UNIT-II

DIFFERENTIAL CALCULUS

Topic Learning Objectives:

Upon Completion of this unit, student will be able to:

- Understand the fundamentals of the differential calculus of functions of one variable.
- Transform the coordinates from rectangular to polar and vice versa.
- Apply concepts of calculus to find angle between polar curves and consequences.
- Visualize Curvature for curves defined in different forms and find circle of curvature.
- Expand the function in power series using Taylor's and Maclaurin's series.
- Simulation using MATLAB.

Recapitulation: Functions of single variable:

The concept of **functions** is vital in calculus because it enables the precise modeling of how one quantity depends on another in real-world situations. The current flowing through an electrical circuit depends on the voltage applied and the resistance present. The height of a projectile at any given time is a function of time, determined by initial speed, gravity, and launch angle; calculus helps predict its position, velocity, and acceleration at any instant. Signal strength in telecommunications changes with distance from the transmitter; signal loss is modelled as a function that describes how power decreases over distance, enabling optimization of network coverage. The temperature at which water boils depends on the elevation above sea level (the boiling point drops as the height increases). The interest paid on a cash investment depends on the length of time the investment is held. The area of a circle depends on the radius of the circle. The distance an object travels from an initial location along a straight-line path depends on its speed. In each case, the value of one variable quantity, which might be called as y, depends on the value of another variable quantity, which might be called x. Since the value of y is completely determined by the value of x, it's said that y is a function of x. Often the value of y is given by a rule or formula that says how to calculate it from the variable x. For instance, $A = \pi r^2$, the equation is a rule that calculates the area A of a circle from its radius r.

A symbolic way to say 'y is a function of x' is by writing $y = f(x)$. In this notation, the symbol f represents the function. The letter x, called the independent variable, represents the input value of f, and y, the dependent variable, represents the corresponding output value of f at x.

Requirement of new coordinate systems:

There is already a familiarity with Cartesian coordinate system for specifying a point in the XY – plane in two-dimensional geometry and $XYZ -$ space in three- dimensional geometry. The requirement to define any new coordinate system is two-fold. One is based on geometry of the problem of practical situation wherein a more suitable coordinate system has to be chosen. For ex., the study of dispersion of a medicine injected in blood flow requires cylindrical coordinate system as the veins are cylindrical in nature. Use of Cartesian system may not be very suitable as it represents a rectangular channel and the corner effects have to be taken care. The second requirement is more of theoretical in nature. A mathematical expression which cannot be simplified in one coordinate system may be solved in simple way by transforming to other coordinate systems. For ex., $\log(x+y)$ cannot be further simplified in Cartesian system whereas it's easier to solve in Polar coordinates.

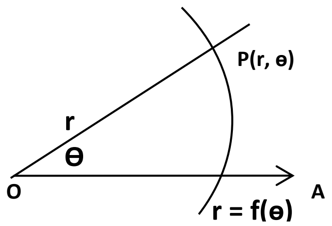
Basics of polar coordinates and polar curves:

A new coordinate system is introduced to understand the concept of polar curves and their properties.

Any point P can be located on a plane with co-ordinates $(\boldsymbol{r}, \boldsymbol{\theta})$ called **polar coordinates** of P where $r =$ **radius vector** OP, (with pole/origin 'O'), θ = projection of OP on the initial line OA.
The equation

$$r = f(\theta) \text{ or } \theta = f(r) \text{ or } f(r,\theta) = c$$

are known as a **polar curve**.

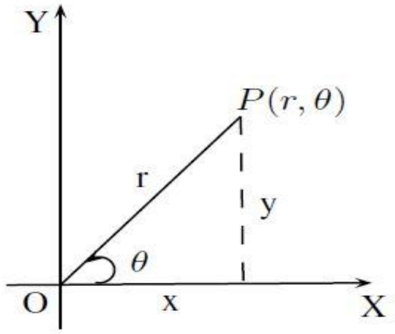


Polar curve

Polar coordinates naturally fit systems with circular/symmetric layouts, like antenna arrays, signal mapping, or electromagnetic field analysis. Most antennas emit signals in patterns best described using polar plots. The directionality or coverage of antennas is commonly represented as a curve $\mathrm{r} = \mathrm{f}(\theta)$.

Relation between Cartesian and polar coordinates:

Consider a point P in the xy-plane. Join the points O (origin) and P. Let r be the length of OP and θ be the angle which OP makes with the (positive) x-axis. The (r,θ) are called the polar coordinates of the point P, and we write $P = (r,\theta)$, or $P(r,\theta)$. In particular, r is called the radial distance and θ is called the polar angle. Also, O is called the pole, the x-axis is called the inital line and OP is called the radius vector.



Let (x, y) be the Cartesian coordinates of the point P. Then we find that

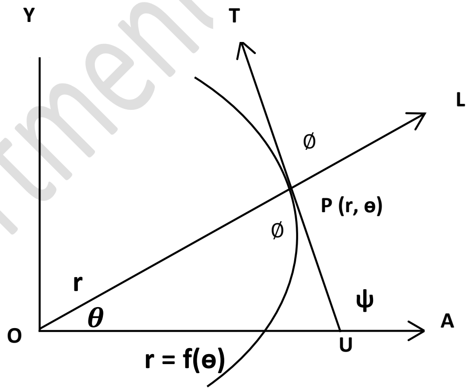
$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \ldots(1)$$
$$x = r\cos\theta,\ y = r\sin\theta. \qquad \ldots(2)$$

Relations (1) enables us to find the polar coordinates (r, θ) when the Cartesian coordinates (x, y) are known. Conversely, relations (2) enable us to find the Cartesian coordinates when the polar coordinates are known. Thus, relations (1) define the transformation from the Cartesian coordinates to polar coordinates and relations (2) defines the inverse transformation.

Angle between the radius vector and the tangent:

With usual notation we can prove that $\tan\varphi = r\frac{d\theta}{dr}$

Let "φ" be the angle between the radius vector OPL and the tangent TPU at the point 'P' on the polar curve $r = f(\theta)$.



Thus, the angle between the radius vector and tangent is given by the expression: $\tan\varphi = \boldsymbol{r\frac{d\theta}{dr}}$
Application

- Angle φ helps analyze the geometry of a path or wave, essential for describing curved signal paths or the course of current in circular circuits.
- Signal Trajectories: In phased array radar or waveguiding, knowing this angle aids in predicting signal direction changes.

Example:

- In analyzing light propagation in optical fibers, the changing direction of the light ray at any point can be interpreted using the angle between the radius and the tangent.

Note:

(i) $\cot\varphi = \frac{1}{r}\frac{dr}{d\theta}$.

(ii) If φ_1 and φ_2 are the angles between the radius vector and the tangents at the point of intersection of two curves $r = f_1(\theta)$ and $r = f_2(\theta)$, then the angle of intersection of the curves is given by $|\varphi_1 - \varphi_2|$.

(iii) Suppose we are not able to obtain φ_1 and φ_2 explicitly then

$$\tan(\varphi_1 - \varphi_2) = \frac{\tan\varphi_1 - \tan\varphi_2}{1+\tan\varphi_1\tan\varphi_2}.$$

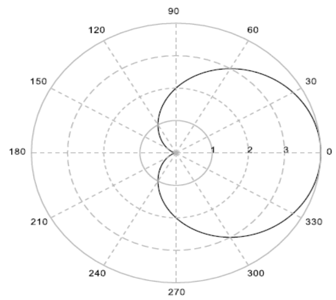
(iv) If $\tan\varphi_1\tan\varphi_2 = -1$, then $\tan(\varphi_1 - \varphi_2) = \infty \Rightarrow \varphi_1 - \varphi_2 = \frac{\pi}{2}$ (condition for the orthogonality of two polar curves).

Examples:

1. Find the angle between the radius vector and the tangent to the following polar curves:

i) $r = a(1+\cos\theta)$

$r = a(1+\cos(\theta))$



[Cardioid $r = a(1+\cos\theta)$ is a curve that is the locus of a point on the circumference of circle rolling round the circumference of a circle of equal radius. Of course the name means 'heart-shaped'. Curve is symmetrical about the initial line.]

Solution:

Consider $r = a(1+\cos\theta)$, differentiating with respect to θ,

$$\frac{dr}{d\theta} = -a\sin\theta$$

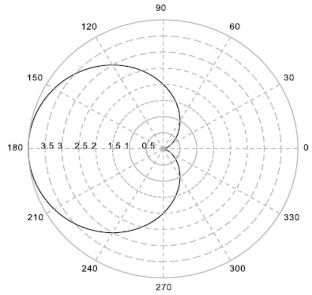
$$\Rightarrow r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta}$$

$$\Rightarrow \tan\phi = -\frac{2\cos^2\left(\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

or

$$\tan\phi = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right) \Rightarrow \phi = \frac{\pi}{2}+\frac{\theta}{2}.$$

ii) Cardioid $r = b(1-\cos\theta)$ in other orientation



Solution:
Consider $r = b(1-cos\theta)$. Differentiating with respect to θ,

$$\frac{dr}{d\theta} = b\sin\theta$$

$$\Rightarrow r\frac{d\theta}{dr} = \frac{b(1-\cos\theta)}{b\sin\theta}$$

$$\Rightarrow \tan\phi = \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

or

$$\tan\phi = \tan\frac{\theta}{2} \Rightarrow \phi = \frac{\theta}{2}.$$

iii) Circle: $r = \sin\theta + \cos\theta$ [This is centred at $\left(\frac{1}{2},\frac{1}{2}\right)$ with a radius of $\sqrt{\frac{1}{2}}$.

