Problem Set 13 - The Cross Product

You should be able to:

- Use the right-hand rule to determine the direction of the cross product $\vec{v} \times \vec{w}$ in relation to the two vectors $\vec{v}$ and $\vec{w}$.
- Compute the cross products of two vectors in $\mathbb{R}^3$ and relate the result to the area of the parallelogram defined by the two vectors.
- Recognize when a cross product can help you find the linear equation of a plane.

1. Find implicit and parametric equations for the plane that passes through $(1, -1, 1)$ and contains the line $x = 4 + 3t, y = 2t, z = t$.

   How can you check your answers?

2. Helen is writing a 3D video game in which the player will shoot a ray gun at rampaging dinosaurs to tranquilize them. Helen's working on the code that will determine whether a player's shot actually hits a dinosaur. As you can see in the picture below, a 3D model of a dinosaur is actually made up of lots and lots of triangles.(1) So, what Helen really needs to be able to do is to decide whether a particular ray gun shot hits a triangle.

   Source: MMKB

   Suppose that the player shoots the ray gun, causing a straight beam to leave the ray gun at $(1, 3, -3)$ in the direction $\langle 2, -1, 1\rangle$. We'd like to know whether this beam hits the triangle $\mathcal{T}$ with vertices $(3, 1, 2)$, $(4, 4, -1)$, and $(5, -3, 1)$. We'll figure this out in a few steps.

   (a) Let $\mathcal{P}$ be the plane containing the triangle $\mathcal{T}$. Find a implicit equation for $\mathcal{P}$.

   (b) Find the point Q where the beam hits the plane $\mathcal{P}$. Explain your reasoning carefully.

   (c) Now, we have to decide whether the point Q is actually inside the triangle $\mathcal{T}$. To accomplish this, it's helpful to write parametric equations for the plane $\mathcal{P}$; do so.

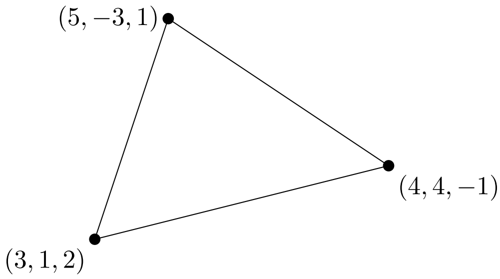
   

   Source: Pluralsight

   (d) What values of the parameters does Q correspond to? Use this to sketch a diagram showing where Q is relative to $(3, 1, 2)$, $(4, 4, -1)$, and $(5, -3, 1)$ and to explain whether Q is inside $\mathcal{T}$ or not.

   Note: In your diagram, you don't need to accurately show where $(3, 1, 2)$, $(4, 4, -1)$, and $(5, -3, 1)$ are in $\mathbb{R}^3$. The important thing is simply that they're 3 points that form a triangle, so you could start with a sketch like this one and then indicate where Q is:

(1) Modern games can involve up to a billion triangles at once! See this video for one example.



(e) Did the player's shot hit the triangle $\mathcal{T}$?

3. Suppose $\vec{v}$ and $\vec{w}$ are two vectors parallel to the plane $x + 2y + 3z = 7$. Suppose furthermore that $\vec{v}$ is perpendicular to $\vec{w}$, $\|\vec{v}\| = 3$, and $\|\vec{w}\| = 4$.

   (a) Based on the given information, what are all possible values of $\vec{v} \times \vec{w}$? As always, be sure to explain your reasoning.

   (b) Give an example of vectors $\vec{v}$ and $\vec{w}$ that satisfy the given information. (Please explain briefly how you came up with your example.)

   (c) Calculate $\vec{v} \times \vec{w}$ for the example you gave in #3(b).

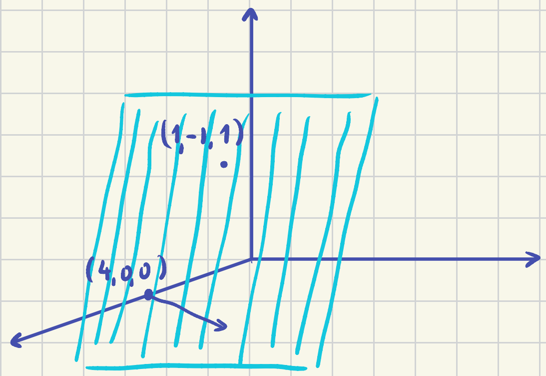
   Feel free to check your answer to #3(c) using WolframAlpha; for example, to evaluate $\langle 1, 1, 1\rangle \times \langle 1, 2, 3\rangle$, you simply enter `<1, 1, 1> x <1, 2, 3>`.

