Deformation analysis

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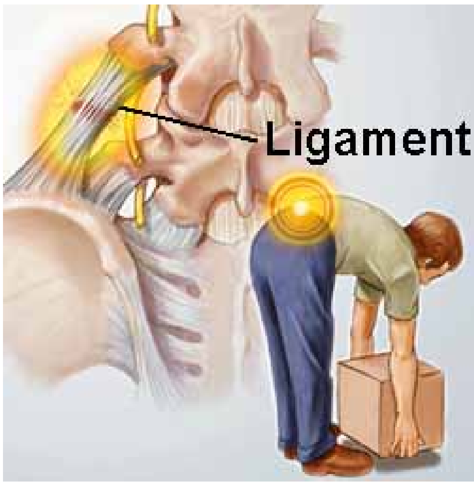
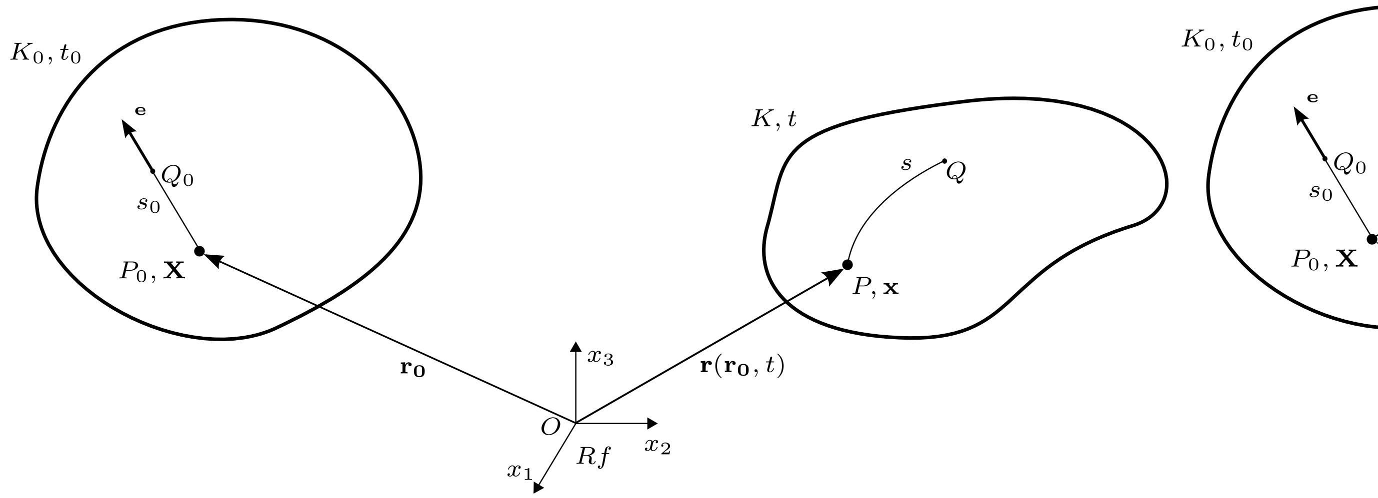
September 4, 2017

Outline

- Strain
  - Used for local deformation in a material
  - i.e. deformation in the neighborhood of a particle
- Strain represents changes in
  - Material lines
  - Angles
  - Volume
- Strain concepts
  - Longitudinal strain (ε)
  - Shear strain (γ)
  - Volumetric strain (ε_V)
- Causes for strain
  - Mechanical stress
  - Temperature changes
  - Swelling and shrinking



Measures of strain



- Longitudinal strain ε

$$\varepsilon = \lim_{s_0 \to 0} \frac{s - s_0}{s_0} = \frac{ds}{ds_0} - 1$$

in direction **e** for particle $\mathbf{r_0}$

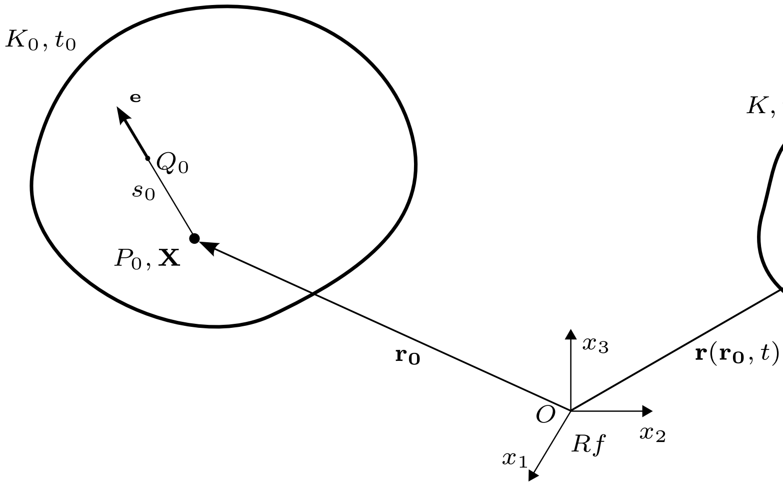
- Shear strain γ
  - Change in $\angle$ between $\perp$ line elements
- Volumetric strain ε_V

