Solution Sketch for Problems 5 and 6

Problem 5:

Let R[i] denote the sum of the elements of row j and C[j] denote the sum of the elements of column j in the matrix X.

Create a graph with the following nodes and edges:

- Add a set of nodes P = {a_1, a_2, …a_p}, one for each of the p rows of the matrix
- Add a set of nodes Q = {b_1, b_2, …b_q}, one for each of the q columns of the matrix
- Add two other nodes s and t
- For each a_i in P, add an edge from s to a_i with capacity R[i]
- For each b_j in Q, add an edge from b_j to t with capacity C[j]
- For each a_i in P, add edges to every b_j in Q with capacity ∞

Now compute the maximum flow in this graph. Since all capacities are integers, there exists an integral flow. The flow assigned on the edge (a_i, b_j) for each (i, j)-pair will give the value of the (i, j)-th element of Y.

Grading remarks: Around 60% on the nodes and edges and 40% on capacities. However, somewhat subjective as answers varied. Alternative solutions possible which are given credit.

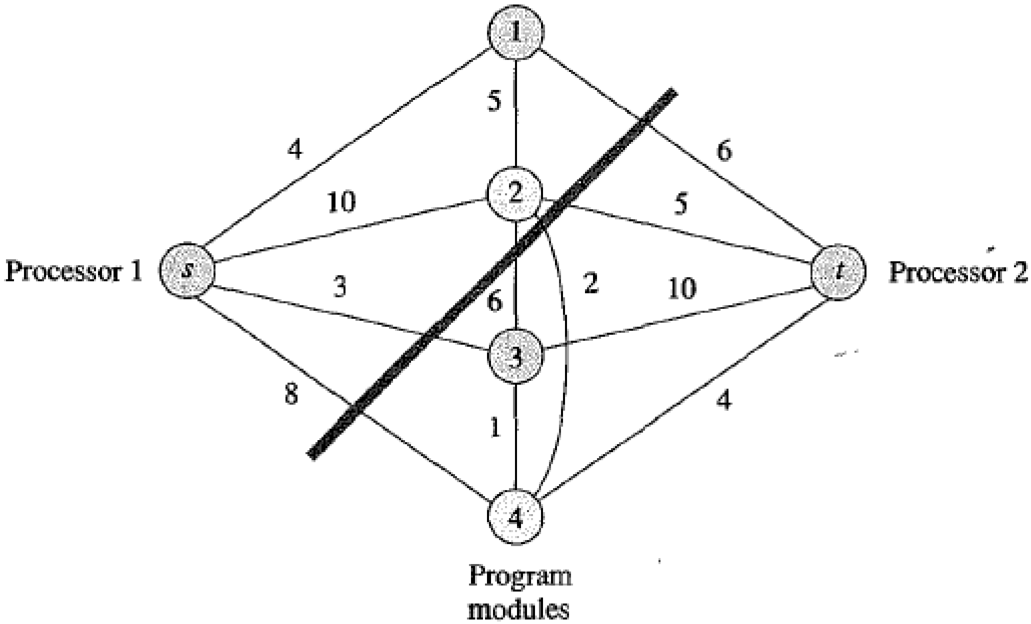
Problem 6:

Create a graph as follows:

- Create one node X_i for each task T_i
- Create one node for M_1 and one node for M_2
- Add an edge between M_1 and each task node X_i with capacity D_i
- Add an edge between M_2 and each task node X_i with capacity C_i
- Add an edge between X_i and X_j if P_{ij} is not equal to 0.

Now find the maximum flow from M_1 to M_2 in this graph to find the minimum cut. Assign a task T_i to M_1 if it is on M_1's side of the cut, and to M_2 if it is on M_2's side of the cut.

Note that we reversed the capacity assignment on edges between M_1 and X_i (assigned D_i instead of C_i) and between M_2 and X_i (assigned C_i instead of D_i). You can also assign capacity C_i to (M_1, X_i) edge and D_i to (M_2, X_i) edge, but then you need to assign the tasks to the machine on the opposite side of the cut.



The picture shows an example graph and a cut with 4 tasks (marked 1 to 4) scheduled on two machines/processors, with $C_1 = 6$, $C_2 = 5$, $C_3 = 10$, $C_4 = 4$, $D_1 = 4$, $D_2 = 10$, $D_3 = 3$, $D_4 = 8$, $P_{24} = 2$, and all other P_{ij}'s = 0. Here Tasks 1 and 2 will be scheduled on Processor 1, and 3 and 4 on Processor 2.

Why is the minimum cut a minimum cost assignment? Firstly, all tasks will be on one or the other side of the cut, so all tasks are assigned to one of the machines. Also, (i) a cut will include exactly one of (M_1, X_i) or (M_2, X_i) edge, so every task is assigned to exactly one machine, and (ii) if two tasks i and j are assigned to different machines and P_{ij} is not equal to 0, the cut must include the edge between X_i and X_j (as they are on opposite sides of the cut, so the cut must cross the edge between them). Hence, the value of the cut will give the total assignment cost. This will be minimum when the cut is minimum.

Grading remarks: 11 for the graph construction, 4 for the argument. Graded hard, as the idea was simple but you need to see it. Also, some parts were easy to guess (like adding edges between M's and T's) without actually seeing the answer, in such cases I looked at what you did with it. Most answers were unnecessarily complex and wrong. For the justification, I accepted leniently as long as you can explain the value of the cut will be the assignment cost. This was quite subjective, so if you have questions on your marks, pls contact me.