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Exam in TKT4150 Biomechanics and TTK4170 Modelling and identification of biological systems

November 30, 2015
Duration: 09.00-13.00

No printed or hand-written support material is allowed.
A specific basic calculator is allowed (D).

Exercise 1: Laminar Pipe Flow

For stationary, laminar flow in a straight, rigid tube (Fig. (1)) the Cauchy equations take the form:

$$\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{zr}\right), \qquad \tau_{zr} = \mu\,\frac{\partial v}{\partial r} \tag{1}$$

where $v(r) = v_z(r)$ is the velocity component in the z-direction at a distance r from the central axis and the only non-zero component of the velocity field in cylindrical coordinates (r,θ,z): $v_\theta = 0$, $v_r = 0$. Further, let d denote the diameter of the vessel and μ the viscosity of the blood.

a) Show that the velocity profile in the vessel may be expressed as:

$$v(r) = -\frac{\partial p}{\partial z}\,\frac{a^2}{4\,\mu}\left(1-(r/a)^2\right), \qquad \text{where} \quad a = d/2. \tag{2}$$

Note this is referred to as Poiseuille flow.

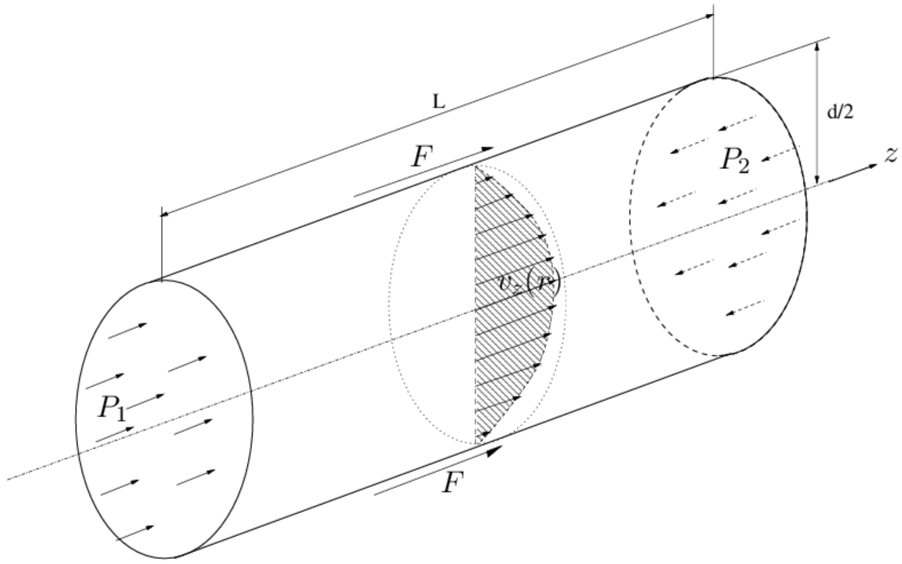

Figure 1: Diagram of stationary Newtonian rigid pipe flow.

Solution. Substituting $\tau_{zr} = \mu \frac{\partial v}{\partial r}$ yeilds

$$\frac{r}{\mu}\frac{\partial p}{\partial z} = \frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) \tag{3}$$

which may be integrated to find

$$\frac{r^2}{2\mu}\frac{\partial p}{\partial z} = r\frac{\partial v}{\partial r} + c_1 \tag{4}$$

and

$$\frac{r}{2\mu}\frac{\partial p}{\partial z} = \frac{\partial v}{\partial r} + \frac{c_1}{r} \tag{5}$$

finally we have

$$\frac{r^2}{4\mu}\frac{\partial p}{\partial z} = v + c_1 \log(r) + c_2. \tag{6}$$

Assuming symmetry we know that $\frac{\partial v}{\partial r}(0) = 0$, and thus $c_1 = 0$. The no slip condition requires that $v(a) = 0$ thus we have

$$\frac{a^2}{4\mu}\frac{\partial p}{\partial z} = c_2. \tag{7}$$

Combining this we have

$$v = -\frac{1}{4\mu}\frac{\partial p}{\partial z}(a^2 - r^2) \tag{8}$$

b) Derive an expression of the volumetric flow Q through the vessel as a function of $\frac{\partial p}{\partial z}$, a, and μ. How is this expression for Q relevant for physiological flow and pressure regulation?

