CS60064 Spring 2022

Computational Geometry

Instructors

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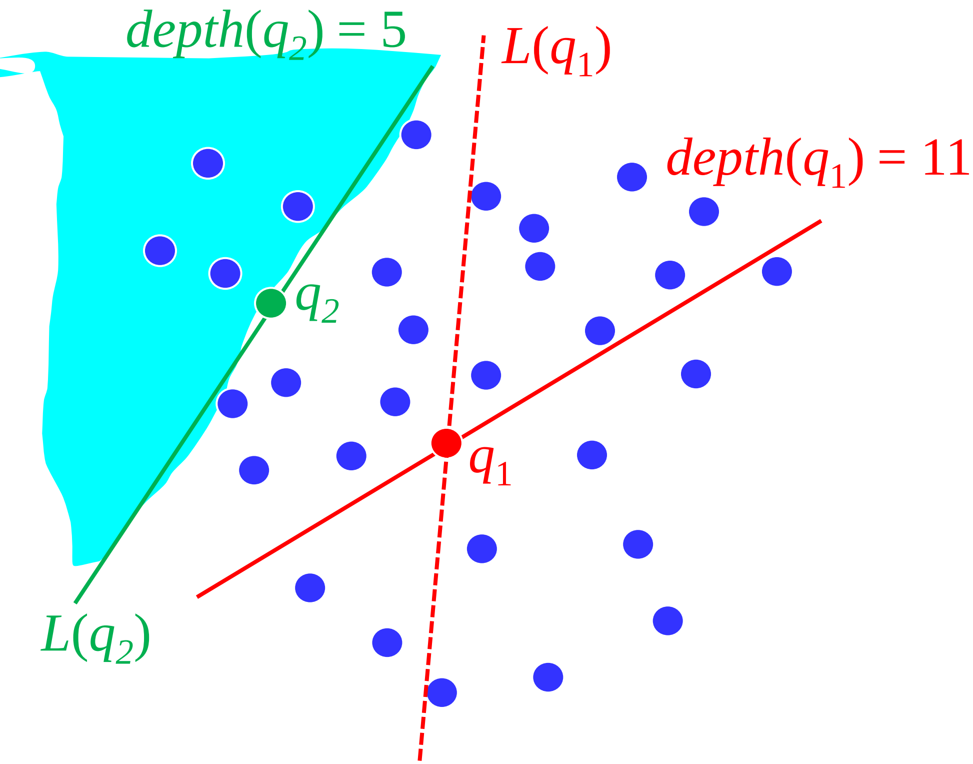
Lecture 04

12 January 2022

Indian Institute of Technology Kharagpur
Computer Science and Engineering

Problem of the Day



depth of query point (q) in 2D:
- imagine a line L passing thru q;
- rotate L around q such that # points appearing on one side of L is *minimized* over all angles;
- Output the number including q;

i.e., the smallest number of points in any closed *half-plane* that contains q (Tukey depth)

Data depth – used in analytics, M/L

Given a cluster of data comprising n points, what is the relative location of a new query point?

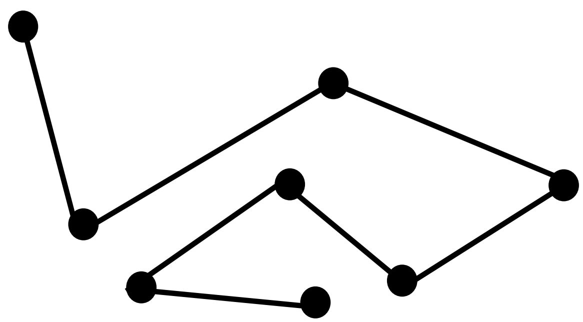
Problem: Design an efficient algorithm for computing the depth of a query point in a 2D cluster of n points

Other measures: use of convex hull; distance of q from the centroid (arithmetic mean) or from the Fermat point

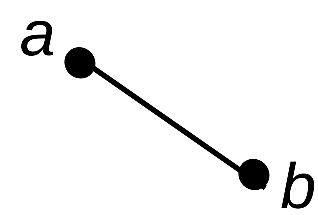
Introducing Polygons

Polygonal Curves

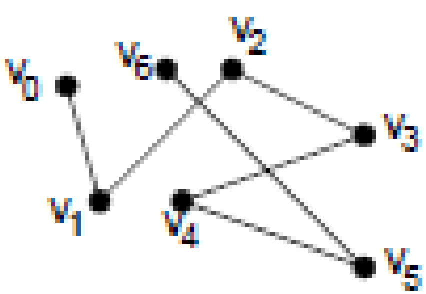
Ref: David Mount, Lecture Notes



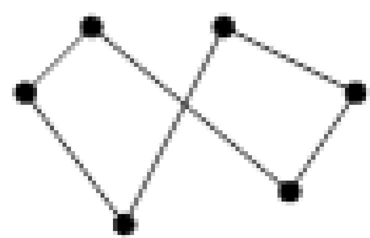
simple polygonal curve (open)



line segment: a subset of a straight-line contained between two end-points *a*, *b* (inclusive), denoted as $\overline{ab}$



open polygonal curve but not simple

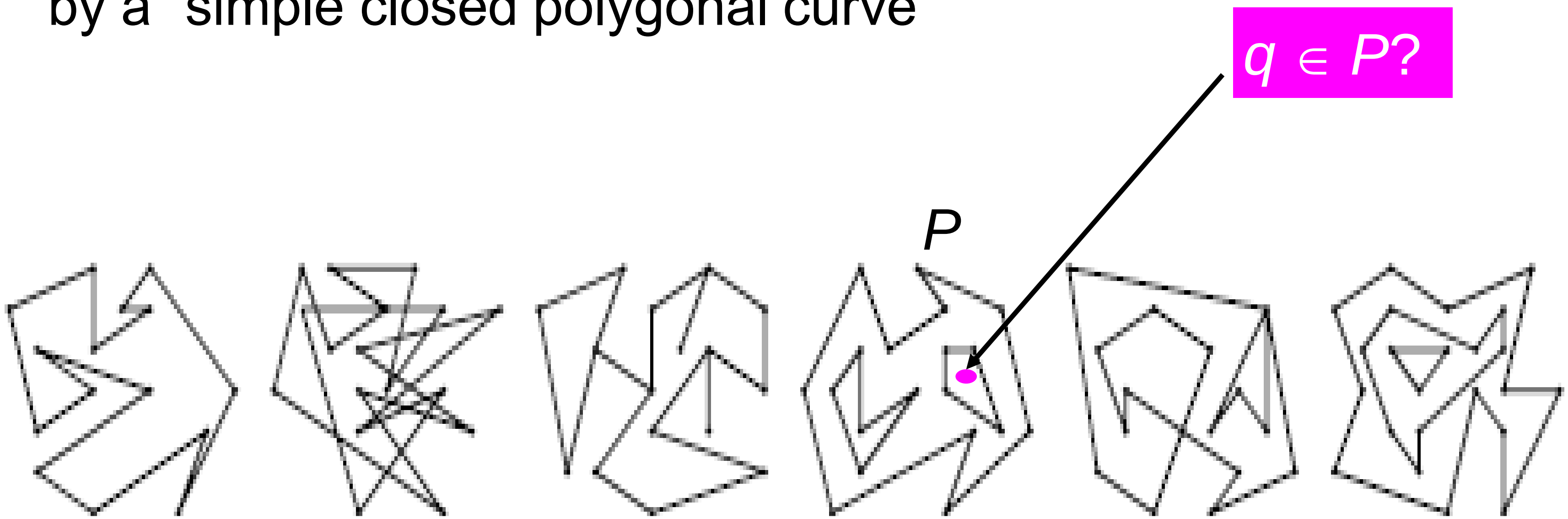


closed polygonal curve but not simple

polygon: closed and simple polygonal curve

Simple Polygons

- *Definition:* A simple polygon *P* is the (closed) region bounded by a "simple closed polygonal curve"

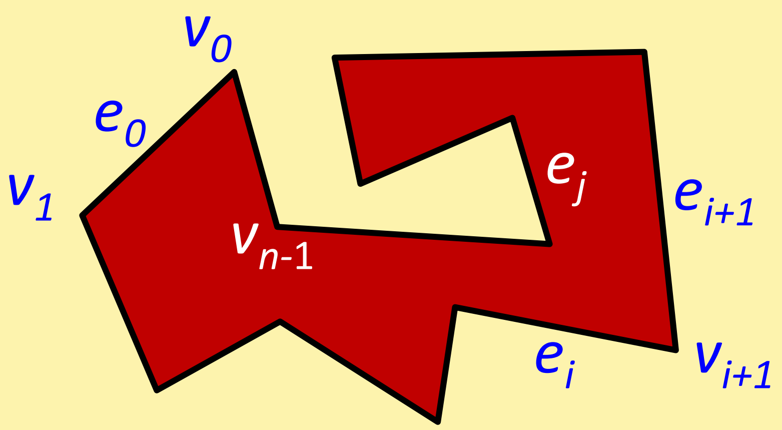


simple polygon | no | PSLG | polygon with holes | no | nested polygons

Simple Polygon

Two non-consecutive edges are disjoint

Two consecutive edges have a single common end-point



Simple Polygons

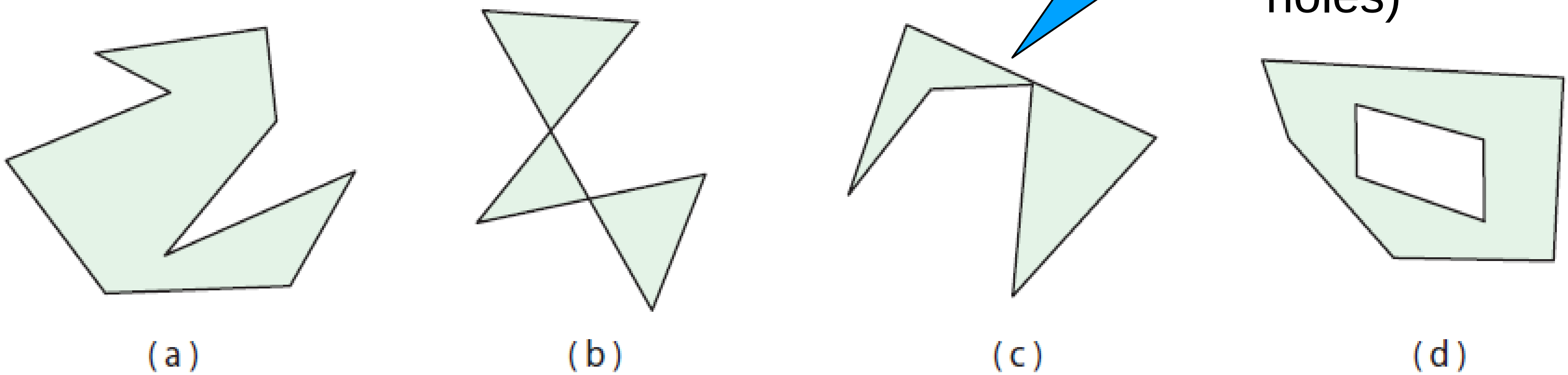
Some definitions would allow this as a "degenerate" simple polygon

yes

no

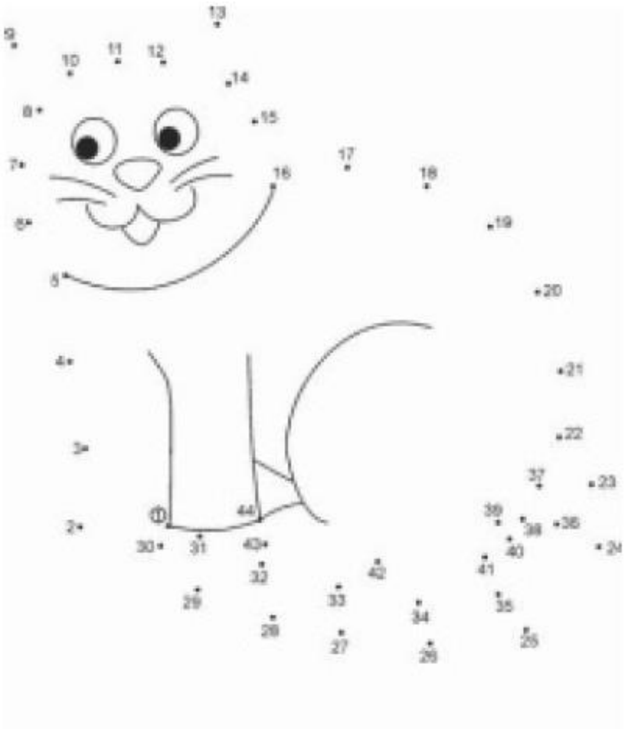
no

no (polygon with holes)

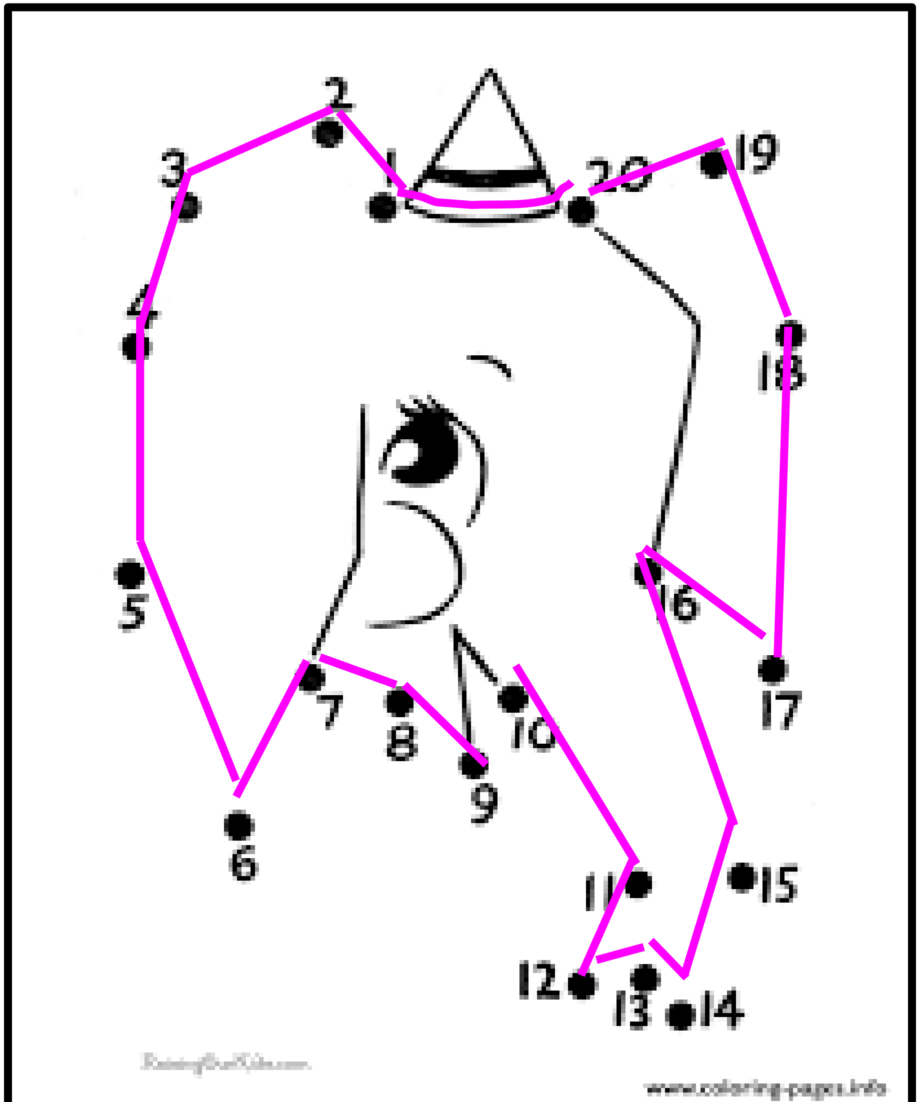


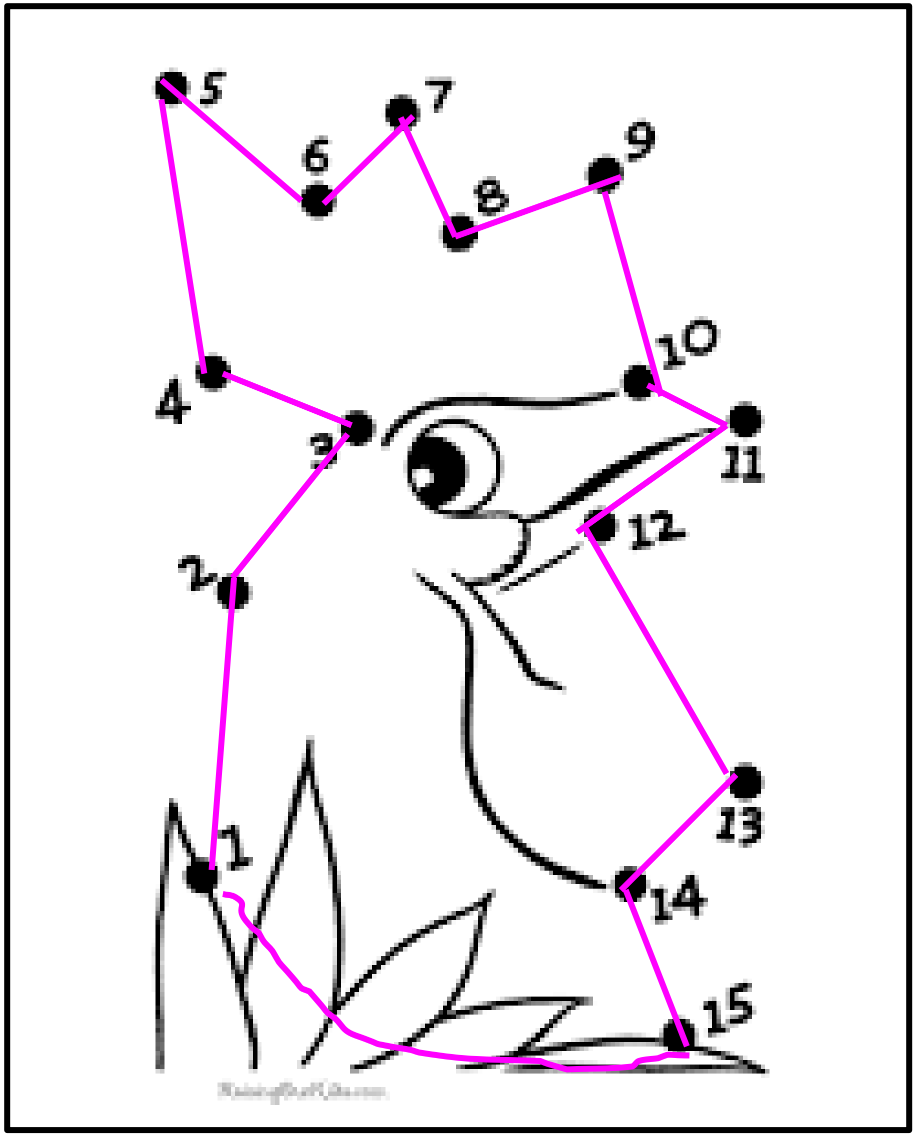
(Polygonal Jordan Curve). *The boundary ∂P of a polygon P partitions the plane into two parts. In particular, the two components of $\mathbb{R}^2 \setminus \partial P$ are the bounded interior and the unbounded exterior.*[2]

