Lecture 6

Linear dependence.

Let $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$. This collection of vectors is called as linearly dependent collection (the vectors are called as linearly dependent) if

$$\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k = 0$$

holds for some $\beta_1, \beta_2, \ldots, \beta_k \in \mathbb{R}$ not all equal to zero.

Let $x_1, \ldots, x_k \in \mathbb{R}^n$ be linearly dependent.

$\Rightarrow$ there exist scalars $\beta_1, \ldots, \beta_k \in \mathbb{R}$, not all zero,

s.t. $\beta_1 x_1 + \cdots + \beta_k x_k = 0$.

Let $\beta_j \neq 0 \Rightarrow x_j = -\frac{\beta_1}{\beta_j} x_1 - \cdots - \frac{\beta_{j-1}}{\beta_j} x_{j-1} - \frac{\beta_{j+1}}{\beta_j} x_{j+1} + \cdots - \frac{\beta_k}{\beta_j} x_k$

Linear dependence of $x_1, \dots, x_k$

$\Rightarrow$ at least one vector, say x_j, is a linear combination of remaining vectors.

Conversely, let $x_1, \dots, x_k$ be a collection of vectors such that one of them is a linear combination of the remaining vectors.

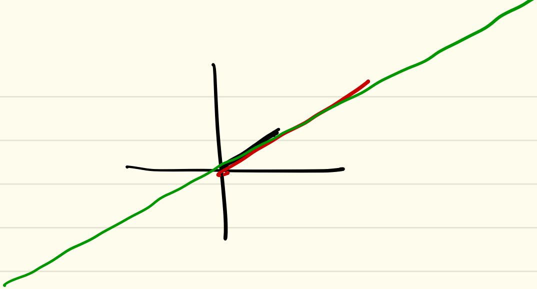
$$x_j = \alpha_1 x_1 + \dots + \alpha_{j-1} x_{j-1} + \alpha_{j+1} x_{j+1} + \dots + \alpha_k x_k$$

$$\Rightarrow \alpha_1 x_1 + \dots + \alpha_{j-1} x_{j-1} + 1 \cdot x_j + \alpha_{j+1} x_{j+1} + \dots + \alpha_k x_k = 0$$

$\Rightarrow$ $x_1, \dots, x_k$ are linearly dependent.

In particular, let $k=2$, then x_1, x_2 linearly dependent

$\Rightarrow$ one is the multiple of the other.



$r = 2b$

Linearly independent vectors.

A collection $x_1, \dots, x_k \in \mathbb{R}^n$ is called as linearly independent if it is NOT linearly dependent.

$\Rightarrow \quad \beta_1 x_1 + \dots + \beta_k x_k = 0$

$\Rightarrow \quad \beta_1 = \beta_2 = \dots = \beta_k = 0$

In general, it is difficult to check if a given collection is linearly dependent or not.

Ex: A collection consisting of a single vector is always linearly independent if & only if the vector is non-zero.

$x_1 \in \mathbb{R}^n$ if $x_1 \neq 0$, then

$\beta_1 x_1 = 0$

$\Rightarrow \beta_1 = 0$

Ex: Any collection of vectors including the zero vector is always linearly dependent.

Ex: $e_1, e_2, \ldots, e_n \in \mathbb{R}^n$; unit vectors are linearly independent.

$$\beta_1 e_1 + \cdots + \beta_n e_n = 0 \Rightarrow \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \beta_i = 0 \ \forall i$$

Ex: $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}$ linearly independent ??

Consider, $\beta_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + \beta_2 \begin{bmatrix}0\\-1\\0\end{bmatrix} + \beta_3 \begin{bmatrix}-1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\Rightarrow \begin{bmatrix}\beta_1 - \beta_3\\ -\beta_2 + \beta_3\\ \beta_3\end{bmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow \beta_3 = 0 \quad \Rightarrow \beta_1 = \beta_2 = 0$$

- Linear combinations of linearly independent vectors.

Let $y \in \mathbb{R}^n$ is a linear combination of $x_1, \dots, x_k \in \mathbb{R}^n$ which are linearly independent.

$$y = \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_k x_k \qquad \text{---} (1)$$

claim: The coefficients in the linear combination are unique

On contrary assume that there exist another set of scalars, $\gamma_1, \dots, \gamma_k \in \mathbb{R}$ s.t.

$$y = \gamma_1 x_1 + \dots + \gamma_k x_k \quad \text{---(2)}$$

(1) - (2)

$$\Rightarrow (\beta_1 - \gamma_1) x_1 + \dots + (\beta_k - \gamma_k) x_k = 0$$

$$\Rightarrow \beta_1 - \gamma_1 = \beta_2 - \gamma_2 = \dots = \beta_k - \gamma_k = 0$$

$$\Rightarrow \gamma_i = \beta_i \quad \text{for } i = 1, \dots, k. \quad \blacksquare$$

Why does this not hold true for linearly dependent vectors.

$x_1, x_2 \in \mathbb{R}^n$ linearly dependent ($\exists \beta \neq 0$ s.t. $x_1 = \beta x_2$)

$$\text{Let } y = \alpha_1 x_1 + \alpha_2 x_2 = \alpha_1 \beta x_2 + \alpha_2 x_2$$

$$= \boxed{(\alpha_1 \beta + \alpha_2)} x_2$$

"γ"

$y = \gamma x_2$

$\gamma = \delta_1 \beta + \delta_2$

$\exists$ infinite choices of δ_1, & δ_2

s.t. $\gamma = \delta_1 \beta + \delta_2$

Note: 1) Any superset of collection of linearly dependent vectors is linearly dependent.

2) Any non empty subset of linearly independent vectors is linearly independent.