Hint. $Q = \int_A v \cdot \mathrm{d}A$

Solution. Integrating (2) across the pipe yields

$$Q = -2\pi \frac{a^2}{4\mu} \frac{\partial p}{\partial z} \int_0^a (1 - (r/a)^2) r dr$$
$$Q = -\frac{\pi a^2}{2\mu} \frac{\partial p}{\partial z} (\frac{a^2}{2} - \frac{a^4}{4a^2})$$
$$Q = -\frac{\pi}{2\mu} \frac{\partial p}{\partial z} (\frac{2a^4}{4} - \frac{a^4}{4})$$

$$Q = -\frac{\pi a^4}{8\mu} \frac{\partial p}{\partial z} \tag{9}$$

This expression relates pressure and flow according to the vessel radius to the fourth power. Thus blood flow and pressure may be regulated very strongly by changing the radius of the vessels.

c) Show that the ratio between Q and the velocity at the center of the vessel $v_0 = v_z(r = 0)$ is equal to $A/2$, where $A = \pi\, a^2$.

Solution. Substituting $r = 0$ into the velocity profile yields the following ratio

$$\frac{\frac{\pi a^4}{8\mu} \frac{\partial p}{\partial z}}{\frac{a^2}{4\mu} \frac{\partial p}{\partial z}} = \frac{\pi a^2}{2} \tag{10}$$

— The constitutive equations for an incompressible Newtonian fluid read:

$$\boldsymbol{T} = -p\mathbf{1} + 2\mu \boldsymbol{D} \text{ or in index notation } T_{ij} = -p\delta_{ij} + 2\mu D_{ij}, \tag{11}$$

where $\boldsymbol{T}$ is the stress tensor, $\boldsymbol{D}$ the strain rate tensor with components: $D_{ij} = 1/2(v_{i,j} + v_{j,i})$. A viscous force F, acts on any cylindrical element of blood in the direction opposite the flow due to slower moving blood outside the element. The magnitude of the viscous force is given by

$$F = S\tau_{rz}, \tag{12}$$

where $S = 2\pi r L$ is the area of the cylindrical element and τ_{rz} is shear stress component that can be derived from eq.(11).

d) Derive an expression for F and calculate the viscous force on the wall of the vessel (at $r = a$) using the result of (12).

Solution. We first find D_{ij} noting that $v_{i,j} = 0$ except for $i = z$ and $j = r$, thus we have $D_i j = 0$ except for $D_{rz} = D_{zr} = \frac{v_{z,r}}{2}$. Thus the shear stress $T_{rz} = \tau_{rz} = 2\mu D_{rz} = 2\mu \frac{r}{4\mu} \frac{\partial p}{\partial z}$. To evaluate the force applied along the edge of the cylinder we subsitute $r = a$ and multiply by the surface area $2\pi a L$

$$F = \pi a^2 \frac{\partial p}{\partial z} L \tag{13}$$

e) Calculate the viscous force on the wall of the vessel (at $r = a$) using the force equilibrium on the fluid inside the vessel of length L and radius a. Compare this expression with the result in the question above.

Solution. Applying force equilibrium we note Euler's first axiom, which indicates the applied force is equal to the rate of change of momentum. Since we assume fully developed flow the change in momentum should be 0, thus the forces on the wall must balance the forces on the inlet and outlet due to pressure. Note, the momentum flux at each end is equal as there is no volume change. Thus we denote the pressure at the inlet p and the pressure at the outlet $p + \frac{\partial p}{\partial z} L$. Thus we have

$$
\begin{aligned}
F = (p + \frac{\partial p}{\partial z} L)A - pA \qquad (14)\\
= AL\frac{\partial p}{\partial z} \qquad (15)\\
= \pi a^2 \frac{\partial p}{\partial z} L \qquad (16)
\end{aligned}
$$

In addition to the assumption of local Poiseuille-flow (see (2)), assume a linear constitutive model:

$$A(p) = A_0 + C\,(p - p_0) \qquad (17)$$

f) What is the physical meaning of C?

