Lecture 3

Linear functionals.

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \longmapsto x_1^2 + x_2^2 + 2x_1x_2x_3 \cdots x_n$$

$$\shortparallel$$

$$f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = f(x)$$

Does NOT satisfy superposition property.

Fix some $a \in \mathbb{R}^n$

$$f_a : \mathbb{R}^n \longrightarrow \mathbb{R} \qquad x \longmapsto a^T x$$

$$f_a(x) = a^T x = \sum_{i=1}^{n} a_i x_i \quad \text{---}(*)$$

For any scalars $\alpha, \beta \in \mathbb{R}$

$$f_a(\alpha x + \beta y) = \alpha f_a(x) + \beta f_a(y)$$

$\longleftrightarrow$ Superposition property.

$$f_a(x+y) = a^T(x+y)$$

$$f_a(\alpha x+\beta y) = a^T(\alpha x+\beta y)$$

$$= a^T(\alpha x) + a^T(\beta y)$$

$$= \alpha(a^T x) + \beta(a^T y)$$

$$f_a(\alpha x+\beta y) = \alpha f_a(x) + \beta f_a(y)$$

<u>Definition</u>: A function $f: \mathbb{R}^n \longrightarrow \underline{\underline{\mathbb{R}}}$ is said to be a linear <u>functional</u> if it satisfies following properties:

1) Additivity : $\forall x, y \in \mathbb{R}^n$: $f(x+y) = f(x) + f(y)$

2) Homogeneity : $\forall \alpha \in \mathbb{R}$, $x \in \mathbb{R}^n$: $f(\alpha x) = \alpha f(x)$

Clearly: $f_a : x \longmapsto a^T x$ is a linear functional from $\mathbb{R}^n \longrightarrow \mathbb{R}$.

Homogeneity + additivity $\Longleftrightarrow$ superposition

$\Rightarrow$ $\qquad\qquad\qquad$ $\alpha = \beta = 1$: Additivity

$\Leftarrow$ $\qquad\qquad\qquad$ $\alpha = 1,\ \beta = 0$: Homogeneity

$f(\alpha x) = \alpha f(x)$

$f(\beta y) = \beta f(y)$

$$f(\alpha x + \beta y) = f(\alpha x) + f(\beta y)$$
$$= \alpha f(x) + \beta f(y)$$

If f satisfies superposition property, then

$$f(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k) = \alpha_1 f(x_1) + \alpha_2 f(x_2) + \cdots + \alpha_k f(x_k)$$

Interesting Q:

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear functional.

Ex: $f : x \longmapsto x_1 + x_2 + \cdots + x_n$ is a linear functional.

Ex: $f : x \longmapsto 0$, Ex: $f_i : x \longmapsto x_i$ for $1 \le i \le n$

Every linear functional (f) is a linear functional f_a defined by $f_a(x) = a^T x$ for some $a \in \mathbb{R}^n$.

$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$ ... linear combination of unit vectors.

$$f(x) = f(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n)$$
$$= x_1 f(e_1) + x_2 f(e_2) + \cdots + x_n f(e_n)$$

Let $a = \begin{bmatrix} f(e_1) \\ \vdots \\ f(e_n) \end{bmatrix} \in \mathbb{R}^n \Rightarrow f(x) = a^T x$

The representation of a linear function $f : \mathbb{R}^n \to \mathbb{R}$ as $\hat{f}(x) = a^T x = f_a(x)$ is "<u>unique</u>."

let $f(x) = a^T x$ and $f(x) = b^T x \quad \forall x \in \mathbb{R}^n$

In particular for $x = e_i$ $\quad f(x) = a^T e_i = a_i = f(e_i)$

$f(x) = b^T e_i = b_i = f(e_i)$

$\Rightarrow a_i = b_i \quad \forall i \in \{1, 2, \cdots, n\}$

$\Rightarrow a = b$

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a = b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$f : \mathbb{R}^n \to \mathbb{R}$ $\longleftrightarrow$ $a \in \mathbb{R}^n$; unique.

