Notations :-

* Vectors are denoted by lower case letters with a tilde below.
* Matrices are denoted by upper case with a tilde below
* Scalars are denoted using lower case alphabets

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→ $\underset{\sim}{x}$ is a n-dimensional vector :-

$$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

eg:- (1) Sequence of number can represent 'n' measurements of a feature like % Urbanization of various countries, heights of students in class

(2) These vectors can also represent displacements, velocity field, electron density, pressure fields on different grid pts in a domain

→ $\underset{\sim}{A}$ (m rows, n columns)

$$\underset{\sim}{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

eg:- (1) In data science, these can be collection of a different features

(2) Discretization of PDEs in a chosen basis can give rise to matrices

eg:- Finite Element method, Finite difference method, Finite volume etc.

(3) Incidence matrix

(4) Adjacency matrix

Matrix - vector multiplication :-

$$* \quad \underset{\sim}{b} = \underset{\sim}{A}\,\underset{\sim}{x} \;; \quad \underset{\sim}{A} \in \mathbb{R}^{m \times n},\ \underset{\sim}{x} \in \mathbb{R}^{n \times 1}$$

$$\underset{\sim}{b} \in \mathbb{R}^{m \times 1}$$

$$b_i = \sum_{j=1}^{n} A_{ij}\, x_j$$

eg:- $\underset{\sim}{A} = \begin{pmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{pmatrix} \quad \underset{\sim}{x} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$

$$\underset{\sim}{A}\,\underset{\sim}{x} = \begin{pmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 2 \times 8 + 3 \times 9 \\ 2 \times 8 + 4 \times 9 \\ 3 \times 8 + 7 \times 9 \end{pmatrix}$$

$$= \begin{pmatrix} \; \end{pmatrix}$$

By rows :-

Inner product of rows of $\underset{\sim}{A}$ with vector $\underset{\sim}{x}$

Defn:- $\underset{\sim}{x} \to \underset{\sim}{A}\underset{\sim}{x}$ is a linear map

i.e for any vectors $\underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^n$

and any scalar $\alpha \in \mathbb{R}$

$$\underset{\sim}{A}(\underset{\sim}{x}+\underset{\sim}{y}) = \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{A}\underset{\sim}{y}$$

$$\underset{\sim}{A}(\alpha \underset{\sim}{x}) = \alpha \underset{\sim}{A}\underset{\sim}{x}$$

By columns:-

$$\underset{\sim}{A}\underset{\sim}{x} = \begin{array}{cc} A & x \\ \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} & \begin{pmatrix} 8 \\ 9 \end{pmatrix} \end{array} = 8\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} + 9\begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}$$

You are viewing $\underset{\sim}{A}\underset{\sim}{x}$ as a linear combination of $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_2$, the two columns of $\underset{\sim}{A}$

$$\underset{\sim}{A}\underset{\sim}{x} = \begin{bmatrix} | & | \\ \underset{\sim}{a}_1 & \underset{\sim}{a}_2 \\ | & | \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \underset{\sim}{a}_1 + x_2 \underset{\sim}{a}_2$$

Fundamental operation of Linear Algebra

$$\underset{\sim}{b} = \underset{\sim}{A}\underset{\sim}{x} = \sum_{j=1}^{n} x_j \underset{\sim}{a}_j$$

This will allow us to define range(A)

Range of a matrix A

Defn:- $\text{range}(A) = \{ y \mid y = Ax \text{ for some } x \}$

Key idea is to take all combinations of columns of A and the set of infinitely many output vectors Ax forms the range of (A).

The space spanned by the columns of A is range (A)

$\hookrightarrow$ column space (A)

Null-space :-

$\text{Null}(A) = \{ x \mid Ax = 0 \}$

i.e set of all vectors x that maps to a zero vector via A

$\rightarrow$ If $x \in \text{Null}(A)$

$Ax = 0 \Rightarrow x_1 a_1 + x_2 a_2 + \dots x_n a_n = 0$

If $a_1, a_2, \dots a_n$ are linearly independent then $x_i = 0$

Rank of a matrix

Column Rank of $\underset{\sim}{A}$:→ Maximum number of linearly indepent columns of $\underset{\sim}{A}$

[Dimension of space spanned by its column]

Row rank of $\underset{\sim}{A}$:→ Maximum number of linearly independent rows of $\underset{\sim}{A}$

[Dim of space spanned by rows]

Result :- Row rank of a matrix equals column rank of a matrix and is called rank of a matrix.

$\underset{\sim}{A} \in \mathbb{R}^{m \times n}$ is said to have a full rank if it has the maximal rank

i.e $\text{rank}(A) = \min(m, n)$

Eg :- $\underset{\sim}{A} = \begin{bmatrix} | & | & & | \\ \underset{\sim}{a_1} & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}_{m \times n}$ $m \geq n$

If such a matrix $\underset{\sim}{A}$ has a full rank it must have 'n' linearly independent columns, rank$(A) = n$

<u>Thm 1:</u> A matrix $\underset{\sim}{A} \in \mathbb{R}^{m \times n}$ with $m \geq n$ maps no two distinct vectors to the same vector if and only if $\underset{\sim}{A}$ is a full rank.

