Module - 4

Integral Calculus

Integration of a function of two and three variables referred as double integrals: $\iint f(x,y)\,dx\,dy$

triple integrals: $\iiint f(x,y,z)\,dx\,dy\,dz$

eg:-

1) $\iint (x+y)\,dx\,dy = \int\left(\frac{x^2}{2}+y\cdot x\right)dy = \frac{x^2}{y}\cdot y + \frac{y^2}{2}\cdot x = \frac{xy}{2}(x+y)$

1st

2nd

2) Evaluate $\int_0^1\int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx$

Soln:-

Let $I = \int_{x=0}^{1}\int_{y=x}^{\sqrt{x}} (x^2+y^2)\,dy\,dx$

$$= \int_0^1 \left[yx^2+\frac{y^3}{3}\right]_{y=x}^{\sqrt{x}} dx$$

$$= \int_0^1 \left[x^2\sqrt{x}+\frac{(\sqrt{x})^3}{3} - x^3 - \frac{x^3}{3}\right]dx$$

$$= \int_0^1 \left(x^{5/2}+\frac{x^{3/2}}{3}-\frac{4x^3}{3}\right)dx$$

$$= \left[\frac{x^{5/2+1}}{5/2+1}+\frac{1}{3}\frac{x^{3/2+1}}{3/2+1}-\frac{4}{3}\frac{x^4}{4}\right]_0^1$$

$$= \left[\frac{x^{7/2}}{7/2}+\frac{1}{3}\frac{x^{5/2}}{5/2}-\frac{4}{3}\cdot\frac{x^4}{4}\right]_0^1$$

$$= \frac{2}{7}+\frac{2}{15}-\frac{1}{3} = \frac{9}{105} = 3/35$$

2) Evaluate: $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y \, dx\, dy$.

Soln:-

we have $I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} x^3 y \, dx\, dy$

$$I = \int_{y=0}^{1} y \left(\frac{x^4}{4}\right)_{x=0}^{\sqrt{1-y^2}} dy.$$

$$= \frac{1}{4}\int_{y=0}^{1} \left(y(\sqrt{1-y^2})^4 - 0\right) dy = \frac{1}{4}\int_{y=0}^{1} y(1-y^2)^2 dy$$

$$= \frac{1}{4}\int_{y=0}^{1} y(1-2y^2+y^4)\, dy$$

$$= \frac{1}{4}\int_{y=0}^{1} (y-2y^3+y^5)\, dy.$$

$$= \frac{1}{4}\left[\frac{y^2}{2} - \frac{\cancel{2}y^4}{\cancel{4}_2} + \frac{y^6}{6}\right]_0^1$$

$$= \frac{1}{4}\left[\cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \frac{1}{6}\right] = 1/24.$$

3) Solve (i) $\int_1^2 \int_3^4 (xy + e^y)\, dy\, dx$ Ans: $\frac{21}{4} + e^4 - e^3$

(ii) $\int_1^2 \int_3^4 (xy + e^y)\, dx\, dy = \int_1^2 \left[\int_3^4 (xy + e^y)\, dx\right] dy$

$$= \int_1^2 \left[\frac{yx^2}{2} + xe^y\right]_3^4 dy$$

$$= \int_1^2 \left[\frac{16y}{2} + 4e^y - \frac{9y}{2} - 3e^y\right] dy$$

$$= \int_1^2 \left(\frac{7y}{2} + e^y\right) dy$$

$$= \left[\frac{7y^2}{4} + e^y\right]_1^2 = \frac{7(\cancel{4})}{\cancel{4}} + e^2 - \frac{7}{4} - e$$

$$= 7 - \frac{7}{4} - (e^2 + e) = \frac{21}{4} - (e^2 + e)$$

Solve $\displaystyle\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2} = \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{1}{(1+x^2)+y^2}\,dy\right] dx$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1}1 - \tan^{-1}0\right] dx$$

$$= \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx = \frac{\pi}{4} \left[\log\left(x + \sqrt{1+x^2}\right)\right]_0^1$$

$$\Rightarrow \frac{\pi}{4} \left[\log\left(1+\sqrt{2}\right) - \log 1\right]$$

$$= \frac{\pi}{4} \log\left(\sqrt{2}+1\right)$$

V.Vimportant

1)
$$\int_0^{\pi/2}\int_0^{a(1+\cos\theta)} r^2\cos\theta \, dr\, d\theta.$$

$$= \int_0^{\pi/2}\left\{\int_0^{a(1+\cos\theta)} r^2\cos\theta \, dr\right\} d\theta = \int_0^{\pi/2}\left[\frac{r^3}{3}\cos\theta\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi/2}\left[\frac{a^3(1+\cos\theta)^3}{3}\cdot\cos\theta - 0\right] \quad (\because (a+b)^3 = a^3+b^3+3ab(a+b))$$

$$= \frac{a^3}{3}\int_0^{\pi/2}\left[1+\cos^3\theta+3\cos\theta(1+\cos\theta)\right]\cos\theta \cdot d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/2}\left(\cos\theta+\cos^4\theta+3\cos^2\theta+3\cos^3\theta\right) d\theta.$$

$$\left[\because \int_0^{\pi/2}\cos^n\theta\, d\theta = \begin{cases}\frac{n-1}{n}\times\frac{n-3}{n-2}\times\frac{n-5}{n-4}\cdots\frac{\pi}{2} & \longrightarrow \text{for } n \text{ is even.}\\ \frac{n-1}{n}\times\frac{n-3}{n-2}\times\frac{n-5}{n-4}\cdots 1 & \longrightarrow \text{for } n \text{ is odd.}\end{cases}\right.$$

$$= \frac{a^3}{3}\left\{\left[\sin\theta\right]_0^{\pi/2} + \left[\frac{4-1}{4}\times\frac{4-3}{4-2}\times\frac{\pi}{2}\right] + 3\left[\frac{2-1}{2}\times\frac{\pi}{2}\right] + \cancel{3}\left[\frac{3-1}{\cancel{3}}\cdot 1\right]\right\}$$

$$= \frac{a^3}{3}\left\{\left[\sin\frac{\pi}{2}-\sin 0\right] + \left[\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}\right] + 3\left[\frac{1}{2}\times\frac{\pi}{2}\right] + [2]\right\}$$

$$= \frac{a^3}{3}\left\{(1-0) + \left(\frac{3}{8}\times\frac{\pi}{2}\right) + \frac{3\pi}{4} + 2\right\}$$

$$= \frac{a^3}{3}\left[\frac{3\pi}{16}+\frac{3\pi}{4}+3\right] = \frac{a^3}{3}\left[\frac{3\pi+12\pi}{16}+3\right]$$

$$= \frac{a^3}{3}\left[\frac{15\pi}{16}+3\right]$$

$$\int_0^1 e^{y/x}\,dy\,dx = \int_0^1 \left[x e^{y/x}\right]_0^x dx$$

$$= \int_0^1 x(e-1)\,dx = \left[\frac{x^2}{2}(e-1)\right]_0^1 = \frac{1}{2}(e-1)$$

$$\int_0^3 \int_0^1 (x^2+3y^2)\,dy\,dx = \int_0^3 \left[x^2 y + y^3\right]_0^1 dx$$

$$= \int_0^3 (x^2+1)\,dx = \left[\frac{x^3}{3}+x\right]_0^3 = 12$$

$$\int_0^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\,d\theta\,dr.$$

$$= \frac{1}{2}\int_0^{\pi/4} \left[\int_0^{\sqrt{\cos 2\theta}} (1+r^2)^{-2}\cdot 2r\,dr\right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \left[-(1-r^2)^{-1}\right]_0^{\sqrt{\cos\theta}} d\theta \qquad \left[\because \int [f(r)]^n f'(r)\,dr = \frac{[f(r)]^{n+1}}{n+1} \quad (n \neq -1)\right]$$

$$= -\frac{1}{2}\int_0^{\pi/4} \left(\frac{1}{1+\cos 2\theta} - 1\right) d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/4} \left(\frac{1}{2\cos^2\theta} - 1\right) d\theta = -\frac{1}{2}\int_0^{\pi/4} \left(\frac{1}{2}\sec^2\theta - 1\right) d\theta$$

$$= \frac{-1}{2}\left[\frac{1}{2}\tan\theta - \theta\right]_0^{\pi/4} = -\frac{1}{2}\left(\frac{1}{2}\tan\frac{\pi}{4} - \frac{\pi}{4} - 0\right)$$

$$= \frac{1}{8}(\pi - 2)$$

$$\int_0 \int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\, dr\, d\theta$$