a linear functional

$\left[\begin{array}{l} \text{The set of all linear functionals} \\ \text{from } \mathbb{R}^n \to \mathbb{R} \end{array} \right] \longleftrightarrow \mathbb{R}^n$

$$(\mathbb{R}^n)^* \longleftrightarrow \mathbb{R}^n$$

Examples:

1) $f(x) = \dfrac{x_1 + x_2 + \cdots + x_n}{n} = \text{avg}(x) = \underbrace{\dfrac{1^T}{n}}_{= a^T} x$

2) $f(x) = x_1 + \cdots + x_n = 1^T x$

3) $f(x) = x_i = e_i^T x$

4) $f(x) = \max \{x_1, \ldots, x_n\}$: Is this a linear functional ?!

No!

Affine functions

$f : \mathbb{R}^n \to \mathbb{R}$ is called as an affine function when

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \quad \text{whenever} \quad \alpha + \beta = 1$$

$\alpha, \beta \in \mathbb{R}$

Every linear function is affine; but converse is <u>NOT</u> true.

<u>Ex:</u> $f : \mathbb{R}^n \to \mathbb{R}$ affine.

Fix $a, b \in \mathbb{R}^n$ and for every $x \in \mathbb{R}^n$

$$f(x) = a^T x + \underline{\underline{b}}$$

$$f(\alpha x + \beta y) = a^T (\alpha x + \beta y) + 1 \cdot b \qquad \because \alpha + \beta = 1$$

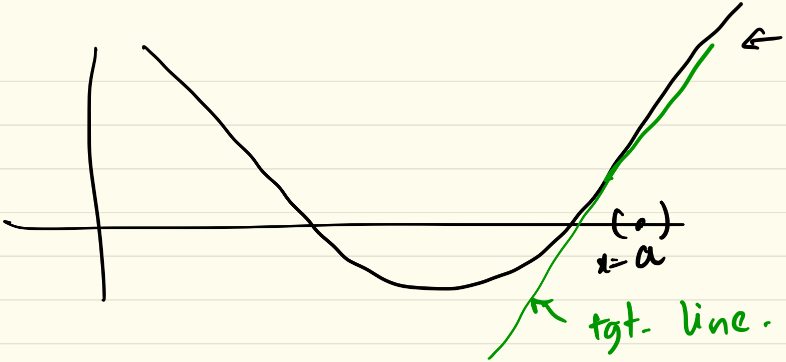
$$= \alpha a^T x + \beta a^T y + (\alpha + \beta) b$$

$$= \alpha a^T x + \alpha b + \beta a^T y + \beta b$$

$$= \alpha (a^T x + b) + \beta (a^T y + b)$$

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

$\Rightarrow f(x) = a^T x + b$ is affine.



nonlinear function.

x

tgt. line. (affine function) approximating a non-linear f^n in a small nbd. around pt a.

Every affine function from $\mathbb{R}^n \to \mathbb{R}$ is of the form $f(x) = a^T x + b$ for some a & b.

$$f(x) = f(0) + x_1\big(f(e_1) - f(0)\big) + \cdots + x_n\big(f(e_n) - f(0)\big)$$

$\underline{a_i = f(e_i) - f(0)}$; $\underline{b = f(0)}$

→ Linear functionals: $\mathbb{R}^n \to \mathbb{R}$

$\left(\begin{array}{l}\text{every linear functional } f \\ \text{from } \mathbb{R}^n \to \mathbb{R}\end{array}\right) \longleftrightarrow a \in \mathbb{R}^n$

$$f(x) = a^T x$$

→ Affine functions from $\mathbb{R}^n \to \mathbb{R}$

$\left(\begin{array}{l}\text{every affine function } f \\ \text{from } \mathbb{R}^n \to \mathbb{R}\end{array}\right) \longleftrightarrow a, b \in \mathbb{R}^n$

$$f(x) = a^T x + b$$

Regression, approximating non-linear functions by an affine function locally. (lazy learning or Just in time modeling)