Linear algebra for AI & ML

Lecture #3
August - 19
(6 video lectures
shared already).

vector space : (over $\mathbb{R}$) $\mathbb{R}^n$

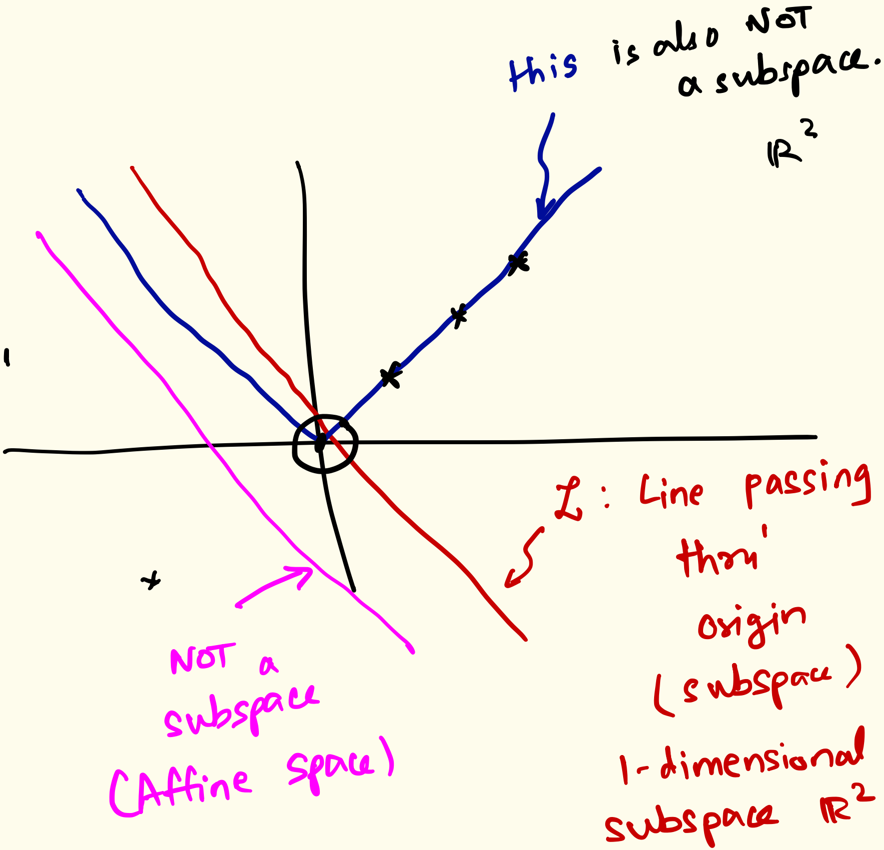
$\mathbb{R}^{m \times n}$

vector subspace :

i) $\{0\}$

ii) Lines passing thru' origin.

iii) $\mathbb{R}^2$



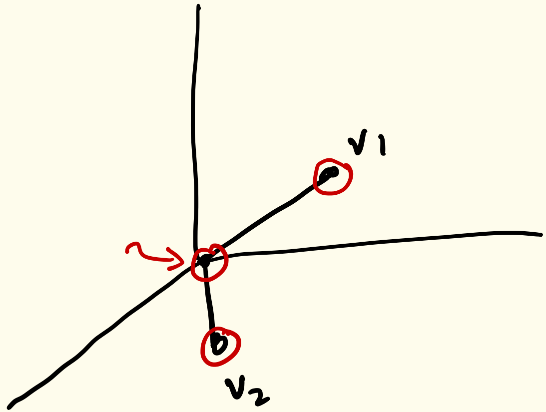
Liner span of $\{v_1, \ldots, v_k\} \subseteq \mathbb{R}^n$

$$= \{\alpha_1 v_1 + \cdots + \alpha_k v_k \mid \alpha_i \in \mathbb{R} ; i = 1, 2, \ldots, k\}$$

Linear span is a subspace of $\mathbb{R}^n$

Take $A = \{v_1, v_2\} \subseteq \mathbb{R}^3$

$$\text{span}(A) = \{\alpha_1 v_1 + \alpha_2 v_2 \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$$



