Fluid mechanics

Leif Rune Hellevik

Department of Structural Engineering
Norwegian University of Science and Technology
Trondheim, Norway

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Outline

Recap

- Viscosity $\Rightarrow$ shear stresses and BL
- Cylinder viscometer and simple shear flows
- Stokes fluids

$$\mathbf{T} = \mathbf{T}[\mathbf{D}, \rho, \theta], \quad \mathbf{T}[\mathbf{0}, \rho, \theta] = -p(\rho, \theta)\mathbf{1}$$

- Linear viscous fluid (Newtonian)

$$\begin{aligned} \mathbf{T} &= -p(\rho, \theta)\mathbf{1} + 2\mu\mathbf{D} + (\kappa - \frac{2\mu}{3})\,(\,\mathrm{tr}\mathbf{D})\,\mathbf{1} \\ T_{ij} &= -p(\rho, \theta)\delta_{ij} + 2\mu D_{ij} + (\kappa - \frac{2\mu}{3})\,D_{kk}\,\delta_{ij} \end{aligned}$$

- Flow between parallel planes
- NS equations
- Laminar stationary pipeflow

Non-Newtonian fluids

- Fluids which do not satisfy

$$\mathbf{T} = -p(\rho, \theta)\mathbf{1} + 2\mu\mathbf{D}$$

- Typical Non-Newtonian fluids
  - Polymer solutions
  - Thermo plastics
  - Drilling fluids
  - Paints
  - Fresh concrete
  - Biological fluids

Generalized Newtonian fluid (GNF) model

- Popular for incompressible non-Newtonian fluids

$$\mathbf{T} = -p\mathbf{1} + 2\eta\mathbf{D} = -p\mathbf{1} + \boldsymbol{\tau}$$

- Stress deviator

$$\boldsymbol{\tau} = 2\eta\mathbf{D}$$

- Viscosity function η

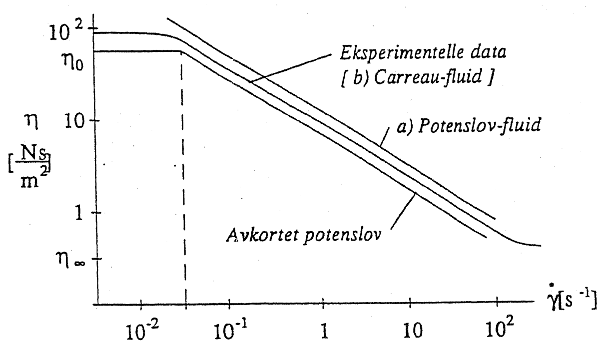
$$\eta = \eta(\dot{\gamma})$$

- Shear measure $\dot{\gamma}$

$$\dot{\gamma} = \sqrt{2D_{ij}D_{ij}} = 2\sqrt{-II_D}$$

- Shear measure reduce to the strain rate for simple shear flow

$$\dot{\gamma} = \frac{\partial v_1}{\partial x_2}$$



Power-law fluid

- Viscosity function

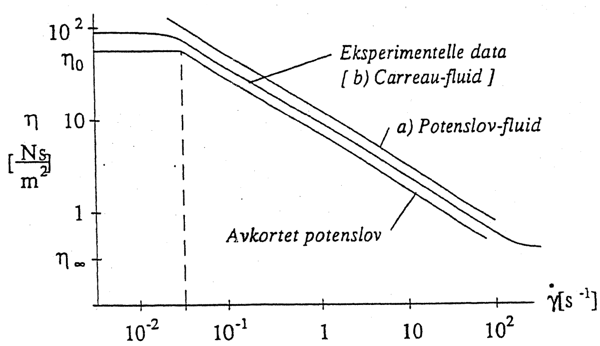
$$\eta = K\dot{\gamma}^{n-1}$$

- Power-law index n
- Consistency parameter

$$K = K_0 \exp(-A(\theta - \theta_0))$$

- Pros and cons
  - Cannot fit η for extremal values of $\dot{\gamma}$
  - Convenient for analytical solutions

- Shear thinning
  - Most real fluids
  - $n < 1$
  - $\eta \downarrow$ as $\dot{\gamma} \uparrow$
- Shear thickening $\eta \uparrow$ as $\dot{\gamma} \uparrow$



Some GNFs

- Carreau-Yasuda model

$$\frac{\eta - \eta_\infty}{\eta_0 - \eta_\infty} = (1 + (\lambda\dot{\gamma})^2)^{n-1}$$

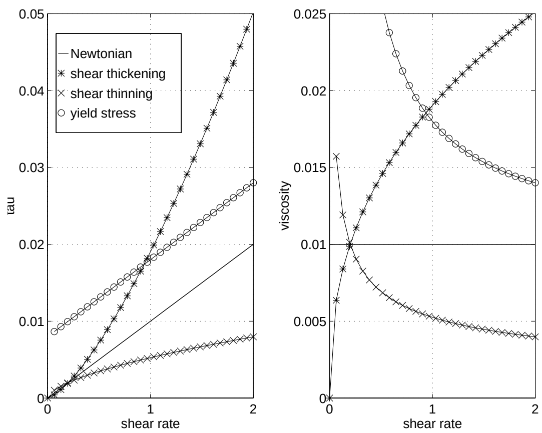
  - Fits experiments well

- Casson model

$$\begin{aligned} \tau^{\frac{1}{m}} &= \tau_0^{\frac{1}{m}} + (\eta_\infty \dot{\gamma})^{\frac{1}{m}} \\ \eta &= \eta_\infty \left[1 + \left(\frac{\tau_0}{\eta_\infty \dot{\gamma}}\right)^{\frac{1}{m}}\right]^m \end{aligned}$$

