Assignment-3

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1. (a) Computing $2x$ as $x+x$

$$\tilde{f}(x) = fl(x) \oplus fl(x)$$

$$= [x(1+\varepsilon)] \oplus [x(1+\varepsilon)] \quad , \quad |\varepsilon| \leq \varepsilon_{machine}$$

$$= [x(1+\varepsilon) + x(1+\varepsilon)](1+\varepsilon_1), \quad |\varepsilon_1| \leq \varepsilon_{machine}$$

$$= 2x(1+\varepsilon+\varepsilon_1+\varepsilon\varepsilon_1)$$

$$= x(1+\varepsilon+\varepsilon_1+\varepsilon\varepsilon_1) + x(1+\varepsilon+\varepsilon_1+\varepsilon\varepsilon_1)$$

We observe $\varepsilon+\varepsilon_1+\varepsilon\varepsilon_1 \leq 2\varepsilon_{machine} + \varepsilon_{machine}^2$

$$= O(\varepsilon_{machine})$$

So $\tilde{f}(x) = x(1+O(\varepsilon_m)) + x(1+O(\varepsilon_m))$

$$= \tilde{x} + \tilde{x} = f(\tilde{x})$$

where $\dfrac{|\tilde{x}-x|}{|x|} = O(\varepsilon_m)$

Hence it's backward stable.

(b) Computing x^2 by $x \times x$

$\tilde{f}(x) = fl(x) \otimes fl(x)$

$= x(1+\varepsilon) \otimes x(1+\varepsilon) \quad , \quad |\varepsilon| \leq \varepsilon_m$

$= [x(1+\varepsilon) \times x(1+\varepsilon)](1+\varepsilon_1) \quad , \quad |\varepsilon_1| \leq \varepsilon_m$

$= [x(1+\varepsilon)\sqrt{1+\varepsilon_1}] \times [x(1+\varepsilon)\sqrt{1+\varepsilon_1}]$

We observe $\dfrac{|(1+\varepsilon)\sqrt{1+\varepsilon_1} - 1|}{|1|} \leq \dfrac{|(1+\varepsilon_m)^2 - 1|}{|1|}$

If $\tilde{x}$ were $x(1+\varepsilon_m)^2 = x(1+2\varepsilon_m + \varepsilon_m^2)$

$= x(1+O(\varepsilon_m))$

, even then $\tilde{x}$ was within $O(\varepsilon_m)$ of x.

Now as $(1+\varepsilon)\sqrt{1+\varepsilon_1}$ is even closer to 1 than $(1+\varepsilon_m)^2$, so $\tilde{x} = x(1+\varepsilon)\sqrt{1+\varepsilon_1}$

$= x(1+O(\varepsilon_m))$

is within $O(\varepsilon_m)$ of x.

Hence $\tilde{f}(x) = f(\tilde{x}) = \tilde{x} \times \tilde{x}$.

Hence it's backward stable.

(c) Computing 1 by x/x

$$\tilde{f}(x) = fl(x) \oslash fl(x)$$

$$= x(1+\varepsilon) \oslash x(1+\varepsilon) \quad , \quad |\varepsilon| \leq \varepsilon_m$$

$$= \frac{x(1+\varepsilon)}{x(1+\varepsilon)} \cdot (1+\varepsilon_1) \quad , \quad |\varepsilon_1| \leq \varepsilon_m$$

$$= 1+\varepsilon_1$$

$$f(\tilde{x}) = \tilde{x}/\tilde{x} = \underline{1}$$

So it's not backward stable

But $\dfrac{|\tilde{f}(x) - f(\tilde{x})|}{|f(\tilde{x})|} = \dfrac{|\varepsilon_1|}{|1|} \leq \varepsilon_{machine}$
$= O(\varepsilon_m)$

Hence it's stable

(d) $A = U \Sigma V^T$, U and V are orthogonal.
Σ is diagonal.

Here input is A and o/p is U, Σ, V^T.

So backward stability means giving exactly right answer to nearly right question.

This means answers $\tilde{U}, \tilde{V}, \tilde{\Sigma}$ should be exactly same as theoritically correct decomposition of $A + \delta A$ where $\frac{\|\delta A\|}{\|A\|} = O(\varepsilon_m)$.

