Numerical Linear Algebra

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Assignment-2

1. $\rho(A) = |\lambda_{max}|$ where $|\lambda_{max}|$ is the largest absolute eigen value of A.

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

$$\Rightarrow A(A\underset{\sim}{x}) = \lambda A\underset{\sim}{x} = \lambda^2 \underset{\sim}{x}$$

$$\Rightarrow A^2\underset{\sim}{x} = \lambda^2 \underset{\sim}{x}.$$

So eigen value of A^k are λ^k where λ is the eigen value of A.

$$\rho(A^k) = |\lambda_{max}^k| = (|\lambda_{max}|)^k = [\rho(A)]^k \quad \text{— ①}$$

For any $\underset{\sim}{x} \in \mathbb{R}^n$,

$$\frac{\|A^k \underset{\sim}{x}\|}{\|\underset{\sim}{x}\|} \leq \|A^k\| \quad \text{— ⓐ}$$

Now if $\underset{\sim}{x}$ is the eigen vector corresponding to λ_{max}^k,

$$\frac{\|A^k \underset{\sim}{x}\|}{\|\underset{\sim}{x}\|} = \frac{\|\lambda_{max}^k \underset{\sim}{x}\|}{\|\underset{\sim}{x}\|} = |\lambda_{max}^k| \leq \|A^k\|$$

($\because$ From ⓐ)

$$\Rightarrow \rho(A^k) \leq \|A^k\| \quad \text{— ②}$$

Now

$$\underset{(k\text{-times})}{\|A \cdots A\|} \leq \|A\| \cdots \|A\| = \|A\|^k$$

(Bounding of norm)

$$\Rightarrow \|A^k\| \leq \|A\|^k \quad — (3)$$

From (1), (2) & (3),

$$[\rho(A)]^k = \rho(A^k) \leq \|A^k\| \leq \|A\|^k$$

— (proved)

2. $\underset{\sim}{x} \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$

(a) $\|\underset{\sim}{x}\|_\infty = \max\limits_{1 \leq i \leq m} |x_i| \Rightarrow \|\underset{\sim}{x}\|_\infty^2 = \left(\max\limits_{1 \leq i \leq m} |x_i|\right)^2$

$$\|\underset{\sim}{x}\|_2^2 = \sum_{i=1}^{m} |x_i|^2$$

We observe that,

$$\|\underset{\sim}{x}\|_2^2 = \left(\max_{1 \leq i \leq m} |x_i|\right)^2 + \underbrace{\sum_{i=1}^{m} |x_i|^2}_{\text{'0' or +ve term.}}$$

$i \neq$ index Corresponding to $\max\limits_{1 \leq i \leq m} |x_i|$.

$$\Rightarrow \|\underset{\sim}{x}\|_\infty^2 \leq \|\underset{\sim}{x}\|_2^2 \Rightarrow \|\underset{\sim}{x}\|_\infty \leq \|\underset{\sim}{x}\|_2$$

— (proved)

e.g. $\underset{\sim}{x} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1}$ for equality.

(b) $\|\underset{\sim}{x}\|_\infty = \max_{1 \le i \le m} |x_i|$

$\|\underset{\sim}{x}\|_2^2 = \sum_{i=1}^{m} |x_i|^2 = |x_1|^2 + \dots + |x_m|^2$

let's say, $\max(|x_1|, \dots, |x_m|) = |x_j|$

Then $\|\underset{\sim}{x}\|_\infty = |x_j|$

$\|\underset{\sim}{x}\|_2^2 = |x_1|^2 + \dots + |x_m|^2 \le |x_j|^2 + \dots + |x_j|^2$

$= m|x_j|^2$

$= m \|\underset{\sim}{x}\|_\infty^2$

$\Rightarrow \|\underset{\sim}{x}\|_2 \le \sqrt{m} \|\underset{\sim}{x}\|_\infty$

e.g. $\underset{\sim}{x} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{m \times 1}$ for equality. — (proved)

(c) We know

$\|\underset{\sim}{x}\|_\infty \le \|\underset{\sim}{x}\|_2 \le \sqrt{m} \|\underset{\sim}{x}\|_\infty$

where ~~$\underset{\sim}{x}$~~ $\underset{\sim}{x} \in R^m$.