Now $\displaystyle\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\, dr$

take $1+r^2 = t$

$2r\,dr = dt \Rightarrow r\,dr = \frac{dt}{2}$

limits

$r = 0 \Rightarrow 1+0 = t \Rightarrow \boxed{t=1}$

$r = \sqrt{\cos 2\theta} \Rightarrow 1+(\sqrt{\cos 2\theta})^2 = 1+\cos 2\theta = t$

$\Rightarrow \boxed{2\cos^2\theta = t}$

$$\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\, dr = \int_1^{2\cos^2\theta} \frac{dt/2}{t^2}$$

$$= \frac{1}{2}\int_1^{2\cos^2\theta} \frac{1}{t^2}\, dt$$

$$= \frac{1}{2}\left[\frac{-1}{t}\right]_1^{2\cos^2\theta} = \frac{1}{2}\left[1 - \frac{1}{2\cos^2\theta}\right]$$

$$\therefore \int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\, dr\, d\theta = \int_0^{\pi/4}\left\{\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\, dr\right\} d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2}\left(1 - \frac{1}{2\cos^2\theta}\right) d\theta.$$

$$= \frac{1}{2}\left[\int_0^{\pi/4} d\theta - \frac{1}{2}\int_0^{\pi/4} \sec^2\theta\, d\theta\right] \quad \left(\because \frac{1}{\cos^2\theta} = \sec^2\theta\right)$$

$$= \frac{1}{2}\left[\theta - \frac{1}{2}\tan\theta\right]_0^{\pi/4} = \frac{1}{2}\left[\pi/4 - \frac{1}{2}\tan\pi/4 - 0\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right] = \pi/8 - 1/4.$$

$$\int_0^x \int_0^x \int_0^{x+y} e^{x+y+z} \, dz \, dy \, dx$$

<u>Sol:-</u>

$$I = \int_{x=0}^{a} \int_{y=0}^{x} \int_{z=0}^{x+y} e^{x+y} \cdot e^{z} \, dz \, dy \, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} e^{x+y} \left[e^{z} \right]_0^{x+y} dy \, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} e^{x+y} \left[e^{x+y} - 1 \right] dy \, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} e^{x+y} \left[e^{x+y} - 1 \right] dy \, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} \left(e^{2x} \cdot e^{2y} - e^{x} \cdot e^{y} \right) dy \, dx$$

$$= \int_{x=0}^{a} \left\{ e^{2x} \left[\frac{e^{2y}}{2} \right]_0^{x} - e^{x} \left[e^{y} \right]_0^{x} \right\} dx$$

$$= \int_{x=0}^{a} \left\{ \frac{e^{2x}}{2} \left(e^{2x} - 1 \right) - e^{x} \left(e^{x} - 1 \right) \right\} dx$$

$$= \int_{x=0}^{a} \left\{ \frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^{x} \right\} dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^{x} \right]_0^{a}$$

$$= \left(\frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} \right) - \left(\frac{1}{8} - \frac{3}{4} + 1 \right)$$

$$= \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} - \frac{3}{8} \Rightarrow \underline{I} = \frac{1}{8} \left(e^{4a} - 6e^{2a} + 8e^{a} - 3 \right)$$

(2). Solve $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a} (x^2+y^2+z^2)\, dz\, dy\, dx$

Sol:-

$$I = \int_{x=-c}^{c}\int_{y=-b}^{b}\int_{z=-a}^{a} (x^2+y^2+z^2)\, dz\, dy\, dx$$

$$= \int_{x=-c}^{c}\int_{y=-b}^{b} \left[x^2 z + y^2 z + \frac{z^3}{3}\right]_{z=-a}^{a} dy\, dx$$

$$= \int_{x=-c}^{c}\int_{y=-b}^{b} \left[x^2(a+a) + y^2(a+a) + \left(\frac{a^3}{3}+\frac{a^3}{3}\right)\right] dy\, dx$$

$$= \int_{x=-c}^{c} \left[2ax^2y + 2a\left(y^3/3\right) + \left(2a^3/3\right)(y)\right]_{y=-b}^{b} dx$$

$$= \int_{x=-c}^{c} \left[2ax^2(b+b) + 2a\left(\frac{b^3}{3}+\frac{b^3}{3}\right) + \left(\frac{2a^3}{3}\right)(b+b)\right] dx$$

$$= \left[(4ab)\left(x^3/3\right) + \left(\frac{4ab^3}{3}\right)(x) + \frac{4a^3b}{3}(x)\right]_{-c}^{c}$$

$$= (4ab)\left(\frac{c^3}{3}+\frac{c^3}{3}\right) + \left(4ab^3/3\right)(c+c) + \left(\frac{4a^3b}{3}\right)(c+c)$$

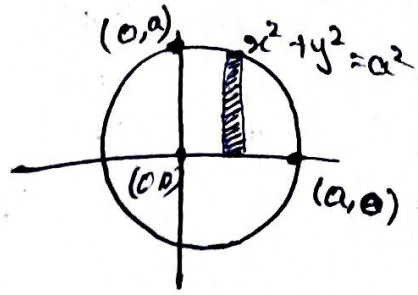
$$= \frac{8abc^3}{3} + \frac{8ab^3c}{3} + \frac{8a^3bc}{3} = \frac{8abc\,(a^2+b^2+c^2)}{3}$$

$$I = \frac{8abc\,(a^2+b^2+c^2)}{3}$$

6b) Solve $\iint xy\,dx\,dy$ over the +ve quadrant of the circle $x^2+y^2=a^2$

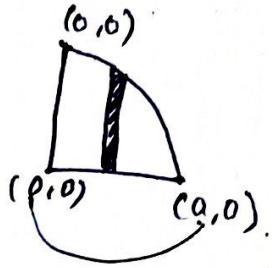
Evaluation of double integral over a given region

Sln:-



$$I = \iint_R xy\,dx\,dy$$

Now $x : 0 \to a$



$x^2+y^2=a^2$

$y^2 = a^2-x^2$

$y = \pm\sqrt{a^2-x^2} \Rightarrow y = \sqrt{a^2-a^2}$ (+ve quad) ~~wrong~~ $x \to 0 \to a$, $y \to 0 \to a$ } square

$\therefore x : 0 \to a$ & $y : 0 \to \sqrt{a^2-x^2}$

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,dx\,dy = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} xy\,dy\,dx$$

$$= \int_0^a x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx = \int_0^a x\left(\frac{a^2-x^2}{2}\right) dx.$$

$$= \frac{1}{2}\int_0^a xa^2 - x^3\,dx = \frac{1}{2}\left\{\frac{x^2a^2}{2} - \frac{x^4}{4}\right\}_0^a$$

$$= \frac{1}{2}\left\{\frac{a^4}{2} - \frac{a^4}{4}\right\} = \frac{1}{2}\left\{\frac{a^4}{4}\right\} = \frac{a^4}{8}$$

5 b) Evaluate $\iint xy(x+y)\,dy\,dx$ taken over the area b/w $y=x^2$ and $y=x$.

