Mid-semester Examination: CS31005: Algorithms II

LTP 3-1-0: Credits 4: Time 2 hours: Autumn 2015: Marks 100
All proofs and arguments must be complete and clear.
Follow lucid writing style, using suitable notation and maintaining rigour.
Make necessary assumption if and wherever required and state your assumptions clearly.

September 14, 2015, 2-5 pm

(1) We are given a set system $G(V, E)$ where V has n items and E has m elements. Here, $E \subseteq 2^V$, is a set of subsets of V. A *hitting set* $H \subseteq V$ is a subset of V such that for each hyperedge $e \in E$, we have a non-empty $E \cap H$.

Design a polynomial time algorithm for computing a small *hitting set* $H \subseteq V$ for G, using the *greedy* approximation algorithm for computing a *set cover*; so you must first create an instance of the *set cover* problem from the given instance of the *hitting set* problem. Derive the approximation ratio, time complexity and space complexity for such a hitting set computation as stated above in terms of n and m. Write precisely and explain with clarity.

[5+3+4+3 marks]

(2) Consider computing a *set cover* $\mathcal{C}$ using the *greedy set cover* heuristic, where we are given a collection $\mathcal{S}$ of subsets of the *universal* set U of *elements*. Let c_x be the *price* assigned to element $x \in U$ in the greedy set cover heuristic. Show that the number of sets in the set cover $\mathcal{C}$ computed by the greedy heuristic for covering all the elements of U is $\sum_{x \in U} c_x$. Also, show that $\sum_{S \subseteq \mathcal{C}^*} \sum_{x \in S} c_x \geq |\mathcal{C}|$, where $\mathcal{C}^*$ is a collection of subsets of U representing a *minimum cardinality* set cover.

[5+5 marks]

(3) Let $G(V, E)$ be an undirected graph with n vertices and m edges. Consider a partition of the vertex set V into three sets $V1$, $V2$ and $V3$. Let $f(V1, V2, V3)$ be the number of edges that run between parts V_i and V_j where $i \neq j$, and $1 \leq i, j \leq 3$. Let OPT be the maximum value of $f(V1, V2, V3)$ over all possible ways of partitioning V into three sets $V1$, $V2$ and $V3$. Now consider the following algorithm for partitioning V into three sets. We randomly, independently and uniformly distribute the vertices into the three sets. Let such a random distribution yield the partition into three sets $W1$, $W2$ and $W3$. Show that the expected value of $f(W1, W2, W3)$ in such a random partition is at least $\frac{2}{3}OPT$.

[10 marks]

(4) Let $\mathcal{S} = \{S_1, S_2, ..., S_m\}$ be a collection of m subsets of a finite set $U = \{e_1, e_2, ..., e_n\}$.

- Write the 0-1 integer program for the *maximum cardinality* set cover problem, where a 0-1 indicator variable x_{S_i} is used to indicate the inclusion of the set $S_i \in \mathcal{S}$ in a set cover $C \subseteq \mathcal{S}$. Clearly mention the objective function for this 0-1 integer program.
- Write the Primal linear program corresponding to the 0-1 integer program above.
- Construct the Dual linear program from the Primal linear program above.
- Show that the greedy maximum cardinality set cover heuristic assigns $price(e_{i_j}) \leq \frac{1}{k-j+1}$, $1 \leq j \leq k$, for each element e_{i_j} in each set $S_i \in \mathcal{S}$, where $|S_i| = k$. Here, each e_{i_j} is an element of U. Also, note that the set S_i may or may not belong to the set cover C computed using the greedy heuristic.

- Show that the quantities $\frac{price(e_j)}{H(n)}$ constitute a feasible solution to the above Dual linear program where $1 \leq j \leq n$.

[5+5+5+5 marks]

(5) Let $G(V, E)$ be an undirected connected graph with n vertices and m edges. For every edge $e \in E$, let $w(e)$ be the positive rational weight of the edge e. Let $MST(G)$ be any *minimum weighted spanning tree* of G. Let $SPT(v, G)$ be the tree resulting from the computation of weighted shortest paths using Dijsktra's algorithm starting at vertex v, where the path $SP(w, v)$, from any vertex w to v in $SPT(v, G)$, is the weighted shortest path from w to v. Let $DFS(v, G)$ be a *depth first search tree* rooted at vertex v in G. Let $PMST(v, G)$ be the $MST(G)$ constructed by Prim's algorithm starting at root v. Answer the following questions generating non-trivial but small examples of graphs as appropriate for each case.

1. Show that $DFS(v, G)$ may have multiple children at root v.
2. Show that $DFS(v, G)$ may have multiple children at a vertex $w \neq v$.
3. Show a small example of a connected undirected weighted graph $G(V, E)$, where $SPT(v, G)$ and $PMST(v, G)$ could have the same set of $n - 1$ edges of G.
4. Show a small example of a connected undirected weighted graph $G(V, E)$, where $SPT(v, G)$ and $PMST(v, G)$ may not have the same set of $n - 1$ edges of G.
5. Suppose G has a Hamiltonian cycle, that is, a cycle with exactly n vertices. Show that the sum of weights in such a cycle must be at least the sum of weights of edges in any $MST(G)$.

[5+5+5+7+3 marks]

(6) Design an algorithm for computing the maximum independent set in a tree. An independent set is a set of vertices that do not share edges in the graph.
[10 marks]

(7) Construct a small but non-trivial connected undirected bipartite graph that has equal number of vertices in both partites and which does not have a perfect matching. A perfect matching is a matching that has as many edges as the number of vertices in each partite.
[10 marks]