Underwater Imaging - Exploring Sub-Nyquist Sampling

Suhas Srinath

Supervisors: Prof. Chandra Shekhar Seelamantula, Sunil R

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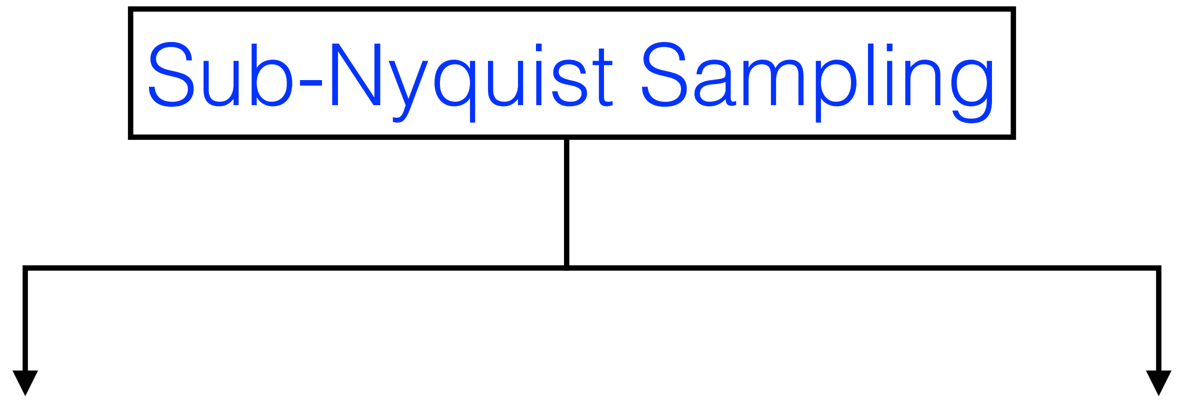
SPECTRUM LAB

Sub-Nyquist Sampling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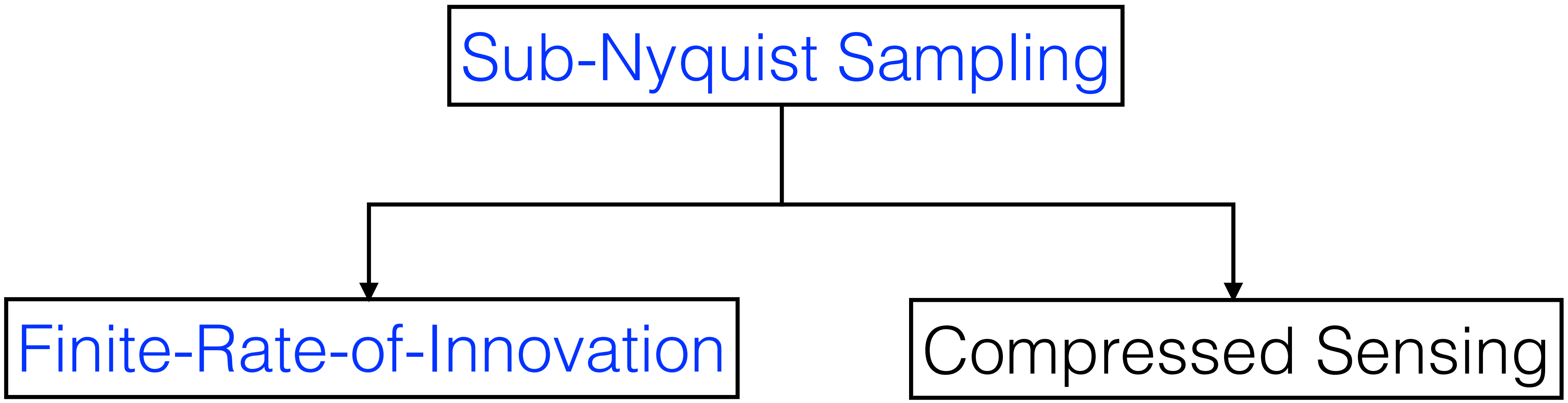
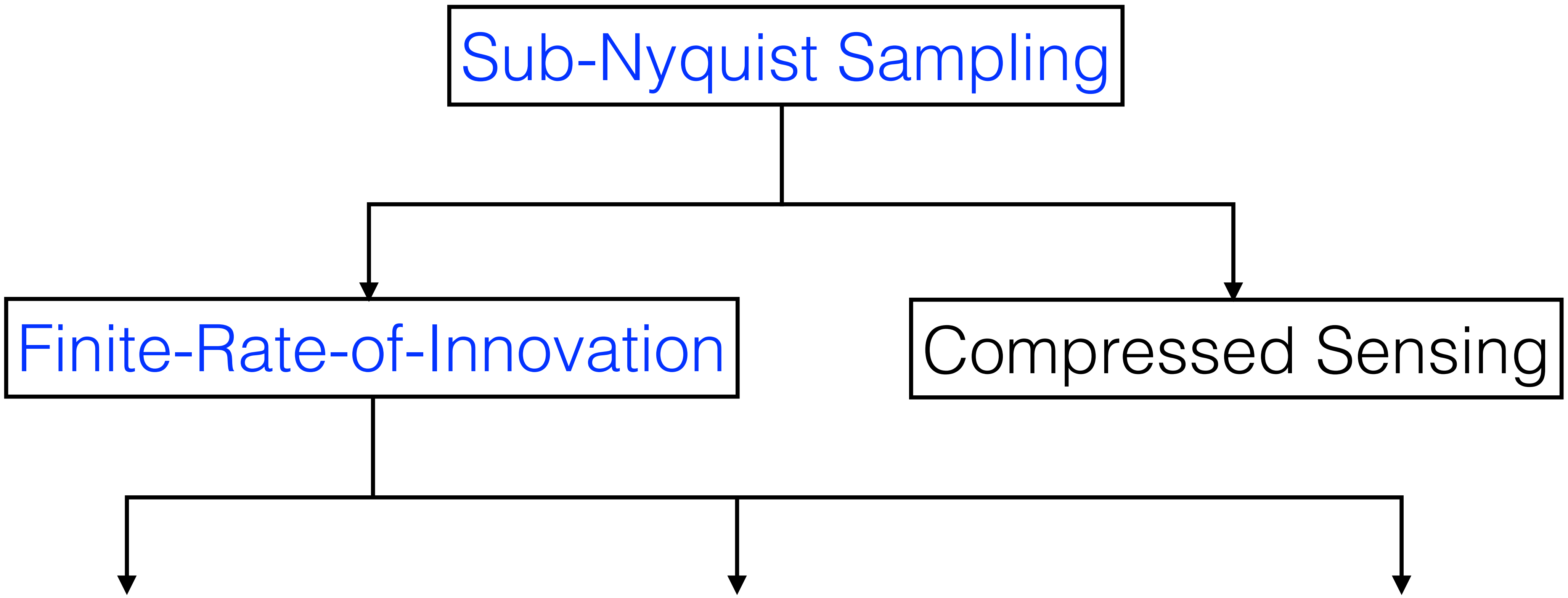

Sub-Nyquist Sampling
Finite-Rate-of-Innovation
Compressed Sensing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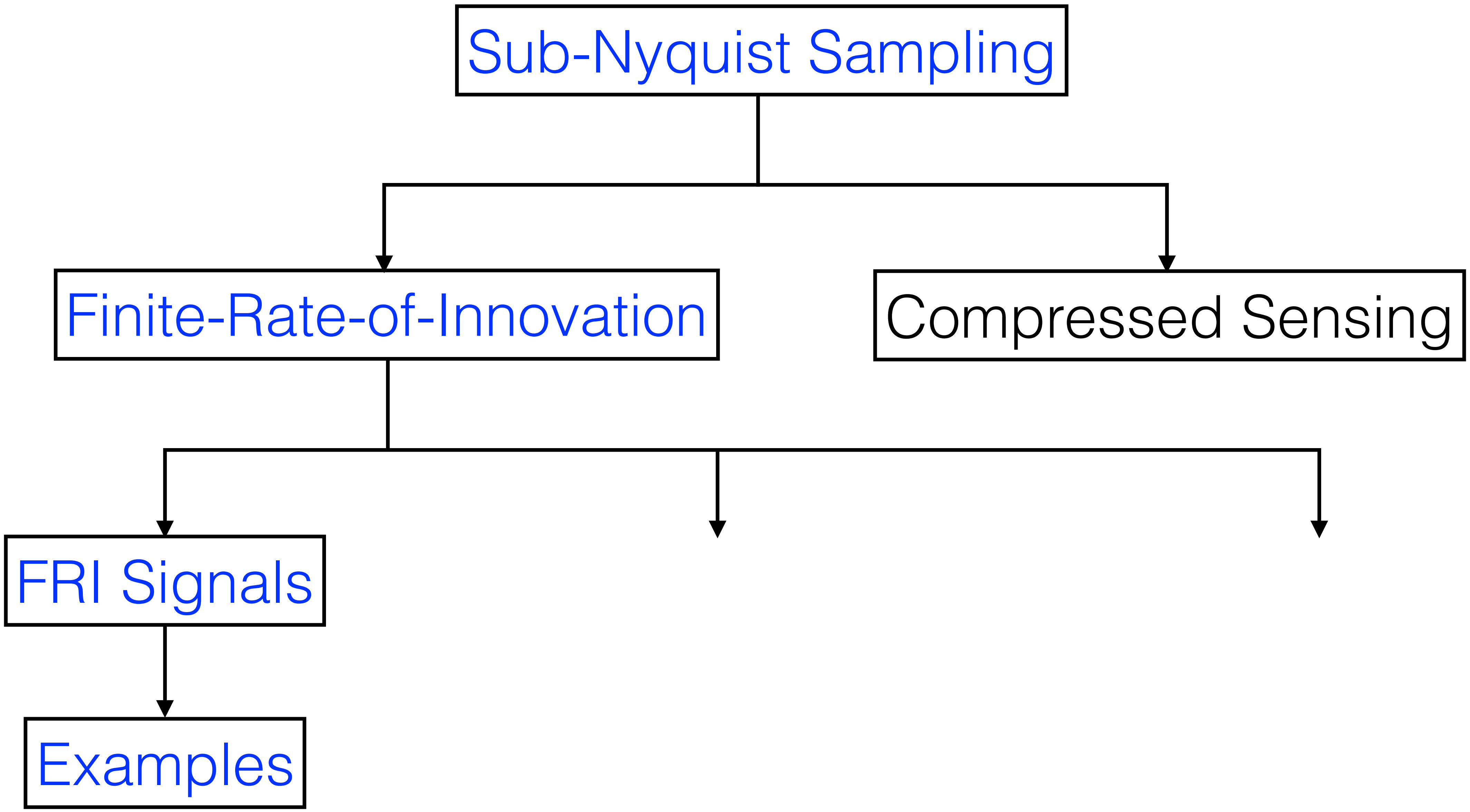
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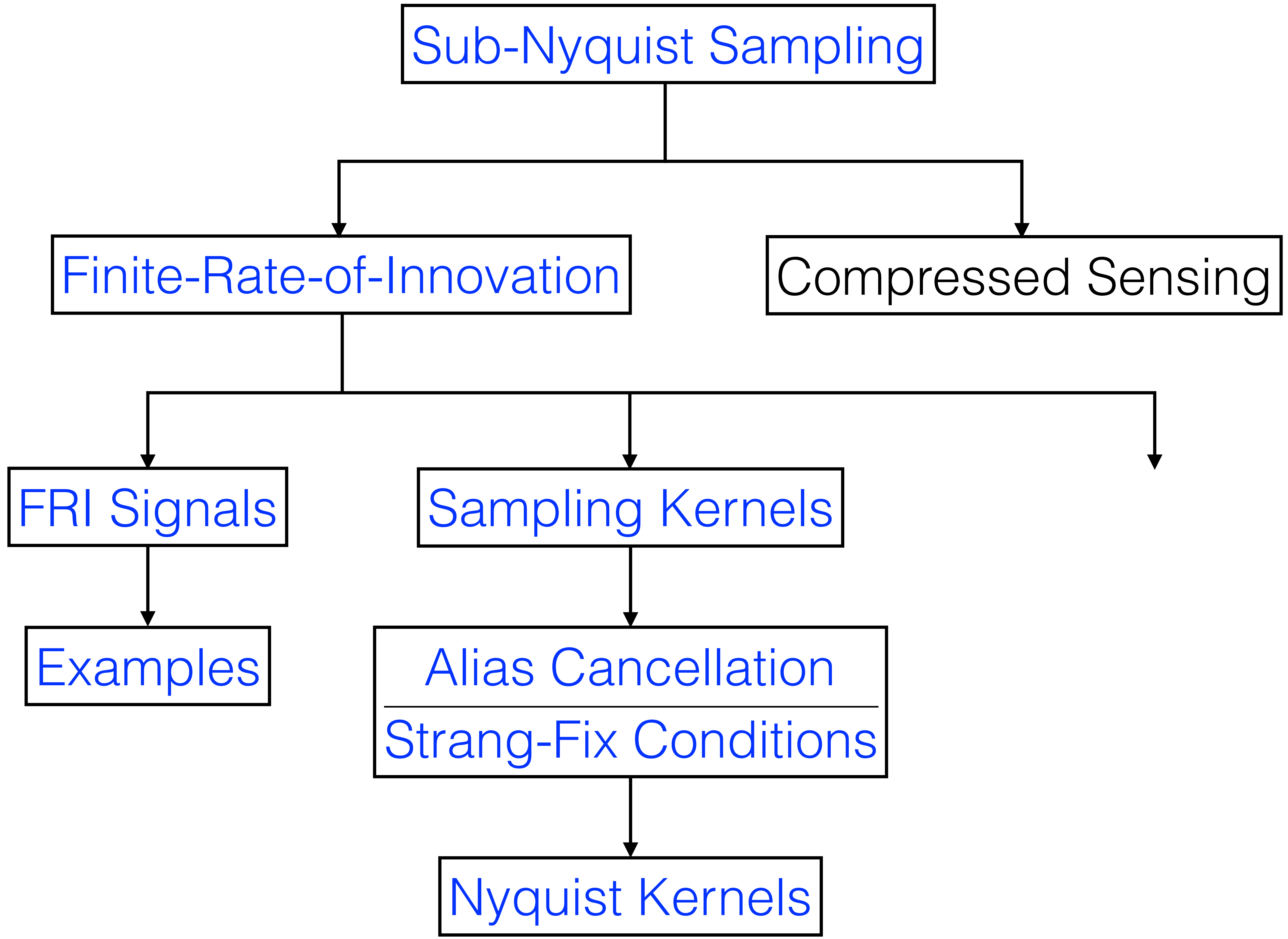
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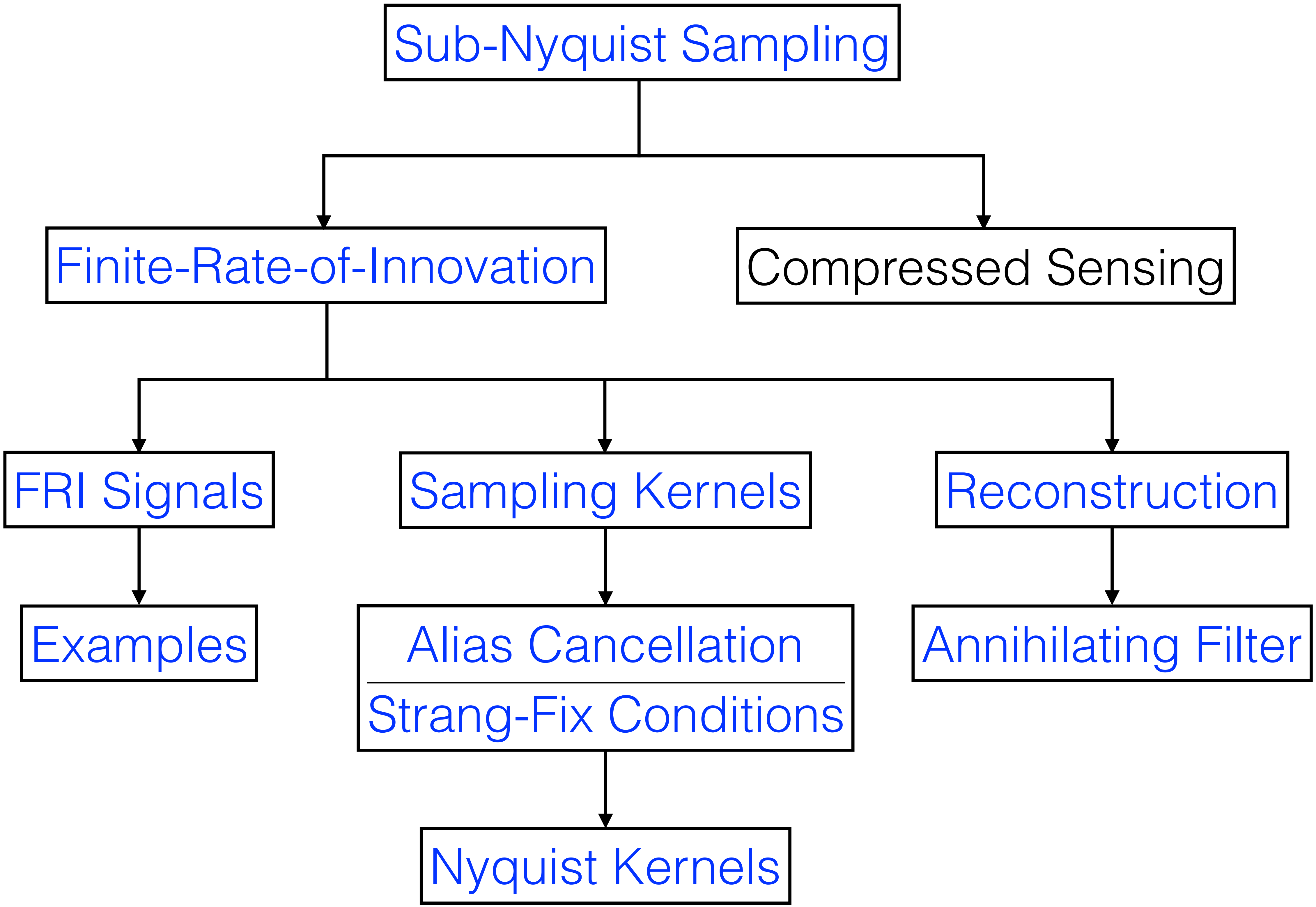
Alias Cancellation
Strang-Fix Conditions
Nyquist Kernels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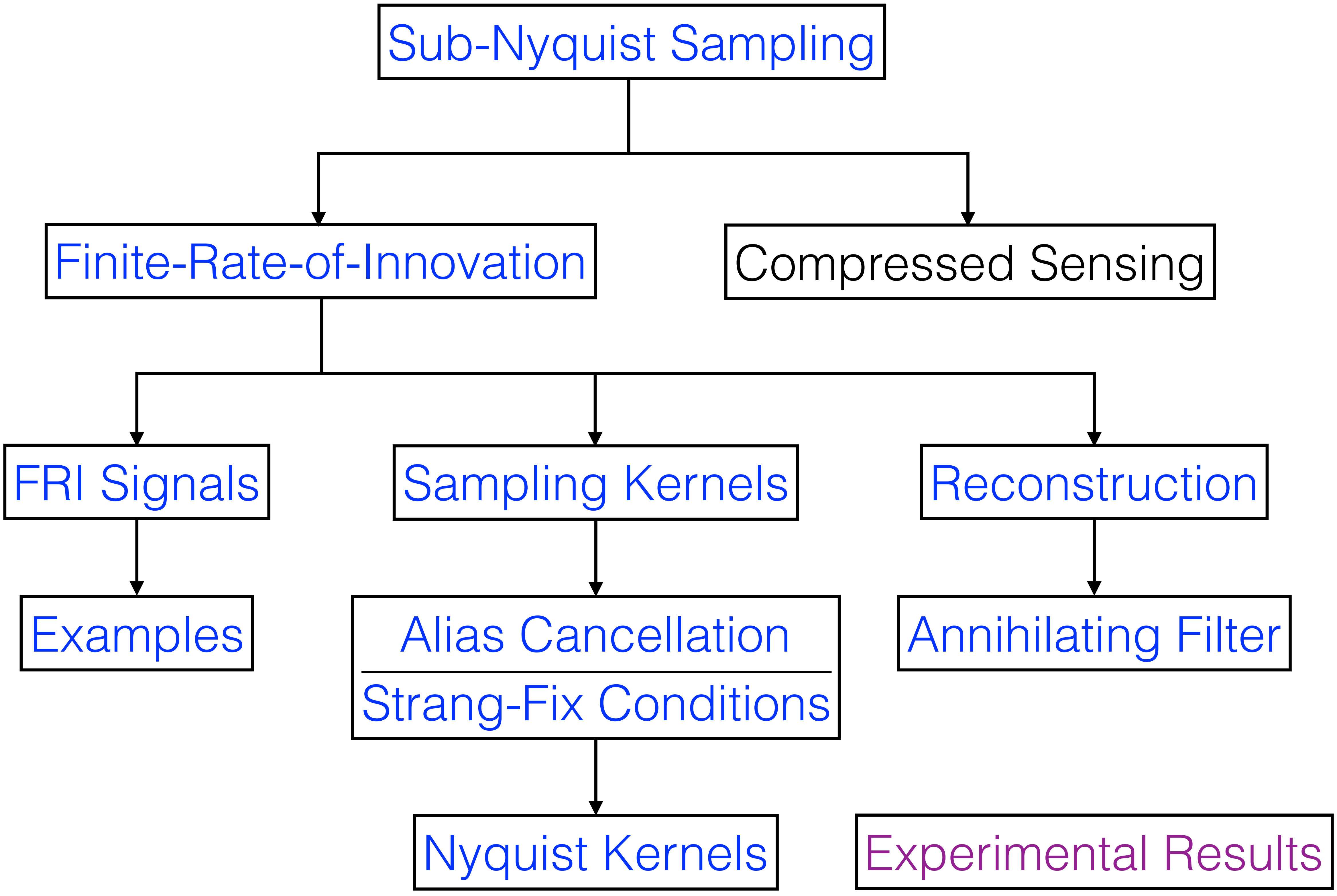