Sln:

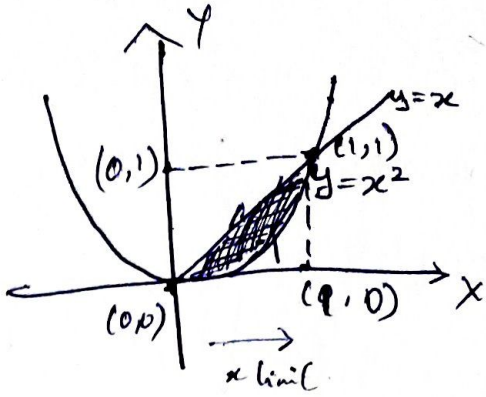
$y=x^2$ & $y=x$

Now, $x^2=x$

$\Rightarrow x^2-x=0 \Rightarrow x(x-1)=0$

$\Rightarrow x=0, x=1$

$\therefore y=0, y=1$



Hence the two curves intersects at the point $(0,0)$ $(1,1)$

$$I = \iint_R xy(x+y)\,dy\,dx$$

$$= \int_0^1 \int_{y=x^2}^{x} (x^2y + xy^2)\,dy\,dx$$

$$= \int_0^1 \left\{ \frac{x^2y^2}{2} + \frac{xy^3}{3} \right]_{y=x^2}^{x} dx = \int_0^1 \left(\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3} \right) dx$$

$$= \left[\frac{x^5}{10} + \frac{x^5}{15} - \frac{x^7}{14} - \frac{x^8}{24} \right]_0^1 = \frac{1}{10} - \frac{1}{14} + \frac{1}{15} - \frac{1}{24} = \frac{3}{56}$$

Note:- We can also write I in the form of

$$I = \int_{y=0}^{1} \int_{x=y}^{\sqrt{y}} xy(x+y)\,dx\,dy \Rightarrow I = 3/56$$

Evaluation of $\iint_R f(x,y)\,dx\,dy$ over the specific region R.

We need to draw the befitting figure from the given description to identify the specific region R. We have to then express.

$$I = \iint_R f(x,y)\,dx\,dy = \int_{x=a}^{b} \int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx. \longrightarrow (1)$$

(or)

$$I = \iint_R f(x,y)\,dx\,dy = \int_{y=c}^{d} \int_{x=x_1(y)}^{x_2(y)} f(x,y)\,dx\,dy. \longrightarrow (2)$$

Method :-

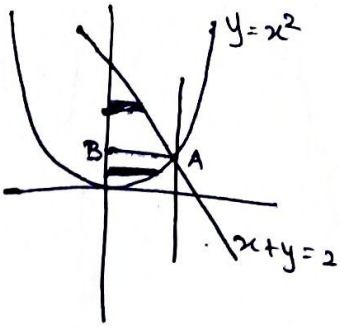
S-1 :- Given the integral in either of the forms (say (1)) we have to identify the region of integration R by writing the figure and express (1) in the form (2).

S-2 :- The evaluation of (2) will be the value of (1) on changing the order of integration. (This can be vice versa)

~~S-3~~ :- ~~P~~

11) a) Change the order of integration and then evaluate (1)

$$\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx.$$



Sln:-

Here the limits for y varies from $y=x^2$ to $y=2-x$ and the limits for x varies from $x=0$ to $x=1$

The region of integration is shown in fig.

Now we have to change the order of integration. First integration w.r.t x then y. Consider horizontal strips (∵ first w.r.t x).

In the region OAB & ABC.

Limits for x	$x=0$, $x=1$	$x=0$, $x=\sqrt{y}$	OAB
Limits for y	$y=x^2$, $y=2-x$	$y=0$, $y=1$.	
Limits for x	$x=0$, $x=1$	$x=0$, $x=2-y$	ABC.
y	$y=x^2$, $y=2-x$	$y=1$, $y=2$	

Region OAB:

$$\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx = \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy = \int_0^1 \left[\frac{x^2y}{2}\right]_0^{\sqrt{y}} dy = \int_0^1 y^2/2\,dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 = 1/6 \quad \longleftrightarrow (1)$$

Region ABC:

$$\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx = \int_1^2 \int_0^{2-y} xy\,dx\,dy = \int_1^2 \left[\frac{x^2y}{2}\right]_0^{2-y} dy = \int_1^2 \frac{(2-y)^2 \cdot y}{2}\,dy$$

$$= \frac{1}{2}\int_1^2 (4+y^2-4y)\,y\,dy$$

$$= \frac{1}{2}\int_1^2 (4+y^2-4y)\,y\,dy$$

$$= \frac{1}{2}\int_1^2 (4y+y^3-4y^2)\,dy$$

$$= \frac{1}{2}\cdot\left[4\cdot\frac{y^2}{2}+\frac{y^4}{4}-4\frac{y^3}{3}\right]_1^2$$

$$= \frac{1}{2}\left[8+4-\frac{32}{3}-2-\frac{1}{4}+\frac{4}{3}\right] = \frac{1}{2}\left[10-\frac{32}{3}+\frac{4}{3}-\frac{1}{4}\right]$$

$$= 5-\frac{16}{3}+\frac{2}{3}-\frac{1}{8} \longrightarrow (2)$$

Add (1) & (2)

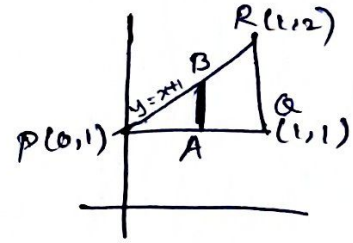
$$\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx = \frac{1}{6}+5-\frac{16}{3}+\frac{2}{3}-\frac{1}{8} = \frac{4+120-128+16-3}{24}$$

$$= \frac{140-131}{24} = 9/24 = 3/8$$

D

5) Evaluate $\iint (x^2-y^2)\,dx\,dy$ over the triangle with vertices $(0,1)$, $(1,1)$, $(1,2)$.

Soln:-



1. The region of integration is ΔPQR.
2. Equation of the line PQ is $y=1$.

   Equation of the line PR is

$$y-1 = \frac{2-1}{1-0}(x-0) = x \Rightarrow y = x+1.$$

3. The integration can be done w.r.t any variable first. Draw a vertical strip AB parallel to y-axis which starts from the line $y=1$ and terminates on the line $y=x+1$.

4. Limits of y : $y=1$ to $y=x+1$

   x : $x=0$ to $x=1$.

$$\therefore I = \iint (x^2-y^2)\,dx\,dy = \int_0^1 \int_1^{x+1} (x^2-y^2)\,dy\,dx$$

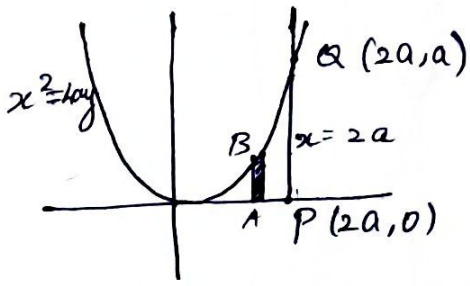
$$= \int_0^1 \left[x^2y - \frac{y^3}{3}\right]_1^{x+1} dx = \int_0^1 \left[x^2(x+1) - \frac{(x+1)^3}{3} - x^2 + \frac{1}{3}\right] dx$$

$$= \left[\frac{x^4}{4} + \frac{x^3}{3} - \frac{(x+1)^4}{12} - \frac{x^3}{3} + \frac{x}{3}\right]_0^1$$

$$= \frac{1}{4} + \frac{1}{3} - \frac{16}{12} + \frac{1}{12} = -2/3.$$

6a) Evaluate $\iint xy\,dx\,dy$ over the region enclosed by the x-axis, the line $x=2a$ and the parabola $x^2=4ay$.

