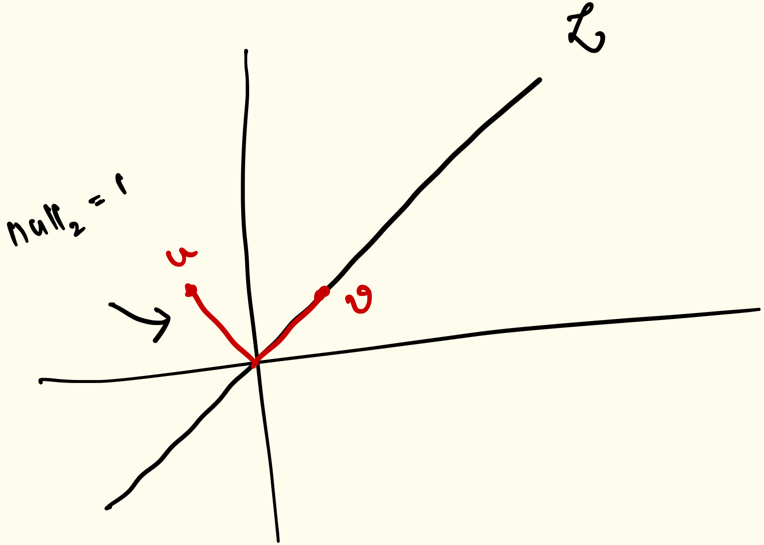
Linear Algebra for AI & ML

September - 3

(Lecture # 8)

$A = QR$ $n = 2$

Q: orthogonal ; R: upper triangular.



$P = uu^T$

$Q = I - 2P$

✓ reflector.

$H_u = \{ v \in \mathbb{R}^n \mid \langle u, v \rangle = 0 \}$: $(n-1)$ dimensional subspace. (hyperplane)

u is <u>NOT</u> a unit vector,

$I - 2P = \underbrace{I - 2uu^T}$

If u is not a unit norm vector,

normalize u !!

$$Q = I - 2\left(\frac{u}{\|u\|_2}\right)\left(\frac{u^T}{\|u\|_2}\right)$$

$$\boxed{Q = I - \frac{2}{\|u\|_2^2}\, uu^T}$$

$$Q^TQ = \underbrace{(I - uu^T)^T (I - uu^T)} = I$$

$$= (I^T - (uu^T)^T)(I - uu^T)$$

$$= I - uu^T - \cancel{uu^T} + u\,\cancel{u^Tu}\,u^T$$

$$u^Tu = 1$$

$$(uu^T)^T$$

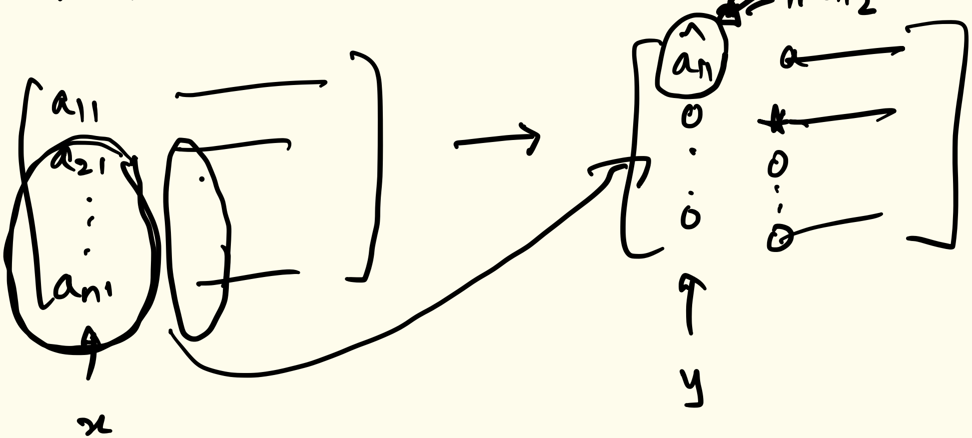
$$= (u^T)^T u^T$$

$$= uu^T$$

Thm: Let $x, y \in \mathbb{R}^n$ with $x \neq y$ and $\|x\|_2 = \|y\|_2$. There exists a reflector Q s.t. $Qx = y$

How does this theorem help us??

