**Exercise 1 Solution**

1. Outset:

$$T = tX^2 \qquad \text{and} \qquad x = (1+t)\,X \qquad \Rightarrow T = \frac{t^2}{1+t}\,x \tag{1}$$

The rate of change in temperature expressed by Lagrangian coordinates follows by differentiation of Eq. (1):

$$\frac{dT}{dt} = 2tX = \frac{2t}{1+t}\,x \tag{2}$$

the latter is the rate of change in Eulerian coordinates which follows by direct substitution of Eq. (1) b) into Eq. (2).

The Eulerian expression may also be found from the canonical definition of the material derivative:

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x_i}\frac{\partial x_i}{\partial t} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x_i}\,v_i = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\,v \tag{3}$$

where the latter equality is due to that we are limited to 1D in this exercise. From Eq. (1) we compute the partial derivatives needed in Eq. (3):

$$\frac{\partial T}{\partial t} = \left(\frac{2t}{1+t} - \frac{t^2}{(1+t)^2}\right)x = \frac{t\,(2+t)}{(1+t)^2}\,x \tag{4a}$$

$$\frac{\partial T}{\partial x} = \frac{t^2}{1+t} \qquad \text{and} \qquad v = \frac{\partial x}{\partial t} = X \tag{4b}$$

By substitution of Eq. (4) into the latter expression of Eq. (3) we get:

$$\dot{T} = \frac{dT}{dt} = \frac{t\,(2+t)}{(1+t)^2}\,x + \frac{t^2}{1+t}X = \frac{t\,(2+t)+t^2}{(1+t)^2}\,x = \frac{2t\,(1+t)}{(1+t)^2}\,x = \frac{2t}{(1+t)}\,x \tag{5}$$

which is equivalent to the expression in Eq. (2).

2. (a) The strain rate matrix is defined by:

$$D_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right) \tag{6}$$

and the velocity components are given by:

$$v_1 = \frac{\alpha x_1}{t-t_0}, \qquad v_2 = -\frac{\alpha x_2}{t-t_0} \tag{7}$$

Partial differentiation of the velocity components in Eq. (7) with respect to $x_1$ and $x_2$ and subsequent substitution into Eq. (6) yields:

$$\mathbf{D} = \begin{bmatrix} \dfrac{\alpha}{t-t_0} & 0 \\ 0 & -\dfrac{\alpha}{t-t_0} \end{bmatrix} \tag{8}$$

(b) As $D_{ii} = 0$, the flow field represents incompressible flow conditions.

3. The velocity field in the fluid cylinder is assumed to be a simple shear flow on the form:

$$v_1 = \frac{v}{h}x_2, \qquad v_2 = v_3 = 0 \tag{9}$$

where the $x_1$ and $v_1$ represent circumferential direction and circumferential fluid velocity, respectively. Whereas, $v$ is the velocity magnitude in the circumferential direction of the inner cylinder $h$ is the radial distance between the inner and outer cylinder. By assuming no slip conditions $v$ is related with the the radius $r$ and the angular velocity $\omega$ of the inner cylinder around the vertical $x_3$-axis by $v = \omega r$. Finally, $x_2$ denotes the radial direction from the wall of the outer cylinder. The radial $v_2$ and vertical components $v_3$ of the velocity field are assumed to be zero.

The only non-zero element in the strain rate matrix is consequently $D_{12}$ and the strain rate may then be expressed:

$$\dot{\gamma} = 2D_{12} = \frac{dv_1}{x_2} = \frac{v}{h} = \frac{\omega r}{h} \tag{10}$$

Consequently, the strain rate may be expressed:

$$\mathbf{D} \quad = \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\dot{\gamma}}{2}$$

with $\dot{\gamma}$ given by Eq. (10).

4. Momentum balance in the circumferential direction yield:

$$(\tau\, r)\,(2\pi r H) = T \tag{11}$$

and thus the shear stress$\tau$ may be expressed by the torque $T$ imposed on the inner cylinder by:

$$\tau = \frac{T}{2\pi\, r^2\, H} \tag{12}$$

5. Assume Newtonian fluid with co-axial strain rate and stress tensors. The only non-zero stress components are then $T_{12} = T_{21} = \tau$. From the constitutive equation for the Newtonian fluid we have:

$$\tau = 2\mu\, D_{12} = \mu\dot{\gamma} = \mu\, \frac{\omega\, r}{h} \tag{13}$$

From Eq. (13) we see that the viscosity may be found from experiments by:

$$\mu = \frac{\tau\, h}{w\, r} \tag{14}$$

as $\tau$ and $\omega$ can be inferred from measurements.