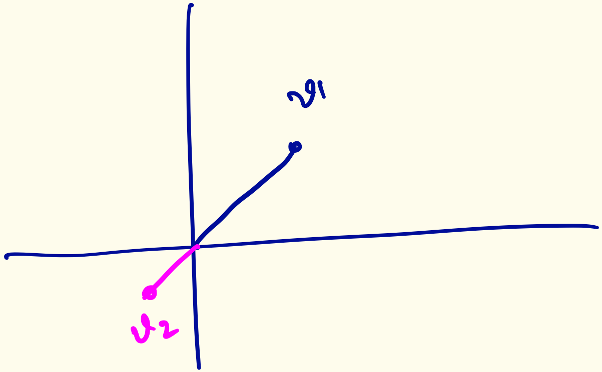
Linear dependence and indepence:

$\{v_1, \dots, v_k\} \subseteq \mathbb{R}^n$

start $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_k = 0$

Linearly independent.



$v_1 = \alpha v_2$

$$\boxed{v_1 - \alpha v_2 = 0}$$

$\alpha_1 = 1, \quad \alpha_2 = -\alpha$

Linear function:

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$v \longmapsto T(v)$$

$$T(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 T(v_1) + \alpha_2 T(v_2)$$

In particular, $m = 1$

Inner product $\rightarrow$ linear $f^{\underline{nal}}_{\cdot\cdot}$.

Affine $f^{\underline{n}}_{\cdot\cdot}$.

$T(x) = Ax + b$ $\quad \forall x \in \mathbb{R}^n, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m$

Linear $f^{\underline{n}}_{\cdot\cdot}$

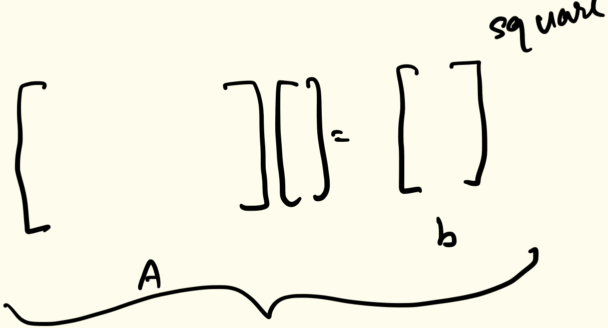
iff $b = 0$

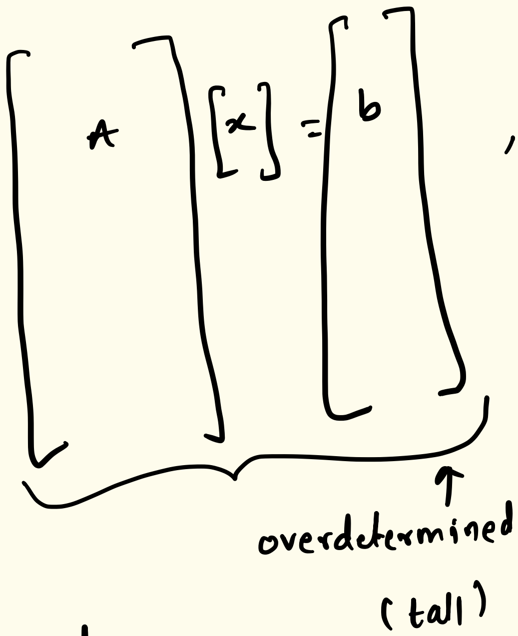
Solving system of linear equations.

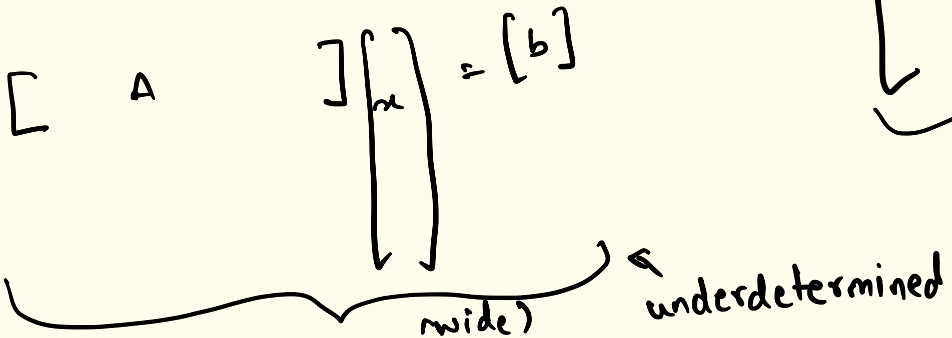
Given: $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^{m}$

Find, if exists, $x \in \mathbb{R}^{n}$ such that

$$Ax = b$$







$A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ find $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

Suppose $\exists\; x \in \mathbb{R}^n$ s.t.

$$Ax = b$$

$$\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b$$

Given $b \in \mathbb{R}^m$, one can write b as a linear combination of columns of A.