$x \to$ first column A



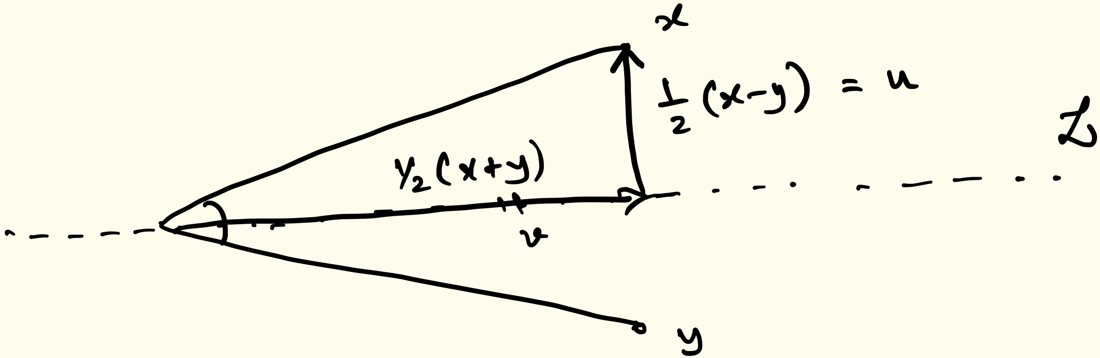
Pf: We want to compute Q s.t.

$$Q = I - \lambda u u^T \qquad \lambda = \frac{2}{\|u\|_2^2}$$

We need to find out u s.t.

$Q = I - \lambda u u^T$ is s.t.

$$Qx = (I - \lambda u u^T)x = y$$



We want to construct Q s.t.

$$Q(x-y) = y-x$$

$$x = \frac{1}{2}(x+y) - \frac{1}{2}(y-x)$$
$$= \frac{1}{2}(x+y) + \frac{1}{2}(x-y)$$

Observe: $x+y$ and $x-y$ are orthogonal to each other.

$$\langle x+y, x-y\rangle = \langle x,x\rangle + \langle x,y\rangle - \langle y,x\rangle - \langle y,y\rangle$$
$$= \langle x,x\rangle - \langle y,y\rangle$$
$$= \|x\|_2^2 - \|y\|_2^2$$
$$= 0$$

$$\Rightarrow x+y \perp x-y$$

$$Qx = \frac{1}{2}Q(x-y) + \frac{1}{2}Q(x+y)$$

Here $\boxed{Q = I - \gamma u u^T}$ where $u = \underbrace{\frac{1}{2}(x-y)}$

$$\Rightarrow \quad Q(x-y) = y - x$$

$$Q(x+y) = x+y$$

$$\Rightarrow \quad Qx = \frac{1}{2}Q(x-y) + \frac{1}{2}Q(x+y)$$

$$= \frac{1}{2}(y-x) + \frac{1}{2}(x+y)$$

$$\boxed{Qx = y}$$

$\blacksquare$

Introducing zero's in a vector using a reflector.

$x, y \in \mathbb{R}^n$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} y_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = y$$

$y_1 = \|x\|_2 \implies \|x\|_2 = \|y\|_2$

Use the previous theorem, to construct a reflector $Q = I - \gamma u u^T$ where $u = x - y$

s.t. $Qx = y$

$$Q \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

QR decomposition with reflectors.

$$A \in \mathbb{R}^{n\times n}$$

$$Q_{n-1}\cdots Q_2 Q_1 \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \overset{\|a_1\|}{*} & * & & \\ 0 & * & & \\ \vdots & * & & \\ & \vdots & & \\ 0 & * & & \end{bmatrix} = R$$

$$Q_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \underbrace{\tilde{Q}_2 \in \mathbb{R}^{(n-1)\times(n-1)}} & \\ 0 & & & \end{bmatrix}$$

where

$$\tilde{Q}_2 \underbrace{\begin{pmatrix} * \\ * \\ * \end{pmatrix}}_{n-1 \text{ entries}} = \begin{pmatrix} * \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solve $Ax = b$

$$\underbrace{Q_{n-1} \cdots Q_2 Q_1 \begin{bmatrix} & \\ & \end{bmatrix}}\, x = Q_{n-1} \cdots Q_2 Q_1 b$$

$$\begin{bmatrix} * & & * \\ & * \ \ddots & \\ 0 & & * \end{bmatrix} x = \tilde{b}$$

Solve: using backward substitution.