**Exercise 2 Solution**

1. $A$ Area $[m^2]$ of a compliant vessel
   $D$ Diameter $[m]$
   $Q$ Flow rate $[m^3/s]$.
   $p$ Pressure $[Pa]$.
   $\rho$ Fluid density $[kg\, m^{-3}]$.
   $\tau$ Wall shear stress of the compliant vessel wall on fluid$[Pa]$.

2. The linearized and inviscid from of the governing equations for a compliant vessel in 1D take the form:

$$C\,\frac{\partial p}{\partial t} + \frac{\partial Q}{\partial z} = 0 \tag{15a}$$

$$\frac{\partial Q}{\partial t} = -\frac{A}{\rho}\frac{\partial p}{\partial z} \tag{15b}$$

when derivatives of the of the cross-sectional area $A$ has been eliminated by introducing a constitutive law, describing the pressure-area relationship by:

$$A(p) = A_0 + \frac{\partial A}{\partial p}\,(p - p_0) = A_0 + C\,(p - p_0) \qquad \text{where } C = \frac{\partial A}{\partial p} \tag{16}$$

where $C$ represents the area-compliance of the compliant vessel. By cross-derivation and subtraction Eq. (15) may be shown to satisfy wave-equations with solutions:

$$p = p_0\; f(z - c\,t) + p_0^*\; g(z + c\,t) \tag{17a}$$

$$Q = Q_0\; f(z - c\,t) + Q_0^*\; g(z + c\,t) \tag{17b}$$

where $f$ and $g$ represents waves traveling with wave speed $c$ forward and backward, respectively.

$$c^2 = \frac{\partial p}{\partial A}\,\frac{A}{\rho} = \frac{1}{C}\,\frac{A}{\rho} \tag{18}$$

By introducing Eq. (17a) and (17b) into Eq. (15b) we obtain:

$$
\begin{aligned}
-Q_0\, c\, f' + Q_0^*\, c\, g' &= -\frac{A}{\rho}\left(p_o f' + p_0^* g'\right) \\
f'\left(p_0 \frac{A}{\rho} - Q_0 c\right) + g'\left(p_0^* \frac{A}{\rho} + Q_0^* c\right) &= 0 \qquad (19)
\end{aligned}
$$

As Eq. (19) must hold for arbitrarily chosen $f$ and $g$, an expression for the *characteristic impedance* $Z_c$ is obtained:

$$Z_c \equiv \frac{p_0}{Q_0} = \frac{\rho\, c}{A} = -\frac{p_0^*}{Q_0^*} \qquad (20)$$

3. (a) Average of $Z_i$
   - Higher frequencies cancel and are damped
   - Average between 4th and 10th harmonic

   (b) Slope of $p$ and $Q$
   - In early part of systole/ejection phase

$$Z_c = \frac{\Delta p/\Delta t}{\Delta Q/\Delta t}$$

Both methods rely on the fact that $Z_c$ is a p-Q relation in absence of reflections ad reflections are small in early systole and at high frequencies.

4.

$$p = p_f + p_b, \quad Q = Q_f + Q_b = \frac{p_f}{Z_c} - \frac{p_b}{Z_c} \qquad (21)$$

which by simple algebraic elimination yield:

$$p_f = \frac{p + Z_c\, Q}{2}, \qquad p_b = \frac{p - Z_c\, Q}{2} \qquad (22a)$$

$$Q_f = \frac{Z_c\, Q + p}{2Z_c}, \qquad Q_b = \frac{Z_c\, Q - p}{2Z_c} \qquad (22b)$$

**Exercise 4 Solution**

**a)** $p$ and $q$ represents the pressure and flow at the inlet of the aorta respectively. $R$ represents the peripheral resistance and $C$ represents the compliance of the aortic wall. The impedance $Z$ is interpreted as the aortic inlet impedance making the model more suitable for rapid pressure changes.

**b)** The equation can be discretized through a first order Taylor expansion. The Taylor expansion is given as

$$f(x+h) = f(x) + hf'(x) + \frac{h}{2} f''(x) + \frac{h}{3!} f'''(x) + \ldots$$

where the expansion is performed around the point $x$ and $h$ is the distance from $x$. A first order expansion neglects all derivatives of higher order than one. This is also called a first order forward difference. A backward difference is also possible to use. Re-arranging the derivative become

$$f'(x) \sim \frac{f(x+h) - f(x)}{h} = \frac{f_{k+1} - f_k}{h}.$$

The discretization in the exercise hence becomes, using $\Delta t = h$

$$\frac{p_{k+1} - p_k}{h} = -\frac{1}{RC} p_k + \left(\frac{1}{C} + \frac{Z}{RC}\right) q_k + Z \frac{q_{k+1} - q_k}{h}$$

which again can be written as

$$p_{k+1} = \left(1 - \frac{h}{RC}\right) p_k + \left(\left(\frac{1}{C} + \frac{Z}{RC}\right) h - Z\right) q_k + Z q_{k+1}$$

where $\alpha_1$, $\alpha_2$ and $\alpha_3$ corresponds to the factors in front of $p_k$, $q_k$ and $q_{k+1}$ respectively.

