Use the Poincar´e-Lindstedt method and find the periodic solution as well as the frequency of the solution ω to several terms.

> $N := 2;\ \text{omega} := 1 + add(w[k]\cdot \epsilon^k, k = 1..N);$

$$N := 2$$

$$\omega := w_2\,\epsilon^2 + w_1\,\epsilon + 1 \tag{1}$$

> $de := \omega^2 \cdot diff(x(t), t, t) + x(t) - \frac{\text{epsilon}\cdot x(t)^3}{6} + \frac{\epsilon^2\cdot x(t)^5}{120}$

$$de := \left(w_2\,\epsilon^2 + w_1\,\epsilon + 1\right)^2 \left(\frac{d^2}{dt^2}\,x(t)\right) + x(t) - \frac{\epsilon\,x(t)^3}{6} + \frac{\epsilon^2\,x(t)^5}{120} \tag{2}$$

> $xp := add(x[k](t)\cdot\epsilon^k, k = 0..N);$

$$xp := x_0(t) + x_1(t)\,\epsilon + x_2(t)\,\epsilon^2 \tag{3}$$

> $\#de1 := expand(subs(x(t) = xp, de));$

> $de1 := convert(taylor(expand(subs(x(t) = xp, de)), \text{epsilon}, N + 1), polynom);$

$$de1 := \frac{d^2}{dt^2}\,x_0(t) + x_0(t) + \left(\frac{d^2}{dt^2}\,x_1(t) - \frac{x_0(t)^3}{6} + 2\left(\frac{d^2}{dt^2}\,x_0(t)\right)w_1 + x_1(t)\right)\epsilon + \left(\frac{d^2}{dt^2}\,x_2(t) + \frac{x_0(t)^5}{120} + \left(\frac{d^2}{dt^2}\,x_0(t)\right)w_1^2 + 2\left(\frac{d^2}{dt^2}\,x_0(t)\right)w_2 + 2\left(\frac{d^2}{dt^2}\,x_1(t)\right)w_1 - \frac{x_0(t)^2\,x_1(t)}{2} + x_2(t)\right)\epsilon^2 \tag{4}$$

> $x0sol := dsolve(\{coeff(de1, \text{epsilon}, 0), x[0](0) = x_0, D(x[0])(0) = 0\}, x[0](t));$

$$x0sol := x_0(t) = x_0\cos(t) \tag{5}$$

> $xp1 := subs(x0sol, xp);$

$$xp1 := x_0\cos(t) + x_1(t)\,\epsilon + x_2(t)\,\epsilon^2 \tag{6}$$

> $de2 := convert(taylor(expand(subs(x(t) = xp1, de)), \text{epsilon}, N + 1), polynom);$

$$de2 := \left(-\frac{x_0^3\cos(t)^3}{6} + \frac{d^2}{dt^2}\,x_1(t) + x_1(t) - 2\,x_0\cos(t)\,w_1\right)\epsilon + \left(\frac{x_0^5\cos(t)^5}{120} + \frac{d^2}{dt^2}\,x_2(t) + 2\left(\frac{d^2}{dt^2}\,x_1(t)\right)w_1 + x_2(t) - x_0\cos(t)\,w_1^2 - 2\,x_0\cos(t)\,w_2 - \frac{x_0^2\cos(t)^2\,x_1(t)}{2}\right)\epsilon^2 \tag{7}$$

>

extract epsilon power 1 seperatly

> $e1 := coeff(de2, \text{epsilon}, 1);$

$$e1 := -\frac{x_0^3\cos(t)^3}{6} + \frac{d^2}{dt^2}\,x_1(t) + x_1(t) - 2\,x_0\cos(t)\,w_1 \tag{8}$$

Now remove the powers of the trignometry functions

> $e1 := combine(e1, trig);$

$$e1 := \frac{d^2}{dt^2} x_1(t) + x_1(t) - \frac{x_0^3 \cos(3\,t)}{24} - \frac{x_0^3 \cos(t)}{8} - 2\, x_0 \cos(t)\, w_1 \tag{9}$$

we now see secular term in the equation which is the coeff of the cos(t) term and solve for w1

