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Exam in TKT4150 Biomechanics and TTK4170 Modelling and identification of biological systems

November 30, 2015
Duration: 09.00-13.00

No printed or hand-written support material is allowed.
A specific basic calculator is allowed (D).

Exercise 1: Laminar Pipe Flow

For stationary, laminar flow in a straight, rigid tube (Fig. (1)) the Cauchy equations take the form:

$$\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{zr}\right), \qquad \tau_{zr} = \mu\,\frac{\partial v}{\partial r} \tag{1}$$

where $v(r) = v_z(r)$ is the velocity component in the z-direction at a distance r from the central axis and the only non-zero component of the velocity field in cylindrical coordinates (r,θ,z): $v_\theta = 0$, $v_r = 0$. Further, let d denote the diameter of the vessel and μ the viscosity of the blood.

a) Show that the velocity profile in the vessel may be expressed as:

$$v(r) = -\frac{\partial p}{\partial z}\,\frac{a^2}{4\,\mu}\left(1-(r/a)^2\right), \qquad \text{where} \quad a = d/2. \tag{2}$$

Note this is referred to as Poiseuille flow.

b) Derive an expression of the volumetric flow Q through the vessel as a function of $\frac{\partial p}{\partial z}$, a, and μ. How is this expression for Q relevant for physiological flow and pressure regulation?

Hint. $Q = \int_A v \cdot \mathrm{d}A$

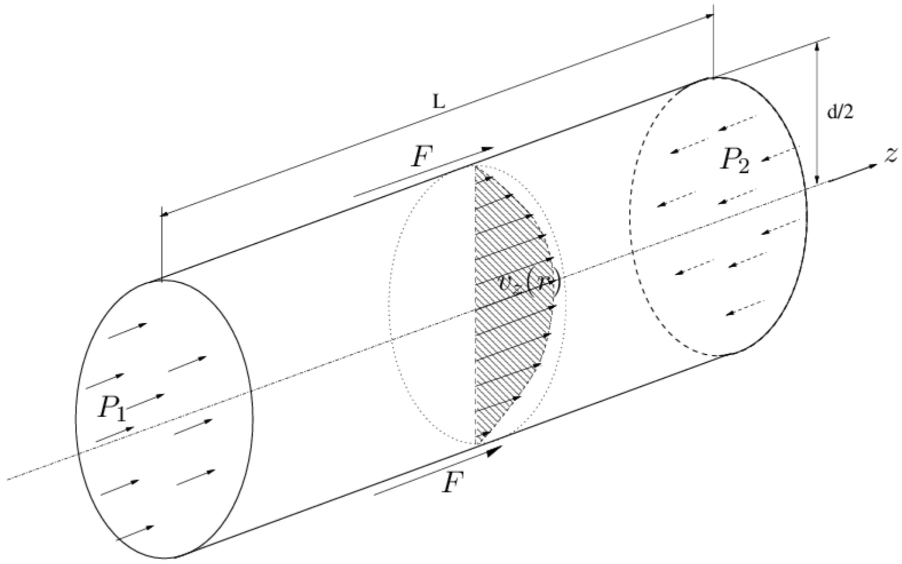

Figure 1: Diagram of stationary Newtonian rigid pipe flow.

c) Show that the ratio between Q and the velocity at the center of the vessel $v_0 = v_z(r = 0)$ is equal to $A/2$, where $A = \pi\, a^2$.

— The constitutive equations for an incompressible Newtonian fluid read:

$$\boldsymbol{T} = -p\mathbf{1} + 2\mu\boldsymbol{D} \text{ or in index notation } T_{ij} = -p\delta_{ij} + 2\mu D_{ij}, \tag{3}$$

where $\boldsymbol{T}$ is the stress tensor, $\boldsymbol{D}$ the strain rate tensor with components: $D_{ij} = 1/2(v_{i,j} + v_{j,i})$. A viscous force F, acts on any cylindrical element of blood in the direction opposite the flow due to slower moving blood outside the element. The magnitude of the viscous force is given by

$$F = S\tau_{rz}, \tag{4}$$

where $S = 2\pi rL$ is the area of the cylindrical element and τ_{rz} is shear stress component that can be derived from eq.(3).

d) Derive an expression for F and calculate the viscous force on the wall of the vessel (at $r = a$) using the result of (4).

e) Calculate the viscous force on the wall of the vessel (at $r = a$) using the force equilibrium on the fluid inside the vessel of length L and radius a. Compare this expression with the result in the question above.

