Class September 23

Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function. Then f is called a linear function if $\forall \alpha, \beta \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

<u>Ex:</u> Fix $A \in \mathbb{R}^{m \times n}$

$$f_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$\underbrace{f_A(x) = Ax} \quad \leftarrow \text{matrix vector multiplication.}$$

To verify: $f_A(\alpha x + \beta y) \overset{?}{=} \alpha f_A(x) + \beta f_A(y)$

$$= A(\alpha x + \beta y) \overset{?}{=} \alpha A x + \beta A y \quad \leftarrow (*) \text{ Already verified in the last class}$$

Ex: $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ $\qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$\underbrace{f: \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_1 \end{pmatrix}} = x \longmapsto Ax$$

Q: Is f linear??

$$\underbrace{A = \begin{bmatrix} 0 & \cdots & \cdots & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & & \\ 1 & 0 & \cdots & 0 \end{bmatrix}} \in \mathbb{R}^{n\times n}$$

$$\underbrace{A = \begin{bmatrix} - & e_n^T & - \\ - & e_{n-1}^T & - \\ & \vdots & \\ - & e_1^T & - \end{bmatrix}} = \begin{bmatrix} | & | & & | \\ e_n & e_{n-1} & \cdots & e_1 \\ | & | & & | \end{bmatrix}$$

f is linear.

Two ways to prove (i) check superposition principle.

Here! (ii) Find $A \in \mathbb{R}^{n\times n}$ s.t. the f^n can be described as action of A.

Ex:

$\boxed{n \geq m}$

$$f : \mathbb{R}^{\underline{n}} \longrightarrow \mathbb{R}^{\underline{m}}$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$f : \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

$$f : \mathbb{R}^{\underline{4}} \longrightarrow \mathbb{R}^{\underline{3}}$$

$$f \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$A \in \mathbb{R}^{\boxed{m \times n}}$$

$$A = \begin{bmatrix} e_1 & e_2 & \cdots & e_m \\ & & & \end{bmatrix}_{n \times m} \times$$

m vectors $\longrightarrow$ n vectors

$$e_i \in \mathbb{R}^n$$

$$A = \begin{bmatrix} e_1 & e_2 & e_3 & \cdots & e_m & \overbrace{0 & \cdots & 0}^{n-m} \end{bmatrix}_{\underline{m \times n}}$$

m-vectors

vectors

$$\begin{bmatrix} e_1 & e_2 & \cdots & e_m & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

The function f is linear.

Ex: $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$f: \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} x_1+1 \\ x_2+2 \\ x_3+3 \\ \vdots \\ x_n+n \end{pmatrix}$$

Not a linear $f^{\underline{n}}$ $\qquad f: 0 \not\mapsto 0$

$\alpha = \beta = 0$

$f(0) = 0$

Ex: $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\qquad n \geq m$

$$f: \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_m^2 \end{pmatrix}$$

Not a linear $f^{\underline{n}}$

Ex: $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

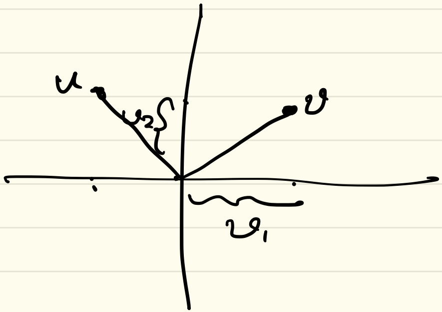
$f : v \longmapsto u$

where u is obtained from v by rotating v in anticlockwise direction by $\frac{\pi}{2}$.

Check: This function is linear.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ -v_1 \end{bmatrix}$$



$$\boxed{f_A(x) = \underline{0}}$$

Given a matrix $A \in \mathbb{R}^{m \times n}$

$$f_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$f_A : x \longmapsto Ax \quad \text{or} \quad f_A(x) = Ax$$

Q: Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear function.

Does there exist $A \in \mathbb{R}^{\boxed{m \times n}}$ s.t. $f(x) = Ax$ $\forall x \in \mathbb{R}^n$.

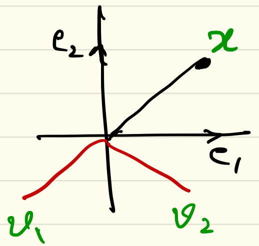
$$e_1, e_2, \cdots, e_n \in \mathbb{R}^n$$

$$f(e_1), f(e_2), \cdots, f(e_n) \in \mathbb{R}^m$$

$$A = \begin{bmatrix} f(e_1) & f(e_2) & \cdots & f(e_n) \end{bmatrix} \in \mathbb{R}^{\boxed{m \times n}}$$

$$x \in \mathbb{R}^n \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$



$$x = \alpha_1 v_1 + \alpha_2 v_2$$

$$x = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

$$f(x) = f(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n)$$

f is linear.

$$= x_1 f(e_1) + x_2 f(e_2) + \cdots + x_n f(e_n)$$

$$= \begin{bmatrix} f(e_1) & \cdots & f(e_n) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

m-vector

$$f(x) = \underbrace{\begin{bmatrix} f(e_1) & \cdots & f(e_n) \end{bmatrix}}_{A \in \mathbb{R}^{m \times n}} x$$

Matrix vector multi.

Ax

length n

$$\begin{bmatrix} - a_1^T - \\ - a_2^T - \\ \vdots \\ - a_m^T - \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_m^T x \end{bmatrix}$$

a_1^T

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

m-vector

$$x_1A_1 + x_2A_2 + \cdots + x_nA_n$$

$$= \underbrace{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} x_2 + \cdots + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} x_n}$$

column 1 of A   column 2 of A   column n of A

Matrix-vector multiplication: Another way to understand