CS213/293 Data Structure and Algorithms 2025

Lecture 3: Stack and queue

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Topic 3.1

Stack

Stack

Definition 3.1

Stack is a container where elements are added and deleted according to the last-in-first-out (LIFO) order.

- Addition is called pushing
- Deleting is called popping

Example 3.1

- Stack of papers in a copier
- Undo-redo features in editors
- Back button on Browser

Interface of stack

Reference: https://en.cppreference.com/w/cpp/container/stack

Stack supports four interface methods.

- `stack<T> s` : allocates new stack s
- `s.push(e)` : Pushes the given element e to the top of the stack.
- `s.pop()` : Removes the top element from the stack.
- `s.top()` : accesses the top element of the stack.

Some support functions

- `s.empty()` : checks whether the stack is empty
- `s.size()` : returns the number of elements

Exercise 3.1

Why define stack when we can use vector for the same effect?

Commentary: Answer: vector in C++ promises to provide efficient random access but stack does not make such a promise. Therefore, the implementations of the stack may make implementation choices that may result in inefficient random access.

Axioms of stack*

Let `s1` and `s` be stacks.

- `Assume(s1 == s); s.push(e); s.pop(); Assert(s1==s);`
- `s.push(e); Assert(s.top()==e);`

`Assume(s1 == s)` means that we assume that the content of s1 and s are the same.
`Assert(s1 == s)` means that we check that the content of s1 and s are the same.

Exercise: action on the empty stack

Exercise 3.2

Let `s` be an empty stack in C++.

- What happens when we run `s.top()`?
- What happens when we run `s.pop()`?

Ask ChatGPT.

Commentary: Answer: `s.top()` will cause a segmentation fault. `s.pop()` will not cause any error and exit without any effect.

Topic 3.2

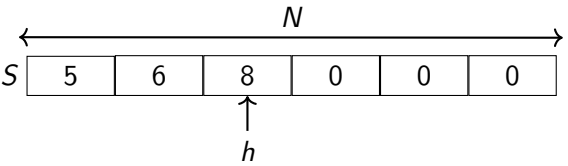
Implementing stack

Array-based stack

Let us look at a simplified array-based implementation of an array of integers.

The stack consists of three variables.

- N specifies the currently available space in the stack
- S is the integer array of size N
- h is the position of the head of the stack



Implementing stack

```
class arrayStack {
  int  N = 2;     // Capacity
  int* S = NULL; // pointer to array
  int  h = -1;    // Current head of the stack
public:
  arrayStack() { S = (int*)malloc(sizeof(int)*N ); }
  int  size()  { return h+1;  }
  bool empty() { return h<0;  }
  int  top()   { return S[h]; } // On empty stack what happens?
  void push(int e) {
    if( size() == N ) expand(); // Expand capacity of the stack
    S[++h] = e;
  }
  void pop() { if( !empty() ) h--; }
```

Commentary: The behavior of the above implementation may not match the behavior of the C++ stack library. To ensure segmentation fault in top() when the stack is empty one may use the following code. `if( empty() ) return *(int*)0; else return S[t];`

Implementing stack (expanding when full)

```
private:
  void expand() {
    int new_size = N*2; // We observed the growth in our lab!!
    int* tmp = (int*) malloc( sizeof(int)*new_size );//New array
    for( unsigned i =0; i < N; i++ ) {// copy from the old array
      tmp[i] = S[i];
    }
    free(S);       // Release old memory
    S = tmp;       // Update local fields
    N = new_size; //
  }
};
```

Exercise 3.3

Can we utilize the DMA (Direct Memory Access) feature of processors to perform the copy?

Efficiency

All operations are performed in $O(1)$ if there is no expansion to stack capacity.

What is the cost of expansion?

Topic 3.3

Why exponential growth strategy?

Growth strategy

Let us consider two possible choices for growth.

- Constant growth: `new_size = N + c` (for some fixed constant c)
- Exponential growth: `new_size = 2*N`

Which of the above two is better?

