Class September 17

Gram-Schmidt algorithm.

Given vectors x & $y \in \mathbb{R}^n$

$$x^T y = \sum_{i=1}^{n} x_i y_i \quad ; \quad x^T x = \|x\|_2^2$$

$$\cos\theta = \frac{x^T y}{\|x\|_2 \|y\|_2}$$

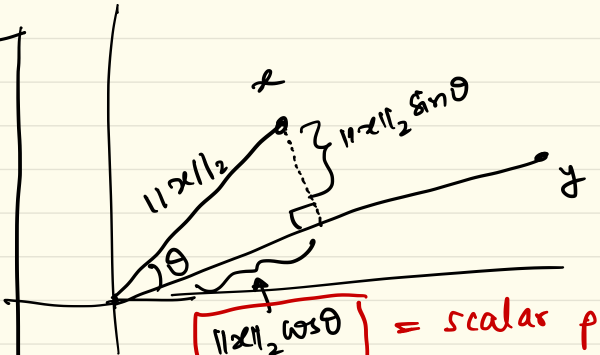
Projection of x_2 on q_1

$$= \|x_2\|_2 \cos\theta \, q_1$$

$$= \cancel{\|x_2\|_2} \frac{x_2^T q_1}{\cancel{\|x_2\|}} q_1$$

$$= (x_2^T q_1) q_1$$

$$= \boxed{(q_1^T x_2) q_1}$$



$\boxed{\|x\|_2 \cos\theta}$ = scalar projection of vector x on y.

$$\boxed{\|x\|_2 \cos\theta \frac{y}{\|y\|_2}} = \text{projection of } x \text{ on } y.$$

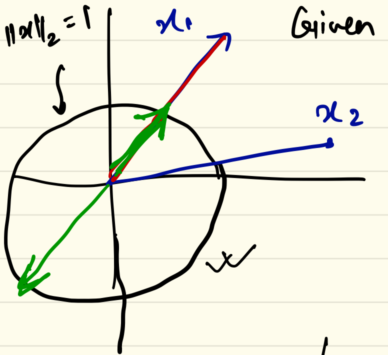
q_1

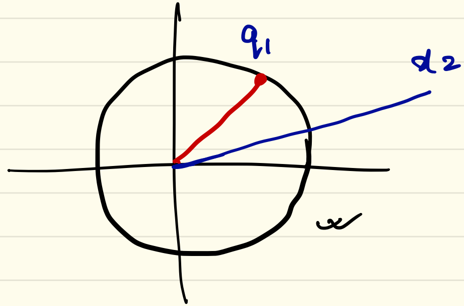
Objective: Given vectors x_1 & x_2 and $\{x_1, x_2\}$ linearly independent.

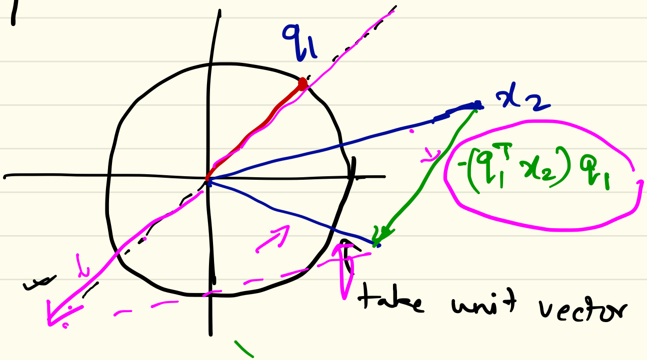
construct q_1, and q_2 s.t.

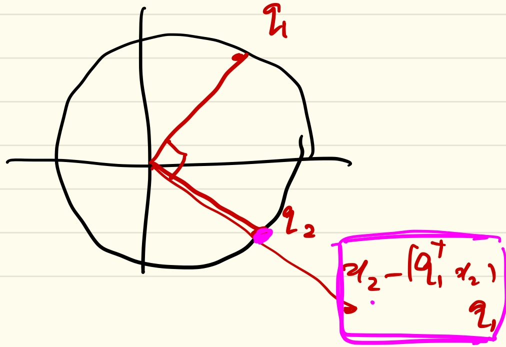
$\text{span}\{x_1\} = \text{span}\{q_1\}$; $\text{span}\{x_1, x_2\} = \text{span}\{q_1, q_2\}$

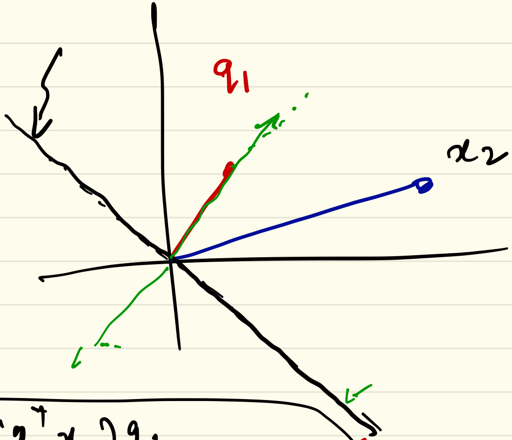
$\{q_1, q_2\}$ is an orthonormal set.











$x_2 =$ linear combination of q_1 & q_2

$$x_2 = \langle x_2, q_1 \rangle q_1 + \langle x_2, q_2 \rangle q_2$$

$$x_2 = (q_1^T x_2)\, q_1 + (q_2^T x_2)\, q_2 \;??$$

$$x_2 - \boxed{(q_1^T x_2)\, q_1} = \boxed{(q_2^T x_2)}\, q_2$$

projection of x_2 on q_1 *

$$\frac{x_2 - (q_1^T x_2)\, q_1}{\|x_2 - (q_1^T x_2)\, q_1\|_2} = q_2$$

normalization process.

$\tilde{q}_2 = \boxed{x_2 - (q_1^T x_2)\, q_1}$ is orthogonal to $\boxed{q_1}$

$$q_1^T \left(x_2 - (q_1^T x_2)\, q_1 \right) = \underline{q_1^T x_2 - q_1^T (q_1^T x_2)\, q_1} = q_1^T x_2 - \underline{(q_1^T x_2)}\, \cancel{q_1^T q_1}^{\,1}$$

$$= 0$$

Algorithm (G.-S. O.)

Given: $x_1, x_2, \cdots, x_k \in \mathbb{R}^n$

for $i = 1, 2, \cdots, k$

1. Orthogonalization $\tilde{q}_i = x_i - (q_1^T x_i) q_1 - \cdots (q_{i-1}^T x_i) q_{i-1}$
2. Test for linear dependence: if $\tilde{q}_i = 0$; quit
3. Normalization: $q_i = \dfrac{\tilde{q}_i}{\|\tilde{q}_i\|_2}$

Let $x_1, \ldots, x_k \in \mathbb{R}^n$ be linearly independent.

i) $\tilde{q}_i \neq 0$

ii) $q_1, \ldots, q_i$ are orthonormal

iii) x_i is a linear combination of $q_1, \ldots, q_i$

iv) q_i is a linear combination of $x_1, \ldots, x_i$