SOFR Bootstrapping
Modeling Methodologies and Issues
(w/ Python and Excel Replicas
of Bloomberg SOFR @ GitHub)

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Abstract

Following the 2008 financial crisis, interest rate market experienced major changes in the ways Libor rate was treated. Since Libor is not a risk free rate, the dual curve bootstrapping (Libor-OIS) has been introduced. The term risk premium (e.g., 3m6m Libor basis) has been handled via newly introduced multi-curve framework. The discovery of the Libor rate manipulations back in 2007 broke Libor's back. Citigroup introduced Nybor in 2008 as an alternative to Libor. Following the Libor crisis, the Alternative Reference Rates Committee (ARRC) has been established to "ensure a successful transition from U.S. dollar (USD) Libor to a more robust reference rate, its recommended alternative, the Secured Overnight Financing Rate (SOFR)." This rippled through other countries with introductions of SONIA[1] (for GBP Libor), ESTER (for Euribor), etc. Although there are a lot of modeling similarities between Libor and SOFR rate bootstrapping, there are a number of differences. In this paper we discuss bootstrapping methodologies along with some modeling differences and new modeling considerations.

1 SOFR instruments

SOFR futures celebrated two-year anniversary this year. The average daily volume grew from under 2,000 contracts back in May of 2018 to more than 60,000 contracts in February of 2020 [1]. SOFR futures contract specifications can be found on CME page here. In our formulas, we will use rates measured in percentage points, e.g. 4% rate will be recorded as 4.00 rather than .04. Leaving out convexity adjustments, the 1-month SOFR (SR1) and 3-month SOFR (SR3) quotes can be represented as follows.

$$100 - SR1 \approx \frac{1}{N_m} \sum_{i=1}^{N_m} SOFR_{t_i}^{Fwd},$$

where N_m represents a number of days in a contract month, t_i represents time corresponding to contract reset dates.

$$100 - SR3 \approx \frac{360}{N_Q} \left(\prod_{i=1}^{Q} \left(SOFR_{t_i}^{Fwd} \frac{dt_i}{360} + 1 \right) - 1 \right)$$

where N_Q is the number of days in a reference quarter, $dt_i = t_{i+1} - t_i$ is the the forward rate tenor.

SOFR swaps are also used for bootstrapping as in case of Libor. SOFR swaps are traded over-the-counter (OTC). Swap contract specifications can be found in [2]. They are consistent with Libor swap specifications except for discounting. SOFR swaps use SOFR for both discounting and projections, while Libor swaps use Libor for projections and OIS for discounting. It is worth noticing that LCH is planning on the OIS-to-SOFR discounting transition starting on October 16th of 2020. The impact of this transition on market risk modeling is discussed in [3].

[1] SONIA was actually introduced in March 1997. Bank of England took control of SONIA in 2016 to support the risk free rate transition in sterling markets.

2 What tenor?

What do SOFR futures quotes represent? In case of Eurodollar futures (EDF), the EDF quote represents a 100 - 3-month termed forward rate.

$$100 - EDF \approx L(T_0, T_1)$$

From the SR1 and SR3 formulas, it is easy to see that $1 - SR1$ represents an average annualized overnight rate and $1 - SR3$ represents an overnight rate compounded over a 3-month period and annualized (using scaling factor $360/N_Q$). Hence, SR1 futures represent overnight rates rather than 1-month rates. Contrarily, SR3 futures represent daily compounded 3-month rate. In both cases, the underlying rate's tenor is overnight.

It is well know that EDF quotes that represent 3-month rates can't be used directly to derive 1-month rates. The tenor basis spread need to be used instead. So, can SOFR futures quotes be used for overnight rate bootstrapping? In actuality, the SOFR futures market participants express their beliefs about future values of the overnight rates during a certain period (1-month or 3-month) by the means of SOFR futures quotes. They are not necessarily expressing their beliefs about 1-month or 3-month SOFR rates. That's why these instruments can be used for overnight curve bootstrapping.

3 Realized, pre-bootstrapped, and unknown rates

A unique feature of SOFR futures, is that they continue trading after the beginning of the reference period. Therefore, some of the overnight rates are realized. For current time t, this can be expressed as follows.

$$100 - SR_1 = \frac{1}{N_m} \sum_{i=1}^{N_m} (SOFR^{Rzd}_{t_i<t} + SOFR^{Fwd}_{t_i \geq t}),$$

where $SOFR^{Rzd}_{t_i<t}$ represents realized overnight SOFR rates.

