Suggested solution:
PROBLEM SET 4

TKT4150 Biomechanics

1 Deformation measures

Assume we have the following homogeneous deformation

$$x_1 = \frac{2}{3}X_1 - 2X_2 + 2X_3 \tag{1}$$

$$x_2 = \frac{4}{3}X_1 + X_2 \tag{2}$$

$$x_3 = -\frac{4}{3}X_1 + X_3 \tag{3}$$

a) *Find the deformation gradient* **F** *and the deformation tensor* **C**.

Deformation gradient **F** and deformation tensor **C**:

$$\mathbf{F} = \frac{\partial \mathbf{r}}{\partial \mathbf{r}_0} \Rightarrow F_{ik} = \frac{\partial x_i}{\partial X_k} \Rightarrow \mathbf{F} = \frac{1}{3}\begin{bmatrix} 2 & -6 & 6 \\ 4 & 3 & 0 \\ -4 & 0 & 3 \end{bmatrix} \tag{4}$$

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} \Rightarrow C_{ij} = F_{ik}F_{kj} \Rightarrow \mathbf{C} = \frac{1}{9}\begin{bmatrix} 2 & -4 & 5 \\ 6 & 3 & 0 \\ -6 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & -6 & 6 \\ 4 & 3 & 0 \\ -4 & 0 & 3 \end{bmatrix} \tag{5}$$

$$\mathbf{C} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & -4 \\ 0 & -4 & 5 \end{bmatrix} \tag{6}$$

b) *Find the principal deformation values* ζ_k *and the principal directions* $\mathbf{n}_k$ *of the deformation tensor* **C**.

Principal values ζ_k and the principal directions $\mathbf{n}_k$:

$$(\mathbf{C} - \zeta\mathbf{1}) \cdot \mathbf{n}_k \Rightarrow det(\mathbf{C} - \zeta\mathbf{1}) = 0 \tag{7}$$

$$det \begin{bmatrix} 4-\zeta & 0 & 0 \\ 0 & 5-\zeta & -4 \\ 0 & -4 & 5-\zeta \end{bmatrix} = 0 \tag{8}$$

$$(4-\zeta) \cdot [(5-\zeta)^2 - 16] = 0 \tag{9}$$

$$(4-\zeta)(9-\zeta)(1-\zeta) = 0 \tag{10}$$

giving

$$\zeta_1 = 9 \tag{11}$$

$$\zeta_2 = 4 \tag{12}$$

$$\zeta_3 = 1 \tag{13}$$

The calculation of the direction $\mathbf{n}_k$ corresponding to each principal value, is only shown for the first prinicipal value:

By inserting $\zeta = \zeta_1$ into

$$\begin{bmatrix} 4-\zeta & 0 & 0 \\ 0 & 5-\zeta & -4 \\ 0 & -4 & 5-\zeta \end{bmatrix} \mathbf{n}_k = \mathbf{0} \tag{14}$$

we can extract the principal direction $\mathbf{n}_1$. This gives

$$\begin{bmatrix} 4-9 & 0 & 0 \\ 0 & 5-9 & -4 \\ 0 & -4 & 5-9 \end{bmatrix} \mathbf{n}_1 = \mathbf{0} \Rightarrow \begin{bmatrix} -5 & 0 & 0 \\ 0 & -4 & -4 \\ 0 & -4 & -4 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{15}$$

The first equation directly reveals that $n_1 = 0$. The second and third equations are, as expected, clearly linearly dependent. Thus, one of these equations provide no additional information, and will give a 'zero row' in the matrix. Therefore, only one equation for the two last unknowns:

$$-4n_2 - 4n_3 = 0 \tag{16}$$

$$\Rightarrow n_2 = -n_3 \tag{17}$$

By choosing $n_2 = 1$, we get

$$\mathbf{n}_1 = \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} \tag{18}$$

Normalization gives

$$\mathbf{n}_1 = \frac{1}{\sqrt{2}}\begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} \tag{19}$$

