3)

$$\|x\|_w = \sqrt{\sum_{i=1}^{n} w_i \cdot x_i^2}$$

(i) Homogenity

$$\|x\|_w = \sqrt{x_1^2 \cdot w_1 + x_2^2 \cdot w_2 \cdots + x_n^2 w_n}.$$

$\|x\|_w \in R$ as. $w_i \in R$ & $w_i > 0$

so each term $w_i x_i^2 \geq 0$

$$\|.\|_w : R^n \to R$$

(ii) Non-negativity

$$\|x\|_w = \sqrt{w_1 x_1^2 + w_2 x_2^2 + \cdots + w_n x_n^2}$$

$w_i > 0$, $x_i^2 \geq 0$

$\|x\|_w \geq 0$.

(iii) Definitiveness

$$\|x\|_w = \sqrt{w_1 x_1^2 + w_2 x_2^2 + \cdots + w_n x_n^2} = 0$$

$$\Rightarrow w_1 x_1^2 = w_2 x_2^2 = w_3 x_3^2 = \cdots = 0.$$

$w_i > 0$

so

$$\Rightarrow x_1 = x_2 = x_3 = \cdots = 0.$$

$x = 0$

(iv) Triangle Inequality.

$f : R^n \to R^n$

Transform a vector x to vector y

$$f : \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \to \begin{bmatrix} \sqrt{w_1}\, x_1 \\ \sqrt{w_2}\, x_2 \\ \vdots \\ \sqrt{w_n}\, x_n \end{bmatrix}$$

$$\|y\|_2 = \sqrt{(\sqrt{w_1}\, x_1)^2 + \cdots + (\sqrt{w_n}\, x_n)^2}$$

$$= \sqrt{w_1 x_1^2 + w_2 x_2^2 + \cdots + w_n x_n^2}$$

$$= \|x\|_w$$

for $\|y\|_2$, we have already proven.

$$\|y_1 + y_2\|_2 \leq \|y_1\|_2$$

So.

$$\|x_1 + x_2\|_w = \|y_1 + y_2\|_2$$

$$\|x_1 + x_2\|_w \leq \|x_1\|_w + \|x_2\|_w$$

$$y_1 + y_2 = \begin{bmatrix} \sqrt{w_1}\,(x_1 + x_2) \\ \sqrt{w_2}\,(x_1 + x_2) \\ \vdots \\ \sqrt{w_n}\,(x_1 + x_2) \end{bmatrix}$$

$\Rightarrow$ $\|\cdot\|$ is a norm (called weighted norm)