For tall systems (overdetermined systems)

$Ax = b$

$$\underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \\ \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}}_{b}$$

$A \in \mathbb{R}^{m \times n}$

$m \geq n$

$b \in \mathbb{R}^{m}$

1) A is full column rank.
(Columns of A form a basis for the col space (A))

2) $b \in \text{colspace}(A)$

$\Rightarrow \exists$ a unique $x \in \mathbb{R}^n$ s.t. $Ax = b$.

$$A = \begin{bmatrix} \quad & \quad \\ & \end{bmatrix}_{m \times n} \longrightarrow \quad R = \begin{bmatrix} \diagdown \\ 0 \\ \hline 0 \end{bmatrix} \Big\} \text{ n-rows}$$

$$Q^T A = \begin{bmatrix} \hat{R} \\ 0 \\ \hline 0 \end{bmatrix} \Big\}$$

orthogonal

$$m = 100 \; ; \; n = 10$$

$$\begin{bmatrix} \diagdown \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Big\} \text{ } 10 \times 10 \text{ matrix}$$

$$R = \begin{bmatrix} \hat{R} \\ \hline 0 \end{bmatrix}$$ $n \times n$: square upper Δlar matrix.

$$\underbrace{Q_{n-1}^T \cdots Q_2^T\, Q_1^T}_{= Q^T} \underbrace{\begin{bmatrix} * & & \\ * & & \\ * & & \\ * & & \\ \vdots & & \\ * & & \end{bmatrix}_{m\times n}}_{A} = \begin{bmatrix} * & * & * & \cdot \\ 0 & * & * & \\ 0 & 0 & * & \\ \vdots & 0 & & \\ \hline \vdots & 0 & & \\ \vdots & 0 & & \\ 0 & 0 & & \\ 0 & 0 & & \end{bmatrix} = \begin{bmatrix} \hat{R}_n \\ \\ \end{bmatrix}$$

$$Q_1^T \in \mathbb{R}^{m\times m}$$

$$\hat{R}_n = \begin{bmatrix} * & & * \\ & * & \\ 0 & * & \\ & & \ddots \end{bmatrix} \in \mathbb{R}^{n\times n}$$

$$\Rightarrow \quad \underset{(n\times m)}{Q^T}\, \underset{(m\times n)}{A} = \underset{(n\times n)}{R} = \hat{R}$$

$$Ax = b$$

$$\Rightarrow Q^T A x = Q^T b$$

$$\Rightarrow \begin{bmatrix} \hat{R} \\ \hline 0 \end{bmatrix} x = Q^T b$$

$$\Rightarrow \begin{bmatrix} \begin{matrix} \diagdown & \\ 0 & \diagdown \end{matrix} \\ \hline 0 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = Q^T b}$$

backward substitution, one can solve for $x_1, x_2, \dots, x_n$.

$A = QR \Rightarrow$ left inverse of $\underset{U}{A} = \boxed{R^{-1} Q^T}$

$$UA = R^{-1} Q^T Q R = I$$

$$Q_3\, Q_2\, Q_1 \underbrace{\begin{bmatrix} a_1 & a_2 & a_3 \\ & & \\ & & \end{bmatrix}_{100\times 3}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \left[\begin{array}{ccc} * & * & * \\ 0 & * & * \\ 0 & 0 & * \\ \hline 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} \hat{R} \\ \\ \\ 0 \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$Ax = b$

→ Solution to this system of linear eq^{ns} ✓

→ computation of this solution ✓

(LU, $\boxed{QR}$)

Gaussian

Left / Right inverses

invertibility /

Gram matrices.

→ Sensitivity analysis.

→ vector / matrix norms ✓

→ magnification ✓

(maximum magnification / minimum magnification)

→ condition number ✓

(We want to start discussing now)

→ derive several results for sensitivity analysis for $Ax = b$. ✓

square system

→ SVD → LS →

Ex:
$$A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix}$$

clearly, $$A^{-1} = \begin{bmatrix} -998 & 999 \\ 999 & -1000 \end{bmatrix}$$

(1) Compute x s.t. $$Ax = \begin{bmatrix} 1999 \\ 1997 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(2) compute y s.t. $$Ay = \begin{bmatrix} 1999.01 \\ 1997.01 \end{bmatrix}$$ $$y = \begin{bmatrix} 1.01 \\ 0.99 \end{bmatrix}$$

3) Compute z s.t.

$$Az = \begin{bmatrix} 1998.99 \\ 1997.01 \end{bmatrix} \leftarrow \tilde{b}$$

$$z = \begin{bmatrix} 20.97 \\ -18.99 \end{bmatrix}$$

<u>Observation:</u> $\underbrace{\text{"Small" changes}}$ in b may lead to $\underbrace{\text{"large" changes}}$ in x !!

Similar problem

$$y = \alpha_0 + \alpha_1 x_1 + \dots + \alpha_n x_n + \varepsilon$$

$$\begin{bmatrix} \\ A \\ \\ \end{bmatrix} [x] = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ \tilde{y}_m \end{bmatrix} \cdot \tilde{b}$$

Regressor matrix

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$\boxed{\text{PCR}}$