## Suggested solution:
# PROBLEM SET 1

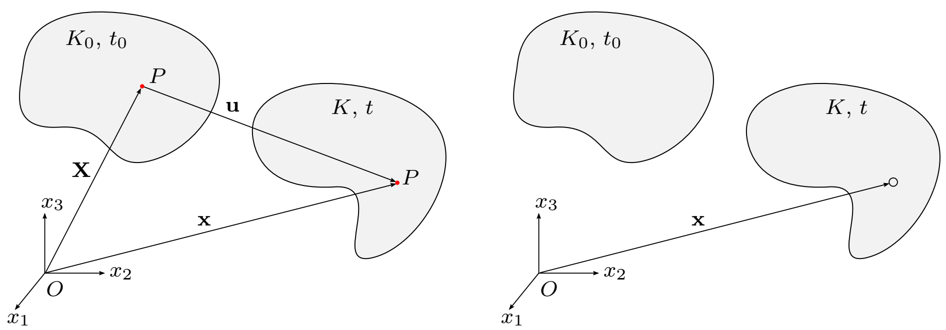
TKT4150 Biomechanics

### 1 Eulerian vs. Lagrangian coordinate systems

a) *Explain the difference between the two coordinate systems, and provide a simple drawing to support it.*

**Lagrangian coordinate system:** Particle coordinates. Used to label material points, which are followed on their journeys.

**Eulerian coordinate system:** Space coordinates. Used to locate a fixed point in space wrt. a fixed basis.




(a) In the Lagrangian coordinate system we follow the particle $P$.

(b) In the Eulerian coordinate system we study fixed points in space, here defined by the vector $\mathbf{x}$.

***Figure 1:*** *Body deformed and displaced from configuration $K_0$ at time $t_0$, to current configuration $K$ at time $t$.*

b) *Based on the different coordinate systems, we get different kinds of derivatives. Show how you can use the chain rule to establish the material derivative.*

The material derivative is the derivative of a particle property $\mathbf{f}(\mathbf{x}(\mathbf{X}, t))$ which originally was in location $\mathbf{X}$. We have the following:

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \tag{1}$$
$$\mathbf{f} = \mathbf{f}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}(\mathbf{X}, t), t) \tag{2}$$

For the material derivative of $\mathbf{f}$, we get from the chain rule

$$\dot{\mathbf{f}} = \frac{d\mathbf{f}}{dt} = \frac{\partial \mathbf{f}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial \mathbf{f}}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial \mathbf{f}}{\partial x_3}\frac{\partial x_3}{\partial t} = \frac{\partial \mathbf{f}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x_i}\frac{\partial x_i}{\partial t} \tag{3}$$

This can be rewritten as

$$\begin{aligned}
\dot{\mathbf{f}} &= \frac{\partial \mathbf{f}}{\partial t} + \frac{\partial x_1}{\partial t}\frac{\partial \mathbf{f}}{\partial x_1} + \frac{\partial x_2}{\partial t}\frac{\partial \mathbf{f}}{\partial x_2} + \frac{\partial x_3}{\partial t}\frac{\partial \mathbf{f}}{\partial x_3} && (4)\\
&= \frac{\partial \mathbf{f}}{\partial t} + v_1\frac{\partial \mathbf{f}}{\partial x_1} + v_2\frac{\partial \mathbf{f}}{\partial x_2} + v_3\frac{\partial \mathbf{f}}{\partial x_3} && (5)\\
&= \frac{\partial \mathbf{f}}{\partial t} + \left(v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2} + v_3\frac{\partial}{\partial x_3}\right)\mathbf{f} && (6)\\
&= \frac{\partial \mathbf{f}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{f} && (7)
\end{aligned}$$

## 2 Reynolds' transport theorem

a) *Explain the differences between an extensive and an intensive property.*

**Extensive property:** A property that is proportional to the amount of material that is in the system, i.e. it is a function of the volume or mass of the body.