$\rightarrow \underset{\sim}{A}\underset{\sim}{x}_1 = \underset{\sim}{y}_1$; $\underset{\sim}{A}\underset{\sim}{x}_2 = \underset{\sim}{y}_2$ $(\underset{\sim}{x}_1 \neq \underset{\sim}{x}_2)$

then $\underset{\sim}{y}_1 \neq \underset{\sim}{y}_2$ if and only if $\underset{\sim}{A}$ is full rank

Pf:-

<u>To show</u>

$\underset{\sim}{x}_1 \in \mathbb{R}^n$, $\underset{\sim}{x}_2 \in \mathbb{R}^n$ and $\underset{\sim}{x}_1 \neq \underset{\sim}{x}_2$

$S_1 \Rightarrow S_2$

$\boxed{\sim S_2 \Rightarrow \sim S_1}$

$\underset{\sim}{A}\underset{\sim}{x}_1 = \underset{\sim}{y}_1$; $\underset{\sim}{A}\underset{\sim}{x}_2 = \underset{\sim}{y}_2$ then $\underset{\sim}{y}_1 \neq \underset{\sim}{y}_2 \Rightarrow A$ is full rank

<u>Proof by contradiction</u>

→ Let us assume $\underset{\sim}{A}$ is not full rank

it has a non-trivial null space, let that vector be $\underset{\sim}{c} \neq 0$ i.e $\underset{\sim}{A}\underset{\sim}{c} = \underset{\sim}{0}$

$$\underset{\sim}{A}\underset{\sim}{x}_1 = \underset{\sim}{y}_1 \;;\; \underset{\sim}{x}_2 = \underset{\sim}{x}_1 + \underset{\sim}{c}$$

$$\underset{\sim}{y}_2 = \underset{\sim}{A}\underset{\sim}{x}_2 = \underset{\sim}{A}(\underset{\sim}{x}_1 + \underset{\sim}{c}) = \underset{\sim}{A}\underset{\sim}{x}_1 = \underset{\sim}{y}_1$$

i.e a contradiction, $\underset{\sim}{A}$ is full rank.

To show

$\underset{\sim}{A}$ is full rank

$\Rightarrow$ $\underset{\sim}{A}\underset{\sim}{x}_1 = \underset{\sim}{y}_1$; $\underset{\sim}{A}\underset{\sim}{x}_2 = \underset{\sim}{y}_2$ where $\underset{\sim}{x}_1 \neq \underset{\sim}{x}_2$ then $\underset{\sim}{y}_1 \neq \underset{\sim}{y}_2$

$\rightarrow$ Consider $\underset{\sim}{x}_1 \neq \underset{\sim}{x}_2$ and assume $\underset{\sim}{y}_1 = \underset{\sim}{y}_2 = \underset{\sim}{y}$

i.e $\underset{\sim}{A}\underset{\sim}{x}_1 = \underset{\sim}{y}$, $\underset{\sim}{A}\underset{\sim}{x}_2 = \underset{\sim}{y}$ $\Rightarrow$ $\underset{\sim}{A}(\underset{\sim}{x}_1 - \underset{\sim}{x}_2) = \underset{\sim}{0}$

$\underset{\sim}{x}_1 - \underset{\sim}{x}_2 = 0$ because $\underset{\sim}{A}$ is full rank.

i.e $\underset{\sim}{x}_1 = \underset{\sim}{x}_2$

(again a contradiction)

Matrix - matrix multiplication :-

$$\underset{\sim}{A} \in \mathbb{R}^{l \times m} \qquad \underset{\sim}{C} \in \mathbb{R}^{m \times n}$$

$$\underset{\sim}{B} = \underset{\sim}{A}\underset{\sim}{C} \qquad \text{i.e } B \in \mathbb{R}^{l \times n}$$

$$B_{ij} = \sum_{k=1}^{m} A_{ik} C_{kj}$$

Inner product form :-

$$B = \underbrace{\begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{C} = \begin{bmatrix} 2a+3c & 2b+3d \\ 3a+4c & 3b+4d \\ 4a+7c & 4b+7d \end{bmatrix}$$

Linear Combination of columns

$$\begin{bmatrix} | & | & & | \\ \underset{\sim}{b_1} & b_2 & \cdots & b_n \\ | & | & & | \end{bmatrix}_{l\times n} = \begin{bmatrix} | & | & & | \\ \underset{\sim}{a_1} & \underset{\sim}{a_2} & \cdots & a_m \\ | & | & & | \end{bmatrix}_{l\times m} \begin{bmatrix} | & | & & | \\ \underset{\sim}{c_1} & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix}_{m\times n}$$

$$B = \underset{\sim}{A} \; \underset{\sim}{C}$$

Columns of $\underset{\sim}{B}$ can be written as linear combination of columns of $\underset{\sim}{A}$ with column vectors of $\underset{\sim}{C}$

$$\boxed{\underset{\sim}{b_j} = \sum_{k=1}^{m} c_{kj} \underset{\sim}{a_k}}$$

as linear combination coefficients

Outer-product form:-

$\underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{R}^m$ outer product of two vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ as a rank 1 matrix $\underset{\sim}{A} \in \mathbb{R}^{m\times m}$

and $\underset{\sim}{A} := \underset{\sim}{u}\underset{\sim}{v}^T$

$$\underset{\sim}{B} = \underset{l\times m}{\underset{\sim}{A}} \; \underset{m\times n}{\underset{\sim}{C}} = \underbrace{\begin{bmatrix} | & | & & | \\ \underset{\sim}{a_1} & \underset{\sim}{a_2} & \cdots & \underset{\sim}{a_m} \\ | & | & & | \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} - & r_1^T & - \\ - & r_2^T & - \\ & \vdots & \\ - & r_m^T & - \end{bmatrix}}_{\underset{\sim}{C}}$$

i^{th} row of $\underset{\sim}{C}$ matrix is denoted as a column vector $\underset{\sim}{r_i} \in \mathbb{R}^n$

$$\underset{\sim}{B} = \underset{\sim}{A}\underset{\sim}{C} = \underset{\sim}{a_1}\underset{\sim}{r_1}^T + \underset{\sim}{a_2}\underset{\sim}{r_2}^T + \underset{\sim}{a_3}\underset{\sim}{r_3}^T + \cdots \underset{\sim}{a_m}\underset{\sim}{r_m}^T$$

Sum of 'm' rank 1 matrices

Four fundamental Subspaces of $\underset{\sim}{A}$

Column space :- $C(\underset{\sim}{A}) \rightarrow$ combination of columns of $\underset{\sim}{A}$

