# QR Algorithm

**Pure QR algorithms** :- $A \in \mathbb{R}^{m \times m}$

$$A^{(0)} = A$$

for $k = 1, 2, \ldots$ do

$\rightarrow Q^{(k)} R^{(k)} = A^{(k-1)}$ % compute QR factorization

$\rightarrow A^{(k)} = R^{(k)} Q^{(k)}$ % Recombine factors in reverse order

end for

Under suitable assumptions, this simple algorithm converges to a Schur form for $A$; upper triangular for arbitrary $A$, or diagonal form if $A$ is real symmetric matrix.

$A^{(k)}$

$$A^{(0)} = A$$

$k = 1$

$$Q^{(1)} R^{(1)} = A^{(0)} \quad \text{--- (1)}$$

$$A^{(1)} = R^{(1)} Q^{(1)} \quad \text{--- (2)}$$

From ① $\underline{R}^{(1)} = \underline{Q}^{(1)T} \underline{A}^{(0)}$

From ② $\underline{A}^{(1)} = \underline{Q}^{(1)T} \underline{A}^{(0)} \underline{Q}^{(1)}$

for generic $k$

$$\underline{R}^{(k)} = (\underline{Q}^{(k)})^T \underline{A}^{(k-1)}$$

then $\underline{A}^{(k)} = \underline{R}^{(k)} \underline{Q}^{(k)}$

$$= \underbrace{(\underline{Q}^{(k)})^T \underline{A}^{(k-1)} \underline{Q}^{(k)}}$$

Unnormalized simultaneous iteration:-

To explain how $\underline{Q}\underline{R}$ algorithm works, we relate to another algorithm called simultaneous iteration (subspace iteration)!

Idea:- Apply power iteration to

several vectors at once.

Suppose we start with a set of $n$ linearly independent vectors at once $\underset{\sim}{A} \in \mathbb{R}^{m \times m}$

$$\underset{\sim}{v}_1^{(0)}, \underset{\sim}{v}_2^{(0)}, \ldots \underset{\sim}{v}_n^{(0)} \qquad n < m$$

The hypothesis is that $\underset{\sim}{A}^k v_1^{(0)}$ converges (as $k \to \infty$) to the eigen vector corresponding to the largest eigenvalue of $\underset{\sim}{A}$, what will happen to space $\langle \underset{\sim}{A}^k \underset{\sim}{v}_1^{(0)}, \underset{\sim}{A}^k \underset{\sim}{v}_2^{(0)}, \ldots \underset{\sim}{A}^k \underset{\sim}{v}_n^{(0)} \rangle$?

Under suitable assumptions, this space $\langle \underset{\sim}{A}^k \underset{\sim}{v}_1^{(0)}, \underset{\sim}{A}^k \underset{\sim}{v}_2^{(0)}, \ldots \underset{\sim}{A}^k v_n^{(0)} \rangle$ converge to the space $\langle \underset{\sim}{q}_1, \underset{\sim}{q}_2, \ldots \underset{\sim}{q}_n \rangle$

corresponding to eigenvectors of $n$ largest eigenvalues of $\underset{\sim}{A}$.

In matrix notation

$$\underset{\sim}{V}^{(0)} = [\underset{\sim}{v}_1^{(0)} | \underset{\sim}{v}_2^{(0)} \cdots | v_n^{(0)}]$$

and define

$$\boxed{\underset{\sim}{V}^{(k)} = \underset{\sim}{A}^k \underset{\sim}{V}^{(0)}}$$

$$= [\underset{\sim}{v}_1^{(k)} | \underset{\sim}{v}_2^{(k)} | \cdots | v_n^{(k)}]$$

Because we are interested in the space spanned by columns of $\underset{\sim}{V}^{(k)}$, we can extract a nice basis for this space from the reduced $\underset{\sim}{Q}\underset{\sim}{R}$

factorization of $\underset{\sim}{V}^{(k)}$

$$\underset{\sim}{\hat{Q}}^{(k)} \underset{\sim}{R}^{(k)} = \underset{\sim}{V}^{(k)}$$

$\underset{\sim}{\hat{Q}}^{(k)} \in \mathbb{R}^{m \times n}$

$\underset{\sim}{R}^{(k)} \in \mathbb{R}^{n \times n}$

Can we hope that as $k \to \infty$ the columns of $\underset{\sim}{\hat{Q}}^{(k)}$ converge to the eigenvectors $(\pm \underset{\sim}{q}_1, \pm \underset{\sim}{q}_2, \ldots \pm \underset{\sim}{q}_n)$?

