

CS105 Endsem: DIC on Discrete Structures

Max marks: 65, Max time: 180min

17 Nov 2024

Instructions:

- Attempt *all* questions. Write all answers and proofs carefully. You can assume results shown in class. However, if you are *making any such assumptions or using results proved in class, state them clearly*. Even if you are unable to solve some part of a question, you can assume it and solve other parts.
 - **Make sure you complete the index in the front page, writing clearly all page numbers where each question has been attempted.**
 - All sets considered below are general sets that can be infinite. Hence, you must not assume that they are finite or countable, unless clearly specified otherwise.
 - Recall that $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$ denote, respectively, the set of real numbers, rational numbers, integers and natural numbers.
 - In this course, one of our aims is to learn how to write good proofs, hence considerable weightage will be given to *clarity and completeness* of proofs.
 - Graphs mentioned below are simple but need not be connected etc, unless otherwise specified.
 - Do not copy or use any other unfair means. Offenders will be reported to the Disciplinary Action Committee.
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Part 1

1. (10 marks = 5*2) True or False. Write a short justification for each.
 - (a) For any predicate P over variables x, y (ranging over natural numbers \mathbb{N}), the statement $\forall x \exists y P(x, y)$ is equivalent to $\exists x \forall y P(x, y)$.
 - (b) Let $f : A \rightarrow B$ be a surjective function and B is countable, then A must be countable.
 - (c) There exists a simple graph with 10 vertices, each vertex having degree 6, which has 28 edges.
 - (d) In any group of 5 people, either there are 3 who know each other, or 3 who don't know each other.
 - (e) In any poset, a chain C and an antichain A can have at most one element in common.

Solution: 1 Mark for True/False. Even without proof, if answer correct, this mark given. However, if true/false is not explicitly written, and proof is wrong, mark not given

- (a) False. Consider, for instance, the predicate that says $y = x + 1$. For all x , there exists a y which equals $x + 1$. However, there does not exist a x such that $P(x, 0)$ is true. Hence it is not the case that $\exists x \forall y P(x, y)$ for this predicate.

1 Mark for correct answer. 0.5 mark for any proper counterexample and 0.5 marks for justification. A common error in the counterexample has been giving one which proves the stronger claim $\exists y \forall x \neg P(x, y)$ instead of $\forall x \exists y \neg P(x, y)$. Marks have not been awarded for justification unless the answer shows how proving this automatically implies proving the claim asked in the question.

- (b) False. Consider the surjection from the reals to the natural numbers, which maps every real number which is also a natural number to itself, and every other real number to 0. We know, however, that \mathbb{R} is an uncountable set while \mathbb{N} is a countable set.

1 Mark for correct answer. 1 mark for any proper counterexample

- (c) False. By the handshake lemma, the sum of the degrees of each vertex is twice the number of edges in a graph. In this case, the sum of the degrees is 60, and hence the number of edges is 30.

1 Mark for correct answer. 1 mark for showing that the number of edges should be 30/the sum of degrees should be 56, regardless of whether the Handshake Lemma has been cited

- (d) False. Let us consider the people as the vertices of a graph with 5 vertices. Let us connect acquaintances by an undirected edge. If the graph is C_5 , observe that there is no triangle and there is no independent set of size 3 (there are no three vertices such that no two of those are connected).

1 Mark for correct answer. 1 mark for either citing $R(3,3) = 6$ or giving a counterexample of size 5.

- (e) True. If there were two distinct elements a, b which were in a chain and an antichain, we would simultaneously have that either $(a < b) \vee (b < a)$ ($<$ instead of \leq since they are distinct) and that it is not possible to relate a and b (hence neither $a < b$ nor $b < a$), which is not possible. 1 Mark for correct answer. 1 mark for citing that if there were two elements in a chain and in an antichain, they would have to be related and not related (very formal answer not required)

2. (10 marks =3+3+4)

- (a) Prove that for all $n \geq 1$, if we divide 3^{2n-1} by 4, we get a remainder 3.
- (b) A string is a palindrome if the string is the same when we read it from the left and from the right, for example: *racecar* or *malayalam*. Consider the set of symbols $S = \{a, b, c, d, e\}$. How many palindromes of length 11 exist that only use symbols from S ? (note: they don't need to make sense in English, e.g., *abcdeaedcba* is such a palindrome.)
- (c) Let $S \subseteq \mathbb{N}$, and for any $x, y \in \mathbb{N}$, define $x \preceq y$ if and only if there exists $z \in S$ such that $x + z = y$. Show that if \preceq is a partial order, then (i) $0 \in S$ and (ii) for any $a, b \in S$, $a + b \in S$.