Row space :- $C(\underset{\sim}{A}^T) \rightarrow$ combination of columns of $\underset{\sim}{A}^T$

Null space :- $N(A) \rightarrow$ all solutions $\underset{\sim}{x}$ to $\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{0}$

Left Null space :- $N(A^T) \rightarrow$ all solutions $\underset{\sim}{y}$ to $\underset{\sim}{A}^T\underset{\sim}{y} = \underset{\sim}{0}$

Inverse of a matrix :-

A full rank square matrix is invertible

i.e., for $\underset{\sim}{A} \in \mathbb{R}^{m\times m}$, '$m$' columns of $\underset{\sim}{A}$ form a basis for $\mathbb{R}^m$.

$$\longleftrightarrow \quad \underset{\sim}{A} = \begin{bmatrix} | & | & & | \\ \underset{\sim}{a}_1 & \underset{\sim}{a}_2 & \cdots & \underset{\sim}{a}_m \\ | & | & & | \end{bmatrix}_{m\times m}$$

$\underset{\sim}{A}$ is a full rank matrix

Hence,

Any vector in $\mathbb{R}^m$ can be expressed as a linear combination of columns of $\underset{\sim}{A}$

$$\underset{\sim}{e}_j = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1} \leftarrow j^{th}$$

$$\underset{\sim}{e}_j = \sum_{i=1}^{m} z_{ij}\underset{\sim}{a}_i$$

$$= \underset{\sim}{A}\underset{\sim}{z}_j \quad — (*)$$

$j = 1, 2, \dots m$

$$\begin{bmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \cdots & \underset{\sim}{e}_m \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ & & & & 1 \end{bmatrix} = I$$

$$\underset{\sim}{I} = \underset{\sim}{A}\underset{\sim}{Z} = \underset{\sim}{A} \begin{bmatrix} | & | & & | \\ \underset{\sim}{z}_1 & z_2 & \cdots & \underset{\sim}{z}_m \\ | & | & & | \end{bmatrix}$$

This matrix is called the inverse of $\underset{\sim}{A}$.

$\underset{\sim}{I} = \underset{\sim}{A}\underset{\sim}{Z} = \underset{\sim}{Z}\underset{\sim}{A}$ (Exercise)

And this $\underset{\sim}{Z}$ is called inverse of matrix $\underset{\sim}{A}$ and denote as $\underset{\sim}{A}^{-1}$

"Every non-singular square matrix $\underset{\sim}{A}$ has a unique inverse $\underset{\sim}{A}^{-1}$ that satisfies"

$$\boxed{\underset{\sim}{A}\underset{\sim}{A}^{-1} = \underset{\sim}{A}^{-1}\underset{\sim}{A} = I}$$

<u>Matrix inverse times vector</u>:-

$\underset{\sim}{A} \in \mathbb{R}^{m \times m}$ (Full rank) ; $\underset{\sim}{b} \in \mathbb{R}^m$; $\underset{\sim}{x} \in \mathbb{R}^m$

$$\boxed{\underset{\sim}{x} = \underset{\sim}{A}^{-1}\underset{\sim}{b}}$$

We can think $\underset{\sim}{x}$ as a unique vector satisfying $\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{b}$

$$\underset{\sim}{b} = \sum_{i=1}^{m} b_i \underset{\sim}{e_i} \quad \text{where } \underset{\sim}{e_i} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \rightarrow i$$

$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ are the coefficients of expansion of $\underset{\sim}{b}$ in $\{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \dots e_m\}$

$$\underset{\sim}{b} = \underset{\sim}{A}\underset{\sim}{x} = \sum_{j=1}^{m} x_j \underset{\sim}{a_j} \quad \text{where } \underset{\sim}{A} = \begin{bmatrix} | & | & & | \\ \underset{\sim}{a_1} & \underset{\sim}{a_2} & \dots & \underset{\sim}{a_m} \\ | & | & & | \end{bmatrix}$$

$$\Downarrow$$

$\underset{\sim}{A}^{-1}\underset{\sim}{b} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$ are the coefficients of expansion of $\underset{\sim}{b}$ in $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \dots \underset{\sim}{a_m}\}$

This means we can think of $\underset{\sim}{A}^{-1}\underset{\sim}{b}$ as a change of basis operation from $\{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \dots \underset{\sim}{e_m}\}$ to $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \dots a_m\}$ i.e column-space of $\underset{\sim}{A}$

<u>Transpose</u> of <u>a</u> <u>matrix</u> ($\underset{\sim}{A}^T$) :-

If $\underset{\sim}{A} \in \mathbb{R}^{m \times n}$ then $\underset{\sim}{A}^T \in \mathbb{R}^{n \times m}$ where $A^T_{ij} = A_{ji}$

Useful properties of A^T and A^{-1} :-

ⓐ $(AB)^T = B^T A^T$ ⓑ $(AB)^{-1} = B^{-1} A^{-1}$

ⓒ $(A^T)^{-1} = (A^{-1})^T$

If $A = A^T$ (Symmetric matrix)

If $A = A^+$ (Hermitian matrix)

A^+ is conjugate transpose of $A \in \mathbb{C}^{m \times n}$

Inner products :→

Let $x, y \in \mathbb{R}^m$, then inner product of x and y is a scalar $(x, y) = x^T y$

$$= \sum_{i=1}^{m} x_i y_i$$

Euclidean length of a vector x :-

$$\|x\| = \sqrt{x^T x} = \left(\sum_{i=1}^{m} x_i^2\right]^{1/2}$$

Angle between vectors x and y is α then $\cos\alpha = \dfrac{x^T y}{\|x\|\|y\|}$

Useful properties of inner products :-

ⓐ $(x_1 + x_2, y) = (x_1, y) + (x_2, y)$

(b) $(\underset{\sim}{x}, \underset{\sim}{y}_1 + \underset{\sim}{y}_2) = (\underset{\sim}{x}, \underset{\sim}{y}_1) + (\underset{\sim}{x}, \underset{\sim}{y}_2)$

