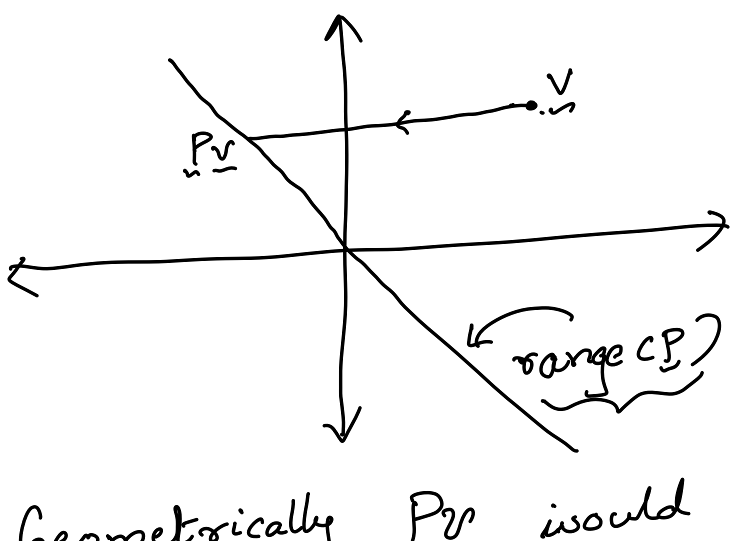
Projectors

A projection on a vector space V is a linear operator $\underset{\sim}{P} : V \rightarrow V$ such that $\underset{\sim}{P}^2 = \underset{\sim}{P}$

In the finite-dimensional case, a square matrix $\underset{\sim}{P}$ is called projector matrix if it is equal to its square i.e $\underset{\sim}{P}^2 = \underset{\sim}{P}$

The condition $\underset{\sim}{P}^2 = \underset{\sim}{P}$ is called idempotent condition!



Geometrically $\underset{\sim}{P}\underset{\sim}{v}$ would be a shadow projected by $\underset{\sim}{v}$ onto range (P) if one were to shine light onto range (P)!

From what direction does the light shine it is from $\underset{\sim}{v}$ to $\underset{\sim}{P}\underset{\sim}{v}$

So, $\underset{\sim}{P}\underset{\sim}{v} - \underset{\sim}{v}$ is the direction of light source

$$\underset{\sim}{P}\left[\underbrace{\underset{\sim}{P}\underset{\sim}{v} - \underset{\sim}{v}}\right] = \underset{\sim}{P}^2\underset{\sim}{v} - \underset{\sim}{P}\underset{\sim}{v}$$
$$= \underset{\sim}{P}\underset{\sim}{v} - \underset{\sim}{P}\underset{\sim}{v} = 0$$

i.e. $\underbrace{\underset{\sim}{P}\underset{\sim}{v} - \underset{\sim}{v}} \in \text{null}(\underset{\sim}{P})$

Remarks :-

1. $\underset{\sim}{P} \in \mathbb{R}^{m \times m}$, $\underset{\sim}{P}^2 = \underset{\sim}{P}$ (Idempotency) is satisfied by a projector

2. $\underset{\sim}{P}\underset{\sim}{v} - \underset{\sim}{v} \in \text{null}(P)$ and is the direction of projection of $\underset{\sim}{v}$ onto range (P)

3. If $\underset{\sim}{P}$ is a projector and vector $\underset{\sim}{x} \in \text{range}(P)$, then $\underset{\sim}{P}\underset{\sim}{x} = \underset{\sim}{x}$

Pf :- If $\underset{\sim}{x} \in$ range (P), then

$\underset{\sim}{x} = \underset{\sim}{P}\underset{\sim}{y}$ for some $\underset{\sim}{y}$

then $\underset{\sim}{P}\underset{\sim}{x} = \underset{\sim}{P}(\underset{\sim}{P}\underset{\sim}{y})$

$= \underset{\sim}{P}^2\underset{\sim}{y} = \underset{\sim}{P}\underset{\sim}{y} = \underset{\sim}{x}$

i.e $\underset{\sim}{x}$ lies exactly in its own shadow.

