Fluid mechanics

Leif Rune Hellevik

Department of Structural Engineering
Norwegian University of Science and Technology
Trondheim, Norway

TKT4150 Biomechanics

Outline

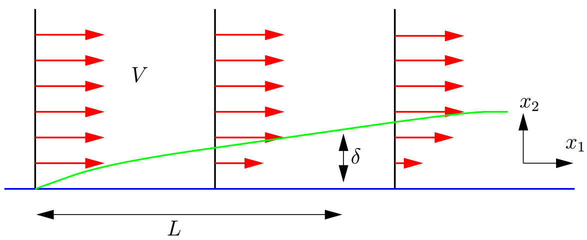
- Recap
- Newtonian fluids
- Navier-Stokes equations
- Examples

Recap

- Conservation equations for fluids
  - Conservation of mass
  - Balance of linear momentum
- General momentum equations (Cauchy's)
- Eulerian equations (compressible/incompressible)
- Sound wave equations

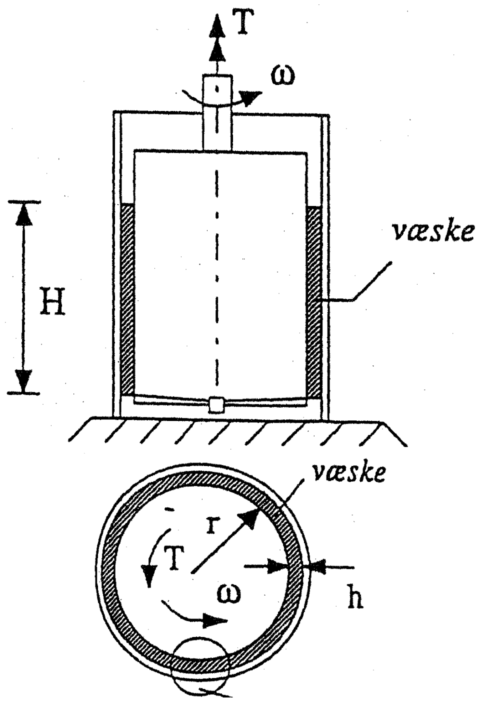
Viscous fluids

- Fluid particles are slowed down in the vicinity of a solid wall
- Shear stresses are present wherever there are velocity gradients
- The influence of (wall) shear stresses (WSS) are most prominent in the vicinity of solid walls



- Flow around rigid bodies
  - External flow field modeled as a perfect fluid
  - BL flow near the rigid surface with asymptotic external flow

Cylinder viscometer



- Shear stress from torque T

$$
\begin{aligned}
(\tau r)(2\pi r H) &= T \\
\tau = \frac{T}{2\pi r^2 H}
\end{aligned}
$$

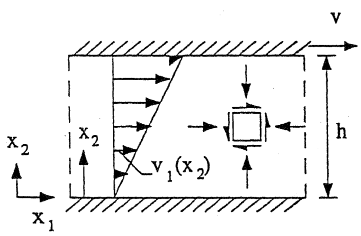
- The velocity field for simple shear flow

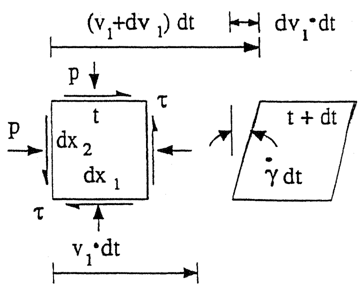
$$v_1 = \frac{v}{h}x_2, v_2 = v_3 = 0$$

- Strain rate

$$\dot{\gamma} = 2D_{12} = \frac{dv_1}{dx_2} = \frac{v}{h} = \frac{\omega r}{h}$$

Simple shear flow





- The velocity field

$$v_1 = \frac{v}{h}x_2, v_2 = v_3 = 0$$

- The rate of deformation

$$
\begin{aligned}
\mathbf{D} &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\dot{\gamma}}{2} \\
\dot{\gamma} &= 2D_{12} = \frac{dv_1}{dx_2} = \frac{v}{h}
\end{aligned}
$$

- From experiments

$$\tau = \mu\dot{\gamma}$$

Generalization from simple shear flow

- From experiments

$$\tau = \mu\dot{\gamma} = 2\mu D_{12}$$

- Newton's law of fluid friction

$$T_{ij} = 2\mu D_{ij} = \mu(v_{i,j} + v_{j,i}) \quad \text{for} \quad i \neq j$$