Solution: Differentiating $r = \sin\theta + \cos\theta$ with respect to θ,

$$\frac{dr}{d\theta} = \cos\theta - \sin\theta$$

$$r\frac{d\theta}{dr} = \frac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta}$$

By dividing numerator and denominator by $\cos\theta$,

$$\tan\phi = \frac{\tan\theta+1}{1-\tan\theta} = \tan\left(\frac{\pi}{4}+\theta\right) \quad \text{or} \quad \phi = \frac{\pi}{4}+\theta.$$

iv) $r = 16\sec^2\frac{\theta}{2}$

Solution:
Consider $r = 16\sec^2\frac{\theta}{2}$.

Differentiating with respect to θ

$$\frac{dr}{d\theta} = 32\sec^2\frac{\theta}{2}\tan\frac{\theta}{2}.\left(\frac{1}{2}\right) = 16\sec^2\frac{\theta}{2}\tan\frac{\theta}{2}$$

$$r\frac{d\theta}{dr} = \frac{16\sec^2\frac{\theta}{2}}{16\sec^2\frac{\theta}{2}\tan\frac{\theta}{2}}$$

$$\tan\varphi = \cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \Rightarrow \varphi = \frac{\pi}{2} - \frac{\theta}{2}.$$

2. An antenna's far-field radiation pattern in polar form is given by $r = 2 + \cos(\theta)$. Calculate the angle between the radius vector and the tangent to the radiation pattern at $\theta = 0$. Explain the engineering significance for antenna design.

Solution: Consider $r = 2 + \cos(\theta)$.

Differentiating with respect to θ, $\frac{dr}{d\theta} = -\sin\theta$. At $\theta = 0, \frac{dr}{d\theta} = 0.$

$\Rightarrow \tan\phi = \frac{2+1}{0}$ or $\Rightarrow \phi = \frac{\pi}{2}.$

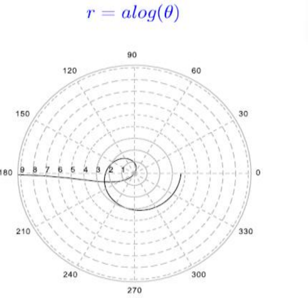
Interpretation:

At $\theta = 0$, the radius vector is perpendicular to the tangent. This means at this angle, the field pattern changes direction most sharply- important for targeting or main lobe adjustment in antenna design.

3. Find the angle between two curves for the following:

i) $r = a\log\theta$ and $r = \frac{a}{\log\theta}$

[$r = a\log\theta$ some kind of 'logarithmic spiral'. The graph comes from negative x-infinity, goes through the origin at θ = 1, and then spirals outwards. It looks like it is heading to a definite limit of the radius but this is an illusion



Solution: Consider $r = a\log\theta$. Differentiating, we get

$$\frac{dr}{d\theta} = \frac{a}{\theta}, \quad r\frac{d\theta}{dr} = \frac{\theta}{a}\,a\log\theta$$

$$\tan\phi_1 = \theta\log\theta$$

Consider $r = \frac{a}{\log\theta}$

$$\log r = \log a - \log(\log\theta) \text{ and } \frac{1}{r}\frac{dr}{d\theta} = -\frac{1}{\theta\log\theta}$$

$$\cot\phi_2 = -\frac{1}{\theta\log\theta}$$

$$\tan\phi_2 = -\theta\log\theta$$

Now consider,

$$\tan(\phi_1 - \phi_2) = \frac{2\theta \log\theta}{1-(\theta\log\theta)^2}.$$

We have to eliminate θ between the given curves,

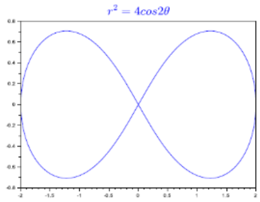
$$r = a\log\theta \quad \text{and} \quad r = \frac{a}{\log\theta}$$

$$(\log\theta)^2 = 1, \quad \log\theta = \pm 1, \quad \theta = e \; or \; \frac{1}{e}$$

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{2e}{1-e^2}.$$

ii) $r = 2(1+\cos\theta)$ and $r^2 = 4\cos 2\theta$

[$r^2 = 4\cos 2\theta$ is Lemniscate. Curve is symmetrical about both the axis]



$$\log r = \log 2 + \log(1 + cos\theta)$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta}$$

$$\cot\phi_1 = \frac{-2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\cos^2\frac{\theta}{2}^2}$$

$$\cot\phi_1 = -\tan\left(\frac{\theta}{2}\right) = \cot\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$$\phi_1 = \frac{\pi}{2} + \frac{\theta}{2}.$$

Consider $r^2 = 4cos2\theta$

$$\frac{dr}{d\theta} = -\frac{4\sin 2\theta}{r}$$

$$\tan\phi_2 = -\tan\left(\frac{\pi}{2} + 2\theta\right)$$

$$\varphi_2 = \frac{\pi}{2} + 2\theta.$$

We have to eliminate θ between the given curves,

$$r = 2(1+\cos\theta) \text{ and } r^2 = 4\cos 2\theta$$

Solving for θ, we get

$$\theta = 1 - \sqrt{3}$$

$$\varphi_1 - \varphi_2 = \frac{3(1-\sqrt{3})}{2} \ .$$

iii) $r = \frac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$

Solution: Consider $r = \frac{a\theta}{1+\theta}$. Taking log both sides

$$\log r = \log(a\theta) - \log(1+\theta)$$

Differentiating w.r.t θ

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{a}{a\theta} - \frac{1}{1+\theta} = \frac{1}{\theta} - \frac{1}{1+\theta} = \frac{1}{\theta(1+\theta)}$$

$$r\frac{d\theta}{dr} = \theta(1+\theta)$$

$$\therefore \tan\varphi_1 = \theta(1+\theta)$$

Consider $r = \frac{a}{1+\theta^2}$

$$\log r = \log a - \log(1+\theta^2)$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-2\theta}{1+\theta^2}$$

$$\cot\varphi = \frac{-2\theta}{1+\theta^2},\ \tan\varphi = \frac{1+\theta^2}{-2\theta}.$$

We have to eliminate θ between the given curves,

$$r = \frac{a\theta}{1+\theta} \quad \text{and} \quad r = \frac{a}{1+\theta^2}$$

$$\theta^3 = 1, \quad \theta = 1$$

$$\Longrightarrow \tan(\phi_1 - \phi_2) = \frac{2-(-1)}{1+(-2)}.$$

$$\Longrightarrow |\phi_1 - \phi_2| = |\tan^{-1} -3\,| = \tan^{-1} 3$$

iv) $r = \cos\theta + \sin\theta\,,\ r = 2\sin\theta.$

Solution: First we find the point of intersection by eliminating r from both the equations,

$$2\sin\theta = \cos\theta + \sin\theta \Rightarrow \cos\theta = \sin\theta \Rightarrow \theta = \frac{\pi}{4}$$

Differentiating $r = \sin\theta + \cos\theta$, we get

$$\frac{dr}{d\theta} = \cos\theta - \sin\theta \Rightarrow \tan\phi_1 = r\frac{d\theta}{dr} = \frac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}$$

Now, differentiating $r = 2\sin\theta$,

$$\frac{dr}{d\theta} = 2\cos\theta \Rightarrow \tan\phi_2 = r\frac{d\theta}{dr} = \tan\theta$$

At $\theta = \frac{\pi}{4}$, $\phi_1 = \frac{\pi}{2}$ and $\phi_2 = \frac{\pi}{4}$.

Thus, angle of intersection of given two curves is $|\phi_1 - \phi_2| = \frac{\pi}{4}$.

v) Prove that the following pairs of polar curves intersect orthogonally,

$$re^{\theta} = a \quad \text{and} \quad r = be^{\theta}$$

Solution: Differentiating $re^{\theta} = a$,

$$re^{\theta} + \frac{dr}{d\theta}e^{\theta} = 0 \Rightarrow \tan\phi_1 = r\frac{d\theta}{dr} = -1.$$

Now, differentiating $r = be^{\theta}$, we get

$$\frac{dr}{d\theta} = be^{\theta} \Rightarrow \tan\phi_2 = r\frac{d\theta}{dr} = 1.$$

Clearly, from above equations,

$$\tan\phi_1 \tan\phi_2 = -1.$$

Thus, the given two curves intersect orthogonally.