Expand $v_j^{(0)}$ and $v_j^{(k)}$ in terms of eigenvectors of $\underset{\sim}{A}$

$$v_j^{(0)} = a_{1j} \underset{\sim}{q}_1 + a_{2j} \underset{\sim}{q}_2 + \ldots a_{mj} \underset{\sim}{q}_m$$

$$v_j^{(k)} = \lambda_1^k a_{1j} \underset{\sim}{q_1} + \lambda_2^k a_{2j} \underset{\sim}{q_2} + \dots \lambda_m^k a_{mj} \underline{q_m}$$

As before like power iteration the convergence happens provided the following two conditions are satisfied :-

① The first $n+1$ eigenvalues are distinct in absolute value :

$$|\lambda_1| > |\lambda_2| > |\lambda_3| \dots > |\lambda_n| > |\lambda_{n+1}| \geq |\lambda_{n+2}| \geq \dots \geq |\lambda_m|$$

② Let $\bar{Q} = [\underset{\sim}{q_1}, \dots \underset{\sim}{q_n}] \in \mathbb{R}^{m \times n}$

i.e $\bar{Q}$ is matrix formed from eigenvectors of $\underset{\sim}{A}$, all the leading principal submatrices of $\tilde{Q}^T \tilde{V}^{(0)}$ are non singular.

(The upper left square submatrices of dimensions $1\times1, 2\times2, \ldots n\times n$) are non singular!

Thm: Let the above two assumptions hold good, suppose we perform iteration for the given $\tilde{V}^{(0)}$,

$$\underbrace{\tilde{V}^{(k)} = \tilde{A}^k \tilde{V}^{(0)}}, \quad \hat{\tilde{Q}}^{(k)} \tilde{R}^{(k)} = \tilde{V}^{(k)}$$

Then $k \to \infty$, the columns of matrices $\hat{Q}^{(k)}$ converge linearly to the eigenvectors of $\tilde{A}$:

$\| q_j^{(k)} - \pm q_j \| = O(C^k)$ for each $j = 1, 2, \ldots n$ where

$$C = \max_{1 \leq k \leq n} \left| \frac{\lambda_{k+1}}{\lambda_k} \right| < 1$$

* As $k \to \infty$, the $v_1^{(k)}, v_2^{(k)}, \ldots v_n^{(k)}$ all converge to multiples of $\underline{q}_1$.

Although $\langle \underline{v}_1^{(k)}, \underline{v}_2^{(k)}, \ldots v_j^{(k)} \rangle$ converges to something useful, these vectors form a highly ill-conditioned basis for that space!

→ Simultaneous iteration as we described is practically not useful!

Hence we orthonormalize at every $k^{th}$ step instead of once at the end, so we do not construct

$\underset{\sim}{V}^{(k)}$ but of a different sequence of matrices $\underset{\sim}{Z}^{(k)}$ with the same column space.

Algo :- choose $\underset{\sim}{Q}^{(0)} \in \mathbb{R}^{m \times n}$ with orthonormal columns.

$$\underline{\underset{\sim}{\hat{Q}}^{(0)} = \underset{\sim}{Q}^{(0)}} \qquad \underline{A Q^{(0)}}$$

for $k = 1, 2, \ldots$ do

$$\checkmark Z^{(k)} = \underset{\sim}{A} \hat{Q}^{(k-1)}.$$

$$\hat{Q}^{(k)} \hat{R}^{(k)} = Z^{(k)}$$ % reduced QR factorization of $Z^{(k)}$

end for

The column space $\hat{Q}^{(k)}$ and $Z^{(k)}$ are the same and also same for $A^k \hat{Q}^{(0)}$

and hence we expect $\hat{Q}^{(k)}$ converge to the largest $n$ eigenvectors under same two conditions as before.

