CS60064 Spring 2022

Computational Geometry

Instructors

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Lecture 10 & Lecture 11

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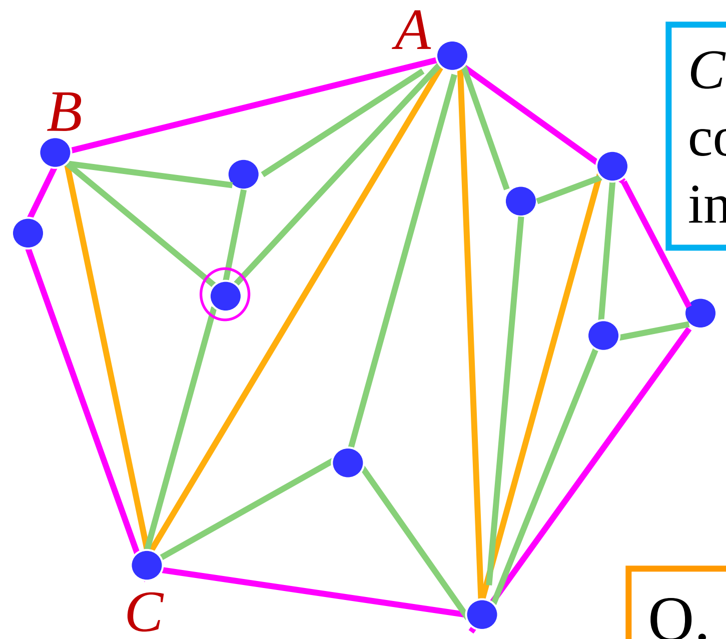
Indian Institute of Technology Kharagpur

Computer Science and Engineering

Problem of the Day

Triangulating a set *S* of points in 2D

S:



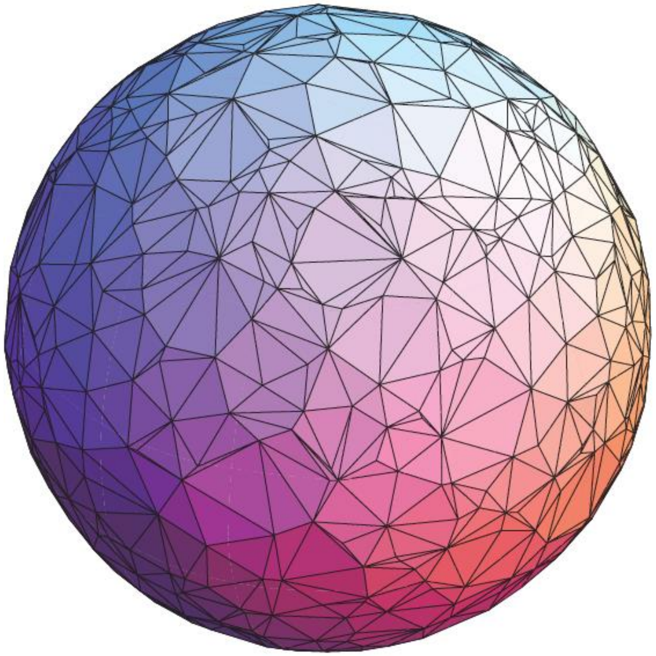
Claim: All edges of the convex hull must be present in any valid triangulation

Q. # triangles when $|S| = n$?

A *triangulation* of a planar point set *S* is a subdivision of the plane defined by a *maximal* set of non-crossing edges whose vertex set is *S*
$\Rightarrow$ all internal faces are triangles

Polygonization through all vertices may not help, because triangulating that polygon may not lead to maximality in a point set!

Triangulating a set S of points in 3D



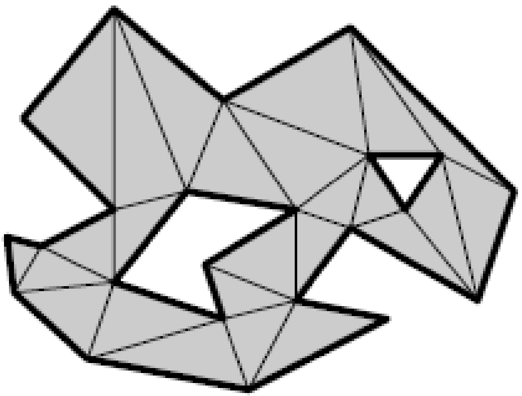
Triangulation of 758 random points on a sphere

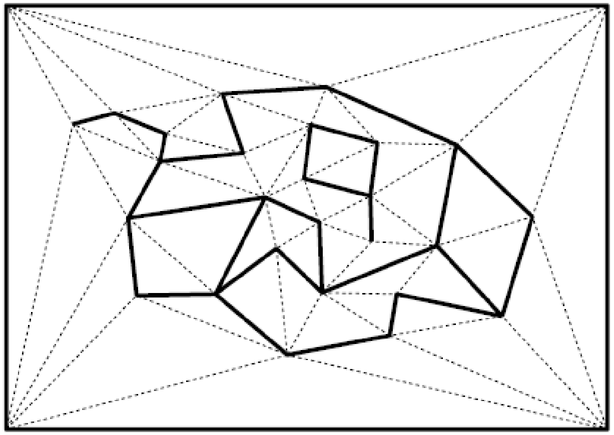
Courtesy: S. L. Devadoss and J. O'Rourke: *Discrete and Computational Geometry*, Princeton University Press, 2011

Summary of Results

Note: The sum of the number of vertices of all monotone pieces is $O(n)$

Theorem: A simple polygon with n vertices can be triangulated in $O(n\log n)$ time with an algorithm that uses $O(n)$ storage





Same complexity results also hold good for polygons with holes as well as for PSLGs

Chronology of Polygon Triangulation

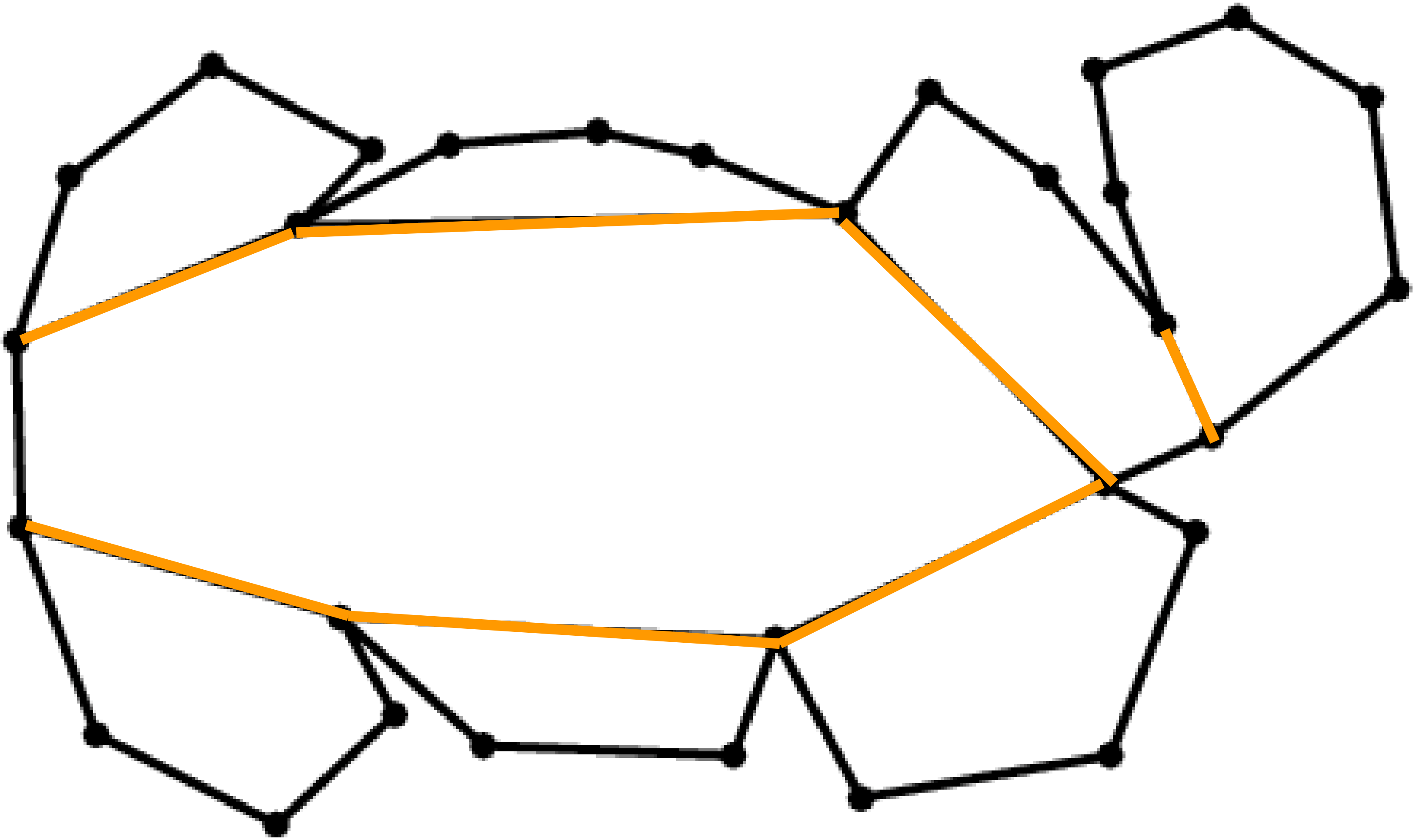
Historical Perspective

- $O(n^2)$: Diagonal insertion
- $O(n \log n)$: Lee and Preparata
  (Monotone decomposition, 1977)
  Avis and Toussaint (1981)
  Chazelle (1982)

  Optimal??