Exercise:

1. Find the angle between radius vector and tangent to the following polar curves:
   i. $r = a\sin^3\left(\frac{\theta}{3}\right)$ **Ans**: $\frac{\theta}{2}$
   ii. $r^m = a^m(\cos m\theta + \sin m\theta)$ **Ans:** $\frac{\pi}{4} + m\theta$
   iii. $r^n = a^n \sin n\theta$ **Ans:** $n\theta$
   iv. $r = 2a\cos^2\left(\frac{\theta}{2}\right)$ **Ans:** $\frac{\pi}{2} + \frac{\theta}{2}$
   v. $\frac{2a}{r} = 1 - \cos\theta$ **Ans:** $\pi - \frac{\theta}{2}$
   vi. $\frac{l}{r} - 1 + e\cos\theta$ Ans: $\tan^{-1}\left[\frac{1+e\cos\theta}{e\sin\theta}\right]$

2. Find the slope of the following curves:
   i. $r^2\cos 2\theta = a^2$ at $\theta = \frac{\pi}{12}$ **Ans:** $2 + \sqrt{3}$
   ii. $r = a\sin 2\theta$ at $\theta = \frac{\pi}{4}$ **Ans:** -1
   iii. $r = a(1 + \sin\theta)$ at $\theta = \frac{\pi}{2}$ **Ans:** 0
   iv. $r\cos^2\theta/2 = a^2$ at $\theta = \frac{2\pi}{3}$ **Ans**: $\frac{\pi}{6}$

3. Prove that the following pairs of polar curves intersect orthogonally
   i. $r = a\sec^2\frac{\theta}{2}$ and $r = b\csc^2\frac{\theta}{2}$
   ii. $r^n = a^n\cos n\theta$ and $r^n = b^n \sin n\theta$
   iii. $r = a(1 + \cos\theta)$ and $r = b(1 - \cos\theta)$
   iv. $r = a\cos\theta$ and $r = a\sin\theta$
   v. $r = ae^{\theta}$ and $re^{\theta} = b$

4. Find the angle of intersection for each of the following pairs of curves:
   i. $r = a\cos\theta$ and $2r = a$ **Ans:** $\frac{\pi}{3}$

ii. $r = a\,(1-\cos\theta)$ and $r = 2a\cos\theta$ **Ans:** $\frac{\pi}{2} + \frac{\cos^{-1}\left(\frac{1}{3}\right)}{2}$

iii. $r = 2(1+\cos\theta)$ and $r = 6\cos\theta$ **Ans:** $\frac{\pi}{2}$

iv. $r = \sin\theta + \cos\theta$ and $r = 2\sin\theta$ **Ans**: $\frac{\pi}{4}$

v. $r^2 \sin 2\theta = 4$ and $r^2 = 16 \sin 2\theta$ **Ans**: $\frac{\pi}{3}$

Curvature and Radius of Curvature:

Curvature is the amount by which a geometric object deviates from being *flat*, or *straight* in the case of a line, but this is defined in different ways depending on the context. In geometry, the **radius of curvature**, *R*, of a curve at a point is a measure of the radius of the circular arc which best approximates the curve at that point. It is the reciprocal of the curvature. The distance from the centre of a circle or sphere to its surface is its radius. For other curved lines or surfaces, the **radius of curvature** at a given point is the radius of a circle that mathematically best fits the curve at that point. In the case of a surface, the radius of curvature is the radius of a circle that best fits a *normal section*.

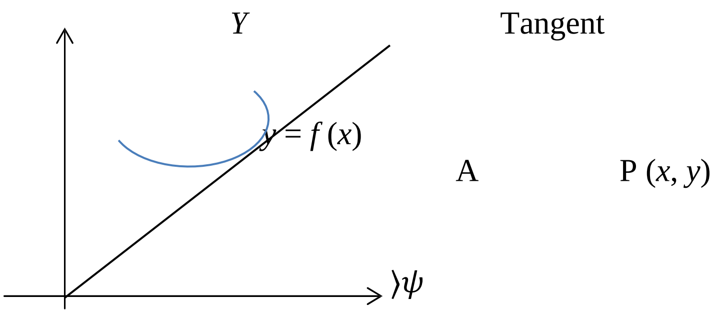
Imagine driving a car on a curvy road on a completely flat plain (so that the geographic plain is a geometric plane). At any one point along the way, lock the steering wheel in its position, so that the car thereafter follows a perfect circle. The car will, of course, deviate from the road, unless the road is also a perfect circle. The circle that the car makes is the circle of curvature; radius and the centre of the circle are radius of curvature and centre of curvature of the curvy road at the point at which the steering wheel was locked. The more sharply curved the road is at the point you locked the steering wheel, the smaller the radius of curvature.

Some of the Applications:

- Radius of curvature is applied to measurements of the stress in the semiconductor structures.
- When engineers design trains track, they need to ensure the curvature of the track to be safe and provide a comfortable ride for the given speed of the trains.
- Curvature (κ) quantifies how sharply a curve bends. In communication circuits, curvature helps in the design of microstrip lines or coils for inductors and transformers, affecting electromagnetic behaviour.
- The radius of curvature (ρ) is the reciprocal of curvature, representing the "tightness" of the bend.

Example: In PCB (Printed Circuit Board) layout, acute bends can cause signal reflections or interference. Curvature analysis helps avoid design flaws in high-speed communication traces.

Let P be a point on the curve $y = f(x)$ at the length 's' from a fixed-point A on it. Let the tangent at 'P' makes are angle ψ with positive direction of x – axis. As the point 'P' moves along curve, both s and ψ vary.



The rate of change ψ w.r.t s, $\frac{d\psi}{ds}$ is called the curvature of the curve at 'P'.
The reciprocal of the Curvature at P is called the radius of curvature at P and is denoted by ρ.

$$\rho = \frac{ds}{d\psi}.$$

Radius of curvature for Cartesian curve $y = f(x)$:

If the curve is given in Cartesian coordinates as $y(x)$, then the radius of curvature is:

$$\rho = \left|\frac{(1+y'^2)^{\frac{3}{2}}}{y''}\right|, \quad \text{where} \quad y' = \frac{dy}{dx},\ y'' = \frac{d^2y}{dx^2}$$

Radius of curvature for parametric equations $x = f(t)$, $y = \phi(t)$:

If the curve is given parametrically by functions $x(t)$ and $y(t)$, then the radius of curvature is

$$\rho = \frac{ds}{d\psi} = \left|\frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}\right|,$$

where $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$ and $\dot{y} = \frac{dy}{dt}$, $\ddot{y} = \frac{d^2y}{dt^2}$.

Radius of curvature for Polar curve $r = f(\theta)$:

The radius of curvature of a polar curve $\boldsymbol{r = f(\theta)}$ is given by:

$$\rho = \frac{(r^2 + {r_1}^2)^{\frac{3}{2}}}{r^2 + 2{r_1}^2 - rr_2}$$

Examples:

1. Find the curvature at any point on the curve $y = x^3$.

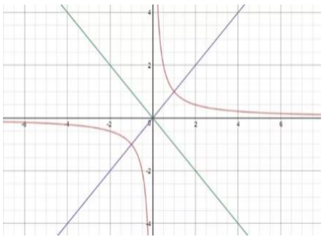
Solution: Curvature

$$\kappa = \frac{y_2}{(1+y_1^2)^{\frac{3}{2}}}.$$

here $y = x^3 \Rightarrow y_1 = 3x^2$ and $y_2 = 6x$

$$\therefore \quad \kappa = \frac{6x}{(1+9x^4)^{\frac{3}{2}}}$$

2. Find the curvature at any point on the rectangular hyperbola
$xy = c^2$



Solution: Curvature $\kappa = \frac{y_2}{(1+y_1^2)^{\frac{3}{2}}}$. Given $xy = c^2 \Rightarrow \quad y = \frac{c^2}{x}$

$$\therefore \; y_1 = -\frac{c^2}{x^2} \quad \text{and} \quad y_2 = \frac{2c^2}{x^3}$$

Thus,

$$\kappa = \frac{\frac{2c^2}{x^3}}{\left\{1+\left(-\frac{c^2}{x^2}\right)^2\right\}^{\frac{3}{2}}} = \frac{2c^2 \times x^6}{x^3\{x^4+c^4\}^{\frac{3}{2}}} = \frac{2c^2 x^3}{(x^4+x^2y^2)^{\frac{3}{2}}} \qquad (\because \;\; xy = c^2) \quad = \frac{2c^2}{(x^2+y^2)^{\frac{3}{2}}}.$$

3. Find the radius of curvature at the origin on $y = x(x-a)^2$.

Solution: Radius of curvature :

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y^2}$$

We need to find $\rho(0,0)$.

$$\text{Given : } \quad y = x(x-a)^2 = x(x^2 - 2ax + a^2)$$
$$= x^3 - 2ax^2 + a^2 x$$

Thus,

$$y_1 = 3x^2 - 4ax + a^2 \; ; \quad y_2 = 6x - 4a$$

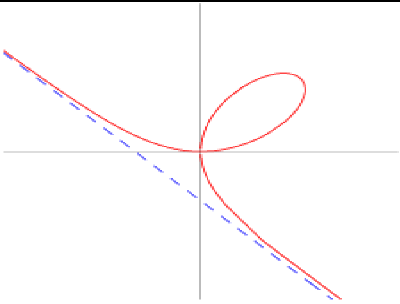
At $(0,0)$, $y_1(0,0) = a^2$ and $y_2(0,0) = -4a$

Therefore, $\rho(0,0) = \left|\frac{(1+a^4)^{\frac{3}{2}}}{-4a}\right| = \frac{(1+a^4)^{\frac{3}{2}}}{4a}$.