(c) $(\alpha \underset{\sim}{x}, \beta \underset{\sim}{y}) = \alpha\beta (\underset{\sim}{x}, \underset{\sim}{y})$

Orthogonality :-

(a) Orthogonal vectors $\underset{\sim}{x}$ and $\underset{\sim}{y}$:→ Angle b/w vectors is 90° i.e $\cos 90^\circ = 0 \Rightarrow (\underset{\sim}{x}, \underset{\sim}{y}) = 0$

or $\underset{\sim}{x}^T \underset{\sim}{y} = 0$

(b) Orthogonal basis for a subspace :-

$\underset{\sim}{S} = \{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots v_n\}$ is said to be an orthogonal basis spanning a n-dimensional space if every pair of vectors in $\underset{\sim}{S}$ is orthogonal i.e $\underset{\sim}{v}_i^T v_j = 0$

orthonormal basis → In addition to orthogonality, $\underset{\sim}{v}_i^T \underset{\sim}{v}_i = 1$ (Length of each vector is 1)

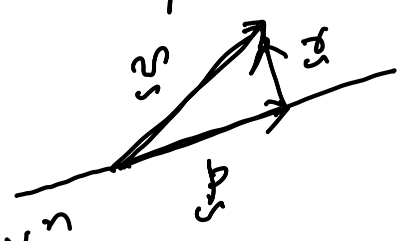
Result: Vectors in a orthogonal set $\underset{\sim}{S}$ are linearly independent

Pf :- exercise

Components of vector :- Consider $\mathbb{R}^m$ and n-dimensional space $\mathbb{W}^n \subseteq \mathbb{R}^m$ spanned by an orthonormal set $\underset{\sim}{Q} = \{\underset{\sim}{q}_1, \underset{\sim}{q}_2, \dots \underset{\sim}{q}_n\}$

(Note :- $n \le m$)

Let $\underset{\sim}{v} \in \mathbb{R}^m$ and let us decompose



$\underset{\sim}{v} = \underset{\sim}{p} + \underset{\sim}{r}$ where $\underset{\sim}{p} \in \mathbb{W}^n$ the space spanned by $\underset{\sim}{Q}$ and $\underset{\sim}{r}$ is orthogonal to $\mathbb{W}^n$

Since $\underset{\sim}{p} \in \mathbb{W}^n$, $\underset{\sim}{p} = \sum_{i=1}^{n} \alpha_i \underset{\sim}{q}_i$ — (*)

$$\underset{\sim}{v} = \underset{\sim}{p} + \underset{\sim}{r} \Rightarrow \underset{\sim}{q}_j^T \underset{\sim}{v} = \underset{\sim}{q}_j^T \underset{\sim}{p} + \cancelto{0}{\underset{\sim}{q}_j^T \underset{\sim}{r}}$$

$$\Rightarrow \underset{\sim}{q}_j^T \underset{\sim}{v} = \underset{\sim}{q}_j^T \left(\sum_{i=1}^{n} \alpha_i \underset{\sim}{q}_i \right) \quad — \square$$

$i \neq j$ will vanish

$$\Rightarrow \boxed{\alpha_j = \underset{\sim}{q}_j^T \underset{\sim}{v}}$$

$$\Rightarrow \underset{\sim}{v} = \underset{\sim}{r} + \sum_{i=1}^{n} (\underset{\sim}{q}_i^T \underset{\sim}{v}) \underset{\sim}{q}_i$$

If $m = n$ then $\underset{\sim}{v} = \sum_{i=1}^{m} (\underset{\sim}{q}_i^T \underset{\sim}{v}) \underset{\sim}{q}_i$

i.e Sum of coefficients of $q_i^T v$ times basis vectors q_i

Also $$v = \sum_{i=1}^{m} q_i (q_i^T v) = \sum_{i=1}^{m} \underbrace{(q_i q_i^T)}_{} v = \sum_{i=1}^{m} P_i v$$

i.e Sum of orthogonal projections of v onto various directions q_i

Ⓒ <u>Orthogonal matrices</u> :-

A square matrix $Q \in \mathbb{R}^{m \times m}$ is called orthogonal matrix if $Q^T = Q^{-1}$

i.e $\boxed{Q^T Q = Q Q^T = I}$ $(AZ = ZA = I)$

$$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_m \end{bmatrix}$$

$$Q^T Q = \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_m^T \end{bmatrix} \begin{bmatrix} q_1 & q_2 & \cdots & q_m \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \end{bmatrix} = I$$

This means $q_i^T q_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

If $Q \in \mathbb{C}^{m\times m}$ then $Q^{*} Q = Q Q^{*} = I$

then Q is called unitary matrix

we saw

Ax and $A^{-1}b$ interpretations

$$A \to Q$$

i.e Qx and $Q^{-1}b \Rightarrow Q^T b$

$Q^T b$ can be thought of change of basis from $\{e_1, e_2, \ldots e_m\}$ to orthogonal basis $\{q_1, q_2, \ldots q_m\}$

i.e columns of Q

(i) <u>Orthogonal matrices preserves inner products</u>

$$(x, y) := x^T y$$

$$(Qx, Qy) = (Qx)^T (Qy) = x^T Q^T Q y$$
$$= x^T y = (x, y)$$

(ii) Orthogonal matrices preserves lengths

$$\|\underset{\sim}{Q}\,\underset{\sim}{x}\| = \|x\| \quad \text{(verify?)}$$

(iii) $\det \underset{\sim}{Q} = \pm 1 \qquad \left[\, \underset{\sim}{Q}^T \underset{\sim}{Q} = I \,\right]$

