Q3)

(a) $A = \begin{pmatrix} -1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \\ -1 & 1 \end{pmatrix} \in \mathbb{R}^{3\times 2}$, $m=3$, $n=2$

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^{2\times 1}$.

$$Ax = \begin{pmatrix} -1/\sqrt{2}\, x_1 \\ -1/\sqrt{2}\, x_2 \\ -x_1 + x_2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where $x_1^2 + x_2^2 = 1$ (unit circle in $\mathbb{R}^2$)

Consider $x_1 = \cos t$, $x_2 = \sin t$
$t \in [0, 2\pi)$

Then we get

$$Ax = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\cos t/\sqrt{2} \\ -\sin t/\sqrt{2} \\ \cos t - \sin t \end{pmatrix}$$ [figure of ellipse obtained attached below]

Thus in $\mathbb{R}^3$ we have an image of the unit circle in $\mathbb{R}^2$, obtained by multiplication from A.

Condition number of $A$ = 2.236

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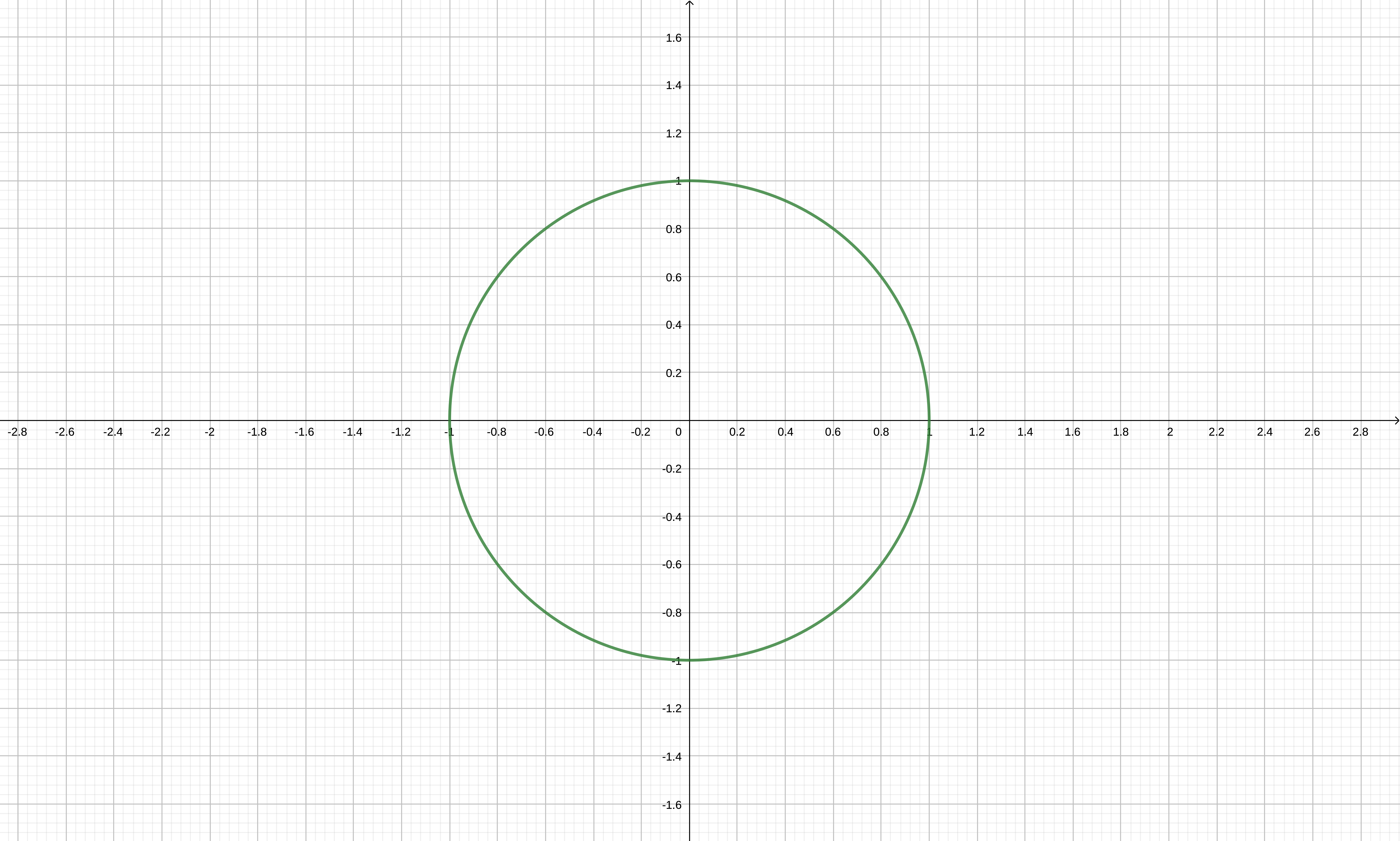
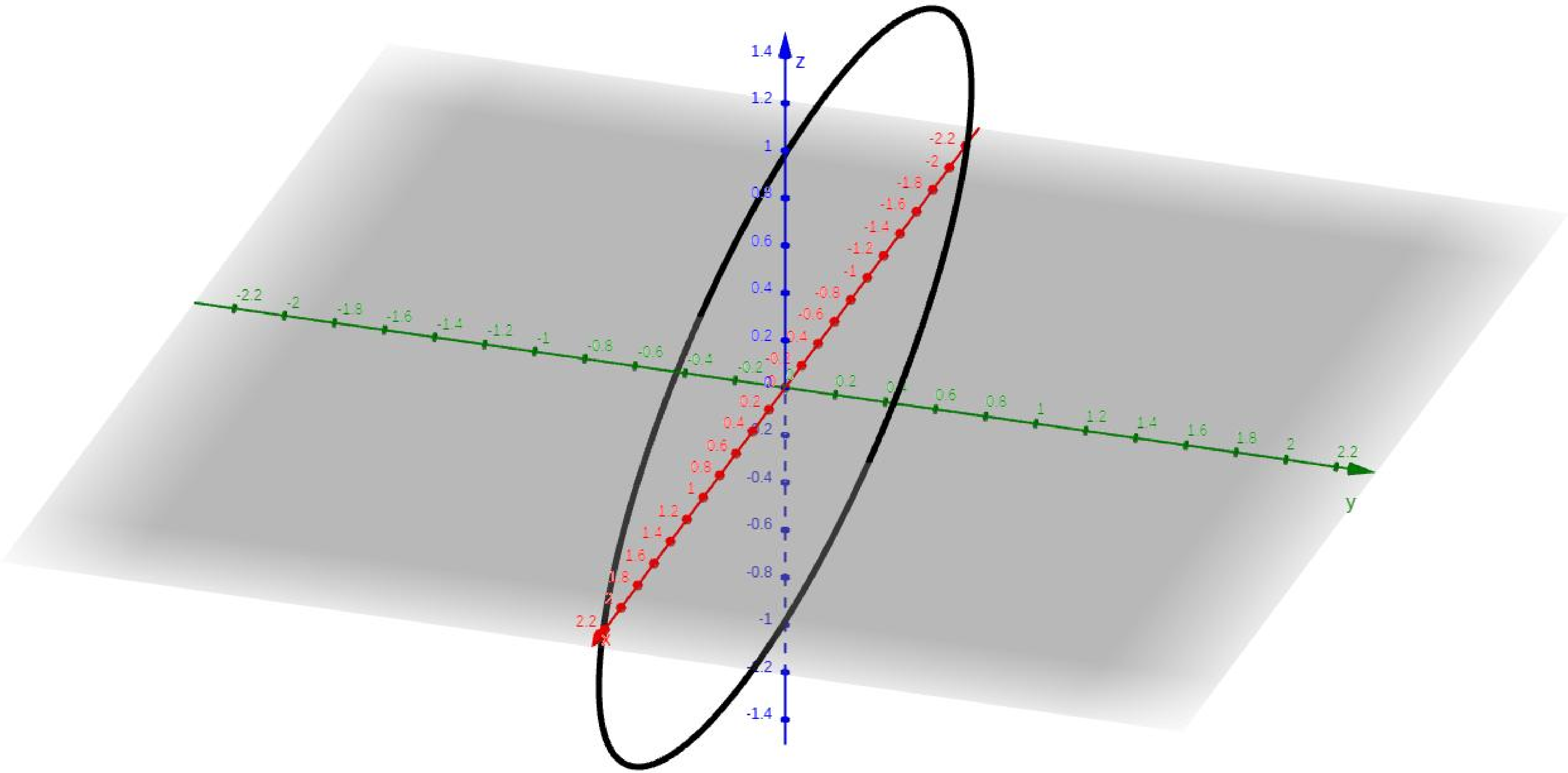
$\boxed{\text{Note}}$: Unit sphere in $\mathbb{R}^2$ is same for parts a, c, d and e, so it is shown only once.


