NTNU
Dept. Structural Engineering

Exam in TKT4150 Biomechanics

May 31, 2008

Duration: 09:00-13:00

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No printed or handwritten aids are permitted (D). Approved calculators are permitted.

Exercise 1 — Equations of motion

The equations of motion in an orthogonal Cartesian coordinate system are:

$$\rho a_i = T_{ik,k} + \rho b_i \tag{1}$$

These equations are normally referred to as Cauchy's equations of motion.

1. Explain the different terms in the equations.
2. Write the equations in xyz-coordinates.
3. The constitutive equations for a linearly viscous fluid Eq. (2a), and the relation between the strain rate tensor and the components of the velocity field Eq. (2b) are:

$$T_{ij} = -p\delta_{ij} + 2\mu D_{ij} + \left(\kappa - \frac{2}{3}\mu\right) D_{kk}\delta_{ij} \tag{2a}$$

$$D_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right) \tag{2b}$$

Derive the Navier-Stokes equations from Eq. (1) and (2).

Exercise 2 — Progressive waves

Consider the governing equations for inviscid, 1D flow in a compliant tube:

$$\begin{aligned} \frac{\partial A^*}{\partial t} + \frac{\partial Q^*}{\partial x} &= 0 \\ \frac{\partial u^*}{\partial t} + u^* \frac{\partial u^*}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \end{aligned} \tag{10}$$

Introduce perturbations on the form:

$$A^* = A_0 + A, \qquad u^* = u_0 + u, \qquad Q^* = Q_0 + Q \tag{11}$$

and Lagrangian coordinates:

$$x' = x - u_0 t, \qquad t' = t \tag{12}$$

1. Show that the transformed governing equations take the form:

$$\frac{\partial A}{\partial t'} = -\frac{\partial Q}{\partial x'} \tag{13a}$$

$$\frac{\partial u}{\partial t'} = -\frac{1}{\rho}\frac{\partial p}{\partial x'} \tag{13b}$$

2. A commonly used constitutive equation for blood vessels is:

$$A^* = A_0 + C(p - p_0), \quad C = \frac{\partial A}{\partial p} \tag{14}$$

where subscript 0 denote a reference state.
   a) How will you classify the constitutive model in Eq. (14)?
   b) Eliminate cross-sectional area A or pressure p from Eq. (13) by introducing the compliance $C = \partial A/\partial p$.
3. Derive an expression for the wave speeds and present them in an Eulerian reference frame. Hint: the long wave assumption implies $\partial A/\partial x \ll 1$.

Exercise 3 — Lumped models for the arterial tree

1. Derive the Windkessel-model for the cardiovascular system.
2. Show that the impedance for the Windkessel-model is:

$$Z = \frac{\hat{p}}{\hat{Q}} = \frac{R_p}{1 + j\omega R_p C} \tag{39}$$

3. The impedance of the Westkessel-model is given by

$$Z = Z_c + \frac{R_p}{1 + j\omega R_p C} \tag{40}$$

Compare the impedances of the Windkessel/Westkessel models (modulus and phase) with a typical arterial input impedance.
4. List pros and cons for the Westkessel model.

Exercise 4 — Art's model of the left ventricle

Arts' model presents a model of the left ventricle as a thick-walled cylinder using the fluid-fiber continuum as a constitutive equation. The muscle fibers are assumed to lie in concentric cylindrical layers with angle α with respect to the cylinder short axis, and fiber stress σ. The stresses on coordinate planes in cylindrical coordinates (R, θ, z) are:

$$\sigma_R = -p, \quad \sigma_\theta = -p + \sigma\cos^2\alpha, \quad \sigma_z = -p + \sigma\sin^2\alpha \tag{48}$$

1. Write the equilibrium in the orthoradial direction (θ) for a shell of thickness dR and height dz. Use the constitutive equation to conclude that

$$\frac{dp}{dR} = -\frac{\sigma\cos^2\alpha}{R} \tag{49}$$

2. a) Write the equilibrium in the longitudinal direction (z).
   b) Finally, show that:

$$\frac{dp}{dR} = -\frac{2}{3}\frac{\sigma}{R} \tag{50}$$

3. Integrate Eq. (50) and express the ratio $\Delta p/\sigma$ as a function of the volume of the ventricular wall V_w and the volume of the left ventricle V_{LV}.

Answer (Exercise 1) — Equations of motion.

