Lecture- 1

Vectors :

objects.

A vector is an ordered $\boxed{\text{finite}}$ list of $\boxed{\text{numbers.}}$

column - vectors $\begin{bmatrix} 0 \\ -1 \\ 3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -1.3 \\ 1.3 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

A vector of length n is called as an n- vector.

entries / elements / coefficients

Notation : a, b, c, x, y, z

a_1 : 1^{st} entry of a $\qquad$ a_1 : number

ambiguity : Suppose a_1, a_2, a_3 : vectors

$(a_1)_j$: j^{th} entry of a_1

Equality of vectors:

Let a & b be two vectors. (n-vectors) same length

$a = b$ (a is equal to b)

$\iff$ $a_i = b_i$ $\forall\, i \in \{1, 2, \dots, n\}$

if & only if

for every

belongs

n- vectors $a, b \in \mathbb{R}^n$

$\mathbb{R}$: set of real numbers. (field) of real numbers.

entries are coming from $\mathbb{R}$

length of the vector.

$\mathbb{F}^n$ length

entries are coming from $\mathbb{F}$

Block vectors: $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^{n_1+n_2+n_3}$

$a_1 \in \mathbb{R}^{n_1}$

$a_2 \in \mathbb{R}^{n_2}$

$a_3 \in \mathbb{R}^{n_3}$

In particular, $a \in \mathbb{R}^n$, $a_i \in \mathbb{R}^1$ for $i = 1, 2, \dots, n$

Indexing: $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

Examples: Zero vector

$$\mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$$

vector

real numbers

ones vector. All entries are 1.

$$\mathbf{1}_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$$

Unit vectors: $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$, ...

$\ldots, e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^n$, $e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n$

$$(e_j)_i = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Examples of vectors:

1) Colour: 3-vector

$\begin{bmatrix} \\ \\ \end{bmatrix}$ ← Red ← Green ← Blue

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow$ red

$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ← yellow

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow$ blue

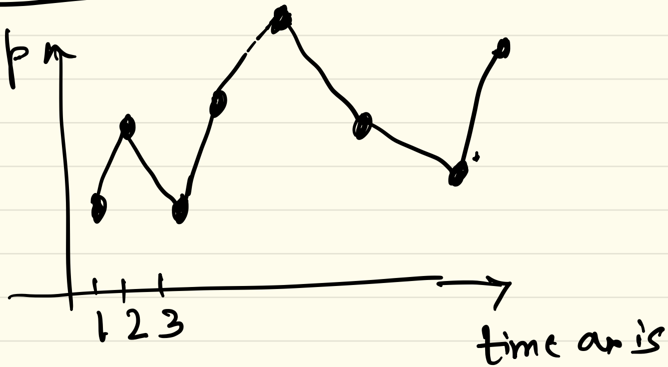
2) <u>Values across a population:</u> population is of size n.

$$\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} \in \mathbb{R}^n$$

↑ n vector

h_i : height of i^{th} person in the population.

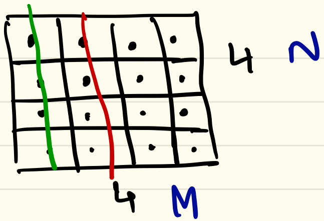
3) <u>Time Series</u> :

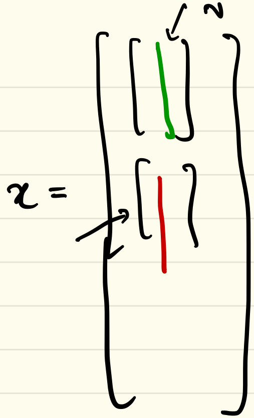


$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

p_i : i^{th} time instant value.

4) Imaging:





$\in \mathbb{R}^{MN}$

5) <u>Word counts:</u> D : Dictionary n - words.

Document:

$$\begin{array}{c} d_1 \\ d_2 \\ \vdots \\ d_n \end{array} \begin{bmatrix} 0 \\ 2 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$$

<u>sparse</u>

Operations of vectors:

1. vector addition
2. scalar multiplication

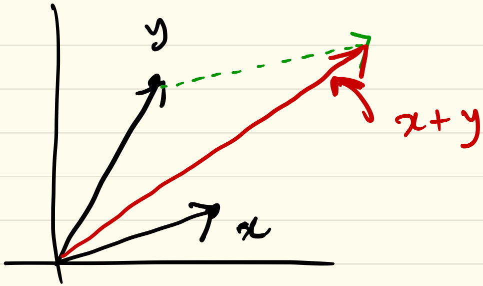
Let $x, y \in \mathbb{R}^n$

$$x+y = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix}$$: addition.

$$x-y = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} - \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1-y_1 \\ \vdots \\ x_n-y_n \end{pmatrix}$$: subtraction

Properties : $x, y, z \in \mathbb{R}^n$

1. Commutative : $x+y = y+x$
2. Associative : $(x+y)+z = x+(y+z)$ $= x+y+z$
3. $0 \in \mathbb{R}^n$: $x+0 = 0+x = x$ [$0 \in \mathbb{R}^n$: additive identity.
4. $\forall\, x \in \mathbb{R}^n$: $x - x = 0$ [$-x$: additive inverse



parallelogram law
of vector addition.