Sln:-



1. The region of integration is OPQ
2. The point of intersection of the parabola $x^2=4ay$ and the line $x=2a$ is obtained as $4a^2=4ay \Rightarrow y=a$.
   Hence $Q:(2a,a)$
3. Here, integration can be done w.r.t any variable first. Draw a vertical strip AB parallel to y-axis, which starts from x-axis and terminates on the parabola $x^2=4ay$.
4. Limits of y: $y=0$ to $y=\frac{x^2}{4a}$
   x: $x=0$ to $x=2a$

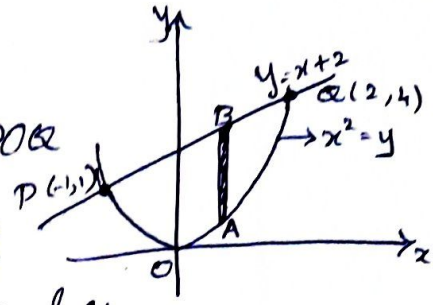
$$I = \iint xy\,dx\,dy = \int_0^{2a} x \int_0^{x^2/4a} y\,dy\,dx$$

$$= \int_0^{2a} x\left[\frac{y^2}{2}\right]_0^{x^2/4a} dx = \int_0^{2a} x \cdot \frac{x^4}{32a^2}\,dx.$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a} = \frac{1}{32a^2}\left[\frac{64a^6}{6}\right] = a^4/3.$$

Q: Evaluate $\iint y\,dx\,dy$ over the region enclosed by the parabola $x^2 = y$ and the line $y = x+2$.

Soln:-



Step: 1 - The region of integration is POQ

2 - The points of intersection of $x^2 = y$ and $y = x+2$ are obtained as

$$x^2 = x+2$$
$$\Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$$
$$x = 2, -1 \text{ and } y = 4, 1$$

$\therefore P(-1,1)$ & $Q(2,4)$

3- Here, integration can be done w.r.t any variable first. Draw a vertical strip AB parallel to y-axis which starts from the parabola $x^2 = y$ and terminates on the line $y = x+2$.

4- Limits of y : $y = x^2$ to $y = x+2$

Limits of x : $x = -1$ to $x = 2$

$$I = \iint y\,dx\,dy = \int_{-1}^{2}\int_{x^2}^{x+2} y\,dy\,dx$$

$$= \int_{-1}^{2}\left[\frac{y^2}{2}\right]_{x^2}^{x+2} dx = \frac{1}{2}\int_{-1}^{2}\left[(x+2)^2 - x^4\right]dx.$$

$$= \frac{1}{2}\left\{\int_{-1}^{2}(x+2)^2 dx - \int_{-1}^{2} x^4\,dx\right\}$$

$$= \frac{1}{2}\left\{\int_{-1}^{2}\left[x^2+4x+4\right]dx - \int_{-1}^{2} x^4\,dx\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{x^3}{3} + \frac{4x^2}{2} + 4x - \frac{x^5}{5}\right]_{-1}^{2}\right\} = 36/5$$

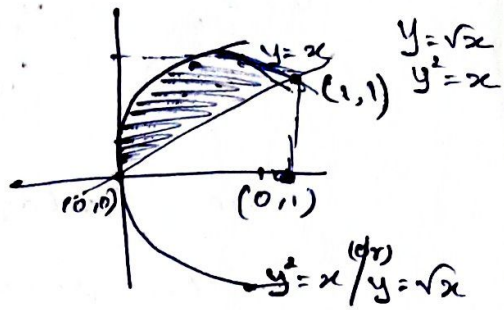
$$= \frac{1}{2}\left\{\frac{64}{3} - \frac{32}{5} - \frac{1}{3} - 1/5\right\}$$

Evaluation of a double integral by changing the order of integration

1) Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx$ by changing the order of integration

Soln:-

$$I = \int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} xy \, dy \, dx$$



$y = x$, $y = \sqrt{x}$

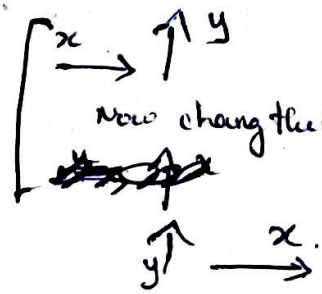
$\Rightarrow x = \sqrt{x} \Rightarrow x^2 = x \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0$

$\Rightarrow x = 0$ & $x = 1$

$\Rightarrow y = 0$ & $y = 1$

$\therefore$ The point of intersection $(0,0)$ & $(1,1)$

On changing the order of integration

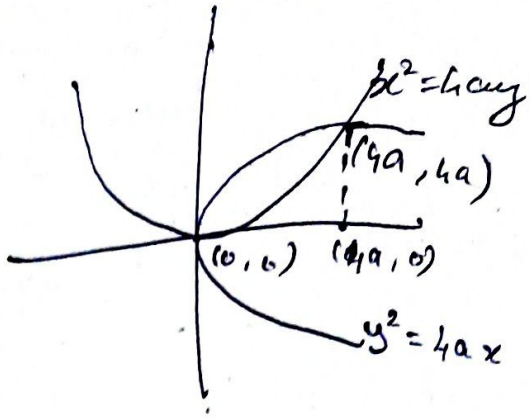


$$I = \int_{y=0}^{1} \int_{x=y^2}^{y} xy \, dx \, dy = \int_{y=0}^{1} y \left[\frac{x^2}{2} \right]_{y^2}^{y} dy.$$

$$= \frac{1}{2} \int_0^1 (y^3 - y^5) \, dy = \frac{1}{2} \left[\frac{y^4}{4} - \frac{y^6}{6} \right]_0^1$$

$$= \frac{1}{2} \left[\frac{1}{4} - \frac{1}{6} \right] = 1/24.$$

2) $I = \int_{x=0}^{4a} \int_{y=x^2/4a}^{2\sqrt{ax}} dx\,dy$



old strip varies from

$x=0$ to $x=4a$

$y = x^2/4a$. $y = 2\sqrt{ax}$

$(x^2=4ay)$ $(y^2=4ax)$

Here $y^2=4ax$ and $x^2=4ay$ ($x \rightarrow \uparrow y$ old)

on equating both

$2\sqrt{ax} = x^2/4a \Rightarrow 4ax = x^4/16a^2 \Rightarrow x^3 = 64a^3$

$\boxed{x=4a} \Rightarrow \boxed{y=4a}$

Now the change of order of interaction

$y=0$ to $y=4a$. ($y \uparrow \xrightarrow{x}$ new)

$x = y^2/4a$, $x = 2\sqrt{ay}$.