(4) If $\underset{\sim}{P}$ is a projector, then $(\underset{\sim}{I} - \underset{\sim}{P})$ is also a projector

$$(\underset{\sim}{I} - \underset{\sim}{P})^2 = (\underset{\sim}{I} - \underset{\sim}{P})(\underset{\sim}{I} - \underset{\sim}{P})$$
$$= \underset{\sim}{I} - \underset{\sim}{P} - \underset{\sim}{P} + \underset{\sim}{P}^2 = \underset{\sim}{I} - \underset{\sim}{P}$$

$\underset{\sim}{I} - \underset{\sim}{P}$ is called complementary projector to $\underset{\sim}{P}$!

onto what space does $(I - P)$ project? range $(I - P)$

Consider any vector in range $(I - P)$

$\rightarrow (\underset{\sim}{I} - P)\underset{\sim}{x} = \underset{\sim}{x} - \underset{\sim}{P}\underset{\sim}{x}$

$\underset{\sim}{P}(\underset{\sim}{x} - \underset{\sim}{P}\underset{\sim}{x}) = 0$

$\Rightarrow \underset{\sim}{x} - \underset{\sim}{P}\underset{\sim}{x} \in null(\underset{\sim}{P})$

This means $\underline{range(\underset{\sim}{I}-\underset{\sim}{P}) \subseteq null(\underset{\sim}{P})}$ — (1)

Similarly let us consider any vector $\underset{\sim}{x}$ in $null(P)$ i.e $\underset{\sim}{P}\underset{\sim}{x} = \underset{\sim}{0}$

then $(\underset{\sim}{I}-\underset{\sim}{P})\underset{\sim}{x}$

$= \underset{\sim}{x} - \underset{\sim}{P}\underset{\sim}{x}$

$= \underset{\sim}{x}$

$\underline{null(\underset{\sim}{P}) \subseteq range(\underset{\sim}{I}-\underset{\sim}{P})}$ — (2)

From (1) and (2) $range(\underset{\sim}{I}-\underset{\sim}{P}) = null(\underset{\sim}{P})$

we can also deduce $range(\underset{\sim}{P}) = null(\underset{\sim}{I}-\underset{\sim}{P})$

(iv) $null(I-P) \cap null(P) = \{0\}$

i.e $range(P) \cap null(P) = \{0\}$

Pf:- Let $\underset{\sim}{v}$ be in both $null(\underset{\sim}{P})$ and $null(\underset{\sim}{I}-\underset{\sim}{P})$

Then $\underset{\sim}{P}\underset{\sim}{v} = (\underset{\sim}{I}-\underset{\sim}{P})\underset{\sim}{v} = \underset{\sim}{0}$

$(\underset{\sim}{I}-\underset{\sim}{P})\underset{\sim}{v} = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{v} - \underset{\sim}{P}\underset{\sim}{v} = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{v} = \underset{\sim}{0}$ $\Rightarrow$ $null(\underset{\sim}{I}-\underset{\sim}{P}) \cap null(\underset{\sim}{P}) = \{0\}$

$\Rightarrow range(P) \cap null(P) = \{0\}$

This say that projector P seperates $\mathbb{R}^m$ into two subspaces.

Orthogonal Projectors :-

An orthogonal projector is one that projects onto a subspace S_1 along subspace S_2 where S_1 and S_2 are orthogonal subspaces.

Thm :- A projector $\underset{\sim}{P}$ is orthogonal projector if and only if $\underset{\sim}{P} = \underset{\sim}{P}^T$

Pf :- step1 If $\underbrace{\underset{\sim}{P} = \underset{\sim}{P}^T}$, we need to show projector $\underset{\sim}{P}$ is orthogonal

Consider an inner product between a vector in S_1 i.e $\underset{\sim}{P}\underset{\sim}{x} \in S_1$ and vector $(\underset{\sim}{I}-\underset{\sim}{P})\underset{\sim}{y} \in S_2$

$$(\underset{\sim}{P}\underset{\sim}{x}, (\underset{\sim}{I}-\underset{\sim}{P})\underset{\sim}{y}) = (\underset{\sim}{P}\underset{\sim}{x})^T (\underset{\sim}{I}-\underset{\sim}{P})\underset{\sim}{y}$$

$$= \underset{\sim}{x}^T \underset{\sim}{P}^T (\underset{\sim}{I}-\underset{\sim}{P})\underset{\sim}{y}$$

$$= \underset{\sim}{x}^T \underset{\sim}{P}(\underset{\sim}{I} - \underset{\sim}{P})\underset{\sim}{y}$$

$$= \underset{\sim}{x}^T [\underset{\sim}{P} - \underset{\sim}{P}^2] \underset{\sim}{y} = \underset{\sim}{0}$$