**Intensive property:** A property that is independent of the amount of material that is in the system, i.e. it is independent of body volume or mass.
Distinction: *Density* when per unit volume and *specific property* when per unit mass.

b) *Derive Reynolds' transport theorem.*

We have the density $b$ of a property $B$:

$$B(t) = \int_{V(t)} b(\mathbf{r},t)dV \tag{8}$$

The definition of the derivative reads out:

$$\dot{B} = \lim_{\Delta t\to 0}\frac{B(t+\Delta t) - B(t)}{\Delta t} \tag{9}$$

Consider $B(t+\Delta t)$:

$$\begin{aligned}
B(t+\Delta t) &= \int_{V(t+\Delta t)} b(\mathbf{r},t+\Delta t)dV && (10)\\
&= \int_{V(t)} b(\mathbf{r},t+\Delta t)dV + \int_{\Delta V} b(\mathbf{r},t+\Delta t)dV && (11)
\end{aligned}$$

By reference to Figure 2, we can further rewrite the last term

$$\int_{\Delta V} b(\mathbf{r}, t + \Delta t) dV = \int_{A(t)} b(\mathbf{r}, t + \Delta t)(\mathbf{v} \cdot \mathbf{n}\Delta t) dA \tag{12}$$

This leads to the following expression for $B(t + \Delta t)$:

$$B(t + \Delta t) = \int_{V(t)} b(\mathbf{r}, t + \Delta t) dV + \int_{A(t)} b(\mathbf{r}, t + \Delta t)(\mathbf{v} \cdot \mathbf{n}\Delta t) dA \tag{13}$$

By insertion back into Equation 9, we get

$$\frac{B(t + \Delta t) - B(t)}{\Delta t} = \int_{V(t)} \frac{b(\mathbf{r}, t + \Delta t) - b(\mathbf{r}, t)}{\Delta t} dV + \int_{A(t)} b(\mathbf{r}, t + \Delta t)(\mathbf{v} \cdot \mathbf{n}) dA \tag{14}$$

Then, finally, by letting $\Delta t \to 0$, we get

$$\dot{B} = \int_{V(t)} \frac{\partial b}{\partial t} dV + \int_{A(t)} b(\mathbf{v} \cdot \mathbf{n}) dA \tag{15}$$

c) *Use Reynolds' transport theorem to establish the control volume version of the mass conservation equation.*

$$\dot{m}_{CV} = \frac{d}{dt} \int_{V(t)} \rho(\mathbf{r}, t) = \int_{V(t)} \frac{\partial \rho}{\partial t} dV + \int_{A(t)} \rho(\mathbf{v} \cdot \mathbf{n}) dA = 0 \tag{16}$$

By enforcing the divergence theorem;

$$\int_{V(t)} (\nabla \cdot \mathbf{F}) dV = \int_{A(t)} (\mathbf{F} \cdot \mathbf{n}) dA \tag{17}$$

we get

$$\dot{m}_{CV} = \int_{V} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) dV = 0 \tag{18}$$

which leads to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{19}$$

For constant density (not specified in this task):