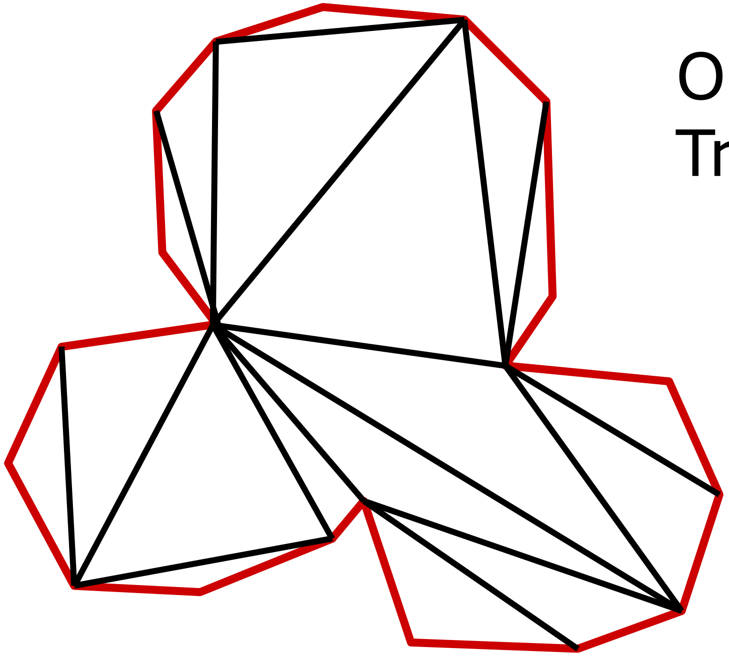
- $O(n \log \log n)$: Tarjan and Van Wyk (1988)
- $O(n \log^* n)$: **Randomized**:
  Clarkson, Tarjan, and Van Wyk (1989)
  Seidel (Trapezoidal decomposition, 1991)
  Devillers (1992)
- $\Theta(n)$: **Optimal** (yet deterministic):
  Chazelle (1991)

Convex Decomposition



Convex Decomposition

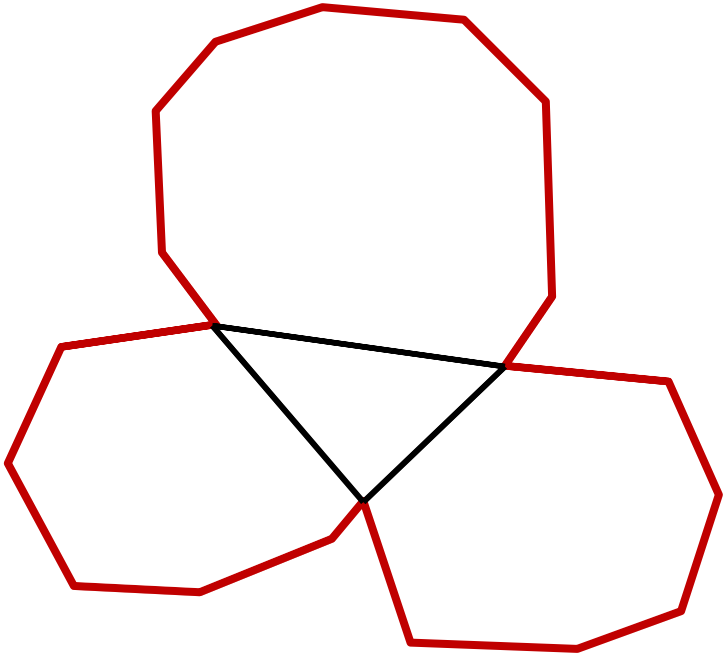
- Partition a simple polygon P into a number of convex pieces



One way to do it:
Triangulate P

Convex Decomposition

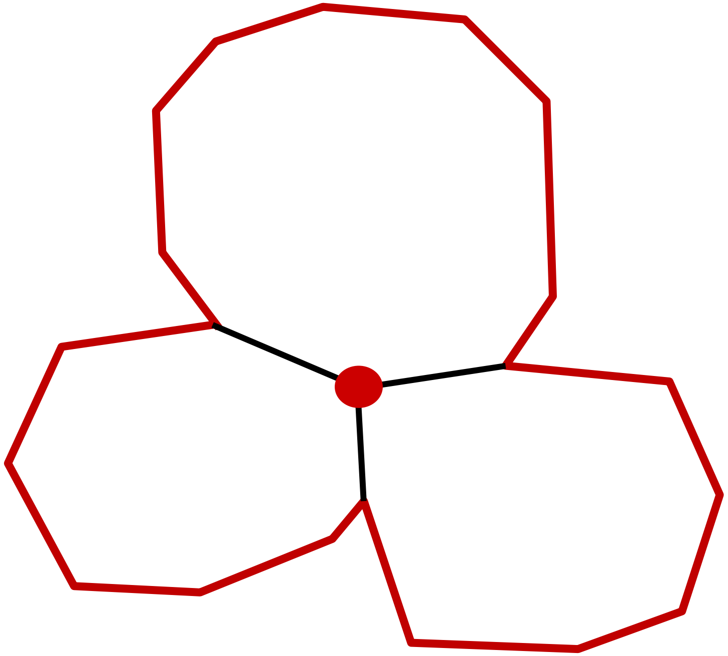
- Partition simple polygon *P* into a small number of convex pieces



Another way to do it: Use diagonals suitably to partition *P* into convex polygons

Convex Decomposition

- Partition simple polygon *P* into a small number of convex pieces

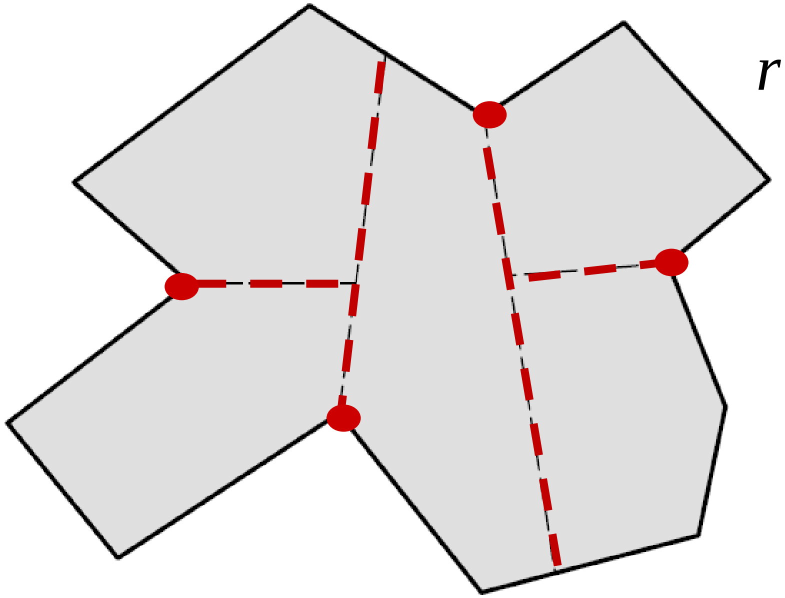


Another way to do it:
Allow “Steiner” points (non-vertices) inside *P*.
Use segments that are not necessarily diagonals.
May lead to fewer pieces!

Role of Relex Vertices

Let φ be the fewest number of convex pieces into which a simple polygon P can be partitioned. Let r denote # reflex vertices in P.

Then $\varphi \leq r + 1$



$r = 4 \Rightarrow$ *at most* five convex pieces

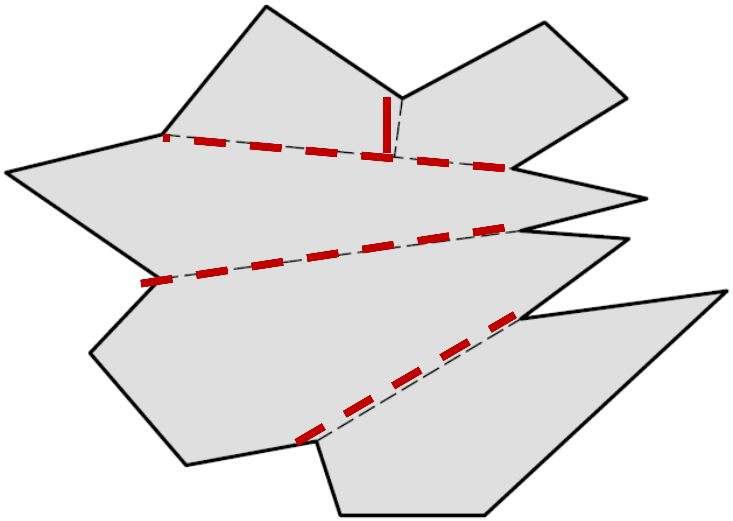
reflex angles cause non-convexity

use bisectors of reflex angles

Role of Relex Vertices

Theorem (Chazelle): Let φ be the fewest number of convex pieces into which a simple polygon P can be partitioned.