From this,

$\|A\underset{\sim}{x}\|_\infty \le \|A\underset{\sim}{x}\|_2$ for any $\underset{\sim}{x} \in R^n$ & $A \in R^{m \times n}$

$\Rightarrow \|A\underset{\sim}{x}\|_\infty \le \|A\underset{\sim}{x}\|_2 \le \|A\| \|\underset{\sim}{x}\|_2$

(Bounding norm)

— (1)

B/c this is true for any $\underset{\sim}{x}$, let, $\|\underset{\sim}{x}\|_\infty = 1$.

So $\|\underset{\sim}{x}\|_2 \leq \sqrt{n}\,\|\underset{\sim}{x}\|_\infty$

$\Rightarrow \|\underset{\sim}{x}\|_2 \leq \sqrt{n}$ (From prev. question)

So $\|A\underset{\sim}{x}\|_\infty \leq \|A\|_2\,\|\underset{\sim}{x}\|_2$ (From eqn. ①)

$\leq \sqrt{n}\,\|A\|_2$ (For any $\underset{\sim}{x} \in R^n$ with $\|\underset{\sim}{x}\|_\infty = 1$)

— ②

So $\|A\|_\infty = \sup_{\substack{\|\underset{\sim}{x}\|_\infty = 1 \\ \underset{\sim}{x} \in R^n}} \|A\underset{\sim}{x}\|_\infty \leq \sqrt{n}\,\|A\|_2$

— (From eqn. ②)

Hence $\|A\|_\infty \leq \sqrt{n}\,\|A\|_2$

— (proved)

(d) e.g. $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ s.t. $\|\underset{\sim}{x}\| = 1$.

$\|A\|_\infty = \sup_{\|\underset{\sim}{x}\| = 1} \|A\underset{\sim}{x}\|$, $\begin{bmatrix} x_1 + x_2 \\ 0 \end{bmatrix} = A\underset{\sim}{x}$, $\max(\|A\underset{\sim}{x}\|) = 2$

when $\underset{\sim}{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

So $\|A\|_\infty = 2$

Now $A\underset{\sim}{x} = \begin{bmatrix} x_1 + x_2 \\ 0 \end{bmatrix}$ $\|A\underset{\sim}{x}\|_2 = |x_1 + x_2|$ Subject to $x_1^2 + x_2^2 = 1$

($\because \|\underset{\sim}{x}\| = 1$)

This max. is achieved when $\underset{\sim}{x} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

So $\|A\|_2 = \frac{2}{\sqrt{2}} = \sqrt{2}$.

We see $\|A\|_\infty = \sqrt{2}\,\|A\|_2 = \sqrt{n}\,\|A\|_2$ (Equality)

(d) We know $\|\underset{\sim}{x}\|_\infty \leq \|\underset{\sim}{x}\|_2 \leq \sqrt{m}\, \|\underset{\sim}{x}\|_\infty$. — (1)

So it follows

$$\|A\underset{\sim}{x}\|_2 \leq \sqrt{m}\, \|A\underset{\sim}{x}\|_\infty \leq \sqrt{m}\, \|A\|_\infty \|\underset{\sim}{x}\|_\infty$$

(for any $\underset{\sim}{x}$)

Now let's say $\|\underset{\sim}{x}\|_2 = 1$. Then $\|\underset{\sim}{x}\|_\infty \leq 1$.

(From (1))

So $\|A\underset{\sim}{x}\|_2 \leq \sqrt{m}\, \|A\|_\infty$ for any $\|\underset{\sim}{x}\|_2 = 1$ & $\underset{\sim}{x} \in R^n$.

