Strain rates and Elasticity

Leif Rune Hellevik

Department of Structural Engineering
Norwegian University of Science and Technology
Trondheim, Norway

TKT4150 Biomechanics

Recap

- Strain measures
  - Longitudinal strain (ϵ)
  - Shear strain (γ)
  - Volumetric strain (ϵ_v)
- Strain tensors
  - Deformation gradient **F**
  - Green's deformation tensor **C**
  - Displacement gradient **H**
  - Green's strain tensor **E**
- All strain measures may be expressed by the Green strain tensor
- The expressions are simplified for small deformations

Strain rates

- Eulerian coordinates for fluid dynamics
- Velocity $\mathbf{v}(\mathbf{r}, t)$ for particle
- Displacement $d\mathbf{u} = \mathbf{v}\, dt$
- Deformations expressed by the displacement gradient tensor

$$dH_{ik} = \frac{\partial v_i}{\partial x_k} dt \equiv v_{i,k}\, dt$$

- Velocity gradient $L_{ik} = v_{i,k}$
- Strain rate tensor

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^{\mathbf{T}}\right)$$

$$D_{ik} = \frac{1}{2}(v_{i,k} + v_{k,i})$$

- General: $\mathbf{E} = \mathbf{D}\, dt$
- Small deformations: $\mathbf{D} = \dot{\mathbf{E}}$

Elasticity

- A material is (Cauchy) elastic if

$$\mathbf{T} = \mathbf{T}(\mathbf{E}, \mathbf{r})$$

- which is a *constitutive* or *material* equation
- Homogeneous if the elastic properties are the same *in every particle*

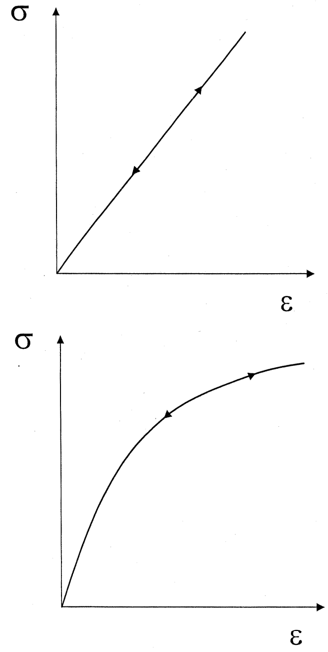
$$\mathbf{T} = \mathbf{T}(\mathbf{E})$$

- Isotropic if the elastic properties are the same *in every direction*
- Linear elastic if stress is linear function of strain

Fundamental properties of elastic materials

- Reversibility
  Identical loading/unloading stress-strain curves
- Path and rate independence
  The stress depends only on the level of strain - not strain history or rate
- Non-dissipative
  The deformation energy may be recovered upon unloading

Uniaxial behavior



Isotropic linear elastic material

- Uniaxial stress $(\sigma_1 \neq 0, \sigma_2 = \sigma_3 = 0)$

$$\epsilon_1 = \frac{\sigma_1}{\eta}, \quad \epsilon_2 = \epsilon_2 = -\nu \frac{\sigma_1}{\eta}$$

  η modulus of elasticity
  ν Poisson's ratio

- Superposition valid due to isotropic and linear stress/strain relationship

$$\epsilon_i = \frac{1+\nu}{\eta}\sigma_i - \frac{\nu}{\eta}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1+\nu}{\eta}\sigma_i - \frac{\nu}{\eta}\ \mathrm{tr}\mathbf{T}$$

The generalized Hooke's law

- Matrix representation in Ox-system with base vectors $\parallel$ to principal directions

$$\epsilon_i \, \delta_{ij} = \frac{1+\nu}{\eta} \, \sigma_i \, \delta_{ij} - \frac{\nu}{\eta} \, \mathrm{tr}\mathbf{T} \, \delta_{ij}$$

- In an arbitrary Ox-system

$$E_{ij} = \frac{1+\nu}{\eta} \, T_{ij} - \frac{\nu}{\eta} \, T_{kk} \, \delta_{ij}$$

- Tensor representation

$$\mathbf{E} = \frac{1+\nu}{\eta} \mathbf{T} - \frac{\nu}{\eta} \, \mathrm{tr}\mathbf{T} \, \mathbf{1}$$