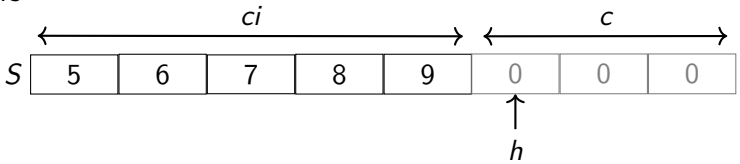
Analysis of constant growth

Let us suppose initially $N = 0$ and there are n consecutive pushes.

After every cth push, there will be an expansion operation.

Therefore, the expansion operation at $(ci + 1)$th push will

- allocate memory of size $c(i + 1)$
- copy ci integers



Cost of ith expansion: $c(2i + 1)$.

Commentary: We are assuming that allocating memory of size k costs k time, which may be more efficient in practice. Bulk memory copy can also be sped up by vector instructions.

Analysis of constant growth(2)

For n pushes, there will be n/c expansions.

The total cost of expansions:

$$c(1 + 3 + ... + (2\frac{n}{c} + 1)) = c(n/c)^2 \in O(n^2)$$

Non-linear cost!

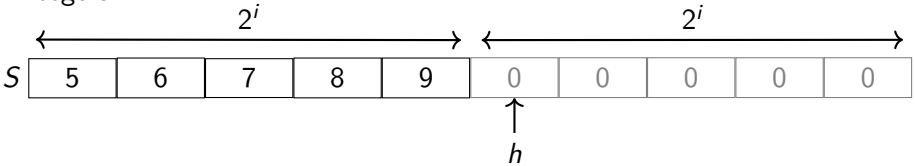
Analysis of exponential growth

Let us suppose initially $N = 1$ and there are $n = 2^r$ consecutive pushes.

The expansion operations will only occur at $(2^i + 1)$th push, where $i \in [0, r-1]$.

The expansion operation at $2^i + 1$th push will

- allocate memory of size 2^{i+1}
- copy 2^i integers



Cost of the expansion: $3 * 2^i$.

Analysis of exponential growth(2)

For 2^r pushes, the last expansion would be at $2^{r-1} + 1$.

The total cost of expansions:

$$3(2^0 + ... + 2^{r-1}) = 3 * (2^r - 1) = 3 * (n - 1)$$

Linear cost! The average cost of push remains $O(1)$.

Exercise 3.4

Why double? Why not triple? Why not 1.5 times? Is there a trade-off?

Commentary: The policy is not the same in all the libraries for vectors. What is the policy in the following implementations of STL?
GCC STL: (libstdc++) https://github.com/gcc-mirror/gcc/blob/master/libstdc%2B%2B-v3/include/bits/stl_vector.h#L2187
Clang STL: (libc++) https://github.com/llvm/llvm-project/blob/main/libcxx/include/__vector/vector.h#L909
MASC STL: https://github.com/microsoft/STL/blob/main/stl/inc/vector#L2006

Topic 3.4

Applications of stack

Stacks are everywhere

Stack is a foundational data structure.

It shows up in a vast range of algorithms.

Example: matching parentheses

Problem:
Given an input text check if it has matching parentheses.

Examples:

- "{a[sic]tik}"✓
- "{a[sic}tik}"✗

```
bool parenMatch(string text ) {
  std::stack<char> s;
  for(char c : text ) {
    if( c == '{' or c == '[' ) s.push(c);
    if( c == '}' or c == ']' ) {
      if( s.empty()       ) return false;
      if( c-s.top() != 2 ) return false;
      s.pop();
    }
  }
  if( s.empty() ) return true;
  return false;
}
```

Topic 3.5

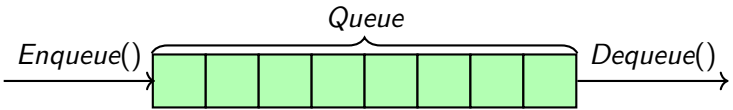
Queue

Queue

Definition 3.2

Queue is a container where elements are added and deleted according to the first-in-first-out (FIFO) order.