1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
-2.8
-2.6
-2.4
-2.2
-2
-1.8
-1.6
-1.4
-1.2
-1
-0.8
-0.6
-0.4
-0.2
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
2.8

# 3. a.



(b)

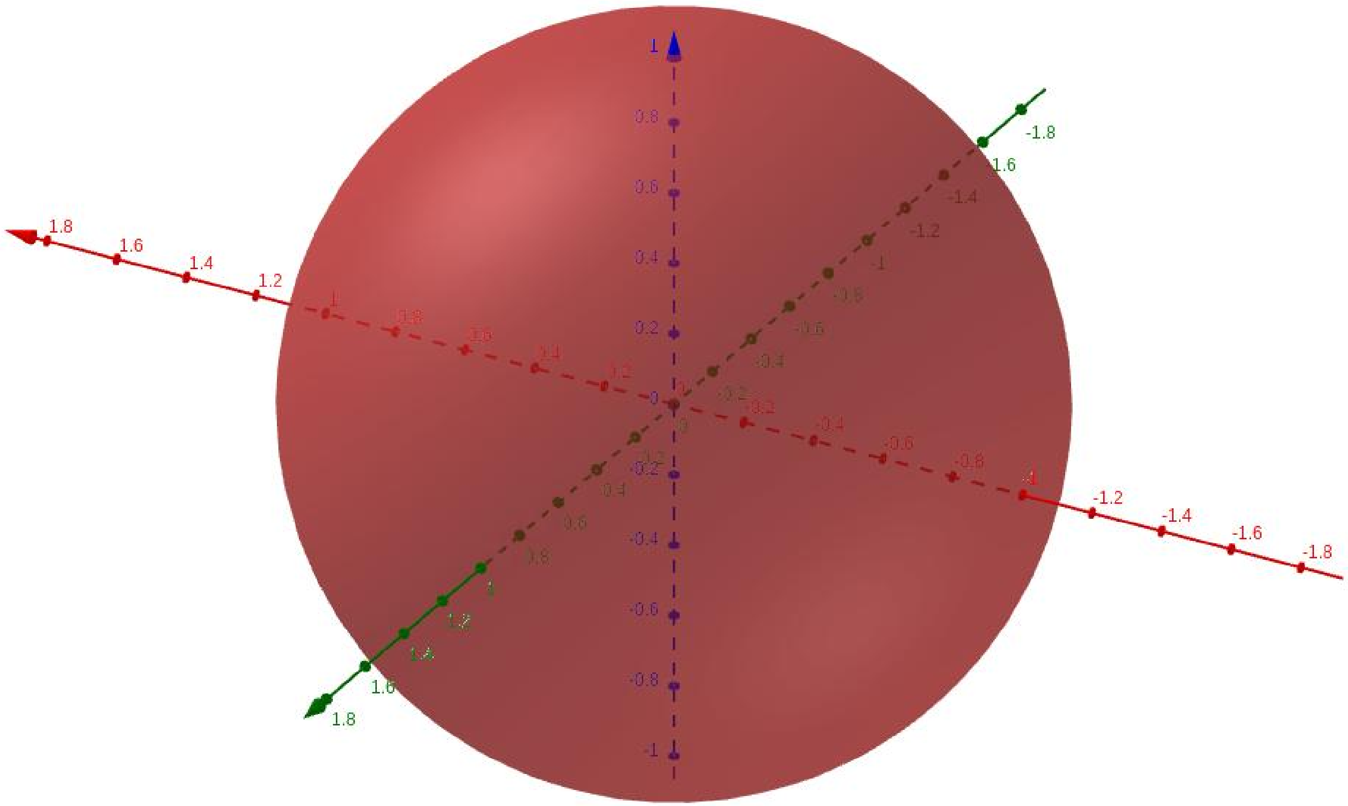
$$A = \begin{pmatrix} -2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 3} \quad m=2, n=3$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ax = \begin{pmatrix} -2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_1 + x_2 + 2x_3 \\ 2x_2 \end{pmatrix}$$

where $x_1^2 + x_2^2 + x_3^2 = 1$ (unit sphere in $\mathbb{R}^3$)

