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Examination paper for TKT4150 - Biomechanics and TTK4170 - Modelling and Identification of Biological Systems

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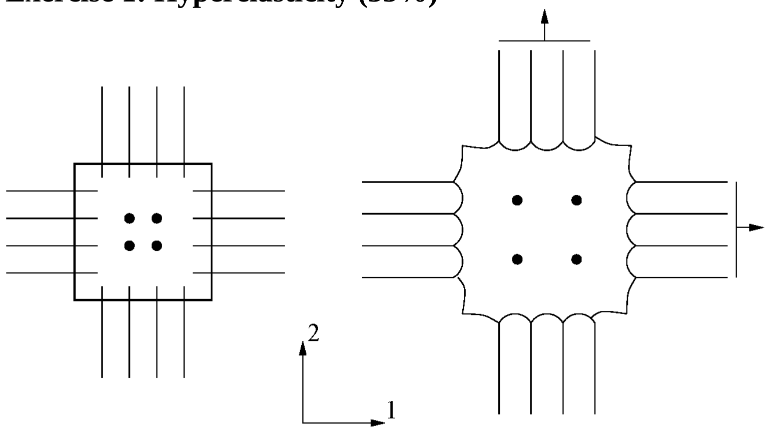
Exercise 1: Hyperelasticity (35%)



A square piece of blood vessel tissue subjected to biaxial stretching. Undeformed configuration Ω_0 on the left and deformed configuration Ω on the right.

A square piece of tissue cut out of a blood vessel is mechanically tested by means of in-plane biaxial stretching test. The specimen is in a state of plane stress. Away from the attachment sites (i.e. in the middle of the specimen) the stress and strain fields are homogeneous. We consider a particle originally at location (X_1, X_2, X_3) in the unloaded reference configuration Ω_0 . The deformations of this particle in the middle of the specimen may be expressed as:

$x_1 = \lambda_1 X_1, x_2 = \lambda_2 X_2, x_3 = \lambda_3 X_3$

where (x_1, x_2, x_3) are the coordinates of the particle in the current configuration Ω . 1 and 2–directions are in–plane and the 3–direction is the out of plane direction.

a) We assume that the tissue can be modeled as an isotropic hyperelastic material. The constitutive equations can be derived from the following strain energy function per unit volume: $\Psi(I_1, J) = c_1(I_1 - 3) - p(J - 1)$, where $I_1 = \mathrm{tr}\mathbf{C}$, $J = \det\mathbf{F}$, p is a Lagrange multiplier and c_1 is a material parameter.

   Derive the second Piola-Kirchhoff stress tensor $\mathbf{S}$ as a function of p , and c_1 and the right Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$.

   (Hint: $\mathbf{S} = 2\dfrac{\partial \Psi}{\partial \mathbf{C}}$, $\dfrac{\partial I_1}{\partial \mathbf{C}} = \mathbf{1}$, $\dfrac{\partial J}{\partial \mathbf{C}} = \dfrac{1}{2} J\mathbf{C}^{-1}$)

For the rest of the problem, the tissue is assumed to be incompressible ($\det\mathbf{F} = 1$) .

b) Express the matrices representing the deformation gradient $\mathbf{F} = \partial\mathbf{x} / \partial\mathbf{X}$, the right and left Cauchy Green tensors $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, respectively, in the (1, 2, 3) orthogonal coordinate system for the biaxial stretching test described above with respect to λ_1 and λ_2 .

c) Derive the Cauchy stress tensor $\mathbf{T}$ as a function of p and c_1 and the left Cauchy-Green tensor $\mathbf{B}=\mathbf{F}\mathbf{F}^{\mathrm{T}}$ (hint: $\mathbf{T}=\frac{1}{J}\mathbf{F}\mathbf{S}\mathbf{F}^{\mathbf{T}}$).

d) Show that $p=\frac{2c_1}{(\lambda_1\lambda_2)^2}$ (hint: use the plane stress condition $T_{i3}=0$, for i=1, 2, 3)

Now, the material is assumed to be anisotropic.

The components of the Cauchy stress tensor $\mathbf{T}$ can now be expressed as: $T_{ij}=2c_1B_{ij}+2k_1(I_4-1)\exp(k_2(I_4-1)^2)a_ia_j-2\frac{c_1}{(\lambda_1\lambda_2)^2}\delta_{ij}$, where c_1, k_1 and k_2 are material parameters, $I_4=n_iC_{ij}n_j$, $\mathbf{a}=\mathbf{F}\mathbf{n}$ where $\mathbf{n}$ is unit vector defined as $n=[\cos\theta \quad \sin\theta \quad 0]^{\mathrm{T}}$ in the (1,2,3) coordinate system.

e) Give the values of θ such as the directions 1 and 2 are principal directions of stress in the middle of the specimen subjected to the biaxial stretching test described above.

f) In this question $\lambda_1=\lambda$ and $\lambda_2=\lambda_3$ in Ω and $\theta=0$. Show that

$$T_{11}=2c_1(\lambda^2-\frac{1}{\lambda})+2k_1\lambda^2(\lambda^2-1)\exp[k_2(\lambda^2-1)^2]$$

And plot T_{11} and T_{22} against λ for $\lambda\in[1,1.25]$. Use c_1=0.01 MPa, k_1=0.1 Mpa and k_2=10.