Question 1

$$\begin{array}{r} (4,0,0) \\ -\ (1,-1,1) \\ \hline \langle 3,1,-1\rangle \end{array}$$

$(x,y,z) = (1,-1,1) + u\langle 3,1,-1\rangle + v\langle 3,2,1\rangle$
parametric

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 3 & 1 & -1 & 3 & 1 \\ 3 & 2 & 1 & 3 & 2 \end{array}$$



$$\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & 6 \\ +2 & -3 & -3 \end{array}$$

$\langle 3,-6,3\rangle$

$\langle 3,-6,3\rangle\langle x-1, y+1, z-1\rangle = 0$
$3(x-1) - 6(y+1) + 3(z-1) = 0$
$3x - 3 - 6y - 6 + 3z - 3 = 0$
$3x - 6y + 3z = 12$
$x - 2y + z = 4$

To verify our answers, we plug in different points on the line
$x = 4 + 3t$; $y = 2t$; $z = t$ to ensure they exist in the plane.

$t = 1$; $x = 7$; $y = 2$; $z = 1$

$\langle 7, 2, 1\rangle$

$7 - 2(2) + 1 = 4$
$7 - 4 + 1 = 4$
$4 = 4$

$(7,2,1) = (1,-1,1) + u\langle 3,1,-1\rangle + v\langle 3,2,1\rangle$

$(6,3,0) = u\langle 3,1,-1\rangle + v\langle 3,2,1$

$u = 1$; $v = 1$

$$\begin{array}{r} \langle 3,1,-1\rangle \\ +\ \langle 3,2,1\rangle \\ \hline \langle 6,3,0\rangle \end{array}$$

hence, verified

Question 2

(a)

$$\begin{array}{r} (4, 4, -1) \\ -\,(3, 1, 2) \\ \hline \langle 1, 3, -3 \rangle \end{array} \qquad \begin{array}{r} (5, -3, 1) \\ -\,(4, 4, -1) \\ \hline \langle 1, -7, 2 \rangle \end{array}$$

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 1 & 3 & -3 & 1 & 3 \\ 1 & -7 & 2 & 1 & -7 \end{array}$$

$$\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 6 & -3 & -7 \\ -21 & -2 & -3 \end{array}$$

$\langle -15, -5, -10 \rangle \Rightarrow 5 \langle 3, 1, 2 \rangle$

$\langle 3, 1, 2 \rangle \langle x-4, y-4, z+1 \rangle = 0$

$3(x-4) + (y-4) + 2(z+1) = 0$

$3x - 12 + y - 4 + 2z + 2 = 0$

$3x + y + 2z = 14$

(b) Beam

$(x, y, z) = (1, 3, -3) + t \langle 2, -1, 1 \rangle$

Parametric : $x = 1 + 2t$
$y = 3 - t$
$z = -3 + t$

The beam travels parametically. Since we broke down the beam into x, y and z components within a common variable. We can substitute these equations into x, y and z variables and simplify.

$3(1+2t) + (3-t) + 2(-3+t) = 14$
$3 + 6t + 3 - t - 6 + 2t = 14$
$5t + 2t = 14$
$7t = 14$
$t = 2$

$$\left.\begin{array}{l} x = 1+4 = 5 \\ y = 3-2 = 1 \\ z = -3+2 = -1 \end{array}\right\} \text{beam hits } (5, 1, -1)$$

(c)

$$\begin{array}{r} (3,1,2) \\ -\ (5,-3,1) \\ \hline \langle -2, 4, 1 \rangle \end{array} \qquad \begin{array}{r} (3,1,2) \\ -\ (4,4,-1) \\ \hline \langle -1, -3, 3 \rangle \end{array}$$

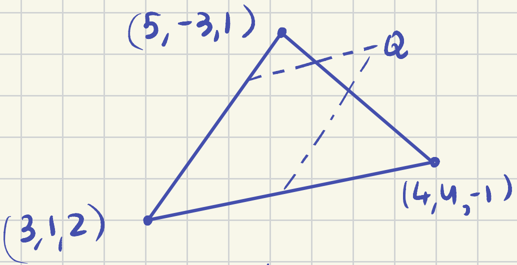
$$(x,y,z) = (3,1,2) + t\langle -2,4,1\rangle + s\langle -1,-3,3\rangle$$

$$x = 3 - 2t - s$$
$$y = 1 + 4t - 3s$$
$$z = 2 + t + 3t$$

(d)

$$5 = 3 - 2t - 5 \qquad (s = -2 - 2t)$$
$$2 = -2t - 5$$
$$1 = 1 + 4t - 3t \qquad \therefore \quad 0 = 4t - 3t$$
$$-1 = 2 + t + 3t$$

$$0 = 4t - 3(-2-2t)$$
$$0 = 4t + 6 + 6t$$
$$-6 = 10t$$
$$-\frac{4}{5} = t$$



$$s = -2 + \frac{8}{5} = -\frac{2}{5}$$

$t = -4/5$
$s = -2/5$

Q is not inside T, because when you add the 2 plane vectors, the resulting point Q is located outside the triangle.

