CS105 Midsem: DIC on Discrete Structures

40 marks, 120min

23 Sep 2023

Instructions:

- Attempt *all* questions. Write all answers and proofs carefully. If you are *making any assumptions or using results proved in class, state them clearly.*
- All sets considered below are general sets that can be infinite. Hence, you must not assume that they are finite or countable, unless clearly specified otherwise.
- Recall that $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$ denote, respectively, the set of real numbers, rational numbers, integers and natural numbers.
- In this course, one of our aims is to learn how to write good proofs, hence considerable weightage will be given to *clarity and completeness* of proofs.
- Do *not* copy or use any other unfair means. Offenders will be reported to the Disciplinary Action Committee.

---

1. [6=2+2+2 marks] True or false. You MUST give a short justification or counterexample for each.
   (a) For any sets A, B, C, $\quad A \cup (B \cap C) = C \cup (B \cap A)$.
   (b) Any subset of an uncountable set is uncountable.
   (c) The set $(\mathbb{R} \cap (\mathbb{Z} \times \mathbb{Q}))$ is countable.
2. [4 marks] Let $f(0) = 1, f(1) = 2$ and for all $n \geq 1$, let $f(n+1) = f(n-1) + 2f(n)$. Prove by induction that for all $n \in \mathbb{N}$, $f(n) \leq 3^n$. Mention clearly if you are using weak or strong induction.
3. [10=3+3+4 marks] Counting and combinatorics
   (a) A class has 20 students, 12 boys and 8 girls. How many ways are there to form a team of 5 students that has at least 1 boy and 1 girl?
   (b) How many functions from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$ are there, that are *not* injective?
   (c) Use double counting (i.e., counting the size of a suitably designed set in two different ways) to show the following identity. Using any other method will fetch partial marks.

$$\sum_{k=0}^{r} \binom{k}{m} \binom{r-k}{n} = \binom{r+1}{m+n+1}$$

4. [7=3+3+1 marks] Consider the relation $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x - y \in \mathbb{Q}\}$ on real numbers $\mathbb{R}$.

   (a) Show that R is an equivalence relation.

   (b) Under R, what are the equivalence classes of $[1]$, $[\frac{1}{2}]$ and $[\pi]$? Recall that the equivalence class $[x]$ under R, for any $x \in \mathbb{R}$, is defined as $[x] = \{y \in \mathbb{R} \mid (x, y) \in R\}$.

   (c) Is the set of equivalence classes of R finite? Why or why not?

5. [12=2+3+3+4 marks] Let $(P, \preceq)$ be a (non-empty) poset with $|P| = n$ and let $Q \subseteq P$. Q is called a *down-set* if whenever $x \in Q, y \in P$ and $y \preceq x$, then $y \in Q$. Let $\mathcal{D}(P)$ denote the set of all down-sets of P and $|\mathcal{D}(P)|$ its cardinality.

   (a) Prove that $(\mathcal{D}(P), \subseteq)$ is a poset.

   (b) What is $|\mathcal{D}(P)|$ if: (i) P is a chain; (ii) P is an anti-chain? Justify.

   (c) We say that R is an *up-set* if whenever $x \in R, y \in P$ and $x \preceq y$, then $y \in R$. Prove or disprove: Q is a down-set of P iff $P \setminus Q$ is an up-set of P.

   (d) Suppose the number of anti-chains in P is k. Then what is $|\mathcal{D}(P)|$ (as a function of k and n)? Justify.

6. [1 mark] What was the most surprising thing/result that you learnt in this course so far? Why?