<u>Thm</u>:- Algorithm for simultaneous iteration with QR decomposition at every $k$ step generates the same matrix $\hat{Q}^{(k)}$ as the iteration without QR decomposition at every step.

" Simultaneous iteration $\Longleftrightarrow$ QR algorithm "

<u>Simultaneous</u> iteration <u>algorithm</u>:-

You start with $n=m$ linearly independent vectors. Infact you start with canonical basis

$$\bar{Q}^{(0)} = I = [e_1 | e_2 | \cdots | e_m]$$

Algo :- $\bar{Q}^{(0)} = I$ ✓

for $k = 1, 2, \ldots$

$$Z = A\bar{Q}^{(k-1)}$$

$$Z = \bar{Q}^{(k)} R_s^{(k)}$$

$$A^{(k)} = [\bar{Q}^{(k)}]^T A \bar{Q}^{(k)}$$

$$\underbrace{\bar{R}_s^{(k)} = R_s^{(k)} R_s^{(k-1)} \cdots R_s^{(1)}}$$

What will be $A^k$?

(i) $A = AI = A\bar{Q}^{(0)} = \bar{Q}^{(1)} R_s^{(1)}$

$$= \underbrace{\bar{Q}^{(1)} \bar{R}_s^{(1)}}$$

(ii) $A^2 = A(A)$

$$= A(\bar{Q}^{(1)} \bar{R}_s^{(1)}) = \bar{Q}^{(2)} R_s^{(2)} \bar{R}_s^{(1)}$$

$$= \underbrace{\bar{Q}^{(2)} \bar{R}_s^{(2)}}$$

(iii) $$\underline{A}^3 = \underline{A} (\underline{A}^2) = \underbrace{\bar{Q}^{(3)} \bar{R}_s^{(3)}}$$

$\vdots$

$$\boxed{\underline{A}^k = \bar{Q}^{(k)} \bar{R}_s^{(k)}} \qquad \boxed{A^{(k)} = \left(\bar{Q}^{(k)}\right)^T A \bar{Q}^{(k)}}$$

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QR Algorithm :-

$$A^{(0)} = A$$

for $k = 1, 2, \ldots$

$$Q^{(k)} R^{(k)} = A^{(k-1)}$$

$$A^{(k)} = R^{(k)} Q^{(k)}$$

$$Q_R^{(k)} = Q^{(1)} Q^{(2)} \cdots Q^{(k)}$$

$$R_R^{(k)} = R^{(k)} R^{(k-1)} \cdots R^{(1)}$$

Defining $Q_R^{(k)}$, $R_R^{(k)}$

What happens to $A^k$ in this QR setting?

(i) $$A = A^{(0)} = Q^{(1)} R^{(1)} = Q_R^{(1)} R_R^{(1)}$$

(ii) $$A^2 = A(A) = A(Q^{(1)} R^{(1)})$$

$$= Q^{(1)} \underbrace{R^{(1)} (Q^{(1)}} R^{(1)})$$

$$= \underbrace{Q^{(1)} Q^{(2)}} \underbrace{R^{(2)} R^{(1)}}$$

$$= Q_R^{(2)} R_R^{(2)}$$

(iii) $$A^3 = A(A^2)$$

$$= A(Q^{(1)} Q^{(2)} R^{(2)} R^{(1)})$$

$$= Q^{(1)} \underbrace{R^{(1)} (Q^{(1)}} \underbrace{Q^{(2)} R^{(2)} R^{(1)}})$$