Then $\lceil r/2 \rceil + 1 \leq \varphi \leq r + 1$, where r: # reflex vertices in P

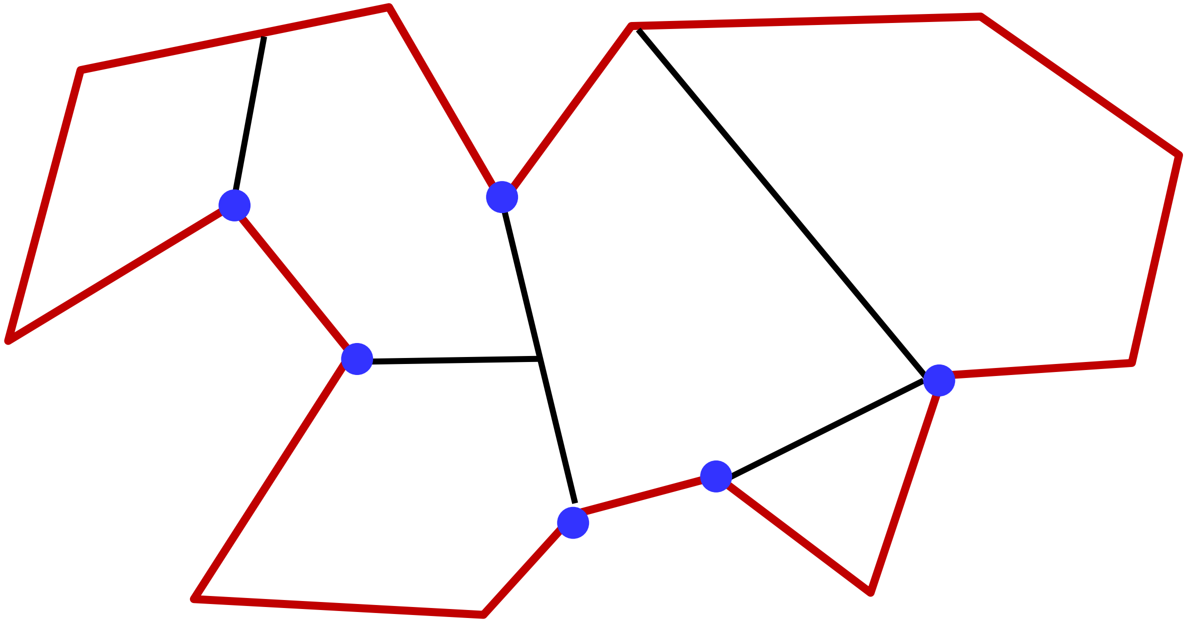


$r = 7 \Rightarrow$ *at least* five convex pieces

All reflex vertices must be resolved; two of them may be neutralized together if a diagonal joins them; hence the proof

J. O'Rourke: *Computational Geometry in C,* Cambridge Univ. Press, 1998

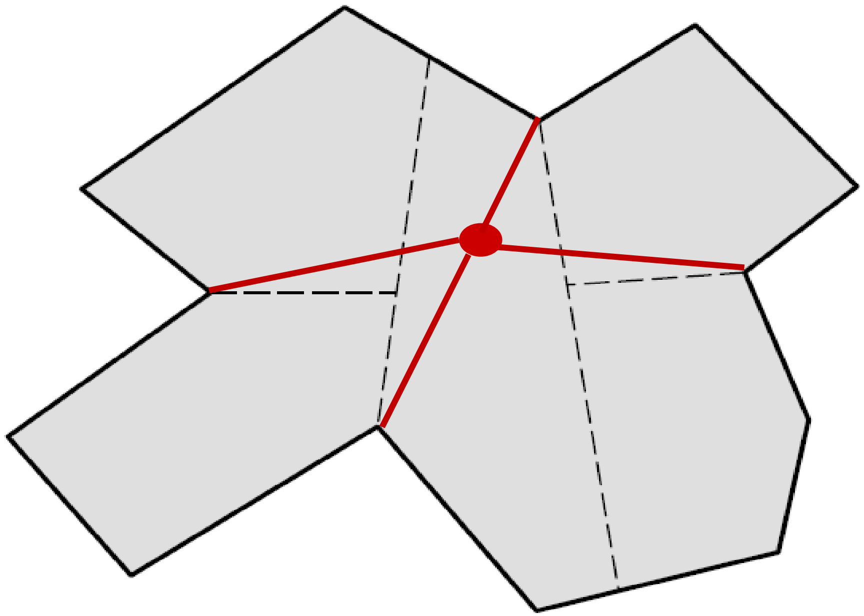
Convex decomposition using matching and bisection: *Example*



r=6

Optimal Convex Decomposition

- Allowing Steiner points



Convex Decomposition

- *Goal:* Partition P into a small number of convex pieces (convex polygons)
- A triangulation is one possible decomposition into convex pieces, but it may have many more pieces than necessary!
- Dynamic programming based algorithms yield optimal solutions for simple polygons (for both Steiner and non-Steiner versions), in roughly $O(n^3)$
- Hertel-Mehlhorn algorithm: 4-approximation in time $O(n)$

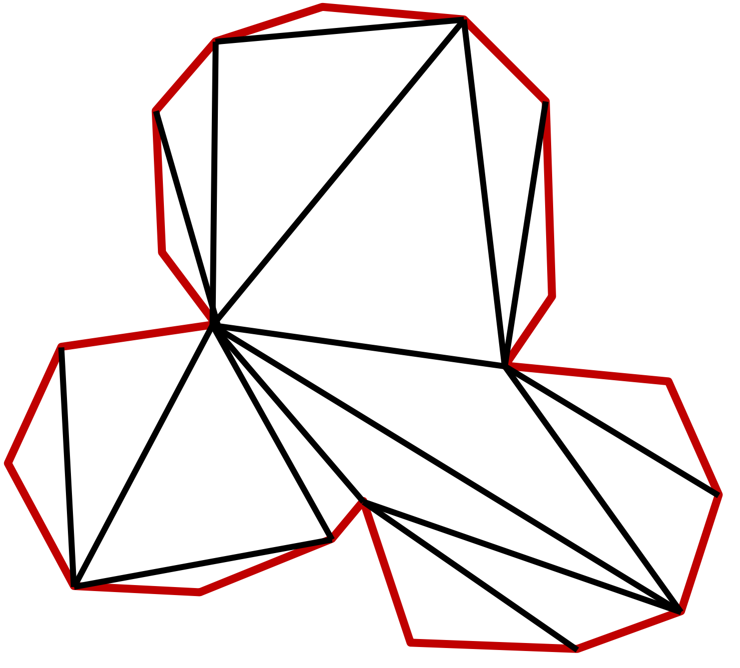
$O(r^2 n \log n)$ without Steiner [Keil'85]; $O(n+r^3)$ allowing Steiner [Chazelle'80]
n:#vertices of P; r:#reflex vertices

Hertel-Mehlhorn Algorithm (uses diagonals for partitioning)

- Start with any triangulation of simple polygon *P*
- Remove inessential* diagonals, in any order
- Repeat until no more diagonals can be deleted

*A diagonal is inessential if its deletion does not impact convexity

H-M Algorithm: Example



H-M Algorithm: Summary

- Start with any triangulation of simple polygon P: $O(n)$ time (Chazelle) or $O(n \log n)$ by trapezoidation
- There are $O(n)$ diagonals in a triangulation
- Check the status of all diagonals (essential or inessential): $O(n)$
- When a diagonal is deleted the status of $O(1)$ diagonals needs to be updated
- Total time complexity for diagonal removal: $O(n)$
- What would be a good data structure that would support such operations?

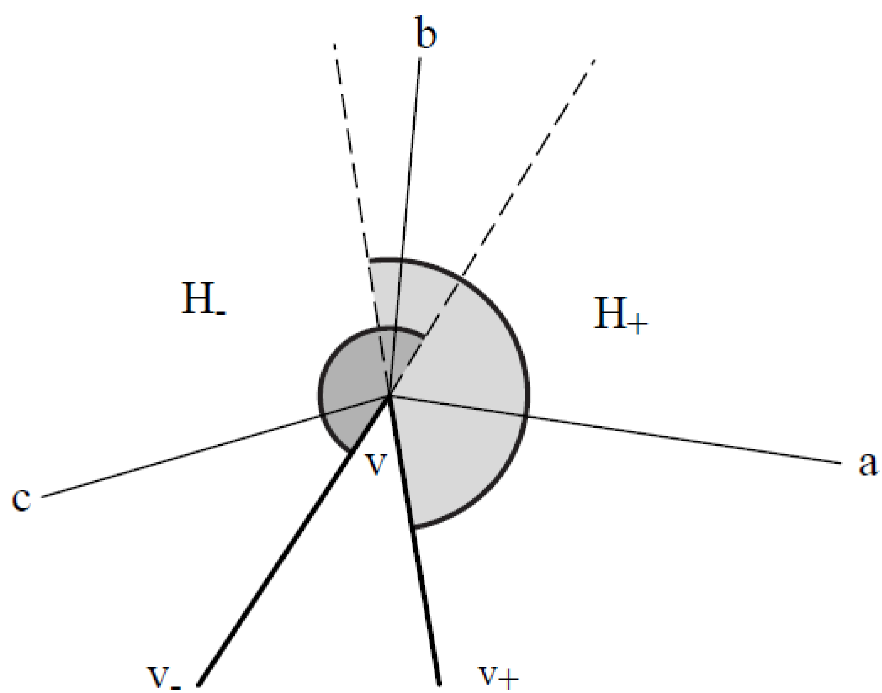
Hertel-Mehlhorn Algorithm

Theorem: The H-M algorithm yields convex decomposition into at most 4*OPT pieces, where OPT is the minimum possible number of pieces in a (Steiner) convex decomposition

Thus, the H-M Algorithm is a "4-approximation algorithm", i.e., not worse than 4-times optimal cost

