W203: Home Work 4 Suhas Gupta

Suhas Gupta

Question 1:

(a)

The outcomes of a coin toss follows a binomial distribution with p.m.f given by:

$$b(x;n,p) = \begin{cases} \binom{n}{k} p^k (1-p)^k & , \quad x \in I \\ 0 & , \quad otherwise \end{cases}$$

where n: number of trials

p: probability of getting heads

$$\rightarrow p = 0.5$$

$$1 - p = 0.5$$

Let h(X) denote the function that represents payout.

Expectation of earnings,

$$E(X) = \sum_{0}^{3} p(x)h(x)$$

$$= p(0)(0) + p(1)(2) + p(2)(4) + p(3)h(3)$$

$$= (1/8)(0) + (3/8)(2) + (3/8)(4) + (1/8)h(3)$$

$$But,\ E(X) = 6(given)$$

Solving for h(3) gives,

$$\rightarrow h(3) = 30$$

Expected earnings from getting 3 heads = **$30**

(b)

$$F(X) = P(X \leq x) = \sum_{y: y \leq x} P(y)$$

$$F(0) = P(y \leq 0) = \sum_{y: y \leq 0} P(y)) = \frac{1}{8}$$

$$F(1) = P(y \leq 1) = \sum_{y: y \leq 1} P(y)) = \frac{1}{2}$$

$$F(2) = P(y \leq 2) = \sum_{y: y \leq 2} P(y)) = \frac{7}{8}$$

$$F(3) = P(y \leq 3) = \sum_{y: y \leq 3} P(y)) = 1$$

Expression for cummulative probability F(Y):

$$F(Y) = \begin{cases} 0 & , \quad y < 0 \\ 1/8 & , \quad 0 \le y < 1 \\ 1/2 & , \quad 1 \le y < 2 \\ 7/8 & , \quad 2 \le y < 3 \\ 1 & , \quad 3 < y \end{cases}$$

Question 2

p.d.f:

$$f(l) = \begin{cases} 0 & , \quad l \le 0 \\ 1/2 & , \quad 0 \le l < 2 \\ 0 & , \quad 2 < l \end{cases}$$

(a)

$$F(l) = P(L \le l) = \int_{-\infty}^{l} f(l)dl = \int_{0}^{l} \frac{l}{2}dl = \frac{l^2}{4}$$

Thus c.d.f of L:

$$F(L) = \begin{cases} 0 & , \quad l \le 0 \\ l^2/4 & , \quad 0 \le l < 2 \\ 1 & , \quad 2 < l \end{cases}$$

(b)

$$\begin{aligned} E(L) &= \int_{-\infty}^{\infty} l \cdot f(l)dl \\ &= \int_{0}^{2} l \cdot \frac{l^2}{4}dl \\ &= \frac{1}{12}l^3\Big|_0^2 \\ &= 0.67 \end{aligned}$$

Question 3

(a)

c.r.v f(t),

$$f(t) = \begin{cases} 1 & , \quad 0 \le x < 1 \\ 0 & , \quad otherwise \end{cases}$$

c.d.f F(T),

$$F(T) = \int_{-\infty}^{t} f(t)dt = \int_{0}^{t} 1 \ dt = t$$

Thus,

$$F(T) = \begin{cases} 0 & , \quad t < 0 \\ t & , \quad 0 \le t < 1 \\ 1 & , \quad 1 \le t \end{cases}$$

$$g(t) = 100(1-t)^{\frac{1}{2}}$$

$$E[g(T)] = \int_{-\infty}^{\infty} g(t) \cdot f(t)dt = \int_0^1 100(1-t)\frac{1}{2} \cdot 1dt = 100\frac{(1-t)^{\frac{3}{2}}}{\frac{3}{2}} \cdot (-1)\Big|_0^1 = \frac{200}{3} = 66.67$$

(b)

