Midterm

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19970

1. (a) True, if we use a backward stable algorithm like householder QR factorization to solve $A\underset{\sim}{x} = \underset{\sim}{b}$. In that case, forward relative error will be $O(\kappa(A)\,\varepsilon_m)$.

Statement is False in general. If unstable algo. is used to solve $A\underset{\sim}{x} = \underset{\sim}{b}$, we can't guarantee anything about accuracy.

(b) True.

If P is orthogonal and $\det(P) = -1$, then it reflects a vector.

So $\exists\, \underset{\sim}{x} \neq \underset{\sim}{0}$ such that $P\underset{\sim}{x} = -\underset{\sim}{x}$

Then $(I+P)\underset{\sim}{x} = \underset{\sim}{0}$

Hence $\underset{\sim}{x} \neq \underset{\sim}{0} \in \text{Null}(I+P)$.

Hence $I+P$ is rank deficient. Not invertible.

(c) $A^T = V\Sigma^T U^T$

$\text{Null}(A^T)$ will be determined by U^T, not V.

$\text{Range}(A^T)$ " by V.

Hence Statement is False.

(d) True.

B/c terms like r_{12}, r_{13}, r_{23} etc. will be zero.

$$A = \begin{bmatrix} \tilde{q}_1 & \cdots & \tilde{q}_n \end{bmatrix} \begin{bmatrix} R \end{bmatrix}$$

B/c columns of A are orthogonal.

$\tilde{q}_1^T \tilde{a}_j = 0$ when $j \neq 1$.

Similarly, $q_i^T \tilde{a}_j = 0$ when $i \neq j$.

Hence all cross terms are 0. So R will be diagonal.

(e) $A^T A \hat{x} = A^T \tilde{b}$. A is full rank, $m > n$.

Hence $A^T A$ is invertible. B/c $A^T A \tilde{x} = \tilde{0}$

$\Rightarrow \tilde{x}^T A^T A \tilde{x} = \tilde{0}$

$\Rightarrow \| A \tilde{x} \|_2^2 = \tilde{0}$

$\Rightarrow A \tilde{x} = \tilde{0}$

$\Rightarrow \tilde{x} = \tilde{0}$ (Full rank).

Hence $A^T A$ is invertible full rank.

So $\hat{x} = (A^T A)^{-1} A^T \tilde{b}$, $A \hat{x} = A (A^T A)^{-1} A^T \tilde{b}$

The statement is false b/c $A \hat{x}$ is not $\tilde{b}$ in general.

$A \hat{x}$ is just a projection of $\tilde{b}$ in Range(A)

(f) $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$P^2 = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} = I_{4\times 4}$

$P \neq P^2$. Hence this is not a projector.

Midterm Exam

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2. (a) $A = \begin{bmatrix} \underset{\sim}{a_1} \mid \underset{\sim}{a_2} \mid \underset{\sim}{a_3} \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$\text{Range}(A) = \{ A\underset{\sim}{x} \mid \underset{\sim}{x} \in R^3 \}$

Range(A) will span R^3 if $\underset{\sim}{a_1}$, $\underset{\sim}{a_2}$ and a_3 ~~are~~ linearly independent. / basis vectors of R^3.

$$A\underset{\sim}{x} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 + x_3 \\ x_2 + x_3 \end{bmatrix}$$

If we can show $\underset{\sim}{a_1}$, $\underset{\sim}{a_2}$ and $\underset{\sim}{a_3}$ are linearly independent and b/c each of them is a vector in R^3, they'll be basis of R^3 and can span R^3.

This comes from theorem that n lin. ind. vector in R^n forms a basis of R^n.

To show if they are lin. independent,

$$A\underset{\sim}{\alpha} = \underset{\sim}{0}$$

$$\Rightarrow \begin{bmatrix} \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_2 + \alpha_3 \end{bmatrix} = \underset{\sim}{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\alpha_1 + \alpha_2 = 0, \quad \alpha_1 + \alpha_2 + \alpha_3 = 0}_{\alpha_3 = 0}, \quad \alpha_2 + \alpha_3 = 0$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 0, \ \alpha_2 + \alpha_3 = 0 \Rightarrow \alpha_1 = 0$$

B/c $\alpha_1 = \alpha_3 = 0$, $\alpha_2 = 0$.