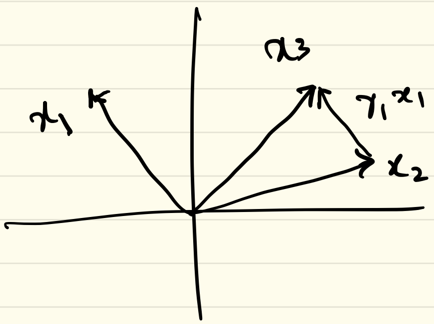
Independence- dimension inequality.

If $x_1, \cdots, x_k \in \mathbb{R}^n$ are linearly independent, then

$k \leq n$

Any collection with $n+1$ vectors from $\mathbb{R}^n$ is linearly dependent.

$\mathbb{R}^2$



Proof: Mathematical Induction.

Base case: $n=1$.

$x_1, \cdots, x_k \in \mathbb{R}^1 = \mathbb{R}$ is a linearly independent collection.

$x_1 \neq 0$. $\quad x_2 = \left(\frac{x_2}{x_1}\right)x_1$, $x_3 = \left(\frac{x_3}{x_1}\right)x_1$, $\cdots$, $x_k = \left(\frac{x_k}{x_1}\right)x_1$

$x_2 = \gamma_2 x_1$, $x_3 = \gamma_3 x_1$, $\cdots$, $x_k = \gamma_k x_1$

$\Rightarrow k \leq 1 \Rightarrow k = 1$

Suppose $n \geq 2$.

Assume that independence dimension inequality holds true for $n-1$. $(*)$

Let $x_1, \ldots, x_k \in \mathbb{R}^n$ be linearly independent.

Then we want to prove : $k \leq n$.

$$x_i = \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix} \in \mathbb{R}^n, \quad i = 1, 2, \ldots, k$$

where $y_i \in \mathbb{R}^{n-1}$

Case i) $\alpha_i = 0$ for $i = 1, \ldots, k$

$\because x_1, \ldots, x_k$ is linearly independent in $\mathbb{R}^n$

$\beta_1 x_1 + \cdots + \beta_k x_k = \underline{\underline{0}}$ $\leftarrow$ n-vectors $\Rightarrow \beta_1 = \cdots = \beta_k = 0$

$\Rightarrow \beta_1 y_1 + \cdots + \beta_k y_k = \underline{\underline{0}}$ $\leftarrow$ $(n-1)$-vectors

$\Rightarrow k \leq n-1 \Rightarrow k \leq n$

Case (ii) At least one of α_i's is not equal to zero.

Assume $\alpha_j \neq 0$ for some $j \in \{1, \ldots, k\}$

$$z_i = b_i - \frac{\alpha_i}{\alpha_j} b_j \qquad i = 1, \ldots, j-1$$

$$z_i = b_{i+1} - \frac{\alpha_{i+1}}{\alpha_j} b_j \qquad i = j, \ldots, k-1$$

Claim: z_i ; $i = 1, \ldots, k-1$ are linearly indep.

$$\sum_{i=1}^{k-1} \beta_i z_i = 0 \rightarrow \mathbb{R}^{n-1}$$

$$\Rightarrow \sum_{i=1}^{j-1} \beta_i \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix} + \gamma \begin{bmatrix} b_j \\ \alpha_j \end{bmatrix} + \sum_{i=j+1}^{k} \beta_{i-1} \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix} = 0 \leftarrow \mathbb{R}^n$$

with $$\gamma = -\frac{1}{\alpha_j} \left(\sum_{i=1}^{j-1} \beta_i \alpha_i + \sum_{i=j+1}^{k} \beta_{i-1} \alpha_i \right)$$

$k-1 \leq n-1$

$\Rightarrow k \leq n \quad \blacksquare$

<u>Basis</u>: A collection of n- linearly independent vectors in $\mathbb{R}^n$ is called as basis. (maximal independent set)

The number n is called as dimension.

<u>Fact</u>: Let $\{x_1, \dots, x_n\}$ be a basis of $\mathbb{R}^n$.

any vector $y \in \mathbb{R}^n$ can be <u>uniquely</u> <u>expressed</u> as a linear combination of $\{x_1, \dots, x_n\}$.

By i-d. inequality, $\{x_1, \dots, x_n, y\}$ is linearly dependent.

$$\beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n + \beta_{n+1} y = 0$$

<u>observe</u>: $\beta_{n+1} \neq 0$

$$y = -\frac{\beta_1}{\beta_{n+1}} x_1 - \frac{\beta_2}{\beta_{n+1}} x_2 - \dots - \frac{\beta_n}{\beta_{n+1}} x_n \quad \blacksquare$$

Expansion of a vector: in Basis B.

$\{x_1, \ldots, x_n\}$: Basis of $\mathbb{R}^n$

$y \in \mathbb{R}^n$

$y = \alpha_1 x_1 + \cdots + \alpha_n x_n$

$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$: coordinate vector of y w.r.t. Basis B.

<u>Ex:</u> $\{e_1, \ldots, e_n\}$: Basis of $\mathbb{R}^n$ std. Cannonical Basis of $\mathbb{R}^n$.

$y \in \mathbb{R}^n$

$y = y_1 e_1 + y_2 e_2 + \cdots + y_n e_n$

$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$: coordinate vector of y w.r.t $B = \{e_1, \ldots, e_n\}$

- Linear dependence / independence. (Examples)
- Independence - dimension inequality.

  $x_1, \ldots, x_k \in \mathbb{R}^n$ which is linearly indep.

  $\Rightarrow k \leq n.$ (using induction).
- Indep - dimension inequality,

  Any collection of $n+1$ vectors in $\mathbb{R}^n$ is linearly dependent.
- Basis - dimension
- Expansion of vectors in a basis.