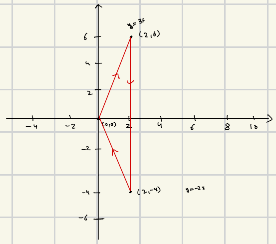
1.



It wouldn't be fun to compute $\int_C \vec{F} \cdot d\vec{r}$ directly as the curve C cannot be parameterised with one parameterisation, instead we would need 3 parameterisations for the 3 separate line segments. This would lead to us calculating three separate integrals and then adding them up.

Using greens theorem to evaluate integral.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R (\mathrm{curl}_2 F) \cdot dA$$

$Q_x = 2x + 2y\cos x$

$P_y = 2y\cos x$

$\mathrm{Curl}_2 F = Q_x - P_y = 2x + 2y\cos x - 2y\cos x$

$= 2x$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^2 \int_{-2x}^{3x} -2x \cdot dy\, dx$$

(If we move along C, the region R is to the right of the curve, hence I take the −ve of the $\mathrm{curl}_2 F$ to account for the orientation.)

$$= -\int_0^2 \int_{-2x}^{3x} 2x \cdot dy \cdot dx$$

$$= -\int_0^2 2x\,[y]_{-2x}^{3x} \cdot dx$$

$$= -\int_0^2 2x\,[3x - (-2x)] \cdot dx$$

$$= -\int_0^2 10x^2 \cdot dx$$

$$= -\left[\frac{10x^3}{3}\right]_0^2$$

$$= -\frac{80}{3}$$

2. $\vec{r}(t) = \langle \cos^3 t, \sin^3 t \rangle \qquad 0 \le t \le 2\pi$

number of rabbits $= \iint_R x^2 \cdot dA$

comparing with green's theorem

$x^2 = \mathrm{Curl}_2 \vec{F} = Q_x - P_y$

Through hit and trial

[Note on orientation, as we move from t=0 to t=2π, the region is always to the left of the curve, hence $\int_C \vec{F} \cdot d\vec{r} = \iint_R \mathrm{curl}_2 F \cdot dA$]

let Q be $\frac{x^3}{3}$ and P be 0 as then $Q_x = x^2$ & $P_y = 0$

$\therefore Q_x - P_y = x^2$

$\therefore \vec{F}(x,y) = \langle 0, \frac{x^3}{3} \rangle$

$\vec{F}(\vec{r}(t)) = \langle 0, \frac{\cos^9 t}{3} \rangle$

$\vec{r}\,'(t) = \langle -3\cos^2(t)\sin(t), 3\sin^2(t)\cos(t) \rangle$

using green's theorem

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R \mathrm{curl}_2 \vec{F} \cdot dA$$

$$\iint_R x^2 \cdot dA = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \cdot dt$$

$$= \int_0^{2\pi} \langle 0, \frac{\cos^9 t}{3} \rangle \cdot \langle -3\cos^2 t \sin t, 3\sin^2 t \cos t \rangle \cdot dt$$

$$= \int_0^{2\pi} \sin^2(t)\cos^{10}(t) \cdot dt$$

(using Wolfram Alpha) $= \frac{21\pi}{512} \approx 0.1288$

number of rabbits $= \left(\int_C \vec{F} \cdot d\vec{r}\right) 1000$

$= 0.1288 \times 1000$

$= 128.8$

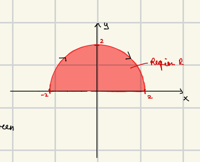
$= 129$

3. (a)

Parameterising $x^2 + y^2 = 4$

$\Rightarrow x = 2\cos t$

$y = 2\sin t$



$\therefore \vec{r}(t) = \langle 2\cos t, 2\sin t \rangle$, where t is between 0 and π

$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t \rangle$

$\vec{F}(\vec{r}(t)) = \langle \cos(2\cos t) + 4\sin^2 t, e^{4\sin^2 t} \rangle$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_\pi^0 \langle \cos(2\cos t) + 4\sin^2 t, e^{4\sin^2 t} \rangle \cdot \langle -2\sin t, 2\cos t \rangle \cdot dt$$

$$= \int_\pi^0 -2\sin t\cos(2\cos t) - 8\sin^3 t + 2\cos t\, e^{4\sin^2 t} \cdot dt$$

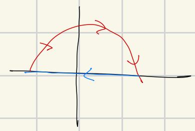
3 (b) We can't use green's theorom as the curve C is not a closed loop, i.e. its start and end points aren't the same. The start point of C is $(-2,0)$ while its end point is $(2,0)$.

