



Indian Institute of Science, Bangalore
Department of Computational and Data Sciences (CDS)
DS284: Numerical Linear Algebra
Quiz 4

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Submission Deadline: Nov 14, 2021 3:55PM

Max Points: 20

Notations: (i) Vectors \mathbf{v} and matrices \mathbf{M} are denoted by bold faced lower case and upper case alphabets respectively. (ii) Set of all real numbers is denoted by \mathbb{R} (iii) Set of all n dimensional vectors is denoted by \mathbb{R}^n and set of all $m \times n$ matrices is denoted by $\mathbb{R}^{m \times n}$. (iv) \mathbf{I}_n denotes the identity matrix of order n . (v) $\mathbf{0}_n$ denotes the null matrix of order $n \times n$

1. Let, $\mathbf{M} \in \mathbb{R}^{n \times n}$ be a non-zero matrix with $n \geq 2$ and $\mathbf{M}^2 = \mathbf{0}$ where, $\mathbf{0}$ is a null matrix. Let, $\alpha \in \mathbb{R} \setminus \{0\}$. Then, (2 Points)
 - (a) α is an eigenvalue of both $(\mathbf{M} + \alpha\mathbf{I})$ and $(\mathbf{M} - \alpha\mathbf{I})$
 - (b) The algebraic multiplicity of the eigenvalue α for $(\alpha\mathbf{I} - \mathbf{M})$ is n
 - (c) $-\alpha$ is one of the k distinct eigenvalues of $(\mathbf{M} - \alpha\mathbf{I})$ with $k > 1$
 - (d) None of the above
2. Let $\mathbf{A} \in \mathbb{R}^{m \times m}$ and $\mathbf{B} \in \mathbb{R}^{m \times m}$ be any two non-zero matrices, then which of the following is true ? (2 Points)
 - (a) If \mathbf{A} is non-singular then (\mathbf{B}) and (\mathbf{ABA}^{-1}) have the same eigenvalues.
 - (b) If \mathbf{A} is real skew-symmetric matrix with λ as eigenvalue then $|\frac{1-\lambda}{1+\lambda}| = 1$.
 - (c) Two diagonalizable matrices \mathbf{A} and \mathbf{B} with the same eigenvalues and eigenvectors must be the same matrix.
 - (d) If λ is an eigenvalue with both algebraic multiplicity and geometric multiplicity to be 3 then the dimension of column space of $\mathbf{A} - \lambda\mathbf{I}_m$ is 3.
3. For any real symmetric matrix \mathbf{A} , the Rayleigh quotient is given by $r(\mathbf{x}) = \frac{\mathbf{x}^T \mathbf{A} \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$. Which of the following is/are correct. (3 Points)
 - (a) For any \mathbf{x} , $r(\mathbf{x})$ is between the minimum and maximum eigenvalues of \mathbf{A} .
 - (b) If the distance between a unit vector \mathbf{x} and the eigenvector \mathbf{q}_J of a real symmetric matrix \mathbf{A} , in the sense of 2 norm is Δ , i.e. $\|\mathbf{x} - \mathbf{q}_J\|_2 = \Delta$.

Then $|\mathbf{x}^T \mathbf{A} \mathbf{x} - \lambda_J| = O(\Delta)$, where λ_J is the eigenvalue corresponding to eigenvector \mathbf{q}_J

- (c) Let $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$. The value $\beta = 5$ minimizes the $\|\mathbf{Ax} - \beta\mathbf{x}\|$
 - (d) Let $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{b} \\ \mathbf{b}^T & \gamma \end{bmatrix}$ (for some symmetric matrix \mathbf{B} , some vector \mathbf{b} , and some real number γ). Then, the smallest eigenvalue of $\mathbf{B} \leq$ smallest eigenvalue of \mathbf{A} .
4. Which of the following is/are true: (2 Points)
- (a) An eigenvalue solver can be designed to compute eigenvalues and eigenvectors of a given matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ in a finite number of steps, using exact arithmetic.
 - (b) An eigenvalue solver designed to compute all eigenvalues and eigenvectors of a symmetric dense matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ requires at most $O(m^3)$ work, if it is not initially reduced to tri-diagonal form in Phase 1.
 - (c) Power iteration produces a sequence of vectors $\mathbf{v}^{(i)}$ that converges to the eigenvector corresponding to the largest eigenvalue of $\mathbf{A} \in \mathbb{R}^{m \times m}$ starting with any initial guess vector $\mathbf{v}^{(0)} \neq \mathbf{0}$.
 - (d) Let $\mathbf{F} \in \mathbb{R}^{m \times m}$ denote the Householder reflector that introduces zeros below the diagonal entry of symmetric matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ in the 1st column when pre-multiplied with \mathbf{A} . Then eigenvalues of $\mathbf{F} \mathbf{A} \mathbf{F}^T$ and \mathbf{A} are the same.
5. Which of the following is/are true: (2.5 Points)
- (a) Pure QR algorithm is equivalent to the Simultaneous iteration applied to an initial guess of vectors which are columns of a square full rank matrix.
 - (b) Pure QR algorithm used to solve the eigenvalues and eigenvectors of a matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ generates a sequence of matrices $\mathbf{Q}^{(k)}$ which converges to the eigenvector matrix of \mathbf{A} as $k \rightarrow \infty$.
 - (c) Pure QR algorithm used to solve the eigenvalues and eigenvectors of a matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ generates QR factorization to be $\mathbf{A}^k = \mathbf{Q}^{(k)} \mathbf{R}^{(k)}$ generated at the k^{th} iteration of the algorithm.
 - (d) Computational complexity for finding the eigenvectors using Pure QR algorithm and the Simultaneous iteration is same.
6. Which of the following is/are true: (2 Points)
- (a) Suppose that matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ and λ be an eigenvalue of \mathbf{A} , then so is $\bar{\lambda}$ (complex conjugate).

- (b) Let $\mathbf{A} \in \mathbb{R}^{m \times m}$ be any non-zero diagonalizable matrix then rank of \mathbf{A} = number of non-zero eigenvalues of \mathbf{A} .
- (c) All orthogonal matrices are normal matrices.
- (d) Consider the matrix $\mathbf{A} = \mathbf{I}_9 - 2\mathbf{u}\mathbf{u}^T$, with $\mathbf{u} = \frac{1}{3}[1, 1, 1, 1, 1, 1, 1, 1, 1]^T$ where \mathbf{I}_9 is a 9×9 identity matrix. If λ and μ are two distinct eigenvalues of \mathbf{A} , then $|\lambda - \mu| = 0$.

7. Given matrix $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 4 & 0 & 3 & 0 & 0 \\ 0 & 8 & 0 & 4 & 2 \\ 0 & 0 & 0 & 8 & 4 \end{bmatrix}$, which of the following is/are true: (2
Points)

- (a) The sum of eigenvalues of \mathbf{A} is 16.
- (b) The sum of eigenvalues of $\mathbf{A}\mathbf{A}^T$ is 194.
- (c) The product of eigenvalues of $\mathbf{A}\mathbf{A}^T$ is 9216.
- (d) Eigenvalues of $\mathbf{A}\mathbf{A}^T$ and $\mathbf{A}^T\mathbf{A}$ are the same.
- (e) The eigenvalues of $\mathbf{A}\mathbf{A}^T$ are always non-zero and positive.