Linear algebra for AI & ML

(August-12)

Linear combination:

Let $x_1, x_2, \ldots, x_m \in \mathbb{R}^n$ $\qquad x_1 = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \in \mathbb{R}^n$

$\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$

new vector: $\underbrace{\alpha_1 x_1}_{\uparrow \text{ scalar multiplication}} + \alpha_2 x_2 + \cdots + \alpha_m x_m$ } linear combination of vectors.

vector addition

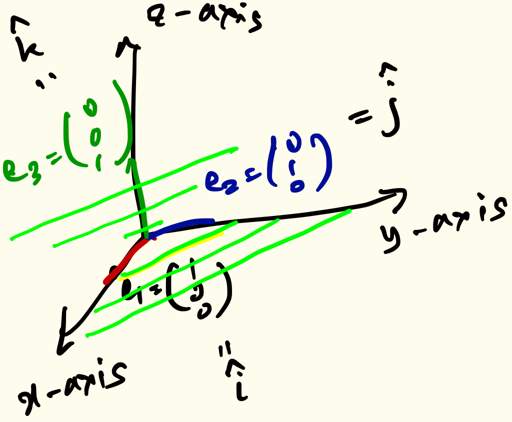
Special vectors:

$\mathbb{1}_n = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$ $\qquad 0_n = 0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$

unit vectors:

$e_1, e_2, \ldots, e_n \in \mathbb{R}^n$ $\qquad e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$ i^{th} entry

$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$



Let $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$

$$a = a_1 e_1 + a_2 e_2 + \dots + a_n e_n$$

$$a = \sum_{i=1}^{n} a_i e_i \in \mathbb{R}^n$$

Every vector in $\mathbb{R}^n$ can be written as a linear combination of unit vectors.

Define a set $S = \operatorname{span}\{x_1, \dots, x_m\} \subseteq \mathbb{R}^n$

$$S = \{\alpha_1 x_1 + \dots + \alpha_m x_m \mid \alpha_1, \dots, \alpha_m \in \mathbb{R}\}$$

$S = \operatorname{span}\{e_1, e_2\} \subseteq \mathbb{R}^3$ $\quad S = \{\alpha_1 e_1 + \alpha_2 e_2 \mid \alpha_1, \alpha_2 \in \mathbb{R}\} = \left\{\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix} : \alpha_1, \alpha_2 \in \mathbb{R}\right\}$

Special linear combinations:

$x_1, \dots, x_m \in \mathbb{R}^n$

1) $\boxed{x_1 + x_2 + \dots + x_m}$ vector in $\mathbb{R}^n$

2) $\frac{1}{m}(x_1 + x_2 + \dots + x_m)$: avg.

$\alpha_1 = \alpha_2 = \dots = \alpha_m = \frac{1}{m}$

particular case of affine as well as convex combination.

Note: $\alpha_1 + \dots + \alpha_m = 1$

3) Affine combination. $\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_m x_m$

$\alpha_1 + \dots + \alpha_m = 1$

4) convex combination.

$\alpha_1 x_1 + \dots + \alpha_m x_m$

* $0 \leq \alpha_i \leq 1$ for $i = 1, 2, \dots, m$

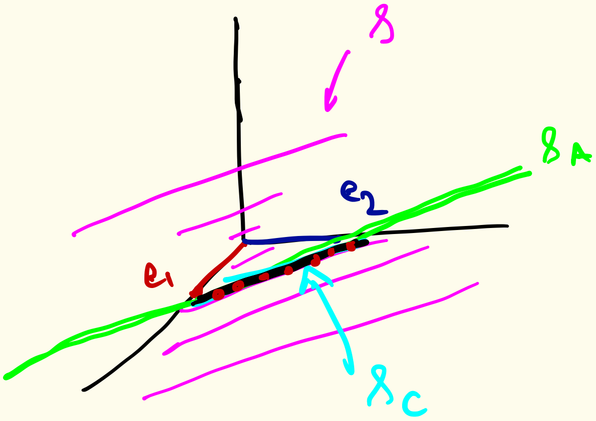
* $\Sigma \alpha_i = 1$

$x_1, \dots, x_m \in \mathbb{R}^n$

$S = \text{span}\{x_1, \dots, x_m\} = \{\alpha_1 x_1 + \dots + \alpha_m x_m \mid \alpha_1, \dots, \alpha_m \in \mathbb{R}\}$

$S_A = \text{affine span}\{x_1, \dots, x_m\} = \{\alpha_1 x_1 + \dots + \alpha_m x_m \mid \alpha_1, \dots, \alpha_m \in \mathbb{R},\ \sum_{i=1}^m \alpha_i = 1\}$