Equivalent forms of Hooke's law

- Strain on LHS

$$\begin{aligned} E_{ij} &= \frac{1+\nu}{\eta} \, T_{ij} - \frac{\nu}{\eta} \, T_{kk} \, \delta_{ij} \\ \mathbf{E} &= \frac{1+\nu}{\eta} \mathbf{T} - \frac{\nu}{\eta} \, \mathrm{tr}\mathbf{T} \, \mathbf{1} \end{aligned}$$

- Stress on LHS

$$\begin{aligned} T_{ij} &= \frac{\eta}{1+\nu} \left(E_{ij} + \frac{\nu}{1-2\nu} \, E_{kk} \, \delta_{ij} \right) \\ \mathbf{T} &= \frac{\eta}{1+\nu} \left(\mathbf{E} + \frac{\nu}{1-2\nu} \, E_{kk} \, \mathbf{1} \right) \end{aligned}$$

Properties of the Hookean solid

- Isotropic linear elastic
- Only two independent material parameters η and ν
- Normal stresses only result in longitudinal strains
- Shear stresses only result in shear strains
- Not the case for anisotropic materials in general

Volumetric strain for the Hookean solid

- ϵ_V obtained from the strain version

  $$\epsilon_V = E_{ii} = \frac{1+\nu}{\eta}\, T_{ii} - \frac{\nu}{\eta}\, T_{kk}\, \delta_{ii} = \frac{1-2\nu}{\eta}\, T_{ii}$$

  which may be represented: $\epsilon_V = \frac{1}{\kappa}\,\sigma^0$
- Mean normal stress: $\sigma^0 = \frac{1}{3}\, T_{ii}$
- Bulk modulus: $\kappa = \frac{\eta}{3\,(1-2\nu)}$
- Possion's ratio: $0 \leq \nu \leq 0.5$
- Incompressible Hookean material $\varepsilon \equiv 0$

  $$\mathbf{T} = -p\mathbf{1} + 2\mu\,\mathbf{E}$$

2D theory of elasticity

Thin plate loaded by

- Body forces **b**
- Contact forces **t** on the boundary A
- $\Rightarrow T_{i3} = 0$ and $T_{\alpha\beta} = T_{\alpha\beta}(x_1, x_2, t)$

Governing equations for thin plate in plane stress

- Cauchy equations of motion[1]

$$\nabla \cdot \mathbf{T} + \rho \mathbf{b} = \rho \ddot{\mathbf{u}} \Leftrightarrow T_{\alpha\beta,\beta} + \rho b_\alpha = \rho \ddot{u}_\alpha$$

- Hooke's law for plane stress

$$T_{\alpha\beta} = 2\mu \left[E_{\alpha\beta} + \frac{\nu}{1-\nu} E_{\rho\rho}\delta_{\alpha\beta} \right], \quad 2\mu = \frac{\eta}{1+\nu}$$

- Green strains for small displacements

$$E_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right)$$

[1] Repeated Greek indices imples sum from 1 to 2

Navier equations for plane stress

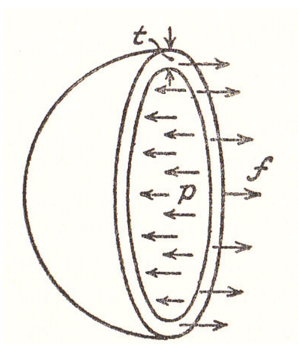
- Hooke's law and $E_{\alpha\beta}$ in Cauchy equations $\Rightarrow$

$$u_{\alpha,\beta\beta} + \frac{1+\nu}{1-\nu} u_{\beta,\beta\alpha} + \frac{\rho}{\mu}\left(b_\alpha - \ddot{u}_\alpha\right) = 0$$

- Boundary conditions

$$\begin{aligned} t_\alpha &= t^*_\alpha, \quad \text{on } A_\sigma \\ u_\alpha &= u^*_\alpha, \quad \text{on } A_u \end{aligned}$$

Spherical shell of steel



- $d_0 = 2000$ mm, $t_0 = 5$ mm $\Rightarrow$ thin walled
- Steel: η = 210 GPa, ν = 0.3
- Load: $p = 1.5$ MPa
- Find Δd and Δt due to the imposed load
- From stress analysis

$$\begin{aligned} \sigma_\phi = \sigma_\theta = \sigma &= \frac{1}{2}\frac{r}{t}p \\ &= \frac{1}{2\left(5 \cdot 10^{-3}\right)} 1.5 \cdot 10^6 = 150 \text{ MPa} \end{aligned}$$

Spherical shell of steel (contd)