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Experimental Results

Band-Limited Sampling

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- Band-limited signals

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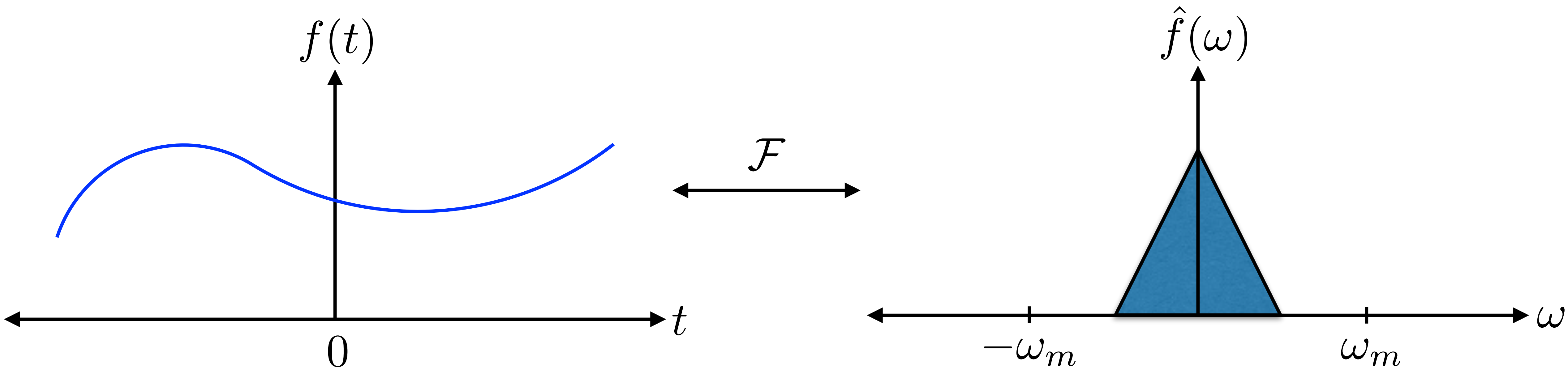
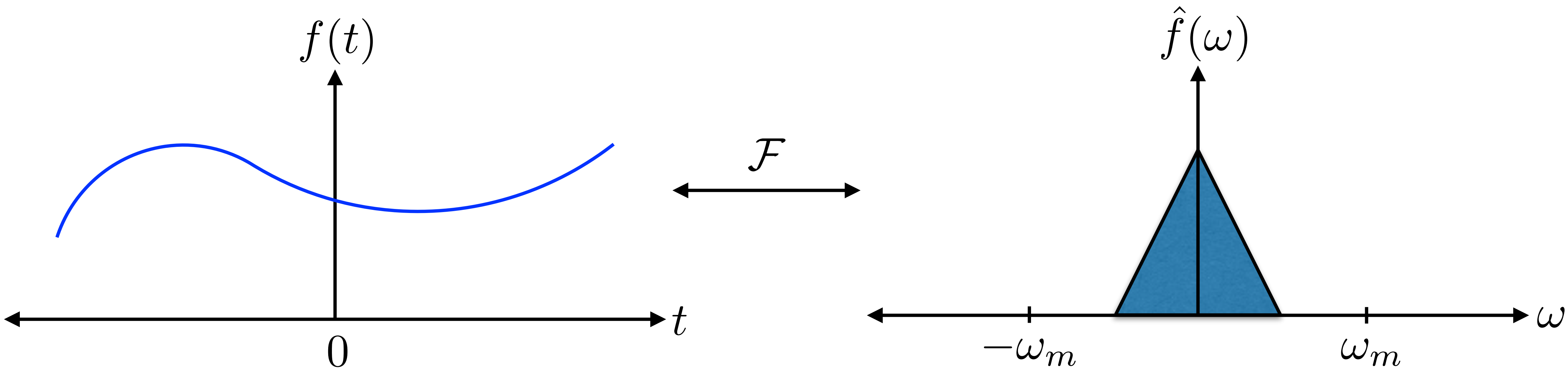


Band-Limited Sampling

- Band-limited signals



- **Shannon-Nyquist theorem:**

Band-Limited Sampling

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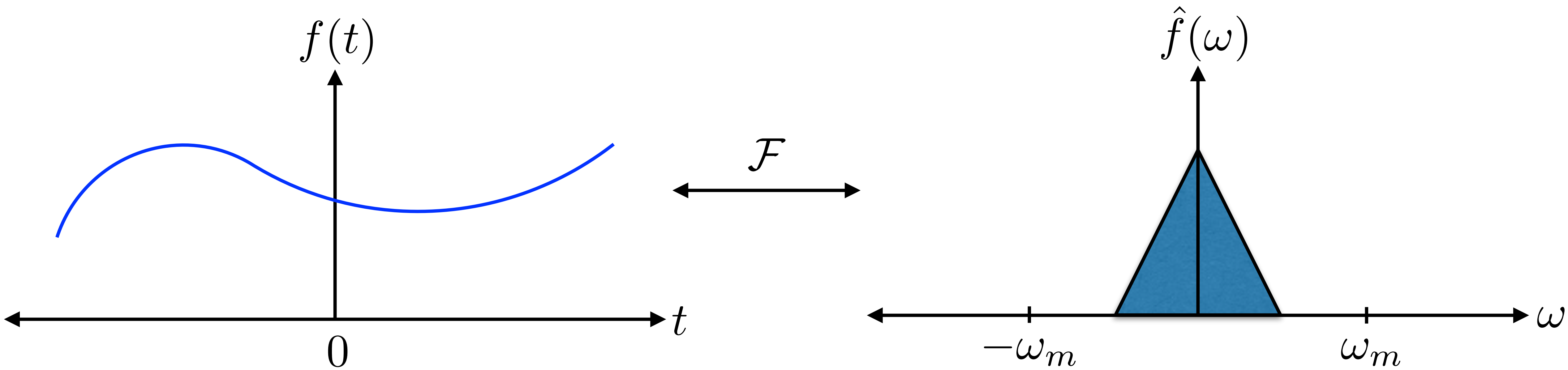

- **Shannon-Nyquist theorem:**

If $f(t)$ has no signal components beyond the frequency ω_m, then it can be determined using samples at intervals spaced not more than $T_s = 2\pi/2\omega_m$.

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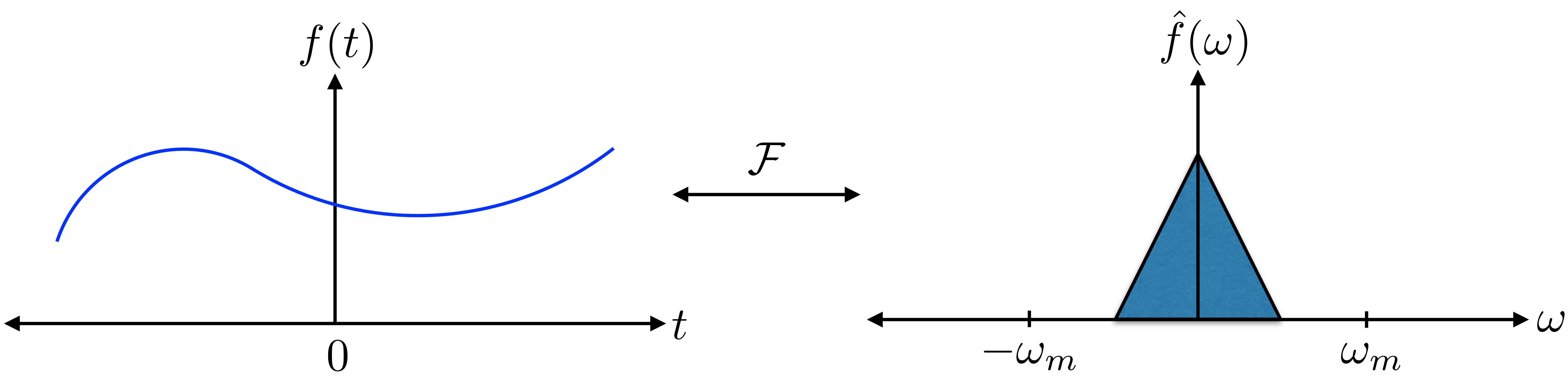

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- Reconstruction - sinc interpolation.

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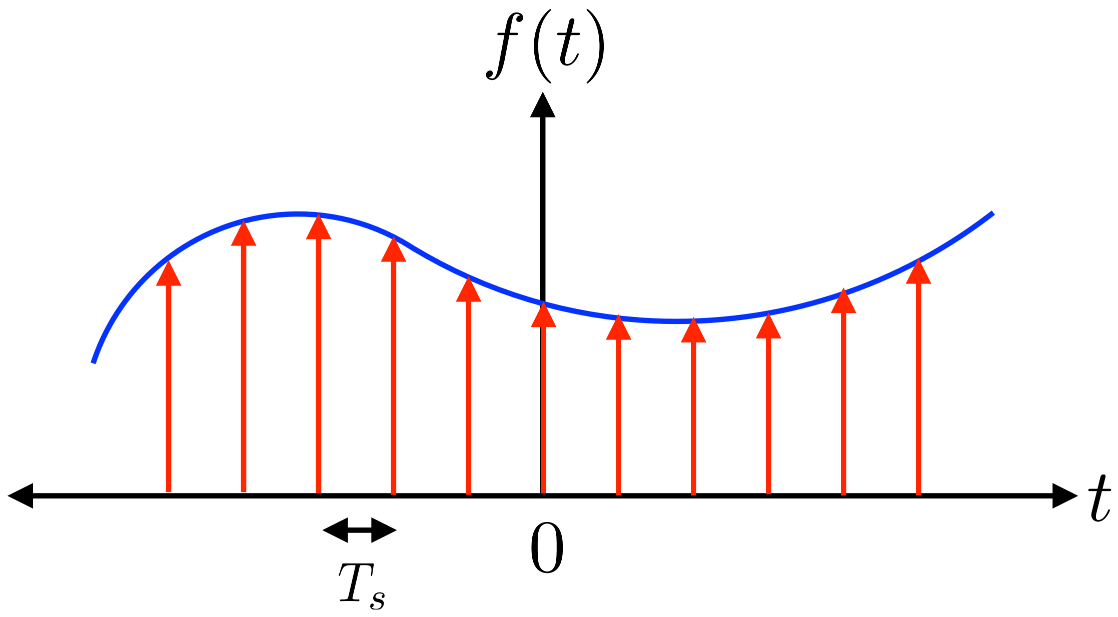

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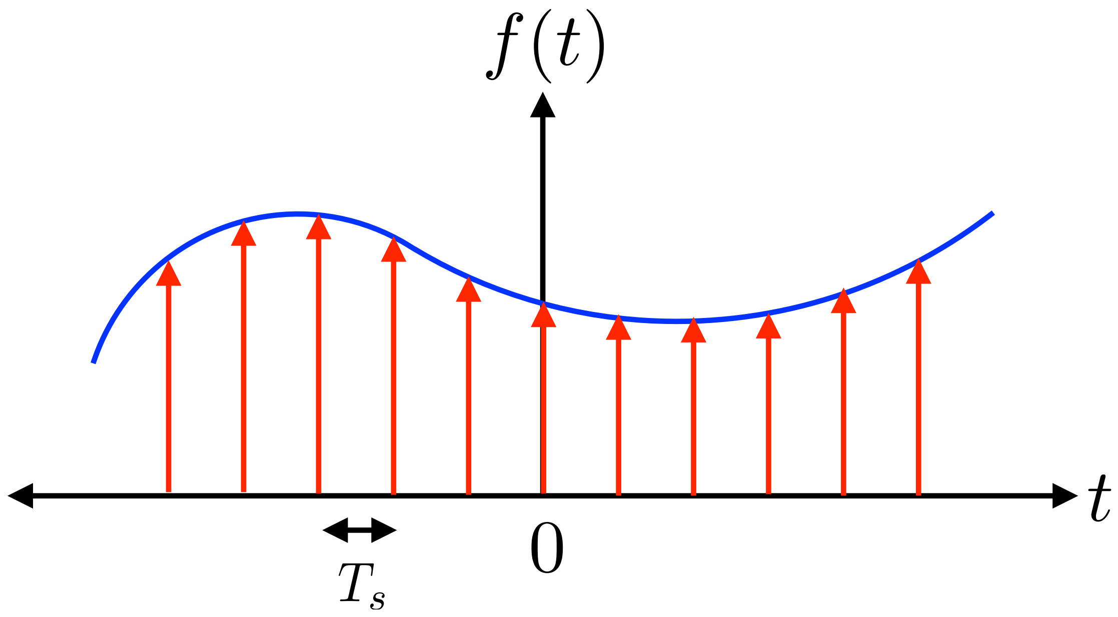

- Real world signals - Finite duration and not band-limited.

