Wave propagation in blood vessels

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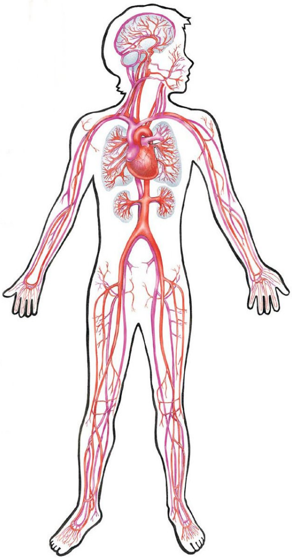
TKT4150 Biomechanics

Outline

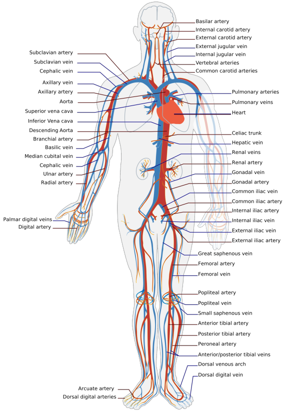
Wave propagation in the arterial system

- The heart contract and relax
- Pressure and flow waves propagate in the arterial system
- The waves are reflected at any discontinuity in impedance
  - Bifurcations
  - Stenoses
  - Aneurysms
  - Abrupt changes in mechanical properties



Reflected waves in the arterial system

- Reflected waves return to the heart
  - Increase aortic pressure
  - Increase the load of the heart ($\Rightarrow$ hypertrophy)
- Reflected waves carry information on the periphery of the arterial system



Reflected waves in the arterial system

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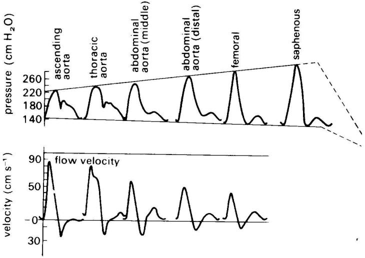
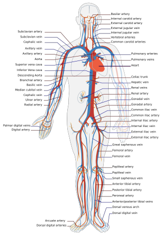
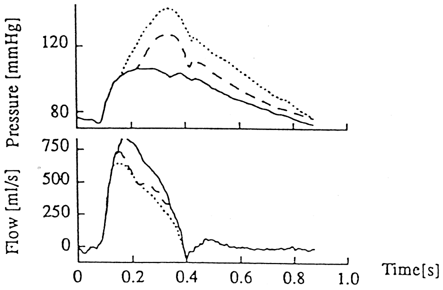
- Example: Spatial evolution of the pressure and velocity wave in arteries as they travel from the heart to the periphery





Wave speed increase with age

- Reflections increase with age due to increased wave speed
- Shorter time needed for reflected waves to reach back to the heart for older people
- Example: Aortic pressure and flow waveforms for subjects of different ages:
  28 years (cont),
  52 years (dashed),
  68 years (dotted)



Wave propagation in the human fetus

- Reflections explain wave phenomena in the human fetal circulation

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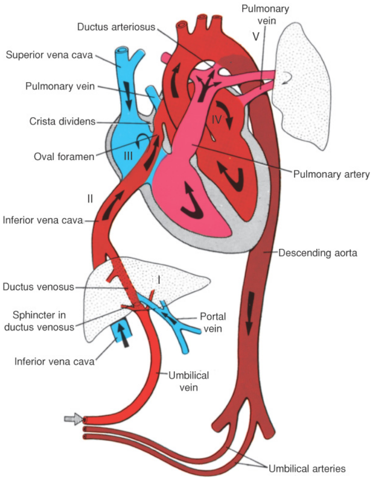
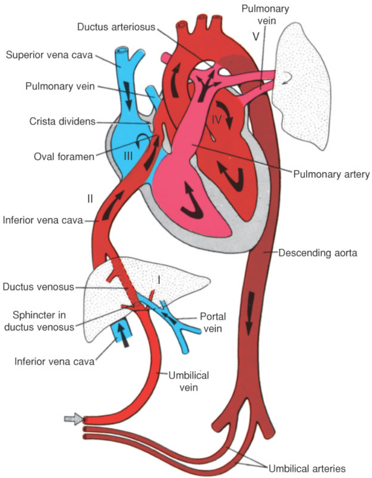


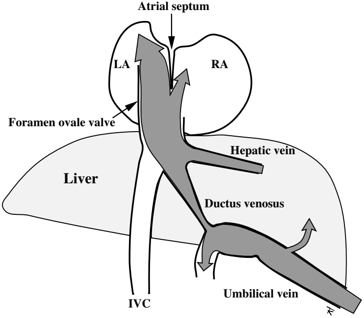
Wave propagation in the human fetus

- Reflections explain wave phenomena in the human fetal circulation
- Oxygenated blood from the placenta leave the umbilical vein (UV) and enter the ductus venosus (DV).