- Hooke's law

$$E_{ij} = \frac{1+\nu}{\eta}\, T_{ij} - \frac{\nu}{\eta}\, T_{kk}\, \delta_{ij}$$

- Azimuthal direction $\sigma_r \ll \sigma \Rightarrow \sigma_r \approx 0$

$$\begin{aligned}\varepsilon_\theta = E_{11} &= \frac{1+\nu}{\eta}\sigma_\theta - \frac{\nu}{\eta}\,(\sigma_\theta + \sigma_\phi) = \frac{1-\nu}{\eta}\,\sigma \\ &= \frac{1-0.3}{210\cdot 10^9}\cdot 150\cdot 10^6 = 0.5\cdot 10^{-3}\end{aligned}$$

- Circumferential strain:

$$\begin{aligned}\varepsilon_\theta &= \frac{\pi d - \pi d_0}{\pi d_0} = \frac{d-d_0}{d_0} = \frac{\Delta d}{d_0} \\ &\Rightarrow \Delta d = \varepsilon_\theta\, d_0 = 0.5\cdot 10^{-3}\, 2000 = 1\,\mathrm{mm}\end{aligned}$$

Spherical shell of steel (contd)

- Radial direction

$$\begin{aligned}\varepsilon_r = E_{33} &= -\frac{\nu}{\eta}\,(\sigma_\theta + \sigma_\phi) = \frac{-2\nu}{\eta}\,\sigma \\ &= \frac{-0.6}{210\cdot 10^9}\cdot 150\cdot 10^6 = -\frac{3}{7}\cdot 10^{-3}\end{aligned}$$

- Radial strain

$$\begin{aligned}\varepsilon_r &= \frac{t-t_0}{t_0} = \frac{\Delta t}{t_0} \\ &\Rightarrow \Delta t = \varepsilon_r t_0 = -\frac{3}{7}\cdot 10^{-3} \approx -2.1\cdot 10^{-3}\mathrm{mm}\end{aligned}$$

Mechanical energy balance

- Work per unit time on volume V

$$P = \int_V \mathbf{b} \cdot \mathbf{v} \, \rho \, dV + \int_A \mathbf{t} \cdot \mathbf{v} \, dA$$

- By using Cauchy's stress theorem and Gauss

$$P = \dot{K} + P_d$$

- Kinetic energy

$$\dot{K} = \int_V \dot{\mathbf{v}} \cdot \mathbf{v} \rho \, dV, \quad K = \int_V \frac{\mathbf{v} \cdot \mathbf{v}}{2} \rho \, dV$$

- Stress power

$$P_d = \int_V \mathbf{T} : \mathbf{D} \, dV$$

Stress work

- Deformation power for a body of volume V

$$P_d = \int_V T_{ij} D_{ij} \, dV$$

- Deformation power per unit volume ω

$$\omega = T_{ij} D_{ij}$$

- For small deformations $\mathbf{D} = \dot{\mathbf{E}}$
- Stress work per unit volume

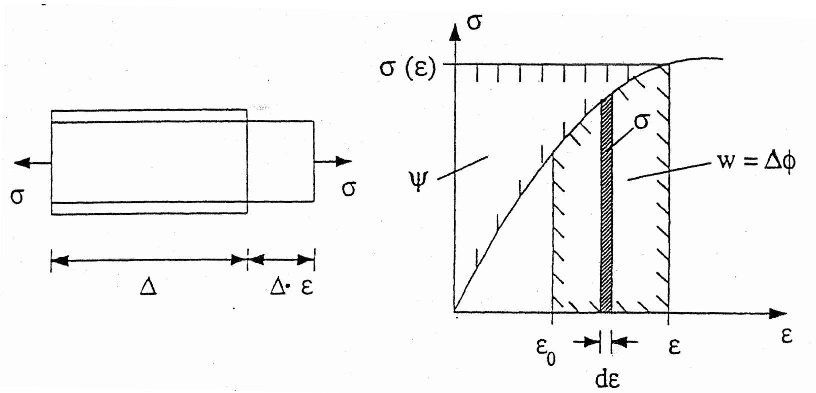
$$w = \int_{t_0}^{t} \omega \, dt = \int_{E_0}^{E} T_{ij} \, dE_{ij}$$

