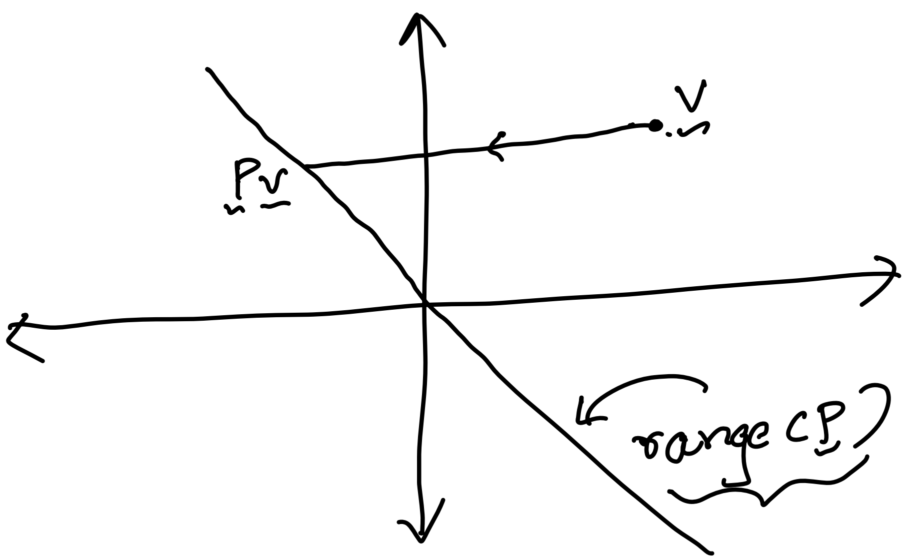
Projectors

A projection on a vector space V is a linear operator $\underset{\sim}{P} : V \to V$ such that $\underset{\sim}{P}^2 = \underset{\sim}{P}$

In the finite-dimensional case, a square matrix $\underset{\sim}{P}$ is called projector matrix if it is equal to its square i.e $\underset{\sim}{P}^2 = \underset{\sim}{P}$

The condition $\underset{\sim}{P}^2 = \underset{\sim}{P}$ is called idempotent condition!



Geometrically $\underset{\sim}{P}\underset{\sim}{v}$ would be a shadow projected by $\underset{\sim}{v}$ onto range (P) if one were to shine light onto range (P)!

From what direction does the light shine it is from $\underset{\sim}{v}$ to $\underset{\sim}{P}\underset{\sim}{v}$

So, $\underset{\sim}{P}\underset{\sim}{v} - \underset{\sim}{v}$ is the direction of light source

$$\underset{\sim}{P}\left[\underbrace{\underset{\sim}{P}\underset{\sim}{v} - \underset{\sim}{v}}\right] = \underset{\sim}{P}^2\underset{\sim}{v} - \underset{\sim}{P}\underset{\sim}{v}$$

$$= \underset{\sim}{P}\underset{\sim}{v} - \underset{\sim}{P}\underset{\sim}{v} = 0$$

i.e $\underbrace{\underset{\sim}{P}\underset{\sim}{v} - \underset{\sim}{v}} \in \text{null}(\underset{\sim}{P})$

Remarks :-

1. $\underset{\sim}{P} \in \mathbb{R}^{m \times m}$, $\underset{\sim}{P}^2 = \underset{\sim}{P}$ (Idempotency) is satisfied by a projector

2. $\underset{\sim}{P}\underset{\sim}{v} - \underset{\sim}{v} \in \text{null}(P)$ and is the direction of projection of $\underset{\sim}{v}$ onto range (P)

3. If $\underset{\sim}{P}$ is a projector and vector $\underset{\sim}{x} \in$ range (P), then $\underset{\sim}{P}\underset{\sim}{x} = \underset{\sim}{x}$

Pf:- If $\underset{\sim}{x} \in \text{range}(P)$, then

$$\underset{\sim}{x} = \underset{\sim}{P}\underset{\sim}{y} \text{ for some } \underset{\sim}{y}$$

then $$\underset{\sim}{P}\underset{\sim}{x} = \underset{\sim}{P}(\underset{\sim}{P}\underset{\sim}{y})$$
$$= \underset{\sim}{P}^2\underset{\sim}{y} = \underset{\sim}{P}\underset{\sim}{y} = \underset{\sim}{x}$$

i.e $\underset{\sim}{x}$ lies exactly in its own shadow.

