# Hyperelastic materials and the Piola-Kirchhoff stress tensors

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# Large deformations and Lagrangian coordinates

- Governing equations are formulated in $K_0$ for large deformations, i.e. $(X, t)$
- Conservation of mass implies $\rho \, dV = \rho_0 \, dV_0$
- Consequently for volume integrals in Cauchy's equations:

$$\int_V \mathbf{b} \, \rho \, dV = \int_{V_0} \mathbf{b} \, \rho_0 \, dV_0, \quad \int_V \mathbf{a} \, \rho \, dV = \int_{V_0} \mathbf{a} \, \rho_0 \, dV_0$$

- However $\int_A \mathbf{t} \, dA = ??$

# Contact forces for Lagrangian coordinates

- We define a stress vector $\mathbf{t}_0$ in $K_0$ by:

$$\mathbf{t}_0\, dA_0 = \mathbf{t}\, dA \quad \Leftrightarrow \quad \int_{A_0} \mathbf{t}_0\, dA_0 = \int_A \mathbf{t}\, dA$$

- which yields: $\mathbf{t}_0 = \frac{dA}{dA_0}\,\mathbf{t}$
- $\mathbf{t}_0$ represents forces in $K$
- $\mathbf{t}_0$ same direction as $\mathbf{t}$ but scaled
- Euler's first axiom:

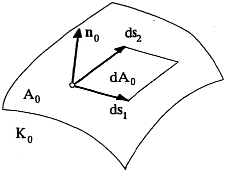
$$\int_{V_0} \mathbf{a}\,\rho_0\, dV_0 = \int_{A_0} \mathbf{t}_0\, dA_0 + \int_{V_0} \mathbf{b}\,\rho_0\, dV_0$$

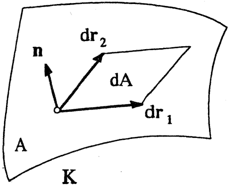
# The Piola-Kirchhoff stress tensors (PKS)

- The first PKS $\mathbf{T}_0$
  - Analogous to the Cauchy stress theorem
  - $\mathbf{t}_0 = \mathbf{T}_0 \cdot \mathbf{n}_0$
  - Invariant linear tensor equation
  - Unit normal $\mathbf{n}_0$ on $dA_0$
  - Relation to the Cauchy stress tensor $\mathbf{T}$

    $$\mathbf{T}_0 = J\mathbf{T}\mathbf{F}^{-T}, \quad J = \det \mathbf{F}$$
  - Not symmetric
  - $\mathbf{T}_0 =$ eng./nominal stress

- The second PKS $\mathbf{S}$
  - $\mathbf{S} = \mathbf{F}^{-1}\mathbf{T}_0 = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}$
  - $\mathbf{S} = \mathbf{S}^T$



# Strain energy and Piola-Kirchhoff stresses

- The Cauchy stress and the strain energy

$$\mathbf{T} = 2\,\rho \mathbf{F} \frac{\partial \psi}{\partial \mathbf{C}} \mathbf{F}^{T}$$

- The second PKS and the strain energy per unit mass $\psi$

$$\mathbf{S} = 2J\rho \frac{\partial \psi}{\partial \mathbf{C}} = 2\rho_0 \frac{\partial \psi}{\partial \mathbf{C}}, \qquad \rho J = \rho_0$$

- The second PKS and the strain energy per unit volume $\phi = \rho_0 \psi$

$$\mathbf{S} = 2 \frac{\partial \phi}{\partial \mathbf{C}}$$

# Isotropic hyperelastic materials

- No influence of directions on material response
- The strain energy can be expressed by the principal invariants of **C**

$$\phi = \phi(I_1, I_2, I_3)$$

- Invariants
  - $I_1 = \mathrm{tr}(\mathbf{C})$
  - $I_2 = \frac{1}{2}((\mathrm{tr}\mathbf{C})^2 - \|\mathbf{C}\|^2)$
  - $I_3 = \det(\mathbf{C}) = \det(\mathbf{F}^2) = J^2$

# Second PKS for isotropic hyperelastic materials

- From the general relation

$$\mathbf{S} = 2\frac{\partial \phi}{\partial \mathbf{C}} = 2\sum_{k=1}^{3} \frac{\partial \phi}{\partial I_k}\,\frac{\partial I_k}{\partial \mathbf{C}}$$

