INDIAN INSTITUTE OF TECHNOLOGY KANPUR

DEPARTMENT OF MECHANICAL ENGINEERING

ME 627 Nonlinear Vibration

2023-2024-II End-semester examination

Time: 1 week
Take Home (assigned on 18-04-2022, due on 25-04-2024) Marks: 100+100

Instructions: I will expect that you all work independently on the problems. If there is a difficulty in any problem feel free to contact me at any time. I would be available via. email at *wahi@iitk.ac.in*. In your solutions, provide enough details to clarify what you have done. This will help me in grading the assignments submitted by you. Provide limited computer output as appropriate and wherever required. To limit the computer output in MAPLE, use colon (:) at the end of the command to suppress the output. All questions carry equal marks.

Work independently on the problems and submit only those problems that you have done on your own. You can discuss the problems with others in the class. Make sure to give due credit to those with whom you have discussed. I will call everyone and ask to explain some key steps of each problem to ascertain whether it has been done by you or not. If I find that you have not done something yourself and have still submitted it as your solution, you will fail the course.

1. Consider the periodic solutions of a pendulum for small amplitudes such that we can approximate $\sin(x) \approx x - x^3/6 + x^5/120$. The resulting non-dimensional equation of motion of the pendulum is

$$\ddot{x} + x - \frac{x^3}{6} + \frac{x^5}{120} = 0\,.$$

Next, introduce the bookkeeping parameter ϵ and write the above equation as

$$\ddot{x} + x - \epsilon\frac{x^3}{6} + \epsilon^2\frac{x^5}{120} = 0\,.$$

Let initial conditions be $x(0) = x_0$ and $\dot{x}(0) = 0$. Use the Poincaré-Lindstedt method and find the periodic solution as well as the frequency of the solution ω to several terms. Finally, set the bookkeeping parameter $\epsilon = 1$, to obtain a series for the time-period of the solution in terms of x_0. Compare the results with values obtained from numerical simulation for $x_0 = 0.1$, $x_0 = 0.5$ and $x_0 = 1$ and comment accordingly.

2. Consider the equation

$$\ddot{x} + \epsilon\dot{x}(1 - \dot{x}x - x^2) + x = 0\,, \quad \text{for } 0 < \epsilon \ll 1\,.$$

Does this system have limit cycles? If yes, find the approximate amplitude of the limit cycle. Find the time-period of the limit cycle (the first correction term). Determine whether or not the limit cycle is stable? If you can not determine the stability, justify your observation based on the method chosen by you. Try other methods to see if you can ascertain the stability of the limit cycle.

3. Consider the equation

$$\ddot{x} + \epsilon \dot{x}^3 + x = 0\,,$$

for $0 < \epsilon \ll 1$. Look for periodic solutions using the Poincaré-Lindstedt method. Point out where it fails. Now add small resonant forcing

$$\ddot{x} + \epsilon \dot{x}^3 + x = \epsilon \sin t\,.$$

Find the amplitude of the resultant solution using the method of averaging.

4. Consider a Van der Pol oscillator with a nonlinear restoring force given by

$$\ddot{x} + \epsilon \dot{x}\left(x^2 - 1\right) + x - \alpha x^2 = 0\,, \quad \text{for } 0 < \epsilon \ll 1 \text{ and } 0 < \alpha \ll 1.$$

Determine the amplitude and frequency of the limit cycle using an appropriate approximate method of your choice.

5. Use harmonic balance to obtain the limit cycle amplitude of the equation

$$\ddot{x} - \left(\frac{1}{10} - \frac{10}{3}\dot{x}^2\right)\dot{x} + x - x^3 = 0\,.$$

In case of more than one answers, choose the correct answer using the Bendixon criterion.

6. Convert the equation

$$\ddot{x} + 2\,A\,\sin t\,\dot{x} + (K + A\,\cos t)x \;=\; 0$$

into a standard Matheiu equation. From this converted equation, infer the stability of the $x = 0$ solution for the equation for $K = 1$ and $A = 0.1$. Also obtain the stability boundary in the K-A plane for the original equation.

7. Consider the damped Mathieu equation

$$\ddot{x} + 2\,\mu\,\dot{x} + (\delta + 2\,\epsilon\,\cos(2\,t))\,x = 0\,.$$

Obtain the equations of the transition curve $\delta(\epsilon)$ corresponding to $\delta = 1$ using either the method of averaging or the method of multiple scales.

Hint: Write $\mu = \epsilon\bar{\mu}$ before applying the usual procedure.

8. Consider the forced Van der Pol equation

$$\ddot{x} + \epsilon \dot{x}\left(x^2 - 1\right) + x = F\cos(\omega\,t)\,, \quad \text{for } 0 < \epsilon \ll 1\,.$$

It is known that in the absence of forcing, i.e., $F = 0$, the response is nearly harmonic with a frequency approximately 1 and an amplitude approximately 2. However, for forcing frequencies ω in the vicinity of 1, the response also has a frequency ω. This phenomenon is called frequency entrainment.

Obtain the equation for determining the approximate amplitude of this frequency-entrained response. In this equation, use the following parameters: the detuning parameter $\nu = \dfrac{\omega^2 - 1}{\epsilon\,\omega}$ and the excitation parameter $\gamma = \dfrac{F}{\epsilon\,\omega}$.

9. Consider the forced Duffing equation with a softening nonlinearity, i.e.,

$$\ddot{x} + \alpha x - \beta x^3 = F\,\cos(\omega t)\,, \quad \text{for } \alpha > 0\,,\ \beta > 0\,.$$

   (a) First non-dimensionalize the system using appropriate time and length scales. Explicitly specify the scales used and provide expressions for the non-dimensional parameters involved in the final equation.

   (b) Obtain the relationship between the amplitude of the periodic response and the forcing amplitude for the non-dimensional equation using the method of averaging.

   (c) Specify the number of possible solutions for different ranges of frequencies and sketch some representative Amplitude-Frequency curves for different values of the forcing amplitude. Mark the unstable ones.

   (d) Does there exist a forcing amplitude beyond which a unique solution is guaranteed for all frequencies? If yes, find this value of the forcing amplitude.

10. Consider the logistic map

$$x_{k+1} = r\,x_k\,(1 - x_k)\ , \quad 0 \le x \le 1\,.$$

   (a) Determine the values of r at which the 1-cycle and the 2-cycle are super stable.

   (b) Next consider the special case of $r = 4$, i.e., the map

$$x_{k+1} = 4\,x_k\,(1 - x_k)\ , \quad 0 \le x \le 1\,.$$

   What are the various types of solutions possible for this special case?
   Hint: You might like to use the substitution $x_k = \sin^2(\theta_k)$ with an appropriate interval for θ_k for uniqueness of the mapping, and obtain the map for θ_k as

$$\theta_{k+1} = g\,(\theta_k)\ .$$

   If you proceed along these lines, also sketch the map for θ_k. Calculate the Lyapunov exponent to justify presence of chaotic solutions if any.

11. Take a paper preferably published in the last 5 years dealing with a nonlinear system and study the system using one of the techniques learnt in this course which is different from the one used in the original paper. Give a print-out of the original paper and a short report of the analysis done by you. This question has 100 marks.