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In addition to the assumption of local Poiseuille-flow (see (2)), assume a linear constitutive model:

$$A(p) = A_0 + C\ (p - p_0) \tag{5}$$

f) What is the physical meaning of C?

g) Use equation (5) to eliminate the pressure p from the pressure flow relation for Poiseuille flow, integrate and express the area $A(z)$ at a given location z as a function of the inlet area $A(0)$, and μ, C, Q and z.

h) Illustrate and discuss how the pressure (area) flow relationship in a compliant vessel differs from a rigid vessel.

Exercise 2: Hyperelastic Aorta

Consider a vessel of internal radius R_i and wall thickness H in its undeformed configuration Ω_0. The vessel has internal pressure p_i, internal radius r_i, and wall thickness h in the deformed configuration Ω.

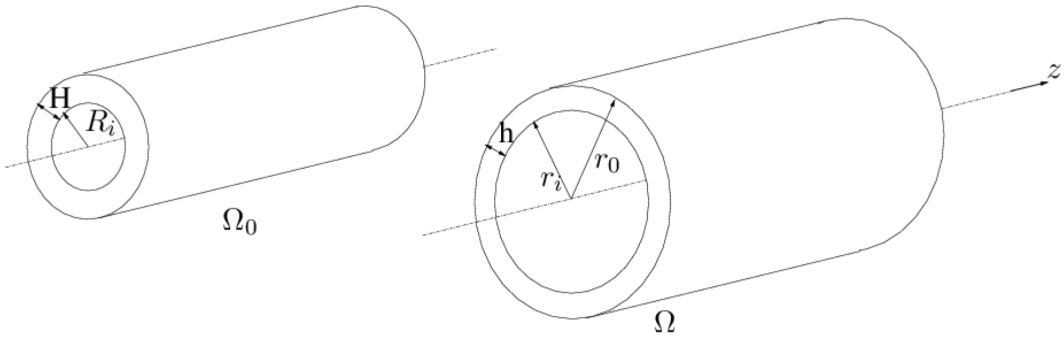

Figure 2: Reference and deformed configurations of a section of blood vessel.

The deformation gradient $\boldsymbol{F}$ can be expressed in the cylindrical coordinate system $(\boldsymbol{e}_{rr}, \boldsymbol{e}_{\theta\theta}, \boldsymbol{e}_{zz})$ as:

$$\boldsymbol{F} = \begin{bmatrix} \lambda_{rr} & 0 & 0 \\ 0 & \lambda_{\theta\theta} & 0 \\ 0 & 0 & \lambda_{zz} \end{bmatrix}, \tag{6}$$

where $\lambda_{rr} = \lambda_r$, $\lambda_{\theta\theta} = \lambda_\theta$, and $\lambda_{zz} = \lambda_z$ are the stretches in the radial, circumferential and longitudinal directions, respectively. We assume that the vessel wall is incompressible: $J = \det\boldsymbol{F} = 1$. The stretch in the longitudinal direction is equal to one, $\lambda_z = 1$.
First, we consider the case where $h \ll r_i$. Therefore, the vessel can be treated as a thin–walled structure and the membrane theory, $T_{rr} = 0$ can be applied to analyze the stresses in the vessel wall.

a) Derive an expression for the internal pressure p_i as a function of r_i, h and $T_{\theta\theta}$ the Cauchy stress in the circumferential direction.

Hint. Use Laplace's law $\left(\frac{\sigma_1}{r_1} + \frac{\sigma_2}{r_2} = \frac{p_i}{h}\right)$.

b) Express h as a function of H and λ_θ and r_i as a function of R_i and λ_θ.

c) Express p_i as a function of R_i, H, $T_{\theta\theta}$ and λ_θ.

The internal radius of a segment of the aorta was measured at diastolic and systolic pressures, 80mmHg and 120mmHg respectively. The values were $r_i(p_i = 80) = 1.2$cm and $r_i(p_i = 120) = 1.5$cm. The thickness of the wall at diastolic pressure was measured to be $h = 1$mm

d) What is the wall thickness h at systolic pressure.

e) Find the ratio between systolic and diastolic circumferential stress ($\frac{T_{\theta\theta}(120)}{T_{\theta\theta}(80)}$)?

