

## Eigenvalue Problems

Let  $A \in \mathbb{R}^{m \times m}$ ,  $\underline{x} \neq 0 \in \mathbb{C}^m$  ( $\mathbb{C}$  is set of complex nos.)

then  $\underline{x}$  is an eigenvector of  $A$  and  $\lambda \in \mathbb{C}$  is its corresponding eigenvalue

if

$$\boxed{A\underline{x} = \lambda \underline{x}}$$

\* The set of all eigenvalues of a matrix  $A$  is called the spectrum of  $A$  denoted by  $\Lambda(A)$

## Application areas :-

- \* Insights into evolution of system
  - vibration analysis
  - study of resonance
  - stability of structure
  - fluid flows subjected to small perturbations

- \* Quantum mechanical modeling of matter  
(Solving Schrödinger equation)
- \* Principal stresses in solid mechanics
- \* PCA in data driven modeling
- \* Page rank algorithm used in search engines is an eigenvalue problem
- \* Eigenvectors of graph Laplacian matrix actually help in construction of efficient fillers Graph Convolution Neural Network!

Eigenvalue decomposition!

An eigenvalue decomposition of

$\underline{A} \in \mathbb{R}^{m \times m}$  is a factorization

$$\underline{A} = \underline{X} \underline{\Lambda} \underline{X}^{-1} \text{ where}$$

$\underline{X}$  is non singular and  $\underline{\Lambda}$  is diagonal with  $\underline{X}$  comprising of eigenvectors of  $\underline{A}$  as columns!

Note: Such decomposition may not always exist!

$$\underline{A} \underline{X} = \underline{X} \underline{\Lambda}$$

$$[\underline{A}] \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & \dots & x_m \\ \vdots & \vdots & \vdots & & \vdots \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & \dots & x_m \\ \vdots & \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\underline{A} \underline{x}_j = \lambda_j \underline{x}_j$$

i.e.  $j^{\text{th}}$  column of  $\underline{X}$  is  $j^{\text{th}}$  eigen vector

and  $c_{j,j}$ ) entry of  $\Lambda$  is corresponding eigenvalue!

→ Geometric multiplicity :- The geometric multiplicity of an eigenvalue  $\lambda$  is the number of linearly independent eigenvectors associated with that eigenvalue  $\lambda$ . If  $\lambda \in \Lambda(A)$ , eigenspace  $E_\lambda$  is an invariant subspace of  $A$

$$\text{i.e. } A E_\lambda \subseteq E_\lambda$$

The dimension of  $E_\lambda$  is the geometric multiplicity of  $\lambda$  i.e. maximum number of linearly independent eigenvectors that can be found for a given  $\lambda$

$$\begin{aligned} x &= \alpha_1 v_1 + \alpha_2 v_2 \\ A x &= A(\alpha_1 v_1 + \alpha_2 v_2) \\ &= \alpha_1 \lambda v_1 + \alpha_2 \lambda v_2 \\ &= \lambda [\alpha_1 v_1 + \alpha_2 v_2] \\ &= \lambda x \end{aligned}$$

## \* Characteristic Polynomial:-

The characteristic polynomial  $p_A$  of  $\underline{A} \in \mathbb{R}^{m \times m}$  is the  $m^{\text{th}}$  degree monic polynomial  $\boxed{p_A(z) = \det(z\mathbb{I} - \underline{A})}$  ✓  
 (coefficient of  $z^m$  is 1 → monic polynomial)

Thm:-  $\lambda$  is eigenvalue of  $\underline{A}$  if and

only if  $p_A(\lambda) = 0$

$$\underline{A}\underline{x} = \lambda\underline{x}$$

Note:-  $\underline{A} \in \mathbb{R}^{m \times m}$   $(\underline{A} - \lambda\mathbb{I})\underline{x} = 0$   
 $\lambda$  can be complex  
 any complex  $\lambda$  must be eigenvector of  $\underline{A}$  lies  
 appears in complex conjugate pairs i.e.  $\lambda = a+ib$  is an eigenvalue  
 $\lambda^* = a-ib$  is an eigenvalue

## \* Algebraic multiplicity :-

Since  $p_A(z)$  is monic m-degree polynomial, it can be written as

$$p_A(z) = (z - \lambda_1)(z - \lambda_2) \dots (z - \lambda_m)$$

for some  $\lambda_j \in \mathbb{C}$  (roots of  $p_A(z)$ )