Connect-the-dots

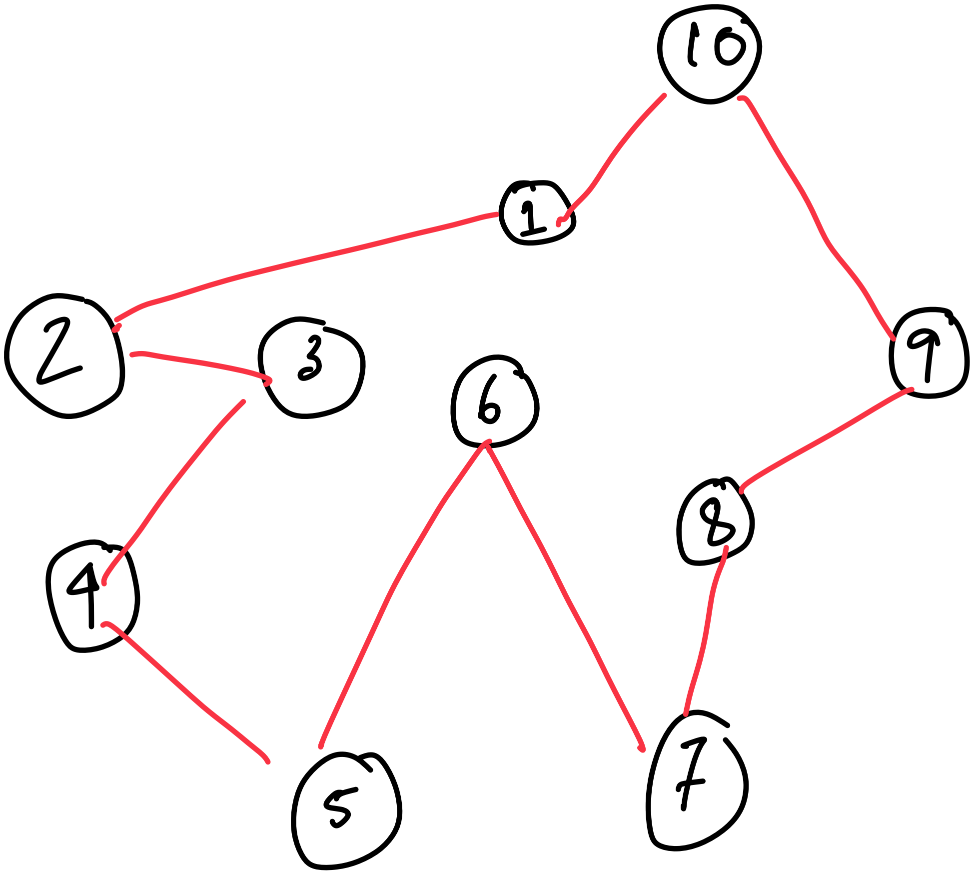


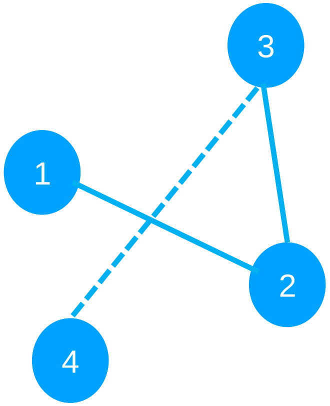
Labels of the vertices are given; draw the polygonal edges in sequence, to reveal …



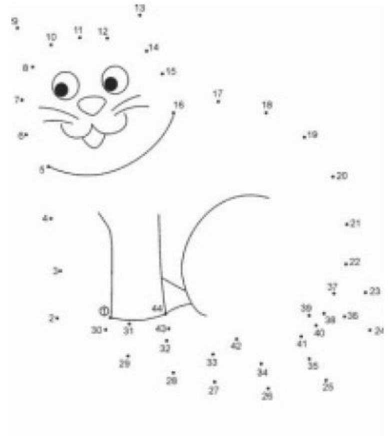


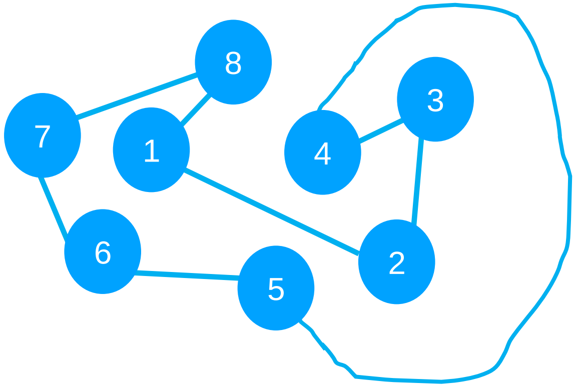
Polygon described as an ordered sequence of vertices



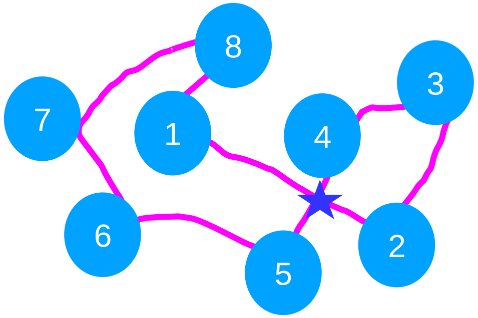


Given labelled points, is it always possible to construct the polygon?





normal



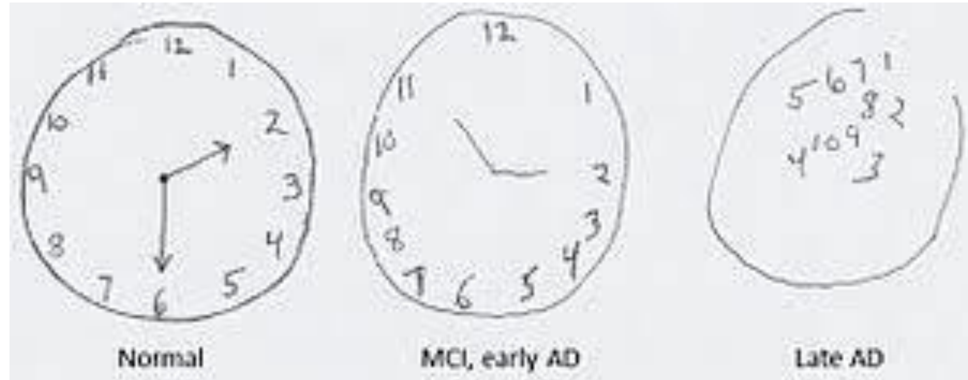
D, P, AD

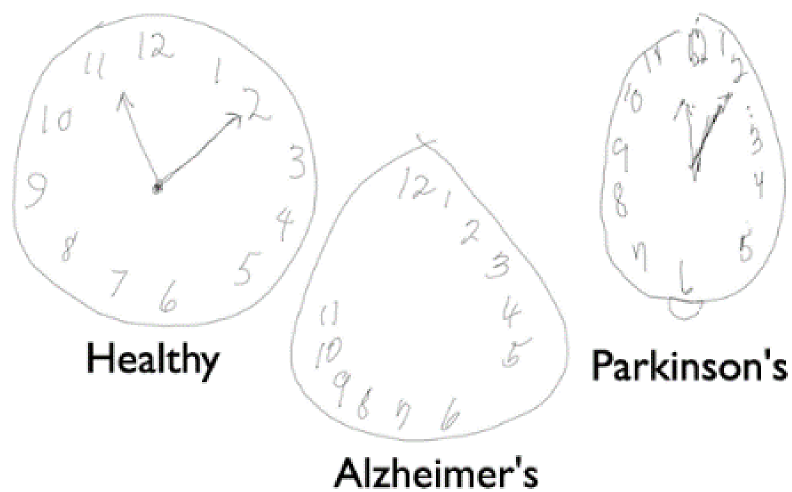
Trail Marking Test:

Often used in testing cognitive decline in dementia, Parkinson’s, Alzheimer's disease

Q. Given points 1, 2, …, *n* on 2D, can you sequentially connect them with a closed curve without crossing?

Geometric Cognition Test

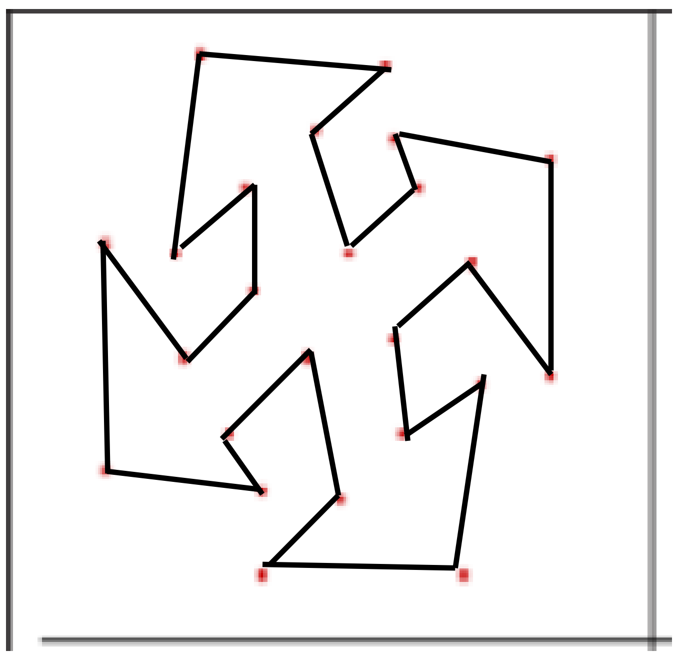




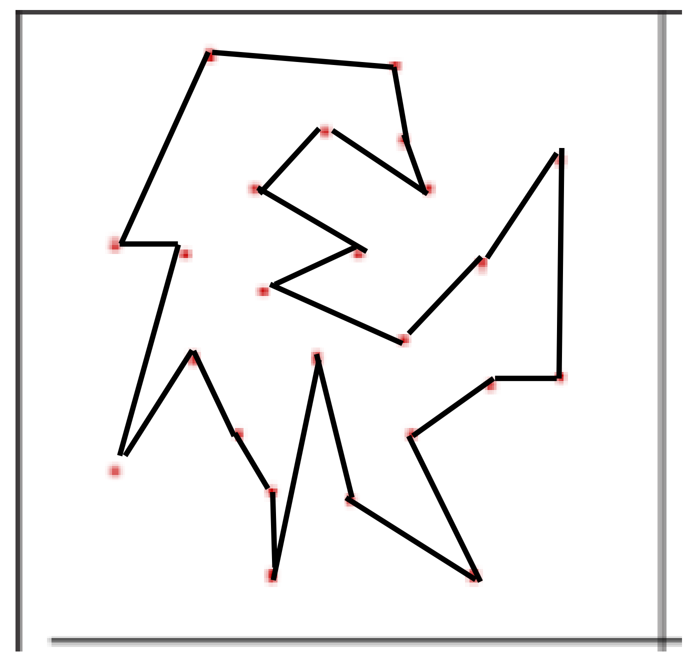
Clock Drawing Test: often used in testing cognitive decline in dementia, Parkinson’s, and Alzheimer's disease
Q. Can you draw the clock with the current time?

Polygonization

Vertices are given but their ordering, i.e., labels are not given; the goal is to construct a simple polygon spanning all vertices



one way of polygonization

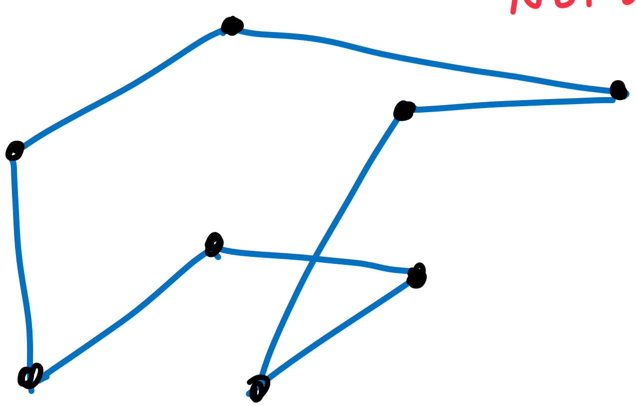


another way of polygonization

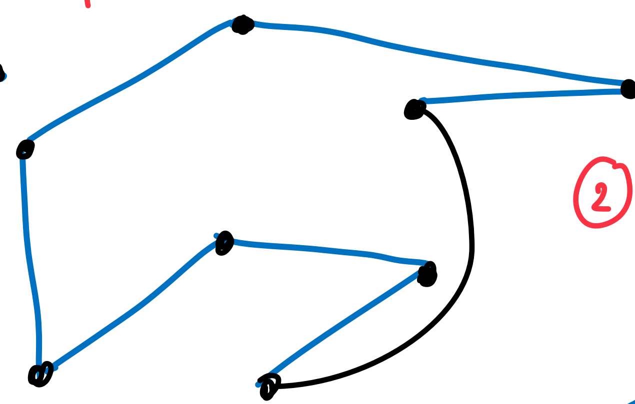
Polygonization

Given a set of points construct a simple polygon that spans all points

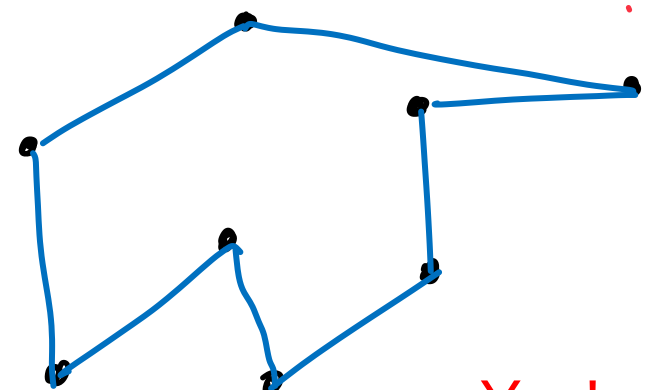
unordered



Not simple



(2) Does it always exist?

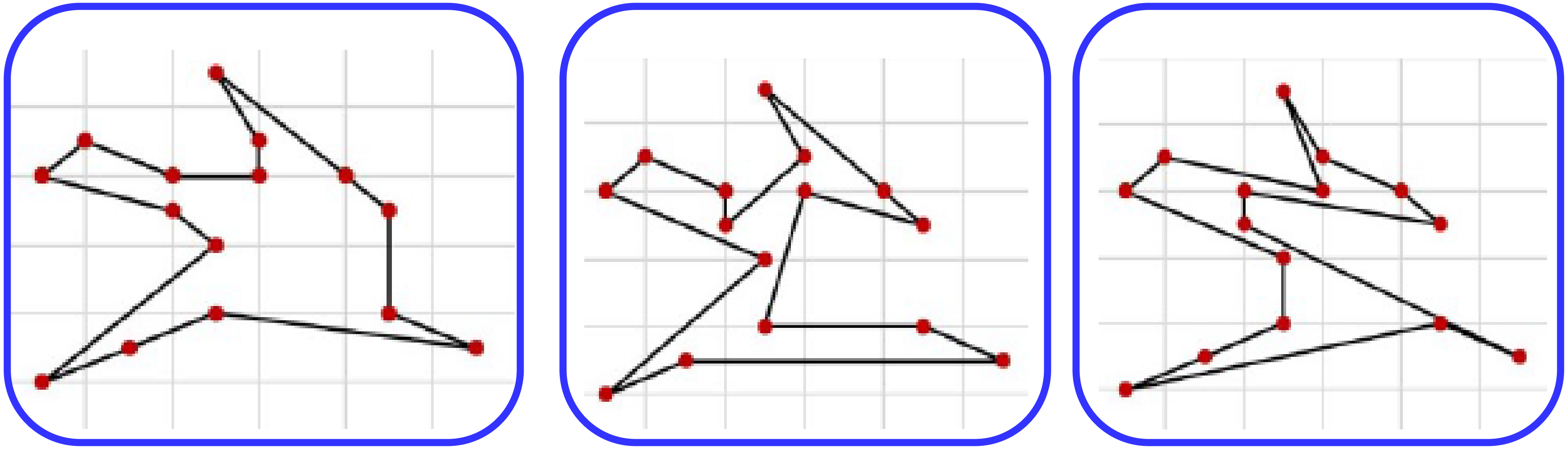


Yes!

(1) How to sequence points so as to guarantee polygonization

How?
HMK - 01

Polygonization



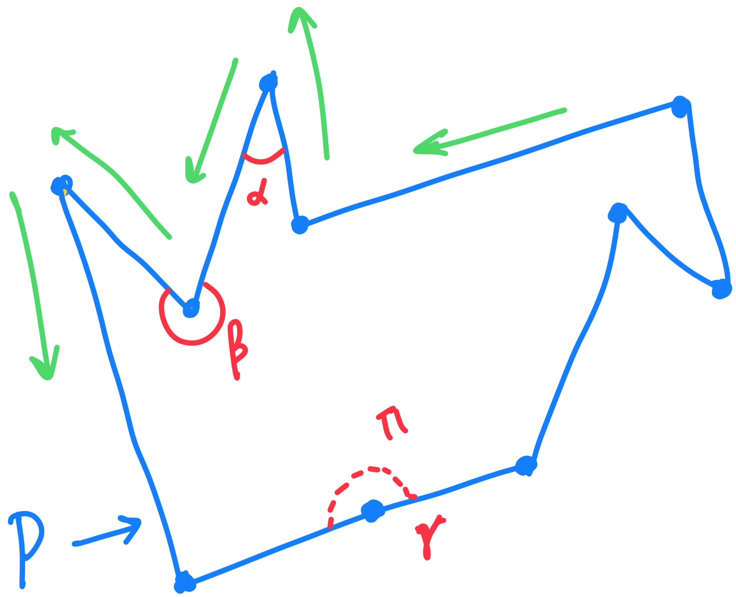
Different polygonizations of the *same set* of points

Q1. Can you find the one with minimum perimeter, area?

Q2. An unordered point set *P* and some edges *E* defined on a subset of *P*, are given. Can you always polygonise such that it includes all edges in *P*?

Q3. An unordered point set *P* and a hole *H* are given. Can you polygonise *P* such that *H* appears as a hole in *P*?

