



Indian Institute of Science, Bangalore
Department of Computational and Data Sciences (CDS)
DS284: Numerical Linear Algebra
Final Exam – Aug 2021 Term
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Duration: 14:00 hrs to 17:00 hrs

Max Points: 100

Notations: (i) Vectors and matrices are denoted by bold faced lower case and upper case alphabets respectively. (ii) Set of all real numbers is denoted by \mathbb{R} (iii) Set of all n dimensional vectors is denoted by \mathbb{R}^n and set of all $m \times n$ matrices is denoted by $\mathbb{R}^{m \times n}$. (iv) \mathbf{I}_m denotes the identity matrix of order m . (v) ϵ_{mach} denotes machine epsilon. (vi) $\mathbf{0}_n$ denotes the n -dimensional column vector with each element being zero.

Start each problem on a new page.

There are 4 pages.

Problem 1

[6x3=18 points]

Assert if the following statements are True or False. Give a detailed reasoning for your assertion. Marks will be awarded only for your reasoning.

- (a) Let $(\mathbf{A} + \mathbf{A}^T)$ be a positive definite matrix for a non-zero matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$. Then \mathbf{A} is always non-singular.
- (b) Oblique projectors have one of its eigenvalues as 0.
- (c) For a matrix $\mathbf{F} = \mathbf{I}_m - 2\mathbf{q}\mathbf{q}^T$, where $\mathbf{q} \in \mathbb{R}^m$ and m is an odd integer. Then the determinant of \mathbf{F} is 1
- (d) A matrix $\mathbf{Q} \in \mathbb{R}^{m \times n}$ has n orthonormal columns, then m has to be always lesser than n .
- (e) If $\hat{\mathbf{x}} \in \mathbb{R}^n$ satisfies the system of equations $\mathbf{A}^T \mathbf{A} \mathbf{x} = \mathbf{A}^T \mathbf{b}$ where $\mathbf{A} \in \mathbb{R}^{m \times n}$ is a full rank matrix with $m > n$ and $\mathbf{b} \in \mathbb{R}^m$, then $\hat{\mathbf{x}}$ always satisfies the system of equations $\mathbf{A} \mathbf{x} = \mathbf{b}$
- (f) Consider a non-zero symmetric matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ and a non-zero vector $\mathbf{b} \in \mathbb{R}^m$. Let $\mathbf{K}_n = \mathbf{Q}_n \mathbf{R}_n$ be the QR factorization of the n -dimensional Krylov subspace $\mathbf{K}_n = [\mathbf{b} \quad \mathbf{Ab} \quad \mathbf{A}^2\mathbf{b} \dots \mathbf{A}^{n-1}\mathbf{b}]$ for $n < m$. As $n \rightarrow m$, columns of \mathbf{Q}_n approach the eigenvectors of \mathbf{A}

Problem 2

[3+3+3 =9 points]

Let us consider some data about a product sold in Walmart. The data is related to normalised price and fiber content of various cornflakes brands available in Walmart. There are 4 brands of cornflakes. A data matrix $\mathbf{X} \in \mathbb{R}^{4 \times 2}$ is constructed with each of these feature vectors forming the two columns of the matrix \mathbf{X} . Upon computing the SVD of \mathbf{X} , one finds that

the matrix \mathbf{U} formed by the left singular vectors to be $\mathbf{U} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$ and the matrix

\mathbf{V} formed by the right singular vectors to be $\mathbf{V} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Further the singular values are found to be $\sqrt{2}$ and 0.5

- (a) Reconstruct the data matrix \mathbf{X} from the singular values and singular vectors given above. Further compute the Frobenius norm of the data matrix \mathbf{X} . What will be the rank of this matrix \mathbf{X} ? Justify.
- (b) Construct the rank one approximation of \mathbf{X} using the dominant singular vector and the corresponding singular value.
- (c) Compute the matrix $\mathbf{C} = \mathbf{X}^T \mathbf{X}$. Verify that the given non-zero singular values of \mathbf{X} are square roots of the eigenvalues of the matrix \mathbf{C} by computing the eigenvalues of the matrix \mathbf{C} .