Correct decomposition of $A + \delta A$, will result in orthogonal U' and V' matrix. But $\tilde{U}, \tilde{V}$ calculated in computer in general won't be orthogonal due to round-off errors. Hence this algorithm can't be backward stable.

Stability in this context means

$$\frac{\|\tilde{f}(x) - f(\tilde{x})\|}{\|f(\tilde{x})\|} = O(\varepsilon_m)$$

$$\Rightarrow \frac{\|\tilde{U} \tilde{\Sigma} \tilde{V}^T - (A + \delta A)\|}{\|A + \delta A\|} = O(\varepsilon_m)$$

So if if you multiply computed $\tilde{U}, \tilde{\Sigma}, \tilde{V}$ and take relative error w.r.t $A + \delta A$, then it has to be of the order of $O(\varepsilon_m)$.

(c) $\underset{\sim}{x}, \underset{\sim}{y} \in R^m$

$$\underset{\sim}{x}^T \underset{\sim}{y} = (x_1 \otimes y_1) \oplus (x_2 \otimes y_2) \oplus \dots$$

Consider when $m = 1$

$$\underset{\sim}{x}^T \underset{\sim}{y} = x_1 \times y_1$$

$$\tilde{f}(x) = fl(x_1) \otimes fl(y_1) = x_1(1+\varepsilon_1) \otimes y_1(1+\varepsilon_2)$$

$$|\varepsilon_1|, |\varepsilon_2| \leq \varepsilon_m$$

$$= [x_1(1+\varepsilon_1) \times y_1(1+\varepsilon_2)](1+\varepsilon_3)$$

$$|\varepsilon_3| \leq \varepsilon_m$$

$$= \left[x_1(1+\varepsilon_1)\sqrt{1+\varepsilon_3}\right] \times \left[y_1(1+\varepsilon_2)\sqrt{1+\varepsilon_3}\right]$$

$$= x_1(1+O(\varepsilon_m)) \times y_1(1+O(\varepsilon_m))$$

$$= \tilde{x}_1 \times \tilde{y}_1 = f(\tilde{x}).$$

Hence it's backward stable for $m = 1$.

When $m = 2$:

$$\underset{\sim}{x}^T \underset{\sim}{y} = (x_1 \times y_1) + (x_2 \times y_2)$$

$$\tilde{f}(x) = (\tilde{x}_1 \times \tilde{y}_1) \oplus (\tilde{x}_2 \times \tilde{y}_2)$$

$$= [(\tilde{x}_1 \times \tilde{y}_1) + (\tilde{x}_2 \times \tilde{y}_2)](1+\varepsilon), \quad |\varepsilon| \leq \varepsilon_m$$

$$= [(x_1(1+O(\varepsilon_m)) \times y_1(1+O(\varepsilon_m))) + (x_2(1+O(\varepsilon_m)) \times y_2(1+O(\varepsilon_m)))](1+\varepsilon)$$

$$= \left[x_1 (1 + O(\varepsilon_m)) \sqrt{1+\varepsilon} \times y_1 (1 + O(\varepsilon_m)) \sqrt{1+\varepsilon} \right] +$$

$$\left[x_2 (1 + O(\varepsilon_m)) \sqrt{1+\varepsilon} \times y_2 (1 + O(\varepsilon_m)) \sqrt{1+\varepsilon} \right]$$

We have prev. seen

$$[1 + O(\varepsilon_m)] \sqrt{1+\varepsilon} = 1 + O(\varepsilon_m)$$

$$= \left[x_1 (1 + O(\varepsilon_m)) \times y_1 (1 + O(\varepsilon_m)) \right] +$$

$$\left[x_2 (1 + O(\varepsilon_m)) \times y_2 (1 + O(\varepsilon_m)) \right]$$

$$= (\tilde{x}_1 \times \tilde{y}_1) + (\tilde{x}_2 \times \tilde{y}_2) = f(\tilde{x})$$

Hence for $m = 2$, it's backward stable.