(iv) Action of $\underset{\sim}{Q} \in \mathbb{R}^{m \times m}$ on a vector is a rigid rotation (if $\det \underset{\sim}{Q} = 1$) or reflection (if $\det \underset{\sim}{Q} = -1$)

Notations :-

* Vectors are denoted by lower case letters with a tilde below.
* Matrices are denoted by upper case with a tilde below
* Scalars are denoted using lower case alphabets

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→ $\underset{\sim}{x}$ is a n-dimensional vector :-

$$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

eg:- (1) Sequence of number can represent 'n' measurements of a feature like % Urbanization of various countries, heights of students in class

(2) These vectors can also represent displacements, velocity field, electron density, pressure fields on different grid pts in a domain

→ $\underset{\sim}{A}$ (m rows, n columns)

$$\underset{\sim}{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix} \in \mathbb{R}^{m\times n}$$

Eg:- (1) In data science, these can be collection of a different features

(2) Discretization of PDEs in a chosen basis can give rise to matrices

Eg:- Finite element method, Finite difference method, Finite volume etc.

Matrix - vector multiplication:-

$$* \quad \underset{\sim}{b} = \underset{\sim}{A}\underset{\sim}{x} \; ; \quad \underset{\sim}{A} \in \mathbb{R}^{m\times n}, \; \underset{\sim}{x} \in \mathbb{R}^{n\times 1}$$

$$\underset{\sim}{b} \in \mathbb{R}^{m\times 1}$$

$$b_i = \sum_{j=1}^{n} A_{ij} x_j$$

Eg:-

$$\underset{\sim}{A} = \begin{pmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{pmatrix} \qquad \underset{\sim}{x} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$$

$$\underset{\sim}{A}\underset{\sim}{x} = \begin{pmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 2\times 8 + 3\times 9 \\ 2\times 9 + 4\times 9 \\ 3\times 8 + 7\times 9 \end{pmatrix}$$

$$= \begin{pmatrix} \; \end{pmatrix}$$

By rows:-

Inner product of rows of $\underset{\sim}{A}$ with vector $\underset{\sim}{x}$

Defn:- $\underset{\sim}{x} \to \underset{\sim}{A}\underset{\sim}{x}$ is a linear map

i.e for any vectors $\underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^n$

and any scalar $\alpha \in \mathbb{R}$

$$\underset{\sim}{A}(\underset{\sim}{x} + \underset{\sim}{y}) = \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{A}\underset{\sim}{y}$$

$$\underset{\sim}{A}(\alpha \underset{\sim}{x}) = \alpha \underset{\sim}{A}\underset{\sim}{x}$$

By columns:-

$$\underset{\sim}{A}\underset{\sim}{x} = \overset{A}{\begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix}} \overset{x}{\begin{pmatrix} 8 \\ 9 \end{pmatrix}} = 8\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} + 9\begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}$$

You are viewing $\underset{\sim}{A}\underset{\sim}{x}$ as a linear combination of $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_2$, the two columns of $\underset{\sim}{A}$

$$\underset{\sim}{A}\underset{\sim}{x} = \begin{bmatrix} | & | \\ \underset{\sim}{a}_1 & \underset{\sim}{a}_2 \\ | & | \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \underset{\sim}{a}_1 + x_2 \underset{\sim}{a}_2$$

Fundamental operation of Linear Algebra

$$\underset{\sim}{b} = \underset{\sim}{A}\underset{\sim}{x} = \sum_{j=1}^{n} x_j \underset{\sim}{a}_j$$

This will allow us to define range(A)

Range of a matrix A

Defn:- $\text{range}(A) = \{ \underset{\sim}{y} \mid \underset{\sim}{y} = \underset{\sim}{A}\underset{\sim}{x} \text{ for some } \underset{\sim}{x} \}$

Key idea is to take all combinations of columns of $\underset{\sim}{A}$ and the set of infinitely many output vectors $\underset{\sim}{A}\underset{\sim}{x}$ forms the range of (A).

The space spanned by the columns of A is range (A)
$\hookrightarrow$ column space (A)

Null-space :-

$\text{Null}(A) = \{ \underset{\sim}{x} \mid \underset{\sim}{A}\underset{\sim}{x} = 0 \}$

i.e set of all vectors $\underset{\sim}{x}$ that maps to a zero vector via $\underset{\sim}{A}$

$\rightarrow$ If $\underset{\sim}{x} \in \text{Null}(A)$

$$\underset{\sim}{A}\underset{\sim}{x} = 0 \Rightarrow x_1 \underset{\sim}{a_1} + x_2 \underset{\sim}{a_2} + \dots x_n \underset{\sim}{a_n} = 0$$

If $\underset{\sim}{a_1}, \underset{\sim}{a_2}, \dots \underset{\sim}{a_n}$ are linearly independent then $x_i = 0$

Rank of a matrix

Column Rank of $\underset{\sim}{A}$:→ Maximum number of linearly indepent columns of $\underset{\sim}{A}$

[Dimension of space spanned by its column]

Row rank of $\underset{\sim}{A}$:→ Maximum number of linearly independent rows of $\underset{\sim}{A}$

[Dim of space spanned by rows]

Result:- Row rank of a matrix equals Column rank of a matrix and is called rank of a matrix.

$\underset{\sim}{A} \in \mathbb{R}^{m \times n}$ is said to have a full rank if it has the maximal rank

i.e $\text{rank}(A) = \min(m, n)$

Eg:- $\underset{\sim}{A} = \begin{bmatrix} | & | & & | \\ \underset{\sim}{a_1} & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}_{m \times n}$ $m \geq n$

If such a matrix $\underset{\sim}{A}$ has a full rank it must have 'n' linearly independent columns, rank $(A) = n$

Thm 1: A matrix $\underset{\sim}{A} \in \mathbb{R}^{m\times n}$ with $m \geq n$ maps no two distinct vectors to the same vector if and only if $\underset{\sim}{A}$ is a full rank.

