Input impedance, reflections and the quarter wave length formula

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September 18, 2017

Outline

Poiseuille flow in compliant tube

Simplified momentum equation:

$$\frac{dp}{dx} = -\frac{8\mu}{\pi a^4}\, Q = -\frac{8\pi\mu}{A^2}\, Q$$

Constitutive model: $A(p) = A_0 + C\,(p - p_0)$

$$\frac{dp}{dx} = \frac{\partial p}{\partial A}\frac{dA}{dx} = \frac{1}{C}\frac{dA}{dx} = -\frac{8\pi\mu}{A^2}\, Q$$

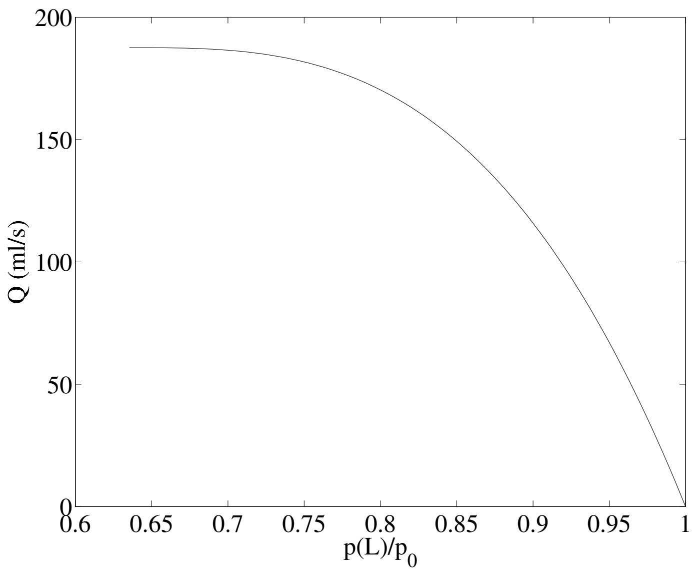
$$A^2\,\frac{dA}{dx} = \frac{1}{3}\,\frac{d}{dx}\left(A^3\right) = -8\pi\mu CQ$$

Integration

$$A(x)^3 = A(0)^3 - 24\pi\mu CQx$$

Pressure and flow for stationary flow in compliant tube

$$Q(x) = \frac{A(0)^3 - A(x)^3}{24\pi\mu Cx}, \quad p(x) = p_0 + \frac{A(x) - A(0)}{C}$$



Progressive waves superimposed on steady flow

- Perturbations

$$A^* = A_0 + A, \quad u^* = u_0 + u, \quad Q^* = Q_0 + Q$$

- Variable transformation: $x' = x - u_o t, \quad t' = t$
- Transformed governing equations

$$\frac{\partial A}{\partial t'} = -\frac{\partial Q}{\partial x'}$$
$$\frac{\partial u}{\partial t'} = -\frac{1}{\rho}\frac{\partial p}{\partial x'}$$

- Solutions on the form

$$p = p_0 f(x' - ct') + p_o' g(x' + ct')$$

- Wave speeds in Eulerian reference frame

$$c_f = c + u_0 \quad c_b = c - u_0$$

Characteristic impedance

- Governing equations

$$\begin{aligned} C\frac{\partial p}{\partial t} &= -\frac{\partial Q}{\partial x} \\ \frac{\partial Q}{\partial t} &= -\frac{A}{\rho}\frac{\partial p}{\partial x} \end{aligned}$$

- Solutions

$$\begin{aligned} p &= p_0\, f(x - ct) + p_0^*\, g(x + ct) \\ Q &= Q_0\, f(x - ct) + Q_0^*\, g(x + ct) \end{aligned}$$

- By subst in momentum equation and collection of terms

$$\left(\frac{A}{\rho} p_0 - cQ_0\right) f' + \left(\frac{A}{\rho} p_0^* + cQ_0^*\right) g' = 0$$

- Must hold for arbitrary f' and g'

$$Z_c = \frac{p_0}{Q_0} = -\frac{p_0^*}{Q_0^*} = \frac{\rho c}{A}$$

Practical estimation of Z_c

- Average of Z_i
  - Higher frequencies cancel and are damped
  - Average between 4th and 10th harmonic
- Slope of p and Q
  - In early part of systole/ejection phase

$$Z_c = \frac{\Delta p/\Delta t}{\Delta Q/\Delta t}$$

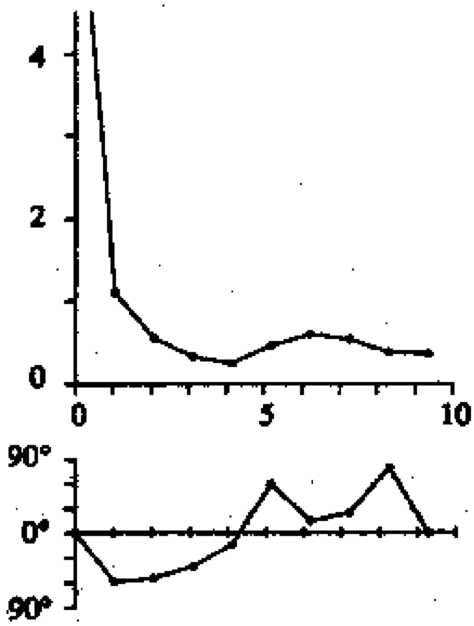
- Both methods rely on the fact that Z_c is a p-Q relation in absence of reflections
- Reflections are small in early systole and at high frequencies