We also observe this is a general trend and will go on forever for any m b/c we can again rewrite this $\tilde{x}$ as $x(1 + O(\varepsilon_m))$ and multiplying it by $\sqrt{1+\varepsilon_i}$ where $|\varepsilon_i| \le \varepsilon_m$, will again result in $1 + O(\varepsilon_m)$.

Hence inner product is backward stable for any m.

(4) $A = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$

Characteristic eqn. $\det(A - \lambda I) = 0$

$\Rightarrow (1-\lambda)^3 = 0$

$\Rightarrow 1 - \lambda^3 + 3\lambda^2 - 3\lambda = 0$

Let's perturb this eqn. slightly.

$-\lambda^3 + 3\lambda^2 - 2.9999\lambda + 0.9999 = 0$

Error is of the order of 10^{-4}.

$\Rightarrow (1-\lambda)(0.99-\lambda)(1.01-\lambda) = 0$

$\Rightarrow \lambda = 1, 0.99, 1.01$

Error in eigen value is of the order of 10^{-2}.

So if error in coefficients in $O(\varepsilon_m)$, then error in eigen value is $O(\sqrt{\varepsilon_m})$ which is bigger than $O(\varepsilon_m)$.

Hence this is unstable algorithm.

2. (a) $G = A^T A \quad , \quad A \in \mathbb{R}^{m \times n}$

$$x^T G x = x^T A^T A x = (Ax)^T (Ax)$$
$$= \|Ax\|_2^2 \geq 0$$

This is b/c norm is non-negative

(b) let, $x \in \text{Null}(A)$. $\quad Ax = 0$

$$\Rightarrow A^T A x = 0$$

Hence $x \in \text{Null}(A^T A)$. $\quad \text{Null}(A) \subseteq \text{Null}(A^T A)$

let, $x \in \text{Null}(A^T A)$, $\quad A^T A x = 0$

$$\Rightarrow \cancel{A}\ x^T A^T A x = 0$$
$$\Rightarrow \|Ax\|_2^2 = 0$$

So $Ax = 0 \Rightarrow x \in \text{Null}(A)$

So $\text{Null}(A^T A) \subseteq \text{Null}(A)$.

Hence $\text{Null}(A) = \text{Null}(A^T A) = p$

$\text{Rank}(A) = n - \text{nullity}(A) = n - p$

$\text{Rank}(A^T A) = n - \text{nullity}(A^T A) = n - p$

Hence $\text{rank}(A) = \text{rank}(A^T A)$.

(c) $A^T A \underset{\sim}{v} = \sigma^2 \underset{\sim}{v}$ (given)

$\Rightarrow \quad A A^T (A \underset{\sim}{v}) = \sigma^2 (A \underset{\sim}{v})$

So $A\underset{\sim}{v}$ is eigen ~~val~~ vector of AA^T with eigen value σ^2.

Now, $\underset{\sim}{v}^T A^T A \underset{\sim}{v} = \sigma^2 \underset{\sim}{v}^T \underset{\sim}{v}$

$\Rightarrow \quad \| A \underset{\sim}{v} \|_2^2 = \sigma^2 \| \underset{\sim}{v} \|_2^2$

So when $\underbrace{\| \underset{\sim}{v} \| = 1}$, then $\| A \underset{\sim}{v} \| = \sigma$.

This is true b/c in SVD, rows of V^T are orthonormal.

We showed $A\underset{\sim}{v}$ is eigen vector of AA^T and $\| A \underset{\sim}{v} \| = \sigma$. So $\dfrac{A\underset{\sim}{v}}{\sigma}$ is unit eigen vector of AA^T.

(d) Construct U as columns $\underset{\sim}{u}_i$.

$$U = \begin{bmatrix} \underset{\sim}{u}_1 & \underset{\sim}{u}_2 & \cdots & \underset{\sim}{u}_m \end{bmatrix}_{m \times m}$$

$$\Sigma = \begin{bmatrix} 1 & \ddots & 0 & \\ 0 & & 1 & 0 \ \ddots \\ & 0 & & \end{bmatrix}_{m \times n}$$

$$\cancel{U\Sigma = \left[\, \sigma_1 \underset{\sim}{u_1} \;\middle|\; \sigma_2 \underset{\sim}{u_2} \;\middle|\; \cdot \;\middle|\; \cdot \;\middle|\; \sigma_m u_m \right]}$$