Problem 3

[3+7+6=16 points]

Let $\mathbf{A} \in \mathbb{R}^{m \times m}$ be a symmetric matrix. Now answer the following questions:

- (a) Show that eigenvectors corresponding to different eigenvalues of \mathbf{A} are orthogonal to each other.
- (b) Let λ_1 be an eigenvalue of \mathbf{A} with algebraic multiplicity 1 and \mathbf{u}_1 be the corresponding normalized eigenvector. We then pick the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{m-1}\}$ in \mathbb{R}^m such that $\mathbf{Q}_1 = [\mathbf{u}_1, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{m-1}]$ is an orthogonal matrix. Show that $\mathbf{Q}_1^T \mathbf{A} \mathbf{Q}_1 = \begin{bmatrix} \lambda_1 & \mathbf{0}_{m-1}^T \\ \mathbf{0}_{m-1} & \mathbf{A}_1 \end{bmatrix}$, where \mathbf{A}_1 is a symmetric matrix of order $m-1$. Furthermore, show that the remaining $m-1$ eigenvalues of \mathbf{A} are the $m-1$ eigenvalues of \mathbf{A}_1 .
- (c) Let $\|\mathbf{A}\|_1$ denote vector induced matrix 1-norm of \mathbf{A} . Show that $\rho(\mathbf{A}) \leq \|\mathbf{A}\|_1$ where, ρ denotes spectral radius i.e. largest absolute eigen value $|\lambda|$ of \mathbf{A} .
 (Hint:- Relate eigenvalues of \mathbf{A} and \mathbf{A}^k subsequently show that
 $|\rho(\mathbf{A})|^k = \rho(\mathbf{A}^k) \leq \|\mathbf{A}^k\|_1 \leq \|\mathbf{A}\|_1^k$)

Problem 4

[3+7=10 points]

Consider the system of linear equations $(\mathbf{A} - \mu \mathbf{I}_m)\mathbf{x} = \mathbf{b}$ with $\mathbf{A} \in \mathbb{R}^{m \times m}$ being a non-zero symmetric matrix and $\mathbf{b} \in \mathbb{R}^m$ is a non-zero vector.

- (a) Show that there exists a unique solution \mathbf{x} for the above system of linear equations if $\mu \neq \lambda_j$, where λ_j is the j^{th} eigenvalue of \mathbf{A} and $1 \leq j \leq m$.
- (b) Show that the solution \mathbf{x} can be written as $\mathbf{x} = \sum_{j=1}^m \frac{\mathbf{u}_j^T \mathbf{b}}{\lambda_j - \mu} \mathbf{u}_j$, where $(\mathbf{u}_j, \lambda_j)$ denotes the j^{th} eigenvector-eigenvalue pair of \mathbf{A} . [Note: Mere substitution of $\mathbf{x} = \sum_{j=1}^m \frac{\mathbf{u}_j^T \mathbf{b}}{\lambda_j - \mu} \mathbf{u}_j$ in $(\mathbf{A} - \mu \mathbf{I}_m)\mathbf{x} = \mathbf{b}$ to show that it satisfies the linear system of equations will not fetch any points]

Problem 5

[3+4+5+7=19 points]

Let $\mathbf{A} \in \mathbb{R}^{m \times m}$ be a full rank matrix, and $\mathbf{A} = \mathbf{QR}$ denote the full QR decomposition of \mathbf{A} , where $\mathbf{Q} \in \mathbb{R}^{m \times m}$ is an orthogonal matrix and $\mathbf{R} \in \mathbb{R}^{m \times m}$ is an upper triangular matrix. Let \mathbf{q}_i and \mathbf{a}_i denote the i^{th} column of the matrices \mathbf{Q} and \mathbf{A} respectively for $1 \leq i \leq m$. Now, answer the following questions with clear arguments:

- (a) Construct the projector matrix \mathbf{P} that orthogonally projects a vector onto a subspace spanned by the set of vectors $\langle \mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_{j-1} \rangle$ for some j ($\leq m$).
- (b) Write down the expression for the complimentary projector to \mathbf{P} obtained above in (a). Show that this complimentary projector is symmetric. Let $\tilde{\mathbf{a}}_j$ be the projection of \mathbf{a}_j obtained using this complementary projector. To this end, write the expression for $\tilde{\mathbf{a}}_j$ in terms of $\langle \mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_{j-1} \rangle$ and \mathbf{a}_j . What subspace does $\tilde{\mathbf{a}}_j$ belong?
- (c) Show that the absolute value of diagonal entry of the \mathbf{R} matrix i.e $|\mathbf{R}_{jj}|$ is related to the 2-norm of the vector $\tilde{\mathbf{a}}_j$ obtained in (b).
- (d) Using the above results and QR decomposition of \mathbf{A} . Show that:
$$|\det(\mathbf{A})| \leq \prod_{j=1}^m \|\mathbf{a}_j\|_2$$

Problem 6

[3+4+8+3+3+7 =28 points]

Consider a symmetric matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$. Answer the following 6 questions:

- (a) Show that the singular values of \mathbf{A} are absolute values of eigenvalues of \mathbf{A} . What can you say about the vector induced matrix norm $\|\mathbf{A}\|_2$ in terms of eigenvalues of \mathbf{A} ? Support your argument.
- (b) Show that $|\mathbf{x}^T \mathbf{A} \mathbf{x}| \leq \|\mathbf{A}\|_2$ for any non-zero unit vector $\mathbf{x} \in \mathbb{R}^m$.
- (c) Let the vector $\mathbf{u} \in \mathbb{R}^m$ be an eigenvector of \mathbf{A} corresponding to an eigenvalue λ i.e. $\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$. Further, let the matrix \mathbf{A} undergo a symmetric matrix perturbation by $\delta\mathbf{A}$ such that $\frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|} = O(\epsilon_{\text{mach}})$. Let, $\tilde{\mathbf{u}} = \mathbf{u} + \delta\mathbf{u}$ and $\tilde{\lambda} = \lambda + \delta\lambda$ be the eigenvector-eigenvalue pair of the perturbed matrix $\tilde{\mathbf{A}} = \mathbf{A} + \delta\mathbf{A}$. Now, show that

$$|\delta\lambda| \leq \|\delta\mathbf{A}\|_2$$

(Hint:- Note that the perturbed matrix $\tilde{\mathbf{A}}$ is symmetric and start with the eigenvalue problem corresponding to $\tilde{\mathbf{A}}$ to first show that $|\delta\lambda| = |\mathbf{u}^T \delta\mathbf{A} \mathbf{u}|$. You may need to use the fact that the eigenvectors of a symmetric matrix are orthogonal and hence form a basis for \mathbb{R}^m)

- (d) Deduce the relative condition number for the problem of computing the eigenvalue λ of our symmetric matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ using the inequality derived in part (c).
- (e) Consider the problem of computing eigenvalues of the matrix $\mathbf{M} = \begin{bmatrix} 2.0 & 0 \\ 0 & 2.0 \end{bmatrix}$. As you can see the eigenvalues of this matrix \mathbf{M} are 2, 2. Find the condition number for the problem of computing the eigenvalue 2 for the above matrix \mathbf{M} using the result obtained in part(d).

- (f) There are two algorithms designed – Algorithm S and Algorithm U to compute the eigenvalues of the above matrix $\mathbf{M} = \begin{bmatrix} 2.0 & 0 \\ 0 & 2.0 \end{bmatrix}$. Algorithm S is designed to be a backward stable algorithm. To this end, comment on the relative forward error incurred in computing an eigenvalue of \mathbf{M} by employing this backward stable Algorithm S . Furthermore, the Algorithm U is designed to compute eigenvalues of the above matrix \mathbf{M} by solving the roots of the characteristic polynomial of the matrix \mathbf{M} i.e. $p(z) = z^2 - 4z + 4$. Compute the forward relative error incurred in computing the eigenvalue 2 of \mathbf{M} using the Algorithm U (**Note:** Assume that the floating point approximation errors incurred in the coefficient of z and constant term in $p(z)$ are both ϵ ($\epsilon \leq \epsilon_{\text{mach}}$)). Using this estimate of error, argue that Algorithm U is unstable with proper reasoning.