  - Viscoplastic model

- Bingham: $m = 1$
- Casson: $m = 2$
  - Originally for pigment/oil mixtures
  - Used for blood for small $\dot{\gamma}$
  - Newtonian as $\dot{\gamma} \quad \uparrow$



Cauchy equations for GNFs

- Generalized Newtonian fluids

$$\mathbf{T} = -p\mathbf{1} + \boldsymbol{\tau}$$

- Cauchy's equations

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla \cdot \mathbf{T} + \rho \mathbf{b}$$

- Cauchy equations for generalized Newtonian fluids

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{b}$$

- Component form

$$\rho(\partial_t v_i + v_k v_{i,k}) = -p_{,i} + \tau_{ik,k} + \rho b_i$$

Cauchy equation in cylinder coordinates for GNF

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} - \frac{\tau_{\theta\theta}}{r} + \rho b_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \rho b_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{zr}) + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \rho b_z$$

Stationary pipeflow for GNF

- Simplifications
  - **v** independent of z and θ
  - Deviatoric stresses are independent of z and θ
  - Symmetry $\Rightarrow \tau_{rz} = \tau_{zr}$
- Cauchy's equations reduces to

$$\begin{aligned} 0 &= -\frac{\partial p}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) - \frac{\tau_{\theta\theta}}{r} \quad \text{and} \quad 0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} \\ 0 &= -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{zr}) \end{aligned}$$

- Constant streamwise pressure gradient (i.e. $\partial_z p = c$) due to

$$\frac{\partial^2 p}{\partial \theta \partial z} = \frac{\partial^2 p}{\partial r \partial z} = \frac{\partial^2 p}{\partial z^2} = 0$$

Stationary pipeflow for GNF (contd)

- From simplified Cauchy equation in z-direction

$$0 = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{zr})$$

- We get

$$\frac{\partial}{\partial r}(r\tau_{zr}) = r\frac{\partial p}{\partial z}$$

- By integration

$$r\tau_{zr} = \frac{r^2}{2}\frac{\partial p}{\partial z} + C_1$$

- As $\tau_{rz}(r=0) = 0 \Rightarrow C_1 = 0$
- Equilibrium equation for stationary pipeflow

$$\tau_{zr} = \frac{r}{2}\frac{\partial p}{\partial z}$$

- May be applied to all GNFs

Power law for steady pipeflow

- Equilibrium

$$\Rightarrow \tau_{rz} = \frac{1}{2}\frac{\partial p}{\partial z}r$$

- Use the power law

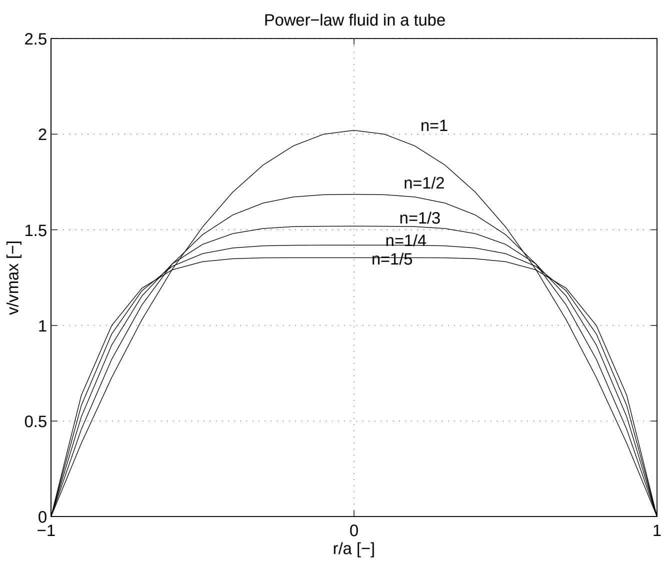
$$\tau_{rz} = \eta\frac{\partial v_z}{\partial r} = K\left|\frac{\partial v_z}{\partial r}\right|^{n-1}\frac{\partial v_z}{\partial r}$$

- After subst ($\partial_z p < 0$)

$$\frac{\partial v_z}{\partial r} = -\left(\frac{1}{2K}\frac{\partial p}{\partial z}r\right)^{\frac{1}{n}}$$

- Integrate and impose BC

$$v_z = \left(\frac{\partial p}{\partial z}\frac{a}{2K}\right)^{\frac{1}{n}}\frac{a}{\frac{1}{n}+1}\left(\left(1-\frac{r}{a}\right)^{\frac{1}{n}+1}\right)$$



Power law and Newtonian fluid for stationary pipeflow

- Velocity profile for a power law fluid

$$v_z = \left(\frac{\partial p}{\partial z}\frac{a}{2K}\right)^{\frac{1}{n}} \frac{a}{\frac{1}{n}+1}\left(1-\left(\frac{r}{a}\right)^{\frac{1}{n}+1}\right)$$

- Newtonian fluid ($K = \mu$ and $n = 1$) in $\eta = K\dot{\gamma}^{n-1}$
- From power law velocity profile

$$\begin{aligned} v_z &= v_0\left(1-\left(\frac{r}{a}\right)^2\right) \\ v_0 &= -\frac{d^2}{16\mu}\frac{\partial p}{\partial z} \end{aligned}$$

- $\Rightarrow$ power law velocity profile reduces to Newtonian expression

Velocity profiles for GNFs

- Velocity profiles for Bingham fluids are obtained with similar procedure
- Based on equilibrium relation for stationary pipeflows
- Velocity profiles