(d) Construct U as columns $\underset{\sim}{u_i}$.

$$U = \begin{bmatrix} \underset{\sim}{u_1} & \big| & \underset{\sim}{u_2} & \big| & \cdots & \big| & \underset{\sim}{u_m} \end{bmatrix}_{m \times m}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ 0 & \ddots & \sigma_n \\ & 0 & \end{bmatrix}_{m \times n}$$

$$U\Sigma = \begin{bmatrix} \sigma_1 \underset{\sim}{u_1} & \big| & \cdots & \big| & \sigma_n \underset{\sim}{u_n} & \big| & 0 & \big| & \cdots & \big| & 0 \end{bmatrix}_{m \times n}$$

Now $\underset{\sim}{u_i} = \dfrac{A \underset{\sim}{v_i}}{\sigma_i}$

So

$$U\Sigma = \begin{bmatrix} A \underset{\sim}{v_1} & \big| & \cdots & \big| & A \underset{\sim}{v_n} & \big| & 0 & \cdots & \big| & 0 \end{bmatrix}_{m \times n}$$

$$= A \begin{bmatrix} \underset{\sim}{v_1} & \big| & \cdots & \big| & \underset{\sim}{v_n} & \big| & 0 & \big| & \cdots & \big| & 0 \end{bmatrix}_{m \times n}$$

$$= AV$$

$\Rightarrow \quad A = U \Sigma V^T$ b/c V is orthogonal.

It has orthonormal columns.

3. (a) For each colour channel, we have 500×500 pixels.

It turns out that by having around 300 singular values of each colour channel, we can reproduce signal.

(b) Number of original pixels $= 500 \times 500 \times 3$

$= 750000$

For 300 singular values to be taken, in reduced SVD, $\tilde{U} \in R^{500 \times 300}$

$\tilde{\Sigma} \in R^{300 \times 300}$, $\tilde{V}^T \in R^{300 \times 500}$

So number of pixels to be transmitted (1 numbers)

$= (500 \times 300) + (300 \times 500) + 300$

(No need to send 0 in diagonal matrix)

$= 300000 + 300$

$= 300300$

(c) 2 norm error $= [55.87, 57.18, 57.74]$

(one for each channel)

Frobenius norm error $= [335.28, 341.32, 351.74]$

It was also verified that,

$$\| A - A_{300} \|_2 = \sigma_{301}$$

$$\| A - A_{300} \|_F = \sqrt{\sigma_{301}^2 + \cdots + \sigma_{500}^2}$$

These singular values were same as matrix norms.

(5) $A = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}$

QR Factorization by Hand

$r_{11} = \sqrt{8^2 + 3^2 + 4^2} = \sqrt{89}$

$q_1 = \frac{1}{\sqrt{89}} \begin{bmatrix} 8 \\ 3 \\ 4 \end{bmatrix}$

$r_{12} = \underset{\sim}{q_1}^T \underset{\sim}{a_1} = \frac{59}{\sqrt{89}}$

$r_{22} = \left\| \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} - \frac{59}{89} \begin{bmatrix} 8 \\ 3 \\ 4 \end{bmatrix} \right\| = 8.24$

$q_2 = \begin{bmatrix} -0.522 \\ 0.365 \\ 0.770 \end{bmatrix}$

Similarly $r_{13} = \frac{73}{\sqrt{89}}$, $r_{23} = 0.9629$

$r_{33} = 4.6509$

$q_3 = \begin{bmatrix} -0.0127 \\ 0.9004 \\ -0.4348 \end{bmatrix}$

QR Factorization From Code

By using numpy QR decomp., we obtained r values $-ve$ of what we got in hand method.
Consequently Q matrix was also $-ve$.

This was possible b/c there was no restriction on R.
If we impose $r_{11}, r_{22}, r_{33} > 0$ rule on R, then R will be uniquely determined.
(r_{ij} are unique b/c of dot product)

As magic matrix is full rank matrix, then uniqueness will also be ensured.

6. (a) $A = [\, 1 \mid x \mid x^2 \mid x^3 \,]_{m \times 4}$

$Q = [\, q_0(x) \mid q_1(x) \mid q_2(x) \mid q_3(x) \,]_{m \times 4}$

$R \in R^{4 \times 4}$ upper triangular.

