (11.6.30) shows that $\mathbf{U}$ is an isotropic tensor = $\lambda\mathbf{1}$. For triclinic material this implies no restrictions on $\mathbf{U}$, which means that any reference configuration may be considered to be an undeformed state. For a material with the symmetry group (11.6.24) in K_o, the stretch tensor part $\mathbf{U}$ of $\mathbf{P}$ must satisfy the condition:

$$\mathbf{U} = \mathbf{Q}_3^{\pi T}\,\mathbf{U}\,\mathbf{Q}_3^{\pi} \tag{11.6.31}$$

which only implies that $\mathbf{U}$ must have $\mathbf{e}_3$ in the x_3–direction as a principle strain direction. A material with the symmetry group (11.6.25) in K_o, must satify the condition:

$$\mathbf{U} = \mathbf{Q}_i^{\pi T}\,\mathbf{U}\,\mathbf{Q}_i^{\pi} \text{ for } i = 1,2,3 \tag{11.6.32}$$

This implies that $\mathbf{e}_i$ in the x_i–directions are principle strain directions for the stretch tensor $\mathbf{U}$.

According to the definition of an isotropic material and the definition of a solid, an isotropic solid must have a reference configuration K_o for which the symmetry group is equal to the orthogonal group o:

$$g_o = o \text{ for a reference configuration } K_o \quad \Leftrightarrow \quad \text{isotropic solid material} \tag{11.6.33}$$

The reference configuration K_o is an undeformed state according to both the definition of an isotropic material and the definition of a solid. The result (11.6.27) shows that an isotropic solid has the isotropy group o as symmetry group with respect to every undeformed state. The two definitions of an undeformed material coincide if both concern the same material. The conditions (11.6.30) imply that the deformation tensor $\mathbf{P}$ must correspond to a form invariant deformation:

$$\mathbf{P} = \mu\mathbf{R} \tag{11.6.34}$$

If the stretch tensor $\mathbf{U}$ to the deformation tensor $\mathbf{P}$ does not satisfy the second condition (11.6.30), then the symmetry group is not a subgroup of the orthogonal group o. The reference configuration is thus according to the definition not an undeformed state. From this conclusion we may assume that:

According to the definition of a solid given above, we should be able to construct material models that cannot support shear stresses when the material is at rest. Experiments with real materials have shown that such models do have relevance. This has the implication that the definition of a fluid presented in Sect. 1.3 and Sect. 8.1 ought to be revised. A better definition would be as follows:

Definition 11.4. A simple material is a fluid if:

$$g_o = u \text{ for all } K_o \tag{11.6.35}$$

i.e. the symmetry group of a fluid is the unimodular group for all reference configurations.

From the definition of an isotropic material and the above definition of a simple fluid it follows that *all simple fluids are isotropic*.

The constitutive equation for a simple fluid may be derivable from the general form (11.6.14). Since $g_o = u$, the left deformation tensor $\mathbf{B}$ should only enter the constitutive equation through its determinant $\det\mathbf{B}$, which by (5.5.22) and (5.5.65) may be expressed by the density of the material:

$$\det \mathbf{B} = \det\left(\mathbf{F}\mathbf{F}^T\right) = (\det \mathbf{F})^2 = \left(\frac{\rho_o}{\rho}\right)^2 \tag{11.6.36}$$

Because $\Pi[\mathbf{B}]$ is an isotropic tensor-valued function of $\mathbf{B}$, the function must according to (4.6.17) be expressed as:

$$\Pi[\mathbf{B}] = -p(\rho)\mathbf{1} \tag{11.6.37}$$

The conclusion is that the general form (11.6.14) of the constitutive equation for simple material has the following reduced form for a simple fluid:

$$\mathbf{T} = -p(\rho)\mathbf{1} + \tilde{\Lambda}\left[\mathbf{C}_t^t;\rho\right], \; \tilde{\Lambda}[\mathbf{1};\rho] = \mathbf{0} \tag{11.6.38}$$

Figure 11.6.3 shows a Venn-diagram for symmetry groups for different types of materials. The elements of the group are supposed be represented by points within the solid line border indicating a group. The symmetry group of a fluid is the unimodular group u, which includes the orthogonal group o. The symmetry group of an isotropic solid is the orthogonal group o. The symmetry group of an anisotropic solid is contained in the orthogonal group o. The last type of materials is represented by *liquid crystals*, briefly presented in Sect. 11.6.5 below. The elements of the symmetry group of a liquid crystal may contain some orthogonal tensors and some non-orthogonal unimodular tensors. Based on the discussion below equation (11.6.34) this type of symmetry groups may also represent a solid with respect to a reference configuration that is not an undeformed state.

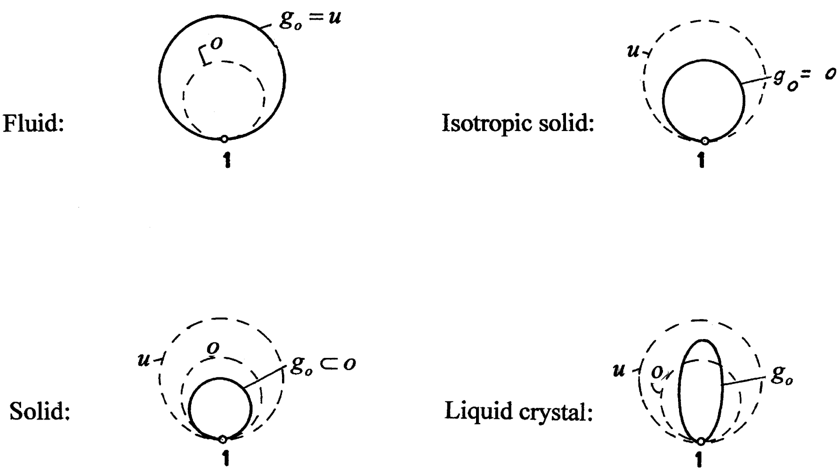


Fig. 11.6.3 Venn-diagram for symmetry groups g_o for fluids, solids, isotropic solid, and liquid crystals. Unimodular group u and orthogonal group o. The unit tensor $\mathbf{1}$

11.6.5 Liquid Crystals

Liquid crystals consist of long organic molecules that within some temperature regions are arranged in a kind of structure. Three main types of structures are discussed in the literature: *nematic structure* in which the molecules are directed along an axis, *cholesteric structure* in which the molecules show a helical pattern, and *smectic structure* in which the molecules form parallel layers. Within each layer in a smectic structure the molecules are parallel with the normal to the layer. The *cholesteric liquid crystals* are derived from cholesterol. However cholesterol does not form liquid crystals. Some species of virus may form liquid crystals, e.g. the tobacco-mosaic- virus (TMV).

The most important feature of liquid crystals is their optical behaviour. Small changes in pressure or temperature, or by subjecting the material to electrical or magnetic fields, may give large changes in the optical anisotropy. Liquid crystals are used in displays in watches, calculators, and screens for PCs and TVs.

Wang [56] has shown that there are 14 different types of liquid crystals characterized by their symmetry group. An isotropic liquid crystal is a simple fluid.

11.7 Thermoelastic Materials

A thermoelastic material is defined by the constitutive equations for the stress tensor $\mathbf{T}$, the heat flux vector $\mathbf{h}$, the internal specific energy ε, and specific entropy s:

$$\begin{aligned}&\mathbf{T} = \mathbf{T}[\mathbf{F}, \theta] \quad , \quad \mathbf{h} = \mathbf{h}[\mathbf{F}, \theta, \mathbf{g}] \\ &\varepsilon = \varepsilon[\mathbf{F}, \theta],\ s = s(\mathbf{F}, \theta),\ \mathbf{g} = \operatorname{grad}\theta \end{aligned} \tag{11.7.1}$$

Truesdell and Toupin [54] have formulated the *principle of equipresence* for constitutive equations:

The independent state variables should be the same in all constitutive equations.

The principle is also called the *principle of maximum ignorance*. The principle is applied in classical thermodynamics by claiming that all dependent state variables are assumed to be functions of the same independent state variables. Special conditions or basic principles of physics may then exclude some of the state variables in some of the equations of state.

The principle of equipresence applied to thermoelastic materials implies that the constitutive equations (11.7.1) should be replaced by:

$$\mathbf{T} = \mathbf{T}[\mathbf{F}, \theta, \mathbf{g}],\ \mathbf{h} = \mathbf{h}[\mathbf{F}, \theta, \mathbf{g}],\ \varepsilon = \varepsilon[\mathbf{F}, \theta, \mathbf{g}],\ \mathbf{g} = \operatorname{grad}\theta \tag{11.7.2}$$

We shall consider (11.7.2) to be the constitutive equations of thermoelastic materials, and then see which consequences the *Clausius-Duhem inequality* (6.4.11) has for these constitutive equations.

We start by computing the material derivatives $\dot{\varepsilon}$ and $\dot{s}$:

$$\dot{\varepsilon} = \frac{\partial \varepsilon}{\partial \mathbf{F}} : \dot{\mathbf{F}} + \frac{\partial \varepsilon}{\partial \theta}\dot{\theta} + \frac{\partial \varepsilon}{\partial \mathbf{g}} \cdot \dot{\mathbf{g}}, \; \dot{s} = \frac{\partial s}{\partial \mathbf{F}} : \dot{\mathbf{F}} + \frac{\partial s}{\partial \theta}\dot{\theta} + \frac{\partial s}{\partial \mathbf{g}} \cdot \dot{\mathbf{g}} \tag{11.7.3}$$

These expressions are substituted into the Clausius-Duhem inequality (6.4.11), and we obtain:

$$\left[\rho\left(\frac{\partial \varepsilon}{\partial \mathbf{F}} - \theta\frac{\partial s}{\partial \mathbf{F}}\right)\mathbf{F}^T - \mathbf{T}\right] : \mathbf{L} + \rho\left(\frac{\partial \varepsilon}{\partial \theta} - \theta\frac{\partial s}{\partial \theta}\right)\dot{\theta} + \rho\left(\frac{\partial \varepsilon}{\partial \mathbf{g}} - \theta\frac{\partial s}{\partial \mathbf{g}}\right)\cdot\dot{\mathbf{g}} + \frac{\mathbf{h}\cdot\mathbf{g}}{\theta} \le 0 \tag{11.7.4}$$

Equation (5.5.28) has been used to replace $\dot{\mathbf{F}}$ by $\mathbf{LF}$. Furthermore $\mathbf{TD} = \mathbf{TL}$. Now we introduce *Helmholtz' free specific energy*:

$$\bar{\varepsilon} = \bar{\varepsilon}(\mathbf{F}, \theta, \mathbf{g}) = \varepsilon - \theta s \tag{11.7.5}$$

and the unequality (11.7.4) may be replaced by:

$$\left[\rho\frac{\partial \bar{\varepsilon}}{\partial \mathbf{F}}\mathbf{F}^T - \mathbf{T}\right] : \mathbf{L} + \rho\left(\frac{\partial \bar{\varepsilon}}{\partial \theta} + s\right)\dot{\theta} + \rho\frac{\partial \bar{\varepsilon}}{\partial \mathbf{g}}\cdot\dot{\mathbf{g}} + \frac{\mathbf{h}\cdot\mathbf{g}}{\theta} \le 0 \tag{11.7.6}$$

The terms in front of the quantities $\mathbf{L}$, $\dot{\theta}$, and $\dot{\mathbf{g}}$ are independent of these quantities, and because the inequality (11.7.6) must be valid for arbitrary values of $\mathbf{L}$, $\dot{\theta}$, and $\dot{\mathbf{g}}$, the mentioned terms must each be zero:

$$\frac{\partial \bar{\varepsilon}}{\partial \mathbf{g}} = \mathbf{0} \quad \Rightarrow \quad \bar{\varepsilon} = \bar{\varepsilon}(\mathbf{F}, \theta) \tag{11.7.7}$$

$$\mathbf{T} = \rho\frac{\partial \bar{\varepsilon}(\mathbf{F}, \theta)}{\partial \mathbf{F}}\mathbf{F}^T, \; s = -\frac{\partial \bar{\varepsilon}(\mathbf{F}, \theta)}{\partial \theta} \tag{11.7.8}$$

The result (11.7.7) shows that the Helmholts free specific energy is independent of temperature gradient $\mathbf{g}$ and thus a function only of the deformation gradient $\mathbf{F}$ and the temperature θ.

Furthermore, it now follows from the inequality (11.7.6) that:

$$\frac{\mathbf{h}\cdot\mathbf{g}}{\theta} \le 0 \tag{11.7.9}$$

This is in agreement with the common notion that heat always flows in the direction of decreasing temperature. It also follows that if the heat flux vector $\mathbf{h}[\mathbf{F}, \theta, \mathbf{g}]$ is a continuous function of the temperature gradient $\mathbf{g}$, then:

$$\mathbf{h}[\mathbf{F}, \theta, \mathbf{0}] = \mathbf{0} \tag{11.7.10}$$

Let $\mathbf{U}$ be the right stretch tensor, $\mathbf{C} = \mathbf{U}^2 =$ the green deformation tensor, $\mathbf{R}$ the rotation tensor derived from the deformation gradient $\mathbf{F}$, and:

$$\tilde{\varepsilon}(\mathbf{C},\theta) \equiv \bar{\varepsilon}(\mathbf{U},\theta),\ \tilde{\mathbf{h}}[\mathbf{C},\theta,\mathbf{g}] \equiv \mathbf{h}[\mathbf{U},\theta,\mathbf{g}] \tag{11.7.11}$$

Then, based on (11.7.5) and (11.7.7, 8) the principle of material objectivity implies:

$$\begin{aligned}&\mathbf{T} = 2\rho\mathbf{F}\frac{\partial\tilde{\varepsilon}}{\partial\mathbf{C}}\mathbf{F}^T,\ s = -\frac{\partial\tilde{\varepsilon}(\mathbf{C},\theta)}{\partial\theta},\ \varepsilon = \tilde{\varepsilon}(\mathbf{C},\theta) - \theta\frac{\partial\tilde{\varepsilon}(\mathbf{C},\theta)}{\partial\theta}\\ &\mathbf{h} = \mathbf{R}\tilde{\mathbf{h}}\left[\mathbf{C},\theta,\mathbf{R}^T\mathbf{g}\right]\end{aligned} \tag{11.7.12}$$

For small deformations we set:

$$\rho = \rho_o \quad , \quad \mathbf{R} \approx \mathbf{1},\ \mathbf{C} = 2\mathbf{E} + \mathbf{1} \approx \mathbf{H} + \mathbf{H}^T + \mathbf{1} \tag{11.7.13}$$

and introduce the energy density:

$$\phi\,(\mathbf{E},\theta) = \rho_o\,\tilde{\varepsilon}\,(\mathbf{C},\theta) \tag{11.7.14}$$

and:

$$\widehat{\mathbf{h}}[\mathbf{E},\theta,\mathbf{g}] = \tilde{\mathbf{h}}[\mathbf{C},\theta,\mathbf{g}] \tag{11.7.15}$$

Then we obtain from (11.7.12):

$$\mathbf{T} = \frac{\partial\phi\,(\mathbf{E},\theta)}{\partial\mathbf{E}},\ \mathbf{h} = \tilde{\mathbf{h}}[\mathbf{E},\theta,\mathbf{g}] \tag{11.7.16}$$

For an isothermal process: $\theta =$ constant, $(11.7.8)_1$ is a constitutive equation for an a hyperelastic material, confer equation (7.10.24):

$$\mathbf{T} = \rho\frac{\partial\bar{\varepsilon}(\mathbf{F})}{\partial\mathbf{F}}\,\mathbf{F}^T \tag{11.7.17}$$

A reasonable slow deformation process in materials with good conductivity will approximately be isothermal.

The temperature θ may be replaced by the entropy density s as independent state variable in (11.7.1), such that the new set of equations are

$$\begin{aligned}&\mathbf{T} = \mathbf{T}[\mathbf{F},s],\ \mathbf{h} = \mathbf{h}[\mathbf{F},s,\mathbf{g}]\\ &\varepsilon = \varepsilon[\mathbf{F},s],\ \theta = s(\mathbf{F},s)\end{aligned} \tag{11.7.18}$$

For simplicity the same symbols for the functions are used in (11.7.1) and (11.7.18). The Clausius-Duhem inequality (6.4.11) now provides the conditions:

$$\mathbf{T} = \rho\frac{\partial\varepsilon(\mathbf{F},s)}{\partial\mathbf{F}}\,\mathbf{F}^T,\ \theta = -\frac{\partial\varepsilon(\mathbf{F},s)}{\partial s} \tag{11.7.19}$$

$$\frac{\mathbf{h}\cdot\mathbf{g}}{\theta} \le 0 \tag{11.7.20}$$

For an isentrop process: $s=$ constant, will $(11.7.19)_1$ be the constitutive equation for a hyperelastic material, confer equation (7.10.24):

$$\mathbf{T}=\rho\frac{\partial\varepsilon(\mathbf{F})}{\partial\mathbf{F}}\,\mathbf{F}^T \tag{11.7.21}$$

Fast deformation processes, e.g. in connection with acoustic waves, may be approximately be considered to be isentropic.

11.8 Thermoviscous Fluids

A *thermoviscous Stokesian fluid* is defined by the constitutive equations for the stress tensor $\mathbf{T}$, the heat flux vector $\mathbf{h}$, the internal specific energy ε, and specific entropy s:

$$\begin{aligned}&\mathbf{T}=\mathbf{T}\left[\rho,\mathbf{D},\theta,\mathbf{g}\right],\ \mathbf{h}=\mathbf{h}\left[\rho,\mathbf{D},\theta,\mathbf{g}\right]\\&\varepsilon=\varepsilon\left[\rho,\mathbf{D},\theta,\mathbf{g}\right],\ s=s\left(\rho,\mathbf{D},\theta,\mathbf{g}\right),\ \mathbf{g}=\text{grad}\,\theta\end{aligned} \tag{11.8.1}$$

$\mathbf{D}$ is the rate of deformation tensor:

$$\mathbf{D}=\frac{1}{2}\left(\mathbf{L}+\mathbf{L}^T\right)\equiv\frac{1}{2}\left(\text{grad}\,\mathbf{v}+\text{grad}\,\mathbf{v}^T\right)\quad\Leftrightarrow\quad D_{ij}=\frac{1}{2}\left(v_{i},_{j}+v_{j},_{i}\right) \tag{11.8.2}$$

In presenting (11.8.1) we have applied the *principle of equipresence* as we did when (11.7.1) were replaced by (11.7.2). We shall now investigate the implications of the *Clausius-Duhem inequality* (6.4.11).

We start by computing the material derivatives $\dot{\varepsilon}$ and $\dot{s}$:

$$\dot{\varepsilon}=\frac{\partial\varepsilon}{\partial\rho}\dot{\rho}+\frac{\partial\varepsilon}{\partial\mathbf{D}}:\dot{\mathbf{D}}+\frac{\partial\varepsilon}{\partial\theta}\dot{\theta}+\frac{\partial\varepsilon}{\partial\mathbf{g}}\cdot\dot{\mathbf{g}},\ \dot{s}=\frac{\partial s}{\partial\rho}\dot{\rho}+\frac{\partial s}{\partial\mathbf{D}}:\dot{\mathbf{D}}+\frac{\partial s}{\partial\theta}\dot{\theta}+\frac{\partial s}{\partial\mathbf{g}}\cdot\dot{\mathbf{g}} \tag{11.8.3}$$

These expressions are substituted into the Clausius-Duhem inequality (6.4.11), and we obtain:

$$\begin{aligned}&\left[\rho^2\left(\frac{\partial\varepsilon}{\partial\rho}-\theta\frac{\partial s}{\partial\rho}\right)\mathbf{1}+\mathbf{T}\right]:\mathbf{D}-\rho\left(\frac{\partial\varepsilon}{\partial\mathbf{D}}-\theta\frac{\partial s}{\partial\mathbf{D}}\right)\dot{\mathbf{D}}-\rho\left(\frac{\partial\varepsilon}{\partial\theta}-\theta\frac{\partial s}{\partial\theta}\right)\dot{\theta}\\&-\rho\left(\frac{\partial\varepsilon}{\partial\mathbf{g}}-\theta\frac{\partial s}{\partial\mathbf{g}}\right)\cdot\dot{\mathbf{g}}-\frac{\mathbf{h}\cdot\mathbf{g}}{\theta}\geq 0\end{aligned} \tag{11.8.4}$$

Helmholtz' free specific energy:

$$\bar{\varepsilon}=\bar{\varepsilon}(\rho,\mathbf{D},\theta,\mathbf{g})=\varepsilon-\theta s \tag{11.8.5}$$

is introduced in the inequality (11.8.4), which becomes:

$$\left[\rho^2\frac{\partial\bar{\varepsilon}}{\partial\rho}+\mathbf{T}\right]:\mathbf{D}-\rho\frac{\partial\bar{\varepsilon}}{\partial\mathbf{D}}\dot{\mathbf{D}}-\rho\left(\frac{\partial\bar{\varepsilon}}{\partial\theta}+s\right)\dot{\theta}-\rho\frac{\partial\bar{\varepsilon}}{\partial\mathbf{g}}\cdot\dot{\mathbf{g}}-\frac{\mathbf{h}\cdot\mathbf{g}}{\theta}\geq 0 \tag{11.8.6}$$

The terms in front of the quantities $\dot{\mathbf{D}}$, $\dot{\theta}$, and $\dot{\mathbf{g}}$ are independent of these quantities, and because the inequality (11.8.6) must be valid for arbitrary values of $\dot{\mathbf{D}}$, $\dot{\theta}$, and $\dot{\mathbf{g}}$, the mentioned terms must each be zero:

$$\frac{\partial \bar{\varepsilon}}{\partial \mathbf{D}} = \mathbf{0}, \ \frac{\partial \bar{\varepsilon}}{\partial \mathbf{g}} = \mathbf{0} \quad \Rightarrow \quad \bar{\varepsilon} = \bar{\varepsilon}(\rho, \theta) \tag{11.8.7}$$

$$s = -\frac{\partial \bar{\varepsilon}}{\partial \theta} = s(\rho, \theta) \tag{11.8.8}$$

It follows from the definition (11.8.5) that also the Helmholtz free specific energy is a function of either density and temperature or of density and entropy:

$$\text{Either: } \bar{\varepsilon} = \bar{\varepsilon}(\rho, \theta) \quad \text{or: } \bar{\varepsilon} = \bar{\varepsilon}(\rho, s) \tag{11.8.9}$$

The Clausius-Duhem inequality (11.8.6) is by the results (11.8.7, 11.8.8) reduced to the inequality:

$$\left[\rho^2 \frac{\partial \bar{\varepsilon}}{\partial \rho} \mathbf{1} + \mathbf{T}\right] : \mathbf{D} - \frac{\mathbf{h} \cdot \mathbf{g}}{\theta} \geq 0 \tag{11.8.10}$$

It is convenient to decompose the stress tensor in an isotropic pressure tensor and a viscous stress tensor:

$$\mathbf{T}^{\nu} = \mathbf{T}^{\nu}(\rho, \mathbf{D}, \theta, \mathbf{g}) = \mathbf{T}(\rho, \mathbf{D}, \theta, \mathbf{g}) + p(\rho, \theta)\mathbf{1} \tag{11.8.11}$$

$p(\rho, \theta)$ is the thermodynamic pressure, which by (8.3.11) is defined by:

$$p = \rho^2 \frac{\partial \bar{\varepsilon}}{\partial \rho} \tag{11.8.12}$$

It follows from (8.3.11) that for an elastic fluid, p=p(ρ), Helmboltz' free energy is equal to the specific elastic energy ψ. The inequality (11.8.10) is now reduced to:

$$\mathbf{T}^{\nu} : \mathbf{D} - \frac{\mathbf{h} \cdot \mathbf{g}}{\theta} \geq 0 \tag{11.8.13}$$

11.9 Advanced Fluid Models

11.9.1 Introduction

Section 8.6.2 presents the simplest and most used models of non-Newtonian fluids, the *generalized Newtonian fluids*. The model is well suited for steady shear flow and in particular steady viscometric flows, see Sect. 8.6.3. The model is also used for unsteady flows of purely viscous fluids. However, a main objection to these models is that they do not reflect normal stress differences in shear flows.

The linearly viscoelastic fluid models in Chap. 9 are only applicable in flows with small deformations, that is small strains and small rotations. Also the strain rates have to be small for these models to apply. The fundamental reasons for these restrictions can be found in the books [2], [4], [26] and [46].

The aim of the present section is to present some of the mostly used advanced models of non-Newtonian fluids. The generalized Newtonian fluids and the fluid models in Chap. 9 will appear as special example of the more advanced fluid models. The references [1], [4], [10] and [26] give a more comprehensive discussion of the models presented in this section, and of other useful but even more complex models.

11.9.2 Stokesian Fluids or Reiner-Rivlin Fluids

The Newtonian fluid and the generalized Newtonian fluid are special versions of *Stokesian fluids* also called *Reiner-Rivlin fluids*. A Stokesian fluid is defined by the constitutive equation:

$$\mathbf{T} = \mathbf{T}[\mathbf{D}, \rho, \theta] \quad , \quad \mathbf{T}[\mathbf{0}, \rho, \theta] = -p(\rho, \theta)\,\mathbf{1} \tag{11.9.1}$$

The stresses are thus functions of the deformation rates, the density of the material, and the temperature. It will be demonstrated below that the stress tensor $\mathbf{T}$ has to be an isotropic function of the deformation rate tensor $\mathbf{D}$. This means that (11.9.1) in general may be presented in the form:

$$\mathbf{T} = -p(\rho, \theta)\,\mathbf{1} + \phi\,\mathbf{1} + 2\eta\,\mathbf{D} + 4\psi_2\,\mathbf{D}^2 \tag{11.9.2}$$

$p(\rho, \theta)$ is the thermodynamic pressure and ϕ, η, and ψ_2 are scalar-valued functions of the principal invariants I_D, II_D, and III_D of $\mathbf{D}$. In the Sects. (8.6.2) and (8.6.4) η is the *viscosity function*, while ψ_2 is the *secondary normal stress coefficient* defined in Sect. (8.6.4) for viscometric flows.

For a incompressible Stokesian fluid equation (11.9.2) is reduced to:

$$\mathbf{T} = -p\,\mathbf{1} + 2\eta\,\mathbf{D} + 4\psi_2\,\mathbf{D}^2 \tag{11.9.3}$$

The constitutive equation (8.4.7) for an incompressible Newtonian fluid and the constitutive equation (8.6.1) for a generalized Newtonian fluid are special cases of (11.9.3).

It will now be shown that the stress in (11.9.1) is an isotropic function of the argument tensor $\mathbf{D}$, i.e. according to (4.6.14): For any orthogonal tensor $\mathbf{Q}$ the stress tensor, in (11.9.4) must satisfy:

$$\mathbf{T}\left[\mathbf{Q}\mathbf{D}\mathbf{Q}^T, \rho, \theta\right] = \mathbf{Q}^T\mathbf{T}[\mathbf{D}, \rho, \theta]\,\mathbf{Q} \tag{11.9.4}$$

By (8.6.13) and Theorem 4.1 the stress tensor $\mathbf{T}$ and the deformation rate tensor $\mathbf{D}$ in the constitutive equation (11.9.1) for a Stokesian fluid are coaxial, which again shows that a *Stokesian fluid is isotropic*. The 4. criterion of Stokes', see Sect. 8.4.1, is automatically satisfied by the constitutive equation (11.9.1).

For a linear Stokesian fluid, i.e. a fluid for which the stresses are linear functions of the deformation rates, we may start with the constitutive equations:

$$T_{ij} = -p\,\delta_{ij} + C_{ijkl}\,D_{kl} \tag{11.9.5}$$

However, the viscous elements C_{ijkl} must be the components of a 4. order isotropic tensor with only two independent viscosities. This fact shows that (11.9.5) may directly be simplified to (8.4.6) for a compressible Newtonian fluid.

11.9.3 Corotational Fluid Models

The constitutive equations for the *corotational fluid models* presented in this section contain the corotational derivative $\partial_r \mathbf{D}$ of the deformation rate tensor $\mathbf{D}$ and the corotational derivative $\partial_r \mathbf{T}$ of the stress tensor $\mathbf{T}$:

$$\partial_r \mathbf{D} = \dot{\mathbf{D}} - \mathbf{WD} + \mathbf{DW} \tag{11.9.6}$$

$$\partial_r \mathbf{T} = \dot{\mathbf{T}} - \mathbf{WT} + \mathbf{TW} \tag{11.9.7}$$

It was shown in Sect. 11.2 that $\mathbf{D}$ is an objective tensor and therefore so is $\partial_r \mathbf{D}$. In order to investigate the behavior of the models in viscometric flows, we need to express the deformation rate tensor $\mathbf{D}$ and the rate of rotation tensor $\mathbf{W}$ in a steady simple shear flow:

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\dot{\gamma}}{2}, \quad W = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\dot{\gamma}}{2} \tag{11.9.8}$$

Then from (11.9.6) we obtain:

$$\partial_r D = \dot{D} - WD + DW = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\ddot{\gamma}}{2} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\dot{\gamma}^2}{2} \tag{11.9.9}$$

We also need the expression for $\mathbf{D}^2$:

$$D^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\dot{\gamma}^2}{4} \tag{11.9.10}$$

A *second-order fluid* is a model that has been used for steady flows. The constitutive equation is:

$$\mathbf{T} = -p\mathbf{1} + 2\mu_1 \mathbf{D} - 2\mu_2 \partial_r \mathbf{D} + 2\mu_3 \mathbf{D}^2 \tag{11.9.11}$$

μ_1, μ_2, and μ_3 are temperature dependent material parameters.

In steady simple shear flow we use the results in (11.9.8, 11.9.9, 11.9.10), and obtain:

$$
\begin{aligned}
&T_{11} = -p + \left(\frac{\mu_3}{2} + \mu_2\right)\dot{\gamma}^2 \quad , \quad T_{22} = -p + \left(\frac{\mu_3}{2} - \mu_2\right)\dot{\gamma}^2,\; T_{12} = \mu_1\,\dot{\gamma} \\
&T_{33} = -p,\; T_{23} = T_{31} = 0
\end{aligned}
\tag{11.9.12}
$$

The viscosity function and the normal stress coefficients are represented by constants:

$$
\eta = \frac{T_{12}}{\dot{\gamma}} = \mu_1,\; \psi_1 = \frac{T_{11} - T_{22}}{\dot{\gamma}^2} = 2\mu_2,\; \psi_2 = \frac{T_{22} - T_{33}}{\dot{\gamma}^2} = \frac{\mu_3}{2} - \mu_2 \tag{11.9.13}
$$

The model may be used for low values of the shear rate.

A better model for steady flows is the CEF fluid, named after W. O. Criminale Jr., J. L. Ericksen, and G. L. Filbey [9]. The constitutive equation is:

$$
\mathbf{T} = -p\,\mathbf{1} + 2\eta\,\mathbf{D} - \psi_1\partial_r\mathbf{D} + (2\psi_1 + 4\psi_2)\,\mathbf{D}^2 \tag{11.9.14}
$$

$\eta\,(\dot{\gamma})\,,\psi_1\,(\dot{\gamma})$, and $\psi_2\,(\dot{\gamma})$ are the viscometric functions for viscometric flows and are functions of the magnitude of shear rate $\dot{\gamma}$ as defined by:

$$
\dot{\gamma} = \sqrt{2\mathbf{D}:\mathbf{D}} = \sqrt{2\,\mathrm{tr}\,\mathbf{D}^2} \tag{11.9.15}
$$

It may be shown that the CEF fluid is a non-linear viscoelastic fluid. The viscoelastic properties are contained in the normal stress coefficients. Unfortunately this fluid model, which by its proper representation of the viscometric functions is perfect for viscometric flows, does only give moderate or even poor results for non-viscometric flows, see R.I. Tanner [46].

The *NIS fluid*, named after H. Norem, F. Irgens, and B. Schieldrop [35], is a viscoelastic-plastic fluid designed to be used for granular materials. A granular material consists of solid particles in a fluid suspension. If the volume fraction of solid particles is small, the material behaves approximately as the suspension fluid, very often as a Newtonian fluid. The flow of the granular material is now characterized as a *macroviscous flow*. For high values of the volume fraction of solid particles, collisions between the particles as the granular material flows and deforms, result in non-Newtonian behavior. In steady simple shear tests represented by the rate of deformation matrix in $(11.9.8)_1$ the shear T_{12} can be proportional to the square of the shear strain rate. Furthermore, it is found that for granular materials with predominantly dry, coarse particles, and at low shear strain rates, the ratio of the shear stress T_{12} and the normal stress T_{22} is approximately constant and independent of the shear strain rate. This relationship may be expressed by:

$$
T_{12} = |T_{22}|\tan\phi\,\frac{\dot{\gamma}}{|\dot{\gamma}|} \tag{11.9.16}
$$

where ϕ is an internal dry friction angle of the granular material.

In its most general form the model is defined by the constitutive equation:

$$\mathbf{T} = -p\,\mathbf{1} + 2\frac{\alpha + \beta p_e}{\dot{\gamma}}\mathbf{D} + 2\eta\,\mathbf{D} - \psi_1 \partial_r \mathbf{D} + (2\psi_1 + 4\psi_2)\,\mathbf{D}^2 \tag{11.9.17}$$

α represents cohesion and β is a dry friction coefficient. p_e, which is called the *effective pressure*, is the part of the total pressure p that represents the direct contact between the solid particles in the suspension. The total pressure consists of the effective pressure and the pore pressure p_o.

A special version of the NIS fluid model has been used in simulations of snow avalanches, landslides, and in submarine slides. The viscometric functions are chosen as power laws:

$$\eta = \mu\,\dot{\gamma}^{n-1},\ \psi_1 = \mu_1\,\dot{\gamma}^{n-2},\ \psi_2 = \mu_2\,\dot{\gamma}^{n-2} \tag{11.9.18}$$

μ, μ_1, μ_2, and n are constant material parameters. Based on experimental evidence, the power law index is chosen to be 2 for a granular material with a high volume fraction of solid particles.

11.9.4 Quasi-Linear Corotational Fluid Models

These models are developed from the linear viscoelastic fluid models discussed in Sect. 9.2.4. The response equations and constitutive equations at a fluid particle of the linear models are presented in a corotational reference for the particle, and then transformed to any convenient reference common to all fluid particles. These new models are therefore called quasi-linear and they satisfy the principle of material objectivity. The quasi-linear fluid models have a much wide application potential then their corresponding linearly viscoelastic counterparts. The reason for this is mainly the fact that the rotations of the principal axes of strains and the principal axes of strain rates are eliminated by introducing the corotational reference

From the Jeffreys fluid the *corotational Jeffreys fluid* is developed. The model is defined by the response equations:

$$\mathbf{T}' + \lambda_1\,\partial_r \mathbf{T}' = 2\tilde{\mu}\,\mathbf{D} + 2\tilde{\mu}\,\lambda_2\,\partial_r \mathbf{D} \tag{11.9.19}$$

$\tilde{\mu}, \lambda_1$, and λ_2 are temperature dependent material parameters. In a corotational reference the response equation (11.9.19) is reduced to the response equations (9.2.81) of the linear viscoelastic Jeffreys fluid.

It follows from the expressions (11.9.6, 11.9.7) for the corotational derivatives of $\mathbf{T}$ and $\mathbf{D}$ that the response equation (11.9.19) is non-linear. The term "quasi-linear" refers to the fact that (11.9.19) is locally linear at a particle if they are related to a corotational reference for that particle.

For the special case $\lambda_2 = 0$ in (11.9.19), we obtain the response equation of the *corotational Maxwell fluid*:

$$\mathbf{T}' + \lambda_1 \, \partial_r \mathbf{T}' = 2\tilde{\mu} \, \mathbf{D} \tag{11.9.20}$$

Compare this equation with the response (9.2.74) for a linearly viscoelastic Maxwell fluid.