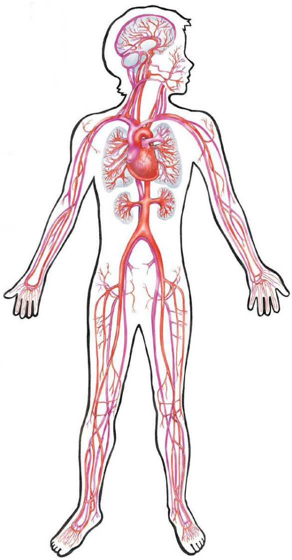
Wave propagation in the human fetus

- Reflections explain wave phenomena in the human fetal circulation
- Oxygenated blood from the placenta leave the umbilical vein (UV) and enter the ductus venosus (DV).
- The DV flow is pulsatile due to the downstream beating heart
- The UV flow is stationary



Governing equations for 1D wave propagation

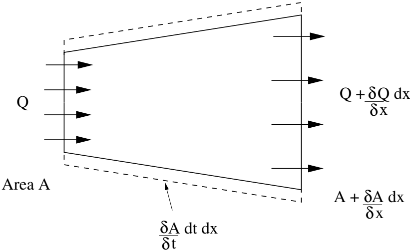
- Arterial system is a complex network
  - Viscoelastic vessels
  - Complex geometries
  - Nonlinear elastic properties
  - Non-Newtonian blood rheology
- Derive 1D governing equations for pressure and flow waves
- Most complicating factors will be disregarded



Conservation of mass

- Conservation equation:
  $\rho \dot{V} = \rho(Q_i - Q_o)$
- Flux: $Q_o = Q_i + \frac{\partial Q}{\partial x}\,\mathrm{dx}$
- Volume rate: $\dot{V} \approx \frac{\partial A}{\partial t}\,\mathrm{dx}$

Mass equation: $\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0$



Forces for momentum equation

Forces

Left: pA

Right:

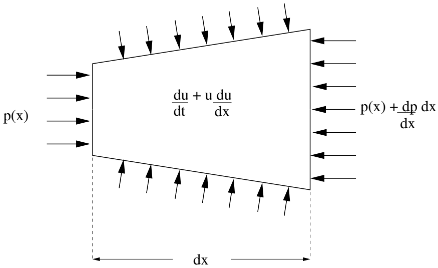
$$-\left(p + \frac{\partial p}{\partial x}\,\mathrm{dx}\right)\left(A + \frac{\partial A}{\partial x}\,\mathrm{dx}\right)$$

Surface:

$$p\frac{\partial A}{\partial x}\,\mathrm{dx}$$

Friction

$$F = \tau\pi D\ \mathrm{dx}$$



Net force

$$F_{tot} = -A\frac{\partial p}{\partial x}\,\mathrm{dx} + \tau\pi D\,\mathrm{dx}$$

Momentum

- Definition

$$\mathbf{p} = \int_{V(t)} \rho \mathbf{v} \, \mathrm{dV}$$

- Reynolds transport theorem for moving control volume with velocity $\mathbf{v}_c$

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt} \int_{V_c(t)} \rho \mathbf{v} \, \mathrm{dV} + \int_{A_c(t)} \rho \mathbf{v} \left(\mathbf{v} - \mathbf{v}_c \right) \cdot \mathbf{n}) \, \mathrm{dA}$$

- For 1D flow field with constant ρ:

$$\begin{aligned} \frac{d\mathbf{p}}{dt} &= \rho A \, \mathrm{dx} \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) \\ &= \rho \left(\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left(\frac{Q^2}{A} \right) \right) \end{aligned}$$

Momentum equation

- Two equivalent forms form the momentum equation
- Mean velocity u as primary variable:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\pi D}{\rho A}\,\tau$$