$r_{11} \, q_0(x) = 1 \implies q_0(x) = 1/r_{11}$

$r_{12} \, q_0(x) + r_{22} \, q_1(x) = x$

$$\implies q_1(x) = \frac{x - r_{12}/r_{11}}{r_{22}}$$

$$r_{13} q_0(x) + r_{23} q_1(x) + r_{33} q_2(x) = x^2$$

$$\Rightarrow \frac{r_{13}}{r_{11}} + \frac{r_{23}}{r_{22}} x - \frac{r_{23} r_{12}}{r_{11} r_{22}} + r_{33} q_2(x) = x^2$$

$$\Rightarrow q_2(x) = \frac{x^2 - \frac{r_{23}}{r_{22}} x + \frac{r_{23} r_{12}}{r_{11} r_{22}} - \frac{r_{13}}{r_{11}}}{r_{33}}$$

$$r_{14} q_0(x) + r_{24} q_1(x) + r_{34} q_2(x) + r_{44} q_3(x) = x^3$$

$$\Rightarrow \frac{r_{14}}{r_{11}} + \frac{r_{24}}{r_{22}} x - \frac{r_{24} r_{12}}{r_{11} r_{22}} + r_{34} q_2(x) + r_{44} q_3(x) = x^3$$

$$\Rightarrow q_3(x) = \frac{x^3 - r_{34} q_2(x) - \frac{r_{24}}{r_{22}} x + \frac{r_{24} r_{12}}{r_{11} r_{22}} - \frac{r_{14}}{r_{11}}}{r_{44}}$$

where $q_2(x)$ is as defined above.

And $$r_{ij} = \int_{-1}^{1} q_{i-1}(x) \cdot x^{j-1} \, dx$$

r_{ii} = norm of $q_{i-1}(x)$ before making it unit.

Depending on even or odd function, it can add up / cancel out and computing it is easy.

$$r_{11} = \int_{-1}^{1} q_0(x)\, x^0 \, dx$$

and $\int_{-1}^{1} q_0(x)\, q_0(x)\, dx = 1$ (orthonormality)

$$\Rightarrow \quad q_0(x) = \frac{1}{\sqrt{2}} \quad , \quad r_{11} = \sqrt{2}$$

$$r_{12} = \int_{-1}^{1} q_0(x)\, x \, dx = \frac{1}{\sqrt{2}} \left[\frac{x^2}{2} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2}} \times 0 = 0$$

$$q_1(x) = \frac{x}{r_{22}}$$

$$\int_{-1}^{1} q_1(x)\, q_1(x)\, dx = 1 \quad \Rightarrow \quad \int_{-1}^{1} x^2 \, dx = r_{22}^2$$

$$\Rightarrow \quad \frac{2}{3} = r_{22}^2$$

$$\Rightarrow \quad r_{22} = \sqrt{\frac{2}{3}}$$

$$q_1(x) = \sqrt{\frac{3}{2}}\, x$$

$$r_{13} = \int_{-1}^{1} q_0(x)\, x^2 \, dx = \sqrt{2}/3$$

$$r_{23} = \int_{-1}^{1} q_1(x)\, x^2 \, dx = 0 \quad \text{(odd function)}$$

$$q_2(x) = \frac{x^2 - \frac{\sqrt{2}}{3} \cdot \frac{1}{\sqrt{2}}}{r_{33}} = \frac{x^2 - \frac{1}{3}}{r_{33}}$$

$$\int_{-1}^{1} [q_2(x)]^2 \, dx = 1$$

$$\Rightarrow \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx = r_{33}^2$$

$$\Rightarrow \int_{-1}^{1} \left(x^4 - \frac{2x^2}{3} + \frac{1}{9}\right) dx = r_{33}^2$$

$$\Rightarrow r_{33}^2 = \frac{2}{5} - \frac{2}{3} \cdot \frac{1}{3} \cdot 2 + \frac{2}{9} = \frac{2}{5} - \frac{2}{9} = \frac{8}{45}$$

$$\Rightarrow r_{33} = \sqrt{8/45}$$

Similarly, $r_{14} = \int q_0(x)\, x^3 \, dx = 0$ (odd Function)

$r_{24} = \int q_1(x)\, x^3 \, dx = \sqrt{\frac{3}{2}} \cdot \frac{1}{\cancel{4}5} \cdot 2 = \frac{\sqrt{6}}{5}$

$r_{34} = \int q_2(x)\, x^3 \, dx = 0$ (odd funct)

Then we can find

$$q_3(x) = \frac{x^3 - \frac{3}{\sqrt{5}} x}{r_{44}}$$

r_{44} can be found similarly.