11.9.5 Oldroyd Fluids

A coordinate system that is embedded in the fluid and moves and deforms with the fluid is called a *convected coordinate system* and will naturally be a curvilinear coordinate system. Another name for a convected coordinates is *codeforming coordinates*.

J. G. Oldroyd [38] defined his fluid models by constitutive equations defined in convected coordinates. Such constitutive equations automatically satisfy the principle of material objectivity: a rigid-body rotation superimposed on any deformation history can not be registered by equations written in convected coordinates. Material derivatives of the components in convected coordinates of an objective tensor represent objective tensors called *convected derivatives* and were defined in Sect. 11.4. A formal presentation of convected derivatives of objective tensors involves application of general curvilinear coordinate analysis and will be postponed till Sect. 13.4.

Transforming the response function of the Jeffreys fluid from convective coordinates to fixed Cartesian coordinates, Oldroyd obtained the response equations of two quasi-linear models: the *lower-convected Jeffreys fluid* or the *Oldroyd A-fluid* defined by:

$$\mathbf{T}' + \lambda_1 \, \mathbf{T}'^{\Delta} = 2\tilde{\mu} \, \mathbf{D} + 2\tilde{\mu} \, \lambda_2 \, \mathbf{D}^{\Delta} \tag{11.9.21}$$

and the *upper-convected Jeffreys fluid* or the *Oldroyd B-fluid* defined by:

$$\mathbf{T}' + \lambda_1 \, \mathbf{T}'^{\nabla} = 2\tilde{\mu} \, \mathbf{D} + 2\tilde{\mu} \, \lambda_2 \, \mathbf{D}^{\nabla} \tag{11.9.22}$$

$\tilde{\mu}, \lambda_1$, and λ_2 are temperature dependent material parameters.

In a steady simple shear flow the following viscometric functions η, ψ_1, and ψ_2 are found, and in a steady uniaxial extensional the following extensional viscosity η_E is found for the Oldroyd A-fluid and the Oldroyd B-fluid:

Oldroyd A-fluid:

$$\begin{aligned} &\eta = \tilde{\mu}, \; \psi_1 = 2\tilde{\mu} \, (\lambda_1 - \lambda_2), \; \psi_2 = -\psi_1 \\ &\eta_E(\dot{\varepsilon}) = 3\tilde{\mu} \frac{1 + \lambda_2 \, \dot{\varepsilon} - 2\lambda_1 \, \lambda_2 \, \dot{\varepsilon}^2}{1 + \lambda_1 \, \dot{\varepsilon} - 2\lambda_1^2 \, \dot{\varepsilon}^2} \end{aligned} \tag{11.9.23}$$

Oldroyd B-fluid:

$$\eta = \tilde{\mu},\ \psi_1 = 2\tilde{\mu}\,(\lambda_1 - \lambda_2),\ \psi_2 = 0$$
$$\eta_E(\dot{\varepsilon}) = 3\tilde{\mu}\,\frac{1 - \lambda_2\,\dot{\varepsilon} - 2\lambda_1\,\lambda_2\,\dot{\varepsilon}^2}{1 - \lambda_1\,\dot{\varepsilon} - 2\lambda_1^2\,\dot{\varepsilon}^2} \tag{11.9.24}$$

The Oldroyd A-fluid overpredicts the secondary normal stress coefficient ψ_2 and has been judged in the literature not to be a useful model. The Oldroyd B-fluid has a zero second normal stress coefficient ψ_2 and can predict a tension-thickening effect, for $\lambda_1 > \lambda_2$, or a constant extensional viscosity $\eta_E = 3\tilde{\mu}$ for $\lambda_1 = \lambda_2$. For $\lambda_1 < \lambda_2$ the Oldroyd B-fluid predicts a negative primary normal stress coefficient ψ_2. Neither the Oldroyd A-fluid nor the Oldroyd B-fluid predict shear-thinning.

If we set $\lambda_2 = 0$ in the Oldroyd A-fluid and Oldroyd B-fluid, two new fluid models are created: the *lower-convected Maxwell fluid*:

$$\mathbf{T}' + \lambda_1\,\mathbf{T}'^{\Delta} = 2\tilde{\mu}\,\mathbf{D} \tag{11.9.25}$$

and the *upper convected Maxwell fluid*:

$$\mathbf{T}' + \lambda_1\,\mathbf{T}'^{\nabla} = 2\tilde{\mu}\,\mathbf{D} \tag{11.9.26}$$

As seen from the formulas (11.9.23) the lower-convected Maxwell fluid has the same drawback as the Oldroyd A-fluid as it overpredicts the second normal stress coefficient ψ_2. As seen from the formulas (11.9.24) the upper-convected Maxwell fluid has a zero second normal stress coefficient ψ_2 and can predict tension thickening effect. None of the convected Maxwell fluids predict shear-thinning.

The most serious drawback with the four fluid models presented above is the fact that none of them are shear-thinning. To remedy this J. L. White and A. B. Metzner [57] proposed a fluid model, later named the *White-Metzner fluid*, and defined by the response equation:

$$\mathbf{T}' + \frac{\eta(\dot{\gamma})}{\mu}\,\mathbf{T}'^{\nabla} = 2\,\eta(\dot{\gamma})\,\mathbf{D} \tag{11.9.27}$$

where $\eta(\dot{\gamma})$ is the shear viscosity function and μ is a shear modulus. The White-Metzner fluid contains both the shear-thinning features of a general Newtonian fluid and the viscoelastic, memory aspects of a Maxwell fluid. If the viscosity function is chosen to have a zero-shear-rate limit, the White-Metzner fluid behaves as an upper convected Maxwell fluid at low shear rates.

Oldroyd proposed a fluid model defined by constitutive equations that added to the response equation for the upper convected Maxwell fluid all possible terms that are linear in $\mathbf{D}$ and $\mathbf{T}'$, and quadratic in $\mathbf{D}$. The response equation of this *Oldroyd 8-constant fluid* is:

$$\begin{aligned}\mathbf{T}' + \lambda_1\,\partial_r\mathbf{T}' + \lambda_3\,[\mathrm{tr}\,\mathbf{T}]\,\mathbf{D} - \lambda_4\,[\mathbf{TD} + \mathbf{DT}] + \lambda_6\,[\mathrm{tr}\,\mathbf{TD}]\,\mathbf{1}&\\ = 2\tilde{\mu}\left\{\mathbf{D} + 2\,\lambda_2\,\partial_r\mathbf{D} - 2\lambda_5\mathbf{D}^2 + \lambda_7\left[\mathrm{tr}\,\mathbf{D}^2\right]\mathbf{1}\right\}&\end{aligned} \tag{11.9.28}$$

$\tilde{\mu}, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$, and λ_7 are 8 temperature dependent material parameters.

Table 11.9.1 Models obtained from the Oldroyd 8-constants fluid model Tanner [46] presents an interesting evaluation of various constitutive equations for non-Newtonian fluid models and their performance

Oldroyd 8-constants fluid model:	$\tilde{\mu}$	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
Newtonian fluid	$\tilde{\mu}$	0	0	0	0	0	0	0
Second-order fluid	$\tilde{\mu}$	0	λ_2	0	λ_1	λ_5	0	0
Corotational Jeffreys fluid	$\tilde{\mu}$	λ_1	λ_2	0	0	0	0	0
Corotational Maxwell fluid	$\tilde{\mu}$	λ_1	0	0	0	0	0	0
Oldroyd A-fluid (lower-convected Jeffreys fluid)	$\tilde{\mu}$	λ_1	λ_2	0	$-\lambda_1$	$-\lambda_2$	0	0
Oldroyd B-fluid (upper-convected Jeffreys fluid)	$\tilde{\mu}$	λ_1	λ_2	0	λ_1	λ_2	0	0
Oldroyd 4-constant fluid	$\tilde{\mu}$	λ_1	λ_2	λ_3	λ_1	λ_2	0	0
Upper-convected Maxwell fluid	$\tilde{\mu}$	λ_1	0	0	λ_1	0	0	0
Lower-convected Maxwell fluid	$\tilde{\mu}$	λ_1	0	0	$-\lambda_1$	λ_2	0	0

In a steady simple shear flow the following viscometric functions η, ψ_1, and ψ_2 are found, and in a steady uniaxial extensional the following extensional viscosity η_E is found for the Oldroyd 8-constants fluid:

$$
\begin{aligned}
\eta(\dot{\gamma}) &= \tilde{\mu}\frac{1+\left[\lambda_1\lambda_2+\lambda_3\left(\lambda_5-\frac{3}{2}\lambda_7\right)-\lambda_4(\lambda_5-\lambda_7)\right]\dot{\gamma}^2}{1+\left[\lambda_1^2+\lambda_3\left(\lambda_4-\frac{3}{2}\lambda_6\right)-\lambda_4(\lambda_4-\lambda_6)\right]\dot{\gamma}^2}\\
\psi_1(\dot{\gamma}) &= 2\lambda_1\eta(\dot{\gamma})-2\lambda_2\tilde{\mu},\ \psi_2(\dot{\gamma}) = (\lambda_4-\lambda_1)\,\eta(\dot{\gamma})-\lambda_5\tilde{\mu}\\
\eta_E(\dot{\varepsilon}) &= 3\tilde{\mu}\frac{1-\lambda_5\dot{\varepsilon}+\left(\frac{3}{2}\lambda_3-2\lambda_4\right)(2\lambda_5-3\lambda_7)\,\dot{\varepsilon}^2}{1-\lambda_4\dot{\varepsilon}+\left(\frac{3}{2}\lambda_3-\lambda_4\right)(2\lambda_4-3\lambda_6)\,\dot{\varepsilon}^2}
\end{aligned}
\tag{11.9.29}
$$

Apart from the CEF fluid, the NIS fluid, and the White-Metzner fluid, all the advanced fluid models presented in this chapter is directly or indirectly represented in the Oldroyd 8-constants fluid model. Table 11.9.1 summarizes these fluid models.

Chapter 12
Tensors in Euclidean Space E_3

12.1 Introduction

This chapter treats scalars, vectors, and tensors in the physical space that we commonly accept as a three-dimensional *Euclidean space* E_3. In this space we may represent analytical operations involving scalars, vectors, and tensors using geometrical pictures. A generalization to abstract n-dimensional spaces becomes fairly straight forward and direct because most of the concepts and operations involved are not confined to E_3. An example of *non-Euclidean geometry* is provided by the geometry of two-dimensional surfaces imbedded in E_3.

The fundamental concepts of tensors and tensor algebra are introduced in Chap. 4, in which the tensors were represented by their components in right-handed orthogonal Cartesian coordinate systems Ox. In the present chapter tensors will be represented by components in general curvilinear coordinate systems.

12.2 General Coordinates. Base Vectors

In many applications of vector and tensor analysis it is convenient to describe the positions of points in space using other parameters than the three coordinates x_i in a right-handed Cartesian Ox-system. Three such parameters or *general coordinates* are necessary. These will be denoted:

$$y_1, y_2, y_3 \;\Leftrightarrow\; y_i \;\; (i = 1, 2, 3)$$

A one-to-one correspondence between points in space and coordinate sets y_i must exist, which implies that reversible dependencies must exist between x_i and y_i:

$$x_i(y) \;\Leftrightarrow\; y_i(x)$$

It may be shown, see Sokolnokoff [43], that sufficient conditions on the functions $y_i(x)$ for this to be the case are:

a) The functions $y_i(x)$ are single-valued, continuous functions that have continuous partial derivatives of order one,
b) The *Jacobian* to the mapping $y_i(x)$ must be non-zero:

$$J_x^y \equiv \det\left[\frac{\partial y_i}{\partial x}\right] \neq 0 \tag{12.2.1}$$

We now assume that these conditions are satisfied in that part of E_3 that is of interest for the problem we are engaged in, except at isolated points, on isolated lines, or on isolated surfaces. Apart from these singular regions, the Jacobian J_x^y is different from zero. We shall see below that the Jacobian either is positive everywhere or negative everywhere.

According to the multiplication theorem for determinants, (2.1.21):

$$\det\left(\frac{\partial x_i}{\partial x_j}\right) = 1 \Rightarrow \det\left(\frac{\partial x_i}{\partial y_k}\frac{\partial y_k}{\partial x_j}\right) = \det\left(\frac{\partial x_i}{\partial y_k}\right)\det\left(\frac{\partial y_k}{\partial x_j}\right) = 1 \Rightarrow$$

$$J_y^x \equiv \det\left(\frac{\partial x_i}{\partial y_k}\right) = \left(\det\frac{\partial y_k}{\partial x_j}\right)^{-1} \equiv \left(J_x^y\right)^{-1} \tag{12.2.2}$$

When one of the y-coordinates is kept constant, the functions $x_i(y)$ describe a *coordinate surface*. For instance, $y_1 = \text{constant} = \alpha_1$ represents a y_1-surface, Fig. 12.2.1:

$$x_i = x_i(\alpha_1, y_2, y_3)$$

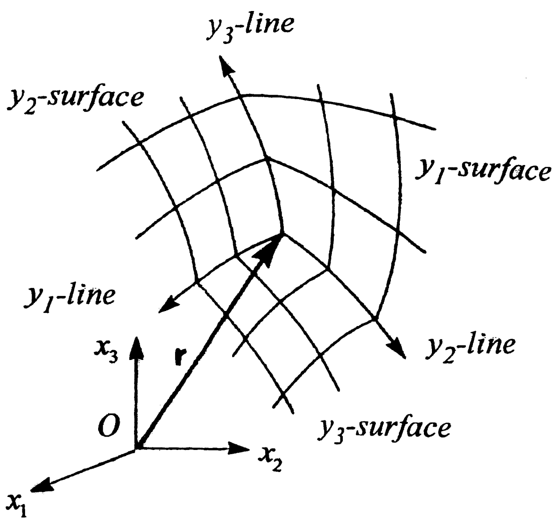


Fig. 12.2.1 Coordinate surfaces and coordinate lines

The intersecting curve between two coordinate surfaces is called a *coordinate line*. The functions $x_i(y)$ describe a coordinate line when two of the y-coordinates are fixed. For example, with $y_1 = \alpha_1$ and $y_2 = \alpha_2$ the functions:

$$x_i = x_i(\alpha_1, \alpha_2, y_3)$$

describe a y_3-line, Fig. 12.2.1. The coordinate lines are in general curved, and the coordinates y_i are therefore called *curvilinear coordinates*. We shall use the symbol y also to denote a *general coordinate system* or a *curvilinear coordinate system*. Likewise we shall, in this chapter, let the symbol x denote a right-handed Cartesian coordinate system.

As will be demonstrated in the examples below, the dimensions of the y-coordinates may be different within one and the same coordinate system. If the coordinate difference between to arbitrary points on a coordinate line is equal to or proportional to the length of the line element between the two points, the coordinate is called a *metric coordinate*.

In *cylindrical coordinates* (R, θ, z), presented in Fig. 12.2.2, we set: $y_1 = R, y_2 = \theta$, $y_3 = z$. The mapping $x_i(y)$ is then:

$$x_1 = R\cos\theta,\ x_2 = R\sin\theta,\ x_3 = z \tag{12.2.3}$$

The R-surfaces are cylindrical surfaces, and the θ- and the z-surfaces are planes. The θ-lines are circles, and the R- and the z-lines are straight lines. R and z are metric coordinates.

In *spherical coordinates* (r, θ, ϕ), presented in Fig. 12.2.3, we set: $y_1 = r, y_2 = \theta$, $y_3 = \phi$. The mapping $x_i(y)$ is then:

$$x_1 = r\sin\theta\cos\phi,\ x_2 = r\sin\theta\sin\phi\ ,\ x_3 = r\cos\theta \tag{12.2.4}$$

The r-surfaces are spheres, the θ-surfaces are cones, and the ϕ-surfaces are planes. The r-lines are straight lines, and the θ – and ϕ – lines are circles. r is a metric coordinate.

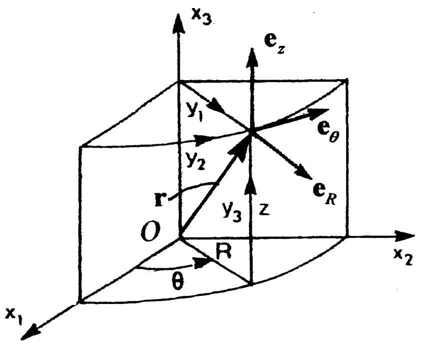


Fig. 12.2.2 Cylindrical coordinates

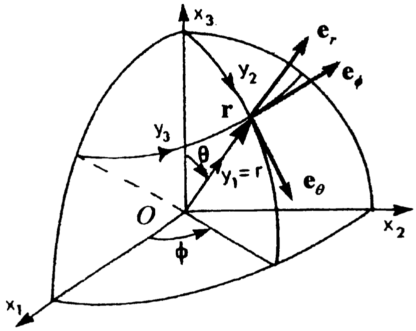


Fig. 12.2.3 Spherical coordinates

Any point P in space may be determined by a set of coordinates in any of the two coordinate systems x or y, or by the *position vector* $\mathbf{r}$ from a reference point O, which in Fig. 12.2.4 happens to be the origin of the x-system. The vector $\mathbf{r}$ may be considered to be a function of either x or y. The *base vectors* in the y-system at the point P are defined as the tangent vectors to the coordinate lines y_i through P:

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial y_i} \tag{12.2.5}$$

The base vectors are not in general unit vectors, as they are if the coordinate system happens to be Cartesian. In a Cartesian x-system:

$$\mathbf{r} = x_i\,\mathbf{e}_i \;\Rightarrow\; \frac{\partial \mathbf{r}}{\partial x_i} = \mathbf{e}_i \;\text{ unit vectors} \tag{12.2.6}$$

12.2.1 Covariant and Contravariant Transformations

When transforming from the y-system to another general coordinate system $\bar{y}$, we find the new base vectors $\bar{\mathbf{g}}_i$ from the relations:

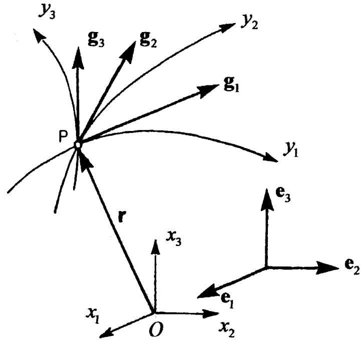


Fig. 12.2.4 Base vectors

$$\bar{\mathbf{g}}_i = \frac{\partial \mathbf{r}}{\partial \bar{y}_i} = \frac{\partial \mathbf{r}}{\partial y_k}\frac{\partial y_k}{\partial \bar{y}_i} \Rightarrow$$
$$\bar{\mathbf{g}}_i = \frac{\partial y_k}{\partial \bar{y}_i}\mathbf{g}_k \Leftrightarrow \mathbf{g}_k = \frac{\partial \bar{y}_i}{\partial y_k}\bar{\mathbf{g}}_i \tag{12.2.7}$$

It follows that the relations between the base vectors in the x- and the y-systems are:

$$\mathbf{g}_i = \frac{\partial x_k}{\partial y_i}\mathbf{e}_k \Leftrightarrow \mathbf{e}_k = \frac{\partial y_i}{\partial x_k}\mathbf{g}_i \tag{12.2.8}$$

The functions $\partial x_k(y)/\partial y_i$ represent the components in the x-system of the base vectors $\mathbf{g}_i$ for the y-system. These functions play the same role as the elements of the transformation matrix Q when transforming from the x-system to another Cartesian $\bar{x}$-system.

The differentials of the functions $y_i(\bar{y})$ and $\bar{y}_i(y)$ are denoted dy^i and $d\bar{y}^i$ respectively. The relationship between the sets of differentials is found as:

$$d\bar{y}^i = \frac{\partial \bar{y}_i}{\partial y_k}dy^k \tag{12.2.9}$$

The formulas $(12.2.7)_1$ and (12.2.9) exemplify two kinds of linear transformations characteristic in general coordinate systems: *covariant transformation* in $(12.2.7)_1$ and *contravariant transformation* in (12.2.9). When operating in general curvilinear coordinates it is convenient to use indices in two positions. An index in the upper position, as for the differentials dy^i and $d\bar{y}^i$, is called a superindex or superscript. When an index is in the lower position, as for the base vectors $\mathbf{g}_i$, it is called a subindex or a subscript.

In coordinate transformations between two Cartesian systems x and $\bar{x}$ (2.3.9, 10):

$$\bar{x}_i = \bar{c}_i + Q_{ik}\, x_k,\; x_k = -c_k + Q_{ik}\, \bar{x}_i \tag{12.2.10}$$

where Q_{ik} are elements of the transformation matrix when transforming from the x-system to the $\bar{x}$-system, the two kinds of transformation coincide, as we can see from the relations:

$$\frac{\partial x_k}{\partial \bar{x}_i} = \frac{\partial \bar{x}_i}{\partial x_k} = Q_{ik} \tag{12.2.11}$$

The base vector $\mathbf{e}_i$ in the x-system represents both a tangent vector to the coordinate line x_i and a normal vector to the coordinate surface $x_i =$constant. In a general curvilinear coordinate system the tangent vectors and the normal vectors need not coincide. It becomes convenient to introduce two sets of base vectors: the tangent vectors $\mathbf{g}_i$ to the y_i-lines, and the normal vectors $\mathbf{g}^i$ to the coordinate surfaces $y_i =$ constants. The normal vectors $\mathbf{g}^i$ are defined by the relations:

$$\mathbf{g}^i \cdot \mathbf{g}_k = \delta^i_k \tag{12.2.12}$$

where $\delta^i_k \equiv \delta_{ik}$ is a *Kronecker delta*. The normal vectors $\mathbf{g}^i$ are called the *reciprocal base vectors* or the *dual base vectors*. If (12.2.12) is multiplied by $\partial y_k/\partial x_j$ and summation over "k" is performed, the result is, after formula $(12.2.8)_2$ has been applied:

$$\mathbf{g}^i \cdot \mathbf{g}_k \frac{\partial y_k}{\partial x_j} = \delta^i_k \frac{\partial y_k}{\partial x_j} = \frac{\partial y_i}{\partial x_j} = \mathbf{g}^i \cdot \mathbf{e}_j \Rightarrow \mathbf{g}^i \cdot \mathbf{e}_j = \frac{\partial y_i}{\partial x_j}$$

The result shows that the symbols $\partial y_i/\partial x_j$ represent the components in the x-system of the reciprocal base vectors $\mathbf{g}^i$:

$$\mathbf{g}^i = \frac{\partial y_i}{\partial x_j}\mathbf{e}_j \Leftrightarrow \mathbf{e}_j = \frac{\partial x_j}{\partial y_i}\mathbf{g}^i \tag{12.2.13}$$

Furthermore, with respect to a $\bar{y}$ – system:

$$\bar{\mathbf{g}}^i = \frac{\partial \bar{y}_i}{\partial x_j}\mathbf{e}_j = \frac{\partial \bar{y}_i}{\partial y_k}\frac{\partial y_k}{\partial x_j}\mathbf{e}_j$$

which with formula $(12.2.13)_1$ gives:

$$\bar{\mathbf{g}}^i = \frac{\partial \bar{y}_i}{\partial y_k}\mathbf{g}^k \Leftrightarrow \mathbf{g}^k = \frac{\partial y_k}{\partial \bar{y}_i}\bar{\mathbf{g}}^i \tag{12.2.14}$$

These relationships show that the reciprocal base vectors follow a contravariant transformation. For this reason it is also customary to call the two sets of base vectors $\mathbf{g}_i$ and $\mathbf{g}^i$ *covariant base vectors* and *contravariant base vectors*, respectively.

The vector product $\mathbf{g}_i \times \mathbf{g}_j$ of any two base vectors is a vector in the direction of the reciprocal base vector $\mathbf{g}^{\mathrm{k}}$, where k is not equal to either i or j. Using the formulas $(12.2.8)_1$, (12.2.13), (2.1.20), and (12.2.2), we obtain:

$$\mathbf{g}_i \times \mathbf{g}_j = \frac{\partial x_r}{\partial y_i}\frac{\partial x_s}{\partial y_j}\mathbf{e}_r \times \mathbf{e}_s = \frac{\partial x_r}{\partial y_i}\frac{\partial x_s}{\partial y_j} e_{rst}\,\mathbf{e}_t = \frac{\partial x_r}{\partial y_i}\frac{\partial x_s}{\partial y_j}\frac{\partial x_t}{\partial y_k} e_{rst}\,\mathbf{g}^k = J^x_y\, e_{ijk}\,\mathbf{g}^k$$

We introduce the symbol:

$$\varepsilon_{ijk} = J^x_y\, e_{ijk} \tag{12.2.15}$$

and may write:

$$\mathbf{g}_i \times \mathbf{g}_j = \varepsilon_{ijk}\,\mathbf{g}^k \tag{12.2.16}$$

Similarly we find:

$$\mathbf{g}^i \times \mathbf{g}^j = \varepsilon^{ijk}\,\mathbf{g}_k \tag{12.2.17}$$

where:

$$\varepsilon^{ijk} = J^y_x\, e_{ijk} \tag{12.2.18}$$

Using the formulas (12.2.12), (12.2.16), and (12.2.17), we obtain the box products:

$$[\mathbf{g}_i\mathbf{g}_j\mathbf{g}_k] = \varepsilon_{ijk}, \quad \left[\mathbf{g}^i\mathbf{g}^j\mathbf{g}^k\right] = \varepsilon^{ijk} \tag{12.2.19}$$

Since it is assumed that the Jacobians J_y^x and J_x^y are different from zero, it follows that the base vectors $\mathbf{g}_i$ and $\mathbf{g}^i$ at a point cannot lie in one and the same plane. We say that the coordinate system y is a *right-handed/left-handed coordinate system* when J_x^y is positive/negative. Furthermore, we say that $\mathbf{g}_i(i = 1,2,3)$ and $\mathbf{g}^i(i = 1,2,3)$ represent *right-handed/left-handed system of vectors* when J_x^y is positive/negative. Note that the results (12.2.16, 12.2.17) are based on the assumption that the x-system is right-handed.

12.2.2 Fundamental Parameters of a Coordinate System

The base vectors $\mathbf{g}_i$ may be decomposed along the $\mathbf{g}^j$-directions, and the base vectors $\mathbf{g}^i$ may be decomposed along the $\mathbf{g}_j$-directions:

$$\mathbf{g}_i = g_{ij}\,\mathbf{g}^j, \; \mathbf{g}^i = g^{ij}\,\mathbf{g}_j \tag{12.2.20}$$

Using the formulas $(12.2.8)_1$ and $(12.2.13)_1$, we obtain:

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \frac{\partial x_k}{\partial y_i}\frac{\partial x_k}{\partial y_j}, \; g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j = \frac{\partial y_i}{\partial x_k}\frac{\partial y_j}{\partial x_k} \tag{12.2.21}$$

The elements g_{ij} and g^{ij} are called the *fundamental parameters* of the y-system. They are symmetric with respect to the indices and represent the magnitudes of the base vectors and the angels between them:

$$\begin{aligned} &|\mathbf{g}_i| = \sqrt{g_{ii}}, \; \left|\mathbf{g}^i\right| = \sqrt{g^{ii}} \\ &\cos(\mathbf{g}_i,\mathbf{g}_j) = \frac{g_{ij}}{\sqrt{g_{ii}\,g_{jj}}}, \; \cos\left(\mathbf{g}^i,\mathbf{g}^j\right) = \frac{g^{ij}}{\sqrt{g^{ii}\,g^{jj}}} \end{aligned} \tag{12.2.22}$$

It will be demonstrated in Sect. 12.4.1 that the fundamental parameters are components of the *unit tensor of 2. order*, also called the *metric tensor*, in the y-system. The fundamental parameters g_{ij} and g^{ij} are related through:

$$g_{ik}\,g^{kj} = \delta_i^j \tag{12.2.23}$$

The determinant of the matrix (g_{ij}) is denoted by the symbol g, and we obtain the results:

$$g \equiv \det(g_{ij}) = \left(J_y^x\right)^2, \; \frac{1}{g} = \det\left(g^{ij}\right) = \left(J_x^y\right)^2 \tag{12.2.24}$$

In Problem 12.1 the reader is asked to derive (12.2.23, 12.2.24), and to show that:

$$g^{ij} = \frac{\operatorname{Co} g_{ij}}{g} \tag{12.2.25}$$

$\operatorname{Co} g_{ij}$ is the cofactor of the matrix element g_{ij}.

12.2.3 Orthogonal Coordinates

It is often convenient in applications to use a coordinate system with orthogonal coordinate lines, which implies:

$$\mathbf{g}_i \cdot \mathbf{g}_j = 0 \text{ and } \mathbf{g}^i \cdot \mathbf{g}^j = 0 \text{ when } i \neq j \tag{12.2.26}$$

Cylindrical coordinates and spherical coordinates are the most common examples of orthogonal coordinates. The two sets of base vectors $\mathbf{g}_i$ and $\mathbf{g}^i$ in orthogonal coordinate systems are parallel sets of vectors. Let h_i denote the *magnitudes of the base vectors* $\mathbf{g}_i$ and let $\mathbf{e}_i^y$ be the unit tangent vectors to the coordinate lines:

$$h_i = \sqrt{\mathbf{g}_i \cdot \mathbf{g}_i} = \sqrt{g_{ii}},\ \mathbf{e}_i^y = \frac{1}{h_i}\mathbf{g}_i \Leftrightarrow \mathbf{g}_i = h_i\, \mathbf{e}_i^y \tag{12.2.27}$$

From the definition (12.2.12) of the reciprocal base vectors and (12.2.27) it follows that:

$$\mathbf{g}^i = \frac{1}{h_i}\mathbf{e}_i^y = \frac{1}{h_i^2}\mathbf{g}_i \tag{12.2.28}$$

It follows from (12.2.26) that:

$$g_{ij} = g^{ij} = 0 \text{ when } i \neq j,\ g^{ii} = \frac{1}{g_{ii}} \text{ (no summation)} \tag{12.2.29}$$

and (g_{ij}) and (g^{ij}) are diagonal matrices.

In *cylindrical coordinates* (R, θ, z), Fig. 12.2.2, we introduce unit vectors $\mathbf{e}_R(\theta)$, $\mathbf{e}_\theta(\theta)$, and $\mathbf{e}_z$ in the directions of the tangents to the coordinate lines, and write for the position vector:

$$\mathbf{r} = R\,\mathbf{e}_R(\theta) + z\mathbf{e}_z \tag{12.2.30}$$

We need the formulas:

$$\frac{d\mathbf{e}_R}{d\theta} = \mathbf{e}_\theta\ ,\ \frac{d\mathbf{e}_\theta}{d\theta} = -\,\mathbf{e}_R \tag{12.2.31}$$

From the definitions of base vectors we now find:

$$\mathbf{g}_1 = \frac{\partial \mathbf{r}}{\partial R} = \mathbf{e}_R\ ,\ \mathbf{g}_2 = \frac{\partial \mathbf{r}}{\partial \theta} = R\frac{\partial \mathbf{e}_R}{\partial \theta} = R\mathbf{e}_\theta\ ,\ \mathbf{g}_3 = \frac{\partial \mathbf{r}}{\partial z} = \mathbf{e}_z$$

$$h_1 = 1,\ h_2 = R,\ h_3 = 1$$

$$\mathbf{g}^1 = \mathbf{e}_R,\ \mathbf{g}^2 = \frac{1}{R}\mathbf{e}_\theta,\ \mathbf{g}^3 = \mathbf{e}_z \tag{12.2.32}$$

$$(g_{ij}) = (\mathbf{g}_i \cdot \mathbf{g}_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \left(g^{ij}\right) = \left(\mathbf{g}^i \cdot \mathbf{g}^j\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/R^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{12.2.33}$$

In *spherical coordinates* (r, θ, ϕ), Fig. 12.2.3, we introduce unit vectors $\mathbf{e}_r(\theta, \phi)$, $\mathbf{e}_\theta(\theta, \phi)$, and $\mathbf{e}_\phi(\theta, \phi)$ in the directions of the tangents to the coordinate lines, and write for the position vector:

$$\mathbf{r} = r\,\mathbf{e}_r(\theta, \phi) \tag{12.2.34}$$

We need the formulas:

$$\begin{aligned}
&\frac{d\mathbf{e}_r}{d\theta} = \mathbf{e}_\theta\ ,\ \frac{d\mathbf{e}_r}{d\phi} = \sin\theta\ \mathbf{e}_\phi\ ,\ \frac{d\mathbf{e}_\theta}{d\theta} = -\ \mathbf{e}_r,\ \frac{d\mathbf{e}_\theta}{d\phi} = \cos\theta\ \mathbf{e}_\phi \\
&\frac{d\mathbf{e}_\phi}{d\theta} = \mathbf{0},\ \frac{d\mathbf{e}_\phi}{d\phi} = -\cos\theta\ \mathbf{e}_\theta - \sin\theta\ \mathbf{e}_r
\end{aligned} \tag{12.2.35}$$

From the definitions of base vectors we find:

$$\begin{aligned}
&\mathbf{g}_1 = \frac{\partial\mathbf{r}}{\partial r} = \mathbf{e}_r\ ,\ \mathbf{g}_2 = \frac{\partial\mathbf{r}}{\partial\theta} = r\frac{\partial\mathbf{e}_r}{\partial\theta} = r\ \mathbf{e}_\theta\ ,\ \mathbf{g}_3 = \frac{\partial\mathbf{r}}{\partial\phi} = r\frac{\partial\mathbf{e}_r}{\partial\phi} = r\sin\theta\ \mathbf{e}_\phi \\
&h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta \\
&\mathbf{g}^1 = \mathbf{e}_r,\ \mathbf{g}^2 = \frac{1}{r}\mathbf{e}_\theta,\ \mathbf{g}^3 = \frac{1}{r\sin\theta}\ \mathbf{e}_\phi
\end{aligned} \tag{12.2.36}$$

$$\begin{aligned}
(g_{ij}) = (\mathbf{g}_i \cdot \mathbf{g}_j) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \\
\left(g^{ij}\right) = \left(\mathbf{g}^i \cdot \mathbf{g}^j\right) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/(r\sin\theta)^2 \end{pmatrix}
\end{aligned} \tag{12.2.37}$$

12.3 Vector Fields

Any vector $\mathbf{a}(y)$ may be decomposed into components a^i along the directions of the base vectors $\mathbf{g}_i$ or into components a_i along the directions of the reciprocal base vectors $\mathbf{g}^i$. This is shown in Fig. 12.3.1 which for simplicity does not show the third dimension. We write:

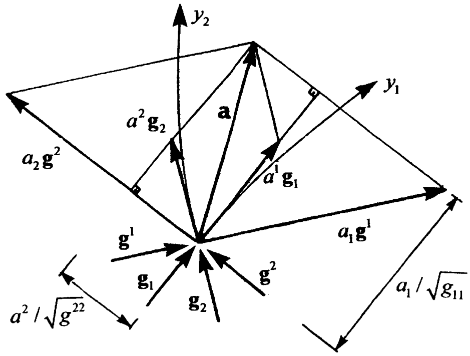
$$\mathbf{a} = a^i\ \mathbf{g}_i = a_i\ \mathbf{g}^i \tag{12.3.1}$$

It follows that:

$$\begin{aligned}
a^i &= \mathbf{g}^i \cdot \mathbf{a} = \mathbf{g}^i \cdot \left(a_k\ \mathbf{g}^k\right) = g^{ik}\,a_k \\
a_i &= \mathbf{g}_i \cdot \mathbf{a} = \mathbf{g}_i \cdot \left(a^k\ \mathbf{g}_k\right) = g_{ik}\,a^k
\end{aligned} \tag{12.3.2}$$

The parameters a^i are called the *contravariant components* of the vector and follows a contravariant transformation rule when changing to a new coordinate system $\bar{y}$:

Fig. 12.3.1 Contravariant and covariant components of a vector **a**



$$\bar{a}^i = \frac{\partial \bar{y}_i}{\partial y_k} a^k \tag{12.3.3}$$

This result is derived from (12.2.14) and (12.3.2). The parameters a_i are called the *covariant components* of the vector and follows a covariant transformation rule:

$$\bar{a}_i = \frac{\partial y_k}{\partial \bar{y}_i} a_k \tag{12.3.4}$$

This result follows from (12.2.7) and (12.3.2).