Solution:

- (a) Notice that $3^{(2n-1)}$ can never have a remainder of 0 or 2 modulo 4, since then it would have been of the form $4k+2$ or $4k$, and no power of an odd number can be even. If $3^{(2n-1)}$ leaves a remainder of 1 for some n , consider 3^{2n} , which would leave a remainder of 3 when divided by 4. However, $3^{2n} = 9^n$, and $9 = 4 * 2 + 1$, hence 9^n would always leave a remainder of 1 when divided by 4, which contradicts the fact that 3^{2n-1} has left a remainder of 1 modulo 4.

Alternatively, one can prove this by induction. For the base case, 3^1 leaves a remainder of 3 modulo 4. If we assume that for some k , 3^{2k-1} is of the form $4m+3$ for some integer m , $3^{2(k+1)-1} = 9 * (3^{2k-1}) = 9 * (4m+3) = 4 * (9m) + 27 = 4 * (9m+6) + 3$ leaves a remainder of 3 modulo 4.

For induction, 1 mark for base case, 2 marks for correct application of induction

- (b) Consider any string of length $a_1 a_2 \dots a_{11}$. It is a palindrome if and only if $a_1 = a_{11}, a_2 = a_{10}, a_3 = a_9, a_4 = a_8, a_5 = a_7$. Note that a_1, a_2, \dots, a_5 can be any of the five symbols from S and for any possible assignment of symbols to these 5 places, the assignment at the places a_{11}, \dots, a_7 is fixed as well. This leaves a_6 which can then take value from any of the five symbols in S . Hence, the total possible palindromes is $5^5 \cdot 5 = 5^6$

1 mark for correct answer, 2 marks for justification

- (c) i. For a partial order, $x \leq x$ for all $x \in \mathbb{N}$. From our conditions, $x \leq x \iff \exists z \in C(x+z=x)$. Thus, $0 \in C$.
- ii. For any natural number x , $x \leq x+a$ since $a \in C$ and $x+a \leq x+a+b$ since $b \in C$. Hence, by transitivity, $x \leq x+a+b$. This is only possible if $\exists z \in C(x+z=x+a+b)$. Thus, $a+b \in C$.

2 marks for proving 0 is in the subset. 2 marks for a+b

3. (5 marks) Consider the statement: every transitive and symmetric relation is reflexive. A student asked chatgpt to prove or disprove the statement. Here is the proof provided by chatgpt.



Theorem:

Every symmetric and transitive relation is reflexive.

Proof:

1. Let R be a symmetric and transitive relation on a set A .
2. **Symmetric:** For all $a, b \in A$, if aRb then bRa .
3. **Transitive:** For all $a, b, c \in A$, if aRb and bRc then aRc .
4. To show R is reflexive, we need to prove that for all $a \in A$, aRa .
5. Consider any $a \in A$.
6. Suppose aRb for some $b \in A$.
7. By symmetry, bRa .
8. By transitivity, aRb and bRa imply aRa .
9. Thus, for any $a \in A$, aRa holds true.
10. Therefore, R is reflexive.

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Message ChatGPT

ChatGPT can make mistakes. Check important info.

- (a) Is the above proof given by chatgpt correct? If not, can you point out the mistake (which line number(s))?
- (b) If it is correct, give an alternate proof by induction, else give a counter-example to the statement.

Solution:

- (a) This proof is not correct. The assumption in line 6 has not been justified and need not be true. However, insofar as it is only an assumption, that is not where the error in the proof lies, since that assumption could have been dealt with later. The error lies in the conclusion on line 9, where we ignore the fact that there has been an assumption and conclude something about all relations.

The assumption proves the claim for a limited scope of relations. If we had also proved the claim for the other relations (proof by cases), The omission of this broader reasoning makes the proof invalid.

[1 mark for stating the proof is incorrect.]

[2 marks for correctly identifying the issue in line 9.]

[Partial marks (1 mark) for explaining that the assumption in line 6 but line 9 is not mentioned.]