(c) $\int_{C_2} \vec{F} \cdot d\vec{r}$

the parameterisation of C_2 is $\vec{r}(t) = \langle 2-4t, 0 \rangle$

$\therefore \ \vec{r}\,'(t) = \langle -4, 0 \rangle$

$\vec{F}(\vec{r}(t)) = \langle \cos(2-4t), 1 \rangle \quad = \langle \cos(2-4t) + 0^2, e^{0^2} \rangle$



$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \cdot dt$$
$$= \int_0^1 \langle \cos(2-4t), 1 \rangle \cdot \langle -4, 0 \rangle \cdot dt$$
$$= -4\int_0^1 \cos(2-4t) \cdot dt$$

let $2-4t = u$
$-4dt = du$
$dt = -\frac{du}{4}$

$$= -4 \int_2^{-2} \frac{-1}{4} \cos u \cdot du$$
$$= 1 \quad [\sin u]_2^{-2}$$
$$= 1\,[-\sin(2) - \sin(2)]$$
$$= \quad -2\sin(2)$$

(d) $\iint_R (\text{curl}_2 \vec{F})(x,y)\, dA$

$\vec{F}(x,y) = \langle \cos x + y^2, e^{y^2} \rangle$

$Q_x = 0$

$P_y = 2y$

$\text{curl}_2 \vec{F} = Q_x - P_y = -2y$

$$-\iint (\text{curl}_2 \vec{F})(x,y) \cdot dA$$

(NOTE on orientation: as the region R is to the right when we move along the curve we need a negative sign, $\therefore \int_C \vec{F} \cdot d\vec{r} = -\iint (\text{curl}_2 \vec{F}) \cdot dA$)

$$= -\iint_R -2y \cdot dx\,dy$$
$$= -\int_0^{\pi}\int_0^{2} -2r\sin\theta \cdot r\,dr \cdot d\theta$$
$$= 2\int_0^{\pi}\int_0^{2} r^2 \sin\theta \cdot dr \cdot d\theta$$
$$= 2\int_0^{\pi} \sin\theta \left[\frac{r^3}{3}\right]_{0=r}^{2=r} \cdot d\theta$$
$$= 2\int_0^{\pi} \sin\theta \left[\frac{8}{3} - 0\right] \cdot d\theta$$
$$= \frac{16}{3}\int_0^{\pi} \sin\theta \cdot d\theta$$
$$= \frac{16}{3}[-\cos\theta]_0^{\pi}$$
$$= \frac{16}{3}[-(-1) - (-1)]$$
$$= \frac{16}{3}(2) \quad = \frac{32}{3}$$

(e) $\int_C \vec{F} \cdot d\vec{r}$ is the work done when going from $(-2,0)$ to $(2,0)$ along the positive half of the circle $x^2+y^2=4$, or the curve C

It is equal to $\int_C \vec{F} \cdot d\vec{r} = \iint_R (\text{curl}_2 \vec{F})(x,y)\,dA - \int_{C_2} \vec{F} \cdot d\vec{r} = \frac{32}{3} - (-2\sin(2))$

$$= \frac{32}{3} + 2\sin(2)$$

We get this is as we divide the boundary of region R into 2 parameterisations and we are now finding the line integral of one of the paths with one of the parameterisations

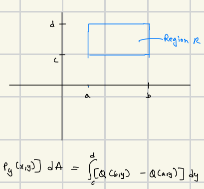
(f) The strategy I used was:

(i) I first found another curve to make a closed loop for the region to use Green's theorem

(ii) I then found the line integral of the other curve I came up with and then I subtracted this from the green's theorem, double integral.

(iii) This then helped me get the line integral over the original curve.

4.



(a) To show:

$$\iint_R [Q_x(x,y) - P_y(x,y)]\, dA = \int_c^d [Q(b,y) - Q(a,y)]\, dy - \int_a^b [P(x,d) - P(x,c)]\, dx$$

Left hand side $= \iint_R [Q_x(x,y) - P_y(x,y)]\, dA$

$$= \iint_R Q_x(x,y) \cdot dA - \iint_R P_y(x,y) \cdot dA$$
$$= \int_c^d\int_a^b Q_x(x,y) \cdot dx \cdot dy - \int_a^b\int_c^d P_y(x,y) \cdot dy\, dx$$

Please turn over.