So

$$\|A\|_2 = \sup_{\substack{\underset{\sim}{x} \in R^n \\ \|\underset{\sim}{x}\|_2 = 1}} \|A\underset{\sim}{x}\|_2 \leq \sqrt{m}\, \|A\|_\infty$$

Hence $\|A\|_2 \leq \sqrt{m}\, \|A\|_\infty$ — (proved)

e.g. $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\|\underset{\sim}{x}\| = 1$.

$A\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$ $\|A\|_\infty = \sup_{\|\underset{\sim}{x}\| = 1} \|A\underset{\sim}{x}\|_\infty = \max(|x_1|, |x_2|)$

$= 1$

when $\underset{\sim}{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ~~or $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$~~ or $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\|A\|_2 = \sup_{\|\underset{\sim}{x}\| = 1} \|A\underset{\sim}{x}\|_2 = \sqrt{x_1^2 + x_1^2} \; \cancel{= 1 \text{ (always when } \|\underset{\sim}{x}\|_2 = 1)}$$

$$= \sqrt{2x_1^2} = \sqrt{2}\, |x_1|$$

Hence $\|A\|_2 = \sqrt{2}$ when $|x_1| = 1$. i.e. $\underset{\sim}{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

So $\|A\|_2 = \sqrt{2}\, \|A\|_\infty = \sqrt{m}\, \|A\|_\infty$ — (Equality).

3. $A \in \mathbb{R}^{m \times n}$

ⓐ $\|A\|_F = \left[\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right]^{1/2}$

Now let's say $C = A^T A$

$$C = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

We observe

$$C_{11} = a_{11}^2 + a_{21}^2 + \cdots + a_{m1}^2$$

$$C_{22} = a_{12}^2 + a_{22}^2 + \cdots + a_{m2}^2$$

$$\vdots$$

$$C_{nn} = a_{1n}^2 + a_{2n}^2 + \cdots + a_{mn}^2$$

$+$

$$\text{trace}(C) = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2$$

$\Rightarrow$ $\text{trace}(A^T A) = \|A\|_F^2$

$\Rightarrow$ $\|A\|_F = \sqrt{\text{trace}(A^T A)}$

— (proved)

(b) Before proving this result, we'll try to show

$$\|\underset{\sim}{x}\|_2 \leq \|\underset{\sim}{x}\|_1 \leq \sqrt{n}\,\|\underset{\sim}{x}\|_2 \,. \quad \underline{x \in R^n}$$ (1)

$$\|\underset{\sim}{x}\|_1 = \sum_{i=1}^{n} |x_i| \qquad \|\underset{\sim}{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

$$\Rightarrow \|\underset{\sim}{x}\|_1^2 = \left(\sum_{i=1}^{n} |x_i|\right)^2, \quad \|\underset{\sim}{x}\|_2^2 = \sum_{i=1}^{n} |x_i|^2$$

From algebra, we know

$$(a_1 + \dots + a_n)^2 \geq a_1^2 + \dots + a_n^2$$

where $a_1, \dots, a_n$ are all +ve number.

Hence $\|\underset{\sim}{x}\|_1^2 \geq \|\underset{\sim}{x}\|_2^2$

$$\Rightarrow \|\underset{\sim}{x}\|_1 \geq \|\underset{\sim}{x}\|_2 \quad \text{—— (a)}$$

Now, $\|\underset{\sim}{x}\|_1 = \sum_{i=1}^{n} |x_i| = \sum_{i=1}^{n} |x_i| \cdot 1$

let's construct a vector $\underset{\sim}{y} = \begin{bmatrix} |x_1| \\ \vdots \\ |x_n| \end{bmatrix}_{n \times 1}$

$$\underset{\sim}{z} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1}$$

So $\|\underset{\sim}{x}\|_1 = \underset{\sim}{y}^T \underset{\sim}{z} \leq \|\underset{\sim}{y}\|_2 \, \|\underset{\sim}{z}\|_2$

(Cauchy-Schwartz)

$$\Rightarrow \|\underset{\sim}{x}\|_1 \leq \sqrt{n}\,\|\underset{\sim}{y}\|_2 = \sqrt{n}\,\|\underset{\sim}{x}\|_2 \quad \text{—— (b)}$$

So from (a) & (b) we have shown vector 1-, 2-norm equivalence.