4. Find the radius of curvature at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on $x^3 + y^3 = 3axy$.

[Name of the curve is Folium Descartes; it is symmetrical about the line y = x.

It is not symmetric about any other line; also it is symmetric about the origin.]



Solution:

It is required to find $\rho\left(\frac{3a}{2},\frac{3a}{2}\right)$ on $x^3+y^3=3axy$

Differentiating with respect to x, we get

$$3x^2+3y^2y_1=3a(xy_1+y)$$

$$\Rightarrow 3(y^2-ax)y_1=3(ay-x^2)$$

$$\Rightarrow\ y_1=\frac{ay-x^2}{y^2-ax}$$

Differentiating again with respect to x, we get

$$y_2=\frac{(y^2-ax)(ay_1-2x)-(ay-x^2)(2yy_1-a)}{(y^2-ax)^2}$$

At $\left(\frac{3a}{2},\frac{3a}{2}\right)$.

$$y_{1\left(\frac{3a}{2},\frac{3a}{2}\right)}=\left(-\frac{(x^2-ay)}{(y^2-ax)}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)}=-1(\text{observe that } x=y \text{ at the point})$$

and

$$y_{2\left(\frac{3a}{2},\frac{3a}{2}\right)}=\frac{\left(\frac{9a^2}{4}-\frac{3a^2}{2}\right)(-a-3a)-\left(\frac{3a^2}{2}-\frac{9a^2}{4}\right)(-3a-a)}{(y^2-ax)^2}$$

$$=\frac{-\frac{3}{4}a^2\times 4a-\frac{3a^2}{4}\times 4a}{\left(\frac{3a^2}{4}\right)^2}=-\frac{6a^3}{\left(\frac{9a^4}{16}\right)}=-\frac{32}{3a}$$

Hence,

$$\rho_{\left(\frac{3a}{2},\frac{3a}{2}\right)}=\frac{\{1+(-1)^2\}^{\frac{3}{2}}}{\left(-\frac{32}{3a}\right)}=-\frac{2\sqrt{2}\times 3a}{32}=-\frac{3a}{8\sqrt{2}}$$

$\therefore$ Radius of curvature at $\left(\frac{3a}{2},\frac{3a}{2}\right)$ is $\frac{3a}{8\sqrt{2}}$

5. Find the radius of curvature at $b^2x^2+a^2y^2=a^2b^2$ at its point of intersection with $y-$ axis

Solution: We have

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y^2}$$

At the point of intersection with $y-$ axis, $x=0$. Thus the equation $b^2x^2 + a^2y^2 = a^2b^2$, reduces to $a^2y^2 = a^2b^2 \Rightarrow y = \pm b$

Therefore, the points are $(0, b)$ and $(0, -b)$.

Now consider $b^2x^2 + a^2y^2 = a^2b^2$

Differentiating with respect to x,

$2b^2x + 2a^2yy_1 = 0$

$\Rightarrow y_1 = -\frac{b^2x}{a^2y}$

Differentiating again with respect to x,

$$y_2 = -\frac{b^2}{a^2}\left(\frac{y - xy_1}{y^2}\right)$$

$$\text{At } (0,b),\ y_1 = -\frac{b^2}{a^2} \times \frac{0}{b} = 0 \text{ and } y_2 = -\frac{b^2}{a^2}\left(\frac{b-0}{b^2}\right) = -\frac{b}{a^2}$$

$$\therefore\ \rho_{(0,b)} = \left|\frac{(1+0)^{\frac{3}{2}}}{\left(-\frac{b}{a^2}\right)}\right| = \frac{a^2}{b}.$$

$$\text{At } (0,-b),\ y_1 = -\frac{b^2}{a^2} \times \frac{0}{-b} = 0 \text{ and } y_2 = -\frac{b^2}{a^2}\left(\frac{-b-0}{b^2}\right) = \frac{b}{a^2}$$

$$\therefore\ \rho_{(0,-b)} = \frac{(1+0)^{\frac{3}{2}}}{\left(\frac{b}{a^2}\right)} = \frac{a^2}{b}.$$

i.e., Radius of curvature at $(0, b)$ and $(0, -b)$ is $\frac{a^2}{b}$.

6. Show that the radius of curvature at any point (x, y) on $x^{2/3} + y^{2/3} = a^{2/3}$ is $3(axy)^{1/3}$.

Solution: We have $\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$.

Given: $x^{2/3} + y^{2/3} = a^{2/3}$

Differentiating with respect to x,

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}} = 0$$

$$\Rightarrow y_1 = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

Differentiating again,

$$y_2 = \frac{-\frac{1}{3}\left\{-x^{\frac{1}{3}}y^{-\frac{2}{3}}\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} - y^{\frac{1}{3}}x^{-\frac{2}{3}}\right\}}{x^{\frac{2}{3}}} = \frac{1}{3}\frac{\left\{\frac{1}{y^{\frac{1}{3}}} + \frac{y^{\frac{1}{3}}}{x^{\frac{2}{3}}}\right\}}{x^{\frac{2}{3}}} = \frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{3x^{\frac{4}{3}}y^{\frac{1}{3}}} = \frac{a^{\frac{2}{3}}}{3x^{\frac{4}{3}}y^{\frac{1}{3}}}$$

$$\text{Now,}\quad 1 + y_1^2 = 1 + \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}} = \frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{x^{\frac{2}{3}}} = \frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}$$

$$\therefore \quad \rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y^2} = -\frac{\left(\frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right)^{\frac{3}{2}}}{\left(\frac{a^{\frac{2}{3}}}{3x^{\frac{4}{3}}y^{\frac{1}{3}}}\right)} = \frac{a \times 3x^{\frac{4}{3}}y^{\frac{1}{3}}}{x \times a^{\frac{2}{3}}} = 3(axy)^{\frac{1}{3}}.$$

7. Show that for ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\rho = \frac{a^2b^2}{p^3}$ where p is the length of the perpendicular from the centre upon the tangent at (x, y) to the ellipse.

Solution: The ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Differentiating,

$\frac{2x}{a^2} + \frac{2yy_1}{b^2} = 0 \Rightarrow y_1 = -\frac{b^2}{a^2}\frac{x}{y}$

Differentiating again,

$$y_2 = -\frac{b^2}{a^2}\left\{\frac{y \times 1 - x \times y_1}{y^2}\right\} = -\frac{b^2}{a^2}\left\{\frac{y + x\frac{b^2}{a^2}\frac{x}{y}}{y^2}\right\}$$

$$= -\frac{b^2}{a^2y^3}\left\{\frac{y^2}{b^2} + \frac{x^2}{a^2}\right\} = -\frac{b^2}{a^2y^3}$$

Now,

$$\rho = \frac{\left(1 + \frac{b^4x^2}{a^4y^2}\right)^{\frac{3}{2}}}{\left(-\frac{b^4}{a^2y^3}\right)} = \left|-\frac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^6y^3} \times \frac{a^2y^3}{b^4}\right| = \frac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^4b^4}$$

The tangent at (x_0, y_0) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$.

Length of perpendicular from $(0,0)$ upon this tangent = $\frac{1}{\sqrt{\left(\frac{x_0}{a^2}\right)^2 + \left(\frac{y_0}{b^2}\right)^2}} = \frac{a^2b^2}{\sqrt{a^4y_0^2 + b^4x_0^2}}$.

So, the length of perpendicular from the origin upon the tangent at (x, y) is

$$p = \frac{a^2b^2}{\sqrt{a^4y^2 + b^4x^2}} \quad \text{(by replacing } x_0 \text{ by } x \text{ \& } y_0 \text{ by } y\text{)}$$

$$\Rightarrow \quad \frac{1}{p^3} = \frac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^6b^6} = \frac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^4b^4} \times \frac{1}{a^2b^2} = \frac{\rho}{a^2b^2}$$

$$\Rightarrow \quad \rho = \frac{a^2b^2}{p^3}.$$

This completes the proof.