Hence $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{a}_3$ are lin. independent, hence basis of R^3 and can span R^3

(b) $Q_1 = F_{3\times3}$

~~$\underset{\sim}{v} = 1x$~~

$$\underset{\sim}{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \|\underset{\sim}{x}\|_2 = \sqrt{2}$$

$$\underset{\sim}{v} = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+\sqrt{2} \\ 1 \\ 0 \end{bmatrix}$$

$$\underset{\sim}{v}^T \underset{\sim}{v} = (1+\sqrt{2})^2 + 1 = 4 + 2\sqrt{2}$$

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\frac{\underset{\sim}{v}\underset{\sim}{v}^T}{\underset{\sim}{v}^T\underset{\sim}{v}}$$

$$\frac{2\underset{\sim}{v}\underset{\sim}{v}^T}{\underset{\sim}{v}^T\underset{\sim}{v}} = \begin{bmatrix} 3+2\sqrt{2} & 1+\sqrt{2} & 0 \\ 1+\sqrt{2} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Big/ (2+\sqrt{2})$$

$$\cancel{Q_1 = F = \begin{bmatrix} -2-2\sqrt{2} & -1-\sqrt{2} & 0 \\ -1-\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Big/ (2+\sqrt{2})}$$

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{3+2\sqrt{2}}{2+\sqrt{2}} & \frac{1+\sqrt{2}}{2+\sqrt{2}} & 0 \\ \frac{1+\sqrt{2}}{2+\sqrt{2}} & \frac{1}{2+\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & F \end{bmatrix}$$

$$\underset{\sim}{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \|\underset{\sim}{x}\|_2 = r_2$$

$$\underset{\sim}{v} = \begin{bmatrix} r_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+r_2 \\ 1 \end{bmatrix}$$

$$\underset{\sim}{v}^T \underset{\sim}{v} = (1+r_2)^2 + 1 = 4 + 2r_2$$

$$\underset{\sim}{v}\,\underset{\sim}{v}^T = \begin{bmatrix} 1+2+2r_2 & 1+r_2 \\ 1+r_2 & 1 \end{bmatrix} = \begin{bmatrix} 3+2r_2 & 1+r_2 \\ 1+r_2 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3+2r_2 & 1+r_2 \\ 1+r_2 & 1 \end{bmatrix} / (2+r_2)$$

$$= \begin{bmatrix} 1 - \frac{3+2r_2}{2+r_2} & -\frac{(1+r_2)}{2+r_2} \\ -\frac{(1+r_2)}{2+r_2} & 1 - \frac{1}{2+r_2} \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 - \frac{3+2r_2}{2+r_2} & -\frac{(1+r_2)}{2+r_2} \\ 0 & -\frac{(1+r_2)}{2+r_2} & 1 - \frac{1}{2+r_2} \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & F \end{bmatrix}$$

$$\underset{\sim}{x} = [1] \qquad \underset{\sim}{v} = [2]$$

$$\frac{2\underset{\sim}{v}\underset{\sim}{v}^T}{\underset{\sim}{v}^T\underset{\sim}{v}} = 2$$

$$F = [1-2] = [-1]$$

$$Q_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} 1 - \frac{3+2r_2}{2+r_2} & -\frac{(1+r_2)}{2+r_2} & 0 \\ -\frac{(1+r_2)}{2+r_2} & 1 - \frac{1}{2+r_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(previously calculated)
halt

3. (a) $A \in \mathbb{R}^{m \times n}$, $m > n$, Rank $r < n$

Given solution $\underset{\sim}{x} = (A^T A)^{-1} A^T \underset{\sim}{b}$

However, this is not the solution when A is rank deficient. This is b/c when A is rank deficient, $A^T A$ won't be invertible. We give a proof below.

let, $A^T A \underset{\sim}{x} = \underset{\sim}{0}$

If $A^T A$ is invertible, then $\underset{\sim}{x} = \underset{\sim}{0}$.
i.e. only trivial null space.

We show that $\exists$ a $\underset{\sim}{x} \neq \underset{\sim}{0}$ such that $A^T A \underset{\sim}{x} = \underset{\sim}{0}$ when A is rank deficient.

$A^T A \underset{\sim}{x} = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{x}^T A^T A \underset{\sim}{x} = 0$

$\Rightarrow (A\underset{\sim}{x})^T A\underset{\sim}{x} = 0$

$\Rightarrow \| A \underset{\sim}{x} \|_2^2 = 0$

$\Rightarrow A \underset{\sim}{x} = \underset{\sim}{0}$ ($\because$ By property of norm)

If A is rank deficient, it has non trivial null space. Hence $\exists$ $\underset{\sim}{x} \neq \underset{\sim}{0}$ such that $A\underset{\sim}{x} = \underset{\sim}{0}$. That same $\underset{\sim}{x}$ also belongs to Null $(A^T A)$ as we have shown.

Hence null space of A^TA is non-trivial i.e.

A^TA is rank deficient and not invertible.

Hence given solution is not valid for rank deficient A.

(b) $A\underset{\sim}{x} = \underset{\sim}{b}$

$\Rightarrow \underset{\sim}{b} = x_1 \underset{\sim}{a_1} + \cdots + x_n \underset{\sim}{a_n}$

However Rank $= r$ i.e. only r of n columns are lin. independent.