Examples:

1) Word counts: D : dictionary, n: number of words in D.

Doc_1 : x

Doc_2 : y

$x+y$

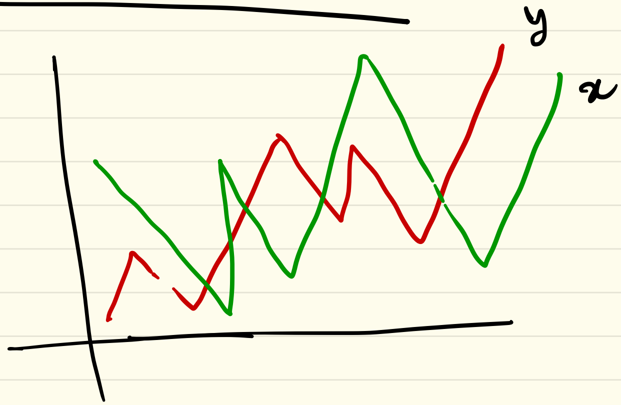
$x-y$

2) Audio addition:

Let a & b audio signals of length n.

$a+b$

3) Time-series addition:



$x+y$

$x-y$

Scalar-vector multiplication.

$\alpha \in \mathbb{R}$, $x \in \mathbb{R}^n$ $\qquad \alpha . x = \alpha x = \alpha \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}$

<u>Properties</u>:

1) $(\alpha + \beta) x = \alpha x + \beta x$ $\qquad$ for $\alpha, \beta \in \mathbb{R}$, $x \in \mathbb{R}^n$

2) $(\alpha \beta) x = \alpha (\beta x)$

3) $\alpha (x + y) = \alpha x + \alpha y$

4) for $0 \in \mathbb{R}$, $x \in \mathbb{R}^n$

$$0 x = 0$$

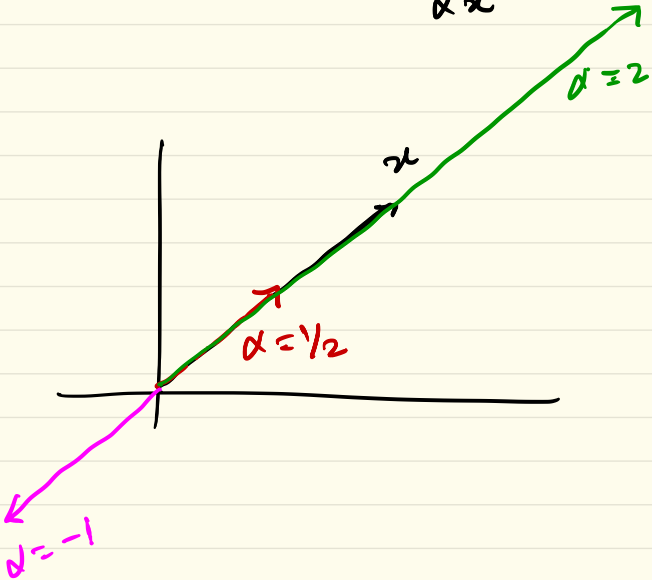
scalar $\qquad$ vector

Audio signal:

$x \in \mathbb{R}^n$, $\alpha > 0$

αx $\alpha = 1/2$ $\downarrow$ quiter audio

$\alpha = 2$ $\uparrow$



Linear combination :

let $x_1, x_2, \ldots, x_n \in \mathbb{R}^n$

$\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$

$$y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = \sum_{i=1}^{n} \alpha_i x_i \in \mathbb{R}^n$$

vector addition

scalar multiplications

Audio signal:

$x_1, x_2, \ldots, x_n$

$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$

Linear combination of unit vectors:

$e_1, e_2, \ldots, e_n \in \mathbb{R}^n$

For any vector $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

Every $x \in \mathbb{R}^n$ is a linear combination (unique) of unit vectors.

Special linear combinations:

(a) $x_1, x_2, \ldots, x_m \in \mathbb{R}^n$

$\alpha_1 = \alpha_2 = \cdots = \alpha_m = 1$

$$y = \sum_{i=1}^{m} \alpha_i x_i = x_1 + \cdots + x_m$$

(•) $x_1, x_2, \ldots, x_m \in \mathbb{R}^n$

$$\alpha_1 = \alpha_2 = \cdots = \alpha_m = \frac{1}{m}$$

$$y = \sum_{i=1}^{m} \alpha_i x_i = \frac{x_1 + x_2 + \cdots + x_m}{m}$$

Average

(•) $x_1, x_2 \in \mathbb{R}^2$

$$y_\alpha = \alpha x_1 + (1-\alpha) x_2 \qquad \alpha \in \mathbb{R}$$

= line (affine linear combination)

$$y_\alpha = \alpha x_1 + (1-\alpha) x_2 \qquad \alpha \in [0, 1]$$

= line segment (convex combination)

Vectors: special vector, examples of vectors.

- vector addition
- scalar multiplication

] listed the properties of these two operations.

Linear combination