Now

$$\|A\underset{\sim}{x}\|_2 \leq \|A\underset{\sim}{x}\|_1 \leq \|A\|_1 \|\underset{\sim}{x}\|_1$$

(For any $\underset{\sim}{x} \in R^n$)

let, $\|\underset{\sim}{x}\|_2 = 1$. Then from ①, $\|\underset{\sim}{x}\|_1 \leq \sqrt{n}$.

So $\|A\underset{\sim}{x}\|_2 \leq \sqrt{n}\,\|A\|_1$ (For $\|\underset{\sim}{x}\|_2 = 1$)

So $\|A\|_2 = \sup\limits_{\substack{\|\underset{\sim}{x}\|_2 = 1 \\ \underset{\sim}{x} \in R^n}} \|A\underset{\sim}{x}\|_2 \leq \sqrt{n}\,\|A\|_1$

So $\|A\|_2 \leq \sqrt{n}\,\|A\|_1$.

Now

$$\|A\underset{\sim}{x}\|_1 \leq \sqrt{m}\,\|A\underset{\sim}{x}\|_2 \leq \sqrt{m}\,\|A\|_2 \|\underset{\sim}{x}\|_2$$

— (From ①)

let, $\|\underset{\sim}{x}\|_1 = 1$. Then $\|\underset{\sim}{x}\|_2 \leq \|\underset{\sim}{x}\|_1 = 1$.

Hence $\|A\underset{\sim}{x}\|_1 \leq \sqrt{m}\,\|A\|_2$ (For $\|\underset{\sim}{x}\|_1 = 1$)

$\|A\|_1 = \sup\limits_{\substack{\|\underset{\sim}{x}\|_1 = 1 \\ \underset{\sim}{x} \in R^n}} \|A\underset{\sim}{x}\|_1 \leq \sqrt{m}\,\|A\|_2$

So $\|A\|_1 \leq \sqrt{m}\,\|A\|_2$

Hence

$$\frac{1}{\sqrt{m}}\|A\|_1 \leq \|A\|_2 \leq \sqrt{n}\,\|A\|_1 .$$

— (proved)

(c) Before proving $\|A\|_2 \leq \sqrt{\|A\|_1 \|A\|_\infty}$, I want to show a result about 2-norm of matrix.

Claim : If $A \in \mathbb{R}^{m \times n}$, $\exists$ a unit 2-norm vector $\underset{\sim}{z} \in \mathbb{R}^n$ s.t. $A^T A \underset{\sim}{z} = \mu^2 \underset{\sim}{z}$ where $\mu = \|A\|_2$.

Proof

Suppose $\|\underset{\sim}{z}\|_2 = 1$ s.t. $\|A\|_2 = \|A\underset{\sim}{z}\|_2$

Basically ~~$\|z\|$~~ $\underset{\sim}{z}$ maximises the function

$$f(\underset{\sim}{x}) = \frac{\|A\underset{\sim}{x}\|}{\|\underset{\sim}{x}\|}$$

or equivalently maximises $\dfrac{\|A\underset{\sim}{x}\|^2}{\|\underset{\sim}{x}\|^2}$

$$= \frac{(A\underset{\sim}{x})^T(A\underset{\sim}{x})}{\underset{\sim}{x}^T\underset{\sim}{x}} = \frac{\underset{\sim}{x}^T A^T A \underset{\sim}{x}}{\underset{\sim}{x}^T \underset{\sim}{x}}$$