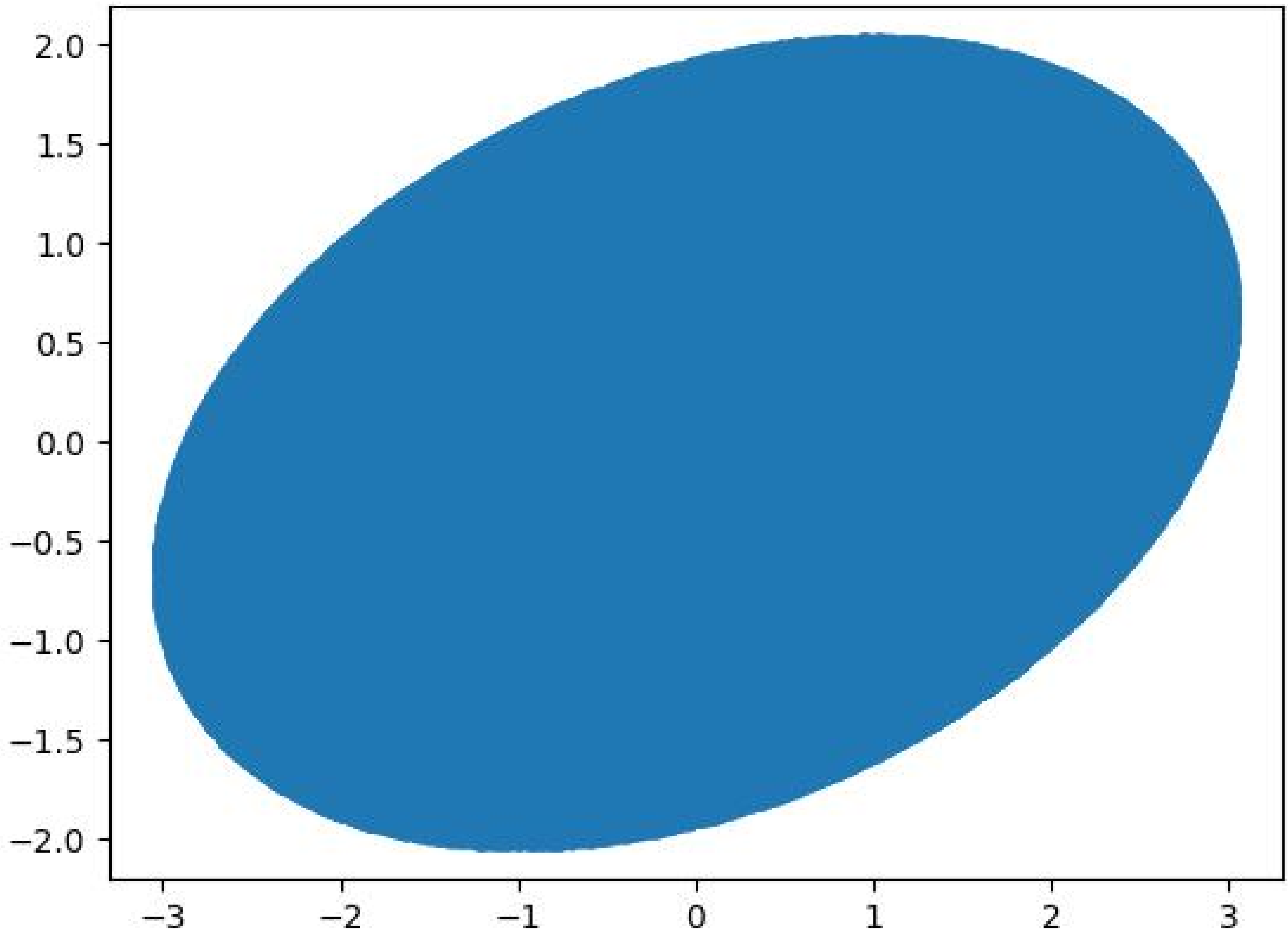
It is not possible to directly obtain the equation of the ellipse as we have to eliminate 3 variables from 2 equations. These equations form a region inside an ellipse. To plot this region we generate 1000s of random points which satisfy the above equations and plot the points. (using python) Code and image of ellipse is attached below.

Condition number of A = 1.7150



1.8
1.6
1.4
1.2
-1.2
-1.4
-1.6
-1.8
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1

```python
import numpy as np
import matplotlib.pyplot as plt

pts = 30000

u = np.linspace(0, np.pi, pts)
v = np.linspace(0, 2 * np.pi, pts)

np.random.shuffle(u)
np.random.shuffle(v)

x = np.sin(u) * np.cos(v)
y = np.sin(u) * np.sin(v)
z = np.cos(u)

x_out = -2 * x + y + 2 * z
y_out = 2 * y

plt.plot(x_out, y_out, 'o')
plt.show()
```



2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−3
−2
−1
0
1
2
3

(c) $A = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 0.8 \end{pmatrix} \in R^{2\times2}$, $m=2$, $n=2$

$$Ax = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 + 0.9x_2 \\ 0.9x_1 + 0.8x_2 \end{pmatrix}$$

$$0.9x - y = 0.9x_1 + 0.81x_2 - (0.9x_1 + 0.8x_2)$$
$$= 0.01x_2$$

$$0.8x - 0.9y = -0.01x_1$$

$x_1^2 + x_2^2 = 1$ (unit circle in $\mathbb{R}^2$)

$$\Rightarrow (0.9x-y)^2 + (0.8x - 0.9y)^2 = 10^{-4}(x_1^2+x_2^2) = 10^{-4}$$

$$\Rightarrow (81x^2+100y^2+64x^2+81y^2) - (180xy + 144xy) = 10^{-2}$$

$$\Rightarrow 145x^2 + 181y^2 - 324xy = 10^{-2}$$ (Equation of ellipse)

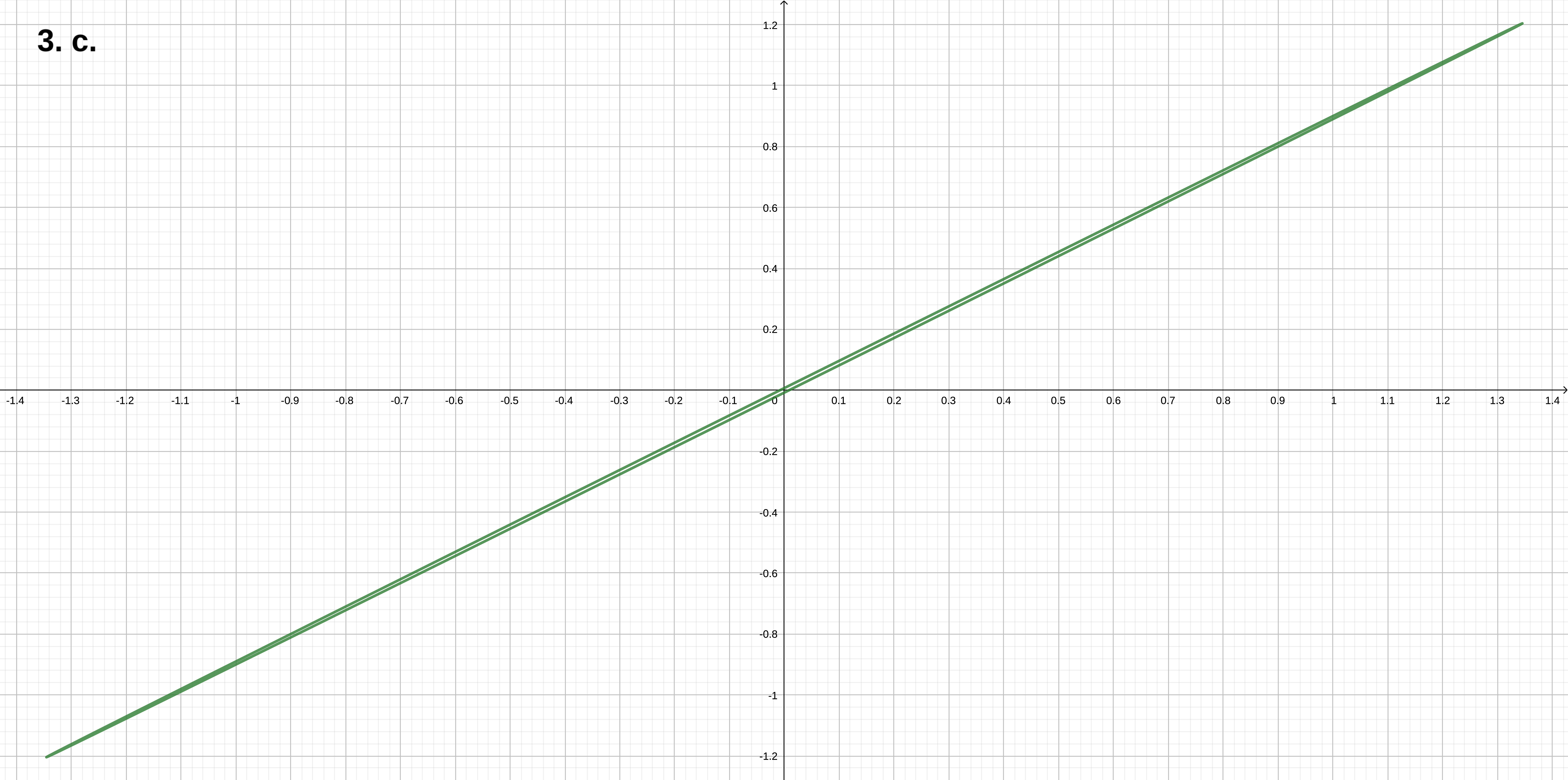
- We map this ellipse in the 2D plane (image is attached below)

Here $m=n$ and coloums of $A$ are linearly independent, so, A is invertible.