[Partial marks (1 mark) for including assumptions in other lines along with line 6 and 9]

[Partial marks (0.5 mark) for those who incorrectly claim line 6 is the main issue without acknowledging the error in line 9.]

- (b) Consider the relation $R = \{(1,2), (2,1), (1,1), (2,2)\}$ on the set $A = \{1,2,3\}$. R is both symmetric and transitive but not reflexive, since $(3,3)$ is not present in R . This counterexample disproves the statement that every transitive and symmetric relation is reflexive. [2 marks for stating the Valid counter-example.]

4. (5 marks) Prove that any simple graph G with n vertices, where every vertex has degree at least $n/2$ must be connected.

Solution: Let us assume for the sake of contradiction that the graph is disconnected. We can thus split it into connected components. Since every vertex has degree at least $\lceil \frac{n}{2} \rceil$, each connected component must have at least $\lceil \frac{n}{2} \rceil + 1$ vertices. However, if there are greater than or equal to 2 connected components, this ensures that we must have at least $2 * \lceil \frac{n}{2} \rceil + 2$ vertices, which exceeds n . Thus, we get a contradiction.

We thus conclude that the graph is connected.

Alternate

Proof. Let G be a simple graph with n vertices such that every vertex has degree at least $\frac{n}{2}$. We will prove that G is connected by contradiction.

Suppose, for the sake of contradiction, that G is disconnected. Then, G can be partitioned into at least two non-empty disjoint connected components. Let us consider two such components, denoted as C_1 and C_2 .

Let n_1 and n_2 be the number of vertices in C_1 and C_2 , respectively. Since C_1 and C_2 are disjoint and non-empty, we have $n_1, n_2 \geq 1$ and $n_1 + n_2 \leq n$.

In C_1 , each vertex is only adjacent to vertices within C_1 (since there are no edges connecting C_1 and C_2). Therefore, the maximum degree of any vertex in C_1 is $n_1 - 1$. Thus, for all $v \in C_1$:

$$\deg(v) \leq n_1 - 1.$$

However, from the given condition, every vertex in G has degree at least $\frac{n}{2}$. Hence, for all $v \in G$:

$$\deg(v) \geq \frac{n}{2}.$$

Combining these inequalities for vertices in C_1 , we get:

$$n_1 - 1 \geq \frac{n}{2}.$$

Rearranging, we find:

$$n_1 \geq \frac{n}{2} + 1.$$

Similarly, for C_2 , we also have:

$$n_2 \geq \frac{n}{2} + 1.$$

Adding these two inequalities, we obtain:

$$n_1 + n_2 \geq n + 2.$$

But from the initial assumption, $n_1 + n_2 \leq n$. This leads to:

$$n + 2 \leq n,$$

which is a contradiction.

Therefore, our assumption is false, and the graph G must be connected. □

Generalized Marking Scheme (Total 5 marks)

Proof Strategy: (1 mark)

Award 1 mark for correctly identifying a valid proof method (e.g., proof by contradiction or direct proof).

Application of Degree Condition: (2 marks)

Award 1 mark for correctly using the minimum degree condition $\delta(G) \geq \frac{n}{2}$.

Award 1 mark for appropriately applying this condition to vertices or connected components.

Logical Reasoning: (1 mark)

Award 1 mark for a logical argument that derives a contradiction if the graph is assumed to be disconnected.

Conclusion: (1 mark)

Award 1 mark for concluding that the graph must be connected based on the reasoning provided.

Part 2

5. (6 marks) Prove that the n^{th} prime number p_n is at most $2^{2^{n-1}}$, i.e., $\forall n \geq 1, p_n \leq 2^{2^{n-1}}$.

Solution We prove this by strong induction.

Base Case: $n = 1$:

$p_1 = 2 \leq 2^{2^0}$, and hence the base case holds true.

Assume now that for a given $n \geq 1$, the hypothesis holds true for all $k \leq n$, that is $p_k \leq 2^{2^{k-1}} \forall 1 \leq k \leq n$. We now show $p_{n+1} \leq 2^{2^n}$.