Solution. C is called the compliance of the pipe, and its value determines how much the pipe may expand for a given increase in pressure. I.e. it is a measure of the stiffness of the pipe. It is also a measure of the vessel's ability to store energy in its deformed state.

g) Use equation (17) to eliminate the pressure p from the pressure flow relation for Poiseuille flow, integrate and express the area $A(z)$ at a given location z as a function of the inlet area $A(0)$, and μ, C, Q and z.

Solution. NOTE: This is simply copied from the chapter in the compendium.

$$\frac{dp}{dx} = -\frac{8\mu}{\pi a^4}\,Q = -\frac{8\pi\mu}{A^2}\,Q \qquad (18)$$

Compliance $\qquad C = \frac{\partial A}{\partial p}$

$$\frac{dp}{dx} = \frac{\partial p}{\partial A}\frac{dA}{dx} = \frac{1}{C}\frac{dA}{dx} = -\frac{8\pi\mu}{A^2}\,Q \qquad (19)$$

$$A^2\frac{dA}{dx} = \frac{1}{3}\frac{d}{dx}\left(A^3\right) = -8\pi\mu C Q \qquad (20)$$

$$A(x)^3 = A(0)^3 - 24\pi\mu C Q x \qquad (21)$$

Thus we have

$$\frac{A(x)}{A(0)} = \left(1 - \frac{24\pi\mu C Q x}{A(0)}\right)^{1/3} \qquad (22)$$

Constitutive model: $A(p) = A_0 + C\,(p - p_0)$
Pressure and flow for stationary flow in compliant vessel

$$Q(x) = \frac{A(0)^3 - A(x)^3}{24\pi\mu C x}, \quad p(x) = p_0 + \frac{A(x) - A(0)}{C} \qquad (23)$$

h) Illustrate and discuss how the pressure (area) flow relationship in a compliant vessel differs from a rigid vessel.

Solution. In a rigid pipe the flow is a linear function of the pressure gradient, thus for an arbitrary increase in the pressure gradient more flow will happen. In a compliant vessel the increased pressure gradient results in a smaller outlet area, thus for a large enough pressure gradient the flow will cease to increase, resulting in "choked" flow (see Figure 2). This may be seend by rearranging the previous result to find an expression for Q Thus we have

$$Q = \frac{1}{24\mu CL}\left(1 - \frac{A(L)^3}{A(0)^3}\right). \tag{24}$$

Since $A(p) = A_0 + C\ (p - p_0)$, A is a monotonic function of pressure, and the qualitative relationship between A and Q is the same as that between p and Q.

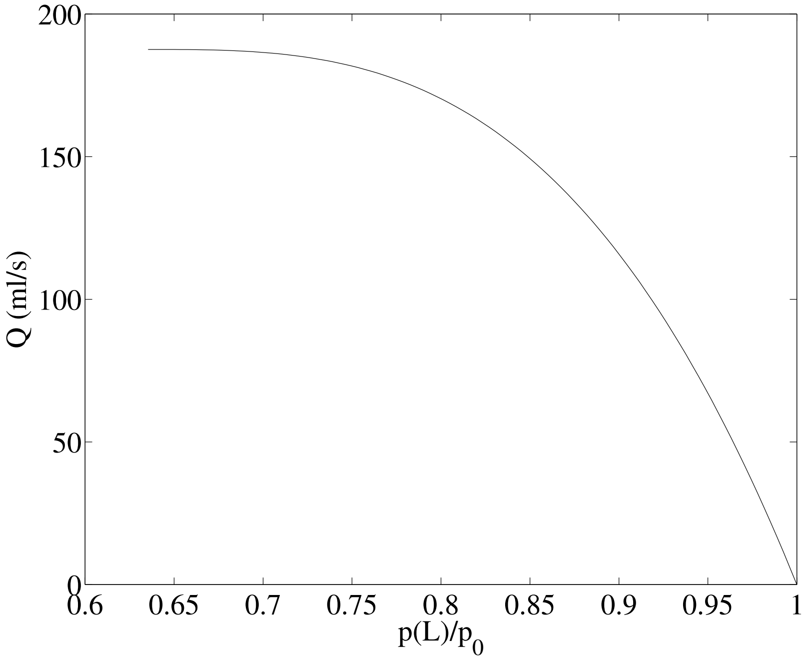

Figure 2: Flow versus normalized outlet pressure for Poiseuille flow in compliant pipe.