- Addition is called enqueue
- Deleting is called dequeue



Example 3.2

- Entry into an airport
- Calling lift in a building (slightly different; it is a priority queue)

Interface of queue

Reference: https://en.cppreference.com/w/cpp/container/queue

Queue supports four main interface methods

- `queue<T> q` : allocates new queue q
- `q.enqueue(e)` : Adds the given element e to the end of the queue. (push)
- `q.dequeue()` : Removes the first element from the queue. (pop)
- `q.front()` : access the first element .

Some support functions

- `q.empty()` : checks whether the queue is empty
- `q.size()` : returns the number of elements

Commentary: All literature uses the terms enqueue and dequeue, but unfortunately the C++ library uses push for enqueue and pop uses for dequeue. Other languages such as Java uses the terms enqueue and dequeue.

Axioms of queue*

1. `queue<T> q; Assert(q.empty() == true);`

2. `q.enqueue(e); Assert(q.empty() == false);`

3. `Assume(q.empty() == true);`
   `q.enqueue(e); Assert(q.front() == e);`

4. `Assume(q.empty() == false && old_q == q);`
   `q.enqueue(e); Assert(old_q.front() == q.front());`

5. `Assume(q.empty() == true && old_q == q);`
   `q.enqueue(e); q.dequeue(); Assert(old_q == q);`

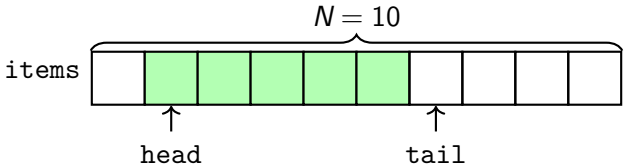
6. `Assume(q.empty() == false && q == q1);`
   `q.enqueue(e);q.dequeue(); q1.dequeue();q1.enqueue(e); Assert(q == q1);`

Topic 3.6

Array implementation of queue

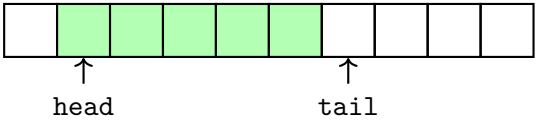
Array-based implementation

- Queue is stored in an array `items` in a circular fashion
- Three integers record the state of the queue
  1. `N` indicates the available capacity (N-1) of the queue
  2. `head` indicates the position of the front of the queue
  3. `tail` indicates position one after the rear of the queue

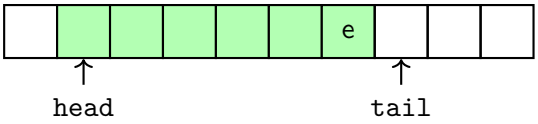


Enqueue operation on array

Consider the state of the queue.

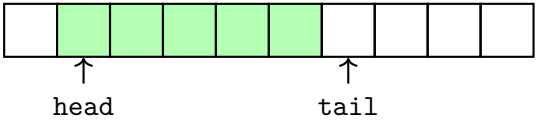


After the enqueue(e) operation:

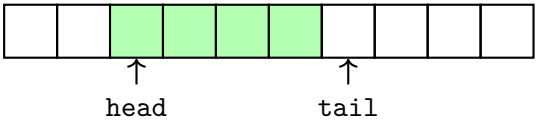


Dequeue operation on array

Consider the state of the queue



After the dequeue() operation:

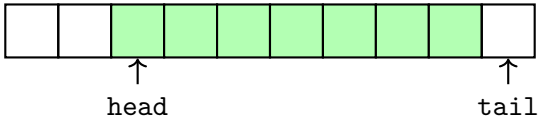


Exercise 3.5

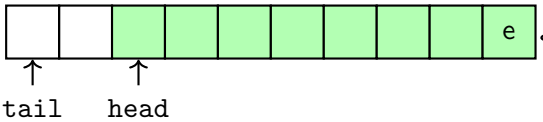
1. Where will front() read from?
2. What is the size of the queue?

Wrap around to utilize most of the array

Consider the state of the queue.