Hence $\nabla f(\underset{\sim}{x})\big|_{\underset{\sim}{z}} = 0$

$$\Rightarrow \frac{A^T A\underset{\sim}{x} \cdot (\underset{\sim}{x}^T\underset{\sim}{x}) - \underset{\sim}{x}^T A^T A \underset{\sim}{x} \cdot \underset{\sim}{x}}{(\underset{\sim}{x}^T\underset{\sim}{x})^2} = 0$$

$$\Rightarrow A^T A \underset{\sim}{x}\big|_{\underset{\sim}{z}} = (\underset{\sim}{x}^T A^T A \underset{\sim}{x})\, \underset{\sim}{x}$$

($\because \|\underset{\sim}{x}^T\underset{\sim}{x}\|_2 = 1$ when $\underset{\sim}{x} = \underset{\sim}{z}$)

$$\Rightarrow A^T A \underset{\sim}{z} = (\underset{\sim}{z}^T A^T A \underset{\sim}{z})\underset{\sim}{z}$$

Now set $\mu = \|A\underset{\sim}{z}\|_2 = \|A\|_2$

Then, we get $A^T A \underset{\sim}{z} = \mu^2 \underset{\sim}{z}$.

Now, we'll prove $\|A\|_2 \leq \sqrt{\|A\|_1 \, \|A\|_\infty}$

If $\underset{\sim}{z}$ is such that $A^T A \underset{\sim}{z} = \mu^2 \underset{\sim}{z}$ with $\mu = \|A\|_2$,
$(\underset{\sim}{z} \neq \underset{\sim}{0})$

Then $\|A^T A \underset{\sim}{z}\|_1 = \|\mu^2 \underset{\sim}{z}\|_1 = \mu^2 \|\underset{\sim}{z}\|_1$

$\Rightarrow \mu^2 \|\underset{\sim}{z}\|_1 \leq \|A^T\|_1 \, \|A\|_1 \, \|\underset{\sim}{z}\|_1$

$\Rightarrow \mu^2 \|\underset{\sim}{z}\|_1 \leq \|A\|_\infty \, \|A\|_1 \, \|\underset{\sim}{z}\|_1$ (Bounding norm)

($\because \|A^T\|_1 = \|A\|_\infty$ b/c in transpose row & column gets altered.
And 1-norm $\rightarrow$ maximum absolute column sum
& ∞-norm $\rightarrow$ maximum absolute row sum)

$\Rightarrow \mu^2 \leq \|A\|_\infty \|A\|_1$

$\Rightarrow \|A\|_2^2 \leq \|A\|_\infty \, \|A\|_1$

$\Rightarrow \|A\|_2 \leq \sqrt{\|A\|_\infty \, \|A\|_1}$

— (proved)

Q.4 will be at the end.

5. Number $= (1.f)_2 \times 2^{\text{exponent} - 1}$

3 bits for f & 2 bits for exponent.

(a) 3 bits for f $\Rightarrow 2^3 = 8$ numbers

3 different exponents possible i.e. 0, 1, 2.

Hence total number of numbers $= 8 \times 3$

$= 24$

(b)

Scientific notation	Binary notation	Decimal
$(1.000) \times 2^{-1}$	$(0.1)_2$	0.5
$(1.001) \times 2^{-1}$	$(0.1001)_2$	0.5625
$(1.010) \times 2^{-1}$	$(0.1010)_2$	0.625
$(1.011) \times 2^{-1}$	$(0.1011)_2$	0.6875
$(1.100) \times 2^{-1}$	$(0.1100)_2$	0.75
$(1.101) \times 2^{-1}$	$(0.1101)_2$	0.8125
$(1.110) \times 2^{-1}$	$(0.1110)_2$	0.875
$(1.111) \times 2^{-1}$	$(0.1111)_2$	0.9375
$(1.000) \times 2^{0}$	$(1)_2$	1
$(1.001) \times 2^{0}$	$(1.001)_2$	1.125
$(1.010) \times 2^{0}$	$(1.010)_2$	1.25
$(1.011) \times 2^{0}$	$(1.011)_2$	1.375