Exercise 2: Hyperelastic Aorta

Consider a vessel of internal radius R_i and wall thickness H in its undeformed configuration Ω_0. The vessel has internal pressure p_i, internal radius r_i, and wall thickness h in the deformed configuration Ω.

The deformation gradient $\boldsymbol{F}$ can be expressed in the cylindrical coordinate system ($\boldsymbol{e}_{rr}$, $\boldsymbol{e}_{\theta\theta}$, $\boldsymbol{e}_{zz}$) as:

$$\boldsymbol{F} = \begin{bmatrix} \lambda_{rr} & 0 & 0 \\ 0 & \lambda_{\theta\theta} & 0 \\ 0 & 0 & \lambda_{zz} \end{bmatrix}, \tag{25}$$

where $\lambda_{rr} = \lambda_r$, $\lambda_{\theta\theta} = \lambda_\theta$, and $\lambda_{zz} = \lambda_z$ are the stretches in the radial, circumferential and longitudinal directions, respectively. We assume that the vessel wall is incompressible: $J = \det\boldsymbol{F} = 1$. The stretch in the longitudinal direction is equal to one, $\lambda_z = 1$.

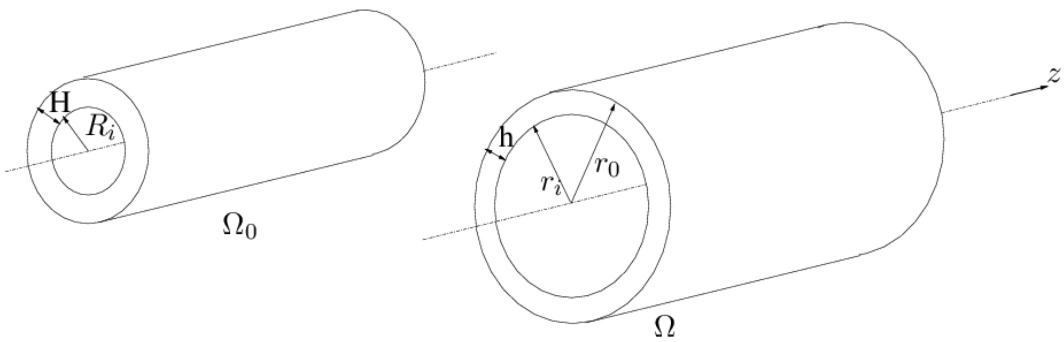

Figure 3: Reference and deformed configurations of a section of blood vessel.

First, we consider the case where $h \ll r_i$. Therefore, the vessel can be treated as a thin–walled structure and the membrane theory, $T_{rr} = 0$ can be applied to analyze the stresses in the vessel wall.

a) Derive an expression for the internal pressure p_i as a function of r_i, h and $T_{\theta\theta}$ the Cauchy stress in the circumferential direction.

Hint. Use Laplace's law $\left(\frac{\sigma_1}{r_1} + \frac{\sigma_2}{r_2} = \frac{p_i}{h}\right)$.

Solution. Since we consider a cylinder the only radius of curvature is the circumferential direction and thus

$$\frac{T_{\theta\theta}}{r} = \frac{p_i}{h}$$

which reduces to

$$\frac{T_{\theta\theta} h}{r} = p_i$$

b) Express h as a function of H and λ_θ and r_i as a function of R_i and λ_θ.

Solution.

$$2\pi r = \lambda_{\theta\theta} 2\pi R$$
$$r = \lambda_{\theta\theta} R$$

Using the fact that $\lambda_{zz} = 1$ and the material is incompressible we know $\lambda_{rr} = \lambda_{\theta\theta}^{-1}$. From this and the definition of λ_{rr} we have

$$h = \lambda_{rr} H = \lambda_{\theta\theta}^{-1} H$$

c) Express p_i as a function of R_i, H, $T_{\theta\theta}$ and λ_θ.

