DS284: NLA Final Exam (25th January, 2021)
9 AM to 12 Noon

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Total: 100 Marks

1. State with clear reason whether the following statements are true or false: [**6 × 3 = 18 Marks**]

   (a) Let $\mathbf{A} \in \mathbb{R}^{m\times n}(m < n)$ be a full rank matrix and $\mathbf{A}$ admits $\mathbf{U\Sigma V}^T$ as its full SVD. If $\mathbf{V} = [\mathbf{V}_1, \mathbf{V}_2]$, where $\mathbf{V}_1 \in \mathbb{R}^{n\times m}$ and $\mathbf{V}_2 \in \mathbb{R}^{n\times(n-m)}$, then the system of equations $\mathbf{V}_2\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} \in \mathbb{R}^{n-m}$ for all vectors $\mathbf{b} \in \mathbb{R}^n$ satisfying $\mathbf{Ab} = \mathbf{0}$

   (b) If $\mathbf{A} \in \mathbb{R}^{m\times n}$ is a full rank matrix, $\mathbf{Ax} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$.

   (c) All projector matrices which are not orthogonal projectors are always rank deficient matrices.

   (d) Let a full rank matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ be decomposed as $\mathbf{A} = \mathbf{TR}$ where $\mathbf{T}$ is a lower triangular matrix (not necessarily unit lower triangular) and $\mathbf{R}$ is an upper triangular matrix. Such a decomposition is always unique.

   (e) If $\mathbf{A} \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix and $\mathbf{X} \in \mathbb{R}^{m\times n}$ is a full rank matrix, then $\mathbf{X}^T\mathbf{AX}$ is always symmetric positive definite.

   (f) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent eigenvectors of $\mathbf{A}$ corresponding to an eigenvalue λ with algebraic multiplicity 3, then all the vectors lying in the subspace spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are eigenvectors of $\mathbf{A}$ for that eigenvalue λ.

2. If $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 3 \end{bmatrix}$, answer the following questions: [**2 + 5 + 5 + 6 = 18 Marks**]

   (a) Write the characteristic equation associated with the above matrix $\mathbf{A}$ and subsequently compute its eigenvalues.

   (b) Set up the Arnoldi iteration with the starting vector $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ to find the orthonormal basis vectors $\{\mathbf{q}_1, \mathbf{q}_2\}$ spanning the two dimensional Krylov subspace $\mathcal{K}_2 =< \mathbf{b}, \mathbf{Ab} >$

   (c) Find the orthogonal projection $\mathbf{H}$ of $\mathbf{A}$ onto $\mathcal{K}_2$ represented in the basis $\{\mathbf{q}_1, \mathbf{q}_2\}$ obtained in (b) above, and then compute the eigenvalues of this $\mathbf{H}$ (also called Ritz values). Find the absolute error between the smallest Ritz value and the smallest eigenvalue of $\mathbf{A}$ and similarly compute the absolute error between the largest Ritz value and largest eigenvalue of $\mathbf{A}$.

   (d) Consider the system of equations $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 3 \end{bmatrix}$ as given above and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ as given in (b). Find the exact solution $\mathbf{x}^*$ which solves $\mathbf{Ax} = \mathbf{b}$ using forward substitution. Subsequently find the vector $\hat{\mathbf{x}} \in \mathcal{K}_2$ that minimizes the norm $\|\mathbf{Ac} - \mathbf{b}\|_2$ over all possible vectors $\mathbf{c} \in \mathcal{K}_2$, where $\mathcal{K}_2$ is the Krylov subspace constructed in (b). Finally, find the norm of error between exact solution $\mathbf{x}^*$ and $\hat{\mathbf{x}}$.

3. If $\mathbf{u}$ and $\mathbf{v}$ are two non-parallel vectors in $\mathbb{R}^n$, then the $n \times n$ matrix $\mathbf{A} = \mathbf{I} + \mathbf{u}\mathbf{v}^T$ is known as the rank one perturbation of the $n \times n$ identity matrix $\mathbf{I}$. [$\mathbf{3 \times 5 = 15}$ **Marks**]

   (a) If $\mathbf{A}$ is non-singular, show that $\mathbf{A}$ has an eigenvalue 1 with geometric multiplicity $n-1$.

   [*Hint*: Consider the space orthogonal to the vector $\mathbf{v}$]

   (b) If $\mathbf{A}$ is non-singular, show that an eigenvector of $\mathbf{A}$ for an eigenvalue $\lambda \neq 1$ must be of the form $c\mathbf{u}$ for some scalar c. Find this eigenvalue λ in terms of the vectors $\mathbf{u}$ and $\mathbf{v}$.