$$\varepsilon_V = \lim_{\Delta V_0 \to 0} \frac{\Delta V - \Delta V_0}{\Delta V_0}$$

The Green strain tensor

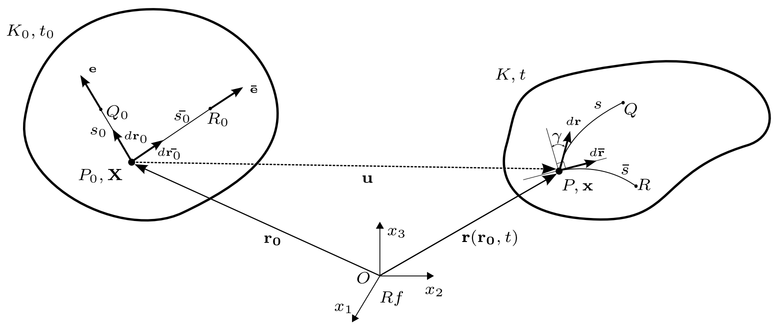
- Objective
  - To use the displacement vector **u** to express the primary measures of strain
- Various tensors will be presented
  - Deformation gradient tensor **F**
  - Green's deformation tensor **C**
  - Displacement gradient tensor **H**
  - Green's strain tensor **E**

Length s of deformed line PQ



- Coordinates of Q_0 and Q
  $Q_0 : \quad X_i + s_0 e_i, \quad \text{with} \quad \mathbf{e} = e_i \mathbf{e}_i$
  $Q : \quad x_i \left(X + s_0 \mathbf{e} \right)$
- Length s with s_0 as curve parameter
- $s = \int_0^{s_0} \sqrt{\frac{\partial x_i}{\partial s_0} \frac{\partial x_i}{\partial s_0}}\; ds_0$
- which yields
  $ds^2 = \left(\frac{\partial x_i}{\partial s_0} \frac{\partial x_i}{\partial s_0} \right) ds_0^2$

Differential relations



- Let ds_0 be the length of dr_0 in direction of $\mathbf{e}$ in K_0

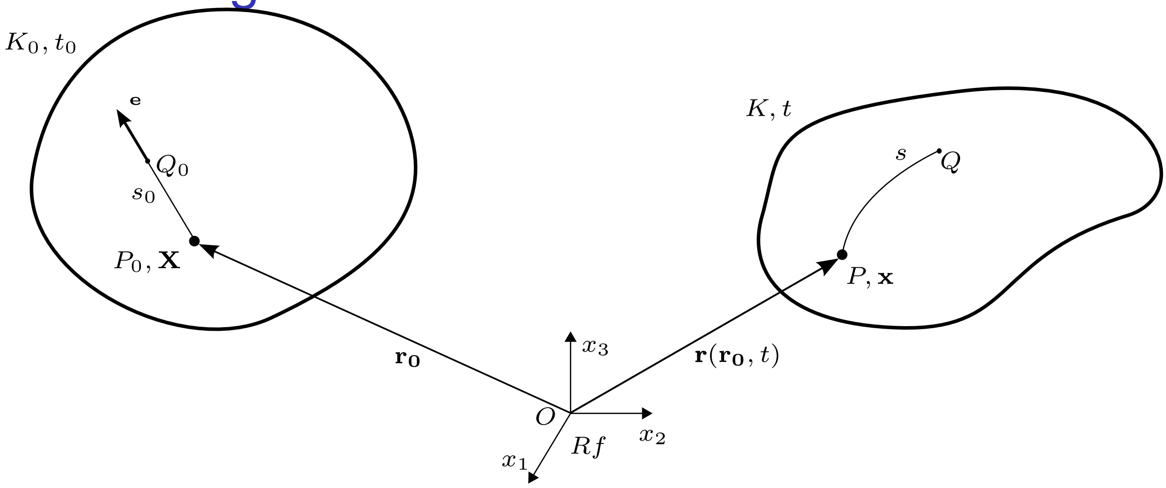
$$d\mathbf{r}_0 = \mathbf{e}\, ds_\mathbf{0} = dX_k \mathbf{e_k} \Rightarrow |d\mathbf{r_0}| = ds_\mathbf{0},\; dX_k = e_k ds_0 \Leftrightarrow e_k = \frac{dX_k}{ds_0}$$