- By algebraic manipulation

$$\frac{\partial I_1}{\partial \mathbf{C}} = \mathbf{1}, \quad \frac{\partial I_2}{\partial \mathbf{C}} = I_1\mathbf{1} - \mathbf{C}, \quad \frac{\partial I_3}{\partial \mathbf{C}} = I_3\mathbf{C}^{-1}$$

- The second PKS

$$\mathbf{S} = 2\left(\frac{\partial \phi}{\partial I_1} + I_1\frac{\partial \phi}{\partial I_2}\right)\mathbf{1} - 2\frac{\partial \phi}{\partial I_2}\mathbf{C} + 2I_3\frac{\partial \phi}{\partial I_3}\mathbf{C}^{-1}$$

# Incompressible isotropic hyperelastic materials

- Incompressibility $\Rightarrow J = 1 \Rightarrow I_3 = 1$
- Modified strain energy function:

$$\bar{\phi} = \phi(\mathbf{C}) + p(J - 1)$$

  - p undetermined Lagrange multiplier
  - determined from boundary conditions
- Second PKS

$$\mathbf{S} = 2\,\frac{\partial\bar{\phi}}{\partial\mathbf{C}} = 2\frac{\partial\phi(\mathbf{C})}{\partial\mathbf{C}} + pJ\mathbf{C}^{-1}$$

- Due to
$\frac{\partial J}{\partial \mathbf{C}} = \frac{\partial J}{\partial I_3}\,\frac{\partial I_3}{\partial \mathbf{C}} = \frac{\partial\sqrt{I_3}}{\partial I_3}\,I_3\mathbf{C}^{-1} = \frac{J}{2}\,\mathbf{C}^{-1}$

# Example: Thin sheet of incompressible material

- $\phi = \phi(I_1, I_2)$
- Zero stress in direction orthogonal to plane

$$\begin{aligned} S_{33} &= 0 \\ &= 2\left(\frac{\partial\phi}{\partial I_1} + I_1\frac{\partial\phi}{\partial I_2}\right) - 2\frac{\partial\phi}{\partial I_2}C_{33} + p\mathbf{C}_{33}^{-1} \end{aligned}$$

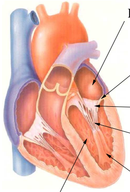
- The Lagrangian multiplier $p$ is then given by

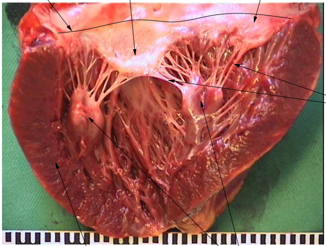
$$p = 2\frac{\partial\phi}{\partial I_2}C_{33}^2 + 2\left(\frac{\partial\phi}{\partial I_1} + I_1\frac{\partial\phi}{\partial I_2}\right)C_{33}$$

# Examples of isotropic incompressible nonlinear hyperelastic materials

- Mooney-Rivlin
  - Used for rubber
  - $\phi = c_1(I_1 - 3) + c_2(I_2 - 3)$
- Neo-Hookean
  - $\phi = c_1(I_1 - 3)$
- Transversely isotropic membrane shells
  - Application to mitral valve mechanics
  - Fiber directions $\mathbf{a}_0$ must be accounted for
  - Additional invariants introduced
  - $I_4 = \mathbf{a}_0\mathbf{C}\mathbf{a}_0$
  - $I_5 = \mathbf{a}_0\mathbf{C}^2\mathbf{a}_0$

# Mitral valve anatomy



# Continuum mechanical framework

- Deformation gradient: $\mathbf{F}$
- Cauchy Green tensor: $\mathbf{C} = \mathbf{F}^{\mathbf{T}}\mathbf{F}$
- Invariants:
  $I_1 = \mathrm{tr}\mathbf{C} = \quad \mathbf{J} = \det\mathbf{F} = \sqrt{\det\mathbf{C}}, \quad \mathbf{I_4} = \mathbf{C} : \mathbf{a_0} \otimes \mathbf{a_0}$
- $\mathbf{a_0}$ is a unit vector corresponding to the fiber orientation in the undeformed configuration