Similarly, the SR3 formula can be re-written in the following manner.

$$100 - SR_3 = \frac{360}{N_Q} \left(\prod_{i=1}^{Q} \left(SOFR^{Rzd}_{t_i<t} \frac{dt_i}{360} + 1 \right) \prod_{i=1}^{Q} \left(SOFR^{Fwd}_{t_i \geq t} \frac{dt_i}{360} + 1 \right) - 1 \right)$$

Some of the unrealized SOFR rates from the front 3-month contract overlap with unrealized SOFR rates from the one or several 1-month contracts. Hence, some of these rates will be bootstrapped using those 1-month futures. Let's denote the first date in the front 3-month contract that does not overlap with the 1-month contract/s by t'. Then we can rewrite the SR3 quote in the following form.

$$1 - SR_3 = \frac{360}{N_Q} \left(\prod_{i=1}^{Q} \left(SOFR^{Rzd}_{t_i<t} \frac{dt_i}{360} + 1 \right) \prod_{i=1}^{Q} \left(SOFR^{Btd}_{t \leq t_i < t'} * \frac{dt_i}{360} + 1 \right) \prod_{i=1}^{Q} \left(SOFR^{Fwd}_{t_i \geq t'} * \frac{dt_i}{360} + 1 \right) - 1 \right),$$

where $SOFR^{Btd}_{t \leq t_i < t'}$ denote unrealized SOFR rates that are pre-bootstrapped using 1-month SOFR futures.

4 Bootstrapping implementation

We have implemented a simple bootstrapping methodology in Python and Excel. Both implementations are freely available at GitHub. For the sake of simplicity we have not included convexity adjustments in the code. The size of the convexity adjustments is small and, therefore, can be ignored without significant loss in replication accuracy. The raw method, the so-called constant forward interpolation method, is coded in SOFR-v2.ipynb. The input files are 1m_futures_data.csv, 3m_futures_data.csv, and rzd_rates.csv. An excel version of this code is captured in SOFR-BBG.xlsm. In the next section, we will describe this simple bootstrapping methodology in details. We have also included a natural cubic spline implementation that is coded in Cubic Splice-SOFR-Git.py. The raw method is consistent with Bloomberg SOFR curve with raw interpolation method.

5 SOFR curve building

Let's start with a high level description of the assumptions and ideas used in our Python code and Excel spreadsheet.

The curve building instrument types include

- SOFR spot rate
- 1-month SOFR futures (SR1)
- 3-month SOFR futures (SR3)
- SOFR swaps (SWP)

It is important to remember that 1M and 3M SOFR futures conventions differ.

The idea is to bootstrap SOFR curve that will utilize

- spot SOFR rate
- 1-month SOFR futures expiring prior to the expiration of the front 3M SOFR futures
- 3-month futures up to 21 months into the future
- SOFR swaps from 2 years and on

Here are some conventions associated with different instruments

- SOFR 1-month futures rates are simple rates
- SOFR 3-month futures rates are compounded daily
- SOFR swap rates are based on the daily resets and annual payments
- SOFR forward rates are recorded in % units (e.g. 0.04% is equivalent to .0004 or 4 bps)

Data inputs In our example, we will bootstrap SOFR forward curve as of 04/30/2020. We will be using realized SOFR rates along with prices of the instruments (such as SR1, SR3, etc.) to bootstrap the unknown SOFR rates. To be able to use this code for a different date, the user will need to modify the inputs accordingly.

In this paper we will use the following notation for SR1 and SR3 futures. The front SR1 and SR3 futures will be denoted as SR1-0 and SR3-0. The other ones, will be sorted/denoted according to their maturity dates SR3-1, SR3-2, etc.

In our case, the front SR1-0 futures expires on 04/30/2020 and is equivalent to the SOFR spot rate. The second SR1 futures (SR1-1) expires on 5/29/2020. And this is the only SR1 futures that we will use since the front SR3-0 futures expires on 06/16/2020. There will be 6 more SR3 futures (SR3-1 through SR3-6) with SR3-6 futures maturing on 12/15/2021.

We will also use SOFR swap rates starting from 2 years through 40 years (SWP-2Y through SWP-40Y).

5.1 Step 1 - spot SOFR rate

Let's define forward SOFR curve and record the spot rate that was observed in the market. This SOFR rate can be found on the FRB page here.

