DS284: Numerical Linear Algebra
Assignment 4

Instructor: Dr. Phani Motamarri
TA: Karan Jeswani
Department of Computational and Data Sciences
Indian Institute of Science, Bangalore

Office Hours: Sat 2 PM to 4 PM
Total: 100 Marks
Posted on Dec 16, 2020 and due on Dec 28, 2020

Problem 1 [20 marks]

In this problem you will test different algorithms for the least squares problem for approximating the polynomial function $f(t) = \sin(10t)$ for $t \in [0, 1]$. To set it up follow the example discussed in the class. Choose the number of points $m = 100$ and $n = 15$ (14th degree polynomial). Determine its least square solutions with the following methods.

(a) Use QR Factorization with your implementation of Modified Gram Schmidt. You should write your own back substitution code for solving the resulting triangular system.

(b) Using QR Factorization with your implementation of Householder factorization.

(c) Using SVD (Computed with any inbuilt libraries in MATLAB/Python/Octave)

(d) Using normal equations, you can use backslash command in MATLAB to solve this system.

Accept the MATLAB/Octave/Python least squares solution (given by backslash "\" in MATLAB) as the truth. Display and plot the approximation given by this "true" solution and compare it with $f(t)$. Compare with the solution given by 4 methods described above. Explain the results.

Problem 2 [10 marks]

Given below is the matrix $\mathbf{A}$ and vector $\mathbf{b}$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \\ 2 & 8 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Find the least square solution for the above. Now do the same for the below matrix also.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \\ 5 & 7 & 9 \end{bmatrix}$$

Were you able to find the least square solution? If not explain why? Must there be a restriction on $\mathbf{A}$ for a least square solution to exist or will it always exist?

Problem 3 [10 marks]

Gaussian Elimination allows us to compute determinant of a square matrix. Recall the following points about the determinant.

- Swapping 2 rows multiplies the determinant by -1.
- Multiplying a row by a non-zero scalar multiplies the determinant by the same scalar.
- Adding one row, which is a scalar multiple of the other does not change the determinant.

Now describe the procedure how Gaussian Elimination with partial pivoting can be used to find the determinant of a general square matrix. Also comment on the number of floating point operations in this procedure!

Problem 4 [15 marks]

Let $\mathbf{A}$ be a matrix defined in MATLAB as:

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A = rand(N)
A = A - diag(diag(A)) + diag(0.001 * ones(N,1))
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Compute LU Decomposition of $\mathbf{A}$ with and without partial pivoting. Plot $\|\mathbf{LU} - \mathbf{A}\|_F$ versus N for $N = 5, 6, 7..., 20$. For LU Decomposition with partial pivoting, use built in LU function. For LU Decomposition without pivoting, write your own function.

Problem 5 [15 marks]

(a) Let $\mathbf{A}$ be a non-singular square matrix and let $\mathbf{A} = \mathbf{QR}$ be its QR factorization. Let also $\mathbf{A}^T\mathbf{A} = \mathbf{U}^T\mathbf{U}$ be the Cholesky factorization of $\mathbf{A}^T\mathbf{A}$. Can you conclude that $\mathbf{R} = \mathbf{U}$? If yes, prove it; if not, why not?

(b) Recall that by $\mathbf{A} \in \mathbb{R}^{m\times m}$, being symmetric and strictly positive definite, we mean $\mathbf{A} = \mathbf{A}^T$ and $\forall \mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq 0$, we have $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$. A symmetric matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ is positive semi-definite if $\forall \mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq 0$, we have $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$

If $\{\phi_i(x)\}_{i=1...m}$ denote m linearly independent basis functions defined over $[-1, 1]$ in an m-dimensional vector space then show that the matrix $\mathbf{M} = \int_{-1}^{1} \phi_i(x)\phi_j(x)dx$ for $i, j = 1, 2\cdots m$ is a symmetric positive definite matrix.

Similarly show that the matrix $\mathbf{K} = \int_{-1}^{1} \frac{d\phi_i(x)}{dx}\frac{d\phi_j(x)}{dx}dx$ for $i, j = 1, 2\cdots m$ is a symmetric positive semi-definite matrix.

(c) Show that a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is symmetric, strictly positive definite if and only if there exists a matrix $\mathbf{B} \in \mathbb{R}^{m\times n}$ of rank n, where $n \leq m$, such that $\mathbf{A} = \mathbf{B}^T\mathbf{B}$. Assuming that $\mathbf{A}$ is of this form, is there a unique such $\mathbf{B}$?

Problem 6 [12 marks]

For each of the following statements prove that it is true or give an example to show that it is false. Assume $\mathbf{A} \in \mathbb{C}^{m\times m}$ unless otherwise indicated.

(a) If λ is an eigenvalue of $\mathbf{A}$ and $\mu \in \mathbb{C}$, then $\lambda - \mu$ is an eigenvalue of $\mathbf{A} - \mu\mathbf{I}$.

(b) If $\mathbf{A}$ is real and λ is an eigenvalue of $\mathbf{A}$ then so it $-\lambda$.

(c) If $\mathbf{A}$ is real and λ is an eigenvalue of $\mathbf{A}$, then so is λ^*. (λ^* is the complex conjugate of λ).

(d) If λ is an eigen value of $\mathbf{A}$ and $\mathbf{A}$ is non-singular, then λ^{-1} is the eigenvalue of $\mathbf{A}^{-1}$.

(e) If all the eigenvalues of $\mathbf{A}$ are zero, then $\mathbf{A} = 0$.

(f) If $\mathbf{A}$ is diagonalizable and all its eigenvalues are equal, then $\mathbf{A}$ is diagonal.

Problem 7 [18 marks]

Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ with entries a_{ij} for $i, j = 1, 2, ..., n$ and define the closed disks $D(a_{ii}, r_i)$ centered at the diagonal entries a_{ii} of $\mathbf{A}$ of radius $r_i = \sum_{j=1}^{n}(1 - \delta_{ij})|a_{ij}|$ for $i = 1, 2, ..., n$. Note that δ_{ij} represents Kronecker delta i.e $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. The above disks are called Greshgorin's disks.

(a) Prove that every eigenvalue of $\mathbf{A}$ lies in a Greshgorin disk.
(*Hint*: Let λ be any eigenvalue of $\mathbf{A}$ and $\mathbf{x}$ be the corresponding eigenvector with largest entry 1.)

(b) Suppose that $\mathbf{A}$ is diagonally dominant i.e. $|a_{ii}| > \sum_{j=1}^{n}(1 - \delta_{ij})|a_{ij}|$ for all $i = 1, 2, ..., n$. Prove that $\mathbf{A}$ is invertible.

(c) Give estimates based on (a), for the eigenvalues of:

$$\mathbf{A} = \begin{bmatrix} 8 & 2 & 0 \\ 1 & 4 & \epsilon \\ 0 & \epsilon & 1 \end{bmatrix} \qquad \text{where } |\epsilon| < 1$$