(4) If $\underset{\sim}{P}$ is a projector, then $(\underset{\sim}{I}-\underset{\sim}{P})$ is also a projector

$$(\underset{\sim}{I}-\underset{\sim}{P})^2 = (\underset{\sim}{I}-\underset{\sim}{P})(\underset{\sim}{I}-\underset{\sim}{P})$$
$$= \underset{\sim}{I}-\underset{\sim}{P}-\underset{\sim}{P}+\underset{\sim}{P}^2 = \underset{\sim}{I}-\underset{\sim}{P}$$

$\underset{\sim}{I}-\underset{\sim}{P}$ is called complementary projector to $\underset{\sim}{P}$!

Onto what space does $(I-P)$ project? range $(I-P)$

Consider any vector in range $(I-P)$

$$\rightarrow (\underset{\sim}{I}-P)\underset{\sim}{x} = \underset{\sim}{x} - \underset{\sim}{P}\underset{\sim}{x}$$
$$\underset{\sim}{P}(\underset{\sim}{x} - \underset{\sim}{P}\underset{\sim}{x}) = 0$$

$\Rightarrow \underline{x} - \underline{P}\underline{x} \in \text{null}(\underline{P})$

This means $\text{range}(\underline{I} - \underline{P}) \subseteq \text{null}(\underline{P})$ — (1)

Similarly let us consider any vector $\underline{x}$ in null (P) i.e $\underline{P}\underline{x} = \underline{0}$

then $(\underline{I} - \underline{P})\underline{x}$

$= \underline{x} - \underline{P}\underline{x}$

$= \underline{x}$

$\text{null}(\underline{P}) \subseteq \text{range}(\underline{I} - \underline{P})$ — (2)

From (1) and (2) $\text{range}(\underline{I} - \underline{P}) = \text{null}(\underline{P})$

we can also deduce $\text{range}(\underline{P}) = \text{null}(\underline{I} - \underline{P})$

IV $\text{null}(I - P) \cap \text{null}(P) = \{0\}$

i.e $\text{range}(P) \cap \text{null}(P) = \{0\}$

Pf:- Let $\underline{v}$ be in both $\text{null}(\underline{P})$ and $\text{null}(\underline{I} - \underline{P})$

Then $\underline{P}\underline{v} = (\underline{I} - \underline{P})\underline{v} = \underline{0}$

$(\underline{I} - \underline{P})\underline{v} = \underline{0}$

$\Rightarrow \underline{v} - \underline{P}\underline{v} = \underline{0}$

$\Rightarrow \underline{v} = \underline{0}$

$\Rightarrow \text{null}(\underline{I} - \underline{P}) \cap \text{null}(\underline{P}) = \{0\}$

$\Rightarrow \text{range}(P) \cap \text{null}(P) = \{0\}$

This say that projector P seperates $\mathbb{R}^m$ into two subspaces.

Orthogonal Projectors:-

An orthogonal projector is one that projects onto a subspace S_1 along subspace S_2 where S_1 and S_2 are orthogonal subspaces.

Thm:- A projector $\underset{\sim}{P}$ is orthogonal projector if and only if $\underset{\sim}{P} = \underset{\sim}{P}^T$

Pf:- step1 If $\underbrace{\underset{\sim}{P} = \underset{\sim}{P}^T}$, we need to show projector $\underset{\sim}{P}$ is orthogonal

Consider an inner product between a vector in S_1 i.e $\underset{\sim}{P}\underset{\sim}{x} \in S_1$ and vector $(\underset{\sim}{I} - \underset{\sim}{P})\underset{\sim}{y} \in S_2$

$$(\underset{\sim}{P}\underset{\sim}{x}, (\underset{\sim}{I}-\underset{\sim}{P})\underset{\sim}{y}) = (\underset{\sim}{P}\underset{\sim}{x})^T(\underset{\sim}{I}-\underset{\sim}{P})\underset{\sim}{y}$$
$$= \underset{\sim}{x}^T\underset{\sim}{P}^T(\underset{\sim}{I}-\underset{\sim}{P})\underset{\sim}{y}$$

$$= \underset{\sim}{x}^T \underset{\sim}{P}(\underset{\sim}{I} - \underset{\sim}{P})\underset{\sim}{y}$$

$$= \underset{\sim}{x}^T [\underset{\sim}{P} - \underset{\sim}{P}^2] \underset{\sim}{y} = \underset{\sim}{0}$$

