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Exam in TKT4150 Biomechanics and TTK 4170 Modelling and identification of biological systems

December 4, 2013
Duration: kl. 15.00-19.00

No printed or handwritten aids are permitted (D). Approved calculators are permitted.

Deadline for examination results: January 6, 2014

Exercise 1

The skin of a rabbit may be modelled with a hyper-elastic model, where the strain energy density (per unit mass) is given by:

$$\begin{aligned}\phi = \frac{1}{2\rho_0}[\alpha_1 E_{11}^2 + \alpha_1 E_{22}^2 + \alpha_3 E_{12}^2 + \alpha_3 E_{21}^2 + 2\alpha_4 E_{11} E_{22} + \\ c \cdot exp(a_1 E_{11}^2 + a_2 E_{22}^2 + a_3 E_{12}^2 + a_3 E_{21}^2 + 2a_4 E_{11} E_{22})]\end{aligned} \tag{1}$$

where:

$$\begin{aligned}&\alpha_1 = 1020Pa, \quad \alpha_3 = 500Pa, \quad \alpha_4 = 254Pa, \quad c = 0.779Pa \\ &a_1 = 3.79, \quad a_2 = 12.7, \quad a_3 = 1.25, \quad a_4 = 0.587\end{aligned} \tag{2}$$

and ρ_0 is the mass density of the skin in the reference configuration. It is further assumed that the rabbit skin is incompressible.

The Second Piola-Kirchhoff stresses are given as:

$$\mathbf{S} = 2J\rho\frac{\partial \phi}{\partial \mathbf{C}} = J\rho\frac{\partial \phi}{\partial \mathbf{E}} = \rho_0\frac{\partial \phi}{\partial \mathbf{E}} \tag{3}$$

The Green deformation tensor $\mathbf{C}$ may be computed from the deformation gradient tensor $\mathbf{F}$ by $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, and their components are given by:

$$F_{ij} = \frac{\partial x_i}{\partial X_j}, \qquad \text{and} \qquad C_{ij} = F_{ki}F_{kj} \tag{4}$$

The Green deformation tensor is related with Green's strain tensor with:

$$\mathbf{C} = \mathbf{1} + 2\mathbf{E} \tag{5}$$

1. Find the expressions for the Piola-Kirchoff stress components S_{ij} as expressions of the strain components E_{ij}.

2. Assume a state of pure strain:

$$x_1 = X_1 + \gamma X_2, \qquad x_2 = X_2, \qquad x_3 = X_3 \tag{6}$$

and compute $\det \mathbf{F}$ and determine whether the derformation is volume conservative or not.

3. (a) Find the components of **C** and **E**.
   (b) Compute S_{11}, S_{12}, and S_{22} and comment on how they depend on the components of E_{ij} for this particular state of pure strain.
   (c) Show that for small deformations the components of **E** reduce to:

$$E = \frac{1}{2}\begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7}$$

   (d) How do the assumption of small deformations affect S_{12}?

Exercise 2
The momentum equation may, subject to certain assumptions, be presented as:

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) \tag{8}$$

1. Present the assumptions one must make in order to render the Navier-Stokes equations in cylindrical coordinates as given in equation (8).

2. Label the various terms in equation (8) according to their physical role.

3. Introduce characteristic scales and argue for by means of an order of magnitude analysis that the boundary layer thickness (with characteristic scale δ) is:

$$\delta = \mathcal{O}\left(\sqrt{\frac{\nu}{\omega}}\right) = \mathcal{O}\left(\frac{a}{\alpha}\right) \tag{9}$$

where the Womersely number has been introduced as:

$$\alpha^2 = \frac{a^2\omega}{\nu} \tag{10}$$

4. What will the velocity profiles look like for small and large Womersley numbers?

Exercise 3
The governing equations for a compliant tube are given by:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{11a}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) = -\frac{A}{\rho}\frac{\partial p}{\partial x} + \frac{\pi D}{\rho}\tau \tag{11b}$$

1. In order to linearize the pressure term in equation (11b), one has to assume:

$$\frac{A}{\rho}\frac{\partial p}{\partial x} \approx \frac{A_0}{\rho}\frac{\partial p}{\partial x} \tag{12}$$

   where $A = A_0 + A'$, A_0 is a constant, and A' a perturbation.

   (a) Find a criterion which has to be fullfilled in order to make this assumption valid. You may use the expression for the pulse wave velocity:

$$c^2 = \frac{A_0}{\rho C} \tag{13}$$

   and the relation for a uni-directional wave $\dfrac{\Delta p}{\Delta v} = \rho c$ in your argumentation.

   (b) Show that this criterion complies with what is commonly know as the 'long wavelength assumption'.

2. Make appropriate assumptions/simplifications and show that the governing equations represent wave phenomena.