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- Mean velocity *u* as primary variable:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\pi D}{\rho A}\tau$$

- Flow rate *Q* as primary variable:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) = -\frac{A}{\rho}\frac{\partial p}{\partial x} + \frac{\pi D}{\rho}\tau$$

Governing equations

- The mass equation and the momentum equation constitute the governing equations

$$
\begin{aligned}
\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} &= 0 \\
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- Three unknowns p, Q and A, but only two equations

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- Three unknowns p, Q and A, but only two equations
- A constitutive equation is needed to close the system

$$
A(p) = A_0 + C\,(p - p_0), \quad C = \frac{\partial A}{\partial p}
$$

Linearized and inviscid governing equations

- Linearized inviscid equations

$$
\begin{aligned}
\frac{\partial A}{\partial t} &= -\frac{\partial Q}{\partial x} \\
\frac{\partial Q}{\partial t} &= -\frac{A}{\rho}\frac{\partial p}{\partial x}
\end{aligned}
$$

- Eliminate area A (or pressure) by

$$
A(p) = A_0 + C\,(p - p_0) \Rightarrow \frac{\partial A}{\partial t} = C\,\frac{\partial p}{\partial t}
$$

- Two equations and two unknowns

$$
\begin{aligned}
C\,\frac{\partial p}{\partial t} &= -\frac{\partial Q}{\partial x} \\
\frac{\partial Q}{\partial t} &= -\frac{A}{\rho}\frac{\partial p}{\partial x}
\end{aligned}
$$

The wave equations

- The linearized governing equations

$$
\begin{aligned}
C\,\frac{\partial p}{\partial t} &= -\frac{\partial Q}{\partial x} \\
\frac{\partial Q}{\partial t} &= -\frac{A}{\rho}\frac{\partial p}{\partial x}
\end{aligned}
$$

- By cross derivation and subtraction

$$
\begin{aligned}
\frac{\partial^2 p}{\partial t^2} - c^2\,\frac{\partial^2 p}{\partial x^2} &= 0 \\
\frac{\partial^2 Q}{\partial t^2} - c^2\,\frac{\partial^2 Q}{\partial x^2} &= 0
\end{aligned}
$$

- Wave speed

$$
c^2 = \frac{dp}{dA}\,\frac{A}{\rho} = \frac{1}{C}\,\frac{A}{\rho}
$$

Solutions to the wave equations

- Solutions:

$$\begin{aligned} p &= p_f + p_b = p_0\, f(x - c\,t) + p_0'\, g(x + c\,t) \\ Q &= Q_f + Q_b = Q_0\, f(x - c\,t) + Q_0'\, g(x + c\,t) \end{aligned}$$

- Forward propagating wave: $f(x - ct)$
- Backward propagating wave: $g(x + ct)$
- From linearized momentum equation

$$p_0 = \frac{\rho\, c}{A}\, Q_0 = \rho\, c\, u_0$$

- Characteristic impedance

$$Z_c \equiv \frac{p_f}{Q_f} = \frac{\rho\, c}{A} = -\frac{p_b}{Q_b}$$

Simple wave equation

- Wave equation
$$\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = 0$$
- Solution
$$v(x,t) = f(x - ct), \quad v(x,0) = f(x)$$

Simple wave equation

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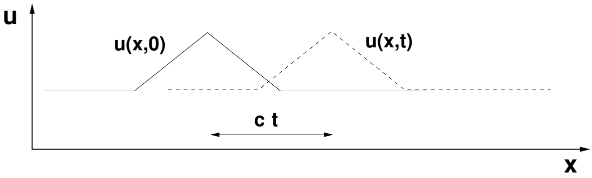
- A wave traveling with wave speed c



General wave equation considerations

- Let $f = f(z) \in C^2$
- Let z be a function of x and t: $\quad z = x - ct$
- Chain rule implies

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial f}{\partial z} \qquad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} = -c\frac{\partial f}{\partial z}$$

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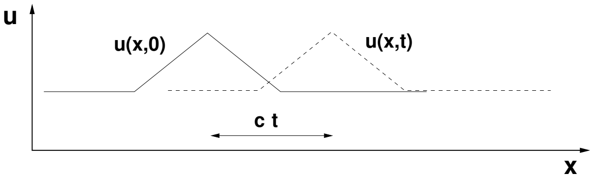
- Last line shows that $f(x - ct)$ satisfies

$$\frac{\partial^2 f}{\partial t^2} = c^2\frac{\partial^2 f}{\partial x^2}$$

General wave equation considerations (contd)

