Norwegian University of Science and Technology, Department of Structural Engineering

Exam in subject TKT4150 Biomechanics

Saturday 29. May 2010
Time: 09:00 - 13:00

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Supporting materials: approved calculator

All answers must be explained, and sufficient calculations must be included so the procedure is obvious from the solution.

grade available 18. June 2010

Problem 1: 20%

A specimen of human cortical bone tissue was subjected to a simple tension test until failure. The test result revealed a stress–strain diagram (see Fig.(1) which has three distinct regions.
These regions are an initial linearly elastic region (O–A), an intermediate nonlinear elastoplastic region

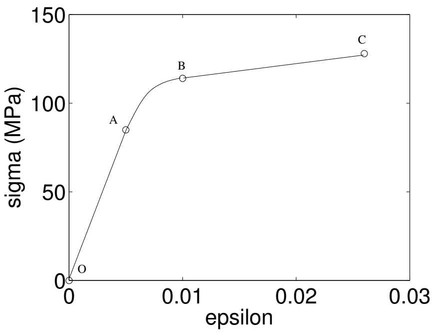


Figure 1: Tensile stress–strain behaviour for cortical bone

(A–B), and a final linearly plastic region (B–C). The stresses and corresponding longitudinal strains at points O, A, B and C are measured (see Table 1).

1) Calculate the elastic modulus (Young's modulus) of the bone specimen.

2) Present briefly Hill's three–element model for cardiac muscle.

Table 1: Measured stresses and strains

Point	Stress σ (MPa)	Strain ϵ
O	0	0.0
A	85	0.005
B	114	0.010
C	128	0.026

Problem 2: 40%

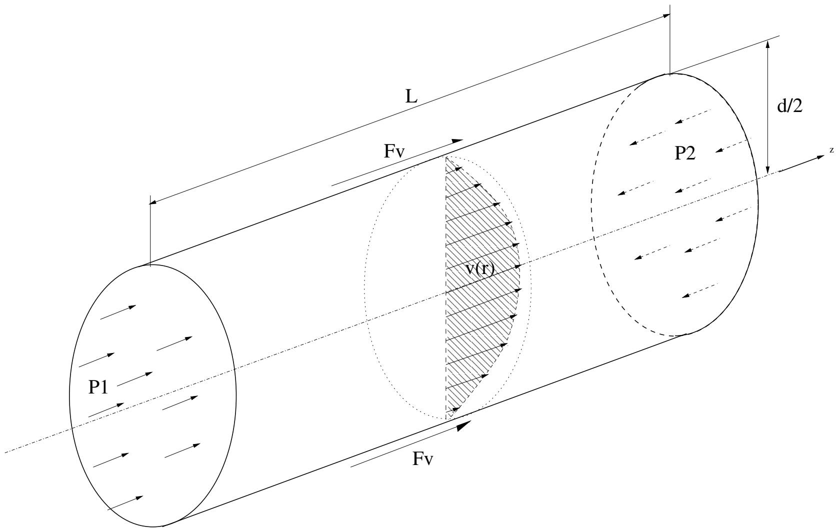
Let's consider a steady, laminar flow of an incompressible Newtonian fluid in a cyclindrical blood vessel. Let L be the length of the blood vessel, d the diameter of the vessel and η the viscosity of the blood. $v_z(r)$ denotes the velocity of the fluid at a distance r from the central axis in the longitudinal direction. $v_z(r)$ is the only non-zero component of the velocity field in cylindrical coordinates (r,θ,z): $v_\theta = 0$, $v_r = 0$, $\mathbf{v} = v_z(r)\mathbf{e}_z$. Then the laminar flow of a blood in a cylindrical vessel can be modelled by

$$v_z(r) = \frac{(\Delta P)(\frac{d^2}{4} - r^2)}{4\eta L}, \tag{1}$$

where r is the distance from the center of the vessel and $\Delta P = P_1 - P_2$.
There is a change of velocity with respect to distance from the center across the vessel, $\frac{dv_z}{dr}$.
The constitutive equations for an incompressible Newtonian fluid read:

$$\mathbf{T} = -p\mathbf{1} + 2\eta\mathbf{D} \text{ or in index notation, } T_{ij} = -p\delta_{ij} + 2\eta D_{ij}, \tag{2}$$

where $\mathbf{T}$ is the stress tensor, $\mathbf{D}$ the strain rate tensor with components: $D_{ij} = 1/2(v_{i,j} + v_{j,i})$.
1) Derive an expression of the volumetric flow Q through the vessel as a function of ΔP, d, η and L



($Q = \int_S v \cdot \mathrm{d}S$).
2) Show that the ratio between Q and the velocity at the center of the vessel $v_0 = v_z(r = 0)$ is equal to $1/2S$, where $S = \pi(d/2)^2$.

A viscous force $F_{viscous} = F$, acts on any cylindrical element of blood in the direction opposite the flow due to slower moving blood outside the element. The magnitude of the viscous force is given by

$$F = -A\tau_{rz}, \tag{3}$$

where $A = 2\pi r L$ is the area of the cylindrical element and τ_{rz} is shear stress component that can be derived from eq.(2).