- Stokes' criteria for stress/velocity relation in viscous fluids
  - **T** is a continuous function of **D**
  - Homogeneous, i.e. **T** independent of particle coordinates
  - $\mathbf{T} = -p(\rho, \theta)\mathbf{1}$ when $\mathbf{D} = 0$
  - Viscosity is an isotropic property (redundant)

Newtonian fluid

- The Stokes criteria imply a constitutive equation for a Stokes fluid of the form

$$\mathbf{T} = \mathbf{T}[\mathbf{D}, \rho, \theta], \quad \mathbf{T}[\mathbf{0}, \rho, \theta] = -p(\rho, \theta)\mathbf{1}$$

- Linear viscous fluid (Newtonian)
  - Linear viscous isotropic properties implies that **T** and **D** are co-axial
  - The constitutive equation may be shown to be

$$\begin{aligned}
\mathbf{T} &= -p(\rho, \theta)\mathbf{1} + 2\mu\mathbf{D} + (\kappa - \frac{2\mu}{3})\,(\,\mathrm{tr}\mathbf{D})\,\mathbf{1} \\
T_{ij} &= -p(\rho, \theta)\delta_{ij} + 2\mu D_{ij} + (\kappa - \frac{2\mu}{3})\,D_{kk}\,\delta_{ij}
\end{aligned}$$

The properties of the Newtonian fluid

- Dynamic shear viscosity
  - $\mu = \mu(\theta)$ (rarely pressure dependent)
  - Relatively simple to determine experimentally
  - Water
    - $\mu = 1.8 \cdot 10^{-3}\ \mathrm{Ns/m^2}$ at 0° C
    - $\mu = 1.0 \cdot 10^{-3}\ \mathrm{Ns/m^2}$ at 20° C
  - Air
    - $\mu = 1.7 \cdot 10^{-5}\ \mathrm{Ns/m^2}$ at 0° C
    - $\mu = 1.8 \cdot 10^{-5}\ \mathrm{Ns/m^2}$ at 20° C
- Bulk viscosity κ
  - Difficult to measure experimentally
  - Resistance toward rapid volume changes
- Incompressible Newtonian fluid

$$\mathbf{T} = -p(\rho, \theta)\mathbf{1} + 2\mu\mathbf{D}$$

The Navier-Stokes equations

- Equations of motion for Newtonian fluids
- Cauchy's equations of motion

$$\dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{\rho}\nabla \cdot \mathbf{T} + \mathbf{b}$$

- Newtonian fluid

$$\mathbf{T} = -p(\rho, \theta)\mathbf{1} + 2\mu\mathbf{D} + (\kappa - \frac{2\mu}{3})\,(\,\mathrm{tr}\mathbf{D})\,\mathbf{1}$$

- The Navier-Stokes (NS) equations are obtained by substitution

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}(\kappa + \frac{\mu}{3})\nabla(\nabla \cdot \mathbf{v}) + \mathbf{b}$$

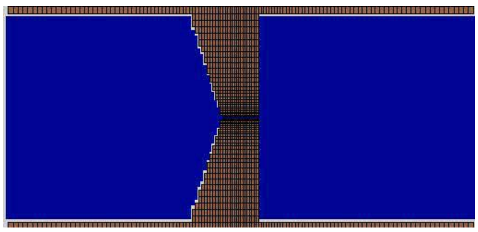
- The NS equations for incompressible fluids

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\mathbf{v} + \mathbf{b}$$

About the NS equations

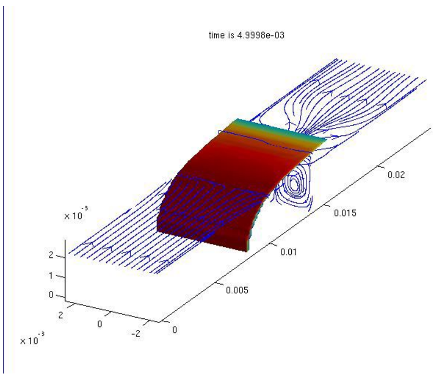
- After Claude-Louis Navier and George Gabriel Stokes
- The most important equations for viscous fluids
- Analytical solutions require major simplifications due to the complexity
- Models for
  - weather forecasts, ocean currents, and pollution
  - stars in galaxies
  - aircraft and car design
  - blood flow
- Coupled with Maxwell's equations to model and study magnetohydrodynamics
- Coupled with Cauchy's equations for solid materials to study fluid structure interaction problems, e.g. blood and vessel wall
- All but the simplest problems must be solved with Computational Fluid Dynamics (CFD) codes