(b) We know $q_i(x)$ are orthonormal.

Also $q_0(x) = \frac{1}{\sqrt{2}}$ from last question.

$$\langle q_0(x), q_{n-1}(x) \rangle = 0$$

$$\Rightarrow \int_{-1}^{1} \frac{1}{\sqrt{2}} q_{n-1}(x)\, dx = 0$$

$$\Rightarrow \int_{-1}^{1} q_{n-1}(x)\, dx = 0$$

— (proved)

7. $A = \begin{bmatrix} 0.70000 & 0.70711 \\ 0.70001 & 0.70711 \end{bmatrix}$

(a) $r_{11} = \sqrt{(0.70000)^2 + (0.70001)^2} = 0.98996$

$\underset{\sim}{q_1} = \underset{\sim}{a_1} / r_{11} = \begin{bmatrix} 0.70710 \\ 0.70711 \end{bmatrix}$

$r_{12} = q_1^T a_2 = 1.0000$

$\underset{\sim}{a_2} - r_{12} \underset{\sim}{q_1} = \begin{bmatrix} 0.00001 \\ 0.00000 \end{bmatrix}$, $r_{22} = 0.00001$

$\underset{\sim}{q_2} = \begin{bmatrix} 1.0000 \\ 0 \end{bmatrix}$

So $Q = \begin{bmatrix} 0.70710 & 1.00000 \\ 0.70711 & 0.00000 \end{bmatrix}$

We observe $\underset{\sim}{q}_1$ and $\underset{\sim}{q}_2$ are not orthogonal.

$$\underset{\sim}{q}_1^T \underset{\sim}{q}_2 = 0.70710 \neq 0.$$

(b) Householder's method

$$Q_2 Q_1 A = R$$

$$Q_1 = F = I - \frac{2\omega\omega^T}{\omega^T\omega}, \quad \cancel{\underset{\sim}{\omega} = \begin{bmatrix} 0 \\ 0.70001 \end{bmatrix}}$$

$$\underset{\sim}{x} = \begin{bmatrix} 0.70000 \\ 0.70001 \end{bmatrix} \qquad \cancel{(B/c \ \underset{\sim}{\omega} = \|x\|}$$

$$\|\underset{\sim}{x}\| = \sqrt{0.98001} = 0.98996$$

$$\underset{\sim}{\omega} = \|\underset{\sim}{x}\| e_1 - \underset{\sim}{x} = \begin{bmatrix} 0.98996 - 0.70000 \\ -0.70001 \end{bmatrix}$$

$$= \begin{bmatrix} 0.28996 \\ -0.70001 \end{bmatrix}$$

$$\underset{\sim}{\omega}\underset{\sim}{\omega}^T = \begin{bmatrix} 0.08408 & -0.20297 \\ -0.20297 & 0.49001 \end{bmatrix} \qquad \underset{\sim}{\omega}^T \underset{\sim}{\omega} = 0.08408 + 0.49001$$

$$= 0.57409$$

$$\frac{2\underset{\sim}{\omega}\underset{\sim}{\omega}^T}{\underset{\sim}{\omega}^T\underset{\sim}{\omega}} = \begin{bmatrix} 0.29291 & -0.70710 \\ -0.70710 & 1.7071 \end{bmatrix}$$

$$F = I - \frac{2\underset{\sim}{\omega}\underset{\sim}{\omega}^T}{\underset{\sim}{\omega}^T\underset{\sim}{\omega}} = \begin{bmatrix} 0.70709 & 0.70710 \\ 0.70710 & -0.70710 \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} 0.70709 & 0.70710 \\ 0.70710 & -0.70710 \end{bmatrix}$$

~~Q_1 is almost~~ $q_1^T q_2 = 0.49998 - 0.49999$

$= -0.00001$

Extremely small. Hence almost orthogonal.