- Solution: $f(x - ct)$
- Disturbance a $t = 0$ is $f(x)$
- The value $f(x - ct)$ is constant for $x - ct$ constant
- Thus $\uparrow t \Rightarrow x = ct$



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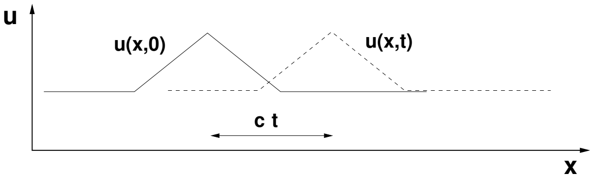

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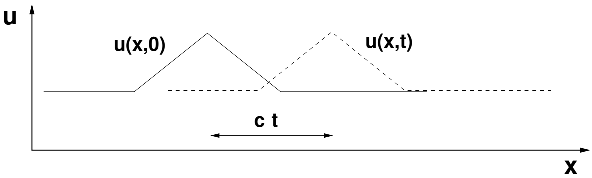
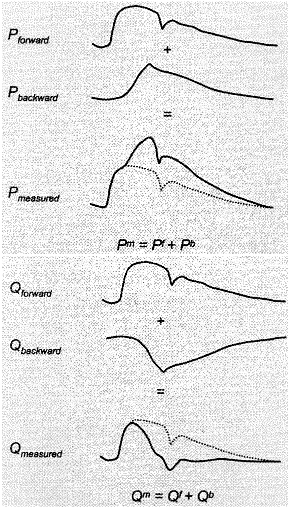


- Lagrangian coordinates z follow the wave
- The solution in the Lagrangian coordinate system is constant

Wave separation



- From solution of linearized wave equations

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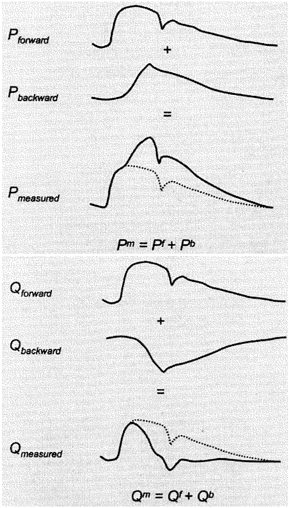
Wave separation



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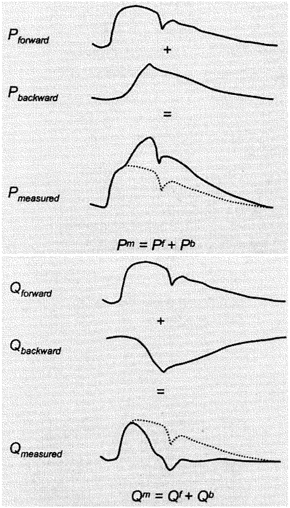

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- Characteristic impedance

$$Z_c \equiv \frac{p_f}{Q_f} = -\frac{p_b}{Q_b}$$

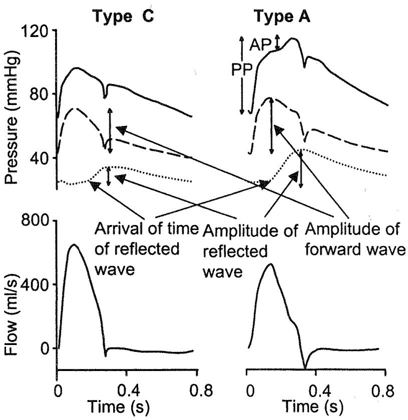
- By algebraic elimination

$$\begin{aligned} p_f &= \frac{p + Z_c\, Q}{2} \\ p_b &= \frac{p - Z_c\, Q}{2} \end{aligned}$$