Determinant of $A = 1\times0.8 - 0.9\times0.9$
$= -0.01$

Condition number of $A = 325.9969$

**3. c.**


1.2
1
0.8
0.6
0.4
0.2
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4 -1.3 -1.2 -1.1 -1 -0.9 -0.8 -0.7 -0.6 -0.5 -0.4 -0.3 -0.2 -0.1 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1 1.1 1.2 1.3 1.4

(d) $A = \begin{pmatrix} 1 & 0 \\ 0 & -10 \end{pmatrix} \in R^{2\times 2}, \quad m=2, n=2$

$$Ax = \begin{pmatrix} 1 & 0 \\ 0 & -10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -10x_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$x_1^2 + x_2^2 = 1$ (Unit circle in $\mathbb{R}^2$)

$x^2 + \left(\frac{y}{10}\right)^2 = 1$

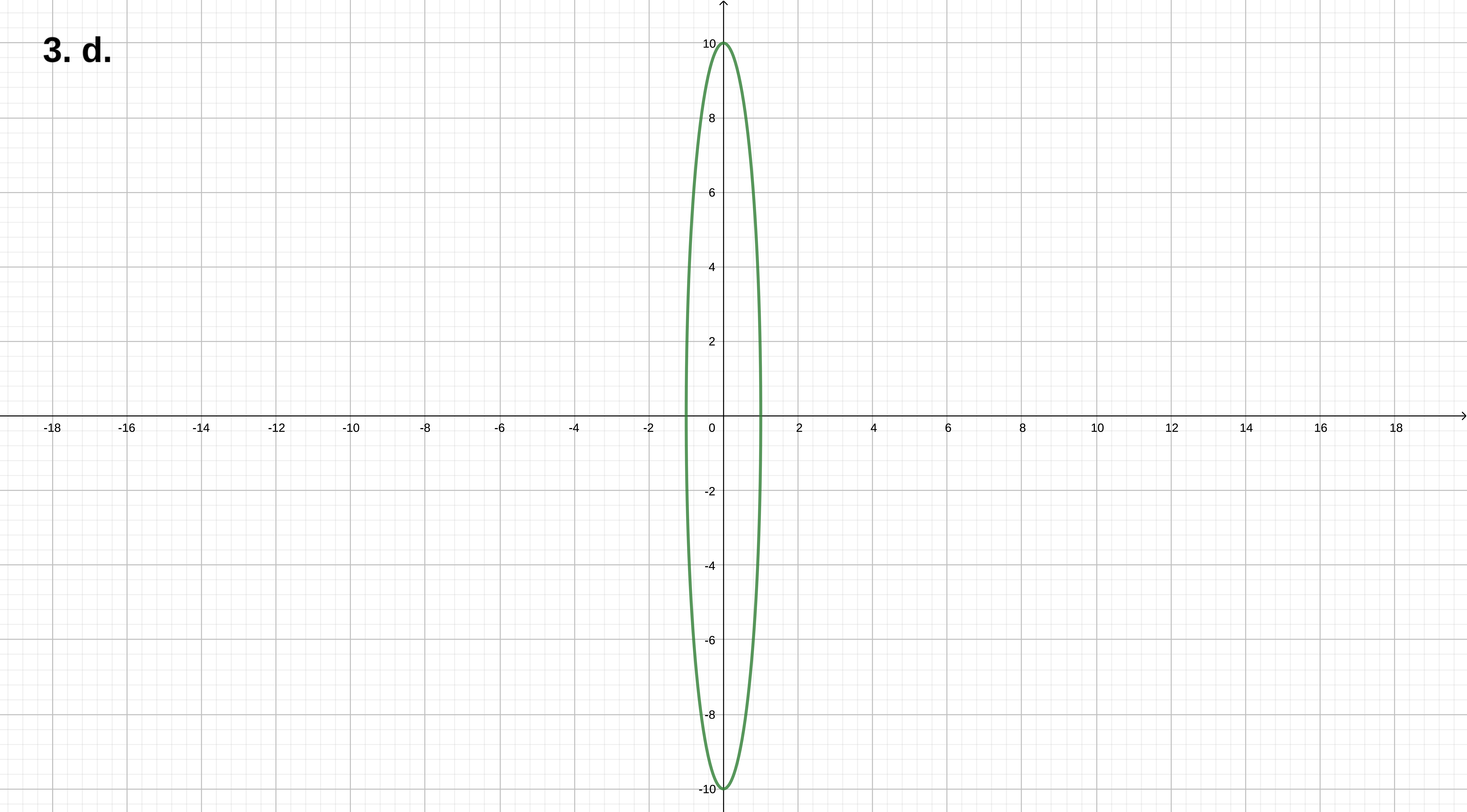
$x^2 + \frac{y^2}{100} = 1$ (Equation of the ellipse)

Here $m = n$ so we need to calculate determinant and invertability.
Coloumns of A are linearly independent
So A is invertible.

~~Det~~ Determinant of $A = 1\times(-10) - 0\times 0$
$= -10$

Condition number of A = 10.0

# 3. d.



(e) $A = \begin{pmatrix} 1 & 1 \\ 1 & \varepsilon \end{pmatrix} \in \mathbb{R}^{2\times 2}$, $m=2$, $n=2$

$$Ax = \begin{pmatrix} 1 & 1 \\ 1 & \varepsilon \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ x_1+\varepsilon x_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$x_1^2+x_2^2 = 1$ ( Unit circle in $\mathbb{R}^2$)

Equation of ellipse is obtained as.

$$(\varepsilon^2+1)x^2 + 2y^2 - 2xy(1+\varepsilon) = (\varepsilon-1)^2$$

We calculate all results for different values of $\varepsilon$. and obtain the results and observe the trend.

$\underline{\varepsilon = 10}$

$$101x^2 + 2y^2 - 22xy = 81$$

$A = \begin{pmatrix} 1 & 1 \\ 1 & 10 \end{pmatrix}$

$m=n$ so we need determinant and invertibility

Coloums of A are linearly independent,

Hence $\underline{A \text{ is invertible}}$.

Determinant of $A = \varepsilon - 1 = 10-1 = \underline{\underline{9}}$

Condition number of $A = \underline{\underline{11.3563}}$.

$\underline{\varepsilon = 5}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}$$

Equation of the ellipse :-

$26x^2 + 2y^2 - 12xy = 16$

Coloumns of A are linearly indopendent,

Hence $\underline{A \text{ is invertible}}$.

Determinant of $A = \varepsilon - 1 = 5 - 1 = \underline{4}$

Condition number of $A = \underline{6.8541}$

$\underline{\varepsilon = 1}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Here the image of the circle does not make an ellipse but flattens to a line $\underline{y = x}$

Condition number of $A$ = infinite.

Coloumns of A are linearly dependent.

Hence matrix $\underline{A \text{ is not invertible}}$.

Determinant of $A = \underline{0}$.

$\varepsilon = 10^{-1}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 10^{-1} \end{pmatrix}$$

Equation of the ellipse :-

$$1.01x^2 + 2y^2 - 2.2xy = 0.81$$

Coloumns of A are linearly independent

Hence A is invertible.

~~Dett~~ Determinant of $A = \varepsilon - 1 = 10^{-1} - 1 = -0.9$

Condition number of $A = 3.0124$.