Simple polygons: Convex and reflex angles



internal angles of P

α: strictly convex $< \pi$

β: strictly concave $> \pi$
(also called reflex)

γ: assume convex $= \pi$

(We often consider collinearity as degenerate cases)

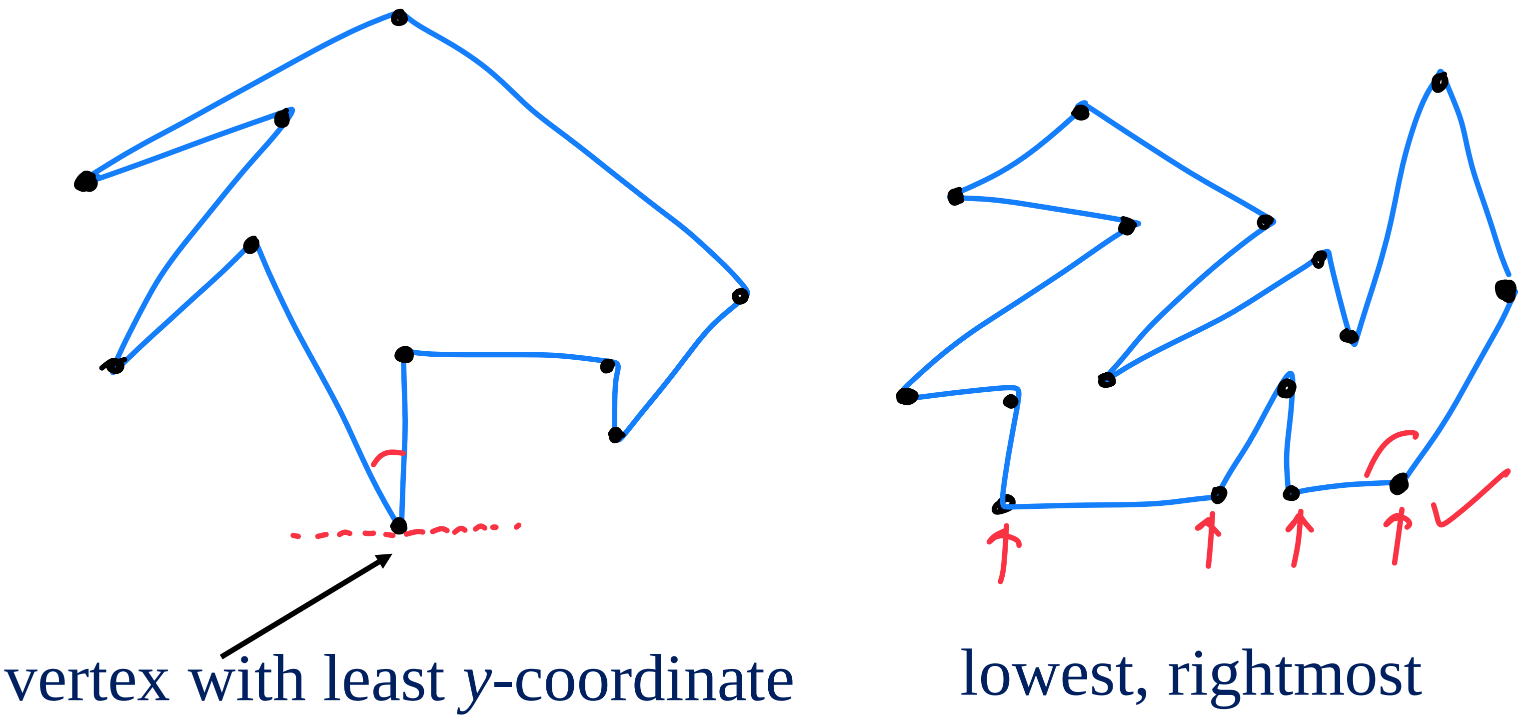
Walk along δP CCW

Left turn $\Rightarrow$ strictly convex

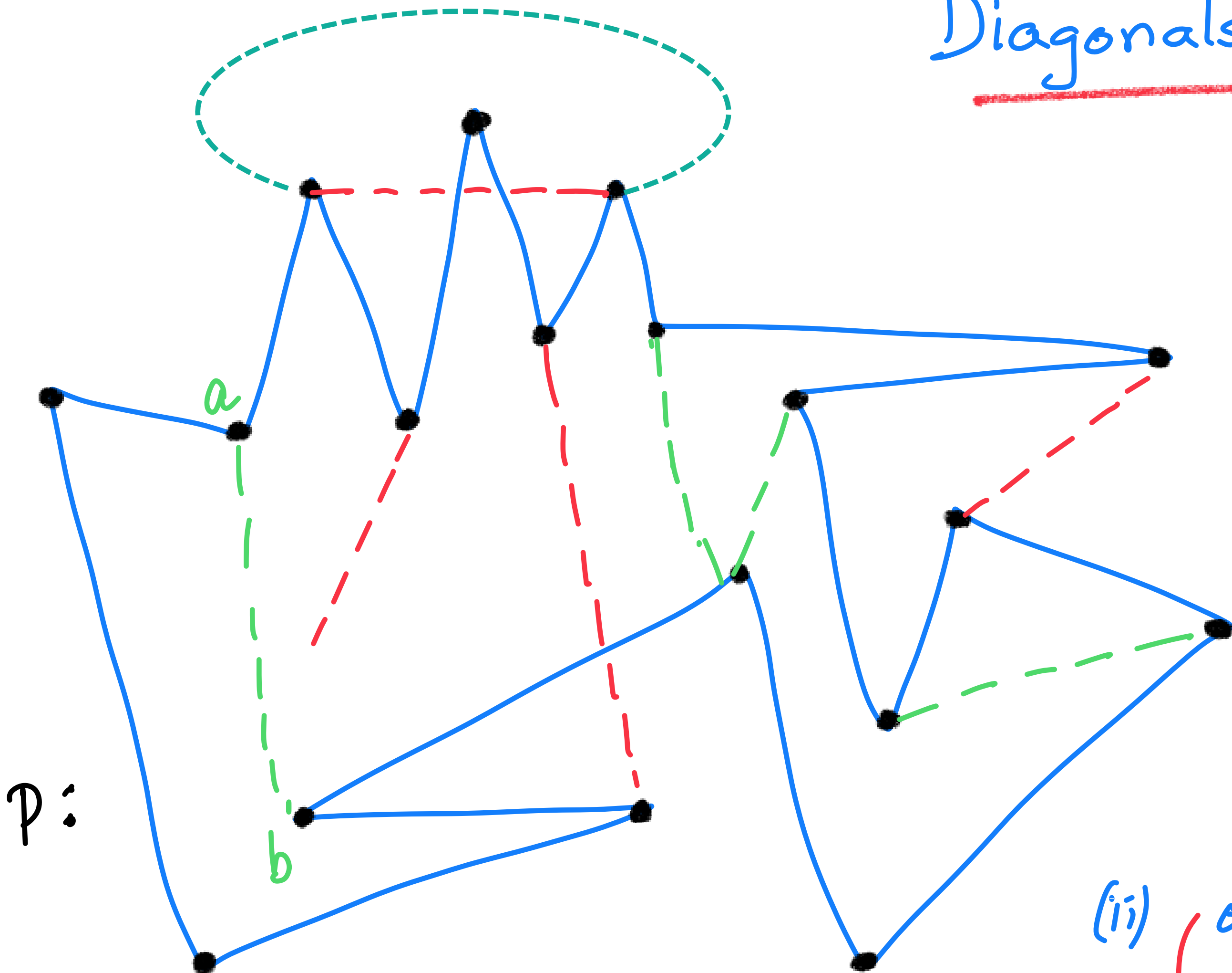
Right turn $\Rightarrow$ strictly reflex

No turn $\Rightarrow$ π

Lemma: Every polygon has at least one strictly convex vertex.



Diagonals in a Polygon



- - - → diagonal

- - - → Not diagonal

straight-line

A line segment $\overline{ab}$ is a diagonal of P if

(i) a, b are two non-adjacent vertices of P

&

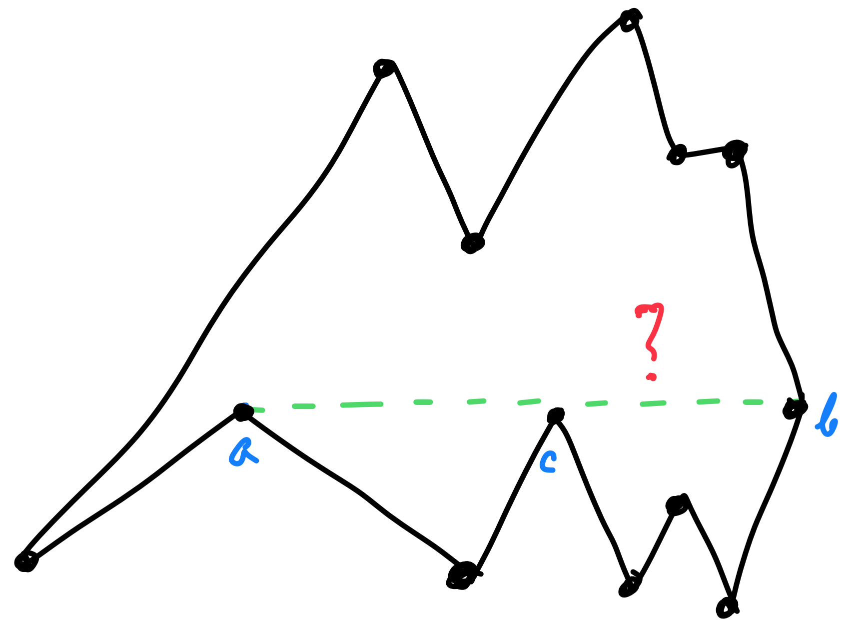
(ii) open $\overline{ab}$ lies completely within P, i.e.,

$\overline{ab} \in P$ &

closed $\overline{ab} \cap \delta P = \{a, b\}$

⇐ (a, b) visible pair

Is $\overline{ab}$ a diagonal?



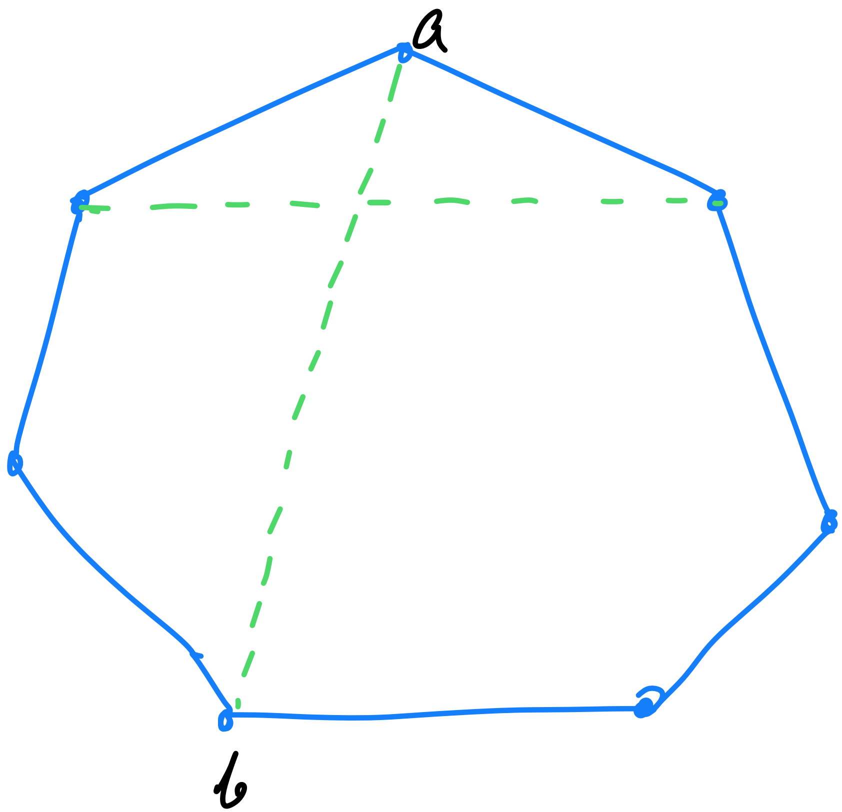
grazing
(degenerate case)

(ii) open $\overline{ab}$ lies completely within P, i.e,

$\overline{ab} \in P$ &

closed $\overline{ab} \cap \delta P = \{a, b\}$

Convex polygons

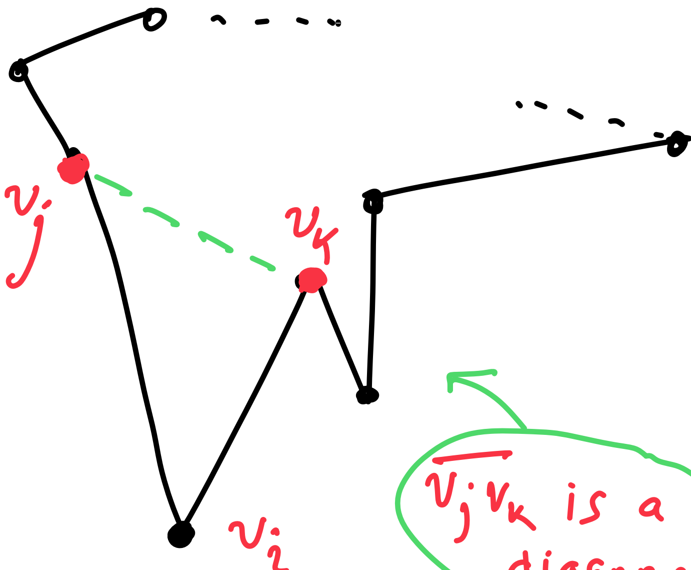


Any line segment $\overline{ab}$ joining two non-adjacent vertices of a convex polygon P is a valid diagonal.

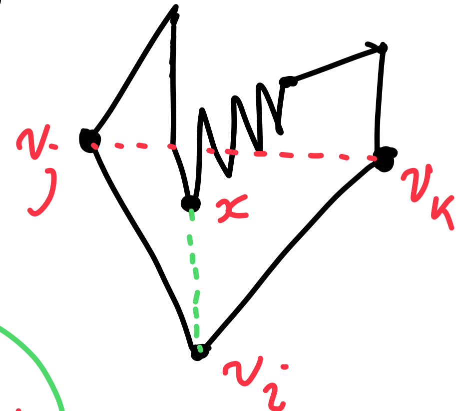
diagonals in a convex P with n vertices

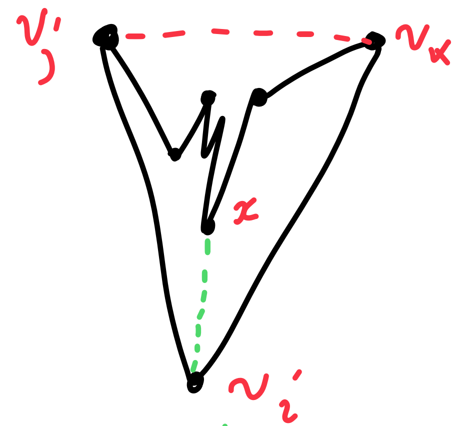
$$\boxed{= \binom{n}{2} - n}$$

Lemma: Every polygon $P(n)$, $n \geq 4$ must have a diagonal



$\overline{v_j v_k}$ is a diagonal





Pick a strictly convex vertex v_i

$v_j, v_k \rightarrow$ immediate neighbor of v_i

$\triangle v_i v_j v_k$ must contain at least one vertex x that is closest to v_i

$\Rightarrow \overline{v_i x}$ is a diagonal

J. O'Rourke: *Computational Geometry in C*, Cambridge Univ. Press, 1998

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Lecture 05 & Lecture 06

14 January 2022

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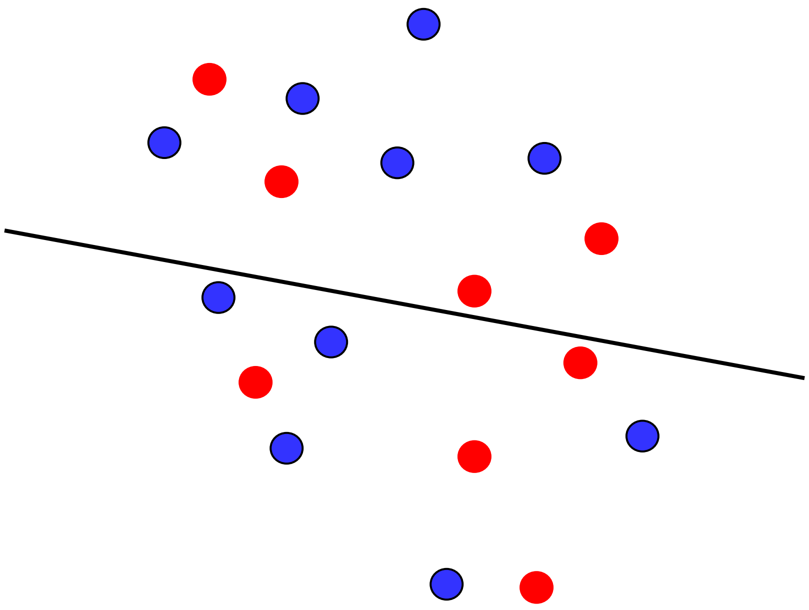
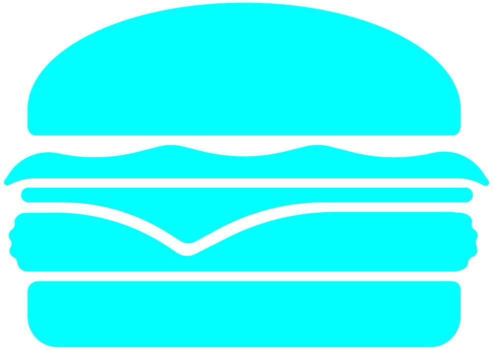
Computer Science and Engineering

Problem of the Day: Magical Cut

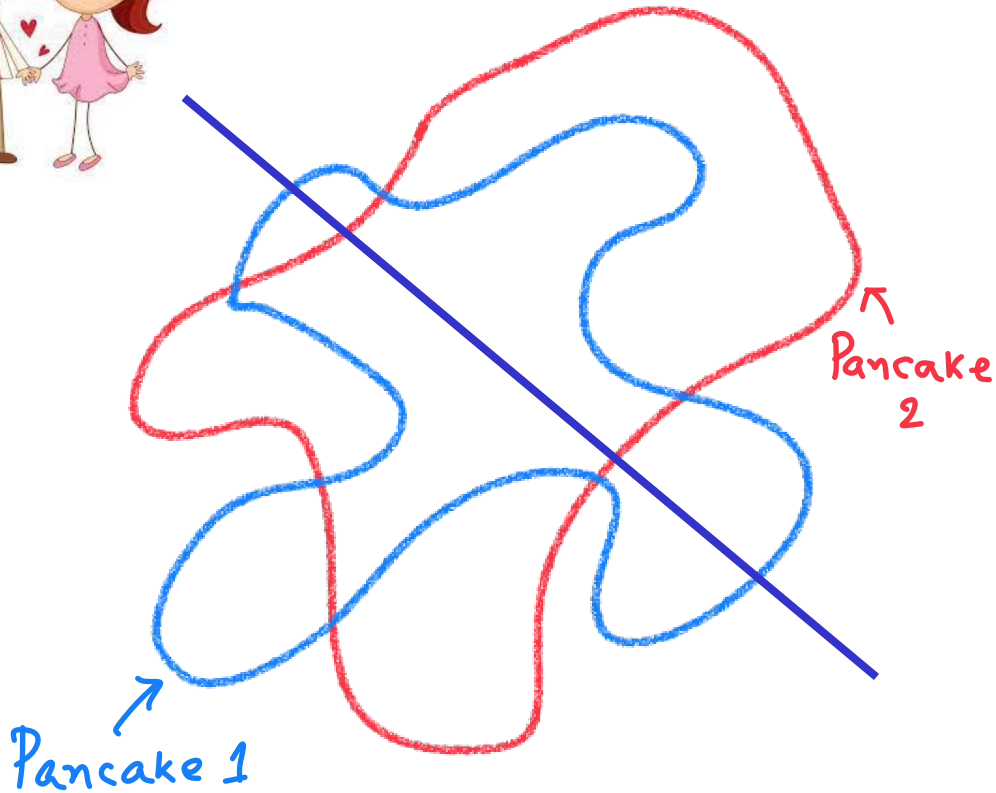


Question: Given $2m$ red points and $2n$ blue points in the plane in general positions, is it possible to divide them in half each, by a *single* straight-line cut?

