# Computer Science & Engineering Department
# I. I. T. Kharagpur

## Principles of Programming Languages: CS40032

*Elective*

**Assignment – 1: $\lambda$-Calculus** *Marks: 25*

Assign Date: *$14^{th}$ January, 2021* Submit Date: *23:55, $21^{st}$ January, 2021*

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**Instructions**: Please solve the questions using pen and paper and scan the images. Every image should contain your roll number and name.

1. Fully parenthesize the following $\lambda$-expressions: **[1.5 * 3 = 4.5]**
   (a) $\lambda x.\ x\ z\ \lambda y.\ x\ y$
   (b) $(\lambda x.\ x\ z)\ \lambda y.\ w\ \lambda w.\ w\ y\ z\ x$
   (c) $\lambda x.\ x\ y\ \lambda x.\ y\ x$

2. Mark the free variables in the following $\lambda$-expressions: **[1.5 * 3 = 4.5]**
   (a) $\lambda x.\ x\ z\ \lambda y.\ x\ y$
   (b) $(\lambda x.\ x\ z)\ \lambda y.\ w\ \lambda w.\ w\ y\ z\ x$
   (c) $\lambda x.\ x\ y\ \lambda x.\ y\ x$

   **BEGIN SOLUTION**

| | |
|---|---|
| λx.x z λy.x y | → (λx.((x z) (λy.(x y)))) |
| (λx. x z) λy. w λw. w y z x | → ((λx.(x z)) (λy.(w (λw.((((w y) z) x)))))) |
| λx. x y λx. y x | → (λx.((x y) (λx.(y x)))) |

   **END SOLUTION**

3. Prove the following using encoding in $\lambda$-calculus: **[2 * 8 = 16]**
   (a) $NOT(NOT\ TRUE) = TRUE$
   Given:
   $$NOT = \lambda x.\ ((x\ FALSE)\ TRUE)$$
   $$TRUE = \lambda x.\ \lambda y.\ x$$
   $$FALSE = \lambda x.\ \lambda y.\ y$$

   **BEGIN SOLUTION**

| | |
|---|---|
| not (not true) | // replacing 1st not w/ encoding |
| = λx.((x false) true) (not true) | // β-reduction: x → not true |
| = ((not true) false) true | // replacing not w/ encoding |
| = ((λx.((x false) true) true) false) true | // β-reduction: x → true |
| = (((true false) true) false) true | // replace true w/ encoding |
| = ((((λx.λy.x) false) true) false) true | // β-reduction: 1st x → false |
| = (((λy.false) true) false) true | // β-reduction: y → true |
| = ((false) false) true | // replace false w/ encoding |
| = ((λx.λy.y) false) true | // β-reduction: x → false |
| = (λy.y) true | // β-reduction: y → true |
| = true | // not (not true) = true |

   **END SOLUTION**

   (b) $OR\ FALSE\ TRUE = TRUE$
   Given:
   $$OR = \lambda x.\ \lambda y.\ ((x\ TRUE)\ y)$$
   $$TRUE = \lambda x.\ \lambda y.\ x$$
   $$FALSE = \lambda x.\ \lambda y.\ y$$

   **BEGIN SOLUTION**

   **END SOLUTION**

```
or false true                              // replacing or w/ encoding
= λx. λy. ((x true) y) false true          // β-reduction: x → false
= λy. ((false true) y) true                // β-reduction: y → true
= (false true) true                        // replace 1st false w/ encoding
= ((λx.λy.y) true) true                    // β-reduction: x → false
= (λy.y) true                              // β-reduction: y → true
= true                                     // or false true = true
```

(c) $SUCC\ 2 = 3$

Given:

$$2 = \lambda f.\ \lambda y.\ f\ (f\ y)$$

$$3 = \lambda f.\ \lambda y.\ f\ (f\ (f\ y))$$

$$SUCC = \lambda z.\ \lambda f.\ \lambda y.\ f\ (z\ f\ y)$$

**BEGIN SOLUTION**

```
succ 2                                     // replacing succ w/ encoding
= (λz.λf.λy.f (z f y)) 2                   // β-reduction: z → 2
= λf.λy.f (2 f y)                          // expanding 2 w/ encoding
= λf.λy.f ((λf.λy.f (f y)) f y)            // β-reduction: 1st f → f
= λf.λy.f ((λy.f (f y)) y)                 // β-reduction: 1st y → y
= λf.λy.f  (f (f y))                       // apply encoding for 3
= 3                                        // succ 2 = 3
```

**END SOLUTION**

(d) $(Y\ FACT)\ 2 = 2$

Given:

$$Y = \lambda f.\ (\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x))$$

$$FACT = \lambda f.\ \lambda n.\ IF\ n = 0\ THEN\ 1\ ELSE\ n\ *\ (f\ (n\ -\ 1))$$

**BEGIN SOLUTION**

```
Given:
    Y = λf.(λx.f (x x)) (λx.f (x x))
    fact = λf. λn.if n = 0 then 1 else n * (f (n-1))
Proof:
    (Y fact) 2                                  // replacing Y w/ encoding
    = (λf.(λx.f (x x)) (λx.f (x x)) fact) 2     // β-reduction: 1st f → fact
    = (λx.fact (x x)) (λx.fact (x x)) 2      // β-reduction: 1st x → λx.fact (x x)
    = (fact ((λx.fact (x x)) (λx.fact (x x)))) 2
            // apply encoding for (Y fact)
            // ((λx.fact (x x)) (λx.fact (x x))) → (Y fact)
            // we know this is the encoding for (Y fact) from 3rd line of proof
    = (fact (Y fact)) 2                         // apply encoding for fact
    = (λf. λn.if n = 0 then 1 else n * (f (n-1)) (Y fact) 2
                                                // β-reduction: 1st f →  (Y fact)
    = (λn.if n = 0 then 1 else n * ((Y fact) (n-1))) 2// β-reduction: n → 2
    = if 2=0 then 1 else 2 * ((Y fact) (2-1))   // apply if
    = 2 * ((Y fact) 1)                          // showed in class (Y fact) 1 = 1
    = 2 * 1                                     // apply *
    = 2
```