$$= Q^{(1)} Q^{(2)} \underbrace{R^{(2)} Q^{(2)}} R^{(2)} R^{(1)}$$

$$= Q^{(1)} Q^{(2)} Q^{(3)} R^{(3)} R^{(2)} R^{(1)}$$

$$= Q_R^{(3)} R_R^{(3)}$$

$$\vdots$$

$$A^k = Q_R^{(k)} R_R^{(k)}$$

Let us look at $A^{(k)}$

$$A^{(k)} = R^{(k)} Q^{(k)}$$

$$A^{(1)} = R^{(1)} Q^{(1)} = I R^{(1)} Q^{(1)}$$

$$= \left(Q^{(1)}\right)^T \underbrace{Q^{(1)} R^{(1)}} Q^{(1)}$$

$$= \left(Q^{(1)}\right)^T A \, Q^{(1)}$$

$$= \underbrace{\left(Q_R^{(1)}\right)^T A \, Q_R^{(1)}}$$

$$A^{(2)} = R^{(2)} Q^{(2)} = \left(Q^{(2)}\right)^T \underbrace{Q^{(2)} R^{(2)}} Q^{(2)}$$

$$= \left(Q^{(2)}\right)^T A^{(1)} Q^{(2)} = \left(Q^{(2)}\right)^T \left(Q_R^{(1)}\right)^T A \, Q_R^{(1)} Q^{(2)}$$

$$= \left(Q_R^{(2)}\right)^T A\, Q_R^{(2)}$$

$\vdots$

$$A^{(k)} = \underbrace{\left(Q_R^{(k)}\right)^T A\, Q_R^{(k)}}$$

<u>Simultaneous iteration</u> :- $A^k = \bar{Q}^{(k)} \bar{R}_s^{(k)}$

<u>QR Algorithm</u> :- $A^k = \underbrace{Q_R^{(k)} R_R^{(k)}}$

Since QR decomposition of $A^k$ has to unique,

$$\bar{Q}^{(k)} = Q_R^{(k)}$$

$$\bar{R}_s^{(k)} = R_R^{(k)} = \bar{R}^{(k)}$$

<u>Simultaneous iteration</u> :- $A^{(k)} = \left(\bar{Q}^{k}\right)^T A \bar{Q}^{(k)}$

<u>QR algo</u> : $A^{(k)} = \left(Q_R^{(k)}\right)^T A\, Q_R^{(k)}$

$$= \left(\bar{Q}^{(k)}\right)^T A \bar{Q}^{(k)}$$

## Convergence of QR algorithm :-

① $\underset{\sim}{A}^k = \bar{\underset{\sim}{Q}}^{(k)} \bar{\underset{\sim}{R}}^{(k)}$ as $k \to \infty$

$\bar{Q}^{(k)}$ converges to eigen subspace

i.e it constructs orthonormal eigenbasis for successive powers of $A^k$

② $\underset{\sim}{A}^{(k)} = (\bar{\underset{\sim}{Q}}^{(k)})^T \underline{A} \, \bar{\underline{Q}}^{(k)}$

→ diagonal elements of $\underset{\sim}{A}^{(k)}$ are the Rayleigh quotients of $\underset{\sim}{A}$ corresponding to the colums of $\bar{Q}^{(k)}$ i.e $\bar{\underset{\sim}{q}}_i^T \underline{A}^{(k)} \underline{q}_i$

columns of $\bar{Q}_k$ → eigenvectors as $k \to \infty$

these rayleigh quotients → eigenvalues

off diagonal entries of $\underset{\sim}{A}^{(k)}$ converge to 0 as $k \to \infty$, as the columnvectors become eigenvectors and the approximation become orthogonal

i.e
$$A_{ii}^{(k)} = \bar{q}_i^{(k)} A^{(k)} q_i^{(k)}$$
$$\Downarrow$$
$$\lambda_i^{(k)}$$

$$A_{ij}^{(k)} = (q_i^{(k)})^T A^{(k)} q_j^{(k)}$$
$$\Downarrow$$
$$0$$

as $q_i^{(k)}$ and $q_j^{(k)}$ become more and more orthogonal as $k \to \infty$

<u>Thm</u> :- Let the pure QR algorithm be applied to a real symmetric matrix $A$ with eigenvalues satisfying $|\lambda_1| > |\lambda_2| \dots > |\lambda_m|$ and whose corresponding eigenvector matrix $Q$

has nonsingular leading principal minors, then as $k \to \infty$,

$$\underset{\sim}{A}^{(k)} \to \operatorname{diag}(\lambda_1, \lambda_2, \dots \lambda_m) \text{ and}$$

$$\underset{\sim}{\underline{Q}}^{(k)} \to Q \text{ at the linear rate } \max_k \frac{|\lambda_{k+1}|}{|\lambda_k|}$$

<u>QR algorithm with shifts</u>

$\underset{\sim}{A} \in \mathbb{R}^{m \times m}$ real and symmetric

we have real eigenvalues $\{\lambda_j\}$ and orthonormal eigenvectors $\{\underset{\sim}{q}_j\}$.

