W203 Home Work 5

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Question 1

$$E(W) = \mu_W = 10$$
$$\sigma_W = 4$$
$$V = 0.5 \cdot W + U$$

Since, U is a standard normal distribution,

$$E(U) = \mu_U = 0$$

and

$$\sigma_U = 1$$

$$E(V) = E(0.5 \cdot W + U)$$
$$= 0.5 \cdot E(W) + E(U)$$
$$= 0.5\mu_W + 0 = 0.5 \cdot 10 = 5$$

Variance of V,

$$\sigma_V^2 = Var(V) = Var(0.5 \cdot W + U) = Var(0.5W) + Var(U) + 2 \cdot cov(0.5W, U)$$
$$= 0.5^2 \cdot Var(W) + Var(U) - 0$$
$$= 0.25 \cdot \sigma_W^2 + 1 = 0.25 \cdot 16 + 1 = 5$$

The covariance of V and W is defined as,

$$cov(V, W) = E(VW) - E(V)E(W) = E(VW) - \mu_W \mu_V$$

Now,

$$E(VW) = E[W(0.5W + U)] = E(0.5W^2 + UW)$$
$$= 0.5E(W^2) + E(UW) = 0.5(\sigma_W^2 + \mu_W^2)$$
$$= 0.5(16 + 100) = 58$$

Substituting these values back in the covariance equation,

$$cov(V, W) = 58 - 5 \cdot 10 = 8$$

Now we can construct the variance-covariance matrix for V and W:

$$\sum = \begin{bmatrix} E[(W - \mu_W)(W - \mu_W)] & E[(W - \mu_W)(V - \mu_V)] \\ E[(V - \mu_V)(W - \mu_W)] & E[(V - \mu_V)(V - \mu_V)] \end{bmatrix}$$

$$\sum = \begin{bmatrix} \sigma_W^2 & cov(W, V) \\ cov(V, W) & \sigma_V^2 \end{bmatrix}$$

$$\sum = \begin{bmatrix} 16 & 8 \\ 8 & 5 \end{bmatrix}$$

Question 2:

(a)

X is a c.r.v with a uniform probability distribution and the marginal p.d.f of X can be written as:

$$f_X(x) = \begin{cases} 1 & , \quad 0 \le x \le 1 \\ 0 & , \quad otherwise \end{cases}$$

Y is a c.r.v whose outcome is conditional on the outcome of X. The conditional p.d.f of Y is also a uniform distribution and can be written as:

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x} & , \quad 0 \le y \le x \\ 0 & , \quad otherwise \end{cases}$$

Using the definition of conditional probability for joint distributions, we have

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} \quad ; -\infty < y < \infty$$

Solving for f(x,y),

$$f(x,y) = \begin{cases} f_{Y|X}(y|x) \cdot f_X(x) = \frac{1}{x} \cdot 1; \quad 0 \le y \le x, \quad 0 \le x \le 1 \\ 0 \end{cases} \quad , \quad otherwise$$

$$f(x,y) = \begin{cases} \frac{1}{x} & ; \quad 0 \le x \le 1 \\ 0 & , \quad otherwise \end{cases}$$

The conditional expectation of Y given X=x can now be computed as:

$$E(Y|X=x) = \int_{-\infty}^{\infty} y \cdot f(y|x)\ dy$$

$$= \int_0^x y \cdot \frac{1}{x} dy = \frac{1}{2x} y^2 \Big|_0^x = \frac{x}{2}$$

(b) To find the unconditional expectation of Y, we can use the law of iterated expectations which states

$$E(Y) = E[E(Y|X)] = \int_{-\infty}^{\infty} E(Y|X) f(x) dx$$

$$= \int_0^1 \frac{x}{2} \cdot 1 dx = \frac{x^2}{4}\Big|_0^1 = \frac{1}{4} = 0.25$$

(c) Using the law of iterated expectations, we can write

$$E(XY) = E_X[E(XY|X)] = E_X[XE(Y|X)]$$

$$= E[X \cdot \frac{X}{2}] = E(\frac{X^2}{2}) = \frac{1}{2}E(X^2)$$

$$= \frac{1}{2}\int_0^1 x^2 \cdot f(x) dx = \frac{1}{2}\int_0^1 x^2 \cdot 1 dx = \frac{1}{6}x^3\Big|_0^1 = \frac{1}{6} = 0.167$$

(d) The covariance of X and Y can be computed from the following formula:

$$cov(X,Y) = E(XY) - E(X)E(Y) = E(XY) - \mu_X \mu_Y$$

Since X has a uniform p.d.f in [0,1], its expectation is given by

$$E(X) = \frac{1}{2}(1+0) = \frac{1}{2}$$

Thus,

$$cov(X,Y) = \frac{1}{6} - \frac{1}{2} \cdot \frac{1}{4} = 0.04167$$

Question 3

(a)

Let X represent the c.r.v for the morning wait time and Y represent the c.r.v for evening wait time.