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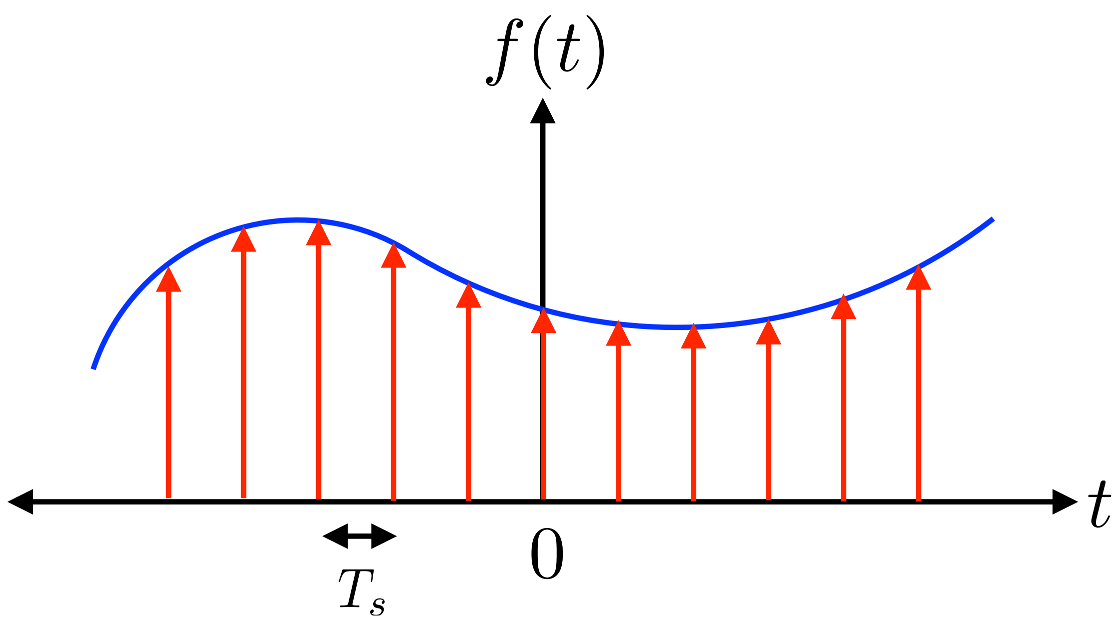

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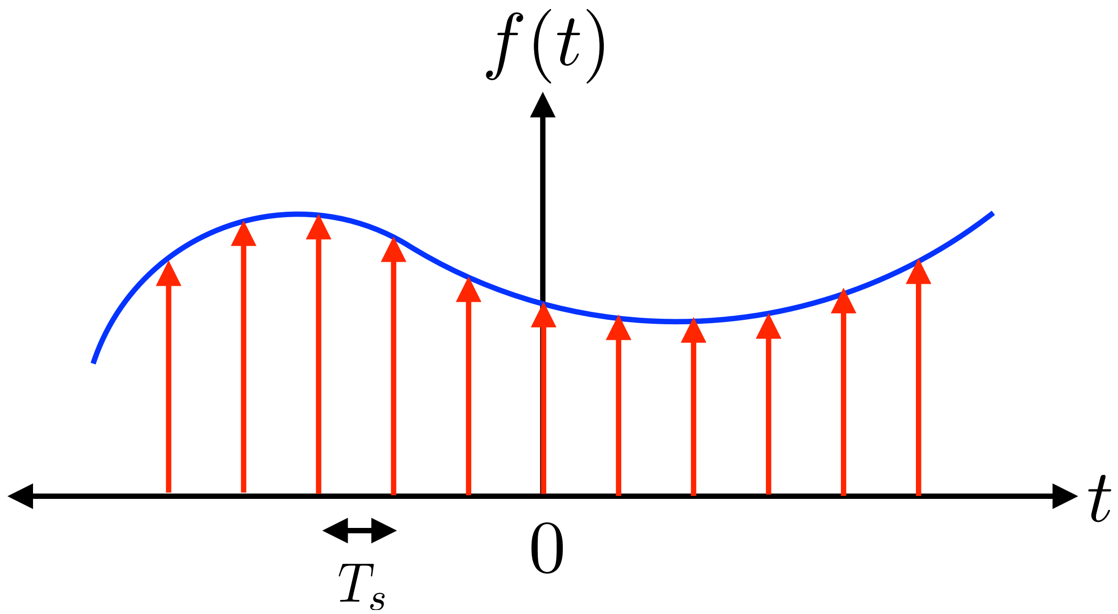

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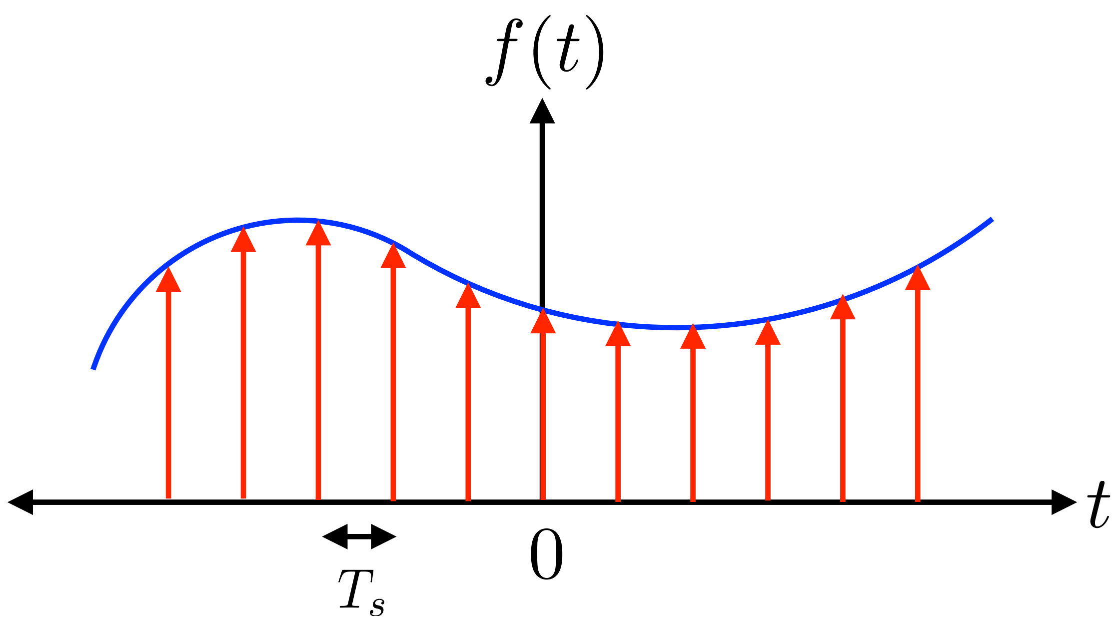

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- Is oversampling the best solution we have?

Going Sub-Nyquist - The FRI model

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$$\boxed{f(t) = \sum_{\ell=1}^{L} a_\ell g(t - t_\ell)}$$

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- Number of parameters = 2L (L amplitudes and L time-delays).

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Some Examples of FRI Signals

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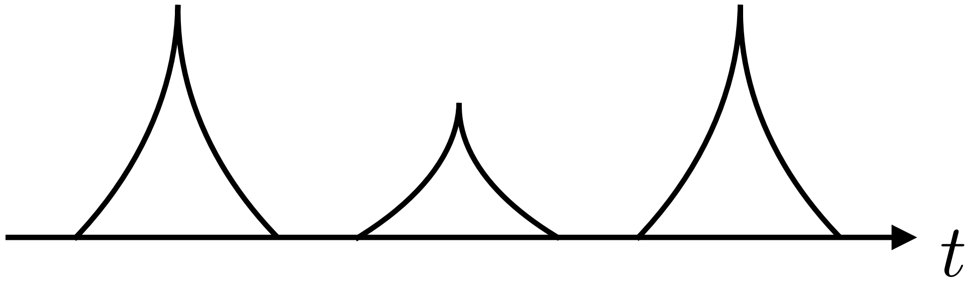

Stream of pulses

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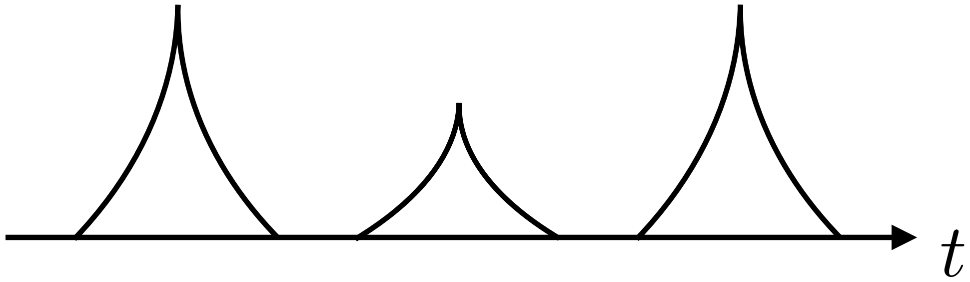

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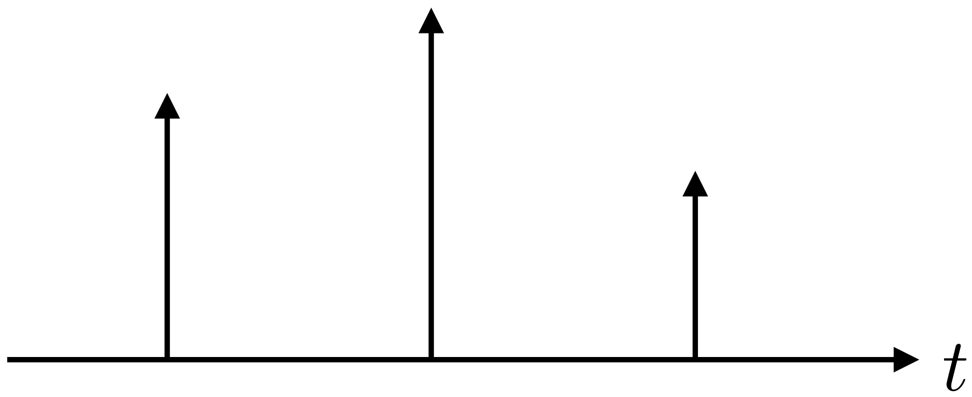

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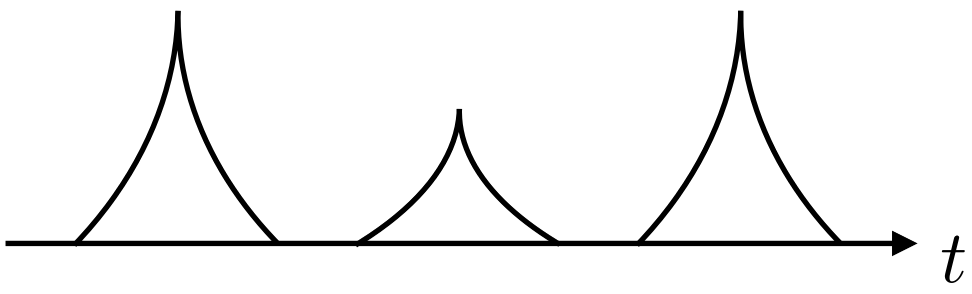

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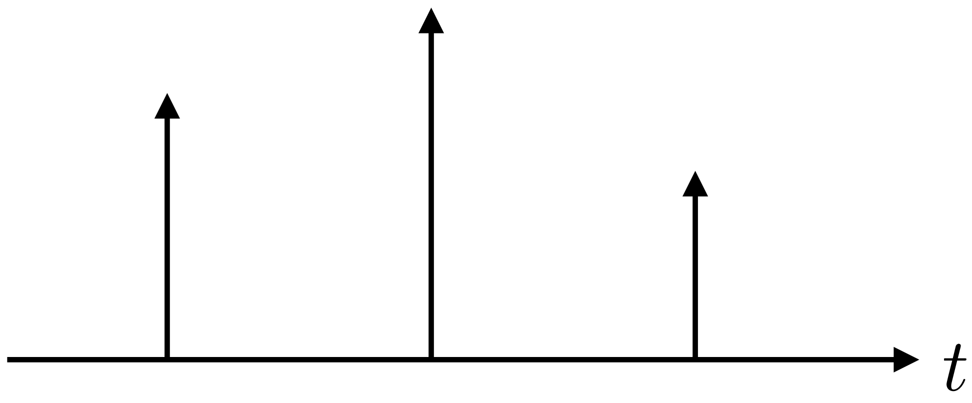

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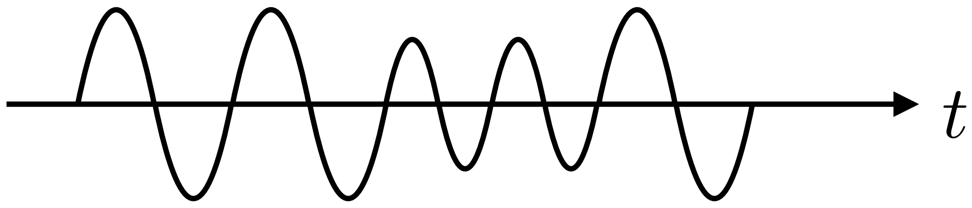

Piecewise sinusoidal

FRI Sampling Scheme

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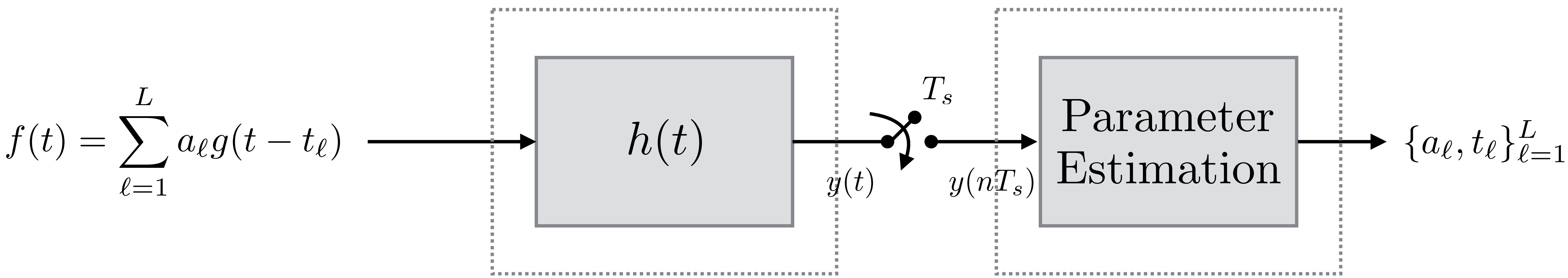

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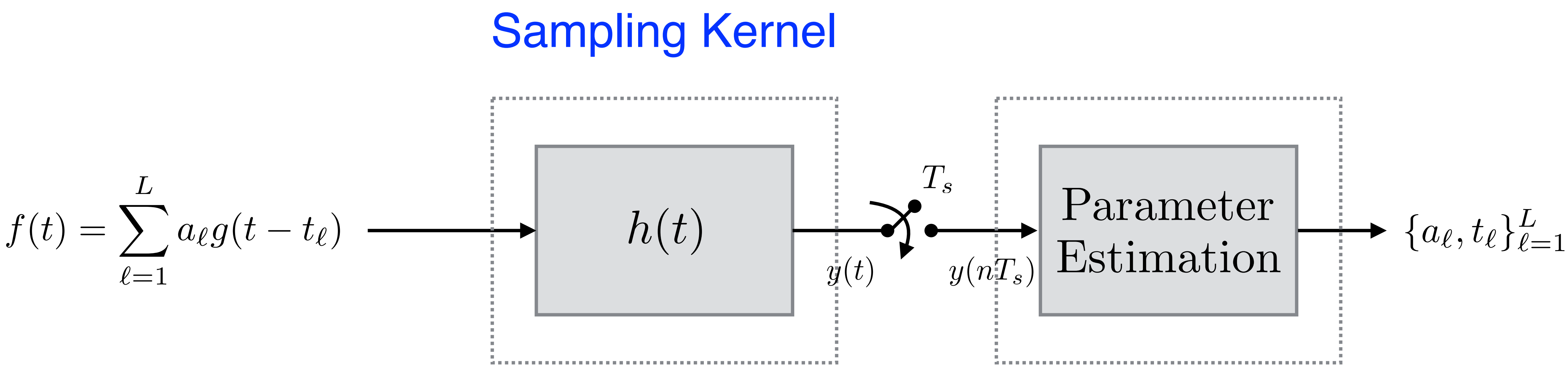

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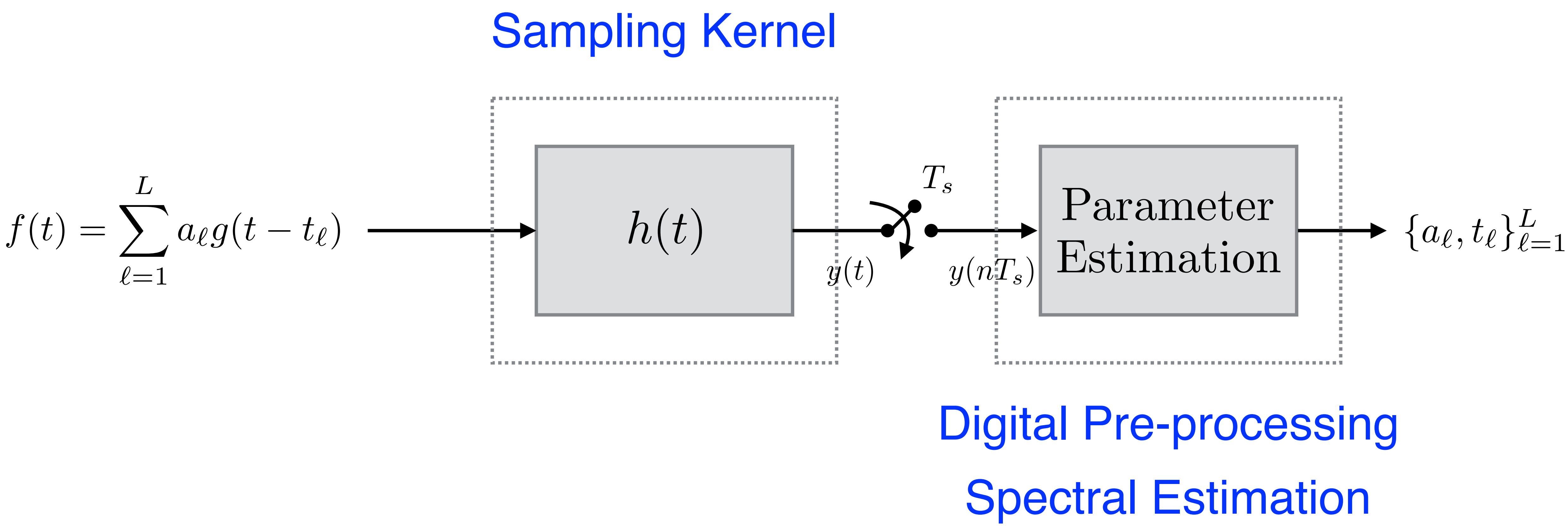

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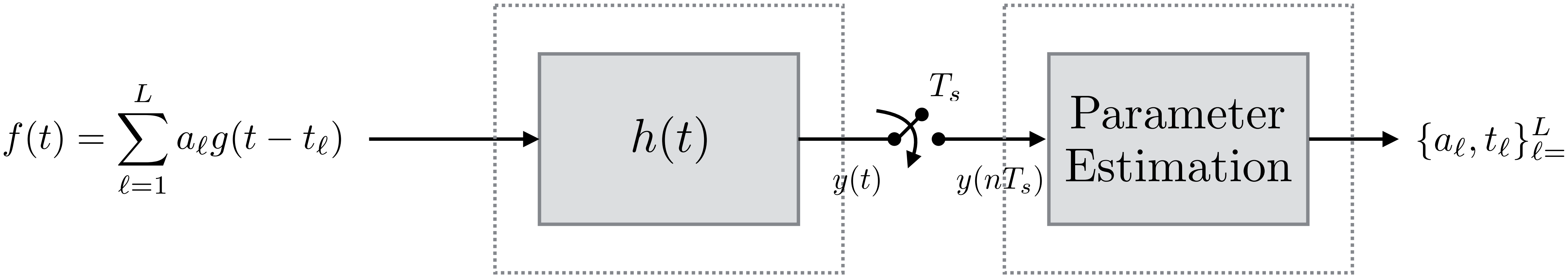