(i)

$$x = 100(1-t)^{\frac{1}{2}} \to (1-t)^{\frac{1}{2}} = \frac{x}{100} \to 1-t = \frac{x^2}{10^4} \to t = 1 - \frac{x^2}{10^4}; 0 \le x \le 100$$

(ii) For

$$X \le x, T \ge t$$

where

$$t = 1 - \frac{x^2}{10^4}$$

$$P(T \ge t) = 1 - P(T \le t) = 1 - F(t) = 1 - t \; ; \quad for \; 0 \le 1$$

(iii)

$$F(X) = P(X \le x) = 1 - t = 1 - \left(1 - \frac{x^2}{10^4}\right) = \frac{x^2}{10^4}$$

$$F(X) = \begin{cases} 0 & , \quad x \le 0 \\ x^2/10^4 & , \quad 0 \le x < 100 \\ 1 & , \quad 100 \le x \end{cases}$$

$$F'(X) = \begin{cases} 2x/10^4 & , \quad 0 \le x \le 100 \\ 0 & , otherwise \end{cases}$$

$$E(X) = \int_0^{100} x \cdot \frac{2x}{10^4}dx = \frac{2}{10^4} \cdot \frac{x^3}{3}\Big|_0^{100} = \frac{200}{3} = 66.67$$

Question 4

$$X; \quad t \in \mathbb{R}$$
$$Y = (X-t)^2$$

(a)

$$E(Y) = \int_{-\infty}^{\infty} (x-t)^2 f(x)dx$$
$$= E[(x-t)^2]$$
$$= E[X^2 + t^2 - 2Xt]$$
$$= E(X^2) + E(t^2) - 2E(Xt)$$
$$= E(X^2) + t^2 - 2tE(X)$$
$$toE(Y) = E(X^2) + t^2 - 2\mu_X t$$

(b)

$$\frac{\partial E(Y)}{\partial t}$$

$$= \frac{\partial E(X^2)}{\partial t} + 2t - 2\mu_X = 2t - 2\mu_X$$

$$For\ minimum\ E(Y),\ 2t - 2\mu_X = 0$$

$$\rightarrow t = \mu_X$$

Question 5

Given, c.r.v X with p.d.f f(x)

& c.r.v Y = h(X)

Let F(X) be the c.d.f of X, i.e. f(x) = F'(X)

Also,

$$X = h^{-1}(X)$$

Let g(Y) be the p.d.f of Y

Then,

$$F_Y(y) = P(Y \leq y) = P(h(X) \leq y) = \int_{h(x)<y} f_X(x)dx$$

If h(X) is increasing, then

$$F_Y(y) = \int_{x \leq h^{-1}(y)} f_X(x)dx = \int_{-\infty}^{h^{-1}(y)} f_X(x)dx = F_X\big(h^{-1}(y)\big)$$

If h(X) is decreasing, thenm

$$F_Y(y) = \int_{x \geq h^{-1}(y)} f_X(x)dx = \int_{h^{-1}(y)}^{\infty} f_X(x)dx = 1 - F_X\big(h^{-1}(y)\big)$$

p.d.f of Y,

$$g(y) = \frac{d}{dy} F_Y(y)$$

$$\rightarrow g(y) = \begin{cases} \frac{d}{dy}\Big(F_X(h^{-1}(y))\Big) & ,\quad if\ h\ is\ increasing \\ \frac{d}{dy}\Big(1 - F_X(h^{-1}(y))\Big) & ,\quad if\ h\ is\ decreasing \end{cases}$$

Applying the chain rule of differentiation, we get

$$g(y) = \begin{cases} f(h^{-1}(y))\Big(\frac{d}{dy}h^{-1}(y)\Big) & ,\quad if\ h\ is\ increasing \\ f(h^{-1}(y))\Big(-\frac{d}{dy}h^{-1}(y)\Big) & ,\quad if\ h\ is\ increasing \end{cases}$$

The above can be written in compact form as,

$$g(y) = f(h^{-1}(y)) \cdot \Big|\frac{d}{dy}h^{-1}(y)\Big|$$