



Indian Institute of Science, Bangalore
Department of Computational and Data Sciences (CDS)

DS284: Numerical Linear Algebra
Previous Years Quiz Questions
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Notations: (i) Vectors \mathbf{v} and matrices \mathbf{M} are denoted by bold faced lower case and upper case alphabets, respectively. (ii) Set of all real numbers is denoted by \mathbb{R} (iii) Set of all n dimensional vectors is denoted by \mathbb{R}^n and set of all $m \times n$ matrices is denoted by $\mathbb{R}^{m \times n}$. (iv) \mathbf{I}_n denotes the identity matrix of order n . (v) 0_n denotes the null matrix of order $n \times n$

1. Let, $\mathbf{M} \in \mathbb{R}^{n \times n}$ such that $\mathbf{M}^3 = \mathbf{I}_n$. Suppose that $\mathbf{M}\mathbf{v} \neq \mathbf{v}$ for any non-zero vector \mathbf{v} . Then, which of the following is/are true?
 - (a) \mathbf{M} has real eigenvalues. [False]
 - (b) $\mathbf{M} + \mathbf{M}^{-1}$ has real eigenvalues. [True]
 - (c) n is always divisible by 2. [True]
 - (d) n is always divisible by 3. [False]
2. Which of the following is/are always true: (Note: $N(\mathbf{A})$ denotes the null space of matrix \mathbf{A}) (2 Points)
 - (a) $\mathbf{A} \in \mathbb{R}^{m \times m}$ is a skew-symmetric matrix and admits a Schur decomposition of the form $\mathbf{A} = \mathbf{Q}\mathbf{T}\mathbf{Q}^{-1}$, then \mathbf{Q} is always an eigenvector matrix corresponding to \mathbf{A} .
 - (b) Let λ be an eigenvalue of $\mathbf{A} \in \mathbb{R}^{m \times m}$ with geometric multiplicity of 3. The dimension of $N(\mathbf{A} - \lambda\mathbf{I}) = m - 3$.
 - (c) If $\mathbf{A} \in \mathbb{R}^{m \times m}$ and non-defective, then the decomposition $\mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^{-1}$ always exists where $\mathbf{Q} \in \mathbb{R}^{m \times m}$ is unitary and \mathbf{D} is diagonal.
 - (d) If $\mathbf{u} \in \mathbb{R}^2$, then the eigenvalues of the matrix $\mathbf{A} = \mathbf{u}\mathbf{u}^T$ are always non-negative.
3. Which of the following is/are always true: (2 Points)
 - (a) Rayleigh quotient of a vector $\mathbf{x} \in \mathbb{R}^m$ for a symmetric matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ is always positive if all the eigenvalues of \mathbf{A} are positive.
 - (b) Let $\mathbf{A} \in \mathbb{R}^{m \times m}$ and λ is an eigenvalue corresponding to matrix \mathbf{A} . Then the matrix $\mathbf{A} - \lambda\mathbf{I}$ has a non-trivial null space of dimension at least 1.
 - (c) Non-singular skew-symmetric real matrices of even dimension always have the product of eigenvalues to be real and positive
 - (d) Let $\mathbf{A} \in \mathbb{R}^{m \times m}$ be symmetric. Let $\mathbf{M} \in \mathbb{R}^{m \times m}$ be the tridiagonal matrix obtained after employing the Phase-1 algorithm among the two phases used for solving the eigenvalue problem corresponding to \mathbf{A} . Then the matrices \mathbf{A} and \mathbf{M} have the same geometric multiplicity corresponding to every eigenvalue of \mathbf{A} and \mathbf{M} .

