PROBLEM SET 6

Main topics

Navier-Stokes equations for velocity profile between parallel plates. Compliant vessels. Material waves.

1 Velocity profile between parallel planes

Assume we have two parallel planes with a given flow between them. The upper plane is moving with a velocity u_0, and the fluid is exposed to a pressure gradient $\nabla p = \frac{\partial p}{\partial x}$. Let the distance between the plates be h. See Figure 1.

a) Derive the generic equation for the velocity profile for flow between parallel planes, and illustrate the velocity profile for various pressure gradients and velocities of the upper plane, when the lower plane is assumed at rest.

b) Consider the case where both plates are at rest. What is the relationship between the volumetric flow, i.e. volume per time, and the pressure gradient?

c) Based on this relationship how could the channel be regulated to perform best at a given pressure, i.e. what parameters would be most effective in modifying the flow corresponding to fixed pressure gradient? Where might such a system be useful in a biological context, e.g. the cardiovascular system.

d) What are some limitations of applying this equation? Are there certain contexts where it might be more reasonable than others?

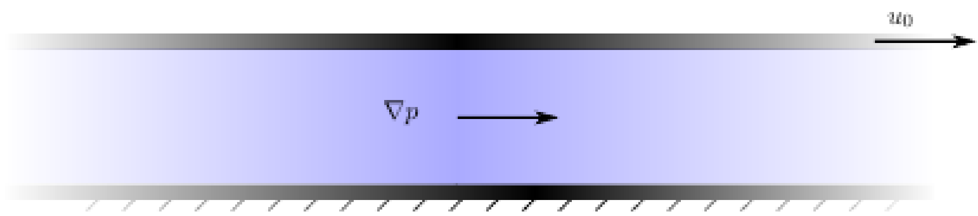

Figure 1: *Fluid between two parallel plates.*

2 Compliant tube

The arterial system consists of compliant vessels.

a) Using basic principles, derive the governing equations for a compliant tube and an incompressible fluid. [Hints: Use Reynold's transport theorem to differentiate the momentum of a control volume (from z to $z + dz$. What does $\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right)$ look like in terms of v and A. Note the key principles are conservation of momentum, and conservation of mass. Consider using the average velocity, $\bar{v}$, over the cross sectional area. For simplicity assume a flat velocity profile, i.e $v(\mathbf{r}) = \bar{v}$. Further consider expanding $f(z + dz) \approx f(z) + \frac{\partial f}{\partial x} dz$.]

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) = -\frac{A}{\rho}\frac{\partial p}{\partial x} + \frac{\pi D}{\rho}\tau$$

b) Show that the general solutions to the linearized and inviscid forms of the 1D governing equations for wave propagation in compliant vessels, (consider $A = A_r + C \cdot (p - p_r)$) read out:

$$p = p_0 f(x - ct) + p_0^* g(x + ct) \tag{1}$$

$$Q = Q_0 f(x - ct) + Q_0^* g(x + ct) \tag{2}$$

Hint: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ has general solutions that look like $f(x - ct)$ and $g(x + ct)$ use this fact to show the result.