Eigenvalue Problems

Let $\underset{\sim}{A} \in \mathbb{R}^{m\times m}$, $\underset{\sim}{x} \neq 0 \in \mathbb{C}^m$ ($\mathbb{C}$ is set of complex nos)

then $\underset{\sim}{x}$ is an eigenvector of $\underset{\sim}{A}$ and $\lambda \in \mathbb{C}$ is its corresponding eigenvalue

if $$\boxed{\underset{\sim}{A}\underset{\sim}{x} = \lambda \underset{\sim}{x}}$$

* The set of all eigenvalues of a matrix $\underset{\sim}{A}$ is called the spectrum of $\underset{\sim}{A}$ denoted by $\Lambda(\underset{\sim}{A})$

Application areas:-

* Insights into evolution of system
  - → Vibration analysis
  - → Study of resonance
  - → Stability of structure
  - → fluid flows subjected to small perturbations

* Quantum mechanical modeling of materi (Solving Schrodinger equation)

* Principal stresses in solid mechanics

* PCA in data driven modeling

* Pagerank algorithm used in search engines is an eigenvalue problem

* Eigenvectors of graph Laplacian matrix actually help in construction of efficient filters Graph Convolution Neural Network!

Eigenvalue decomposition!

An eigenvalue decomposition of

$\underset{\sim}{A} \in \mathbb{R}^{m \times m}$ is a factorization

$$\underset{\sim}{A} = \underset{\sim}{X} \underset{\sim}{\Lambda} \underset{\sim}{X}^{-1}$$ where

$\underset{\sim}{X}$ is non singular and $\underset{\sim}{\Lambda}$ is diagonal with $\underset{\sim}{X}$ comprising of eigenvectors of $\underset{\sim}{A}$ as columns!

<u>Note</u>:- Such decomposition may not always exist!

$$\underset{\sim}{A} \underset{\sim}{X} = \underset{\sim}{X} \underset{\sim}{\Lambda}$$

$$[A] \begin{bmatrix} | & | & | & & | \\ \underset{\sim}{x_1} & \underset{\sim}{x_2} & x_3 & \cdots & x_m \\ | & | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & | & \\ \underset{\sim}{x_1} & x_2 & x_3 & \cdots \\ | & | & | & \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{bmatrix}$$

$$\underset{\sim}{A} \underset{\sim}{x_j} = \lambda_j x_j$$

i.e j^{th} column of $\underset{\sim}{X}$ is j^{th} eigen vector and (j,j) entry of Λ is corresponding eigenvalue!

→ Geometric multiplicity :- The geometric multiplicity of an eigenvalue λ is the number of linearly independent eigenvectors associated with that eigen value λ. If $\lambda \in \Lambda(A)$, eigenspace E_λ

E_λ is an invariant subspace of $\underset{\sim}{A}$

i.e $\underset{\sim}{A} E_\lambda \subseteq E_\lambda$

$\underset{\sim}{v_1}, \underset{\sim}{v_2}$ λ

$$\underset{\sim}{x} = \alpha_1 \underset{\sim}{v_1} + \alpha_2 \underset{\sim}{v_2}$$

$$\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{A}(\alpha_1 \underset{\sim}{v_1} + \alpha_2 \underset{\sim}{v_2})$$
$$= \alpha_1 \lambda \underset{\sim}{v_1} + \alpha_2 \lambda \underset{\sim}{v_2}$$
$$= \lambda [\alpha_1 \underset{\sim}{v_1} + \alpha_2 \underset{\sim}{v_2}$$
$$= \lambda \underset{\sim}{x}$$

The dimension of E_λ is the geometric multiplicity of λ i.e maximum number of linearly independent eigenvectors that can be found for a given λ

* Characteristic Polynomial :-

The characteristic polynomial p_A of $\underset{\sim}{A} \in \mathbb{R}^{m \times m}$ is the m^{th} degree monic polynomial $\boxed{p_A(z) = \det(z\underline{I} - \underset{\sim}{A})}$ ✓

(coefficient of z^m is 1 $\rightarrow$ monic polynomial)