Hertel-Mehlhorn Algorithm

Lemma: There can be at most two diagonals essential for any reflex vertex *v*



4-Approximation Algorithm

Assume H-M algorithm produces M convex pieces at the end

- There are at most $2r$ diagonals left (by previous lemma) $\Rightarrow 2r+1 \geq M$
- Let φ be the optimum number of convex pieces (even by using Steiner's vertices)

$\Rightarrow \varphi \geq \lceil r/2 \rceil + 1$

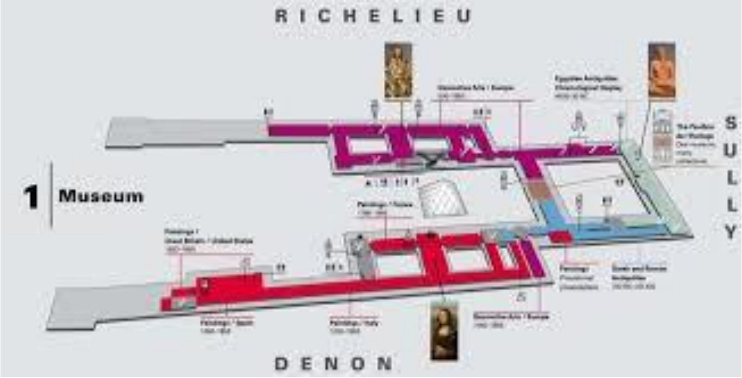
$\Rightarrow 4\varphi \geq 2r + 4 \geq M$

$\Rightarrow M \leq 4\varphi$

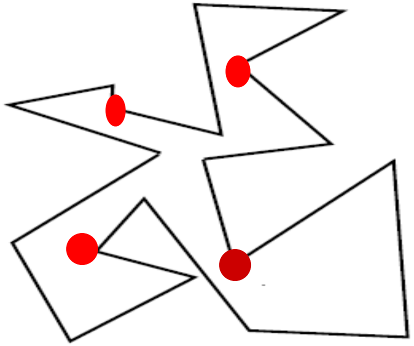
Visibility and Art-Gallery Problem

Art Gallery Problem

Louvre Museum, Paris



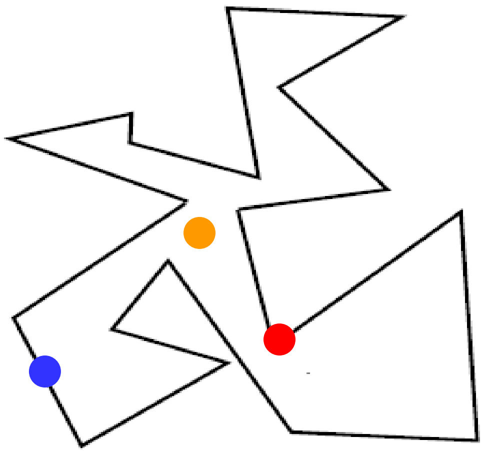
Victor Klee (1973)



Vasek Chvátal (1975)

Given the floorplan of a weirdly shaped art gallery having *n* straight sides, how many guards will have to be posted, in the worst case, so that the entire wall is visible to them?

Art Gallery Problem: Assumptions and Questions



Question: Given a polygonal gallery with *n* vertices, where should we position guards/watchmen so that the assets are protected from theft? In other words, every asset should be visible to at least one guard.

Floodlight illumination problem (or surveillance problem): Where should we put light sources (or cameras) so that the entire floor, the interior of a polygon be illuminated (or be under watch)?

Assumptions

Positions of guards: vertex, edge, or anywhere in the interior? Static or roaming? Guards are assumed to be points and have continous 360° vision; cannot see through obstructions; they do not block visibility of each other

Coverage: Wall (boundary coverage) or area coverage? How about 3D?

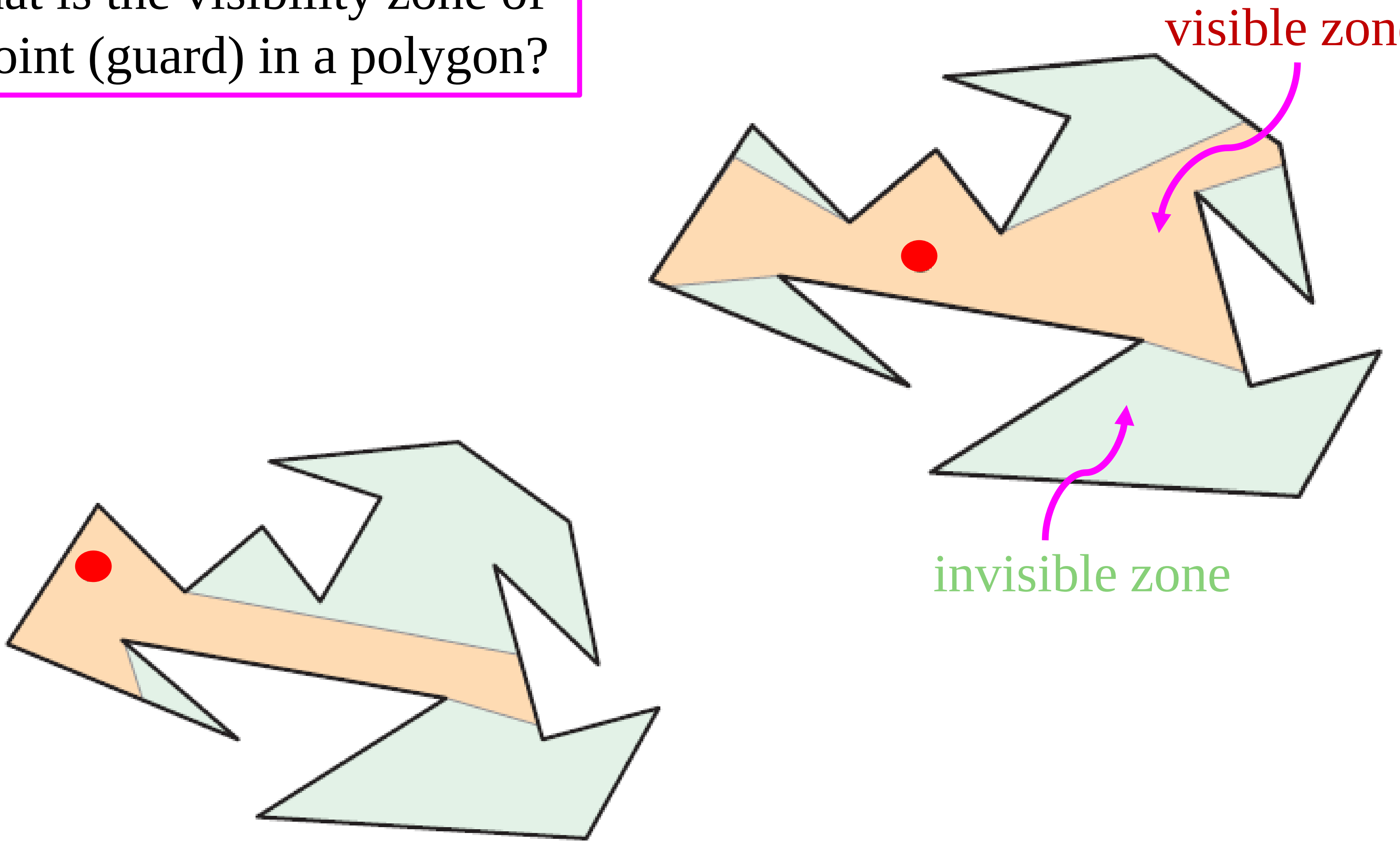
Optimization issue: What is the minimum number of guards necessary to cover a given polygon? For boundary coverage or area coverage?

Existential issue: For a given value of *n*, in the worst case, how many guards will be needed?

Generality: How about polygons with holes?

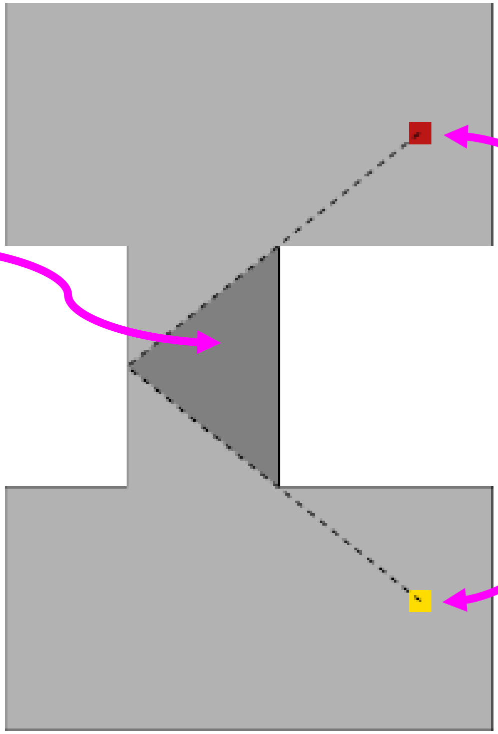
Visibility Zones

What is the visibility zone of a point (guard) in a polygon?