$\therefore I = \int_{y=0}^{4a} \int_{x=y^2/4a}^{2\sqrt{ay}} dx\,dy = \int_{y=0}^{4a} \left[y \right]_{y^2/4a}^{2\sqrt{ay}} dx$

$= \int_0^{4a} \left[2\sqrt{ay} - y^2/4a\right] dx = \left[2\sqrt{a}\, \frac{y^{1/2+1}}{1/2+1} - \frac{y^3}{4a} \right]_0^{4a}$

$= 2\sqrt{a}\left(\frac{(4a)^{3/2}}{3/2} - \frac{(4a)^3}{4a} \right) = \left(\frac{2}{3}\right) 2\sqrt{a}\left[(4a)^{3/2} - (4a)^2\right]$

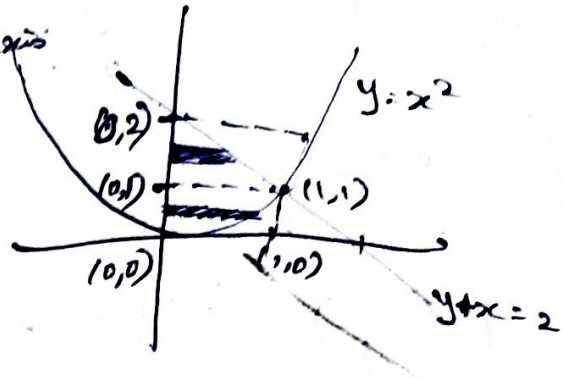
$= \frac{4}{3}\sqrt{a}\left[4^{3/2} a^{3/2} - 4^2 \cdot a^2\right]$

(Ⅱ)

3) Evaluate $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$ by changing the order of integration.

Sln:- Given limits / Region, $y = x^2$ to $y = 2 - x$ $(x + y = 2)$ $\left\{ \begin{matrix} x=0 & , x=1 \\ y=0 & y=1 \end{matrix} \right\}$

Now change the strip, parallel to x-axis



$\therefore$ $y = 0$ to $y = 1$

$x = 0$ to $x = \sqrt{y}$

Also

$y = 1$ to $y = 2$

$x = 0$ to $x = 2 - y$

$$\therefore \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy\,dy\,dx + \int_{y=1}^{2} \int_{x=0}^{2-y} xy\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2 \int_0^{2-y} xy\,dx\,dy = \left\{ \int_0^1 \left[\frac{x^2}{2} y\right]_0^{\sqrt{y}} dy + \int_1^2 \left[\frac{x^2}{2} y\right]_0^{2-y} dy \right.$$

$$= \int_0^1 \frac{y^2}{2}\,dy + \int_1^2 \frac{y(2-y)^2}{2}\,dy = \int_0^1 \frac{y^2}{2}\,dy + \int_1^2 \frac{4y - 4y^2 + y^3}{2}\,dy.$$

$$= \frac{1}{2}\left\{ \left[\frac{y^3}{3}\right]_0^1 + \left[\frac{4y^2}{2} - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2 \right\}$$

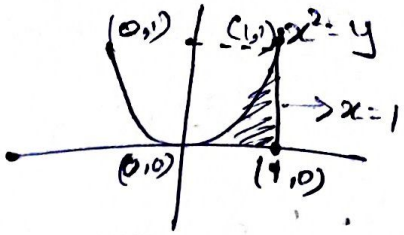
$$= \frac{1}{2}\left\{ \frac{1}{3} + \frac{4(4)}{2} - \frac{4(8)}{3} + \frac{16}{4} - \frac{4}{2} + \frac{4}{3} - \frac{1}{4} \right\}$$

$$= \frac{1}{6} + \frac{5}{24} = 3/8$$

1) Change the order of the integration and hence evaluate $\int_0^1 \int_{\sqrt{y}}^1 dx\,dy$

Sol^n:

$$I = \int_{y=0}^{1} \int_{x=\sqrt{y}}^{1} dx\,dy$$



$y = 0$ to $y = 1$

$x = \sqrt{y}$ to $x = 1$

$(x^2 = y)$

on changing the order of integration,

$$\int_{x=0}^{1} \int_{y=0}^{x^2} dy\,dx \qquad \left\{ \begin{array}{l} x = 0 \text{ to } 1 \\ y = 0 \text{ to } y = x^2 \end{array} \right\}$$

$$= \int_{x=0}^{1} [y]_0^{x^2}\,dx = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = 1/3$$

4) Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\,dy$ by changing polar coordinates.

Sol:-

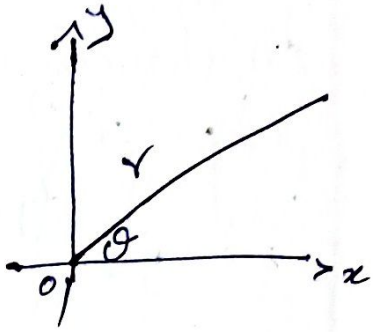
In polar, we have $x = r\cos\theta$, $y = r\sin\theta$

$x^2+y^2 = r^2$. and $dx\,dy = r\,dr\,d\theta$

$\because x \to 0$ to ∞ & $y \to 0$ to ∞

$\Rightarrow r \to 0$ to ∞.

In first quadrant $\theta \to 0$ to $\pi/2$



$$\therefore I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2}\, r\, dr\, d\theta$$

take $r^2 = t \Rightarrow r\,dr = dt/2 \Rightarrow t \to 0$ to ∞.

$$I = \int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} e^{-t} \frac{dt}{2}\, d\theta.$$

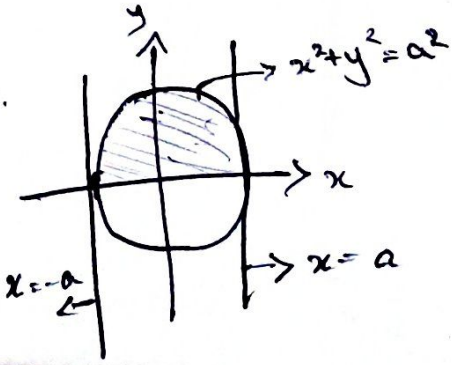
$$= \frac{1}{2}\left\{ \int_{\theta=0}^{\pi/2} \left[-e^{-t}\right]_t^\infty d\theta = \frac{1}{2}\int_{\theta=0}^{\pi/2} (0-1)\, d\theta = \frac{1}{2}\left[\theta\right]_0^{\pi/2}\right.$$

$$= \pi/4$$

5) Change the integral $\int_{-a}^{a} \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$ into polars

12/6 and hence evaluate the same.

Sol:- The region of integration is given.



Clearly $\theta \to 0$ to π

$x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2 = r^2$

ie, $a^2 = r^2 \Rightarrow r = a$

$\therefore r \to 0$ to a

$dx\,dy = r\,dr\,d\theta$

$$\therefore I = \int_{\theta=0}^{\pi} \int_{r=0}^{a} r \cdot r \cdot dr\, d\theta$$

$$= \int_{\theta=0}^{\pi} \left[\frac{r^3}{3}\right]_0^a d\theta = \frac{a^3}{3}\left[\theta\right]_0^{\pi} = \frac{a^3}{3}(\pi - 0) = \frac{\pi a^3}{3}$$

5) Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\, dx\, dy$ by changing into polar

12b

Soln:

$y \to 0$ to a

$x \to 0$ to $\sqrt{a^2-y^2}$

$\Rightarrow x = 0, \quad x = \sqrt{a^2-y^2}$

$(x^2 = a^2 - y^2 \Rightarrow x^2+y^2 = a^2)$ circle

$\therefore y \to 0$ to a, the region of integration in 1st quadrant

w.k.t $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2 = r^2$

$\Rightarrow r^2 = a^2 \Rightarrow r = a$

Also $x = 0, y = 0 \Rightarrow r = 0$ and to $r = a$

In first quad $\theta \to 0$ to $\pi/2$

$dx\,dy = r\,dr\,d\theta$

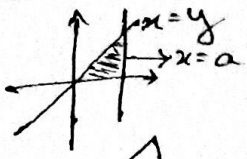
$$\therefore I = \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} r\sin\theta \cdot r \cdot r\, dr\, d\theta = \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} r^3 \sin\theta\, dr\, d\theta$$

$$= \int_{r=0}^{a} r^3 \left(-\cos\theta\right)_0^{\pi/2} dr = \int_{r=0}^{a} -r^3(0-1)\, dr$$

$$= \left[\frac{r^4}{4}\right]_0^a = \frac{a^4}{4}$$

6) Evaluate the double integral $\int_0^a\int_0^a \frac{x}{x^2+y^2}\,dx\,dy$ by changing into polar.