Solution. Substitute the expressions for H and R into the relation derived from Laplace's law to show

$$T_{\theta\theta} = p_i \frac{\lambda_\theta^2 R}{H}, \tag{26}$$

which may be rearranged to find

$$p_i = T_{\theta\theta}\frac{H}{\lambda_\theta^2 R} \tag{27}$$

The internal radius of a segment of the aorta was measured at diastolic and systolic pressures, 80mmHg and 120mmHg respectively. The values were $r_i(p_i = 80) = 1.2$cm and $r_i(p_i = 120) = 1.5$cm. The thickness of the wall at diastolic pressure was measured to be $h = 1$mm

d) What is the wall thickness h at systolic pressure.

Solution. We treat the diastolic configuration as the refrence configuration and thus $\lambda_\theta = \frac{1.5}{1.2} = 1.25$. From this we use formula for wall thickness to find $h(p_i = 120) = \frac{h(p_i=80)}{1.25} = \frac{0.1}{1.25} = 0.08$ mm.

e) Find the ratio between systolic and diastolic circumferential stress ($\frac{T_{\theta\theta}(120)}{T_{\theta\theta}(80)}$)?

Solution. From the equation (26) we have the ratio

$$\frac{T_{\theta\theta}(120)}{T_{\theta\theta}(80)} = \frac{120(1.25)^2}{80} = 2.34375. \tag{28}$$

Now, we assume that the vessel wall can be modeled as an incompressible isotropic hyperelastic material. The second Piola–Kirchhoff stress tensor $\boldsymbol{S}$ can be expressed as:

$$\boldsymbol{S} = 2f(I_1)\mathbf{1} + q\boldsymbol{C}^{-1}, \tag{29}$$

where f is a scalar function, $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$ is the right Cauchy–Green tensor, $\mathbf{1}$ is the second order identity tensor, q is a Lagrange multiplier, and $I_1 = \text{tr}\boldsymbol{C}$ is the first invariant of $\boldsymbol{C}$.

f) The Cauchy stress $\boldsymbol{T}$ can be found by pushing forward the second Piola-Kirchhoff stress tensor $\boldsymbol{S}$ into the current configuration: $\boldsymbol{T} = J^{-1}\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T$. Give the expression of the Cauchy stress tensor $\boldsymbol{T}$.

Solution. First note that $J = 1$ as the material is incompressible. Since $\boldsymbol{F}$ is a diagonal matrix, it commutes and thus $\boldsymbol{T} = \boldsymbol{S}\boldsymbol{F}^\top\boldsymbol{F}$. Thus we now have

$$\boldsymbol{T} = (2f(I_1)\mathbf{1} + q\boldsymbol{C}^{-1})\boldsymbol{C} \tag{30}$$
$$= 2f(I_1)\boldsymbol{C} + q\mathbf{1}. \tag{31}$$

In index notation this is $T_{ij} = (2f(I_1)\lambda_{ij}^2 + q)\delta_{ij}$ with no summation.

In the case where $h \ll r_i$, the stress in the radial direction can be neglected: $T_{rr} = 0$ or $S_{rr} = 0$. Hence, the vessel is in a plane stress state.

g) Use the plane stress condition to derive an expression of the Lagrange multiplier q (as a function of $f(I_1)$ and λ_θ).

Solution. From the index notation $T_{ij} = (2f(I_1)\lambda_{ij}^2 + q)\delta_{ij}$ we consider the case of the radial stress

$$T_{rr} = 2f(I_1)\lambda_{rr}^2 + q = 0, \tag{32}$$

thus $q = -2f(I_1)\lambda_\theta^{-2}$

h) Assuming $\lambda_\theta > 1$, find the principal stresses and their directions in terms of $\boldsymbol{e}_{rr}$, $\boldsymbol{e}_{\theta\theta}$ and $\boldsymbol{e}_{zz}$.