$$\nabla \cdot \mathbf{v} = 0 \tag{20}$$

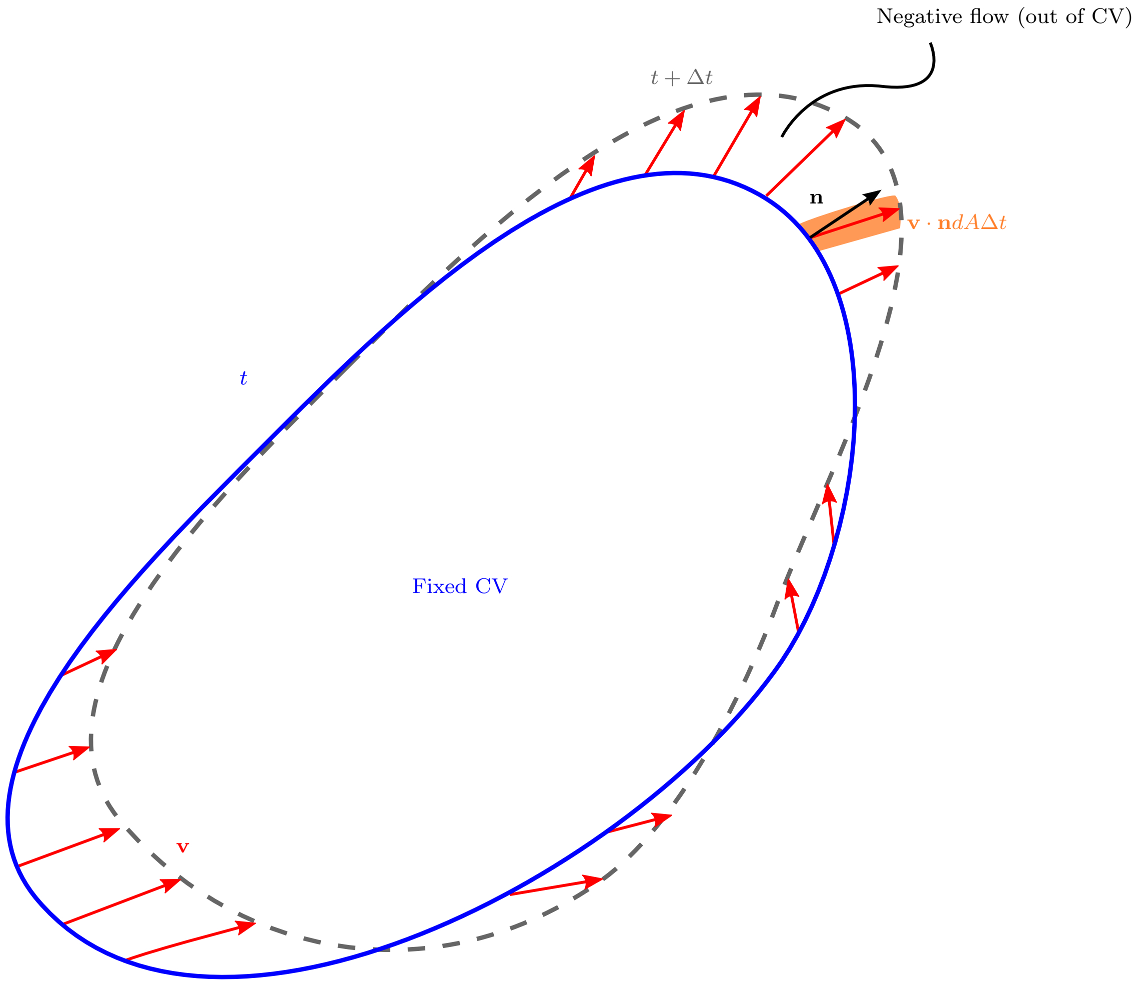



***Figure 2:*** *Flux over control surface of control volume.*

d) Consider Figure 3. Apply Reynolds' transport theorem applied to momentum to determine the pressure drop from $A_1$ to $A_2$. Show your control volume. You may neglect viscous effects along the wall of the pipe and assume laminar flow.

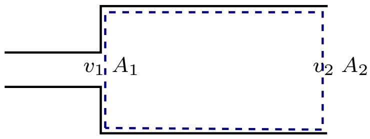



***Figure 3:*** *Fluid flows through a narrow pipe with area $A_1$ at velocity $v_1$ into a larger pipe with area $A_2$ with a velocity $v_2$ shortly after the junction.*

Choosing a control volume extending immediately after the opening to further down stream. We formulate the integral of interest

$$\dot{\mathbf{p}} = \int_V \frac{\partial \rho \mathbf{v}}{\partial t} dV + \int_A \rho \mathbf{v}\mathbf{v} \cdot \mathbf{n} dA \tag{21}$$

The assumption of steady flow requires that $\frac{\partial \rho \mathbf{v}}{\partial t} = 0$, thus we have

$$\dot{\mathbf{p}} = \int_A \rho \mathbf{v}(\mathbf{v} \cdot \mathbf{n} dA \tag{22}$$