Exercise 2: Flow in a compliant tubes (35%)

The governing equations for flow in a compliant tube are

$$\begin{aligned} \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} &= 0 \\ \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) &= -\frac{A}{\rho}\frac{\partial p}{\partial x} + \frac{\pi D}{\rho}\tau \end{aligned} \tag{1}$$

a) What basic principles do the two equations represent physically? What does each variable and parameter represent and what are their dimensions?

b) Let $C = \frac{\partial A}{\partial p}$ and show that

$$\begin{aligned} p &= p_0 f(x-ct) + p_0^* g(x+ct) \\ Q &= Q_0 f(x-ct) + Q_0^* g(x+ct) \end{aligned} \tag{2}$$

are general solutions to the governing equations under suitable assumptions, where c is a constant.

1. List the necessary assumptions and show that the general solutions satisfy the governing equations under these assumptions.
2. What is the physical meaning of C?
3. What is the physical meaning of c? Provide an expression for c in terms of the other variables and parameters.
4. Explain how the pressure and flow in (2) should be interpreted with respect the pressure and flow in the orignal equations (1)

c) The characteristic impedance Z_c is defined as the ratio between pressure and flow in a unidirectional wave. Show that $Z_c = \frac{\rho c}{A}$ for waves in a compliant tube.

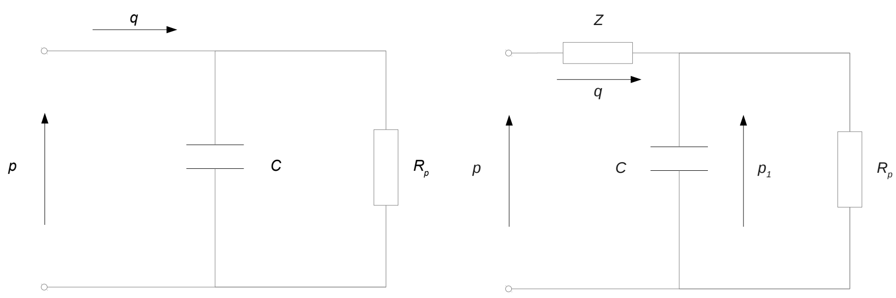
d) Describe how wave speed c could be estimated using the following measurements:

1. ultrasound measurements of area A and flow Q (Hint: Consider how changes in flow ΔQ could be related to a deviation in volume observed $\Delta V = \Delta x \Delta A$ in a section of artery with a length Δx.)
2. pressure measurments p in the arm and leg.

Discuss the differences and limitations of these two methods.

When would one be better than the other in assessing a person's health?

Exercise 3: Modelling and parameter estimation (30%)



(a) Two element model

(b) Three element model

Figure 1: Windkessel models

a) Figure 1a and 1b shows the two element Windkessel model (2WK) and the three element Windkessel model (3WK) respectively. Explain the parameters in the models.

b) How does the 3WK model improve upon the 2WK model, i.e. how does the model input impedances compare to the measured input impedance?

c) As humans grow older, their arteries get stiffer. Calculate the input impedance of the 2WK model, and use this to discuss how age influences the pressure-flow ratio.

d) The 3WK model is described by the following set of equations:

$$\frac{\partial p_1(t)}{\partial t} = -\frac{1}{RC}p_1(t) + \frac{1}{C}q(t) \tag{3}$$

$$p(t) = p_1(t) + Zq(t) \tag{4}$$

Show that a discretization of $p(t)$ kan be written as

$$p_{k+1} = \alpha_1 p_k + \alpha_2 q_k + \alpha_3 q_{k+1} \tag{5}$$

And further in matrix form as

$$\hat{\mathbf{p}} = \mathbf{A}\alpha \tag{6}$$

where

$$\alpha_1 = 1 - \frac{\Delta t}{RC} \tag{7}$$

$$\alpha_2 = (\frac{1}{C} + \frac{Z}{RC})\Delta t - Z \tag{8}$$

$$\alpha_3 = Z \tag{9}$$

e) Assume that you have a set of measurements $\mathbf{p} = [p_1, p_2, \ldots, p_N]$ and $\mathbf{q} = [q_1, q_2, \ldots, q_N]$. Find the (one-step) least squares estimate of the parameter vector α from equation 6. Show your calculations.

f) Define observabillity and robustness of parameter estimation. What is the sensitivity matrix (S-matrix) of a model? How is the elements of the S-matrix defined? How can this matrix be used to analyse observability and robustness of the parameter estimation?

g) Given the matrix

$$\begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}$$

where a is always in the range [0, 1]. Calculate the eigenvalues and the ratio between the largest and the smallest eigenvalue. What is the requirement on this ratio if we want a robust inverse?