Exercise 1: Velocity profile between parallel planes

Assume we have two parallel planes with a given flow between them. The upper plane is moving with a velocity u_0, and the fluid is exposed to a pressure gradient $\nabla p = \frac{\partial p}{\partial x}$. See Figure 1.

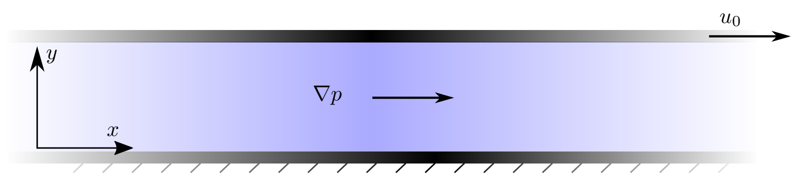

Figure 1: Fluid between two parallel plates.

a) Derive the generic equation for the velocity profile for flow between parallel planes, and illustrate the velocity profile for various pressure gradients and velocities of the upper plane, when the lower plane is assumed at rest.

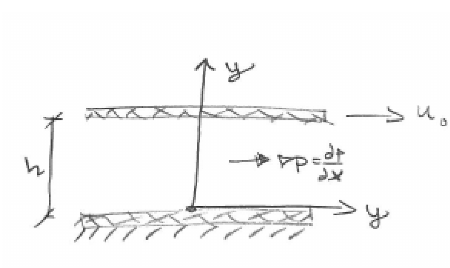

Figure 2: Coordinates and dimensions used in solution.

Solution.

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \mathbf{v} \tag{1}$$

Assuming steady flow we know $\dot{\mathbf{v}} = \mathbf{0}$ and thus

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{0} \tag{2}$$

leaving

$$\frac{1}{\rho}\nabla p = \frac{\mu}{\rho}\nabla^2 \mathbf{v}. \tag{3}$$

Eliminating ρ

$$\nabla p = \mu \nabla^2 \mathbf{v} \tag{4}$$

The gradient of flow should be 0 in the direction along the plates thus

$$\mathbf{v} = \begin{bmatrix} u(y) & 0 & 0 \end{bmatrix}^\top . \tag{5}$$

So we have

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} \tag{6}$$

and thus

$$\frac{\partial^2 u}{\partial y^2} = \mu^{-1} \frac{\partial p}{\partial x} \tag{7}$$

Integrating over y yields

$$\frac{\partial u}{\partial y} = \mu^{-1} y \frac{\partial p}{\partial x} + C_1 \tag{8}$$

and again

$$u = \frac{y^2}{2\mu} \frac{\partial p}{\partial x} + C_1 y + C_2. \tag{9}$$

The problem statement gives us that $u(0) = 0$ implying $C_2 = 0$. Additionally $u(h) = u_0$ yields

$$u_0 = \mu^{-1} \frac{h^2}{2} \frac{\partial p}{\partial x} + C_1 h \tag{10}$$

which solving for C_1 gives

$$C_1 = \frac{\mu^{-1} \frac{h^2}{2} \frac{\partial p}{\partial x} - u_0}{h}. \tag{11}$$

Thus the general form for u is

$$u = \frac{y^2}{2\mu} \frac{\partial p}{\partial x} + \frac{\left(u_0 - \frac{h^2}{2\mu} \frac{\partial p}{\partial x} \right)}{h} y \tag{12}$$

b) Consider the case where both plates are at rest. What is the relationship between the volumetric flow, i.e. volume per time, and the pressure gradient?

Solution. We note that the volumetric flow is simply the integral of the $\rho\mathbf{u}$ over the cross section of the channel. Let D denote the depth of the channel and assume $\mathbf{u}$ doesn't depend on depth then:

$$Q = D \int_0^h u(y) dy = \frac{-Dh^3}{12\mu} \frac{\partial p}{\partial x} \tag{13}$$

c) Based on this relationship how could the channel be regulated to perform best at a given pressure, i.e. what parameters would be most effective in modifying the flow corresponding to fixed pressure gradient? Where might such a system be useful in a biological context, e.g. the cardiovascular system.

Solution. As Q is related to the cube of the channel width h, variations in this parameter will typically have a greatest effect on Q for a fixed pressure gradient. The depth and viscoscity only have linear effects. If channel flow were an accurate model of blood vessels, the flow could be efficiently regulated by varying the vessel width. This could be achieved by muscle activity to stretch or compress the vessel.

d) What are some limitations of applying this equation? Are there certain contexts where it might be more reasonable than others?

Solution. This problem relies on assuming a 2D flow, or that the depth of the channel is much greater than its width. It could perhaps describe the flow of interstitial between two tissues or two layers of cells.