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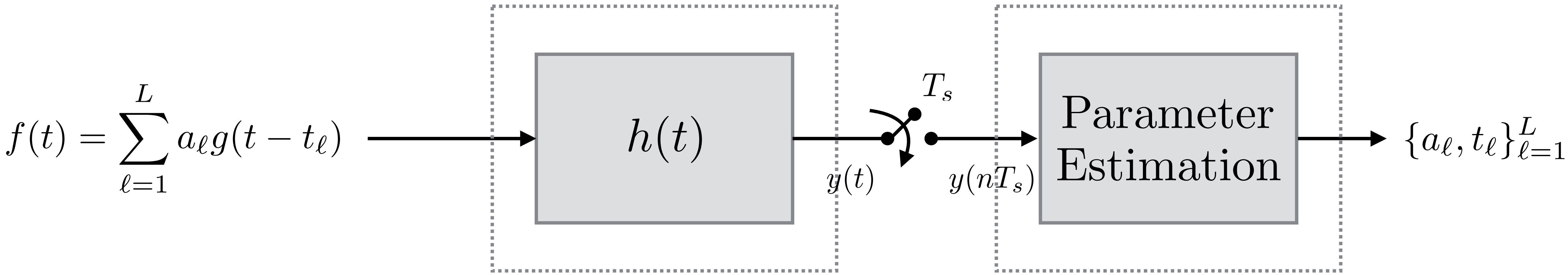
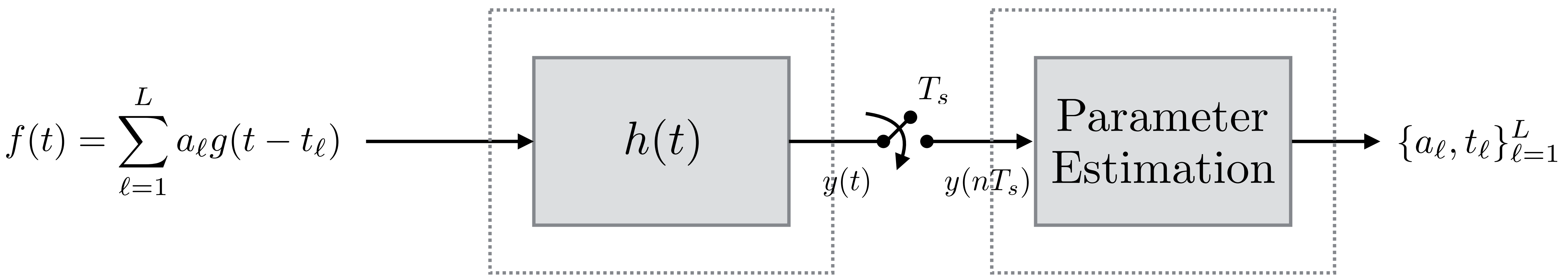


- Sampling Kernel - Pre-filter before sampling

FRI Sampling Scheme



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$$y(t) = f(t) * h(t)$$

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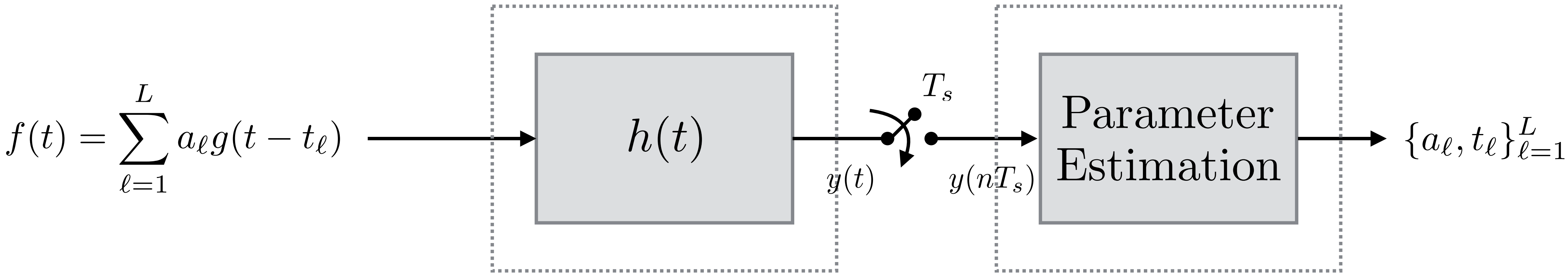

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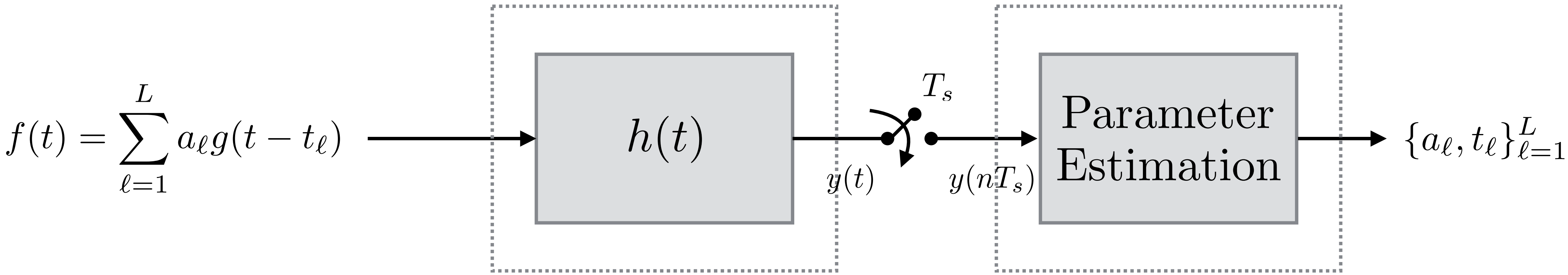

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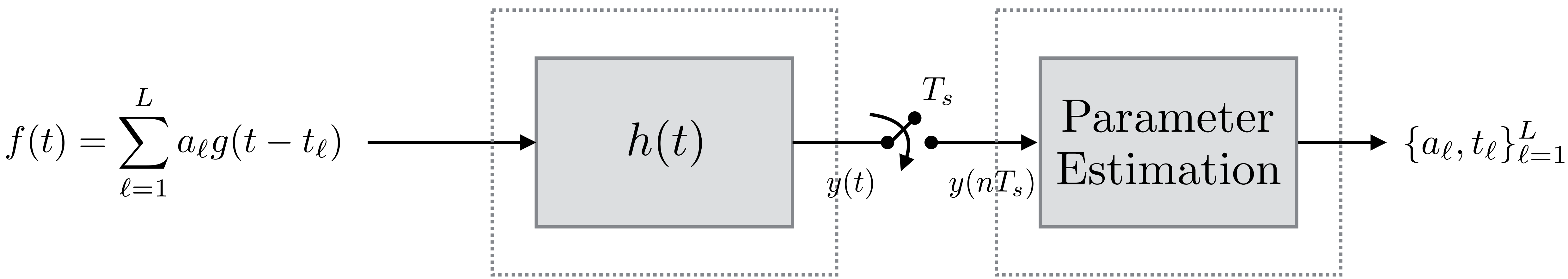

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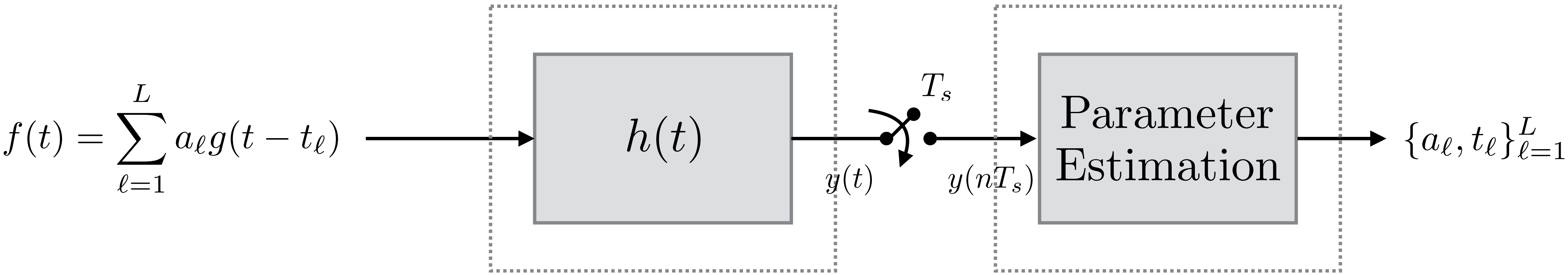

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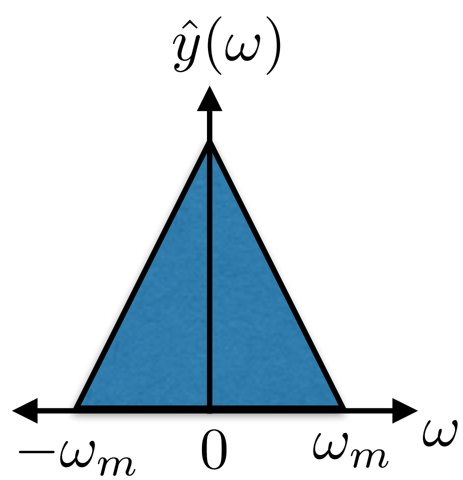

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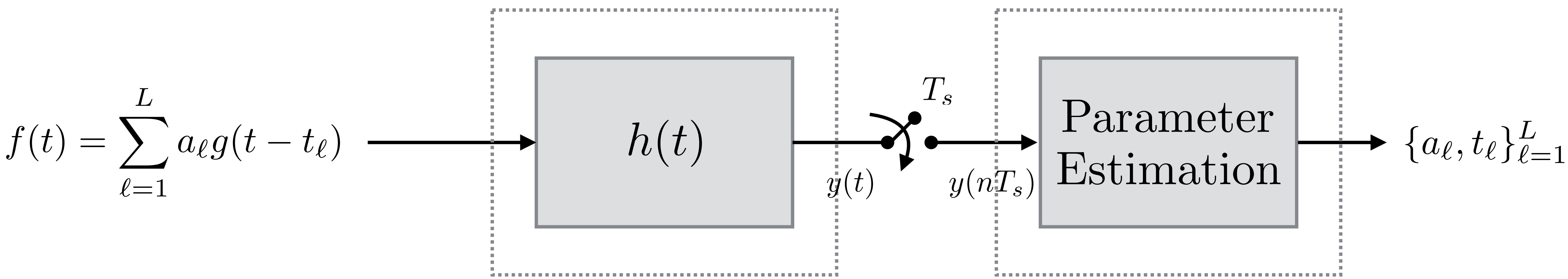

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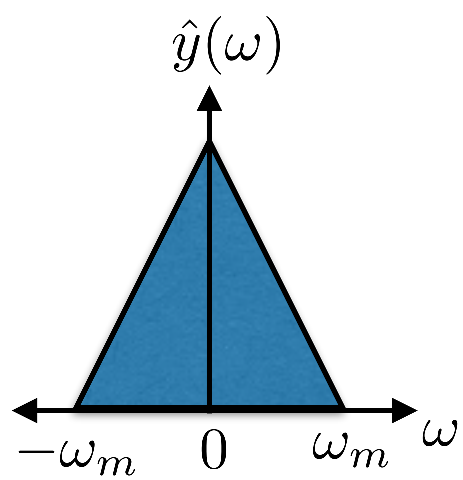

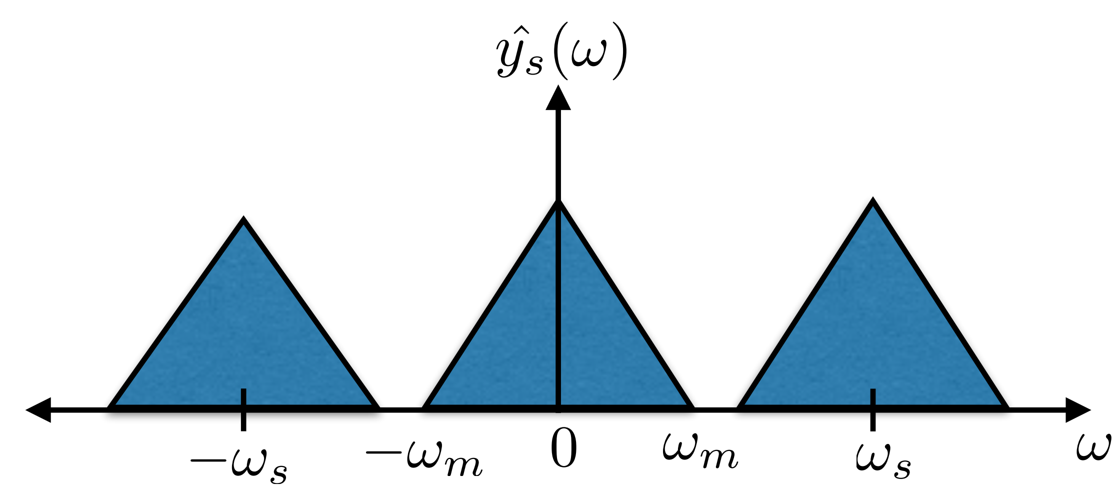

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Alias-Cancelling Conditions

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- Ideal sampling kernel - Sum of Sincs (SoS)

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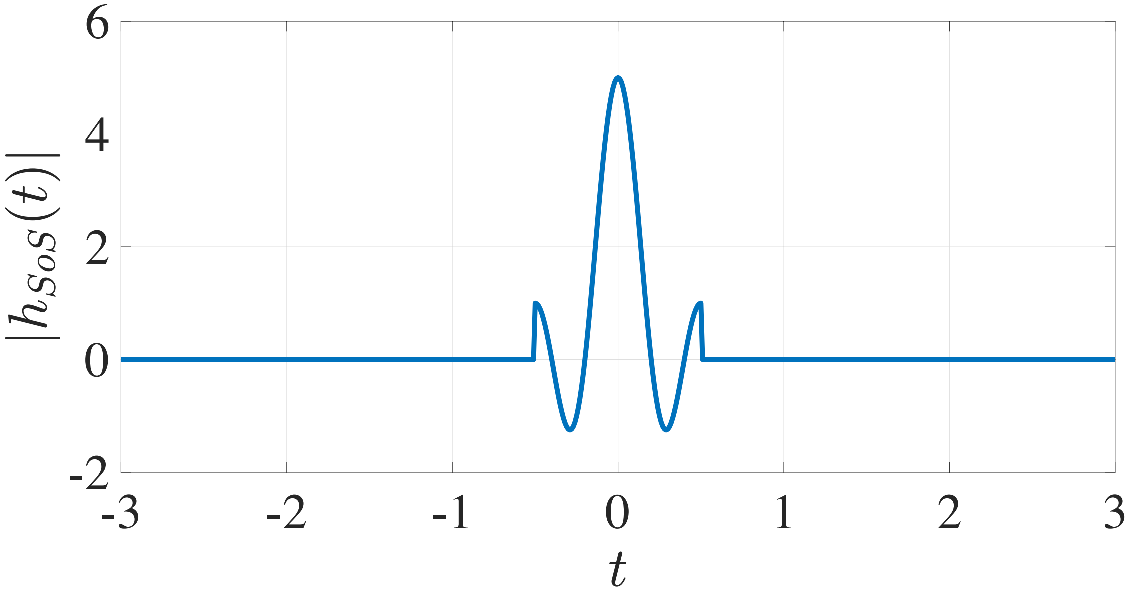
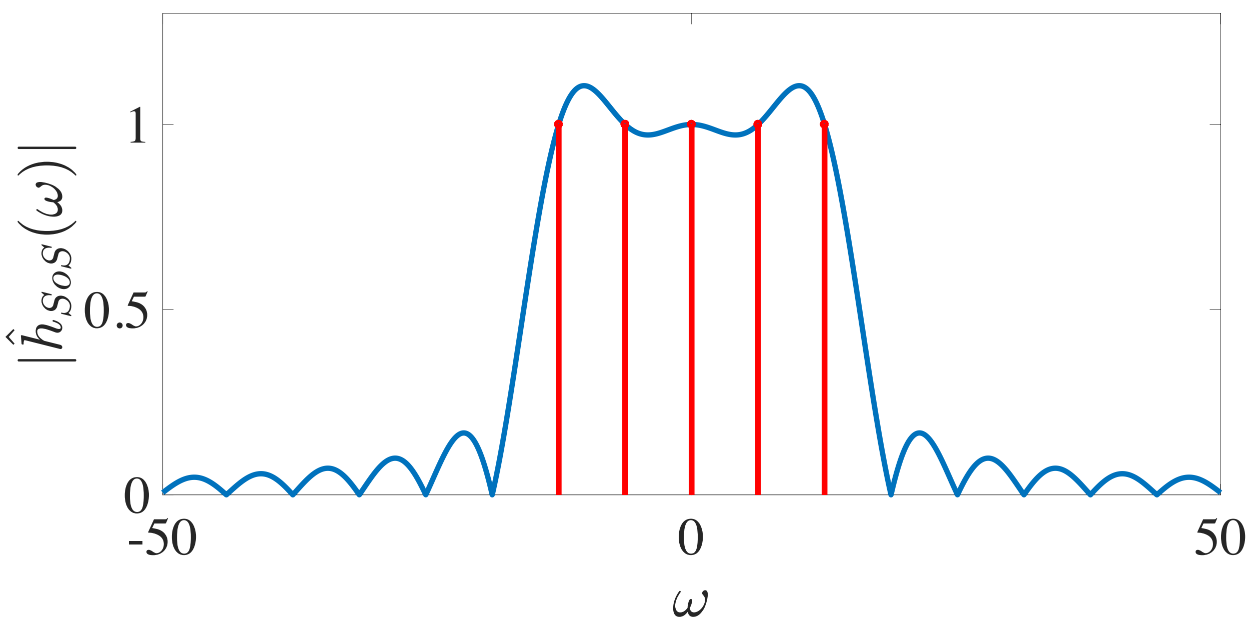


Frequency response (Magnitude)



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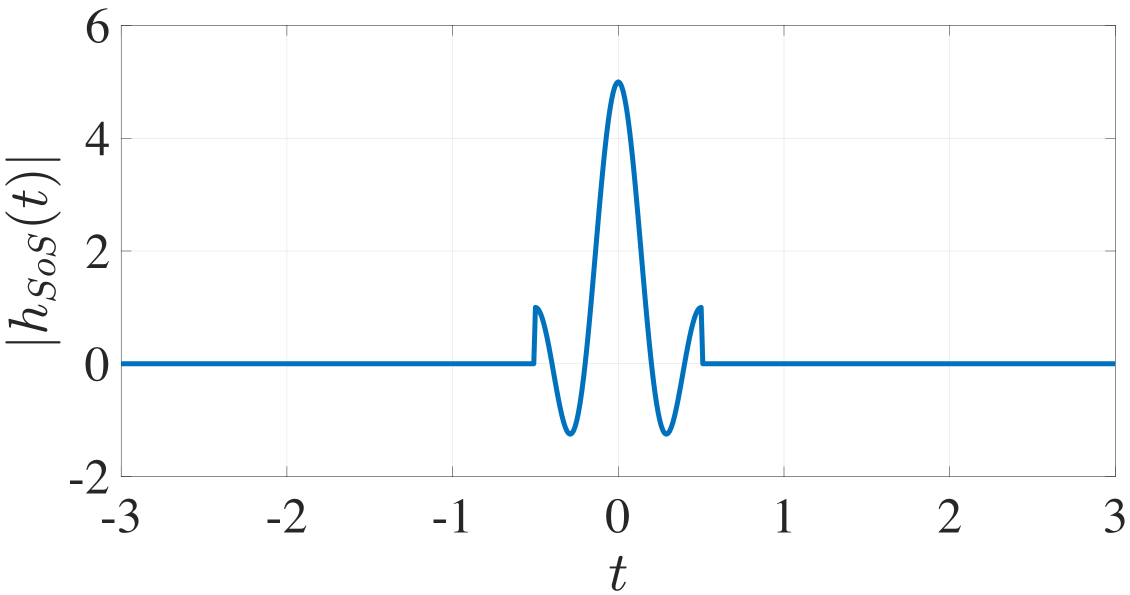