- Let $d\mathbf{r}$ be a line element from P in K

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial s_0}\, ds_\mathbf{0} \quad \Leftrightarrow \quad dx_i = \frac{\partial x_i}{\partial s_0} ds_\mathbf{0}$$

- Thus $ds^2 = \left(\frac{\partial x_i}{\partial s_0} \frac{\partial x_i}{\partial s_0} \right) ds_0^2 \Rightarrow |d\mathbf{r}| = ds$

Deformation gradient



- The relation between $d\mathbf{r}$ and $d\mathbf{r_0}$

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \mathbf{r}_0} \cdot \; d\mathbf{r}_0 = \mathbf{F} \cdot \; d\mathbf{r}_0 \quad \Leftrightarrow \quad dx_i = \frac{\partial x_i}{\partial X_k} dX_k = F_{ik} dX_k$$

- The deformation gradient $\mathbf{F}$

$$\mathbf{F} = \frac{\partial \mathbf{r}}{\partial \mathbf{r}_0} \quad \Leftrightarrow \quad F_{ik} = \frac{\partial x_i}{\partial X_k}$$

Green's deformation tensor

- Directional derivative

$$\left.\frac{\partial x_i \left(X + s_0 \mathbf{e}, t\right)}{\partial s_0}\right|_{s_0=0} = \frac{\partial x_i \left(X, t\right)}{\partial X_k} \left.\frac{d\left(X_k + s_0 e_k, t\right)}{ds_0}\right|_{s_0=0}$$

- Which yields by use of the $ds_0 - e$-relation

$$\frac{\partial x_i}{\partial s_0} = F_{ik} \frac{dX_k}{ds_0} = F_{ik} e_k$$

- Then by using the $ds_0 - ds$-relation

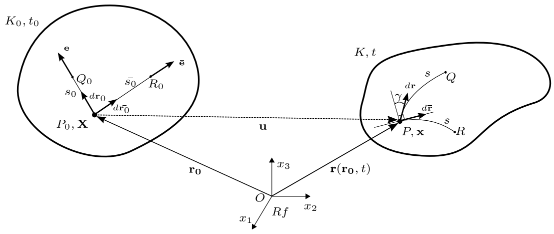
$$\left(\frac{ds}{ds_0}\right)^2 = \frac{\partial x_i}{\partial s_0} \frac{\partial x_i}{\partial s_0} = F_{ij} e_j F_{ik} e_k = \mathbf{e} \cdot \left(\mathbf{F}^T \mathbf{F}\right) \cdot \mathbf{e} = \mathbf{e} \cdot \mathbf{C} \cdot \mathbf{e}$$

- Symmetric
- 2. order

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} \quad \Leftrightarrow \quad C_{ij} = F_{ki} F_{kj}$$

Displacement gradient tensor

$$\mathbf{r} \equiv \mathbf{r}(\mathbf{r_0}, t) = \mathbf{r_0} + \mathbf{u}(\mathbf{r_0}, t) \quad \Leftrightarrow \quad x_i(X, t) = X_i + u_i(X, t)$$



- $$\frac{\partial x_i}{\partial X_k} = \delta_{ik} + \frac{\partial u_i}{\partial X_k}$$
- Displacement gradient tensor

$$H_{ik} = \frac{\partial u_i}{\partial X_k} \quad \Leftrightarrow \quad \mathbf{H} = \frac{\partial \mathbf{u}}{\partial \mathbf{r}_0}$$

- Relation between **F** and **H**

$$F_{ik} = \delta_{ik} + H_{ik} \quad \Leftrightarrow \quad \mathbf{F} = \mathbf{1} + \mathbf{H}$$

- Relation between **C** and **H**

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \left(\mathbf{1} + \mathbf{H}^T\right)(\mathbf{1} + \mathbf{H}) = \mathbf{1} + \mathbf{H} + \mathbf{H}^T + \mathbf{H}^T\mathbf{H}$$

