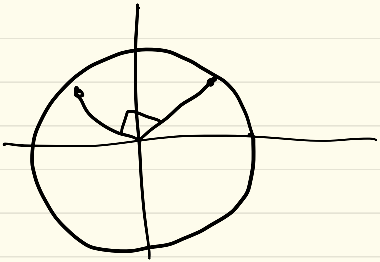
Class Sept. 18



Vector spaces

$$\boxed{x \in \mathbb{R}^n} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

norm, inner product, linear dependence, basis, orthogonality, G-S. algorithm.

Matrices :

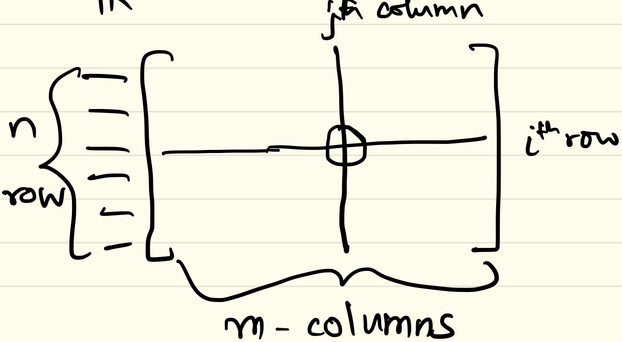
Matrix :

└ 2-d list of objects / real numbers $\qquad \mathbb{R}^{m\times n}$

$\mathbb{R}^{n\times m}$ ← rows, ← columns

$\mathbb{R}^n$ — $\mathbb{R}^{n\times 1}$ — vectors

$$x \in \mathbb{R}^n \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$



$$A \in \mathbb{R}^{n\times m}$$

$$\underline{A_{ij} = a_{ij} = A(i,j)}$$

$$x = (x_i)_{i=1}^{n}$$

$$A = [a_{ij}]_{i,j} \in \mathbb{R}^{n\times m}$$

$$1 \le i \le n; \quad 1 \le j \le m.$$

$$A = \begin{bmatrix} | & | & | & \\ | & | & | & \cdots \\ | & | & | & \end{bmatrix}_{n\times m} \in \mathbb{R}^{n\times m}$$

has nm - entries.

Image - matrix 28×28

$n = 28,\ m = 28$

square matrix

$n > m$

tall matrix.

$n < m$

wide matrix.

$$\mathrm{vec}(A) = \begin{bmatrix} | \\ | \\ | \\ | \\ | \\ | \end{bmatrix} \in \mathbb{R}^{nm}$$

$$\mathbb{R}^{n\times m} \xrightarrow[\text{Vectorization operation}]{} \mathbb{R}^{nm}$$

<u>Matrix addition</u>: $A, B \in \mathbb{R}^{n \times m}$

$$C = A + B \quad \text{where} \quad C_{ij} = A_{ij} + B_{ij}$$

for $1 \le i \le n$
$1 \le j \le m$

<u>Scalar multiplication</u>: for any $\alpha \in \mathbb{R}$, $A \in \mathbb{R}^{n \times m}$

$$(\alpha A)_{ij} = \alpha A_{ij}$$

<u>Check</u>: $\mathbb{R}^{n \times m}$ is a real vector space. (with matrix addition & scalar multi.)

$0 \in \mathbb{R}^{n \times m}$: Additive identity (zero "vector") matrix.

Similarly for $-1 \in \mathbb{R}$

$(-1 \cdot A)_{ij} = -A$: Additive inverse.

Ex: $\mathbb{R}^{2\times 3}$: real vector space.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \in \mathbb{R}^{2\times 3}$$

$$E_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\} = B$$

i) $\boxed{\mathrm{span}(B) = \mathbb{R}^{2\times 3}}$

ii) Is B linearly independent?

$$\beta_{11} E_{11} + \beta_{12} E_{12} + \beta_{13} E_{13} + \beta_{21} E_{21} + \beta_{22} E_{22} + \beta_{23} E_{23} = 0$$

0 matrix of size 2×3

$$\Rightarrow \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Consider $\{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}, Y\}$: $\boxed{\text{linearly dependent.}}$

where $Y \in \mathbb{R}^{2\times 3}$ is any arbitrary matrix.

