CSCI 567 HW # 5

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Sol. 1.1 Given

$$D = \sum_{n=1}^{N}\sum_{k=1}^{K} r_{nk}\|x_n - \mu_k\|_2^2$$

taking partial derivative w.r.t. μ_k, we get

$$\frac{\partial D}{\partial \mu_k} = \sum_{n=1}^{N} r_{nk}[-2(x_n - \mu_k)] = 0$$

$$= \sum_{n=1}^{N} r_{nk}x_n - \mu_k \sum_{n=1}^{N} r_{nk} = 0$$

$$\mu_k = \frac{\sum_{n=1}^{N} r_{nk}x_n}{\sum_{n=1}^{N} r_{nk}}$$

Sol. 1.2 Given

$$D = \sum_{n=1}^{N}\sum_{k=1}^{K} r_{nk}\|x_n - \mu_k\|_1$$

We want find an optimal μ_k, such that it minimizes D and that optimal μ is equal to the median of $x_n = [x_{n1}, x_{n2} \dots x_{nD}]$.

Now, let us assume that, μ_k is optimal and for a given vector $x_n, x_n \in R^D$, so fro the definition of "closeness" in the text we get that

$$L = \sum_{i=1}^{D} |x_{ni} - \mu_k|$$

, Now if there are l numbers to the left of μ_k and r number to the right, then if we were to shift μ_k by a distance d to the left, the L increases by $(l-r)d \quad if(l > r)$. Similarly if, we were to shift μ_k to the right by a distance of d, then again the measurement of L increases by $(r-l)d \quad if(r > l)$. Therefore L will achieve if minimum value when $l = r$, i.e. μ_k is the median of the vector x_n.

Now we derive the above conclusion to multi-dimensionalities. We can see only when is the elementwise median of cluster k, $L_k = \sum_{i=1}^{N_k} |x_i - \mu_k|$ has the minimal value. Suppose there are K clusters, the overall loss is

$$L = \sum_{k=1}^{K} L_k = \sum_{k=1}^{K}\sum_{n=1}^{N} |x_n - \mu_k|$$

which will also be minimal, which equals to

$$D = \sum_{k=1}^{K} L_k = \sum_{k=1}^{K}\sum_{n=1}^{N} r_{nk}|x_n - \mu_k|$$

Sol 1.3 Given

$$\tilde{D} = \sum_{n=1}^{N}\sum_{k=1}^{K} r_{nk}\|\phi(x_n) - \tilde{\mu}_k\|_2^2 \tag{1}$$

where,

$$\tilde{\mu}_k = \frac{1}{N_k}\sum_{n=1}^{N} r_{nk}\phi(x_n) \tag{2}$$

We can write $\frac{r_{nk}}{N_k} = \gamma_{nk}$, then we can rewrite $\tilde{\mu}$ as

$$\tilde{\mu}_k = \sum_{n=1}^{N} \gamma_{nk}\phi(x_n)$$

Then,

$$\|\phi(x_n) - \tilde{\mu}\|^2 = \|\phi(x_n) - \sum_{n=1}^{N}\gamma_{nk}\phi(x_n)\|^2$$

$$= [\phi(x_n) - \sum_{n=1}^{N}\gamma_{nk}\phi(x_n)] \cdot [\phi(x_n) - \sum_{n=1}^{N}\gamma_{nk}\phi(x_n)]$$

$$= K(x,x) - 2\sum_{i=1}^{N}\gamma_{ik}K(x,x_i) + \sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{ik}\gamma_{jk}K(x_i,x_j)$$

Therefore, we can write

$$\tilde{D} = \sum_{n=1}^{N}\sum_{k=1}^{K} r_{nk}\big[K(x_n,x_n) - 2\sum_{i=1}^{N}\gamma_{ik}K(x_n,x_i) + \sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{ik}\gamma_{jk}K(x_i,x_j)\big]$$

To assign a point to a cluster k, we initialize (randomly or through other methods) $\tilde{\mu}_k$ and for each iteration, assign, x_n to k where

$$argmax_{k\in K} K(x_n,x_n) - 2\sum_{i=1}^{N}\gamma_{ik}K(x_n,x_i) + \sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{ik}\gamma_{jk}K(x_i,x_j)$$