Green's strain tensor

- Defined by

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T\mathbf{H}\right)$$

- Such that $\mathbf{C} = \mathbf{1} + 2\mathbf{E}$
- By substitution into previous expression

$$\left(\frac{ds}{ds_0}\right)^2 = 1 + 2\,\mathbf{e} \cdot \mathbf{E} \cdot \mathbf{e}$$

- Longitudinal strain ε

$$\varepsilon = \frac{ds}{ds_0} - 1 = \sqrt{1 + 2\,\mathbf{e} \cdot \mathbf{E} \cdot \mathbf{e}} - 1$$

Shear strain and volumetric strain

- The shear strain γ may be expressed by the Green strain tensor

$$\sin\gamma = \frac{2\bar{\mathbf{e}}\cdot\mathbf{E}\cdot\mathbf{e}}{\sqrt{(1+2\bar{\mathbf{e}}\cdot\mathbf{E}\cdot\bar{\mathbf{e}})\,(1+2\mathbf{e}\cdot\mathbf{E}\cdot\mathbf{e})}} = \frac{2\bar{\mathbf{e}}\cdot\mathbf{E}\cdot\mathbf{e}}{(1+\bar{\varepsilon})\,(1+\varepsilon)}$$

  where $\bar{\mathbf{e}}\perp\mathbf{e}$ and $\bar{\varepsilon}$ is the corresponding longitudinal strain
- The volumetric strain ε_V may also be expressed by the Green strain tensor

$$\varepsilon_V = \frac{dV - dV_0}{dV_0} = \det\mathbf{F} - \mathbf{1} = \det(\mathbf{1}+\mathbf{H}) - \mathbf{1} = \sqrt{\det(\mathbf{1}+\mathbf{2E})} - \mathbf{1}$$

Relevance of the Green strain tensor

- All strain measures may be expressed by the Green strain tensor
  - Longitudinal strain (ε)
  - Shear strain (γ)
  - Volumetric strain ε_V
- Green strain tensor is related with
  - Deformation gradient tensor **F**
  - Displacement gradient tensor **H**

Small strains

- From the general expression:
  $\varepsilon = \frac{ds}{ds_0} - 1 = \sqrt{1 + 2\,\mathbf{e} \cdot \mathbf{E} \cdot \mathbf{e}} - 1$
- By rearrangement $(1+\varepsilon)^2 = 1 + 2\varepsilon + \varepsilon^2 = 1 + 2\,\mathbf{e} \cdot \mathbf{E} \cdot \mathbf{e}$
- Longitudinal strain for small strains

$$\varepsilon = \ \mathbf{e} \cdot \mathbf{E} \cdot \mathbf{e} = e_i E_{ij} e_j$$

- Shear strain for small strains

$$\gamma = \ 2\,\bar{\mathbf{e}} \cdot \mathbf{E} \cdot \mathbf{e} = 2\,\bar{e}_i E_{ij} e_j$$

- Volumetric strain for small strains

$$\begin{aligned} 1 + 2\varepsilon_V + \varepsilon_V^2 &= \det(\mathbf{1} + 2\mathbf{E}) \\ \Rightarrow \quad \varepsilon_V &= tr(\mathbf{E}) \end{aligned}$$

Small deformations

- Small deformations

$$\Leftrightarrow \quad \left|H_{ij}\right| = \left|\frac{\partial u_i}{\partial X_j}\right| \ll 1 \quad \Leftrightarrow \quad norm\,(\mathbf{H}) \ll 1$$

- Small deformations imply
  - Small strains
  - Small rotations
- For an arbitrary field

$$\frac{\partial f}{\partial X_j} = \frac{\partial f}{\partial x_k}\frac{\partial x_k}{\partial X_j} = \frac{\partial f}{\partial x_k}\left(\delta_{ki} + \frac{\partial u_k}{\partial X_i}\right) \approx f_{,i}$$

- The Green strain tensor

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right) \quad \Leftrightarrow \quad E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

- Volumetric strain $\varepsilon = tr(\mathbf{E})$

Compatibility criteria for the Green strain tensor

- Necessary and sufficient conditions for **E** to correspond to a unique and continuous $\mathbf{u}(\mathbf{x}, \mathbf{t})$
- For small deformations/strains

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right) \quad \Leftrightarrow \quad E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

- Compatibility equations

$$E_{ij,kl} + E_{kl,ij} - E_{il,jk} - E_{jk,il} = 0$$

- Symmetry of **E** reduce the number of independent equations from $3^4 = 81$ to 6