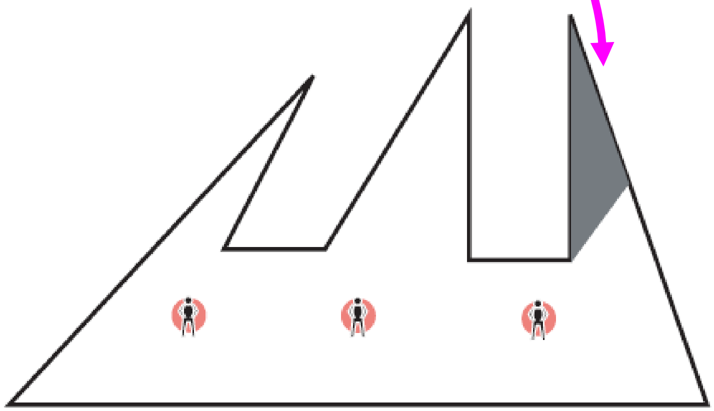


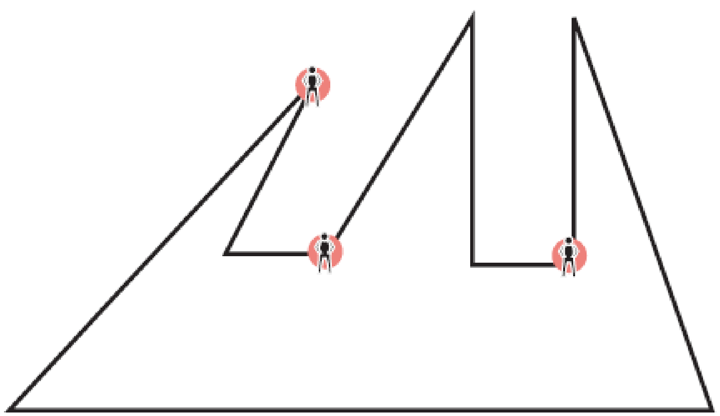
Art Gallery Problem: Visible Zones

zone invisible to two guards or cameras as placed



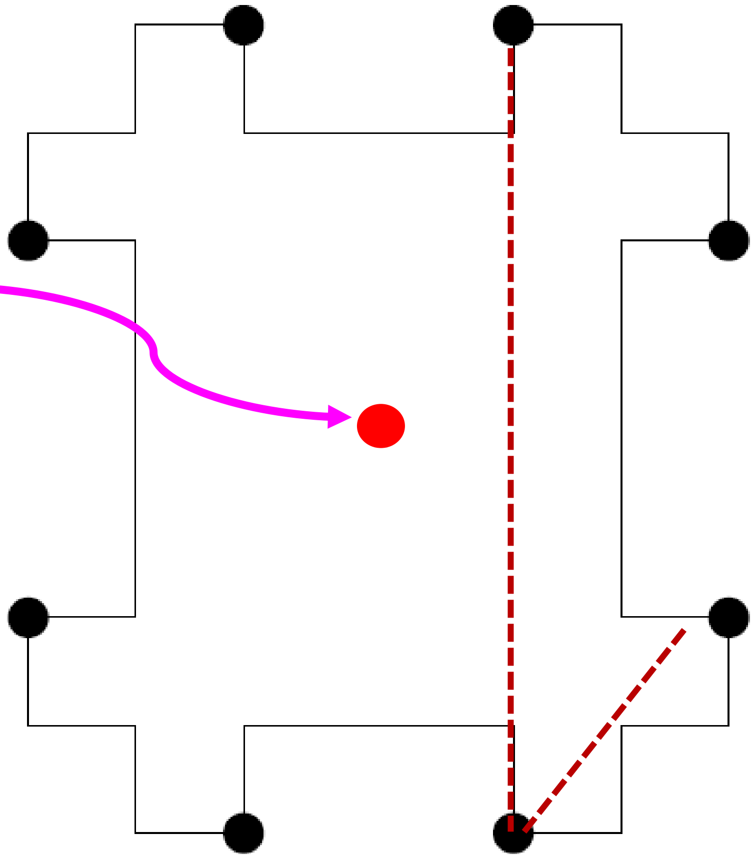
guards





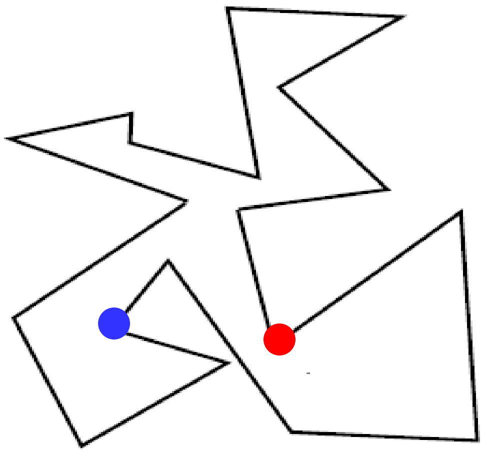
Boundary *versus* Area Visibility

The entire boundary (walls) of the polygon is visible in the interior but the central zone is invisible to the guards



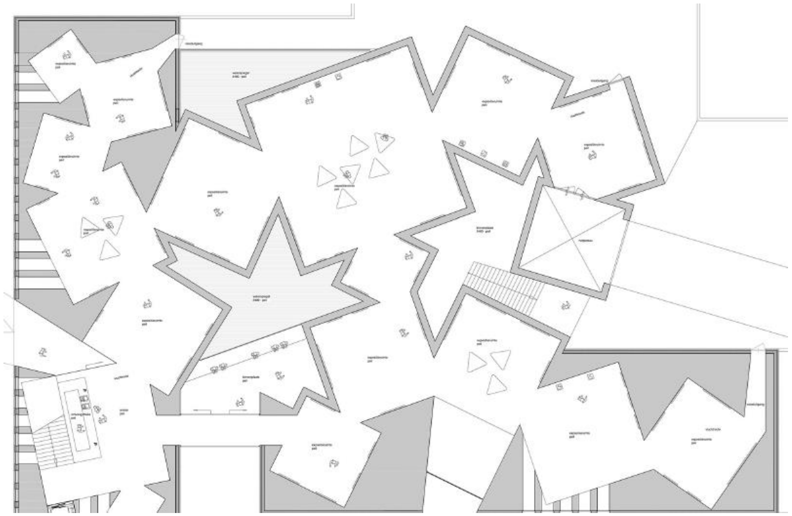
- vertex guards
- invisible zone

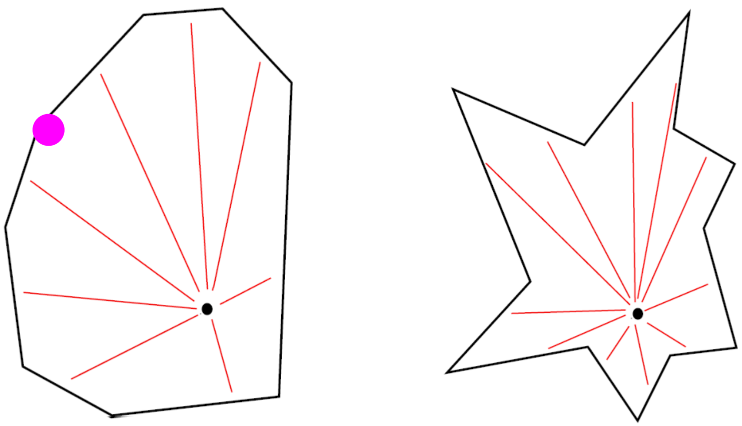
Art Gallery Problem: Examples



Question: Given a polygonal gallery with *n* vertices, where should we position guards/watchmen so that the assets are protected from theft? In other words, every asset should be visible to at least one guard.

