Indian Institute of Science, Bangalore
Department of Computational and Data Sciences (CDS)

DS284: Numerical Linear Algebra

Quiz 2 [Sept 27, 2024]

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भारतीय विज्ञान संस्थान

Exam duration: 5:45 PM - 6:45 AM | Max Points: 30

Notations: Important Information: (i) Vectors $\mathbf{x}$ and matrices $\mathbf{A}$ are denoted by bold faced lower case and upper case alphabets respectively. (ii) Set of all real numbers is denoted by $\mathbb{R}$. (iii) Set of all n dimensional vectors is denoted by $\mathbb{R}^n$ and set of all $m \times n$ matrices are denoted by $\mathbb{R}^{m\times n}$. (iv) $\|\mathbf{A}\|_F$ denotes the Frobenius norm of a matrix $\mathbf{A}$

1. Answer whether the following statements are True or False (Yes or No) with appropriate justifications. **Marks will be awarded only for justification (10 $\times$ 3 = 30 points)**

   (a) Let $fl : \mathbb{R}^+ \rightarrow \mathbb{F}$ be a function that takes positive real numbers $x \in \mathbb{R}^+$ as input and returns the **nearest** floating point number approximation $x' \in \mathbb{F}$ (Assume that all inputs are representable as normalized floating point numbers and there is no risk of overflow/underflow in any of the operations). For $a, b, c \in \mathbb{R}^+$, assert individually if the following are true/false

   - $a + b = c \implies fl(fl(a) + fl(b)) = fl(c)$
   - $a \geq b \implies fl(a) \geq fl(b)$

   (b) Consider the mathematical problem of computing the scalar $f(\mathbf{x}) = x_1x_2$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. If the relative condition number of this problem is denoted by $\kappa(\mathbf{x})$ measured in $\|.\|_1$ norm, the problem becomes ill-conditioned when $|x_1| \gg |x_2|$ or $|x_2| \gg |x_1|$

   (c) Algorithms designed on a computer performing arithmetic operations with greater precision for solving not-so-well-conditioned mathematical problems are needless and wasteful.

   (d) Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ $(m < n)$ be a full rank matrix and $\mathbf{A}$ admits $\mathbf{U\Sigma V}^T$ as its full SVD. If $\mathbf{V} = [\mathbf{V}_1\mathbf{V}_2]$, where $\mathbf{V}_1 \in \mathbb{R}^{n\times m}$ and $\mathbf{V}_2 \in \mathbb{R}^{n\times(n-m)}$. The system of equations $\mathbf{V}_2\mathbf{x} = \mathbf{b}$, may not have a solution $\mathbf{x} \in \mathbb{R}^{(n-m)}$ for some right hand side vector $\mathbf{b} \in \mathbb{R}^n$ satisfying $\mathbf{Ab} = 0$

   (e) Let the rank of a non-symmetric matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ be $p(< m)$ and this matrix satisfies $\mathbf{A}(\mathbf{I} - \mathbf{A}) = \mathbf{0}$. Further, let $\{\mathbf{v_i}\}_{i=1}^{p}$ form an orthonormal basis for range($\mathbf{A}$) and $\{\mathbf{w_i}\}_{i=1}^{m-p}$ form an orthonormal basis for range($\mathbf{I} - \mathbf{A}$). Now let us construct a square matrix $\mathbf{Q}$ with first p columns to be $\{\mathbf{v_i}\}_{i=1}^{p}$ and the remaining $m-p$ columns to be $\{\mathbf{w_i}\}_{i=1}^{m-p}$. Can we always claim (i) $\mathbf{Q}$ is a full rank matrix, (ii) $\mathbf{QQ}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$?

   (f) If $\mathbf{A} \in \mathbb{R}^{m\times m}$ is a rank 1 matrix with trace($\mathbf{A}$) = 1, then $\mathbf{A}$ has to be a projector matrix.

(g) $\mathbf{A}$, $\mathbf{B} \in \mathbb{R}^{m \times m}$ are two matrices such that $\mathbf{A} = \mathbf{Q}\mathbf{B}\mathbf{Q}^T$, where $\mathbf{Q} \in \mathbb{R}^{m \times m}$ is an orthogonal matrix, then $\mathbf{A}$ and $\mathbf{B}$ will have same singular values.

(h) If the matrix $\mathbf{A}_r$ is the best r rank approximation of $\mathbf{A} \in \mathbb{R}^{m \times m}$ $(r < m)$ in the 2-norm sense, then $\|\mathbf{A} - \mathbf{A}_r\|_F = \sigma_{r+1}$, the $(r+1)^{th}$ singular value (assume singular values are arranged in decreasing order as per convention)

(i) If $\sigma_{\max}$ and $\sigma_{\min}$ are the maximum and minimum singular values of a full rank matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$, then the condition number $\kappa(\mathbf{A}^T\mathbf{A}) = \frac{\sigma_{\max}}{\sigma_{\min}}$.

(j) Let $\mathbf{P}$ be an orthogonal projector onto a subspace $\mathbb{S} \subset \mathbb{R}^m$. Can I claim that the vector $\mathbf{Px}$ is the best approximation in the subspace $\mathbb{S}$ to a given vector $\mathbf{x} \in \mathbb{R}^m$ in the 2-norm sense? In other words, verify whether $\|\mathbf{Px} - \mathbf{x}\|_2 \leq \|\mathbf{y} - \mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{S}$. is true or not.