Indian Institute of Science, Banglore
Department of Computational and Data Sciences (CDS)

DS284: Numerical Linear Algebra

Quiz 1 [Aug 31, 2023]

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Exam duration: 10:05 AM - 11:20 AM Max Points: 30

Important Information: (i) Vectors $\mathbf{x}$ and matrices $\mathbf{A}$ are denoted by bold faced lower case and upper case alphabets respectively. (ii) Set of all real numbers is denoted by $\mathbb{R}$. (iii) Set of all rational numbers is denoted by $\mathbb{Q}$. (iv) Set of all n dimensional vectors is denoted by $\mathbb{R}^n$ and set of all $m \times n$ matrices are denoted by $\mathbb{R}^{m\times n}$ (v) Matrices $\mathbf{P}$ and $\mathbf{Q}$ are said to commute with each other if $\mathbf{PQ} = \mathbf{QP}$

1. Assert if the following statements are True or False. Give a short reasoning for your assertion. Marks will be awarded only for your reasoning. **$7 \times 3 = 21$ points**

   (a) Given a matrix $\mathbf{B} \in \mathbb{R}^{m\times n}$ $(m > n)$ with columns of $\mathbf{B}$ being orthonormal. Then $\mathbf{B}\mathbf{B}^T = \mathbf{I}$ is always true

   (b) Suppose $\mathbf{A} \in \mathbb{R}^{n\times n}$ be such that $rank(\mathbf{A}) = rank(\mathbf{A}^2)$. Then $null(\mathbf{A}) \cap range(\mathbf{A}) = \{0\}$. i.e an empty set.

   (c) If $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a skew-symmetric matrix ($\mathbf{A} = -\mathbf{A}^T$) and n is odd, then $\mathbf{A}$ may be invertible.

   (d) Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\|\mathbf{A}\|_p$ denotes induced matrix norm discussed in class. If $\lambda \in \mathbb{R}$ is an eigen value of $\mathbf{A}$, then $|\lambda| \leq \|\mathbf{A}\|_p$. (**Hint:** Recall the definition of eigenvalue problem to be $\mathbf{Ax} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq 0$.)

   (e) We have a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, and further we also know that for a matrix $\mathbf{C} \in \mathbb{R}^{n\times n}$, $\mathbf{AC} = \mathbf{I}$. Then the matrices $\mathbf{A}$ and $\mathbf{C}$ should always commute with each other.

   (f) A real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is constructed with $f(\mathbf{v})$ defined as the number of non-zero entries in a given vector $\mathbf{v}$. The function f defines a valid norm on $\mathbb{R}^n$.

   (g) The Vikram moon lander conducted experiments on the south pole of the moon and recorded 100 entries, which are all real numbers. Instead of sending the full data, the moon lander evaluated three quantities T_1, T_2, T_3 associated with this data and sent these quantities to Earth. T_1 is computed as sum of the absolute values and the value is 2.5. T_2 is computed as the square root of the sum of the square of the entries and the value is 2.891. T_3 is computed as the max absolute entry in the data and the value is 1.2. A scientist on Earth receives only the values of T_1, T_2 and T_3 and concludes that the analysis done by the moon lander is error-free. Do you agree with this conclusion?

2. Starting with an initial guess $\mathbf{x}^{(0)}$, a fixed point iteration of the form $\mathbf{x}^{(k+1)} = \mathbf{T}\mathbf{x}^{(k)} + \mathbf{v}$ can be devised to solve a linear system of equations $\mathbf{Ax} = \mathbf{b}$ ($\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{x}, \mathbf{b} \in \mathbb{R}^n$)

involving the matrix $\mathbf{T}(\in \mathbb{R}^{n\times n})$ and a vector $\mathbf{v}(\in \mathbb{R}^n)$. The iterations are repeated till $\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\| \leq \epsilon$, where ϵ is a specified tolerance value.

Answer the following: **3+4+2 = 9 points**

(a) Under what conditions (on $\mathbf{T}, \mathbf{v}$) does the fixed point problem $\mathbf{x} = \mathbf{T}\mathbf{x} + \mathbf{v}$ has a solution. Note that a fixed-point solution, $\mathbf{x}^*$ should satisfy $\mathbf{x}^* = \mathbf{T}\mathbf{x}^* + \mathbf{v}$.

(b) Assume that the above condition is satisfied, i.e. a fixed point solution exists. Under what condition on $\mathbf{T}$ does the fixed point iteration $\mathbf{x}^{(k+1)} = \mathbf{T}\mathbf{x}^{(k)} + \mathbf{v}$ converge to the fixed point solution for any initial guess $\mathbf{x}^{(0)}$?
[**Hint:** Recall that $\mathbf{x}^*$ is a fixed point solution, and for convergence, the error between the trial solution at k^{th} iteration and $\mathbf{x}^*$ should tend to zero with increasing k]

(c) Under what conditions (on $\mathbf{T}, \mathbf{v}$) does the fixed point problem converge to a unique solution for any initial value?