Step 2 :-

To prove :→ An orthogonal projector $\underset{\sim}{P} \in \mathbb{R}^{m \times m}$ ($\underset{\sim}{P}$ projects onto S_1 along S_2 where $S_1 \perp S_2$)

satisfies $\underset{\sim}{P} = \underset{\sim}{P}^T$

Let S_1 have dimension $n < m$ and let $\{\underset{\sim}{q}_1, \underset{\sim}{q}_2, \dots \underset{\sim}{q}_m\}$ be the basis for $\mathbb{R}^m$ where $\{\underset{\sim}{q}_1, \underset{\sim}{q}_2, \dots \underset{\sim}{q}_n\}$ be the basis for S_1 and $\{\underset{\sim}{q}_{n+1}, \dots \underset{\sim}{q}_m\}$ be the basis for S_2.

Let us try to construct SVD for $\underset{\sim}{P}$.

for $j \le n$, $\underset{\sim}{P}\underset{\sim}{q}_j = \underset{\sim}{q}_j$ ✓

and $j > n$, $\underset{\sim}{P}\underset{\sim}{q}_j = \underset{\sim}{0}$

Now let us construct a matrix $\underset{\sim}{Q}$

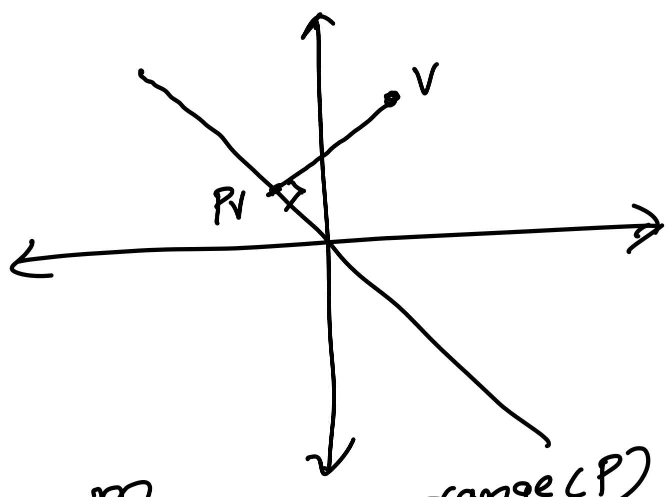
$$\underset{\sim}{Q} = \begin{bmatrix} | & | & | & & | \\ \underset{\sim}{q_1} & \underset{\sim}{q_2} & q_3 & \cdots & q_m \\ | & | & | & & | \end{bmatrix}$$

$$\underset{\sim}{P}\underset{\sim}{Q} = \begin{bmatrix} | & | & & | & | & | & & | \\ \underset{\sim}{q_1} & q_2 & \cdots & q_n & 0 & 0 & \cdots & 0 \\ | & | & & | & | & | & & | \end{bmatrix}$$

$$\underset{\sim}{Q}^T \underset{\sim}{P} \underset{\sim}{Q} = \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{bmatrix} = \underset{\sim}{\Sigma}$$

$$\underset{\sim}{P} = \underset{\sim}{Q} \underset{\sim}{\Sigma} \underset{\sim}{Q}^T \quad \text{we constructed}$$

SVD of $\underset{\sim}{P}$ $\Rightarrow$ $\underset{\sim}{P} = \underset{\sim}{P}^T$



Orthogonal Projectors corresponding to a subspace spanned by orthonormal basis

$V^n \subseteq \mathbb{R}^m$

Let us consider an n-dimensional subspace in $\mathbb{R}^m$ and $\{\underset{\sim}{q_1}, \underset{\sim}{q_2}, \ldots \underset{\sim}{q_n}\}$ be the set of n orthonormal vectors in $\mathbb{R}^m$ spanning our n-dimensional subspace.