Story of Pancake and Sandwich



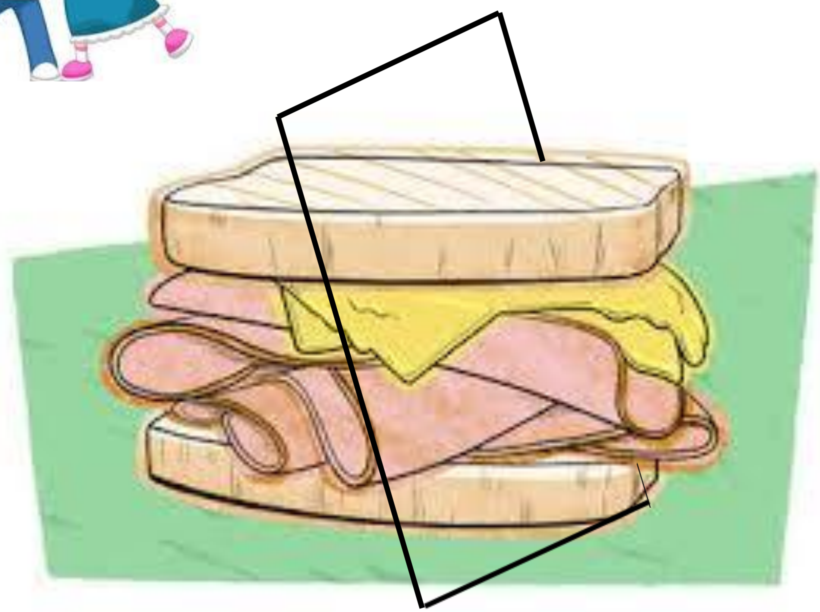
Story of Pancake

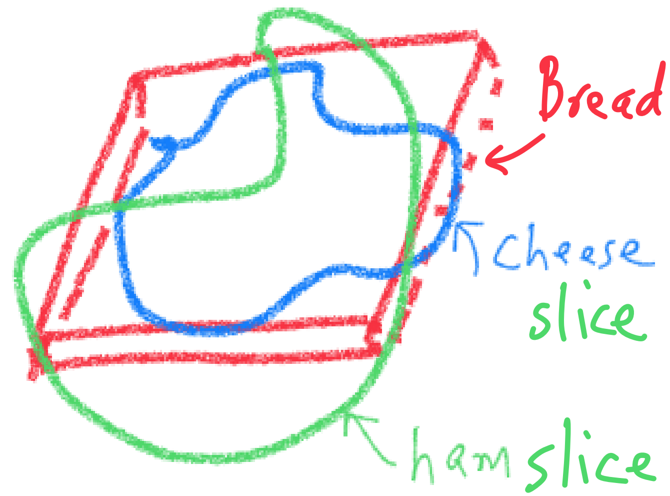


Cut by a straight knife (straight line cut) 2D

Pancake Theorem: It is *always possible* to cut the stack of two arbitrarily-shaped pancakes into two equal-size (area) portions each, by a single straight knife-cut, without moving them relative to each other

Story of Sandwich





cut by a 2D hyperplane (3D)

Ham-Sandwich Theorem: Given *n* measurable objects in *n*-dimensional Euclidean space, it is possible to divide them in half each, by a single (*n* -1)-dimensional hyperplane

Today’s Agenda

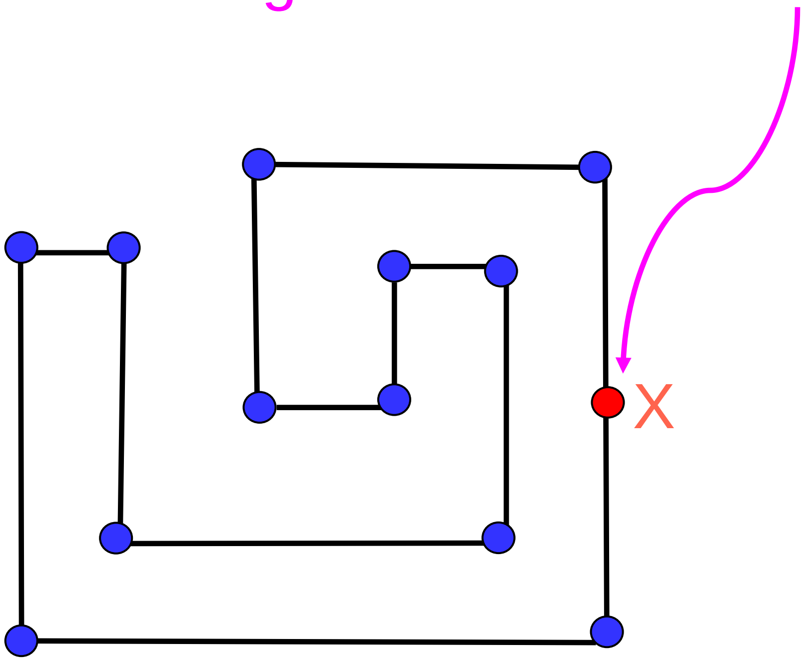
1. Orthogonal polygonization
2. Triangulation of simple polygons

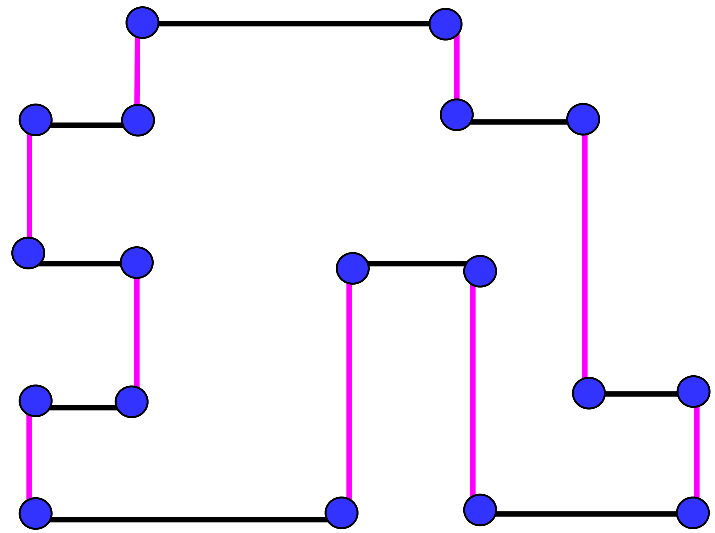
Orthogonal Polygons

All edges are axis-parallel

In other words, internal turn angles are either $\pi/2$ or $3\pi/2$

Avoid degenerate cases where internal angle is π





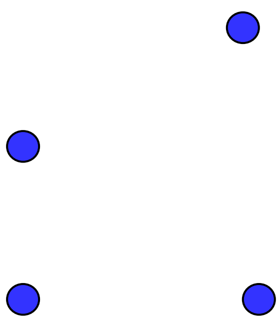
orthogonal polygonization
(from unlabelled vertices to polygon)

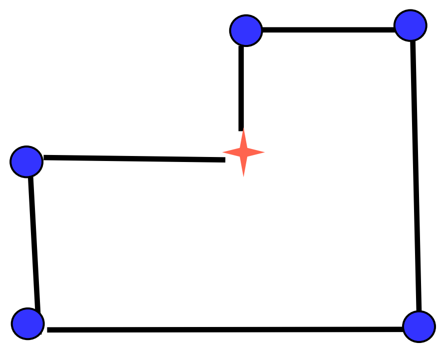
Polygonization: Vertices are given but their ordering, i.e., labels are not given; the goal is to construct an orthogonal polygon spanning all vertices

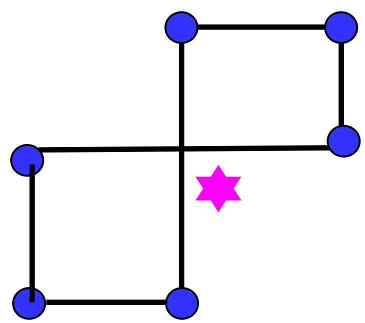
Orthogonal Polygonization

Every horizontal row or vertical column must have even number of vertices (assuming no degeneracy with internal angle π)

not sufficient

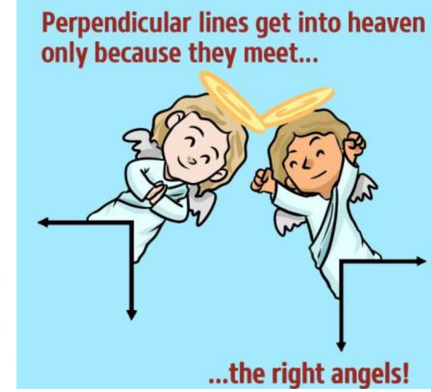


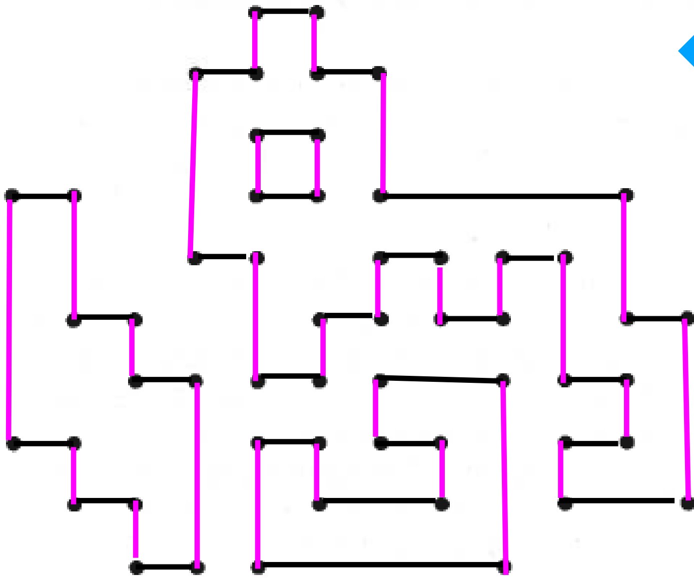


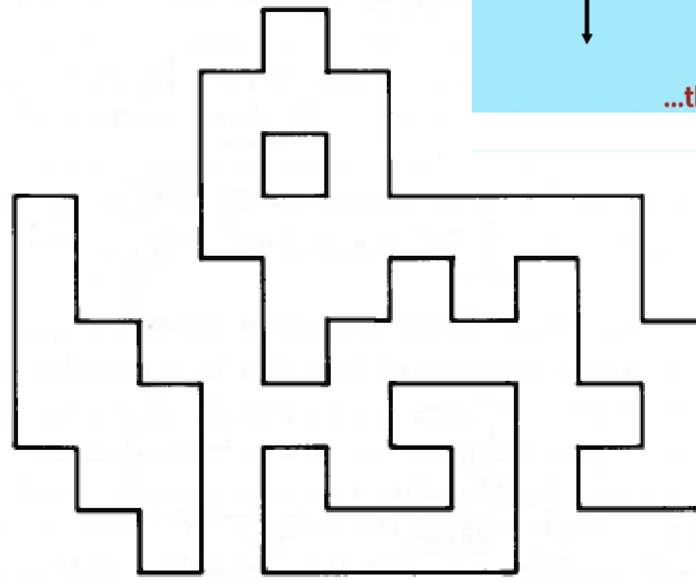


Instances which are not ortho-polygonizable

Orthogonal Polygonization







From point set to ortho-polygonization

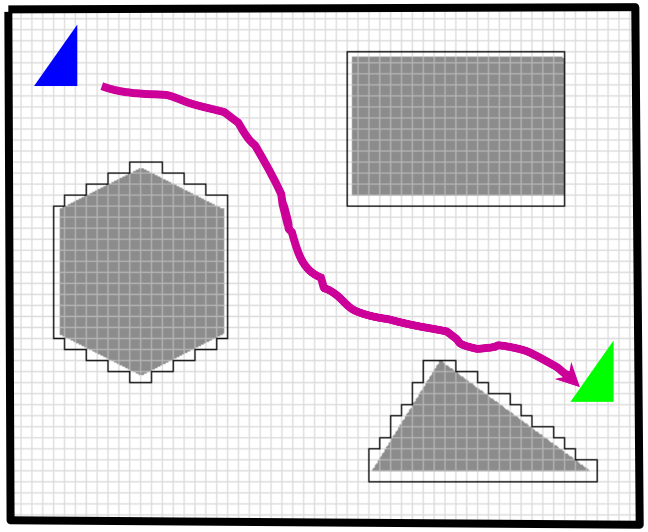
orthogonal polygons with holes

If polygonization is possible, then the solution is unique; may generate multiple polygons and with holes;

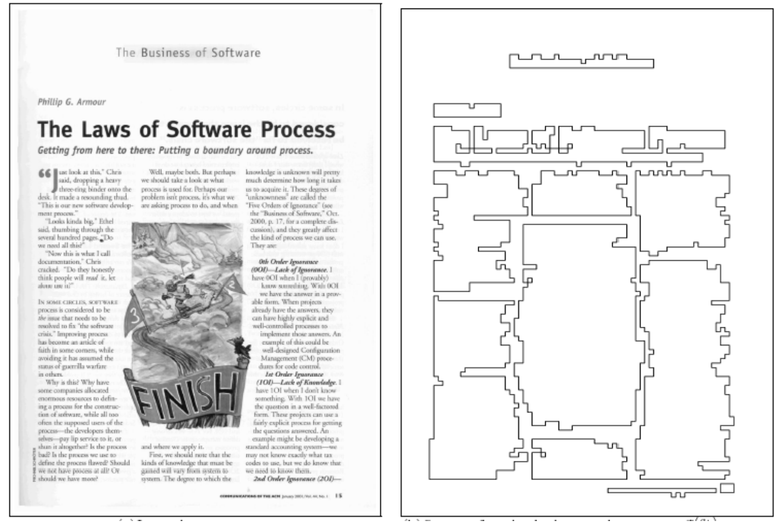
Can be accomplished in

$O(n \log n)$ time, where n is the number of points

Why Orthogonal?