Step 2 :-

To prove :→ An orthogonal projector $\underset{\sim}{P} \in \mathbb{R}^{m \times m}$ ($\underset{\sim}{P}$ projects onto S_1 along S_2 where $S_1 \perp S_2$)

satisfies $\underset{\sim}{P} = \underset{\sim}{P}^T$

Let S_1 have dimension $n < m$ and let $\{\underset{\sim}{q}_1, \underset{\sim}{q}_2, \dots \underset{\sim}{q}_m\}$ be the basis for $\mathbb{R}^m$ where $\{\underset{\sim}{q}_1, \underset{\sim}{q}_2, \dots \underset{\sim}{q}_n\}$ be the basis for S_1 and $\{\underset{\sim}{q}_{n+1}, \dots \underset{\sim}{q}_m\}$ be the basis for S_2.

Let us try to construct SVD for $\underset{\sim}{P}$.

for $j < n$, $\underset{\sim}{P}\,\underset{\sim}{q}_j = \underset{\sim}{q}_j$ ✓

and $j > n$, $\underset{\sim}{P}\,\underset{\sim}{q}_j = \underset{\sim}{0}$

Now let us construct a matrix $\underset{\sim}{Q}$

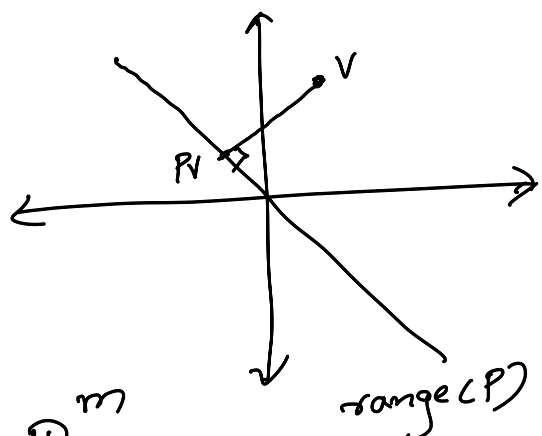
$$\underset{\sim}{Q} = \begin{bmatrix} | & | & | & & | \\ q_1 & q_2 & q_3 & \cdots & q_m \\ | & | & | & & | \end{bmatrix}$$

$$\underset{\sim}{P}\underset{\sim}{Q} = \begin{bmatrix} | & | & & | & | & | & & | \\ q_1 & q_2 & \cdots & q_n & 0 & 0 & \cdots & 0 \\ | & | & & | & | & | & & | \end{bmatrix}$$

$$\underset{\sim}{Q}^T \underset{\sim}{P} \underset{\sim}{Q} = \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{bmatrix} = \underset{\sim}{\Sigma}$$

$\underset{\sim}{P} = \underset{\sim}{Q}\,\underset{\sim}{\Sigma}\,\underset{\sim}{Q}^T$ we constructed

SVD of $\underset{\sim}{P}$ $\Rightarrow$ $\underset{\sim}{P} = \underset{\sim}{P}^T$



Orthogonal Projectors corresponding to a subspace spanned by orthonormal basis $V^n \subseteq \mathbb{R}^m$

Let us consider an n-dimensional subspace in $\mathbb{R}^m$ and $\{\underset{\sim}{q}_1, \underset{\sim}{q}_2, \ldots \underset{\sim}{q}_n\}$ be the set of n orthonormal vectors in $\mathbb{R}^m$ spanning our n-dimensional subspace.