> $e1\_sec := coeff(e1, \cos(t));$

$$e1\_sec := -\frac{1}{8} x_0^3 - 2\, x_0\, w_1 \tag{10}$$

> $ans := solve(\%, w[1])$

$$ans := -\frac{x_0^2}{16} \tag{11}$$

> $\omega := subs(w[1] = ans, \omega);$

$$\omega := \epsilon^2 w_2 - \frac{1}{16} \epsilon\, x_0^2 + 1 \tag{12}$$

> $de3 := collect(expand(subs(w[1] = ans, de2)), \text{epsilon});$

$$de3 := \left(-\frac{x_0^3 \cos(t)^3}{6} + \frac{d^2}{dt^2} x_1(t) + x_1(t) + \frac{x_0^3 \cos(t)}{8} \right) \epsilon + \left(\frac{x_0^5 \cos(t)^5}{120} + \frac{d^2}{dt^2} x_2(t) - \frac{\left(\frac{d^2}{dt^2} x_1(t) \right) x_0^2}{8} + x_2(t) - \frac{x_0^5 \cos(t)}{256} - 2\, x_0 \cos(t)\, w_2 - \frac{x_0^2 \cos(t)^2 x_1(t)}{2} \right) \epsilon^2 \tag{13}$$

Now solve for the equation of epsilon

> $e1 := coeff(de3, \text{epsilon}, 1);$

$$e1 := -\frac{x_0^3 \cos(t)^3}{6} + \frac{d^2}{dt^2} x_1(t) + x_1(t) + \frac{x_0^3 \cos(t)}{8} \tag{14}$$

> $e1 := dsolve(\{e1, x[1](0) = 0, D(x[1])(0) = 0\}, x[1](t));$

$$e1 := x_1(t) = \frac{\cos(t) \sin(t)^2 x_0^3}{48} \tag{15}$$

> $xp2 := subs(e1, xp1);$

$$xp2 := x_0 \cos(t) + \frac{\cos(t) \sin(t)^2 x_0^3 \epsilon}{48} + x_2(t)\, \epsilon^2 \tag{16}$$

> $de4 := collect(combine(expand(subs(e1, de3)), trig), \text{epsilon});$

$$de4 := \left(\frac{3\, x_0^5 \cos(5\,t)}{2560} - \frac{x_0^5 \cos(3\,t)}{384} + \frac{x_0^5 \cos(t)}{1536} - 2\, x_0 \cos(t)\, w_2 + \frac{d^2}{dt^2} x_2(t) + x_2(t) \right) \epsilon^2 \tag{17}$$

> $e2 := coeff(de4, epsilon, 2);$

$$e2 := \frac{3\, x_0^5 \cos(5\,t)}{2560} - \frac{x_0^5 \cos(3\,t)}{384} + \frac{x_0^5 \cos(t)}{1536} - 2\, x_0 \cos(t)\, w_2 + \frac{d^2}{dt^2} x_2(t) + x_2(t) \tag{18}$$

> $e\_2 := coeff(e2, \cos(t));$

(19)

$$e\_2 := \frac{1}{1536} x_0^5 - 2 x_0 w_2 \tag{19}$$

> $ans1 := solve(\%, w[2]);$

$$ans1 := \frac{x_0^4}{3072} \tag{20}$$

> omega := $subs(w[2] = ans1$, omega);

$$\omega := \frac{1}{3072} \epsilon^2 x_0^4 - \frac{1}{16} \epsilon x_0^2 + 1 \tag{21}$$

>

>

>

We gona find the solution of the equation :

> omega := $subs(w[2] = ans1$, omega);

$$\omega := \frac{1}{3072} \epsilon^2 x_0^4 - \frac{1}{16} \epsilon x_0^2 + 1 \tag{22}$$

> $de5 := subs(w[2] = ans1 , e2)$

$$de5 := \frac{3 x_0^5 \cos(5 t)}{2560} - \frac{x_0^5 \cos(3 t)}{384} + \frac{d^2}{dt^2} x_2(t) + x_2(t) \tag{23}$$

> $eq2 :=$ $dsolve(\{de5 , x[2](0) = 0 , \mathrm{D}(x[2])(0) = 0 \}, x[2](t));$

$$eq2 := x_2(t) = \frac{\sin(t)^2 x_0^5 \left(-\cos(t)^3 + \frac{23 \cos(t)}{12}\right)}{1280} \tag{24}$$

> $eq\_final := subs(eq2 , xp2);$

$$eq\_final := x_0 \cos(t) + \frac{\cos(t) \sin(t)^2 x_0^3 \epsilon}{48} + \frac{\sin(t)^2 x_0^5 \left(-\cos(t)^3 + \frac{23 \cos(t)}{12}\right) \epsilon^2}{1280} \tag{25}$$

>

>

Set the bookkeeping parameter $\epsilon = 1$, to obtain a series for the time-period of the solution in terms of x0.

