Indian Institute of Science Banglore
Department of Computational and Data Sciences (CDS)

DS284: Numerical Linear Algebra

Mid-semester Exam 2023

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Duration: 15:05 hrs to 17:00 hrs, Oct 8 2023 Max Points: 60

Notations: (i) Vectors and matrices are denoted by bold faced lower case and upper case alphabets respectively. (ii) Set of all real numbers is denoted by $\mathbb{R}$ (iii) Set of all n dimensional vectors is denoted by $\mathbb{R}^n$ and set of all $m \times n$ matrices is denoted by $\mathbb{R}^{m\times n}$. (iv) $\mathbf{I}_n$ denotes the identity matrix of order n. (v) $\mathbf{0}_n$ denotes the null matrix of order $n \times n$.

Problem 1 [6x3=18 points]

Assert if the following statements are True or False. Give a detailed reasoning for your assertion. Marks will be awarded only for your reasoning.

(a) Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ with $m > n$ be a matrix with n orthogonal columns. Let $\mathbf{A} = \hat{\mathbf{Q}}\hat{\mathbf{R}}$ be its reduced **QR** factorization. Then $\hat{\mathbf{R}}$ is a diagonal matrix.

(b) There can exist a projector $\mathbf{P} \in \mathbb{R}^{m\times m}$ that is an orthogonal matrix but not a symmetric matrix.

(c) If $\mathbf{P} \in \mathbb{R}^{m\times m}$ is an orthogonal projector projecting a vector $\mathbf{v} \in \mathbb{R}^m$ to an n dimensional subspace of $\mathbb{R}^m$ ($n < m$), then $\mathbf{I} - \mathbf{P}$ has n non-zero singular values.

(d) Two matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ are related to each other such that $\mathbf{A} = \mathbf{U}\mathbf{B}\mathbf{V}^{\mathbf{T}}$ for some orthogonal matrices $\mathbf{U}$ and $\mathbf{V}$. Then $\mathbf{A}$ and $\mathbf{B}$ should have the same singular values.

(e) Suppose $\mathbf{A} \in \mathbb{R}^{102\times 102}$ is such that $||\mathbf{A}||_2 = 50$ and $||\mathbf{A}||_F = 51$. Then the 2-norm condition number $\kappa(\mathbf{A})$ is always greater than or equal to 50.

(f) If a nonzero row is added to $\mathbf{A} \in \mathbb{R}^{m\times n}$ to obtain a new matrix $\mathbf{B} \in \mathbb{R}^{(m+1)\times n}$, then the largest singular value of $\mathbf{B}$ is always greater than or equal to the largest singular value of $\mathbf{A}$.

Problem 2 [1+6+5 = 12 points]

Consider the three vectors $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$; $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; $\mathbf{a}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

(a) Verify the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are linearly independent.

(b) Choosing $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ as the basis vectors that span $\mathbb{R}^3$, express the vector $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ in the basis of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ by first setting up the required system of equations to be solved and then solve this system of equations by QR factorization.

(c) Construct the projector matrix $\mathbf{P}$ that projects any vector $\mathbf{v} \in \mathbb{R}^3$ orthogonally to the 2-dimensional subspace spanned by the vectors $\mathbf{a}_1, \mathbf{a}_2$. Subsequently find the orthogonal projection of $\mathbf{a}_3$ onto this subspace spanned by $\mathbf{a}_1, \mathbf{a}_2$.

Problem 3 [3+4+7+2+2+2+6+4 = 30 points]

Consider a symmetric matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$. Answer the following 8 questions:

(a) Show that the singular values of $\mathbf{A}$ are absolute values of eigenvalues of $\mathbf{A}$. What can you say about the vector-induced matrix norm $\|\mathbf{A}\|_2$ in terms of eigenvalues of $\mathbf{A}$? Support your argument.

(b) Show that $|\mathbf{x}^T \mathbf{A} \mathbf{x}| \leqslant \|\mathbf{A}\|_2$ for any non-zero unit vector $\mathbf{x} \in \mathbb{R}^m$.

(c) Let the vector $\mathbf{u} \in \mathbb{R}^m$ be an eigenvector of the above symmetric matrix $\mathbf{A}$ corresponding to an eigenvalue λ i.e. $\mathbf{A}\mathbf{u} = \lambda \mathbf{u}$. Further, let the matrix $\mathbf{A}$ undergo a symmetric matrix perturbation by $\delta \mathbf{A}$ such that $\dfrac{\|\delta \mathbf{A}\|}{\|\mathbf{A}\|} = O(\epsilon_{mach})$. Let, $\tilde{\mathbf{u}} = \mathbf{u} + \delta \mathbf{u}$ and $\tilde{\lambda} = \lambda + \delta\lambda$ be the eigenvector-eigenvalue pair of the perturbed matrix $\tilde{\mathbf{A}} = \mathbf{A} + \delta \mathbf{A}$. Now, show that

$$|\delta\lambda| \leqslant \|\delta \mathbf{A}\|_2$$

(Hint:- Note that the perturbed matrix $\tilde{\mathbf{A}}$ is symmetric and start with the eigenvalue problem corresponding to $\tilde{\mathbf{A}}$ to first show that $|\delta\lambda| = |\mathbf{u}^T\, \delta \mathbf{A}\, \mathbf{u}|$. You may also assume that $\mathbf{A}$ is full rank and eigenvector-eigenvalue perturbations caused due to the symmetric perturbations in $\mathbf{A}$ are small and in the order of $||\delta \mathbf{A}||_2$.)

(d) Deduce the relative condition number for the problem of computing the eigenvalue λ of our symmetric matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ using the inequality derived in part (c).

(e) We now consider the problem of computing eigenvalues of the matrix $\mathbf{M} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, where a is a non-zero real number. As you can see, the two eigenvalues of this matrix $\mathbf{M}$ are a, a. Find the relative condition number for the mathematical problem of computing the eigenvalues for the above matrix $\mathbf{M}$ using the result obtained in part(d).

(f) An Algorithm S is designed to compute the eigenvalues of the above matrix $\mathbf{M} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ on a computer using floating point arithmetic. Algorithm S is designed to be a backward stable algorithm. To this end, comment on the relative forward error incurred in computing an eigenvalue of $\mathbf{M}$ by employing this backward stable Algorithm S.

(g) Furthermore, another Algorithm U is designed to compute eigenvalues of the above matrix $\mathbf{M}$ by solving the roots of the characteristic polynomial of $\mathbf{M}$ i.e. $p_{\mathbf{M}}(z) = \det(\mathbf{M} - z\mathbf{I}) = 0$. Assume that in $\mathbf{M} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, $a \in \mathbb{F}$, i.e. a does not have a floating point error when represented on a computer. List the steps of the Algorithm U required to compute eigenvalues of $\mathbf{M}$ on a computer. In doing so, deduce the floating-point approximation errors incurred both in evaluating the coefficients of $p_{\mathbf{M}}(z)$ and also, in the expressions employed to compute the roots of $p_{\mathbf{M}}(z)$ involving these coefficients. When doing this exercise, you may assume all the relative errors arising in floating point approximations to be the maximum possible relative error i.e. ϵ_M, the machine epsilon.

(h) Using the information in part (g), compute the forward relative error incurred in computing the eigenvalue a of $\mathbf{M}$ using the above Algorithm U on a computer and using this estimate, argue that the Algorithm U is unstable. (Hint: An Algorithm G is unstable if it is not both backward stable and stable. Also note that if the Algorithm G is backward stable or stable, then the relative forward error in the solution is $O(\kappa\epsilon_M)$ where κ is the condition number of the problem)