$\varepsilon = 10^{-2}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 10^{-2} \end{pmatrix}$$

Equation of the ellipse :-

$$1.0001x^2 + 2y^2 - 2.02xy = 0.9801$$

Coloumns of A are linearly independent

Hence A is invertible.

Determinant of $A = \varepsilon - 1 = 10^{-2} - 1 = -0.99$

Condition number of $A = 2.6535$

$\underline{\underline{\varepsilon = 10^{-4}}}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 10^{-4} \end{pmatrix}$$

Equation of the ellipse:--

$1.00000001\, x^2 + 2y^2 - 2.0002xy = 0.99980001$

As coloumns of A are linearly independent

Hence A is invertible

Determinant of $A = \varepsilon - 1 = 10^{-4} - 1 = \underline{\underline{-0.9999}}$

Condition number of $A = \underline{\underline{2.6183}}$

$\underline{\underline{\varepsilon = 0}}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Equation of the ellipse:

$x^2 + 2y^2 - 2xy = 1$

Coloumns of A are linearly indopendent

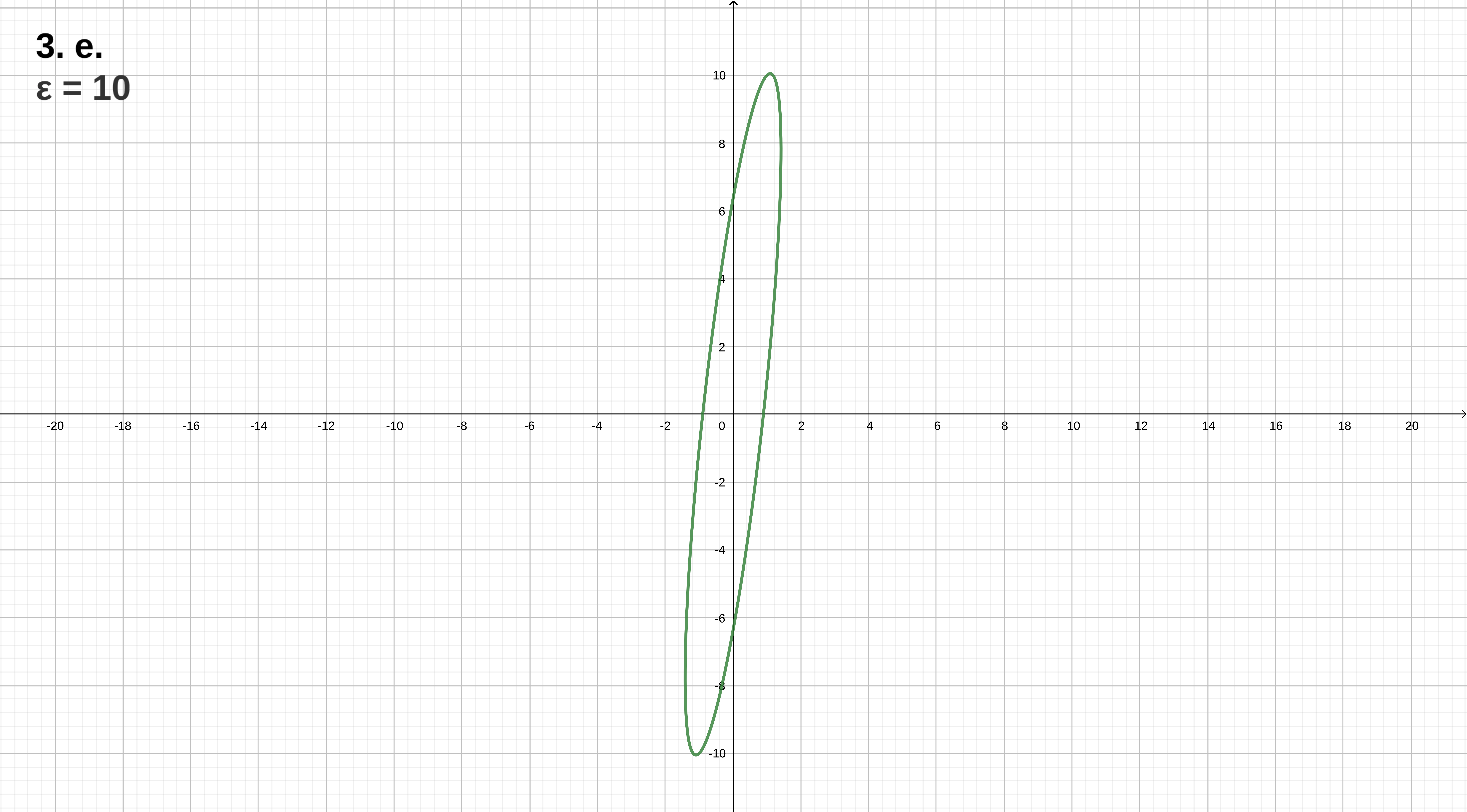
Hence A is invertible.