Here, $a_1, a_2, \ldots, a_n \in \mathbb{R}^m$ are columns of A.

Matrix-Vector product

$$\begin{array}{cc} & \\ A & \begin{bmatrix} \text{---} \; r_1^T \; \text{---} \\ r_2^T \\ \vdots \\ r_m^T \end{bmatrix} \end{array} \begin{array}{c} x \\ \begin{bmatrix} | \\ | \\ | \end{bmatrix} \end{array} = \begin{bmatrix} r_1^T x \\ r_2^T x \\ \vdots \\ r_m^T x \end{bmatrix} = x_1 \begin{bmatrix} | \\ a_1 \\ | \end{bmatrix} + x_2 \begin{bmatrix} | \\ a_2 \\ | \end{bmatrix} + \cdots + x_n \begin{bmatrix} | \\ a_n \\ | \end{bmatrix}$$

Linear span of columns of a matrix $A \in \mathbb{R}^{m \times n}$ is called as column space of A (estimation space).

Since column space is a linear span, it is a subspace of $\mathbb{R}^m$.

dim (column space of A) = rank(A)

If rank$(A) = m$

$\Rightarrow$ column space of $A = \mathbb{R}^m$

$\Rightarrow$ $\forall\, b \in \mathbb{R}^m$, one can always find an $x \in \mathbb{R}^n$ s.t. $Ax = b$

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

column space of $A = \mathbb{R}^2$

for $b \in \mathbb{R}^2$

$Ax = b$ where $x \in \mathbb{R}^3$

$$x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{span} \underbrace{\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}}_{\text{basis of } \mathbb{R}^2} = \mathbb{R}^2$$

$$\alpha_1 e_1 + \alpha_2 e_2 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$\alpha = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

$\alpha = 1$

$$x + \alpha \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix} \leftarrow z$$

$$x = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$$

$\alpha = -1$

$A \in \mathbb{R}^{m \times m}$

$\text{col}(A) = \mathbb{R}^m$

columns of A $= \underbrace{\{a_1, \dots, a_m\}}_{\text{a basis of } \mathbb{R}^m}$ span $\mathbb{R}^m$

for any $b \in \mathbb{R}^m$, there will exist a unique linear combination $[x_1, \dots, x_m]^T$ s.t.

$Ax = b$.

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}_{m\times n} \begin{bmatrix} x \end{bmatrix}_{n\times 1} = \begin{bmatrix} b \\ \\ \end{bmatrix}_{m\times 1}$$

$$\begin{bmatrix} a_1 & \cdots & a_n \\ & & \\ & & \end{bmatrix}_{m\times n}$$

$m \geq n$

$(m >> n)$

Ex: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$

$\left\{ \begin{pmatrix} \alpha_1 \\ 0 \\ \alpha_1 + \alpha_2 \end{pmatrix} : \alpha_1, \alpha_2 \in \mathbb{R} \right\}$

$y = 0$: xz - plane

$Ax = b$ $\quad A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

i) if $b \in col(A)$, then $\exists$ $\boxed{\text{a}}$ solution $x \in \mathbb{R}^n$ s.t. $Ax = b$

ii) If columns of A form a basis of $col(A)$, then for $b \in col(A)$, $\boxed{\text{the}}$ solution $x \in \mathbb{R}^n$ is unique.

Let $z \in \mathbb{R}^n$ be s.t.

$$Az = 0$$

[$z = 0$ is obvious]

Use linearity of A:

Let $x \in \mathbb{R}^n$ be s.t.

$$Ax = b$$

$$\Rightarrow A(x+z) = b$$

$$\Rightarrow A(\underbrace{x + \alpha z}) = b$$

$$\mathcal{N}(A) = \{ z \in \mathbb{R}^n \mid Ax = 0 \}$$

particular soln + $\mathcal{N}(A)$