We have just seen

Pure QR algo $\to$ Simultaneous iteration applied to $\underset{\sim}{I}$

$\Rightarrow$ first column evolves as a power iteration applied to $\underset{\sim}{e}_1$

$\boxed{\text{We can also view pure QR as a simultaneous inverse iteration applied to a flipped identity matrix.}}$

As before let $\underset{\sim}{Q}^{(k)}$ be the orthogonal matrix generated at step k of the QR algorithm!

$$\boxed{\bar{\underset{\sim}{Q}}^{(k)} = \underbrace{\prod_{j=1}^{k} \underset{\sim}{Q}^{(j)}} = \left[ \underset{\sim}{q}_1^{(k)} \,\middle|\, \underset{\sim}{q}_2^{(k)} \,\middle|\, \cdots \,\middle|\, \underset{\sim}{q}_m^{(k)} \right]}$$

is same as the <u>orthogonal matrix</u> at step (k) of the simultaneous iteration.

$$\boxed{\underset{\sim}{A}^{k} = \bar{\underset{\sim}{Q}}^{(k)} \bar{\underset{\sim}{R}}^{(k)}}$$

$$\Rightarrow (\underset{\sim}{A}^{k})^{-1} = \left( \bar{\underset{\sim}{Q}}^{(k)} \bar{\underset{\sim}{R}}^{(k)} \right)^{-1}$$

$$\Rightarrow \underset{\sim}{A}^{-k} = \left(\bar{R}^{(k)}\right)^{-1}\left(\bar{Q}^{(k)}\right)^{-1}$$

$$= \underbrace{\left(\bar{R}^{(k)}\right)^{-1}\left(\bar{Q}^{(k)}\right)^{T}}$$ ✓

Using symmetry of $A^{-k}$, we can write

$$\underset{\sim}{A}^{-k} = \left(\underset{\sim}{A}^{-k}\right)^{T} = \left(\underbrace{\left(\bar{R}^{(k)}\right)^{-1}\left(\bar{Q}^{(k)}\right)^{T}}\right)^{T}$$

$$= \underbrace{\underset{\sim}{\bar{Q}}^{(k)}\left(\bar{R}^{(k)}\right)^{-T}}$$

$$\boxed{\underset{\sim}{A}^{-k} = \underset{\sim}{\bar{Q}}^{(k)}\left(\bar{R}^{(k)}\right)^{-T}} \quad — \text{(1)}$$

Now Define $P \in \mathbb{R}^{m \times m}$, a flipped identity matrix

$$\boxed{\underset{\sim}{P} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & 1 & 0 \\ 0 & \cdot^{\cdot^{\cdot}} & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \qquad \underset{\sim}{P}^{2} = \underset{\sim}{I}}$$

using (1)

$$\underset{\sim}{A}^{-k}\underset{\sim}{P} = \bar{Q}^{(k)}\left(\bar{R}^{(k)}\right)^{-T}P$$

$$= [\bar{Q}^{(k)}] I (\bar{R}^{(k)})^{-T} P \qquad A[\;]$$

$$A^k[\;] = \bar{Q}^{(k)} \bar{R}$$

$$\boxed{\underline{A}^{-k} \underline{P} = \left(\underbrace{\bar{Q}^{(k)} P}\right) \boxed{\left(\underbrace{P(\bar{R}^{(k)})^{-T} P}\right)}}$$