Claim: $p_n < p_{n+1} \leq p_1 p_2 \cdots p_n + 1$

Proof:

We make cases on $\alpha = p_1 p_2 \cdots p_n + 1$:

Case 1: α is a prime number:

In this case, $\alpha = p_{n'}$ for some $n' > n$ (since $\alpha > p_n$). Hence $p_n < p_{n+1} \leq p_{n'}$

Case 2: α is a composite number:

In this case, $\alpha = p \cdot r$ for some prime p . By the definition of α , $p_i \nmid \alpha$ for all $1 \leq i \leq n$. Hence the prime p cannot belong to the first n prime numbers. Hence $p = p_{n'}$ for some $n' > n$. Since $p < \alpha$, we have $p_n < p_{n+1} \leq p < \alpha$

We now use induction hypothesis to conclude that,

$$\begin{aligned} p_{n+1} &\leq p_1 p_2 \cdots p_n + 1 \\ &\leq 2^{2^{1-1}} \cdot 2^{2^{2-1}} \cdots 2^{2^{n-1}} + 1 \\ &\leq 2^{1+2+\cdots+2^{n-1}} + 1 \\ &\leq 2^{2^n - 1} + 1 \text{ (Sum of geometric series)} \end{aligned}$$

Since $2^{2^n} > 2 \forall n \geq 1$, $2^{2^n} > 1 + 2^{2^n - 1} \implies 2^{2^n} > 1 + 2^{2^n - 1}$. Therefore,

$$p_{n+1} \leq 2^{2^n}$$

By strong induction we conclude that $p_n \leq 2^{2^{n-1}}$ for all $n \geq 1$

Marking scheme : 0.5 mark for Base Case, 1 mark for strong induction step, 4 marks for identification and proof of claim (1 mark for prime case and 3 marks for composite case), 0.5 mark for calculation and conclusion.

6. (12marks = 6+6) Graph theory

- Consider the set $\{1, \dots, n\}$ for some $n \in \mathbb{N}$ and $k \leq n/2$. Show that there exists a bijection f from the subsets of size k to the subsets of size $(n-k)$ (of set $\{1, \dots, n\}$) such that for every k -element subset S , we have $S \subseteq f(S)$.
- If G is a graph with a maximum matching of size $2k$, what is the smallest possible size of a maximal matching in G ? Justify your answer and also show that your answer is indeed achievable.

Solution:

- (a) Consider constructing a bipartite graph $(V_1 \cup V_2, E)$, with vertices in V_1 representing subsets of size k , that is $V_1 = \{S \mid |S| = k, S \subseteq [n]\}$, and vertices in V_2 representing subsets of size $(n-k)$, $V_2 = \{S \mid |S| = (n-k), S \subseteq [n]\}$. We draw an edge from subset $S_1 \in V_1$ to $S_2 \in V_2$ if $S_1 \subseteq S_2$.

Observe that any perfect matching of the graph corresponds to a bijection f from the two sets of subsets which satisfy that $S \subseteq f(S)$.

For any vertex $S_1 = \{a_1, \dots, a_k\} \in V_1$, S_1 is adjacent to vertices $\{a_1, \dots, a_k, b_1, \dots, b_{n-2k}\}$ for all choices of b_1, \dots, b_{n-2k} from the remaining $n-k$ elements of $[n]$. Hence, the degree of each vertex in V_1 is $\binom{n-k}{n-2k}$.

Similarly, from any subset $S_2 = \{b_1, \dots, b_{n-k}\} \in V_2$, we may choose any k elements to obtain a subset $S_1 \in V_1$ such that $S_1 \subseteq S_2$. Hence, each vertex in V_2 is adjacent to $\binom{n-k}{k} = \binom{n-k}{n-2k}$ vertices. Hence the graph is a regular bipartite graph with degree of each vertex $\binom{n-k}{n-2k} > 0$. Since such a regular bipartite graph has a perfect matching, there exists a perfect matching in the above constructed graph. Hence there exists a bijection f which satisfies the condition of $S \subseteq f(S)$.

2 marks for the construction of the bipartite graph. 2 mark for proving that it is k -regular. 1 mark for regular implying perfect matching using Hall's theorem, 1 for saying perfect matching implies bijection

- (b) **Claim:** The maximal matching in G must be at-least of size k .

