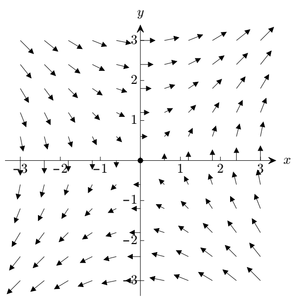
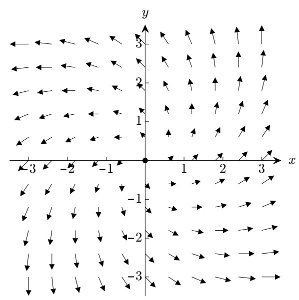
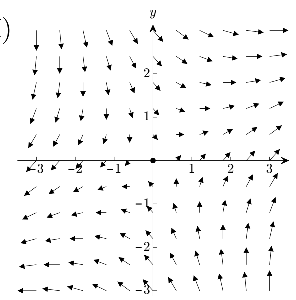
Problem Set 21 - Line Integrals

You should be able to:

- Qualitatively describe the computation underlying a line integral, and determine the sign of $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ from a picture of a curve $\mathcal{C}$ in a vector field $\vec{F}$.
- Use parameterized curves to compute line integrals, with attention to the orientation of the curve.

1. Let $\vec{F}(x, y) = \langle x - y, x + y \rangle$.

   (a) Without using technology, decide which of the following pictures shows $\vec{F}$. Explain how you made your decision.

   (I)
   
   (II)
   
   (III)
   

   (b) Let C be the straight-line path from $(-1, -1)$ to $(-1, 1)$. Without calculating, is $\int_C \vec{F} \cdot d\vec{r}$ positive, negative, or 0? How do you know?

   (c) If C is instead the path along $y = x^3$ from $(-1, -1)$ to $(1, 1)$, is $\int_C \vec{F} \cdot d\vec{r}$ positive, negative, or 0?

   (d) Find two paths C_1 and C_2 from $(2, 2)$ to $(-2, -2)$ with the property that $\int_{C_1} \vec{F} \cdot d\vec{r} \neq \int_{C_2} \vec{F} \cdot d\vec{r}$.

   Please show your calculations of both $\int_{C_1} \vec{F} \cdot d\vec{r}$ and $\int_{C_2} \vec{F} \cdot d\vec{r}$.

   Hint: Use the picture to help you pick good paths!

2. The vector field in #2(b) on the worksheet was $\vec{F}(x, y) = \langle y, x \rangle$. In this problem, you'll look at it a bit more.

   (a) Let C be the path along $y = x^2 - 2x - 2$ from $(-1, 1)$ to $(2, -2)$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.

   (b) Open this GeoGebra applet, which allows you to enter a vector field $\vec{F}$, draw a curve C, and see the value of $\int_C \vec{F} \cdot d\vec{r}$. The vector field $\vec{F}(x, y) = \langle y, x \rangle$ is already entered, as are the starting point $(-1, 1)$ and ending point $(2, -2)$ that you looked at in in class.

   Click and drag the red point to draw a path C from $(-1, 1)$ to $(2, -2)$, and then hit "Calculate line integral" to see $\int_C \vec{F} \cdot d\vec{r}$. Hit "Draw new path" to start a new path.

Draw several different paths C from $(-1,1)$ to $(2,-2)$, and look at the values of $\int_C \vec{F} \cdot d\vec{r}$. What do you notice?

(c) Now, pick a different starting and ending point (whatever you like). Enter these new points in the applet, and then use the applet to look at $\int_C \vec{F} \cdot d\vec{r}$ for several paths from your starting point to your ending point. What do you notice?

(d) (Exploratory[(1)]) In complete sentences, summarize what you've observed about this vector field. Does the vector field in #1 behave similarly?

3. Here's some information about an unknown vector field $\vec{F}(x,y)$:

- $\int_{C_1} \vec{F} \cdot d\vec{r} = 5$, where C_1 is the path along $y = \sin x$ from $(0,0)$ to $(\pi, 0)$.
- $\int_{C_2} \vec{F} \cdot d\vec{r} = -3$, where C_2 is the line segment from $(0,0)$ to $(\pi, 0)$.

Find $\int_C \vec{F} \cdot d\vec{r}$, where C is the path which goes along $y = \sin x$ from $(0,0)$ to $(\pi, 0)$ and then returns to $(0,0)$ along a straight line.

4. A particle is moving through the force field $\vec{F}(x,y,z) = \langle 0, 0, z \rangle$.

(a) Find the work done on the particle as it moves along the straight line from $(2,6,2)$ to $(5,7,-4)$.

(b) Your friend doesn't understand graphically why the answer to #4(a) has the sign that it does. Explain why. (Remember you picked the picture of $\vec{F}$ in Problem Set 21, #1(b).)

5. **Reflect Back.** We've now studied several types of integrals, and it's important to keep track of how they're different. Fill out the following table. You should also spend some time thinking about how to interpret each type of integral and when you would use one vs. another.

[(1)] The goal of exploratory problems is to get you thinking about issues we'll study more in the future. Because they are often open-ended, you will receive full credit as long as you show evidence of real thought and effort (even if your answers are not completely correct.)

Type of integral	**Notation**	**Is the integrand[(2)] a scalar-valued function or a vector field?**	**What's the dimension of the domain of integration?**	**Does the domain of integration live in $\mathbb{R}^2$ or $\mathbb{R}^3$, or could it be in either?**
double integral	$\iint_R f(x,y)\,dA$			
triple integral				
scalar surface integral				
line integral				

6. **Reflect Back.** Let $\vec{F}(x,y)$ be a vector field on $\mathbb{R}^2$. Suppose that (a,b) is a point at which

$$(\mathrm{curl}_2\,\vec{F})(a,b) > 0.$$

Let C be a tiny loop around (a,b), traveled clockwise. Do you expect $\int_C \vec{F}\cdot d\vec{r}$ to be positive, negative, or 0? Explain.

We'll see next time that the line integral around such a loop is closely related to the 2-dimensional curl!

(2) The integrand is the function that appears in the integral. For example, in $\int_C \vec{F}\cdot d\vec{r}$, the integrand is $\vec{F}$ and the domain is C.