It may be seen from Fig. 12.3.1 and from the formulas (12.3.1) and (12.3.2) that the contravariant component a^i represents both the vector component, with magnitude $a^i\sqrt{g_{ii}}$, in the direction of the base vector $\mathbf{g}_i$, but also the normal projection of $\mathbf{a}$, with magnitude $a^i/\sqrt{g^{ii}}$, onto the direction of the reciprocal base vector $\mathbf{g}^i$. Likewise it follows that the covariant component a_i represents both the vector component, with magnitude $a_i\sqrt{g^{ii}}$, in the direction of the reciprocal base vector $\mathbf{g}^i$, but also the normal projection of $\mathbf{a}$, with magnitude $a_i/\sqrt{g_{ii}}$, onto the direction of the base vector $\mathbf{g}_i$. The magnitudes of the vectors $a^i\mathbf{g}_i$ (no summation), are called the *physical components* of the vector $\mathbf{a}$, and are denoted by $a(i)$:

$$a(i) \equiv a^i\sqrt{g_{ii}}, \; \mathbf{a} = \sum_i a(i)\, \frac{\mathbf{g}_i}{\sqrt{g_{ii}}} \tag{12.3.5}$$

The vectors:

$$\mathbf{e}_i^y \equiv \frac{\mathbf{g}_i}{\sqrt{g_{ii}}} \tag{12.3.6}$$

are unit vectors tangent to the coordinate lines as in (12.2.27) for orthogonal coordinate systems.

In a Cartesian coordinate system we find that:

$$a(i) = a^i = a_i \tag{12.3.7}$$

In cylindrical and spherical coordinates the physical components of a vector $\mathbf{a}$ are related to the contravariant and covariant components through:

$$a_R = a^1 = a_1,\ a_\theta = Ra^2 = \frac{1}{R}a_2,\ a_z = a^3 = a_3$$
$$a_r = a^1 = a_1,\ a_\theta = ra^2 = \frac{1}{r}a_2,\ a_\phi = r\sin\theta\ a^3 = \frac{a_3}{r\sin\theta} \tag{12.3.8}$$

When adding or subtracting vectors, we find that:

$$\mathbf{a}+\mathbf{b}=\mathbf{c} \Leftrightarrow a^i+b^i=c^i \Leftrightarrow a_i+b_i=c_i \tag{12.3.9}$$

Adding/subtracting the covariant components of one vector to/from the contravariant components of another vector has no meaning.

The *scalar product* of two vectors may be developed thus:

$$\mathbf{a}\cdot\mathbf{b} = \left(a^i\,\mathbf{g}_i\right)\cdot\left(b_j\,\mathbf{g}^j\right) = a^i b_j\left(\mathbf{g}_i\cdot\mathbf{g}^j\right) = a^i b_j\ \delta_i^j = a^i b_i$$

Similarly we find: $\mathbf{a}\cdot\mathbf{b} = a_i b^i$. Thus:

$$\mathbf{a}\cdot\mathbf{b} = a^i\ b_i = a_i\ b^i \tag{12.3.10}$$

If we write:

$$a_i\ b^i = \sum_i\left[\frac{a_i}{\sqrt{g_{ii}}}\right]\left[b^i\ \sqrt{g_{ii}}\right] \tag{12.3.11}$$

we may interpret the scalar product $\mathbf{a}\cdot\mathbf{b}$ as the sum of the products of the normal projections of the vector $\mathbf{a}$ onto the base vectors $\mathbf{g}_i$ and the physical components of $\mathbf{b}$.

The covariant and contravariant components of the *vector product* $\mathbf{b}\times\mathbf{c}$ may be found by using the formulas (12.2.16) and (12.2.17). The result is:

$$\mathbf{d}=\mathbf{b}\times\mathbf{c} \Leftrightarrow d_i = \varepsilon_{ijk}\,b^j\,c^k \Leftrightarrow d^i = \varepsilon^{ijk}b_j\,c_k \tag{12.3.12}$$

The *scalar triple product*, or the *box product*, may be computed from:

$$[\mathbf{abc}] \equiv \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \varepsilon_{ijk}\,a^i\,b^j\,c^k = \varepsilon^{ijk}a_i\,b_j\,c_k \tag{12.3.13}$$

In some literature the definitions (12.2.15) and (12.2.18) are replaced by:

$$\varepsilon_{ijk} = \sqrt{g}\,e_{ijk},\ \varepsilon^{ijk} = \frac{1}{\sqrt{g}}\ e_{ijk} \tag{12.3.14}$$

When these expressions are used in the formulas (12.3.12) for the components of the vector product, and the component formulas are used to define the vector product, the vector product changes direction when transforming from a right-handed system to a left-handed system and the other way around. The vector product is now what is called an *axial vector*. We shall demonstrate this be transforming from a right-handed Cartesian system x to a *left-handed Cartesian system* y:

$$y_1 = -x_1,\ y_2 = x_2,\ y_3 = x_3 \Rightarrow$$
$$\mathbf{g}_1 = \mathbf{g}^1 = -\mathbf{e}_1,\ \mathbf{g}_2 = \mathbf{g}^2 = \mathbf{e}_2,\ \mathbf{g}_3 = \mathbf{g}^3 = \mathbf{e}_3$$

Then:

$$J_y^x = \det\left[\frac{\partial x_k}{\partial y_i}\right] \equiv \det\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -1,\ g = \left(J_y^x\right)^2 = 1 \ \Rightarrow\ \sqrt{g} = 1$$

A vector $\mathbf{a}$ with components $a(k)$ in the x-system will have these components in the y-system:

$$a^i = \frac{\partial y_i}{\partial x_k} a(k) \ \Rightarrow\ a^1 = -a(1),\ a^2 = a(2),\ a^3 = a(3)$$

According to different presentations of vector algebra in the literature, the vector product $\mathbf{a} \times \mathbf{b}$ may be represented as either of the following two vectors. Using $\varepsilon_{ijk} = J_y^x\, e_{ijk}$ from the definition (12.2.15), we get the first alterative:

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \varepsilon_{ijk} a^j b^k \mathbf{g}^i = J_y^x e_{ijk} a^j b^k \mathbf{g}^i = -\det\begin{pmatrix} -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -a(1) & a(2) & a(3) \\ -b(1) & b(2) & b(3) \end{pmatrix} = e_{ijk} a(j) b(k) \mathbf{e}_i \tag{12.3.15}$$

Using the definition $(12.3.14)_1$, we get the second alternative:

$$\mathbf{c}_{\text{alt}} = \mathbf{a} \times \mathbf{b} = \varepsilon_{ijk} a^j b^k \mathbf{g}^i = \sqrt{g}\, e_{ijk} a^j b^k \mathbf{g}^i = \det\begin{pmatrix} -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -a(1) & a(2) & a(3) \\ -b(1) & b(2) & b(3) \end{pmatrix}$$
$$= -\, e_{ijk} a(j) b(k) \mathbf{e}_i \tag{12.3.16}$$

The vector $\mathbf{c}_{\text{alt}}$ has the opposite direction of the vector $\mathbf{c}$.

In the present presentation the vector product is defined geometrically by the formula (2.2.10) as a coordinate invariant quantity. This implies that if we do not intend to limit our choices of coordinate systems to only right-handed systems, or only left-handed systems, we need to keep the definitions (12.2.15) and (12.2.18) for the ε-symbols. Then the vector $\mathbf{c}$ in (12.3.15) is the proper vector product of the vectors $\mathbf{a}$ and $\mathbf{b}$. By using the definitions (12.2.15) and (12.2.18) rather then the definitions (12.3.14), the vector product defined through the definition of its components in (12.3.15) is independent of any choice of coordinate system.

In Problem 12.2 the reader is asked to show that the scalar product and the vector product of the vectors $\mathbf{a}$ and $\mathbf{b}$ when expressed in physical components are respectively:

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i,k} \frac{a(i)\ b(k)\ g_{ik}}{\sqrt{g_{ii}\ g_{kk}}} \tag{12.3.17}$$

$$\mathbf{a} \times \mathbf{b} = \mathbf{c} \ \Leftrightarrow\ c(i) = \sum_{j,k,l} e_{ljk}\ a(j)\ b(k)\ g^{il} \sqrt{\frac{g_{ii}}{g_{jj} g_{kk}}}\, J_y^x \tag{12.3.18}$$

In orthogonal coordinates these formulas reduce to:

$$\mathbf{a} \cdot \mathbf{b} = a(i)\ b(i),\ \mathbf{a} \times \mathbf{b} = \mathbf{c} \Leftrightarrow c(i) = e_{ijk}\, a(j)\, b(k) \tag{12.3.19}$$

12.4 Tensor Fields

12.4.1 Tensor Components. Tensor Algebra

In Sect. 4.1 a *tensor of order n* is defined as a *multilinear scalar-valued function of n argument vectors*. Let **C** be tensor of 2. order. Then the scalar value of **C** for the argument vectors **a** and **b** is:

$$\alpha = \mathbf{C}[\mathbf{a}, \mathbf{b}] \tag{12.4.1}$$

In a Cartesian coordinate system x with base vectors $\mathbf{e}_i$ the tensor is represented by its components:

$$C_{ij} = \mathbf{C}[\mathbf{e}_i, \mathbf{e}_j] \tag{12.4.2}$$

In a general coordinate system y the tensor **C** is represented by either of four associated sets of components:

$$\begin{aligned} &C_{ij} = \mathbf{C}[\mathbf{g}_i, \mathbf{g}_j] \text{ covariant components} \\ &C^{ij} = \mathbf{C}\left[\mathbf{g}^i, \mathbf{g}^j\right] \text{ contravariant components} \\ &C_i^{\ j} = \mathbf{C}\left[\mathbf{g}_i, \mathbf{g}^j\right],\ C^i{}_j = \mathbf{C}\left[\mathbf{g}^i, \mathbf{g}_j\right] \text{ mixed components} \end{aligned} \tag{12.4.3}$$

Using the formulas (12.2.20) for the base vectors in the expressions (12.4.3), we find the relationships between four the sets of components of the tensor. For example:

$$C^i{}_j = g^{ik} C_{kj} = g_{jk} C^{ik},\ C^{ij} = g^{ik} g^{jl} C_{kl} \tag{12.4.4}$$

Such operations are called "*raising and lowering of indices*".

A tensor **C** of 2. order is symmetric/antisymmetric if $\mathbf{C}[\mathbf{a}, \mathbf{b}] = \mathbf{C}[\mathbf{b}, \mathbf{a}] / -\mathbf{C}[\mathbf{b}, \mathbf{a}]$. Symmetry implies that $C_{ij} = C_{ji}$, $C^{ij} = C^{ji}$, and $C^i{}_j = C_j{}^i$. For a symmetric tensor we introduce the notation:

$$C^i_j \equiv C^i{}_j \equiv C_j{}^i \tag{12.4.5}$$

The scalar value of the *unit tensor* **1** of 2. order for any two argument vectors **a** and **b** is equal to the scalar product of the vectors, confer the formulas (4.1.22):

$$\mathbf{1}[\mathbf{a}, \mathbf{b}] = \mathbf{a} \cdot \mathbf{b} \tag{12.4.6}$$

The components of the tensor in a Cartesian coordinate system x with base vectors $\mathbf{e}_i$ are δ_{ij}:

$$\mathbf{1}[\mathbf{a}, \mathbf{b}] = a_i b_j \delta_{ij} = a_i b_i \text{ in an } x-\text{system} \tag{12.4.7}$$

From the Definition (12.4.6) it follows that:

$$\mathbf{1}[\mathbf{g}_i,\mathbf{g}_j] = \mathbf{g}_i \cdot \mathbf{g}_j = g_{ij},\ \mathbf{1}\left[\mathbf{g}^i,\mathbf{g}^j\right] = \mathbf{g}^i \cdot \mathbf{g}^j = g^{ij}$$
$$\mathbf{1}\left[\mathbf{g}_i,\mathbf{g}^j\right] = \mathbf{g}_i \cdot \mathbf{g}^j = \mathbf{1}\left[\mathbf{g}^j,\mathbf{g}_i\right] = \mathbf{g}^j \cdot \mathbf{g}_i = \delta_i^j \tag{12.4.8}$$

The result shows that the fundamental parameters of a general coordinate system y represent the components of the unit tensor in that system. Equations (12.4.4) may be considered as component expressions for the identities $\mathbf{C} = \mathbf{1C}$ and $\mathbf{C} = \mathbf{11C}$. The scalar product (12.4.6) may alternatively be expressed by:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{1}[\mathbf{a},\mathbf{b}] = a^i b_j \mathbf{1}\left[\mathbf{g}_i,\mathbf{g}^j\right] = a^i b_j\, \delta_i^j = a^i b_i = a_i b^i \tag{12.4.9}$$

as in the formulas (12.3.10).

When the components of a 2. order tensor $\mathbf{C}$ in the y – system is known, we may find the components of the tensor in another $\bar{y}$ – system by the use of (12.2.7) and (12.2.14). For example:

$$\bar{C}_{ij} = \mathbf{C}[\bar{\mathbf{g}}_i,\bar{\mathbf{g}}_j] = \frac{\partial y_k}{\partial \bar{y}_i}\frac{\partial y_l}{\partial \bar{y}_j}\ \mathbf{C}[\mathbf{g}_k,\mathbf{g}_l] = \frac{\partial y_k}{\partial \bar{y}_i}\frac{\partial y_l}{\partial \bar{y}_j}\, C_{kl}$$

Using similar procedures we obtain the results:

$$\bar{C}_{ij} = \frac{\partial y_k}{\partial \bar{y}_i}\frac{\partial y_l}{\partial \bar{y}_j}\, C_{kl} \text{ covariant transformation}$$
$$\bar{C}^{ij} = \frac{\partial \bar{y}_i}{\partial y_k}\frac{\partial \bar{y}_j}{\partial y_l}\, C^{kl} \text{ contravariant transformation}$$
$$\bar{C}_i{}^j = \frac{\partial y_k}{\partial \bar{y}_i}\frac{\partial \bar{y}_j}{\partial y_l}\, C_k{}^l,\ \bar{C}^i{}_j = \frac{\partial \bar{y}_i}{\partial y_k}\frac{\partial y_l}{\partial \bar{y}_j}\, C^k{}_l \text{ mixed transformations} \tag{12.4.10}$$

A generalization of the results above to tensors of higher order is straight forward. A tensor of order n has in every y-system 2^n associated sets of components. The algebra of tensors using components in general coordinate systems is very similar to what apply in Cartesian coordinates. *Addition* and *subtraction* of tensor of the same order must be done with components of the same kind. For example, the sum of two tensors $\mathbf{A}$ and $\mathbf{B}$ of 2. order is a tensor $\mathbf{C}$ of 2. order, which may be represented by either of the following sets of components:

$$C_{ij} = A_{ij} + B_{ij},\ C^{ij} = A^{ij} + B^{ij},\ C^i{}_j = A^i{}_j + B^i{}_j,\ C_i{}^j = A_i{}^j + B_i{}^j \tag{12.4.11}$$

The *tensor product* of a tensor $\mathbf{A}$ of order 2 and a tensor $\mathbf{B}$ of order 3 is a tensor $\mathbf{C}$ of order $2+3=5$:

$$\mathbf{C} = \mathbf{A} \otimes \mathbf{B} \Leftrightarrow C_{ij}{}^k{}_{lp} = A_{ij} B^k_{lp} \text{ etc.} \tag{12.4.12}$$

Contraction of a tensor must be performed with respect to indices of opposite kind: one superindex and one subindex. For example, a vector $\mathbf{a}$ may be constructed from the tensor $\mathbf{C}$ of 3. order in the following manner:

$$a_i = C_{ik}{}^k,\ a^i = C^{ik}{}_k \tag{12.4.13}$$

The *scalar product*, the *dot product*, and *composition* of tensors are illustrated by the following examples:

$$\alpha = \mathbf{A} : \mathbf{B} = A^{ij} B_{ij}, \ \mathbf{a} = \mathbf{A}\mathbf{b} \equiv \mathbf{A} \cdot \mathbf{b} \ \Leftrightarrow \ a_i = A_{ik} b^k$$
$$\mathbf{d} = \mathbf{A} : \mathbf{C} \ \Leftrightarrow \ d_k = A_{ij} C^{ij}{}_k, \ \mathbf{D} = \mathbf{A}\mathbf{B} \ \Leftrightarrow \ D_{ij} = A_{ik} B^k{}_j \tag{12.4.14}$$

The component form of tensor equations in a Cartesian coordinate system x are readily translated into component equations in a general coordinate system y as shown by the following example. The component form of the tensor equation:

$$\mathbf{a} = \mathbf{A} \cdot \mathbf{b} + \mathbf{B} : \mathbf{C} \tag{12.4.15}$$

in the x-system is:

$$a_i = A_{ij}\, b_j + B_{ijk}\, C_{jk} \ (x - \text{system}) \tag{12.4.16}$$

One possible representation in the y-system is:

$$a_i = A_{ij}\, b^j + B_{ijk}\, C^{jk} \ (y - \text{system}) \tag{12.4.17}$$

The *trace*, the *norm*, and the *determinant* of a 2. order tensor $\mathbf{B}$ are by components in a y-system given by:

$$\operatorname{tr} \mathbf{B} = B_i{}^i = B^i{}_i \tag{12.4.18}$$

$$|\mathbf{B}| \equiv \operatorname{norm} \mathbf{B} = \sqrt{\operatorname{tr} \mathbf{B}\mathbf{B}^T} = \sqrt{\mathbf{B} : \mathbf{B}} = \sqrt{B_{ij}\, B^{ij}} \tag{12.4.19}$$

$$\det \mathbf{B} = \det\left(B_{ij}/g\right) = \det\left(B_i{}^j\right) = \det\left(B^i{}_j\right) \tag{12.4.20}$$

If $\det \mathbf{B} \neq 0$, the *inverse tensor* $\mathbf{B}^{-1}$ may be found from:

$$\mathbf{B}^{-1}\, \mathbf{B} = \mathbf{1} \ \Leftrightarrow \ B^{-1}{}_{ik}\, B^{kj} = \delta_i^j \tag{12.4.21}$$

The derivation of the formula (12.4.20) from the definition (4.3.16) of $\det \mathbf{B}$ is given as Problem 12.3.

It may be shown, Problem 12.4, that the symbols ε^{ijk} and ε_{ijk} are components of the *permutation tensor* $\mathbf{P}$, and, Problem 12.5, that the following identity holds:

$$\varepsilon^{ijk}\, \varepsilon_{rsk} = \delta_r^i \delta_s^j - \delta_s^i \delta_r^j \tag{12.4.22}$$

12.4.2 Symmetric Tensors of 2. Order

Let $\mathbf{S}$ be a symmetric 2. order tensor and $\mathbf{a}$ and $\mathbf{b}$ two orthogonal unit vectors. To find the principal values and the principal directions for the tensor we proceed as in Sect. 4.3.1. The vector:

$$\mathbf{s} = \mathbf{S} \cdot \mathbf{a} \ \Leftrightarrow \ s^i = S_k^i\, a^k \tag{12.4.23}$$

has the *normal component* σ in the direction $\mathbf{a}$:

$$\sigma = \mathbf{a} \cdot \mathbf{s} = \mathbf{a} \cdot \mathbf{S} \cdot \mathbf{a} = a_i\, S^i_k\, a^k \tag{12.4.24}$$

and the *shear component* τ with respect to the two orthogonal directions $\mathbf{a}$ and $\mathbf{b}$:

$$\tau = \mathbf{b} \cdot \mathbf{s} = \mathbf{b} \cdot \mathbf{S} \cdot \mathbf{a} = b_i\, S^i_k\, a^k \tag{12.4.25}$$

The principal values $\sigma_i = \sigma$ and the principal directions $\mathbf{a}_i = \mathbf{a}$ for the tensor $\mathbf{S}$ are determined from the condition:

$$\mathbf{s} = \mathbf{S} \cdot \mathbf{a} = \sigma\, \mathbf{a} \;\Leftrightarrow\; \left(\sigma\, \delta^i_k - S^i_k\right) a_i = 0 \tag{12.4.26}$$

The condition for a solution of these equations is that:

$$\det\left(\sigma\, \delta^i_k - S^i_k\right) a_i = 0 \;\Leftrightarrow\; \sigma^3 - I\sigma^2 + II\sigma - III = 0 \tag{12.4.27}$$

I, II, and III are the *principal invariants* of the tensor $\mathbf{S}$:

$$\begin{aligned} &I = \operatorname{tr}\mathbf{S},\; III = \det\mathbf{S} = \det\left(S^i_i\right) \\ &II = \frac{1}{2}\left[(\operatorname{tr}\mathbf{S})^2 - (\operatorname{norm}\mathbf{S})^2\right] = \frac{1}{2}\left[S^i_i\, S^k_k - S^i_k\, S^k_i\right] \end{aligned} \tag{12.4.28}$$

The principal values σ_i are determined from (12.4.27), after which the principal directions $\mathbf{a}_i$ may be computed from (12.4.26).

12.4.3 Tensors as Polyadics

In Sect. 4.2.2 tensors were expressed as polyadics, i.e. linear combinations of polyads. For instance, a tensor $\mathbf{B}$ of 2. order with Cartesian components B_{kl} may be expressed as:

$$\mathbf{B} = B_{kl}\, \mathbf{e}_k \otimes \mathbf{e}_l \tag{12.4.29}$$

Let the contravariant components of the tensor $\mathbf{C}$ in a general y-system be C^{ij}. Using (12.4.10) to compute the Cartesian components of the tensor and $(12.2.8)_2$ to express the base vectors $\mathbf{e}_k$ by the base vectors of the y-system, we obtain from (12.4.29):

$$\begin{aligned} \mathbf{B} = \frac{\partial x_k}{\partial y_i}\frac{\partial x_l}{\partial y_j} B^{ij}\left(\frac{\partial y_r}{\partial x_k}\mathbf{g}_r\right) \otimes \left(\frac{\partial y_s}{\partial x_l}\mathbf{g}_s\right) &= B^{ij}\frac{\partial y_r}{\partial y_i}\frac{\partial y_s}{\partial y_j}\mathbf{g}_r \otimes \mathbf{g}_s = B^{ij}\delta^r_i\; \delta^s_j \mathbf{g}_r \otimes \mathbf{g}_s \\ &= B^{ij}\mathbf{g}_i \otimes \mathbf{g}_j \;\Rightarrow \mathbf{B} = B^{ij}\; \mathbf{g}_i \otimes \mathbf{g}_j \end{aligned} \tag{12.4.30}$$

Similar expressions may be obtained by raising and lowering of indices:

$$\mathbf{B} = B^j_i\; \mathbf{g}^i \otimes \mathbf{g}_j = B_{ij}\; \mathbf{g}^i \otimes \mathbf{g}^j = B^i{}_j\; \mathbf{g}_i \otimes \mathbf{g}^j \tag{12.4.31}$$

The results obtained for 2. order tensors are readily extended to higher order tensors. The following operations are easily verified:

$$\mathbf{D}\cdot\mathbf{a}=\mathbf{B}\Leftrightarrow\left(D^i{}_{jk}\,\mathbf{g}_i\otimes\mathbf{g}^j\otimes\mathbf{g}^k\right)\cdot\left(a^l\mathbf{g}_l\right)=D^i{}_{jk}\,a^l\mathbf{g}_i\otimes\mathbf{g}^j\left(\mathbf{g}^k\cdot\mathbf{g}_l\right)$$
$$=D^i{}_{jk}\,a^l\mathbf{g}_i\otimes\mathbf{g}^j\delta^k_l=D^i{}_{jk}\,a^k\mathbf{g}_i\otimes\mathbf{g}^j=B^i{}_j\,\mathbf{g}_i\otimes\mathbf{g}^j \tag{12.4.32}$$

12.5 Differentiation of Tensors

12.5.1 Christoffel Symbols

The base vectors in a general curvilinear coordinate system are point functions: $\mathbf{g}_i(y)$ and $\mathbf{g}^i(y)$. We shall now find the changes of the base vectors along the coordinate lines. Using the comma notation when differentiating with respect to the coordinates y_i we write:

$$\frac{\partial\mathbf{g}_i}{\partial y_j}\equiv\mathbf{g}_{i},_{j}=\Gamma_{ijk}\mathbf{g}^k=\Gamma^k_{ij}\mathbf{g}_k \tag{12.5.1}$$

The components Γ_{ijk} and Γ^k_{ij} are called the *Christoffel symbols of 1. and 2. kind* respectively. These symbols have their names after Elwin Bruno Christoffel [1829–1900]. Using the formulas (12.2.8) and $(12.2.13)_2$ we obtain the results:

$$\mathbf{g}_{i},_{j}=\frac{\partial^2x_r}{\partial y_j\partial y_i}\mathbf{e}_r=\frac{\partial^2x_r}{\partial y_j\partial y_i}\frac{\partial x_r}{\partial y_k}\mathbf{g}^k\quad\text{or}=\frac{\partial^2x_r}{\partial y_j\partial y_i}\frac{\partial y_k}{\partial x_r}\mathbf{g}_k$$

By comparing the result with the definitions (12.5.1), we conclude that:

$$\Gamma_{ijk}=\frac{\partial^2x_r}{\partial y_i\partial y_j}\frac{\partial x_r}{\partial y_k}=\Gamma_{jik},\;\Gamma^k_{ij}=\frac{\partial^2x_r}{\partial y_i\partial y_j}\frac{\partial y_k}{\partial x_r}=\Gamma^k_{ji} \tag{12.5.2}$$

In Problem 12.6 the reader is asked to prove the following identities:

$$\Gamma^k_{ij}=g^{kl}\,\Gamma_{ijl},\;\Gamma_{ijk}=g_{kl}\Gamma^l_{ij} \tag{12.5.3}$$

$$g_{ij},_{k}=\Gamma_{ikj}+\Gamma_{jki},\;\Gamma_{ijk}=\frac{1}{2}\left(g_{ik},_{j}+g_{jk},_{i}-g_{ij},_{k}\right) \tag{12.5.4}$$

$$\mathbf{g}^k,_{j}=-\Gamma^k_{ij}\,\mathbf{g}^i \tag{12.5.5}$$

In Problem 12.7 the reader is asked to use the formulas (12.5.1, 3, 4) and (2.1.22) in order to show that:

$$\Gamma^k_{ik}=\frac{1}{2g}\,g,_{i}=\frac{1}{\sqrt{g}}\left(\sqrt{g}\right),_{i} \tag{12.5.6}$$

In orthogonal coordinate systems, it follows from formulas $(12.5.4)_2$ and $(12.5.3)_1$ that:

$$\Gamma_{ijk} = \Gamma_{ij}^{k} = 0 \ \text{ when } i \neq j \neq k \neq i \tag{12.5.7}$$

In *cylindrical coordinates* (R, θ, z) the non-zero Christoffel symbols are:

$$\Gamma_{122} = -\Gamma_{221} = R,\ \Gamma_{12}^{2} = \frac{1}{R},\ \Gamma_{22}^{1} = -R \tag{12.5.8}$$

In *spherical coordinates* (r, θ, ϕ) the non-zero Christoffel symbols are:

$$\begin{aligned}
&\Gamma_{122} = -\Gamma_{221} = -\Gamma_{22}^{1} = r,\ \Gamma_{133} = -\Gamma_{331} = -\Gamma_{33}^{1} = r\sin^2\theta \\
&\Gamma_{233} = -\Gamma_{322} = r^2 \sin\theta\cos\theta\ ,\ \Gamma_{12}^{2} = \Gamma_{13}^{3} = \frac{1}{r},\ \Gamma_{23}^{3} = \cot\theta \\
&\Gamma_{33}^{2} = -\sin\theta\cos\theta
\end{aligned} \tag{12.5.9}$$

The Christoffel symbols Γ_{ijk} and Γ_{ij}^{k} do not represent a tensor. This is seen from the transformation formula relating $\bar{\Gamma}_{ij}^{k}$ and Γ_{ij}^{k} in two general coordinate systems $\bar{y}$ and y:

$$\bar{\Gamma}_{ij}^{k} = \frac{\partial y_r}{\partial \bar{y}_i}\frac{\partial y_s}{\partial \bar{y}_j}\frac{\partial \bar{y}_k}{\partial y_t}\Gamma_{rs}^{t} + \frac{\partial^2 y_r}{\partial \bar{y}_i \partial \bar{y}_j}\frac{\partial \bar{y}_k}{\partial y_r} \tag{12.5.10}$$

This formula, to be derived as Problem 12.8, clearly shows that the Christoffel symbols of the second kind do not represent en tensor.

12.5.2 Absolute and Covariant Derivatives of Vector Components

A curve in space may described by the coordinates functions $y_i(p)$, where p is a parameter, in a general curvilinear coordinate system y. The same curve may be represented by the coordinate functions $x_i(p)$ in a Cartesian coordinate system x, or by the position vector $\mathbf{r}(p) = x_i(p)\mathbf{e}_i$ from the origin of the system x to a point on the curve identified by the parameter value p. The length of this curve, from a point represented by the parameter value $p = p_o$ to any point represented by parameter value p, is given by the *arc length formula*, see (5.2.1):

$$s(p) = \int_{p_o}^{p} \sqrt{\frac{d\mathbf{r}}{d\bar{p}} \cdot \frac{d\mathbf{r}}{d\bar{p}}}\, d\bar{p} = \int_{p_o}^{p} \sqrt{\frac{dx_i}{d\bar{p}}\frac{dx_i}{d\bar{p}}}\, d\bar{p} \tag{12.5.11}$$

To obtain the representation of the arc length formula in the y-system, we set:

$$\frac{d\mathbf{r}}{dp} = \frac{\partial \mathbf{r}}{\partial y_i}\frac{d\,(y_i)}{dp} = \mathbf{g}_i \frac{dy^i}{dp} \tag{12.5.12}$$

The differentials $dy^i \equiv d(y_i)$ are the contravariant components of the vector differential $d\mathbf{r}$:

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial y_i}\, dy^i = \mathbf{g}_i\, dy^i \tag{12.5.13}$$

The arc length formula (12.5.11) now takes the form:

$$s(p) = \int_{p_o}^{p} \sqrt{\frac{d\mathbf{r}}{d\bar{p}} \cdot \frac{d\mathbf{r}}{d\bar{p}}}\, d\bar{p} = \int_{p_o}^{p} \sqrt{g_{ij}\, \frac{dy^i}{d\bar{p}}\, \frac{dy^j}{d\bar{p}}}\, d\bar{p} \tag{12.5.14}$$

From this formula it follows that:

$$(ds)^2 = g_{ij}\, dy^i\, dy^j \text{ and } |d\mathbf{r}| = ds \tag{12.5.15}$$

The quadratic form $g_{ij}\, dy^i dy^j$ is called the *metric of the space* E_3.

The *tangent vector* **t** to the curve is defined as the unit vector:

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} \Leftrightarrow t^i = \frac{dy^i}{ds} \tag{12.5.16}$$

The tangent vector $d\mathbf{r}/d\mathrm{p}$ to the curve $y_i(p)$ may now be expressed by:

$$\frac{d\mathbf{r}}{dp} = \mathbf{t}\, \frac{ds}{dp} \Leftrightarrow \frac{dy^i}{dp} = t^i \frac{ds}{dp} \tag{12.5.17}$$

We shall now choose the arc length as the curve parameter: $p = s$. Let $\alpha(y)$ be a scalar field. The change of the field along the curve $y_i(p) \equiv y_i(s)$ may be expressed by the derivative $d\alpha/ds$:

$$\frac{d\alpha}{ds} = \alpha_{,i}\, \frac{dy^i}{ds} = \alpha_{,i}\, t^i \tag{12.5.18}$$

Because $d\alpha/ds$ is a scalar the formula shows that $\alpha,_i$ represent covariant components of a vector "grad α", the *gradient of the scalar field* α, first defined by the formulas (2.4.8, 2.4.9):

$$\text{grad}\, \alpha \equiv \frac{\partial \alpha}{\partial \mathbf{r}} \equiv \nabla \alpha = \mathbf{g}^i \frac{\partial \alpha}{\partial y_i} \equiv \mathbf{g}^i\, \alpha_{,i} \tag{12.5.19}$$

The *del-operator* is a coordinate invariant scalar operator defined in a general coordinate system y by:

$$\nabla(\ ) = \mathbf{g}^i \frac{\partial(\ )}{\partial y_i} \tag{12.5.20}$$

The definition is in accordance with the formulas (2.4.10), (4.4.13, and (4.4.32). To see that the operator is indeed coordinate invariant, we use the transformation (12.2.14) for reciprocal base vectors and obtain:

$$\nabla(\ ) = \mathbf{g}^i \frac{\partial(\ )}{\partial y_i} = \left[\frac{\partial y_i}{\partial \bar{y}_k}\, \bar{\mathbf{g}}^k\right] \frac{\partial(\ )}{\partial \bar{y}_j} \frac{\partial \bar{y}_j}{\partial y_i} = \bar{\mathbf{g}}^k \frac{\partial(\ )}{\partial \bar{y}_j} \delta_k^j = \bar{\mathbf{g}}^j \frac{\partial(\ )}{\partial \bar{y}_j} \tag{12.5.21}$$

It is convenient to write:

$$\text{grad}\,\alpha = \alpha|_i \mathbf{g}^i = \alpha|^i \mathbf{g}_i \tag{12.5.22}$$

where we have introduced the notations:

$$\alpha|_i \equiv \alpha,_i, \ \ \alpha|^i = g^{ik}\alpha,_k \tag{12.5.23}$$

$\alpha|^i$ are the contravariant components of the vector grad α. Equation (12.5.18) may now be presented as:

$$\frac{d\alpha}{ds} = \text{grad}\,\alpha \cdot \mathbf{t} \tag{12.5.24}$$

The derivative of vector field $\mathbf{a}(y)$ along the curve $y_i(s)$ is a new vector:

$$\frac{d\mathbf{a}}{ds} = \frac{d\left(a^i \mathbf{g}_i\right)}{ds} = \frac{da^i}{ds}\mathbf{g}_i + a^i \mathbf{g}_i,_k \frac{dy^k}{ds} = \frac{d\left(a^i \mathbf{g}_i\right)}{ds} = \frac{da_i}{ds}\,\mathbf{g}^i + a_i\,\mathbf{g}^i,_k \frac{dy^k}{ds}$$

Using (12.5.1) and (12.5.5), we may rewrite this to:

$$\frac{d\mathbf{a}}{ds} = \left[\frac{da^i}{ds} + a^j \Gamma^i_{jk}\, t^k\right]\mathbf{g}_i = \left[\frac{da_i}{ds} - a_j \Gamma^j_{ik}\, t^k\right]\mathbf{g}^i \tag{12.5.25}$$

We now define *absolute derivatives* of vector components as the contravariant and covariant components of the vector $d\mathbf{a}/ds$:

$$\frac{\delta a^i}{\delta s} = \frac{d\mathbf{a}}{ds} \cdot \mathbf{g}^i, \ \frac{\delta a_i}{\delta s} = \frac{d\mathbf{a}}{ds} \cdot \mathbf{g}_i \tag{12.5.26}$$

such that:

$$\frac{d\mathbf{a}}{ds} = \frac{\delta a^i}{\delta s}\,\mathbf{g}_i = \frac{\delta a_i}{\delta s}\,\mathbf{g}^i \tag{12.5.27}$$

From (12.5.25) it follows that:

$$\frac{\delta a^i}{\delta s} = \frac{da^i}{ds} + a^j \Gamma^i_{jk}\, t^k, \ \frac{\delta a_i}{\delta s} = \frac{da_i}{ds} - a_j \Gamma^j_{ik}\, t^k \tag{12.5.28}$$

In Cartesian coordinates the Christoffel symbols are zero and therefore:

$$\frac{\delta a^i}{\delta s} = \frac{\delta a_i}{\delta s} = \frac{da_i}{ds} \text{ in Cartesian coordinate systems} \tag{12.5.29}$$

If we write:

$$\frac{da^i}{ds} = a^i,_k \frac{dy^k}{ds} = a^i,_k\ t^k, \ \frac{da_i}{ds} = a_i,_k \frac{dy^k}{ds} = a_i,_k\ t^k \tag{12.5.30}$$

the expressions for the absolute derivatives of the vector components in (12.5.28) take the forms:

$$\frac{\delta a^i}{\delta s} = a^i|_k\, t^k, \ \frac{\delta a_i}{\delta s} = a_i|_k\, t^k \tag{12.5.31}$$

where:

$$a^i\big|_k = a^i{}_{,k} + a^j\Gamma^i_{jk}, \;\; a_i|_k = a_{i,k} - a_j\Gamma^j_{ik} \tag{12.5.32}$$

and are called *covariant derivatives* of the vector components a^i and a_i respectively. In Cartesian coordinate systems the covariant derivatives reduce to partial derivatives:

$$a^i\big|_k = a^i{}_{,k} = a_i|_k = a_{i,k} \text{ in Cartesian coordinate systems} \tag{12.5.33}$$

The covariant derivatives of vector components are components of the tensor we have called the *gradient to the vector field* **a**. Equations (12.5.31) are now seen to express the coordinate invariant relation between the vector $d\mathbf{a}/ds$, the 2. order tensor grad **a**, and the tangent vector **t**:

$$\frac{d\mathbf{a}}{ds} = \operatorname{grad}\mathbf{a}\cdot\mathbf{t} \tag{12.5.34}$$

We may express grad **a** as the polyadics:

$$\operatorname{grad}\mathbf{a} = a^i\big|_k\, \mathbf{g}_i \otimes \mathbf{g}^k = a_i|_k\, \mathbf{g}^i \otimes \mathbf{g}^k \tag{12.5.35}$$

When we use any parameter p to describe the curve $y_i(p)$ rather than the arc length $s(p)$, we obtain from (12.5.24) and (12.5.34):

$$\frac{d\alpha}{dp} = \operatorname{grad}\alpha\cdot\mathbf{t}\,\frac{ds}{dp} = \operatorname{grad}\alpha\cdot\frac{d\mathbf{r}}{dp} \Leftrightarrow \frac{d\alpha}{dp} = \alpha_{,i}\,\frac{dy^i}{dp} \tag{12.5.36}$$

$$\frac{d\mathbf{a}}{dp} = \operatorname{grad}\mathbf{a}\cdot\mathbf{t}\,\frac{ds}{dp} = \operatorname{grad}\mathbf{a}\cdot\frac{d\mathbf{r}}{dp} \Leftrightarrow \frac{\delta a^i}{\delta p} = a^i\big|_k\,\frac{dy^k}{dp} \tag{12.5.37}$$

Let the space curve be the y_i-coordinate line with $p = y_i$. Then we obtain from (12.5.36, 12.5.37):

$$\alpha_{,i} = \operatorname{grad}\alpha\cdot\mathbf{g}_i, \;\; \mathbf{a}_{,i} = \operatorname{grad}\mathbf{a}\cdot\mathbf{g}_i = a^k\Big|_i\,\mathbf{g}_k = a_k|_i\,\mathbf{g}^k \tag{12.5.38}$$

The first result also follows directly from (12.5.19), while the result for $\mathbf{a}_{,i}$ may also be obtained directly by differentiating $\mathbf{a} = a^k\mathbf{g}_k = a_k\mathbf{g}^k$ with respect to y_i.