The pseudo-code

```
Kernel_Kmeans():
  G = GramMatrix(X)
  Assignment = init_random_assigmnment()
  Means = random.sample(X, k)
  for max_iterations:
        for k in K:
          for i, x in enumerate(X):
                distance[i, k] = G[x, x]
                distance[i, k] -= (2 * sum(G[(Assignment == k), i])
                distance[i, k] += sum(G[Assignment == k, Assignment == k])
    Assignment = argmin(distance)
    for k in K:
      NewMeans[k] = mean(X[Assignment == k])
    if convereged(Means, NewMeans):
      return Assignment, NewMeans
    else:
      Means = NewMeans
```

Sol 2.1 We can write the likelihood function as

$$p(x|\alpha) = \frac{\alpha}{\sqrt{2\pi}} exp(-\frac{1}{2}x^2) + \frac{1-\alpha}{\sqrt{\pi}} exp(-x^2)$$

and for observed sample x_1, we can write

$$p(x_1|\alpha) = (\frac{1}{\sqrt{2\pi}} exp(-\frac{1}{2}x_1^2) - \frac{1}{\sqrt{(\pi)}} exp(-x_1^2))\alpha + \frac{1}{\sqrt{\pi}} exp(-x^2)$$

We observe the likelihood function for an observed sample x_1 is a linear function of α where the slope of the line is determined by the values of the gaussian probabilities. So, for maximum likelihood, if the gaussian probability $N(x_1|0,1) > N(x_1|0,0.5)$ then we choose $\alpha = 1$ for maximum likelihood, else we choose $\alpha = 0$.

Sol 3.1 Let us define the hidden variable as $z_i = 1$ when a person in the sample has taken insurance, and $z_i = 0$ otherwise. Now, if $x_i > 0$, then clearly $z_i = 1$, however, when $x_i = 0$, then $z_i = 1$ or 0.

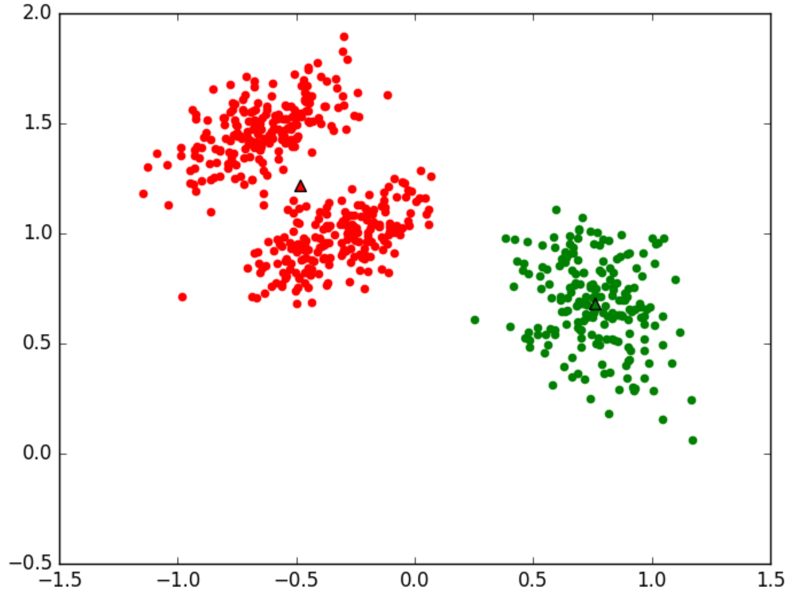
Therefore our likelihood function can be given as

$$L = \prod_{i=1}^{N} \pi^{1-u_i} \big[(1-\pi)\frac{e^{-\lambda}\lambda^{x_i}}{x_i!}\big]^{u_i}$$

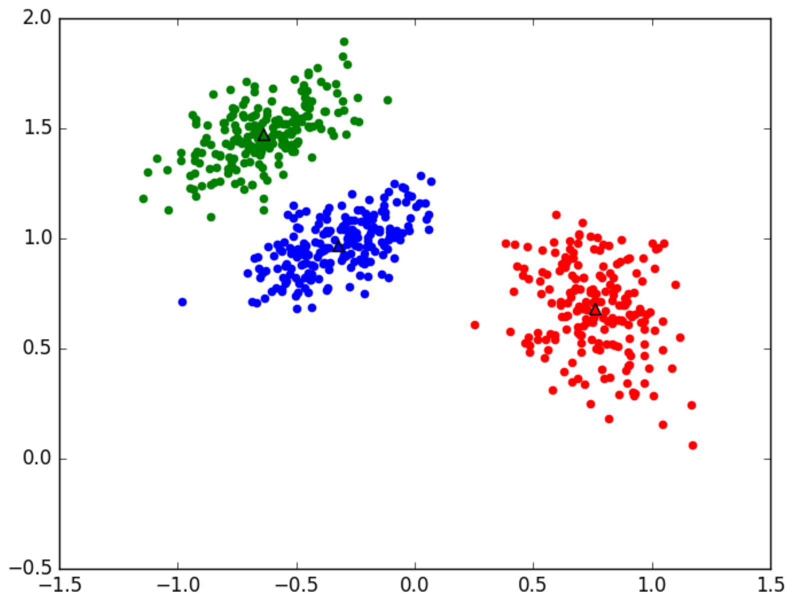
where $u_i = 1$ if $x_i > 0$ and $u_i = z_i$ if $x_i = 0$.