Immediately after the expansion the fluid velocity is nonzero only across the area $A_1$, i.e. no sufficient distance for the flow to diverge from what it was in the narrower pipe, whereas we assume a uniform $v_2$ across all of $A_2$, thus

$$\dot{\mathbf{p}} = -A_1 \rho v_1^2 \mathbf{e}_1 + A_2 \rho v_2^2 \mathbf{e}_1. \tag{23}$$

Now the external forces acting on the control volume are due to the pressures along the boundary. We assume symmetry about the midline, thus the top and bottom pressures are equal and have no net effect. We further assume the pressure is uniform along the radial axis, i.e. the pressure is equal regardless of how far out from the center line. This is necessary for *laminar flow* along a pipe. Thus we have a pressure $P_1$ acting on on an area $A_2$ at the entrance and a pressure $P_2$ acting on an area $A_2$ down stream as well, these are opposite so we have $\dot{p} = (P_1 A_2 - P_2 A_2)\mathbf{e}_1$. Combining the integral and external forces we see that

$$P_1 - P_2 = \rho \left( v_2^2 - \frac{A_1}{A_2} v_1^2 \right) \tag{24}$$

## 3 Velocity field

a) *Show that the flow is volume preserving, or isochoric.*

Note that $\nabla \cdot \mathbf{v} = 0$ is sufficient to show that a **flow v** is volume preserving. Apply the Gauss integral theorem to the continuity equation we have $\int_V \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v})\, dV = 0$. Note the following uses index summation notation. The integrand may be rewritten as $\frac{\partial \rho}{\partial t} + \rho_{,i} v_i + \rho v_{i,i}$ and $\dot{\rho} = \frac{\partial \rho}{\partial t} + \rho_{,i} v_i$. Thus by conservation of mass $\dot{\rho} + \rho v_{i,i} = 0$, which implies that $\dot{rho} = 0$, i.e. volume is preserved, if $v_{i,i} = 0$

The divergence $\nabla \cdot \mathbf{v}$ is in Cartesian coordinates defined as

$$\nabla \cdot \mathbf{v} = \frac{\partial v_i}{\partial x_i} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \tag{25}$$

Here, Einstein's summation convention is assumed. For the given velocity field, this gives

$$\nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \frac{-\alpha}{t_0 - t} + \frac{\alpha}{t_0 - t} + 0 = 0 \tag{26}$$

Since the flow has zero divergence for all coordinate values, the flux (the sum of the divergence inside a volume) is zero. Thus, it is volume preserving.

b) *Determine the local acceleration, the convective acceleration and the particle acceleration* $\dot{\mathbf{v}}$*.*

The material derivative gives the following acceleration:

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \mathbf{a} = \dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \tag{27}$$

For each component:

$$\begin{aligned}
a_1 &= \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + \underbrace{v_2 \frac{\partial v_2}{\partial x_2}}_{=0} + \underbrace{v_3 \frac{\partial v_3}{\partial x_3}}_{=0} && (28) \\
&= -1 \cdot \frac{\alpha x_1}{(t_0 - t)^2}(-1) + \frac{-\alpha x_1}{t_0 - t} \frac{\alpha}{t_0 - t} && (29) \\
&= -\frac{\alpha x_1}{(t_0 - t)^2} + \frac{\alpha^2 x_1}{(t_0 - t)^2} && (30) \\
&= \frac{x_1}{(t_0 - t)^2}(\alpha^2 - \alpha) && (31) \\
a_2 &= \frac{\partial v_2}{\partial t} + \underbrace{v_1 \frac{\partial v_2}{\partial x_1}}_{=0} + v_2 \frac{\partial v_2}{\partial x_2} + \underbrace{v_3 \frac{\partial v_2}{\partial x_3}}_{=0} && (32) \\
&= \frac{x_2}{(t_0 - t)^2}(\alpha^2 + \alpha) && (33) \\
a_3 &= 0 && (34)
\end{aligned}$$