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Exam in TKT4150 Biomechanics and TTK 4170 Modelling and identification of biological systems

December 9, 2014
Duration: kl. 09.00-13.00
No printed or handwritten aids are permitted (D). Approved calculators are permitted.

Deadline for examination results: January 9, 2015

Exercise 1

1. Consider the elongation of a uni-axial bar expressed by the transformation $x = (1+t)X$, where x denotes Eulerian coordinates and X Lagrangian coordinates (see Fig. 1). The bar is exposed to a temperature elevation given by $T = X t^2$, and the corresponding rate of change of temperature may be expressed by the material derivative of the temperature in Lagrangian coordinates:

$$\dot{T} = 2t X \tag{1}$$

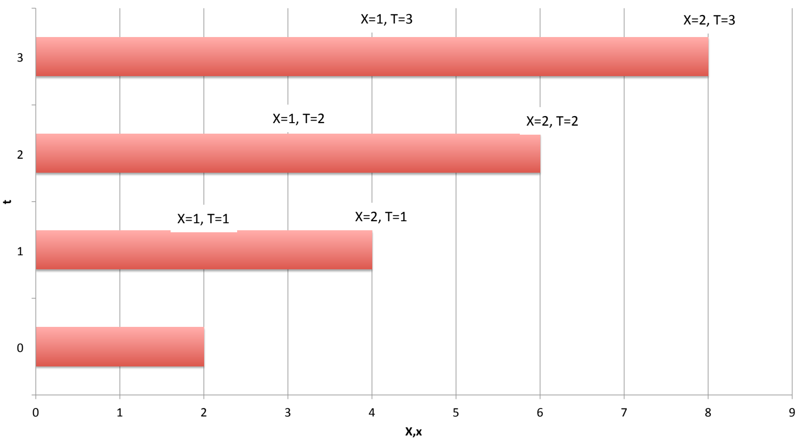

Figure 1: Elongation of uni-axial bar.

Compute the rate of change in temperature (i.e. material derivative) in Eulerian coordinates and compare with the expression in Lagrangian coordinates Eq. (1).

2. The strain rate matrix is defined by:

$$D_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right) \tag{2}$$

   (a) Determine the strain rate matrix $[D_{ij}]$ for the velocity field:

$$v_1 = \frac{\alpha x_1}{t - t_0}, \qquad v_2 = -\frac{\alpha x_2}{t - t_0} \tag{3}$$

   where α and t_0 are constants.

   (b) Argue for whether or not the flow field represents incompressible flow conditions.

3. Consider a cylinder viscometer as illustrated in Fig. 2.

   (a) Make appropriate assumptions of the velocity field in the fluid and show that the strain rate matrix may be expressed as:

$$\mathbf{D} \;=\; \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\dot{\gamma}}{2}$$

   and provide the expression for the strain rate $\dot{\gamma} = 2D_{12}$.

   (b) Express the shear stress τ on the inner cylinder in Fig. 2 by the imposed torque T.

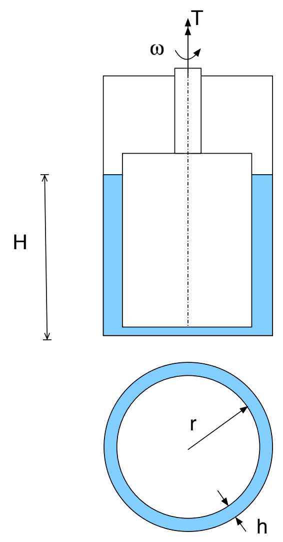

Figure 2: Cylinder viscometer. Sideview (upper panel) and topview (bottom panel). The fluid has the cyan color.

   (c) Assume that the fluid is Newtonian and incompressible at iso-thermal conditions, and thus may be expressed by the constitutive relation:

$$\mathbf{T} = -p\,\mathbf{1} + 2\mu\mathbf{D} \qquad \Leftrightarrow \qquad T_{ij} = -p\,\delta_{ij} + 2\mu\,D_{ij} \tag{4}$$

Explain how the dynamic viscosity μ may be found from the measurements with the cylinderviscometer.