Similar calculations for the second and third direction gives:

$$\mathbf{n}_2 = \frac{1}{\sqrt{2}}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} \tag{20}$$

$$\mathbf{n}_3 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \tag{21}$$

Note: the 'correct' (right-hand-coordinates) direction of the third vector (-1 or +1) could be found by enforcing the cross product between the two first normal vectors.

c)*Evaluate the shear strain components γ_{ij} from Green's deformation tensor $\mathbf{C}$, when*

$$sin\gamma_{ij} = \frac{\mathbf{e}_i \cdot \mathbf{C} \cdot \mathbf{e}_j}{\sqrt{(\mathbf{e}_i \cdot \mathbf{C} \cdot \mathbf{e}_i)(\mathbf{e}_j \cdot \mathbf{C} \cdot \mathbf{e}_j)}} \tag{22}$$

Remember that tensor components, C_{ij}, of $\mathbf{C}$ are found by $C_{ij} = \mathbf{e}_i \cdot \mathbf{C} \cdot \mathbf{e}_j$.

The deformation matrix is symmetric, which means that $C_{ij} = C_{ji}$. From deformation tensor $\mathbf{C}$ found in a), we also see that $C_{12} = C_{13} = 0$. Therefore, only $C_{23} = -4$ gives a contribution to the shear strain γ_{ij}. $i = 2$ and $j = 3$ inserted in Equation 22 gives

$$sin(\gamma_{23}) = \frac{C_{23}}{\sqrt{C_{22}C_{33}}} \Rightarrow \gamma_{23} = arcsin\frac{-4}{\sqrt{5 \cdot 5}} \approx -53.1\,\mathrm{deg} \tag{23}$$

d) *Express the volumetric strain ε_v with Green's deformation tensor, and evaluate the answer.*

Volumetric strain ε_v from Greens deformation tensor $\mathbf{C}$:

$$\varepsilon_v = det\mathbf{F} - 1 = \sqrt{det\mathbf{C}} - 1 = \sqrt{36} - 1 = 5 \tag{24}$$

2 Hooke's law

Hooke's law

For a uni-axially, say in coordinate direction x_1, loaded material the stress-strain relation is given by $\epsilon_1 = \frac{\sigma_1}{\eta}$, and $\epsilon_2 = \epsilon_3 = \frac{\sigma_1 \nu}{\eta}$.

a) *By using superposition, show that we get these general linear elastic normal strains:*

$$\varepsilon_i = \frac{1+\nu}{\eta}\sigma_i - \frac{\nu}{\eta}tr\mathbf{T} \tag{25}$$

where ν is the Poisson ratio, η is the Young's modulus and $\mathbf{T}$ the stress matrix.

$$\sigma_1 \Rightarrow \varepsilon_1 = \frac{\sigma_1}{\eta} \tag{26}$$
$$\varepsilon_2 = -\frac{\sigma_1 \cdot \nu}{\eta} \tag{27}$$
$$\varepsilon_3 = -\frac{\sigma_1 \cdot \nu}{\eta} \tag{28}$$

$$\sigma_2 \Rightarrow \varepsilon_1 = -\frac{\sigma_2 \cdot \nu}{\eta} \tag{29}$$
$$\varepsilon_2 = \frac{\sigma_2}{\eta} \tag{30}$$
$$\varepsilon_3 = -\frac{\sigma_2 \cdot \nu}{\eta} \tag{31}$$

$$\sigma_3 \Rightarrow \varepsilon_1 = -\frac{\sigma_3 \cdot \nu}{\eta} \tag{32}$$
$$\varepsilon_2 = -\frac{\sigma_3 \cdot \nu}{\eta} \tag{33}$$
$$\varepsilon_3 = \frac{\sigma_3}{\eta} \tag{34}$$