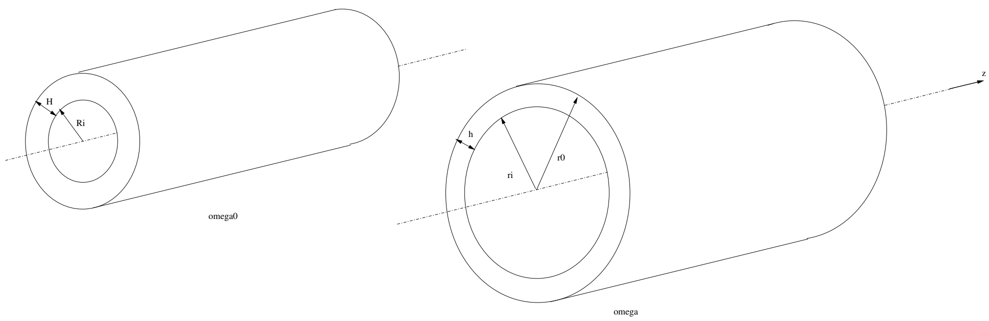
3) Derive the expression of F as a function of $v_z(r)$, A and η.

4) Calculate the viscous force on the wall of the vessel (at $r = d/2$) using the result of question 3).

5) Calculate the viscous force on the wall of the vessel (at $r = d/2$) using the force equilibrium on the fluid inside the vessel of length L and radius $d/2$.

Problem 3: 40%

Let's consider a vessel of internal radius R_i and wall thickness H in its undeformed configuration Ω_0. The vessel is subjected to an internal pressure p_i and has an internal radius r_i and a wall thickness h in the deformed configuration Ω.



The deformation gradient $\mathbf{F}$ can be expressed in the cylindrical coordinate system ($\mathbf{e}_{rr}$, $\mathbf{e}_{\theta\theta}$, $\mathbf{e}_{zz}$) as:

$$[\mathbf{F}] = \begin{bmatrix} \lambda_{rr} & 0 & 0 \\ 0 & \lambda_{\theta\theta} & 0 \\ 0 & 0 & \lambda_{zz} \end{bmatrix}, \tag{4}$$

$$\tag{5}$$

where $\lambda_{rr} = \lambda_r$, $\lambda_{\theta\theta} = \lambda_\theta$, $\lambda_{zz} = \lambda_z$ are the stretches in the radial, circumferential and longitudinal directions, respectively. We assume that the vessel wall is incompressible: $J = \det\mathbf{F} = \lambda_r \lambda_\theta \lambda_z = 1$. The stretch in the longitudinal direction is equal to one, $\lambda_z = 1$.
First, we consider the case where $h << r_i$. Therefore, the vessel can be treated as a thin–walled structure and the membrane theory can be apply to the vessel wall.

1) Use Laplace's law $\left(\frac{\sigma_1}{r_1} + \frac{\sigma_2}{r_2} = \frac{p_i}{h}\right)$ to express the internal pressure p_i as a function of r_i, h and $\sigma_{\theta\theta}$ the Cauchy stress in the circumferential direction.
2) Express h as a function of H and λ_θ and r_i as a function of R_i and λ_θ.
3) Express p_i as a function of R_i, H, $\sigma_{\theta\theta}$ and λ_θ.

Now, we assume that the vessel wall can be modeled as an incompressible isotropic hyperelastic material. The second Piola–Kirchhoff stress tensor $\mathbf{S}$ can be expressed as:

$$\mathbf{S} = 2f(I_1)\mathbf{1} + q\mathbf{C}^{-1}, \tag{6}$$

where f is a scalar function, $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is the right Cauchy–Green tensor, $\mathbf{1}$ is the second order identity tensor, q is a Lagrange multiplier and $I_1 = \mathrm{tr}\mathbf{C}$ is the first invariant of $\mathbf{C}$.

4) The Cauchy stress $\boldsymbol{\sigma}$ can be found by pushing forward the second Piola-Kirchhoff stress tensor $\mathbf{S}$ into the current configuration: $\boldsymbol{\sigma} = J^{-1}\mathbf{F}\mathbf{S}\mathbf{F}^T$. Give the expression of the Cauchy stress tensor $\boldsymbol{\sigma}$ (note that $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is the left Cauchy Green tensor).

In the case where $h << r_i$, the stress in the radial direction can be neglected: $\sigma_{rr} = 0$ or $S_{rr} = 0$. Hence, the vessel is in a plane stress state.

5) Use the plane stress condition to derive an expression of the Lagrange multiplier q (as a function of $f(I_1)$ and λ_θ).

In many cases, the condition $h << r_i$ is not fulfilled and the vessel as to be treated as a thick–walled cylindrical structure. Then the Cauchy equations of motion can be used to write the cross–sectional equilibrium,

$$\mathrm{div}\boldsymbol{\sigma} + \rho\mathbf{b} = \rho\mathbf{a}, \tag{7}$$

where ρ is the density of the structure, $\mathbf{b}$ the body forces and $\mathbf{a}$ the acceleration.
The Cauchy equation in cylindrical coordinates (r,θ,z) in the radial direction is,

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} + \rho b_r = \rho a_r, \tag{8}$$

where σ_{rr} and $\sigma_{\theta\theta}$ are normal stresses and $\tau_{r\theta}$ and τ_{zr} are shear stresses.

6) If the normal stresses σ_{rr}, $\sigma_{\theta\theta}$ and σ_{zz} are also principal stresses, give the value of the shear stresses.

7) Simplify eq.(8) when body forces and accelerations are excluded and $\mathbf{e}_{rr}$, $\mathbf{e}_{\theta\theta}$ and $\mathbf{e}_{zz}$ are principal stress directions.

8) Now, let's consider the vessel subjected to an internal pressure p_i. Using the result from question 7) and the boundary conditions $p_i = -\sigma_{rr}|_{r=r_i}$ and $\sigma_{rr}|_{r=r_0} = 0$ show that:

$$p_i = \int_{r_i}^{r_0} \frac{\sigma_{\theta\theta} - \sigma_{rr}}{r} dr, \quad r_i \leq r \leq r_0, \tag{9}$$

where r_0 is the outer radius of the vessel in Ω.