Proof: Let M be the maximum matching of size $2k$, and let M' be a maximal matching in G . Since M' is a maximal matching, no edge from M can be added to M' to extend it to a larger matching. Hence, for each edge $uv \in M$, either u or v is covered by some edge in M' . Assuming $M = \{u_1v_1, u_2v_2, \dots, u_{2k}v_{2k}\}$ with vertices u_i, v_i distinct, for each i atleast one of the two u_i, v_i must be covered by M' . Hence, M' covers at least $2k$ vertices. Hence, M' has a size at least k .

We now give examples of graphs where the size of maximum matching is $2k$ having maximal matching of size k for any $k > 0$.

Consider the line graph $a-b-c-d$. The maximum matching in this is of size 2: $\{a-b, c-d\}$, while a maximal matching of size 1 is $\{b-c\}$. To extend this example for graphs with maximum matching of size $2k$, create k disjoint copies of the above graph. This will have a maximum matching of size $2k$, and a maximal matching of size k . 2 marks for the correct example (which should be family of graphs dependant on k). 1 marks for the correct answer (k). 3 marks for the proof (1 mark for either of the vertices is covered by M , 2 marks for M' being maximal and hence no edge being added to M')

7. (6 marks) A child is playing with 10 sticks. The length of each stick is strictly greater than 1 cm but (strictly) lesser than 55 cm. The child wants to make a triangle using three of these 10 sticks. Will the child succeed? That is, among these 10 sticks, are there always 3 that form a triangle? Why or why not?

Solution: The triangle inequality gives necessary and sufficient conditions for a triangle to be formed having lengths $\{a, b, c\}$ for some real numbers a, b, c . We order the 10 lengths in increasing order as $\{a_1, \dots, a_{10}\}$.

For the sake of contradiction, we will try to construct a sequence of 10 lengths such that no 3 lengths form a triangle. Having ordered the 10 lengths as above, we note that if $a_i + a_j > a_k$ for any $i < j < k$, we can certainly form a triangle (note that the other triangle inequalities, $(a_j + a_k > a_i, a_i + a_k > a_j)$ are trivially true since $a_k \geq a_j$ and $a_k \geq a_i$ and $a_j, a_i > 0$). Also note that the above condition we derived can be reduced further to saying that $a_{k-2} + a_{k-1} \leq a_k$, as this automatically implies $a_i + a_j \leq a_k$ for any $i < j < k$. Hence, $a_1 + a_2 \leq a_3$, $a_2 + a_3 \leq a_4$ and so on.

$$\begin{aligned} a_2 &\geq a_1 \\ a_3 &\geq a_2 + a_1 \geq 2a_1 \\ a_4 &\geq a_3 + a_2 \geq 3a_1 \\ &\dots \\ a_{10} &\geq a_9 + a_8 \geq 55a_1 \end{aligned}$$

(Notice the Fibonacci number pattern occurring above)

However, $a_1 > 1$ and $a_{10} < 55$ are constraints given to us, hence it cannot be true that $a_{10} \geq 55a_1 > 55$. Due to this contradiction, we conclude that it is impossible to construct 10 lengths satisfying the given constraints such that no triangle can be formed.

1 mark for mentioning triangle inequality (0.5 mark if inequality is incorrect).

Marking scheme for proof (out of 5 marks):

- (a) 5 marks for complete proof.
- (b) Maximum of 4 marks if lengths of sticks are assumed to be natural.
- (c) Maximum of 3 marks if lengths of sticks are assumed to be distinct naturals.

In case of optimal sequence construction, if proof of optimality is not given then -2 marks.

8. (12 marks = 3+6+3) A subgraph F of a graph G without isolated vertices and whose edges $E(F)$ form a perfect matching of G is called a *factor* of G . A graph $G = (V, E)$ is called *factorizable* if G contains factors F_1, \dots, F_r such that $\{E(F_1), \dots, E(F_r)\}$ is a partition of E . The partition is called a factorization of G . (Note that even if you are unable to solve some sub problem, you can assume it and solve others.)

- (a) Prove that every factorizable graph is r -regular for some positive number r and has an even number of vertices. Recall: an r -regular graph is a graph each of whose vertices have degree r .
- (b) For every even number $n \geq 2$, show that K_n is factorizable by giving a factorization for any K_n . Recall that K_n is the complete graph on n vertices. (hint: try for K_4, K_6)
- (c) From a group of 8 chess players C_1, \dots, C_8 , we want to schedule 4 chess matches per day for seven days so that no one has two chess matches on the same day and everyone plays a chess match against all of the other 7 players. Is this possible? If yes, give a proof, else give a counter-example.

