Suggested solution:
PROBLEM SET 5

TKT4150 Biomechanics

Main topics: Hyper-elasticity.

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(1) Mechanics of hyper-elastic rabbit skin

The skin of a rabbit is modelled using a hyper-elastic model, where $S_{ij} = \rho_0 \frac{\partial \phi}{\partial E_{ij}}$. *Assume the strain energy density (per unit mass) is given by:*

$$\phi = \frac{1}{2\rho_0}[\alpha_1 E_{11}^2 + \alpha_1 E_{22}^2 + \alpha_3 E_{12}^2 + \alpha_3 E_{21}^2 + 2\alpha_4 E_{11}E_{22} + \tag{1}$$
$$c \cdot exp(a_1 E_{11}^2 + a_2 E_{22}^2 + a_3 E_{12}^2 + a_3 E_{21}^2 + 2a_4 E_{11}E_{22})] \tag{2}$$

where

$$\alpha_1 = 1020Pa, \quad \alpha_3 = 500Pa, \quad \alpha_4 = 254Pa, \quad c = 0.779Pa \tag{3}$$
$$\alpha_1 = 3.79, \quad a_2 = 12.7, \quad a_3 = 1.25, \quad a_4 = 0.587 \tag{4}$$

and ρ_0 *is the mass density of the skin. It is further assumed that the rabbit skin is incompressible.*

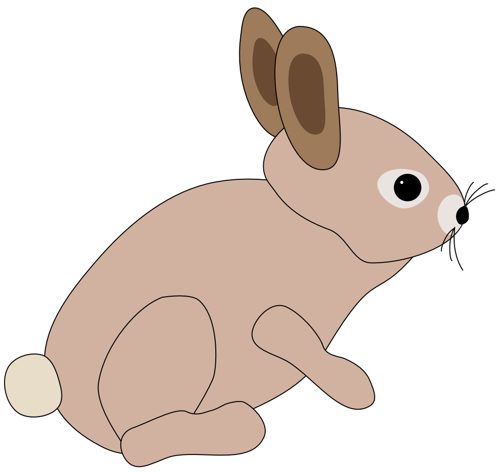

Figure 1: *Rabbit.*

a) *Establish the expressions for the Piola-Kirchoff stress components* S_{ij} *as expressions of the strain components* E_{ij}.

We have these relations:

$$\mathbf{T} = 2\rho\mathbf{F}\frac{\partial\psi}{\partial\mathbf{C}}\mathbf{F}^T = \frac{1}{J}\mathbf{F}\mathbf{S}\mathbf{F}^T \tag{5}$$

$$\mathbf{S} = 2J\rho\frac{\partial\psi}{\partial\mathbf{C}} = J\rho\frac{\partial\psi}{\partial\mathbf{E}} = \rho_0\frac{\partial\psi}{\partial\mathbf{E}} \tag{6}$$

This gives:

$$\kappa = a_1E_{11}^2 + a_2E_{22}^2 + a_3E_{12}^2 + a_3E_{21}^2 + 2a_4E_{11}E_{22} \tag{7}$$

$$S_{11} = \alpha_1E_{11} + \alpha_1E_{22} + c(a_1E_{11} + a_4E_{22})exp(\kappa) \tag{8}$$

$$S_{22} = \alpha_1E_{22} + \alpha_4E_{11} + c(a_2E_{22} + a_4E_{11})exp(\kappa) \tag{9}$$

$$S_{12} = \alpha_3E_{12} + ca_3E_{12}exp(\kappa) \tag{10}$$

b) *Choose $E_{12} = E_{22} = 0$. Determine the Cauchy stresses T_{11} and T_{22}, as functions of the stretch λ_1 and the strain component E_{11}.*

Let $x_1 = \lambda_1X_1$, $x_2 = X_2$ and $x_3 = \lambda_3X_3$.