   (c) If $\mathbf{v}^T\mathbf{u} = -1$, show that $\mathbf{A}$ is singular. What can you say about the null space of $\mathbf{A}$?

4. The matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ are called orthogonally similar matrices if there is an orthogonal matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$ such that $\mathbf{A} = \mathbf{Q}\mathbf{B}\mathbf{Q}^T$. [$\mathbf{4 \times 5 = 20}$ **Marks**]

   (a) If $\mathbf{A}$ and $\mathbf{B}$ are orthogonally similar matrices, show that (i) $\mathbf{A}^T\mathbf{A}$ is similar to $\mathbf{B}^T\mathbf{B}$ and (ii) $\mathbf{A}\mathbf{A}^T$ is similar to $\mathbf{B}\mathbf{B}^T$.

   (b) If $\mathbf{A}$ and $\mathbf{B}$ are orthogonally similar, show that they have the same singular values.

   (c) If $\mathbf{B} \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, show that $\mathbf{AB}$ and $\mathbf{BA}$ are orthogonally similar matrices.

   (d) If $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ are two symmetric matrices with same eigenvectors, then show that $\mathbf{AB} = \mathbf{BA}$.

5. Let $\mathbf{A} = \mathbf{U\Sigma V}^T$ be the full SVD of $\mathbf{A} \in \mathbb{R}^{m\times m}$. Use this SVD of $\mathbf{A}$ to obtain the eigen-decomposition of the $2m \times 2m$ symmetric matrix $\mathbf{B} = \begin{bmatrix} 0 & \mathbf{A}^T \\ \mathbf{A} & 0 \end{bmatrix}$. [**Hint**: Let λ be the eigenvalue of $\mathbf{B}$ and denote the coresponding eigenvector to be $\begin{bmatrix} \mathbf{x} \\ \boldsymbol{\zeta} \end{bmatrix}$, where $\mathbf{x} \in \mathbb{R}^m$ and $\boldsymbol{\zeta} \in \mathbb{R}^m$] [**15 Marks**]

6. If $\mathbf{A} \in \mathbb{R}^{m\times m}$ be a symmetric matrix with one eigenvalue much smaller than others, i.e. if $\lambda_1, \lambda_2, ..., \lambda_m$ are eigenvalues of $\mathbf{A}$ in increasing order, then $\lambda_1 <<< \lambda_2 \leq \lambda_3 ... \leq \lambda_m$ (This means that $\mathbf{A}$ is an ill-conditioned matrix). Let $\mathbf{q}_1, \mathbf{q}_2, ...\mathbf{q}_m$ be the corresponding eigenvectors. Now answer the following questions: [**4 + 5 + 5=14 Marks**]

   (a) If the system of equations $\mathbf{Aw} = \mathbf{v}$ is solved using a backward stable algorithm for some $\mathbf{v} \in \mathbb{R}^m$ yielding a computed vector $\widetilde{\mathbf{w}} = \mathbf{w} + \delta\mathbf{w}$, show that $\delta\mathbf{w} = -(\mathbf{A} + \delta\mathbf{A})^{-1}(\delta\mathbf{A})\mathbf{w}$ for some $\delta\mathbf{A}$ such that $\|\delta\mathbf{A}\| = O(\epsilon_M)\|\mathbf{A}\|$.

   (b) Assuming that $\mathbf{v}$ is a vector with components in the directions of all eigenvectors $\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_m$ of $\mathbf{A}$, show that $\frac{\mathbf{w}}{\|\mathbf{w}\|} \approx \mathbf{q}_1$, where $\mathbf{w} = \mathbf{A}^{-1}\mathbf{v}$ is the exact solution of the system of equations given in (a). [**Hint**: First show that $\mathbf{w}$ is approximately in the direction of $\mathbf{q}_1$]

   (c) Using the Taylor series expansion $(\mathbf{A}+\delta\mathbf{A})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}(\delta\mathbf{A})\mathbf{A}^{-1} + O(\epsilon_M^2)$, show that $\frac{\widetilde{\mathbf{w}}}{\|\widetilde{\mathbf{w}}\|} \approx \mathbf{q}_1$. [**Hint**: Using the expression $\delta\mathbf{w}$ derived in (a) and the fact $\delta\mathbf{A}$ is random roundoff perturbation, show that $\delta\mathbf{w}$ is in the direction of $\mathbf{q}_1$]

   [This sequence of steps in (a), (b) and (c) show that, though the computed solution $\widetilde{\mathbf{w}}$ is far away from $\mathbf{w}$ for the ill-conditioned system $\mathbf{Aw} = \mathbf{v}$, $\frac{\widetilde{\mathbf{w}}}{\|\widetilde{\mathbf{w}}\|}$ need not be far from $\frac{\mathbf{w}}{\|\mathbf{w}\|}$.]