Since both X & Y have a uniform probability distribution, the probability density function for each can be written as:

$$f(x) = \begin{cases} \frac{1}{5} & , \quad 0 \le x \le 5 \\ 0 & , \quad otherwise \end{cases}$$

$$f(y) = \begin{cases} \frac{1}{10} & , \quad 0 \le x \le 10 \\ 0 & , \quad otherwise \end{cases}$$

We can define a new random variable Z that represents the total wait time for 5 days including morning and evenings. Then,

$$Z = 5X + 5Y$$

Expectation,

$$E(Z) = E(5X+5Y) = 5 \cdot E(X+Y) = 5 \cdot [(E(X) + E(Y)]$$

The expectation of a uniform probability distribution in [A,B] is given by

$$\frac{1}{2}(B+A)$$

Thus, expectaction of total wait time on all 5 days including mornings and evenings is,

$$E(Z) = 5 \cdot \frac{1}{2} \cdot (5+10) = 37.5$$

(b)

Variance of total wait time,

$$Var(Z) = Var(5X+5Y)$$

Using the properties of variance,

$$Var(Z) = 5^2 \cdot Var(X+Y)$$
$$= 25 \cdot [Var(X) + Var(Y) + 2 \cdot cov(X,Y)]$$

Since X & Y are independent,

$$cov(X,Y) = 0$$

Also, variance of a random probability distribution in [A,B] is given by

$$Var = \frac{1}{12}(B-A)^2$$

Thus, variance of total wait time is

$$Var(Z) = 25 \cdot \frac{1}{12} \cdot (5^2 + 10^2) = 260.4167$$

(c)

Let us define a c.r.v K = 5X - 5Y (difference between total morning and evening wait times).

Mean value of difference in wait times is the absolute value of expectation for K:

$$MD = E(|K|) = E(|5X - 5Y|)$$
$$= 5|E(X - Y)| = 5|E(X) - E(Y)| = 5|\frac{5}{2} - \frac{10}{2}| = 12.5$$

(d)

$$Var(K) = Var(|X - Y|) = Var(X) + Var(Y) - 2 \cdot cov(X, Y)$$

$$= 25 \cdot \frac{1}{12} \cdot (5^2 + 10^2) = 260.4167$$

Question 4

Given random variables X and Y, we have

$$Y = aX + b$$

where

$$a \neq 0$$

Correlation of two random variables is given by

$$\rho_{X,Y} = \frac{cov(X, Y)}{\sigma_X \sigma_Y}$$

Now,

$$cov(X, Y) = E(XY) - E(X)E(Y)$$
$$= E[X(aX + b)] - E(X) \cdot E(aX + b)$$
$$= E(aX^2 + bX) - \mu_X \cdot (aE(X) + b)$$
$$= aE(X^2) + bE(X) - a\mu_X E(X) - b\mu_X$$
$$= a(\sigma_X^2 + \mu_X^2) + b\mu_X - a\mu_X^2 - b\mu_X = a\sigma_X^2$$

Also using the properties of variance, we can write

$$Var(Y) = \sigma_Y^2 = Var(aX + b) = a^2 Var(X) = a^2 \sigma_X$$

The standard deviation of Y,

$$\sigma_Y = \sqrt{a^2 \sigma_X^2}$$

$$\sigma_Y = \begin{cases} a\sigma_X & , a < 0 \\ -a\sigma_X & , a > 0 \end{cases}$$

Then, correlation

$$\rho_{X,Y} = \frac{a\sigma_X^2}{\sigma_X \cdot \pm a\sigma_X}$$

$$\rho_{X,Y} = \begin{cases} -1 & ,a < 0 \\ +1 & ,a > 0 \end{cases}$$

Question 5

Given a poisson random variable M, with p.m.f given by

$$P_M(m) = \begin{cases} \frac{\alpha^m}{m!}e^{-\alpha} & ,m = 0, 1, 2, ... \\ 0 & ,otherwise \end{cases}$$

N is d.r.v with uniform probability distribution that can take m+1 discrete values, conditional on M=m.

$$p_{N|M}(n|M = m) = \begin{cases} \frac{1}{m+1} & ,n = 0, 1, 2, 3, ...m \\ 0 & ,otherwise \end{cases}$$

(a)

We have from the definition of conditional probability for joint distributions,

$$p_{N|M}(n|m) = \frac{p(m, n)}{p_M(m)}$$

The joint probability distribution for M and N can then be written as:

$$\rightarrow p(m, n) = p_{N|M}(n|m) \cdot p_M(m)$$

$$p(m, n) = \begin{cases} \frac{1}{m+1} \cdot \frac{\alpha^m}{m!}e^{-\alpha} = \frac{\alpha^m}{m+1!}e^{-\alpha} & ,n \leq m \\ 0 & ,otherwise \end{cases}$$

(b) The marginal probability distribution of N is defined as

$$p_N(n) = \sum_{m:P(m,n)>0}^{\infty} p(m, n) \quad ,n \leq m$$

$$= \sum_{k=n}^{\infty} \frac{\alpha^k}{k+1!}e^{-\alpha}$$

$$= \frac{e^{-\alpha}}{\alpha} \sum_{n}^{\infty} \frac{\alpha^{k+1}}{k+1!}$$

$$= \frac{e^{-\alpha}}{\alpha} \Big[\sum_{0}^{\infty} \frac{\alpha^k}{k!} - \sum_{0}^{n} \frac{\alpha^k}{k!}\Big]$$

Using Maclaurin series expansion, the above reduces to

$$p_N(n) = \frac{e^{-\alpha}}{\alpha} \Big[e^{\alpha} - \sum_{0}^{n} \frac{\alpha^k}{k!}\Big]$$

$$p_N(n) = \frac{1}{\alpha} \Big[1 - \sum_{0}^{n} \frac{\alpha^k}{k!}e^{-\alpha}\Big]$$

(c) For large values of N, the bernoulli process can be approximated with a poisson process. The selection of numbers on the real number line for N will be evenly spaced and as N increases, the probability for each number will shrink signifying a rare event.