Sol 3.2 N/A

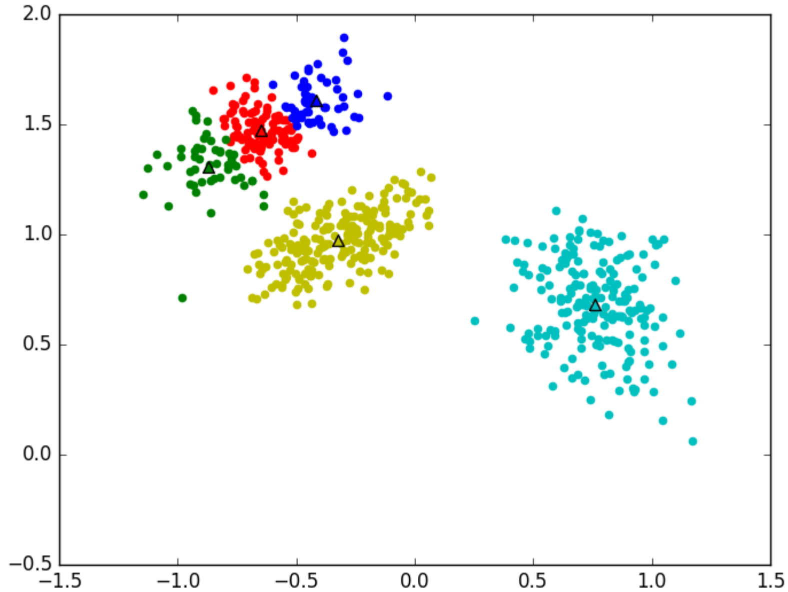
Sol 4.1



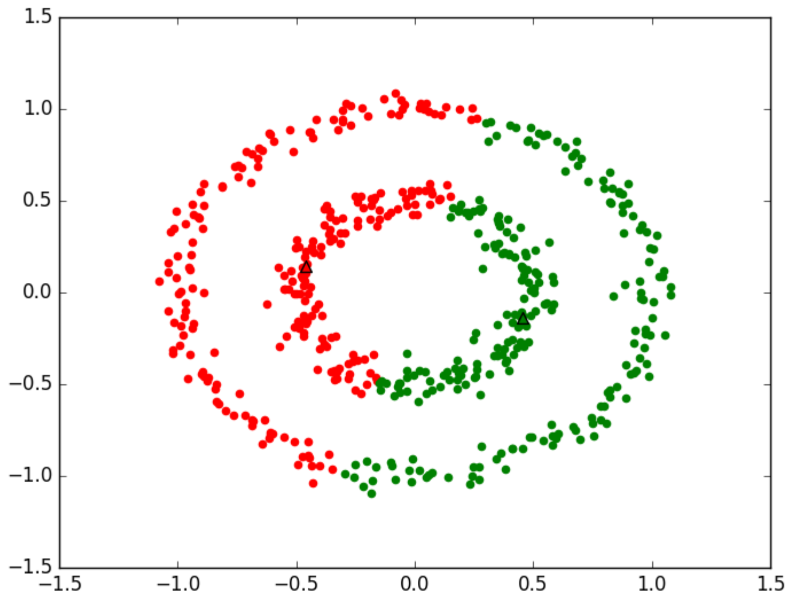
Blob, K = 2



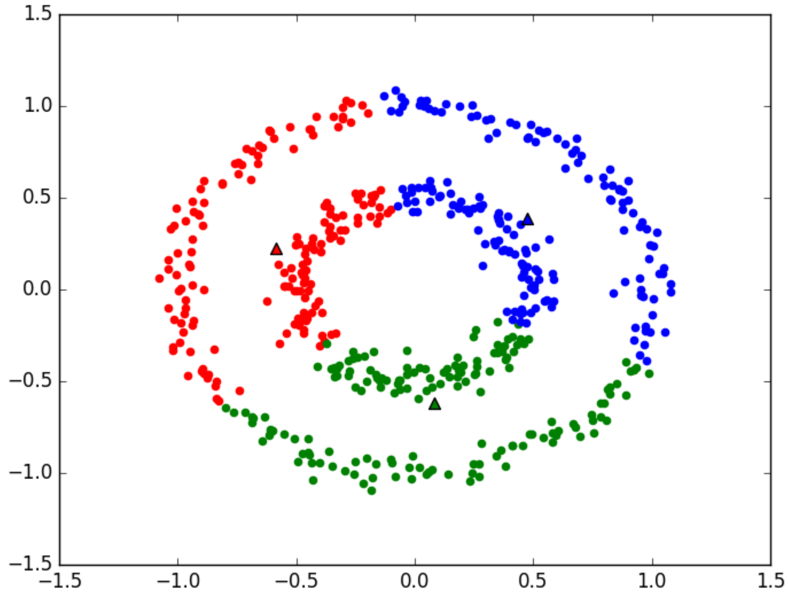
Blob, K = 3



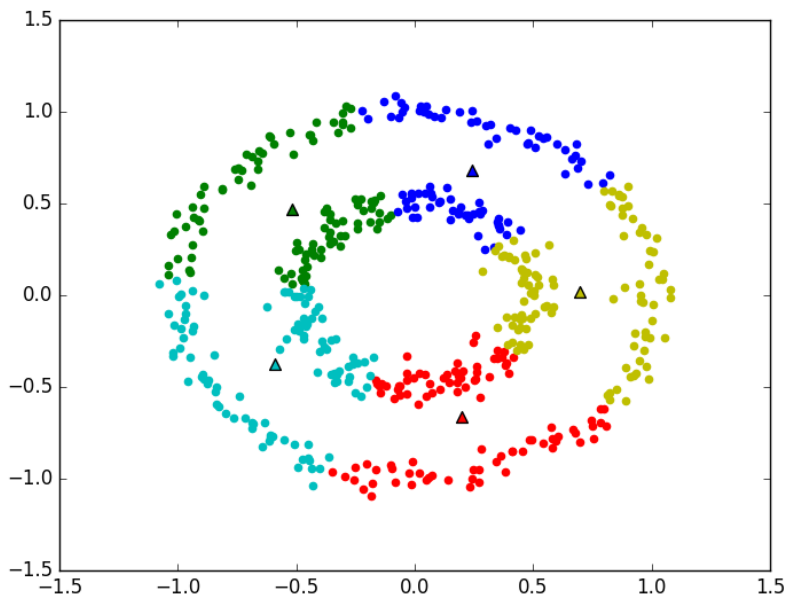
Blob, K = 5



Circle, K = 2



Circle, K = 3