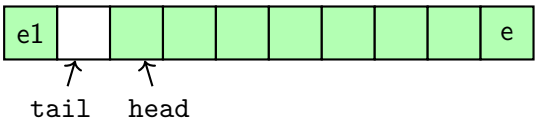


After the enqueue(e) operation, we move the tail to 0.



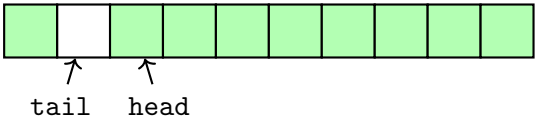
Wrap-around allows us to use the array repeatedly.

After another `enqueue(e1)` operation:

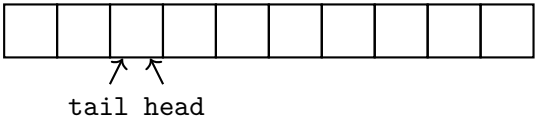


Full and empty queue

Full queue:



Empty queue:



Exercise 3.6

Can we use all N cells for storing elements?

Array implementation

The code is not written in C++; We will slowly move towards pseudo code to avoid clutter on slides.

```
int head = 0, tail=0, N = INITIAL_CAPCITY;

Object items[N];                    //Some initial size

bool empty()   { return (head == tail);  }

bool size()    { return (N+tail-head)%N; }

Object front() { return items[head];     }
```

Array implementation

```
void dequeue() {
  if( empty() ) throw Empty;              // Queue is empty
  free(items[head]);items[head] = NULL; // Clear memory
  head = (head+1)%N;      //Remove an element
}

void enqueue( Object x ) {
  if ( size() == N-1 ) expand(); // Queue is full;expand
  items[tail] = x;
  tail = (tail+1)%N; // insert element
}
```

Exercise 3.7

In our stack implementation, we did not invoke free in pop, but we invoke free in dequeue. Why?

Commentary: In the stack implementation, we were only handling a stack of int. Here, we are handling queue of **arbitrary objects**. If the object has dynamic size then it cannot live on the queue itself. We will store a reference to the object on the queue and allocate the object somewhere else. Therefore, we must free the objects on dequeue (the above syntax of free is not in C++; It will only work when items[head] is a reference). However, there is no need to free an int because it has a fixed size (32 bits $\leq$ size of pointers=64bits) and it was stored on the array of the stack itself.

Topic 3.7

Queue via linked lists

Can we avoid expansion?

Arrays need expansion whenever they are full.

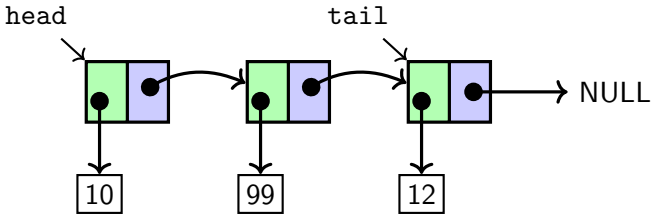
Can we expand without the need for copying?

We may use linked lists to achieve this.

Linked lists

Definition 3.3

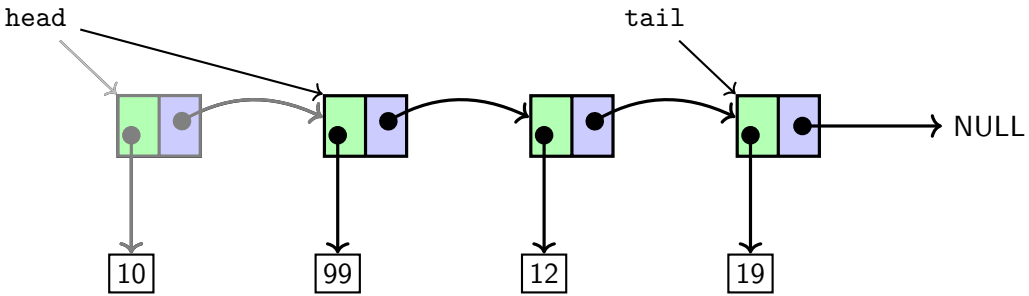
A linked list consists of nodes (small pieces of memory) with two fields data and next pointer. The nodes form a chain via the next pointer. The data pointers point to the objects that are stored on the linked list.



