PROBLEM SET 4

TKT4150 Biomechanics

Main topics: Deformation measures. Elasticity. Hooke's law + Green strain + Cauchy equations ⇒ Navier's equations. Hyperelasticity. Piola-Kirchoff stress.

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(1) Deformation measures

Assume we have the following homogeneous deformation

$$x_1 = \frac{2}{3}X_1 - 2X_2 + 2X_3 \tag{1}$$
$$x_2 = \frac{4}{3}X_1 + X_2 \tag{2}$$
$$x_3 = -\frac{4}{3}X_1 + X_3 \tag{3}$$

a) Find the deformation gradient $\mathbf{F}$ and the deformation tensor $\mathbf{C}$.

b) Find the principal deformation values ζ_k and the principal directions $\mathbf{n}_k$.

c) Evaluate the shear strain components γ_{ij} from Green's deformation tensor $\mathbf{C}$, when

$$sin\gamma_{ij} = \frac{\mathbf{e}_i \cdot \mathbf{C} \cdot \mathbf{e}_j}{\sqrt{(\mathbf{e}_i \cdot \mathbf{C} \cdot \mathbf{e}_i)(\mathbf{e}_j \cdot \mathbf{C} \cdot \mathbf{e}_j)}} \tag{4}$$

Remember that tensor components, C_{ij}, of $\mathbf{C}$ are found by $C_{ij} = \mathbf{e}_i \cdot \mathbf{C} \cdot \mathbf{e}_j$.

d) Express the volumetric strain ε_v with Green's deformation tensor, and evaluate the answer.

(2) Hooke's law

Hooke's law

a) By using superposition, show that we get these general linear elastic normal strains:

$$\varepsilon_i = \frac{1+\nu}{\eta}\sigma_i - \frac{\nu}{\eta}tr\mathbf{T} \tag{5}$$

where ν is the Poisson ratio, η is the Young's modulus and $\mathbf{T}$ the stress matrix.

b) The following shear stress-strain relation is assumed:

$$E_{ij} = \frac{1+\nu}{\eta} T_{ij} \quad i \neq j \tag{6}$$

From this, establish the index notation (2nd order tensor) version of Hooke's law, with strain on left hand side, that incorporates both normal and shear components.

c) Show how the answer from b) can be reformulated to:

$$T_{ij} = \frac{\eta}{1+\nu}\left(E_{ij} + \frac{\nu}{1-2\nu} E_{kk}\delta_{ij}\right) \tag{7}$$

d) Let's introduce plane stress conditions. This means that

$$T_{i3} = 0, \quad i = 1,2,3 \tag{8}$$

Show that Equation 7 now can be written

$$T_{\alpha\beta} = \frac{\eta}{1+\nu}\left(E_{\alpha\beta} + \frac{\nu}{1-\nu} E_{\rho\rho}\delta_{\alpha\beta}\right) \quad \alpha = 1,2 \tag{9}$$

3 Navier's equations

The Cauchy equations read out

$$T_{ij,j} + \rho b_i = \rho \ddot{u}_i \tag{10}$$

Compatibility requires

$$E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{11}$$

By introducing Hooke's law and Green's strain tensor, the Cauchy equations are simplified. The result is called Navier's equations.

a) Develop Navier's equation from Equations 7, 10 and 11.

4 Hyperelasticity

A material is hyperelastic if there exists a potential function $\phi = \phi(E_{ij})$ *that satisfies* $T_{ij} = \frac{\partial \phi}{\partial E_{ij}}$, *i.e. the stress is defined by a potential function of the strain. An equivalent definition of hyperelasticity requires that the strain power* ω *may be derived from a scalar valued potential* $\phi(E_{ij})$*:*

$$\omega = \dot{\phi} = \frac{\partial \phi}{\partial E_{ij}} \dot{E}_{ij} \tag{12}$$

a) Briefly explain the term *strain power*.

b) Show why the two mentioned definitions are equivalent.

(5) Piola-Kirchoff stress

a) Briefly explain the main idea behind the different Piola-Kirchoff stress tensors.

b) Write down the tensor form relations between the Cauchy, the first and second Piola-Kirchoff stresses.

c) Show that the second Piola-Kirchhoff stress tensor is symmetric:

$$\mathbf{S} = \mathbf{S}^T \tag{13}$$