$\rightarrow$ $\underset{\sim}{A}\,\underset{\sim}{x_1} = \underset{\sim}{y_1}$; $\underset{\sim}{A}\,\underset{\sim}{x_2} = \underset{\sim}{y_2}$ $(\underset{\sim}{x_1} \neq \underset{\sim}{x_2})$

then $\underset{\sim}{y_1} \neq \underset{\sim}{y_2}$ if and only if $\underset{\sim}{A}$ is full rak

Pf:- Given $\underset{\sim}{A} \in \mathbb{R}^{m\times n}$ maps no two distinct vector to the same vector.

$S_1 \Rightarrow S_2$

$\boxed{\sim S_2 \Rightarrow \sim S_1}$

$\underset{\sim}{x_1} \in \mathbb{R}^n$, $\underset{\sim}{x_2} \in \mathbb{R}^n$ and $\underset{\sim}{x_1} \neq \underset{\sim}{x_2}$

$\underline{\underset{\sim}{A}\,\underset{\sim}{x_1} = \underset{\sim}{y_1}}$; $\underset{\sim}{A}\,\underset{\sim}{x_2} = \underset{\sim}{y_2} \Rightarrow \underset{\sim}{y_1} \neq \underset{\sim}{y_2}$

Now we need to show $\underset{\sim}{A}$ is full rank.

$\rightarrow$ Let us assume $\underset{\sim}{A}$ is not full rank it has a non-trivial null space, let that vector be $\underset{\sim}{c} \neq 0$ i.e $\underset{\sim}{A}\,\underset{\sim}{c} = \underset{\sim}{0}$

$$\underset{\sim}{A}\,\underset{\sim}{x_1} = \underset{\sim}{y_1} \;;\; \underset{\sim}{x_2} = \underset{\sim}{x_1} + \underset{\sim}{c}$$

$\underset{\sim}{y}_2 = \underset{\sim}{A}\underset{\sim}{x}_2 = \underset{\sim}{A}(\underset{\sim}{x}_1 + \underset{\sim}{c}) = \underset{\sim}{A}\underset{\sim}{x}_1 = \underset{\sim}{y}_1$

i.e a contradiction, $\underset{\sim}{A}$ is full rank.

Now we need to show if $\underset{\sim}{A}$ is full rank

$\underset{\sim}{A}\underset{\sim}{x}_1 = \underset{\sim}{y}_1$; $\underset{\sim}{A}\underset{\sim}{x}_2 = \underset{\sim}{y}_2$ where $\underset{\sim}{x}_1 \neq \underset{\sim}{x}_2 \Rightarrow \underset{\sim}{y}_1 \neq \underset{\sim}{y}_2$

→ Consider $\underset{\sim}{x}_1 \neq \underset{\sim}{x}_2$ i.e $\underset{\sim}{A}\underset{\sim}{x}_1 = \underset{\sim}{y}$ and $\underset{\sim}{A}\underset{\sim}{x}_2 = \underset{\sim}{y}$

i.e $\underset{\sim}{A}(\underset{\sim}{x}_1 - \underset{\sim}{x}_2) = \underset{\sim}{0}$

$\underset{\sim}{x}_1 - \underset{\sim}{x}_2 = 0$ because $\underset{\sim}{A}$ is full rank.

i.e $\underset{\sim}{x}_1 = \underset{\sim}{x}_2$

(again a contradiction)

Matrix - matrix multiplication :-

$\underset{\sim}{A} \in \mathbb{R}^{l \times m}$ $\quad \underset{\sim}{C} \in \mathbb{R}^{m \times n}$

$\underset{\sim}{B} = \underset{\sim}{A}\underset{\sim}{C}$ i.e $\underset{\sim}{B} \in \mathbb{R}^{l \times n}$

$$B_{ij} = \sum_{k=1}^{m} A_{ik} C_{kj}$$

Inner product form :-

$$\underset{\sim}{B} = \underbrace{\begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 7 \end{bmatrix}}_{\underset{\sim}{A}} \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\underset{\sim}{C}} = \begin{bmatrix} 2a+3c & 2b+3d \\ 3a+4c & 3b+4d \\ 4a+7c & 4b+7d \end{bmatrix}$$

Linear Combination of columns

$$\begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_n \\ | & | & & | \end{bmatrix}_{l \times n} = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_m \\ | & | & & | \end{bmatrix}_{l \times m} \begin{bmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix}_{m \times n}$$

$$B = A\, C$$

Columns of B can be written as linear combination of columns of A with column vectors of C

$$\boxed{b_j = \sum_{k=1}^{m} c_{kj}\, a_k}$$

as linear combination coefficients

Outer-product form:-

$u, v \in \mathbb{R}^m$ outer product of two vectors u and v as a rank 1 matrix $A \in \mathbb{R}^{m \times m}$

and $A := u v^T$

$$B = \underset{l \times m}{A}\ \underset{m \times n}{C} = \underbrace{\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_m \\ | & | & & | \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} - & r_1^T & - \\ - & r_2^T & - \\ & \vdots & \\ - & r_m^T & - \end{bmatrix}}_{C}$$

i^{th} row of C matrix is denoted as a column vector $r_i \in \mathbb{R}^n$

$$B = A\,C = a_1 r_1^T + a_2 r_2^T + a_3 r_3^T + \ldots a_m r_m^T$$

Sum of 'm' rank 1 matrices

Four fundamental Subspaces of $\underset{\sim}{A}$

Column space :- $C(\underset{\sim}{A}) \rightarrow$ combination of columns of $\underset{\sim}{A}$

Row space :- $C(\underset{\sim}{A}^T) \rightarrow$ combination of columns of $\underset{\sim}{A}^T$

Null space :- $N(\underset{\sim}{A}) \rightarrow$ all solutions $\underset{\sim}{x}$ to $\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{0}$

