Linear algebra for AI and ML

August - 26

Lecture # 5

$Ax = b \qquad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m$

$a^{-1}b = ba^{-1} = \frac{1}{a}b = b\frac{1}{a}$

$ax = b$ ← scalar eqⁿ.

$x = a^{-1}b = \frac{1}{a}b \qquad (a \neq 0)$ ↔ linear independence of columns.

For tall matrices, left inverse can be the concept which generalises inverse of scalars.

Right inverse:

A matrix X satisfies $AX = I$, then X is called as right inverse of A.

Let A has right inverse B.

$$AB = I$$

$$\Rightarrow \quad B^T A^T = I^T = I$$

$\Rightarrow$ B^T is a left inverse of A^T.

(Right inverse exists) $\Leftrightarrow$ (Rows of the matrix are lin. indep.)

Application to solving system of linear eqn:

$$Ax = b$$

Suppose A is right invertible.

Let B be a right inverse.

$\underbrace{x = Bb}$ $\Rightarrow$ $Ax = A(Bb) = (AB)b = Ib = b$

$$A = \begin{bmatrix} -3 & -4 \\ \boxed{\begin{matrix} 4 & 6 \\ 1 & 1 \end{matrix}} \end{bmatrix}$$

$$B = \frac{1}{9}\begin{bmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{bmatrix} \quad ; \quad C = \frac{1}{2}\begin{bmatrix} 0 & \boxed{\begin{matrix} -1 & 6 \\ 1 & -4 \end{matrix}} \\ 0 & \end{bmatrix}$$

1) For $b = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$, $Ax = b$ has a unique solution.

$$x = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad x = \underbrace{B\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}} = C\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

2) $b = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\quad Ax = b$ does <u>NOT</u> solution.

<u>overdetermined</u>

$$x = B\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

3) $A^T y = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ under-determined

$$\begin{bmatrix} -3 & 4 & 1 \\ -4 & 6 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Since B and C are left inverses of A, B^T & C^T are right inverses of A^T.

$y = B^T \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $y = C^T \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Inverse:

Suppose a matrix A is left as well as right invertible.

$$AX = I \quad \text{and} \quad YA = I$$

right inverse   left inverse

$$X = IX = (YA)X \overset{\text{associative}}{=} Y(AX) = YI = Y \quad -(*)$$

Every right inverse is the left inverse.

Suppose $\tilde{X}$ be another right inverse.

$$A\tilde{X} = I, \quad YA = I, \quad AX = I$$

$$\tilde{X} = I\tilde{X} = (YA)\tilde{X} = Y(A\tilde{X}) = YI = Y = X \quad -(**)$$

Right inverse is unique.

A^{-1} : unique left as well as right inverse.

A : invertible or non-singular (if the inverse exists)

<u>Solving the system of eq^{ns}:</u>

$$Ax = b$$

choose. $x = A^{-1}b$ if A is invertible.

Let $A \in \mathbb{R}^{n \times n}$ be a matrix which is left invertible.

$\Rightarrow$ columns of A are linearly independent.

$\Rightarrow$ these columns form a basis of $\mathbb{R}^n$.

$\Rightarrow$ Every vector in $\mathbb{R}^n$ can be written as a unique linear combination of columns of A.

$\Rightarrow$ In particular, e_i : i^{th} unit vector (i^{th} column of the identity matrix)

$$e_i = Ab_i \quad \text{for } i = 1, 2, \ldots, n$$

$\Rightarrow$

- Construct $B = [\, b_1 \;\; b_2 \cdots \;\; b_n \,] \in \mathbb{R}^{n \times n}$

$$AB = A[\, b_1 \;\; b_2 \cdots \;\; b_n \,] = [e_1 \;\; e_2 \cdots \; e_n] = I$$

$\Rightarrow$ B is a right inverse of A.

<u>For square matrices.</u>

left invertibility $\Rightarrow$ column independence $\Rightarrow$ right invertibility

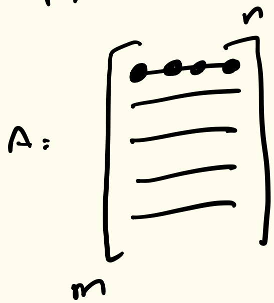
&

right invertibility $\Rightarrow$ row independence $\Rightarrow$ left invertibility

Pseudo-inverse:

Gram-matrix

We first show that columns of A are linearly independent if and only if its $n \times n$ Gram-matrix $A^T A$ is invertible.



Pf: First assume columns of A are linearly independent.

T.S. $A^T A$ is invertible.

Let x be a vector s.t. $(A^T A)x = 0$

$$0 = x^T 0 = x^T (A^T A) x = (x^T A^T)(Ax) = (Ax)^T (Ax)$$

$$= \|Ax\|_2^2 \Rightarrow Ax = 0 \Rightarrow x = 0$$

$\Rightarrow$ $A^T A$ has linearly indep. columns

$\Rightarrow$ $A^T A$ is invertible ($\because$ $A^T A$ is square)

<u>Conversely,</u>

T.S. $\begin{pmatrix} A^T A \text{ is} \\ \text{invertible} \end{pmatrix} \Rightarrow \begin{pmatrix} \text{columns of } A \\ \text{are lin. indep} \end{pmatrix}$

$\Leftrightarrow$ $\begin{pmatrix} \text{columns of} \\ A \text{ are lin. dep.} \end{pmatrix} \Rightarrow \begin{pmatrix} A^T A \text{ is not} \\ \text{invertible} \end{pmatrix}$

cols. of lin. dep. $\Rightarrow$ $\exists\, x \neq 0$ s.t. $Ax = 0$

$0 = A^T 0 = A^T (Ax) = (A^T A)x$

$\Rightarrow$ Cols. of $A^T A$ are lin. dep. $\Rightarrow$ $A^T A$ is <u>NOT</u> invertible ∎

Suppose $A \in \mathbb{R}^{m \times n}$ and cols. of A

lin. indep.

$\Rightarrow$ $A^T A$ is invertible.

$$B = \underbrace{(A^T A)^{-1} A^T}$$ = pseudo inverse of A.

Is B a left inverse of A??

$$BA = (A^T A)^{-1} (A^T A) = I$$