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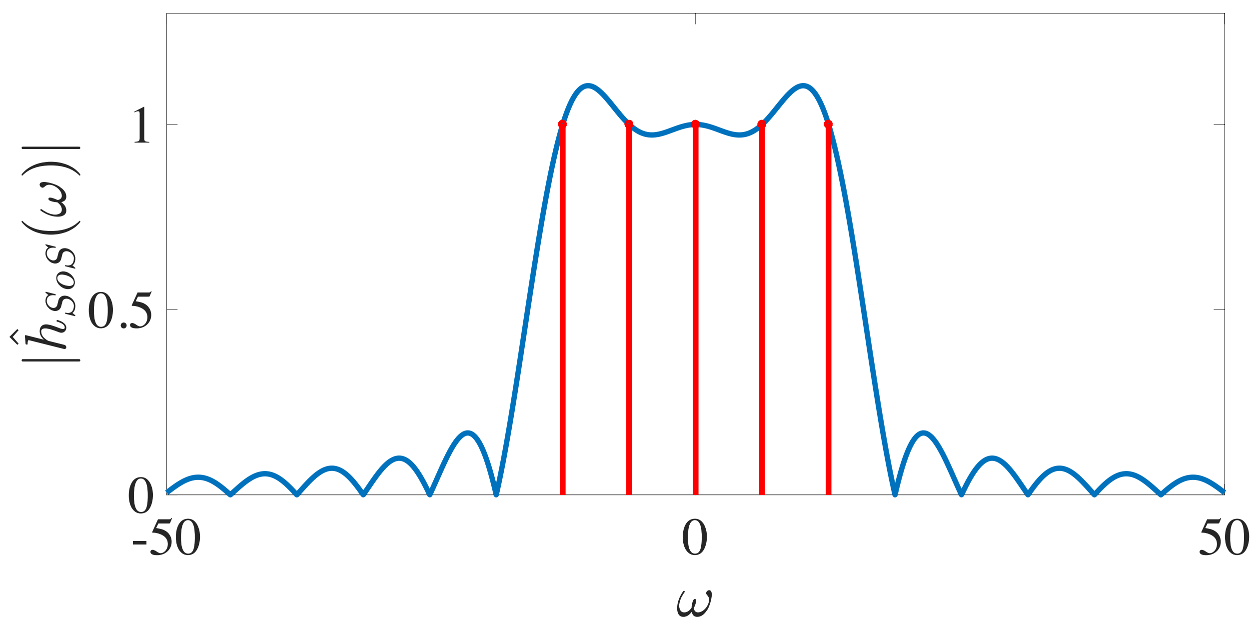

L = 2, $\mathcal{K}$ = {-2,-1,0,1,2}

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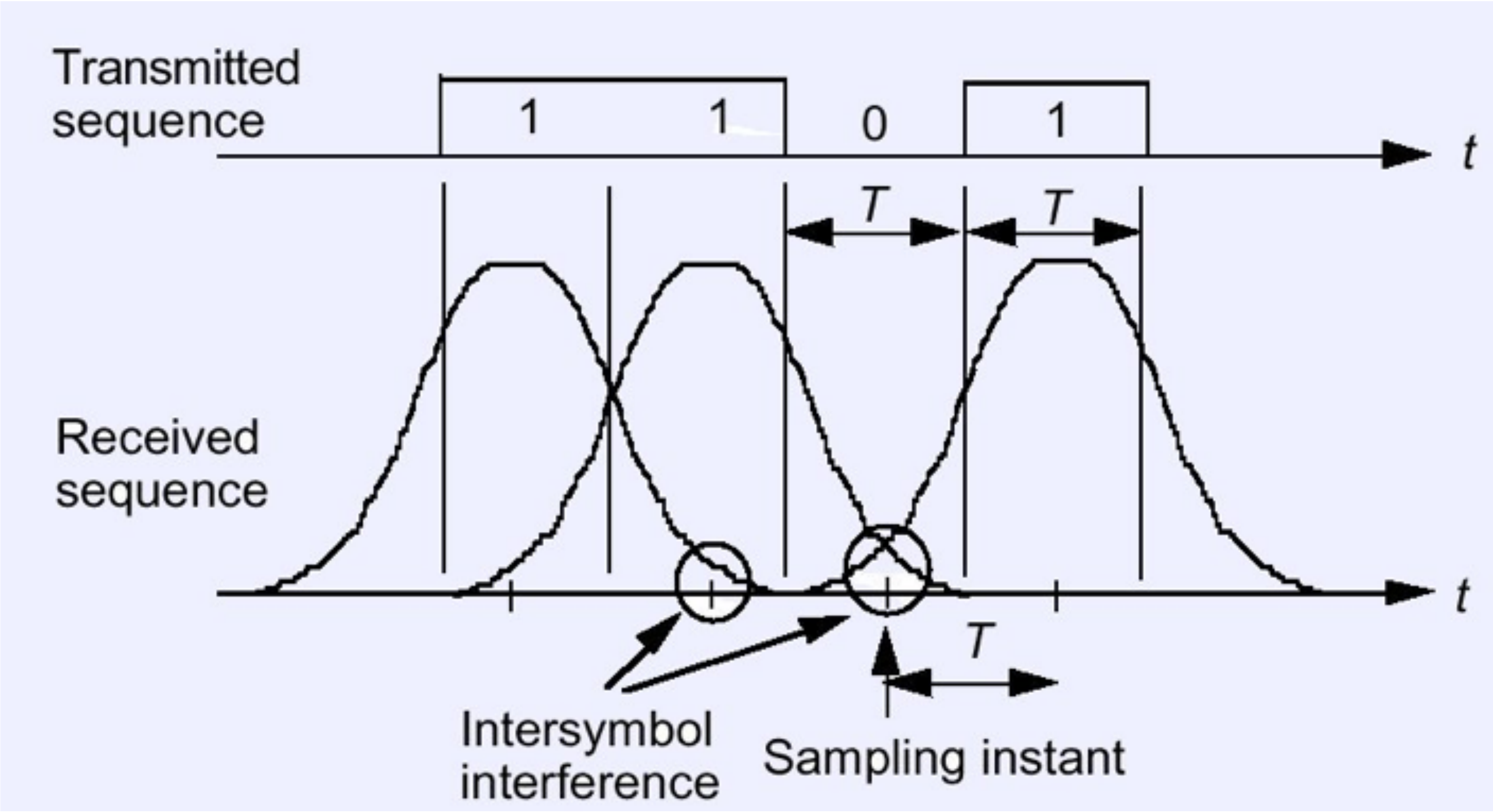

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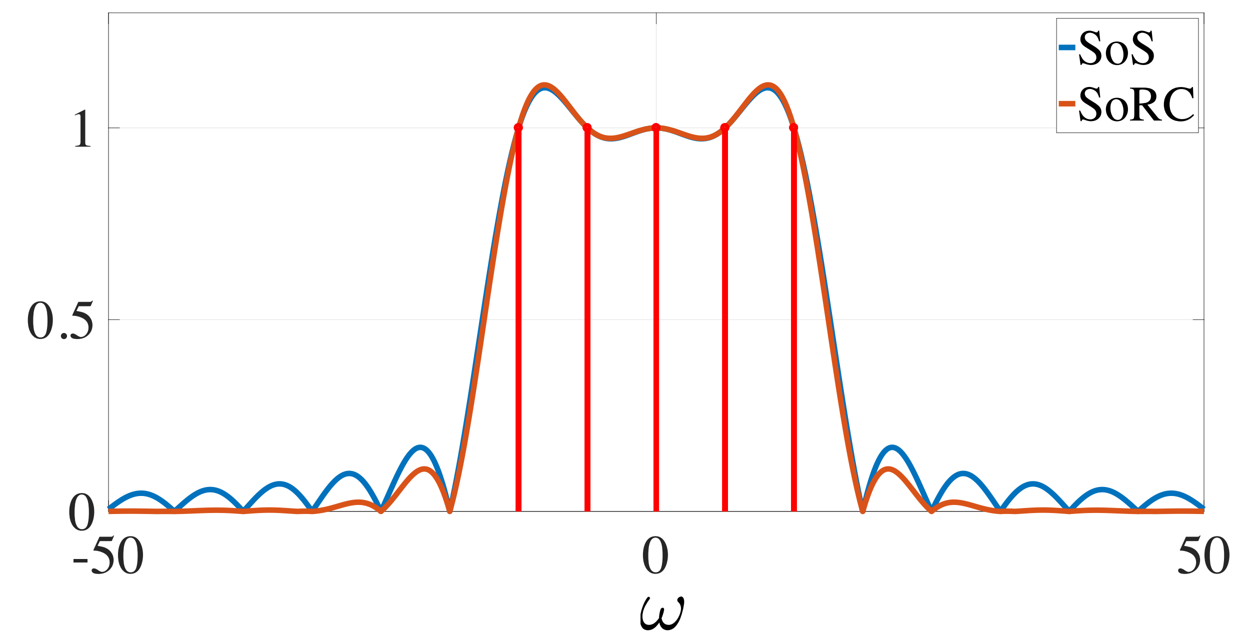

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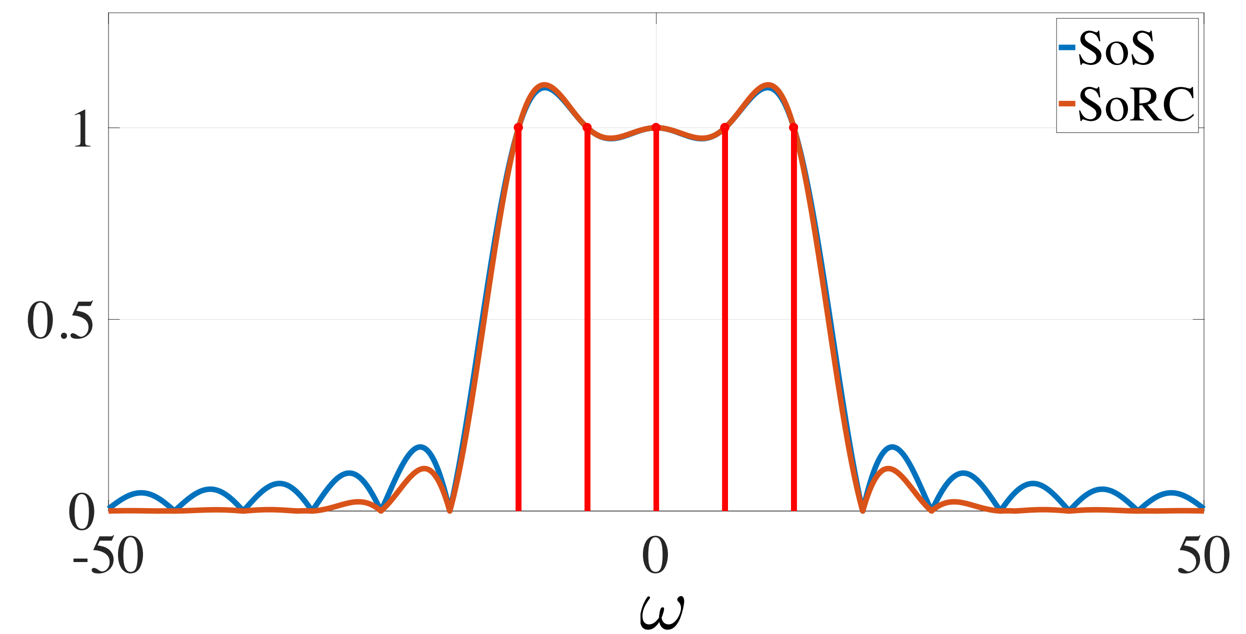

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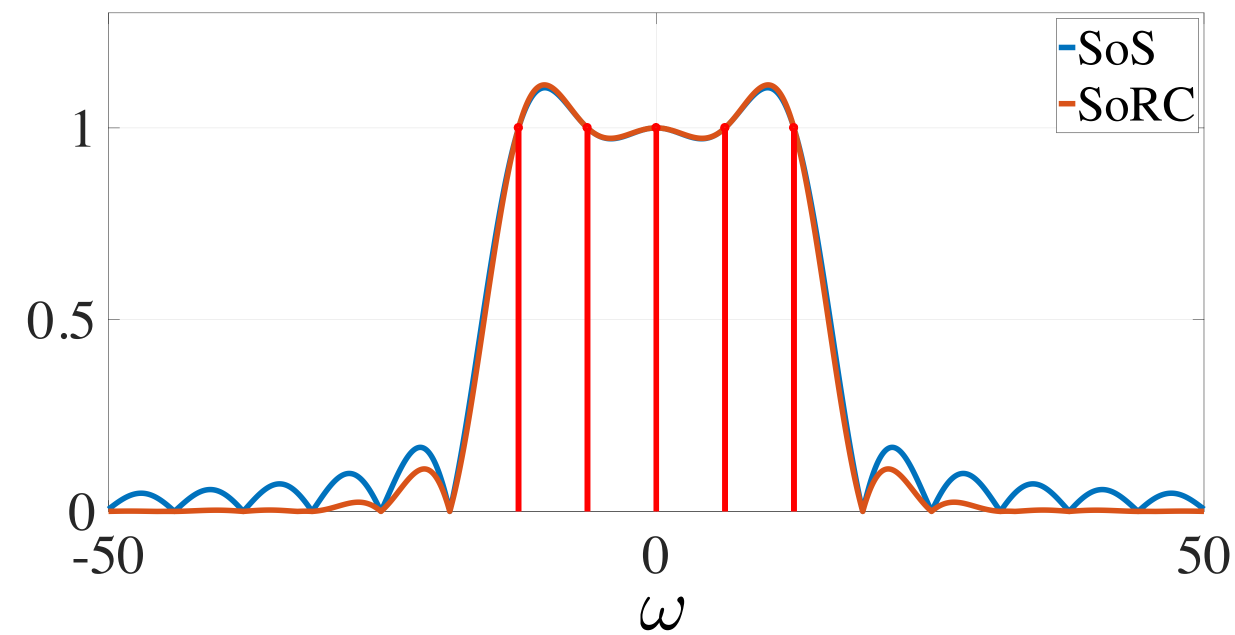

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$$\hat{h}_{RC}(\omega) = \sum_{k \in \mathcal{K}} \frac{\operatorname{sinc}\left(\frac{\omega}{\omega_0} - k\right) \cos\left[\pi\xi\left(\frac{\omega}{\omega_0} - k\right)\right]}{1 - \left[2\xi\left(\frac{\omega}{\omega_0} - k\right)\right]^2}$$

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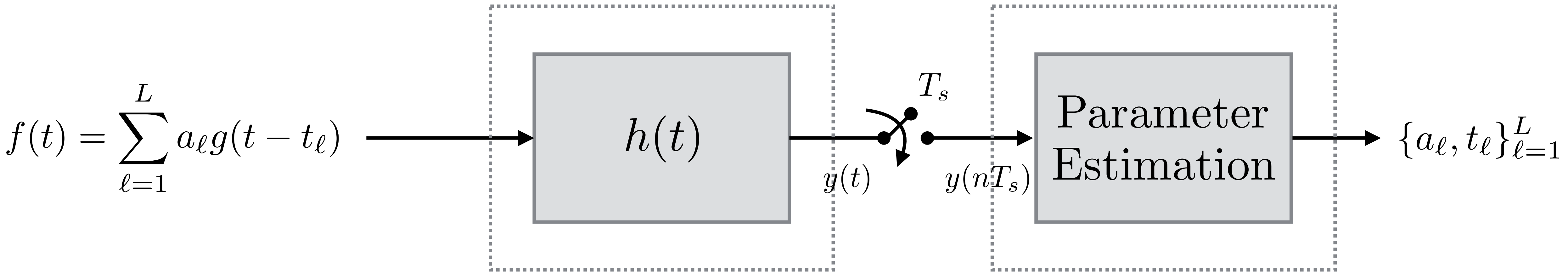

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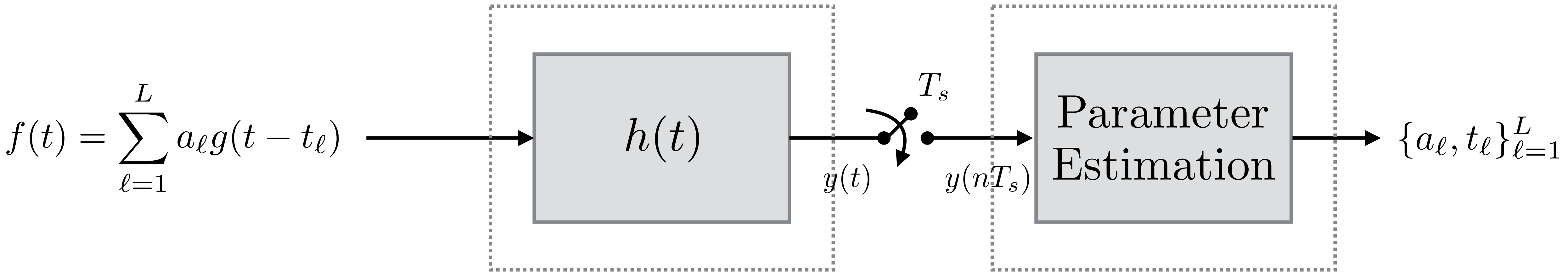

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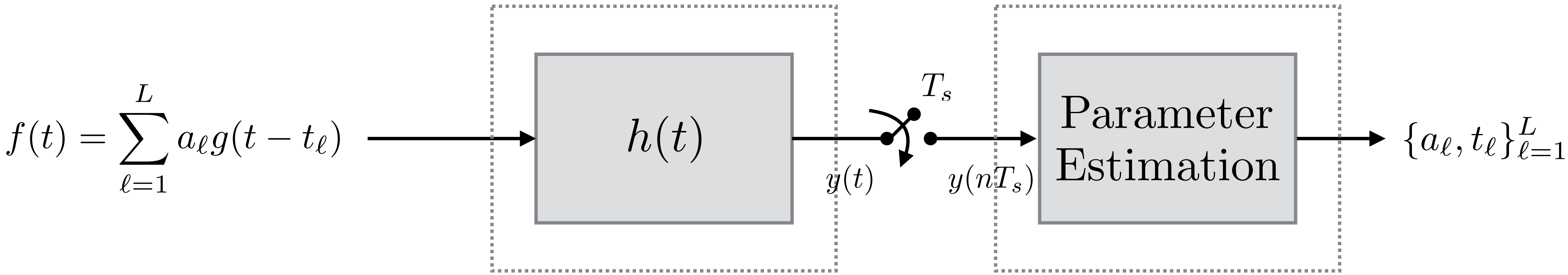