1. Explain the different terms in Cauchy's equations of motion
   ρa_i: acceleration term
   $T_{ik,k}$: surface forces
   ρb_i: body forces
2. Let 1,2,3 denote x,y and z directions, respectively:

$$\begin{array}{ll} \text{x-dir:} & \rho a_1 = T_{11,1} + T_{12,2} + T_{13,3} + \rho b_1 \\ \text{y-dir:} & \rho a_2 = T_{21,1} + T_{22,2} + T_{23,3} + \rho b_2 \\ \text{x-dir:} & \rho a_3 = T_{31,1} + T_{32,2} + T_{33,3} + \rho b_3 \end{array}$$

3. Derivation of Navier Stokes equations
   The equations of motion are given by

$$\rho a_i = T_{ik,k} + \rho b_i \tag{3}$$

and the constitutive equation for a Newtonian fluid:

$$T_{ij} = -p\delta_{ij} + 2\mu D_{ij} + (K - \frac{2}{3}\mu)D_{kk}\delta_{ij} \tag{4}$$

with the strainrate tensor:

$$D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \Rightarrow D_{kk} = v_{k,k} \tag{5}$$

The solution is found by differentiating Eq. (4):

$$T_{ik,k} = T_{ij,j} = \underbrace{-p_{,j}\delta_{ij}}_{=-p_{,i}} + \mu(v_{i,jj} + \underbrace{v_{j,ij}}_{=v_{k,ki}}) + (\kappa - \frac{2}{3}\mu)\underbrace{v_{k,kj}\delta_{ij}}_{v_{k,ki}} \tag{6}$$

$$= -p_{,i} + \mu v_{i,jj} + (\kappa + \frac{\mu}{3})v_{k,ki} \tag{7}$$

and substitution of Eq. (6) into Eq. (3):

$$\rho a_i = \rho\left(\frac{\partial v}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right) = -p_{,i} + \mu v_{i,jj} + (\kappa + \frac{\mu}{3})v_{k,ki} \tag{8}$$

or equivalent on vector form:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p + \mu\nabla^2\mathbf{v} + (\kappa + \frac{\mu}{3})\nabla(\nabla\cdot\mathbf{v}) \tag{9}$$

Answer (Exercise 2) — Progressive waves

1. The governing equations are given by:

$$\begin{aligned} \frac{\partial A^*}{\partial t} + \frac{\partial Q^*}{\partial x} &= 0 \\ \frac{\partial u^*}{\partial t} + \underbrace{u^*\frac{\partial u^*}{\partial x}}_{\text{non-linear term}} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \end{aligned} \tag{15}$$

which after introduction of

$$A^* = A_0 + A, \qquad u^* = u_0 + u, \qquad Q^* = Q_0 + Q \tag{16}$$

yields:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{17a}$$

$$\frac{\partial u}{\partial t} + \underbrace{u_0 \frac{\partial u}{\partial x}}_{\text{linearization}} = \frac{1}{\rho} \frac{\partial p}{\partial x} \tag{17b}$$

From the variable transformation:

$$x' = x - u_0 t, \qquad t' = t \Rightarrow dt' = dt \tag{18}$$

we introduce the differential operators:

$$\frac{\partial ()}{\partial t} = \frac{\partial ()}{\partial t'} \frac{\partial t'}{\partial t} + \frac{\partial ()}{\partial x'} \frac{\partial x'}{\partial t} = \frac{\partial ()}{\partial t'} - u_0 \frac{\partial ()}{\partial x'} \tag{19a}$$

$$\frac{\partial ()}{\partial x} = \frac{\partial ()}{\partial t'} \frac{\partial t'}{\partial x} + \frac{\partial ()}{\partial x'} \frac{\partial x'}{\partial x} = \frac{\partial ()}{\partial x'} \tag{19b}$$

By using the differential operators defined in Eq. (19) on Eq. (17a) we get:

$$\frac{\partial A}{\partial t'} - u_0 \frac{\partial A}{\partial x'} + \frac{\partial Q}{\partial x'} = 0 \qquad Q^* = (A_0 + A)(u_0 + u) \tag{20}$$

$$\frac{\partial A}{\partial t'} - u_0 \frac{\partial A}{\partial x'} + u_0 \frac{\partial A}{\partial x'} + A_0 \frac{\partial u}{\partial x'} = 0 \qquad \frac{\partial Q}{\partial x'} \approx u_0 \frac{\partial A}{\partial x'} + A_0 \frac{\partial u}{\partial x'} \tag{21}$$

$$\frac{\partial A}{\partial t} + A_0 \frac{\partial u}{\partial x'} = 0 \tag{22}$$

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad \text{(long wave assump.)} \tag{23}$$

and similarly Eq. (17b) is transformed to:

$$\frac{\partial u}{\partial t'} - u_0 \frac{\partial u}{\partial x'} + u_0 \frac{\partial u}{\partial x'} = -\frac{1}{\rho} \frac{\partial p}{\partial x'} \tag{24}$$

$$\Rightarrow \frac{\partial u}{\partial t'} = -\frac{1}{\rho} \frac{\partial p}{\partial x'} \tag{25}$$

And thus the governing equations are transformed to:

$$\frac{\partial A}{\partial t'} = -\frac{\partial Q}{\partial x'} \tag{26}$$

$$\frac{\partial u}{\partial t'} = -\frac{1}{\rho} \frac{\partial p}{\partial x'} \tag{27}$$

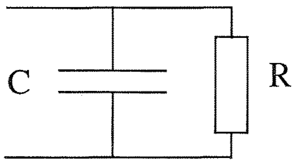
which is identical with the linearized equations.

