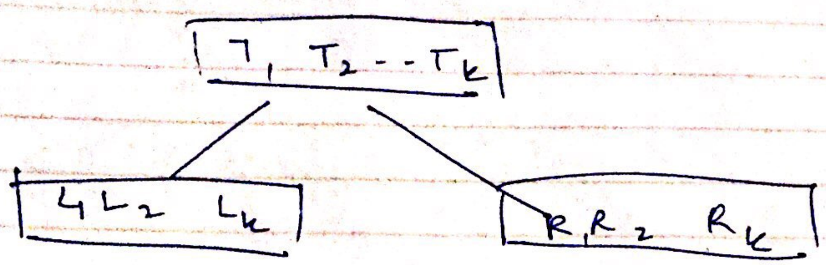
PROBLEM 4



T_i = label i count before split, k labels

$T_i = L_i + R_i$

L_i = label i count on left branch

R_i = label i count on right branch

$\sum T_i = T$, $\sum L_i = L$, $\sum R_i = R$

$$H_T = H_{before} = \sum \frac{T_i}{T} \log\left(\frac{T}{T_i}\right)$$

$$H_s = H_{after} = \frac{L}{T} \sum \frac{L_i}{L} \log\left(\frac{L}{L_i}\right) + \frac{R}{T} \sum \frac{R_i}{R} \log\left(\frac{R}{R_i}\right)$$

want to prove that $H_{before} - H_{after} \leq 1$

$$\sum_{i=1}^{k} \frac{T_i}{T} \log \frac{T}{T_i} - \sum_{i=1}^{k} \frac{L_i}{T} \log \frac{L}{L_i} - \sum_{i=1}^{k} \frac{R_i}{T} \log \frac{R}{R_i} \leq 1$$

$$\sum_i \frac{T_i}{T}\left(\log\left(\frac{T}{T_i}\right) - \frac{L_i}{L} \log \frac{L}{L_i} - \frac{R_i}{T_i} \log \frac{R}{R_i} \right) \leq 1$$

$$\sum_i \frac{T_i}{T} \left[\log \frac{T}{T_i} + \frac{L_i}{T_i} \log \frac{L_i}{L} + \frac{R_i}{T_i} \log \frac{R_i}{R} \right] \leq 1$$

$$\sum_i \frac{T_i}{T} \left[\log(T) - \log(T_i) + \frac{L_i}{T_i} \log(L_i) - \frac{L_i}{T_i} \log L + \frac{R_i}{T_i} \log R_i - \frac{R_i}{T_i} \log R \right] \leq 1$$

Maximum Entropy is when $T_i = L_i + R_i$ where

$L_i = \frac{T_i}{2}$ $R_i = \frac{T_i}{2}$

$$\sum \frac{T_i}{T}\left[\log \frac{T}{T_i} + \frac{\cancel{T_i}}{2\cancel{T_i}} \log \frac{L_i}{L} + \frac{\cancel{T_i}}{2\cancel{T_i}} \log \frac{R_i}{R}\right] \leq 1$$

$$\sum \frac{T_i}{T}\left[\log \frac{T}{T_i} + \frac{1}{2}\log \frac{L_i}{L} + \frac{1}{2}\log \frac{R_i}{R}\right] \leq 1$$

We know that $\sum T_i = \sum R_i + \sum L_i$

$T = R + L$

$\therefore R = \frac{T}{2}$ & $L = \frac{T}{2}$

$$\sum \frac{T_i}{T}\left[\log \frac{T}{T_i} + \frac{1}{2}\log\left(\frac{T_i (2)}{2 (T)}\right) + \frac{1}{2}\log\left(\frac{T_i}{2}\left(\frac{2}{T}\right)\right)\right] \leq 1$$

$$\sum \frac{T_i}{T}\left[\log T - \log T_i + \frac{1}{2}\log T_i - \frac{1}{2}\log T + \frac{1}{2}\log T_i - \frac{1}{2}\log T\right] \leq 1$$

$$\sum \frac{T_i}{T}\left[\log T - \log T_i + \log T_i - \log T\right] \leq 1$$

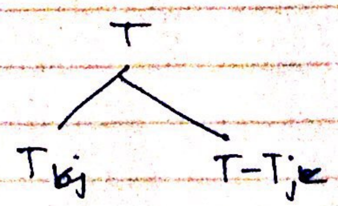
$$0 \leq 1$$

Thus, we see that $H_{before} - H_{after} = 0$

i.e there is no change in entropy

Scanned with CamScanner

Now consider the case where Righ has pure labels and left has the rest.