By adding together all contributions to the first normal strain, we get

$$\varepsilon_1 = \frac{\sigma_1}{\eta} - \frac{\sigma_2}{\eta}\nu - \frac{\sigma_3}{\eta}\nu \tag{35}$$
$$\varepsilon_1 = \frac{\sigma_1}{\eta}(1+\nu) - (\sigma_1 + \sigma_2 + \sigma_3) \cdot \frac{\nu}{\eta} \tag{36}$$

This will be similar for the two other components as well. In general, this gives

$$\varepsilon_i = \frac{\sigma_i(1+\nu)}{\eta} - \frac{\nu}{\eta} tr(\mathbf{T}) \tag{37}$$

b) *The following shear stress-strain relation is assumed:*

$$E_{ij} = \frac{1+\nu}{\eta} T_{ij} \quad i \neq j \tag{38}$$

From this, establish the index notation tensor version of Hooke's law, with strain on left hand side, that incorporates both normal and shear components.

We then have the following relations between stress and strain:

$$E_{ij} = \begin{cases} \frac{1+\nu}{\eta} T_{ij} & i \neq j \\ \frac{1+\nu}{\eta} T_{ij} - \frac{\nu}{\eta} T_{kk} & i = j \end{cases} \tag{39}$$

These can be combined by introducing the Kroenecker-Delta:

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{40}$$

This finally gives

$$E_{ij} = \frac{1+\nu}{\eta} T_{ij} - \frac{\nu}{\eta} T_{kk} \delta_{ij} \tag{41}$$

c) *Show how the answer from b) can be reformulated to:*

$$T_{ij} = \frac{\eta}{1+\nu} \left(E_{ij} + \frac{\nu}{1-2\nu} E_{kk} \delta_{ij} \right) \tag{42}$$

We have

$$E_{ij} = \frac{1+\nu}{\eta} T_{ij} - \frac{\nu}{\eta} T_{kk} \delta_{ij} \tag{43}$$

By solving this w.r.t. T_{ij} we get

$$T_{ij} = \frac{\eta}{1+\nu} E_{ij} + \frac{\eta}{1+\nu} \frac{\nu}{\eta} T_{kk} \delta_{ij} \tag{44}$$

By summing over $i = j$ in Equation 43 we get

$$E_{ll} = \frac{1+\nu}{\eta} T_{ll} - \frac{\nu}{\eta} T_{kk} \delta_{ll} \tag{45}$$

Using that $\delta_{kk} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$, we get:

$$\begin{aligned} E_{kk} &= \frac{1+\nu}{\eta} T_{kk} - 3\frac{\nu}{\eta} T_{kk} && (46) \\ &= \frac{1+\nu-3\nu}{\eta} T_{kk} && (47) \\ &= \frac{1-2\nu}{\eta} T_{kk} && (48) \end{aligned}$$

This yields

$$T_{kk} = \frac{\eta}{1-2\nu} E_{kk} \tag{49}$$

By introducing this into Equation 44, we get

$$\begin{aligned} T_{ij} &= \frac{\eta}{1+\nu} E_{ij} + \frac{\eta}{1+\nu} \frac{\nu}{\eta} \frac{\eta}{1-2\nu} E_{kk} \delta_{ij} && (50) \\ &= \frac{\eta}{1+\nu} \left[E_{ij} + \frac{\nu}{1-2\nu} E_{kk} \delta_{ij} \right] && (51) \end{aligned}$$

d) *Let's introduce plane stress conditions. This means that*

$$T_{i3} = 0, \quad i = 1, 2, 3 \tag{52}$$

Show that Equation 42 now can be written

$$T_{\alpha\beta} = \frac{\eta}{1+\nu}\left(E_{\alpha\beta} + \frac{\nu}{1-\nu}E_{\rho\rho}\delta_{\alpha\beta}\right) \quad \alpha = 1, 2 \tag{53}$$

Since $T_{33} = 0$, $T_{kk} = T_{\alpha\alpha}$, thus we can replace the result from part b) with

$$E_{\alpha\beta} = \frac{1+\nu}{\eta}T_{\alpha\beta} - \frac{\nu}{\eta}T_{\gamma\gamma}\delta_{\alpha\beta}. \tag{54}$$