**END SOLUTION**

(e) Given: $mul = \lambda n.\lambda m.\lambda x.\ (n\ (m\ x))$

Solve: $mul\ \overline{3}\ \overline{3}$

(f) Solve: $add\ \overline{8}\ \overline{1}$

Given: $add = \lambda n.\lambda m.\lambda f.\lambda x.\ n\ f\ (m\ f\ x)$

**BEGIN SOLUTION**

shown for $add\ \overline{7}\ \overline{1}$

**END SOLUTION**

(g) $IF\ FALSE\ THEN\ x\ ELSE\ y = y$

Given:

$$IF\ a\ THEN\ b\ ELSE\ c = a\ b\ c$$

$$TRUE = \lambda x.\ \lambda y.\ x$$

Solve : add $\bar{7}\ \bar{1}$

Given : add $= \lambda n.\lambda m.\lambda f.\lambda x.\ n f (m f x)$

add $\bar{7}\ \bar{1}$

$= (\lambda n.\lambda m.\lambda f.\lambda x\ n f (m f x))\ \bar{7}.\bar{1}$

$= \lambda f\ \lambda x.\bar{7}\ f((\lambda g.\lambda y.gy)\ f\ x)$

$= \lambda f\ \lambda x.\bar{7}\ f(f x)$

$= \lambda f\ \lambda x.((\lambda g.\lambda y.g^7(y))f)(f x)$

$= \lambda f \lambda x.(f^7(f x))$

$= \lambda f \lambda x\ (f^8(x))$

$= \bar{8}$

$$FALSE = \lambda x.\ \lambda y.\ y$$

**BEGIN SOLUTION**

IF FALSE THEN x ELSE y = y. (to show)

Given : IF a THEN b ELSE c = a b c

TRUE = $\lambda x.\lambda y\ x$

FALSE = $\lambda x.\ \lambda y\ y$

IF FALSE THEN x ELSE y

= FALSE x y

$= (\lambda f.\lambda g\ g)\ x\ y$

$= y$

= RHS

**END SOLUTION**

(h) Prove: *add* and *mul* are commutative

**BEGIN SOLUTION**

<u>add</u>

add $\bar{p}\ \bar{q}$

$= (\lambda n. \lambda m\ \lambda f\ \lambda x.\ nf\ (m f\ x))\ \bar{p}\ \bar{q}$

$= \lambda f\ \lambda x.\ \bar{p}\ f\ (\bar{q}\ f\ x)$

$= \lambda f\ \lambda x.\ (\lambda g. \lambda h\ g^{p}(h)\ f)\ (\bar{q}\ f\ x)$

$= \lambda f\ \lambda x\ .\quad f^{p}\ (\bar{q}\ f\ x)$

$= \lambda f\ \lambda x.\quad f^{p}\ (\lambda g. \lambda h\ .\ g^{q}(h).\ f\ x))$

$= \lambda f. \lambda x.\quad f^{p}\ (f^{q}(x))$

$= f^{p+q}(x)$

Similiarly, we could arrive at add $\bar{q}\ \bar{p} = f^{q+p}(x)$

However, $f^{p+q}(x) = f^{q+p}(x)$

$\therefore$ add $\bar{p}\ \bar{q}$ = add $\bar{q}\ \bar{p}$

Thus, add is commutative.

<u>mul</u>

mul $\bar{p}\ \bar{q}$

$= \lambda n. \lambda m. \lambda x\ (n(mx))\ \bar{p}\ \bar{q}$

$= \lambda x.\ \bar{p}\ (\bar{q}\ x)$

$= \lambda x.\ \bar{p}\ ((\lambda g \lambda h.\ g^{q}(h)).\ x)$

$= \lambda x.\ \bar{p}\ (\lambda h.\ x^{q}(h))$

$= \lambda x.\ (\lambda f\ \lambda g.\ f^{p}g)\ (\lambda h.\ x^{q}(h))$

$= \lambda x.\ (\lambda g.\ (\lambda h.\ x^{q}(h))^{p}\ g)$

$= \lambda x.\ (\lambda g.\ (\lambda h.\ x^{q}h)^{p-1}\ ((\lambda h. x^{q}h)\ g))$

$= \lambda x.\ (\lambda g.\ (\lambda h.\ x^{q}h)^{p-1}\ (x^{q}g))$

$= \lambda x.\ (\lambda g.\ (\lambda h.\ x^{q}h)^{p-2}\ ((\lambda h. x^{q}g)\ (x^{q}g)))$

$= \lambda x.\ (\lambda g.\ (\lambda h.\ x^{q}h)^{p-2}\ (x^{2q}g))$

$\vdots$

$= \lambda x.\ (\lambda g.\ x^{pq}\ g)$

Similiarly, we can show mul $\bar{q}\ \bar{p} = \lambda x. (\lambda h.\ x^{qp}h)$
$= \lambda x.\ (\lambda g.\ x^{qp}g)$

Since $x^{pq} = x^{qp}$

mul $\bar{p}\ \bar{q}$ = mul $\bar{q}\ \bar{p}$

Thus, mul is commutative.

**END SOLUTION**

Given:

$mul = \lambda n. \lambda m. \lambda x.\ (n\ (m\ x))$

$add = \lambda n. \lambda m. \lambda f. \lambda x.\ n\ f\ (m\ f\ x)$