(Bonus Qn (try after exam!): Is converse of (a) true? Can you come up with a characterization/condition for factorizability?)

Solution:

- (a) Note that every factor must contain every vertex of the original graph, for its edges to form a perfect matching in the original graph. If not, and there was a vertex $v \in V$ but not in the set of vertices of a factor F , it would not be mapped to any vertex by the factor, and hence the matching would not be perfect. Secondly, every vertex must have one and exactly one edge connected to it in a factor for that factor to be a perfect matching.

Thus, if the edges of G can be partitioned into the edges of r factors, each vertex must have exactly one edge in each of these r factors which is distinct for each factor as they are a partition which implies no common edges, and hence the degree of each vertex is at least r . If the degree of any vertex exceeds r , then there would have been an edge not in any of the factors' edge sets, which contradicts the fact that $\{E(F_1), \dots, E(F_r)\}$ is a partition of E (union of all elements of the partition should give the whole set).

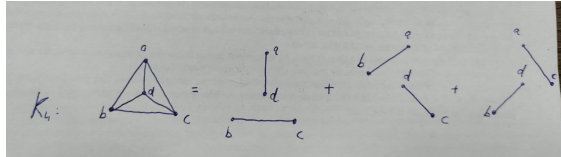
Hence, G is r -regular. Since G has a perfect matching it has an even number of vertices.

1 mark for atleast r edges due to no common edges

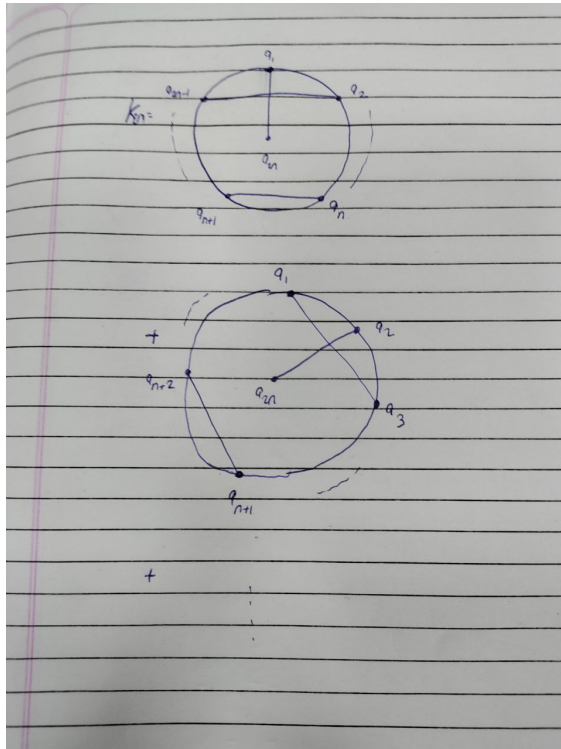
1 mark for no edges left due to partition

1 mark for even

- (b) We show K_{2k} is factorizable by providing a factorization. As an example, consider K_4 :



We generalize this for K_{2k} as:



An intuitive way of looking at this factorization is that we fix a_{2k} at the center and place the other vertices in a circle around this. For factor F_i , we match a_{2k} to a_i , and the vertices which are symmetrically opposite about the "line segment" $a_i - a_{2k}$ are matched together.

6 marks if correct construction

Partial marking:

1 mark for K_4

2 marks for K_6

- (c) This follows from question 8(b). If we take the players as the vertices of K_8 , and we get a factorization of the graph with factors $\{F_1, \dots, F_7\}$ (notice that 7 comes here because each factor has 4 edges and there are a total of 28 edges). Each factor corresponds to a set of 4 matches scheduled for one day (a perfect matching between players), and since the edges of the factors cover the entire graph, we ensure that every potential matchup occurs at least once. Also, as in a perfect matching, each vertex has only 1 edge, no player plays more than once each day.

1 mark for applying 8(b)

1 mark for everyone plays atmost once each day

1 mark for all matches covered in 7 days

Alternate answer: 3 marks Giving a valid schedule

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9. (1 mark) If there is one result that you learned in this course that you would tell your (possibly hypothetical) younger brother/sister/cousin to make them excited to know more about discrete structures, what would that result be? why? Answer in 1-2 sentences.
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The end.