From this we note that the index used only counts two values, $\alpha = 1, 2$, thus $\delta_{\rho\rho} = 1 + 1 = 2$. By using this in Equation 45, we get

$$E_{\rho\rho} = \frac{1+\nu-2\nu}{\eta}T_{\rho\rho} = \frac{1-\nu}{\eta}T_{\rho\rho} \Rightarrow T_{\rho\rho} = \frac{\eta}{1-\nu}E_{\rho\rho} \tag{55}$$

Again, replacing T_{kk} with $T_{\rho\rho}$ in Equation 44, and yields

$$T_{\alpha\beta} = \frac{\eta}{1+\nu}\left[E_{\alpha\beta} + \frac{\nu}{1-\nu}E_{\rho\rho}\delta_{\alpha\beta}\right] \tag{56}$$

Alternatively note that $E_{kk} = E_{\alpha\alpha} + E_{33}$. $E_{33} = -\frac{\nu}{\eta}T_{\alpha\alpha}$ and thus $E_{kk} = \frac{1-\nu}{\eta}T_{\alpha\alpha} - \frac{\nu}{\eta}T_{\alpha\alpha} = \frac{1-2\nu}{\eta}T_{\alpha\alpha}$. Thus E_{kk} is equal in both formulations.

3 Navier's equations

The Cauchy equations read out

$$T_{ij,j} + \rho b_i = \rho\ddot{u}_i \tag{57}$$

Compatibility requires

$$E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{58}$$

By introducing Hooke's law and Green's strain tensor, the Cauchy equations are simplified. The result is called Navier's equations.

a) Develop Navier's equation from Equations 42, 57 and 58.

We have Hooke's law:

$$T_{ij,j} = \frac{\eta}{1+\nu}\left[E_{ij,j} + \frac{\nu}{1-2\nu}E_{kk,j}\delta_{ij}\right] \tag{59}$$

$E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ $E_{ii} = \frac{1}{2}\left(u_{i,i} + u_{i,i}\right) = u_{i,i}$ no summation over i. Thus $E_{ij,j} = \frac{1}{2}\left(u_{i,jj} + u_{j,ij}\right)$ and $E_{ii,j} = u_{i,ij}$

Introduce $E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ into this:

$$T_{ij,j} = \frac{\eta}{1+\nu}\left[\frac{1}{2}(u_{i,jj} + u_{j,ij}) + \frac{\eta}{1-2\nu}\underbrace{(u_{k,kj}\delta_{ij})}_{u_{k,ki}}\right] \tag{60}$$

Furthermore, we use this in Cauchy:

$$\begin{align}
\rho\ddot{u}_i &= \frac{\eta}{1+\nu}\left[\frac{1}{2}(u_{i,jj} + u_{j,ij}) + \frac{\nu}{1-2\nu}u_{k,ki}\right] + \rho b_i \tag{61}\\
\Leftrightarrow \frac{2(1+\nu)}{\eta}\rho\ddot{u}_i &= u_{i,jj} + u_{j,ij} + \frac{2\nu}{1-2\nu}\cdot u_{k,ki} + \frac{2(1+\nu)}{\eta}\rho b_i \tag{62}\\
\Leftrightarrow \frac{2(1+\nu)}{\eta}\rho(\ddot{u}_i - b_i) &= u_{i,jj} + u_{j,ij} + \frac{2\nu}{1-2\nu}u_{k,ki} \tag{63}\\
&= u_{i,kk} + u_{k,ik} + \frac{2\nu}{1-2\nu}u_{k,ik} \tag{64}\\
&= u_{i,kk} + u_{k,ik} + \frac{2\nu}{1-2\nu}u_{k,ik} \tag{65}\\
&= u_{i,kk} + u_{k,ik}\frac{1-2\nu+2\nu}{1-2\nu} \tag{66}\\
\frac{2(1+\nu)}{\eta}\rho(\ddot{u}_i - b_i) &= u_{i,kk} + u_{k,ik}\frac{1}{1-2\nu} \tag{67}
\end{align}$$