5.2 Step 2 - SR1 futures

SR1 quotes can be pulled from CME page here. SOFR futures are traded at CME and include 1M and 3M tenors. The 3M futures and traditional Eurodollar (ED) futures share similar reference quarters but different re-setting convention. SR1 pricing formula can be seen below. As stated earlier, our coding does not take into consideration convexity adjustments. Hence, the formulas that are given below in this section are approximate.

$$1 - SR_1 = \frac{1}{N_m} \sum_{i=1}^{N_m} SOFR_{t_i}^{Fwd},$$

where N_m represents a number of days in a contract month. Since we are using constant forward assumption, all $SOFR_{t_i}^{Fwd}$ rates are assumed to be the same and we obtain the following formula.

$$SOFR_{t_i}^{Fwd} = N_m(1 - SR_1)$$

SOFR futures pricing is straightforward, however, the major difference between ED futures and SOFR futures is on the front contract. The SOFR realized rates up to the current time t in the reference month(SR1-0)/quarter(SR3-0) need to be considered in the pricing formula.

$$1 - SR_1 = \frac{1}{N_m} \sum_{i=1}^{N_m} (SOFR_{t_i<t}^{Rzd} + SOFR_{t_i\geq t}^{Fwd})$$

On 04/30/2020, the SR1-0 futures expires and the last forward rate is determined. Hence, we will simply use the spot SOFR rate without bootstrapping this info from the expiring front SR1-0 futures. However, we use the next SR1-1 futures to bootstrap rates through May of

2020.

The spot SOFR rate 0f .04% (4bps) implies SR1-0 price of 99.98. The actual market quote was 99.981 which leads to the SOFR spot of .029% (3bps). We will use the market spot SOFR rate of .04%.

5.3 Step 3 - SR3 futures

SR3 quotes can be pulled from CME page here. The first 3M SOFR futures (SR3-0) that we will use for bootstrapping starts on 03/18/2020, the last one we will be using ends on 12/15/2021. As discussed earlier, it matures at least 3 month prior to the start of the 2-year SOFR swap.

The first futures price can be calculated as follows.

$$1-SR_3 = \frac{360}{N_Q}\left(\prod_{i=1}^{Q}\left(SOFR^{Rzd}_{t_i<t}\frac{dt_i}{360}+1\right)\prod_{i=1}^{Q}\left(SOFR^{Btd}_{t\le t_i<t'}\frac{dt_i}{360}+1\right)\prod_{i=1}^{Q}\left(SOFR^{Fwd}_{t_i\ge t'}\frac{dt_i}{360}+1\right)-1\right),$$

where N_Q is the number of days in a reference quarter. Notice that dt_i is not necessarily 1/360 due to weekends and holidays. For example, on Friday it will be 3/360. As one can see in the formula above, there are three types of SOFR forward rates - realized rates (that were determined in the past $t_i < t$) and unrealzied rates (spot rate at $t_i = t$ and future rates at $t_i > t$). Some of the unrealized SOFR rates have been bootstraped in the earlier steps using spot SOFR and SR1 (at $t \le t_i < t'$). We will use those bootstrapped values.

Starting from the second futures and on (SR1-1 through SR1-6), one does not need to deal with the realized and pre-bootstrapped rates. Hence, their prices can be calculated using a simpler formula.

$$100 - SR_3 = \frac{360}{N_Q}\left(\prod_{i=1}^{Q}\left(SOFR^{Fwd}_{t_i}\frac{dt_i}{360}+1\right)-1\right)$$

5.4 Step 4 - SOFR Swaps

For a swap starting at time $t = T_0$ and maturity T, resets occur daily at times t_i, while payments occur annually at times T_j. The swap rate K is a solution of the following equation.

$$\sum_j\left(\prod_i\left(1+SOFR_{T_{j-1}<t_i\le T_j}\frac{dt_i}{360}\right)-1\right)\frac{360}{dT_j}P(0,T_j) = \sum_j KP(0,T_j),$$

$SOFR_{T_{j-1}<t_i\le T_j}$ represents all the bootstrapped and forward SOFR rates within year T_j ($t_i \in (T_{j-1}, T_j]$), $dT_j = T_j - T_{j-1}$. It is easy to see that $\sum_{T_{j-1}<t_i\le T_j} t_i = T_j$.
As before, one needs to keep track of the rates that have been bootstrapped earlier. For the first two-year SOFR swap (SWP-2Y), all rates in year one as well as some rates from year 2 have been bootstrapped earlier. We will need to calculate those first.