$\underbrace{\bar{Q}^{(k)}} P$ is orthogonal matrix and $\underbrace{P(\bar{R}^{(k)})}^{-T} P$ is also upper triangular, we have a QR factorization for $\underline{A}^{-k} \underline{P}$

$$\underline{(A^{-1})^k \underline{P}} = \left[\underbrace{\bar{Q}^{(k)} \underline{P}}\right] \left[\underbrace{\underline{P}(\bar{R}^{(k)})^{-T} \underline{P}}\right]$$

i.e we are carrying out simultaneous iteration on $\underline{A}^{-1}$ applied to intial matrix $\underline{P}$ (Flipped identity matrix)

This means, we are doing simultaneous inverse iteration $\underset{\sim}{A}$.

In particular, first column of $\bar{\underset{\sim}{Q}}^{(k)} \underset{\sim}{P}$ which is the last column of $\bar{\underset{\sim}{Q}}^{(k)}$ evolves as k-steps of inverse iteration to $\underset{\sim}{e}_m$.

<u>Shifted inverse iteration within the framework of QR algorithm</u>:-

QR algo is both simultaneous and inverse simultaneous iteration

<u>Algo</u>:- $[Q^{(0)}]^T \underset{\sim}{A}^{(0)} Q^{(0)} = \underset{\sim}{A}$ ($\underset{\sim}{A}^{(0)}$ is tridiagonal reduction of A)

for $k = 1, 2 \ldots$

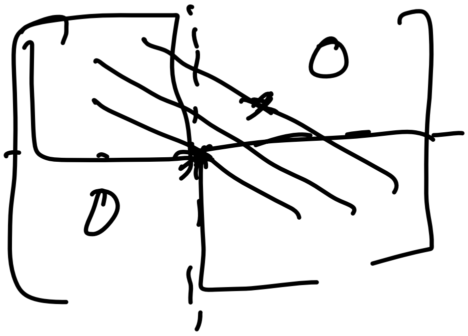
Pick a shift $\mu^{(k)}$ eg:- $\mu^{(k)} = A_{mm}^{(k-1)}$

$\rightarrow \underset{\sim}{Q}^{(k)} \underset{\sim}{R}^{(k)} = \underset{\sim}{A}^{(k-1)} - \mu^{(k)} \underset{\sim}{I}$

$\rightarrow \underset{\sim}{A}^{(k)} - \mu^{(k)} \underset{\sim}{I} = R^{(k)} Q^{(k)}$

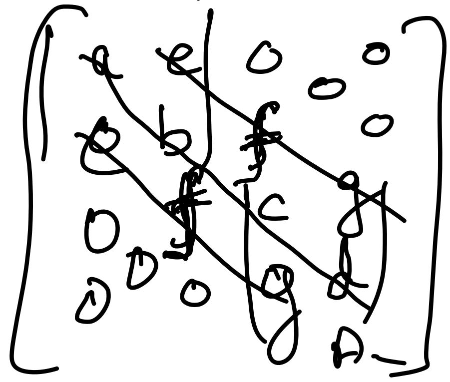
$\rightarrow$ If any offdiagonal $\left[\underline{A}^{(k)} = \underline{R}^{(k)} \underline{Q}^{(k)} + \mu^{(k)} \underline{I}\right]$ element $A^{(k)}_{j,j+1}$ is sufficiently close to 0, we set

$$A_{j,j+1} = A_{j+1,j} = 0$$



$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = A^{(k)}$$

$A_1$



$A_2$

and keep QR algo to $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$

$$\left.\begin{aligned} &\rightarrow \left(\underset{\sim}{A}^{(k-1)} - \mu^{(k)} \underset{\sim}{I}\right) = \underset{\sim}{Q}^{(k)} \underset{\sim}{R}^{(k)} \\ &\rightarrow \underset{\sim}{A}^{(k)} = R^{(k)} Q^{(k)} + \mu^{(k)} \underset{\sim}{I} \end{aligned}\right\}$$

$$\Rightarrow \quad \underset{\sim}{A}^{(k)} = \left(\underset{\sim}{Q}^{(k)}\right)^T \underset{\sim}{A}^{(k-1)} \underset{\sim}{Q}^{(k)}$$

$$\Rightarrow \quad \boxed{\underset{\sim}{A}^{(k)} = \left(\bar{\underset{\sim}{Q}}^{(k)}\right)^T \underset{\sim}{A}^{(0)} \bar{\underset{\sim}{Q}}^{(k)}}$$