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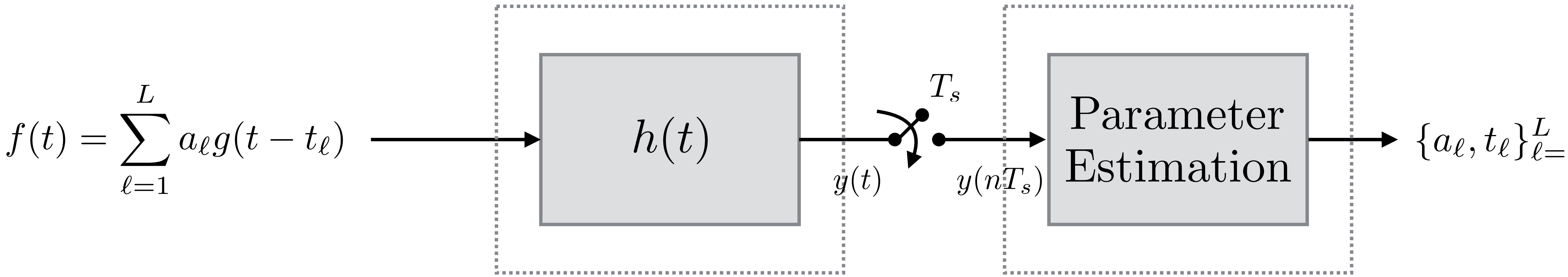

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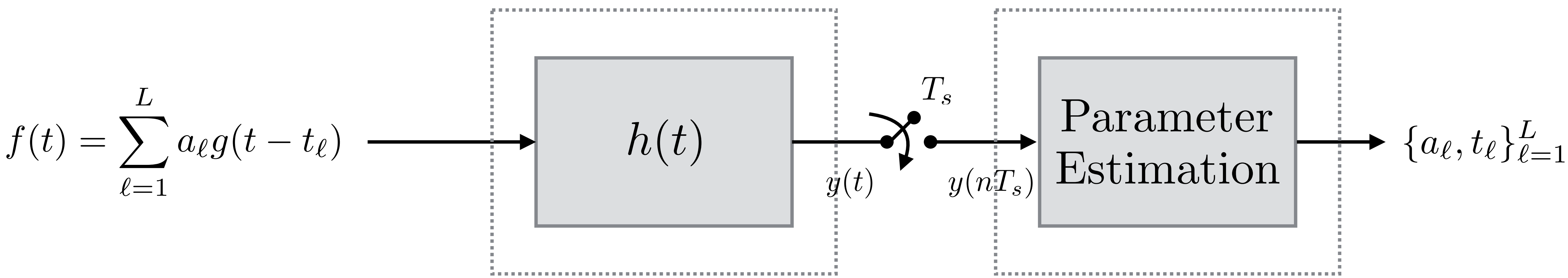

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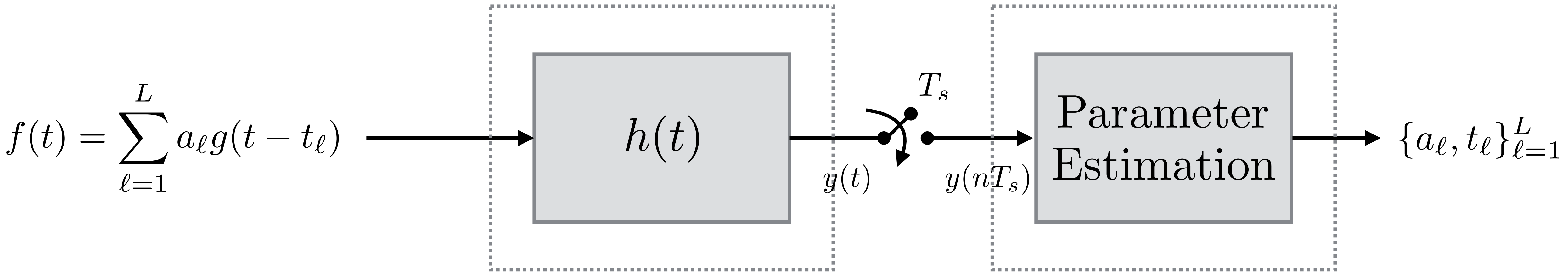

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- Solved using Prony's Annihilating Filter, ESPRIT, MUSIC, etc.

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- If N values of *s* are available, and we choose N > 2K, a solution will exist when rank of **S** = K.

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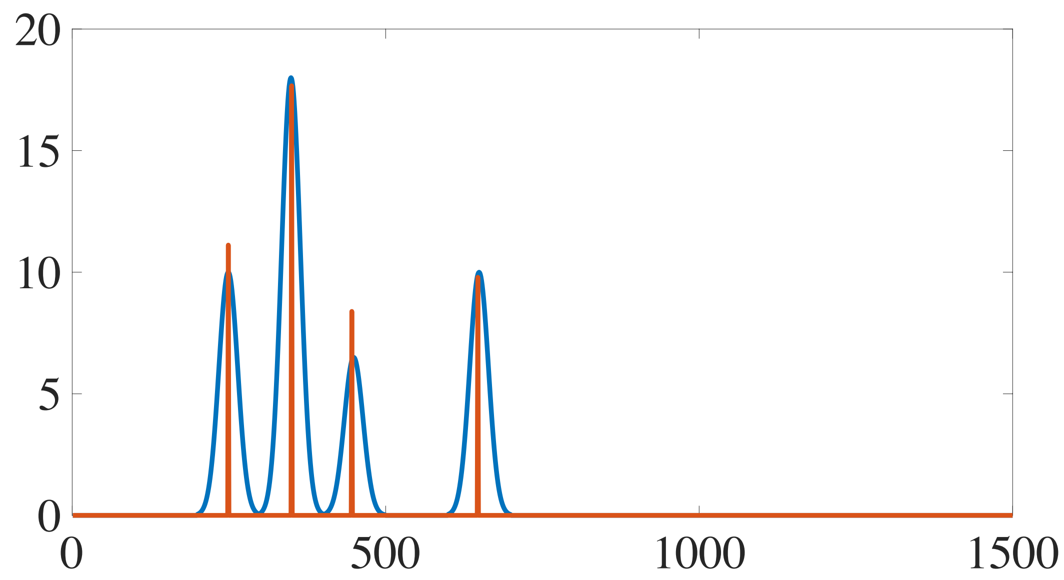

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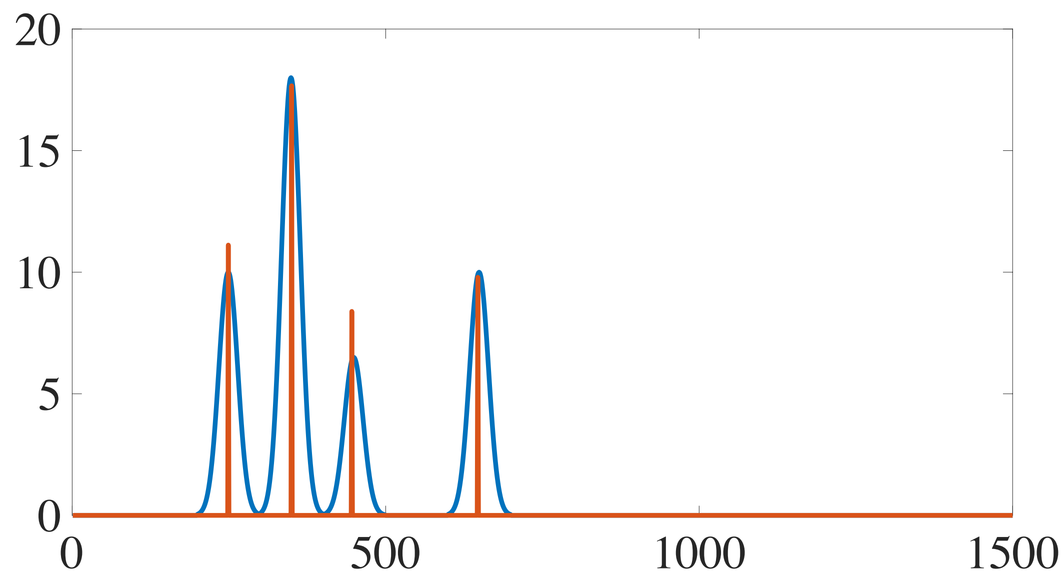
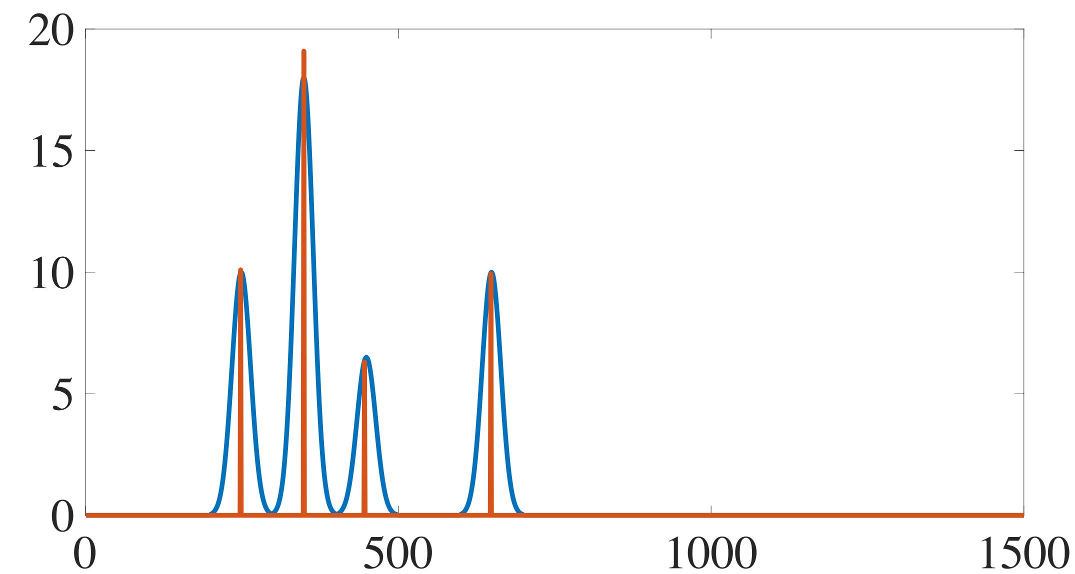


10 dB SNR



Underwater Imaging Set-up

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- Each of these reconstructions are used to form an image of the target after a delay-sum beam forming.

Underwater Imaging - Formulation

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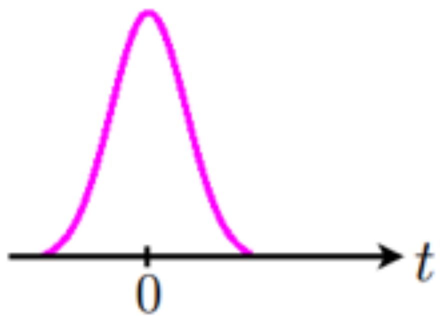

Transmitted pulse

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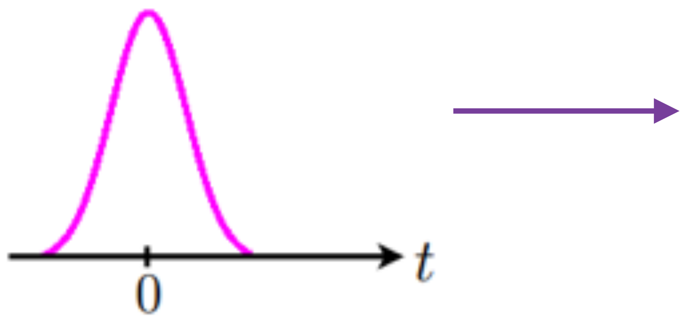

Transmitted pulse

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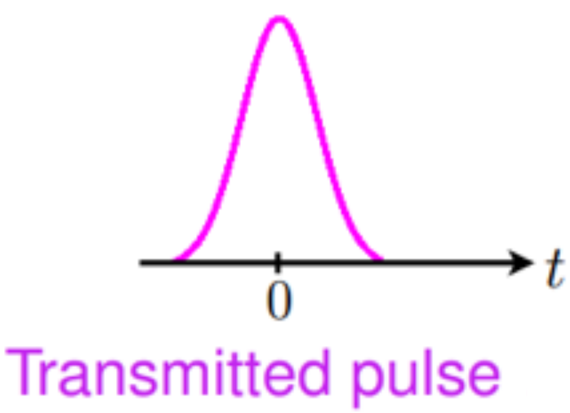

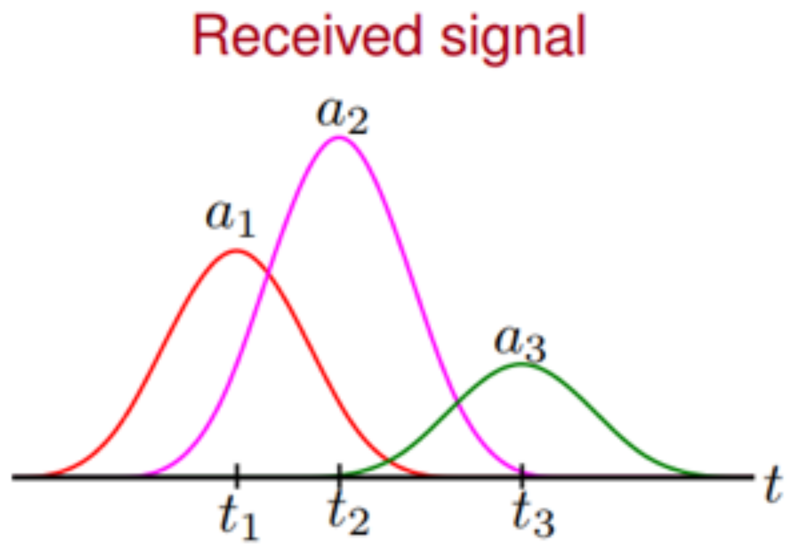

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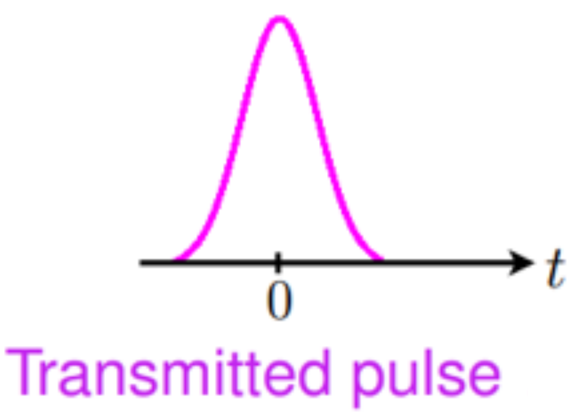

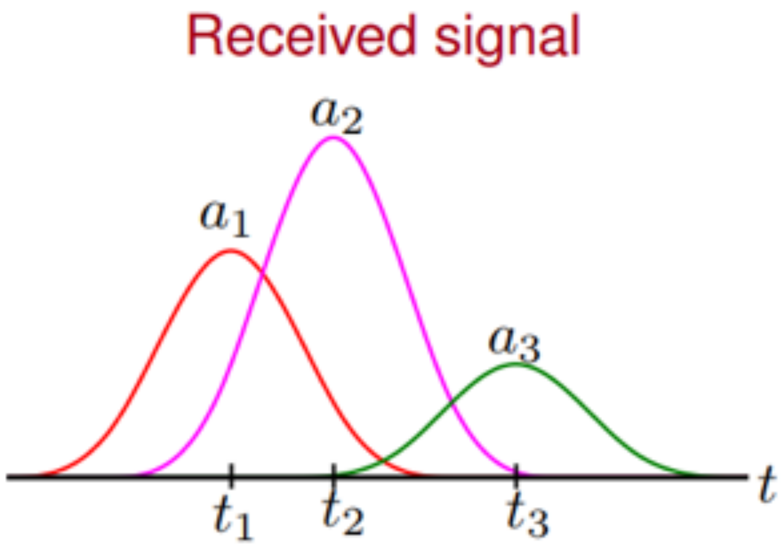

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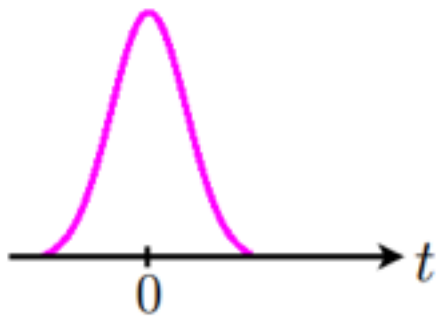
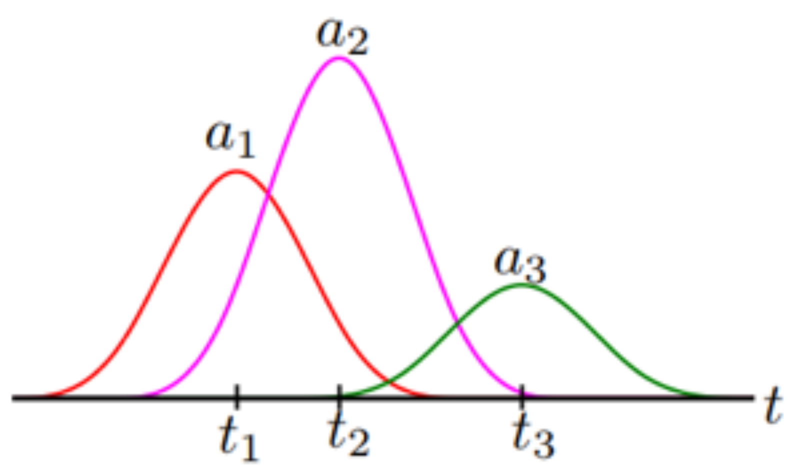


