Q6) $P_{ij} = (x_i, y_j)$, $i = 1, 2 \dots M$, $j = 1, 2, \dots N$

$$f(u, v) = \theta_1 + \theta_2 u + \theta_3 v + \theta_4 uv$$

$$f(P_{ij}) = f_{ij}$$

(a) $A\theta = b$

$$\underbrace{\begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_1 & y_2 & x_1 y_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_1 & y_n & x_1 y_n \\ 1 & x_2 & y_1 & x_2 y_1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_m & y_1 & x_m y_1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_m & y_n & x_m y_n \end{bmatrix}}_{\mathbb{R}^{MN \times 4}} \underbrace{\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}}_{\mathbb{R}^{4 \times 1}} = \underbrace{\begin{bmatrix} f_{11} \\ f_{12} \\ \vdots \\ f_{1N} \\ \vdots \\ f_{MN} \end{bmatrix}}_{\mathbb{R}^{MN \times 1}}$$

(b) For unique solution to the equation $A\theta = b$ firstly for the existance of solution $b \in \text{colspan}(A)$. Secondly for it to be unique coloums of A should form the basis of colspace (A). This implies that colums of A should be linearly ~~inded~~ independently.

Hence, no. of rows $\geq$ no. of coloums

No. of rows = MN

No. of coloums = 4

Hence $MN \geq 4$

Since we want to minimise M and N, we will choose $MN = 4$.

Now for M and N individually, we have three possibilities.

a) $M = 1$ $N = 4$

b) $M = 2$ $N = 2$

c) $M = 4$ $N = 1$

Consider

(a) $M=1$, $N=4$, second coloum becomes all x, and thus linearly dependant because first cloum is all 1. Hence the solution will not be unique in this case.

(c) If we consider $M=4$ $N=1$, then similarly third coloum will become all y, and ~~thus~~ thus linearly dependent. resulting in no unique solutions.

Thus minimum values of M and N such that $A\theta = b$ may expect a unique solution are

$$\boxed{M = 2 \text{ and } N = 2}$$