$(1.100) \times 2^0$	$(1.100)_2$	1.5
$(1.101) \times 2^0$	$(1.101)_2$	1.675
$(1.110) \times 2^0$	$(1.110)_2$	1.75
$(1.111) \times 2^0$	$(1.111)_2$	1.875
$(1.000) \times 2^1$	$(10.00)_2$	2
$(1.001) \times 2^1$	$(10.01)_2$	2.25
$(1.010) \times 2^1$	$(10.10)_2$	2.5
$(1.011) \times 2^1$	$(10.11)_2$	2.75
$(1.100) \times 2^1$	$(11.00)_2$	3
$(1.101) \times 2^1$	$(11.01)_2$	3.25
$(1.110) \times 2^1$	$(11.10)_2$	3.5
$(1.111) \times 2^1$	$(11.11)_2$	3.75

$\mathcal{E}_m = 0.0625$ (Derived later)

(c) minimum real number $\Rightarrow 0.5 - x = \mathcal{E}_m \cdot x \Rightarrow x = 0.4705$

Maximum real number $\Rightarrow 3.75 = (1-\mathcal{E}_m)x \Rightarrow x = 4$

(d) From table, we see absolute gap betn. numbers increases with increase in magnitude of number being represented. Only relative gap is constant. in ~~a range of 2^j to 2^{j+1}~~.
(i.e. normalized with 2^j, preceding number)

(e) For every $x \in R$, $\exists\ x' \in F$ s.t. $\dfrac{|x - x'|}{|x|} \leq \mathcal{E}_{machine}$

Alternatively, we can think of it as half the gap betn. 1 & succeding floating point number.

Hence $\mathcal{E}_{machine} = \dfrac{1.125 - 1.00}{2} = 0.0625$

6. $A \in \mathbb{R}^{n \times n}$ be an invertible matrix

$$Ax = b \ , \quad b \neq 0 .$$

(a) $\quad x = A^{-1} b$

Condition number of the problem when b is perturbed.

~~$\hat{k}$ = Relative condition number~~

$$\cancel{= \max_{\delta b} \left[\frac{\frac{\| x + \delta x \|}{\| x \|}}{} \right.}$$

$\hat{k}$ = Relative condition number

$$= \max_{\delta b} \left[\frac{\dfrac{\| A^{-1}(b + \delta b) - A^{-1} b \|}{\| A^{-1} b \|}}{\dfrac{\| b + \delta b - b \|}{\| b \|}} \right]$$

$$= \max_{\delta b} \left[\frac{\| A^{-1} \delta b \|}{\| \delta b \|} \cdot \frac{\| b \|}{\| A^{-1} b \|} \right]$$

$$= \frac{\| A^{-1} \| \, \| b \|}{\| A^{-1} b \|}$$

Now $\quad \hat{k} = \dfrac{\| A^{-1} \| \, \| b \|}{\| A^{-1} b \|} .$

(b) Finding lower bound of $\hat{k}$.

$$\hat{k} = \frac{\|A^{-1}\| \, \|\underset{\sim}{b}\|}{\|A^{-1}\underset{\sim}{b}\|} \geq \frac{\|A^{-1}\underset{\sim}{b}\|}{\|A^{-1}\underset{\sim}{b}\|} = 1.$$

So $\hat{k} \geq 1$. We've to show that it's a tight lower bound.

B/c norm of A^{-1} exists, there exists a vector $\underset{\sim}{x}^*$ s.t. $\dfrac{\|A^{-1}\underset{\sim}{x}^*\|}{\|\underset{\sim}{x}^*\|}$ is maximum.

Hence $\|A^{-1}\| = \dfrac{\|A^{-1}\underset{\sim}{x}^*\|}{\|\underset{\sim}{x}^*\|}$

Now set $\underset{\sim}{b} = \underset{\sim}{x}^*$.

