DS284: Numerical Linear Algebra
Assignment 5

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Problem 1 [10 marks]

If you encounter a zero pivot in Gaussian elimination without partial pivoting for a matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$, then which of the following is true:

(a) The matrix $\mathbf{A}$ is always singular.

(b) The matrix $\mathbf{A}$ is always non-singular.

(c) The matrix $\mathbf{A}$ may be singular or non-singular.

(d) The matrix $\mathbf{A}$ cannot be column-diagonally dominant.

Problem 2 [10 marks]

Which of the following is/are true?

(a) **QR** factorization and **LU** factorization can turn out to be identical for certain full rank matrices $\mathbf{A} \in \mathbb{R}^{m \times m}$.

(b) **LU** factorization is always unique for any full rank matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$.

(c) Gaussian elimination algorithm with partial pivoting will not have row interchanges for row-diagonally dominant full rank matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$.

(d) **LU** factorization using Gaussian elimination with partial pivoting algorithm results in **L** having $\|\mathbf{L}\| = O(1)$.

(e) Gaussian elimination algorithm without partial pivoting is always unstable.

Problem 3 [10 marks]

Which of the following is/are true?

(a) Recall that a given matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ can always be written as $\mathbf{A} = \mathbf{A}_\text{s} + \mathbf{A}_\text{ss}$, such that $\mathbf{A}_\text{s} = \mathbf{A}_\text{s}^T$ and $\mathbf{A}_\text{ss} = -\mathbf{A}_\text{ss}^T$. If $\mathbf{A}_\text{s}$ is symmetric positive definite, then $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$.

(b) Let $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & -4 & 1 \end{bmatrix}$ and $\mathbf{U} = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 2 & -8 \\ 0 & 0 & -67 \end{bmatrix}$ be the $\mathbf{LU}$ factorization of a given matrix $\mathbf{A}$. Then this matrix $\mathbf{A}$ has to be a symmetric positive definite matrix.

(c) If $\mathbf{R}$ is an exact Cholesky factor of a symmetric positive definite matrix $\mathbf{A}$ and $\tilde{\mathbf{R}}$ is the Cholesky factor obtained on a finite precision computer for the same matrix $\mathbf{A}$. Then $\frac{\|\tilde{\mathbf{R}}^T\tilde{\mathbf{R}}-\mathbf{A}\|}{\|\mathbf{A}\|} \leq \frac{\|\tilde{\mathbf{R}}-\mathbf{R}\|}{\|\mathbf{R}\|}$

(d) Applying Gaussian elimination algorithm without partial pivoting and Cholesky factorization algorithm on a given symmetric positive definite matrix gives the same $\mathbf{L}$ and $\mathbf{U}$ $(= \mathbf{L}^T)$ factors and takes same number of floating point operations.

(e) If $\mathbf{A} \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix and $\mathbf{X} \in \mathbb{R}^{m\times m}$ is any matrix, then $\mathbf{X}^T\mathbf{A}\mathbf{X}$ is always symmetric positive definite.

Problem 4 [10 marks]

If $\mathbf{A} \in \mathbb{R}^{m\times n}$ $(m > n)$ is a full rank matrix and let $\mathbf{M} = \mathbf{A}^T\mathbf{A}$ be $n \times n$ matrix. Then which of the following is true. ($\kappa(.)$ denotes the condition number)

(a) $\mathbf{M}$ is always full rank.

(b) $\kappa(\mathbf{A}) = \|\mathbf{A}\|\|\mathbf{M}^{-1}\mathbf{A}\|$.

(c) $\kappa(\mathbf{A}) = \|\mathbf{A}\|\|\mathbf{M}^{-1}\mathbf{A}^T\|$.

(d) If the system of equations $\mathbf{M}\mathbf{x} = \mathbf{A}^T\mathbf{b}$ are solved using a backward stable algorithm, then forward relative error in $\mathbf{x}$ i.e. $\frac{\|\tilde{\mathbf{x}}-\mathbf{x}\|}{\|\mathbf{x}\|} = O(\kappa(\mathbf{A})^2\epsilon_M)$.