Solution. Principal stresses are defined as the stresses in the three directions where there are no shear stresses. We note that $\boldsymbol{T}$ is already diagonalized, and thus we must only determine the values of the stresses. σ_θ and σ_z are both positive as the material is under tension for $\lambda > 1$, thus $\sigma_3 = \sigma_r$ as found in the previous exercise to be 0, and with direction $\boldsymbol{e}_{rr}$.
$\sigma_\theta = p_i \frac{\lambda_\theta^2 R}{H} = 2f(I_1)\lambda_\theta^2(1-\lambda_\theta^{-4})$ is a principal stress in direction $\boldsymbol{e}_{\theta\theta}$. NOTE: $f(I_1) = \frac{p_i R}{2H(1-\lambda_\theta^{-4})}$.
Finally, we have σ_z in direction $\boldsymbol{e}_{zz}$ with value $2f(I_1)\lambda_z^2 - 2f(I_1)\lambda_\theta^{-2}$, but we note that $\lambda_z = 1$ and thus we have $2f(I_1)(1-\lambda_\theta^{-2})$, thus since $\lambda_\theta > 1$ we know that $\sigma_\theta > \sigma_z$ and thus $\sigma_1 = \sigma_\theta$ and $\sigma_2 = \sigma_z$

i) Using the expressions for $\boldsymbol{T}$ and q calculate the ratio between systolic and diastolic axial stress ($\frac{T_{zz}(120)}{T_{zz}(80)}$) using the reported data for the section of the aorta.

Solution. First we find the symoblic form for T_{zz}, which is simply $2f(I_1)\lambda_{zz}^2 - 2f(I_1)\lambda_\theta^{-2}$, or more conveniently $2f(I_1)(1-\lambda_\theta^{-2})$. Thus the ratio of $T_{zz}(120)$ to $T_{zz}(80)$ is simply $\frac{2f(I_1(120))(1-\lambda_\theta^{-2}(120))}{2f(I_1(80))(1-\lambda_\theta^{-2}(80))}$.
The key unknown is thus the ratio of $f(I_1(120))$ to $f(I_1(80))$. However we may use the previous results from Laplace's law to determine this ratio. In particular we use two formulas for $T_{\theta\theta}$:
$\sigma_\theta = p_i \frac{\lambda_\theta^2 R}{H} = 2f(I_1)\lambda_\theta^2(1-\lambda_\theta^{-4})$, which implies that

$$f(I_1) = \frac{p_i R}{2H(1-\lambda_\theta^{-4})}. \tag{33}$$

thus the ratio between the pressures 120 and 80, assuming a reference $R = 1$ is

$$\frac{\frac{120R}{2H(1-1.5^{-4})}}{\frac{80R}{2H(1-1.2^{-4})}} \tag{34}$$

$$\frac{\frac{120}{(1-1.5^{-4})}}{\frac{80}{(1-1.2^{-4})}} = 0.9677884615384615. \tag{35}$$

Thus we have

$$\frac{2f(I_1(120))(1-\lambda_\theta^{-2}(120))}{2f(I_1(80))(1-\lambda_\theta^{-2}(80))} = 0.9677994615384615\frac{1-1.5^{-2}}{1-1.2^{-2}} \tag{36}$$

$$= 1.7596153846153844 \tag{37}$$

Consider the previously found ratio of $T_{\theta\theta}$ at systolic and diastolic pressures. We may express this ratio for the given constitutive law as

$$\frac{T_{\theta\theta}(120)}{T_{\theta\theta}(80)} = \frac{120(1.25)^2}{80} = 2.34375 \tag{38}$$

$$= \frac{2f(I_1(120))(\lambda_\theta^2(120)-\lambda_\theta^{-2}(120))}{2f(I_1(80))(\lambda_\theta^2(80)-\lambda_\theta^{-2}(80))}. \tag{39}$$