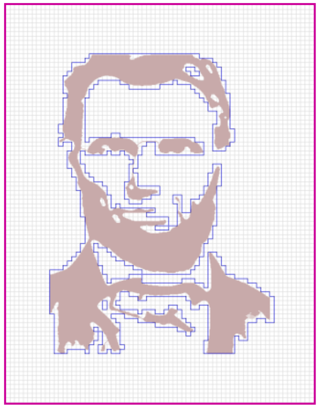


Robot-path configuration

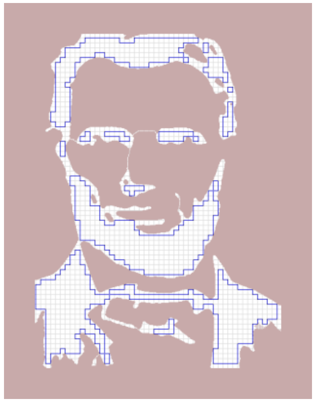


Document image segmentation

Original



Outer approximation



Inner approximation

huge space savings

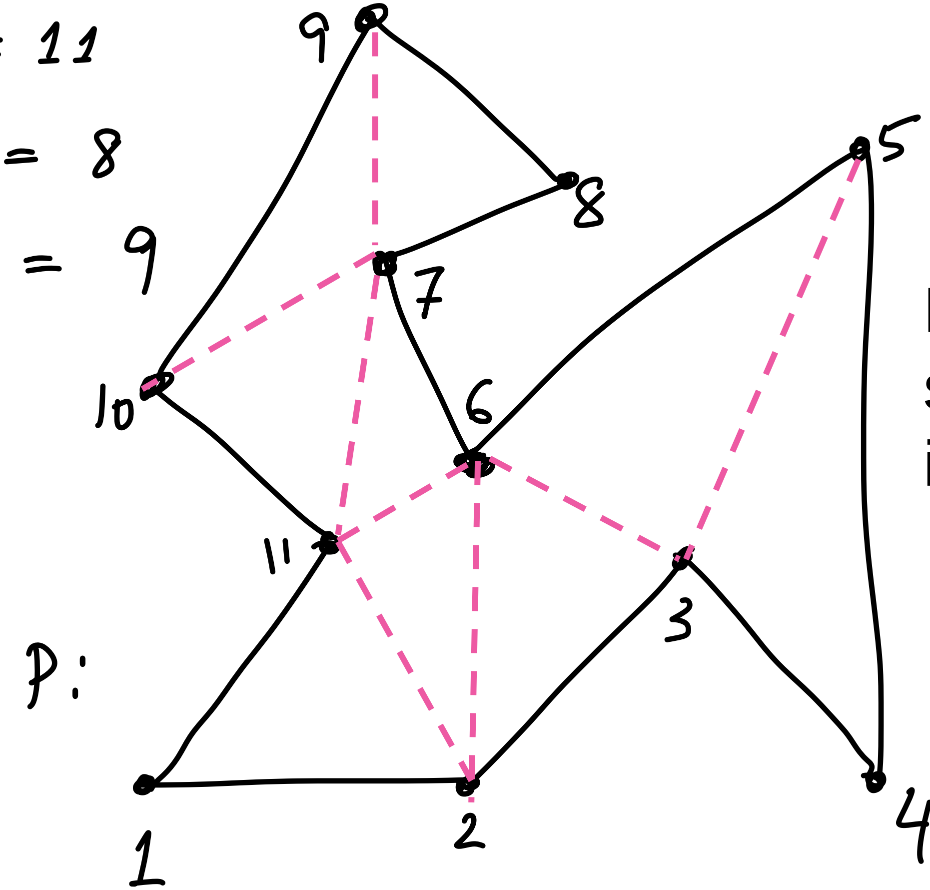
Triangulation of Simple Polygons

Triangulation of a simple polygon

$n = 11$

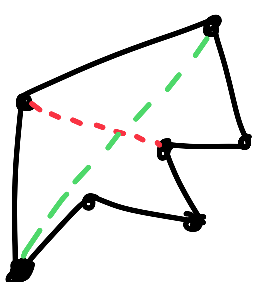
$\# D = 8$

$\# T = 9$



Keep on adding diagonals so that the P is partitioned into triangles

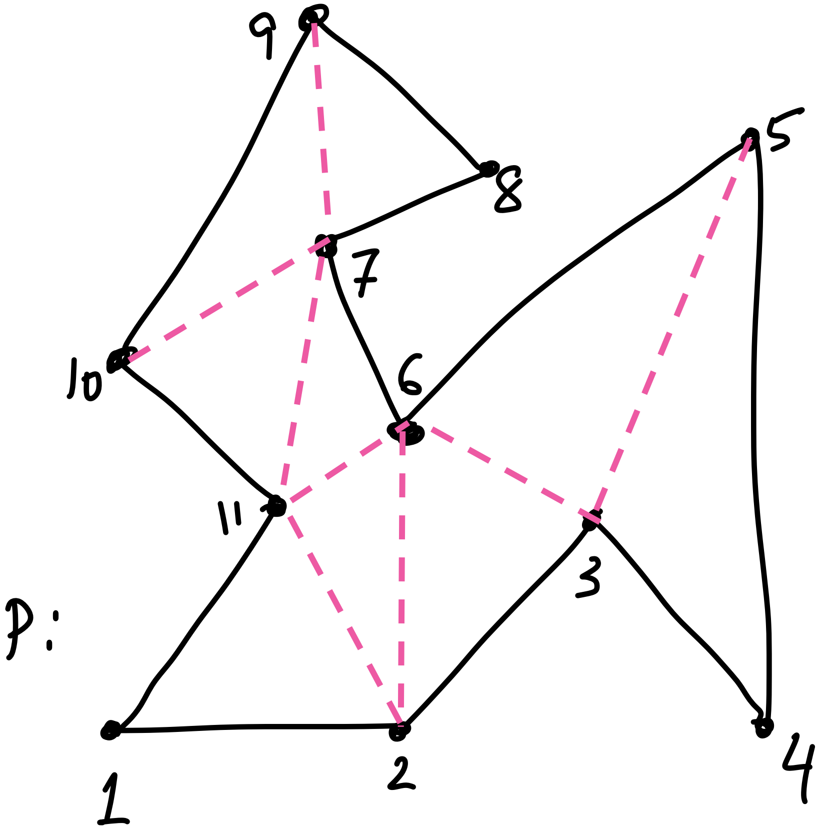
Claim: Triangulation is always possible

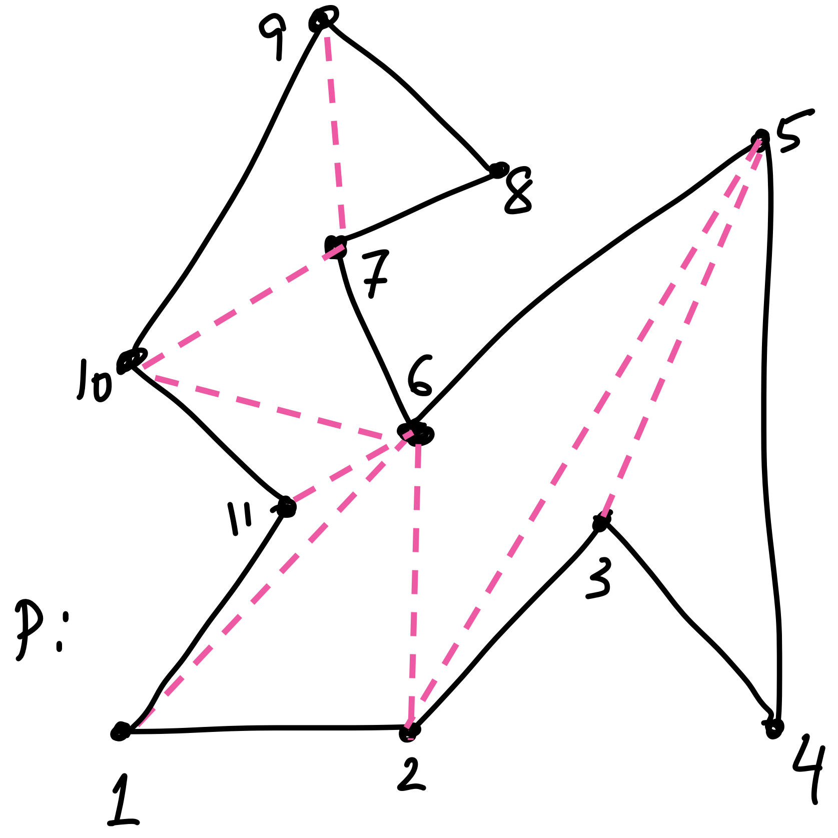


No crossing diagonals allowed

vertices of all Δs must be the vertices of polygon

Triangulation: A partition of *P* into triangles by a *maximal* set of non-crossing diagonals.

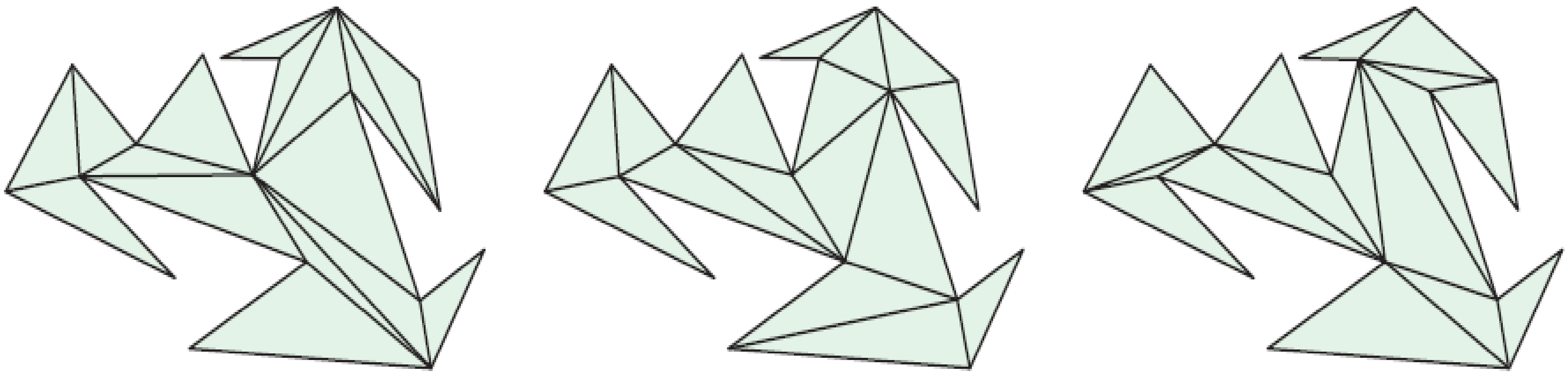




Triangulation is not unique

Triangulation of a simple polygon

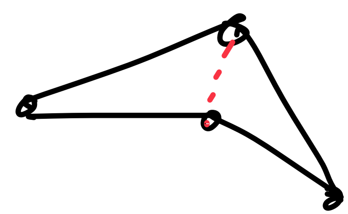
Triangulation is not unique



However, #D = n-3; #T = n-2, for all cases

Theorem: A simple polygon $P(n)$ $n \geq 4$ can always be triangulated using exactly $(n-3)$ diagonals that partition $P(n)$ into $(n-2)$ triangles.

Proof: Basis



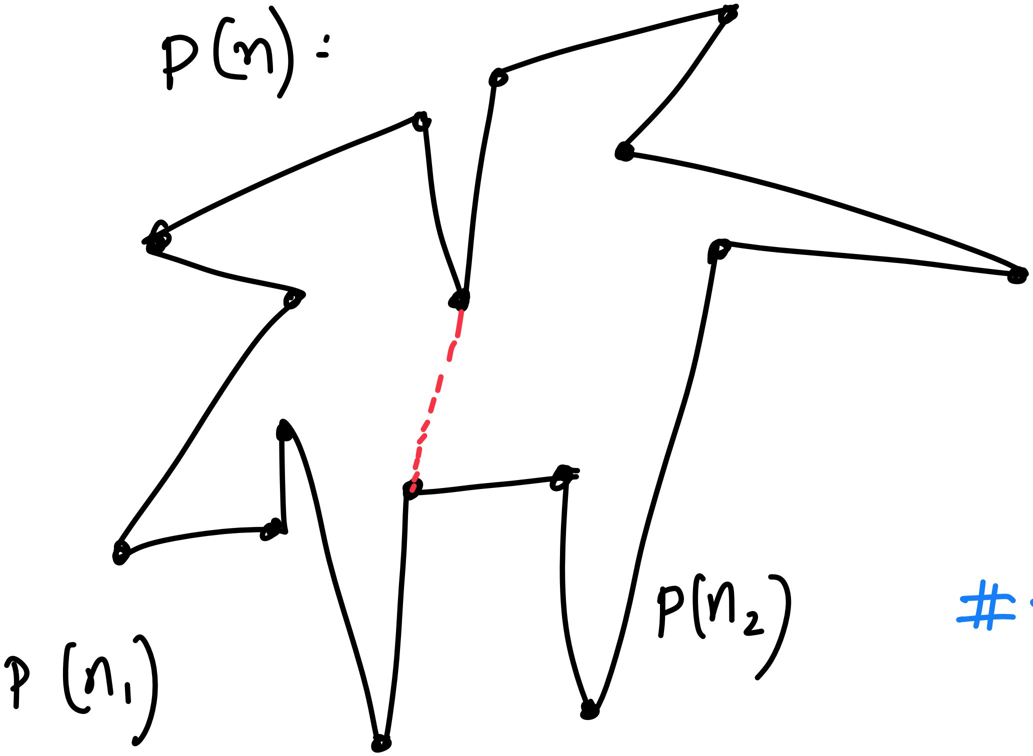
$n = 4$

$\# D = 1$

$\# T = 2$

Theorem holds for base case

Assume it holds for $k < n$



By induction hypothesis,

$\#$ diagonals

$= (n_1 - 3) + (n_2 - 3) + 1$

$= n_1 + n_2 - 5 = \boxed{n - 3}$

$\#$ triangles

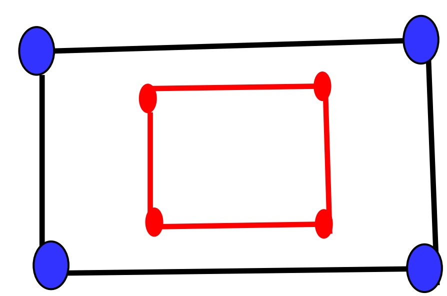
$= (n_1 - 2) + (n_2 - 2)$

$= n_1 + n_2 - 4 = \boxed{n - 2}$

$n_1 + n_2 = n + 2$

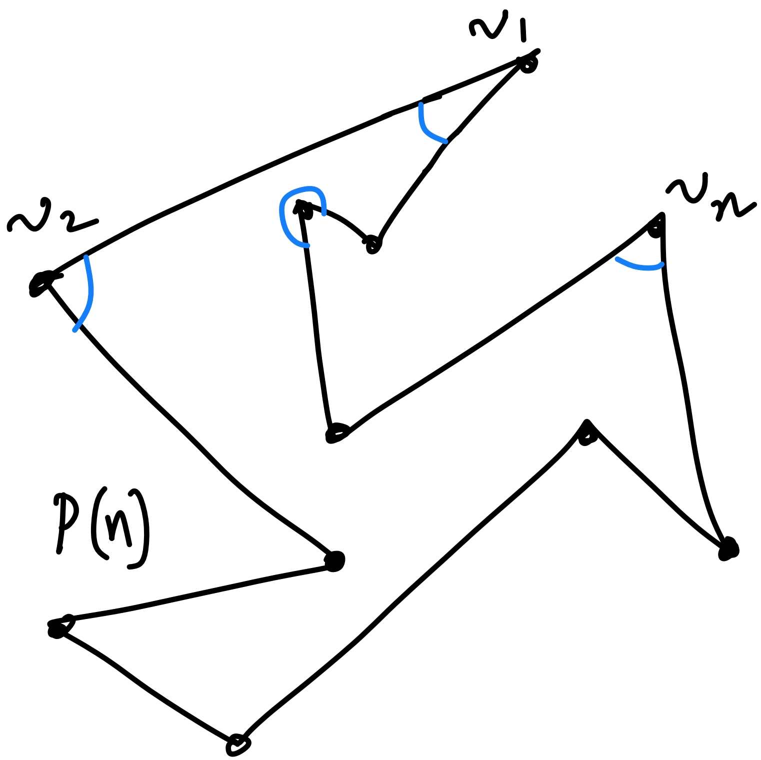
(*) A valid triangulation always exists

To prove: # D = n – 3; # T = n – 2



Same argument *does not* hold good for polygons with holes!

Corollary:

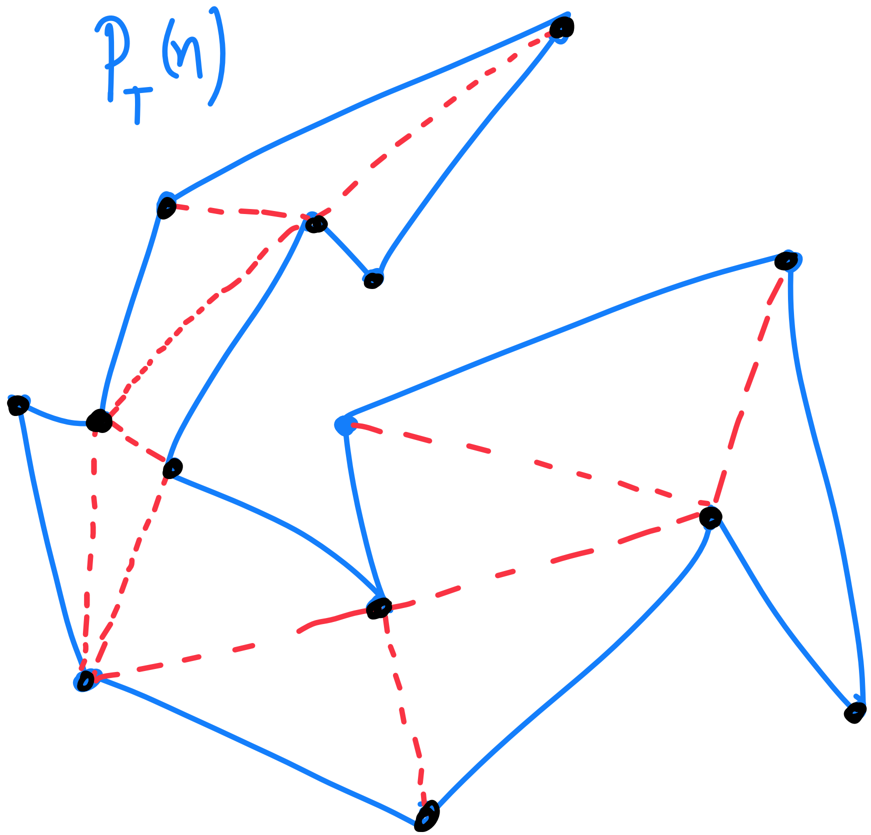


$$\sum_{1}^{n} \alpha_i = (n-2)\pi$$

$\alpha_i \in$ internal angles of $P(n)$

$$\text{Area of } P(n) = \sum_{i=1}^{n-2} A(T_i)$$

Triangulated polygon as a graph



$P_T(n) \Longrightarrow G(V, E)$

$|V| = n, \quad |E| = n + (n-3)$
$= 2n - 3$

$|F| = (n-2) + 1 = n - 1$

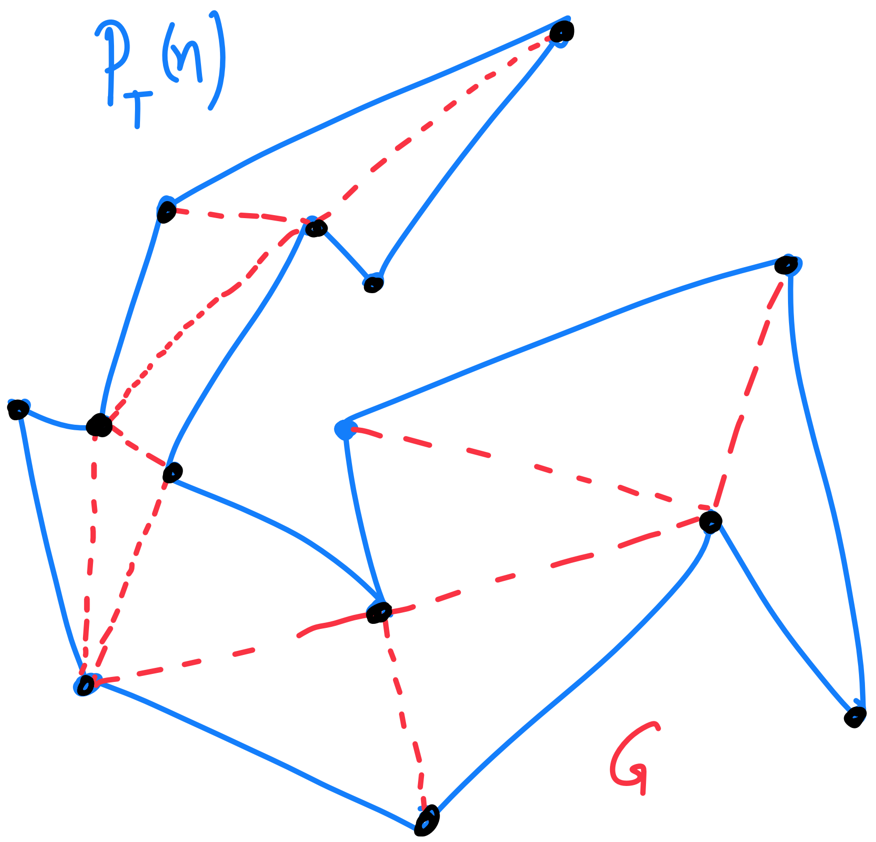
external face

G is planar

$|V| - |E| + |F| = 2$ ✓

D = n – 3; # T = n – 2

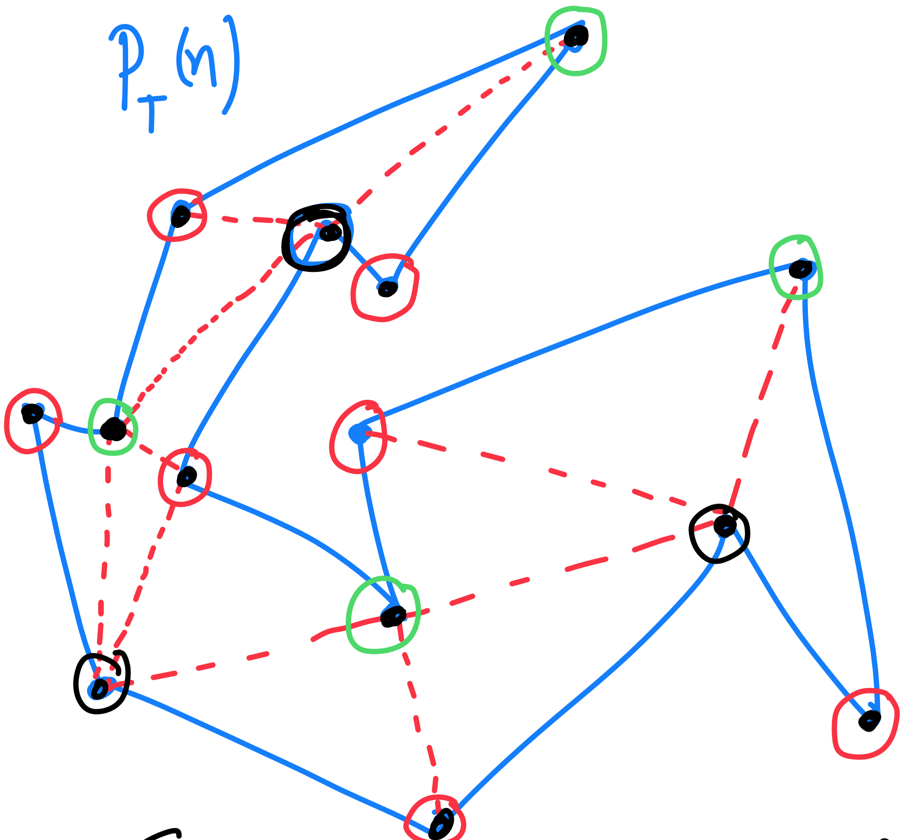
Triangulated polygon as a graph



G is a maximal outerplanar graph

G is 3-colorable

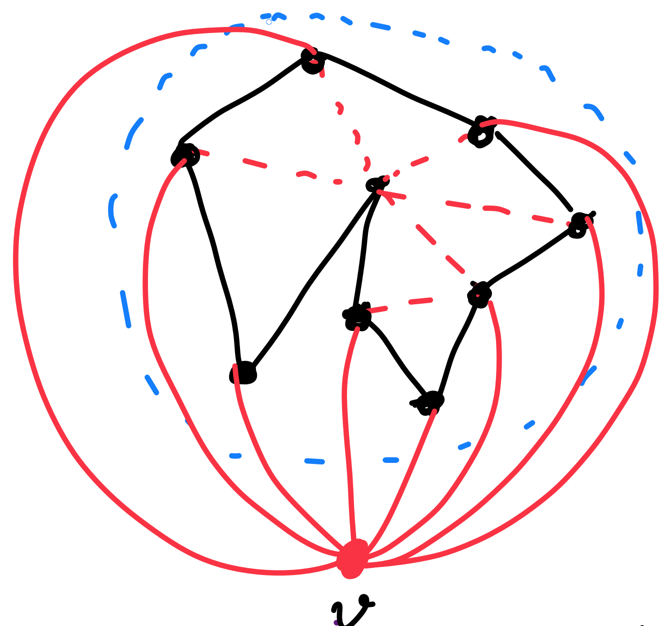
Triangulated polygon as a graph



G is 3-colorable, i.e., $\chi(G) = 3$

<u>Proof</u>: Suppose not, i.e., $\chi(G) = 4$

G' is also planar

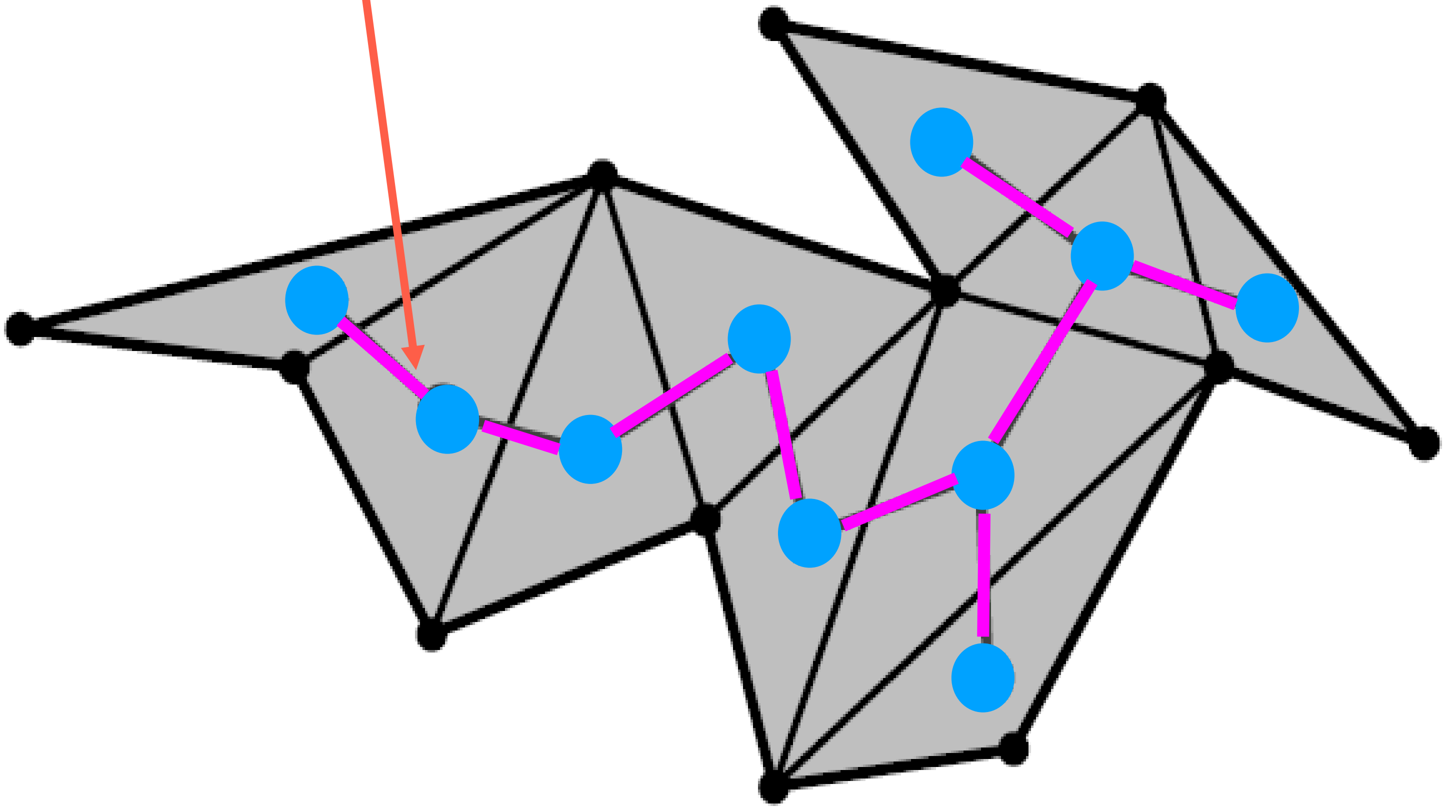


Consider $G' = G \cup (v)$ as shown

$\chi(G') = 5$

contradiction

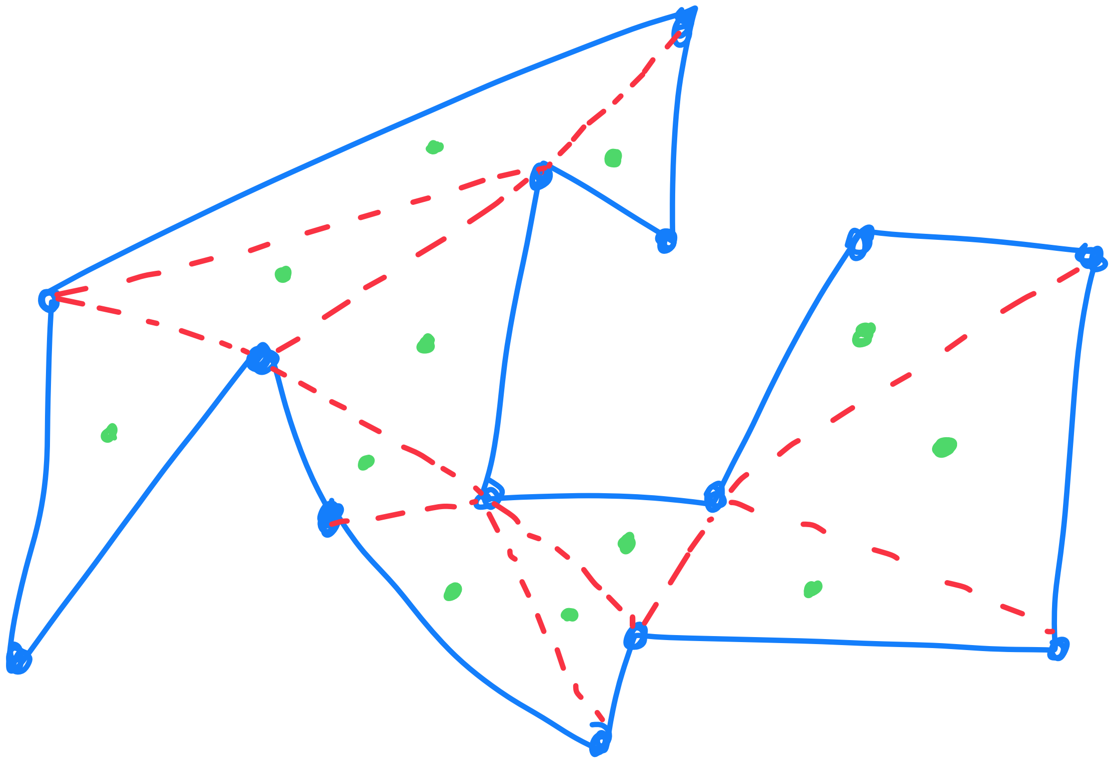
Dual of a triangulated polygon



each triangle (face) → a vertex in dual graph

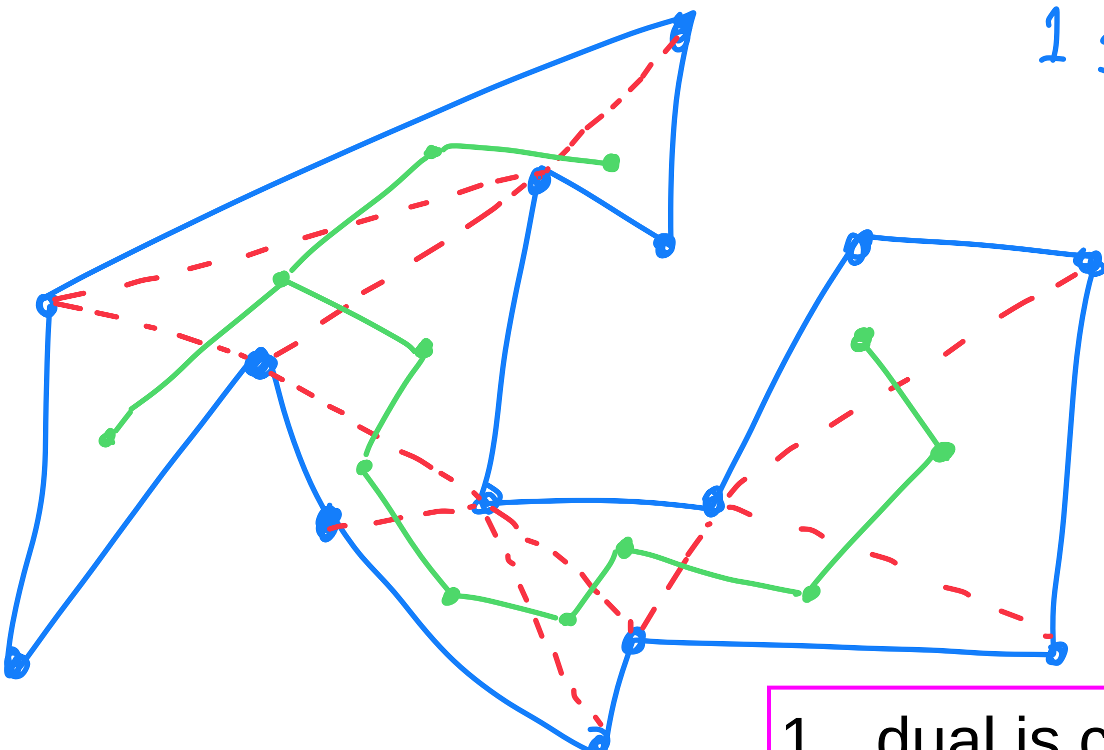
If two triangles share a diagonal, put an edge between the two corresponding vertices in the dual graph

The internal dual of a triangulated simple polygon is a tree



The internal dual of a triangulated simple polygon is a tree T

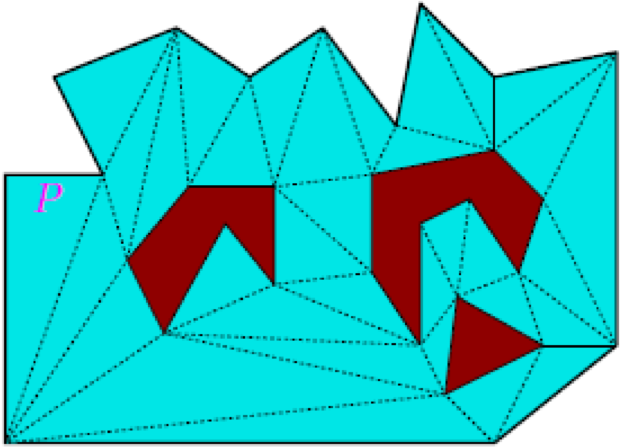
$$1 \le d(v)\big|_T \le 3 \quad \forall v$$



D = n – 3; # T = n – 2

1. dual is connected
2. # vertices(dual) = #T = n – 2;
3. # edges(dual) = #D = n – 3

Triangulations of a polygon with holes



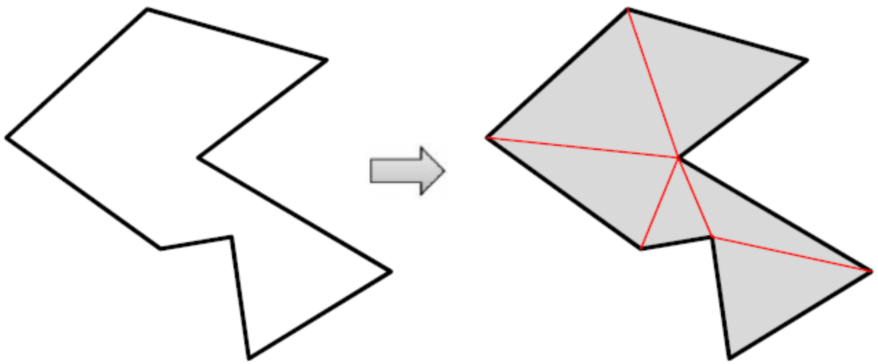
Every triangulation of a polygon with *h* holes with a total of *n* vertices uses $n+3h-3$ diagonals and has $n+2h-2$ triangles

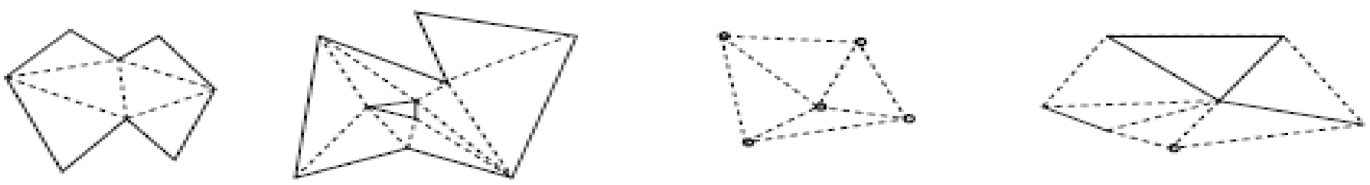
The dual graph of a triangulation of a polygon with holes must have a *cycle*

Self study

Summary and generalizations: Triangulation

Select a *maximal* set of non-intersecting diagonals or edges that subdivide the interior into triangles





polygon

polygon with hole

point set

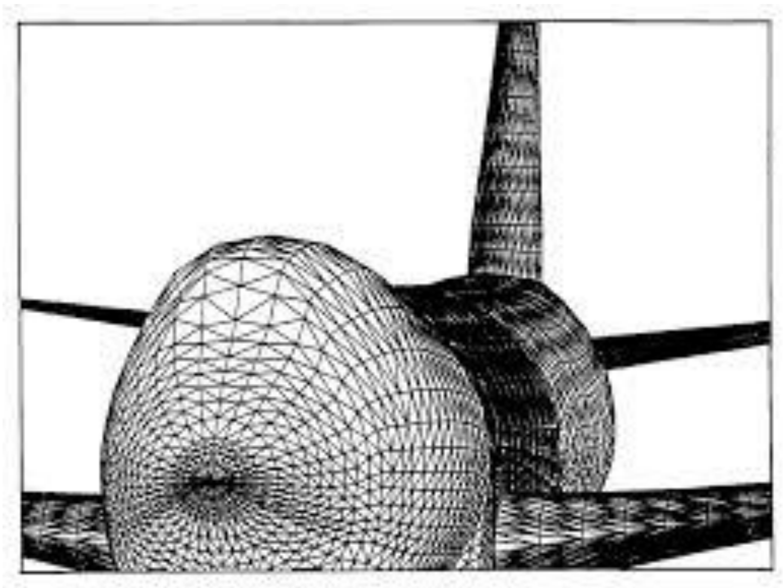
PSLG

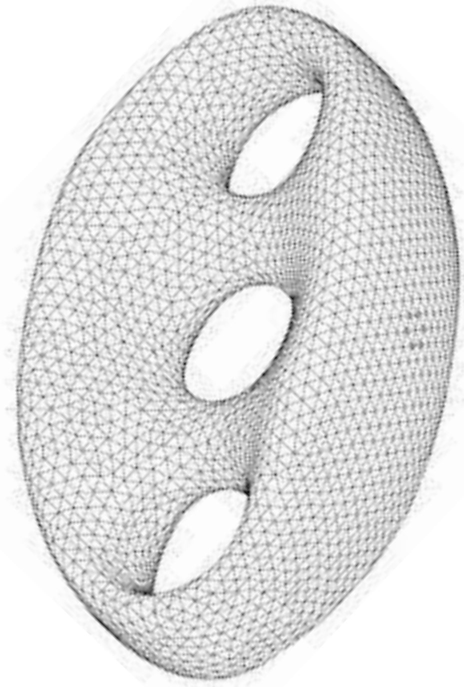
Triangulation is a general concept applicable to many instances

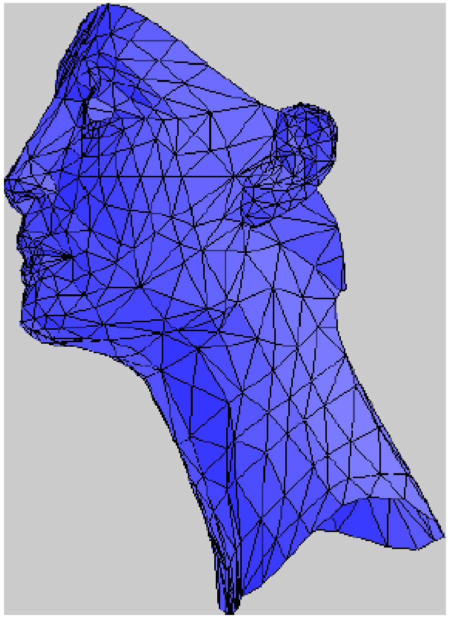
Summary:

- A line segment l joining any two visible vertices of a polygon is called a *diagonal* of the polygon provided l lies completely within P
- Every triangulation of a simple polygon P of n vertices uses $n - 3$ diagonals and has $n - 2$ triangles
- The sum of the internal angles of a simple polygon of n vertices is $(n - 2)\pi$
- The dual of a triangulation of a simple polygon P is a tree, while that of a polygon with holes contains cycles

Surface Triangulation



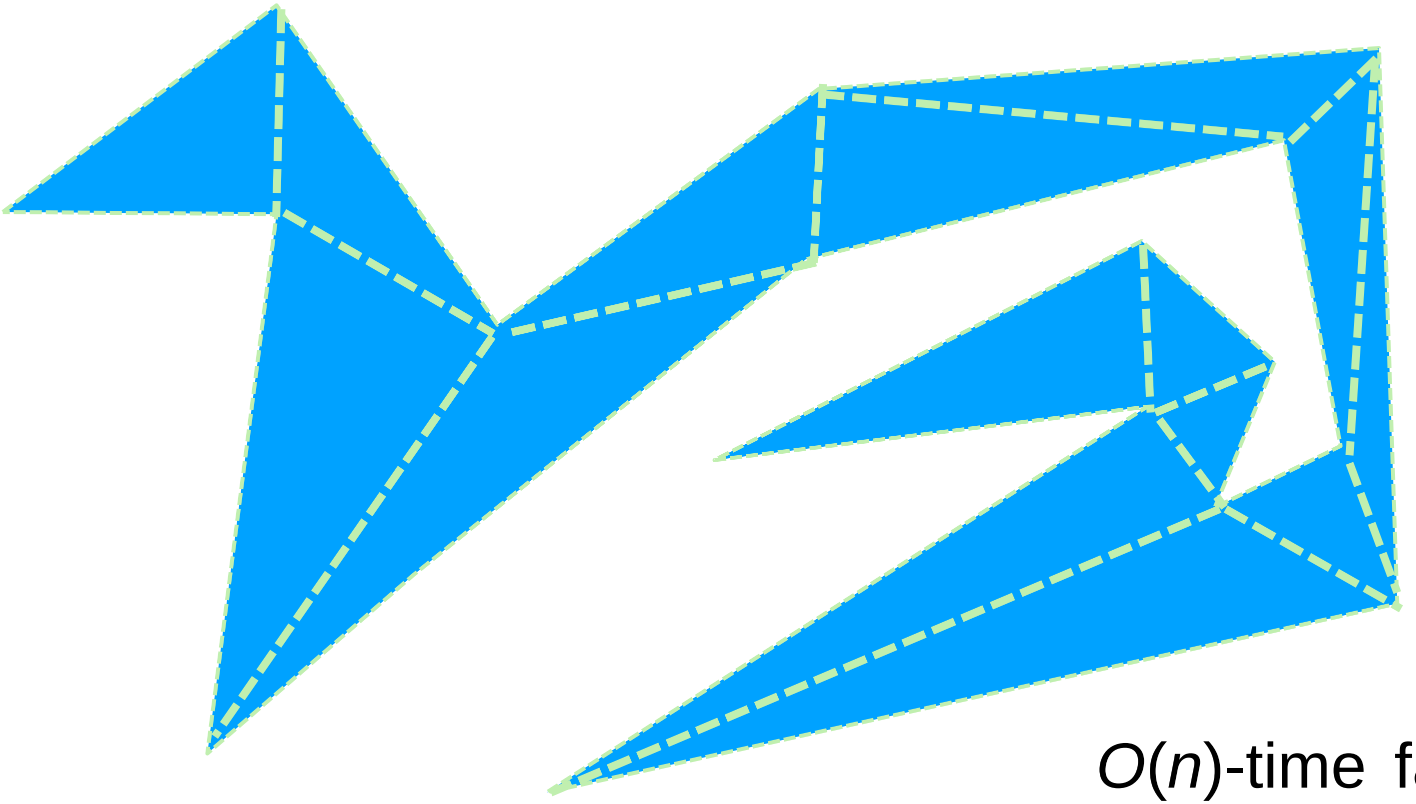


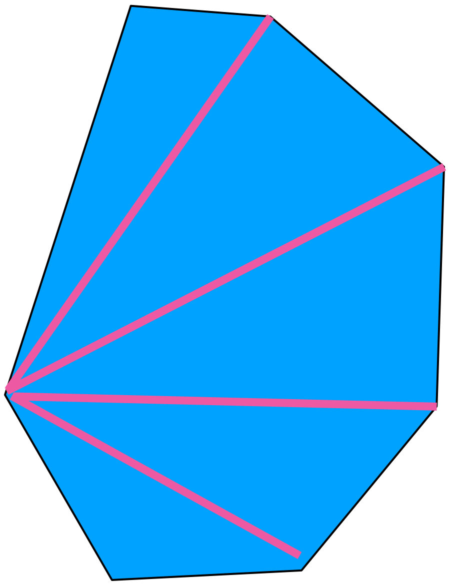


Surface triangulation of 3D objects, mesh generation, 3D modeling, visualization, computer graphics

Surface Triangulation

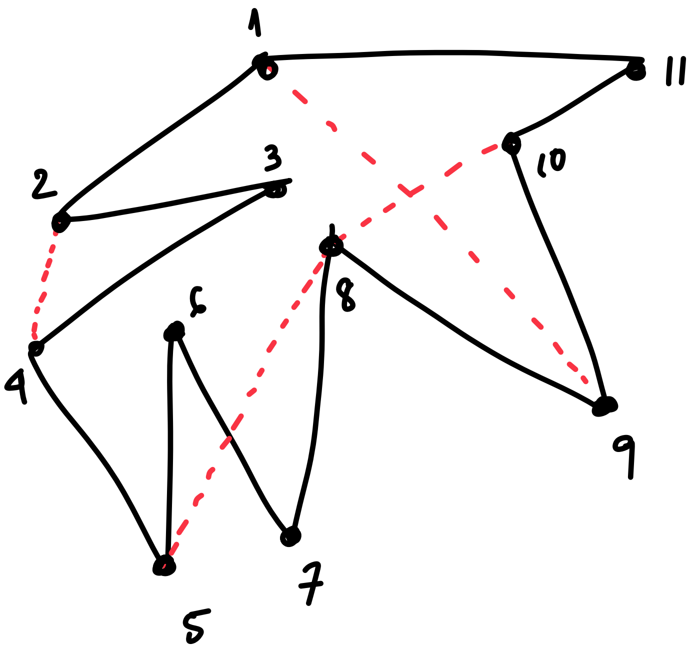
Algorithms for triangulation





$O(n)$-time fan algorithm for convex polygons

Challenges in triangulation: P_n



$D(2,4)$
$D(5,8)$
$D(1,9) \cup D(8,10)$
not allowed

1. Select $(n-3)$ sticks $\overline{ab}$ s.t.
   $a, b \in$ vertices of P_n
2. $\overline{ab} \in P_n$
3. $\overline{ab} \cap \delta P_n = \{a, b\}$

i.e. $\overline{ab}$ is a diagonal

4. $\overline{ab}$ must not intersect with any other previously selected diagonals

$\Rightarrow$ Result: $T(P_n)$, i.e., partition of P_n into $(n-2)$ Δs.

Every simple polygon P_n can be partitioned into triangles by adding $(n-3)$ diagonals

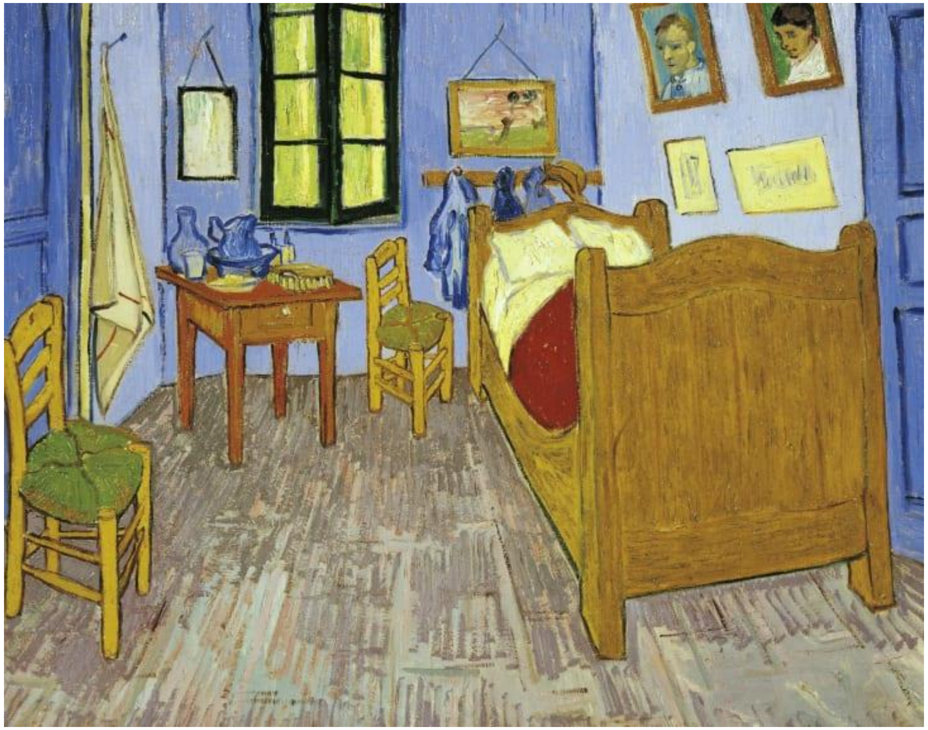
Triangulation Algorithm (Naive)

1. $\binom{n}{2} = O(n^2)$ diagonal candidates

2. Test for diagonal $\rightarrow O(n)$

Hence, to insert $(n-3)$ diagonals $\Rightarrow O(n^4)$

Interesting Fact: Triangulation algorithm time complexity $O(n^4) \rightarrow O(n^3) \rightarrow O(n^2) \rightarrow O(n\log n) \rightarrow O(n)$



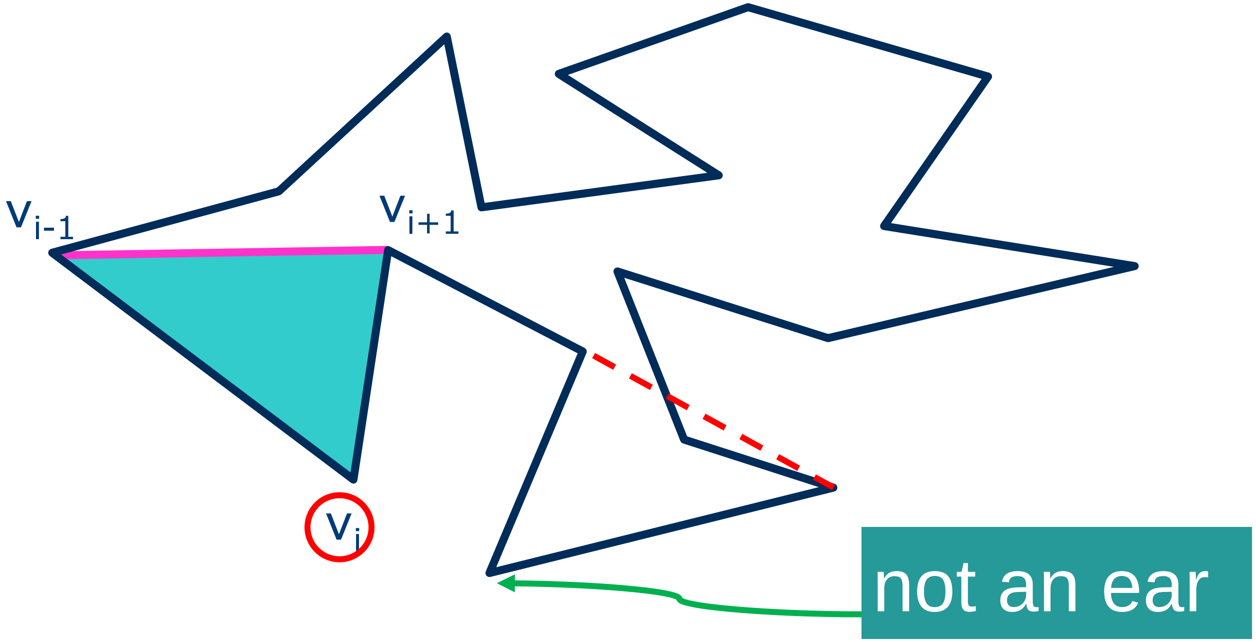
Room where van Gogh chopped off his own ear (1888)

Self-Portrait by Vincent van Gogh (1853-1890)

Polygon Triangulation by Ear-Clipping Algorithm

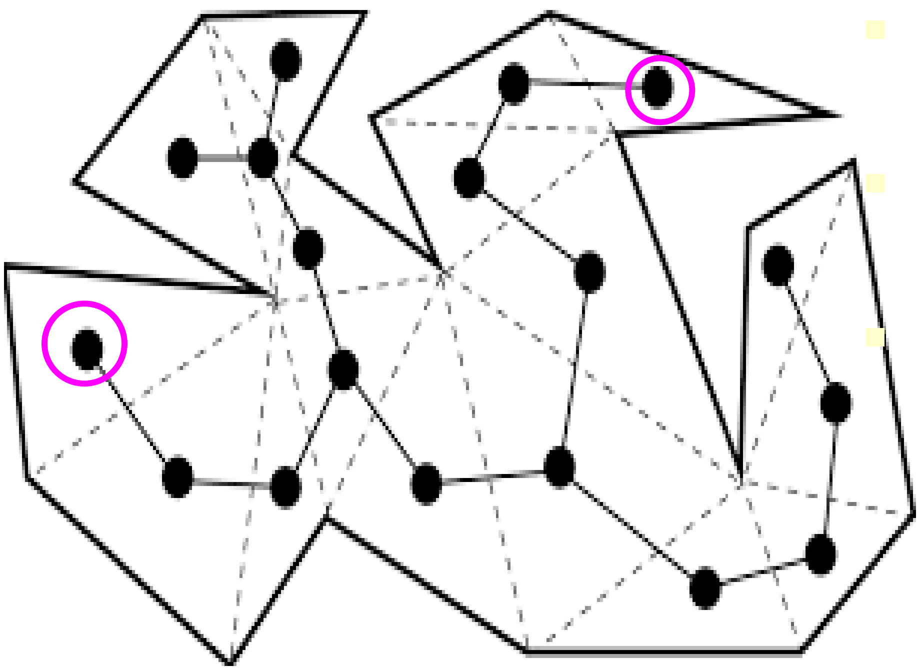
Ears in a Polygon

- Three consecutive vertices v_{i-1} v_i v_{i+1} of a polygon is an *ear* if $v_{i-1}v_{i+1}$ is a diagonal; and v_i is the ear-tip of the triangle
- There are at most *n* ears
- (a convex polygon has exactly *n* ears)



Meister’s Two-Ear Theorem

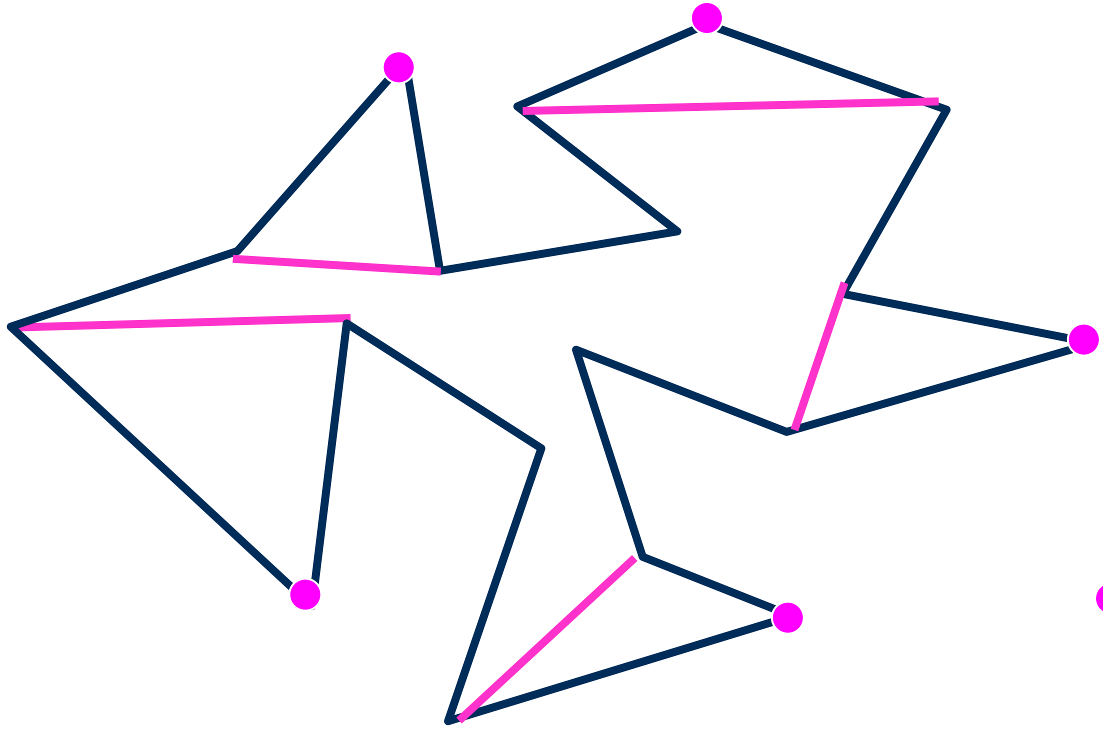
- Every polygon with n > 4 vertices has at least *two* non-overlapping ears

