Linear algebra for AI and ML

September - 9

(Lecture - 10)

Sensitivity analysis: A : square ; A is invertible.

$$Ax = b$$

$$A(x + \delta x) = b + \delta b$$

$$\frac{\|\delta x\|_2}{\|x\|_2} \leq \|A\|_2 \|A^{-1}\|_2 \frac{\|\delta b\|_2}{\|b\|_2}$$

$$\|A\|_2 \|A^{-1}\|_2 = k_2(A)$$

If $k_2(A) = \|A\|_2 \|A^{-1}\|_2$ is "small", then

$\frac{\|\delta b\|_2}{\|b\|_2}$ "small" $\Rightarrow$ $\frac{\|\delta x\|_2}{\|x\|_2}$ is also small.

This implies that the solution is not sensitive, in this case, to perturabations in b.

On the other hand, if $\kappa_2(A)$ is "large", then small perturbations $\frac{\|\delta b\|_2}{\|b\|_2}$ may result in "large" perturabations $\frac{\|\delta x\|_2}{\|x\|_2}$.

(Refer to the example we discussed. We will calculate $\kappa_2(A)$ where $A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix}$.)

This implies, the solution x may be very sensitive to "small" perturabations $\frac{\|\delta b\|_2}{\|b\|_2}$.

Notice: $\kappa_2(I) = \|I\|_2 \|I^{-1}\|_2$

$= \|I\|_2 \|I\|_2$

$= 1$

$\|I\|_2 = \max_{x \neq 0} \frac{\|Ix\|_2}{\|x\|_2}$

$= \max_{x \neq 0} 1$

$= 1$

for any invertible matrix $A \in \mathbb{R}^{n \times n}$, note that

$I = AA^{-1}$

$\Rightarrow 1 = \|I\|_2 = \|AA^{-1}\|_2 \leq \|A\|_2 \|A^{-1}\|_2 = \kappa_2(A)$

$\Rightarrow$ $\boxed{\kappa_2(A) \geq 1}$

"Best possible" condition number is 1.

(Note that orthogonal matrices have "best possible" condition number).

Geometrical Interpretation.

$$K_2(A) = \|A\|_2 \|A^{-1}\|_2$$

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

$$= \max_{x \neq 0} \left\| A \frac{x}{\|x\|_2} \right\|_2$$

for $x \neq 0$

$\frac{x}{\|x\|_2}$ is a unit norm vector.

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

$$= \max_{x \neq 0} \left\| A \frac{x}{\|x\|_2} \right\|_2$$

$$\boxed{\|A\|_2 = \max_{\|y\|_2 = 1} \|Ay\|_2}$$

$\alpha = \|x\|_2$

$$= \max_{x \neq 0} \frac{\|Ax\|_2}{\alpha}$$

$$= \max_{x \neq 0} \left\| \frac{1}{\alpha} Ax \right\|_2$$

$$= \max_{x \neq 0} \left\| A \frac{x}{\alpha} \right\|_2$$

maximum magnification of A

$$\text{max mag}(A) = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\|x\|_2 = 1} \|Ax\|_2$$

minimum magnification of A

$$\text{min mag}(A) = \min_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \min_{\|x\|_2 = 1} \|Ax\|_2$$