Let, those r columns are denoted by $\underset{\sim}{a}_{k1}, \ldots, \underset{\sim}{a}_{kr}$

B/c $\underset{\sim}{b} \in$ Range (A)

$\Rightarrow \underset{\sim}{b} = \alpha_1 \underset{\sim}{a}_{k1} + \cdots + \alpha_r \underset{\sim}{a}_{kr}$

This $(\alpha_1, \ldots, \alpha_r)$ is unique b/c $(\underset{\sim}{a}_{k1}, \ldots, \underset{\sim}{a}_{kr})$ are basis vectors.

However other $n-r$ columns of A, can be expressed as linear comb. of $(\underset{\sim}{a}_{k1}, \ldots, \underset{\sim}{a}_{kr})$

Hence there are multiple $(x_1, \ldots, x_n)$ corresponding to unique $(\alpha_1, \ldots, \alpha_r)$. i.e. ~~they ca~~ coefficients $(\alpha_1, \ldots, \alpha_r)$ can be split in multiple ways to obtain $(x_1, \ldots, x_n)$. There are $(n-r)$ degrees of freedom.

~~Hence solution $\underset{\sim}{x}$ won't be~~ unique

(b) $A\underset{\sim}{x} = \underset{\sim}{b}$

B/c $\underset{\sim}{b} \in$ Range(A), $\exists$ at least one $\underset{\sim}{x}$ such that $A\underset{\sim}{x} = \underset{\sim}{b}$.

Further b/c A is rank deficient, $\exists$ a $\underset{\sim}{c} \neq \underset{\sim}{0}$ such that $A\underset{\sim}{c} = \underset{\sim}{0}$

Hence $A(\underset{\sim}{x} + \underset{\sim}{c}) = \underset{\sim}{b}$. B/c $\underset{\sim}{c} \neq \underset{\sim}{0}$, $\underset{\sim}{x} + \underset{\sim}{c} \neq \underset{\sim}{x}$

Similarly, we can construct infinitely many solutions $\underset{\sim}{x} + 2\underset{\sim}{c}$, $\underset{\sim}{x} + 3\underset{\sim}{c}$ .....

Hence there are infinite no. of soln. for this.

(c) $\underset{\sim}{x} = V_1 \Sigma_1^{-1} U_1^T \underset{\sim}{b}$

$A = U_1 \Sigma_1 V_1^T$

$A\underset{\sim}{x} = U_1 \Sigma_1 V_1^T V_1 \Sigma_1^{-1} U_1^T \underset{\sim}{b}$

$V_1 \in R^{n \times r}$, $V_1^T V_1 \in R^{r \times r}$

let, $V_1 = \begin{bmatrix} \underset{\sim}{q}_1 & \cdots & \underset{\sim}{q}_r \end{bmatrix}$ $V_1^T = \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_r^T \end{bmatrix}$

B/c $(q_1, \dots, q_r)$ are orthonormal,

$$q_i^T q_i = 1 \quad \text{and} \quad q_i^T q_j = 0 \;, \; i \neq j$$

Hence $$V_1^T V_1 = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}_{r \times r} = I_{r \times r}$$

Now $$A\underset{\sim}{x} = U_1 \Sigma_1 V_1^T V_1 \Sigma_1^{-1} U_1^T \underset{\sim}{b}$$

$$= U_1 \Sigma_1 \Sigma_1^{-1} U_1^T \underset{\sim}{b} \qquad (\because V_1^T V = I)$$

$$= U_1 U_1^T \underset{\sim}{b} \qquad (\because \Sigma_1 \Sigma_1^{-1} = I)$$

$U_1 \in R^{m \times r}$, $U_1 U_1^T \in R^{m \times m}$

$$U_1 = \begin{bmatrix} \underset{\sim}{q_1} & \cdots & \underset{\sim}{q_r} \end{bmatrix}_{m \times r}$$ These may be different vectors than that of V_1.

$$U_1^T = \begin{bmatrix} q_1^T \\ \vdots \\ q_r^T \end{bmatrix}$$

$$U_1 U_1^T = \underset{\sim}{q_1} \underset{\sim}{q_1}^T + \cdots + \underset{\sim}{q_r} q_r^T$$

$$U_1 U_1^T \underset{\sim}{b} = \underset{\sim}{q_1} \underset{\sim}{q_1}^T \underset{\sim}{b} + \cdots + \underset{\sim}{q_r} \underset{\sim}{q_r}^T \underset{\sim}{b}$$

~~$q_1 q_1^T b$~~ $U_1 U_1^T = (q_1 q_1^T + \cdots + q_r q_r^T)$ will act as a projector to space spanned by r orthogonal basis vectors forming a subspace V^r.

B/c b lies in range (A) and Rank$(A) = r$, b belongs to the r-dimensional subspace within R^m. [$\because A = U \Sigma V^T$, Hence Range$(A)$ = Range(U) = lin. comb. of r orthonormal columns of U]

B/c $b \in$ Range$(U_1 U_1^T)$, Hence $U_1 U_1^T b = b$

projector projects to the same vector if vector is already in range of projector.

and hence Range(A) = Range(UU^T)

Hence $Ax = U_1 U_1^T b = b$.

— (proved)