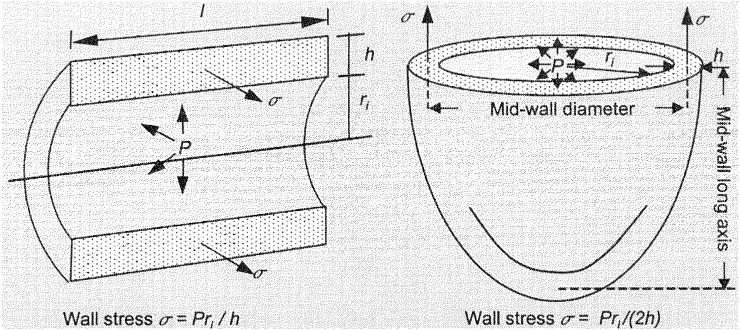
Stress analysis



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Outline

- Recap
  - Reynolds transport theorem
  - Cauchy stress theorem
  - Governing equations
- Principal stress theorem
- LaPlace law for membranes
- Physiological relevance

Reynolds and Cauchy

- Reynolds transport theorem

$$\dot{B} = \int\limits_{V(t)} \frac{\partial b}{\partial t}\, dV + \int\limits_{A(t)} b\,(\mathbf{v} \cdot \mathbf{n})\, dA$$

- Material derivative of an extensive property

$$\dot{B} = \frac{d}{dt} \int\limits_{V(t)} \beta\, \rho\, dV = \int\limits_{V(t)} \dot{\beta}\, \rho\, dV$$

- Cauchy stress theorem

$$t_i = T_{ik} n_k \Leftrightarrow \mathbf{t} = \mathbf{T}\mathbf{n}$$

Governing equations

- Conservation of mass

$$\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i} = 0$$

- Cauchy's equations of motion

$$\rho a_i = T_{ik,k} + \rho b_i$$

Principal stress theorem

- There are three orthogonal planes without shear stresses
- The planes are called principal stress planes
- The principal directions are the unit normals $\mathbf{n}_i$
- The principal stresses σ_i are the normal stresses on these planes

Show that we may find $\mathbf{t} \parallel \mathbf{n}$

- $\mathbf{t} \parallel \mathbf{n}_i \quad \Rightarrow \mathbf{t} = \sigma\,\mathbf{n}$
- Cauchy stress theorem: $\mathbf{t} = \mathbf{T} \cdot \mathbf{n}$
- $\mathbf{T} \cdot \mathbf{n} = \sigma\,\mathbf{n}$
  i.e. an eigenvalue problem
- Equivalent representations of the eigenvalue problem

$$(\sigma \mathbf{I} - \mathbf{T}) \cdot \mathbf{n} = 0 \Leftrightarrow (\sigma\,\delta_{ik} - T_{ik})\,n_k = 0$$

Solutions to the eigenvalue problem

- The eigenvalue problem $(\sigma\mathbf{I} - \mathbf{T}) \cdot \mathbf{n} = 0$ has only non-trivial solutions if

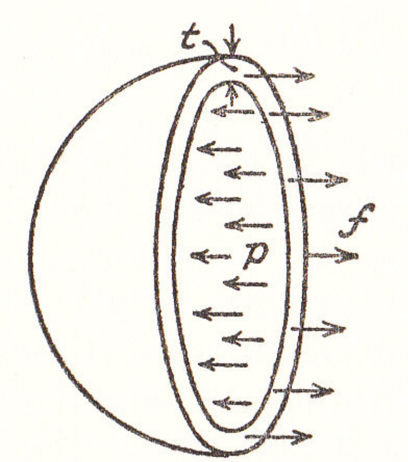
$$\det(\sigma\mathbf{I} - \mathbf{T}) = 0$$

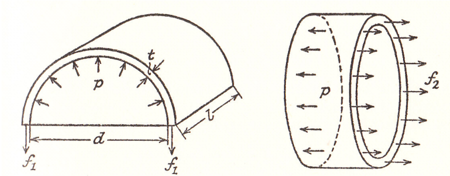
- Characteristic equation for the stress tensor

$$\sigma^3 - I\,\sigma^2 + II\,\sigma - III = 0$$

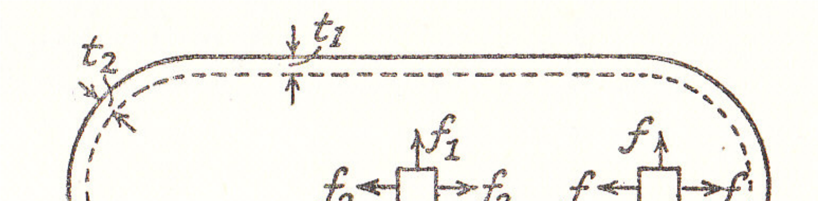
- The principal invariants of **T** are denoted: I, II, and III
- It may be shown that the three σ_i are real since $\mathbf{T} = \mathbf{T}^T$
- Orthogonal basis ($\mathbf{n}_i \cdot \mathbf{n}_j = 0$) if $\sigma_i \neq \sigma_j$

