Linear algebra for AI and ML

August 25

(Lecture #4)

$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

Q: Does there exist $x \in \mathbb{R}^n$ s.t.

$$Ax = b$$

x: solution of this system of linear eq^{ns}.

$\underline{Ax} \rightarrow x$ - linear combination of columns of A.

$= a_1 x_1 + \dots + a_n x_n$ where $A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$\text{rank}(A) = \dim(\text{colspan}(A))$

$b \in \text{colspace}(A) = \text{span}\{a_1, \dots, a_n\}$: Existence

If columns of A form basis for colspace(A), then $\exists$ a unique x s.t. $Ax = b$: uniqueness

System of linear eqns:

i) Understanding existence / uniqueness of solns

ii) Computation of solution. (LU, QR)

iii) Sensitivity analysis.

(square, tall)

$$Ax = b \quad —(1)$$

1/3

(SVD)

$$\underbrace{(A+\Delta A)}\tilde{x} = \underbrace{(b+\Delta b)} \quad —(2)$$

Matrix norms:

scalar eqn:

$$ax = b \qquad a \in \mathbb{R},\ b \in \mathbb{R} \qquad m = n = 1$$

$$x = \frac{1}{a} b \qquad a \neq 0$$

$$x = a^{-1} b$$

generalize

$$Ax = b \qquad A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^{m}$$

Left inverse of a matrix.

A matrix X that satisfies $XA = I$ is called left inverse of A.

$$A \in \mathbb{R}^{m \times n}\ ;\ X \in \mathbb{R}^{n \times m}$$

Ex:

1) $A = [a]$, $x = \left[\frac{1}{a}\right] = a^{-1}$, $I = [1]$

2) $A = \begin{bmatrix} \\ \\ \end{bmatrix} \in \mathbb{R}^{n \times 1}$ $\qquad xA = I = [1]$

$A = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ $\qquad \boxed{x = \frac{1}{3}\left[1 \quad \frac{1}{2} \quad \frac{1}{3}\right]}$

$xA = 1$

$\boxed{\frac{1}{a_j} e_j^T}$ LI

Left inverse is <u>NOT</u> unique.

3) $A \in \mathbb{R}^{m \times n}$ such that

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \qquad a_j \in \mathbb{R}^m \qquad j = 1, 2, \ldots, n$$

The columns of A are orthonormal.

$$a_i^T a_j = \langle a_i, a_j \rangle = \delta_{ij} \qquad i, j = 1, 2, \ldots, n$$

$$\delta_{ij} = 1 \quad \text{if } i = j \qquad\qquad A^T A = I_{n \times n}$$
$$= 0 \quad \text{if } i \neq j$$

4) $A = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}$, $B = \frac{1}{9}\begin{bmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$

$$\boxed{C = \frac{1}{8}\begin{bmatrix} -11 & -10 & 15 \\ 7 & 8 & -12 \end{bmatrix}} \times$$

Left inverse and column independence.

If A has left inverse C, then the columns of A are linearly indep.

Pf: Assume that $Ax = 0$

To prove: $x = 0$

Given: $CA = I$

$$0 = C0 = C(Ax) = (CA)x = Ix$$
$$= x$$

$$\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\Rightarrow x_1 \begin{bmatrix} a_1 \end{bmatrix} + x_2 \begin{bmatrix} a_2 \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_n \end{bmatrix} = 0$$

$$\Rightarrow x_1 = 0;\ x_2 = 0,\ \ldots,\ x_n = 0$$

$$\overset{p}{\begin{bmatrix} A \text{ has left inverse} \end{bmatrix}} \Rightarrow \overset{q}{\begin{bmatrix} \text{cols of } A \text{ are linearly indep.} \end{bmatrix}}$$

$p \Rightarrow q$

$\equiv$

$-q \Rightarrow -p$

This observation implies that wide matrices (matrices with more number of columns than rows) can not have left inverse.

How can one use left inverse to compute the solution of $Ax = b$??

Assume: A is left invertible and C is a left inverse of A.

$Ax = b$ has a solution.

Guess: $Cb = x$

$$Cb = C(Ax) = (CA)x = Ix = x$$

Example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad ; \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{Ax = b}$$

$$A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\tilde{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

A^T is a left inverse.

$$\underbrace{Cb}_{} = A^T b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = x$$

$$Ax = b$$

Left inverse:

1) Given A, x is a left inverse if

$$xA = I$$

2) Wide matrices can not have left inverse.

3) $\left(\begin{array}{c}A \text{ is left} \\ \text{invertible}\end{array}\right) \Rightarrow \left(\begin{array}{c}\text{cols of } A \\ \text{are lin. indep.}\end{array}\right)$

Is the converse true??

4) $Ax = b$ is s.t. soln exists. ($b \in$ colspace A)

Then $x = Cb$ is a soln where $CA = I$

5) Left inverse can be used to determine if a soln to $Ax = b$ exists or not.