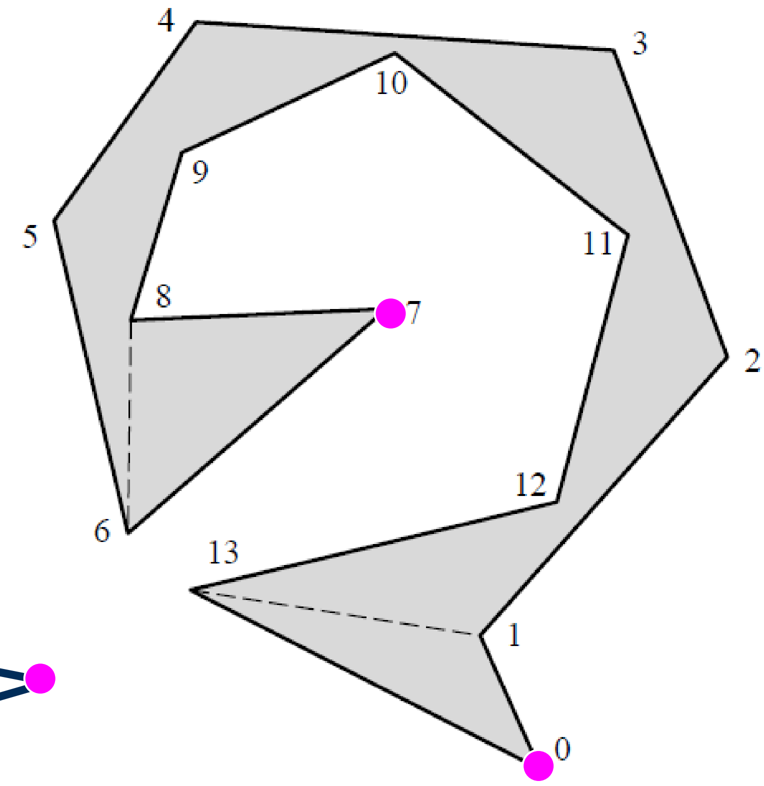


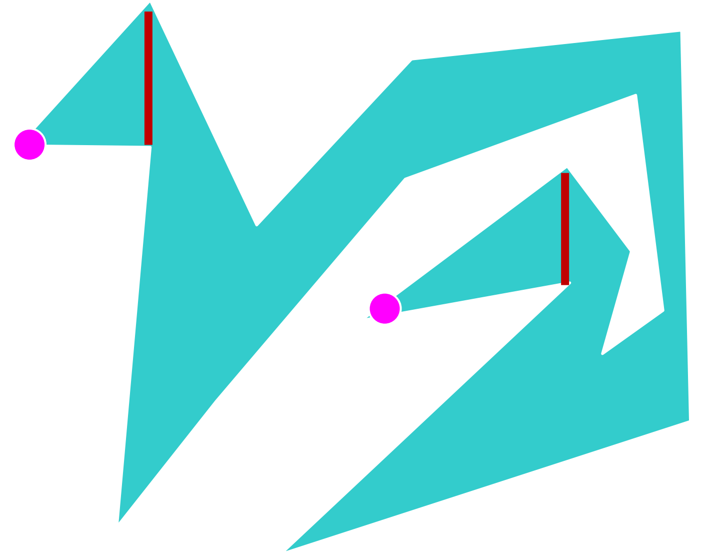
- The dual of a polygon is a tree
- A tree has at least two leaves
- The face (triangle) containing a leaf must be an ear

J. O'Rourke: *Computational Geometry in C,* Cambridge Univ. Press, 1998

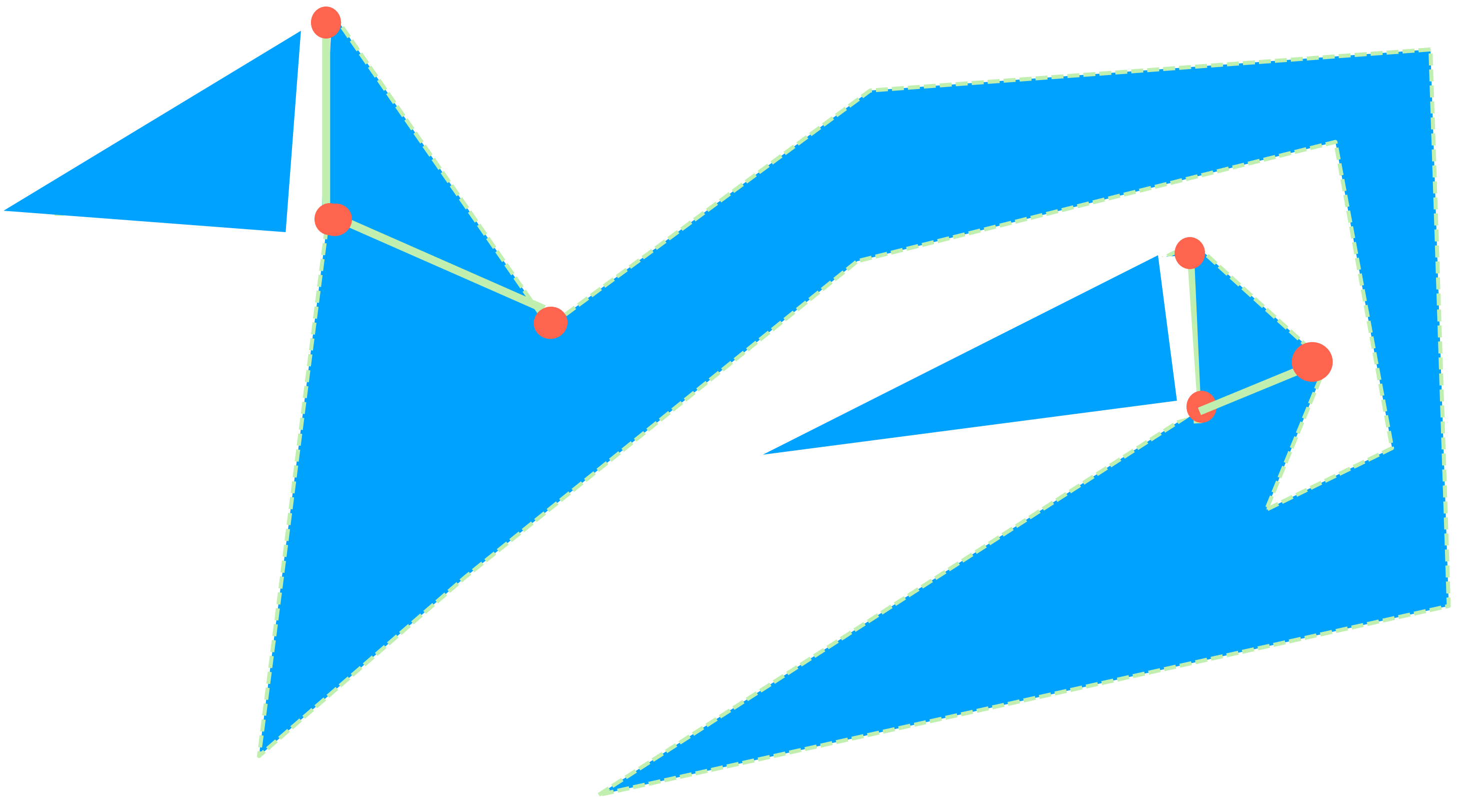
How many ears?







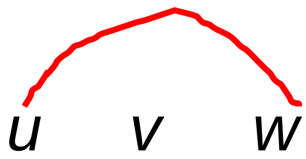
Triangulation: Ear-Clipping Algorithm



Every polygon has at least two ears!

Find an ear, fix a diagonal, chop the ear and iterate

Triangulation by ear-clipping

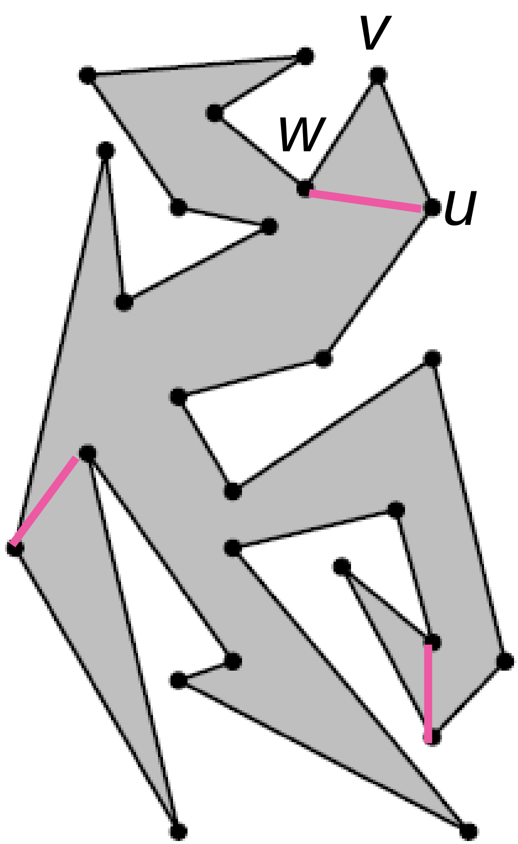


Using the two-ears theorem:
(an ear consists of three consecutive vertices u, v, w where $\overline{uw}$ is a diagonal)

Find an ear, cut it off with a diagonal, triangulate the rest iteratively

Question: Why does every simple polygon have an ear?

Question: How efficient is this algorithm?

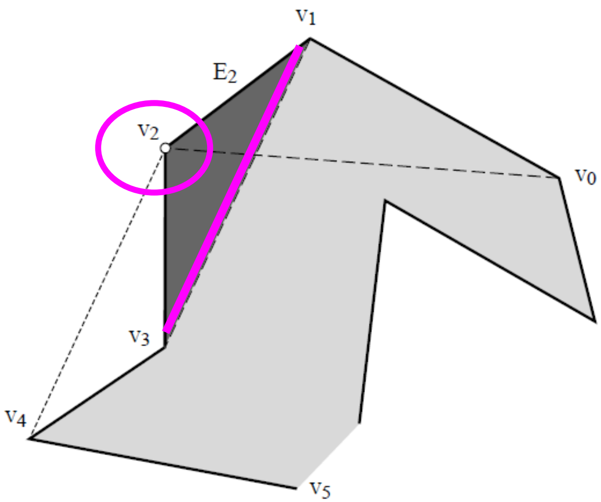


There are $O(n)$ ear candidates; checking each for ear $O(n)$; (n – 3) diagonals; Total time complexity $O(n^3)$

Can we improve it to $O(n^2)$

Smarter approach

- When clipping ear with tip v_i the only ear tip statuses that can change are at v_{i-1} and v_{i+1}



Both v_1 and v_2 are possible ear-tips; When v_2 is clipped and the diagonal is inserted, v_1 no longer remains an ear-tip. So, only the two neighbors of v_2 need to be checked for change of status. “Ear”-ities of other vertices are not affected. Thus, only $O(1)$ updates are needed after inserting one diagonal!

Naive: $O(n^3)$

There are $O(n)$ ear candidates; checking each for ear $O(n)$; Total $\Rightarrow O(n^2)$
Update $O(1)$; There will be $O(n)$ iterations

Total: $O(n^2)$

Ear-clipping algorithm

Triangulation

Initialize the ear tip status of each vertex.

while $n > 3$ do

  Locate an ear tip v_2.

  Output diagonal $v_1 v_3$.

  Delete v_2.

  Update the ear tip status of v_1 and v_3.

Initially determine "ear-tip status" of each v_i, $O(n^2)$

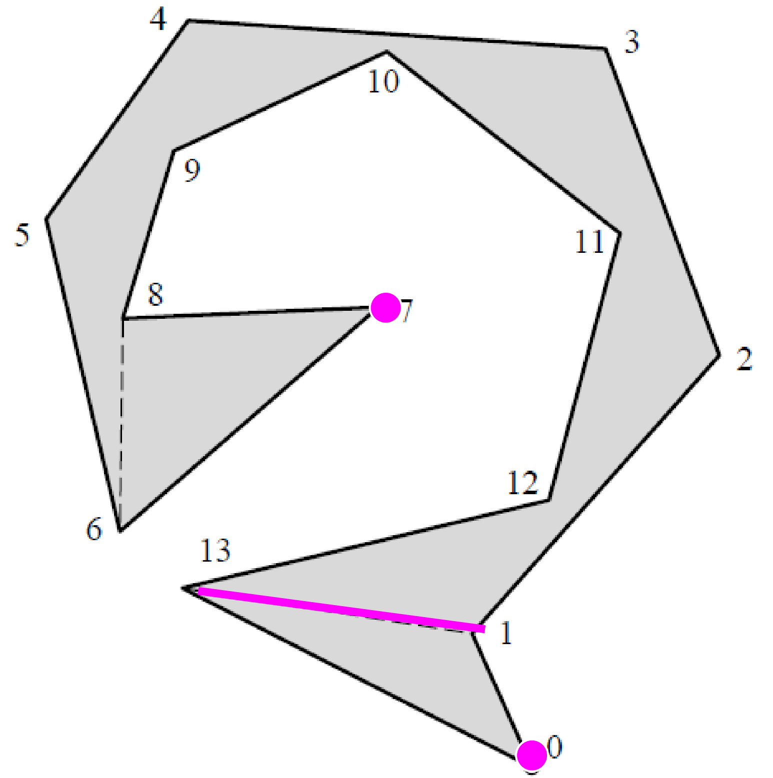
Update of each ear status requires $O(1)$; ear-tip tests @ $O(n)$ per test; $n - 3$ diagonals

Total: $O(n^2)$

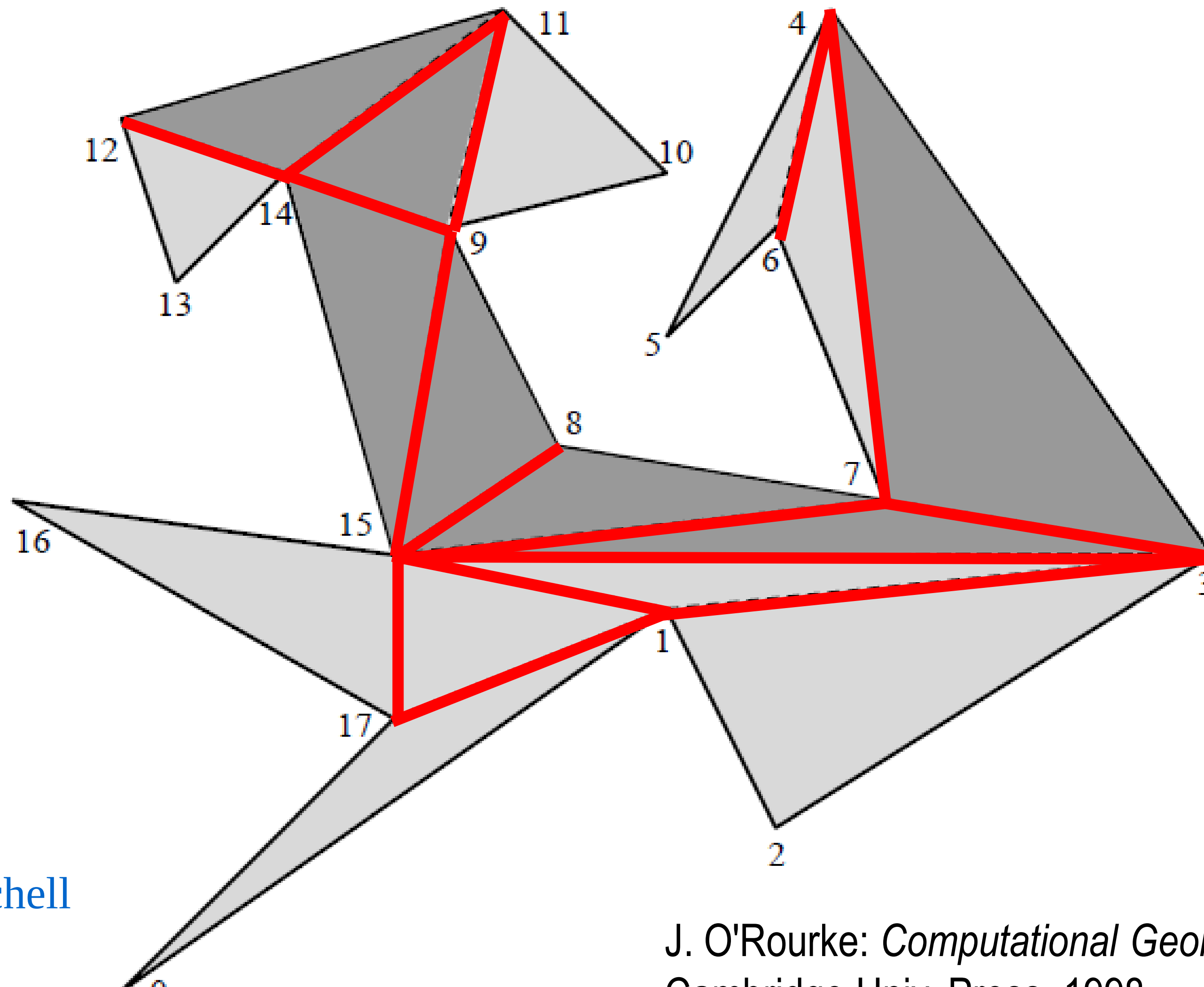
J. O'Rourke: *Computational Geometry in C*, Cambridge Univ. Press, 1998

After inserting one diagonal, the search for the next ear may take $O(n)$ time

Total: $\Omega(n^2)$



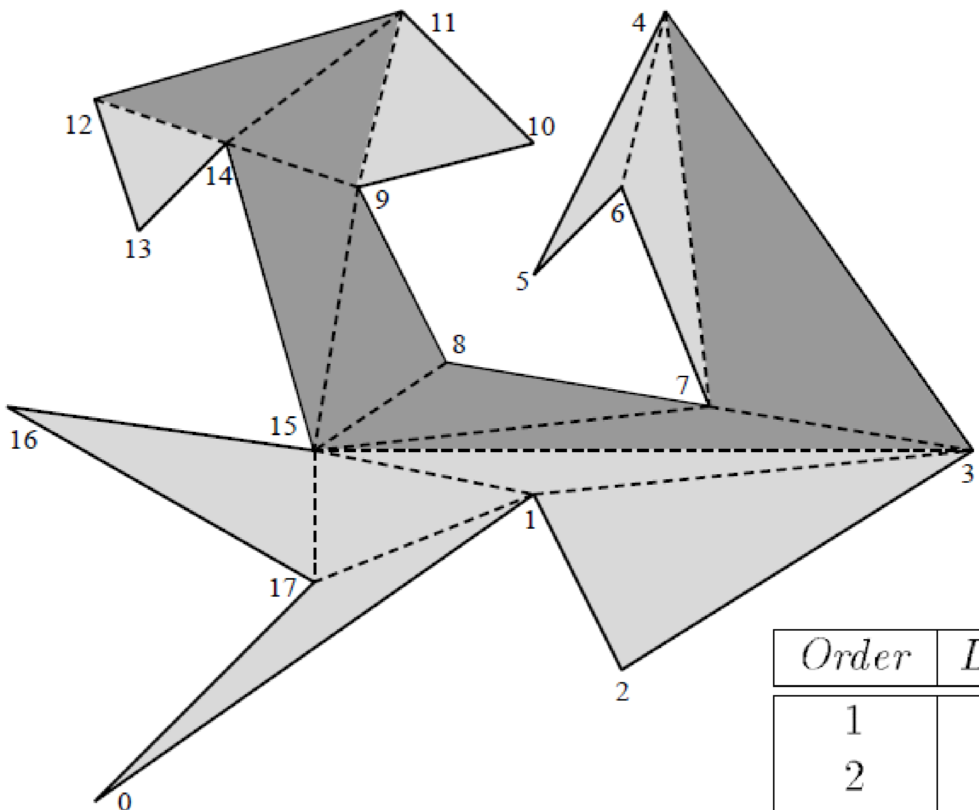
Example: Triangulate



Courtesy: Joe Mitchell

J. O'Rourke: *Computational Geometry in C*, Cambridge Univ. Press, 1998

Example: Output



Order	*Diagonal indices*	*Order*	*Diagonal indices*
1	$(17, 1)$	10	$(3, 7)$
2	$(1, 3)$	11	$(11, 14)$
3	$(4, 6)$	12	$(15, 7)$
4	$(4, 7)$	13	$(15, 8)$
5	$(9, 11)$	14	$(15, 9)$
6	$(12, 14)$	15	$(9, 14)$
7	$(15, 17)$		
8	$(15, 1)$		
9	$(15, 3)$		

Courtesy: Joe Mitchell