Sln: $x \to r\cos\theta,\ y \to r\sin\theta,\ dx\,dy \to r\,dr\,d\theta$



$$= \int_{\theta=0}^{\theta=\pi/4}\int_{r=0}^{r=a\sec\theta} \frac{\cancel{r}\cos\theta}{\cancel{r^2}[\cos^2\theta+\sin^2\theta]}\,\cancel{r}\,dr\,d\theta = \int\int \cos\theta\,dr\,d\theta$$

$x=y$
$r\cos\theta = r\sin\theta$
$\tan\theta = 1$
$\theta = \pi/$

$x=a$
$r = \frac{a}{\cos\theta}$
$r = a\sec\theta$

$$= \int_0^{\pi/4} \cos\theta\,[r]_0^{a\sec\theta}\,d\theta$$

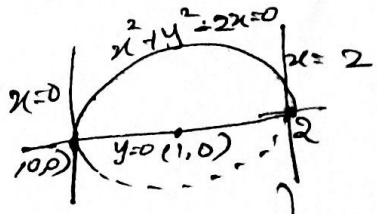
$$= \int_0^{\pi/4} \cos\theta\,(a\sec\theta - 0)\,d\theta$$

$$= a\,[\theta]_0^{\pi/4} \Rightarrow a\pi/4.$$

7). By changing into polar co-ordinates, evaluate, $\int_0^2\int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\,dx\,dy$

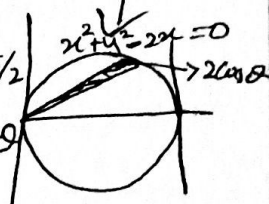
Sln: $x \to r\cos\theta,\ y \to r\sin\theta,\ dx\,dy \to r\,dr\,d\theta.$

$y=0,\ y=\sqrt{2x-x^2} \Rightarrow y^2 = 2x - x^2;\ \Rightarrow x=0,\ x=2.$



$$x^2+y^2-2x=0$$
$$(x^2-2x)+y^2=0$$
$$x^2-2x.1+1^2-1^2+y^2=0.$$
$$(x-1)^2+(y-0)^2=1. \Rightarrow c=(1,0),\ r=1.$$
$$(\Rightarrow (x-a)^2+(y-b)^2=r^2 \Rightarrow (a,b),\ r)$$

$\theta = 0$ to $\pi/2$
$r = 0$ to $2\cos\theta$



$$\int_0^{\pi/2}\int_0^{2\cos\theta} (\cos\theta)\,dr\,d\theta = \int_0^{\pi/2} \cos\theta\,[r]_0^{2\cos\theta}\,d\theta$$

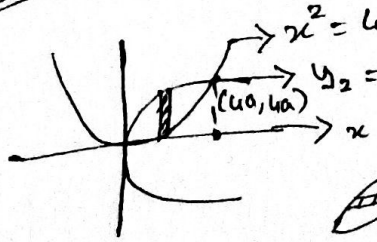
$$= 2\int_0^{\pi/2} \cos^2\theta\,d\theta = 2\left[\tfrac{1}{2}\cdot\pi/2\right]$$

$$= \pi/2 \,//.$$

$x^2+y^2-2x=0$
$r^2[\cos^2\theta+\sin^2\theta]-2r\cos\theta=0$
$r^2-2r\cos\theta=0$
$r-2\cos\theta=0$
$r=2\cos\theta.$

$$\left[\because \int_0^{\pi/2}\cos^n\theta\,d\theta = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\tfrac{1}{2}\cdot\pi/2,\ \text{if } n \text{ is even}\right].$$

H.W Change the order of int, $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx.$



$\left[\begin{array}{l} y_1 = x^2/4a \\ x^2 = 4ay_1 \end{array} \middle/ \begin{array}{l} y_2 = 2\sqrt{ax} \Rightarrow (4a, 4a) \\ y_2^2 = 4ax \end{array}\right]$

Change L.L $x = y^2/4a,\ \Rightarrow x^2 = 4ay$
U.P $x = 2\sqrt{ay}$

$$\rightarrow \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dx\,dy = \frac{32a^2}{3} - \frac{16}{3}a^2$$
$$= \boxed{16a^2/3}$$

20. a) Calculate the volume of the solid bounded by the planes $x=0$, $y=0$, $x+y+z=1$ and $z=0$

Ans:- Formula volume $= V = \iiint dz\,dy\,dx$

Given $x+y+z=1 \Rightarrow z = 1-x-y \rightarrow (1)$

$\therefore$ limits for z is from 0 to $1-x-y$.

And put $z=0$ in (1), $0 = 1-x-y \Rightarrow y = 1-x$

$\therefore$ limits for y is from 0 to $1-x$.

Also put $z=0$, $y=0$ in (1), $0 = 1-x-0 \Rightarrow x = 1$

$\therefore$ limits from x is from 0 to 1

$$V = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} dz\,dy\,dx = \int_0^1\int_0^{1-x}\int_0^{1-x-y} dy\,dx.$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} (1-x-y-0)\,dy\,dx = \int_0^1\int_0^{1-x} (1-x-y)\,dy\,dx$$

$$= \int_0^1 \left(y - xy - \frac{y^2}{2}\right)_{y=0}^{1-x} dx = \int_{x=0}^{1}\left[1-x-x(1-x)-\frac{(1-x)^2}{2}-0\right]dx$$

$$= \int_{x=0}^{1}\left(1-x-x+x^2-\frac{(1+x^2-2x)}{2}\right)dx = \int_{x=0}^{1}\left(\frac{x^2}{2}-x+\frac{1}{2}\right)dx$$

$$= \frac{x^3}{6}-\frac{x^2}{2}+\frac{x}{2}\Big]_{x=0}^{1} = \frac{1}{6}-\frac{1}{2}+\frac{1}{2}-0 = \frac{1}{6}.$$

14 b) $\displaystyle\int_0^\infty\int_0^\infty\int_0^\infty \frac{dx\,dy\,dz}{(1+x^2+y^2+z^2)^2}$

Sln:- It is difficult to integrate in Cartesian form.

Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$,

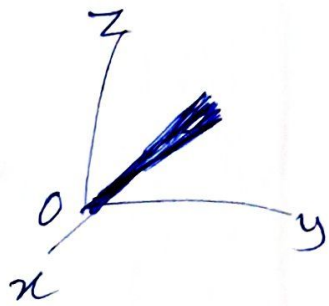
the integral changes to spherical form.