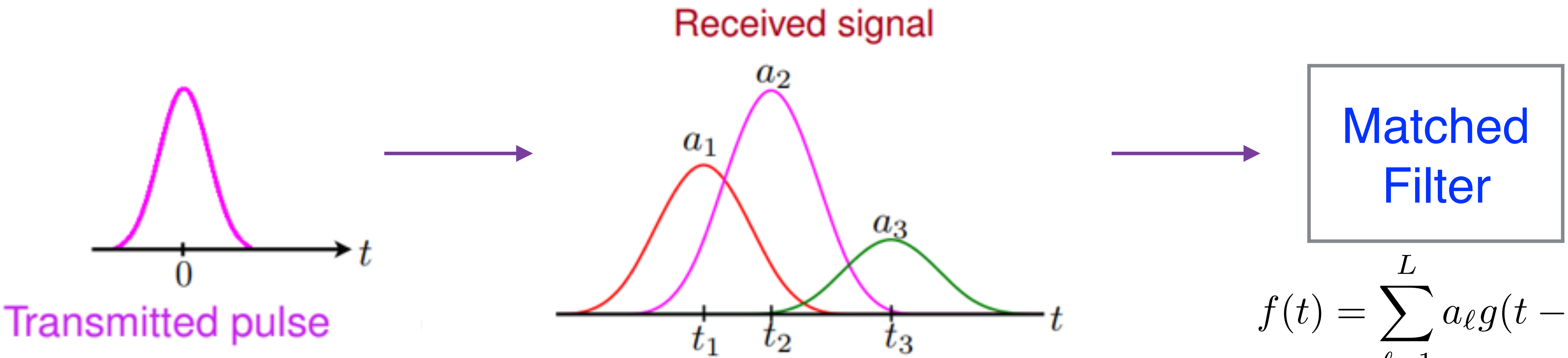
Transmitted pulse

Received signal



Matched Filter

Underwater Imaging - Formulation



$$f(t) = \sum_{\ell=1}^{L} a_\ell g(t - t_l)$$

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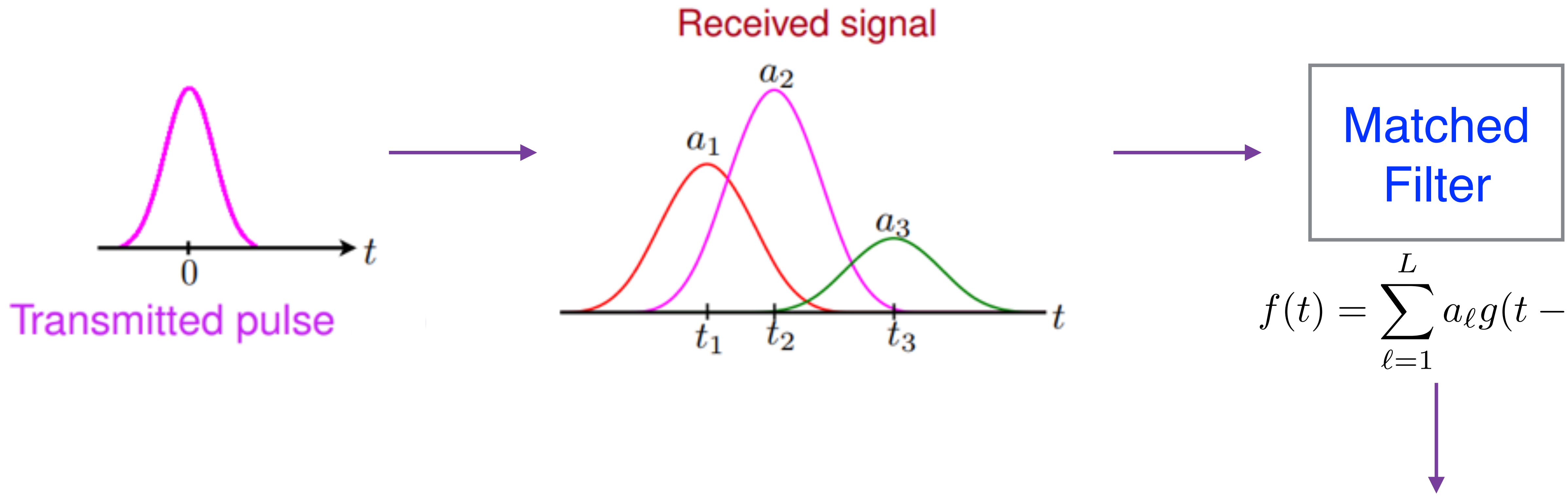

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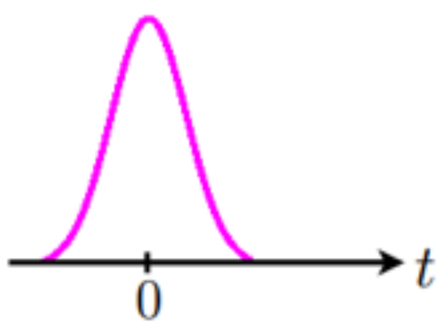

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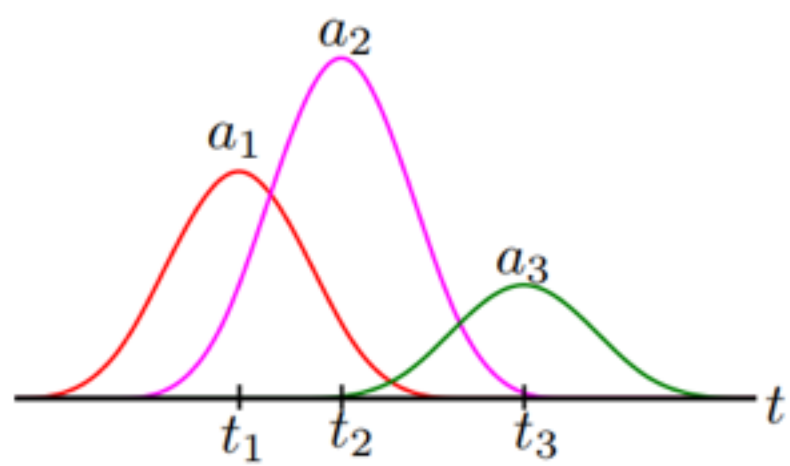

Matched Filter

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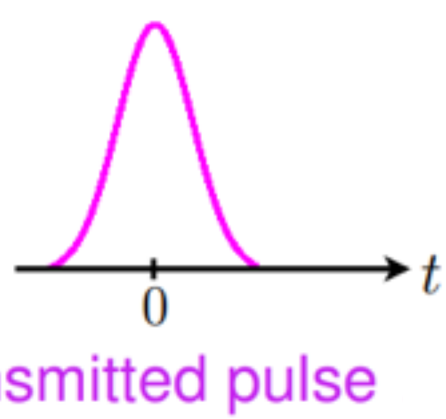
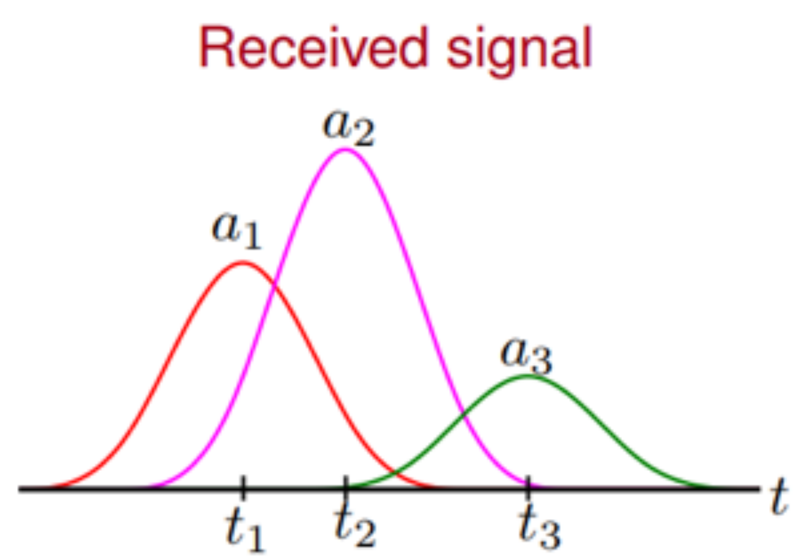
Compute Approximate CTFT Values

Underwater Imaging - Formulation





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$$\approx \hat{f}(k\omega_0)$$

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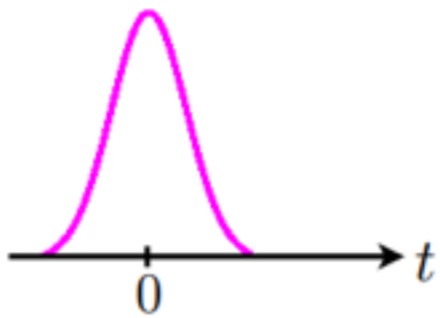
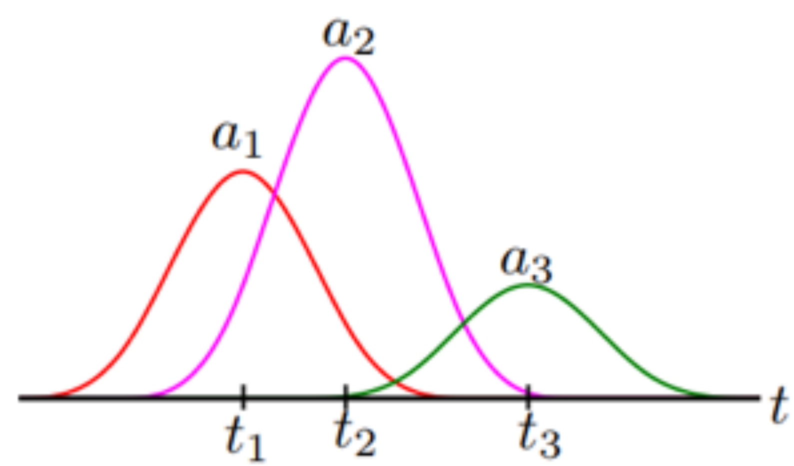


Transmitted pulse

Received signal



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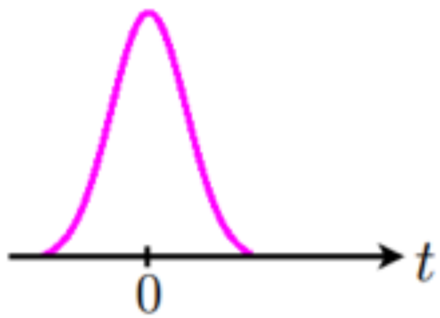

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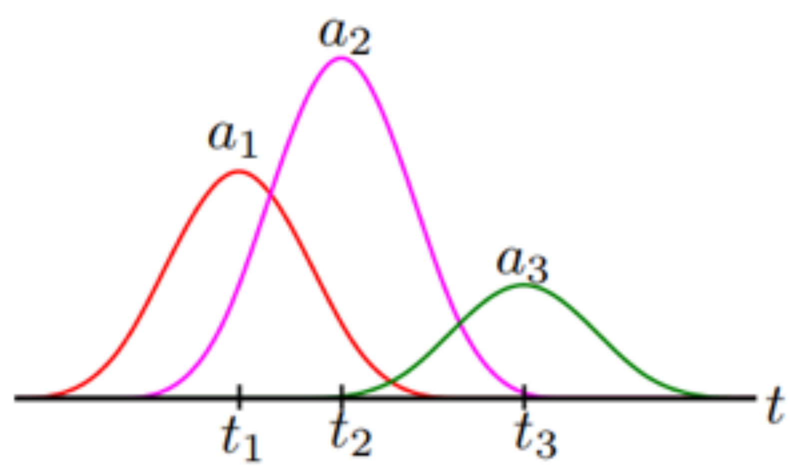

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Estimate parameters

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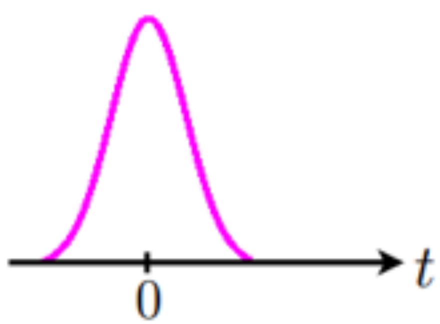
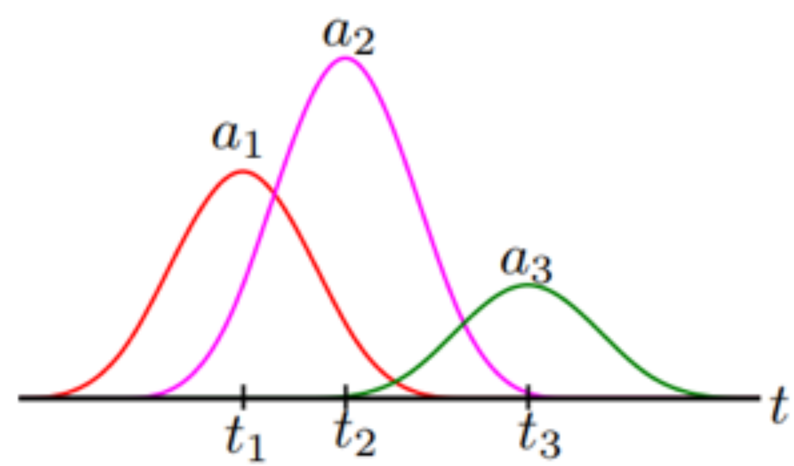


Transmitted pulse

Received signal



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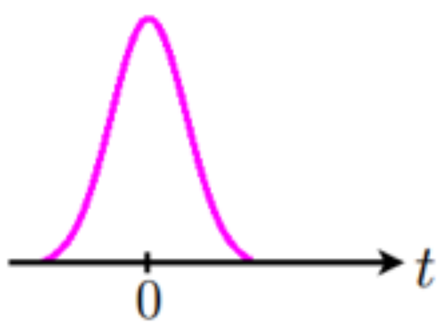

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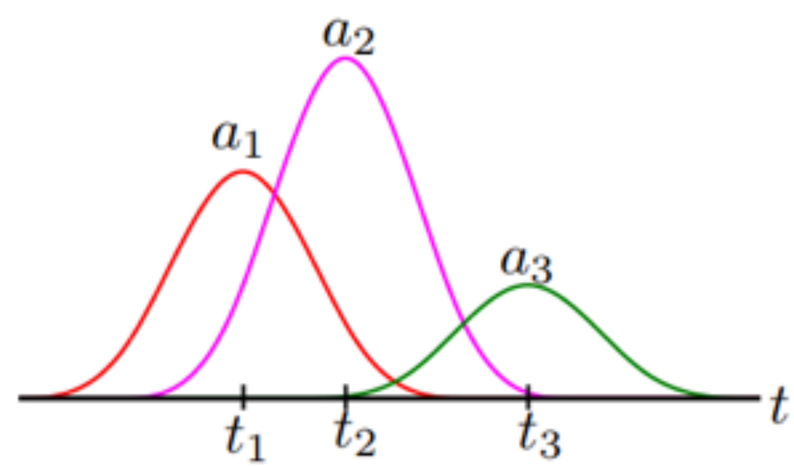

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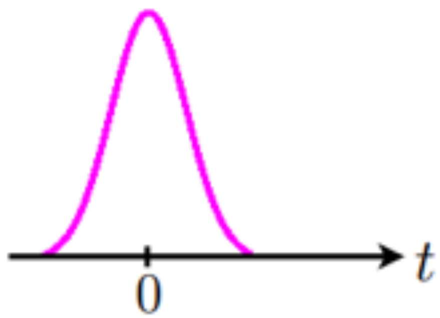

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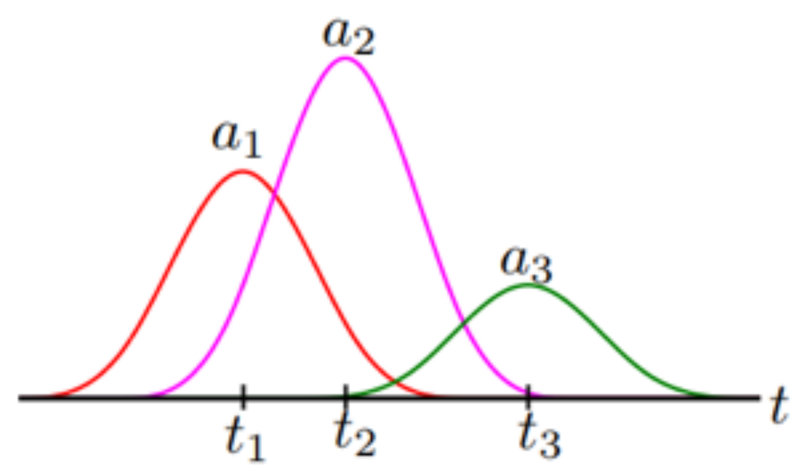

→ **Matched Filter**

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↓

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← **Estimate parameters**

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← **Reconstruct Image**

Reconstruction Results

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- 4 cell mesh-grid slanted at 30^0, FRI reconstruction done using 141 samples.

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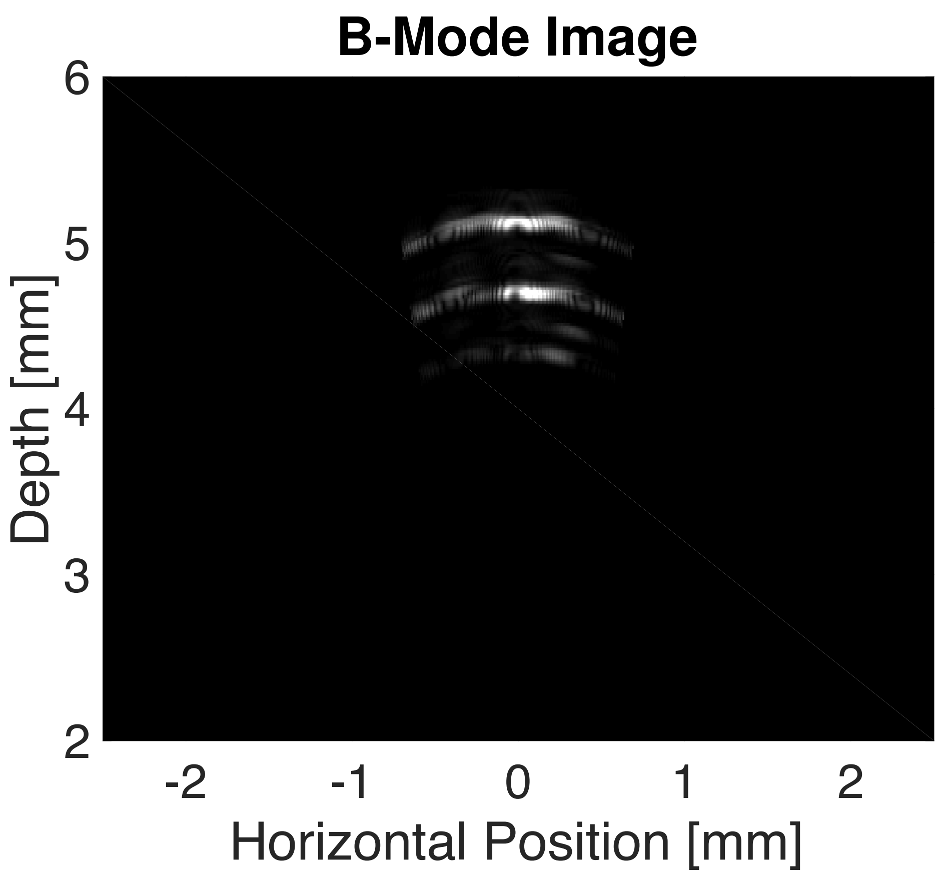