8. Find the radius of curvature at the point θ on the curve

$$x = a \log \sec\theta \text{ and } y = a(\tan\theta - \theta)$$

Solution:

$$x = a\log\sec\theta \Rightarrow \frac{dx}{d\theta} = a \times \frac{1}{\sec\theta} \times \sec\theta\tan\theta = \mathrm{a}\tan\theta$$

$$y = a(\tan\theta - \theta) \Rightarrow \frac{dy}{d\theta} = a(\sec^2\theta - 1) = a\tan^2\theta$$

$$\therefore \quad y_1 = \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\tan^2\theta}{\mathrm{a}\tan\theta} = \tan\theta$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}(\tan\theta) = \frac{d}{d\theta}(\tan\theta) \times \frac{d\theta}{dx} = \sec^2\theta \times \frac{1}{a\tan\theta} = \frac{\sec^2\theta}{a\tan\theta}.$$

$$\text{Now,} \quad \rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{(1+\tan^2\theta)^{\frac{3}{2}}}{\left(\frac{\sec^2\theta}{a\tan\theta}\right)} = \frac{\sec^3\theta}{\sec^2\theta} \times a\tan\theta = a\sec\theta\tan\theta.$$

9. Show that the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$ is $4a\cos\left(\frac{\theta}{2}\right)$.

Solution: Differentiating the given equation,

$$\frac{dx}{d\theta} = a(1 + \cos\theta), \frac{dy}{d\theta} = a\sin\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \cdot \frac{d\theta}{dx} = \frac{1}{2}\sec^2\frac{\theta}{2} \cdot \frac{1}{a(1+\cos\theta)} = \frac{\frac{1}{2}\sec^2\frac{\theta}{2}}{2a\cos^2\frac{\theta}{2}} = \frac{1}{4a}\sec^4\frac{\theta}{2}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{4a\left(1 + \tan^2\frac{\theta}{2}\right)^{3/2}}{\sec^4\frac{\theta}{2}} = 4a\cos\frac{\theta}{2}.$$

10. Find radius of curvature at a point for the curves

$$x = 6t^2 - 3t^4, \quad y = 8t^3.$$

Solution: Differentiating, we get

$$\dot{x} = \frac{dx}{dt} = 12t - 12t^3 = 12t(1-t^2); \quad \dot{y} = 24t^2$$

$$\ddot{x} = \frac{d^2x}{dt^2} = 12 - 36t^2 = 12(1-3t^2); \quad \ddot{y} = 48t$$

$$\rho = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}} = \frac{\{[12t^2(1-t^2)]^2 + (24t^2)^2\}^{3/2}}{12t(1-t^2)48t - 24t^2(12-36t^2)}$$

$$= \frac{\{144t^4(1-2t^2+t^4) + 24^2t^4\}^{3/2}}{48 \times 12t^2 - 12 \times 48t^3 - 24 \times 12t^2 - 24 \times 36t^4}$$

$$= \frac{\{(1+t^2)^2\}^{3/2} \times 6t}{1+t^2} = \frac{\{(1+t^2)^3\}^3 \times 6t}{1+t^2} = 6t(1+t^2)^2.$$

11. Show that the radius of curvature at any point of the cardiod $r = a(1 - cos\,\theta)$ varies as $\sqrt{r}$

Solution: Differentiating w.r.t θ we get:

$$r_1 = a\sin\theta\,, r_2 = a\cos\theta$$

$$(r^2 + {r_1}^2)^{\frac{3}{2}} = [a^2(1-\cos\theta)^2 + a^2\sin^2\theta]^{\frac{3}{2}}$$

$$r^2 + 2{r_1}^2 - rr_2 = a^2(1-\cos\theta)^2 - a^2(1-\cos\theta)\cos\theta + 2a^2\sin^2\theta = 3a^2(1-\cos\theta)$$

Now, $\rho = \frac{(r^2+{r_1}^2)^{\frac{3}{2}}}{r^2+2{r_1}^2-rr_2} = \frac{a^3 2\sqrt{2}(1-\cos\theta)^{\frac{3}{2}}}{3a^2(1-\cos\theta)} = \frac{2a\sqrt{2}}{3}\left(\frac{r}{a}\right)^{\frac{1}{2}} \propto \sqrt{\boldsymbol{r}}$

12. Find the radius of curvature at any point for the curve

$$r^n = a^n \cos n\theta$$

Solution: Taking log on both sides

$$n\log r = n\log a + \log\cos n\theta$$

Differentiating w.r.t $\theta,$ we get

$$\frac{n}{r}\frac{dr}{d\theta} = -\frac{\sin n\theta}{\cos n\theta}.n$$

$$r_1 = \frac{dr}{d\theta} = -r\tan n\theta$$

Differentiating again w.r.t $\theta,$ we get

$$r_2 = \frac{d^2r}{d\theta^2} = r\tan^2 n\theta - nr\sec^2 n\theta$$

Now,

$$\rho = \frac{(r^2+{r_1}^2)^{\frac{3}{2}}}{r^2+2{r_1}^2-rr_2} = \frac{(r^2+r^2tan^2n\theta)^{\frac{3}{2}}}{r^2+2r^2tan^2n\theta - r^2tan^2n\theta + nr^2sec^2n\theta}$$

$$= \frac{(r^3 sec^3 n\theta)^{\frac{3}{2}}}{(n+1)r^2 sec^2 n\theta} = \frac{r}{(n+1)\cos n\theta} = \frac{r.a^n}{(n+1)r^n} = \frac{a^n r^{1-n}}{(n+1)} .$$

13. If ρ_1 and ρ_2 be the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax,$ then show that $\rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}.$

Solution: The parametric equation of the given parabola is $x = at^2$ and $y = 2at$

Differentiating w.r.t $t,$

$$\frac{dx}{dt} = 2at,\ \frac{dy}{dt} = 2a,\ \frac{d^2x}{dt^2} = 2a \text{ and } \frac{d^2y}{dx^2} = 0.$$

Radius of curvature at any point $(at^2, 2at)$ is

$$\rho = \left|\frac{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}\right| = \left|\frac{(4a^2t^2 + 4a^2)^{\frac{3}{2}}}{-4a^2}\right| = 2a(1 + t^2)^{3/2}.$$

If $P(t_1)$ and $Q(t_2)$ are the extremities of the focal chord of the parabola, then

$$t_1 t_2 = -1 \quad \text{or} \quad t_2 = -1/t_1$$

We find ρ at these points:

ρ_1 at $P(t_1)$ is $2a(1 + t_1^2)^{3/2}$ and ρ_2 at $Q(t_2)$ is $2a(1 + t_2^2)^{3/2}$

Thus, $\quad \rho_1^{-\frac{2}{3}} + \rho_2^{-\frac{2}{3}} = (2a)^{-\frac{2}{3}}((1 + t_1^2)^{-1} + (1 + t_2^2)^{-1})$

$$= (2a)^{-\frac{2}{3}}\left(\frac{1}{1 + t_1^2} + \frac{t_1^2}{1 + t_1^2}\right) = (2a)^{-\frac{2}{3}}.$$

This completes the proof.

14. For the curve $y = \frac{ax}{a+x}$, prove that $\left(\frac{2\rho}{a}\right)^{2/3} = \frac{y^2}{x^2} + \frac{x^2}{y^2}$

where ρ is the radius of curvature of the curve at its point (x, y).

Solution: Differentiating $y = \frac{ax}{a+x}$,

$$\frac{dy}{dx} = \frac{(a + x)a - ax}{(a + x)^2} = \frac{a^2}{(a + x)^2}\ ;\ \frac{d^2y}{dx^2} = \frac{(-2a^2)}{(a + x)^3}$$

Now,

$$\rho = \frac{(1 + y_1^2)^{\frac{3}{2}}}{y_2} = \left(1 + \frac{a^4}{(a + x)^4}\right)^{\frac{3}{2}} \times \frac{(a + x)^3}{2a^2}$$

Therefore,

$$\rho^{2/3} = \left(1 + \frac{a^4}{(a + x)^4}\right) \times \frac{(a + x)^2}{2^{2/3}a^{4/3}}$$

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left(1 + \frac{a^4}{(a+x)^4}\right) \times \frac{(a+x)^2}{2^{2/3}a^{4/3}} \times \frac{2^{2/3}}{a^{2/3}}$$

$$= \frac{1}{a^2}\left(1 + \frac{a^4}{(a+x)^4}\right)(a+x)^2$$

$$= \frac{(a+x)^2}{a^2} + \frac{a^2}{(a+x)^2} = \frac{x^2}{y^2} + \frac{y^2}{x^2}.$$

15. Show that the radius of curvature at the point (r, θ) of the curve $r^2 \cos 2\theta = a^2$ is $\frac{r^3}{a^2}$.

Solution: Given $r^2 = a^2 \sec 2\theta$. Differentiating, we get

$$2rr_1 = 2a^2 \sec 2\theta \tan 2\theta \Rightarrow r_1 = r \tan 2\theta$$

Again, differentiating, we get

$$r_2 = 2r \sec^2 2\theta + r_1 \tan 2\theta = 2r \sec^2 2\theta + r \tan^2 2\theta$$

Now, $\rho = \left|\frac{(r^2+r_1{}^2)^{\frac{3}{2}}}{r^2+2r_1{}^2-rr_2}\right|$

$$\Rightarrow \rho = \left|\frac{(r^2 + r^2 \tan^2 2\theta)^{\frac{3}{2}}}{r^2 + 2r^2 \tan^2 2\theta - 2r^2 \sec^2 2\theta - r^2 \tan^2 2\theta}\right| = \left|\frac{r^3 \sec^3 2\theta}{r^2 - 2r^2 - r^2 \tan^2 2\theta}\right|$$

$$= \left|\frac{r^3 \sec^3 2\theta}{-r^2 \sec^2 2\theta}\right| = r \sec 2\theta = r\frac{r^2}{a^2} = \frac{r^3}{a^2}.$$

16. A segment of the railway track curve is modelled by $y = 5x^2 - x + 14$. Calculate the radius of curvature at $x = 2$ to check if the curve is safe for high-speed trains.