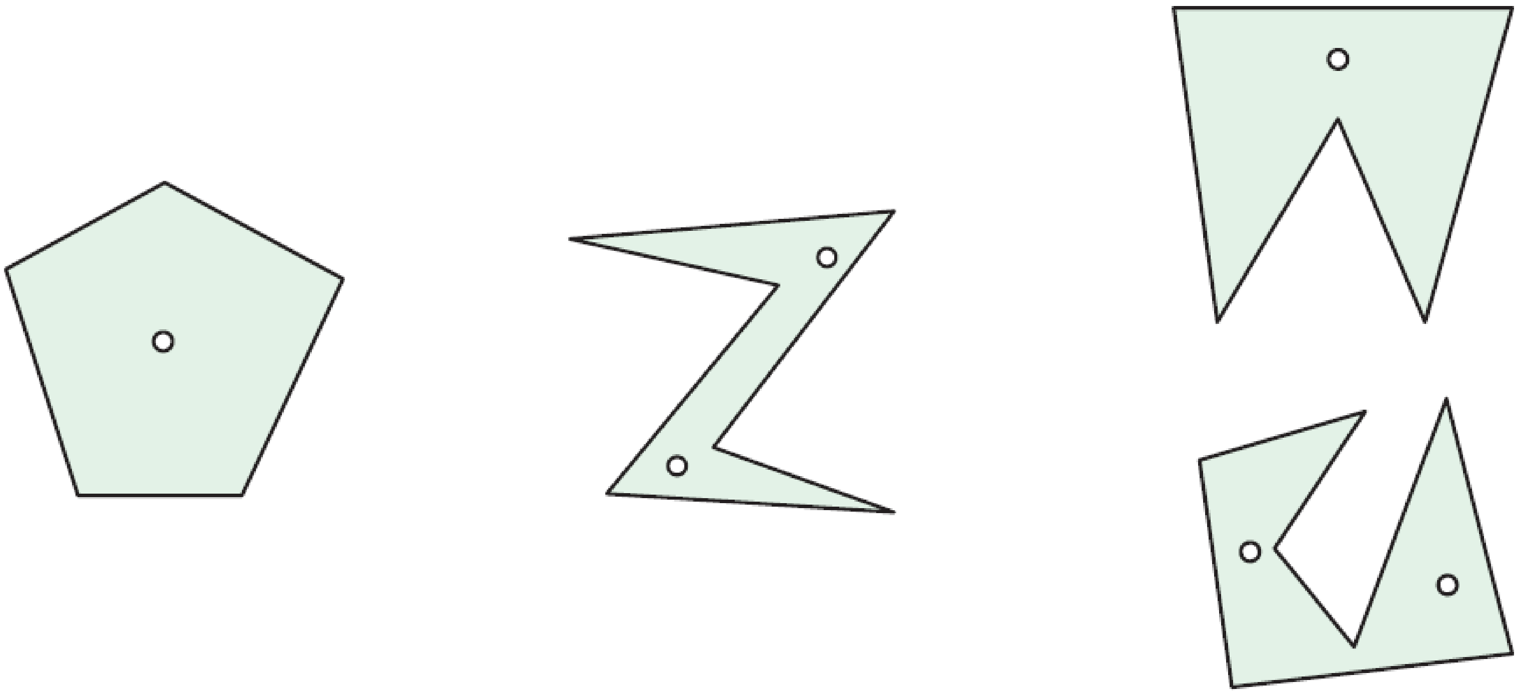
Assume: boundary visibility



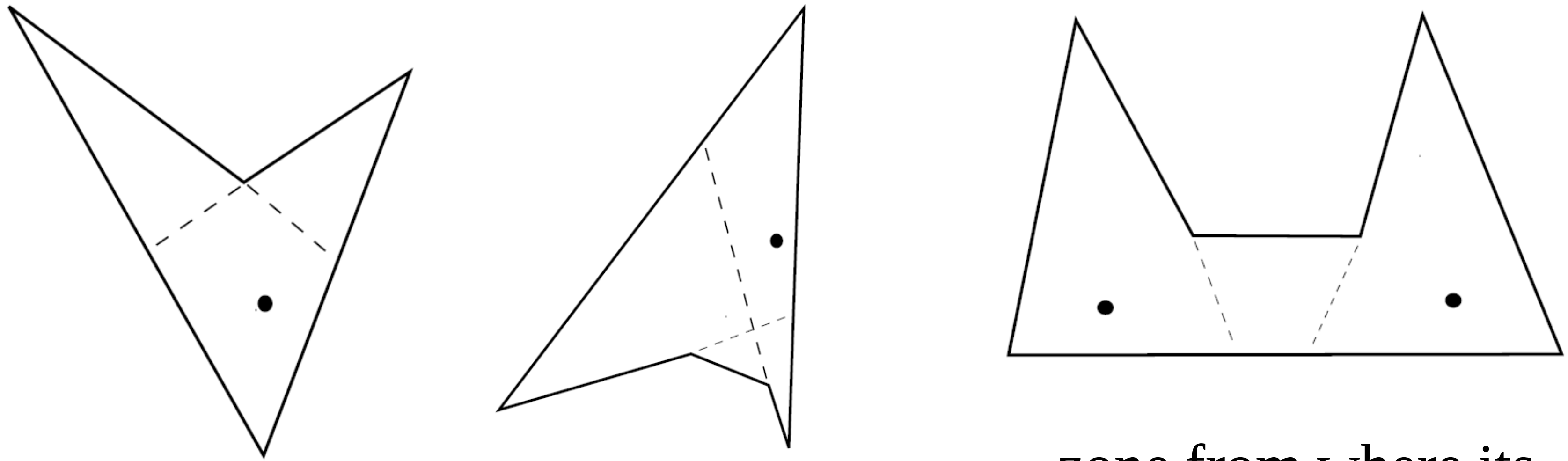


A convex gallery requires only one vertex guard (or anywhere in the interior) ;
A star-shaped polygon needs only one "interior" guard

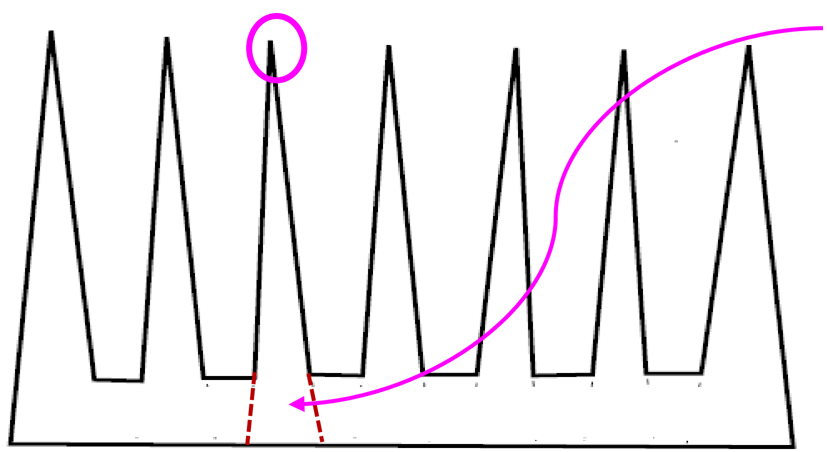
Art Gallery Problem: Area Visibility



Art Gallery Problem: Examples



$n = 21$



zone from where its prong is visible

There are seven prongs

7-prong comb; each wedge needs at least one guard

$\Rightarrow$ Minimum #guards = 7 = $\lfloor n/3 \rfloor$

Art Gallery Problem: Examples

Question (V. Klee, 1973)
How many guards does an *n*-sided polygonal gallery need? Is the comb the worst case?

Theorem (V. Chvátal, 1975)
Every *n*-sided gallery (simple polygon) needs at most $\lfloor n/3 \rfloor$ vertex guards (sufficiency), and there are situations (comb-shaped) where these many guards are required (necessary).

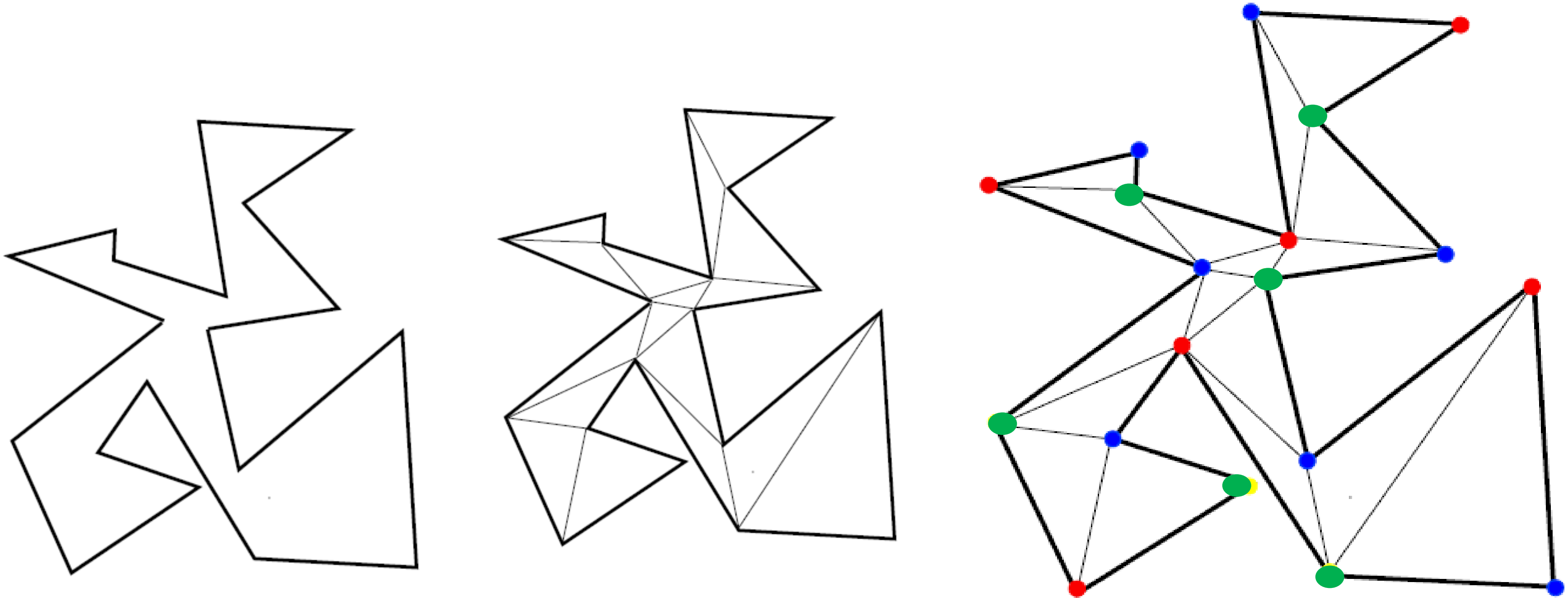
Art Gallery Problem: Examples

Theorem (V. Chvátal, 1975)

$\lfloor n/3 \rfloor$ vertex guards are always sufficient and sometimes necessary (e.g., in comb-shaped polygons) to guard a simple polygon with n vertices

Proof by Steve Fisk (JCT, B, 1978)

Triangulated polygon $\Rightarrow$ maximal outerplanar graph $\Rightarrow$ 3-colorable