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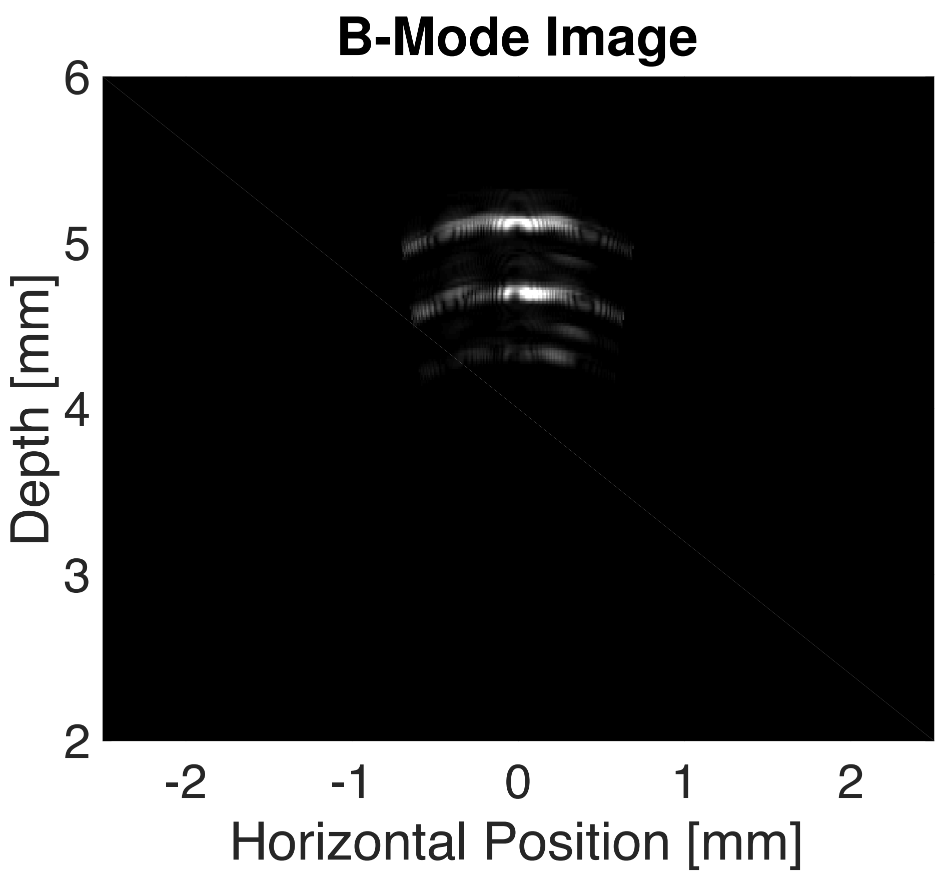

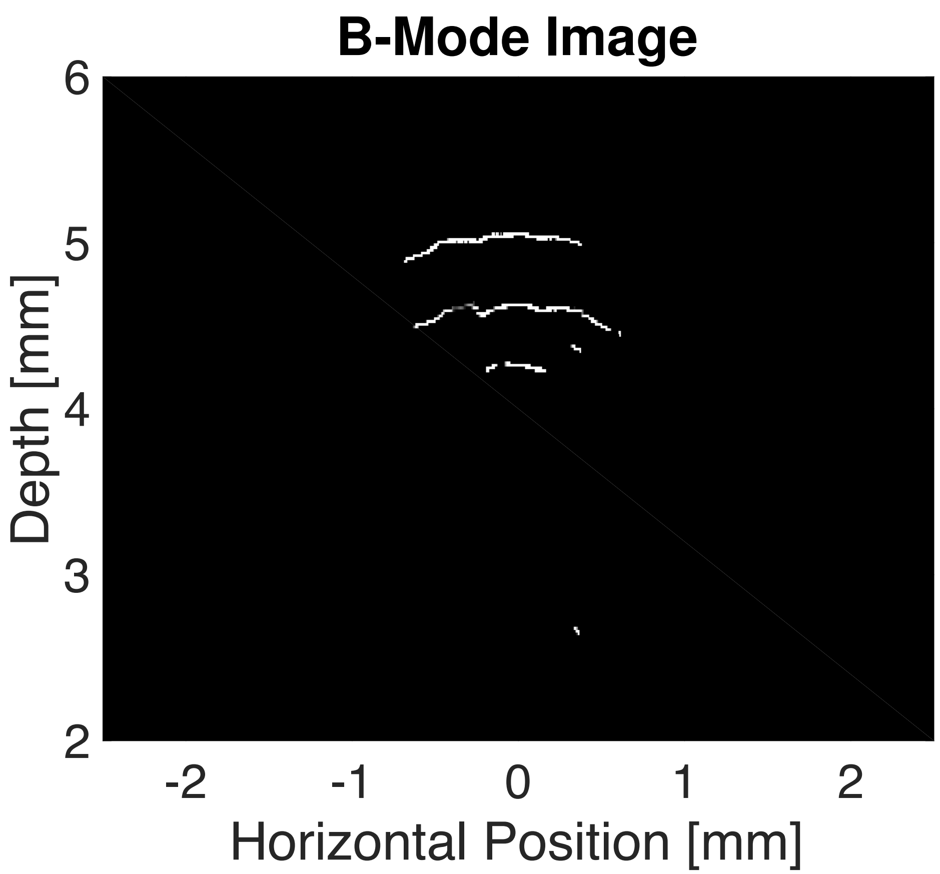

Reconstruction Results

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- 16 cell mesh-grid slanted at 30^0, FRI reconstruction done using 141 samples.

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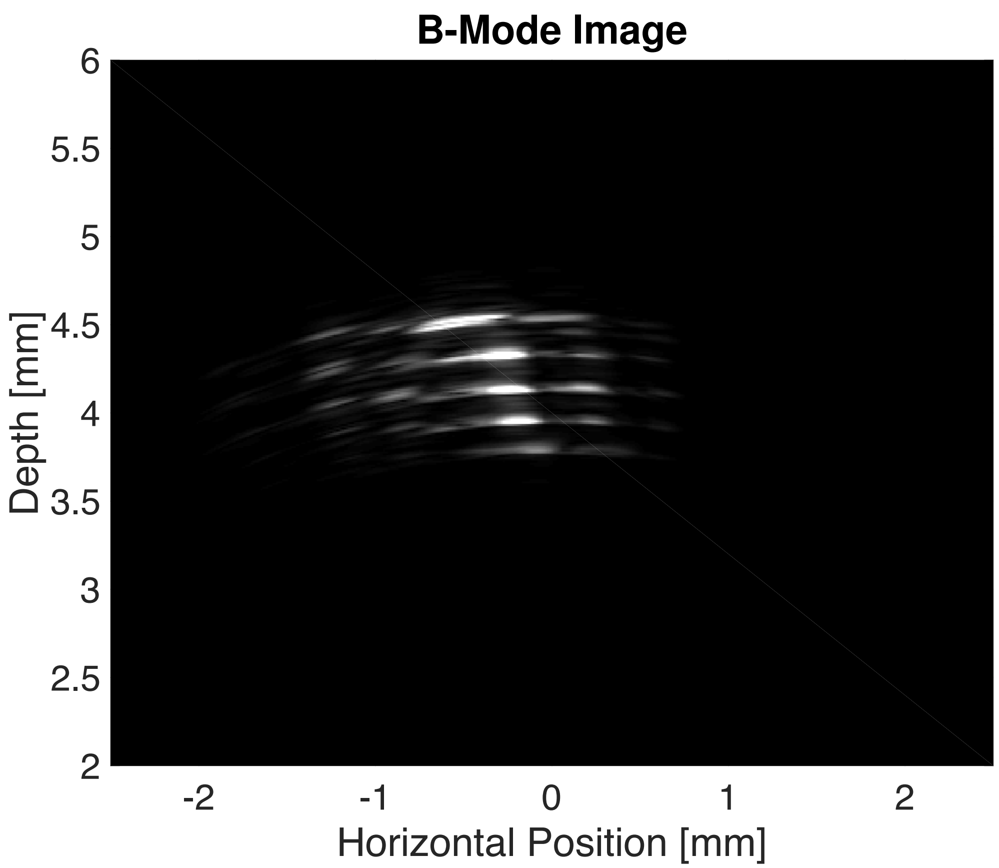

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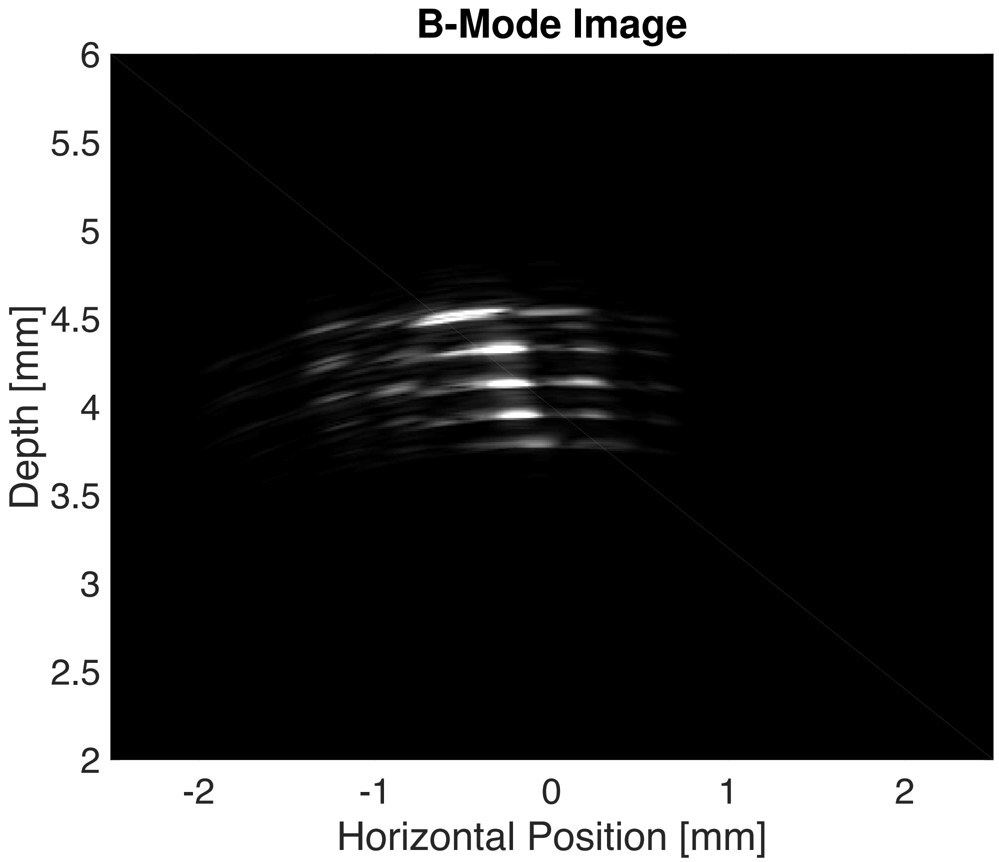

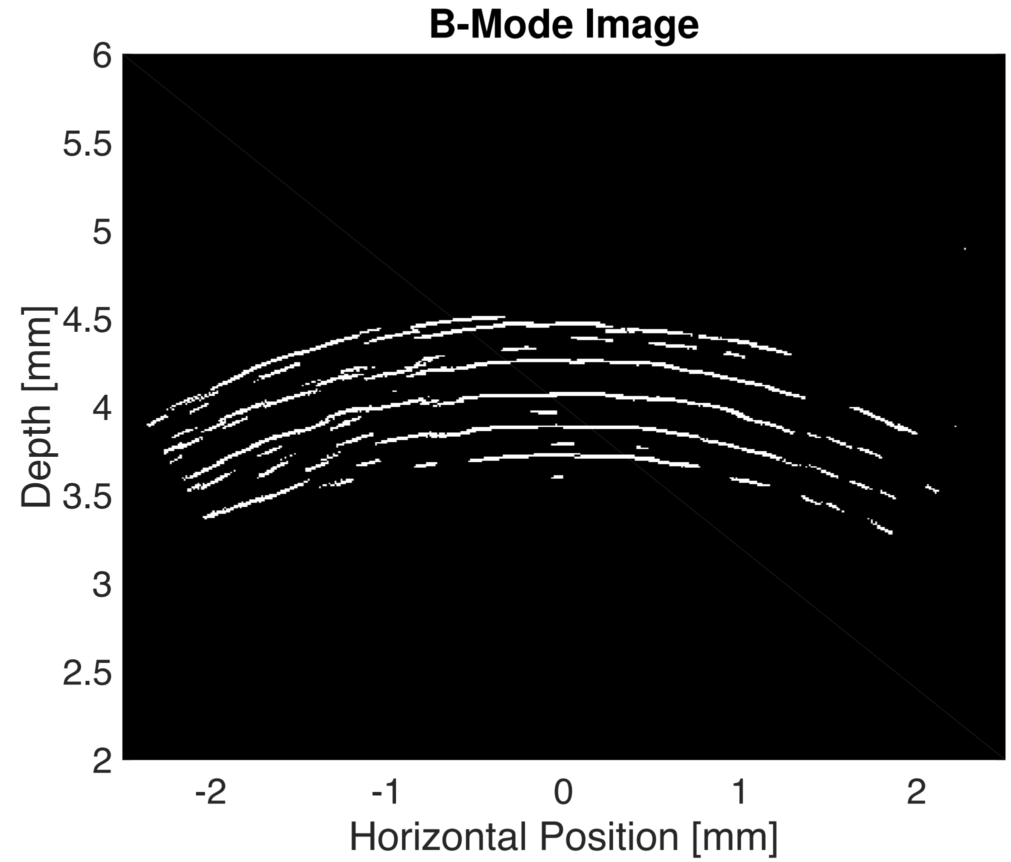

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- 16 cell mesh-grid slanted at 30^0, FRI reconstruction done using 141 samples for a particular steering angle (0^0 azimuth) of the transducer.

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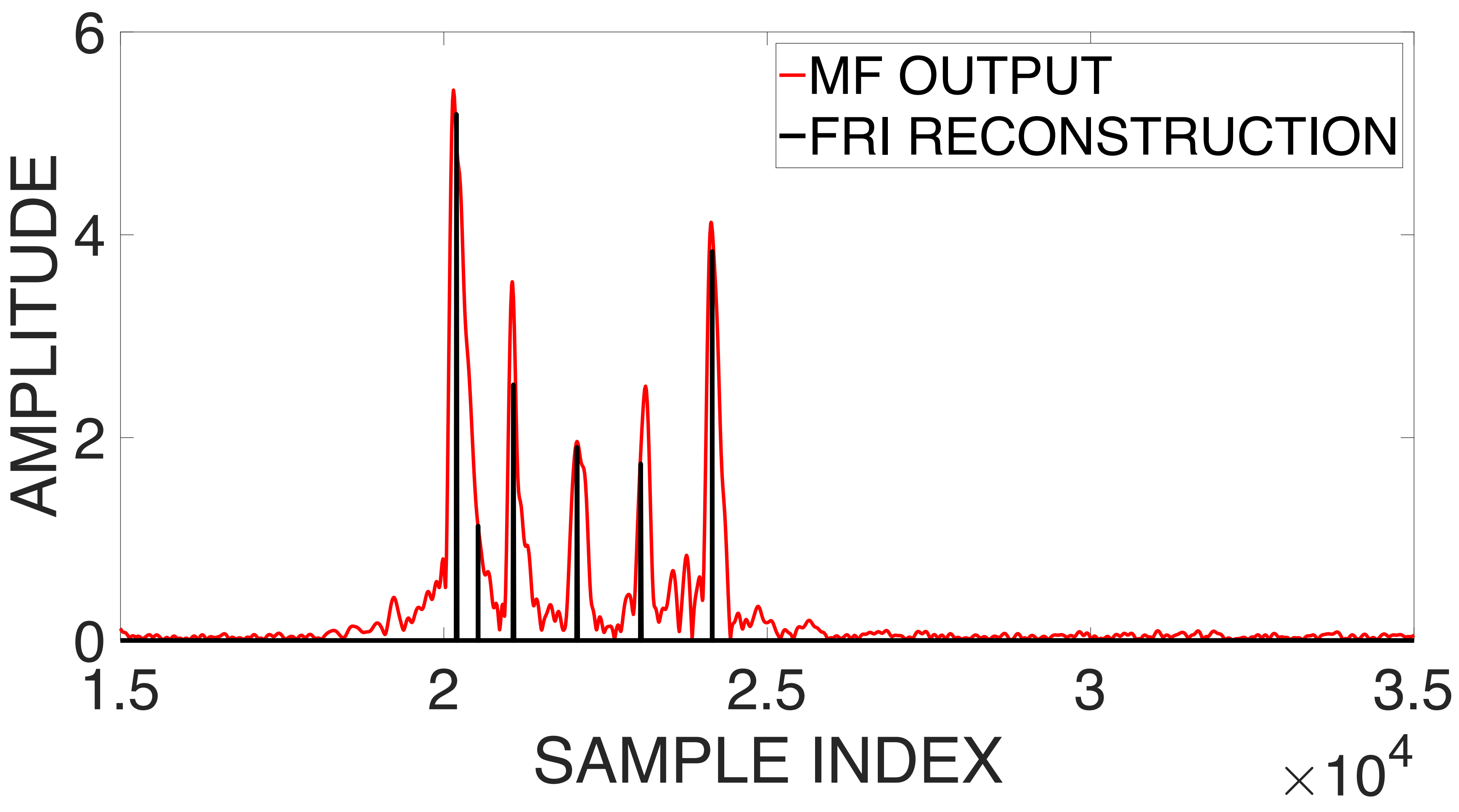

Challenges in Underwater Imaging

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- Poor reconstructions at low SNR.

Future Scope

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- Learn/estimate the response of the medium.
- Develop better beam-forming techniques.

References

[1] M. Vetterli, P. Marziliano, and T. Blu, “Sampling signals with finite rate of innovation,” IEEE Transactions on Signal Processing, vol. 50, no. 6, pp. 1417-1428, Jun. 2002.

[2] P. Stoica and R. L. Moses, *Introduction to Spectral Analysis,* Prentice Hall Upper Saddle River, NJ, 1997, vol. 1.

[3] R. Roy and T. Kailath, “Esprit-estimation of signal parameters via rotational invariance techniques,” IEEE Transactions on Acoustics, Speech, and Signal Processing, vol. 37, no. 7, pp. 984-995, Jul. 1989.

[4] J. L. Stewart and E. C.Westereld, “A theory of active sonar detection,” Proceedings of the IRE, vol. 47, no. 5, pp. 872-881, May. 1959.

[5] J. P. Hermand and W. I. Roderick, “Acoustic model-based matched filter processing for fading time-dispersive ocean channels: theory and experiment,” IEEE Journal of Oceanic Engineering, vol. 18, no. 4, pp.447-465, Oct. 1993.

[6] L. Zou, K. Tan, and J. Zha, “Active sonar detection using adaptive timefrequency feature," in 2016 IEEE/OES China Ocean Acoustics (COA), Jan 2016, pp. 1-4.

Thank you

(Questions, suggestions and corrections)