Linear algebra for AI and ML

September 2

Lecture #7

$Ax = b$: solve ($n \times n$ case)

→ QR decomposition of A

Gram-Schmidt orthogonalization

$A = QR$

Q : orthogonal

R : upper triangluar

$$Ax = b \Rightarrow QRx = b \Rightarrow Rx = Q^T b$$

backward substitution.

<u>Rotators</u> :

$Q \in \mathbb{R}^{2\times 2}$; $Q = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$

$c = \cos\theta$
$s = -\sin\theta$

$$Q \in \mathbb{R}^{n\times n} \; ; \; Q = \begin{array}{cl} \begin{array}{ccc} & i^{th}\text{ column} \quad & j^{th}\text{ column} \end{array} & \\ \begin{bmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & c & & & & -s & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ & & & s & & & & c & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \leftrightarrow i^{th}\text{ row} \\ \\ \\ \\ \leftrightarrow j^{th}\text{ row} \\ \\ \\ \\ \end{array} \end{array} \in \mathbb{R}^{n\times n}$$

Rotation is happening in $x_i x_j$ plane.

$$\Rightarrow \quad Qx = \begin{array}{c} \begin{array}{ccccccccc} & & & & & j^{th}\text{ column} & & & \end{array} \\ \begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & c & & -s & & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & 1 & & & \\ & & & s & & & c & & \\ & & & & & & & 1 & \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{bmatrix} \\ \begin{array}{ccccccccc} & & & i^{th}\text{ column} & & & & & \end{array} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{i-1} \\ * \\ x_{i+1} \\ \vdots \\ x_{j-1} \\ * \\ x_{j+1} \\ \vdots \\ x_n \end{bmatrix} \begin{array}{l} \\ \\ \\ \\ \leftarrow \sqrt{x_i^2+x_j^2} \\ \\ \\ \\ \leftarrow 0 \\ \\ \\ \\ \end{array}$$

<u>Check:</u> Qx only changes i^{th} and j^{th} entry of x.

In general, $A \in \mathbb{R}^{n\times n}$; $QA \Leftarrow$ only i^{th} and j^{th} rows of A will change.

$Q[a_1 \ldots a_n]$

Thm: Let $A \in \mathbb{R}^{n\times n}$. Then there exists an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$. (A invertible)

Pf:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\underbrace{Q_{n1}^T \, Q_{n-1\,1}^T \cdots Q_{41}^T \, Q_{31}^T \, Q_{21}^T}_{(n-1)\text{ number}} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{n1} \end{bmatrix} \rightarrow \begin{bmatrix} * \\ 0 \\ 0 \\ 0 \\ \vdots \\ a_{n1} \end{bmatrix}$$

$$\underbrace{Q_{n1}^T \cdots}_{(n-2)} Q_{21}^T A = \begin{bmatrix} * & \text{—} & & \\ 0 & * & \text{—} & \\ 0 & 0 & \text{—} & \\ \vdots & \vdots & \text{—} & \\ 0 & 0 & 0 & \text{—} \end{bmatrix}$$

$$\underbrace{Q_1 Q_2}_{n^3} \underbrace{b}_{n^2} \qquad (Q_1(Q_2 b))$$

$(n-1) + (n-2) + \cdots + 1$: total no. rotators.

$$Ax = b$$

$$\underbrace{\cdots Q_{32}^T Q_{n1}^T \cdots Q_{31}^T Q_{21}^T A}\, x = \underbrace{\cdots Q_{32}^T Q_{n1}^T \cdots Q_{31}^T Q_{21}^T b}$$