2)

Limits of x: $x = 0$ to $x \to \infty$

$y = 0$ to $y \to \infty$

$z = 0$ to $z \to \infty$



The region of integration is the +ve octant of the plane.

r: $r = 0$ to ∞

$\theta = 0$ to $\pi/2$

$\phi = 0$ to $\pi/2$

$$I = \int_0^\infty\int_0^\infty\int_0^\infty \frac{dx\,dy\,dz}{(1+x^2+y^2+z^2)^2}$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^\infty \frac{r^2\sin\theta}{(1+r^2)^2}\,dr\,d\theta\,d\phi.$$

$r = \tan t$, $dr = \sec^2 t\,dt$

Where $r = 0$, $t = 0$

$r \to \infty$, $t = \pi/2$

$$= \frac{\pi}{2}\cdot 1\cdot \frac{1}{2}B\left(\frac{3}{2},\frac{1}{2}\right)$$

$$= \frac{\pi}{4}\,\frac{\frac{3}{2}!\,\frac{1}{2}!}{2!}$$

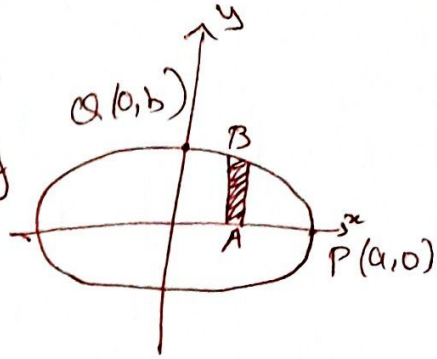
$$= \frac{\pi}{4}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}$$

$$I = \int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta\,d\theta \int_0^{\pi/2} \frac{\tan^2 t}{\sec^4 t}\cdot\sec^2 t\,dt = |\phi|_0^{\pi/2}\left[-\cos\theta\right]_0^{\pi/2}\int_0^{\pi/2}\sin^2 t\,dt$$

Find the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ above x-axis.

Sln:-

(1) The region is symmetric about the y-axis, total area $= 2$ (Area bounded by the ellipse in the first quadrant)



(2) Draw a vertical strip AB in the region which lies in the first quadrant. AB starts from the x-axis and terminates on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Limits of y: $y=0$ to $y=b\sqrt{1-\frac{x^2}{a^2}}$.

x : $x=0$ to $x=a$

$$\text{Area } A = 2\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} dy\,dx$$

$$= 2\int_0^a [y]_0^{b\sqrt{1-x^2/a^2}} dx = 2\int_0^a b\sqrt{1-x^2/a^2}\,dx.$$

$$= \frac{2b}{a}\int_0^a \sqrt{a^2-x^2}\,dx = \frac{2b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{b}\right]_0^a$$

$$= \frac{2b}{a}\left(\frac{a^2}{2}\sin^{-1}1\right) = \frac{2b}{a}\left(\frac{a^2}{2}\cdot\frac{\pi}{2}\right) = \frac{\pi ab}{2}.$$

(X) Evaluate $\iint r^2 \sin\theta \, dr\, d\theta$, over the cardioid $r = a(1+\cos\theta)$, (intial line)

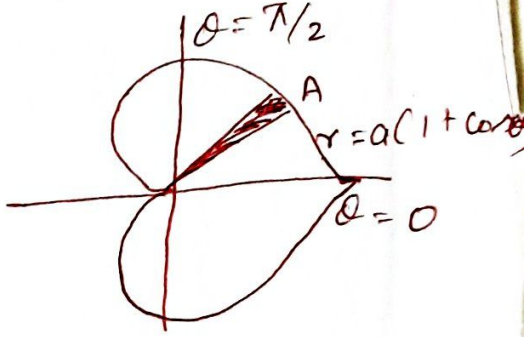
Soln:-

(1) The region of integration is the part of the Cardioid $r = a(1+\cos\theta)$ above the intial line $(\theta = 0)$

(2) Draw an elementary radius vector OA which starts from the origin and terminates on the cardioid $r = a(1+\cos\theta)$

(3) Limits of r: $r = 0$ to $r = a(1+\cos\theta)$

θ: $\theta = 0$ to $\theta = \pi$.



$$I = \iint r^2 \sin\theta \, dr\, d\theta = \int_0^{\pi} \sin\theta \int_0^{a(1+\cos\theta)} r^2 \, dr\, d\theta$$

$$= \int_0^{\pi} \sin\theta \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} d\theta = \frac{1}{3}\int_0^{\pi} \sin\theta \cdot a^3 (1+\cos\theta)^3 \, d\theta$$

$$= \frac{-a^3}{3} \int_0^{\pi} (1+\cos\theta)^3 (-\sin\theta)\, d\theta$$

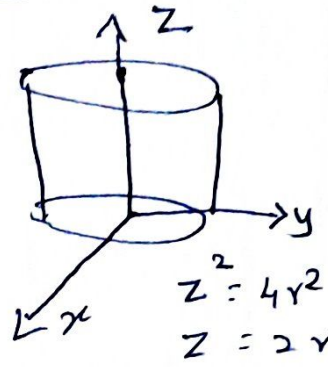
$$= -\frac{a^3}{3} \left[\frac{(1+\cos\theta)^4}{4}\right]_0^{\pi} \qquad \left[\because \int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$= \frac{-a^2}{12}(0-16) = \frac{4}{3}a^2$$

(17) a) Use cylindrical co-ordinate to evaluate $\iiint x^2\,dV$ where 'c' is the solid that lies within the cylinder $x^2+y^2=1$ above the plane $z=0$, and below the cone $z^2=4y^2+4x^2$

Soln:-

$$\iiint_C x^2\,dv = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{2r} r^2\cos^2\theta\, r\, dz\,dr\,d\theta$$



$$= \int_0^{2\pi}\int_0^{1}\left[r^3\cos^2\theta\,(z)\right]_0^{2r} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1} 2r^4\cos^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{2r^5}{5}\right)_0^1\cos^2\theta\,d\theta = \frac{2}{5}\int_0^{1}\frac{(1+\cos 2\theta)}{2}\,d\theta$$

$$= \frac{1}{5}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{1}{5}\left[2\pi+\frac{\sin 2(2\pi)}{2}-0\right]$$

$$= \frac{1}{5}\left[2\pi+0\right] = 2\pi/5$$

(19) a) b) Find the value of $\iiint z\,dx\,dy\,dz$ where v is the hemisphere $x^2+y^2+z^2=a^2$, $z\geq 0$

Soln:- $$\iiint_V z\,dx\,dy\,dz = \int_{z=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} z\,dz\,dy\,dx.$$

$$\Rightarrow \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(\frac{z^2}{2}\right)_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= \frac{1}{2} \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (a^2-x^2-y^2)\,dy\,dx = \frac{1}{2}\int_{-a}^{a} \left[a^2y - x^2y - \frac{y^3}{3}\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx.$$

$$= \frac{1}{2}\int_{-a}^{a} \left[(a^2-x^2)y - \frac{y^3}{3}\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx = \frac{1}{2}\int_{-a}^{a} \left\{(a^2-x^2)\left[\sqrt{a^2-x^2} - (-\sqrt{a^2-x^2})\right] - \frac{(a^2-x^2)^{3/2}}{3} - \frac{[-(a^2-x^2)]^{3/2}}{3}\right\} dx$$

$$= \frac{1}{2}\int_{-a}^{a} \left[(a^2-x^2)\,2\sqrt{a^2-x^2} - \frac{2(a^2-x^2)^{3/2}}{3}\right] dx.$$

$$= \frac{1}{2}\int_{-a}^{a} (a^2-x^2)^{3/2}\left[2 - \frac{2}{3}\right] dx$$

$$= \frac{\cancel{4}^{\,2}}{\cancel{6}_{\,3}} \int_{-a}^{a} (a^2-x^2)^{3/2}\,dx$$

$$= \frac{2}{3}\cdot 2\int_{0}^{a} (a^2-x^2)^{3/2}\,dx$$

Now take $x = a\sin\theta$,
(assume). $dx = a\cos\theta\,d\theta$

θ varies from 0 to $\pi/2$

$$= \frac{4}{3}\int_{\theta=0}^{\pi/2} (a^2 - a^2\sin^2\theta)^{3/2}\, a\cos\theta\, d\theta$$

$$= \frac{4}{3}\int_{0}^{\pi/2} \left[a^2(1-\sin^2\theta)\right]^{3/2} a\cos\theta\, d\theta$$

$$= \frac{4}{3}\int_{0}^{\pi/2} (a^2\cos^2\theta)^{3/2}\, a\cos\theta\, d\theta$$