A 2-year SOFR swap can be viewed as a sum of present values of the year 1 and year 2 cash flows (floating leg minus fixed leg). The year 1 value is pre-determined due to the pre-bootstrapped rates. When we run optimizer, the unknown rate should make the value of the year 2 equal to the negative of the year 1 value. So that the total value of the swap is equal to zero. This is exactly what is happening in the code. The year 1 and year 2 values offset each other.

In the following steps, we will use simplified SOFR swap pricing formula assuming no weekends. In other words, $dt_i = 1$ and $dT_j = 365$. We then obtain.

$$\sum_j \left(\left(1 + SOFR_{T_{j-1} < t_i \leq T_j} \frac{1}{360} \right)^{365} - 1 \right) \frac{360}{365} P(0, T_j) = K \sum_j P(0, T_j)$$

Let's denote floating leg cash flow corresponding to year T_j as $Flt(T_j)$, then the formula can be re-written as follows.

$$\sum_j Flt(T_j) P(0, T_j) = K \sum_j P(0, T_j)$$

Now, it is easy to see that in order to find $Flt(T_{j+1})$, one needs to solve the following equation.

$$\sum_j Flt(T_j) P(0, T_j) + Flt(T_{j+1}) P(0, T_j) P(T_j, T_{j+1}) = K \sum_j P(0, T_j) + K P(0, T_j) P(T_j, T_{j+1})$$

Notice, that swap rate K will change when moving from one tenor to another.

6 Convexity adjustment complications

Let's use $P(t, T)$ to denote a time t value of \$1 obtained at time T, and $F(t; T_1, T_2)$ denoting the forward rate between T_1 and T_2 estimated at time t. Note, that $F(T_1; T_1, T_2)$ can also be interpreted as futures rate or future spot rate. For the sake of simplicity, we will use $F(T_1, T_2)$ notation for $F(T_1; T_1, T_2)$. The SR3 futures quote allows one to calculate the futures rate.

$$1 - \frac{SR3}{100} = \frac{1}{\tau} \mathbb{E}_Q \left[\frac{1}{\frac{1}{1 + \tau F(T_1, T_2)}} - 1 \right] = \mathbb{E}_Q [F(T_1, T_2)]$$

It is worth noticing that futures implied rate $\mathbb{E}_Q[F(T_1, T_2)]$ is different from the forward rate $F(t; T_1, T_2)$ estimated at time t.

$$1 - \frac{SR3}{100} = \mathbb{E}_Q [F(T_1, T_2)] \neq \frac{1}{\tau} \left[\frac{\mathbb{E}_Q [P(t, T_1)]}{\mathbb{E}_Q [P(t, T_2)]} - 1 \right] = \frac{1}{\tau} \left[\frac{P(t, T_1)}{P(t, T_2)} - 1 \right] = F(t; T_1, T_2)$$

In other words, the forward rate is implied from the expected discount factors (under risk neutral measure) and is different from the expected forward rate (under risk neutral measure as well). This is due to a non-linear relationship between forward rates and discount factors. In simple terms, this concept can be expressed using the following inequality.

$$\mathbb{E}_Q \left[\frac{1}{1 + \tau F(T_1, T_2)} \right] > \frac{1}{1 + \tau \mathbb{E}_Q [F(T_1, T_2)]}$$

The SR3 convexity adjustment can be defined as follows.

$$CA_{SR3} = \mathbb{E}_Q[F(T_1, T_2)] - F(t; T_1, T_2)$$

Once the convexity adjustment is calculated, we can plug it to obtain accurate rather than approximate forward rate.

$$1 - \frac{SR3}{100} - CA_{SR3} = 1 - \frac{SR3}{100} - \left[1 - \frac{SR3}{100} - F(t, T_1, T_2)\right] = F(t, T_1, T_2)$$

There are two complications associated with convexity adjustments for SOFR futures. One is due to the fact that SOFR futures can be traded beyond forward start date T_1. As we will show later, this leads to additional term associated with future rates uncertainty between T_1 and T_2[2]. This uncertainty does not exist in case of EDF, since forward rates are fixed at time T_1.

The second complication is related to the fact that SR1 does not represent a 1-month SOFR rate. It represents an average overnight rate. Therefore, the total convexity adjustment in this case, can be viewed as a sum of overnight convexity adjustments. Let's discuss this issues in details.