Validity of linearization

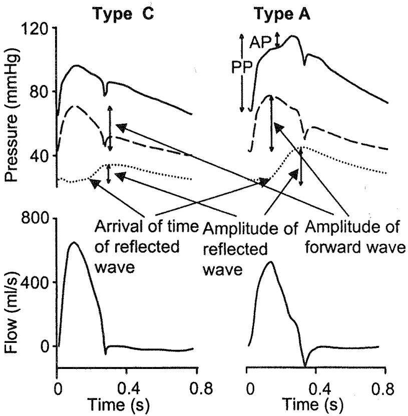
- Linear wave separation simple when Z_c is real
- Valid
  - when wall friction and viscoelasticity of the vessel wall may be neglected
  - OK for conduit arteries
- Smaller arteries
  - wall friction and viscoelasticity may not be neglected
  - The same analysis holds but
  - Z_c must be calculated in the frequency domain (complex number)

Wave separation: physiological and clinical relevance



- Useful for quantification of of wave reflection

Wave separation: physiological and clinical relevance



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  - Late arrival
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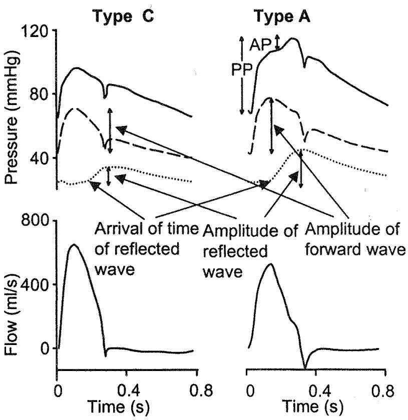
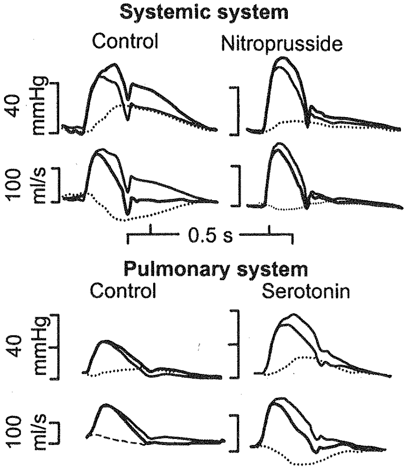


- Useful for quantification of of wave reflection
- Young healthy adult (C)
  - Small amplitude of reflected wave
  - Late arrival
  - No significant increase in late systole
- Old subject (A)
  - Higher amplitude of reflected wave
  - Arrives earlier
  - Significant late systolic peak

Impact of vascular bed on reflections



- Reflections less significant in pulmonary circulation than in the systemic arterial tree

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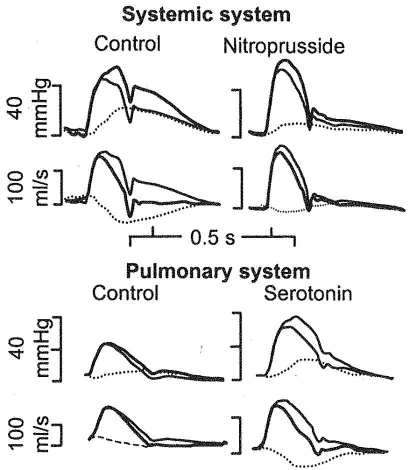

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- Nitroprusside
  - Dilates *systemic* bed
  - Reflections decrease

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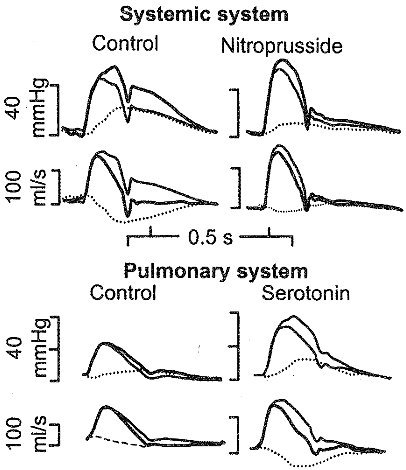

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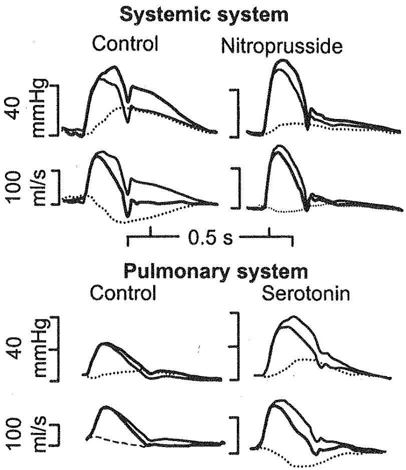

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- Pressure and flow waves are similar when reflections are small

Summary