Solution: We have $\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$.

Given: $y = 5x^2 - x + 14$

Differentiating with respect to x,

$$y_1 = \frac{dy}{dx} = 10x - 1. \text{ At } x = 2, y_1 = 10(2) - 1 = 19$$

Differentiating again,

$$y_2 = \frac{d^2y}{dx^2} = 10$$

$$\therefore \quad \rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y^2} = \frac{(1+19^2)^{\frac{3}{2}}}{10} = 688.12.$$

Interpretation: For railway tracks, a larger radius implies a gentler curve, allowing trains to maintain higher speeds safely. Such calculations are used to determine speed limits and banking angles for curved sections to prevent derailments and ensure passenger comfort.

Exercise:

1. For the curve $y = c\log\left(\sec\left(\frac{x}{c}\right)\right)$ find the radius of curvature at a point (x, y).
2. Find radius of curvature at a point for the curves
$x = a\left(\cos t + \log\left(\tan\left(\frac{t}{2}\right)\right)\right)$ and $y = a\sin t$.
3. For the curve $y = 4\sin x - \sin 2x$ find the radius of curvature at a point $x = \frac{\pi}{2}$.
4. Find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $\left(\frac{a}{4}, \frac{a}{4}\right)$.
5. Find the radius of curvature at the point $(a, 0)$ on $xy^2 = a^3 - x^2$.
6. Find the radius of curvature at any point on $y = \text{acosh}\left(\frac{x}{a}\right)$.
7. Find the radius of curvature of $y^2 = \frac{a^2(a-x)}{x}$ at $(a, 0)$.
8. Find the radius of curvature of the curve $r^2 = a^2\sin 2\theta$ at any point (r, θ)
9. For the curve $\frac{2a}{r} = 1 + \cos\theta$ find the radius of curvature at a point (r, θ).
10. Find the radius of curvature at any point on $r(1 + \cos\theta) = a$.
11. Show that at the point of intersection of the curves $r = a\theta$ and $r\theta = a$, the curvature are in the ration $3:1$ $(0 < \theta < 2\pi)$.

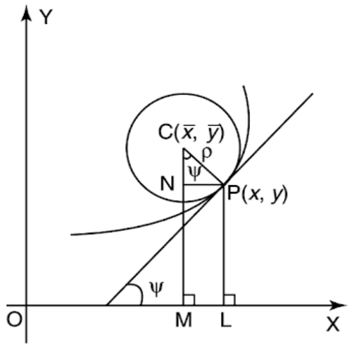
Answers:

1. $c\sec\left(\frac{x}{c}\right)$,
2. $a\cot t$
3. $5\sqrt{5}/4$
4. $a/\sqrt{2}$
5. $3a/2$
6. y^2/a
7. $a/2$
8. $a^2/3r$
9. $2\sqrt{\frac{r^3}{a}}$
10. $\frac{(2r)^3}{\sqrt{a}}$

Centre of curvature:

In geometry, the **center of curvature** of a curve is found at a point that is at a distance from the curve equal to the radius of curvature lying on the normal vector. It is the point at infinity if the curvature is zero. The osculating circle to the curve is centered at the center of curvature. Cauchy defined the center of curvature C as the intersection point of two infinitely close normal lines to the curve. Centre of curvature at any point $P(x, y)$ on the curve

$y = f(x)$ is given by: $\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$, $\bar{y} = y + \frac{(1+y_1^2)}{y_2}$



Let $C(\bar{x}, \bar{y})$ be the centre of curvature and ρ be the radius of curvature at $P(x, y)$. Draw PL and CM perpendicular to OX and PN perpendicular to CM. Let the tangent at P makes an angle ψ with the $x -$ axis. Then $\angle NCP = 90 - \angle NPC = \psi$.

$$\bar{x} = OM = OL - ML = OL - Np = \rho \sin\psi = x - \frac{y_1(1 + y_1^2)}{y_2}$$

Similarly,

$$\bar{y} = MC = MN + NC = LP + \rho cos\psi = y + \frac{(1 + y_1^2)}{y_2}$$

The center of curvature tells where the "best fitting circle" that approximates the curve locally is centered. This helps in designing curved components (like micro strip antenna elements) where precise curvature affects signal propagation. For example, knowing the center of curvature helps optimize reflection angles on curved surfaces such as reflectors in antennas or waveguides.

Note: Equation of the circle of curvature at $P(x, y)$ is

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$$

Examples:

1. Find the coordinates of the center of curvature at any point of the parabola $y^2 = 4ax$

Solution: Given $y^2 = 4ax$,

Differentiating, $y_1 = \frac{2a}{y}$ and $y_2 = -\frac{4a^2}{y^3}$

Center of curvature:

$$\bar{x} = x - \frac{y_1(1 + y_1^2)}{y_2}, \qquad \bar{y} = y + \frac{(1 + y_1^2)}{y_2}$$

That is,

$$\bar{x} = 3x + 2a, \quad \bar{y} = -\frac{2x^{\frac{3}{2}}}{\sqrt{a}}.$$

2. Find circle of curvature at $(1,0)$ on $y = x^3 - x^2$.

Solution: At (x_0, y_0) the equation of the circle of curvature is

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$$

Given: $y = x^3 - x^2$.

Differentiating,

$$\frac{dy}{dx} = 3x^2 - 2x; \quad y_{1(1,0)} = 1$$

$$\frac{d^2y}{dx^2} = 6x - 2; \quad y_{2(1,0)} = 4$$

$$(1 + {y_1}^2)_{(1,0)} = 2, \ \bar{x}_{(1,0)} = \frac{1}{2}; \quad \bar{y}_{(1,0)} = \frac{1}{2}; \quad \rho_{(1,0)} = \frac{1}{\sqrt{2}}$$

Circle of curvature is

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2} \Rightarrow x^2 + y^2 - x - y = 0.$$

3. Determine the radius, centre, and circle of curvature at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$ of the curve, folium of Descartes, defined by the equation $x^3 + y^3 = 3xy$.

Solution: Given $x^3 + y^3 = 3xy$
Differentiating w r t x,

$$y_1 = \frac{y - x^2}{y^2 - x}$$

Differentiating again w r t x

$$y_2 = \frac{(y^2 - x)(y_1 - 2x) - (y - x^2)(2yy_1 - 1)}{(y^2 - x)^2}$$

At $\left(\frac{3}{2}, \frac{3}{2}\right)$, $y_1 = -1$, $y_2 = -\frac{32}{3}$ and

radius of curvature

$$\rho = \left|\frac{(1 + y_1^2)^{\frac{3}{2}}}{y_2}\right| = \frac{3}{8\sqrt{2}}$$

Centre of curvature is $(\overline{x}, \overline{y})$, where $\overline{x} = x - \frac{y_1(1+y_1^2)}{y_2} = \frac{21}{16}$ and

$$\overline{y} = y + \frac{1 + y_1^2}{y_2} = \frac{21}{16}$$

Circle of curvature is $\left(x - \frac{21}{16}\right)^2 + \left(y - \frac{21}{16}\right)^2 = \frac{9}{128}$.

4. Find the radius, centre, and circle of curvature of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$ at the point, where $\theta = \pi/4$.

Solution: Given $x = a\cos^3\theta$, $y = a\sin^3\theta$
Differentiating w r t θ,

$$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta \quad \text{and} \quad \frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta.$$

$$\frac{dy}{dx} = -\tan\theta \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{1}{3a\cos^4\theta\sin\theta}$$

At $\theta = \frac{\pi}{4}$, $y_1 = -1$, $y_2 = -\frac{4\sqrt{2}}{3a}$ and radius of curvature $\rho = \left|\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}\right| = \frac{3a}{2}$

Centre of curvature is $(\bar{x}, \bar{y})$ where

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = \sqrt{2}a \quad \text{and} \quad \bar{y} = y + \frac{1+y_1^2}{y_2} = \sqrt{2}a$$

Circle of curvature is $\left(x - \sqrt{2}a\right)^2 + \left(y - \sqrt{2}a\right)^2 = \frac{9a^2}{4}$

5. A micro strip antenna trace on a PCB is designed as a curve $y = x^3$. Find the coordinates of the center of curvature at the point, where $x = 1$. How does knowing the centre of curvature help in optimizing the antenna's signal integrity?

Solution:

$$y_1 = \frac{dy}{dx} = 3x^2 = 3(1)^2 = 3$$

$$y_2 = \frac{d^2y}{dx^2} = 6x = 6(1) = 6$$

Centre of curvature is $(\bar{x}, \bar{y})$ where

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = 1 - \frac{3(1+9)}{6} = -4 \text{ and}$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = 1 + \frac{1+9}{6} = 2.6667$$

Coordinates of centre of curvature: $(-4, 2.6667)$.