**c) One-step:** One-step method are based on estimating the parameters from a set of measurements (or estimates). Assuming that the value is correct $p_n$ when estimating $p_{n+1}$, a one-step method will find the set of parameters that minimizes the total, local error, that is: minimize the error of all steps used for estimation.

**Ballistic:** A ballistic method aim to minimize the global error of the model. Based on the first measurement, $p_0$, the other estimates are calculated based on this and the parameters used. The error of a full run of the model is then compared to the measurements and the parameters are adjusted to make the error smaller. After each adjustment, a full run of the model must be performed.

One-step method requires linear models in the parameter $\boldsymbol{\alpha}$. Ballistic method may be nonlinear. Often parameters estimated with a one-step method can be used as an initial condition to the ballistic method.

**d)** Given a set of cyclic measurements $\mathbf{p} = [p_1, \cdots, p_N]^T$ and $\mathbf{q} = [q_1, \cdots, q_N]^T$. A set of equations $\hat{\mathbf{p}} = \mathbf{A}\boldsymbol{\alpha}$ can be formed as

$$\begin{bmatrix} \hat{p}_2 \\ \vdots \\ \hat{p}_N \\ \hat{p}_1 \end{bmatrix} = \begin{bmatrix} p_1 & q_1 & q_2 \\ \vdots & \vdots & \vdots \\ p_{N-1} & q_{N-1} & q_N \\ p_N & q_N & q_1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

where $\hat{\mathbf{p}}$ are the estimated values for $p$ in each step. A least squares approximation can from this be expressed as a functional $J(\boldsymbol{\alpha})$ that can be minimized with respect to $\boldsymbol{\alpha}$ as

$$J(\boldsymbol{\alpha}) = [\hat{\mathbf{p}}(\boldsymbol{\alpha}) - \mathbf{p}]^T [\hat{\mathbf{p}}(\boldsymbol{\alpha}) - \mathbf{p}] = [\mathbf{A}\boldsymbol{\alpha} - \mathbf{p}]^T [\mathbf{A}\boldsymbol{\alpha} - \mathbf{p}].$$

By re-writing and differentiate with respect to $\boldsymbol{\alpha}$ we get

$$\frac{dJ}{d\boldsymbol{\alpha}} = 2\mathbf{A^T}\mathbf{A}\boldsymbol{\alpha} - 2\mathbf{A^T}\mathbf{p} = 0$$

where $\boldsymbol{\alpha}$ now can be found by inverting the $\mathbf{A^T A}$ matrix as

$$\boldsymbol{\alpha} = (\mathbf{A^T A})^{-1}\mathbf{A^T p}.$$

The parameters $R$ and $C$ can be found from $\boldsymbol{\alpha}$. This least squares method is a step-wise method and requires the model to be linear in the parameter $\boldsymbol{\alpha}$.

**e)** We start by normalizing the row vectors in the matrix, obtaining

$$s = \begin{bmatrix} \frac{a}{\sqrt{a^2+c^2}} & \frac{b}{\sqrt{b^2+d^2}} \\ \frac{c}{\sqrt{a^2+c^2}} & \frac{d}{\sqrt{b^2+d^2}} \end{bmatrix}$$

the normalized sensitivity matrix is then given as

$$s^T s = \begin{bmatrix} 1 & \frac{ab+cd}{\sqrt{b^2+d^2}\sqrt{a^2+c^2}} \\ \frac{ca+db}{\sqrt{b^2+d^2}\sqrt{a^2+c^2}} & 1 \end{bmatrix}$$

**f)** Putting in the values we obtain a sensitivity matrix

$$s^T s = \begin{bmatrix} 1 & -0.8 \\ -0.8 & 1 \end{bmatrix}$$

as we can see the values on the off-diagonal are close to 1. This means that inversion of the matrix can become troublesome. This system is observable, but not particularly robust.

**g)** Changing the value of $d$ and recomputing the sensitivity matrix we obtain

$$s^T s = \begin{bmatrix} 1 & -0.1414 \\ -0.1414 & 1 \end{bmatrix}$$

the off-diagonal elements have decreased in value, making the system more robust.