Exercise 3.8

a. If we use a linked list for implementing a queue, which side should be the front of the queue?
b. What is the fundamental difference between an array and a linked list?

Dequeue in linked lists

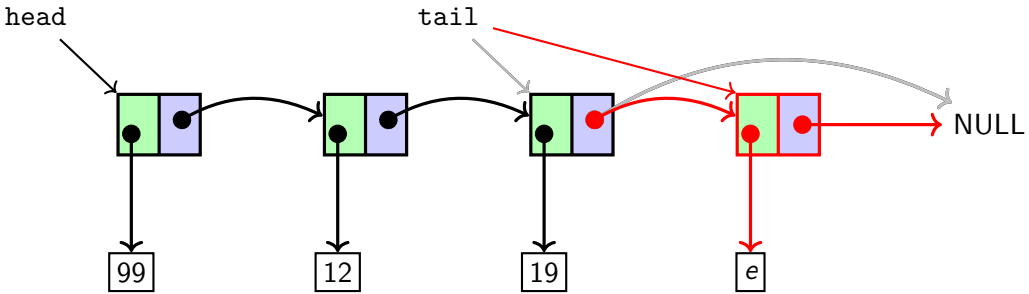


Exercise 3.9

What happens to the object containing 10?

Commentary: Answer: There are several choices. The object is deallocated, the reference to the object is returned to the caller, or the copy of the object is returned to the caller. Different implementations may do it differently. This behavior is not part of the specification of the queue.

Enqueue(e) in linked lists



Exercise 3.10

a. Which one is better: array or linked list?

b. Do we need the tail pointer?

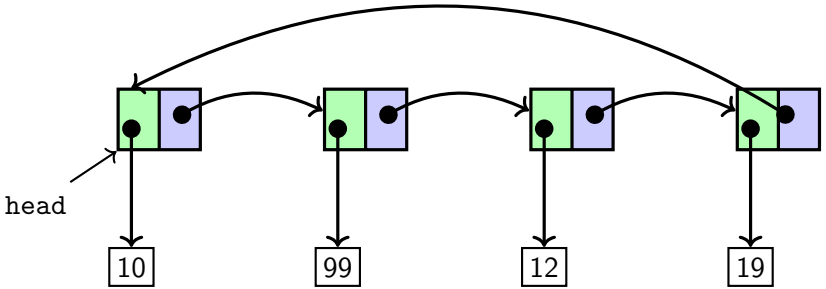
Topic 3.8

Circular linked list

Circular linked lists

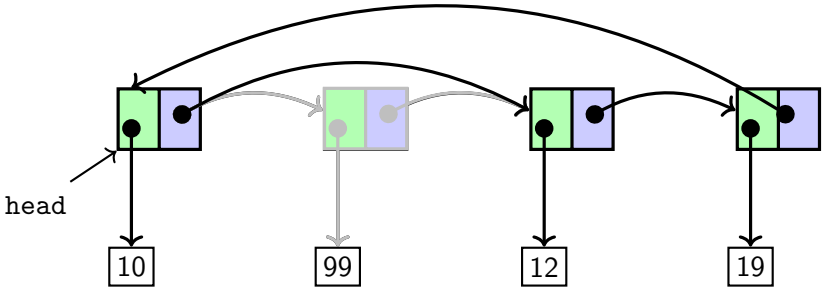
Definition 3.4

In a circular linked list, the nodes form a circular chain via the next pointer.