Now, we assume that the vessel wall can be modeled as an incompressible isotropic hyperelastic material. The second Piola–Kirchhoff stress tensor $\boldsymbol{S}$ can be expressed as:

$$\boldsymbol{S} = 2f(I_1)\mathbf{1} + q\boldsymbol{C}^{-1}, \tag{7}$$

where f is a scalar function, $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$ is the right Cauchy–Green tensor, $\mathbf{1}$ is the second order identity tensor, q is a Lagrange multiplier, and $I_1 = \mathrm{tr}\boldsymbol{C}$ is the first invariant of $\boldsymbol{C}$.

f) The Cauchy stress $\boldsymbol{T}$ can be found by pushing forward the second Piola-Kirchhoff stress tensor $\boldsymbol{S}$ into the current configuration: $\boldsymbol{T} = J^{-1}\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T$. Give the expression of the Cauchy stress tensor $\boldsymbol{T}$.

In the case where $h \ll r_i$, the stress in the radial direction can be neglected: $T_{rr} = 0$ or $S_{rr} = 0$. Hence, the vessel is in a plane stress state.

g) Use the plane stress condition to derive an expression of the Lagrange multiplier q (as a function of $f(I_1)$ and λ_θ).

h) Assuming $\lambda_\theta > 1$, find the principal stresses and their directions in terms of $\boldsymbol{e}_{rr}$, $\boldsymbol{e}_{\theta\theta}$ and $\boldsymbol{e}_{zz}$.

i) Using the expressions for $\boldsymbol{T}$ and q calculate the ratio between systolic and diastolic axial stress ($\frac{T_{zz}(120)}{T_{zz}(80)}$) using the reported data for the section of the aorta.

In many cases, the condition $h \ll r_i$ is not fulfilled and the vessel must be treated as a thick–walled cylindrical structure. The Cauchy equations of motion can be used to write the cross–sectional equilibrium,

$$\mathrm{div}\boldsymbol{T} + \rho\boldsymbol{b} = \rho\boldsymbol{a}, \tag{8}$$

where ρ is the density of the structure, $\boldsymbol{b}$ the body forces and $\boldsymbol{a}$ the acceleration. The Cauchy equation in cylindrical coordinates (r,θ,z) in the radial direction is,

$$\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} + \rho b_r = \rho a_r, \tag{9}$$

where T_{rr} and $T_{\theta\theta}$ are normal stresses and $\tau_{r\theta}$ and τ_{zr} are shear stresses.

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j) Simplify eq.(9) when body forces and accelerations are excluded and $\boldsymbol{e}_{rr}$, $\boldsymbol{e}_{\theta\theta}$ and $\boldsymbol{e}_{zz}$ are principal stress directions.

k) Now, suppose the internal pressure of the vessel is p_i. Using the result from the previous question and the boundary conditions $p_i = -T_{rr}|_{r=r_i}$ and $T_{rr}|_{r=r_0} = 0$, show

$$p_i = \int_{r_i}^{r_0} \frac{T_{\theta\theta} - T_{rr}}{r} dr, \quad r_i \leq r \leq r_0, \tag{10}$$

where r_0 is the outer radius of the vessel in Ω.

Exercise 3: Estimation

a) Draw the circuit equivalent of the two-element Windkessel model, and explain the physical background of each parameter.

Q $\qquad\qquad$ Q_p

Figure 3: The rate of change of arterial volume equals the difference between aortic inflow (Q) and outflow (Q_p) towards the periphery.

b) Using conservation of mass, derive the differential equation describing the model.

Hint. See figure (3).

c) Discretize the differential equation from b, and show that it can be written in matrix form as

$$\hat{\boldsymbol{P}} = \boldsymbol{A}\boldsymbol{\alpha} \tag{11}$$

where $\hat{\boldsymbol{P}}$ is the model pressure. Define $\boldsymbol{A}$ and $\boldsymbol{\alpha}$.

d) Find the least squares estimate of the parameter vector $\boldsymbol{\alpha}$ from (11).

e) When using a ballistic estimation method, only the initial pressure is assumed known, and thus the model can't be written in the matrix form given in (11). Instead, a common method is to use a least square adaption of a ballistic simulation of the model to the measurements. Present a functional based on a ballistic simulation of the model and describe a scheme for a ballistic least square estimate of the parameter vector using the sensitivity matrix.

f) How can the observability and robustness of the estimation be analysed from the sensitivity matrix?