6.1 Convexity period extension

As discussed in the previous section, the futures quote does not give us a forward rate and one needs to calculate convexity adjustment. In order to calculate convexity adjustment, one needs to make an assumption about distribution of forward rates. In this paper, we will assume Ho-Lee model. In this model, the short rate process (r_t) can be represented as follows.

$$dr_t = \theta_t dt + \sigma dW_t,$$

where W_t is a Wiener process.

Under the risk neutral measure Q we have.

$$P(t, T) = \mathbb{E}_Q\left[e^{-\int_t^T r_s ds} | \mathcal{F}_t\right]$$

For the sake of simplicity, we will introduce another process $\hat{R}_t$, such that.

$$\ln \hat{R}_t = \ln\left[\frac{P(t, T_1)}{P(t, T_2)} \frac{1}{\tau}\right]$$

Using definition of $\hat{R}_t$ we get.

$$d \ln \hat{R}_t = d \ln\left[\frac{P(t, T_1)}{P(t, T_2)} \frac{1}{\tau}\right] = d \ln\left[\frac{P(t, T_1)}{P(t, T_2)}\right]$$

Applying Ito's formula, and using $dP(t, T) = P(t, T)(r_t dt - \sigma(T - t)dW_t)$ expression we can derive.

[2]Without loss of generality we will use T_1 and T_2 notation to represent the start and end points of any SOFR futures.

$$
\begin{aligned}
d\ln[\hat{R}_t] = d\ln\left[\frac{P(t,T_1)}{P(t,T_2)}\right] &= d\ln[P(t,T_1)] - d\ln[P(t,T_2)] \\
= \frac{dP(t,T_1)}{P(t,T_1)} - \frac{dP(t,T_2)}{P(t,T_2)} + \frac{1}{2}&\left[-\frac{(dP(t,T_1))^2}{P^2(t,T_1)} + \frac{(dP(t,T_2))^2}{P^2(t,T_2)}\right] \\
&= (r_t dt - \sigma(T_1-t)dW_t) - (r_t dt - \sigma(T_2-t)dW_t) \\
+\frac{1}{2}[-(r_t dt - \sigma(T_1-t)dW_t)^2 &+ (r_t dt - \sigma(T_2-t)dW_t)^2] \\
&= \sigma\tau dW_t + \frac{1}{2}\sigma^2[(T_2-t)^2 - (T_1-t)^2],
\end{aligned}
$$

where $\tau = T_2 - T_1$. Notice, that $P(t,T_1)$ is known for $t > T_1$ and can be viewed as a deterministic process $dP(t,T) = P(t,T)r_t dt$. Therefore, the formula above simplifies to the following form.

$$
d\ln[\hat{R}_t] = \frac{1}{2}\sigma^2(T_2-t)^2 dt + \sigma(T_2-t)dW_t
$$

This can be rewritten as follows.

$$
\ln(\hat{R}_{T_2}) - \ln(\hat{R}_t) = \frac{1}{2}\sigma^2\int_t^{T_2}(T_2-t)^2 dt + \sigma\int_t^{T_2}(T_2-t)dW_t
$$

In the latter case ($t > T_1$), the right hand side is normally distributed with mean $\frac{1}{2}\sigma^2(T_2-t)^2$ and variance $\sigma^2(T_2-t)^2$. Taking exponential and using Gaussian moment generating function we obtain.

$$
\mathbb{E}_Q[\hat{R}_{T_2}|\mathcal{F}_t] = \hat{R}_t \exp\left\{\int_t^{T_2}\sigma^2(T_2-t)^2 dt\right\} = \exp\left\{\frac{1}{3}\sigma^2(T_2-t)^3\right\}
$$

Using $\hat{R}_t = F(t;T_1,T_2) + \frac{1}{\tau}$, we obtain.

$$
\mathbb{E}_Q[\hat{R}(t,w)|F_t] = \mathbb{E}_Q[F(T_1,T_2)(w)|F_t] + \frac{1}{\tau} = \left(F(t;T_1,T_2) + \frac{1}{\tau}\right)\exp\left\{\frac{1}{3}\sigma^2(T_2-t)^3\right\}
$$

After moving $\frac{1}{\tau}$ to the right side and subtracting $F(t;T_1,T_2)$ from both side, we get.