Exercise 2

The governing equations for 1D flow in a compliant pipe may be presented as:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{5a}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) = -\frac{A}{\rho}\frac{\partial p}{\partial x} + \frac{\pi D}{\rho}\tau \tag{5b}$$

1. Explain the symbols in Eq. (5) and the physical meaning of each equation.
2. The characteristic impedance Z_c, is defined as the ratio of pressure and flow for a unidirectional wave. Make appropriate simplifications of Eq. (5) and show that:

$$Z_c = \frac{\rho c}{A} \tag{6}$$

3. How can the characteristic impedance be estimated Z_c?
4. The solutions to the linearized, inviscid form of Eq. (5) flow in a compliant pipe may be written:

$$p = p_f + p_b \qquad \text{and} \qquad Q = Q_f + Q_b \tag{7}$$

   Introduce the characteristic impedance Z_c and derive expressions for the separated waves p_f and p_b.

Exercise 3

1. In Fig. 3 an electrical circuit diagram of the three element Windkessel model is shown. What does the different parameters p, q, C, Z and R represent?

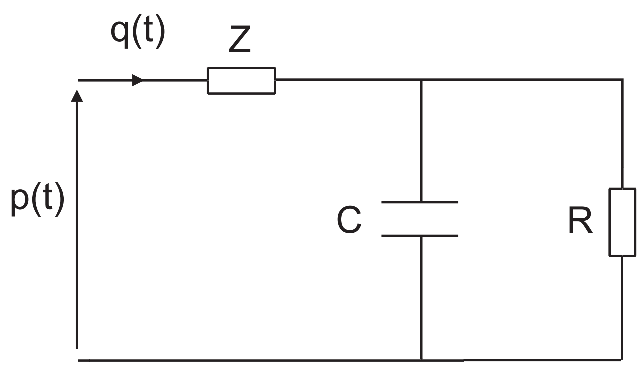

Figure 3: Three-element Windkessel model.

2. The differential equation for this system can be written as

$$\frac{dp}{dt} = -\frac{1}{RC}p + \left(\frac{1}{C} + \frac{Z}{RC}\right)q + Z\frac{dq}{dt}. \tag{8}$$

   Discretize eq.(8) by a first order Taylor expansion and show that the result can be written:

$$p_{k+1} = \alpha_1 p_k + \alpha_2 q_k + \alpha_3 q_{k+1}$$

   where k and $k+1$ denotes different time steps in the discretization. Define the parameters α_1, α_2 and α_3 in relation to the parameters R, C and Z.

3. Explain the difference between *step-wise* and *ballistic* estimation methods. How can the two be efficiently combined?

4. Assume you have performed a set of measurements, $\mathbf{p} = [p_1, \cdots, p_N]^T$ and $\mathbf{q} = [q_1, \cdots, q_N]^T$ and wish to perform a parameter fit of the model values to the measurements. Assume that the measurements are cyclic, $p_{N+1} = p_1$ and $q_{N+1} = q_1$.

   Based on the results in 2., express the model predictions by:

$$\hat{\mathbf{p}} = A\boldsymbol{\alpha},$$

   where $\hat{\mathbf{p}}$ is the model pressure at time $k+1$ and $\boldsymbol{\alpha}$ is the model's parameter vector.

   Find the least squares estimate of the parameter vector, $\hat{\boldsymbol{\alpha}}$. Is this estimation method *step-wise* or *ballistic*?

5. Given a quadratic system with perturbation matrix,

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{9}$$

   calculate the normalized sensitivity matrix $s^T s$ by means of column norms.

6. Set $a = 1$, $b = 2$, $c = -3$ and $d = 6$. Is the system observable? Is the estimation procedure robust?

7. Set $d = 1$. What effect has this on the robustness of the estimation procedure?

8. How would you use the sensitivity matrix to form a ballistic estimation approach?