__Thm__ :- λ is eigenvalue of $\underset{\sim}{A}$ if and only if $p_A(\lambda) = 0$

$\underset{\sim}{A}\underset{\sim}{x} = \lambda \underset{\sim}{x}$

__Note__ :- $\underset{\sim}{A} \in \mathbb{R}^{m \times m}$

$(\underset{\sim}{A} - \lambda \underline{I})\underset{\sim}{x} = 0$

λ can be complex

any comple λ must appear in complex conjugate pairs

Eigenvector of $\underline{A}$ lies in the null space of $(\underline{A} - \lambda \underline{I})$

i.e $\lambda = a + ib$ is an eigenvalue

$\lambda^* = a - ib$ is an eigenvalue

* Algebraic multiplicity :-

Since $p_A(z)$ is monic m-degree polynomial, it can be written as

$$p_A(z) = (z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_m)$$

for some $\lambda_j \in \mathbb{C}$ (roots of $p_A(z)$)

Each λ_j is an eigenvalue and in general may be repeated

"The multiplicity of λ as a root of $p_A(z)$ is the algebraic multiplicity of an eigenvalue λ"

Remark:- (a) If $\underline{A} \in \mathbb{R}^{m \times m}$ then $\underline{A}$ has m eigenvalues counting algebraic multiplicity. In particular if roots of $p_A(z)$ are simple, then $\underline{A}$ has m distinct eigenvalues.

(b) The algebraic multiplicity of an eigenvalue λ is always atleast as large as its geometric multiplicity.

* <u>Similarity transformation</u> :-

If $\underset{\sim}{X} \in \mathbb{R}^{m \times m}$ is non singular, then

$$\underset{\sim}{A} \rightarrow \underset{\sim}{X}^{-1} \underset{\sim}{A} \underset{\sim}{X}$$ is called a

similarity transformation of $\underset{\sim}{A} \in \mathbb{R}^{m \times m}$

We say that two matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are similar if there is a similarity transformation of one to another

i.e if there is a nonsingular $\underset{\sim}{X} \in \mathbb{R}^{m \times m}$ such that $\underset{\sim}{B} = \underset{\sim}{X}^{-1} \underset{\sim}{A} \underset{\sim}{X}$

then $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are said to similar!

Thm:- If X is nonsingular, then A and $X^{-1}AX$ have the same characteristic polynomial, eigenvalues and algebraic multiplicity and geometric multiplicity!

Pf:-

$$p_{X^{-1}AX}(z) = \det(zI - X^{-1}AX)$$
$$= \det(zX^{-1}X - X^{-1}AX)$$
$$= \det(X^{-1}(zI - A)X)$$
$$= \left(\det X^{-1}\right)\left[\det(zI - A)\right](\det X)$$
$$= (\det X)^{-1}(\det(zI - A))(\det X)$$
$$= \det(zI - A) = p_A(z)$$

Hence A, $X^{-1}AX$ have the same eigenvalues (same roots of $p_A(z)$) and hence same algebraic multiplicity

Build a matrix E_λ whose column vectors span the eigenspace for the matrix $\underset{\sim}{A}$, corresponding to the eigen value λ

$$(\underset{\sim}{X}^{-1} \underset{\sim}{A} \underset{\sim}{X})\underbrace{(\underset{\sim}{X}^{-1} E_\lambda)} = \underset{\sim}{X}^{-1} \underset{\sim}{A} \underset{\sim}{E}_\lambda$$
$$= \underbrace{\underset{\sim}{X}^{-1} \underset{\sim}{E}_\lambda} \Lambda$$

$\underset{\sim}{X}^{-1} E_\lambda$ is the eigenspace for $\underline{X}^{-1}\underline{A}\,\underline{X}$ corresponding to the eigenvalue λ.

Hence geometric multiplicity of $\underset{\sim}{A}$ and $\underset{\sim}{X}^{-1}\underset{\sim}{A}\underset{\sim}{X}$ are the same.

($\because E_\lambda$ and $\underline{X}^{-1} E_\lambda$ has the same rank)

<u>Defective eigenvalues and matrices</u>

* A generic matrix need not have distinct eigenvalues i.e algebraic multiplicity need not be 1 and geometric multiplicity need not be 1 as well and need not be equal to algebraic multiplicity as well)

$$\underset{\sim}{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} ; \; B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Both $\underset{\sim}{A}$ and $\underset{\sim}{B}$ eigenvalue

$\lambda = 2$.

Algebraic multiplicity of $\lambda = 2$ for A?

$\longrightarrow$ 3

Algebraic multiplicity of $\lambda = 2$ for B?