Let $R = 1$ be the refrence length, thus $\lambda_\theta(80) = 1.2$ and $\lambda_\theta(80) = 1.5$ and

$$\frac{2f(I_1(120))}{2f(I_1(80))} = 2.34375\frac{(\lambda_\theta^2(80)-\lambda_\theta^{-2}(80))}{(\lambda_\theta^2(120)-\lambda_\theta^{-2}(120))} \tag{40}$$

$$= 2.34375\frac{1.2^2-1.2^{-2}}{1.5^2-1.5^{-2}} = \tag{41}$$

$$\tag{42}$$

In many cases, the condition $h \ll r_i$ is not fulfilled and the vessel must be treated as a thick–walled cylindrical structure. The Cauchy equations of motion can be used to write the cross–sectional equilibrium,

$$\mathrm{div}\boldsymbol{T} + \rho\boldsymbol{b} = \rho\boldsymbol{a}, \tag{43}$$

where ρ is the density of the structure, $\boldsymbol{b}$ the body forces and $\boldsymbol{a}$ the acceleration. The Cauchy equation in cylindrical coordinates (r,θ,z) in the radial direction is,

$$\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} + \rho b_r = \rho a_r, \tag{44}$$

where T_{rr} and $T_{\theta\theta}$ are normal stresses and $\tau_{r\theta}$ and τ_{zr} are shear stresses.

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j) Simplify eq.(44) when body forces and accelerations are excluded and $\boldsymbol{e}_{rr}$, $\boldsymbol{e}_{\theta\theta}$ and $\boldsymbol{e}_{zz}$ are principal stress directions.

Solution. First we note that the cylindrical coordinates align with the principal stresses of the system. Thus $\tau_{ij} = 0\delta_{ij}$, and the equation reduces to

$$\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0 \tag{45}$$

$$\frac{\partial T_{rr}}{\partial r} = \frac{T_{\theta\theta} - T_{rr}}{r} \tag{46}$$

k) Now, suppose the internal pressure of the vessel is p_i. Using the result from the previous question and the boundary conditions $p_i = -T_{rr}|_{r=r_i}$ and $T_{rr}|_{r=r_0} = 0$, show

$$p_i = \int_{r_i}^{r_0} \frac{T_{\theta\theta} - T_{rr}}{r} dr, \quad r_i \leq r \leq r_0, \tag{47}$$

where r_0 is the outer radius of the vessel in Ω.

Solution. From the previous result we integrate both sides

$$\int_{r_i}^{r_o} \frac{\partial T_{rr}}{\partial r} dr = \int_{r_i}^{r_o} \frac{T_{\theta\theta} - T_{rr}}{r} dr \tag{48}$$

$$T_{rr}|_{r=r_0} - T_{rr}|_{r=r_i} = \int_{r_i}^{r_o} \frac{T_{\theta\theta} - T_{rr}}{r} dr \tag{49}$$

$$0 - (-p_i) = \int_{r_i}^{r_o} \frac{T_{\theta\theta} - T_{rr}}{r} dr \tag{50}$$

$$p_i = \int_{r_i}^{r_o} \frac{T_{\theta\theta} - T_{rr}}{r} dr \tag{51}$$

$$\tag{52}$$

Exercise 3: Estimation

a) Draw the circuit equivalent of the two-element Windkessel model, and explain the physical background of each parameter.

b) Using conservation of mass, derive the differential equation describing the model.

Q Q_p

Figure 4: The rate of change of arterial volume equals the difference between aortic inflow (Q) and outflow (Q_p) towards the periphery.

Hint. See figure (4).

c) Discretize the differential equation from b, and show that it can be written in matrix form as

$$\hat{\boldsymbol{P}} = \boldsymbol{A}\boldsymbol{\alpha} \tag{53}$$

where $\hat{\boldsymbol{P}}$ is the model pressure. Define $\boldsymbol{A}$ and $\boldsymbol{\alpha}$.

d) Find the least squares estimate of the parameter vector $\boldsymbol{\alpha}$ from (53).

e) When using a ballistic estimation method, only the initial pressure is assumed known, and thus the model can't be written in the matrix form given in (53). Instead, a common method is to use a least square adaption of a ballistic simulation of the model to the measurements. Present a functional based on a ballistic simulation of the model and describe a scheme for a ballistic least square estimate of the parameter vector using the sensitivity matrix.

f) How can the observability and robustness of the estimation be analysed from the sensitivity matrix?