$\mathbb{R}^{n\times m}$ over $\mathbb{R}$

$B = \{E_{ij}\}$

$(P)_{l,k}$

matrix

$$(E_{ij})_{l,k} = \begin{cases} 1 & \text{if } i = l,\ j = k \\ 0 & \text{otherwise} \end{cases}$$

$1 \le i, l \le n$

$1 \le j, k \le m$

$$(e_i)_j = \begin{cases} 1 & i = j \\ 0 & i \ne j \end{cases}$$

unit vectors in $\mathbb{R}^n$, $1 \le i, j \le n$

$\mathbb{R}^{n\times n}$: over $\mathbb{R}$

$$I_n = I_{n\times n} = I = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

$$(I)_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \qquad 1 \le i,j \le n$$

Matrix vector multiplication.

Let $A \in \mathbb{R}^{n\times m}$, $b \in \mathbb{R}^{m}$ $\qquad y = \underline{Ab} \in \mathbb{R}^{n}$

Define matrix vector multiplication.

$$\underbrace{\begin{bmatrix} \text{———} \; a_1^T \; \text{———} \\ \text{———} \; a_2^T \; \text{———} \\ \text{———} \; a_3^T \; \text{———} \\ \vdots \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} | \\ | \\ | \end{bmatrix}}_{b} = \begin{bmatrix} \leftarrow \text{scalar} \\ \leftarrow \text{scalar} \\ \leftarrow \text{scalar} \end{bmatrix}$$

$$= (y)_i = a_i^T b$$

$$\begin{bmatrix} - a_1^T - \\ - a_2^T - \\ \vdots \\ - a_n^T - \end{bmatrix}_{n\times m} [b]_{m\times 1} = \begin{bmatrix} a_1^T b \\ a_2^T b \\ \vdots \\ a_n^T b \end{bmatrix}_{n\times 1}$$

A $\qquad$ Ab

a_i^T : i^{th} row of matrix A.

$$(Ab)_i = a_i^T b \quad \text{for } i = 1, 2, \dots, n$$

Fix a matrix $A \in \mathbb{R}^{n\times m}$

for any vectors $x, y \in \mathbb{R}^m$ and a scalar $\alpha \in \mathbb{R}$

$$A(x+y) = \begin{bmatrix} a_1^T(x+y) \\ a_2^T(x+y) \\ \vdots \\ a_n^T(x+y) \end{bmatrix} = \underbrace{\begin{bmatrix} a_1^T x + a_1^T y \\ a_2^T x + a_2^T y \\ \vdots \\ a_n^T x + a_n^T y \end{bmatrix}} = \underbrace{\begin{bmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_n^T x \end{bmatrix} + \begin{bmatrix} a_1^T y \\ a_2^T y \\ \vdots \\ a_n^T y \end{bmatrix}}$$

$$\left.\begin{aligned} A(x+y) &= Ax + Ay \\ A(\alpha x) &= \alpha A x \end{aligned}\right\} (*)$$

In general, $\underbrace{\underset{\mathbb{R}^{n\times m}}{A}(\underbrace{\alpha x+\beta y}_{\mathbb{R}^m})}_{\mathbb{R}^n} \overset{??}{=} \underbrace{\alpha \underset{\mathbb{R}^n}{Ax} + \beta \underset{\mathbb{R}^n}{Ay}}_{\mathbb{R}^n}$ for a fixed $A \in \mathbb{R}^{n\times m}$

for any $x, y \in \mathbb{R}^m$ & $\alpha, \beta \in \mathbb{R}$

$$\underbrace{A(\alpha x+\beta y) = \alpha Ax + \beta Ay}$$: superposition property.

$$[a_{ij}]_{n\times m} \quad [b_j]_{\mathbb{R}^m} \longrightarrow [Ab]_n$$