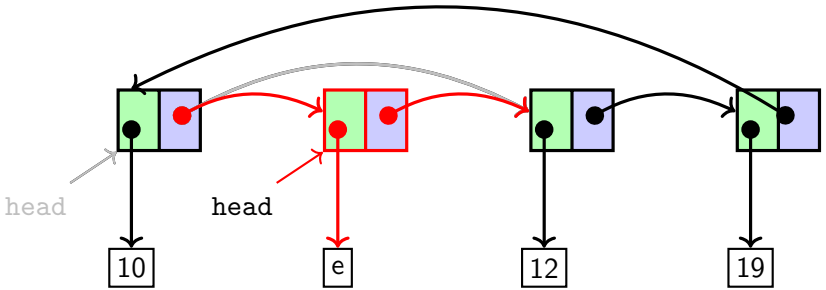


A head pointer points at some node of the circular list. A single pointer can do the job of the head and tail.

Dequeue in circular linked lists



Enqueue(e) in circular linked lists



Exercise 3.11

a. Which element should be returned by front()?

b. Give pseudo code of the implementation of queue using circular linked list. (Midsem 2023)

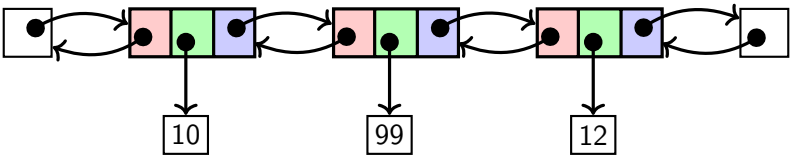
Topic 3.9

Deque via a doubly linked list

Doubly linked lists

Definition 3.5

A doubly linked list consists of nodes with three fields prev, data, and next pointer. The nodes form a bidirectional chain via the prev and next pointer. The data pointers point to the objects that are stored on the linked list.

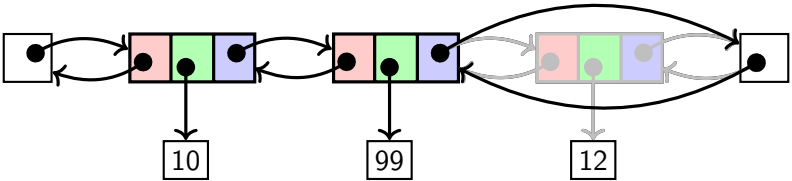


At both ends, two dummy or sentinel nodes do not store any data and are used to store the start and end points of the list.

Exercise 3.12

Why do we need sentinel nodes at the ends?

Deleting a node in a doubly linked list



Deque (Double-ended queue)

Definition 3.6

Deque is a container where elements are added and deleted according to both last-in-first-out (LIFO) and first-in-first-out (FIFO) order.

Interface of Deque

Reference: https://en.cppreference.com/w/cpp/container/deque

Queue supports four main interface methods

- `deque<T> q` : allocates new queue q
- `q.push_back(e)` : Adds the given element e to the back.
- `q.push_front(e)` : Adds the given element e to the front.
- `q.pop_front()` : Removes the first element from the queue.
- `q.pop_back()` : Removes the last element from the queue.
- `q.front()` : access the first element .
- `q.back()` : access the first element .

Some support functions

- `q.empty()` : checks whether the stack is empty
- `q.size()` : returns the number of elements

We can implement the Deque data structure using the doubly linked lists.

Stack and queue via Deque

We can implement both stack and queue using the interface of deque.

Exercise 3.13

- Which functions of deque implement stack?
- Which functions of deque implement queue?

All modification operations are implemented in $O(1)$.

Exercise 3.14

Can we implement `size` in $O(1)$ in a doubly linked list?

Topic 3.10

Discussion

Point of discussion: vector instructions for fast expansion*

Exercise 3.15

Save the following code in `copy.cpp` and compile with the following options.

```
void copy(int* dst, int* src) {
  for( unsigned i =0; i < 32; i++ ) {
    dst[i] = src[i];
  }
}
```

- Compile with: `g++ copy.cpp -O3 -S`
- Compile with: `g++ copy.cpp -O3 -mavx512f -S` (Enables vector instructions!)

Read the assembly generated in the `copy.s` file. Observe the difference.

Point of discussion: store values or references in the collection*

A collection like a queue has two choices, either store values or the references to the values.