Now $Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & F \end{bmatrix}$

$\underset{\sim}{x} = [0.70711]$ $\quad \underset{\sim}{\omega} = [0]$

$$F = I - \frac{2\underset{\sim}{\omega}\underset{\sim}{\omega}^T}{\underset{\sim}{\omega}^T\underset{\sim}{\omega}} = [1] \qquad Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Q = Q_1^T Q_2^T = \begin{bmatrix} 0.70709 & 0.70710 \\ 0.70710 & -0.70710 \end{bmatrix}$$

As shown above $\underset{\sim}{q}_1^T q_2 = -0.00001$ (Extremely small)

Hence Householder method produces closer to orthogonal Q.

4 (a) P is orthogonal projector.

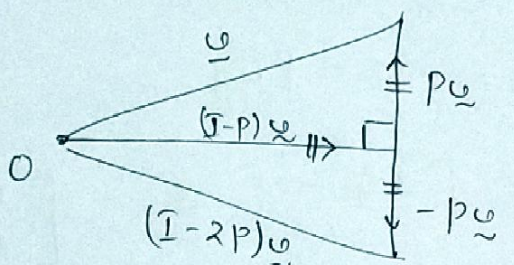
$$\mathbf{v} = (I-P)\mathbf{v} + P\mathbf{v}$$

$$(I-2P)\mathbf{v} = (I-P)\mathbf{v} - P\mathbf{v}$$

$$\begin{aligned}[(I-P)\mathbf{v}]^T (P\mathbf{v}) &= \mathbf{v}^T (I-P^T) P(\mathbf{v}) \\ &= \mathbf{v}^T (I-P) P\mathbf{v} \qquad , \quad P^T = P \\ &= \mathbf{v}^T (P-P^2)\mathbf{v} = 0\end{aligned}$$

So $(I-P)\mathbf{v}$ and $P\mathbf{v}$ are orthogonal.

Geometry.



So it's clear from this triangle that

$$\|\mathbf{v}\|_2 = \|(I-2P)\mathbf{v}\| \quad \text{(pythagoreans theorem)}$$

Further $\mathbf{v}$ is just rotated to get $(I-2P)\mathbf{v}$.

<u>Algebraically</u>

$$\begin{aligned}(I-2P)^T (I-2P) &= (I-2P)^2 \qquad (\because P = P^T) \\ &= I - 4P + 4P^2 \\ &= I \qquad (\text{B/c } P = P^2).\end{aligned}$$

Similarly $(I-2P)(I-2P)^T = I$.

So $I-2P$ is orthogonal matrix

(b) $A \in \mathbb{R}^{m \times n}$ full rank. P is orthogonal projector onto Range (A).

$P\underset{\sim}{\omega} = \underset{\sim}{y}$, $\underset{\sim}{y} \in$ Range (A), $\underset{\sim}{y} - \underset{\sim}{\omega} \in [\text{Range}(A)]^{\perp}$

$\Rightarrow A^T(A\underset{\sim}{x} - \underset{\sim}{\omega}) = 0$

$\Rightarrow A^T A \underset{\sim}{x} = A^T \underset{\sim}{\omega} \Rightarrow \underset{\sim}{x} = \underbrace{(A^TA)^{-1}}_{\text{Non singular when A has full rank.}} A^T \underset{\sim}{\omega}$

$\underset{\sim}{y} = A\underset{\sim}{x} = A(A^TA)^{-1}A^T\underset{\sim}{\omega}$

$= P\underset{\sim}{\omega}$

$$\boxed{P = A(A^TA)^{-1}A^T}$$

Given P, we can find Null (A^T), not Null (A).

We see, let, $x \in$ Null (P)

$\Rightarrow A(A^TA)^{-1}A^T\underset{\sim}{x} = 0$

$\Rightarrow (A^TA)(A^TA)^{-1}A^T\underset{\sim}{x} = 0$

$\Rightarrow A^T\underset{\sim}{x} = 0 \Rightarrow \underset{\sim}{x} \in$ Null (A^T)

So Null$(P) \subseteq$ Null (A^T).

