# PROBLEM SET 12

TKT4150 Biomechanics

**Main topics:** Arterial stenosis. Dimensional analysis with the Buckingham Π-theorem.

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## 1 Arterial stenosis

*A patient is suffering from a narrowing of the aorta, due to a calcium build-up. The dimensions and modelling assumptions of the stenosis are shown in Figure 1.*

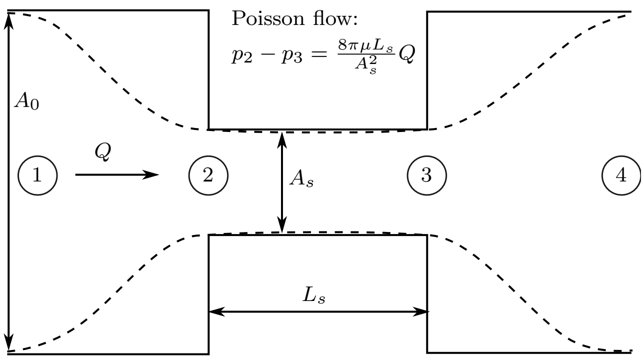


***Figure 1:*** *Modelling of an arterial stenosis. Assumptions made in different regions are:* $1 \rightarrow 2$*: Bernoulli,* $2 \rightarrow 3$*: Poisson flow and* $3 \rightarrow 4$*: Conservation of momentum.*

a) *Using the assumptions indicated in the figure, show that the pressure drop over the section is:*

$$\Delta p = \frac{8\pi\mu L_s}{A_s^2} Q + \frac{\rho}{2A_0^2}\left(\frac{A_0}{A_s} - 1\right)^2 Q^2 \tag{1}$$

For the first section ($1 \rightarrow 2$) we have Bernoulli:

$$\frac{v_1^2}{2} + \frac{p_1}{\rho} = \frac{v_2^2}{2} + \frac{p_2}{\rho} \tag{2}$$

We have $v_1 = Q/A_0$ and $v_2 = Q/A_s$. Therefore:

$$p_2 - p_1 = \frac{\rho}{2A_0^2}\left[1 - \left(\frac{A_0}{A_s}\right)^2\right] Q^2 \tag{3}$$

Further, for the second section ($2 \rightarrow 3$), we have Poisson flow. This gives pressure drop as stated in the figure:

$$p_3 - p_2 = -\frac{8\pi\mu L_s}{A_s^2} Q \tag{4}$$

Finally, for the third section ($3 \rightarrow 4$), we can utilize the momentum equation. This gives:

$$\begin{aligned}
(p_3 - p_4)A_0 &= (\dot{m}v)_4 - (\dot{m}v)_3 && (5)\\
&= \left(\rho Q \frac{Q}{A_0}\right) - \left(\rho Q \frac{Q}{A_s}\right) && (6)\\
(p_4 - p_3) &= \frac{\rho}{A_0}\left[\frac{1}{A_s} - \frac{1}{A_0}\right] Q^2 && (7)\\
(p_4 - p_3) &= \frac{\rho}{A_0^2}\left[\frac{A_0}{A_s} - 1\right] Q^2 && (8)
\end{aligned}$$

The total pressure difference can be expressed as:

$$\begin{aligned}
\Delta p = p_4 - p_1 &= (p_4 - p_3) + (p_3 - p_2) + (p_2 - p_1) && (9)\\
&= \frac{\rho}{A_0^2}\left[\frac{A_0}{A_s} - 1\right] Q^2 - \frac{8\pi\mu L_s}{A_s^2} Q + \frac{\rho}{2A_0^2}\left[1 - \left(\frac{A_0}{A_s}\right)^2\right] Q^2 && (10)\\
&= -\frac{\rho Q^2}{2A_0^2}\left[-2\frac{A_0}{A_s} + 2 - 1 + \left(\frac{A_0}{A_s}\right)^2\right] - \frac{8\pi\mu L_s}{A_s^2} Q && (11)\\
&= -\left[\frac{8\pi\mu L_s}{A_s^2} Q + \frac{\rho}{2A_0^2}\left(\frac{A0}{A_s} - 1\right)^2 Q^2\right] \qquad \text{q.e.d.} && (12)
\end{aligned}$$

b) *During systole, the patient has an average blood flow of 54 ml/s. Blood is assumed to have density and viscosity $\rho = 1060 kg/m^3$ and $\mu = 3.5 \cdot 10^{-3} Pa \cdot s$. Further, the dimensions characterising the stenosis are given in Table 1. The flow during systole is assumed steady (a fairly crude assumption). What pressure increase is needed, for the flow rate to be maintained?*

MATLAB gives:

```
>> dp=@(rho,Q,A0,As,Ls,mu) -rho*Q^2/(2*A0^2)*((A0/As)-1)^2-8*pi*mu*Ls/(As^2)*Q

dp =

    @(rho,Q,A0,As,Ls,mu)-rho*Q^2/(2*A0^2)*((A0/As)-1)^2-8*pi*mu*Ls/(As^2)*Q

>> dp(1060,5.4E-5,3.5E-5,7.5E-6,0.01,3.5E-3)

ans =

  -1.7806e+04
```

Thus, the stenosis gives a pressure drop of $17.8kPa \approx 134mmHg$. To maintain the same flow rate, the heart pressure has to increase with 134 mmHg! This is not possible, and the stenosis will therefore result in both a lowered blood flow and a higher load on the heart.