4. Consider the normal equation $\mathbf{A}^T \mathbf{A} \mathbf{x} = \mathbf{A}^T \mathbf{b}$ where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $m > n$, is a full rank matrix. Which one of the following algorithms would you use to find an accurate solution with least computational cost? (3 Points)
- Compute Cholesky decomposition of $\mathbf{A}^T \mathbf{A}$ as $\mathbf{L} \mathbf{L}^T$ and solve $\mathbf{L} \mathbf{y} = \mathbf{A}^T \mathbf{b}$ using forward substitution and $\mathbf{L}^T \mathbf{x} = \mathbf{y}$ using back substitution.
 - Compute QR factorization of \mathbf{A} , using the Householder algorithm, as $\mathbf{Q} \mathbf{R}$ and solve $\mathbf{R} \mathbf{x} = \mathbf{Q}^T \mathbf{b}$ using back substitution.
 - Compute QR factorization of \mathbf{A} , using the modified Gram-Schmidt algorithm, as $\mathbf{Q} \mathbf{R}$ and solve $\mathbf{R} \mathbf{x} = \mathbf{Q}^T \mathbf{b}$ using back substitution.
 - Compute reduced SVD of \mathbf{A} as $\mathbf{U} \mathbf{\Sigma} \mathbf{V}^T$ and evaluate $\mathbf{x} = \mathbf{V} \mathbf{\Sigma}^{-1} \mathbf{U}^T \mathbf{b}$.
5. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix with eigenvalues $\mu_1 \leq \mu_2 \leq \dots \leq \mu_n$ and $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ as the corresponding eigenvectors. Let $\mathbf{x} \in \mathbb{R}^{n \times 1}$ be a non-zero vector, defining a matrix $\mathbf{B} = \begin{bmatrix} \mathbf{A} & \mathbf{x} \\ \mathbf{x}^T & 0 \end{bmatrix}$ with eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{n+1}$ and the corresponding eigenvectors are $\begin{bmatrix} \mathbf{v}_1 \\ 1 \end{bmatrix}, \begin{bmatrix} \mathbf{v}_2 \\ 1 \end{bmatrix}, \dots, \begin{bmatrix} \mathbf{v}_{n+1} \\ 1 \end{bmatrix}$ where $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n+1} \in \mathbb{R}^{n \times 1}$ are non-zero vectors. Which of the following is/are true? (3 Points)
- \mathbf{B} is always invertible.
 - $\lambda_n = \mathbf{v}_n^T \mathbf{x}$
 - $(\mathbf{A} - \lambda_1 \mathbf{I})$ can be singular.
 - $\lambda_{n+1} > \mu_n$
6. Let, $\mathbf{M} \in \mathbb{R}^{n \times n}$ be a non-zero matrix with $n \geq 2$ and $\mathbf{M}^2 = \mathbf{0}$ where, $\mathbf{0}$ is a null matrix. Let, $\alpha \in \mathbb{R} \setminus \{0\}$. Then, (2 Points)
- α is an eigenvalue of both $(\mathbf{M} + \alpha \mathbf{I})$ and $(\mathbf{M} - \alpha \mathbf{I})$
 - The algebraic multiplicity of the eigenvalue α for $(\alpha \mathbf{I} - \mathbf{M})$ is n
 - $-\alpha$ is one of the k distinct eigenvalues of $(\mathbf{M} - \alpha \mathbf{I})$ with $k > 1$
 - None of the above
7. Let $\mathbf{A} \in \mathbb{R}^{m \times m}$ and $\mathbf{B} \in \mathbb{R}^{m \times m}$ be any two non-zero matrices, then which of the following is true? (2 Points)
- If \mathbf{A} is non-singular then (\mathbf{B}) and $(\mathbf{A} \mathbf{B} \mathbf{A}^{-1})$ have the same eigenvalues.
 - If \mathbf{A} is real skew-symmetric matrix with λ as eigenvalue then $|\frac{1-\lambda}{1+\lambda}| = 1$.
 - Two diagonalizable matrices \mathbf{A} and \mathbf{B} with the same eigenvalues and eigenvectors must be the same matrix.
 - If λ is an eigenvalue with both algebraic multiplicity and geometric multiplicity to be 3 then the dimension of column space of $\mathbf{A} - \lambda \mathbf{I}_m$ is 3.

8. For any real symmetric matrix \mathbf{A} , the Rayleigh quotient is given by $r(\mathbf{x}) = \frac{\mathbf{x}^T \mathbf{A} \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$. Which of the following is/are correct. (3 Points)

- (a) For any \mathbf{x} , $r(\mathbf{x})$ is between the minimum and maximum eigenvalues of \mathbf{A} .
- (b) If the distance between a unit vector \mathbf{x} and the eigenvector \mathbf{q}_J of a real symmetric matrix \mathbf{A} , in the sense of 2 norm is Δ , i.e. $\|\mathbf{x} - \mathbf{q}_J\|_2 = \Delta$. Then $|\mathbf{x}^T \mathbf{A} \mathbf{x} - \lambda_J| = O(\Delta)$, where λ_J is the eigenvalue corresponding to eigenvector \mathbf{q}_J .
- (c) Let $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$. The value $\beta = 5$ minimizes the $\|\mathbf{A}\mathbf{x} - \beta\mathbf{x}\|$.
- (d) Let $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{b} \\ \mathbf{b}^T & \gamma \end{bmatrix}$ (for some symmetric matrix \mathbf{B} , some vector \mathbf{b} , and some real number γ). Then, the smallest eigenvalue of $\mathbf{B} \leq$ smallest eigenvalue of \mathbf{A} .

9. Which of the following is/are true: (2 Points)

- (a) Suppose that matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ and λ be an eigenvalue of \mathbf{A} , then so is $\bar{\lambda}$ (complex conjugate).
- (b) Let $\mathbf{A} \in \mathbb{R}^{m \times m}$ be any non-zero diagonalizable matrix then rank of \mathbf{A} = number of non-zero eigenvalues of \mathbf{A} .
- (c) All orthogonal matrices are normal matrices.
- (d) Consider the matrix $\mathbf{A} = \mathbf{I}_9 - 2\mathbf{u}\mathbf{u}^T$, with $\mathbf{u} = \frac{1}{3}[1, 1, 1, 1, 1, 1, 1, 1, 1]^T$ where \mathbf{I}_9 is a 9×9 identity matrix. If λ and μ are two distinct eigenvalues of \mathbf{A} , then $|\lambda - \mu| = 0$.

10. Given matrix $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 4 & 0 & 3 & 0 & 0 \\ 0 & 8 & 0 & 4 & 2 \\ 0 & 0 & 0 & 8 & 4 \end{bmatrix}$, which of the following is/are true: (2 Points)

- (a) The sum of eigenvalues of \mathbf{A} is 16.
- (b) The sum of eigenvalues of $\mathbf{A}\mathbf{A}^T$ is 194.
- (c) The product of eigenvalues of $\mathbf{A}\mathbf{A}^T$ is 9216.
- (d) Eigenvalues of $\mathbf{A}\mathbf{A}^T$ and $\mathbf{A}^T\mathbf{A}$ are the same.
- (e) The eigenvalues of $\mathbf{A}\mathbf{A}^T$ are always non-zero and positive.