Determinant of $A = \varepsilon - 1 = 0 - 1 = \underline{\underline{-1}}$

Condition number of $A = \underline{\underline{2.6180}}$

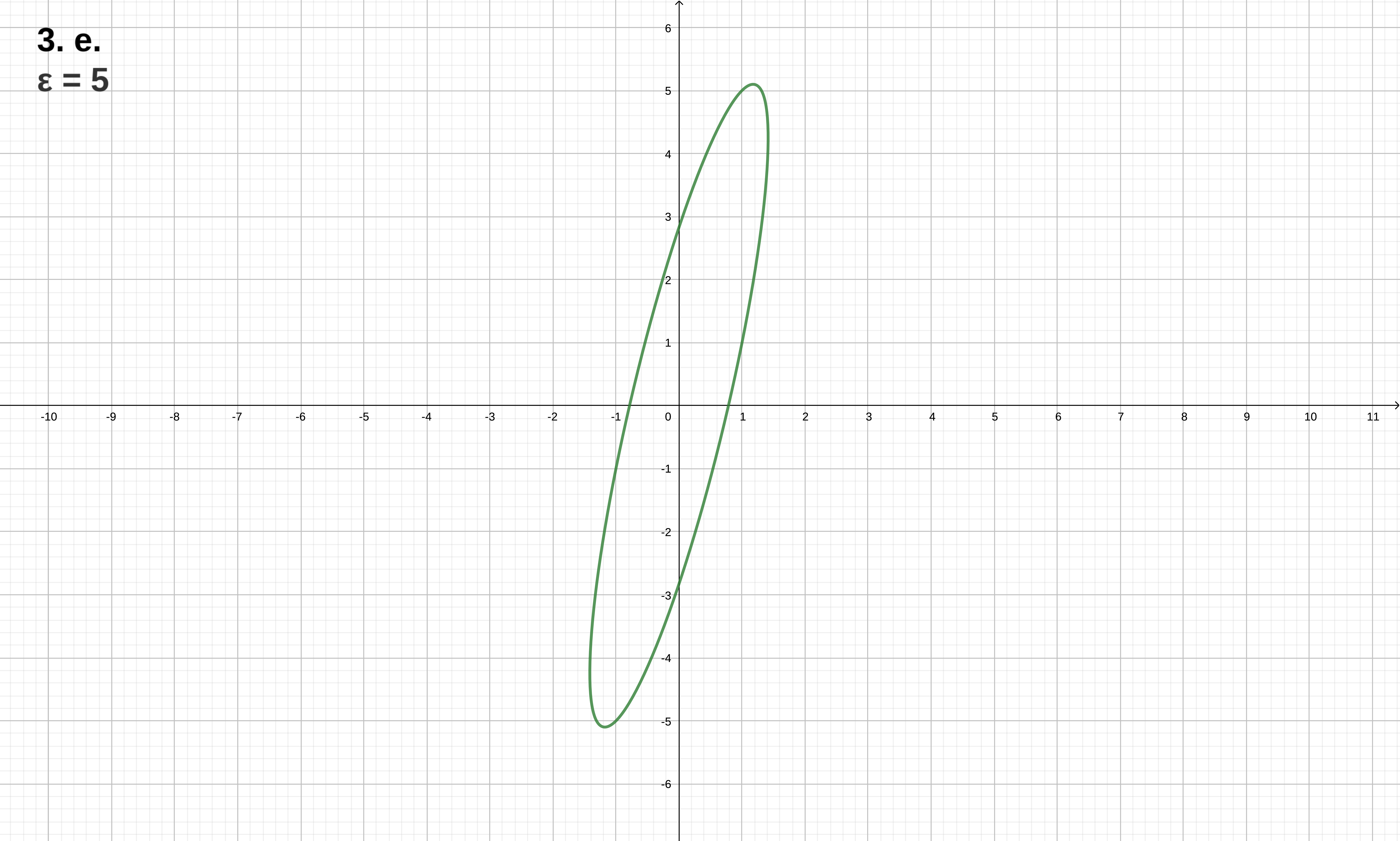
**3. e.**

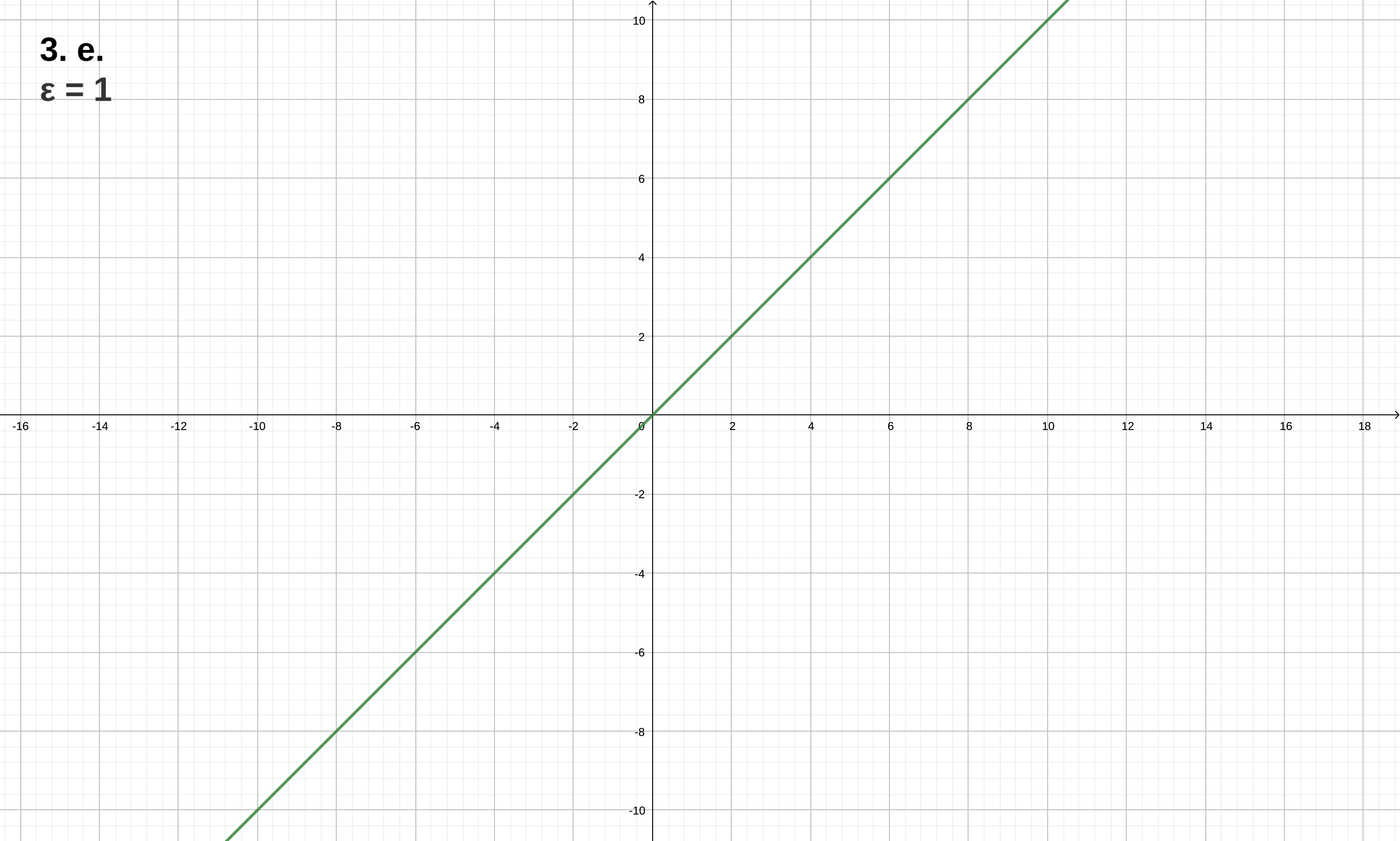
**ε = 10**

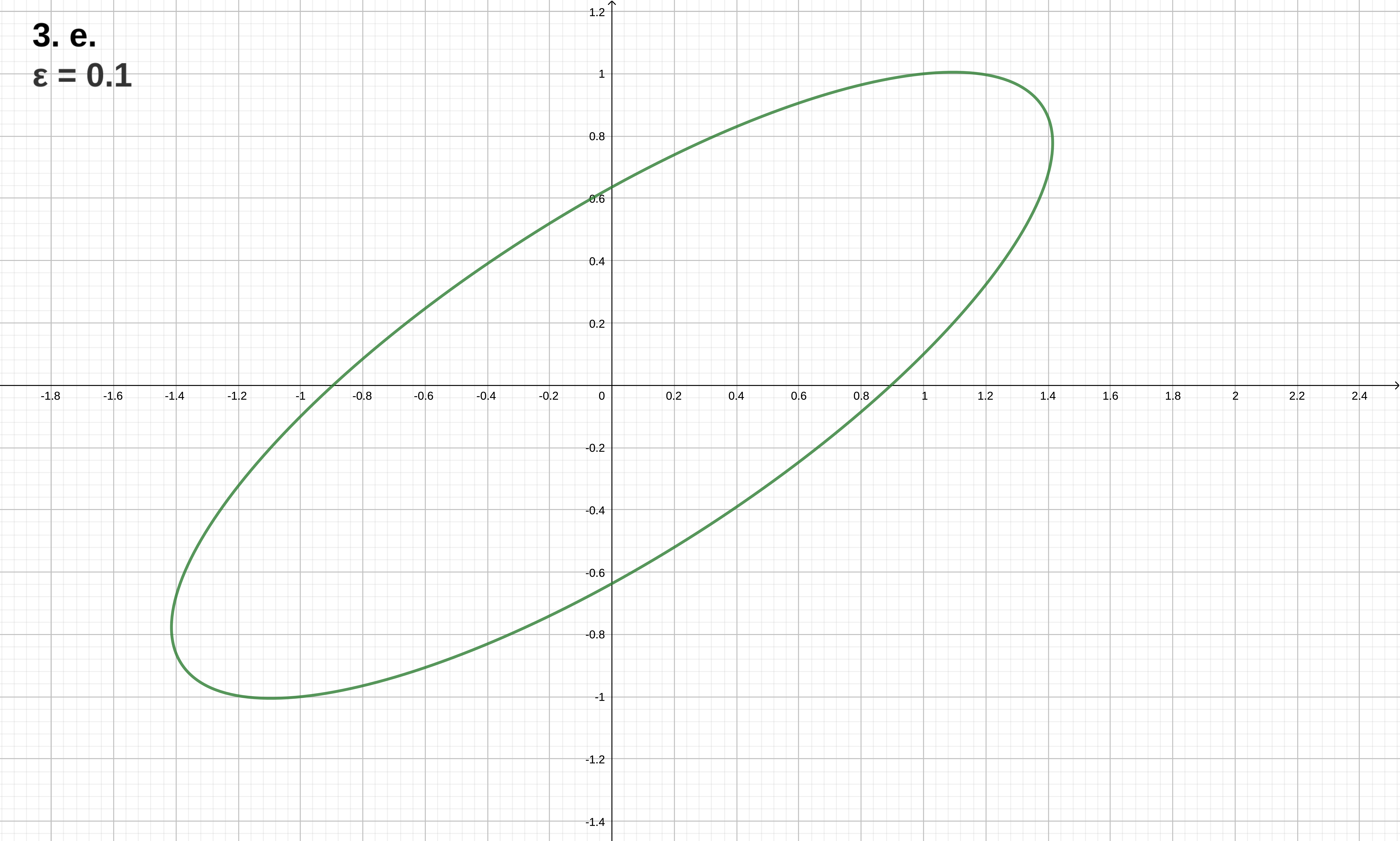


## 3. e.

$\varepsilon = 5$

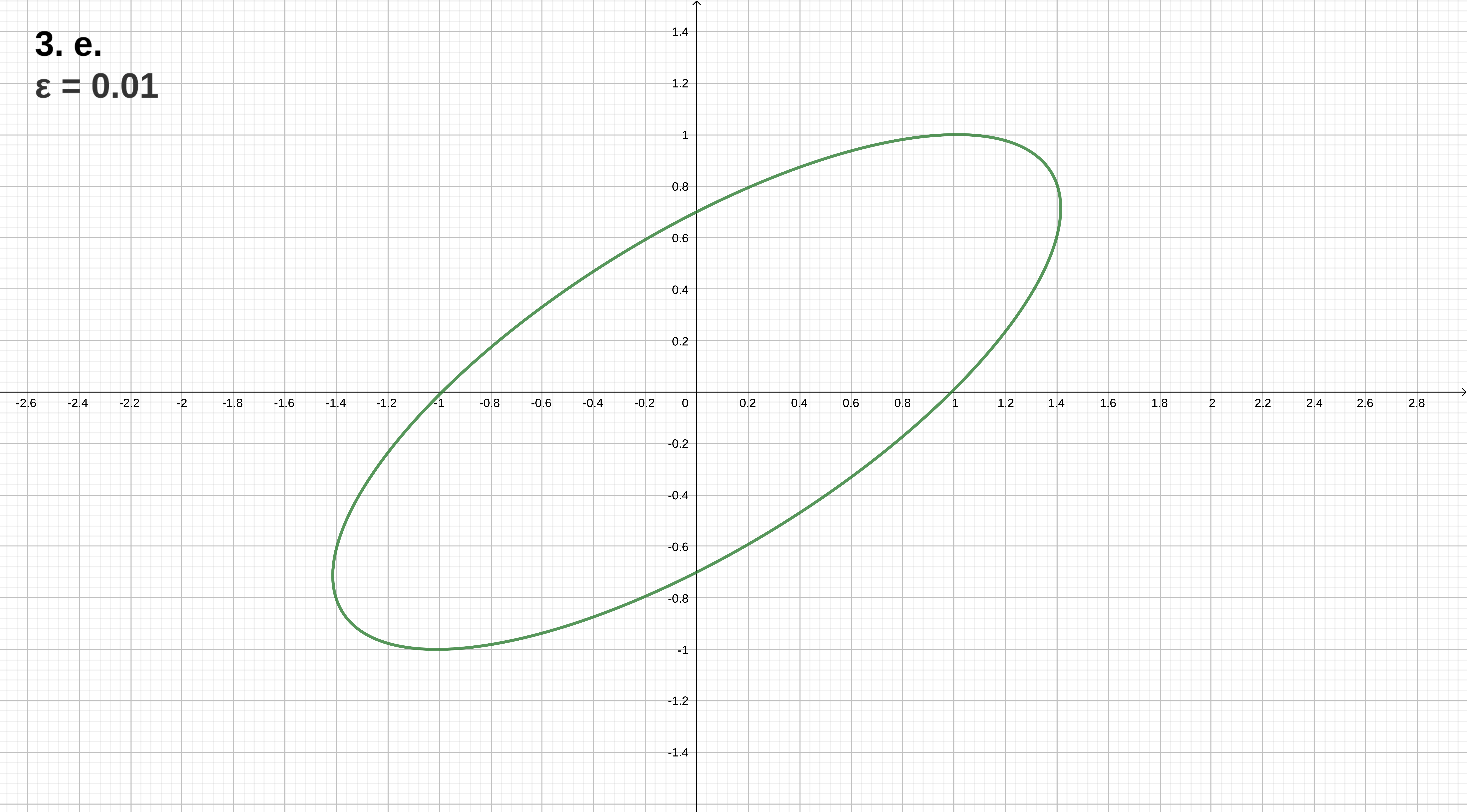


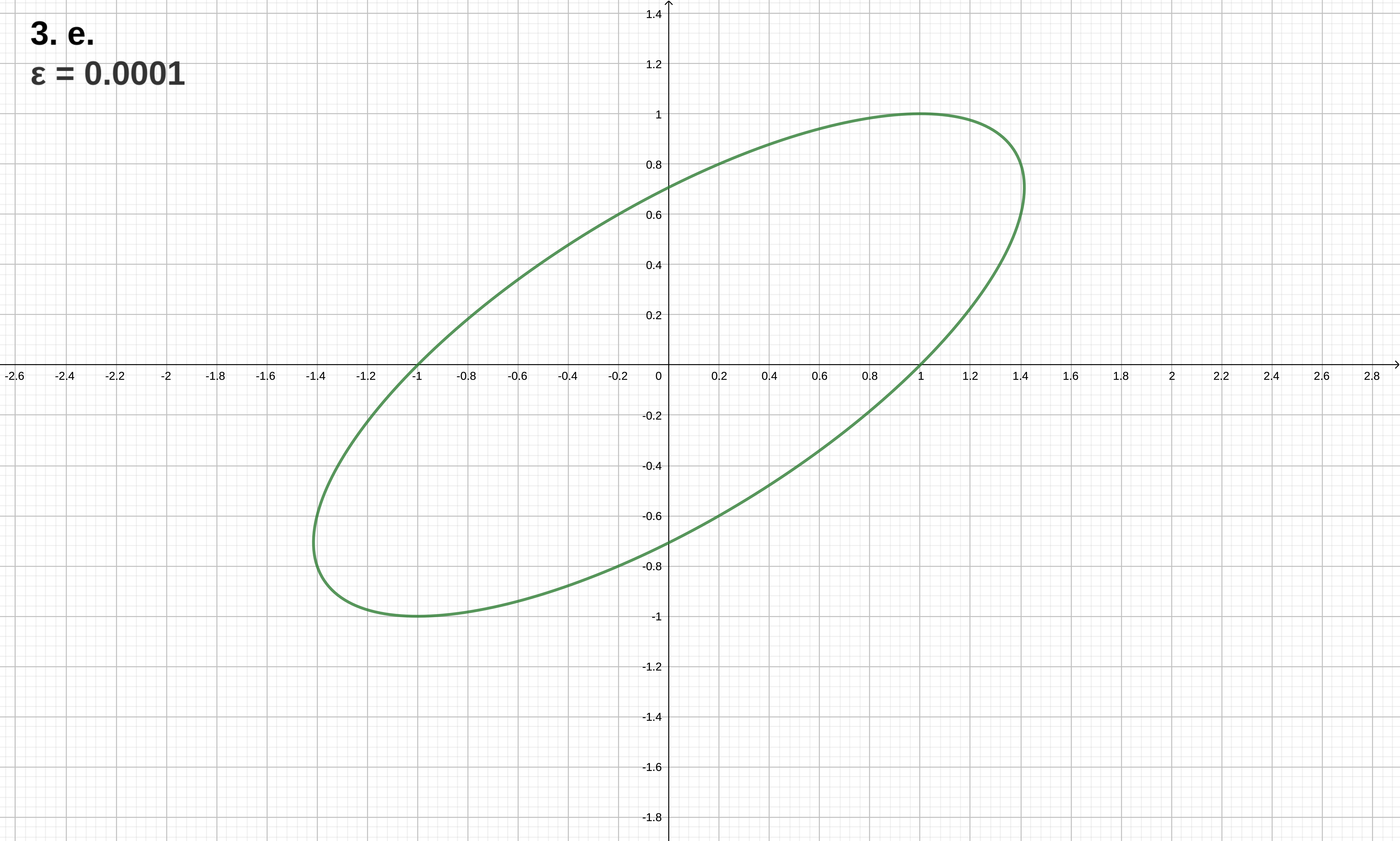

3. e.
ε = 1
10
8
6
4
2
-2
-4
-6
-8
-10
-16
-14
-12
-10
-8
-6
-4
-2
0
2
4
6
8
10
12
14
16
18


3. e.
ε = 0.1
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.8
-1.6
-1.4
-1.2
-1
-0.8
-0.6
-0.4
-0.2
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4