> epsilon := *1*

$$\epsilon := 1 \tag{26}$$

> omega

$$\frac{1}{3072} x_0^4 - \frac{1}{16} x_0^2 + 1 \tag{27}$$

>

> eq_final

$$x_0 \cos(t) + \frac{\cos(t) \sin(t)^2 x_0^3}{48} + \frac{\sin(t)^2 x_0^5 \left(-\cos(t)^3 + \frac{23 \cos(t)}{12}\right)}{1280} \tag{28}$$

>

Plot for comparision

> $f := t \to eq\_final;$

$$f := t \mapsto eq\_final \quad \textbf{(29)}$$

> $x_0 := 0.1$

$$x_0 := 0.1 \quad \textbf{(30)}$$

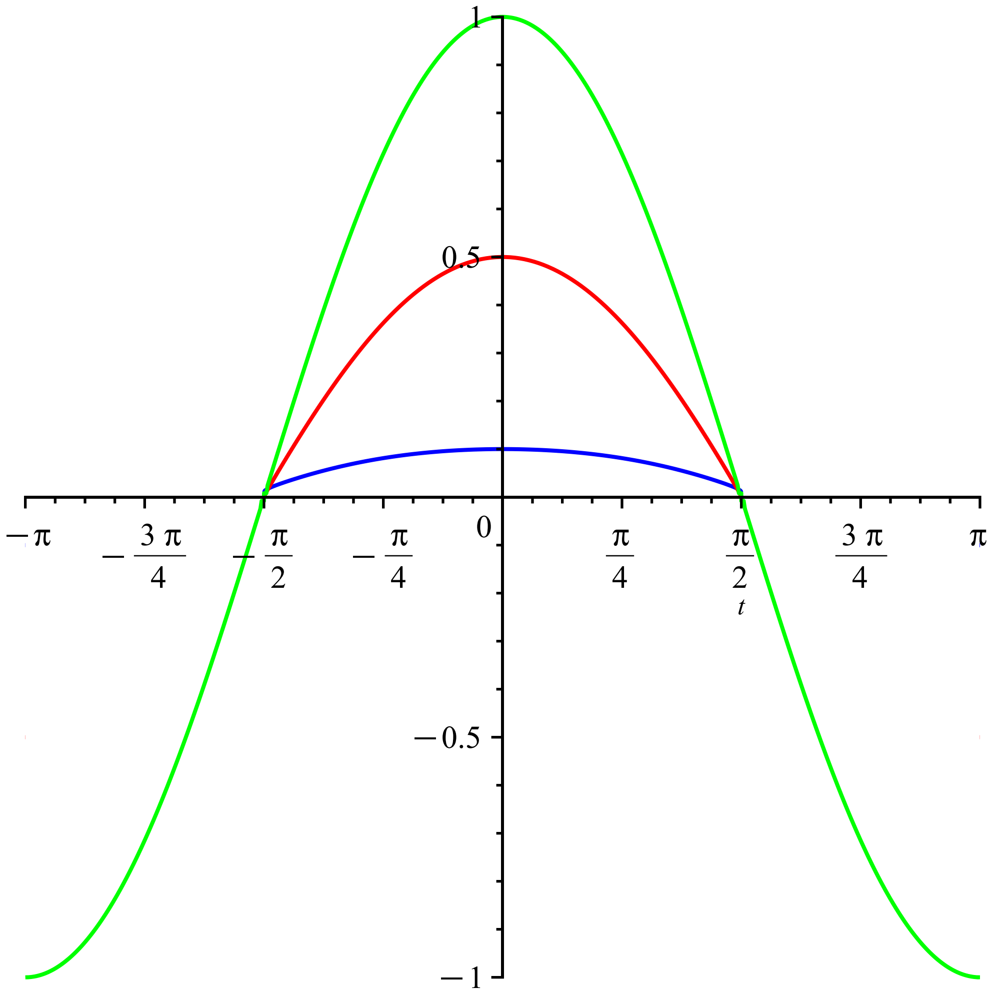
> $eq\_final1 := subs(x_0 = 0.1, eq\_final); eq\_final2 := subs(x_0 = 0.5, eq\_final); eq\_final3 := subs(x_0 = 1, eq\_final);$

$$eq\_final1 := 0.1 \cos(t) + \frac{\cos(t)^{0.1} \sin(t)^2}{48} + \frac{\sin(t)^2 \left(-\cos(t)^3 + \frac{23 \cos(t)}{12}\right)^{0.1}}{1280}$$

$$eq\_final2 := 0.5 \cos(t) + \frac{\sqrt{\cos(t)} \sin(t)^2}{48} + \frac{\sin(t)^2 \sqrt{-\cos(t)^3 + \frac{23 \cos(t)}{12}}}{1280}$$

$$eq\_final3 := \cos(t) + \frac{\cos(t) \sin(t)^2}{48} + \frac{\sin(t)^2 \left(-\cos(t)^3 + \frac{23 \cos(t)}{12}\right)}{1280} \quad \textbf{(31)}$$

> $plot([eq\_final1, eq\_final2, eq\_final3], t = -\pi .. \pi, color = [blue, red, green]);$


1
0.5
−0.5
−1
−π
−3π/4
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−π/4
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