Left Null space :- $N(A^T) \rightarrow$ all solutions $\underset{\sim}{y}$ to $\underset{\sim}{A}^T\underset{\sim}{y} = \underset{\sim}{0}$

Inverse of a matrix :-

A full rank square matrix is invertible

i.e., for $\underset{\sim}{A} \in \mathbb{R}^{m\times m}$, '$m$' columns of $\underset{\sim}{A}$ form a basis for $\mathbb{R}^m$.

$$\longleftrightarrow \quad \underset{\sim}{A} = \begin{bmatrix} | & | & & | \\ \underset{\sim}{a}_1 & \underset{\sim}{a}_2 & \cdots & \underset{\sim}{a}_m \\ | & | & & | \end{bmatrix}_{m\times m}$$

$\underset{\sim}{A}$ is a full rank matrix

Hence,

Any vector in $\mathbb{R}^m$ can be expressed as a linear combination of columns of $\underset{\sim}{A}$

$$\underset{\sim}{e}_j = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1} \leftarrow j^{th}$$

$$\underset{\sim}{e}_j = \sum_{i=1}^{m} z_{ij}\,\underset{\sim}{a}_i \checkmark$$

$$= \underset{\sim}{A}\,\underset{\sim}{z}_j \quad - (*)$$

$j = 1, 2, \dots m$

$$\begin{bmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \cdots & \underset{\sim}{e}_m \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ & & & & 1 \end{bmatrix} = I$$

$$\underset{\sim}{I} = \underset{\sim}{A}\,\underset{\sim}{Z} = \underset{\sim}{A} \begin{bmatrix} | & | & & | \\ z_1 & z_2 & \cdots & z_m \\ | & | & & | \end{bmatrix}$$

This matrix is called the inverse of $\underset{\sim}{A}$.

$\underset{\sim}{I} = \underset{\sim}{A}\,\underset{\sim}{Z} = \underset{\sim}{Z}\,\underset{\sim}{A}$ (Exercise)

And this $\underset{\sim}{Z}$ is called inverse of matrix $\underset{\sim}{A}$ and denote as $\underset{\sim}{A}^{-1}$

" Every non-singular square matrix $\underset{\sim}{A}$ has a unique inverse $\underset{\sim}{A}^{-1}$ that satisfies "

$$\boxed{\underset{\sim}{A}\,\underset{\sim}{A}^{-1} = \underset{\sim}{A}^{-1}\underset{\sim}{A} = \underset{\sim}{I}}$$

Matrix inverse times vector:-

$\underset{\sim}{A} \in \mathbb{R}^{m \times m}$ (Full rank) ; $\underset{\sim}{b} \in \mathbb{R}^m$; $\underset{\sim}{x} \in \mathbb{R}^m$

$$\boxed{\underset{\sim}{x} = \underset{\sim}{A}^{-1}\underset{\sim}{b}}$$

We can think $\underset{\sim}{x}$ as a unique vector satisfying $\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{b}$

$$\underset{\sim}{b} = \sum_{i=1}^{m} b_i \underset{\sim}{e_i} \quad \text{where } \underset{\sim}{e_i} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \rightarrow i$$

$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ are the coefficients of expansion of $\underset{\sim}{b}$ in $\{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \dots \underset{\sim}{e_m}\}$

$$\underset{\sim}{b} = \underset{\sim}{A}\underset{\sim}{x} = \sum_{j=1}^{m} x_j \underset{\sim}{a_j} \quad \text{where } \underset{\sim}{A} = \begin{bmatrix} | & | & & | \\ \underset{\sim}{a_1} & \underset{\sim}{a_2} & \cdots & \underset{\sim}{a_m} \\ | & | & & | \end{bmatrix}$$

$$\Downarrow$$

$\underset{\sim}{A}^{-1}\underset{\sim}{b} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$ are the coefficients of expansion of $\underset{\sim}{b}$ in $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \dots \underset{\sim}{a_m}\}$

This means we can think of $\underset{\sim}{A}^{-1}\underset{\sim}{b}$ as a change of basis operation from $\{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \dots \underset{\sim}{e_m}\}$ to $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \dots a_m\}$ i.e column-space of $\underset{\sim}{A}$

<u>Transpose</u> of a <u>matrix</u> ($\underset{\sim}{A}^T$) :-

If $\underset{\sim}{A} \in \mathbb{R}^{m \times n}$ then $\underset{\sim}{A}^T \in \mathbb{R}^{n \times m}$ where $A^T_{ij} = A_{ji}$

Useful properties of A^T and A^{-1} :-

ⓐ $(AB)^T = B^T A^T$ ⓑ $(AB)^{-1} = B^{-1} A^{-1}$

ⓒ $(A^T)^{-1} = (A^{-1})^T$

If $A = A^T$ (Symmetric matrix)

If $A = A^{+}$ (Hermitian matrix)

A^{+} is conjugate transpose of $A \in \mathbb{C}^{m \times n}$

Inner products :→

Let $x, y \in \mathbb{R}^m$, then inner product of x and y is a scalar $(x, y) = x^T y$

$$= \sum_{i=1}^{m} x_i y_i$$

Euclidean length of a vector x :-

$$\|x\| = \sqrt{x^T x} = \left(\sum_{i=1}^{m} x_i^2\right)^{1/2}$$

Angle between vectors x and y is α then $\cos\alpha = \dfrac{x^T y}{\|x\|\|y\|}$

Useful properties of inner products :-

ⓐ $(x_1 + x_2, y) = (x_1, y) + (x_2, y)$

(b) $(\underset{\sim}{x}, \underset{\sim}{y}_1 + \underset{\sim}{y}_2) = (\underset{\sim}{x}, \underset{\sim}{y}_1) + (\underset{\sim}{x}, \underset{\sim}{y}_2)$