$\longrightarrow$ 3

For $\underset{\sim}{A}$, we can choose 3 linearly independent eigenvectors $\underset{\sim}{e_1}, \underset{\sim}{e_2}, \underset{\sim}{e_3}$ and geometric multiplicity is also 3.

For B, we can only have only linearly independent eigenvector $\underset{\sim}{e_1}$, the geometric multiplicity of $\underset{\sim}{B}$ is 1

* An eigenvalue whose algebraic multiplicity is greater than its geometric multiplicity is called a defective eigenvalue!

* A matrix that has atleast one defective eigenvalue is a defective matrix i.e it does not possess a full set of m linearly independent eigenvectors

* A diagonal matrix is non-defective

(algebraic multiplicity of eigenvalue λ = geometric multiplicity of λ)

Diagonalizability

<u>Thm</u>: $\underset{\sim}{A} \in \mathbb{R}^{m \times m}$ is non-defective

if and only if it has eigenvalue decomposition $\underset{\sim}{A} = \underset{\sim}{X} \underset{\sim}{\Lambda} \underset{\sim}{X}^{-1}$ where $\underset{\sim}{X} \in \mathbb{R}^{m \times m}$ non singular matrix.

→ A non-defective matrix is diagonalizable

Proof: To show given $\underset{\sim}{A} = \underset{\sim}{X} \underset{\sim}{\Lambda} \underset{\sim}{X}^{-1}$, $\underset{\sim}{A}$ is non-defective!

$\underset{\sim}{A} = \underset{\sim}{X} \underset{\sim}{\Lambda} \underset{\sim}{X}^{-1} \rightarrow \underset{\sim}{A}$ is similar to $\underset{\sim}{\Lambda}$

$\underset{\sim}{A}$ and $\underset{\sim}{\Lambda}$ has the same eigenvalues and same multiplicities.

Since $\underset{\sim}{\Lambda}$ is a diagonal matrix, it is non-defective and thus $\underset{\sim}{A}$ is non-defective.

→ To show if $\underset{\sim}{A}$ is non-defective ↑

then $\underset{\sim}{A} = \underset{\sim}{X} \underset{\sim}{\Lambda} \underset{\sim}{X}^{-1}$

* Since $\underset{\sim}{A}$ is non-defective, it must have m linearly independent eigenvectors!

If $\underset{\sim}{X} = \left[\underset{\sim}{x}_1 \middle| \underset{\sim}{x}_2 \middle| \cdots \middle| \underset{\sim}{x}_m \right]$

comprises of these m linearly independent vectors, X is full rank

$$\underset{\sim}{A}\,\underset{\sim}{X} = \underset{\sim}{A} \left(\underset{\sim}{x}_1 \middle| \underset{\sim}{x}_2 \middle| \cdots \middle| \underset{\sim}{x}_m \right)$$

$$= \left[\underset{\sim}{x}_1 \middle| \underset{\sim}{x}_2 \middle| \cdots \middle| x_m \right] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{bmatrix}$$

$$= \underset{\sim}{X}\,\underset{\sim}{\Lambda}$$

$$\underset{\sim}{A}\,\underset{\sim}{X} = \underset{\sim}{X}\,\underset{\sim}{\Lambda}$$

$$\boxed{\underset{\sim}{A} = \underset{\sim}{X}\,\underset{\sim}{\Lambda}\,\underset{\sim}{X}^{-1}}$$

<u>Thm</u>:- Trace of $\underset{\sim}{A} \in \mathbb{R}^{m \times m}$

$$tr(\underset{\sim}{A}) = \sum_{j=1}^{m} a_{jj} = \sum_{j=1}^{m} \lambda_j$$

Determinant of $\underset{\sim}{A} \in \mathbb{R}^{m \times m}$

$$det(\underset{\sim}{A}) = \prod_{j=1}^{m} \lambda_j$$

<u>Unitary</u> / <u>orthogonal</u> <u>diagonalization</u>:-

* $\underset{\sim}{Q} \in \mathbb{C}^{m \times m}$ is a unitary matrix

if $\underset{\sim}{Q}^{+}\underset{\sim}{Q} = \underset{\sim}{Q}\,\underset{\sim}{Q}^{+} = \underset{\sim}{I}$

when $\underset{\sim}{Q}^{+}$ is conjugate transpose of $\underset{\sim}{Q}$

* Unitary matrix reduces to orthogonal matrices if $\underline{Q}$ is real i.e $\underline{Q}^T\underline{Q} = \underline{Q}\underline{Q}^T = \underline{I}$ is satisfied!