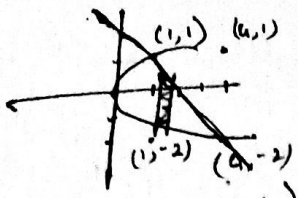
$$= \frac{4a^4}{3}\int_{0}^{\pi/2} \cos^4\theta\, d\theta$$

$$= \frac{4a^4}{3}\left(\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$= \frac{4a^4}{\cancel{3}}\left(\frac{\cancel{3}\pi}{\cancel{4}\cdot 2}\right) = \frac{a^4\pi}{4}$$

1) Find the center of gravity of the area in first quadrant bounded by the parabola $y^2 = x$ and the line $x+y=2$. Assume uniform density $\rho = 1$ → int fn

Sln:-

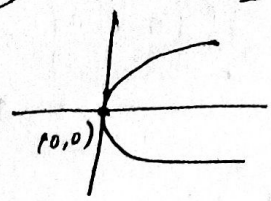


$x = y^2$

$x = 2 - y$

$\Rightarrow y^2 + y - 2 = 0$

$(y+2)(y-1) = 0$



$(1,1), (4,-2)$ point of intersection. $y = 1, -2$ $\therefore y = 1, -2$

$(1,1)\ (1,2)$
$(4,1)\ (4,-2)$

$x = 1,\ x = 4$

$x = 2-1,\ x = 2-(-2) = 4$

$$M = \iint_R 1 \cdot dA = \int_0^1 \int_{y^2}^{2-y} 1 \cdot dy\, dx.$$

$$= \int_0^1 \left(2 - y - \frac{y^2}{2}\right) dy = \left(2y - \frac{y^2}{2} - \frac{y^3}{6}\right)_0^1 = 7/6$$

$$\bar{x} = \frac{1}{M} \iint_R x\, dA = \frac{1}{M} \int_0^1 \int_{y^2}^{2-y} x\, dx\, dy = \frac{7}{6} \int_0^1 \left[(2-y)^2 - y^4\right] dy$$

$$= \frac{6}{7} \int_0^1 (4 - 4y + y^2 - y^4)\, dy = 16/15$$

$$\bar{y} = \frac{1}{M} \iint_R y\, dA = \frac{1}{M} \int_0^1 \int_{y^2}^{2-y} dx\, dy = \frac{6}{7} \int_0^1 (2y - y^2 - y^3)\, dy = \frac{6}{7} \times \frac{5}{12} = \frac{5}{14}$$

2) Find the center of gravity of the area in the first quad bounded by $y = x^2$, $y = x$. Assume uniform density $\rho = 1$.

Sln:-

$$M = \iint dA = \int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 (x - x^2)\, dx = 1/6$$

$$\bar{x} = \frac{1}{M} \iint_R x\, dA = \frac{1}{M} \int_0^1 \int_{x^2}^{x} x\, dy\, dx = \frac{1}{M} \int_0^1 (x^2 - x^3)\, dx = 6 \times \frac{1}{12} = 1/2$$

$$\bar{y} = \frac{1}{M} \iint_R y\, dA = \frac{1}{M} \int_0^1 \int_{x^2}^{x} y\, dy\, dx = \frac{1}{2M} \int_0^1 (x^2 - x^4)\, dx = \frac{6}{15} = 2/5$$

3) Find the centroid of the area bounded by the parabola $y^2 = 20x$, $x^2 = 20y$, Assume uniform density $\rho = 1$

Sln:- $M = \iint_A dA = \int_0^{20} \int_{x^2/20}^{\sqrt{20x}} dy\,dx = \int_0^{20} \left(\sqrt{20x} - x^2/20\right) dx = \left[\sqrt{20}\,\frac{2}{3} x^{3/2} - x^3/60\right]_0^{20}$

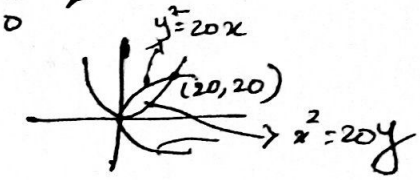
$= \frac{800}{3} - \frac{400}{3} = 400/3.$

$\bar{x} = \frac{1}{M} \iint_R x\,dA = \frac{3}{400} \int_0^{20} \int_{y^2/20}^{\sqrt{20y}} x\,dx\,dy = \frac{1}{2 \times 400} \int_0^{20} \left[20y - y^4/400\right] dy$

$= \frac{3}{800} \left[10y^2 - y^5/2000\right]_0^{20} = 9$

$\bar{y} = \frac{1}{M} \iint_R x\,dA = \frac{3}{400} \int_0^{20} \int_{x^2/20}^{\sqrt{20x}} y\,dy\,dx = \frac{3}{2 \times 400} \int_0^{20} \left[20x - x^4/400\right] dx = 9$

$\therefore$ center of gravity $(\bar{x}, \bar{y})$ is $(9, 9)$.



4). Find the center of gravity $(\bar{x}, \bar{y})$ of the area of the circle $x^2 + y^2 = a^2$ in the first quadrant. Assume uniform density $\rho = 1$.

Sln:- $M = \iint_R f(x,y)\,dx\,dy = \int_0^{\infty} \int_0^{\infty} r\,dr\,d\theta = \frac{1}{2} \int_0^{\infty} a^2\,d\theta = a^2/2\,\pi$

$\bar{x} = \frac{1}{M} \iint_R x f(x,y)\,dx\,dy = \frac{2}{a^2 \pi} \int_0^{\pi} \int_0^{a} r \cos\theta\,dr\,d\theta$

$= \frac{2}{a^2 \pi} \int_0^{\pi} \cos\theta\,d\theta \times \int_0^{a} r^2\,dr = \left(\frac{2}{a^2 \pi}\right) \left(\sin\theta\right)_0^{\pi} \cdot \left(r^3/3\right)_0^{a} = 0$

$\bar{y} = \frac{1}{M} \iint_R y f(x,y)\,dx\,dy = \frac{2}{a^2 \pi} \int_0^{\pi} \int_0^{a} r \sin\theta\,dr\,d\theta$

$= \frac{2}{a^2 \pi} \int_0^{\pi} \sin\theta\,d\theta \times \int_0^{a} r^2\,dr = \left(\frac{2}{a^2 \pi}\right) \left(-\cos\theta\right)_0^{\pi} \left(r^3/3\right)_0^{a} = \frac{4a}{3\pi}$

$\therefore (\bar{x}, \bar{y}) = (0, 4a/3\pi)$.

H.W Determine the center of gravity of the triangular lamina bounded by the coordinate axis and the line $x + y = 1$ $\rho = 1$.

Ans: $M = 1/2$, $(\bar{x}, \bar{y}) = (1/3, 1/3)$.