Norms & Distances $\qquad$ 2-norm / Euclidean distance.

standard - deviation <-> norm
of a vector

Angle between the vectors.

Let $x \in \mathbb{R}^n$, $\quad \text{avg}(x) = \frac{\mathbb{1}_n^T}{n} x = \frac{x_1 + x_2 + \cdots + x_n}{n}$

Define $\hat{x} = x - \text{avg}(x) \mathbb{1}_n = x - (\mathbb{1}^T x / n) \mathbb{1}$

$$\text{std}(x) = \text{rms}(\hat{x})$$

$$\text{std}(x) = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \text{avg}(x))^2}{n}}$$

$$\text{std}(x) = \frac{\| x - (\mathbb{1}^T x / n) \mathbb{1} \|_2}{\sqrt{n}}$$

Note: $\mathrm{std}(x) = \dfrac{\|\hat{x}\|_2}{\sqrt{n}}$ ← $n-1$

Computational complexity:

$x^T y = \sum_{i=1}^{n} x_i y_i$

n - multiplications
$n-1$ - additions
$\overline{}$
$2n-1$

$\|x\|_2 = \sqrt{x^T x}$

$x^T x$: $\underset{\approx}{}$ $2n-1$ operations ≈ n operations
then square root : 1 operations
$\overline{}$
$2n$ operations

$\mathrm{avg}(x)$: $\dfrac{1^T x}{n}$

$1^T x$: $2n-1$ operations n
division by n 1 operation.

$(n+1)$: operations ≈ n operations

$$\mathrm{rms}(x) = \sqrt{\frac{\Sigma x_i^2}{n}} = \frac{\sqrt{x^T x}}{\sqrt{n}}$$

2n : operations
1 : divison
1 : square root.

$$\mathrm{std}(x) = \frac{\| x - (1^T x)/n \cdot 1 \|_2}{\sqrt{n}}$$

Avg: $\simeq n$
subtraction $\simeq n$
norm $\approx 2n$ operations
1 sq. root
1 divison by $\sqrt{n}$

Average, RMS value, SD. For $x \in \mathbb{R}^n$

$$[\text{rms}(x)]^2 = [\text{avg}(x)]^2 + [\text{std}(x)]^2 \quad \text{— (A)}$$

$$[\text{std}(x)]^2 = \frac{1}{n} \| x - (\mathbf{1}^T x)/n \, \mathbf{1} \|_2^2$$

$$= \frac{1}{n} \Big[x^T x - 2x^T \boxed{(\mathbf{1}^T x)/n} \, \mathbf{1}$$

$$+ \Big(\frac{\mathbf{1}^T x}{n} \mathbf{1} \Big)^T \Big(\frac{\mathbf{1}^T x}{n} \mathbf{1} \Big) \Big]$$

$$= \frac{1}{n} \Big(x^T x - \frac{2}{n} (\mathbf{1}^T x)^2 + n \Big(\frac{\mathbf{1}^T x}{n} \Big)^2$$

$x^T \mathbf{1} = \mathbf{1}^T x$ $n = \mathbf{1}^T \mathbf{1}$

$$= \frac{x^T x}{n} - \Big(\frac{\mathbf{1}^T x}{n} \Big)^2$$

$$[\text{std}(x)]^2 = [\text{rms}(x)]^2 - [\text{avg}(x)]^2$$

1. Interprete this formula.
2. Computation complexity.

Chebyshev Inequality:

at least

Let. $x \in \mathbb{R}^n$ & let there be k entries in x

s.t. $|x_i| > a$ for some $a > 0$. Then

$$\frac{k}{n} \leq \left(\frac{\text{rms}(x)}{a}\right)^2$$

Apply Chebyshev inequality on $\hat{x} = x - \text{avg}(x)\,\mathbb{1}$

For some $a > 0$, there are k entries in $\hat{x}$

$$|x_i - \text{avg}(x)| \geq a$$

$$\text{CI} \Rightarrow \quad \frac{k}{n} \leq \left(\frac{\text{std}(x)}{a}\right)^2$$

$a = 3\,\text{std}(x)$

$$|x_i - \text{avg}(x)| > 3\,\text{std}(x)$$

$$\frac{k}{n} \leq \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

$= (11 \sim 12)\%$

(ii) $\mathrm{avg}(x) - 3\,\mathrm{std}(x) \le x_i \le \mathrm{avg}(x) + 3\,\mathrm{std}(x)$

How many such x_i ?? at least $1 - \frac{1}{9} = \frac{8}{9}$

$\approx 89\%$

Standardization of $x \in \mathbb{R}^n$.

$$\tilde{x} = \frac{x - \mathrm{avg}(x)\,\mathbf{1}}{\mathrm{std}(x)} \in \mathbb{R}^n$$

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix} = \text{score vector}$$

$\tilde{x}_i = -1.3$

$\tilde{x}_j = 0.2$

Ex: Computational complexity of $\tilde{x}$??