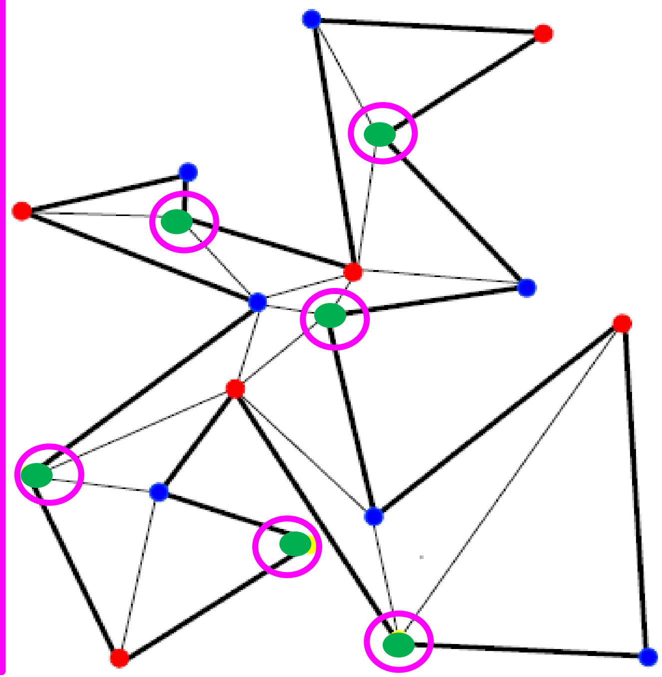


Theorem (V. Chvátal, 1975) – Fisk's Proof 1978

$\lfloor n/3 \rfloor$ vertex guards are always sufficient and sometimes necessary to guard a simple polygon with n vertices

Triangulated polygon ⇒ maximal outerplanar graph ⇒ 3-colorable

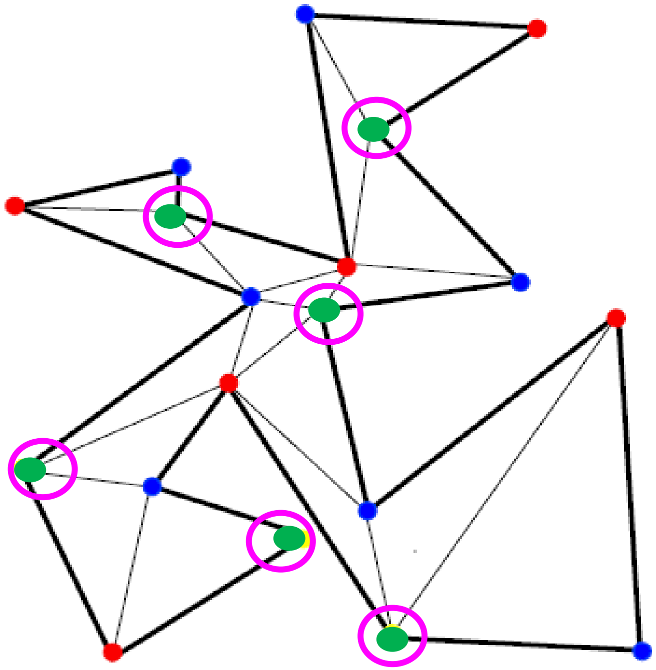
In a triangle, every vertex receives a different color;
Since each of the n vertices is assigned to a color out of three, by Pigeon-Hole Principle, there exists at least one color which has been assigned to at most $\lfloor n/3 \rfloor$ vertices.
Post guards at these vertices. This subset of vertices is also a min dominating set.



In this example, n = 19; pick up the color that is least used, e.g., green (# =6) ⇒six guards posted therein are sufficient.

Complexity of Fisk's Method

Given a simple polygon, the vertices at which the $\lfloor n/3 \rfloor$ cameras should be placed to guard the polygon, can be determined in $O(n)$ time



Placement of Guards

Note that Fisk's proof gives an explicit placement of guards on the vertices (corners) of the polygon.

Vertex guard means the guard should be positioned on a vertex.

Other variations on guards include:

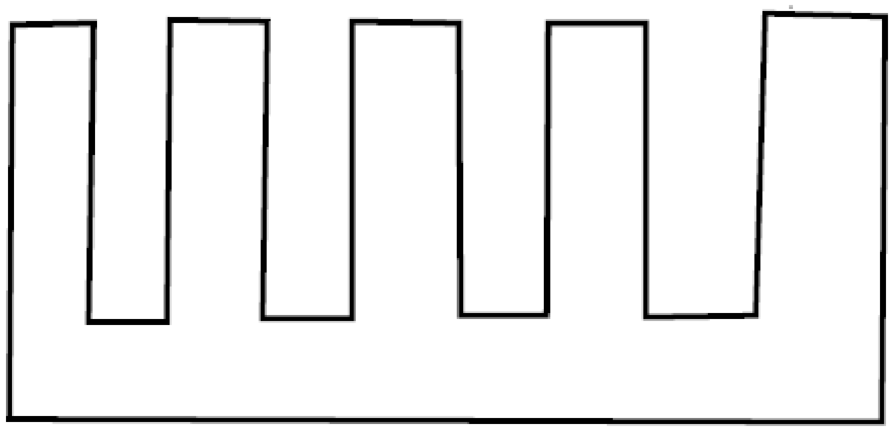
Point guard means the guard can be placed anywhere in the polygon.

Edge guard means the guard can be placed anywhere along an edge of the polygon.

Mobile guard means the guard is allowed to patrol along a line segment lying in the polygon.

Orthogonal (Isothetic) Gallery

$n = 20$

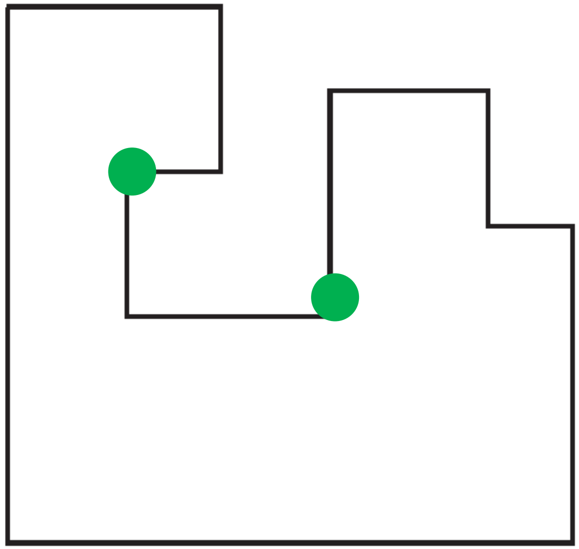


Theorem (O'Rourke 1983)

Every n-sided simple orthogonal polygon can be guarded with at most $\lfloor n/4 \rfloor$ vertex guards (sufficiency), and there are situations (comb-shaped) where these many guards are required (necessary).

Proof: Self study

Orthogonal (Isothetic) Gallery



reflex vertices = 3

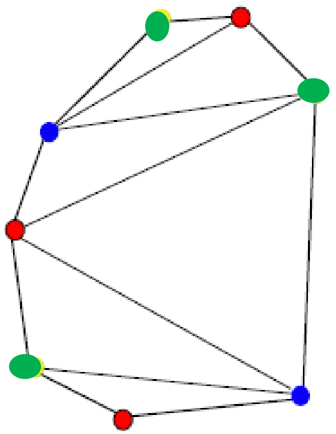
Theorem

Any orthogonal polygon with r reflex vertices can always be guarded by $\lfloor r/2 \rfloor + 1$ vertex guards

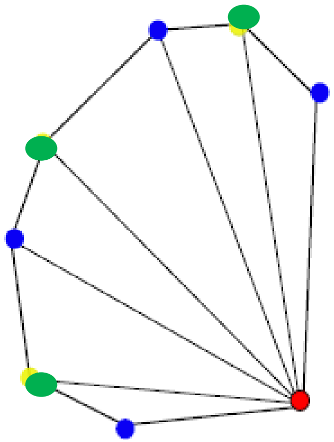
Proof: Self study

Art Gallery Problem

How good is the triangulation method?



#least-used colors
(blue) $\Rightarrow$ two guards

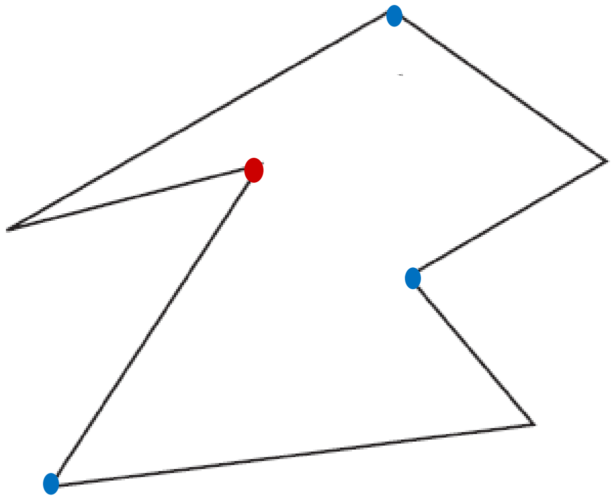


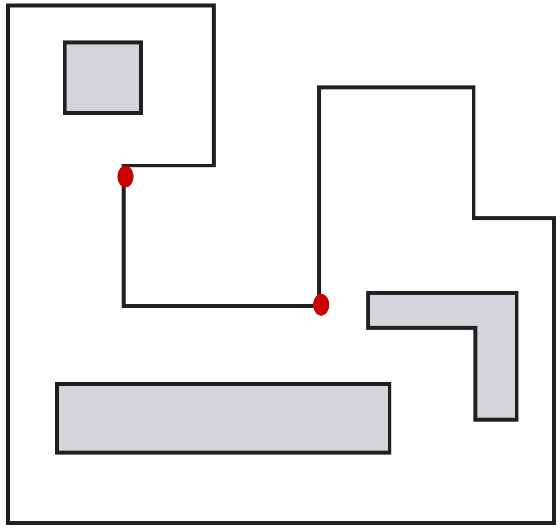
#least-used colors
(red) $\Rightarrow$ one guard

Provides sufficiency!