Operations on the collection need to behave differently. For example,

Operation	Storing values	Storing references
`enqueue(Object e)`	store a copy of the e	store a reference to e
`front()`	return a copy	return a reference
`dequeue()`	delete the object	remove the reference

Exercise 3.16

What are the advantages or disadvantages of methods of storing collections?

Point of discussion: where are the libraries in my computer?

Exercise 3.17

How does C++ compiler finds the libraries? Where are they stored in computer?

Topic 3.11

Tutorial problems

Exercise: Use of stack

Exercise 3.18

The span of a stock's price on *ith* day is the maximum number of consecutive days (up to *ith* day) the price of the stock has been less than or equal to its price on day i.

Example: for the price sequence 2 4 6 3 5 7 of a stack, the span of prices is 1 2 3 1 2 6.

Give a linear-time algorithm that computes s_i for a given price series.

Exercise: flipping Dosa

Exercise 3.19

There is a stack of dosas on a tava, of distinct radii. We want to serve the dosas of increasing radii. Only two operations are allowed: (i) serve the top dosa, (ii) insert a spatula (flat spoon) in the middle, say after the first k, hold up this partial stack, flip it upside-down, and put it back. Design a data structure to represent the tava, input a given tava, and produce an output in sorted order. What is the time complexity of your algorithm?
This is also related to the train-shunting problem.

Exercise: exponential growth

Exercise 3.20

a. Analyze the performance of exponential growth if the growth factor is three instead of two. Does it give us better or worse performance than doubling policy?

b. Can we do a similar analysis for growth factor 1.5?

Exercise: reversing a linked list (Quiz 2024)

Exercise 3.21

Give an algorithm to reverse a linked list. You must use only three extra pointers.

Exercise: middle element

Exercise 3.22

Give an algorithm to find the middle element of a singly linked list.

Exercise: stack and queue (Endsem 2023)

Exercise 3.23

Given two stacks $S1$ and $S2$ (working in the LIFO method) as black boxes, with the regular methods: "Push", "Pop", and "isEmpty", you need to implement a Queue (specifically : Enqueue and Dequeue working in the FIFO method). Assume there are n Enqueue/ Dequeue operations on your queue. The time complexity of a single method Enqueue or Dequeue may be linear in n, however the total time complexity of the n operations should also be $\Theta(n)$.

Topic 3.12

Problems

True or False

Exercise 3.24

Mark the following statements as True/False and also justify.

1. The element at the top of the stack has been in the stack for the longest time.
2. A stack does not allow random access to its elements.
3. O(1) is the worst-case time complexity of dequeue in a queue containing m elements implemented using an array.
4. The pop operation on a stack is always $O(1)$.
5. Stack can be implemented using a doubly-linked list as well as a singly-linked list
6. We can traverse a singly linked list backward starting from the last node to the first node.
7. In C++, `std::queue<T>` has method `push_back`.
8. In C++ `std::vector<T>`, the buffer size is doubled whenever there is a call to `push_back` after the vector is full.

Problem: messy queue

Exercise 3.25

The mess table queue problem: There is a common mess for k hostels. Each hostel has some $N_1,...,N_k$ students. These students line up to pick up their trays in the common mess. However, the queue is implemented as follows: If a student sees a person from his/her hostel, she/he joins the queue behind this person. This is the "enqueue" operation. The "dequeue" operation is as usual, at the front. Think about how you would implement such a queue. What would be the time complexity of enqueue and dequeue? Do you think the average waiting time in this queue would be higher or lower than a normal queue? Would there be any difference in any statistic? If so, what?

Merge sorted queues (Quiz 2023)

Exercise 3.26

Write a time and space efficient algorithm to merge k sorted-linked list in sorted order, each containing the same no of elements?

Exercise: axioms of queue**

Exercise 3.27

Using axioms of queue, show that the assert in the following does not fail.

```
queue<int> q,q1;
q.enqueue(2);
q.enqueue(0);
q.enqueue(7);
q.dequeue();
q.dequeue();
q.enqueue(3);

q1.enqueue(7);
q1.enqueue(3);
Assert(q == q1);
```

End of Lecture 3