## 3. e.

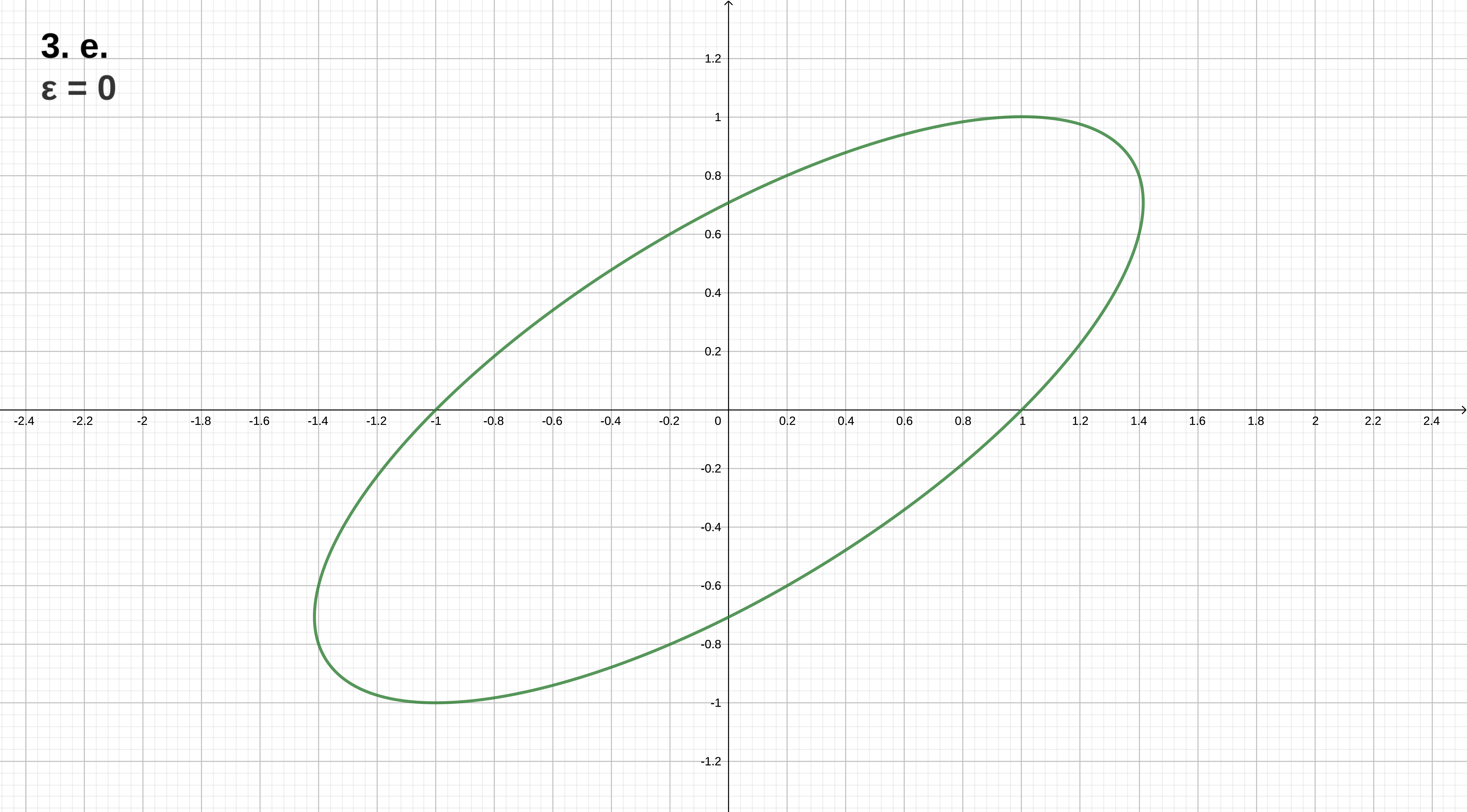
## $\varepsilon = 0.01$




3. e.
ε = 0.0001
-2.6 -2.4 -2.2 -2 -1.8 -1.6 -1.4 -1.2 -1 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2 2.2 2.4 2.6 2.8
1.4 1.2 1 0.8 0.6 0.4 0.2 -0.2 -0.4 -0.6 -0.8 -1 -1.2 -1.4 -1.6 -1.8

**3. e.**

**ε = 0**



1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-2.4
-2.2
-2
-1.8
-1.6
-1.4
-1.2
-1
-0.8
-0.6
-0.4
-0.2
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4

Relationship between value of determinant and condition number.

$\Rightarrow$ If determinant of a matrix is very small then that matrix is mostly an ill-conditioned matrix ( Condition number very high)

Observation

In (C) determinant of A was small (-0.01) which resulted in high condition number (325.99)

We can also notice this relationship in other examples.

Reason.

High condition number. $\Rightarrow \min_{\|x\|=1} \|Ax\|_2$ is "very very small"

$\Rightarrow \|Ax\|_2$ is "almost zero!"

$\Rightarrow Ax$ is "almost zero"!

$\Rightarrow$ Coloumns of A are "almost linearly dependent".

$\Rightarrow$ matrix A is "almost singular"

$\Rightarrow$ Almost singular ~~...~~ means very small determinant.

$\Rightarrow$ We can also observe when determinant $=0$; (linearly dependent coloumns) $\Rightarrow$ condition number goes to infinity.