2. a) The constitutive equation may be classified as an elastic const. eqn. In general it is non-linear as C may be a function of p. However, if C is const. it is a linear elastic const. eqn.
b) The transient area term in the mass eqn. may be reformed by using the const. eqn:
$$\frac{\partial A}{\partial t'} = \frac{\partial A}{\partial p}\frac{\partial p}{\partial t'} = C\frac{\partial p}{\partial t'} \tag{28}$$
The governing equations then take the form:
$$C\frac{\partial p}{\partial t'} = -\frac{\partial Q}{\partial x'} \tag{29}$$
$$\frac{\partial u}{\partial t'} = -\frac{1}{\rho}\frac{\partial p}{\partial x'} \tag{30}$$

3. In a Lagrangian reference frame the governing equations take the form:
$$C\frac{\partial p}{\partial t'} + \frac{\partial Q}{\partial x'} = 0 \approx C\frac{\partial p}{\partial t'} + A_0\frac{\partial u}{\partial x'} \tag{31}$$
$$\frac{\partial u}{\partial t'} + \frac{1}{\rho}\frac{\partial p}{\partial x'} = 0 \tag{32}$$
where we have used the long wave assumption in the mass equation. By cross-derivation, subtraction, and introduction of the wavespeed:
$$c^2 = \frac{A}{\rho C} \tag{33}$$
we get either:
$$\frac{\partial^2 p}{\partial t'^2} - c^2\frac{\partial^2 p}{\partial x'^2} = 0 \tag{34}$$
or:
$$\frac{\partial^2 u}{\partial t'^2} - c^2\frac{\partial^2 u}{\partial x'^2} = 0 \tag{35}$$
which both are canonical wave equations with solutions:
$$p = p_f\, f(x' - c\,t') + p_b\, g(x' + c\,t') \tag{36}$$
A representation in the Eulerian reference is obtained by using the definition Eq. (12) in Eq. (36) which yields:
$$\begin{aligned} p &= p_f\, f(x - (c + u_0)t) + p_b\, g(x + (c - u_0)t) \\ &= p_f\, f(x - c_f\, t) + p_b\, g(x + c_b\, t) \end{aligned} \tag{37}$$
which naturally introduces the forward and backward wave speeds
$$c_f = c + u_0, \qquad c_b = c - u_0 \tag{38}$$
respectively.

Answer (Exercise 3) — Derivation of the Windkessel model[1].
The vascular is modeled by a total arterial compliance $C = \frac{\partial V}{\partial p}$ and a peripheral resistance R_p (see fig). In this setting mass conservation (equivalent to Kirchhoff's law for electrical circuits) may be expressed by:

$$Q = Q_a + Q_p \tag{41}$$



where Q is the inflow in the aorta, Q_a is the flow which expands the compliant vessels and Q_p is the stationary flow toward the periphery. Mathematically we represent the contributions to the flow split:

$$Q_a = \frac{\partial V}{\partial p}\frac{\partial p}{\partial t} = C\frac{\partial p}{\partial t} \tag{42a}$$

$$Q_p = \frac{p}{R_p} \tag{42b}$$

which by substitution into Eq. (41) yields:

$$Q = C\frac{\partial p}{\partial t} + \frac{p}{R_p} \tag{43}$$

The impedance of the Windkessel is obtained by introduction of Fourier components:

$$p = \hat{p}\, e^{j\omega t}, \quad Q = \hat{Q}\, e^{j\omega t} \tag{44}$$

which by substitution into Eq. (43) yields:

$$\hat{Q} = j\omega C\,\hat{p} + \frac{\hat{p}}{R_p} \tag{45}$$

The impedance may the be expressed:

$$Z = \frac{\hat{p}}{\hat{Q}} = \frac{R_p}{1 + j\omega R_p C} = \frac{R_p}{1 + (\omega R_p C)^2}(1 - j\omega R_p C) \tag{46a}$$

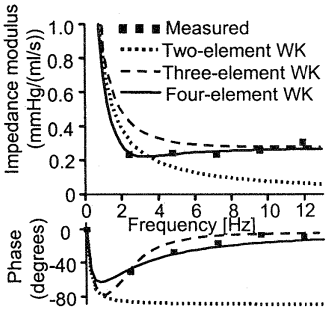
$$|Z| = \frac{R_p}{1 + (\omega R_p C)^2}\sqrt{1 + (\omega R_p C)^2} = \frac{R_p}{\sqrt{1 + (\omega R_p C)^2}} \tag{46b}$$