So $\|A^{-1}\| = \dfrac{\|A^{-1}\underset{\sim}{b}\|}{\|\underset{\sim}{b}\|}$

Hence $\hat{k} = \dfrac{\|A^{-1}\| \, \|\underset{\sim}{b}\|}{\|A^{-1}\underset{\sim}{b}\|} = 1$

for the choice of $\underset{\sim}{b}$ which maximises $\dfrac{\|A^{-1}\underset{\sim}{b}\|}{\|\underset{\sim}{b}\|}$.

So $\hat{k} = 1$ is a tight lower bound.

7. $A \in \mathbb{R}^{n\times n}$ be an invertible matrix

$$A\underset{\sim}{x} = \underset{\sim}{b}, \qquad \underset{\sim}{b} \neq 0$$

$$k(A)\,\|\Delta A\| < \|A\|$$

$$\Rightarrow \|A\|\|A^{-1}\|\,\|\Delta A\| < \|A\|$$

$$\Rightarrow \|A^{-1}\|\,\|\Delta A\| < 1$$

$$\Rightarrow \|A^{-1}\Delta A\| \leq \|A^{-1}\|\,\|\Delta A\| < 1 \quad \text{— } ①$$

Before proving the actual result, I want to claim a statement.

Claim: $B \in \mathbb{R}^{n\times n}$. $\|B\| < 1$. Then $\|(I-B)^{-1}\| \leq \dfrac{1}{1-\|B\|}$

proof: For $\forall\, \underset{\sim}{x} \in \mathbb{R}^n$ and $\underset{\sim}{x} \neq 0$

$$\|(I-B)\underset{\sim}{x}\| \geq \|\underset{\sim}{x}\| - \|B\underset{\sim}{x}\| \geq \|\underset{\sim}{x}\| - \|B\|\,\|\underset{\sim}{x}\|$$
$$= (1-\|B\|)\,\|\underset{\sim}{x}\|$$

As $\|B\| < 1$, $\|(I-B)\underset{\sim}{x}\| > 0$.

So $(I-B)\underset{\sim}{x} = 0$ has unique solution $\underset{\sim}{x} = 0$

Hence $I-B$ is invertible.

Now $\|(I-B)^{-1}(I-B)\| = 1$

$$\Rightarrow \|(I-B)^{-1} - (I-B)^{-1}B\| = 1$$

$$\Rightarrow \|(I-B)^{-1}\| - \|(I-B)^{-1}B\| \leq 1$$

$$\Rightarrow \|(I-B)^{-1}\| - \|(I-B)^{-1}\|\,\|B\| \leq 1$$

$$\Rightarrow (1-\|B\|)\,\|(I-B)^{-1}\| \leq 1$$

$$\Rightarrow \|(I-B)^{-1}\| \leq \frac{1}{1-\|B\|} \quad \text{— (proved)} \quad \text{— (2)}$$

Now, we move to actual proof.

$$(A+\Delta A)(\underset{\sim}{x}+\Delta \underset{\sim}{x}) = \underset{\sim}{b} + \Delta \underset{\sim}{b}$$

$$\Rightarrow A(I + A^{-1}\Delta A)(\underset{\sim}{x}+\Delta\underset{\sim}{x}) = \underset{\sim}{b} + \Delta\underset{\sim}{b}$$

$$\Rightarrow A(I+A^{-1}\Delta A)\Delta\underset{\sim}{x} + A\underset{\sim}{x} + \Delta A\underset{\sim}{x} = \underset{\sim}{b} + \Delta\underset{\sim}{b}$$

$$\Rightarrow A(I+A^{-1}\Delta A)\Delta\underset{\sim}{x} = \Delta\underset{\sim}{b} - \Delta A\underset{\sim}{x} \quad (\because A\underset{\sim}{x} = \underset{\sim}{b})$$

$$\Rightarrow \Delta\underset{\sim}{x} = (I+A^{-1}\Delta A)^{-1}\left[A^{-1}\Delta\underset{\sim}{b} - A^{-1}\Delta A\underset{\sim}{x}\right]$$

$$\Rightarrow \|\Delta\underset{\sim}{x}\| \leq \|(I+A^{-1}\Delta A)^{-1}\|\,\|A^{-1}\|\left(\|\Delta\underset{\sim}{b}\| + \|\Delta A\|\,\|\underset{\sim}{x}\|\right)$$

$$\leq \frac{1}{1-\|A^{-1}\Delta A\|}\|A^{-1}\|\left(\|\Delta\underset{\sim}{b}\| + \|\Delta A\|\,\|\underset{\sim}{x}\|\right)$$