Interpretation: For micro strip antennas, sharp bends cause impedance discontinuities leading to signal loss/reflection. Knowing the centre of curvature helps designers calculate how "tight" the bend is and allows for adjustment to minimize signal degradation. Thus, the centre of curvature aids in tailoring PCB trace layouts for optimal electromagnetic behaviour, improving antenna efficiency and communication quality.

Exercise:

1. Find circle of curvature of $x + y = ax^2 + by^2 + cx^3$ at the origin.
[Answer: $(a+b)(x^2+y^2) = 2(x+y)$.]

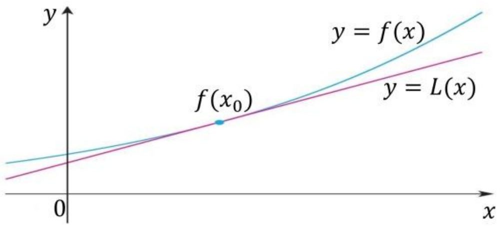
2. Find circle of curvature of $x^3 + y^3 = 3xy$ at $\left(\frac{3}{2}, \frac{3}{2}\right)$.
[Answer: $x^2 + y^2 - \frac{21}{8}(x+y) + \frac{432}{128} = 0$.]

3. Find the centre of curvature for the curve $xy = a^2$ at (a, a).
[Answer: $(2a, 2a)$]

4. Find the circle of curvature for the curve $\frac{x^2}{4} + \frac{y^2}{9} = 2$.
[Answer: $\left(x + \frac{5}{4}\right)^2 + \left(y - \frac{5}{6}\right)^2 = \frac{13^2}{12^2}$]

Taylor and Maclaurin Series

Recall that, for a differentiable function $f(x)$, at a point $x = x_0$, the tangent line is a close approximation of the function in a neighbourhood of the point x_0.
We call the equation of the tangent the linearization of the function



A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

is known as power series, where x is a variable and c_n's are constants called the coefficients of the series.

The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$$

whose domain is the set of all x for which the series converges.

More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

is known as **power series in $(x-a)$,** or **a power series cantered at a or a power series about a.**

Example 1: Geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

which converges when $-1 < x < 1$ and diverges when $|x| \geq 1$.

Suppose that f is any function that can be represented by a power series

$$f(x) = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \cdots \qquad (1)$$

Let's try to determine the coefficients c_n in terms of f. Note that if we put $x = a$ in equation 1, we get $f(a) = c_0$.

Differentiate the series in equation (1) term by term:

$$f'(x) = c_1 + 2c_2 (x-a) + 3c_3 (x-a)^2 + 4c_4 (x-a)^3 + \cdots$$

Substituting $x = a$ gives $f'(a) = c_1$.

If we continue this procedure, we get $f^{(n)}(a) = n!\, c_n$, solving for c_n gives

$$c_n = \frac{f^{(n)}(a)}{n!}, \ n = 0,1,2, \ldots$$

Theorem: If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \ |x-a| < R$$

Then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Substituting this formula for c_n back into the series, we see that if f has a power series expansion at a, then it must be of the following form.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \qquad (2)$$

The series in Equation (2) is called the Taylor series of the function f at a (or about a or centered at a). For the special case $a = 0$ the Taylor series is known as Maclaurin series, given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \qquad (3)$$

Taylor Polynomial:

$$T_1(x) = f(a) + \frac{f'(a)}{1!}(x-a) \ ;$$

this is called 1st degree Taylor Polynomial.

$$T_2(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 \ ;$$

this is called 1st degree Taylor Polynomial.

$\vdots$

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Notice that $T_n(x)$ is a polynomial of degree n called the nth-degree Taylor polynomial of f at a.

In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n\to\infty} T_n(x).$$

If we let $R_n(x) = f(x) - T_n(x)$ so that $f(x) = T_n(x) + R_n(x)$
then $R_n(x)$ is called the remainder of order n of the Taylor series.

Lagrange's form of remainder term, states that there is a number c between x and a such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x. This version is an extension of the Mean Value Theorem (which is the case $n = 0$).

Advantages of using Taylor series and Maclaurin series

- Taylor series are studied because polynomial functions are easy and if one could find a way to represent complicated functions as series (infinite polynomials) then one can easily study the properties of difficult functions.

- Evaluating definite Integrals: Some functions have no anti derivative which can be expressed in terms of familiar functions. This makes evaluating definite integrals for some functions difficult because the Fundamental Theorem of Calculus cannot be used. If we have a polynomial representation of a function, we can oftentimes use that to evaluate a definite integral.
- Understanding asymptotic behaviour: Sometimes, a Taylor series can tell us useful information about how a function behaves in an important part of its domain.
- Understanding the growth of functions
- Solving differential equations
- Used for approximating nonlinear functions in signal processing, systems analysis, and filter design in electronics.

Example: When analysing nonlinear components (like a diode's I-V characteristic), Maclaurin or Taylor expansions allow engineers to model the component for small voltages around an operating point to facilitate linear circuit analysis.

Examples:

1. Use the Taylor series expansion of e^x about $x = 0$ (Maclaurin series) to approximate $e^{0.3}$ up to the 4th term. Explain how this approximation is useful in electrical engineering.

Solution: We have

$$f(x) = e^x \implies f^{(n)}(x) = e^x \implies f^{(n)}(0) = 1.$$

Substituting these values in the definition of Maclaurin series, we get

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\text{For } x = 0.3,\ e^{0.3} = 1 + 0.3 + \frac{(0.3)^2}{2} + \frac{(0.3)^3}{6} + \frac{(0.3)^4}{24} + \cdots \approx 1.3498$$

Interpretation :
Taylor series approximations like this are widely used in circuit design and signal processing where exponential terms arise (e.g., charging of capacitors, attenuation in transmission lines).

- Instead of computationally intensive exponential functions, engineers use polynomial approximations for faster calculations in embedded circuits or real-time systems.
- It simplifies complex nonlinear relationships into manageable linear/quadratic components, enabling efficient simulations and control system design.

Thus, Taylor series provide engineers a practical tool for quick, accurate approximations essential in system modeling and analysis.

In particular,

$$e = \sum \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

1) Find the Maclaurin series of $f(x) = \sin x$.

Solution: Consider

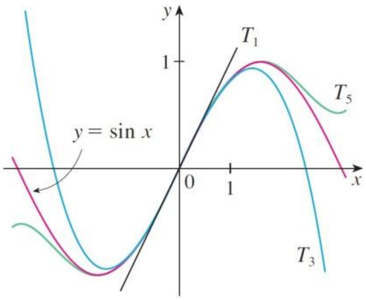
$$f(x) = \sin x\,, \qquad f(0) = 0$$
$$f'(x) = \cos x\,, \qquad f'(0) = 1$$
$$f''(x) = -\sin x\,, \qquad f''(0) = 0$$
$$f'''(x) = -\cos x\,, \quad f'''(0) = -1$$
$$f^{(4)}(x) = \sin x\,, \quad f^{(4)}(0) = 0$$

Then the Maclaurin series is given by,

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$
$$= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

First few Taylor polynomials at $x = 0$ of $f(x) = \sin x$ are given by

$$T_1(x) = x; \quad T_3(x) = x - \frac{x^3}{3!}\,, T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}\,; \quad T_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$



Note that as n increases $T_n(x)$ becomes better approximation to $\sin x$, which is evident from the figure.

2) Obtain the Taylor's series of $f(x) = \log x$ about the point $x = 1$ upto the terms containing fourth degree and hence evaluate $\log(1.1)$

Solution: we have the Taylor series expansion about $x = a$ given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots.$$

Given that $f(x) = \log x$ and $a = 1$. Thus, $f(a) = \log 1 = 0$.
Differentiating $f(x)$ successively we get

$$f'(x) = \frac{1}{x}, \qquad \Rightarrow f'(1) = 1$$
$$f''(x) = -\frac{1}{x^2} \qquad \Rightarrow f''(1) = -1$$
$$f'''(x) = \frac{2}{x^3} \qquad \Rightarrow f'''(1) = 2$$

$$f^{(iv)}(x) = -\frac{6}{x^4} \quad \Rightarrow f^{(iv)}(1) = -6$$

The Taylor series up to fourth degree term is given by

$$\log x \approx (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}.$$

We shall substitute $x = 1.1$ in the above expansion to obtain $\log(1.1)$

$$\log(1.1) \approx (0.1) - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} = 0.0953.$$

3) Determine the Maclaurin series for $f(x) = (1+x)^k$, where k is any real number.

Solution: We have

$$\begin{aligned} f(x) &= (1+x)^k, & f(0) &= 1 \\ f'(x) &= k(1+x)^{k-1}, & f'(0) &= k \\ f''(x) &= k(k-1)(1+x)^{k-2}, & f''(0) &= k(k-1) \\ &\vdots \end{aligned}$$

$$f^{(n)}(x) = k(k-1)\dots(k-n+1)(1+x)^{k-n},$$

$$f^{(n)}(0) = k(k-1)\cdots(k-n+1)$$

Therefore, the Maclaurin series for $f(x) = (1+x)^k$ is given by

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n.$$

This series is called the binomial series.

