Class September - 16

Orthonormal vectors.

A collection of vectors $x_1, x_2, \ldots, x_k$ is called as orthogonal (mutually orthogonal) if $\langle x_i, x_j \rangle = 0$

$$x_i^T x_j = 0 \quad \text{for } 1 \leq i \neq j \leq k$$

If further, each x_i is such that $\underline{\|x\|_2 = 1,}$ then the collection is called as orthonormal collection of vectors. for $1 \leq i, j \leq k$

$$x_i^T x_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

<u>Ex:</u> $e_1, e_2, \ldots, e_n \in \mathbb{R}^n$ is an orthonormal collection.

$$\left(\begin{array}{lll} e_i^T e_j = 1 & \text{for } i \neq j & 1 \leq i, j \leq n. \\ = 0 & \phantom{\text{for }} i \neq j & \end{array} \right)$$

Ex: $\begin{pmatrix}1\\0\\0\end{pmatrix}, \underbrace{\begin{pmatrix}0\\1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}, \begin{pmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}}$ is an orthonormal collection.

Ex: $\boxed{\begin{pmatrix}0\\0\\0\end{pmatrix}}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ is an orthogonal collection but <u>Not</u> orthonormal.

Linear independence and orthonormal collection.

Let $x_1, x_2, \dots, x_k \in \mathbb{R}^n$ be an orthonormal collection.

$$x_i^T x_j = 1 \quad \text{for } i = j$$
$$= 0 \quad \text{for } i \neq j$$

To check: Whether $x_1, \dots, x_k$ is a linearly independent collection??

$$\left[\beta_1 x_1 + \beta_2 x_2 + \dots + \beta_k x_k = 0 \Rightarrow \beta_1 = \dots = \beta_k = 0\right]$$

Consider $\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k = 0$ ----- vector eq^n.

$x_j^T(\beta_1 x_1 + \cdots + \beta_k x_k) = x_j^T 0$. for any $\underbrace{j \in \{1, \ldots, k\}}$ ---- scalar eq^n.

$\Rightarrow \beta_1 x_j^T x_1 + \beta_2 x_j^T x_2 + \cdots + \beta_k x_j^T x_k = 0$

$\Rightarrow \beta_j x_j^T x_j = 0$

$\Rightarrow \beta_j = 0$

$\Rightarrow \beta_1 = \beta_2 = \cdots = \beta_k = 0$

$\Rightarrow \{x_1, \ldots, x_k\}$ is a linearly independent set.

Linear combination of orthonormal vectors.

Let $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$ be an orthonormal collection.

Let $y \in \text{span}\{x_1, \ldots, x_k\}$

$\exists$ scalars, $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ such that

$$y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k$$

In order to compute $\alpha_1, \ldots, \alpha_k$,

$$x_j^T y = \alpha_1 x_j^T x_1 + \cdots + \alpha_k x_j^T x_k \quad \text{for } j = 1, 2, \ldots, k$$

$$\Rightarrow \quad \underline{\underline{\alpha_j = x_j^T y}}$$

Orthonormal basis:

Let $x_1, \ldots, x_n \in \mathbb{R}^n$ be an orthonormal collection.

$\Rightarrow \{x_1, \ldots, x_n\}$ is a basis with the property that $\{x_1, \ldots, x_n\}$ is an orthonormal set

Orthonormal basis.

For any $y \in \mathbb{R}^n$,

$$y = \underbrace{(y^T x_1)}_{\beta_1} x_1 + \underbrace{(y^T x_2)}_{\beta_2} x_2 + \cdots + \underbrace{(y^T x_n)}_{\beta_n} x_n$$

Gram-Schmidt Orthonormalisation algorithm. $\underline{\underline{(\mathbb{R}^n)}}$

$\{x_1, \cdots, x_k\}$ orthonormal set $\Rightarrow$ $\{x_1, \cdots, x_k\}$ linearly independent

$\overset{??}{\Leftarrow}$ is the converse true??

In $\mathbb{R}^3$ $\underbrace{\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}}$

span x_1 $\longleftrightarrow$ span q_1

span $\{x_1, x_2\}$ $\longleftrightarrow$ span $\{q_1, q_2\}$

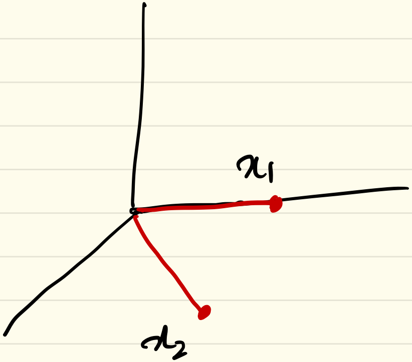
span $\{x_1, x_2, x_3\}$ $\longleftrightarrow$ span $\{q_1, q_2, q_3\}$

$\vdots$

span $\{x_1, \dots, x_k\}$ $\longleftrightarrow$ span $\{q_1, \dots, q_k\}$

$x_1, \dots, x_k$ — l.i.

$q_1, \dots, q_k$ — orthonormal.



G. S.

$$x_1 \longleftrightarrow q_1$$

$$x_1, x_2 \longleftrightarrow q_1, q_2$$

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad q_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{span}\left\{ x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \qquad q_2 = ??$$

$$= \text{span}\{q_1, q_2\}$$

$$q_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Ex: 2) $x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$