(e) If σ_1 and σ_n are the maximum and minimum singular values of $\mathbf{A}$, then $\kappa(\mathbf{M}) = \frac{\sigma_1}{\sigma_n}$.

Problem 5 [10 marks]

Which of the following is/are true?

(a) For real symmetric matrices $\mathbf{A} \in \mathbb{R}^{m\times m}$ and $\mathbf{B} \in \mathbb{R}^{m\times m}$, $\mathbf{AB}$ and $\mathbf{BA}$ always have the same eigenvalues.

(b) For matrices $\mathbf{A} \in \mathbb{R}^{m\times m}$ and $\mathbf{B} \in \mathbb{R}^{m\times m}$, if $\mathbf{B}$ is invertible, then $\mathbf{AB}$ and $\mathbf{BA}$ are similar matrices.

(c) Two diagonalizable matrices $\mathbf{A}$ and $\mathbf{B}$ with the same eigenvalues and eigenvectors must be the same matrix.

(d) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are the linearly independent eigenvectors of $\mathbf{A}$ corresponding to an eigenvalue λ with algebraic multiplicity 3, then all vectors lying in the subspace spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are eigenvectors of $\mathbf{A}$ for that eigenvalue λ.

(e) If $\mathbf{A}$ is real and λ is an eigenvalue of $\mathbf{A}$, then so is $\bar{\lambda}$ (complex conjugate).

Problem 6 [10 marks]

Which of the following is/are true:

(a) If $\mathbf{A} \in \mathbb{R}^{m\times m}$ has a repeated eigenvalue λ with algebraic multiplicity $k(< m)$ and rest $(m-k)$ eigenvalues are distinct. If the dimension of the null space of $(\mathbf{A} - \lambda\mathbf{I})$ is also k. Then the system of equations $\mathbf{Px} = \mathbf{b}$ always has a unique solution for $\mathbf{x}$ where $\mathbf{P} \in \mathbb{R}^{m\times m}$ is the eigenvector matrix of $\mathbf{A}$.

(b) If the matrices $\mathbf{A} \in \mathbb{R}^{m\times m}$ and $\mathbf{B} \in \mathbb{R}^{m\times m}$ are symmetric and are related by a similarity transformation, then the row space of $\mathbf{Q}$ and $\overline{\mathbf{Q}}$ are the same where $\mathbf{Q} \in \mathbb{R}^{m\times m}$ is the eigenvector matrix of $\mathbf{A}$ and $\overline{\mathbf{Q}} \in \mathbb{R}^{m\times m}$ is the eigenvector matrix of $\mathbf{B}$.

(c) $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 2 & 5 \end{bmatrix}$, one of the eigenvalues of $\mathbf{A}^2 - 7\mathbf{A} + 13\mathbf{I}$ is 5.

(d) All orthogonal matrices are normal matrices.

(e) Let $\mathbf{A} \in \mathbb{C}^{m\times m}$ satisfy the property $\mathbf{A} = \mathbf{A}^\dagger$ where $\mathbf{A}^\dagger$ is complex conjugate transpose of $\mathbf{A}$. Let this $\mathbf{A}$ admit the Schur form $\mathbf{A} = \mathbf{QTQ}^\dagger$, then the eigenvectors of $\mathbf{A}$ are the columns of $\mathbf{Q}$ matrix.

Problem 7 [10 marks]

Which of the following is/are true:

(a) Power Iteration with a shift, when applied to symmetric matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$, where the shift μ is close to an eigenvalue λ_J of $\mathbf{A}$, converges to an eigenvector corresponding to eigenvalue λ_J.

(b) If the distance between a unit vector $\mathbf{x}$ and the eigenvector $\mathbf{q}_J$ of a real symmetric matrix $\mathbf{A}$, in the sense of 2 norm is Δ, i.e. $\|\mathbf{x} - \mathbf{q}_J\|_2 = \Delta$, then $|\mathbf{x}^T\mathbf{Ax} - \lambda_J| = O(\Delta^2)$.

