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19CS30048

1) Given:

$P_n(R)$ = {Set of all polynomials in $x$ with real coeff}

$P_n(R) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

$a_0, a_1, a_2, a_3 \ldots a_n \in R$

(a) Proof that $P_n(R)$ is a vector space.

Addition.

(i) $p = a_0 + a_1 x + \cdots a_n x^n$, $p \in P_n(R)$

$q = b_0 + b_1 x + \cdots b_n x^n$, $q \in P_n(R)$

$p+q = (a_0+b_0) + (a_1+b_1)x + \cdots + (a_n+b_n)x^n$

$p+q \in P_n(R)$

(ii) $[a_0 + a_1 x + \cdots + a_n x^n] + \underbrace{[-a_0 - a_1 x - \cdots - a_n x^n]}_{\text{Additive Inverse}} = 0$

(iii) $(p+q)+r = p+(q+r)$

$[(a_0+b_0) + (a_1+b_1)x + \cdots (a_n+b_n)x^n] + (c_0 + c_1 x + \cdots c_n x^n]$

$= [a_0 + a_1 x + \cdots + a_n x^n] + [(b_0+c_0) + (b_1+c_1)x + \cdots + (b_n+c_n)x^n]$

(iv) $(a_0 + a_1 x + \cdots a_n x^n) + 0 = a_0 + a_1 x + \cdots + a_n x^n$.

(V) $(a_0+b_0) + (a_1+b_1)x + \cdots + (a_n+b_n)x^n = (b_0+a_0) + (b_1+a_1)x + \cdots (b_n+a_n)x^n$.

Scalar Multiplication.

(i) $\alpha(a_0 + a_1x + \dots + a_nx^n) = (\alpha a_0) + (\alpha a_1)x + \dots + (\alpha a_n)x^n$

(ii) $(\alpha\beta)(a_0 + a_1x + \dots + a_nx^n) = (\alpha\beta a_0) + (\alpha\beta a_1)x + \dots + (\alpha\beta a_n)x^n$

$= \alpha(\beta a_0 + \beta a_1 x + \dots + \beta a_n x^n) = \beta(\alpha a_0 + \alpha a_1 x + \dots + \alpha a_n x^n)$

(iii) $1 * (a_0 + a_1x + \dots a_nx^n) = a_0 + ax + \dots + a_nx^n$.

(iv) $(\alpha+\beta)(a_0 + a_1x + \dots + a_nx^n) = (\alpha a_0 + \alpha a_1 x + \dots + \alpha a_n x^n) + (\beta a_0 + \beta a_1 x + \dots + \beta a_n x^n)$

(v) $\alpha(a_0 + ax + \dots + a_nx^n + b_0 + b_1x + \dots + b_nx^n) = (\alpha a_0 + \alpha a_1 x + \dots \alpha a_n x^n) + (\beta b_0 + \beta b_1 x + \dots \beta b_n x^n)$

(b) $f(p(x)) = \frac{d}{dx} p(x) \Big|_{x=0}$

$f(\alpha p(x) + \beta q(x)) = \frac{\alpha\, dp(x)}{dx}\Big|_{x=0} + \frac{\beta\, dq(x)}{dx}\Big|_{x=0}$

$= \cancel{\alpha\beta q}$

$= \alpha f(p(x)) + \beta f(q(x))$

$\Rightarrow f(p(x))$ is a linear functional

Hence Proved.

(c) $p(x) = a_0 + a_1 x + \cdots + a_n x^n$

$$p = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$f(p(x)) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} \Big|_{x=0}$$

$$= a_1$$

$$\boxed{f[p(x)] = e_1^T \cdot p}$$