For a non-defective matrix $\underline{A} \in \mathbb{R}^{m\times m}$, it is possible not only to have m linearly independent eigenvectors, but these vectors can be orthogonal as well!

[If $A \in \mathbb{R}^{m\times m}$ eigenvalues can be complex and corresponding eigenvectors have to be complex as well]

Now in general for a non-defective $\underline{A} \in \mathbb{R}^{m\times m}$, we define unitary diagonalizability i.e there exists a unitary matrix $\underset{\sim}{Q}$, such that

$$\underset{\sim}{A} = \underset{\sim}{Q} \underset{\sim}{\Lambda} \underset{\sim}{Q}^{+} \quad \text{where}$$

Q^{+} is conjugate transpose of $\underset{\sim}{Q}$ and columns of $\underset{\sim}{Q}$ are eigenvectors of $\underset{\sim}{A}$

Unitary diagonalizability reduces to orthogonal diagonalizability if $\underset{\sim}{Q}$ is real matrix and λ are real

i.e. $\underset{\sim}{A} = \underset{\sim}{Q} \underset{\sim}{\Lambda} \underset{\sim}{Q}^{T}$

Eg:- A symmetric matrix $\underset{\sim}{S} \in \mathbb{R}^{m \times m}$ satisfies $\underset{\sim}{S} = \underset{\sim}{S}^{T}$ & has real eigen values

$$\rightarrow \quad \underset{\sim}{S} \underset{\sim}{x} = \lambda \underset{\sim}{x}$$

Take complex conjugate both sids $\Rightarrow$ $\underbrace{\underset{\sim}{S} \underset{\sim}{x}^{*} = \lambda^{*} \underset{\sim}{x}^{*}}_{\text{where } \lambda^{*} \text{ is complex conjugate of } \lambda}$

$$Sx = \lambda x \Rightarrow x^{*T} S x = \lambda x^{*T} x$$

$$\Rightarrow \underbrace{(Sx)^T} x^* = \lambda x^{*T} x \quad — (1)$$

$$Sx^* = \lambda^* x^* \Rightarrow x^T S x^* = \lambda^* x^T x^*$$

$$\Rightarrow x^T S^T x^* = \lambda^* x^T x^*$$

$$\Rightarrow \underbrace{(Sx)^T} x^* = \lambda^* x^T x^* \quad — (2)$$

$$(1) - (2) \Rightarrow 0 = \lambda x^{*T} x - \lambda^* x^T x^*$$

$$= (\lambda - \lambda^*)(x^T x^*)$$

Since $x^T x^* \neq 0 \Rightarrow \lambda - \lambda^* = 0$

$$\Rightarrow \boxed{\lambda = \lambda^*}$$

Hence eigenvalues of a symmetric matrix real.

Consider the case where S is symmetric (λ_1, x_1) and (λ_2, x_2) are two eigenpairs of S where $\lambda_1 \neq \lambda_2$

$$S x_1 = \lambda_1 x_1, \quad S x_2 = \lambda_2 x_2$$

$$\Rightarrow \underline{x_2^T S x_1 = \lambda_1 x_2^T x_1}, \quad \underline{x_1^T S x_2 = \lambda_2 x_1^T x_2}$$

$\underline{x}_2^T \underline{S}\, \underline{x}_1 = \lambda_1 \underline{x}_2^T \underline{x}_1$

$\Rightarrow \underline{x}_2^T \underline{S}^T \underline{x}_1 = \lambda_1 \underline{x}_2^T \underline{x}_1$

$\Rightarrow (\underline{S}\,\underline{x}_2)^T \underline{x}_1 = \lambda_1 \underline{x}_2^T \underline{x}_1 \quad - (1)$

$\underline{x}_1^T \underline{S}\, \underline{x}_2 = \lambda_2 \underline{x}_1^T \underline{x}_2$

$(\underline{S}\,\underline{x}_2)^T \underline{x}_1 = \lambda_2 \underline{x}_1^T \underline{x}_2 \quad - (2)$

(1) − (2)

$0 = (\lambda_1 - \lambda_2)\, \underline{x}_1^T \underline{x}_2$

$\lambda_1 - \lambda_2 \neq 0 \Rightarrow \underline{x}_1^T \underline{x}_2 = 0$

i.e the eigenvectors $\underline{x}_1, \underline{x}_2$ are orthogonal to each other.