$$
CA_{SR3} = \mathbb{E}_Q[F(T_1,T_2)|\mathcal{F}_t] - F(t;T_1,T_2) = \left(F(t;T_1,T_2) + \frac{1}{\tau}\right)\left(\exp\left\{\frac{1}{3}\sigma^2(T_2-t)^3\right\} - 1\right)
$$

This leads us to the following expression for the futures rate.

$$
\mathbb{E}_Q[F(T_1,T_2)|\mathcal{F}_t] = \exp\left\{\frac{1}{3}\sigma^2(T_2-t)^3\right\}F(t;T_1,T_2) + \frac{1}{\tau}\left(\exp\left\{\frac{1}{3}\sigma^2(T_2-t)^3\right\} - 1\right)
$$

Once again, when $t = T_1$ we obtain the following expression.

$$
\mathbb{E}_Q[F(T_1,T_2)|\mathcal{F}_{T_1}] = \exp\left\{\frac{1}{3}\sigma^2\tau^3\right\}F(T_1;T_1,T_2) + \frac{1}{\tau}\left(\exp\left\{\frac{1}{3}\sigma^2\tau^3\right\} - 1\right)
$$

For $t \leq T_1$ we can take expectation with respect to information $\mathcal{F}_{T_1}$ first and $\mathcal{F}_t$ second.

$$\mathbb{E}_Q[\mathbb{E}_Q[\hat{R}(t,w)|\mathcal{F}_{T_1}]|\mathcal{F}_t] = \exp\left\{\frac{1}{3}\sigma^2\tau^3\right\}\left(\mathbb{E}_Q[\hat{R}_{T_1}|\mathcal{F}_t] + \frac{1}{\tau}\right) + \frac{1}{\tau}\left(\exp\left\{\frac{1}{3}\sigma^2\tau^3\right\} - 1\right)$$

Let's derive expression for $\mathbb{E}_Q[\hat{R}_{T_1}|\mathcal{F}_t]$ for $t < T_1$. It has been shown earlier that.

$$d\ln[\hat{R}_t] = \frac{1}{2}\sigma^2[(T_2-t)^2 - (T_1-t)^2] + \sigma\tau dW_t$$

This can be rewritten as follows.

$$\ln(\hat{R}_{T_1}) - \ln(\hat{R}_t) = \frac{1}{2}\sigma^2\int_t^{T_1}(T_2-t)^2 - (T_1-t)^2 dt + \sigma\int_t^{T_1}\tau dW_t$$

Let's calculate the first integral on the right side, but let's rewrite the integrand first.

$$\begin{aligned}(T_2-t)^2 - (T_1-t)^2 &= (T_2^2 - 2T_2t + t^2) - (T_1^2 - 2T_1t + t^2) = T_2^2 - T_1^2 - 2T_2t + 2T_1t \\ &= \tau(T_2+T_1) - 2t\tau = \tau(T_2 - t + T_1 - t) = \tau(2T_2 - 2t - T_2 + T_1) = \tau(2(T_2-t) - \tau)\end{aligned}$$

Now we can easily integrate it.

$$\begin{aligned}\frac{1}{2}\sigma^2\tau\int_t^{T_1}2(T_2-u) - \tau du &= \sigma^2\tau\int_t^{T_1}(T_2-u) - \frac{1}{2}\tau du \\ &= \sigma^2\tau\left[T_2(T_1-t) - \frac{u^2}{2}\Big|_t^{T_1} - \frac{1}{2}\tau(T_1-t)\right] \\ &= \sigma^2\tau\left[(T_1-t)\left(T_2 - \frac{1}{2}\tau\right) - \frac{1}{2}(T_1^2 - t^2)\right] \\ &= \sigma^2\tau\left[(T_1-t)\left(T_2 - \frac{1}{2}\tau - \frac{1}{2}(T_1+t)\right)\right] \\ &= \frac{1}{2}\sigma^2\tau(T_1-t)(T_2-t)\end{aligned}$$

We get the following expression for $\ln(\hat{R}_{T_1})$.

$$\ln(\hat{R}_{T_1}) - \ln(\hat{R}_t) = \frac{1}{2}\sigma^2\tau(T_1-t)(T_2-t) + \sigma\tau(W_{T_1} - W_t)$$

The right hand side is normally distributed with mean $\frac{1}{2}\sigma^2\tau(T_1-t)(T_2-t)$ and variance $\sigma^2\tau^2(T_1-t)$. Taking exponential and using Gaussian moment generating function we obtain.