$S_C = \text{convex span}\{x_1, \dots, x_m\} = \{\alpha_1 x_1 + \dots + \alpha_m x_m \mid \alpha_1, \dots, \alpha_m \in \mathbb{R},\ \sum_{i=1}^m \alpha_i = 1,\ 0 \le \alpha_i \le 1\}$



$\boxed{\alpha_2 = 1 - \alpha_1}$

$S_A = \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix} : \alpha_1 + \alpha_2 = 1 \right\}$

$S_C = \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix} : \begin{array}{l} \alpha_1 + \alpha_2 = 1 \\ 0 \le \alpha_1, \alpha_2 \le 1 \end{array} \right\}$

$S_C \subseteq S_A \subseteq S$

linear span

Ex: Audio mixing:

$x_1, \dots, x_m \in \mathbb{R}^n$ audio signals (over same time)

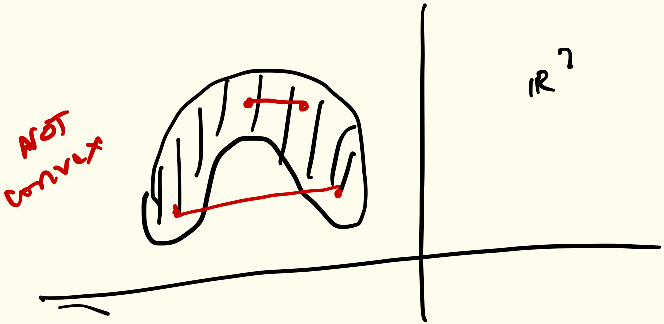
simultaneously recorded

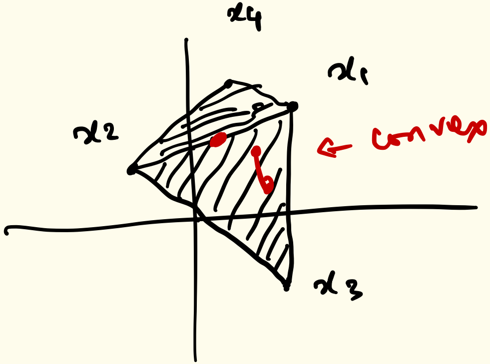
$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_m x_m$ (mix)

$\mathbb{R}^2$, $x_1, x_2, x_3 \in \mathbb{R}^2$

convex span $\{x_1, x_2, x_3\}$

$\mathbb{R}^2$





convex span $\{x_1, \dots, x_4\}$

Inner product; (dot product)

$x, y \in \mathbb{R}^n$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}; \quad \mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

bold face

$$x^T = (x_1 \cdots x_n)$$

$$x^T y = (x_1 \ x_2 \ \cdots \ x_n)_{1\times n} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}_{n\times 1} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

$$= \sum_{i=1}^{n} x_i y_i$$

outer product: $x, y \in \mathbb{R}^n$

$$x y^T = \begin{bmatrix} \\ \\ \end{bmatrix}_{n\times 1} \begin{bmatrix} & & \end{bmatrix}_{1\times n}.$$

$= n \times n$ matrix

Properties of inner product:

1) Commutativity: $x, y \in \mathbb{R}^n$, $x^T y = y^T x = \langle x, y \rangle$

2) Associativity with scalar multiplication:

$$(\alpha x)^T y = \alpha (x^T y)$$

3) Distributivity over vector addition

$$(x+y)^T z = x^T z + y^T z$$

Simple computation:

$$(x+y)^T (z+w) = x^T z + x^T w + y^T z + y^T w$$

Ex:

1) Inner product with unit vectors.

$x \in \mathbb{R}^n$

$\langle e_i, x \rangle = e_i^T x = x_i$ (i^{th} component)

$$\boxed{x = \sum_{i=1}^{n} \langle e_i, x \rangle \, e_i}$$

i^{th} unit vector

$$x = \sum_{i=1}^{n} \underbrace{(e_i^T x)}_{i^{th} \text{ component}} e_i$$

x is a linear combination of e_i's.

$\alpha_i = e_i^T x$ for $i = 1, 2, \ldots, n$

2) $1_n^T x$ for $x \in \mathbb{R}^n$ $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$= \boxed{x_1 + x_2 + \cdots + x_n} \quad \text{scalar}$$

3) $x \in \mathbb{R}^n$, $1_n \in \mathbb{R}^n$

$$\frac{1_n^T}{n} x = \frac{x_1 + \cdots + x_n}{n} \quad \text{(scalar)}$$

$a = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$a^T x$

4) Sum of squares

$$x^T x = x_1^2 + x_2^2 + \cdots + x_n^2$$

$a \in \mathbb{R}^n$

5) Selective sum : a = only 1's and 0's

$a^T x$ = sum of only those components in x where a is 1.