Theorem: The minimum vertex, point, and edge guard problem for polygons with or without holes including orthogonal polygons, is NP-hard

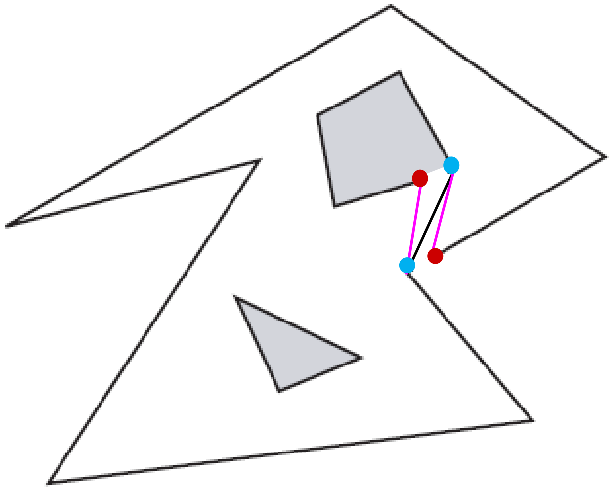
Polygons with holes





Guards cannot see through holes

Polygons with holes

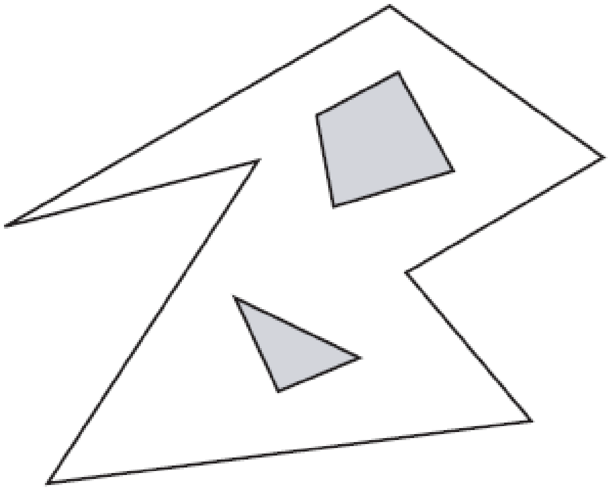


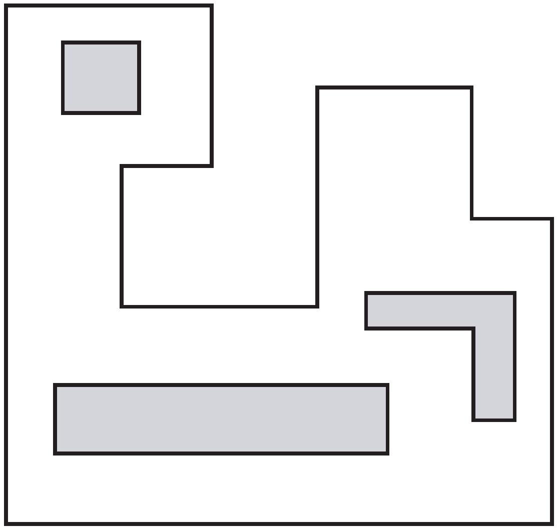
modified polygon P' will have $n + 2h$ vertices;
P' is devoid of any hole;
apply Chvátal's theorem

Guards cannot see through holes

Theorem:
Any polygon with n vertices and h holes can always be guarded by $\lfloor (n + 2h)/3 \rfloor$ vertex guards
[n vertices include vertices on holes as well]

Polygons with holes



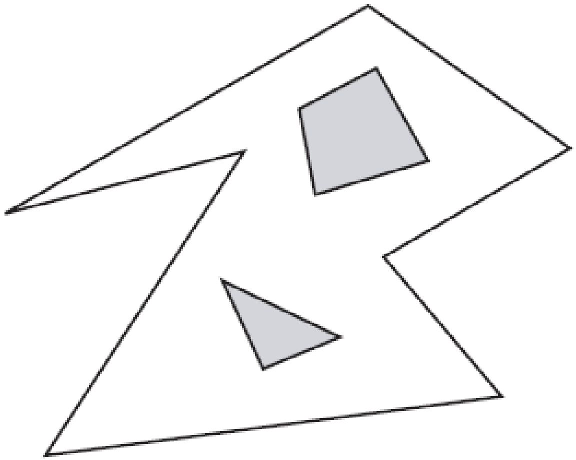


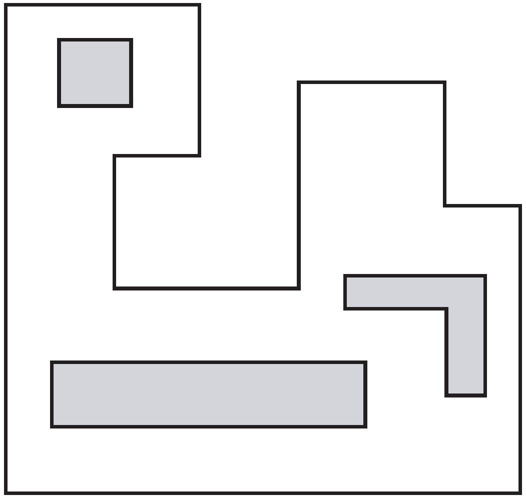
Guards cannot see through holes

Shermer's conjecture (1982)
Any polygon with n vertices and h holes can be guarded by
$\lfloor (n + h)/3 \rfloor$ vertex guards

The conjecture is proved for $h = 1$; still open for $h > 1$

Polygons with holes



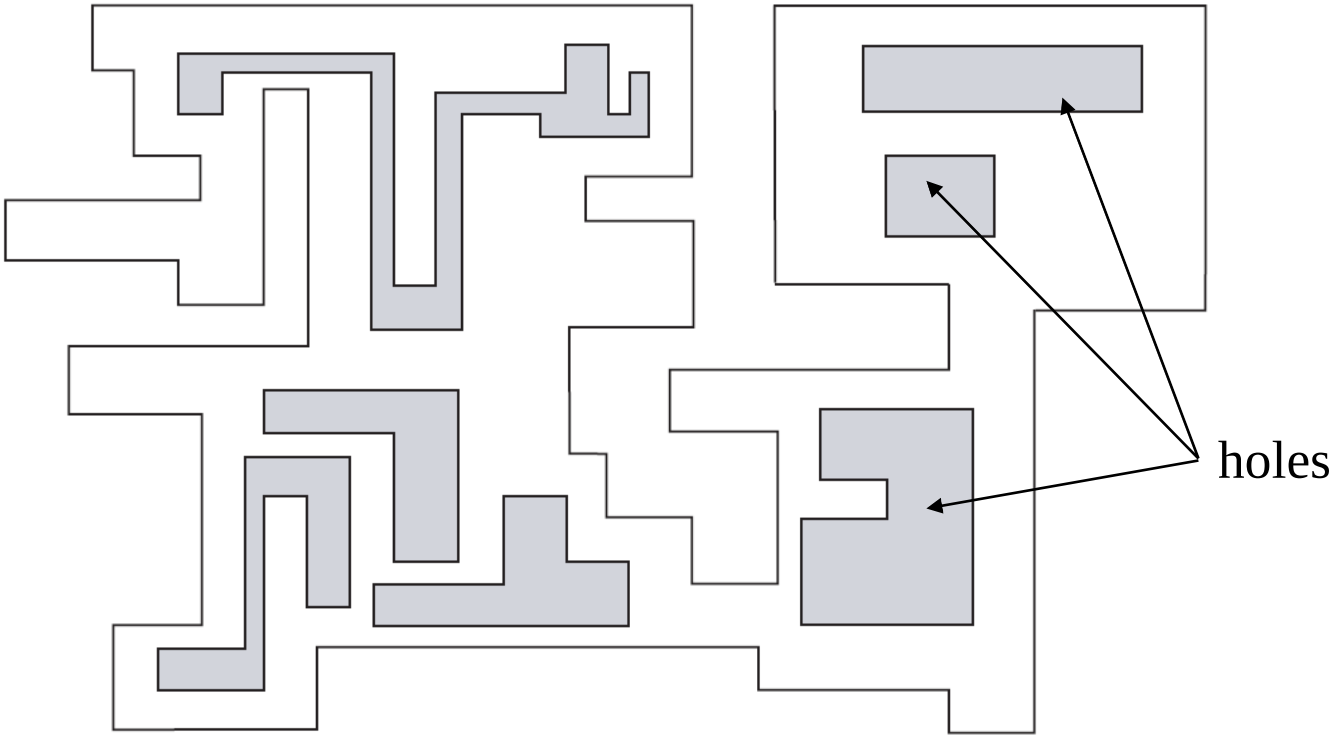


Theorem (O'Rourke, 1987): Any orthogonal polygon with n vertices and h holes can be guarded by $\lfloor (n + 2h)/4 \rfloor$ vertex guards

Shermer's Conjecture (1982): Any orthogonal polygon with n vertices and h holes can be guarded by $\lfloor (n + h)/4 \rfloor$ vertex guards

The conjecture is proved for $h = 1$ (Agarwal, 1984); still open for $h > 1$

Practice Problem: Polygons with holes



Show a solution with minimum number of vertex guards that are sufficient to watch the interior of the above orthogonal polygon with seven holes