A vector field $\mathbf{a}(y)$ is said to be *uniform* along a curve $y_i(p)$ if $\mathbf{a}(y)$ has the same direction and magnitude at all points on the curve. This implies that:

$$\frac{d\mathbf{a}}{dp} = \mathbf{0} \Leftrightarrow \frac{\delta a^i}{\delta p} = 0 \text{ along the space curve } \; y_i(p) \tag{12.5.39}$$

If the vector field $\mathbf{a}(y)$ is uniform everywhere in a region R in space, then:

$$\operatorname{grad}\mathbf{a} = \mathbf{0} \Leftrightarrow a^i\big|_k = 0 \text{ everywhere in the region } R \tag{12.5.40}$$

12.5.3 The Frenet-Serret Formulas of Space Curves

A space curve may be given by the coordinate functions $y_i(s)$, where s represents the arc length from some reference point on the curve, or by a position vector $\mathbf{r}(s)$. The *tangent vector* to the curve is defined as the unit vector $\mathbf{t}$ given by formula (12.5.16). The *principal normal* to the curve is defined by the unit vector:

$$\mathbf{n} = \frac{1}{\kappa}\frac{d\mathbf{t}}{ds} = \frac{1}{\kappa}\frac{d^2\mathbf{r}}{ds^2} \Leftrightarrow n^i = \frac{1}{\kappa}\frac{\delta t^i}{\delta s} \tag{12.5.41}$$

κ is the *curvature* to the space curve:

$$\kappa = \left|\frac{d\mathbf{t}}{ds}\right| = \sqrt{g_{ij}\frac{\delta t^i}{\delta s}\frac{\delta t^j}{\delta s}} \tag{12.5.42}$$

The *binormal* $\mathbf{b}$ to the curve is defined by the unit vector:

$$\mathbf{b} = \mathbf{t}\times\mathbf{n} \Leftrightarrow b_i = \varepsilon_{ijk}t^j n^k \tag{12.5.43}$$

The *torsion* τ of the curve is defined by the formula:

$$\frac{d\mathbf{b}}{ds} = -\tau\mathbf{n} \tag{12.5.44}$$

It follows that:

$$\begin{aligned}
&\text{a) } \mathbf{b}\cdot\mathbf{n} = 0 \Rightarrow \mathbf{b}\cdot\frac{d\mathbf{n}}{ds} = -\frac{d\mathbf{b}}{ds}\cdot\mathbf{n} = \tau\\
&\text{b) } \mathbf{n}\cdot\mathbf{n} = 0 \Rightarrow \frac{d\mathbf{n}}{ds}\cdot\mathbf{n} = 0 \Rightarrow \left[\frac{d\mathbf{n}}{ds} + \kappa\,\mathbf{t}\right]\cdot\mathbf{n} = 0\\
&\text{c) } \mathbf{n}\cdot\mathbf{t} = 0 \Rightarrow \frac{d\mathbf{n}}{ds}\cdot\mathbf{t} = -\mathbf{n}\cdot\frac{d\mathbf{t}}{ds} = -\kappa \Rightarrow \left[\frac{d\mathbf{n}}{ds} + \kappa\,\mathbf{t}\right]\cdot\mathbf{t} = 0
\end{aligned} \tag{12.5.45}$$

From the result (12.5.45a) we obtain the formula for the torsion of the space curve:

$$\tau = \mathbf{b}\cdot\frac{d\mathbf{n}}{ds} = (\mathbf{t}\times\mathbf{n})\cdot\frac{d\mathbf{n}}{ds} = \left[\mathbf{t}\mathbf{n}\frac{d\mathbf{n}}{ds}\right] = \varepsilon_{ijk}t^i n^j\frac{\delta n^k}{\delta s} \tag{12.5.46}$$

The results (12.5.45) prove that:

$$\frac{d\mathbf{n}}{ds} + \kappa\,\mathbf{t} = \tau\,\mathbf{b} \tag{12.5.47}$$

We summarize the above results in the three *Frenet-Serret formulas* for a space curve:

$$\begin{aligned}
\frac{d\mathbf{t}}{ds} &= \kappa\mathbf{n} \Leftrightarrow \frac{\delta t^i}{\delta s} = \kappa n^i \\
\frac{d\mathbf{n}}{ds} &= \tau\mathbf{b} - \kappa\,\mathbf{t} \Leftrightarrow \frac{\delta n^i}{\delta s} = \tau b^i - \kappa t^i \\
\frac{d\mathbf{b}}{ds} &= -\tau\mathbf{n} \Leftrightarrow \frac{\delta b^i}{\delta s} = -\tau n^i
\end{aligned} \tag{12.5.48}$$

These formulas are named after Jean Frederic Frenet [1816–1900] and Joseph Alfred Serret [1819–1885].

Example 12.1. Kinematics of a Particle Moving on a Space Curve
The path of a mass particle is given by the space curve $y_i(t)$, where t is the time. Let $s(t)$ be the path length along the curve, measured from some reference point on the path. The velocity of the particle is given by:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \dot{s}\,\mathbf{t} \Leftrightarrow v^i = \frac{dy^i}{dt} = \frac{dy^i}{ds}\frac{ds}{dt} = \dot{s}\,t^i \tag{12.5.49}$$

The acceleration of the particle is:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{s}\,\mathbf{t} + \kappa\dot{s}^2\,\mathbf{n} \tag{12.5.50}$$

The first term on the right-hand side is the *tangential acceleration* and the second term is the *normal acceleration*. The contravariant components of the acceleration vector are, according to (12.5.27) and $(12.5.31)_1$:

$$a^i = \frac{\delta v^i}{\delta t} = \frac{\delta v^i}{\delta s}\frac{ds}{dt} = v^i\big|_k t^k\dot{s} = v^i\big|_k v^k = \left[v^i{,}_k + v^j\Gamma^i_{jk}\right] v^k = \frac{dv^i}{dt} + v^j\Gamma^i_{jk}v^k \tag{12.5.51}$$

Using results from the formulas (12.2.33), (12.3.5), and (12.5.8), we obtain the following contravariant and physical components of the velocity vector and the physical components of the acceleration in cylindrical coordinates (R,θ,z):

$$\begin{aligned}
&\left(v^i\right) = \left(\dot{R},\ \dot{\theta},\ \dot{z}\right),\ v(i) = v^i\sqrt{g_{ii}} \Leftrightarrow (v(i)) = \left(\dot{R},\ R\dot{\theta},\ \dot{z}\right) \\
&a(i) = a^i\sqrt{g_{ii}} \Leftrightarrow (a(i)) = \left(\ddot{R} - R\,\dot{\theta}^2,\ R\ddot{\theta} + 2\dot{R}\dot{\theta},\ \dot{z}\right)
\end{aligned} \tag{12.5.52}$$

12.5.4 Divergence and Rotation of a Vector Field

The definition of the del-operator ∇ in a general coordinate system y has been introduced in formula (12.5.20) and may be used in the expressions for the *divergence of a vector field* $\mathbf{a}(y)$ and the *rotation of a vector field* $\mathbf{a}(y)$.

The *divergence* of a vector field $\mathbf{a}(y)$ is the scalar field:

$$\operatorname{div}\mathbf{a} = \nabla\cdot\mathbf{a} = \mathbf{g}^i\cdot\frac{\partial\mathbf{a}}{\partial y_i} = \mathbf{g}^i\cdot a^k\Big|_i\mathbf{g}_k = a^i\big|_i = a_i\big|^i \tag{12.5.53}$$

The *rotation* of a vector field $\mathbf{a}$ is the vector field:

$$\begin{aligned}\text{rot}\,\mathbf{a} \equiv \text{curl}\,\mathbf{a} \equiv \nabla \times \mathbf{a} = \mathbf{g}^j \times \frac{\partial \mathbf{a}}{\partial y_j} = \mathbf{g}^j \times a_k|_j\, \mathbf{g}^k \\ = \varepsilon^{ijk}\, a_k|_j\, \mathbf{g}_i = \varepsilon^{ijk} a_{k,j}\, \mathbf{g}_i = \frac{1}{\sqrt{g}} \det \begin{pmatrix} \mathbf{g}_1 & \mathbf{g}_2 & \mathbf{g}_3 \\ \frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} & \frac{\partial}{\partial y_3} \\ a_1 & a_2 & a_3 \end{pmatrix}\end{aligned} \tag{12.5.54}$$

In Problem 12.9 the reader is asked to prove the last two equalities in (12.5.54).

The second covariant derivatives of a scalar field α is written as:

$$\alpha|_i|_j = \alpha|_{ij} \tag{12.5.55}$$

and represent a symmetric tensor of 2. order: the *gradient of the gradient of the scalar field*. The symmetry implies that we may write:

$$\alpha|^i_{\;j} = \alpha|_j^{\;i} = \alpha|^i_j \tag{12.5.56}$$

For div grad $\alpha = \nabla^2\alpha$ we get, using the formulas (12.5.22) and (12.5.57):

$$\text{div grad}\,\alpha = \nabla^2\alpha \equiv \nabla \cdot \nabla\alpha = \left(\alpha|^i\right)\Big|_i = \alpha|^i_i \tag{12.5.57}$$

In Problem 12.10 the reader is asked to use formula (12.5.6) to derive the formulas:

$$\text{div}\,\mathbf{a} \equiv \nabla \cdot \mathbf{a} = \frac{1}{\sqrt{g}}\left(\sqrt{g}\, a^i\right)_{,i},\ \nabla^2\alpha = \frac{1}{\sqrt{g}}\left(\sqrt{g}\, g^{ij}\alpha_{,j}\right)_{,i} \tag{12.5.58}$$

These formulas may be utilized to develop expressions for $\nabla \cdot \mathbf{a}$ and $\nabla^2\alpha$ in a particular coordinate system.

12.5.5 Orthogonal Coordinates

In orthogonal coordinate system we introduce the unit tangent vectors $\mathbf{e}_i^y$ and the symbols h_i defined in (12.2.27). We also apply the symbol:

$$h = \sqrt{g} = h_1 h_2 h_3 \tag{12.5.59}$$

The del-operator (12.5.20) becomes:

$$\nabla(\ ) = \mathbf{g}^i \frac{\partial(\ )}{\partial y_i} = \sum_i \frac{1}{h_i} \mathbf{e}_i^y \frac{\partial(\ )}{\partial y_i} \tag{12.5.60}$$

From the general formulas (12.5.52–12.5.58) we obtain for orthogonal coordinates:

$$\text{grad}\,\alpha \equiv \nabla\alpha = \sum_i \frac{1}{h_i}\,\alpha_{,i}\,\mathbf{e}_i^y \tag{12.5.61}$$

$$\text{div}\,\mathbf{a} \equiv \nabla\cdot\mathbf{a} = \frac{1}{h}\sum_i \frac{\partial}{\partial y_i}\left(h\,\frac{a(i)}{h_i}\right) \tag{12.5.62}$$

$$\begin{aligned}\text{rot}\,\mathbf{a} \equiv \text{curl}\,\mathbf{a} \equiv \nabla\times\mathbf{a} &= \frac{1}{h}\sum_{i,j,k} e_{ijk}\frac{\partial}{\partial y_j}\left(h_k\,a(k)\right)h_i\,\mathbf{e}_i^y \\ &= \frac{1}{h}\det\begin{pmatrix} h_1\,\mathbf{e}_1^y & h_2\,\mathbf{e}_2^y & h_3\,\mathbf{e}_3^y \\ \frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} & \frac{\partial}{\partial y_3} \\ h_1\,a(1) & h_2\,a(2) & h_3\,a(3)\end{pmatrix}\end{aligned} \tag{12.5.63}$$

See the Problems 12.11 and 12.12.

For the Christoffel symbols of 2. kind we obtain from $(12.5.4)_2$ and (12.5.3):

$$\begin{aligned}&\Gamma_{ik}^i = \frac{1}{h_i}h_{i,k} \;\text{ no summation }\;,\; \Gamma_{ii}^k = -\frac{h_i}{h_i^2}h_{i,k}\; k\neq i \;\text{ no summation} \\ &\Gamma_{ij}^k = 0 \; i\neq j\neq k\neq i\end{aligned} \tag{12.5.64}$$

The covariant derivatives of the vector components a^i and a_i become:

$$a^i\big|_k = \frac{1}{h_i}\frac{\partial a(i)}{\partial y_k} - \frac{1}{h_i^2}\frac{\partial h_k}{\partial y_i}a(k) \quad i\neq k \tag{12.5.65}$$

$$a^i\big|_i = \frac{\partial}{\partial y_i}\left(\frac{a(i)}{h_i}\right) + \frac{1}{h_i}\sum_k \frac{1}{h_k}\frac{\partial h_i}{\partial y_k}a(k) \text{ no summation w.r.t. to } i \tag{12.5.66}$$

$$a_i|_k = h_i\frac{\partial a(i)}{\partial y_k} - \frac{\partial h_k}{\partial y_i}a(k)\; i\neq k \tag{12.5.67}$$

$$a_i|_i = h_i^2\,a^i|_i \text{ no summation w.r.t. to } i \tag{12.5.68}$$

Example 12.2. Covariant Derivatives in Cylindrical Coordinates

A vector $\mathbf{a}$ is expressed in physical, contravariant, and covariant components in cylindrical coordinates (R,θ,z) in (12.3.8) and we write:

$$\begin{aligned}&\mathbf{a} = a_R\,\mathbf{e}_R + a_\theta\,\mathbf{e}_\theta + a_z\,\mathbf{e}_z = a^i\,\mathbf{g}_i = a_i\,\mathbf{g}^i \;\Leftrightarrow \\ &\left[a^1,a^2,a^3\right] \equiv \left[a_R,\frac{1}{R}a_\theta,a_z\right],\;[a_1,a_2,a_3] \equiv [a_R,Ra_\theta,a_z]\end{aligned} \tag{12.5.69}$$

We shall derive expressions for the covariant derivatives of the contravariant vector components a^k and covariant vector components a_k in cylindrical coordinates (R,θ,z).

Using the formula $(12.5.32)_1$ and the expressions (12.5.8) for Γ_{jk}^i, we obtain the following matrix for the covariant derivatives of the contravariant vector components:

$$\left(a^i\big|_k\right)=\left(a^i{}_{,k}+a^j\Gamma^i_{jk}\right)=\begin{pmatrix}\frac{\partial a_R}{\partial R} & \frac{\partial a_R}{\partial\theta}-a_\theta & \frac{\partial a_R}{\partial z}\\ \frac{1}{R}\frac{\partial a_\theta}{\partial R} & \frac{1}{R}\frac{\partial a_\theta}{\partial\theta}+\frac{a_R}{R} & \frac{1}{R}\frac{\partial a_\theta}{\partial z}\\ \frac{\partial a_z}{\partial R} & \frac{\partial a_z}{\partial\theta} & \frac{\partial a_z}{\partial z}\end{pmatrix} \tag{12.5.70}$$

The matrix for the covariant derivatives of the covariant vector components may be obtained from the formula $(12.5.32)_2$:

$$\left(a_i\big|_k\right)=\left(a_{i,k}-a_j\Gamma^j_{ik}\right)=\begin{pmatrix}\frac{\partial a_R}{\partial R} & \frac{\partial a_R}{\partial\theta}-a_\theta & \frac{\partial a_R}{\partial z}\\ R\frac{\partial a_\theta}{\partial R} & R\frac{\partial a_\theta}{\partial\theta}+R\,a_R & R\frac{\partial a_\theta}{\partial z}\\ \frac{\partial a_z}{\partial R} & \frac{\partial a_z}{\partial\theta} & \frac{\partial a_z}{\partial z}\end{pmatrix} \tag{12.5.71}$$

This matrix may also be obtained from the matrix (12.5.70) through the relationship:

$$a_i\big|_k=g_{ij}\,a^j\big|_k \tag{12.5.72}$$

In Fig. 12.5.1 the components $a_1\big|_2$ and $a_2\big|_2$ are shown geometrically. For simplicity we have chosen $a_z=0$ and then constructed:

$$\mathbf{a}_{,2}=\frac{\partial\mathbf{a}}{\partial\theta}=a_1\big|_2\,\mathbf{g}^1+a_2\big|_2\,\mathbf{g}^2=\left[\frac{\partial a_R}{\partial\theta}-a_\theta\right]\mathbf{e}_R+\left[\frac{\partial a_\theta}{\partial\theta}+a_R\right]\mathbf{e}_\theta$$

12.5.6 Absolute and Covariant Derivatives of Tensor Components

The results derived above for scalar and vector fields, i.e. for tensors of order 0 and 1, will now be generalized to tensor fields of any order. We define:

> Let $\mathbf{N}$ be a tensor field of order n and $y_i(p)$ a space curve, p being a curve parameter. The derivative of a tensor field $\mathbf{N}$ of order n with respect to p is a new tensor field $d\mathbf{N}/dp$ of order n, which in a Cartesian coordinate system x is represented by the derivatives $dN_{i..j}/dp$ of the components $N_{i..j}$ of the tensor $\mathbf{N}$. In a general coordinate system y the tensor $d\mathbf{N}/dp$ is represented by the *absolute derivatives* $\delta N_{i..j}/\delta p$ of the tensor components $N_{i..j}$.

and

> The *gradient* of a tensor field $\mathbf{N}$ of order n is a new tensor field grad $\mathbf{N}$ of order $(n+1)$, which in a Cartesian coordinate system is represented by the partial derivatives of the tensor components $N_{i..j,k}$. In a general coordinate system y the new tensor is represented by the *covariant derivatives* $N_{i..j}\big|_k$ of the tensor components $N_{i..j}$.

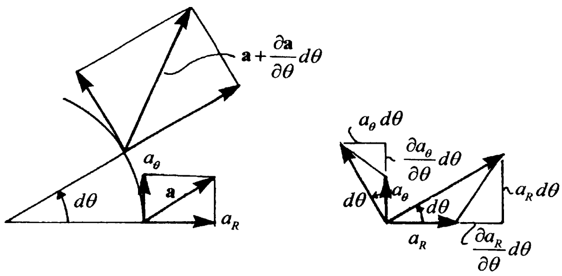


Fig. 12.5.1 Geometrical representation of the covariant derivatives $a_1|_2$ and $a_2|_2$

The components of the two new tensors: $d\mathbf{N}/dp$ of order n and grad $\mathbf{N}$ of order $(n+1)$, are well-defined in any Cartesien coordinate system. It will now be demonstrated how we may obtain the expressions for the components in any general coordinate system. First we note that the following tensor relation is valid in a Cartesian coordinate system, and due to the definitions above, is also valid in a general coordinate system y:

$$\frac{d\mathbf{N}}{dp} = \text{grad}\,\mathbf{N}\cdot\mathbf{t}\,\frac{ds}{dp} \Leftrightarrow \frac{\delta N_{i..j}}{\delta p} = N_{i..j}\big|_k t^k \frac{ds}{dp} \tag{12.5.73}$$

Confer (12.5.37) for tensors of first order. It will be demonstrated below how we may obtain the expressions for the covariant derivatives of the tensor components. Analogous to the formulas (12.5.38) for scalar and vector fields, we in general have for tensor of any order:

$$\mathbf{N}_{,i} = \text{grad}\,\mathbf{N}\cdot\mathbf{g}_i \tag{12.5.74}$$

From the definitions above it also follows that the rules of ordinary and partial differentiation also apply to absolute and covariant differentiation. For example, from the tensor equation:

$$\mathbf{A} = \mathbf{b}\otimes\mathbf{c} \Leftrightarrow A^i{}_j = b^i\,c_j \tag{12.5.75}$$

we may compute:

$$\frac{d\mathbf{A}}{dp} = \frac{d\mathbf{b}}{dp}\otimes\mathbf{c}+\mathbf{b}\otimes\frac{d\mathbf{c}}{dp} \Leftrightarrow \frac{\delta A^i{}_j}{\delta p} = \frac{\delta b^i}{\delta p}\,c_j + b^i\frac{\delta c_j}{\delta p} \tag{12.5.76}$$

$$\text{grad}\,\mathbf{A} = \text{grad}\;\mathbf{b}\otimes\mathbf{c}+\mathbf{b}\otimes\,\text{grad}\,\mathbf{c} \Leftrightarrow A^i{}_j\big|_k = b^i\big|_k\,c_j + b^i\,c_j\big|_k \tag{12.5.77}$$

We now seek the expressions for covariant derivatives of the component sets of a tensor field, which by definition represent the component sets of the gradient of the tensor field. Having found these expressions, the absolute derivatives of the tensor components may be found from (12.5.73). In order to simplify the presentation, we take as an example a tensor field $\mathbf{C}$ of 2. order, and we shall find the expressions:

$$C^i{}_j\big|_k = C^i_{j,k} + C^l{}_j\,\Gamma^i_{lk} - C^i{}_l\,\Gamma^l_{jk} \tag{12.5.78}$$

$$C_{ij}\big|_k = C_{ij,k} - C_{lj}\,\Gamma^l_{ik} - C_{il}\,\Gamma^l_{jk} \tag{12.5.79}$$

$$C^{ij}\big|_k = C^{ij}{}_{,k} + C^{lj}\,\Gamma^i_{lk} + C^{il}\,\Gamma^j_{lk} \tag{12.5.80}$$

$$C_i{}^j\big|_k = C_i{}^j{}_{,k} - C_l^j\,\Gamma^l_{ik} + C_i^l\,\Gamma^j_{lk} \tag{12.5.81}$$

We may use either one of three methods:

1. By application of the transformation rules for tensor components from the partial derivatives in an x-system to the covariant derivatives in a general y-system.
2. By partial and covariant differentiations of the component form of the scalar-valued function that the tensor represents.
3. By partial differentiation of the tensor presented as a polyadic.

The first method is straight forward but leads to some lengthy manipulations, and for that reason the method will not be demonstrated here. The second method may be applied to find the covariant derivative (12.5.78) as follows. Let **a** and **b** be two vector fields, such that according to the formulas (12.5.32):

$$a_{i,k} - a_i\big|_k = a_l\,\Gamma^l_{ik}\,,\; b^j{}_{,k} - b^j\big|_k = -\,b^l\,\Gamma^j_{lk} \tag{12.5.82}$$

The scalar value of the 2. order tensor **C** for the argument vectors **a** and **b** is:

$$\alpha = \mathbf{C}\,[\mathbf{a},\mathbf{b}] = C^i{}_j\,a_i\,b^j$$

Partial differentiation of this equation yields:

$$\alpha_{,k} = C^i{}_{j,k}\;a_i\,b^j + C^i{}_j\,a_{i,k}\;b^j + C^i{}_j\,a_i\,b^j{}_{,k} \tag{12.5.83}$$

Since partial differentiation coincides with covariant differentiation in a Cartesian coordinate system, (12.5.83) may also be considered to be the Cartesian form of the more general tensor equation:

$$\alpha\big|_k = C^i{}_j\big|_k\;a_i\,b^j + C^i{}_j\;a_i\big|_k\;b^j + C^i{}_j\,a_i\;b^j\big|_k \tag{12.5.84}$$

Now since $\alpha_{,k} \equiv \alpha\big|_k$, we obtain from (12.5.83, 12.5.84) that:

$$C^i{}_j\big|_k\;a_i\,b^j - \left[C^i{}_{j,k}\;a_i\,b^j + C^i{}_j\left(a_{i,k} - a_i\big|_k\right)b^j + C^i{}_j a_i\left(b^j{}_{,k} - b^j\big|_k\right)\right] = 0$$

Using (12.5.82) and renaming indices appropriately, we obtain the equation:

$$\left\{C^i{}_j\big|_k - \left[C^i{}_{j,k} + C^l{}_j\,\Gamma^i_{lk} - C^i{}_l\,\Gamma^l_{jk}\right]\right\}a_i\,b^j = 0$$

Because the vectors **a** and **b** may be chosen arbitrarily, it follows that the coefficients $\{\}$ must be zero. Thus:

$$C^i{}_j\big|_k = C^i{}_{j,k} + C^l{}_j\,\Gamma^i_{lk} - C^i{}_l\,\Gamma^l_{jk} \;\Rightarrow\quad (12.5.78)$$

The result indicates a general recipe on how to obtain covariant derivatives of tensor components as a sum of the partial derivative of the component and contracted products of the components and the Christoffel symbols, one product for each component index. The products are added with a positive/negative sign according to whether the contracted tensor index is a superscript or a subscript. This rule applies to tensors of any order.

The third method for finding the covariant derivatives of tensor components will now be illustrated. The tensor **C** and grad **C** are presented as:

$$\mathbf{C} = C^i{}_j \, \mathbf{g}_i \otimes \mathbf{g}^j \, , \; \text{grad}\, \mathbf{C} = C^i{}_j\big|_l \, \mathbf{g}_i \otimes \mathbf{g}^j \otimes \mathbf{g}^l \tag{12.5.85}$$

Substituting the expression for grad **C** into formula (12.5.41), we get:

$$\mathbf{C}_{,k} = \text{ grad}\, \mathbf{C} \cdot \mathbf{g}_k = C^i{}_j\big|_l \, \mathbf{g}_i \otimes \mathbf{g}^j \otimes \mathbf{g}^l \cdot \mathbf{g}_k = C^i{}_j\big|_k \mathbf{g}_i \otimes \mathbf{g}^j \tag{12.5.86}$$

The expression for $\mathbf{C}_{,k}$ is also obtained by differentiation of equation in $(12.5.79)_1$:

$$\mathbf{C}_{,k} = C^i{}_{j,k} \, \mathbf{g}_i \otimes \mathbf{g}^j + C^i{}_j \, \mathbf{g}_{i,k} \otimes \mathbf{g}^j + C^i{}_j \, \mathbf{g}_i \otimes \mathbf{g}^j{}_{,k}$$

By use of the formulas (12.5.1) and (12.5.5), this may be rewritten to:

$$\mathbf{C}_{,k} = \left[C^i{}_{j,k} + C^l{}_j \, \Gamma^i_{lk} - C^i{}_l \, \Gamma^l_{jk} \right] \mathbf{g}_i \otimes \mathbf{g}^j \tag{12.5.87}$$

By comparing the two expressions for $\mathbf{C}_{,k}$ in (12.5.86) and (12.5.87), we get the expression for the covariant derivatives presented in (12.5.78).

A tensor field **N** is said to be *uniform* along a space curve $y_i(p)$ if its components in a Cartesian coordinate system are constants at all points on the curve. This implies that:

$$\frac{d\mathbf{N}}{dp} = \mathbf{0} \;\Leftrightarrow\; \frac{\delta N^i{}_{\cdot\cdot j}}{\delta p} = 0 \text{ along the space curve } \; y_i(p) \tag{12.5.88}$$

If the tensor field **N(y)** is uniform everywhere in a region R in space, then:

$$\text{grad}\, \mathbf{N} = \mathbf{0} \;\Leftrightarrow\; N^i{}_{\cdot\cdot j}\big|_k = 0 \text{ everywhere in the region } R \tag{12.5.89}$$

The unit tensor **1** and the permutation tensor **P** are uniform tensor fields everywhere in E_3. Therefore:

$$\frac{\delta g_{ij}}{\delta p} = \frac{\delta g^{ij}}{\delta p} = 0 \, , \; g_{ij}\big|_k = g^{ij}\big|_k = 0 \tag{12.5.90}$$

$$\frac{\delta \varepsilon_{ijk}}{\delta p} = \frac{\delta \varepsilon^{ijk}}{\delta p} = 0, \; \varepsilon_{ijk}\big|_l = \varepsilon^{ijk}\Big|_l = 0 \tag{12.5.91}$$

Second covariant derivatives of the tensor components $C^i{}_j$ are presented by:

$$C^i{}_j\big|_k\big|_l \equiv C^i{}_j\big|_{kl} = C^i{}_j\big|_{lk} \tag{12.5.92}$$

The symmetry with respect to the last two indices follows from the fact that this symmetry is true in Cartesian coordinate systems. This type of symmetry is an inherent property of Euclidean spaces.

The *divergence* of a tensor $\mathbf{N}$ of order n is a new tensor div $\mathbf{N}$ of order $(n-1)$ and defined by the components $N_i..^k\big|_k$, confer equation (4.4.9):

$$\text{div}\,\mathbf{N} = N_i..^k\Big|_k \mathbf{g}^i \otimes .. \tag{12.5.93}$$

The *rotation* of a tensor $\mathbf{N}$ of order n is a new tensor rot $\mathbf{N}$ of order n and defined by the components $\varepsilon^{ijk} N_{r..k}\big|_j$, confer equation (4.4.11):

$$\text{rot}\,\mathbf{N} \equiv \text{curl}\,\mathbf{N} = \varepsilon^{ijk}\; N_{r}..{}_k\big|_j\, \mathbf{g}^i \otimes \mathbf{g}^r \otimes .. \tag{12.5.94}$$

The *Laplace-operator* ∇^2 may be used to express the divergence to the gradient of a tensor $\mathbf{N}$ of order n, confer equation (4.4.12):

$$\nabla^2\mathbf{N} \equiv \text{div grad}\,\mathbf{N} = N^i..{}_j\big|_k^k\, \mathbf{g}_i \otimes .. \otimes \mathbf{g}^j \tag{12.5.95}$$

In the Navier equations (13.6.4) in the linear theory of elasticity and in the Navier-Stokes equations (13.7.17) in Fluid Mechanics the divergence to the gradient of a vector and the gradient to the divergence of a vector appear. For a vector field $\mathbf{a}(y)$:

$$\text{div grad}\,\mathbf{a} \equiv \nabla^2\mathbf{a} = a^i\big|_k^k\; \mathbf{g}_i \;,\; \text{grad div}\,\mathbf{a} = a^k\Big|_k^i\; \mathbf{g}_i \tag{12.5.96}$$

We shall present formulas for these to vectors that are useful in applications. With the help of (12.5.22) and $(12.5.58)_1$ we obtain the result:

$$\text{grad div}\,\mathbf{a} = a^k\Big|_k^i\; \mathbf{g}_i = \frac{\partial}{\partial y_i}\left[\frac{1}{\sqrt{g}}\frac{\partial}{\partial y_k}\left(\sqrt{g}\,a^k\right)\right]\mathbf{g}^i \tag{12.5.97}$$

The following identity from Problem 2.9b:

$$\text{div grad}\,\mathbf{a} = \text{ grad div}\,\mathbf{a} - \text{rot rot}\,\mathbf{a} \;\Leftrightarrow\; \nabla^2\mathbf{a} = \nabla(\nabla\cdot\mathbf{a}) - \nabla\times(\nabla\times\mathbf{a}) \tag{12.5.98}$$

and the expression, see Problem 12.14:

$$\nabla\times(\nabla\times\mathbf{a}) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial y_k}\left[\sqrt{g}g^{kr}g^{is}\left(a_{r,s} - a_{s,r}\right)\right]\mathbf{g}_i \tag{12.5.99}$$

may be used to compute div grad $\mathbf{a}$.

12.6 Integration of Tensor Fields

The volume integral of any field $f(x) = f(x(y))$ over a region in space of volume V is shown by Theorem C.8 in Appendix C to be expressed by:

$$\int_V f\,dV = \int_V f(x)\,dx_1 dx_2 dx_3 = \int_V f(x(y))\,\left|J_y^x\right|\,dy^1\,dy^2\,dy^3 \tag{12.6.1}$$

The element of volume dV may be interpreted as the parallelepiped shown in Fig. 12.6.1 and is given by the box product of the three tangent vectors $d\mathbf{s}_i$ to the coordinate lines.

$$d\mathbf{s}_i = \frac{\partial \mathbf{r}}{\partial y_i} dy^i = \mathbf{g}_i\,dy^i = \frac{\partial x_k}{\partial y_i}\,dy^i\,\mathbf{e}_k \tag{12.6.2}$$

Hence:

$$dV = [d\mathbf{s}_1\,d\mathbf{s}_2\,d\mathbf{s}_3] = \left|J_y^x\right|\,dy^1 dy^2 dy^3 = \sqrt{g}\,dy^1 dy^2 dy^3 \tag{12.6.3}$$

Only the absolute value of the Jacobian J_y^x appears in equation (12.6.3) to make sure that the volume element dV is always positive, also when the y-system is left-handed. In Cartesian coordinate systems $g = 1$ and $dV = dx_1 dx_2 dx_3$.