If $\underset{\sim}{x} \in Null(A^T)$, $\quad A^T \underset{\sim}{x} = 0$

$$\Rightarrow P\underset{\sim}{x} = \underset{\sim}{0} \Rightarrow \underset{\sim}{x} \in Null(P)$$

So $Null(P) = Null(A^T)$.

Eigen values of P

$$P\underset{\sim}{x} = \lambda \underset{\sim}{x} \Rightarrow A(A^TA)^{-1}A^T\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

$$\Rightarrow A^T\underset{\sim}{x} = \lambda A^T \underset{\sim}{x} \quad \text{(Multiplying } A^T\text{)}$$

$$\Rightarrow 1-\lambda = 0 \Rightarrow \lambda = 1$$

So eigen value of P is $\underline{1}$.

(c) $P \in R^{m\times m}$ be non-zero projector.

$$\|P^2\underset{\sim}{x}\|_2 \leq \|P\|_2 \, \|P\underset{\sim}{x}\|_2 \quad \text{(Bounding norm)}$$

$$\Rightarrow \|P\|_2 \geq \frac{\|P^2\underset{\sim}{x}\|_2}{\|P\underset{\sim}{x}\|_2} \qquad \text{But } P^2 = P$$

So $\|P\|_2 \geq \underline{1}$.

Now if P is orthogonal, the its SVD shows max singular value of $\underline{1}$. (Done in class)

So $\|P\|_2 = \sigma_{max} = 1$ for P = orthogonal projector.

Now, we have to show

8. (a) $\bar{r}_{12} = fl(q_1^T a_2)$

$$\bar{r}_{12} = fl(q_{11} a_{21}) \oplus fl(q_{12} a_{22}) \oplus \cdots$$

$$= q_{11} a_{21} (1+\varepsilon_1) \oplus q_{12} a_{22} (1+\varepsilon_2) \oplus \cdots$$

$$\leq (q_{11} a_{21} \oplus q_{12} a_{22} \oplus \cdots\cdots)(1+\varepsilon_m)$$

B/c $|\varepsilon_1| < \varepsilon_m$.

Now observe $a \oplus b = (a+b)(1+\varepsilon) \leq (a+b)(1+\varepsilon_m)$

$$a \oplus b \oplus c \leq [(a+b)(1+\varepsilon_m) + c][1+\varepsilon_m]$$

$$= (a+b+c)\left[1 + \frac{a+b}{a+b+c}\varepsilon_m\right]\left[1+\varepsilon_m\right]$$

$$= (a+b+c)\left[1 + \varepsilon_m + \frac{a+b}{a+b+c}\varepsilon_m + O(\varepsilon_m^2)\right]$$

$$\leq (a+b+c)\left[1+2\varepsilon_m + O(\varepsilon_m^2)\right]$$

So if we go on for 'm-1' additions, ~~a+b+c~~ then we'll have $(m-1)\,\varepsilon_m$ and $O(\varepsilon_m^2)$ term.

Finally, $\bar{r}_{12} \leqslant r_{12}\,(1+(m-1)\,\varepsilon_m + O(\varepsilon_m^2))\,(1+\varepsilon_m)$

$$= r_{12}\left[1+m\,\varepsilon_m + (m-1)\,\varepsilon_m^2 + O(\varepsilon_m^2)\right]$$

$$= r_{12}\left[1+m\,\varepsilon_m + O(\varepsilon_m^2)\right]$$

So $|\bar{r}_{12} - r_{12}| \leq |\,r_{12}\,(m\,\varepsilon_m + O(\varepsilon_m^2))\,|$

$$\Rightarrow \frac{|\bar{r}_{12} - r_{12}|}{|r_{12}|} \leq m\,\varepsilon_m + O(\varepsilon_m^2)$$

$|r_{12}| = |q_1^T a_2| \leq \|q_1\|_2\,\|a_2\|_2 = \underline{1}$

So $|r_{12}| \leq \underline{1}$.

Hence $|\bar{r}_{12} - r_{12}| \leq m\,\varepsilon_m + O(\varepsilon_m^2)$

— (proved)