Stress analysis for thin walled containers





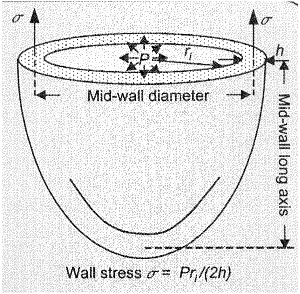
(a) (b)



Thin-walled spherical shell

- Plane-isotropic state of stress
- Force equilibrium analysis yields:

$$\sigma = \sigma_\theta = \sigma_\phi = \frac{r}{2t}p$$

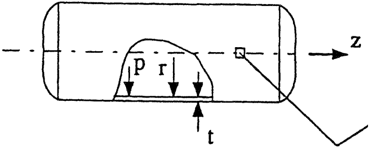


- Any direction parallel to shell surface is a principal direction ($\sigma_\theta = \sigma_\phi$)
- Stress matrix in spherical coordinates (r, θ, ϕ)

$$\mathbf{T} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sigma$$

Biaxial state of stress

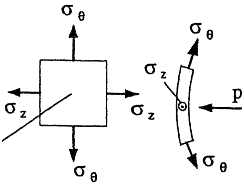
- A thin-walled circular vessel is subjected to internal pressure p
- The mid-wall radius is r
- Wall thickness is $t \ll r$



- LaPlace equations by force equilibrium analysis:

$$\sigma_z = \frac{r}{2t} p \quad \text{and} \quad \sigma_\theta = \frac{r}{t} p$$

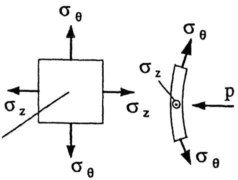
- Radial stress $\sigma_R \in [-p, 0]$, i.e. $\sigma_R \ll (\sigma_z, \sigma_\theta)$



Biaxial state of stress (cont)

- Stress matrix in cylindrical coordinates (R, θ, z)

$$\mathbf{T} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{r}{2t} p$$



Law of LaPlace

- A generalization of the previous expressions
- A membrane of thickness t and main radii of curvature r_1 and r_2
- σ_1 and σ_2 denote normal stresses in directions of main curvature
- Transmural pressure p

$$\frac{\sigma_1}{r_1} + \frac{\sigma_2}{r_2} = \frac{p}{t}$$

Models for heart wall-stress estimation

Several models have been proposed for heart wall-stress. We will come back to these later (chap 6)

- Ellipsoidal model

$$\frac{p}{D} = \frac{D}{2t}\left(1 - \frac{t}{D} - \frac{D^2}{2l^2}\right)$$

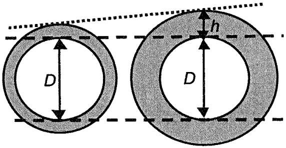
- The Arts model for fiber stress σ_f

$$\frac{p}{\sigma_f} = \frac{1}{3}\ln\left(1 + \frac{V_w}{V}\right)$$

where V_w is LV wall volume

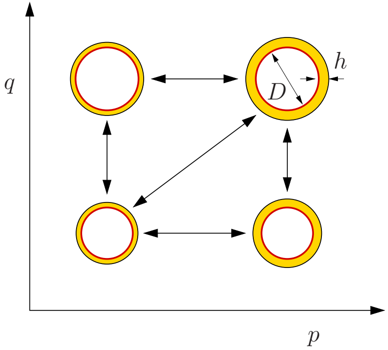
Physiological relevance of the law of LaPlace

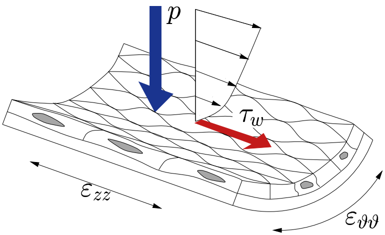
- Great conceptual importance, although valid for simple geometries only
- r/t main determinant for wall stress
- Hypertension (high blood pressure)
  - Cardiac muscle cells grow and increase wall thickness
  - $r/t \downarrow$ while $p \uparrow$
  - Wall stress $\sigma \propto \frac{r}{t} p$ remain similar
- Same effect for arteries which hypertrophy in hypertension



- How stresses in cells are sensed is still largely unknown

Important parameters for vascular hemodynamics[1]





- pressure $p \uparrow$ $\Rightarrow$ wall strain $\varepsilon_{\theta\theta} \uparrow$ $\Rightarrow$ wall thickness $h \uparrow$
- flow $q \uparrow$ $\Rightarrow$ wall shear stress $\tau_w \uparrow$ $\Rightarrow$ diameter $D \uparrow$

[1] vd Vosse 2008

Summary

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