# Two–dimensional formulation

- Transversely isotropic strain–energy function for mitral tissue:

$$\Psi(I_1, I_4) = c_0[exp^{c_1(I_1-3)^2+c_2(I_4-1)^2} - 1] + p(J-1)$$

  - $p$: Lagrange multiplier
  - $c_0,\ c_1 \text{ and } c_2$ are material parameters (see May-Newman and Yin (1998))

- Second Piola–Kirchhoff stress tensor **S** and Cauchy stress tensor $\boldsymbol{\sigma}$:

$$\mathbf{S} = 2\frac{\partial \Psi}{\partial \mathbf{C}} \quad \boldsymbol{\sigma} = \frac{1}{J}\mathbf{FSF^T}$$

# Two–dimensional formulation

- Plane stress state, $S_{33} = 0$ gives:
- Material elasticity tensor:

$$\mathbb{C} = 2\frac{\partial \mathbf{S}}{\partial \mathbf{C}}$$

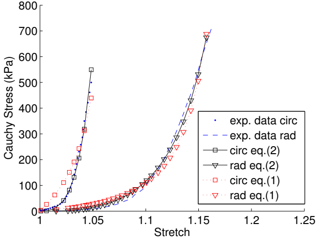
- Spatial elasticity tensor for incompressible material ($J = 1$) (push-forward of $\mathbb{C}$): $\mathbb{c} = \chi_* (\mathbb{C})\,, \quad c_{ijkl} = F_{iI}F_{jJ}F_{kK}F_{lL}C_{IJKL}$

$$\mathbb{c} = 4\psi_{11}\mathbf{B}\otimes\mathbf{B} + 4\psi_{14}(\mathbf{B}\otimes\mathbf{a}\otimes\mathbf{a} + \mathbf{a}\otimes\mathbf{a}\otimes\mathbf{B}) + 4\psi_{44}\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{a} + \mathbf{2}\,\mathbf{1}\otimes\left(\mathbf{F}\frac{\partial \mathbf{p}}{\partial \mathbf{C}}\mathbf{F}^{\mathbf{T}}\right) - \mathbf{2p}\mathbb{I}$$

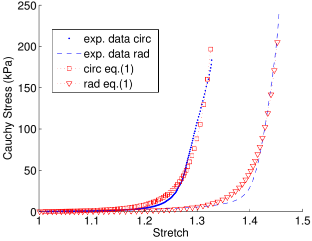
- Implemented in UMAT ABAQUS/standard
- Cauchy Stress components and the tangent stiffness have to be computed

# Experimental results (Leaflets) and optimization

Healthy anterior leaflet




HOCM anterior leaflet



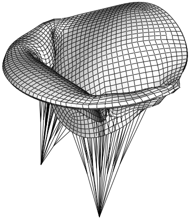

Strain–energy functions used to capture transversely isotropic response of the leaflet tissues:

$$\Psi = \mu(\bar{I}_1 - 3) + c_0\left(e^{c_1(\bar{I}_1-3)^2 + c_2(\bar{I}_4-1)^2} - 1\right) + \kappa(J-1)^2, (1)$$

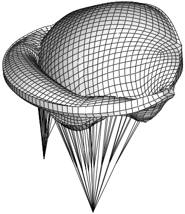
$$\tilde{\Psi} = \tilde{\mu}(\bar{I}_1 - 3) + \tilde{c}_0\left(e^{\tilde{c}_1(\bar{I}_1-3)^2 + \tilde{c}_2(\bar{I}_4-1)^4} - 1\right) + \kappa(J-1)^2, (2)$$

# Physiological versus pathological states

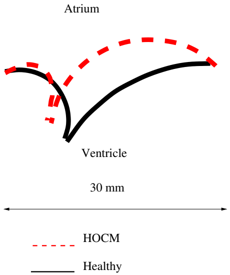
Mitral valves at peak systole (120 mmHg)
Healthy valve



HOCM valve



Deformed profiles



# Animation of deceased response

# Mitral valve mechanics

- The work of V. Prot and B. Skallerud on material valve mechanics was presented
- The model is a transversely isotropic membrane shell model as outlined in the previous slide
- In press for Int J Numer Meth Engng

# Summary