Let
$$\hat{Q} = \begin{bmatrix} | & | & | & & | \\ q_1 & q_2 & q_3 & \cdots & q_n \\ | & | & | & & | \end{bmatrix}_{m \times n}$$

Let $v \in \mathbb{R}^m$ can be decomposed into a component in the column space $\hat{Q}$ plus a component r perpendicular to column space of $\hat{Q}$

$$v = r + \underbrace{\sum_{i=1}^{n} (q_i^T v)\, q_i}$$

The map $v \mapsto \sum_{i=1}^{n} (q_i^T v)\, q_i$ is an orthogonal projection onto range $(\hat{Q})$

$$y = Pv = \sum_{i=1}^{n} \underbrace{(q_i^T v)\, q_i}$$

$$y = Pv = \sum_{i=1}^{n} \underbrace{q_i (q_i^T v)}$$

$$= \underbrace{\sum_{i=1}^{n} q_i q_i^T}_{P} v$$

$$y = \hat{Q}\hat{Q}^T v$$

where $\hat{Q} = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}$

$$\boxed{P = \hat{Q}\hat{Q}^T}$$

$$\tilde{Q} = \begin{bmatrix} | & | & | & & | \\ \tilde{q}_1 & \tilde{q}_2 & \tilde{q}_3 & \cdots & \tilde{q}_n \\ | & | & | & & | \end{bmatrix}$$

Show $P = \tilde{P}$? $\qquad \tilde{P} = \tilde{Q}\tilde{Q}^T$

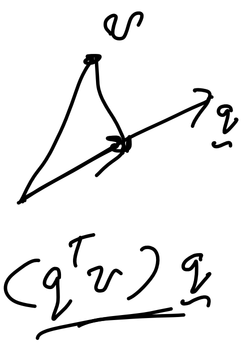
Exercise

* Complement of orthogonal projector is also orthogonal projector.

i.e $P = P^T$ then $(I - P)^T = (I - P)$

The complements projects onto space orthogonal to range ($\underset{\sim}{P}$)

* Eg:- Rank 1 orthogonal projector that isolates component of a vector $\underset{\sim}{v}$ in a single direction $P_q = \underset{\sim}{q}\,\underset{\sim}{q}^T$



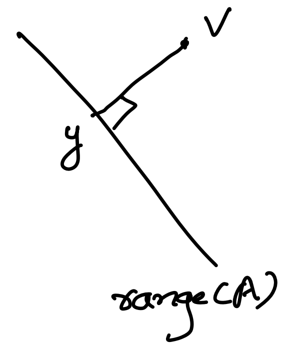
$$(\underset{\sim}{q}\,\underset{\sim}{q}^T)\,\underset{\sim}{v} = \bar{v}$$

$$= (\underset{\sim}{q}^T \underset{\sim}{v})\,\underset{\sim}{q}$$

<u>Projection onto</u> n-dimen<u>sional</u> <u>subspace</u> represented <u>by arbitrary</u> <u>basis</u> :-

Let the subspace be spanned by the linearly independent vectors $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots \underset{\sim}{a_n}\}$. $\underset{\sim}{A} \in \mathbb{R}^{m \times n}$ have the columns $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots \underset{\sim}{a_n}\}$

$$A = \begin{bmatrix} | & | & | & & | \\ a_1 & a_2 & a_3 & \cdots & a_n \\ | & | & | & & | \end{bmatrix}_{m\times n}$$



Let $y \in \text{range}(A)$ be the projected vector. We know that $(y - v) \perp \text{range}(A)$

$$a_j^T (y - v) = 0 \qquad \text{— (1)}$$

for every $j = 1 \ldots n$

Since $y \in \text{range}(A)$, we can write

$y = Ax$ for some x.

and hence (1) becomes

$$a_j^T (Ax - v) = 0 \quad \forall\, j = 1 \ldots n$$

$$A^T [Ax - v] = 0$$

$$\Rightarrow A^T A x = A^T v$$

$$x = (A^T A)^{-1} A^T v$$

(show $(A^T A)^{-1}$ exists?)

$$y = Pv = Ax$$

$$\Rightarrow Pv = A (A^T A)^{-1} A^T v \quad \forall\, v \in \mathbb{R}^m$$

$$\Rightarrow \boxed{\underset{\sim}{P} = \underset{\sim}{A}(\underset{\sim}{A}^T\underset{\sim}{A})^{-1}\underset{\sim}{A}^T}$$

" Show that this $\underset{\sim}{P}$ is same as $\underset{\sim}{P}$ obtained by $\hat{Q}\hat{Q}^T$ before ? "

Exercise!