Let us say Algebraic multiplicity of $\underline{S} \in \mathbb{R}^{m\times m}$ is $n < m$ for an eigenvalue λ you can find $(m-n)$ distinct eigenvalues and the corresponding eigenvectors are orthogonal. It is possible to find 'n' linearly independent eigenvectors corresponding to the eigenvalue λ whose algebraic multiplicity is n. This means I can find 'n' orthogonal vectors which span this 'n' dimensional invariant subspace which are the eigenvectors for my $\underset{\sim}{S}$.

Thm:- A symmetric matrix $\underset{\sim}{S} \in \mathbb{R}^{m \times m}$ is always non-defective and is orthogonally diagonalizable with real eigenvalues i.e

$$\underset{\sim}{S} = \underset{\sim}{Q} \underset{\sim}{\Lambda} \underset{\sim}{Q}^T$$

where $\underset{\sim}{Q}$ is orthogonal matrix
$\underset{\sim}{\Lambda}$ is a diagonal matrix

Thm:- A skew symmetric matrix $\underset{\sim}{W} \in \mathbb{R}^{m \times m}$ which satisfies $\underset{\sim}{W} = -\underset{\sim}{W}^T$. This skew symmetric is non-defective and has purely imaginary eigenvalues.

and is unitarily diagonalizable

i.e $\underset{\sim}{W} = \underset{\sim}{Q} \underset{\sim}{\Lambda} \underset{\sim}{Q}^{+}$

It turns out that there is a class of unitarily diagonalizable matrices and these set of matrices satisfy the property $\underset{\sim}{A}^T \underset{\sim}{A} = \underset{\sim}{A} \underset{\sim}{A}^T$

* A matrix is normal if

$$\underset{\sim}{A}^T \underset{\sim}{A} = \underset{\sim}{A} \underset{\sim}{A}^T$$

Thm:- A matrix is unitarily diagonazable if and only if $\underset{\sim}{A}^T \underset{\sim}{A} = \underset{\sim}{A} \underset{\sim}{A}^T$ for $\underset{\sim}{A} \in \mathbb{R}^{m \times m}$

Pf:- Direction 1

If a matrix is unitary diagonalizable then we need to show $\underset{\sim}{A}^T \underset{\sim}{A} = \underset{\sim}{A} \underset{\sim}{A}^T$

Since the given matrix is unitarily diagonalizable we know

$$\underset{\sim}{A} = \underset{\sim}{Q} \underset{\sim}{\Lambda} \underset{\sim}{Q}^{+}$$

$$\underset{\sim}{A}^{T} \underset{\sim}{A} - \underset{\sim}{A} \underset{\sim}{A}^{T} = \underset{\sim}{A}^{+} \underset{\sim}{A} - \underset{\sim}{A} \underset{\sim}{A}^{+}$$

$$= (Q \Lambda Q^{+})^{+} (\underline{Q} \underline{\Lambda} Q^{+}) - (Q \Lambda Q^{+})(Q \Lambda Q^{+})^{+}$$

$$= (Q^{+})^{+} \Lambda^{+} \underbrace{Q^{+} Q} \Lambda Q^{+} - Q \Lambda \underbrace{Q^{+} (Q^{+})^{+}} \Lambda^{+} Q^{+}$$

$$= Q \Lambda^{+} I \Lambda Q^{+} - Q \Lambda \underbrace{Q^{+} Q} \Lambda^{+} Q^{+}$$

$$= Q \Lambda^{+} \Lambda Q^{+} - Q \Lambda \Lambda^{+} Q^{+}$$

Since $\Lambda^{+} \Lambda = \Lambda \Lambda^{+}$ as Λ is a diagonal matrix

$$= 0$$

$$\Rightarrow \boxed{A^{T} A - A A^{T} = 0}$$

To show that $A^T A = A A^T$ implies unitary diagonalizability.