4 Hyperelasticity

A material is hyperelastic if there exists a potential function $\phi = \phi(E_{ij})$ *that satisfies* $T_{ij} = \frac{\partial\phi}{\partial E_{ij}}$*, i.e. the stress is defined by a potential function of the strain. An equivalent definition of hyperelasticity requires that the stress power* ω *may be derived from a scalar valued potential* $\phi(E_{ij})$*:*

$$\omega = \dot{\phi} = \frac{\partial\phi}{\partial E_{ij}}\dot{E}_{ij} \tag{68}$$

a) *Briefly explain the term stress power.*

The stress power of a body is the power per unit volume expended to perform deformation of the body.

For a *small* 1D deformation:

$$\omega = \sigma\dot{\varepsilon} \tag{69}$$

b) *Show why the two mentioned definitions are equivalent.*

Stress power is in general expressed by:

$$\omega = T_{ij}\dot{E}_{ij} \tag{70}$$

For a hyperelastic material:

$$\omega = \frac{\partial \phi}{\partial E_{ij}} \dot{E}_{ij} \tag{71}$$

Thus:

$$\omega = T_{ij}\dot{E}_{ij} = \frac{\partial \phi}{\partial E_{ij}} \dot{E}_{ij} \Rightarrow T_{ij} = \frac{\partial \phi}{\partial E_{ij}} \tag{72}$$

Because $\frac{\partial \phi}{\partial E_{ij}}$ is not dependent on the strain rate $\dot{E}_{ij}$, the stress power is a linear function of $\dot{E}_{ij}$. This implies that the stresses T_{ij} also are independent of $\dot{E}_{ij}$.

5 Piola-Kirchoff stress

a) *Briefly explain the main idea behind the different Piola-Kirchoff stress tensors.*

The first Piola-Kirchoff tensor relates the force acting in the deformed (current) configuration to a surface element in the undeformed (reference) configuration. First Piola-Kirchoff stress is equivalent to engineering stress. Non-symmetric tensor.

The second Piola-Kirchoff tensor relates the force mapped to the undeformed (reference) configuration to a surface element in the undeformed (reference) configuration. It has a less clear physical meaning. It is found to be a suitable stress measure, much because it gives the work or energy when multiplied with the Green strain tensor - i.e. it is the energy conjugate of Green strain. Symmetric tensor.

b) *Write down the tensor form relations between the Cauchy, the first and second Piola-Kirchoff stresses.*

$$\mathbf{T}_0 = J\mathbf{T}\mathbf{F}^{-T}, \quad J = det(\mathbf{F}) \tag{73}$$

$$\mathbf{S} = \mathbf{F}^{-1}\mathbf{T}_0 = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T} \tag{74}$$

c) *Show that the second Piola-Kirchhoff stress tensor is symmetric:*

$$\mathbf{S} = \mathbf{S}^T \tag{75}$$

This yields

$$\begin{aligned}
\mathbf{S} &= J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T} && (76)\\
\mathbf{F}\mathbf{S} &= J\mathbf{T}\mathbf{F}^{-T} && (77)\\
(\mathbf{F}\mathbf{S})^T &= J(\mathbf{T}\mathbf{F}^{-T})^T && (78)\\
\mathbf{S}^T\mathbf{F}^T &= J\mathbf{F}^{-1}\mathbf{T}^T && (79)\\
\mathbf{S}^T &= J\mathbf{F}^{-1}\mathbf{T}^T\mathbf{F}^{-T} && (80)\\
\mathbf{S}^T &= J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T} && (81)\\
& && (82)
\end{aligned}$$

By comparing this to Equation 74, this gives

$$\mathbf{S} = \mathbf{S}^T \tag{83}$$