(e) The beams hits on the outside so, no.

Question 3

(a) $\vec{v} \times \vec{w} = t\langle 1,2,3\rangle$

$\|\langle 1,2,3\rangle\| = \sqrt{1+4+9} = \sqrt{14}$

$\vec{u} = \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$

$\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\|\vec{w}\|\sin\theta$

$3 \cdot 4(1) = 12$

$\vec{v} \times \vec{w} = \pm 12 \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$

When we find the implicit equation, we find the dot product of normal vector $\vec{n}$ to $\langle x-a, y-b, z-c\rangle$ where (a,b,c) is an arbitrary point. Because normal vector is perpendicular, $\langle n_1, n_2, n_3\rangle\langle x-a, y-b, z-c\rangle = 0$ i.e $n_1 x + n_2 y + n_3 z = an_1 + bn_2 + cn_3$. Since the coefficients are the same and the implicit equation is $x + 2y + 3z = 7$, $\vec{v} \times \vec{w}$ will be a scaled factor of $\langle 1,2,3\rangle$

(b) $12\left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle \langle v_1, v_2, v_3\rangle = 0$

$\frac{12}{\sqrt{14}} v_1 + \frac{24}{\sqrt{14}} v_2 + \frac{36}{\sqrt{14}} v_3 = 0$

$v_1 = 2 \qquad v_2 = -1 \qquad v_3 = 0$

$\langle 2, -1, 0\rangle$

unit vector : $\sqrt{2^2+1^2+0} = \sqrt{5}$

$3 \cdot \left\langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}, 0 \right\rangle = \left\langle \frac{6}{\sqrt{5}}, \frac{-3}{\sqrt{5}}, 0 \right\rangle = \vec{v}$

$\frac{12}{\sqrt{14}} w_1 + \frac{24}{\sqrt{14}} w_2 + \frac{36}{\sqrt{14}} w_3 = 0$

$\langle 2, -1, 0\rangle\langle w_1, w_2, w_3\rangle = 0$

$2w_1 - w_2 = 0$

$2w_1 = w_2$

$$\frac{12}{\sqrt{14}} w_1 + \frac{24}{\sqrt{14}} \cdot 2w_1 + \frac{36}{\sqrt{14}} w_3 = 0$$

$$\frac{60}{\sqrt{14}} w_1 + \frac{36}{\sqrt{14}} w_3 = 0$$

$$w_1 = 3 \qquad w_2 = 6 \qquad w_3 = -5$$

$$\sqrt{3^2+6^2+(-5)^2} = \sqrt{9+36+25} = \sqrt{70}$$

$$\frac{4}{\sqrt{70}} \langle 3, 6, -5 \rangle$$

$$\vec{w} = \left\langle \frac{12}{\sqrt{70}}, \frac{24}{\sqrt{70}}, \frac{-20}{\sqrt{70}} \right\rangle$$

(c)

$$\begin{array}{ccccc}
\vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\
\frac{4}{\sqrt{5}} & \frac{-3}{\sqrt{5}} & 0 & \frac{6}{\sqrt{5}} & \frac{-3}{\sqrt{5}} \\
\frac{12}{\sqrt{70}} & \frac{24}{\sqrt{70}} & \frac{-20}{\sqrt{70}} & \frac{12}{\sqrt{70}} & \frac{24}{\sqrt{70}}
\end{array}$$

$$\begin{array}{ccc}
\vec{i} & \vec{j} & \vec{k} \\
\frac{60}{\sqrt{350}} & 0 & \frac{144}{\sqrt{350}} \\
0 & \frac{120}{\sqrt{350}} & \frac{36}{\sqrt{350}}
\end{array}$$

$$\sqrt{350} = 5\sqrt{14}$$

$$\vec{v} \times \vec{w} = \left\langle \frac{12}{\sqrt{14}}, \frac{24}{\sqrt{14}}, \frac{36}{\sqrt{14}} \right\rangle$$