Angle:

C-S inequality:

For any $x, y \in \mathbb{R}^n$

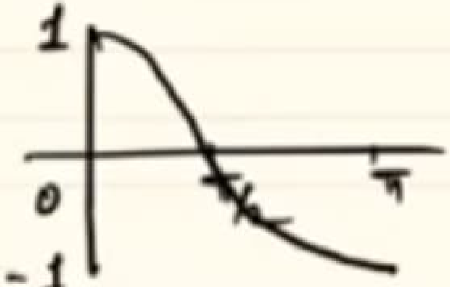
$$|x^T y| \leq \|x\|_2 \|y\|_2$$

$$\Rightarrow \frac{|x^T y|}{\|x\|_2 \|y\|_2} \leq 1$$

$$\Rightarrow -1 \leq \frac{x^T y}{\|x\|_2 \|y\|_2} \leq 1$$

The angle between the vectors x & $y \in \mathbb{R}^n$ is defined as

$$\theta = \arccos\left(\frac{x^T y}{\|x\|_2 \|y\|_2}\right)$$



$$\Rightarrow \cos\theta = \frac{x^T y}{\|x\|_2 \|y\|_2}$$

$$\Rightarrow x^T y = \|x\|_2 \|y\|_2 \cos\theta$$

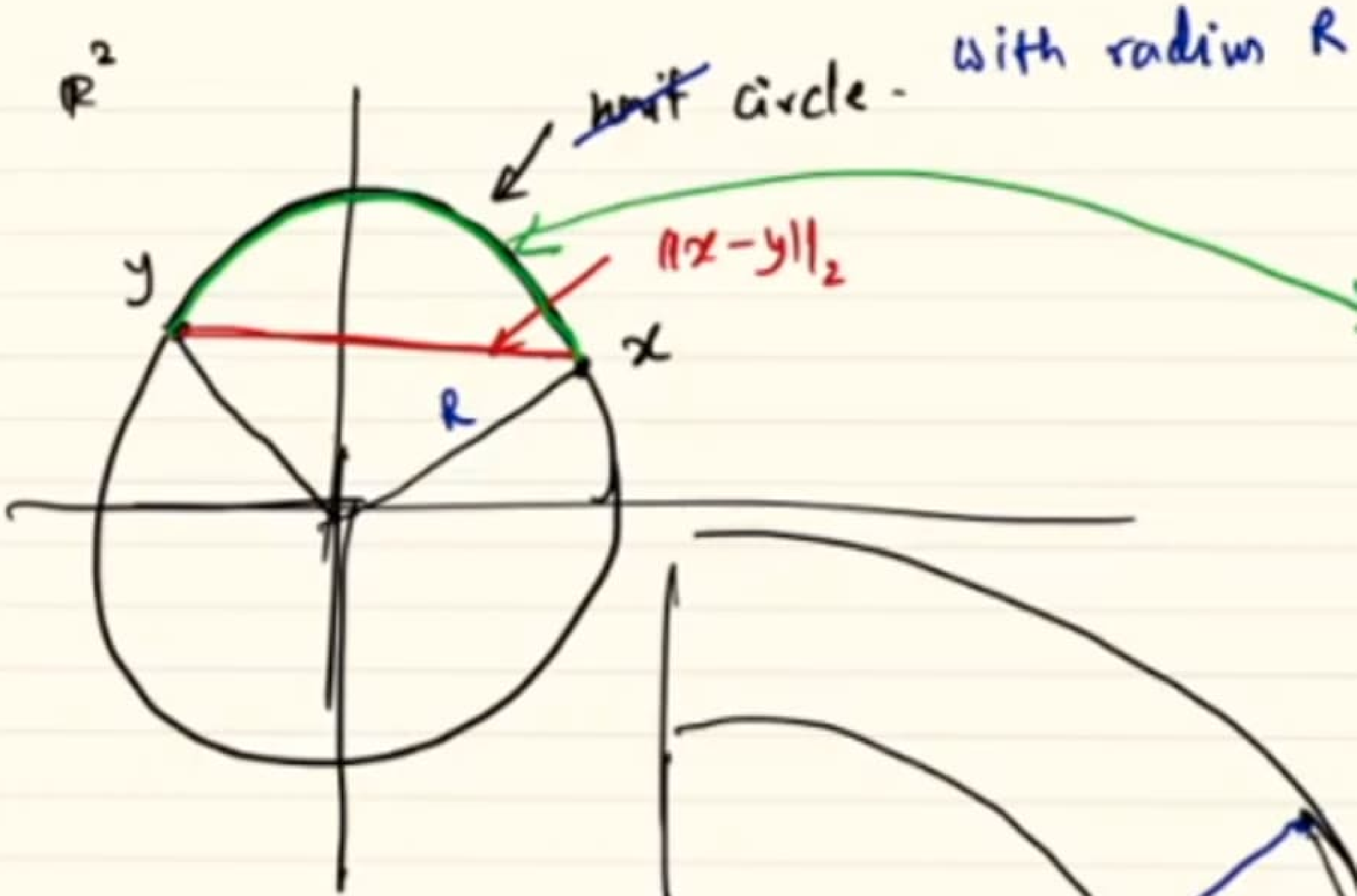
1) $x^T y = 0 \Rightarrow \theta = \pi/2$ orthogonal / perpendicular vectors.

2) If $\theta = 0$, $\cos\theta = 1 \Rightarrow x^T y = \|x\|_2 \|y\|_2$

3) If $\theta = \pi$, $\cos\theta = -1 \Rightarrow x^T y = -\|x\|_2 \|y\|_2$

4) $0 < \theta < \pi/2$: acute angle.

5) $\pi/2 < \theta < \pi$: obtuse angle.



circular distance $= \left(\text{angle between } x \,\&\, y\right) R$

$x \in \mathbb{R}^n$; $\hat{x} = x - \text{avg}(x)\mathbf{1}$

$\text{std}(x)$: $\text{rms}(\hat{x})$ $\qquad \tilde{x} = \dfrac{\hat{x}}{\text{std}(x)}$: stadardization

Chebyshev inequality for $\hat{x}$.

Computational complexity :

Angle:

distance based on angles between the vectors.