PROBLEM 5.

a) Prove that $P(A|B,C) = \frac{P(B|A,C) \cdot P(A|C)}{P(B|C)}$

Ans. By definition, we know that

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \quad \text{——— (1)}$$

$$\therefore \quad P(A|(B\cap C)) = \frac{P((A\cap B)|C)}{P(B|C)} \quad \text{——— (2)}$$

Now let $D = B\cap C$,

$P(A|(B\cap C)) = P(A|D)$ which is same as (1)

$$P(A|(B\cap C)) = P(A|D) = \frac{P(A\cap D)}{P(D)}$$

$$= \frac{P(A\cap(B\cap C))}{P(B\cap C)} = \frac{P(A\cap B\cap C)}{P(B\cap C)} \quad \text{——— (3)}$$

Now re-arranging the final result of eq (3) & applying eq (1), we have

$$\frac{P(A\cap B\cap C)}{P(B\cap C)} = \frac{P(B\cap(A\cap C))}{P(B\cap C)}$$

Now applying eq (1), we have

$$P(B \cap (A \cap C)) = P(B \mid (A \cap C)) \cdot P(A \cap C)$$

$\therefore$ We have,

$$P\left(\frac{B \cap (A \cap C)}{P(B \cap C)}\right) = P\left(\frac{B \mid (A \cap C)) \cdot P(A \cap C)}{P(B \cap C)}\right)$$

Since we got the eq (3) from eq (2), we now have,

~~We now~~

$$P(A \mid (B \cap C)) = \frac{P(B \mid (A \cap C)) \cdot P(A \cap C)}{P(B \cap C)}$$

thus, proved.

$$P(A \mid B, C) = \frac{P(B \mid A, C) \cdot P(A \mid C)}{P(B \mid C)}$$

PROBLEM 5

b) You are given a coin which is either fair or double-headed. You believe that priori odds of it being fair are F to 1 i.e you believe that the priori probability of the coin being fair is $\frac{F}{F+1}$.

As a function of F, how many heads in a row would you need to see before convinced that there is better than even chance that coin is double headed

Ans. let probability of coin being fair $= P_f = \frac{F}{F+1}$

then, probability of it being unfair (biased) $= \sim P_f$

$$= 1 - \frac{F}{F+1}$$

$$= \frac{F+1-F}{F+1} = \frac{1}{F+1}$$

let data of observing $x$ heads under each assumption be $D$.

Therefore,

if the coin is fair $P(D|P_f) = \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2}$

$$= \frac{1}{2^x} = 2^{-x}$$

Now if the coin is double headed,

$$P(D \mid \sim P_f) = 1$$

According to bayes theorem,

$$P(\sim P_f \mid D) = \frac{P(D \cap \sim P_f)}{P(D)}$$

$$= \frac{P(D \mid \sim P_f) \cdot P(\sim P_f)}{P(D \mid \sim P_f) \cdot P(\sim P_f) + P(D \mid P_f) \cdot P(P_f)}$$

$P(D \mid \sim P_f) = 1$ & $P(D \mid P_f) = 2^{-x}$

$$\therefore \; P(\sim P_f \mid D) = \frac{1 \cdot P(\sim P_f)}{\left[1 \cdot P(\sim P_f)\right] + \left[2^{-x} P(P_f)\right]}$$

$$\frac{1 \cdot \left(\frac{1}{F+1}\right)}{1 \cdot \left(\frac{1}{F+1}\right) + 2^{-x}\left(\frac{F}{F+1}\right)} \underset{=}{\text{want}} 1.$$

$$\frac{\cancel{\frac{1}{F+1}}}{\cancel{\frac{1}{F+1}}\left(1 + 2^{-x} F\right)} = 1.$$

$$\frac{1}{1+2^{-x}F} = 1$$

$$\frac{1}{1} = 1+2^{-x}F$$

$$\therefore \quad 2^{-x}F = 0$$

We know that F cannot be zero, thus.

$2^{-x} \approx 0$. This is possible when $x$ is a very large number. Fore example if $x$ is 1 million $2^{-10^6} \approx 0$.

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