Note: When $\boldsymbol{k}$ **is a nonnegative integer**, the binomial series is **finite.** This is because all the derivatives of f after $(k+1)^{\text{th}}$ derivative are zero. Thus

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots + x^k.$$

We use the following notation when k is a nonnegative integer to denote binomial co-efficient.

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

Hence,

$$(1+x)^k = \sum_{r}^{k} \binom{k}{n} x^r, \quad k \in \mathbb{Z}^+.$$

If $\boldsymbol{k}$ **is not a positive integer or zero**, the series is **infinite** and converges for $|x| < 1$.

4) Find the first four terms of the binomial series for the function $f(x) = \sqrt{1+x}$.

Solution: With $k = 1/2$, the binomial series gives

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \cdots$$
$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots.$$

Hence $\sqrt{1+x}$ can be approximated by taking first few terms of the series.

5) Obtain the binomial series for the functions $\frac{1}{1+x}$ and $\frac{1}{1-x}$.

Solution: Here $k = -1$, and the binomial coefficients are given by

$$\frac{-1(-2)(-3)\cdots(-1-k+1)}{k!} = (-1)^k \frac{k!}{k!} = (-1)^k$$

Therefore

$$\frac{1}{1+x} = (1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^k x^k = 1 - x + x^2 - x^3 + \cdots$$

If we replace x by $-x$ in the above equation, we get

$$\frac{1}{1-x} = (1-x)^{-1} = \sum_{n=0}^{\infty} (-1)^k (-x)^k = 1 + x + x^2 + x^3 + \cdots$$

6) Approximate the integral $I = \int_0^1 e^{-x^2} dx$.

Solution: Recall the Maclaurin series of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Replacing x by $-x^2$ in the above formula, we get

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

The integral I take the form

$$\int_0^1 e^{-x^2} dx = \int_0^1 \left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right] dx$$
$$= \left[x - \frac{x^3}{3} + \frac{x^5}{5 \times 2!} - \frac{x^7}{7 \times 3!} + \cdots\right]_0^1$$
$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \cdots \approx 0.7475$$

Euler identity: A complex number is a number of the form $a + ib$, where a and b are real numbers and $i = \sqrt{-1}$. If we substitute $x = i\theta$ (θ real) in the Taylor series for e^x and use the relations

$$i^2 = -1, \; i^3 = i^2 i = -i, \; i^4 = i^2 i^2 = 1, \; i^5 = i^4 i = i,$$

and so on, we obtain

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} e^{i\theta}$$
$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!}\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!}\right) e^{i\theta} = \cos\theta + i\sin\,\theta$$

Hence we define $e^{i\theta} = \cos\theta + i\sin\,\theta$, for any real number θ.

7) Obtain a Taylor's expansion for $f(x) = \sin x$ in the ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to the fourth-degree term.

Solution: The Taylor's expansion for $f(x)$ about $\frac{\pi}{4}$ is

$$f(x) = f\left(\frac{\pi}{4}\right) + \left(x - \frac{\pi}{4}\right) f'\left(\frac{\pi}{4}\right) + \frac{\left(x - \frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right) + \frac{\left(x - \frac{\pi}{4}\right)^3}{3!} f'''\left(\frac{\pi}{4}\right)$$
$$+ \frac{\left(x - \frac{\pi}{4}\right)^4}{4!} f^{(iv)}\left(\frac{\pi}{4}\right) + \cdots \text{ ------------------------ } (1)$$

$$f(x) = \sin x \quad \Rightarrow f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\ ;$$

$$f'(x) = \cos x \quad \Rightarrow f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f''(x) = -\sin x \ \Rightarrow f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$f'''(x) = -\cos x \Rightarrow f'''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$f^{(iv)}(x) = \sin x \ \Rightarrow f^{(iv)}\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

Substituting these in (1) we obtain the required Taylor's series in the form

$$f(x) = \frac{1}{\sqrt{2}} + \left(x - \frac{\pi}{4}\right)\left(\frac{1}{\sqrt{2}}\right) + \frac{\left(x - \frac{\pi}{4}\right)^2}{2}\left(-\frac{1}{\sqrt{2}}\right) + \frac{\left(x - \frac{\pi}{4}\right)^3}{6}\left(-\frac{1}{\sqrt{2}}\right)$$
$$+ \frac{\left(x - \frac{\pi}{4}\right)^4}{24}\left(\frac{1}{\sqrt{2}}\right)\cdots.$$

Using Taylor's Theorem, expand $\log(\sin x)$ in ascending powers of $(x-3)$.

Solution: $f(x) = \log(\sin x)$, $a = 3$ and $f(3) = \log(\sin 3)$

$$f'(x) = \cot(x), \qquad f'(3) = \cot(3)$$
$$f''(x) = -\csc^2 x, \qquad f''(3) = -\csc^2 3$$
$$f'''(x) = 2\csc^2 x \cot x, \quad f'''(3) = 2\csc^2 3 \cot 3$$

Taylor series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \frac{(x-a)^3}{6}f'''(x) \ldots$$

$$\log(\sin x) = \log(\sin(3)) + (x-3)\cot(3) - \frac{(x-3)^2}{2}\operatorname{cosec}^2(3)$$
$$+\frac{(x-3)^3}{3}\operatorname{cosec}^2(3)\cot(3) + \cdots$$

8) Obtain Taylor's expansion of the function $\cos\left(\frac{\pi}{4}+h\right)$ in ascending powers of h up to the terms containing h^4.

Solution: Taylors's expansion of $f(x+h)$ is given by

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{iv}(x) + \ldots$$

At $x = \frac{\pi}{4}$

$$f(x) = \cos x \quad \Rightarrow \quad f(\pi/4) = \cos\pi/4 = \tfrac{1}{\sqrt{2}}$$
$$f'(x) = -\sin x \Rightarrow \quad f'(\pi/4) = -\sin\pi/4 = -\tfrac{1}{\sqrt{2}}$$
$$f''(x) = -\cos x \Rightarrow \quad f''(\pi/4) = -\cos\pi/4 = -\tfrac{1}{\sqrt{2}}$$
$$f'''(x) = \sin x \quad \Rightarrow \quad f'''(\pi/4) = \sin\pi/4 = \tfrac{1}{\sqrt{2}}$$

Then we have

$$\cos\left(\frac{\pi}{4}+h\right) = \frac{1}{\sqrt{2}} - \frac{h}{1!}\frac{1}{\sqrt{2}} - \frac{h^2}{2!}\frac{1}{\sqrt{2}} + \frac{h^3}{3!}\frac{1}{\sqrt{2}} + \frac{h^4}{4!}\frac{1}{\sqrt{2}} + \cdots$$
$$= \frac{1}{\sqrt{2}}\left[1 - h - \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!} + \ldots\right].$$

9) Obtain expansion of $f(x) = \frac{x}{\sin x}$ up to the term containing x^4

Solution: Maclaurin series expansion of $\sin x$ is given by

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\therefore f(x) = \frac{x}{\sin x} = x\left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right\}^{-1} = xx^{-1}\left\{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right\}^{-1}$$
$$= \left\{1 - \left(\frac{x^2}{3!} - \frac{x^4}{5!} - \ldots\right)\right\}^{-1}$$

By Binomial expansion

$$\frac{x}{\sin x} = 1 + \left(\frac{x^2}{3!} - \frac{x^4}{5!} - \ldots\right) + \left(\frac{x^2}{3!} - \frac{x^4}{5!} - \ldots\right)^2 + \cdots$$

$$= 1 + \frac{x^2}{3!} - \frac{x^4}{5!} + \left(\frac{x^2}{3!}\right)^2 + \cdots$$

$$= 1 + \frac{x^2}{6} - \frac{x^4}{120} + \frac{x^4}{36} + \cdots$$

$$= 1 + \frac{x^2}{6} - \frac{7x^4}{360} + \cdots$$

Exercise:

1) Expand a^x in ascending powers of x.
2) Expand $\log(\sec x)$ in ascending powers of x up to and including the term in x^6 and hence deduce the expansion of $\tan x$.
3) Show that $\sin^{-1}(3x - 4x^3) = 3\left[x + \frac{x^3}{6} + 3\frac{x^5}{40} + \ldots\right]$

   Hint: put $x = \sin\theta$
4) Obtain Maclaurin series for $\cos x$ and $\log(1 + x)$.

Answers:

1) $1 + x\log a + \frac{x^2}{2}(\log a)^2 + \frac{x^3}{6}(\log a)^3 + \cdots$
2) $\frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \cdots$
4) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$

   $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

Video Links:

1. **Polar Co-ordinates: https://youtu.be/aSdaT62ndYE**
2. **Curvature & Radius of curvature:**

   https://youtu.be/VGcJv8tLPTU
3. **Taylor Series: https://www.youtube.com/watch?v=3d6DsjIBzJ4**

   https://www.youtube.com/watch?v=epgwuzzDHsQ

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