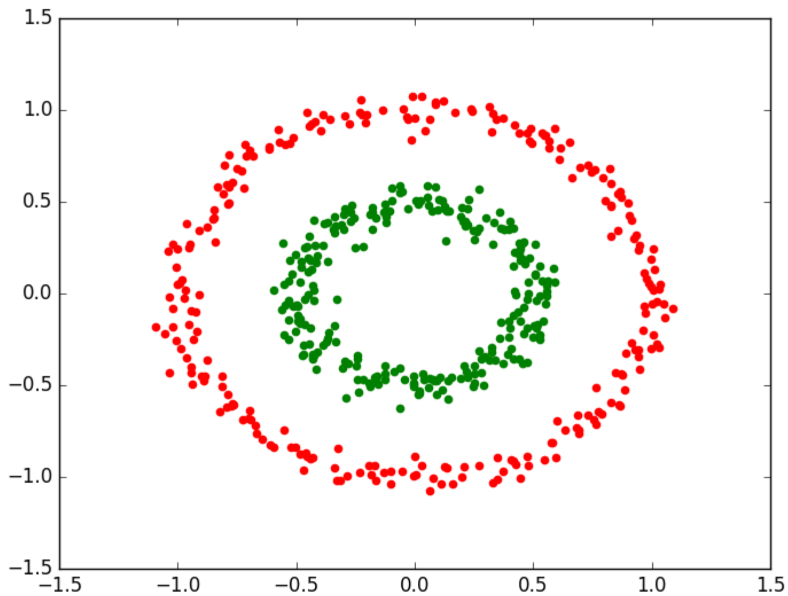


Circle, K = 5

For `circle.csv` and $K = 2$, since, both the clusters are concentric circles and their centroid falls at the same point and hence for a point the difference between minimum distances is small or equal.