(From eqn. (2))

$$\leq \frac{1}{1-\|A^{-1}\|\,\|\Delta A\|}\cdot\|A^{-1}\|\cdot\left(\|\Delta\underset{\sim}{b}\| + \|\Delta A\|\,\|\underset{\sim}{x}\|\right)$$

$$= \frac{\|A\|\,\|A^{-1}\|}{\|A\| - \|A\|\,\|A^{-1}\|\,\|\Delta A\|}\left[\|\Delta\underset{\sim}{b}\| + \|\Delta A\|\,\|\underset{\sim}{x}\|\right]$$

$$= \frac{k(A)}{\|A\| - k(A)\|\Delta A\|}\left[\|\Delta\underset{\sim}{b}\| + \|\Delta A\|\,\|\underset{\sim}{x}\|\right]$$

So

$$\|\Delta x\| \leq \frac{k(A)}{1 - k(A)\frac{\|\Delta A\|}{\|A\|}} \cdot \left[\frac{\|\Delta b\|}{\|A\|} + \frac{\|\Delta A\| \, \|x\|}{\|A\|} \right]$$

$$\Rightarrow \frac{\|\Delta x\|}{\|x\|} \leq \frac{k(A)}{1 - k(A)\frac{\|\Delta A\|}{\|A\|}} \cdot \left[\frac{\|\Delta b\|}{\|A\| \, \|x\|} + \frac{\|\Delta A\|}{\|A\|} \right]$$

$$\leq \frac{k(A)}{1 - k(A)\frac{\|\Delta A\|}{\|A\|}} \left[\frac{\|\Delta b\|}{\|b\|} + \frac{\|\Delta A\|}{\|A\|} \right]$$

($\because$ $Ax = b$ and $\|Ax\| \leq \|A\| \, \|x\|$)

Hence the theorem is proved.

4. I completed this assignment using Octave. Following is the results. (Attaching the code at the end)

p-norm	$\|Ax\|$	$\|A\|$	Difference
1	86.429	86.726	-0.2963
2	10.824	10.824	-2.149×10^{-6}
∞	5.2236	5.2456	-0.02

Slight diffe. is due to insufficient Sampling.

Hence we verify that $\sup_{\|x\|=1} \|Ax\| = \|A\|$.

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Norm_practice.m

```
A = randn(100, 2);
one_norm = [];
two_norm = [];
inf_norm = [];

for p = [1, 2, inf]
  for i = 1:1000
    x = randn(2, 1);
    x = x / norm(x, p);
    norm_of_Ax = norm(A*x, p);
    if p == 1
      one_norm = [one_norm, norm_of_Ax];
    elseif p == 2
      two_norm = [two_norm, norm_of_Ax];
    elseif p == inf
      inf_norm = [inf_norm, norm_of_Ax];
    endif
  endfor
endfor

max_Ax_one = max(one_norm);
max_Ax_two = max(two_norm);
max_Ax_inf = max(inf_norm);

norm_A_one = norm(A, 1);
norm_A_two = norm(A, 2);
norm_A_inf = norm(A, inf);

disp(max_Ax_one - norm_A_one)
disp(max_Ax_two - norm_A_two)
disp(max_Ax_inf - norm_A_inf)

disp(max_Ax_one)
disp(max_Ax_two)
disp(max_Ax_inf)
disp(norm_A_one)
disp(norm_A_two)
disp(norm_A_inf)
```

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