$$\mathbb{E}_Q[\hat{R}_{T_1}|\mathcal{F}_t] + \frac{1}{\tau} = \hat{R}_t\exp\left\{\frac{1}{2}\sigma^2\tau(T_1-t)(T_2-t) + \frac{1}{2}\sigma^2\tau^2(T_1-t)\right\}$$

Using the earlier formula for $\mathbb{E}_Q[\mathbb{E}_Q[\hat{R}(t,w)|\mathcal{F}_{T_1}]|\mathcal{F}_t]$ we obtain.

$$\begin{aligned}\mathbb{E}_Q[\mathbb{E}_Q[\hat{R}(t,w)|\mathcal{F}_{T_1}]|\mathcal{F}_t] = \exp\left\{\frac{1}{3}\sigma^2\tau^3\right\}&\left(\hat{R}_t\exp\left\{\frac{1}{2}\sigma^2\tau(T_1-t)(T_2-t) + \frac{1}{2}\sigma^2\tau^2(T_1-t)\right\}\right) \\ &+\frac{1}{\tau}\left(\exp\left\{\frac{1}{3}\sigma^2\tau^3\right\} - 1\right)\end{aligned}$$

Finally, using $\hat{R}_t = F(t;T_1,T_1) + \frac{1}{\tau}$ and after some rearranging we obtain the following expression for the convexity adjustment $CA_{SR3} = \mathbb{E}_Q[R_t|\mathcal{F}_t] - F(t;T_1,T_2)$.

$$CA_{SR3} = \left(F(t;T_1,T_2) + \frac{1}{\tau}\right)\left(e^{\frac{1}{3}\sigma^2\tau^3 + \frac{1}{2}\sigma^2\tau(T_1-t)(T_2-t) + \frac{1}{2}\sigma^2\tau^2(T_1-t)} - 1\right)$$

It is worth noticing that when dealing with Eurodollar futures rather than SOFR futures, there is no convexity adjustment for $t > T_1$ since the rate is fixed at time T_1. When $t < T_1$, the SOFR convexity has extra term $e^{\frac{1}{3}\sigma^2\tau^3}$ that accounts for the volatility over the $[T_1, T_2]$ time period.

6.2 Convexity additive property

Now we are moving to the SR1 futures. Using definition of SR1 futures and forward rate notation defined in the previous section, we can write SR1 futures price in the following form.

$$SR1 = \frac{1}{N_m}\sum_{i=1}^{N_m} F(t;t_i,t_{i+1})$$

Notice that for $t_i > t$, $F(t;t_i,t_{i+1}) = SOFR^{Fwd}_{t_i}$; and for $t_i \leq t$, $F(t;t_i,t_{i+1}) = SOFR^{Rzd}_{t_i}$. For unrealized rates, using results from the previous section, we get the following expression for the total convexity adjustment CA_{SR1} by adding up and averaging daily convexity adjustment $\mathbb{E}_Q[F(t_i,t_{i+1})|\mathcal{F}_t] - F(t;t_i,t_{i+1})$.

$$CA_{SR1} = \frac{1}{N_m}\sum_{i=1}^{N_m}\left(F(t;t_i,t_{i+1}) + \frac{1}{\tau}\right)\left(e^{\frac{1}{2}\sigma^2\tau(t_i-t)(t_{i+1}-t) + \frac{1}{2}\sigma^2\tau^2(t_i-t)} - 1\right)$$

In case of SR1, $\tau = t_{i+1} - t_i$ and it can vary depending on day of the week and holidays. It is also important to remember that SOFR futures are based on the ACT/360 day counting convention, therefore τ used for SOFR forward rates scaling and τ used in the expectation calculations above may differ.

7 SOFR term structure model calibration without options?

The SOFR option market is still in the development phase. Hence, the Ho-Lee model as well as any other term structure model can't be calibrated to the option instruments yet. The only option that is available at the moment is to calibrate these models under physical measure based on the historical data. Under Ho-Lee model, the short rate is normally distributed $N(\theta_t, \sigma^2)$. We will assume constant drift $\theta_t \equiv \theta$. The daily change in the SOFR rate then follows $N(\theta\tau, \sigma^2\tau)$ distribution. The MLE estimators are given by the following equations.

$$\hat{\theta} = \frac{1}{N}\sum_{i=1}^{N}\frac{\Delta SOFR_{t_i}}{\tau}$$

and

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{N}\frac{(\Delta SOFR_{t_i} - \hat{\theta}\tau)^2}{\tau}$$