$$Rx = \tilde{b}$$

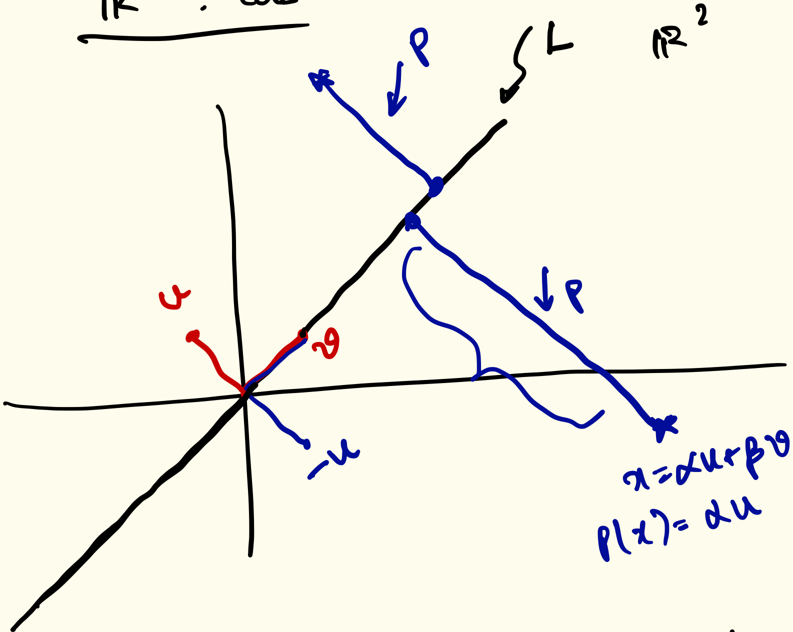
back.substitution.

No need to explicitely calculate QR decomposition while solving $Ax = b$

Reflectors:

$\mathbb{R}^2$: case :

$Q \in \mathbb{R}^{2\times 2}$



L: a line passing through origin.

Q: reflect every vector in $\mathbb{R}^2$ through this line L.

v is a unit vector in L s.t. every vector in L is a multiple of v.

u is a unit vector in $\mathbb{R}^2$ which is orthogonal to L. $\{u, v\}$ forms a basis of $\mathbb{R}^2$.

Q acts on u & v in the following way.

$Qv = v$

$\boxed{Qu = -u}$

For any vector $x \in \mathbb{R}^2$; $\exists$ unique $\alpha, \beta \in \mathbb{R}$

s.t. $x = \alpha u + \beta v$

$$Qx = Q(\alpha u + \beta v) = \alpha Q(u) + \beta Q(v)$$
$$= -\alpha u + \beta v$$

<u>Aim</u>: To construct Q with this property.

Construct $P = uu^T \in \mathbb{R}^{2\times 2}$

$$\boxed{Pu = (uu^T)u = u(u^Tu) = u}$$

$(\|u\|_2^2 = u^Tu = 1)$

$$Pv = (uu^T)v = u(u^Tv) = 0 \quad \because u \,\&\, v \text{ are orthogonal.}$$

Can we construct Q from P??

$$Q = I - 2P$$

we want $Qu = -u$

$Qv = v$

$$Qu = (I-2P)u = (I - 2uu^T)u = u - 2u\underbrace{u^Tu}$$
$$= -u$$

$$Qv = (I-2P)v = (I-2uu^T)v = v - 2u(\cancel{u^Tv})$$
$$= v$$

$$(I-2P)^T(I-2P) = (I-2uu^T)^T(I-2uu^T) = I$$

This process can be generalized to $\mathbb{R}^n$.

Let $\mathcal{L}$ be an $(n-1)$-dimensional subspace of $\mathbb{R}^n$.

Let $\{v_1, v_2, \dots, v_{n-1}\}$ be an (orthonormal) basis of $\mathcal{L}$.

Let u be s.t. $\|u\|_2 = 1$ and u is orthogonal to $\mathcal{L}$. (Geometrically u is a unit vector in the direction of normal to this plane $\mathcal{L}$).

Construct orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ s.t. every $x \in \mathbb{R}^n$ Qx is reflection of x thru' $\mathcal{L}$.

How does reflectors help us introduce zero's in a vector?

$$\underset{\sim}{Q}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \longrightarrow \begin{pmatrix} * \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$