By definition, $\|A\|_2 = \text{max mag}(A)$

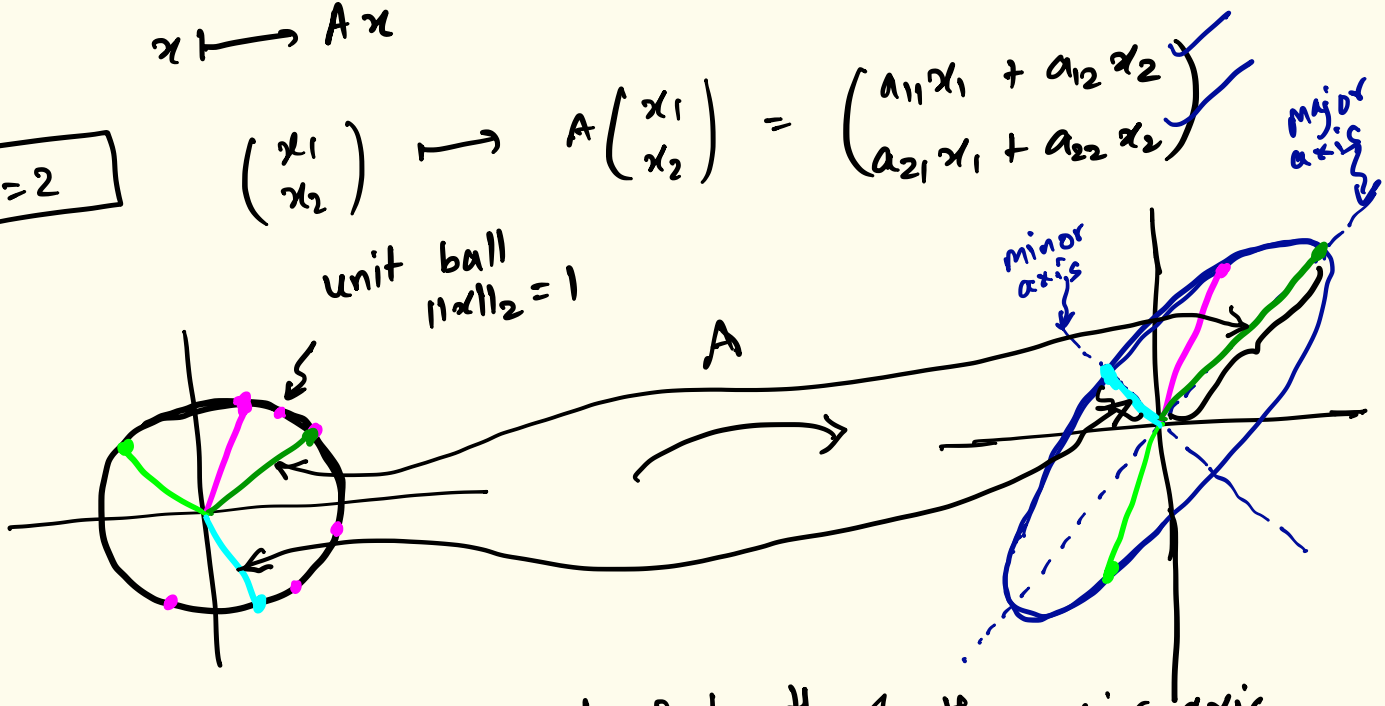
Note: For orthogonal matrices, max mag (Q) as well as min mag (Q) are 1.

$A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ ($A \in \mathbb{R}^{n \times n}$, invertible)

$x \longmapsto Ax$

$n = 2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}$$



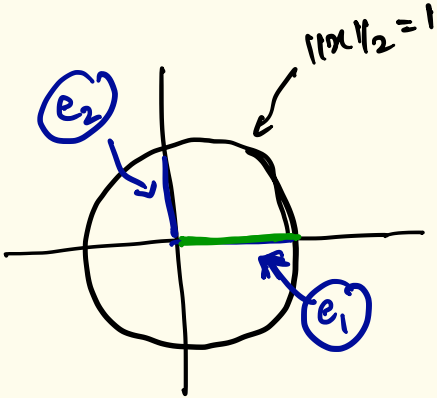
Geometrically, max mag(A) = half length of the major axis.

min mag(A) = half length of the minor axis.

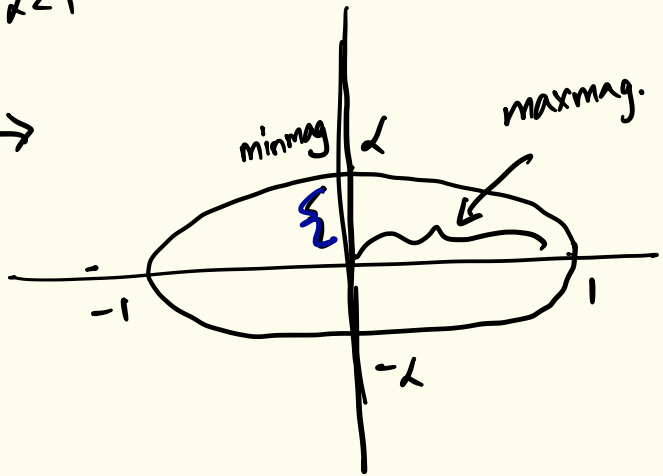
$$A = \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix}$$

$\alpha \neq 0$; $\alpha > 0$

$\alpha < 1$



A



domain

$$x_1^2 + x_2^2 = 1$$

$$Ax = \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \alpha x_1 \end{bmatrix}$$

$$Ae_1 = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$$

$$Ae_2 = \begin{bmatrix} 0 \\ \alpha \end{bmatrix} \alpha = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$$