Incompressible material leads to $J = det(\mathbf{F}) = 1 = \lambda_1 \cdot 1 \cdot \lambda_3 \Rightarrow \lambda_3 = \frac{1}{\lambda_1}$

This gives:

$$F_{ij} = \left[\frac{\partial x_i}{\partial X_j}\right] = \begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{\lambda_1}\end{bmatrix} \quad \Rightarrow \quad F_{ij}^{-1} = \begin{bmatrix}\frac{1}{\lambda_1} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \lambda_1\end{bmatrix} \tag{11}$$

We know that:

$$\mathbf{T} = 2\rho\mathbf{F}\frac{\partial\psi}{\partial\mathbf{C}}\mathbf{F}^T = \frac{1}{J}\mathbf{F}\mathbf{S}\mathbf{F}^T \tag{12}$$

$$\mathbf{S} = 2J\rho\frac{\partial\psi}{\partial\mathbf{C}} = J\rho\frac{\partial\psi}{\partial\mathbf{E}} = \rho_0\frac{\partial\psi}{\partial\mathbf{E}} \tag{13}$$

$$\mathbf{T} = \mathbf{F}\mathbf{S}\mathbf{F}^T \tag{14}$$

Combining this with Equations 7-10, we get:

$$T_{11} = \lambda_1^2S_{11}, \quad T_{22} = S_{22} \tag{15}$$

The definition of longitudinal strain in direction $\mathbf{e}$ is given as:

$$\left(\frac{\partial s}{\partial s_0}\right)^2 - 1 = 2e_iE_{ij}e_j \tag{16}$$

The stretch λ_1 along $\mathbf{e} = \mathbf{e}_1$ can be inserted to give:

$$\lambda_1^2 - 1 = 2E_{11} \quad \Leftarrow \quad \lambda_1^2 = 1 + 2E_{11} \tag{17}$$

Combining this with Equation 15, we get:

$$T_{11} = (1 + 2E_{11})S_{11}, \quad T_{22} = S_{22} \tag{18}$$

c) *Assume a state of pure shear:*

$$x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \quad x_3 = X_3 \tag{19}$$

Determine the Cauchy stresses T_{ij} expressed by the Piola-Kirchoff stresses S_{ij} for this deformation state. Which components of the Green strain tensor affect the resulting Piola-Kirchoff and Cauchy stresses? What happens to the Cauchy stresses when we assume small deformations?

We see that:

$$\mathbf{F} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{F}^T = \begin{bmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad J = det(\mathbf{F}) = 1 \tag{20}$$

$$\mathbf{C} = \mathbf{F}^T\mathbf{F}, \quad \mathbf{C} = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & \gamma^2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \mathbf{1} + 2\mathbf{E} \tag{21}$$

This gives, from relation between strain tensor $\mathbf{E}$ and the deformation tensor $\mathbf{C}$:

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}) = \frac{1}{2}\begin{bmatrix} 0 & \gamma & 0 \\ \gamma & \gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{22}$$

Thus, for pure shear and *large deformation*, we have:

$$E_{22} \neq 0 \Rightarrow S_{12} = \rho_0 \frac{\partial \psi}{\partial E_{12}} = f(E_{12}, E_{22}) \tag{23}$$

and

$$S_{22} = g(E_{12}, E_{22}) \tag{24}$$

where $f(\cdot)$ and $g(\cdot)$ denote functions. For $\gamma << 1$ we get:

$$\mathbf{E} = \frac{1}{2}\begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E_{ii} = \varepsilon_v = 0 \tag{25}$$

The Piola-Kirchoff stress and the deformation gradient are for large deformations:

$$\mathbf{S} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad J = det(\mathbf{F}) = 1 \tag{26}$$

and the Cauchy stresses are found from:

$$\mathbf{T} = 2\rho\mathbf{F}\frac{\partial \psi}{\partial \mathbf{C}} = \frac{1}{J}\mathbf{F}\mathbf{S}\mathbf{F}^T = \begin{bmatrix} S_{11} + 2\gamma S_{12} + \gamma^2 S_{22} & S_{12} + \gamma S_{22} & 0 \\ S_{12} + \gamma S_{22} & S_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{27}$$

