Proving Lyapunov CLT using ChF

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Last week, we mentioned that an important idea behind simulation is CLT. I am thinking why not try to prove for a sequence of independent but not identically distributed random variables? This a good review on probability theory anyway.

Lyapunov condition

Assuming $\mathbb{E}X_j = 0$, one has

$$B_n^{-3}\sum_{n=1}^{\infty}\mathbb{E}\left|X_j\right|^3 \to 0 \text{ as } n \to \infty$$

where $B_n^2 := \mathbb{V}(S_n) = \sum_{j=1}^{n}\mathbb{E}X_j^2$

Note that here we assume X_j's all have zero mean, in the proof below we need to standardize accordingly.

Lyapunov CLT statement

For a series of independent but not necessarily identically distributed random variables $X_1, X_2, X_3, X_4, ...$, suppose that $\mathbb{E}\left|X_j\right| < \infty$ and that $\mathbb{E}X_j^2 < \infty$ and $\sigma_j^2 := \mathbb{V}(X_j) > 0\ \forall j \in \mathbb{N}$, then:

$$Y_n := \frac{\sum_{j=1}^{n}X_j - \sum_{j=1}^{n}\mu_j}{\sqrt{\sum_{j=1}^{n}\sigma_j^2}} \xrightarrow{d} Z \sim N(0,1) \text{ as } n \to \infty$$

assuming Lyapunov condition holds.

Proof

We first standardize X_j's by setting $\tilde{X}_j := X_j - \mu_j$.

Then we have $B_n^2 := \mathbb{V}(S_n) = \sum_{j=1}^{n}\mathbb{E}\tilde{X}_j^{\,2}$

Thus, we have:

$$Y_n = \frac{\tilde{S_n}}{B_n}$$

where $\tilde{S_n} = \sum_{j=1}^{n}\tilde{X}_j$

Let $\varphi_X(t) := \mathbb{E}(e^{itX})$ be the characteristic function for an arbitrary random variable X, then by the properties of ChF, we have:

$$\varphi_{Y_n}(t) = \varphi_{\frac{\tilde{S_n}}{B_n}}(t) = \varphi_{\tilde{S_n}}(t/B_n) = \prod_{j=1}^{n}\varphi_{\tilde{X_j}}(t/B_n) \tag{1}$$

let $s := \frac{t}{B_n}$, by Taylor expansion, we have that:

$$\varphi_{\tilde{X_j}}(s) = \varphi_{\tilde{X_j}}(0) + \varphi'_{\tilde{X_j}}(0)s\ + \frac{1}{2}\varphi''_{\tilde{X_j}}(0)s^2\ + \frac{1}{6}\varphi'''_{\tilde{X_j}}(0)s^3\ + o(s^3) \tag{2}$$

By the properties of ChF

$$\varphi_{\tilde{X}_j}(0) = 1 \tag{3}$$

$$\varphi'_{\tilde{X}_j}(0) = 0 \tag{4}$$

$$\varphi''_{\tilde{X}_j}(0) = -\mathbb{E}\tilde{X}_j^{\ 2} = -\mathbb{V}\tilde{X}_j^{\ 2} = \sigma_j^2 \tag{5}$$

$$\varphi'''_{\tilde{X}_j}(0) = i^3\mathbb{E}\tilde{X}_j^{\ 3} \tag{6}$$

Combining the results from equations (3), (4), (5) and (6), we have:

$$\varphi_{\tilde{X}_j}(s) = 1 - \frac{\sigma_j^2}{2}(s^2) + \frac{i^3\mathbb{E}\tilde{X}_j^{\ 3}}{6}(s^3) + o(s^3) \tag{7}$$

Utilizing result from (7), we go ahead to show the convergence of ChF of Y_n:

$$\begin{aligned}
\varphi_{Y_n}(t) &= \prod_{j=1}^{n} \varphi_{\tilde{X}_j}(t/B_n) \\
&= \prod_{j=1}^{n} \left(1 - \frac{\sigma_j^2}{2}(s^2) + \frac{i^3\mathbb{E}\tilde{X}_j^{\ 3}}{6}(s^3) + o(s^3)\right) \qquad s := t/B_n \\
&= \prod_{j=1}^{n} \left(1 - \frac{\mathbb{E}\tilde{X}_j^{\ 2}}{2}(s^2) + o(s^2)\right) \\
&= \prod_{j=1}^{n} \exp\left(\ln\left(1 - \frac{\mathbb{E}\tilde{X}_j^{\ 2}}{2}(s^2) + o(s^2)\right)\right) \\
&= \prod_{j=1}^{n} \exp\left(-\frac{\mathbb{E}\tilde{X}_j^{\ 2}}{2}(s^2) + o(s^2) + a(o(1))\right) \qquad a := -\frac{\mathbb{E}\tilde{X}_j^{\ 2}}{2}(s^2) + o(s^2) \\
&= \exp\left(-\sum_{j=1}^{n} \frac{\mathbb{E}\tilde{X}_j^{\ 2}}{2}(s^2) - \sum_{j=1}^{n} o(s^2) + a\sum_{j=1}^{n}(o(1))\right) \\
&= \exp\left(-\sum_{j=1}^{n} \frac{\mathbb{E}\tilde{X}_j^{\ 2}}{2B_n^2}(t^2) - \sum_{j=1}^{n} o(s^2) + a\sum_{j=1}^{n}(o(1))\right) \\
&= \exp\left(-\frac{t^2\sum_{j=1}^{n}\mathbb{E}\tilde{X}_j^{\ 2}}{2\sum_{j=1}^{n}\mathbb{E}\tilde{X}_j^{\ 2}} - \sum_{j=1}^{n} o(s^2) + a\sum_{j=1}^{n}(o(1))\right) \\
&= \exp\left(-\frac{t^2}{2} - \sum_{j=1}^{n} o(s^2) + a\sum_{j=1}^{n}(o(1))\right) \\
&\to e^{-t^2/2} \quad \text{as } n \to \infty
\end{aligned}$$

This end result of the convergence is the ChF of standard normal distribution. It can be shown that convergence in ChF guarantees the convergence in distribution. □