CS105 Endsem: DIC on Discrete Structures

85 marks, 3 hrs

20 Nov 2023

Instructions:

- Attempt *all* questions. Write all answers and proofs carefully. If you are *making any assumptions or using results proved in class, state them clearly.* Even if you are unable to solve some part of a question, you can assume it and solve other parts.
- Make sure you complete the index in the front page, writing clearly all page numbers where each question has been attempted.
- All sets considered below are general sets that can be infinite. Hence, you must not assume that they are finite or countable, unless clearly specified otherwise.
- Also in what follows, any graph is considered to be **simple**, i.e, no self-loops or multiple edges.
- Recall that $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$ denote, respectively, the set of real numbers, rational numbers, integers and natural numbers.
- In this course, one of our aims is to learn how to write good proofs, hence considerable weightage will be given to *clarity and completeness* of proofs.
- Do <u>*not*</u> copy or use any other unfair means. Offenders will get **FR** grade and be reported to the Disciplinary Action Committee.

Part A

1. (5*2=10 marks) True or False. Give a **short** justification for your answer.
   (a) For any $n \in \mathbb{N}$, if n^3 is odd, then so is n^2.
   (b) If a graph G has 5 vertices and 10 edges, then some vertex must have degree 4.
   (c) If a set A is infinite, then there is a bijection from $\mathbb{N}$ to $A \cup \mathbb{N}$.
   (d) Any two maximal matchings in a graph have the same size, i.e., number of edges.
   (e) Any theorem that can be shown by strong induction can also be proved using induction.
2. (2*2.5=5 marks) Give examples for the following, if they exist. If not, explain why they don't exist with a short proof.
   (a) A graph G with at most 10 vertices and two matchings M, M' in G such that (i) there is an M-augmenting path in G and (ii) size of M' is strictly larger than size of M.

(b) Draw all non-isomorphic graphs with 3 vertices.

3. (5 marks) Prove or disprove: Every connected graph G with at least one edge has a closed walk in which every edge of G occurs exactly twice.

4. (14+6=20 marks) Recurrences and counting

   (a) (5+6+3=14 marks) Consider a row of n numbered chairs, where a child is sitting on each chair. Each child may move by at most one seat (to left or right) or stay in place. Let a_n be the number of ways in which these children can be rearranged in the chairs (i.e., each chair has exactly one child, no chair is empty or has two children in it).

      i. Write a recurrence for a_n, along with its initial conditions.

      ii. Now, suppose the n chairs are placed in a circle (with each child in a chair as before). Let b_n be the number of ways they can be arranged now? Write a recurrence for b_n and solve it.

      iii. Using the recurrences, or otherwise, compute values of b_7 and a_8. You may assume any facts about Fibonacci, Catalan numbers etc that were taught in class.

   (b) (6 marks) Over the alphabet $S = \{a, b, c, d\}$, how many n-length strings are there, which have an even number of a's? Express your answer as a function of n, in as simple a form as possible.

Part B

5. (24 marks) A poset $(S, \preceq)$ is called a *well-partial order* if every infinite sequence of elements $x_0, x_1 \ldots$ from S contains at least one pair of elements x_i, x_j for $0 \leq i < j$ such that $x_i \preceq x_j$.

   (a) Are the following posets well-partial orders? Why or why not? (2 + 3 marks)

      (i) Natural numbers with the divisibility relation $(\mathbb{N}, |\ )$

      (ii) Natural numbers with the usual ordering relation $(\mathbb{N}, \leq)$

   (b) Prove or disprove: every infinite sequence of $(\mathbb{N}, \leq)$ has an *infinite* non-decreasing subsequence. Recall that a sequence $x_0, x_1 \ldots$ is non-decreasing if $x_i \leq x_{i+1}$ for all $i \geq 0$. (5 marks)

   (c) For any $k \in \mathbb{N}, k \geq 1$, consider the set $\mathbb{N}^k$ of vectors of k natural numbers and $\leq^k$ the component-wise ordering, defined by, $(a_1, \ldots, a_k) \leq^k (b_1, \ldots, b_k)$ if $a_i \leq b_i$ for all $1 \leq i \leq k$. (4+5 marks)

      i. Show that $(\mathbb{N}^k, \leq^k)$ is a poset. Is it a total order? Why or why not?

      ii. Show that $(\mathbb{N}^k, \leq^k)$ is a well-partial order.

   (d) *Show that a poset is a well-partial order iff it has no infinite strictly decreasing sequence and no infinite anti-chain. Recall that an anti-chain is a set of elements in which any pair are incomparable. (5 marks)

6. (9 marks) A table with m rows and n columns is filled with nonnegative integers such that each row and each column contains at least one positive integer. Moreover, if a row and a column intersect in a positive integer, then the sums of their integers are the same. Using Hall's theorem or otherwise, prove that $m = n$.

7. (6+6=12 marks) Let $T(n)$ denote the maximum number of edges that a graph with n vertices can have, if it does not contain a triangle (i.e., a subgraph isomorphic to K_3).

   (a) Prove that for all $n \in \mathbb{N}$, $T(n) \leq \lfloor \frac{n^2}{4} \rfloor$.

   (b) If G is a graph with n vertices and at least $\lfloor \frac{n^2}{4} \rfloor + 1$ edges, show that it contains at least $\lfloor \frac{n}{2} \rfloor$ triangles.