Voice research



M.P. de Vries - RuG Biomedical Engineering

- Elastic vocal chords interacting with air
- A constant pressure drop over the pipe, vocal chords oscillate



MM Hegeman - RuG Biomedical Engineering

- Oscillatory flow past 3D voice-producing element

Flow between parallel planes

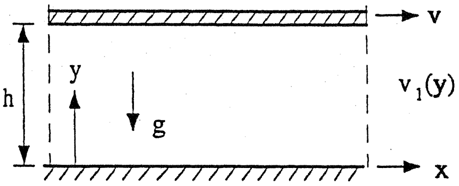
- *Saint-Venant's semi-inverse method*
  - Unknown functions are partly assumed known
  - Employ the governing equations and BCs to determine completely
- Assume steady state and velocity field

$$v_1 = v_1(y), v_2 = v_3 = 0$$

- Incompressibility $\nabla \cdot \mathbf{v} = 0$
- The constitutive equation yields

$$T_{11} = T_{22} = T_{33} = -p, \quad T_{12} = \mu v_{1,2}$$

- Both pressure gradient and upper plate are driving forces for the flow



Flow between parallel planes (contd)

- For convenience $-p_{,1} = c$
- By substitution of the stress field in Cauchy's equation

$$\begin{aligned} 0 &= c + \mu v_{1,22} \\ 0 &= -p_{,2} - \rho g \\ 0 &= -p_{,3} \end{aligned}$$

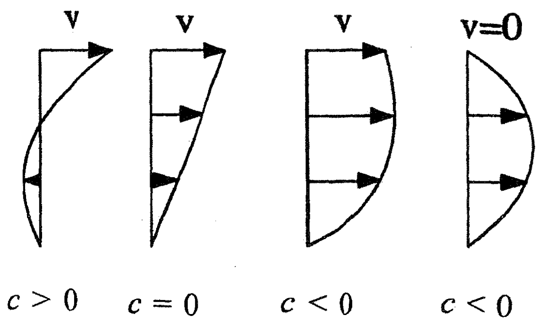
- By integration

$$\begin{aligned} p &= -\rho g y - c x \\ v_1 &= -\frac{c}{2\mu} y^2 + B y + C \end{aligned}$$

B and C constants

- Use BCs $v_1 = 0$ at $y = 0$ and $v_1 = v$ at $y = h$ to find constants
- The velocity field

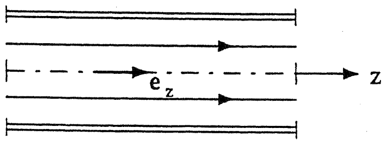
$$v_1(y) = \frac{ch^2}{2\mu}\left[\frac{y}{h} - \left(\frac{y}{h}\right)^2\right] + v\frac{y}{h}$$



Laminar pipe flow

- Incompressible Newtonian flow
- Pipe with diameter d
- Flow driven by pressure gradient in z-direction
- Laminar, steady flow field

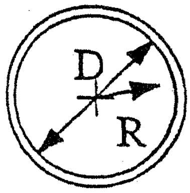
$$v_z = v(R), \quad v_R = v_\theta = 0$$



- NS in cylindrical coordinates

$$\begin{aligned} -\frac{\partial p}{\partial R} = 0, \quad \frac{1}{R}\frac{\partial p}{\partial \theta} &= 0 \\ -\frac{1}{R}\frac{\partial}{\partial R}\left(R\mu\frac{\partial v}{\partial R}\right) - \frac{\partial p}{\partial z} &= 0 \end{aligned}$$

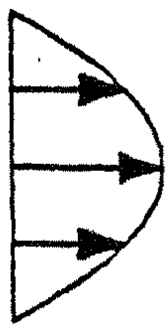
- BCs: $v(D/2) = 0$ and $v(0) \neq \infty$



Velocity field for laminar pipe flow

- Parabolic velocity field

$$\begin{aligned} v &= v_0\left(1 - \left(\frac{2R}{d}\right)^2\right) \\ v_0 &= \frac{d^2}{16\mu}c \\ c &= -\frac{\partial p}{\partial z} \end{aligned}$$



v(R)

Summary

- Newtonian (viscous) fluids
- Cylinder viscometer to measure viscosity
- Simple shear flow
- Constitutive equation for Newtonian fluids
- Navier-Stokes equations
- Flow between parallel planes
- Laminar pipe flow