$L_i = T_j$
$L = T_j$
$R = T - T_j$

$$\sum_i \frac{T_i}{T}\left[\log\frac{T}{T_i} + \frac{L_i}{T_i}\log\frac{L_i}{L} + \frac{R_i}{T_i}\log\frac{R_i}{R}\right] \le 1$$

$$\sum_i \frac{T_i}{T}\left[\log\frac{T}{T_i} + \frac{R_i}{T_i}\log\frac{R_i}{R}\right] \le 1$$

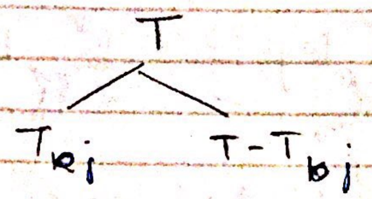
$$\sum_i \frac{T_i}{T}\left[\log\frac{T}{T_i} + \cancel{\frac{T_i}{T_i}}\log\frac{T_i}{T}\right]$$

$$\sum_i \frac{T_i}{T}\left[\log\left(\frac{R+L}{R_i+L_i}\right) + \frac{L_i}{(L_i+R_i)}\log\left(\frac{\cancel{R_i}L_i}{L} + \frac{R_i}{R_i+L_i}\log\frac{R_i}{R}\right]$$

$\log_2 x = 1$

$2^1 \to x$

Now consider the case ~~when~~ when Left and Right divisions have pure labels



$\therefore L_i = T_j$

$R_j = 0$

$R = \sum T_i - L_j$

$L = T_j \qquad R = T - L_j$

$$\sum \frac{T_i}{T}\left[\log \frac{T}{T_i} + \frac{L_i}{T_i} \log \frac{L_i}{L} + \frac{R_i}{T_i} \log \frac{R_i}{R}\right]$$

$$\sum \frac{T_i}{T}\left[\log \frac{T}{T_i} + \frac{T_i}{T_i} \log \frac{T_i}{T_i} + \frac{0}{T_i} \log \left(\quad \right)\right]$$

$$\sum \frac{T_i}{T}\left[\log \frac{T}{T_i} + \log 1 + 0\right]$$

$$= \sum \frac{T_i}{T} \log \frac{T}{T_i} \qquad \sum_{i \neq j} \frac{T_i}{T}\left[\log \frac{T}{T_i} + \frac{L_i}{T_i} \log \frac{L_i}{L} + \cancel{\frac{R_i}{T_i}} \log \frac{R_i}{R}\right] +$$

$$\sum \frac{T_i}{T}\left[\log \frac{T}{T_i} + \frac{L_i}{T_i} \log \frac{L_i}{L} + \frac{R_i}{T_i} \log \frac{R_i}{R}\right] \frac{T_j}{T}\left[\log \frac{T}{T_j} + \frac{L_j}{T_j} \log \frac{L_j}{L} + \cancel{\frac{R_j}{T_j}}\right.$$

$$\sum \frac{T_i}{T} \log \frac{T}{T_i} + \sum \frac{\cancel{T_i}}{T} \cdot \frac{L_i}{\cancel{T_i}} \log \frac{L_i}{L} + \sum \frac{\cancel{T_i}}{T} \cdot \frac{R_i}{\cancel{T_i}} \log \frac{R_i}{R}$$

$$\sum \frac{T_i}{T} \log \frac{T}{T_i} + \sum \frac{L_i}{T} \log \frac{L_i}{L} + \sum \frac{R_i}{T} \log \frac{R_i}{R}$$

$$\sum \frac{T_i}{T} \log \frac{T}{T_i} + \sum \frac{L_i}{T} \log \frac{L_i}{L} + \frac{R}{T} \log \frac{R}{R}\ ]$$

$$\sum \frac{T_i}{T} \log \frac{T}{T_i} + \sum_{i \neq j} \frac{T_i}{T} \log \frac{T_i}{L}$$