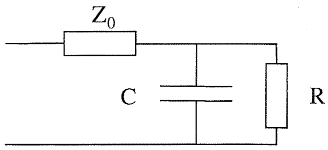
$$\angle Z = -\arctan \omega R_p C \tag{46c}$$

Limitations of the Windkessel

- At high frequencies $|Z| \to 0$, not Z_c
- At high frequencies $\angle Z \to -90°$, not 0
- High frequency information not captured

[1] Otto Frank 1899





The Westkessel model[2] (3-elt WK) was introduced to correct for hight frequency problems with the 2-elt WK. The impedance of the Westkessel models is given by:

$$Z = Z_c + \frac{R_p}{1 + j\omega R_p C} \tag{47}$$

- Pros
  - Good high frequency $|Z|$
  - Good high frequency $\angle Z$
  - Good fit of p and Q
- Cons
  - Bad compliance estimates
  - Bad low frequency est
  - Only monotonous decay in $|Z|$

Answer (Exercise 4) — 1. Equilibrium in the orthoradial direction:

$$2r\,dz\,p = 2(r + dr)dz(p + dp) + 2\sigma_\theta\,dr\,dz \tag{51}$$

$$r\,p = (r + dr)(p + dp)\sigma_\theta\,dr \tag{52}$$

which by discarding higher order terms reduces to:

$$0 = r\,dp + dr\,p + \sigma_\theta\,dr \tag{53}$$

$$\frac{dp}{dr} = -\frac{\sigma_\theta + p}{r} = -\frac{\sigma\cos^2\alpha}{r} \tag{54}$$

2. Equilibrium in the longitudinal direction

$$p\pi r^2 = (p + dp)\pi(r + dr)^2 + 2\sigma_z r\,dr \tag{55}$$

$$pr^2 = (p + dp)(r^2 + 2r\,dr + dr^2) + 2\sigma_z r\,dr \tag{56}$$

and by discarding higher order terms:

$$0 = 2pr\,dr + dp\,r^2 + 2\sigma_z r\,dr \tag{57}$$

$$\frac{1}{2}\frac{dp}{dr} = -\frac{\sigma_z + p}{r} = -\frac{\sigma\sin^2\alpha}{r} \tag{58}$$

[2]Westerhof

3. Summation of Eq. (54) and (56) yields:

$$\frac{3}{2}\frac{dp}{dr} = -\frac{\sigma}{r} \tag{59}$$

and the pressure difference may be found by integration:

$$\Delta p = -\frac{2}{3}\int_{r_i}^{r_o} \frac{\sigma}{r}\,dr = -\frac{2}{3}\sigma \ln\left(\frac{r_o}{r_i}\right) = -\frac{\sigma}{3}\ln\left(\frac{\pi r_o^2 L}{\pi r_i^2 L}\right) = -\frac{\sigma}{3}\ln\left(\frac{V_w + V_{LV}}{V_{LV}}\right) \tag{60}$$

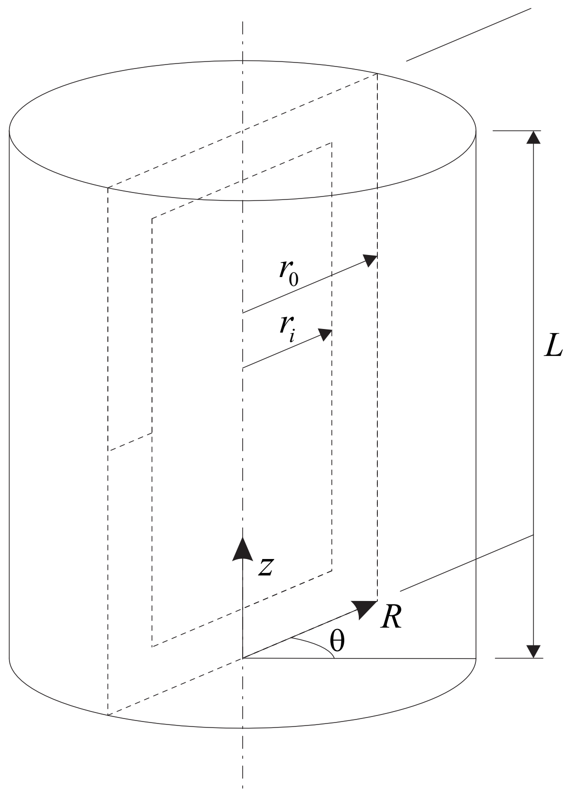


Figure 1: Schematic of Art's model of the left ventricle