4. $= \int_c^d \int_a^b Q_x(x,y) \cdot dx \cdot dy - \int_a^b \int_c^d P_y(x,y) \cdot dy\, dx$

$= \int_c^d [Q(b,y) - Q(a,y)] \cdot dy - \int_a^b [P(x,d) - P(x,c)] \cdot dx$

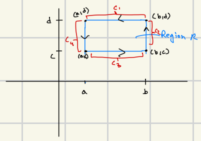
$= RHS$

$\therefore$ as $LHS = RHS$

$\therefore$ shown $\iint_R [Q_x(x,y) - P_y(x,y)] \cdot dA = \int_c^d [Q(b,y) - Q(a,y)] \cdot dy - \int_a^b [P(x,d) - P(x,c)] \cdot dx$

(b) To show: $\int_C \vec{F} \cdot d\vec{r} = \int_c^d [Q(b,y) - Q(a,y)] \cdot dy - \int_a^b [P(x,d) - P(x,c)] \cdot dx$

(LHS) Left hand side $= \int_C \vec{F} \cdot d\vec{r}$



using $C = C_1 + C_2 + C_3 + C_4$ (Look at figure drawn)

$\therefore LHS = \int_C \vec{F} \cdot d\vec{r}$

$= \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \int_{C_3} \vec{F} \cdot d\vec{r}$

$+ \int_{C_4} \vec{F} \cdot d\vec{r}$

$= \int_{C_1} \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t) \cdot dt + \int_{C_2} \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t) \cdot dt$

$+ \int_{C_3} \vec{F}(\vec{r}_3(t)) \cdot \vec{r}_3{}'(t) \cdot dt + \int_{C_4} \vec{F}(\vec{r}_4(t)) \cdot \vec{r}_4{}'(t)\, dt$

[$\vec{r}_1(t) = \langle b+t(a-b), d \rangle$
$\vec{r}_1{}'(t) = \langle a-b, 0 \rangle$
$\vec{r}_2(t) = \langle b, c+t(d-c) \rangle$
$\vec{r}_2{}'(t) = \langle 0, d-c \rangle$
$\vec{r}_3(t) = \langle a+t(b-a), c \rangle$
$\vec{r}_3{}'(t) = \langle b-a, 0 \rangle$
$\vec{r}_4(t) = \langle a, d+t(d-c) \rangle$
$\vec{r}_4{}'(t) = \langle 0, d-c \rangle$

$= \int_{C_1} \langle P(b+t(a-b),d), Q(b+t(a-b),d) \rangle \cdot \langle a-b, 0 \rangle \cdot dt$

$+ \int_{C_2} \langle P(b, c+t(d-c)), Q(b, c+t(d-c)) \rangle \cdot \langle 0, d-c \rangle \cdot dt$

$+ \int_{C_3} \langle P(a+t(b-a), c), Q(a+t(b-a), c) \rangle \cdot \langle b-a, 0 \rangle \cdot dt$

$+ \int_{C_4} \langle P(a, d+t(d-c)), Q(a, d+t(d-c)) \rangle \cdot \langle 0, d-c \rangle$

$= \int_0^1 (a-b) P(b+t(a-b), d) \cdot dt + \int_0^1 (Q(b, c+t(d-c)))(d-c) \cdot dt$

$+ \int_0^1 (b-a) P(a+t(b-a), c) + \int_0^1 (Q(a, d+t(d-c)))(d-c)$

for C_1 let $a+t(b-a) = x$, $(b-a)\,dt = dx$ $\therefore \int_0^1 (a-b) P(b+t(a-b), d) \cdot dt = -\int_a^b P(x,d)\, dx$

for C_2 let $c+t(d-c) = y$, $(d-c) \cdot dt = dy$ $\therefore \int_0^1 (Q(b, c+t(d-c)))(d-c) \cdot dt = \int_c^d Q(b,y) \cdot dy$

for C_3 let $a+t(b-a) = x$, $dx = (b-a)\,dt$ $\therefore \int_0^1 (b-a) P(a+t(b-a), c) = \int_a^b P(x,c) \cdot dx$

for C_3 let $d+t(d-c) = y$, $(d-c)dt = dy$ $\therefore \int_0^1 (Q(a, d+t(d-c)))(d-c) = -\int_c^d Q(a,a) \cdot dy$

substituting these

$\therefore \int_0^1 (a-b) P(b+t(a-b), d) \cdot dt + \int_0^1 (Q(b, c+t(d-c)))(d-c) \cdot dt$

$+ \int_0^1 (b-a) P(a+t(b-a), c) + \int_0^1 (Q(a, d+t(d-c)))(d-c)$

$= -\int_a^b P(x,d)\, dx + \int_c^d Q(b,y) \cdot dy + \int_a^b P(x,c) \cdot dx - \int_c^d Q(a,a) \cdot dy$

$= \int_c^d [Q(b,y) - Q(a,y)] \cdot dy - \int_a^b [P(x,d) - P(x,c)] \cdot dy$