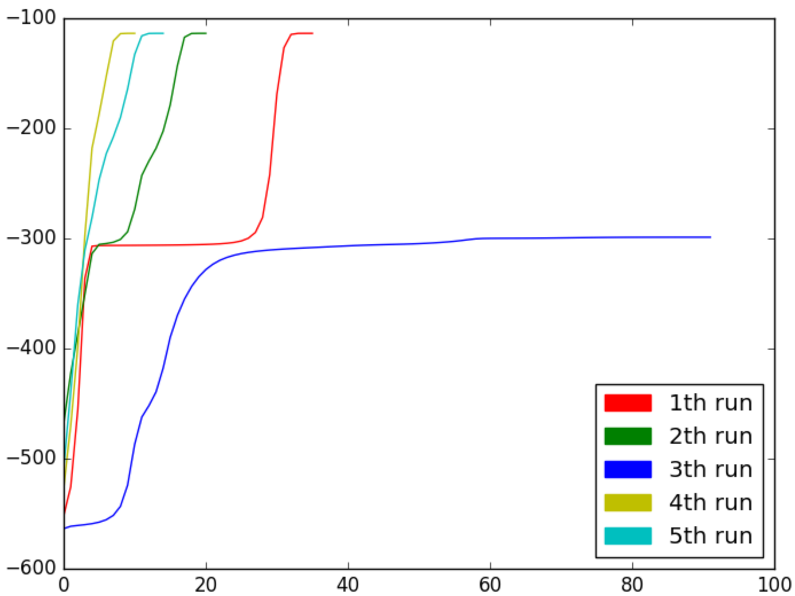
Sol 4.2 Using the feature transformation

$$\phi(x, y) = [x, y, 2(x^2 + y^2)]$$

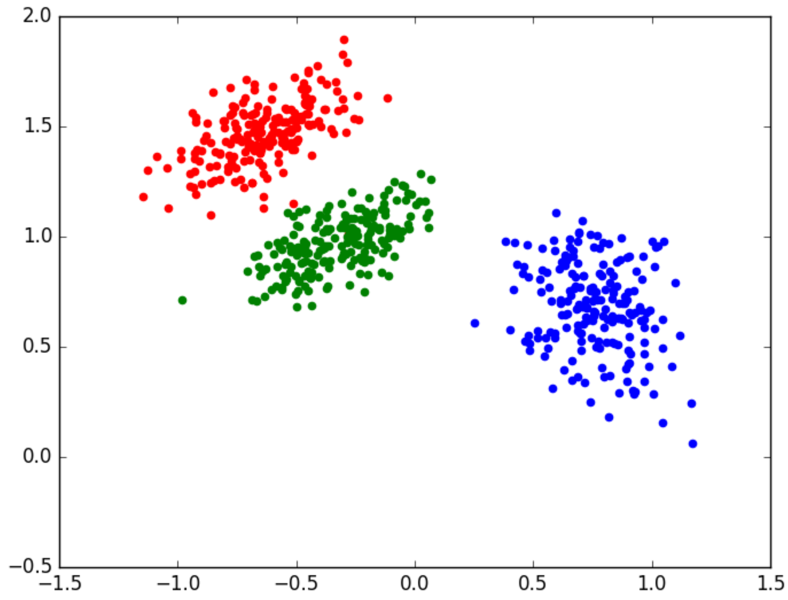


k = 2

Sol 4.3



Log-Likehood Plot



Cluster Assignment Plot

$$Means = \begin{bmatrix} K = 1 & 0.75896032 & 0.67976983 \\ K = 2 & -0.32591595 & 0.97133268 \\ K = 3 & -0.63946222 & 1.47460006 \end{bmatrix}$$

$$Covariance[k = 1] = \begin{bmatrix} 0.02717056 & -0.00840045 \\ -0.00840045 & 0.040442 \end{bmatrix}$$

$$Covariance[k = 2] = \begin{bmatrix} 0.03604869 & 0.01463998 \\ 0.01463998 & 0.01629099 \end{bmatrix}$$

$$Covariance[k = 3] = \begin{bmatrix} 0.03596703 & 0.01549264 \\ 0.01549264 & 0.01935347 \end{bmatrix}$$