Volume and surface integrals of invariants like scalar, vector, and tensor fields become scalars, vectors, and tensors respectively. In Cartesian coordinate systems the volume and surface integral of components of tensor fields become components of tensors. This follows from the fact that the transformation matrix Q is constant when transforming from one Cartesian coordinate system to another. In general coordinates this is in general not so because the transformation elements in (12.2.7, 12.2.8) are not constants. Obviously it is difficult to give a reasonable meaning to integration of general tensor components because these components are related to base vectors that may change from one space point to the next.

The general integration theorem in E_3 is the *Gauss theorem* presented as Theorem C. 3 in Appendix C. Let V be the volume of a region in space and A the area of the surface of that volume. For any field $f(x(y))$ the theorem states that:

$$\int_V f_{,i}\,dV = \int_A f\,n_i\,dA \quad \text{in Cartesian systems} \tag{12.6.4}$$

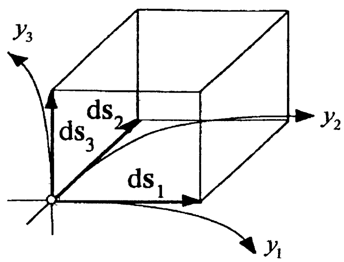


Fig. 12.6.1 Element of volume dV in a general coordinate system y

For a tensor field $\mathbf{N}$ of order n we then have with respect to a Cartesian coordinate system x:

$$\int_V N_{i\cdot\cdot j,k}\, dV = \int_A N_{i\cdot\cdot j}\; n_k\, dA \text{ in Cartesian systems} \tag{12.6.5}$$

This may be called the x-representation of the *gradient theorem*, which in coordinate invariant form is:

$$\int_V \operatorname{grad} \mathbf{N}\, dV = \int_A \mathbf{N} \otimes \mathbf{n}\, dA \tag{12.6.6}$$

From this theorem the *general divergence theorem* may be derived:

$$\int_V \operatorname{div} \mathbf{N}\, dV = \int_A \mathbf{N} \cdot \mathbf{n}\, dA \tag{12.6.7}$$

Special versions of the divergence theorem are:

$$\int_V a^i\big|_i dV = \int_A a^i\, n_i\, dA, \quad \int_V B^{ij}\big|_j\; \mathbf{g}_i\, dV = \int_A B^{ij}\, n_j\, \mathbf{g}_i\, dA \tag{12.6.8}$$

The *Stokes theorem*, Theorem C.5, has the following representation in general coordinates:

$$\int_A \mathbf{n} \cdot \operatorname{rot} \mathbf{a}\, dA \equiv \int_A \mathbf{n} \cdot \nabla \times \mathbf{a}\, dA = \oint_C \mathbf{a} \cdot d\mathbf{r} \Leftrightarrow \int_A n_i\, \varepsilon^{ijk}\, a_k\big|_j dA = \oint_C a_k\, dy^k \tag{12.6.9}$$

A is a surface bordered by the curve C and $\mathbf{n}$ is a unit vector normal to the surface. The positive direction of integration along C is determined such that the vector $\mathbf{n} \times d\mathbf{r}$ points to the side of C connected to A.

12.7 Two-Point Tensor Components

The content in this section will be applied in the deformation analysis in Sect. 13.3 and 13.4. With reference to Fig. 12.7.1 we introduce three coordinate systems: y and Y are in general curvilinear coordinate systems, while x is a Cartesian system. The coordinate sets Y_i and y_i may represent two different places in E_3. The place y has the position vector $\mathbf{r}$ and Cartesian coordinates x_i, while the place Y has the position vector $\mathbf{r}_o$ and Cartesian coordinates X_i. The base vectors in the Y-system are $\mathbf{g}_K$ and $\mathbf{g}^K$, and in the y-system $\mathbf{g}_i$ and $\mathbf{g}^i$. Upper case and lower case letter indices shall refer to the Y- and the y-systems respectively.

The base vectors $\mathbf{g}_K$ and $\mathbf{g}^K$ at the place Y are moved to the place y and decomposed there. We write:

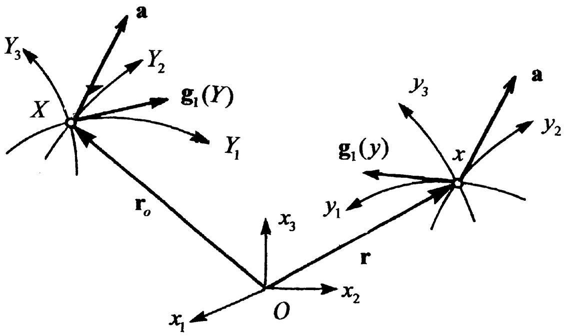


Fig. 12.7.1 Three coordinate systems: y, Y and x

$$\mathbf{g}_K = g_K^i\,\mathbf{g}_i = g_{Ki}\,\mathbf{g}^i\ ,\ \mathbf{g}^K = g^{Ki}\,\mathbf{g}_i = g_i^K\,\mathbf{g}^i \tag{12.7.1}$$

The components g_K^i, g_{Ki}, g^{Ki}, and g_i^{Ki} are called *Euclidean shifters* and represent *two-point components* of the unit tensor $\mathbf{1}$:

$$g_K^i = \mathbf{g}_K \cdot \mathbf{g}^i = \mathbf{1}\left[\mathbf{g}_K,\ \mathbf{g}^i\right] = \frac{\partial X_j}{\partial Y_K}\frac{\partial y_i}{\partial x_j} \tag{12.7.2}$$

$$g_{Ki} = g_{iK} = \mathbf{g}_K \cdot \mathbf{g}_i = \mathbf{1}\left[\mathbf{g}_K,\ \mathbf{g}_i\right] = \frac{\partial X_j}{\partial Y_K}\frac{\partial x_j}{\partial y_i} \tag{12.7.3}$$

$$g_i^K = \mathbf{g}^K \cdot \mathbf{g}_i = \mathbf{1}\left[\mathbf{g}^K,\ \mathbf{g}_i\right] = \frac{\partial Y_K}{\partial X_j}\frac{\partial x_j}{\partial y_i} \tag{12.7.4}$$

$$g^{Ki} = g^{iK} = \mathbf{g}^K \cdot \mathbf{g}^i = \mathbf{1}\left[\mathbf{g}^K,\ \mathbf{g}^i\right] = \frac{\partial Y_K}{\partial X_j}\frac{\partial y_i}{\partial x_j} \tag{12.7.5}$$

When the base vectors $\mathbf{g}_i$ and $\mathbf{g}^i$ at the point y are shifted to the point Y, we get:

$$\mathbf{g}_i = g_i^K\,\mathbf{g}_K = g_{iK}\,\mathbf{g}^K\ ,\ \mathbf{g}^i = g^{iK}\,\mathbf{g}_K = g_K^i\,\mathbf{g}^K \tag{12.7.6}$$

A vector $\mathbf{a}$ may be decomposed at the places Y or y with respect to either set of base vectors:

$$\mathbf{a} = a^K\,\mathbf{g}_K = a_K\,\mathbf{g}^K = a^i\,\mathbf{g}_i = a_i\,\mathbf{g}^i \tag{12.7.7}$$

From these expressions we get the following relations between the component sets:

$$\begin{aligned} a^K &= g_i^K\,a^i = g^{Ki}a_i,\ a_K = g_K^i\,a_i = g_{Ki}\,a^i \\ a^i &= g_K^i\,a^K = g^{iK}a_K,\ a_i = g_i^K\,a_K = g_{iK}\,a^K \end{aligned} \tag{12.7.8}$$

We conclude that the Euclidean shifters have the property of moving a vector from one place in space to another place in space. The Euclidean shifters are functions of both y- and Y-coordinates.

Let $\mathbf{C}$ be tensor field of 2. order and $\mathbf{a}$ and $\mathbf{b}$ two argument vectors fields. The scalar field $\mathbf{C}[\mathbf{a,b}]$ may now be calculated from alternative formulas, as for instance:

$$\alpha = \mathbf{C}\left[\mathbf{a},\mathbf{b}\right] = C_{Ki}\,a^K\,b^i = C^K{}_i\,a_K\,b^i = C_{iK}\,a^i\,b^K \text{ etc.} \tag{12.7.9}$$

The components:

$$C_{Ki} = \mathbf{C}[\mathbf{g}_K, \mathbf{g}_i], \; C^K{}_i = \mathbf{C}\left[\mathbf{g}^K, \mathbf{g}_i\right] \; \text{etc.} \tag{12.7.10}$$

are called *two-point components* of the tensor $\mathbf{C}$. The component sets are related through formulas of the type:

$$C_{Ki} = g^j_K \, C_{ji}, \; C^K{}_i = g^L_i \, C^K{}_L \tag{12.7.11}$$

Two-point components of a 2. order tensor $\mathbf{C}$ may be used to transform a vector $\mathbf{a}$ at one point to a new vector $\mathbf{b}$ at another point. For example, let $\mathbf{a}$ be a vector at the place $\mathbf{r}_o$. Then:

$$b^i = C^i{}_K \, a^K \tag{12.7.12}$$

are the components of a vector $\mathbf{b}$ at the place $\mathbf{r}$. For each distinct value of the index K the components $C^i{}_K$ and C_{iK} represent a vector at the place $\mathbf{r}$. Likewise, for each distinct value of the index i the components $C^i{}_K$ and C_{iK} represent a vector at the place $\mathbf{r}_o$. In the literature the two-point tensor components are therefore often presented as the components of a *two-point tensor* or a *double vector*. In the present exposition we shall not distinguish between tensors with argument vectors at one and the same place or at two different places.

A vector field $\mathbf{a} = \mathbf{a}(y, Y)$ is called a *two-point vector field*, or a *two-point tensor field of order 1*. The components of the vector in both the y- and the Y-system are in general functions of the six variables y_i and Y_i, with special cases when $\mathbf{a} = \mathbf{a}(y)$ or $\mathbf{a} = \mathbf{a}(Y)$.

$$\begin{aligned} \mathbf{a}(y,Y) &\Leftrightarrow a_i(y,Y) \text{ and } a_K(y,Y) \\ \mathbf{a}(y) &\Leftrightarrow a_i(y) \text{ and } a_K(y,Y) \\ \mathbf{a}(Y) &\Leftrightarrow a_i(y,Y) \text{ and } a_K(Y) \end{aligned} \tag{12.7.13}$$

Two-point tensor fields of higher order are defined similarly.

Let $\mathbf{C}(y, Y)$ be a two-point tensor field of 2. order. Since the components $C^i{}_K$ behave as vector components at the places y and Y, we may calculate covariant derivatives at these places from (12.5.32):

$$C^i{}_K\big|_j = C^i{}_{K,j} + C^k{}_K \, \Gamma^i_{kj}, \; C^i{}_K\big|_L = C^i{}_{K,L} - C^i{}_N \, \Gamma^N_{KL} \tag{12.7.14}$$

The Christoffel symbols Γ^i_{kj} and Γ^N_{KL} refer to the systems y and Y respectively. The expressions (12.7.14) are called *partial-covariant derivatives* of the two-point components of $\mathbf{C}$. If the coordinate systems y and Y are chosen to be identical to the Cartesian system x, the partial-covariant derivatives reduce to:

$$C^i_K\big|_j = \frac{\partial C^i_K}{\partial x_j}, \; C^i_K\big|_L = \frac{\partial C^i_K}{\partial X_L} \tag{12.7.15}$$

If a one-to-one mapping between the places y and Y is given:

$$y = y(Y) \Leftrightarrow Y = Y(y) \tag{12.7.16}$$

the *total-covariant derivatives* of tensor components may be defined:

$$A^i_K\big\|_j = A^i_K\big|_j + A^i_K\big|_L \frac{\partial Y_L}{\partial y_j},\ A^i_K\big\|_L = A^i_K\big|_L + A^i_K\big|_j \frac{\partial y_j}{\partial Y_L} \tag{12.7.17}$$

12.8 Relative Tensors

The tensor fields presented so far in this book will in the present section be called *absolute tensors*, *absolute vectors*, or *absolute scalar fields*. In the extended definition to *relative tensor fields* the tensors lose some of their characteristic coordinate invariance. Since relative tensors are used in the literature to some extend, they will be given a brief introduction here.

An *NP-scalar field* is defined as a quantity that in every coordinate systems y is represented by a magnitude α, such that:

$$\alpha = \left|J^x_y\right|^N \left(\text{ sign } J^x_y\right)^P \alpha_o \tag{12.8.1}$$

N is an integer, $P = 0$ or $= 1$, and α_o is a scalar field. If y is a right-handed Cartesian system x, it follows that $\alpha_o = \alpha$. For the case $P = 0$, the quantity is called a *relative scalar field of weight N*. A relative scalar field of weight $N = 1$ is called a *scalar density*. From (12.2.24) it follows that $g = \det(g_{ij})$ is a relative scalar of weight 2, and that $\sqrt{g}$ is a scalar density. For $N = 0$ and $P = 1$ the quantity α in (12.8.1) is called an *axial scalar*. Note that the magnitude α of an axial scalar changes sign by a transformation from a right-handed/left-handed coordinate system and to a left-handed/right-handed coordinate system. For $N = P = 0$ the *NP*-scalar is an *absolute scalar*. From (12.8.1) it finally follows that the magnitudes α and $\bar{\alpha}$ in two coordinate systems y and $\bar{y}$ respectively, are related through the formula:

$$\bar{\alpha} = \left|J^y_{\bar{y}}\right|^N \left(\text{ sign } J^y_{\bar{y}}\right)^P \alpha \tag{12.8.2}$$

An *NP-vector* $\mathbf{b}$ is defined as a linear *NP*-scalar-valued function of a vector $\mathbf{a}$:

$$\alpha = \mathbf{b}[\mathbf{a}] \tag{12.8.3}$$

In the coordinate system y the *NP*-vector $\mathbf{b}$ is represented by the component sets:

$$b_i = \mathbf{b}[\mathbf{g}_i],\ b^i = \mathbf{b}\left[\mathbf{g}^i\right] \tag{12.8.4}$$

It now follows from (12.8.2-4) that the components sets of $\mathbf{b}$ in two coordinate system y and $\bar{y}$ are related through the formulas:

$$\bar{b}_i = \left|J^y_{\bar{y}}\right|^N \left(\text{ sign } J^y_{\bar{y}}\right)^P \frac{\partial y_k}{\partial \bar{y}_i} b_k\ ,\ \bar{b}^i = \left|J^y_{\bar{y}}\right|^N \left(\text{sign } J^y_{\bar{y}}\right)^P \frac{\partial \bar{y}_i}{\partial y_k} b^k \tag{12.8.5}$$

For $P = 0$ the *NP*-vector is called a *relative vector of weight N*, and for $N = 0$ and $P = 1$ the *NP*-vector is called an *axial vector*. For $N = P = 0$ the *NP*-vector is an *absolute vector*. If the vector product of two absolute vectors **a** and **b** are defined by the components $\sqrt{g}e_{ijk}a^j b^k$ rather then by the components $\varepsilon_{ijk}a^j b^k$, the vector product becomes an axial vector. Confer the discussion in Sect. 12.3.

An *NP-tensor of order n* is defined as a multilinear *NP*-scalar–valued function of n absolute vectors. In the coordinate system y the *NP*-tensor is represented by component sets defined similarly to the components of absolute tensors. For $P = 0$ the *NP*-tensor is called a *relative tensor of weight N*, and for $N = 0$ and $P = 1$ the *NP*-tensor is called an *axial tensor*. For $N = P = 0$ the *NP*-tensor is an *absolute tensor*. The components $\sqrt{g}e_{ijk}$ define an axial tensor of 3. order. Confer the discussion in Sect. 12.3.

The algebra of *NP*-tensors follows the rules applying for absolute tensors. Addition has only meaning for tensors of equal weight N and of the same value of P. By tensor multiplication the weights of the tensors are added. The P-value of the product of two tensors is = 0 if the P-values of the factors are equal, 0 or 1, otherwise = 1.

Problems

Problem 12.1. Derive (12.2.23, 12.2.24, 12.2.25).

Problem 12.2. Develop the expressions (12.3.17) and (12.3.18) for the scalar product and the vector product of two vectors. Show also the reduction from (12.3.17, 12.3.18) to the formulas (12.3.19).

Problem 12.3. Derive formula (12.4.20) for the determinant of a tensor of 2. order from the definition (4.3.16).

Problem 12.4. Show that the symbols ε_{ijk} and ε^{ijk} defined by the formulas (12.2.15) and (12.2.18) respectively are components of the permutation tensor **P**.

Problem 12.5. Prove the identity (12.4.22).

Problem 12.6. Prove the formulas (12.5.3, 12.5.4, 12.5.5).

Problem 12.7. Use the formulas (12.2.25), (12.5.3,4), and (2.1.22) to prove formula (12.5.6).

Problem 12.8. Derive the transformation rule (12.5.10) for the Christoffel symbols of 2. kind.

Problem 12.9. Prove the last two equalities in (12.5.54).

Problem 12.10. Derive the formulas (12.5.58) by using (12.5.6).

Problem 12.11. Use the formulas (12.5.61) and (12.5.62) to develop the expressions for grad α, div **a**, and $\nabla^2\alpha$ in cylindrical coordinates. Appendix A provides the answers.

Problem 12.12. Use the formulas (12.5.61) and (12.5.62) to develop the expressions for grad α, div **a**, and $\nabla^2\alpha$ in spherical coordinates. Appendix A provides the answers.

Problem 12.13. In Section 4.4.1 the gradient of a vector **a** is defined through its Cartesian components $\partial a_i/\partial x_j$. Use the transformation formulas (12.4.10) for tensor components to show that the (12.5.32) represent the components of grad **a** in a general coordinate system y.

Problem 12.14. Derive formula (12.5.99).

Chapter 13
Continuum Mechanics in Curvilinear Coordinates

13.1 Introduction

In this chapter the basic equations of continuum mechanics are presented in general curvilinear coordinates. In Sect. 13.3 on deformation analysis the use of material coordinates are applied. Convective time derivatives of tensors were briefly introduced in Sect. 11.4, but is now properly defined and presented in Sect. 13.4. General equations of motion are given in Sect. 13.5. The basic equations in Elasticity are the subject of Sect. 13.6, and the equations of Fluid Mechanics are presented in Sect. 13.7.

13.2 Kinematics

In order to describe the motion and deformation of a continuum a reference configuration K_o of the material body is introduced. This configuration will usually be a real configuration of the body, representing the body at a reference time t_o, but not necessarily. In K_o a general coordinate system Y is introduced. The coordinates Y_K are used to identify particles of the body, and we use the expression *particle* Y. To identify places in space we introduce another general coordinate system y. In the *present configuration* K of the body, at the present time t, the particle Y is at the *place* y, where y_i are the coordinates of the particle in the coordinate system y. In special cases it is convenient to consider the Y-system as a *material coordinate system* that moves and deforms with the body. Y_K are then *material coordinates*.

A third coordinate system is the *Cartesian coordinate system* x with origin O. The particle Y has the Cartesian coordinates X_i, while the place y has the Cartesian coordinates x_i. All three coordinate systems Y, at the reference time t_o, y, and x are fixed in the same reference *Rf*, such that one-to one correspondence between the coordinate sets Y and X, and between y and x exist:

$$X_i = X_i(Y) \;\Leftrightarrow\; Y_K = Y_K(X),\; x_i = x_i(y) \;\Leftrightarrow\; y_i = y_i(x) \tag{13.2.1}$$

Note that lower case Latin (or Greek) letters are used as indices in the x- and the y-systems, while upper case Latin (or Greek) letters are used as indices in the Y-system. We may identify one and the same particle by either Y or X, and one and the same place by either y or x.

The positions of a particle Y in the configurations K_o and K are given by the place vectors $\mathbf{r}_o$ and $\mathbf{r}$ respectively, as shown in Fig. 13.2.1. The *motion* of the body at the present time t may be given by the place vector $\mathbf{r}$(Y,t), the coordinate functions $y_i(Y,t)$, or by the displacement vector:

$$\mathbf{u}(Y,t) = \mathbf{r}(Y,t) - \mathbf{r}_o(Y) \tag{13.2.2}$$

The velocity $\mathbf{v}$ and the acceleration $\mathbf{a}$ of a particle are:

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial \mathbf{r}(Y,t)}{\partial t}, \; v^i = \frac{\partial y_i}{\partial t}, \; v^K = v^i g_i^K \tag{13.2.3}$$

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{\partial \mathbf{v}(Y,t)}{\partial t} = \ddot{\mathbf{r}} \tag{13.2.4}$$

The symbols g_i^K are *Euclidean shifters* as defined by the formulas (12.7.4).

In the *reference description* we use the *Lagrangian coordinates* (Y_K,t) as independent variables. The *material derivative* of a tensor $\mathbf{N}(Y,t)$ of order n is a new tensor of the same order and defined by:

$$\dot{\mathbf{N}} = \frac{\partial \mathbf{N}(Y,t)}{\partial t} \Leftrightarrow \dot{N}_{K..}{}^{L} \equiv \dot{\mathbf{N}}\left[\mathbf{g}_K,..,\mathbf{g}^L\right] = \frac{\partial}{\partial t}\left(N_{K..}{}^{L}\right) \tag{13.2.5}$$

In the *spacial description* we use the *Eulerian coordinates* (y_i,t) as independent variables. The *material derivative* of a tensor $\mathbf{N}(y,t)$ of order n is a new tensor of the same order and defined by:

$$\dot{\mathbf{N}} = \frac{\partial}{\partial t}\mathbf{N}(y(Y,t),t) = \partial_t \mathbf{N}(y,t) + \operatorname{grad} \mathbf{N}(y,t) \cdot \mathbf{v}(y,t) \; \Leftrightarrow$$

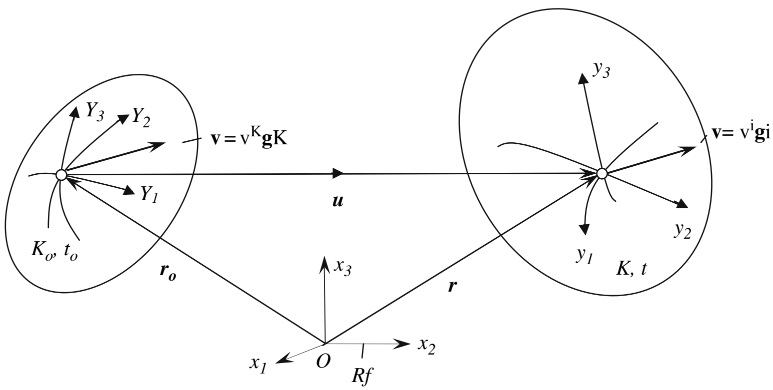


Fig. 13.2.1 General coordinate systems Y and y, and Cartesian coordinate system x

$$\dot{N}_i{}^{..j} \equiv \dot{\mathbf{N}}\left[\mathbf{g}_i, .., \mathbf{g}^j\right] = \partial_t N_i{}^{..j} + N_i{}^{..j}\big|_k v^k \tag{13.2.6}$$

Note that $\dot{N}_i{}^{..j}$ are components of the tensor $\dot{\mathbf{N}}$ and not the material derivatives of the tensor components $N_i{}^{..j}$. The latter are given by:

$$\frac{\partial}{\partial t}\left[N_i{}^{..j}(y(Y,t),t)\right] = \frac{\partial}{\partial t}\left[N_i{}^{..j}(y,t)\right] + \frac{\partial}{\partial y_k}\left[N_i{}^{..j}(y,t\right] v^k \tag{13.2.7}$$

and do not represent a tensor in general coordinates Y, only when the coordinate system Y is Cartesian.

Material differentiation follows the standard differentiation rules. For example, the material derivative of the vector $\mathbf{a} = \mathbf{Bc}$ has the components:

$$\dot{a}_i = \dot{B}_i^k c_k + B_i^k \dot{c}_k \tag{13.2.8}$$

This result is easy to prove by use of (13.2.6).

Because the unit tensor $\mathbf{1}$ and the permutation tensor $\mathbf{P}$ are constant tensors it follows that:

$$\dot{g}_{ij} = \dot{g}^{ij} = 0, \; \dot{\varepsilon}_{ijk} = \dot{\varepsilon}^{ijk} = 0 \tag{13.2.9}$$

The definition (13.2.6) may be used to find the following expression for the particle acceleration $\mathbf{a}(y,t)$:

$$\mathbf{a} = \dot{\mathbf{v}} = \partial_t \mathbf{v} + \operatorname{grad}\mathbf{v} \cdot \mathbf{v} = \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} \;\Leftrightarrow\; a^i = \partial_t v^i + v^i\big|_k v^k \tag{13.2.10}$$

If $\mathbf{N}$ is a steady tensor field: $\mathbf{N} = \mathbf{N}(y)$, the material derivative may be expressed by the absolute derivative with respect to time:

$$\dot{\mathbf{N}} = \frac{d\mathbf{N}}{dt} \;\Leftrightarrow\; \dot{N}_i{}^{..j} = \frac{\delta N_i{}^{..j}}{\delta t} \tag{13.2.11}$$

13.3 Deformation Analysis

13.3.1 Strain Measures

The section is based on the presentation in Chap. 5 Deformation Analysis and is subdivided into five parts: 1. Small strains and small deformations, 2. Deformation kinematics, 3. General deformation analysis, 4. Deformation analysis in convective coordinates, and 5. Deformation of material surfaces.

It has been shown in Sect. 5.2 that the deformation in the neighborhood of a particle Y, or X, is determined by the *deformation gradient* $\mathbf{F}$ defined by:

$$d\mathbf{r} = \mathbf{F} \cdot d\mathbf{r}_o, \; \mathbf{F} = \operatorname{Grad} \mathbf{r} = \frac{\partial \mathbf{r}}{\partial \mathbf{r}_o} \tag{13.3.1}$$

$d\mathbf{r}_o$ and $d\mathbf{r}$ represent a material line element in the reference configuration K_o and in the present configuration K. In the x-system the relations (13.3.1) are represented by:

$$dx_i = F_{ik}\, dX_k, \; F_{ik} = \frac{\partial x_i}{\partial X_k} \tag{13.3.2}$$

In a general coordinate system y the motion is given by $y(Y,t)$ and the relations (13.3.1) have the component representations:

$$dy^i = F^i_K\, dY^K, \; F^i_K = \frac{\partial y_i}{\partial Y_K} \tag{13.3.3}$$

F^i_K are *two-point-components of the deformation gradient* $\mathbf{F}$.

The lengths ds_o and ds of the material line element in K_o and K are found from:

$$(ds_o)^2 = d\mathbf{r}_o \cdot d\mathbf{r}_o = g_{KL}\, dY^K\, dY^L, \; (ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = g_{ij}\, dy^i\, dy^j \tag{13.3.4}$$

Using (13.3.1) and (13.3.3) we may write:

$$\begin{aligned} (ds)^2 &= d\mathbf{r} \cdot d\mathbf{r} = (\mathbf{F} \cdot d\mathbf{r}_o) \cdot (\mathbf{F} \cdot d\mathbf{r}_o) \;\Rightarrow \\ (ds)^2 &= d\mathbf{r}_o \cdot \mathbf{C} \cdot d\mathbf{r}_o = C_{KL}\, dY^K\, dY^L \end{aligned} \tag{13.3.5}$$

The *Green deformation tensor* $\mathbf{C}$ is defined by:

$$\mathbf{C} = \mathbf{F}^T\, \mathbf{F} \;\Leftrightarrow\; C_{KL} = F^i{}_K\, F_{iL} \tag{13.3.6}$$

If we imagine that the Y-system is imbedded in the continuum and moves and deforms with the body, it follows from (13.3.5) that the components C_{KL} also represent the fundamental quantitites of this convective Y-system. This interpretation of C_{KL} will be utilized below.

The *Green strain tensor* $\mathbf{E}$ is defined by:

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}) \;\Leftrightarrow\; E_{KL} = \frac{1}{2}(C_{KL} - g_{KL}) \tag{13.3.7}$$

The strain measures at a particle Y: the *longitudinal strain* ε in the direction $\mathbf{e}$, the *volumetric strain* ε_v, and the *shear strain* γ with respect to two orthogonal directions $\mathbf{e}$ and $\bar{\mathbf{e}}$, were defined in Sect. 5.1. The expressions for these in general coordinates are:

$$\begin{aligned} \varepsilon &= \frac{ds - ds_o}{ds_o} = \sqrt{\mathbf{e} \cdot \mathbf{C} \cdot \mathbf{e}} - 1 = \sqrt{e^K\, C_{KL}\, e^L} - 1 \\ &= \sqrt{1 + 2\mathbf{e} \cdot \mathbf{E} \cdot \mathbf{e}} - 1 = \sqrt{1 + 2e^K\, E_{KL}\, e^L} - 1 \end{aligned} \tag{13.3.8}$$

$$\begin{aligned} \varepsilon_v &= \frac{dV - dV_o}{dV_o} = \det\mathbf{F} - 1 = \sqrt{\det\mathbf{C}} - 1 = \sqrt{\det(\mathbf{1} + 2\mathbf{E})} - 1 \\ &= \sqrt{\det\left(C^L_K\right)} - 1 = \sqrt{\det\left(\delta^L_K + 2\mathbf{E}^L_K\right)} - 1 \end{aligned} \tag{13.3.9}$$

$$\sin\gamma = \frac{\bar{\mathbf{e}}\cdot\mathbf{C}\cdot\mathbf{e}}{\sqrt{(\bar{\mathbf{e}}\cdot\mathbf{C}\cdot\bar{\mathbf{e}})\,(\mathbf{e}\cdot\mathbf{C}\cdot\mathbf{e})}} = \frac{\bar{e}^K C_{KL}\, e^L}{\sqrt{(\bar{e}^M C_{MN}\,\bar{e}^N)\,(e^P C_{PQ}\, e^Q)}}$$

$$= \frac{2\bar{\mathbf{e}}\cdot\mathbf{E}\cdot\mathbf{e}}{\sqrt{(1+2\bar{\mathbf{e}}\cdot\mathbf{E}\cdot\bar{\mathbf{e}})\,(1+2\mathbf{e}\cdot\mathbf{E}\cdot\mathbf{e})}} = \frac{2\,\bar{e}^K E_{KL}\, e^L}{\sqrt{(1+\bar{\varepsilon})\,(1+\varepsilon)}} \qquad (13.3.10)$$

$\bar{\varepsilon}$ is the longitudinal strain in the direction $\bar{\mathbf{e}}$.

When the motion is expressed by the displacement vector: $\mathbf{u}(\mathbf{r}_o,t) = \mathbf{r}(\mathbf{r}_o,t) - \mathbf{r}_o$, the deformation will be defined in terms of the *displacement gradient* $\mathbf{H}$:

$$\mathbf{H} = \frac{\partial\mathbf{u}}{\partial\mathbf{r}_o} = \mathbf{F} - \mathbf{1},\; H_{KL} \equiv u_K|_L = F_{KL} - g_{KL} \qquad (13.3.11)$$

It now follows that:

$$\mathbf{C} = \mathbf{1} + \mathbf{H} + \mathbf{H}^T + \mathbf{H}^T\mathbf{H} \;\Leftrightarrow\; C_{KL} = g_{KL} + u_K|_L + u_L|_K + u^N\big|_K\, u_N|_L \qquad (13.3.12)$$

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T\mathbf{H}\right) \;\Leftrightarrow\; E_{KL} = \frac{1}{2}\left(u_K|_L + u_L|_K + u^N\big|_K\, u_N|_L\right) \qquad (13.3.13)$$

13.3.2 Small Strains and Small Deformations

The special but very important case of small strains has been discussed in Sect. 5.3. Small strains may be defined by the inequality: norm $\mathbf{E} << 1$. The expressions (13.3.8–13.3.10) for the primary strain measures are reduced to:

$$\varepsilon = \mathbf{e}\cdot\mathbf{E}\cdot\mathbf{e} = e^K E_{KL}\, e^L,\; \gamma = 2\bar{\mathbf{e}}\cdot\mathbf{C}\cdot\mathbf{e} = 2\,\bar{e}^K E_{KL}\, e^L,\; \varepsilon_v = \operatorname{tr}\mathbf{E} = E_K^K \qquad (13.3.14)$$

Small deformations imply small strains and small rotations. The condition of small deformations is defined by:

$$\text{norm}\,\mathbf{H} = \sqrt{\operatorname{tr}\left(\mathbf{H}\mathbf{H}^T\right)} = \sqrt{u^K|_L\, u^L|_K} \ll 1 \qquad (13.3.15)$$

Note that the components $u^K|_L$ are dimensionless. The expressions (13.3.12, 13.3.13) for the Green deformation tensor and the Green strain tensor may be approximated by:

$$\mathbf{C} = \mathbf{1} + \mathbf{H} + \mathbf{H}^T \;\Leftrightarrow\; C_{KL} = g_{KL} + u_K|_L + u_L|_K \qquad (13.3.16)$$

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right) \;\Leftrightarrow\; E_{KL} = \frac{1}{2}\left(u_K|_L + u_L|_K\right) \qquad (13.3.17)$$

If we further can assume that the displacements are small, such that the K_o and K are configurations close to one another, the place coordinates y_i may be used as particle coordinates. The expressions (13.3.17) are now written as:

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H}+\mathbf{H}^T\right) \Leftrightarrow E_{ij} = \frac{1}{2}\left(u_i\big|_j + u_j\big|_i\right) \tag{13.3.18}$$

The three characteristic strain measures at a particle Y: the *longitudinal strain* ε in the direction $\mathbf{e}$, the *shear strain* γ with respect to two orthogonal directions $\mathbf{e}$ and $\bar{\mathbf{e}}$, and the *volumetric strain* ε_v are now expressed by:

$$\varepsilon = \mathbf{e}\cdot\mathbf{E}\cdot\mathbf{e} = e^i\, u_i\big|_j e^j,\ \gamma = 2\bar{\mathbf{e}}\cdot\mathbf{E}\cdot\mathbf{e} = \bar{e}^i\left(u_i\big|_j + u_j\big|_i\right)e^j,\ \varepsilon_v = \mathrm{tr}\,\mathbf{E} = u^i\big|_i \tag{13.3.19}$$

The *rotation tensor for small deformations* $\tilde{\mathbf{R}}$ is defined by:

$$\tilde{\mathbf{R}} = \frac{1}{2}\left(\mathbf{H}-\mathbf{H}^T\right) \Leftrightarrow \tilde{R}_{ij} = \frac{1}{2}\left(u_i\big|_j - u_j\big|_i\right) \tag{13.3.20}$$

The principal directions of the strain tensor $\mathbf{E}$ rotates a small angle determined by the rotation vector:

$$\mathbf{z} = \frac{1}{2}\nabla\times\mathbf{u} \Leftrightarrow z^i = \frac{1}{2}\varepsilon^{ijk}\ u_k\big|_j = \frac{1}{2}\varepsilon^{ijk}\,u_{k,j} = \frac{1}{2}\varepsilon^{ijk}\,\tilde{R}_{kj} \tag{13.3.21}$$

The small strain tensor field $\mathbf{E}(y,t)$ has to satisfy the *compatibility equations*, which in Cartesian coordinates are presented in Sect. 5.3.9 as (5.3.40). Since each term in these equations are components of tensors, the equations may be directly transformed to general coordinates y:

$$E_{ij}\big|_{kl} + E_{kl}\big|_{ij} - E_{il}\big|_{jk} - E_{jk}\big|_{il} = 0 \tag{13.3.22}$$

Written out in detail these equations become very long and quite complex. Since they will not be used directly in the following, we shall refrain from presenting them. Malvern [28] presents a complete listing of the compatibility equations in orthogonal coordinates and in cylindrical coordinates.

