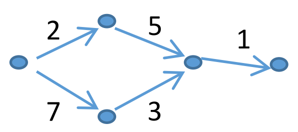
Solution Sketch for Tutorial-2

1. Suppose a graph, in addition to capacities on edges, has a capacity on every vertex such that the total inflow (and therefore the total outflow from the node) cannot exceed the capacity of the vertex. The other constraints remain the same as in the standard maximum-flow problem. How will you find the maximum flow in this graph?

   [Solution Sketch]
   - Replace each vertex v with two vertices v1 and v2, with a directed edge from v1 to v2 with capacity equal to the vertex capacity.
   - For each edge (u, v), replace (u, v) with (u,v1) with same capacity
   - For each edge (v, w), replace (v, w) with (v2, w) with same capacity
   - Now you have a graph with only edge capacities. Find the maximum flow in it.

2. Suppose all capacities in a graph are distinct. Will the max-flow be unique?

   [Solution Sketch] No. Consider the following graph. It is obvious that the max flow is not unique (can use either the 2-5 capacity path or the 7-3 capacity path to achieve the max flow of 1.



3. Suppose that a maximum flow is already computed in a graph with integral capacities. Then the capacity of exactly one edge is increased by 1. Which of the following is true about the complexity of finding the new maximum flow?
   a) O(1), as the max flow will not change
   b) O(1), as the capacity of only 1 edge is increased by 1
   c) O(V+E)
   d) O(VE)

   (c) is true. This is because increasing the capacity of an edge by 1 can either make the maxflow remain the same or increase the maxflow by at most 1 (Can you prove this now that you know that maximum flow value = capacity of minimum cut? Just think whether the edge whose capacity is increased belongs to the minimum cut or not). So the new max flow will be found by at most one step of finding the augmenting path and augmenting capacity as the flows are integral, and if an augmenting path is found, it must increase the flow value by at least 1. This can be done in O(V+E) time. (Technically, O(VE) is also true, but we are looking for the tightest bound possible among the given ones).

4. A company running a factory has to arrange for extra maintenance staff to work over vacation periods. There are K vacation periods in the year, numbered from 1 to K, with the i-th vacation period having d_i days.There are N maintenance staff available numbered from 1 to N, with the j-th maintenance being available for a total of c_j days over all vacation periods. In addition, for each vacation period i, each staff can only work on a subset of the d_i days (for example, if a vacation period spans over Friday, Saturday and Sunday, staff 1 may be available for only Friday and Saturday but not Sunday, staff 2 may be available for only Friday and Sunday but not Saturday, and staff 3 may be available for all days). What is the maximum no. of vacation days that can be covered by the company with its available staff?

   [Solution Sketch] Create a bipartite graph with two partitions X and Y. X contains one node for each maintenance staff (say node j for staff j). Y contains one node for each vacation day of each vacation period (basically all vacation days, so if you have 3 vacation periods with 2, 4, and 3 days respectively, you will have 2+4+3 = 9 nodes in Y). Add an edge from node j in X to node i in Y if maintenance staff j can work on the vacation day represented by node i. Now add two vertices s and t. Add an edge from s to each node j in X with capacity c(i). Add an edge from each node i in Y to t with capacity 1. Now find the maximum flow from s to t. The value of the flow is the required answer.

5. A college has N students $X_1, X_2, \ldots, X_N$, M departments $D_1, D_2, \ldots, D_M$, and P societies $S_1, S_2, \ldots, S_P$. Each student is enrolled in exactly one department, and is a member of at least one society. The college has a student association (like your gymkhana) with one member from each society (You can assume that every society has at least one member). However, the society members have to be chosen such that the student association has at most Q_k members from any department D_k. Design an algorithm to form the association. Will you be able to form such an association always? You must model the problem as a maximum flow problem.

   [Solution Sketch] Create a graph as follows:
   - For each society S_i, add a node $V(S_i)$
   - For each student X_j, add a node $V(X_j)$
   - For each department D_k, add a node $V(D_k)$
   - Add a source node s and a sink node t
   - Add an edge from s to each node $V(S_i)$ with capacity 1
   - Add an edge from a node $V(S_i)$ to a node $V(X_j)$ with capacity 1 if the student X_j is a member of society S_i
   - Add an edge from a node $V(X_j)$ to a node $V(D_k)$ with capacity 1 if the student X_j belongs to department D_k
   - Add an edge from each node $V(D_k)$ to t with capacity Q_k

Compute the maximum flow in this graph. If the maximum flow value is = P, the number of societies, then the answer is yes, and in that case, the edges $(V(S_i), V(X_j))$ for which the flow is 1 will indicate that the student X_j is added to the student association as a representative of society S_i.