Every $A \in \mathbb{R}^{m \times m}$ can be decomposed as $A = A_s + A_{ss}$

$$A_s = \frac{1}{2}(A + A^T)$$

$$A_{ss} = \frac{1}{2}(A - A^T)$$

What does

$A^T A - A A^T = 0$ result in ?

$$A^T A - A A^T = (A_s + A_{ss})^T (A_s + A_{ss}) - (A_s + A_{ss})(A_s + A_{ss})^T$$

$$= (A_s - A_{ss})(A_s + A_{ss}) - (A_s + A_{ss})(A_s - A_{ss})$$

$$= 2\underbrace{(A_s A_{ss} - A_{ss} A_s)}$$

We have $\underset{\sim}{A}^T \underset{\sim}{A} - \underset{\sim}{A}\underset{\sim}{A}^T = 0$

$$(A_s A_{ss} - A_{ss} A_s) = 0$$

$$\Rightarrow \boxed{A_s A_{ss} = A_{ss} A_s}$$

(This means A_s and A_{ss} commute with each other)

Note :-

* Two matrices commute if and only if they have same eigenvectors!

Here Q is a unitary matrix

$$\underset{\sim}{A_s} = \underset{\sim}{Q} \underset{\sim}{\Lambda_s} \underset{\sim}{Q}^{\dagger} \quad , \quad \underset{\sim}{A_{ss}} = \underset{\sim}{Q} \underset{\sim}{\Lambda_{ss}} \underset{\sim}{Q}^{\dagger}$$

$$\Rightarrow \underset{\sim}{\Lambda_s} = \underbrace{\underset{\sim}{Q}^{\dagger} \underset{\sim}{A_s} Q} \qquad \Rightarrow \underset{\sim}{\Lambda_{ss}} = \underbrace{\underset{\sim}{Q}^{\dagger} \underset{\sim}{A_{ss}} \underset{\sim}{Q}}$$

Consider

$$\underset{\sim}{Q}^{\dagger} \underset{\sim}{A} \underset{\sim}{Q} = \underset{\sim}{Q}^{\dagger} (\underset{\sim}{A_s} + \underset{\sim}{A_{ss}}) \underset{\sim}{Q}$$

$$= \underset{\sim}{\Lambda_s} + \underset{\sim}{\Lambda_{ss}} = \underset{\sim}{\Lambda}$$

Hence $\underset{\sim}{Q}^{+}\underset{\sim}{A}\underset{\sim}{Q} = \underset{\sim}{\Lambda}$

$$\Rightarrow \boxed{\underset{\sim}{A} = \underset{\sim}{Q}\underset{\sim}{\Lambda}\underset{\sim}{Q}^{+}}$$

Schur factorization

A factorization of $\underset{\sim}{A} \in \mathbb{C}^{m\times m}$ of the form $\underset{\sim}{A} = \underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^{+}$ where $\underset{\sim}{Q}$ is unitary $(Q^{+}Q = I)$ and $\underset{\sim}{T}$ is upper triangular is called Schur factorization

Note:- Since $\underset{\sim}{A}$ and $\underset{\sim}{T}$ are similar, the eigenvalues of $\underset{\sim}{A}$ and $\underset{\sim}{T}$ are same!

Thm:- Every square matrix $\underset{\sim}{A} \in \mathbb{C}^{m \times m}$ has a Schur factorization of the form

$$\underset{\sim}{A} = \underset{\sim}{Q}\,\underset{\sim}{T}\,\underset{\sim}{Q}^{+}$$ as discussed before.

If $\underset{\sim}{A} \in \mathbb{R}^{m \times m}$, a factorization with real matrices exist $\underset{\sim}{A} = \underset{\sim}{U}\,\underset{\sim}{T}\,\underset{\sim}{U}^{T}$ where $\underset{\sim}{U}^{T}\underset{\sim}{U} = \underset{\sim}{U}\,\underset{\sim}{U}^{T} = I$ where $\underset{\sim}{T}$ is quasi upper triangular.

Quasi upper triangular matrices are those with diagonals with 1×1 block or 2×2 block

Summary:-

(i) A diagonalization $\underset{\sim}{A} = \underset{\sim}{X}\,\underset{\sim}{\Lambda}\,\underset{\sim}{X}^{-1}$ exists if and only if $\underset{\sim}{A}$ is non-defective

(ii) A unitary diagonalization $\underset{\sim}{A} = \underset{\sim}{Q}\,\Lambda\,\underset{\sim}{Q}^{+}$

exists if $\underset{\sim}{A}$ is normal

(iii) A unitary triangularization
(Schur Factorization)

$\underset{\sim}{A} = \underset{\sim}{Q}\,\underset{\sim}{T}\,Q^{\dagger}$ always exists