8 To smooth or not?

The federal funds target rate is determined by a meeting of the members of the Federal Open Market Committee (FOMC) which occurs eight times a year. FOMC may also hold additional meetings and implement target rate changes outside of its normal schedule. The overnight rates are subject to the FOMC policy changes. Hence, one may argue that forward rates should not be smooth to reflect FOMC's discreet interventions. However, the market participants start pricing in anticipated changes into the futures ahead of time and the current market typically morphs as well. Therefore, the futures rates may start increasing prior to the FOMC meeting if the market participants all agree that FOMC will lift rates. So, the forward rate smoothness assumption is still reasonable. There is typically still a lot of uncertainty about timing and magnitude of the FOMC rate changes. Some of the changes are not anticipated, like the most recent rate cut due to COVID-19. All in all, potential FOMC rate changes have to be somehow accounted for during bootstrapping.

One of the ways to deal with the problem was described in [4]. The authors proposed to use two assumptions when taking into consideration the impact of FOMC changes on the rate - policy gradualism and six months jump window.

Let's describe the first assumption - policy gradualism. As the name suggests, this assumption assumes gradual rather than abrupt changes in rates. In simple terms, the SOFR-implied increase in rates follows the pattern that minimizes the absolute size of the largest individual jump. In other words, if two FOMC meetings are captured under one SOFR futures contract that is pointing to a 50 bps increase, one would assume two 25 bps increases during each FOMC meeting rather than one 50 bps increase. This approach does not eliminate rate jumps, but rather minimizes the size of the jumps. And in this regard, it is "consistent" (at least in spirit) with the smoothness assumption.

The six months jump window assumption suggests that no jumps will occur beyond 6-month point (starting from the current time). The authors argue that such distant projections are not reliable. In this case, the rates will increase naturally and the regular smoothness assumptions will apply. Although the probabilities of future distant increases can be high, the timing is hard to predict. Alternatively, one could utilize the dot plots to make assumptions beyond 6-month point.

9 Conflicting rates?

It is a common issue, when different instruments imply different rates for the same time period. For example, three consecutive 1-month SOFR futures may disagree (in terms of implied rates) with the 3-month SOFR futures. This may occur due to a number of different reasons - liquidity, dealer pricing, execution type/platform, etc. In this case, there are two approaches to close the gap - instrument prioritization and pricing error minimization.

One way to deal with this problem is to impose prioritization rules. For example, one may prefer a rate implied from the 3-month SOFR futures. This can be done by using 3-month futures and assuming interpolated rate within the futures time period or implying the rates from the first two 1-month contracts and the 3-month contract. In the latter case, one takes into consideration information implied from some, but not all 1-month futures.

Alternatively, one can try minimizing the pricing error for both instruments. In this case, it is impossible to find a rate that will allow one to price 1-month and 3-month contracts correctly. However, one would be able to price both of them closely assuming that different rates are not far apart.

The selection of the "right" methodology may depend on the usage. For example, for a trader using 3-month futures as a hedge, building the curve that prioritizes 3-month rate over 1-month rate might be the right approach. However, from the risk management perspective, this may lead to the heightened risk due to potential mispricing that currently exists in the market.

10 Historical rates proxy

Some of the models (e.g., prepayment models) are built on the Libor rate and will need to transition to the SOFR rate. These models typically utilize historical information. If replacing the Libor rate with SOFR rate, the existing historical data time series for SOFR rate may not be sufficient. Hence, there is a need for the proxy rate. FRB publisjed a paper [5] that discusses this issue in details.

The author argues that historical SOFR rates can be approximated using survey rate and DTCC's GCF rate in the following manner.

$$SOFR = SurveyRate + .38(GCF - SurveyRate - .05)$$

Please, refer to the paper for further details.

References

[1] CME, *SOFR Futures Celebrate Two-Year Anniversary*, May 13, 2020. Available here.

[2] CME, *OTC SOFR Swaps Clearing*, Q4 2018. Available here.

[3] Sun, P., *SOFR Discounting Analysing the Market Impact*, Journal of Risk, February 19, 2020.

[4] Heitfield, E., and Park, Y., *Inferring Term Rates from SOFR Futures Prices*, 2019. Available here.

[5] Bowman, D., *Historical Proxies for the Secured Overnight Financing Rate*, 2019. Availabe here.