DS284: Numerical Linear Algebra
Assignment 1

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Office Hours: Fri 2 PM to 4 PM

Total: 100 Marks
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Problem 1 [42 marks]

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times p}$, $R(.)$ denotes the range of the matrix, $N(.)$ denotes the null space of a given matrix, $dim(.)$ denotes the dimension of a vector space, then prove the following:

(a) $dim[R(\mathbf{AB})] \leq dim[R(\mathbf{A})]$ [5 marks]

(b) If the matrix $\mathbf{B}$ is non-singular then $dim[R(\mathbf{AB})] = dim[R(\mathbf{A})]$ [5 marks]

(c) $dim[N(\mathbf{AB})] \leq dim[N(\mathbf{A})] + dim[N(\mathbf{B})]$ [7 marks]

(d) $dim[R(\mathbf{A})] + dim[N(\mathbf{A})] = n$ [7 marks]

(e) $rank(\mathbf{A}) + rank(\mathbf{B}) - n \leq rank(\mathbf{AB}) \leq min(rank(\mathbf{A}), rank(\mathbf{B}))$ [8 marks]

Hint: Use the result in (a)

(f) Given a vector $\mathbf{u} \in \mathbb{R}^n$, $rank\,(\mathbf{u}\mathbf{u}^T)$ is 1. [5 marks]

Hint: Use the result in (a)

(g) Row rank always equals column rank. [5 marks]

Problem 2 [16 marks]

Suppose there always exists a set of real coefficients $c_1, c_2, c_3, ...c_{10}$ for any set of real numbers $d_1, d_2, d_3, ...d_{10}$

$$\sum_{j=1}^{10} c_j f_j(i) = d_i \qquad\qquad \text{for } \; i \in \{1, 2, ...10\}$$

where $f_1, f_2, f_3, ...f_{10}$ are a set of functions defined on the interval [1,10]

(a) Use the theorems discussed in class to show that $d_1, d_2, d_3, ...d_{10}$ determine $c_1, c_2, c_3, ...c_{10}$ uniquely. [8 marks]

(b) Let $\mathbf{A}$ be a 10×10 matrix representing the linear mapping from data $d_1, d_2, d_3, ...d_{10}$ to coefficients $c_1, c_2, c_3, ...c_{10}$. What is the i,j th entry of $\mathbf{A}^{-1}$? [8 marks]

Problem 3 [42 marks]

Let $\mathbf{A} \in \mathbb{R}^{m \times m}$ be a square matrix. An eigenvector of $\mathbf{A}$ is a nonzero vector $\mathbf{x}$ such that $\mathbf{Ax} = \lambda\mathbf{x}$ for scalar λ, the corresponding eigen value.

Prove the following:

(a) If $\mathbf{A}$ is symmetric, eigenvalues will be real. [6 marks]

(b) If $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors of a symmetric matrix $\mathbf{A}$, corresponding to distinct eigenvalues, then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal. [6 marks]

(c) If $\mathbf{S}$ is skew-symmetric, then eigenvalues must be complex. [6 marks]

(d) Given any skew-symmetric matrix $\mathbf{S}$, the matrix $\mathbf{I} - \mathbf{S}$ must always be non singular. [6 marks]

(e) The matrix $\mathbf{Q} = (\mathbf{I} - \mathbf{S})^{-1}(\mathbf{I} + \mathbf{S})$ is orthogonal for any skew-symmetric matrix $\mathbf{S}$. [6 marks]

(f) Note that a symmetric matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ can be decomposed as $\mathbf{QDQ}^T$ where $\mathbf{Q}$ is an orthogonal matrix and $\mathbf{D}$ is diagonal matrix. Using this result, show that $\mathbf{u}^T\mathbf{Au} = 0 \;\forall\; \mathbf{u} \in \mathbb{R}^m$, if and only if $\mathbf{A} = 0$. [6 marks]

(g) Show that "$\mathbf{u}^T\mathbf{Su} = 0 \;\forall\; \mathbf{u} \in \mathbb{R}^m$, if and only if $\mathbf{S}$ is a skew-symmetric matrix." [6 marks]