The equations of compatibility for small deformations were presented and discussed in the Sect. 5.3.9 and 5.3.10. An interesting but mathematically rather complex derivation of the compatibility equations, relevant also for large deformations, is based on the following arguments. First we introduce a convected coordinate system Y, formally to be presented in Sect. 13.3.5. This coordinate system is imbedded in the continuum, which implies that the system moves and deforms with the material. The compatibility equations result from the requirement that the convected Y-system shall at all times represent a Euclidean space. This implies that a one-to-one correspondence exists between any coordinate set x in a fixed Cartesian system and a set Y in the convected material coordinate system at all times. However, the compatibility equations for large deformations seem to be of little importance in applications and will not be presented in the present exposition.

13.3.3 Rates of Deformation, Strain, and Rotation

The *velocity gradient tensor* **L**, the *rate of deformation tensor* **D**, and the *rate of rotation tensor* **W** at the time t are defined by:

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{r}} \equiv \operatorname{grad}\mathbf{v} \;\Leftrightarrow\; L_{ik} = v_i\big|_k \tag{13.3.23}$$

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{L}+\mathbf{L}^T\right) \;\Leftrightarrow\; D_{ik} = \frac{1}{2}\left(v_i\big|_k + v_k\big|_i\right) \tag{13.3.24}$$

$$\mathbf{W} = \frac{1}{2}\left(\mathbf{L}-\mathbf{L}^T\right) \;\Leftrightarrow\; W_{ik} = \frac{1}{2}\left(v_i\big|_k - v_k\big|_i\right) = \frac{1}{2}\left(v_{i,k} - v_{k,i}\right) \tag{13.3.25}$$

The last equality follows from the fact that the *rate of rotation tensor* is antisymmetric.

The *longitudinal strain rate* $\dot{\varepsilon}$ in the direction **e**, the *shear strain rate* $\dot{\gamma}$ with respect to two orthogonal directions **e** and $\bar{\mathbf{e}}$, and the *volumetric strain rate* $\dot{\varepsilon}_v$ have been defined in Sect. 5.4. The expressions in general coordinates are:

$$\dot{\varepsilon} = \mathbf{e}\cdot\mathbf{D}\cdot\mathbf{e} = e^i D_{ik} e^k = e^i \; v_i\big|_k \, e^k \tag{13.3.26}$$

$$\dot{\gamma} = 2\bar{\mathbf{e}}\cdot\mathbf{D}\cdot\mathbf{e} = 2\bar{e}^i D_{ik} e^k = e^i \left(v_i\big|_k + v_k\big|_i\right) e^k \tag{13.3.27}$$

$$\dot{\varepsilon}_v = \operatorname{tr}\mathbf{D} = \operatorname{div}\mathbf{v} = D^i_i = v^i\big|_i \tag{13.3.28}$$

The *principal directions* of **D** rotate with the angular velocity:

$$\mathbf{w} = \frac{1}{2}\nabla\times\mathbf{v} \;\Leftrightarrow\; w^i = \frac{1}{2}\varepsilon^{ijk} v_{k,j} = \frac{1}{2}\varepsilon^{ijk} W_{kj} \tag{13.3.29}$$

13.3.4 Orthogonal Coordinates

This section will present coordinate strains for small deformations, coordinate strain rates, and rates of rotation in orthogonal coordinates. First we present the physical components of the displacement vector **u** and the velocity vector **v**:

$$\mathbf{u} = u(i)\,\mathbf{e}_i^y \;\Leftrightarrow\; u(i) = \mathbf{u}\cdot\mathbf{e}_i^y,\; \mathbf{v} = v(i)\,\mathbf{e}_i^y \;\Leftrightarrow\; v(i) = \mathbf{v}\cdot\mathbf{e}_i^y \tag{13.3.30}$$

where $\mathbf{e}_i^y$ are the unit tangent vectors to coordinate lines in the orthogonal system y:

$$\mathbf{e}_i^y = \frac{1}{h_i}\mathbf{g}_i = h_i\,\mathbf{g}^i \tag{13.3.31}$$

The *physical components* of the small strain tensor **E** and the rate of deformation tensor **D** are defined by:

$$\varepsilon_{ii} = E(ii) = \mathbf{e}_i^y \cdot \mathbf{E} \cdot \mathbf{e}_i^y = \frac{1}{h_i^2} E_{ii} = \frac{1}{h_i^2}\, u_i|_i \text{ no summation}$$
$$\gamma_{ij} = 2E(ij) = 2\mathbf{e}_i^y \cdot \mathbf{E} \cdot \mathbf{e}_j^y = \frac{2}{h_i h_j} E_{ij} = \frac{1}{h_i h_j}\left(u_i|_j + u_j|_i\right)\ i \neq j \tag{13.3.32}$$

$$\dot{\varepsilon}_{ii} = D(ii) = \mathbf{e}_i^y \cdot \mathbf{D} \cdot \mathbf{e}_i^y = \frac{1}{h_i^2} D_{ii} = \frac{1}{h_i^2}\, v_i|_i \text{ no summation}$$
$$\dot{\gamma}_{ij} = 2D(ij) = 2\mathbf{e}_i^y \cdot \mathbf{D} \cdot \mathbf{e}_j^y = \frac{2}{h_i h_j} D_{ij} = \frac{1}{h_i h_j}\left(v_i|_j + v_j|_i\right)\ i \neq j \tag{13.3.33}$$

Using the formulas (12.5.67, 12.5.68) we obtain from the formulas (13.3.32):

$$\varepsilon_{ii} = E(ii) = \frac{1}{h_i^2} E_{ii} = \left[\frac{\partial}{\partial y_i}\left(\frac{u(i)}{h_i}\right) + \sum_j \frac{h_{i,j}}{h_i h_j} u(j)\right] \text{ no summation w. r. to } i$$
$$\gamma_{ij} = 2E(ij) = \frac{2}{h_i h_j} E_{ij} = \frac{h_i}{h_j}\frac{\partial}{\partial y_j}\left(\frac{u(i)}{h_i}\right) + \frac{h_j}{h_i}\frac{\partial}{\partial y_i}\left(\frac{u(j)}{h_j}\right)\ i \neq j$$
$$\varepsilon_v = \operatorname{tr}\mathbf{E} = E(ii) = \frac{1}{h}\sum_i \frac{\partial}{\partial y_i}\left(\frac{h u(i)}{h_i}\right) \tag{13.3.34}$$

Similarly, using the formulas (12.5.67, 12.5.68) we obtain from the formulas (13.3.33):

$$\dot{\varepsilon}_{ii} = D(ii) = \frac{1}{h_i^2} D_{ii} = \left[\frac{\partial}{\partial y_i}\left(\frac{v(i)}{h_i}\right) + \sum_j \frac{h_{i,j}}{h_i h_j} v(j)\right] \text{ no summation w. r. to } i$$
$$\dot{\gamma}_{ij} = 2D(ij) = \frac{2}{h_i h_j} D_{ij} = \frac{h_i}{h_j}\frac{\partial}{\partial y_j}\left(\frac{v(i)}{h_i}\right) + \frac{h_j}{h_i}\frac{\partial}{\partial y_i}\left(\frac{v(j)}{h_j}\right)\ i \neq j$$
$$\dot{\varepsilon}_v = \operatorname{tr}\mathbf{D} = D(ii) = \frac{1}{h}\sum_i \frac{\partial}{\partial y_i}\left(\frac{h v(i)}{h_i}\right) \tag{13.3.35}$$

The physical components of the vorticity vector $\mathbf{c} = \text{rot}\,\mathbf{v}$ are according to formula (12.5.63):

$$c(i) = \frac{h_i}{h}\sum_{j,k} e_{ijk}\left[h_k v(k)\right]_{,j} \tag{13.3.36}$$

In Cartesian coordinate systems the vorticity vector **c** and the rate of rotation tensor **W** are according to (5.4.12), (5.4.15), and (4.3.10) related through:

$$c_i = e_{ijk} W_{kj} \ \Leftrightarrow\ W_{ij} = -\frac{1}{2} e_{ijk} c_k \text{ in Cartesian coordinate systems} \tag{13.3.37}$$

In orthogonal coordinate systems we define the physical components of **W** through the similar relation:

$$c(i) = e_{ijk} W(kj) \;\Leftrightarrow\; W(ij) = -\frac{1}{2} e_{ijk} c(k) \tag{13.3.38}$$

From (13.3.36) and (13.3.38) we obtain:

$$W(ij) = \frac{1}{2h_i h_j}\left[(h_i v(i))_{,j} - (h_j v(j))_{,i}\right] \tag{13.3.39}$$

The results of applying (13.3.32–13.3.39) in cylindrical coordinates and spherical coordinates are presented in the formulas (5.3.13–5.3.16) and (5.4.18 –5.4.23).

13.3.5 General Analysis of Large Deformations

The displacement and deformation of a differential material line element from the reference configuration K_o to the present configuration K may be decomposed into a deformation of pure strain and a rigid-body motion as discussed in Sect. 5.5 and illustrated in Fig. 5.5.3. The decomposition, which does not necessarily represent the actual displacement and deformation of the material, may be considered in two alternative ways. First, let the material be subjected to pure strain through the *right stretch tensor* **U**, transforming the line element $d\mathbf{r}_o$ emanating from the particle Y to the line element $d\bar{\mathbf{r}}$:

$$d\bar{\mathbf{r}} = \mathbf{U} \cdot d\mathbf{r}_o \;\Leftrightarrow\; d\bar{Y}^L = U^L_K \, dY^K \tag{13.3.40}$$

Then the line element $d\bar{\mathbf{r}}$ is rotated to give the element $d\tilde{\mathbf{r}}$:

$$d\tilde{\mathbf{r}} = \mathbf{R} \cdot d\bar{\mathbf{r}} \;\Leftrightarrow\; d\tilde{Y}^N = R^N{}_L \, d\bar{Y}^L \tag{13.3.41}$$

Finally, the line element $d\tilde{\mathbf{r}}$ is given the displacement **u** to its final position at place y:

$$d\mathbf{r} = d\tilde{\mathbf{r}} \;\Leftrightarrow\; dy^i = g^i_N \, d\tilde{Y}^N \;\Rightarrow \tag{13.3.42}$$

$$d\mathbf{r} = d\tilde{\mathbf{r}} = \mathbf{RU} \cdot d\mathbf{r}_o \;\Leftrightarrow\; dy^i = g^i_N R^N{}_L U^L_K dY^K \tag{13.3.43}$$

The alternative displacement and deformation of the material element starts with a displacement **u**, followed by a rotation **R**, and then a stretch given by the left stretch tensor **V**:

$$d\mathbf{r} = \mathbf{VR} \cdot d\mathbf{r}_o \;\Leftrightarrow\; dy^i = V^i_j R^j{}_k g^k_K \, dY^K = V^i_j R^j{}_K \, dY^K \tag{13.3.44}$$

When the present configuration K is used as reference configuration it is convenient to introduce the *inverse deformation gradient*:

$$\mathbf{F}^{-1} \equiv \frac{\partial \mathbf{r}_o}{\partial \mathbf{r}} \;\Leftrightarrow\; F^{-1K}{}_i = \frac{\partial Y_K}{\partial y_i} \tag{13.3.45}$$

such that:

$$d\mathbf{r}_o = \mathbf{F}^{-1} \cdot d\mathbf{r} \tag{13.3.46}$$

The *inverse deformation tensor*, also called the *Cauchy deformation tensor*, is defined by:

$$\mathbf{B}^{-1} = \mathbf{F}^{-T}\mathbf{F}^{-1} \;\Leftrightarrow\; B^{-1}{}_{ij} = F^{-1K}{}_i F^{-1}{}_{Kj} \tag{13.3.47}$$

It follows that:

$$(ds_o)^2 = d\mathbf{r}_o \cdot d\mathbf{r}_o = d\mathbf{r} \cdot \mathbf{B}^{-\mathbf{1}} \cdot d\mathbf{r} = dy^i B^{-1}{}_{ij}\, dy^j \tag{13.3.48}$$

The *Euler strain tensor*, also called the *Almansi strain tensor* is defined by the expression:

$$(ds)^2 - (ds_o)^2 = 2\, d\mathbf{r} \cdot \tilde{\mathbf{E}} \cdot d\mathbf{r} \tag{13.3.49}$$

under the condition that the tensor is symmetric. From $(13.3.4)_2$ and (13.3.48, 13.3.49) we obtain:

$$\tilde{\mathbf{E}} = \frac{1}{2}\left(\mathbf{1} - \mathbf{B}^{-1}\right),\; \tilde{E}_{ij} = \frac{1}{2}\left(\delta_{ij} - B^{-1}{}_{ij}\right) \tag{13.3.50}$$

The following formulas are to be derived in Problem 13.1:

$$\tilde{E}_{ij} = \frac{1}{2}\left(\mathbf{g}_i \cdot \mathbf{u}_{,j} + \mathbf{g}_j \cdot \mathbf{u}_{,i} - \mathbf{u}_{,i} \cdot \mathbf{u}_{,j}\right) = \frac{1}{2}\left(u_i\big|_j + u_j\big|_i - u_k\big|_i\, u^k\Big|_j \right) \tag{13.3.51}$$

$$\mathbf{E} = \mathbf{F}^T\, \tilde{\mathbf{E}}\, \mathbf{F} \;\Leftrightarrow\; E_{KL} = \tilde{E}_{ij} F^i{}_K F^j{}_L \tag{13.3.52}$$

13.3.6 Convected Coordinates

The coordinate system Y introduced in the reference configuration K_o is now assumed to be imbedded in the continuum, which implies that the coordinate system moves and deforms with the material. The system is called a convected coordinate system. The base vectors and the fundamental parameters for the Y-system are place and time functions. The base vectors are denoted:

$$\mathbf{c}_K(Y,t) = \frac{\partial \mathbf{r}(Y,t)}{\partial Y_K} \;\Rightarrow\; \mathbf{c}_K(Y,t_o) = \mathbf{g}_K(Y) \tag{13.3.53}$$

The reciprocal base vectors $\mathbf{c}^K(Y,t)$ are defined by:

$$\mathbf{c}^K \cdot \mathbf{c}_L = \delta^K_L \tag{13.3.54}$$

The fundamental parameters for the Y-system are:

$$C_{KL} = C_{KL}(y,t) = \mathbf{c}_K \cdot \mathbf{c}_L,\; C^{KL} = C^{KL}(Y,t) = \mathbf{c}^K \cdot \mathbf{c}^L \tag{13.3.55}$$

The length ds_o and ds of a material line element in the two configurations K_o and K are respectively given by:

$$(ds_o)^2 = g_{KL}\, dY^K\, dY^L,\ (ds)^2 = C_{KL}\, dY^K\, dY^L \tag{13.3.56}$$

The components $C_{KL}(Y,t)$ now represent two tensors in the Y-system: In (13.3.5, 13.3.6) $C_{KL}(Y,t)$ are components in K_o of the deformation tensor $\mathbf{C}$. In $(13.3.56)_2$ $C_{KL}(Y,t)$ are fundamental parameters in the Y-system and components of the unit tensor $\mathbf{1}$ in K. In particular:

$$C_{KL}(Y,t_o) = g_{KL}(Y) \tag{13.3.57}$$

The components $C^{KL}(Y,t)$ also represent two tensors in the Y-system: contravariant components in K_o of the deformation tensor $\mathbf{C}$ and fundamental parameters in the Y-system and components of the unit tensor $\mathbf{1}$ in K. In particular:

$$C^{KL}(Y,t_o) = g^{KL}(Y) \tag{13.3.58}$$

It follows from the expressions (13.3.7) that in convected coordinates the components E_{KL} of the strain tensor represent half of the change in the fundamental parameters from g_{KL} in K_o to C_{KL} in K for the convected Y-system.

If the y-system in K is chosen such that the coordinate system coincides with the convective Y-system at time t then:

$$F^i{}_K = \frac{\partial y_i}{\partial Y_K} = \delta^i_K \tag{13.3.59}$$

and (13.3.59) shows that the matrix in of the Green strain tensor and the matrix in the Euler strain tensor become identical.

If the motion from K_o to K is given by the displacement vector $\mathbf{u}(Y,t)$ in (13.2.2), then:

$$\mathbf{c}_K = \frac{\partial \mathbf{r}}{\partial Y_K} = \frac{\partial \mathbf{r}_o}{\partial Y_K} + \frac{\partial \mathbf{u}}{\partial Y_K} = \mathbf{g}_K + \mathbf{u}_{,K} \tag{13.3.60}$$

$$C_{KL} = \mathbf{c}_K \cdot \mathbf{c}_L = \mathbf{g}_K \cdot \mathbf{g}_L + \mathbf{g}_K \cdot \mathbf{u}_{,L} + \mathbf{u}_{,K} \cdot \mathbf{g}_L + \mathbf{u}_{,K} \cdot \mathbf{u}_{,L} \ \Rightarrow$$

$$C_{KL} = g_{KL} + u_K|_L + u_L|_K + u^N\big|_K\, u_N|_L \tag{13.3.61}$$

Note that the displacement components u_K and u^K are related to the Y-system in K_o:

$$u_K = \mathbf{u} \cdot \mathbf{g}_K,\ u^K = \mathbf{u} \cdot \mathbf{g}^K \tag{13.3.62}$$

and that covariant differentiation is to be performed in K_o with Christoffel symbols based on the fundamental quantities g_{KL}.

Alternatively we may use the displacement components u_K and u^K related to the Y-system in K:

$$u_K = \mathbf{u} \cdot \mathbf{c}_K,\ u^K = \mathbf{u} \cdot \mathbf{c}^K \tag{13.3.63}$$

For simplicity we use the same symbols for the displacement components here. Using the result (13.3.60), we obtain:

$$g_{KL} = \mathbf{g}_K \cdot \mathbf{g}_L = (\mathbf{c}_K - \mathbf{u}_{,K}) \cdot (\mathbf{c}_L - \mathbf{u}_{,L}) = \mathbf{c}_K \cdot \mathbf{c}_L - \mathbf{c}_K \cdot \mathbf{u}_{,L} - \mathbf{u}_{,K} \cdot \mathbf{c}_L + \mathbf{u}_{,K} \cdot \mathbf{u}_{,L}$$
$$\Rightarrow C_{KL} = g_{KL} + u_K\|_L + u_L\|_K - u^N\|_K\, u_N\|_L \tag{13.3.64}$$

Covariant differentiation is now marked by a double vertical line to indicate that it is to be performed in K and with Christoffel symbols computed from the fundamental quantities C_{KL}. The strain components E_{KL} in the formulas (13.3.13) will have to different forms:

$$E_{KL} = \frac{1}{2}\left(u_K|_L + u_L|_K + u^N|_K\, u_N|_L\right) \tag{13.3.65}$$
$$E_{KL} = \frac{1}{2}\left(u_K\|_L + u_L\|_K - u^N\|_K\, u_N\|_L\right) \tag{13.3.66}$$

Note the two sets of the strain components E_{KL} in (13.3.65) and (13.3.66) are identical, while the displacement components u_K and the covariant differentiations are not the same. Comparing the formulas (13.3.66) for the components of the Green strain tensor with formulas (13.3.51) for the components of the Euler strain tensor, we see that the two components set are identical if we choose a y-system that coincides with the Y-system at the present time t.

Example 13.1. Simple Shear

The deformation simple shear may be described by:

$$x_1(Y_1, Y_2, t) = Y_1 + \beta(t) Y_2,\ x_2(Y_2) = Y_2\ ,\ x_3(Y_3) = Y_3,\ \beta(t_o) = 0 \tag{13.3.67}$$

x_i are Cartesian coordinates and Y_K are convected coordinates chosen such that $Y_K = x_i \delta_K^i$ at the reference time t_o. Figure 13.3.1 shows the deformation of a block of the material. The Green strain tensor and the coordinate strains will be determined.

It follows from (13.3.67) that:

$$\left(\frac{\partial x_i}{\partial Y_K}\right) = \begin{pmatrix} 1 & \beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \left(\frac{\partial Y_K}{\partial x_i}\right) = \begin{pmatrix} 1 & -\beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

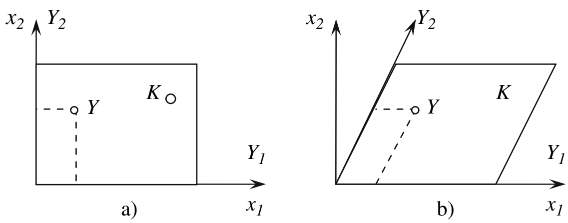


Fig. 13.3.1 A material block in K_o at time t_o and in K at time t. Convected coordinates Y

$$(C_{KL}) = \left(\frac{\partial x_i}{\partial Y_K}\frac{\partial x_i}{\partial Y_L}\right) = \begin{pmatrix} 1 & \beta & 0 \\ \beta & 1+\beta^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\left(B^{-1}{}_{KL}\right) = \left(\frac{\partial Y_K}{\partial x_i}\frac{\partial Y_L}{\partial x_i}\right) = \begin{pmatrix} 1+\beta^2 & -\beta & 0 \\ -\beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(E_{KL}) = \left(\frac{1}{2}\left(C_{KL} - g_{KL}\right)\right) = \frac{1}{2}\begin{pmatrix} 0 & \beta & 0 \\ \beta & \beta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The non-zero coordinate strains are obtained from (5.2.17) and (5.2.20):

$$\begin{aligned}
\varepsilon_{22} &= \sqrt{1 + 2\mathbf{e}_2 \cdot \mathbf{E} \cdot \mathbf{e}_2} - 1 = \sqrt{1 + 2\frac{\mathbf{g}_2}{\sqrt{g_{22}}} \cdot \mathbf{E} \cdot \frac{\mathbf{g}_2}{\sqrt{g_{22}}}} - 1 \\
&= \sqrt{1 + 2\frac{E_{22}}{g_{22}}} - 1 = \sqrt{1+\beta^2} - 1 \\
\sin\gamma_{12} &= \frac{2\mathbf{e}_1 \cdot \mathbf{E} \cdot \mathbf{e}_2}{\sqrt{1 + 2\mathbf{e}_1 \cdot \mathbf{E} \cdot \mathbf{e}_1}\sqrt{1 + 2\mathbf{e}_2 \cdot \mathbf{E} \cdot \mathbf{e}_2}} \\
&= \frac{2\dfrac{\mathbf{g}_1}{\sqrt{g_{11}}} \cdot \mathbf{E} \cdot \dfrac{\mathbf{g}_2}{\sqrt{g_{22}}}}{\sqrt{1 + 2\dfrac{\mathbf{g}_1}{\sqrt{g_{11}}} \cdot \mathbf{E} \cdot \dfrac{\mathbf{g}_1}{\sqrt{g_{11}}}}\sqrt{1 + 2\dfrac{\mathbf{g}_2}{\sqrt{g_{22}}} \cdot \mathbf{E} \cdot \dfrac{\mathbf{g}_2}{\sqrt{g_{22}}}}} \\
&= \frac{2\dfrac{E_{12}}{\sqrt{g_{11}g_{22}}}}{\sqrt{1 + 2\dfrac{E_{11}}{g_{11}}}\sqrt{1 + 2\dfrac{E_{22}}{g_{22}}}} = \frac{2\dfrac{\beta}{2}}{\sqrt{1 + 2\dfrac{1}{2}\beta^2}} \Rightarrow \sin\gamma_{12} = \frac{\beta}{\sqrt{1+\beta^2}}
\end{aligned}$$

For small deformations: $\beta \ll 1 \Rightarrow \beta^2 \ll \beta$, the strain matrix and the non-zero coordinate strain are:

$$(E_{KL}) \approx (E_{ij}) = \frac{1}{2}\begin{pmatrix} 0 & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \gamma_{12} = \beta$$

13.4 Convected Derivatives of Tensors

A tensor quantity that is independent of the choice of reference, is called a *reference invariant tensor* or *objective tensor*. Tensor quantities dependent of the choice of reference, are called *reference related tensors*. These concepts were introduced and discussed in Sect. 11.2. It was pointed out that the material derivative of an objective tensor not necessarily is objective. One form of time derivative of an objective tensor leading to a new objective tensor is the corotational derivative of the objective tensor, defined in Sect. 11.3. Other objective time derivatives are the lower-convected and upper-convected derivatives of objective tensors, briefly presented in Sect. 11.4. We are now in the position to give a proper definition of these convected derivatives in the setting of convected coordinates.

We apply convected coordinates Y with base vectors $\mathbf{c}_K(\mathrm{y},\mathrm{t})$. The material derivatives of these base vectors are:

$$\dot{\mathbf{c}}_K = \frac{\partial}{\partial t}\frac{\partial \mathbf{r}}{\partial Y_K} = \frac{\partial}{\partial Y_K}\frac{\partial \mathbf{r}}{\partial t} = \frac{\partial \mathbf{v}}{\partial Y_K} \equiv \mathbf{v}_{,K} \tag{13.4.1}$$

$\mathbf{v}$ is the particle velocity. The components of the material derivatives of the base vectors are covariant derivatives of the velocity components:

$$\dot{\mathbf{c}}_K = v^L\big\|_K \mathbf{c}_L \tag{13.4.2}$$

It is straight forward to show that, see Problem 13.2.:

$$\dot{\mathbf{c}}^K = -\, v^K\big\|_L \mathbf{c}^L \tag{13.4.3}$$

Let $\mathbf{a}(Y,t)$ be a convected vector field, i.e. a vector field associated with the particles in the material we are considering:

$$\mathbf{a} = a^K \mathbf{c}_K = a_K \mathbf{c}^K \tag{13.4.4}$$

The material derivative of $\mathbf{a}$ is:

$$\dot{\mathbf{a}} = \dot{a}^K \mathbf{c}_K = \left[\frac{\partial a^K}{\partial t} + a^L\, v^K\big\|_L\right]\mathbf{c}_K,\ \dot{\mathbf{a}} = \dot{a}_K \mathbf{c}^K = \left[\frac{\partial a_K}{\partial t} - a_L\, v^L\big\|_K\right]\mathbf{c}^K \tag{13.4.5}$$

$\dot{\mathbf{a}}$ is a reference related vector field, while the two vectors defined by the components:

$$\partial_c a^K \equiv \frac{\partial a^K}{\partial t},\ \partial_c a_K \equiv \frac{\partial a_K}{\partial t} \tag{13.4.6}$$

are objective. The expressions (13.4.6) are called convected differentiated vector components. The vectors defined in (13.4.5) are two different vector fields. In order to determine the components of these vectors in a reference fixed coordinate system y, we rearrange (13.4.5):

$$\partial_c a^K = \dot{a}^K - a^L\, v^K\big\|_L,\ \partial_c a_K = \dot{a}_K + a_L\, v^L\big\|_K \tag{13.4.7}$$

Because these equations are tensor equations, they may directly be transformed to the fixed y-system:

$$\partial_c a^i = \dot{a}^i - a^k\, v^i\big|_k = \frac{\partial a^i}{\partial t} + a^i\big|_k v^k - a^k\, v^i\big|_k \tag{13.4.8}$$

$$\partial_c a_i = \dot{a}_i + a_k\, v^k\Big|_i = \frac{\partial a_i}{\partial t} + a_i\big|_k v^k + a_k\, v^k\Big|_i \tag{13.4.9}$$

Note that the convective derivatives of the contravariant and the covariant vector components do not result in one and the same vector. The vector defined by (13.4.8) is called the *upper-convected derivative* of the vector **a** and the vector defined by (13.4.9) is called the lower-convected derivative of **a**. The two derivatives are also presented as:

$$\mathbf{a}^{\nabla} = \dot{\mathbf{a}} - \mathbf{L}\mathbf{a} \Leftrightarrow a^{\nabla i} = \dot{a}^i - v^i\big|_k a^k \text{ upper-convected derivate of } \mathbf{a} \tag{13.4.10}$$

$$\mathbf{a}^{\Delta} = \dot{\mathbf{a}} + \mathbf{L}^T\mathbf{a} \Leftrightarrow a_i^{\Delta} = \dot{a}_i + v^k\Big|_i a_k \text{ lower-convected derivate of } \mathbf{a} \tag{13.4.11}$$

Confer the definitions (11.4.1–11.4.2) in Cartesian coordinates.

Convected differentiated tensor components are defined by their representation in a convected Y-system in which they are given directly by the material derivatives of the tensor components. As an example we consider a 2. order tensor **B**. The convected derivatives of the components $B^K{}_L$ in the Y-system are defined by:

$$\partial_c B^K{}_L \equiv \frac{\partial}{\partial t} B^K{}_L \tag{13.4.12}$$

If **B** is an objective tensor then so is the tenor defined by the components (13.4.12). One set of components of this tensor in the y-system are denoted $\partial_c B^i{}_j$ and are found as follows. Let **a** and **b** be two objective vectors. The scalar $\alpha = \mathbf{B}[\mathbf{a},\, \mathbf{b}]$ may alternatively be computed from:

$$\alpha = B^K{}_L a_K b^L, \; \alpha = B^i{}_j a_i b^j \tag{13.4.13}$$

Then we may write:

$$\dot{\alpha} = \partial_c B^K{}_L \, a_K b^L + B^K{}_L \, \partial_c a_K \, b^L + B^K{}_L \, a_K \, \partial_c b^L \tag{13.4.14}$$

$$\dot{\alpha} = \dot{B}^i{}_j \, a_i b^j + B^i{}_j \, \dot{a}_i \, b^j + B^i{}_j \, a_i \, \dot{b}^j \tag{13.4.15}$$

The equations (13.4.14) are transformed to the y-system:

$$\dot{\alpha} = \partial_c B^i{}_j \, a_i b^j + B^i{}_j \, \partial_c a_i \, b^j + B^i{}_j \, a_i \, \partial_c b^j \tag{13.4.16}$$

The formulas (13.4.8, 13.4.9) are used for $\partial_c a_i$ and $\partial_c b^j$ in (13.4.16), and then (13.4.15) is subtracted from (13.4.16). The result is:

$$\left[\partial_c B^i{}_j - \dot{B}^i{}_j + B^k{}_j \, v^i\big|_k - B^i{}_k \, v^k\Big|_j\right] a_i b^j = 0$$

Because the vectors **a** and **b** may be chosen arbitrarily, the expression in the brackets must be zero. Thus we have the result:

$$\partial_c B^i{}_j = \dot{B}^i{}_j - B^k{}_j \, v^i\big|_k + B^i{}_k \, v^k\Big|_j \tag{13.4.17}$$

When the material derivative and the covariant derivatives in (13.4.17) are written out in detail, we shall see that the terms with Christoffel symbols are eliminated, and that the result is, see Problem 13.3:

$$\partial_c B^i{}_j = \frac{\partial}{\partial t} B^i{}_j + B^i{}_{j,k}\, v^k - B^k{}_j\, v^i{}_{,k} + B^i{}_k\, v^k{}_{,j} \tag{13.4.18}$$

Oldroyd [37] derived the formula (13.4.18) by evaluating the material derivative of the transformation equation:

$$B^K{}_L = \frac{\partial Y_K}{\partial y_i} \frac{\partial y_j}{\partial Y_L} B^i{}_j \tag{13.4.19}$$

Followed by a substitution of the result into the transformation equation:

$$\partial_c B^i{}_j = \frac{\partial y_i}{\partial Y_K} \frac{\partial Y_L}{\partial y_j} \partial_c B^K{}_L \tag{13.4.20}$$

See Problem 13.4.

Now we define two tensors by their components in the fixed y-system and the corresponding components in the convected Y-system:

$$\partial_c B_{ij} \text{ in the} y\text{-system} \;\Leftrightarrow\; \frac{\partial}{\partial t} B_{KL} \text{ in the } Y\text{-system} \tag{13.4.21}$$

$$\partial_c B^{ij} \text{ in the} y\text{-system} \;\Leftrightarrow\; \frac{\partial}{\partial t} B^{KL} \text{ in the } Y\text{-system} \tag{13.4.22}$$

Then the following formulas may be derived, see Problem 13.5:

$$\partial_c B_{ij} = \dot{B}_{ij} + B_{kj}\, v^k\Big|_i + B_{ik}\, v^k\Big|_j \tag{13.4.23}$$

$$\partial_c B^{ij} = \dot{B}^{ij} - B^{kj}\, v^i\big|_k - B^{ik}\, v^j\big|_k \tag{13.4.24}$$

The results (13.4.8, 13.4.9) and (13.4.17, 13.4.23, 13.4.24) show a pattern for constructing the convected derivatives of objective tensors of any order. Note that the convective derivatives of the tensor components of different types do not result in one and the same tensor. For a 2. order tensor **B** it is customary to let the components (13.4.23) defined the *lower-convected derivative* of the tensor B, while the components (13.4.24) defined the *upper-convected derivative* of the tensor **B**. Special symbols are introduced for these two tensors:

$$\begin{aligned} \mathbf{B}^{\Delta} &= \dot{\mathbf{B}} + \mathbf{L}^T\mathbf{B} + \mathbf{B}\mathbf{L} \;\Leftrightarrow \\ B^{\Delta}{}_{ij} &= \dot{B}_{ij} + v^k\big|_i B_{kj} + B_{ik}\, v^k\big|_j \end{aligned} \quad \text{lower-convected derivative of } \mathbf{B} \tag{13.4.25}$$

$$\begin{aligned} \mathbf{B}^{\nabla} &= \dot{\mathbf{B}} - \mathbf{L}\mathbf{B} - \mathbf{B}\mathbf{L}^T \;\Leftrightarrow \\ B^{\nabla ij} &= \dot{B}^{ij} - v^i\big|_k B^{kj} - B^{ik}\, v^j\big|_k \end{aligned} \quad \text{upper-convected derivative of } \mathbf{B} \tag{13.4.26}$$

Confer the definitions (11.4.3, 11.4.4) in Cartesian coordinates.

13.5 Stress Tensors. Equations of Motion

13.5.1 Physical Stress Components

The stress vector **t** on a surface through a particle with unit normal **n** is determined by the Cauchy stress theorem:

$$\mathbf{t} = \mathbf{T} \cdot \mathbf{n} \Leftrightarrow t^i = T^i_k n^k \tag{13.5.1}$$

T is the Cauchy stress tensor. The normal stress σ and the shear stress τ on the surface are given by:

$$\sigma = \mathbf{n} \cdot \mathbf{t} = \mathbf{n} \cdot \mathbf{T} \cdot \mathbf{n} = n^i T^i_k n^k, \ \tau = \sqrt{\mathbf{t} \cdot \mathbf{t} - \sigma^2} = \sqrt{t^i t_i - \sigma^2} \tag{13.5.2}$$

The stress vector $\mathbf{t}_i$, normal stress σ_i, and shear stress τ_i on a coordinate surface $y_i = \text{constant}$ are:

$$\mathbf{t}_i = \mathbf{T} \cdot \frac{\mathbf{g}^i}{\sqrt{g^{ii}}} = \frac{T^{ki}}{\sqrt{g^{ii}}} \mathbf{g}_k \tag{13.5.3}$$

$$\sigma_i = \frac{\mathbf{g}^i}{\sqrt{g^{ii}}} \cdot \mathbf{T} \cdot \frac{\mathbf{g}^i}{\sqrt{g^{ii}}} = \frac{T^{ii}}{g^{ii}}, \ \ \tau_i = \sqrt{\frac{T^{ki}T^i_k}{g^{ii}} - (\sigma_i)^2} \tag{13.5.4}$$

The shear stress τ_i has two components on the surface $y_i = \text{constant}$:

$$(\tau_i)_k = \frac{\mathbf{g}_k}{\sqrt{g_{kk}}} \cdot \mathbf{t}_i = \frac{\mathbf{g}_k}{\sqrt{g_{kk}}} \cdot \mathbf{T} \cdot \frac{\mathbf{g}^i}{\sqrt{g^{ii}}} = \frac{T^i_k}{\sqrt{g_{kk}g^{ii}}} \ k \neq i \tag{13.5.5}$$

The physical components of the stress vector $\mathbf{t}_i$ are according to the general definition (12.3.5):

$$\tau_{ki} = T^{ki}\sqrt{\frac{g_{kk}}{g^{ii}}} \tag{13.5.6}$$

Figure 13.5.1 shows the physical components τ_{12} and τ_{22} of the stress vector $\mathbf{t}_2$. Green and Zerna [19] call the components (13.5.6) *physical stress components*. As seen from Fig. 13.5.1, τ_{12} (and τ_{32}) are shear stresses, while τ_{22} represents in general both a shear stress and a normal stress.