(c) Let $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$. The value $\beta = 5$ minimizes the $\|\mathbf{Ax} - \beta\mathbf{x}\|_2$.

(d) Given $\mathbf{A} \in \mathbb{R}^{m\times m}$ and $\mathbf{x} \in \mathbb{R}^m$. Then normal equations for solving least square problem for minimizing $\|\mathbf{Ax} - \alpha\mathbf{x}\|_2$ with respect to α is given by $(\mathbf{A}^T\mathbf{A})\mathbf{x} = \mathbf{A}^T(\alpha\mathbf{x})$.

(e) If $\mathbf{A} \in \mathbb{R}^{m\times m}$ is symmetric and tri-diagonal, then the number of floating point operations in each matrix vector multiplication step of Power Iteration is $O(m^2)$.

Problem 8 [10 marks]

Which of the following is/are true:

(a) Pure QR algorithm is equivalent to the Simultaneous iteration applied to an initial guess of vectors which are columns of a square full rank matrix.

(b) Pure QR algorithm used to solve the eigenvalues and eigenvectors of a matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ generates a sequence of matrices $\mathbf{Q}^{(k)}$ which converges to the eigenvector matrix of $\mathbf{A}$ as $k \to \infty$.

(c) Pure QR algorithm used to solve the eigenvalues and eigenvectors of a matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ generates QR factorization of $\mathbf{A}^k$ with $\mathbf{Q}^{(k)}$ and $\mathbf{R}^{(k)}$ generated at the k^{th} iteration of the algorithm.

(d) Power iteration of $\mathbf{A} \in \mathbb{R}^{m\times m}$ applied to a bunch of n linearly independent vectors followed by QR factorization at every step converges to the space spanned by n eigenvectors corresponding to the n largest eigenvalues of $\mathbf{A}$.

(e) Computational complexity for finding the eigenvectors using Pure QR algorithm and the Simultaneous iteration is same.

Problem 9 [10 marks]

Which of the following is/are true:

(a) An eigenvalue solver can be designed to compute eigenvalues and eigenvectors of a given matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ in a finite number of steps, using exact arithmetic.

(b) An eigenvalue solver designed to compute all eigenvalues and eigenvectors of a symmetric dense matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ requires at most $O(m^3)$ work, if it is not initially reduced to tri-diagonal form in Phase 1.

(c) Power iteration produces a sequence of vectors $\mathbf{v}^{(i)}$ that converges to the eigenvector corresponding to the largest eigenvalue of $\mathbf{A} \in \mathbb{R}^{m\times m}$ starting with any initial guess vector $\mathbf{v}^{(0)} \neq \mathbf{0}$.

(d) Let $\mathbf{F} \in \mathbb{R}^{m\times m}$ denote the Householder reflector that introduces zeros below the diagonal entry of symmetric matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ in the 1st column when pre-multiplied with $\mathbf{A}$. Then eigenvalues of $\mathbf{FAF}^T$ and $\mathbf{A}$ are the same.

(e) If $\{\mathbf{Q}_k\}_{k=1...m}$ denotes the sequence of matrices arising from Householder QR factorization which triangularizes symmetric matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$. Then $\mathbf{Q}^T\mathbf{AQ}$ will be tri-diagonal, where $\mathbf{Q} = \mathbf{Q}_1\mathbf{Q}_2...\mathbf{Q}_m$.

Problem 10 [10 marks]

Given matrix $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 4 & 0 & 3 & 0 & 0 \\ 0 & 8 & 0 & 2 & 0 \\ 0 & 0 & 0 & 9 & 5 \end{bmatrix}$, which of the following is/are true:

(a) The sum of eigenvalues of $\mathbf{A}$ is 12.

(b) The sum of eigenvalues of $\mathbf{AA}^T$ is 204.

(c) The product of eigenvalues of $\mathbf{AA}^T$ is 3600.

(d) Eigenvalues of $\mathbf{A}$, $\mathbf{AA}^T$ and $\mathbf{A}^T\mathbf{A}$ are the same.

(e) The eigenvalues of $\mathbf{AA}^T$ are non-negative (≥ 0).