***Table 1:*** *Dimensions characterizing the stenosis.*

| $A_s$ | $A_0$ | $L_s$ |
|---|---|---|
| $0.75cm^2$ | $3.5cm^2$ | $1cm$ |

c) *What consequences do stenoses imply for affected patients? How does the area of a stenosis affect the pressure drop?*

Normally, a stenosis in the aorta leads to heightened blood pressure in the upper body, and a lowered blood pressure in the lower body and legs. A stenosis can result in a decreased blood flow, which might lead to *syncope* (low blood flow to the brain).

Equation 1 gives:

$$\Delta p \propto (\frac{A_0}{A_s} - 1)^2 \tag{13}$$

$$\Delta p \propto Q^2 \tag{14}$$

Thus, the area of the stenosis affects the pressure drop quadratically. The flow rate affects the pressure drop caused by a given stenosis very much as well $\Rightarrow$ less severe when at rest.

## 2 Dimensional analysis

*A pipe has length $L$, diameter $D$ and wall roughness $\epsilon$, and is filled with a fluid with density $\rho$ and viscosity $\mu$. The flow has the average velocity $V$. The pressure drop can be assumed a function like this:*

$$\Delta p = f(\rho, \mu, V, D, L, \epsilon) \tag{15}$$

a) *Use the Buckingham $\Pi$-theorem to determine three non-dimensional terms which constitute the variables of a non-dimensional version of Equation 15.*

The units of the variables by their basic dimensions are shown in Table 2.

***Table 2:*** *Basic dimensions of variables.*

| | $\Delta p$ | $\rho$ | $\mu$ | $V$ | $D$ | $L$ | $\epsilon$ |
|---|---|---|---|---|---|---|---|
| L | -1 | -3 | -1 | 1 | 1 | 1 | 1 |
| M | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| T | -2 | 0 | -1 | -1 | 0 | 0 | 0 |

The repeating variables are chosen (need 3) as $\rho$, $V$ and $D$. These are dimensionally independent (can't express the units of either of them using the units of the others) and all basic dimensions are included. Emphasis on simplicity when chosing these are also made, so that the equations to solve afterwards get as simple as possible.

This gives these Π-terms:

$$\Pi_1 = \Delta p \cdot \rho^{a_1} V^{b_1} D^{c_1} \tag{16}$$
$$\Pi_2 = \mu \cdot \rho^{a_2} V^{b_2} D^{c_2} \tag{17}$$
$$\Pi_3 = L \cdot \rho^{a_3} V^{b_3} D^{c_3} \tag{18}$$
$$\Pi_4 = \epsilon \cdot \rho^{a_4} V^{b_4} D^{c_4} \tag{19}$$
$$\tag{20}$$

The potential constants yield the following equation systems for each $Pi$-term:

$$\Pi_1 : \qquad L^0 M^0 T^0 = (L^{-1} M^1 T^{-2})(L^{-3} M^1 T^0)^{a_1} (L^1 M^0 T^{-1})^{b_1} (L^1 M^0 T^0)^{c_1} \tag{21}$$
$$\Downarrow$$
$$-1 - 3a_1 + b_1 + c_1 = 0$$
$$1 + a_1 = 0$$
$$-2 - b_1 = 0$$
$$\Downarrow$$
$$a_1 = -1, \quad b_1 = -2, \quad c_1 = 0 \tag{22}$$
$$\Downarrow$$
$$\Pi_1 = \Delta p \cdot \rho^{-1} V^{-2} D^0 = \frac{\Delta p}{\rho V^2} \tag{23}$$

$$\Pi_2 : \qquad L^0 M^0 T^0 = (L^{-1} M^1 T^{-1})(L^{-3} M^1 T^0)^{a_2} (L^1 M^0 T^{-1})^{b_2} (L^1 M^0 T^0)^{c_2} \tag{24}$$
$$\Downarrow$$
$$-1 - 3a_2 + b_2 + c_2 = 0$$
$$1 + a_2 = 0$$
$$-1 - b_2 = 0$$
$$\Downarrow$$
$$a_2 = -1, \quad b_2 = -1, \quad c_2 = -1 \tag{25}$$
$$\Downarrow$$
$$\Pi_2 = \mu \rho^{-1} V^{-1} D^{-1} = \frac{\mu}{\rho V D} \tag{26}$$

$$\Pi_3: \qquad L^0M^0T^0 = (L^1M^0T^0)(L^{-3}M^1T^0)^{a_3}(L^1M^0T^{-1})^{b_3}(L^1M^0T^0)^{c_3} \tag{27}$$

$$\Downarrow$$

$$1 - 3a_3 + b_3 + c_3 = 0$$

$$a_3 = 0$$

$$b_3 = 0$$

$$\Downarrow$$

$$a_3 = 0, \quad b_3 = 0, \quad c_3 = -1 \tag{28}$$

$$\Downarrow$$

$$\Pi_3 = L\rho^0V^0D^{-1} = \frac{L}{D} \tag{29}$$

$$\Pi_4: \qquad L^0M^0T^0 = (L^1M^0T^0)(L^{-3}M^1T^0)^{a_3}(L^1M^0T^{-1})^{b_3}(L^1M^0T^0)^{c_3} \tag{30}$$

$$\Downarrow$$

$$1 - 3a_4 + b_4 + c_4 = 0$$

$$a_4 = 0$$

$$b_4 = 0$$

$$\Downarrow$$

$$a_4 = 0, \quad b_4 = 0, \quad c_4 = -1 \tag{31}$$

$$\Downarrow$$

$$\Pi_4 = L\rho^0V^0D^{-1} = \frac{\epsilon}{D} \tag{32}$$

Equations 23, 26, 29 and 32 result in:

$$\frac{\Delta p}{\rho V^2} = f\left(\frac{\mu}{\rho V D}, \frac{L}{D}, \frac{\epsilon}{D}\right) \tag{33}$$