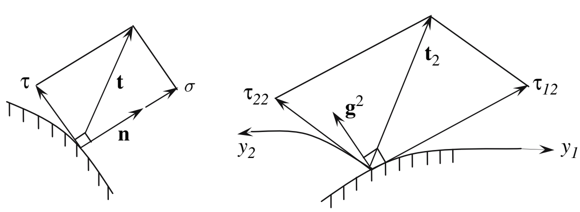


Fig. 13.5.1 Stresses on a surface and on a coordinate surface $y_2 = \text{constant}$

Truesdell [50] defines *physical stress components* differently from Green and Zerna. First the stress tensor **t** and the unit normal surface vector **n** are expressed in physical components:

$$t(k) = t^k \sqrt{g_{kk}}\,, \quad n(i) = n^i \sqrt{g_{ii}} \tag{13.5.7}$$

Then the Cauchy stress theorem (13.5.1) is expressed as:

$$t(k) = T(ki)\,n(i)\,, \quad T(ki) = T_i^k \sqrt{\frac{g_{kk}}{g_{ii}}} \tag{13.5.8}$$

Since the physical components $n(i)$ are dimensionless and the physical components t(k) have dimension force per unit area, the dimension of the components $T(ki)$ is also force per unit area. $T(ki)$ are called *physical stress components* because they are the proper coefficients in the linear relations (13.5.8) between the physical components of the unit normal vector and the physical components of the stress vector. Note that the components $T(ki)$ are not in general symmetric. It may be shown, Problem 13.6, that the two sets of physical components defined by (13.5.6) and (13.5.8) are related through:

$$\tau_{ki} = \sum_j T(kj)\,g^{ij} \sqrt{\frac{g_{jj}}{g^{ii}}} \tag{13.5.9}$$

In orthogonal coordinate systems:

$$g_{ij} = g^{ij} = 0 \text{ for } i \neq j\,, \quad g_{ii} = \frac{1}{g^{ii}} = h_i^2$$

Hence:

$$\tau_{ki} = T(ki) = h_k h_i T^{ki} = \frac{h_k}{h_i} T_i^k \tag{13.5.10}$$

$T(ki)$ for $k = i$ is the normal stress σ_i on the coordinate surface $y_i =$ constant, while $T(ki)$ for $k \neq i$ are orthogonal components of the shear stress τ_i on the same surface. In the following the physical stress components are only used in orthogonal coordinates, and for practical reasons we shall use the notation $T(ki)$. Note that:

$$T(ki) = \mathbf{e}_k^y \cdot \mathbf{T} \cdot \mathbf{e}_i^y \tag{13.5.11}$$

$\mathbf{e}_i^y$ are unit tangent vectors to the coordinate lines.

Example 13.2. Physical Stress Components in Cylindrical Coordinates

The physical components of the stress tensor in cylindrical coordinates are expressed with alternative symbols as shown below. Expressed in terms of the mixed tensor components and the contravariant tensor components we find:

$$(T(ij)) \equiv \begin{pmatrix} \sigma_R & \tau_{R\theta} & \tau_{Rz} \\ \tau_{\theta R} & \sigma_\theta & \tau_{\theta z} \\ \tau_{zR} & \tau_{z\theta} & \sigma_z \end{pmatrix} \equiv \begin{pmatrix} T_{RR} & T_{R\theta} & T_{Rz} \\ T_{\theta R} & T_{\theta\theta} & T_{\theta z} \\ T_{zR} & T_{z\theta} & T_{zz} \end{pmatrix} \text{ as shown in Fig. 13.5.2}$$

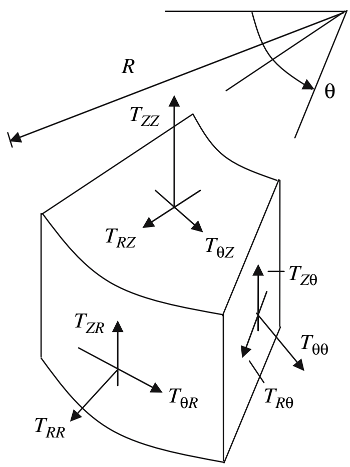


Fig. 13.5.2 Physical stress components in cylindrical coordinates

$$= \begin{pmatrix} T_1^1 & T_2^1/R & T_3^1 \\ R\,T_1^2 & T_2^2 & R\,T_3^2 \\ T_1^3 & T_2^3/R & T_3^3 \end{pmatrix} = \begin{pmatrix} T^{11} & R\,T^{12} & T^{13} \\ R\,T^{21} & R^2\,T^{22} & R\,T^{23} \\ T^{31} & R\,T^{31} & T^{33} \end{pmatrix} \tag{13.5.12}$$

13.5.2 Cauchy Equations of Motion

The Cauchy equations of motion (3.2.35) have the following representation in a general curvilinear coordinates:

$$T^{ik}\Big|_k + \rho\, b^i = \rho\, a^i \;\Leftrightarrow\; T^{ik}{}_{,k} + T^{lk}\,\Gamma^i_{lk} + T^{il}\,\Gamma^k_{lk} + \rho\, b^i = \rho\, a^i \tag{13.5.13}$$

By using the result (12.5.6), we obtain the alternative form:

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial y_k}\left(\sqrt{g}\,T^{ik}\right) + T^{lk}\,\Gamma^i_{lk} + \rho\, b^i = \rho\, a^i \tag{13.5.14}$$

In *orthogonal coordinates* we use the formulas (12.5.59), (13.5.10), and (12.5.64) and can rewrite (13.5.14) to:

$$\sum_k \frac{1}{h}\frac{\partial}{\partial y_k}\left(\frac{h}{h_k}T(ik)\right) + \frac{h_{i,k}}{h_i h_k}T(ik) - \frac{h_{k,i}}{h_i h_k}T(kk) + \rho\, b(i) = \rho\, a(i) \tag{13.5.15}$$

The Cauchy equations in cylindrical coordinates and in spherical coordinates are presented in (3.2.39, 3.2.40, 3.2.41) and (3.2.42, 3.2.43, 3.2.44).

13.6 Basic Equations in Elasticity

The classical theory of elasticity has been presented in Chap. 7. The most important equations of the theory will now be transformed to general coordinates.

In the Cauchy equations of motion represented now by (13.5.13, 13.5.14, 13.5.15) the particle acceleration will be expressed through the displacement vector **u**(y,t):

$$\mathbf{a} = \ddot{\mathbf{u}} \Leftrightarrow a^i = \ddot{u}^i \tag{13.6.1}$$

The constitutive equations for isotropic, linearly elastic material are represented by Hooke's law, which in tensorial form, i.e. invariant form, and in Cartesian form, is represented by (7.2.8). The invariant form and the form in general coordinates are:

$$\mathbf{T} = \frac{\eta}{1+\nu}\left[\mathbf{E} + \frac{\nu}{1-2\nu}(\mathrm{tr}\,\mathbf{E})\,\mathbf{1}\right] \Leftrightarrow T^i_j = \frac{\eta}{1+\nu}\left[E^i_j + \frac{\nu}{1-2\nu}E^k_k\delta^i_j\right] \tag{13.6.2}$$

The strain tensor may be expressed by the displacement gradients:

$$E_{ij} = \frac{1}{2}\left(u_i\big|_j + u_j\big|_i\right) \tag{13.6.3}$$

The Navier equations, i.e. the equations of motion expressed through the displacement **u**(y,t), are in invariant form and Cartesian form given by (7.6.28). The invariant form and the form in general coordinates are:

$$\nabla^2\mathbf{u} + \frac{1}{1-2\nu}\nabla(\nabla\cdot\mathbf{u}) + \frac{\rho}{\mu}(\mathbf{b}-\mathbf{a}) = 0 \Leftrightarrow u^i\big|^k_k + \frac{1}{1-2\nu}\,u^k\big|^i_k + \frac{\rho}{\mu}\left(b^i - a^i\right) = 0 \tag{13.6.4}$$

In any orthogonal coordinate system the Navier equations (13.6.4) may be expanded in details by application of formula $(12.5.98)_2$ for $\nabla^2\mathbf{u}$ and formula (12.5.97) for $\nabla(\nabla\cdot\mathbf{u})$.

As an example of the Navier equations in special coordinate system, the following equations apply in cylindrical coordinates:

$$\begin{aligned}
&\nabla^2 u_R - \frac{u_R}{R^2} - \frac{2}{R^2}\frac{\partial u_\theta}{\partial\theta} + \frac{1}{1-2\nu}\frac{\partial}{\partial R}(\nabla\cdot\mathbf{u}) + \frac{\rho}{\mu}(b_R - \ddot{u}_R) = 0\\
&\nabla^2 u_\theta + \frac{2}{R^2}\frac{\partial u_R}{\partial\theta} - \frac{u_\theta}{R^2} + \frac{1}{1-2\nu}\frac{1}{R}\frac{\partial}{\partial\theta}(\nabla\cdot\mathbf{u}) + \frac{\rho}{\mu}(b_\theta - \ddot{u}_\theta) = 0\\
&\nabla^2 u_z + \frac{1}{1-2\nu}\frac{\partial}{\partial z}(\nabla\cdot\mathbf{u}) + \frac{\rho}{\mu}(b_z - \ddot{u}_z) = 0
\end{aligned} \tag{13.6.5}$$

where:

$$\nabla^2 = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}\,, \quad \nabla\cdot\mathbf{u} = \frac{1}{R}\frac{\partial}{\partial R}(Ru_R) + \frac{1}{R}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z} \tag{13.6.6}$$

13.7 Basic Equations in Fluid Mechanics

This section will only present the most important equations pertaining to classical fluid mechanics. The equations presented in invariant form and in terms of Cartesian components in Chap. 8 will now be transformed to general coordinates and in particular to orthogonal coordinates.

The *continuity equation for a control volume V with control surface A* is:

$$\int_V \partial_t \rho \, dV + \int_A \rho \, v^i n_i \, dA = 0 \tag{13.7.1}$$

The *continuity equation for a place* becomes:

$$\dot{\rho} + \rho \, v^i\big|_i = 0 \;\Leftrightarrow\; \partial_t \rho + \left(\rho \, v^i\right)\big|_i = 0 \tag{13.7.2}$$

The *Reynolds transport theorem* takes the form:

$$\dot{B} = \int_V \dot{\beta} \, \rho \, dV = \int_V \partial_t \left(\beta \, \rho\right) dV + \int_A \beta \, \rho \, v^i n_i \, dA \tag{13.7.3}$$

The equation of continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \;\Leftrightarrow\; \frac{\partial \rho}{\partial t} + \left(\rho \, v^i\right)\big|_i = 0 \tag{13.7.4}$$

By application of formula (12.5.58) we may rewrite the component form to:

$$\frac{\partial \rho}{\partial t} + \frac{1}{\sqrt{g}} \left(\sqrt{g} \, \rho \, v^i\right),_i = 0 \tag{13.7.5}$$

Irrotational flow, defined by the condition:

$$\mathbf{c} \equiv \nabla \times \mathbf{v} = \mathbf{0} \;\Leftrightarrow\; v_k\big|_i = v_i\big|_k \;\Leftrightarrow\; v_{k},_{i} = v_{i},_{k} \tag{13.7.6}$$

implies potential flow:

$$\mathbf{v} = \nabla \phi \;\Leftrightarrow\; v_i = \phi,_i \;\Leftrightarrow\; v^i = \phi\big|^i \tag{13.7.7}$$

For potential flow of an incompressible fluid the velocity potential ϕ must satisfy the Laplace equation:

$$\nabla^2 \phi = 0 \;\Leftrightarrow\; \phi\big|_i^i = 0 \tag{13.7.8}$$

The thermal energy balance equation (6.3.14) contains the heat flux vector:

$$\mathbf{h} = -\tilde{\kappa} \nabla \theta \;\Leftrightarrow\; h_i = -\tilde{\kappa} \, \theta,_i \tag{13.7.9}$$

$\tilde{\kappa}$ is the thermal conductivity and θ is the temperature. The heat flux vector gives the heat flux $q(\mathbf{n},\mathbf{r},\mathbf{t})$ per unit area through a material surface at the place $\mathbf{r}$ with unit normal $\mathbf{n}$:

$$q = -\mathbf{h}\cdot\mathbf{n} = -h_i n^i \tag{13.7.10}$$

The stress power per unit volume is:

$$\omega \equiv \mathbf{T}:\mathbf{D} = \mathbf{T}:\mathbf{L} = T^{ik}\ v_i|_k \tag{13.7.11}$$

The thermal energy balance equation (6.3.14) may now be presented:

$$\rho\,\dot{\varepsilon} = -\nabla\cdot\mathbf{h} + \mathbf{T}:\mathbf{D} \Leftrightarrow \rho\,\dot{\varepsilon} = -\left.h^i\right|_i + T^{ik}\ v_i|_k \Leftrightarrow$$

$$\rho\,\dot{\varepsilon} = -\frac{1}{\sqrt{g}}\left(\sqrt{g}\,h^i\right)_{,i} + T^{ik}\left(v_{i,k} - v_j\Gamma^j_{ik}\right) \Leftrightarrow$$

$$\rho\,\dot{\varepsilon} = \frac{1}{\sqrt{g}}\left(\tilde{\kappa}\sqrt{g}\,g^{ik}\theta_{,k}\right)_{,i} + T^{ik}\left(v_{i,k} - v_j\Gamma^j_{ik}\right) \tag{13.7.12}$$

For an incompressible fluid we may introduce $c_{\text{in}}(\theta)$ as the specific heat, such that:

$$\dot{\varepsilon} = c_{\text{in}}\,\dot{\theta} \tag{13.7.13}$$

13.7.1 Perfect Fluids ≡ Eulerian Fluids

The Eulerian fluid is defined by the constitutive equation:

$$\mathbf{T} = -p\mathbf{1}\,,\ \ p = p(\rho,\theta) \Leftrightarrow T^{ij} = -p\,g^{ij} \tag{13.7.14}$$

With this expression for the stress tensor the Cauchy equations of motion (13.5.13) give the *Euler equations*:

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{b} \Leftrightarrow \frac{\partial v^i}{\partial t} + v^k\,\left.v^i\right|_k = -\frac{1}{\rho}\,p|^i + b^i \Leftrightarrow$$

$$\frac{\partial v^i}{\partial t} + v^k\left(v^i{}_{,k} + v^l\,\Gamma^i_{lk}\right) = -\frac{1}{\rho}g^{ik}\,p_{,k} + b^i \tag{13.7.15}$$

13.7.2 Linearly Viscous Fluids ≡ Newtonian Fluids

The Newtonian fluid is defined by the constitutive equation (8.4.6), which we now present as:

$$\mathbf{T} = -p(\rho,\theta)\,\mathbf{1} + 2\mu\,\mathbf{D} + \left(\kappa - \frac{2\mu}{3}\right)(\mathrm{tr}\,\mathbf{D})\,\mathbf{1} \;\Leftrightarrow$$
$$T^i_j = -p(\rho,\theta)\,\delta^i_j + 2\mu\,D^i_j + \left(\kappa - \frac{2\mu}{3}\right) D^k_k\,\delta^i_j \;\Leftrightarrow$$
$$T^i_j = -p(\rho,\theta)\,\delta^i_j + \mu\left(v^i\big|_j + v^j\big|_i\right) + \left(\kappa - \frac{2\mu}{3}\right) v^k\Big|_k\,\delta^i_j \tag{13.7.16}$$

When the constitutive equation is substituted into the Cauchy equation of motion (13.5.13) we get the *Navier-Stokes equations*:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\frac{\mu}{3}+\kappa\right)\nabla(\nabla\cdot\mathbf{v}) + \mathbf{b} \;\Leftrightarrow$$
$$\frac{\partial v^i}{\partial t} + v^k\,v^i\big|_k = -\frac{1}{\rho}\,p\big|^i + \frac{\mu}{\rho}\,v^i\big|^k_k + \frac{1}{\rho}\left(\frac{\mu}{3}+\kappa\right) v^k\Big|^i_k + b^i \;\Leftrightarrow$$
$$\frac{\partial v^i}{\partial t} + v^k\left(v^i{}_{,k} + v^l\,\Gamma^i_{lk}\right) = -\frac{1}{\rho}g^{ik}\,p_{,k} + b^i$$
$$+\frac{\mu}{\rho}g^{lk}\left[v^i{}_{,lk} + v^r\left(\Gamma^i_{rl,k} + \Gamma^s_{rl}\Gamma^i_{sk} - \Gamma^i_{rs}\Gamma^s_{lk}\right) + v^r{}_{,k}\Gamma^i_{rl} + v^r{}_{,l}\Gamma^i_{rk} - v^i{}_{,r}\Gamma^r_{lk}\right]$$
$$+\frac{1}{\rho}\left(\frac{\mu}{3}+\kappa\right) g^{il}\left[\frac{1}{\sqrt{g}}\left(\sqrt{g}v^k\right)_{,k}\right]_{,l} \tag{13.7.17}$$

13.7.3 Orthogonal Coordinates

The basic formulas will now be presented in general orthogonal coordinates. We shall apply formulas developed in Sect. 12.5.

The equation of continuity is:

$$\frac{\partial \rho}{\partial t} + \frac{1}{h}\sum_i\left(\frac{h}{h_i}\rho\,v(i)\right)_{,i} = 0 \tag{13.7.18}$$

$v(i)$ are the physical velocity components $v(i) = v^i\,h_i$.

We use the expressions (12.5.65–12.5.68) for the covariant derivatives of vector components and obtain the formulas for the physical components of the particle acceleration in the Euler equations and in the Navier-Stokes equations:

$$a(i) = \frac{Dv(i)}{Dt} + \sum_k \frac{v(k)}{h_i\,h_k}\left[h_{i,k}v(i) - h_{k,i}v(k)\right] \tag{13.7.19}$$

where we have introduced the operator:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum_k \frac{v(k)}{h_k} \frac{\partial}{\partial y_k} \tag{13.7.20}$$

In cylindrical coordinates the physical acceleration components become:

$$\begin{aligned}
a_R &= \frac{\partial v_R}{\partial t} + v_R \frac{\partial v_R}{\partial R} + \frac{v_\theta}{R}\frac{\partial v_R}{\partial \theta} + v_z \frac{\partial v_R}{\partial z} - \frac{v_\theta^2}{R} \\
a_\theta &= \frac{\partial v_\theta}{\partial t} + v_R \frac{\partial v_\theta}{\partial R} + \frac{v_\theta}{R}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_R v_\theta}{R} \\
a_z &= \frac{\partial v_z}{\partial t} + v_R \frac{\partial v_z}{\partial R} + \frac{v_\theta}{R}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}
\end{aligned} \tag{13.7.21}$$

In spherical coordinates the physical acceleration components become:

$$\begin{aligned}
a_r &= \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \\
a_\theta &= \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{\cot\theta}{r} v_\phi^2 \\
a_\phi &= \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi}{r}\cot\theta
\end{aligned} \tag{13.7.22}$$

The constitutive equations of the Newton fluid in terms of physical components are:

$$T(ij) = -p(\rho,\theta)\,\delta_{ij} + 2\mu D(ij) + \left(\kappa - \frac{2\mu}{3}\right) D(kk)\,\delta_{ij} \tag{13.7.23}$$

The physical components $T(ij)$ and $D(ij)$ are given by the formulas (13.5.10) and (13.3.33) respectively.

In order to obtain the physical components of the Navier-Stokes equations we need the expressions for $\nabla^2 \mathbf{v}$ and $\nabla(\nabla\cdot\mathbf{v})$ presented respectively by the formulas $(12.5.98)_2$ and (12.5.97). Then the Navier-Stokes equations become:

$$\begin{aligned}
a(i) = &-\frac{1}{\rho h_i} p_{,i} + b(i) + \frac{1}{\rho}\left(\frac{\mu}{3} + \kappa\right)\frac{1}{h_i}\left\{\frac{1}{h}\sum_k \left[\frac{h}{h_k} v(k)\right]_{,k}\right\}_{,i} \\
&+ \frac{\mu}{\rho}\left\{\nabla^2 v(i) + \sum_k \left[\frac{h_i}{h}\left(\frac{h h_{i,k}}{h_i^2 h_k^2}\right)_{,k} v(i)\right.\right. \\
&+ \left.\left(\frac{1}{h_i}\left(\frac{1}{h}\left(\frac{h}{h_k}\right)_{,k}\right)_{,i} - \frac{h_i}{h}\left(\frac{h h_{i,k}}{h_i^2 h_k^2}\right)_{,k}\right) v(k)\right] \\
&+ \left.\sum_k \left[\frac{2}{h_i^2 h_k^2} h_i \left(h_k h_{i,k}\, v(k)_{,i} - h_i h_{k,i}\, v(k)_{,k}\right)\right]\right\}
\end{aligned} \tag{13.7.24}$$

where:

$$\nabla^2() = \frac{1}{h}\sum_k \left[\frac{h}{h_k^2}(),k\right],k \tag{13.7.25}$$

The Euler equations for inviscid fluids are incorporated in these equations.
The thermal energy balance equation (6.3.14) becomes:

$$\rho\frac{D\varepsilon}{Dt} = -\frac{1}{h}\sum_k \left[\frac{h}{h_k}h(k)\right]_{,k} + T(ij)\,D(ij) \tag{13.7.26}$$

If Fourier's law applies, the thermal energy balance equation becomes:

$$\rho\frac{D\varepsilon}{Dt} = \frac{1}{h}\sum_k \left[\tilde{\kappa}\frac{h}{h_k^2}\theta_{,k}\right]_{,k} + T(ij)\,D(ij) \tag{13.7.27}$$

Problems

Problem 13.1. Derive the formulas (13.3.51, 13.3.52).

Problem 13.2. Derive (13.4.3).

Problem 13.3. Show that (13.4.18) follows from (13.4.17).

Problem 13.4. Derive the expression (13.4.18) for the convective derivatives of the components of an objective tensor of 2. order tensor **B** by applying the Oldroyd method described in connection with (13.4.19, 13.4.20).

Problem 13.5. Derive (13.4.23, 13.4.24).

Problem 13.6. Show that the two sets of physical components defined by (13.5.6) and (13.5.8) are related through (13.5.9).

Problem 13.7. Use (13.5.15) to develop the Cauchy equations in cylindrical coordinates and in spherical coordinates. The results are listed as (3.2.39, 3.2.40, 3.2.41) and (3.2.42, 3.2.43, 3.2.44).

Appendices

Appendix A Del-Operator

This appendix presents the formulas for the gradient of a scalar field $\alpha(\mathbf{r})$ and the divergence and rotation of a vector field $\mathbf{a}(\mathbf{r})$ in the two most commonly used orthogonal coordinate systems: cylindrical coordinates and spherical coordinates. A detailed development of the bases for these formulas is given in Sect. 12.5 by (12.5.61, 12.5.62) and (12.5.63).

CYLINDRICAL COORDINATES (R, θ, z).

$$\text{grad } \alpha \equiv \nabla\alpha = \mathbf{e}_R \frac{\partial \alpha}{\partial R} + \mathbf{e}_\theta \frac{1}{R}\frac{\partial \alpha}{\partial \theta} + \mathbf{e}_z \frac{\partial \alpha}{\partial z}$$

$$\text{div grad } \alpha \equiv \nabla^2\alpha = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \alpha}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 \alpha}{\partial \theta^2} + \frac{\partial^2 \alpha}{\partial z^2}$$

Let: $\mathbf{a} = a_R\mathbf{e}_R + a_\theta\mathbf{e}_\theta + a_z\mathbf{e}_z$. Then:

$$\text{div}\,\mathbf{a} \equiv \nabla\cdot\mathbf{a} = \frac{1}{R}\frac{\partial}{\partial R}(Ra_R) + \frac{1}{R}\frac{\partial a_\theta}{\partial \theta} + \frac{\partial a_z}{\partial z}$$

$$\text{rot}\,\mathbf{a} \equiv \nabla\times\mathbf{a} = \left[\frac{1}{R}\frac{\partial a_z}{\partial \theta} - \frac{\partial a_\theta}{\partial z}\right]\mathbf{e}_R + \left[\frac{\partial a_R}{\partial z} - \frac{\partial a_z}{\partial R}\right]\mathbf{e}_\theta + \left[\frac{1}{R}\frac{\partial}{\partial R}(Ra_\theta) - \frac{1}{R}\frac{\partial a_R}{\partial \theta}\right]\mathbf{e}_z$$

SPHERICAL COORDINATES (r, θ, ϕ).

$$\text{grad } \alpha \equiv \nabla\alpha = \mathbf{e}_r \frac{\partial \alpha}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial \alpha}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r\sin\theta}\frac{\partial \alpha}{\partial \phi}$$

$$\text{div grad } \alpha \equiv \nabla^2\alpha = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \alpha}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \alpha}{\partial \theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2 \alpha}{\partial \phi^2}$$

Let: $\mathbf{a} = a_r \mathbf{e}_r + a_\theta \mathbf{e}_\theta + a_\phi \mathbf{e}_\phi$. Then:

$$\operatorname{div}\mathbf{a} \equiv \nabla \cdot \mathbf{a} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 a_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, a_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial \phi}$$

$$\operatorname{rot}\mathbf{a} \equiv \nabla \times \mathbf{a} = \frac{1}{r\sin\theta}\left[\frac{\partial a_z}{\partial \theta}\left(\sin\theta\, a_\phi\right) - \frac{\partial a_\theta}{\partial \phi}\right]\mathbf{e}_r + \left[\frac{1}{r\sin\theta}\frac{\partial a_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}\left(r a_\phi\right)\right]\mathbf{e}_\theta$$
$$+ \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r a_\theta\right) - \frac{1}{r}\frac{\partial a_r}{\partial \theta}\right]\mathbf{e}_\phi$$

Appendix B The Navier – Stokes Equations

The Navier-Stokes equations are the Cauchy equations of motion for linearly viscous fluids $\equiv$ Newtonian fluids. The equations are presented in Sect. 8.4.2, in general coordinate invariant form in (8.4.19) and in Cartesian coordinates by (8.4.20). Section 13.7.2 presents the Navier-Stokes equations in general curvilinear coordinates in (13.7.17). Section 13.7.3 presents The Navier-Stokes equations in orthogonal coordinates by (13.7.19, 13.7.20) and (13.7.24).

This appendix presents the Navier-Stokes equations for incompressible fluids in cylindrical coordinates and in spherical coordinates.

CYLINDRICAL COORDINATES (R, θ, z) (incompressible fluid)

$$\frac{\partial v_R}{\partial t} + v_R \frac{\partial v_R}{\partial R} + \frac{v_\theta}{R}\frac{\partial v_R}{\partial \theta} + v_z \frac{\partial v_R}{\partial z} - \frac{v_\theta^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial R} + \frac{\mu}{\rho}\left(\nabla^2 v_R - \frac{v_R}{R^2} - \frac{2}{R^2}\frac{\partial v_\theta}{\partial \theta}\right) + b_R$$

$$\frac{\partial v_\theta}{\partial t} + v_R \frac{\partial v_\theta}{\partial R} + \frac{v_\theta}{R}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_R v_\theta}{R} = -\frac{1}{\rho R}\frac{\partial p}{\partial \theta} + \frac{\mu}{\rho}\left(\nabla^2 v_\theta - \frac{v_\theta}{R^2} + \frac{2}{R^2}\frac{\partial v_R}{\partial \theta}\right) + b_\theta$$

$$\frac{\partial v_z}{\partial t} + v_R \frac{\partial v_z}{\partial R} + \frac{v_\theta}{R}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\nabla^2 v_z + b_z$$

where:

$$\nabla^2 = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

SPHERICAL COORDINATES (r, θ, ϕ) (incompressible fluid)

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}$$
$$= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\mu}{\rho}\left(\nabla^2 v_r - \frac{2v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2\cot\theta}{r^2} v_\theta - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \theta}\right) + b_r$$

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2}{r}\cot\theta$$
$$= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \frac{\mu}{\rho}\left(\nabla^2 v_\theta - \frac{v_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}\right) + b_\theta$$

$$\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} - \frac{v_\theta v_\phi}{r}\cot\theta$$
$$= -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial \phi} + \frac{\mu}{\rho}\left(\nabla^2 v_\phi - \frac{v_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial \phi}\right) + b_\phi$$

where:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

Appendix C Integral Theorems

Throughout this appendix we use a Cartesian coordinate system Ox. A place, i.e. a point in space, is defined by a place vector $\mathbf{r}$ and the Cartesian coordinates x_i, such that:

$$\mathbf{r} = x_i \mathbf{e}_i \quad , \quad \mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial x_i} \tag{C.1}$$

where $\mathbf{e}_i$ are the *base vectors of the Ox-system.*

A curve C in space is given by the place function $\mathbf{r}(s)$, where s is the arc length parameter. The integral of a function $f(\mathbf{r})$ along the curve from a place $\mathbf{r}_1 = \mathbf{r}(s_1)$ to a place $\mathbf{r}_2 = \mathbf{r}(s_2)$ is defined by:

$$\int_C f(\mathbf{r})\, ds \equiv \int_{s_1}^{s_2} f(\mathbf{r}(s))\, ds \tag{C.2}$$

A common form for the function $f(\mathbf{r})$ is $g(\mathbf{r})\, dx_i/ds$, and we write:

$$\int_C g(\mathbf{r})\, \frac{dx_i}{ds}\, ds = \int_C g(\mathbf{r})\, dx_i \tag{C.3}$$

Theorem C.1. *Integration Independent of the Integration Path*
Let $\mathbf{a}(\mathbf{r})$ *be a vector field and* $\mathbf{r}_1$ *and* $\mathbf{r}_2$ *two places in space. Then:*

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{a} \cdot d\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} a_i \, dx_i \quad \textit{is independent of the integration path} \tag{C.4}$$

between $\mathbf{r}_1$ *and* $\mathbf{r}_2$ *if and only if* $\mathbf{a}$ *may be expressed as the gradient of a scalar field* $\alpha(\mathbf{r})$*:*

$$\mathbf{a} = \nabla\alpha \tag{C.5}$$

Proof. (C.5) $\Rightarrow$ (C.4): From $\mathbf{a} = \nabla\alpha$ it follows that $\mathbf{a} \cdot d\mathbf{r} = \alpha_{,i}\ dx_i = d\alpha$, such that the integral from $\mathbf{r}_1$ to $\mathbf{r}_2$ becomes:

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} d\alpha = \alpha(\mathbf{r}_2) - \alpha(\mathbf{r}_1) \tag{C.6}$$

thus only dependent on the choice of the places $\mathbf{r}_1$ and $\mathbf{r}_2$.

Proof. (C.4) $\Rightarrow$ (C.5): Let $\mathbf{r}_1$ be a fixed place, $\mathbf{r}_2 = \mathbf{r} = x_i\mathbf{e}_i$, and $\mathbf{r}_3 = \mathbf{r}_2 + h\mathbf{e}_1$. Then the integral (C.4) is a scalar field $\alpha(\mathbf{r})$, $\mathbf{a} \cdot d\mathbf{r} = d\alpha$, and:

$$\int_{\mathbf{r}_1}^{\mathbf{r}_3} \mathbf{a} \cdot d\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}} \mathbf{a} \cdot d\mathbf{r} + \int_{\mathbf{r}}^{\mathbf{r}_3} \mathbf{a} \cdot d\mathbf{r} \quad \Rightarrow$$

$$\alpha(x_1 + h, x_2, x_3) - \alpha(\mathbf{r}_1) = [\alpha(x_1, x_2, x_3) - \alpha(\mathbf{r}_1)] + \int_{x_1}^{x_1+h} a_1 dx_1$$

From this result it follows that:

$$\left. \frac{\partial\alpha(x_1 + h, x_2, x_3)}{\partial h} \right|_{h=0} = \frac{\partial\alpha(\mathbf{r})}{\partial x_1} = a_1$$

which is easily generalized to (C.5).

For integration of a function $f(\mathbf{r})$ along a closed curve C in the x_1x_2-plane we introduce clockwise as positive direction when looking in the positive x_3-direction, and use the symbol:

$$\oint_C f(\mathbf{r})\, ds \tag{C.7}$$

For a surface integral of a function $f(\mathbf{r})$ over a region A in the x_1x_2-plane we use the symbols:

$$\int_A f(\mathbf{r})\, dA = \int_A f(x)\, dx_1\, dx_2 \tag{C.8}$$

dA represents $dx_1 dx_2$ or a differential of area in a curvilinear coordinate system, discussed below.

Theorem C.2. *Gauss' Integral Theorem in a Plane*

Let A be a surface in the x_1x_2-plane and $\mathbf{n}$ *a unit normal to the curve C bordering the surface A. The unit normal* $\mathbf{n}$ *lies in the x_1x_2-plane and pointing outward from the surface. Then, for any function $f(x_1,x_2)$:*

$$\oint_C f n_\alpha = \int_A f,_\alpha \, dA \tag{C.9}$$

Proof. The proof will be given for the index value $\alpha = 1$. First we consider the case in Fig. C.1, where a straight line parallel to the x_1-axis only intersects the curve C in two points $\bar{x}_1(x_2)$ and $\bar{\bar{x}}_1(x_2)$. Then:

$$\int_A f,_1 \, dA = \int_{P_1}^{P_2} \left[\int_{\bar{x}_1}^{\bar{\bar{x}}_1} f,_1 \, dx_1 \right] dx_2 = \int_{P_1}^{P_2} [f(\bar{\bar{x}}_1(x_2),x_2) - f(\bar{x}_1(x_2),x_2)] \, dx_2$$
$$= \oint_C f(x_1,x_2) \, dx_2 = \oint_C f n_1 \, ds$$

If a straight line parallel to the x_1-axis intersects C in more than two points, the surface A is divided into parts, as shown in Fig. C.2, each of which satisfies the condition of only two points of intersection. The contributions from the lines dividing lines add up to zero. Thus in general:

$$\int_A f,_1 \, dA = \oint_C f n_1 \, ds$$

Let the function $f(x_1,x_2)$ be replaced by a vector field $\mathbf{a}(x_1,x_2)$. Then Theorem C.2 implies that:

$$\int_A a_\alpha,_\alpha \, dA = \oint_C a_\alpha n_\alpha \, ds \quad \Leftrightarrow \quad \int_A \operatorname{div} \mathbf{a} \, dA = \oint_C \mathbf{a} \cdot \mathbf{n} \, ds \tag{C.10}$$

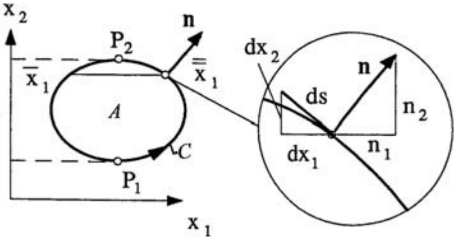


Fig. C.1 Surface A in the X_1 X_2-plane bordered by the curve C

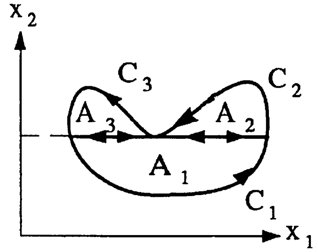


Fig. C.2 Surface $A = A_1 + A_2 + A_3$ bordered by the curve $C = C_1 + C_2 + C_3$

The result represents the *divergence theorem in a plane*. Equation (C.9) also provides a proof for the *Stokes' theorem in a plane*:

$$\int_A (\nabla \times \mathbf{a}) \cdot \mathbf{e}_3 \, dA = \oint_C \mathbf{a} \cdot \mathbf{t} ds \tag{C.11}$$

where $\mathbf{e}_3$ is the base vector in the x_3-direction, i.e. a unit normal to the plane, and $\mathbf{t}$ is the tangent vector to the curve C. To see that (C.11) follows from (C.9), we write:

$$(\nabla \times \mathbf{a}) \cdot \mathbf{e}_3 = a_{2,1} - a_{1,2} \text{ and } \mathbf{a} \cdot \mathbf{t} = a_\alpha \frac{dx_\alpha}{ds} = a_2 n_1 - a_1 n_2$$

By this result and (C.9) it follows that the left-hand side and the right-hand side in (C.11) are equal.