We can also show shifted QR algorithm

$$(\underset{\sim}{A} - \mu^{(k)} \underset{\sim}{I})(\underset{\sim}{A} - \mu^{(k-1)} \underset{\sim}{I}) \cdots (\underset{\sim}{A} - \mu^{(1)} \underset{\sim}{I}) = \bar{Q}^{(k)} \bar{R}^{(k)}$$

i.e $\bar{Q}^{(k)} = \prod_{j=1}^{k} Q^{(j)}$ is an

orthogonalization of $\prod_{j=k}^{1} (\underset{\sim}{A} - \mu^{(j)} \underset{\sim}{I})$

The first column of $\bar{Q}^{(k)}$ is the result of applying shifted power iteration to $\underset{\sim}{e}_1$ with shifts $\mu^{(j)}$ and

Last column of $\bar{Q}^{(k)}$ evolves as shifted inverse iteration applied to $\underset{\sim}{e}_m$ with shifts $\mu^{(j)}$.

If these shifts are good eigenvalue estimates, the last column of $\bar{Q}^{(k)}$ converge quickly to ~~an~~ an eigenvector!

$\rightarrow$ Choose shifts in the spirit of rayleigh quotient iteration applied to the last column of $\bar{Q}^{(k)}$.

$$\mu^{(k)} = \frac{\left(q_m^{(k)}\right)^T A\, q_m^{(k)}}{q_m^{(k)T} q_m^{(k)}}$$

$$= \boxed{\left(q_m^{(k)}\right)^T A\, q_m^{(k)}} \checkmark$$

If we choose this shift at every step $\mu^{(k)}$, $q_m^{(k)}$ are the same as those computed by Rayleigh quotient iteration starting with $\underset{\sim}{e_m}$.

QR algorithm has cubic convergence in the sense $q_m^{(k)}$ converge cubically to the eigen vector!

$r(q_m^{(k)}) \longrightarrow m,m$ entry of $A^{(k)}$

$A_{mm}^{(k)}$

$$\boxed{A_{mm}^{(k)}} = \underline{e}_m^T \underline{A}^{(k)} \underline{e}_m$$
$$= \underset{\sim}{e_m}^T (\underline{Q}^{(k)})^T \underline{A}\, \underline{Q}^{(k)} e_m$$
$$= (\underline{q}_m^{(k)})^T \underline{A}\, \underline{q}_m^{(k)}$$

This is called Rayleigh quotient shift. (RQS)

<u>Wilkinson shifts</u> are usually used in case RQS fails to converge.

<u>Thm</u>:- Let $\underset{\sim}{A} \in \mathbb{R}^{m \times m}$ a real, symmetric and tridiagonal. Suppose we diagonalize $\underset{\sim}{A}$ by the QR algorithm. Let $\tilde{\Lambda}$ be computed diagonalized $\underset{\sim}{A}$, and $\tilde{\underset{\sim}{Q}}$ be the exact orthogonal matrix resulting

Then $\tilde{\underset{\sim}{Q}} \tilde{\underset{\sim}{\Lambda}} \tilde{\underset{\sim}{Q}}^T = \underset{\sim}{A} + \underset{\sim}{\delta A}$

where $\dfrac{\|\delta A\|}{\|A\|} = O(\epsilon_M)$

for some $\underset{\sim}{\delta A} \in \mathbb{R}^{m \times m}$

i.e QR algorithm produces an

exact solution to a slightly perturbed problem!

We can conclude that tridigonal reduction followed by QR algorithm is a backward stable way of computing eigenvalues of a matrix!

Also computed eigenvalues $\tilde{\lambda}_j$ satisfy

$$\boxed{\frac{|\tilde{\lambda}_j - \lambda_j|}{\|A\|} = O(\epsilon_M)}$$