1. Let P and Q be two problems in NP such that the same polynomial-time reduction f can be used in the reductions $P \leqslant Q$ and $Q \leqslant P$. Prove/Disprove: f must be a bijection.

False. Let G be an undirected graph. Consider the following problems on G.

EVC: Decide whether G has a minimal vertex cover of even size
OVC: Decide whether G has a minimal vertex cover of odd size

(The covers are minimal in the sense that removing any vertex from the cover fails to cover all the edges. Minimum covers are minimal, but not necessarily conversely. These problems are clearly in NP.)

Consider the following reduction that works for both EVC $\leqslant$ OVC and OVC $\leqslant$ EVC. This is clearly not a bijection (every application increases the number of vertices by two).

2. Every instance of a problem in NP can be encoded in binary. Without loss of generality, we can therefore assume that the space of input instances of all problems in NP is $\{0,1\}^*$. Invalid encodings can be assumed to belong to the REJECT set.

The intersection $P \wedge Q$ of two problems P and Q in NP is the problem having

$$\mathrm{Accept}(P \wedge Q) = \mathrm{Accept}(P) \cap \mathrm{Accept}(Q).$$

(a) Prove/Disprove: The class NP is closed under intersection.

A cetificate for I in Accept($P \wedge Q$) is a certificate of I in Accept(P) concatenated with a certificate of I in Accept(Q).

(b) Prove/Disprove: The class of NP-complete problems is closed under intersection.

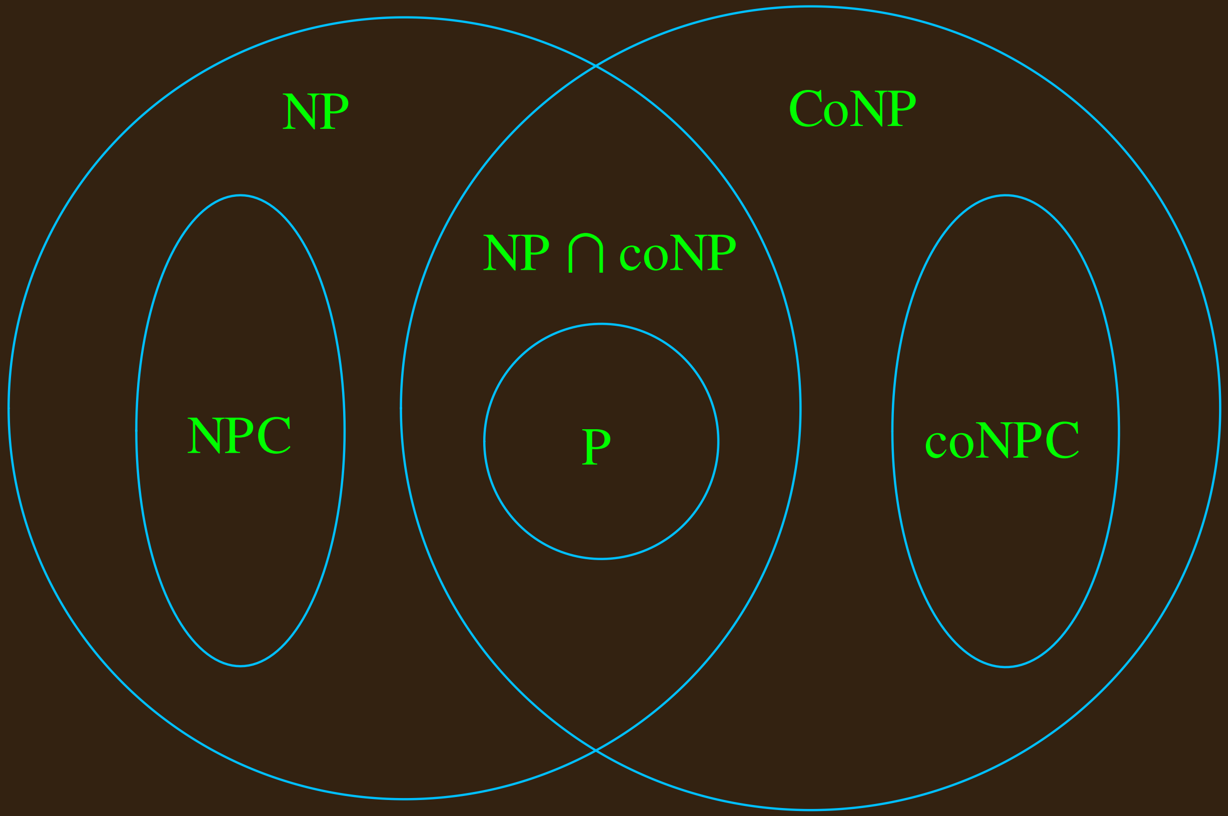
False. Let G be an undirected graph with n vertices. Consider the following problems.

LARGE_CLIQUE: Decide whether G has a clique of size $\geqslant n/2 + 1$
LARGE_IS: Decide whether G has an independent set of size $\geqslant n/2 + 1$

First, prove that these problems are NP-complete (use reductions from CLIQUE and IND_SET).

Then, note that the intersection of these problems is trivial, that is, $\text{Accept}(P \wedge Q) = \varnothing$ and $\text{Reject}(P \wedge Q) = \{0, 1\}^*$.

Hierarchy of complexity classes



3. Let φ and ψ be two Boolean formulas of the same variables. Let SAME be the problem of deciding whether φ and ψ evaluate to the same truth values for all assignments of the variables. What type of problem is SAME?

A truth assigment of the variables for which the two formulas evaluate to different truth values is a succinct disqualification for an instance in Reject(SAME). So SAME is in coNP.

It is coNP-complete too. Use the reduction TAUTOLOGY $\leqslant$ SAME that maps φ to $(\varphi, 1)$.