For small strains ($\gamma << 1$), this reduces to:

$$\mathbf{T} = \begin{bmatrix} 0 & S_{12} & 0 \\ S_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{28}$$

Thus, for small strains, $T_{12} = S_{12}$.

d) *Establish an expression for the shear strain* γ_{12} *(expressed by* γ*), given the same deformation state.*

Shear strain γ between the two $\perp$ directions $\mathbf{e}$ and $\bar{\mathbf{e}}$:

$$sin\gamma = \frac{2\bar{\mathbf{e}}\mathbf{E}\mathbf{e}}{(1+\bar{\varepsilon})(1+\varepsilon)} \tag{29}$$

where

$$\bar{\varepsilon} = \sqrt{1 + 2\bar{\mathbf{e}}\mathbf{E}\bar{\mathbf{e}}} - 1, \quad \varepsilon = \sqrt{1 + 2\mathbf{e}\mathbf{E}\mathbf{e}} - 1 \tag{30}$$

We have this Green's strain matrix:

$$\mathbf{E} = \frac{1}{2}\begin{bmatrix} 0 & \gamma & 0 \\ \gamma & \gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & E_{12} & 0 \\ E_{12} & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{31}$$

From Equation 30, we get:

$$\bar{\varepsilon} = \sqrt{1 - 2E_{22}} - 1 = \sqrt{1+\gamma^2} - 1 \tag{32}$$

$$\varepsilon = 0 \tag{33}$$

This gives:

$$2\bar{\mathbf{e}}\mathbf{E}\mathbf{e} = 2E_{12} = \gamma \tag{34}$$

The shear strain γ_{12} for large deformation is given by:

$$sin\gamma_{12} = \frac{2\bar{\mathbf{e}}\mathbf{E}\mathbf{e}}{(1+\bar{\varepsilon})(1+\varepsilon)} = \frac{2E_{12}}{\sqrt{1+E_{22}}} = \frac{\gamma}{\sqrt{1+\gamma^2}} \tag{35}$$

(For small strains: $sin\gamma_{12} \approx \gamma_{12}$)

e) *Choose* $E_{12} = E_{22} = 0$*. Plot the function* $T_{11}(\lambda_1)$ *in the interval* $1 < \lambda_1 < 2$.

See Figure 2 for plot produced by MATLAB.

```
clear all; clc; clf; close all

%PARAMTERS
alpha1=1020;
a1=3.79;
c=0.779;

%GAMMA-AXIS
gamma=1:0.001:2;

%CALCULATIONS
E11=0.5.*(gamma.^2-1);
S11=alpha1.*E11+c.*exp(a1.*E11.^2).*a1.*E11;
T11=gamma.^2.*S11;

%PLOT
plot(gamma,T11)
ylabel('T_{11} [Pa]')
xlabel('\lambda_1')
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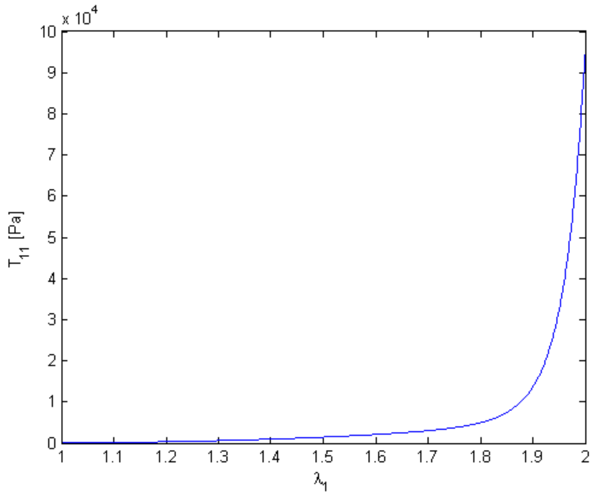

Figure 2: *Plot of* $T_{11}(\lambda_1)$.