(c) $(\alpha \underset{\sim}{x}, \beta \underset{\sim}{y}) = \alpha\beta\, (\underset{\sim}{x}, \underset{\sim}{y})$

Orthogonality :-

(a) Orthogonal vectors $\underset{\sim}{x}$ and $\underset{\sim}{y}$:→ Angle b/w vectors is 90° i.e $\cos 90^\circ = 0 \Rightarrow (\underset{\sim}{x}, \underset{\sim}{y}) = 0$

or $\underset{\sim}{x}^T \underset{\sim}{y} = 0$

(b) Orthogonal basis for a subspace :-

$\underset{\sim}{S} = \{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots \underset{\sim}{v}_n\}$ is said to be an orthogonal basis spanning a n-dimensional space if every pair of vectors in $\underset{\sim}{S}$ is orthogonal i.e $\underset{\sim}{v}_i^T \underset{\sim}{v}_j = 0$

orthonormal basis → In addition to orthogonality, $\underset{\sim}{v}_i^T \underset{\sim}{v}_i = 1$ (Length of each vector is 1)

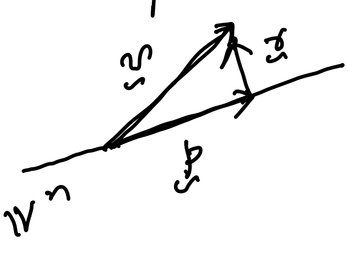
Result: Vectors in a orthogonal set $\underset{\sim}{S}$ are linearly independent

Pf :- exercise

Components of vector :- Consider $\mathbb{R}^m$ and n-dimensional space $\mathbb{W}^n \subseteq \mathbb{R}^m$ spanned by an orthonormal set $\mathcal{Q} = \{ \underset{\sim}{q}_1, \underset{\sim}{q}_2, \ldots \underset{\sim}{q}_n \}$

(Note :- $n \leq m$)

Let $\underset{\sim}{v} \in \mathbb{R}^m$ and let us decompose



$\underset{\sim}{v} = \underset{\sim}{p} + \underset{\sim}{r}$ where $\underset{\sim}{p} \in \mathbb{W}^n$ the space spanned by $\mathcal{Q}$ and $\underset{\sim}{r}$ is orthogonal to $\mathbb{W}^n$

Since $\underset{\sim}{p} \in \mathbb{W}^n$, $\underset{\sim}{p} = \sum_{i=1}^{n} \alpha_i \underset{\sim}{q}_i$ — ⊛

$$\underset{\sim}{v} = \underset{\sim}{p} + \underset{\sim}{r} \Rightarrow \underset{\sim}{q}_j^T \underset{\sim}{v} = \underset{\sim}{q}_j^T \underset{\sim}{p} + \cancelto{0}{\underset{\sim}{q}_j^T \underset{\sim}{r}}$$

$$\Rightarrow \underset{\sim}{q}_j^T \underset{\sim}{v} = \underset{\sim}{q}_j^T \left(\sum_{i=1}^{n} \alpha_i \underset{\sim}{q}_i \right) \quad — \square$$

$i \neq j$ will vanish

$$\Rightarrow \boxed{\alpha_j = \underset{\sim}{q}_j^T \underset{\sim}{v}}$$

$$\Rightarrow \underset{\sim}{v} = \underset{\sim}{r} + \sum_{i=1}^{n} (\underset{\sim}{q}_i^T \underset{\sim}{v}) \underset{\sim}{q}_i$$

If $m = n$ then $\underset{\sim}{v} = \sum_{i=1}^{m} (\underset{\sim}{q}_i^T \underset{\sim}{v}) \underset{\sim}{q}_i$

i.e Sum of coefficients of $q_i^T v$ times basis vectors q_i

Also $\underset{\sim}{v} = \sum_{i=1}^{m} q_i (q_i^T v) = \sum_{i=1}^{m} \underbrace{(q_i q_i^T)}_{} v$

$= \sum_{i=1}^{m} P_i v$

i.e Sum of orthogonal projections of v onto various directions q_i

ⓒ <u>Orthogonal matrices</u> :-

A square matrix $Q \in \mathbb{R}^{m \times m}$ is called orthogonal matrix if $Q^T = Q^{-1}$

i.e $\boxed{Q^T Q = Q Q^T = I}$ $(AZ = ZA = I)$

$$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_m \end{bmatrix}$$

$$Q^T Q = \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_m^T \end{bmatrix} \begin{bmatrix} q_1 & q_2 & \cdots & q_m \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \end{bmatrix} = I$$

This means $q_i^T q_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

If $Q \in \mathbb{C}^{m \times m}$ then $Q^\dagger Q = Q Q^\dagger = I$

then Q is called unitary matrix

we saw

Ax and $A^{-1}b$ interpretations

$$A \rightarrow Q$$

i.e Qx and $Q^{-1}b \Rightarrow Q^T b$

$Q^T b$ can be thought of change of basis from $\{e_1, e_2, \dots e_m\}$ to orthogonal basis $\{q_1, q_2, \dots q_m\}$

i.e columns of Q

(i) Orthogonal matrices preserves inner products

$$(x, y) := x^T y$$

$$(Qx, Qy) = (Qx)^T (Qy) = x^T Q^T Q y$$
$$= x^T y = (x, y)$$

(ii) Orthogonal matrices preserves lengths

$$\|\underset{\sim}{Q}\underset{\sim}{x}\| = \|\underset{\sim}{x}\| \quad (\text{verify?})$$

(iii) $\det \underset{\sim}{Q} = \pm 1 \qquad [\underset{\sim}{Q}^T \underset{\sim}{Q} = \underset{\sim}{I}]$

(iv) Action of $\underset{\sim}{Q} \in \mathbb{R}^{m \times m}$ on a vector is a rigid rotation (if $\det \underset{\sim}{Q} = 1$) or reflection (if $\det \underset{\sim}{Q} = -1$)