Arterial input impedance

- Provides a (complete) and comprehensive description of the arterial system [a]
- Impedance
  - A measure of oppostion to flow
  - Frequency dependent resistance
  - Resistance for non-oscillatory or steady motion
- Definition
  - The ratio of harmonic terms of pressure and corresponding harmonic terms of flow

$$p = |\hat{p}|e^{j(\omega t+\phi)}, \quad q = |\hat{q}|e^{j(\omega t+\beta)}$$

$$Z_i = \frac{|\hat{p}|}{|\hat{q}|}e^{j\theta}, \qquad \theta = \phi - \beta$$



[a] Snapshots of hemodynamics

Applicability of Fourier analysis

- Used to relate hemodynamic variables such as pressure and flow
- Not meaningful to relate time signals
  - Diastolic pressure and flow
  - Division of pressure by zero flow does not provide meaningful information
- Produce a mean and multiples of the heart rate
- Each harmonic has an amplitude and a phase angle
- Impedance
  - Relate pressure and flow
  - Ohm's law is applied for each frequency
  - Only valid for a linear relation between pressure and flow
- Aortic input impedance
  - Venous pressure may be neglected
  - Aortic pressure and flow gives a sufficiently accurate approximation of the input impedance

Limitations to the use of Fourier analysis

- May only be used for periodic signals. The value of the signal at start and end should be the same
- The relation of two signals should be linear
- Despite nonlinear pressure-flow relation, the nonlinearity is not so strong in many cases that large errors result
- The scatter in modulus and phase has been attributed to nonlinearities
- Higher harmonics with smaller amplitudes are more exposed to noise than lower harmonics
- High frequency information should be considered with care.

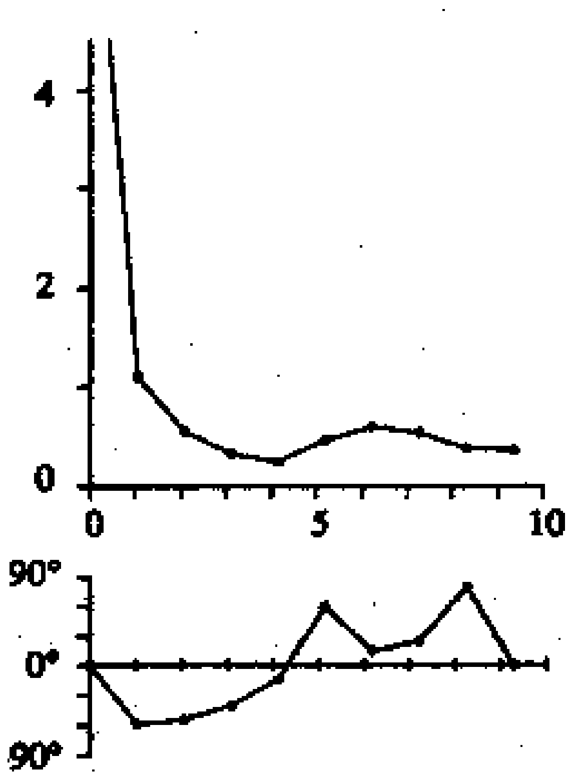
Reflection factor

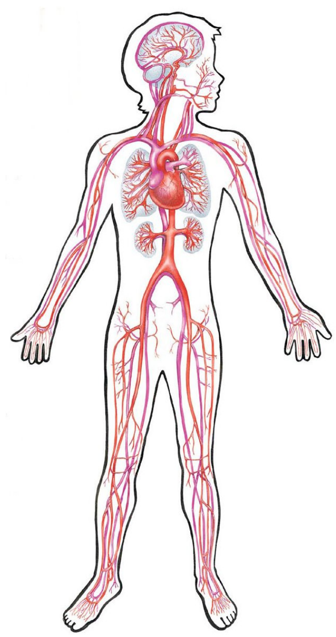
- Occur at any point where there is an abrupt change in characteristic impedance (mismatch in impedance)
- Oscillations at origin: the reflections will mix with the original pulse
- Spatial variations in amplitude and different wave pattern in flow and pressure are indicators for reflections
- Wave separation
$$p = p_f + p_b$$
- Reflection factor
$$\Gamma \equiv \frac{p_b}{p_f} = -\frac{Q_b}{Q_f}$$
- Easy to show
$$\Gamma = \frac{Z_{in} - Z_c}{Z_{in} + Z_c}$$

The quarter wavelength formula

- Forward waves
$$p_f = p_0\ e^{j\omega t}, \quad Q_f = \frac{p_0}{Z_c}\ e^{j\omega t}$$
- Reflected waves ($\Gamma = 1$)
$$p_b = p_0\ e^{j\omega(t-2L/c)}, \quad Q_b = -\frac{p_0}{Z_c}\ e^{j\omega(t-2L/c)}$$
- Input impedance
$$Z_{in} = \frac{p_f + p_b}{Q_f + Q_b} = Z_c\ \frac{e^{j\omega t} + e^{j\omega(t-2L/c)}}{e^{j\omega t} - e^{j\omega(t-2L/c)}}$$
- $Z_{in} = 0 \Rightarrow \dfrac{2\omega L}{c} = \pi \Rightarrow L = \frac{\lambda}{4}$

Example: The quarter wavelength formula and Z_{in}





min $|Z_{in}|$ at f = 3.8 Hz

$\Rightarrow L \approx 0.33m$

Summary