Hyperelastic materials and strain energy

- Hyperelastic material
  - If ω and w may derived from a scalar valued potential $\phi(\mathbf{E})$ such that
  $$\omega = \dot{\phi} = \frac{\partial \phi}{\partial E_{ij}} \dot{E}_{ij}$$
  - and
  $$w = \int_{t_0}^{t} \omega \, dt = [\phi]_{t_0}^{t} = \phi(\mathbf{E}) - \phi(\mathbf{E}_0)$$
- $\phi(\mathbf{E})$ is called *elastic energy* or *strain energy* per unit volume

Example: Uniaxial stress

$$w = \int_{t_0}^{t} \omega \, dt = \int_{t_0}^{t} \sigma \dot{\epsilon} \, dt = \int_{\epsilon_0}^{\epsilon} \sigma \, d\epsilon$$



Strain energy and stress tensor

- The stress power is represented by two expressions for a hyperelastic material

$$\omega = T_{ij}\dot{E}_{ij} = \frac{\partial \phi}{\partial E_{ij}}\,\dot{E}_{ij}$$

- As the strain energy is a function of strain only (i.e. $\phi(\mathbf{E})$) and not the strain rate $\dot{E}_{ij}$, we get:

$$T_{ij} = \frac{\partial \phi}{\partial E_{ij}}$$

- Thus $\mathbf{T} = \mathbf{T}(\mathbf{E})$
- Hyperelasticity $\Rightarrow$ elasticity

Large deformations and hyperelasticity

- Use elastic energy per *mass unit* $\psi(\mathbf{C})$ rather than per volume unit $\phi(\mathbf{E})$
- Hyperelastic condition
  - The stress power $\omega\, dV$ must be derived from the potential $\psi\,\rho\, dV$ in the following manner
  - $\omega\, dV = \frac{d}{dt}\,(\psi\rho\, dV) = \dot{\psi}\,\rho dV$
- Consequently

$$\omega = \rho\dot{\psi} = \rho\frac{\partial \psi}{\partial C_{ij}}\dot{C}_{ij}$$

Rate of the Green deformation tensor

- Rate of deformation gradient, velocity gradient and strain rate

$$F_{ik} = \frac{\partial x_i}{\partial X_k} \quad \text{and} \quad L_{il} = v_{i,l} \quad \text{and} \quad D_{ik} = \frac{1}{2}(v_{i,k} + v_{k,i})$$
$$\dot{F}_{ik} = \frac{\partial^2 x_i}{\partial t \partial X_k} = \frac{\partial}{\partial X_k}\left(\frac{\partial x_i}{\partial t}\right) = \frac{\partial}{\partial X_k} v_i = \frac{\partial v_i}{\partial x_l}\frac{\partial x_l}{\partial X_k}$$

- Which yields: $\dot{\mathbf{F}} = \mathbf{L}\cdot\mathbf{F}$ and $\dot{\mathbf{F}}^T = \mathbf{F}^T \cdot \mathbf{L}^T$
- Green's deformation tensor $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$
- Rate of the Green deformation tensor

$$\begin{aligned} \dot{\mathbf{C}} &= \dot{\mathbf{F}}^T \cdot \mathbf{F} + \mathbf{F}^T \cdot \dot{\mathbf{F}} \\ &= 2\,\mathbf{F}^T \cdot \mathbf{D} \cdot \mathbf{F} \end{aligned}$$

Elastic energy and stress tensor

- From the definition of stress power and the conditions for hyperelasticity

$$\omega = T_{ij} D_{ij} = \rho\dot{\psi} = \rho\frac{\partial \psi}{\partial C_{ij}}\dot{C}_{ij}$$

- By using the expression for $\dot{C}_{ij}$

$$\dot{\psi} = \frac{\partial \psi}{\partial C_{ij}}\dot{C}_{ij} = \frac{\partial \psi}{\partial C_{ij}}\left(2F_{ki}D_{kl}F_{lj}\right) = 2\left(\mathbf{F}\frac{\partial \psi}{\partial \mathbf{C}}\mathbf{F}^T\right) : \mathbf{D}$$

- Thus by substitution $\omega = \mathbf{T} : \mathbf{D} = \left(2\,\rho\mathbf{F}\frac{\partial\psi}{\partial\mathbf{C}}\mathbf{F}^T\right) : \mathbf{D}$
- As ψ is independent of $\mathbf{D}$

$$\mathbf{T} = 2\,\rho\mathbf{F}\frac{\partial \psi}{\partial \mathbf{C}}\mathbf{F}^T$$

Stress tensors for large deformations

- The stress and strain energy relations may be simplified
- Introduce the second Piola-Kirchhoff's stress tensor

Summary