A place vector given as a function of two variables u_1 and u_2:

$$\mathbf{r} = \mathbf{r}(u_1, u_2) \quad , \quad x_i = x_i(u_1, u_2) \tag{C.12}$$

describes a surface A in space. The parameters u_1 and u_2 are called surface parameters or *surface coordinates*. If one surface coordinate is given a definite value, the function $\mathbf{r}(u_1, u_2)$ describes space curves on the surface called *coordinate curves*: u_1 =constant gives u_2-curves, and u_2 =constant gives u_1-curves. Tangent vectors to the coordinate curves, defined by:

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial u_\alpha} = \frac{\partial \mathbf{r}}{\partial x_i} \frac{\partial x_i}{\partial u_\alpha} = \mathbf{e}_i \frac{\partial x_i}{\partial u_\alpha} \tag{C.13}$$

are the *base vectors to the coordinate system u*. Line elements along the coordinate curves are defined by the vectors:

$$d\mathbf{s}_\alpha = \mathbf{a}_\alpha du^\alpha \tag{C.14}$$

du^1 and du^2 are ordinary differentials of the coordinates u_α. The reason why super indices are used, is explained in detail in Chap. 12.

The *element of area* is represented by the vector, see Fig. C.3:

$$d\mathbf{A} = d\mathbf{s}_1 \times d\mathbf{s}_1 = \mathbf{n} dA \tag{C.15}$$

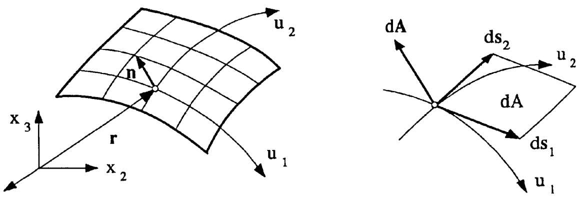


Fig. C.3 Surface A in space. Unit normal **n**. Coordinate curves: u_1-curve and u_2-curve. Line elements ds_2. Element of area dA

dA is the area differential and **n** is the unit normal to the surface A:

$$dA = |\mathbf{a}_1 \times \mathbf{a}_2| du^1 du^2 \quad , \quad \mathbf{n} = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|} \tag{C.16}$$

The direction of **n** defines the positive side A^+ of the surface. The *fundamental parameters of first order* to the surface are defined by the elements:

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta = \frac{\partial x_i}{\partial u_\alpha} \frac{\partial x_i}{\partial u_\beta} \tag{C.17}$$

Because:

$$\cos(\mathbf{a}_1, \mathbf{a}_2) = \frac{\mathbf{a}_1 \cdot \mathbf{a}_2}{|\mathbf{a}_1||\mathbf{a}_2|} = \frac{a_{12}}{\sqrt{a_{11}a_{22}}} \quad , \quad |\mathbf{a}_1 \times \mathbf{a}_2| = \sqrt{a_{11}a_{22}} \sin(\mathbf{a}_1, \mathbf{a}_2)$$

we obtain the result:

$$|\mathbf{a}_1 \times \mathbf{a}_2| = \sqrt{a_{11}a_{22} - (a_{12})^2} = \sqrt{\alpha} \quad \text{(no summation)} \tag{C.18}$$

where:

$$\sqrt{\alpha} = \det(a_{\alpha\beta}) \tag{C.19}$$

Then from (C.16):

$$dA = \sqrt{\alpha}\, du^1\, du^2 \tag{C.20}$$

Let A be a surface in the x_1x_2-plane, see Fig. C.4, and J the Jacobian to the mapping $x_\alpha(u_1, u_2)$, i.e.:

$$J = \det\left(\frac{\partial x_\alpha}{\partial u_\beta}\right) = \det(a_{\alpha\beta}) = \sqrt{\alpha} \tag{C.21}$$

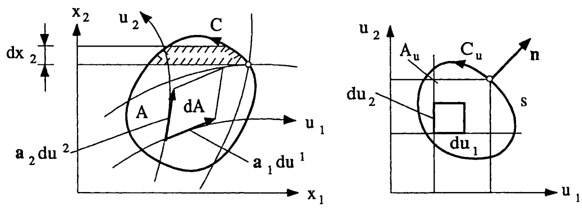


Fig. C.4 Surface A in the X_1 X_2-plane. Image A_u of A in the "Cartesian" u-system

Then:

$$dA = \sqrt{\alpha}\, du^1\, du^2 = J\, du^1\, du^2 \tag{C.22}$$

It will now be shown that the area of the surface is given by:

$$A = \int_A dA = \int_{A_u} J\, du^1\, du^2 \tag{C.23}$$

where A_u is the image of the area A in the "Cartesian" u-system as shown in Fig. C.4.

We introduce the curve C surrounding the area A in the x_1x_2-plane, the image C_u of C in the "Cartesian" u-system as shown in Fig. C.4, and the arc length s of C_u in the "Cartesian" u-system. Then from Fig. C.4:

$$A = \oint_C x_1\, dx_2$$

When the arc length parameter s in the u-system is introduced, we obtain:

$$A = \oint_C x_1\, dx_2 = \oint_{C_u} \left[x_1 \frac{\partial x_2}{\partial u_\alpha} \frac{du_\alpha}{ds} \right] ds = \oint_{C_u} \left[-x_1 \frac{\partial x_2}{\partial u_1} n_2 + -x_1 \frac{\partial x_2}{\partial u_2} n_1 \right] ds$$

Applying Theorem C.2, we get:

$$A = \int_{A_u} \left[-\frac{\partial x_1}{\partial u_2} \frac{\partial x_2}{\partial u_1} + \frac{\partial x_1}{\partial u_1} \frac{\partial x_2}{\partial u_2} \right] du^1\, du^2 = \int_{A_u} J\, du^1\, du^2$$

The surface A is subdivided into small parts A_n with δ as the maximum diameter, and x_n as an arbitrarily chosen point in A_n. The general definition of the double integral of a function $f(x)$ on a surface A is given by:

$$\int_A f(x)\, dx_1\, dx_2 = \lim_{\delta \to 0} \sum_n f(x_n) A_n \tag{C.24}$$

This definition will now be used to show that:

$$\int_A f\, dA = \int_A f(x)\, dx_1\, dx_2 = \int_{A_u} f(x(u))\, J\, du_1\, du_2 \tag{C.25}$$

Let $f_{\min}$ be the minimum value and $f_{\max}$ the maximum value of $f(x)$ in A_n. Then it follows that:

$$f_{\min} A_n = f_{\min} \int_{A_n} J\, du_1\, du_2 \leq \int_{A_n} f(x(u))\, J\, du_1\, du_2 \leq f_{\max} \int_{A_n} J\, du_1\, du_2 = f_{\max} A_n$$

and:

$$\sum_n f_{\min} A_n \leq \sum_n f(x_n) A_n \leq \sum_n f_{\max} A_n$$

This result and (C.24) show that (C.25) is true, because the sum in (C.24) must lie between the bounds:

$$\sum_n f_{\min} A_n \text{ and} \sum_n f_{\max} A_n$$

and these bounds approach each other as $\delta \to 0$.

Based on what has just been shown, it seems reasonable to define the area of a curved surface by the following integral.

$$A = \int_A dA = \int_{A_u} \sqrt{\alpha}\, du^1\, du^2 \qquad \text{(C.26)}$$

In addition to comply with the concept of area of plane surfaces, this definition of area satisfies, as we shall see, the natural condition that the integral expressing the area is coordinate invariant with respect both to space coordinates and to surface coordinates. From the definitions of the base vectors $\mathbf{a}_\alpha$, the fundamental parameters $a_{\alpha\beta}$, and α it follows directly that the integral on the right-hand side in (C.26) is coordinate invariant with respect to space coordinates. To see that the integral also is coordinate invariant with respect to surface coordinates, we start by introducing alternative surface coordinates $\bar{u}_\alpha$ on the surface. Then we find:

$$\bar{a}_{\alpha\beta} = \frac{\partial x_i}{\partial \bar{u}_\alpha}\frac{\partial x_i}{\partial \bar{u}_\beta} = \frac{\partial u_\gamma}{\partial \bar{u}_\alpha}\frac{\partial u_\lambda}{\partial \bar{u}_\beta} a_{\gamma\lambda} \quad , \quad \bar{\alpha} = \det(\bar{a}_{\alpha\beta}) = \underline{J}^2 \alpha \qquad \text{(C.27)}$$

where $\underline{J}$ is the *Jacobian* to the mapping $u_\gamma(\bar{u})$:

$$\underline{J} = \det\left(\frac{\partial u_\gamma}{\partial \bar{u}_\alpha}\right)$$

It now follows from (C.26) and (C.27) that:

$$A = \int_A dA = \int_{A_u} \sqrt{\alpha}\, du^1\, du^2 = \int_{A_{\bar{u}}} \sqrt{\alpha}\, J d\bar{u}^1\, d\bar{u}^2 = \int_{A_{\bar{u}}} \sqrt{\bar{\alpha}}\, d\bar{u}^1\, d\bar{u}^2$$

This result proves that the formula (C.26) is coordinate invariant with respect to surface coordinates.

The integral of $f(x)$ on the surface A is defined as:

$$\int_A f\, dA = \int_{A_u} f(x(u))\sqrt{\alpha}\, du^1\, du^2 \qquad \text{(C.28)}$$

which seems reasonable based on the result (C.25).

For the integral of $f(x)$ in the volume V we write:

$$\int_V f(x)\, dx_1\, dx_2\, dx_3 = \int_V f\, dV$$

The volume element dV may be represented by $dx_1dx_2dx_3$ or a volume differential in a convenient curvilinear coordinate system. We shall return to the last alternative in connection with Theorem C.8.

Theorem C.3. *Gauss' Integral Theorem in Space*

Let V be a volume with surface A and **n** *as an outward unit vector to A. Then for any field $f(x)$:*

$$\int_V f_{,i}\, dV = \int_A f n_i\, dA \tag{C.29}$$

Proof for $i = 3$. Consider first the case, Fig. C.5, where a straight line parallel to the x_3-axis intersects the surface A in only two points: $\bar{x}_3(x_1, x_2)$ and $\tilde{x}_3(x_1, x_2)$. The sets of points $\bar{x}_3$ and $\tilde{x}_3$ describe respectively the surfaces $\bar{A}$ and $\tilde{A}$, where $\bar{A} + \tilde{A} = A$. Then with $dV = dx_1dx_2dx_3$:

$$\int_V f_{,3}\, dV = \int_A \left(\int_{\bar{x}}^{\tilde{x}} f_{,3}\, dx_3 \right) dx_1\, dx_2 = \int_{\tilde{A}} f\, dx_1\, dx_2 - \int_{\bar{A}} f\, dx_1\, dx_2$$

From (3.2.24) we can deduce that:

$$dx_1\, dx_2 = n_3\, dA \text{ on } \tilde{A} \text{ and } dx_1\, dx_2 = -n_3\, dA \text{ on } \bar{A}$$

Hence we have found that:

$$\int_V f_{,3}\, dV = \int_A f n_3\, dA$$

If lines parallel with the x_3-axis intersect the surface A in more than two points, Fig. C.6, the volume V may be divided into parts that each satisfies the condition about only two points of intersection with lines parallel with the x_3-axis. In the sum of surface integrals the integrals over the boundary surfaces between the volume parts will cancel each other, since they appear with opposite signs.

Theorem C.4. *The Divergence Theorem in Space follows directly from the Gauss theorem, Theorem C.3, if the function $f(x)$ is replaced by the vector components*

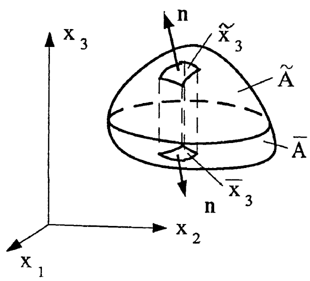


Fig. C.5 Volume V with surface $A = \bar{A} + \tilde{A}$

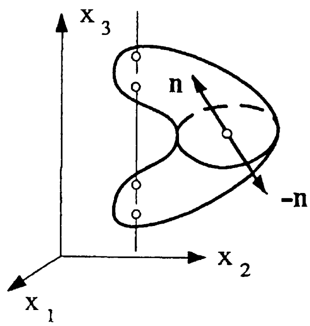


Fig. C.6 Lines parallel with X_3-axis intersect the surface in more than two points

$a_i(x)$ of a vector field $\mathbf{a}(x)$:

$$\int_V a_{i,i}\, dV = \int_A a_i n_i dA \quad \Leftrightarrow \quad \int_V div\,\mathbf{a}\, dV = \int_A \mathbf{a}\cdot\mathbf{n}\, dA \tag{C.30}$$

Theorem C.5. *Stokes' Theorem for a Curved Surface*

Let $\mathbf{b}(x)$ be any vector field and A a surface bordered by a curve C. Then:

$$\int_A (\mathrm{rot}\,\mathbf{b})\cdot\mathbf{n}dA \equiv \int_A (\nabla\times\,\mathbf{b})\cdot\mathbf{n}dA = \oint_C \mathbf{b}\cdot d\mathbf{r} \tag{C.31}$$

n *is unit vector to the surface. The positive direction of integration along C is determined such that the vector $\mathbf{n}\times d\mathbf{r}$ points to the side of C connected to A.*

Proof. Let:

$$\mathbf{n}\, dA = \mathbf{a}_1\times\mathbf{a}_2 du^1\, du^2 = \mathbf{e}_i\times\mathbf{e}_j\frac{\partial x_i}{\partial u_1}\frac{\partial x_j}{\partial u_2}du^1\, du^2 \quad , \quad \mathrm{rot}\,\mathbf{b} = e_{krs}b_{s,r}\,\mathbf{e}_k$$

where (C.13) has been applied. Then, since $(\mathbf{e}_i\times\mathbf{e}_j)\cdot\mathbf{e}_k = e_{ijk}$:

$$(\mathrm{rot}\,\mathbf{b})\cdot\mathbf{n}\, dA = e_{ijk}\, e_{krs}\, b_{s,r}\,\frac{\partial x_i}{\partial u_1}\frac{\partial x_j}{\partial u_2}\, du^1\, du^2$$

The Gauss' integration theorem in the u_1u_2-plane and the identity (2.1.17) now give:

$$\int_A (\mathrm{rot}\,\mathbf{b})\cdot\mathbf{n}dA = \oint_{\bar{C}}\left(b_i\frac{\partial x_i}{\partial u_2}\bar{n}_1 - b_j\frac{\partial x_j}{\partial u_1}\bar{n}_2\right) ds$$

where $\bar{n}_1$ and $\bar{n}_2$ are the components of the unit normal vector to the image $\bar{C}$ of the curve C in a "Cartesian" coordinate system u. We may refer to Fig. C.1 and write:

$$\bar{n}_1 = \frac{du^2}{ds} \quad , \quad \bar{n}_2 = -\frac{du^1}{ds}$$

Then we have obtained the result:

$$\int_A (\mathrm{rot}\,\mathbf{b}) \cdot \mathbf{n}\,dA = \oint_{\bar{C}} \left(b_i \frac{\partial x_i}{\partial u_2} \bar{n}_1 - b_i \frac{\partial x_i}{\partial u_1} \bar{n}_2 \right) ds = \oint_{\bar{C}} b_i \frac{\partial x_i}{\partial u_\alpha} du^\alpha$$
$$= \oint_{\bar{C}} b_i\, dx_i = \oint_{\bar{C}} \mathbf{b} \cdot d\mathbf{r}$$

which proves Theorem C.5.

Theorem C.6. *The Mean Value Theorem*

Let $f(x)$ and $g(x)$ be two continuous field functions and V a volume in space. Then at least one point $\bar{x}$ in V exists such that:

$$\int_V f(x)\, g(x)\, dV = f(\bar{x}) \int_V g(x)\, dV \tag{C.32}$$

$f(\bar{x})$ is called the mean value of $f(x)$ in the volume V. The theorem has analogous formulations and proofs for line and surface integrals.

Proof. Let $f_{\min}$ and $f_{\max}$ be the maximum and minimum values of $f(x)$ in V. Then:

$$f_{\min} \int_V g(x)\, dV \le \int_V f(x)\, g(x)\, dV = \varepsilon \int_V g(x)\, dV \le f_{\max} \int_V g(x)\, dV$$
$$\text{where} f_{\min} \le \varepsilon \le f_{\max}$$

Since $f(x)$ is a continuous function, a point $\bar{x}$ must exist such that $f(\bar{x}) = \varepsilon$, which proves the theorem.

Theorem C.7. *Let $f(x)$ and $g(x)$ be two continuous field functions. Then:*

$$\textit{If for any volume } V: \int_V f(x)\, dV = \int_V g(x)\, dV\,, \textit{ it follows that:} f(x) \equiv g(x) \tag{C.33}$$

The theorem applies with analogous formulations for line and surface integrals.

Proof. Let: $h(x) = f(x) - g(x)$. Then:

$$\int_V h(x)\, dV = 0 \text{ for any volume } V \tag{C.34}$$

If it is assumed that $h(x) \neq 0$, let us say greater than zero, in a point $\bar{x}$ in V, then according to the condition of continuity for the functions $f(x)$ and $g(x)$ the integrand $h(x)$ must be greater than zero also in a small region ΔV surrounding $\bar{x}$. This implies that:

$$\int_{\Delta V} h(x)\, dV > 0$$

in contradiction to (C.34). The supposition $h(x) \neq 0$ is thus impossible, and $h(x) = 0$, which means that $f(x) = g(x)$, and the theorem is proved.

Theorem C.8. *Change of Variable in the Volume Integral*

Let y represent a curvilinear coordinate system in space. The relationship between the coordinates y and the coordinates x in a Cartesian system of a space point is given by the one-to-one mapping:

$$y_i = y_i(x) \quad \Leftrightarrow \quad x_i = x_i(y)$$

It is assumed that the coordinates y_i are ordered such that the Jacobian to the mapping $x_i(y)$ is positive:

$$J \equiv \det\left(\frac{\partial x_i}{\partial y_j}\right) > 0 \tag{C.35}$$

Let $f(x)$ be a field function and V a volume in space, and V_y the mapping of V onto an orthogonal coordinate system y. Then:

$$\int_V f\, dV = \int_V f(x)\, dx_1\, dx_2\, dx_3 = \int_{V_y} f(x(y))\, J\, dy^1\, dy^2\, dy^3 \tag{C.36}$$

The volume element dV may alternatively be considered to be:

$$\begin{aligned} dV &= dx_1\, dx_2\, dx_3 \text{ in Caresian coordinates} \\ dV &= J\, dy^1\, dy^2\, dy^3 \text{ in curvilinear coordinates} \end{aligned} \tag{C.37}$$

Proof. Analogous to the two-dimensional plane case presented in Fig. C.4 where the area element dA alternatively is considered as $dA = dx_1 dx_2$ in Cartesian coordinates as in (C.8), or as $dA = J du_1 du_2$ in curvilinear coordinates as in (C.22), we may consider the volume element to alternatively be a orthogonal parallelepiped with sides dx_1, dx_2, and dx_3 in Cartesian coordinates, or the parallelepiped shown in Fig. C.7 in general curvilinear coordinates y and with sides given by the line elements $d\mathbf{s}_i$ along the coordinate curves:

$$d\mathbf{s}_i = \frac{\partial \mathbf{r}}{\partial y_i} dy^i = \frac{\partial x_k}{\partial y_i}\, dy^i\, \mathbf{e}_k \tag{C.38}$$

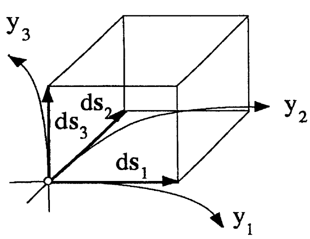


Fig. C.7 Volume element dV in curvilinear coordinates

The volume of the element shown in Fig. C.7 becomes:

$$dV = [d\mathbf{s}_1 d\mathbf{s}_2 d\mathbf{s}_3] = J\, dy^1\, dy^2\, dy^3 \tag{C.39}$$

We have shown that the integral of a function $f(x)$ over a plane surface may be given by either form C.8 or form C.28. Similarly we shall find that the two forms in (C.36) is true.

As an application of Theorem C.8 the transformation from Cartesian coordinates x to cylindrical coordinates (R, θ, z) will be demonstrated. The mapping is:

$$x_1 = R\cos\theta \quad , \quad x_2 = R\sin\theta \quad , \quad x_3 = z$$

and the Jacobian becomes:

$$J = \det\begin{pmatrix} \cos\theta & -R\sin\theta & 0 \\ \sin\theta & R\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = R$$

Hence:

$$\int\limits_V f(x)\, dx_1\, dx_2\, dx_3 = \int\limits_{V_y} f(x(y))\, R\, dR\, d\theta\, dz$$

Theorem C.9. *A body of a continuum has the volume V and surface A with outward unit normal $\mathbf{n}$. Let $f(\mathbf{r},t)$ be an intensive quantity per unit volume at the place $\mathbf{r}$ in V, and let $g(\mathbf{r},t,\mathbf{n})$ be an intensive property at the place $\mathbf{r}$ on A and defined per unit area of A. If then the following integral equation is true for any volume V with surface A:*

$$\int\limits_V f\, dV = \int\limits_A g\, dA \tag{C.40}$$

then:

$$g(\mathbf{r},t,\mathbf{n}) = g_i n_i \quad \textit{and} \quad f(\mathbf{r},t) = g_{i,i} \tag{C.41}$$

where g_i are the three field functions:

$$g_i = g(\mathbf{r},t,\mathbf{e}_i) \tag{C.42}$$

Proof. Let any volume V be subdivided into V_1 and V_2 by the surface A', see Fig. C.8. Equation (C.40) is applied to the volumes V, V_1, and V_2, and the contributions from V_1 and V_2 are subtracted from the contribution from V. The result is:

$$\int\limits_{A'} [g(\mathbf{r},t,\mathbf{n}) + g(\mathbf{r},t,-\mathbf{n})]\, dA = 0$$

Because V and thus A' may be chosen arbitrarily, Theorem C.7, modified for surface integrals, implies that:

$$g(\mathbf{r},t,\mathbf{n}) = -g(\mathbf{r},t,-\mathbf{n}) \tag{C.43}$$

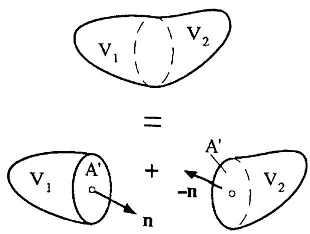


Fig. C.8 Volume V subdivided into V_1 and V_2 by the interface A

Now we choose for the volume V a tetrahedron as shown in Fig. C.9. It follows that:

$$3V = Ah = A_i h_i \text{ (no summation)} \quad \text{and } h_i = \frac{h}{n_i} \Rightarrow$$

$$A_i = An_i \quad \text{and} \quad V = \frac{Ah}{3} \tag{C.44}$$

The integral equation (C.40) applied to the tetrahedron yields:

$$\int_V f(\mathbf{r},t)\, dV = \int_A g(\mathbf{r},t,\mathbf{n})\, dA + \sum_i \int_{A_i} g(\mathbf{r},t,-\mathbf{e}_i)\, dA$$

Let f, g, and – g_i represent mean values over the volume V and the surfaces A and A_i respectively. Then:

$$fV = gA + g_i A_i$$

Here we use the relations (C.44), then divide by A and let $h \to 0$, and by (C.43) we have obtained the result:

$$g(\mathbf{r},t,\mathbf{n}) = g(\mathbf{r},t,\mathbf{e}_i)\, n_i \equiv g_i n_i$$

Hence the first part of (C.41) is proved. The result is substituted into (C.40) and the surface integral is transformed using Gauss' theorem:

$$\int_V f\, dV = \int_A g\, dA = \int_A g_i n_i\, dA = \int_V g_{i,i}\, dV \quad \Rightarrow \quad \int_V [f - g_{i,i}]\, dV = 0$$

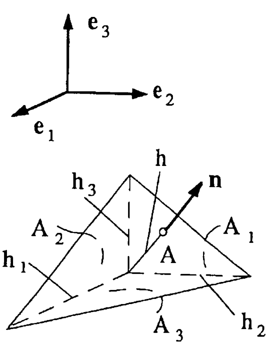


Fig. C.9 Tetrahedron of volume V and with surface $A_1 + A_2 + A_3 + A$

This integral l equation is true for any volume V. Hence: $f = g_{i,i}$ and the second part of (C.41) is proved.

The functions f and g may represent scalars, or components of vectors and tensors. The functions $g_i = g(\mathbf{r},t,\mathbf{e}_i)$ will always represent a tensor of order one higher than the order of the tensors represented by f and g. A special case of the theorem has already been applied to Euler's first axiom in Sect. 3.2.4 to obtain the Cauchy's stress theorem.

Theorem C.10. *A necessary and sufficient condition for a vector field* $\mathbf{a}(x,t)$ *to be irrotational, i.e.* $\nabla \times \mathbf{a} = \mathbf{0}$*, is that it is a* gradient vector field*:*

$$\nabla \times \mathbf{a}(x,t) = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{a}(x,t) = \nabla\phi(x,t) \tag{C.45}$$

The theorem also applies to tensor fields of higher order.

Proof of $\nabla \times \mathbf{a}(x,t) = \mathbf{0} \Leftarrow \mathbf{a}(x,t) = \nabla\phi(x,t)$*. Because* $\phi_{,jk} = \phi_{,kj}$ *and* $e_{ijk} = -e_{ikj}$*:*

$$\nabla \times \mathbf{a} = e_{ijk}\, a_{k,j}\, \mathbf{e}_i = e_{ijk}\, \phi_{,kj}\, \mathbf{e}_i = \mathbf{0}$$

Proof of $\nabla \times \mathbf{a}(x,t) = \mathbf{0} \Rightarrow \mathbf{a}(x,t) = \nabla\phi(x,t)$*. Let A be any surface bordered by the curve C. Then, according to Theorem C.5:*

$$\int_C \mathbf{a} \cdot d\mathbf{r} = \int_A (\nabla \times \mathbf{a}) \cdot \mathbf{n}\, dA = 0$$

which by theorem C.1 implies that a scalar field $\phi(x,t)$ *exists such that* $\mathbf{a}(x,t) = \nabla\phi(x,t)$*.*

Theorem C.11. *Material-Derivative of an Extensive Quantity*

Let $f(\mathbf{r},t)$ *be a place function representing an intensive physical quantity per unit mass, with* $F(t)$ *as the corresponding extensive quantity for a material body of volume* $V(t)$*, such that:*

$$F(t) = \int_{V(t)} f\rho\, dV \tag{C.46}$$

ρ *is the mass density of the material in the body. The material-derivative of the extensive quantity* $F(t)$ *may then be given by:*

$$\dot{F}(t) = \int_{V(t)} \dot{f}\rho\, dV \tag{C.47}$$

Proof. Section 3.1.3 presented a physical argument for the theorem. Here we shall use Theorem C.8 to present a somewhat more stringent mathematical proof of the theorem.

The motion of the body may be interpreted as a one-to-one mapping:

$$x_i = x_i(X,t) \quad \Leftrightarrow \quad X_i = X_i(x,t)$$

between the place coordinates X_i of the particles in the reference configuration K_o and the place coordinates x_i of the particles in the present configuration K. The integral in (C.46) is by Theorem B.8 transformed to:

$$F(t) = \int_{V_o} f\rho\, J\, dV_o \tag{C.48}$$

where J is the Jacobian to the deformation gradient $\mathbf{F}$. If we in particular choose $f = 1$ everywhere, then F represents the mass of the body, since ρ is mass per unit volume in K, while ρJ according to (C.48) is mass per unit volume in K_o. This means that the quantity ρJ is independent of time, and thus:

$$\frac{d}{dt}(\rho\, J) = 0$$

Because the volume V_o is independent of time, the material-derivative of the integral in (C.48) may be obtained by differentiation of the integrand. Hence:

$$\dot{F}(t) = \int_{V_o} \left[\dot{f}\rho\, J + f\frac{d}{dt}(\rho\, J)\right] dV_o = \int_{V_o} \dot{f}\rho\, J\, dV_o = \int_{V} \dot{f}\rho\, dV$$

This proves the Theorem 11.

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Symbols

$A, B, \ldots$	matrices
A_{ij}	tensor components
$\mathbf{A}, \mathbf{B}, \ldots$	tensors
$a, b, \ldots$	vector matrices
a_i	vector components
$\mathbf{a}, \mathbf{b}, \ldots$	vectors
$\mathbf{b}$	binormal of a space curve
$\mathbf{a}$	acceleration
$\mathbf{B}$	left deformation tensor
$\mathbf{b}$	body force
C	center of mass
$\mathbf{C}$	Green's deformation tensor, right deformation tensor
c	wave velocity, volume fraction
c_p, c_v	specific heat at constant pressure, and at constant density
$\mathbf{c}$	vorticity
$\mathbf{D}, D_{ij}$	rate of deformation tensor
E	modulus of elasticity, internal energy
$\mathbf{E}$	Green strain tensor
$e_{ijk}\ e_{\alpha\beta}$	permutation symbol
$\mathbf{e}$	unit tensor
$\mathbf{e}_i$	base vector
$\mathbf{F}$	deformation gradient
f_y	yield stress
f_u	tensile strength, ultimate stress
$\mathbf{f}, f$	force
g	gravitational force per unit mass
g_{ij}, g^{ij}	fundamental parameters of a coordinate system
$g_{iK}, g^{iK}\ g^i_K$	Euclidean shifters
$\mathbf{g}_i, \mathbf{g}^i$	base vectors and reciprocal base vectors
H	Heaviside unit step function
$\mathbf{H}$	displacement gradient tensor

h	specific enthalpy
$\mathbf{h}$	heat flux vector
I, I_p	second moment of area, polar moment of area
$\mathbf{I}$	inertia tensor
J	Jacobian, torsion constant
K	present configuration, kinetic energy, consistency parameter
K_o	reference configuration
$\mathbf{K}$	compliance tensor, flexibility tensor
k	heat conduction coefficient
$\mathbf{l}$	angular momentum
m	mass
$\mathbf{m}$	moment
$\mathbf{n}$	principal curve normal, unit normal to a space curve, unit surface normal
P	mechanical power
P^d	stress power
P^e	external virtual mechanical power
P^i	internal virtual mechanical power
p	pressure, thermodynamic pressure
$\mathbf{P}$	permutation tensor
Q	transformation matrix, supplied heat, volumetric flow
$\mathbf{Q}$	rotation tensor
q	heat flux
R	radius, gas constant
$\mathbf{R}$	rotation tensor
$\tilde{\mathbf{R}}$	rotation tensor for small deformations
r	specific heat source
$\mathbf{r}$	place vector
S	entropy, stiffness matrix, elasticity matrix
$\mathbf{S}$	stiffness tensor, elasticity tensor
	symmetric tensor, second Piola-Kirchhoff stress tensor
s	arc length, specific entropy
$\mathbf{T}$	Cauchy's stress tensor
$\mathbf{T}_{\mathrm{o}}$	first Piola-Kirchhoff stress tensor
t	time
$\mathbf{t}$	stress vector, unit tangent vector of a space curve
U	potential energy
$\mathbf{U}$	right stretch tensor
$\mathbf{u}$	displacement vector
$\mathbf{V}$	left stretch tensor
$\mathbf{v}$	velocity
W	work
w	work per unit volume
$\mathbf{W}$	rate of rotation tensor
$\mathbf{w}$	angular velocity

$x,\ x_i,\ X,\ X_i$	Cartesian coordinates
Z	Zener-Hollomon parameter
$\mathbf{z}$	rotation vector
$\alpha, \beta, \ldots$	scalars
α	linear coefficient of thermal expansion
$\alpha(t)$	creep function
α_g, α_e	glass compliance, equilibrium compliance
β	force potential
$\beta(t)$	relaxation function
β_g, β_e	glass modulus, equilibrium modulus
Γ	circulation
$\Gamma^i_{jk}, \Gamma_{ijk}$	Christoffel symbols
γ	shear strain
$\dot{\gamma}$	shear strain rate, magnitude of shear rate
δ_{ij}, δ^i_j	Kronecker delta
$\delta(t)$	Dirac delta function
$\varepsilon, \dot{\varepsilon}$	strain, longitudinal strain, strain rate
ε^p	plastic strain
$-\varepsilon'$	thermal strain
$\varepsilon, \bar{\varepsilon}$	specific internal energy, Helmholtz free specific energy
$\varepsilon_v, \dot{\varepsilon}_v$	volumetric strain, volumetric strain rate
η	modulus of elasticity
$\tilde{\eta}$	viscosity
$\eta(\dot{\gamma})$	viscosity function
$\eta_E(\dot{\varepsilon})$	extensional viscosity, Trouton viscosity
$\eta_{EB}(\dot{\varepsilon}), \eta_{EP}(\dot{\varepsilon})$	biaxial extensional viscosity, planar extensional viscosity
θ	temperature, state variables
κ	bulk modulus, bulk viscosity, curvature of a space curve
λ	Lamé constant, relaxation time, retardation time, stretch
μ	shear modulus, viscosity, Lamé constant
$\mu, \tilde{\mu}$	viscosity
ν	Poisson's ratio, kinematic viscosity
ρ	density
σ	normal stress
$\sigma_M, \sigma_{\mathrm{oct}}$	Mises stress, octahedral normal stress
τ, τ_y	torsion of a space curve,shear stress, yield shear stress
τ_{yM}, τ_{yT}	Mises yield shear stress, Tresca yield shear stress
τ_{oct}	octahedral shear stress
Φ, ϕ	elastic energy, elastic energy per unit volume angle of twist
ϕ_c	complementary energy per unit volume
ϕ	force potential, strain potential, velocity potential
ψ	elastic energy per unit mass, warping function
ψ_1, ψ_2	primary and secondary normal stress coefficient
Ω	Prandtl's stress function

ω	deformation power per unit volume, angular frequency
Ψ	Airy's stress function
R, θ, z	cylindrical coordinates
r, θ, ϕ	spherical coordinates
I, II, III	principal invariants
0	zero matrix
$\mathbf{0}$	zero tensor
1	unit matrix
$\mathbf{1}$	unit tensor

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