Each  $\lambda_j$  is an eigenvalue and in general may be repeated

"The multiplicity of  $\lambda$  as a root of  $p_A(z)$  is the algebraic multiplicity of an eigenvalue  $\lambda$ "

Remark:- @ If  $A \in \mathbb{R}^{m \times m}$  then A has m eigenvalues counting algebraic multiplicity. In particular if roots of  $p_A(z)$  are simple, then A has m distinct eigenvalues.

b) The algebraic multiplicity of an eigenvalue  $\lambda$  is always at least as large as its geometric multiplicity.

\* Similarity transformation :-

If  $X \in \mathbb{R}^{m \times m}$  is non singular, then

$A \rightarrow X^{-1}AX$  is called a

similarity transformation of  $A \in \mathbb{R}^{m \times m}$

We say that two matrices  $A$  and  $B$  are similar if there is a similarity transformation of one to another

i.e. if there is a nonsingular  $X \in \mathbb{R}^{m \times m}$  such that  $B = X^{-1}AX$

then  $A$  and  $B$  are said to be similar!

Thm:- If  $\underline{X}$  is nonsingular, then  $\underline{A}$  and  $\underline{X}^{-1}\underline{A}\underline{X}$  have the same characteristic polynomial, eigenvalues and algebraic multiplicity and geometric multiplicity.

$$\begin{aligned}
 \text{Pf:- } p(z) &= \det(z\underline{I} - \underline{X}^{-1}\underline{A}\underline{X}) \\
 \underline{X}^{-1}\underline{A}\underline{X} &= \det(z\underline{X}^{-1}\underline{X} - \underline{X}^{-1}\underline{A}\underline{X}) \\
 &= \det(\underline{X}^{-1}(z\underline{I} - \underline{A})\underline{X}) \\
 &= (\det \underline{X}^{-1})(\det(z\underline{I} - \underline{A}))(\det \underline{X}) \\
 &= (\det \underline{X})^{-1}(\det(z\underline{I} - \underline{A}))(\det \underline{X}) \\
 &= \det(z\underline{I} - \underline{A}) = p_A(z)
 \end{aligned}$$

Hence  $\underline{A}$ ,  $\underline{X}^{-1}\underline{A}\underline{X}$  have the same eigenvalues (same roots of  $p_A(z)$ ) and

hence same algebraic multiplicity

Build a matrix  $E_\lambda$  whose column vectors span the eigenspace for the matrix  $A$ , corresponding to the eigen value  $\lambda$ .

$$\begin{aligned} (\underbrace{X^{-1} A X}_{\text{A}}) (\underbrace{X^{-1} E_\lambda}_{\text{E}}) &= \underbrace{X^{-1} A E_\lambda}_{\text{E}} \\ &= \underbrace{X^{-1} E_\lambda}_{\text{E}} \lambda \end{aligned}$$

$X^{-1} E_\lambda$  is the eigenspace for  $X^{-1} A X$  corresponding to the eigenvalue  $\lambda$ .

Hence geometric multiplicity of

$A$  and  $X^{-1} A X$  are the same.

( $\because E_\lambda$  and  $X^{-1} E_\lambda$  has the same rank)

Defective eigenvalues and matrices

\* A generic matrix need not have distinct eigenvalues i.e algebraic multiplicity need not be 1 and geometric multiplicity need not be 1 as well and need not be equal to algebraic multiplicity as well)

$$\underline{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \underline{B} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Both  $\underline{A}$  and  $\underline{B}$  eigenvalue

$$\lambda = 2.$$

Algebraic multiplicity of  $\lambda = 2$  for  $A$ ?

$$\xrightarrow{3} 3$$

Algebraic multiplicity of  $\lambda = 2$  for  $B$ ?

$$\xrightarrow{3} 3$$

For  $\underline{A}$ , we can choose 3 linearly independent eigenvectors  $e_1, e_2, e_3$  and geometric multiplicity is also 3.

For  $\underline{B}$ , we can only have only linearly independent eigenvectors  $e_1$ , the geometric multiplicity of  $\underline{B}$  is 1

\* An eigenvalue whose algebraic multiplicity is greater than its geometric multiplicity is called a defective eigenvalue!