Let $$\hat{Q} = \begin{bmatrix} | & | & | & & | \\ q_1 & q_2 & q_3 & \cdots & q_n \\ | & | & | & & | \end{bmatrix}_{m \times n}$$

Let $v \in \mathbb{R}^m$ can be decomposed into a component in the column space $\hat{Q}$ plus a component r perpendicular to column space of $\hat{Q}$

$$v = r + \underbrace{\sum_{i=1}^{n} (q_i^T v)\, q_i}$$

The map $v \mapsto \sum_{i=1}^{n} (q_i^T v)\, q_i$ is an orthogonal projection onto range $(\hat{Q})$

$$y = Pv = \sum_{i=1}^{n} \underbrace{(q_i^T v)\, q_i}$$

$$\underset{\sim}{y} = \underset{\sim}{P}\underset{\sim}{v} = \sum_{i=1}^{n} \underbrace{\underset{\sim}{q}_i (\underset{\sim}{q}_i^T \underset{\sim}{v})}$$

$$= \underbrace{\sum_{i=1}^{n} \underset{\sim}{q}_i \underset{\sim}{q}_i^T}_{P} \underset{\sim}{v}$$

$$\underset{\sim}{y} = \hat{\underset{\sim}{Q}} \hat{\underset{\sim}{Q}}^T \underset{\sim}{v}$$

where $\hat{\underset{\sim}{Q}} = \begin{bmatrix} | & | & & | \\ \underset{\sim}{q}_1 & \underset{\sim}{q}_2 & \cdots & \underset{\sim}{q}_n \\ | & | & & | \end{bmatrix}$

$$\boxed{\underset{\sim}{P} = \hat{\underset{\sim}{Q}} \hat{\underset{\sim}{Q}}^T}$$

$$\tilde{\underset{\sim}{Q}} = \begin{bmatrix} | & | & | & & | \\ \tilde{\underset{\sim}{q}}_1 & \tilde{\underset{\sim}{q}}_2 & \tilde{\underset{\sim}{q}}_3 & \cdots & \tilde{\underset{\sim}{q}}_n \\ | & | & | & & | \end{bmatrix}$$

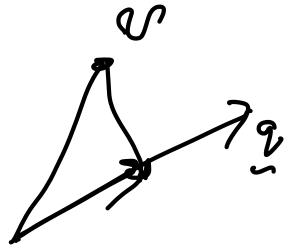
Show $P = \tilde{P}$? Exercise $\quad \tilde{P} = \tilde{Q}\tilde{Q}^T$

* Complement of orthogonal projector is also orthogonal projector.

i.e $\underset{\sim}{P} = \underset{\sim}{P}^T$ then $(\underset{\sim}{I} - \underset{\sim}{P})^T = (\underset{\sim}{I} - \underset{\sim}{P})$

The complements projects onto space orthogonal to range (P)

* Eg:- Rank 1 orthogonal projector that isolates component of a vector v in a single direction $P_q = q q^T$



$(q^T v) q$

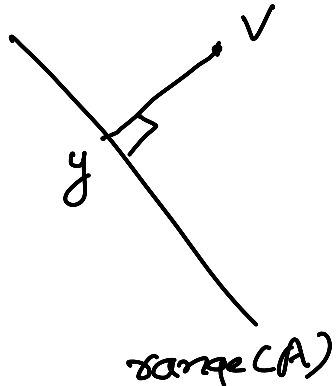
$$(q q^T) v = \bar{v}$$
$$= (q^T v) q$$

Projection onto n-dimensional subspace represented by arbitrary basis :-

Let the subspace be spanned by the linearly independent vectors $\{a_1, a_2, \dots a_n\}$. $A \in \mathbb{R}^{m \times n}$ have the columns $\{a_1, a_2, \dots a_n\}$

$$A = \begin{bmatrix} | & | & | & & | \\ a_1 & a_2 & a_3 & \cdots & a_n \\ | & | & | & & | \end{bmatrix}_{m \times n}$$

Let $y \in \text{range}(A)$ be the projected vector. We know that $(y - v) \perp \text{range}(A)$



$$a_j^T (y - v) = 0 \quad \text{— (1)}$$

for every $j = 1 \ldots n$

Since $y \in \text{range}(A)$, we can write

$$y = A x \text{ for some } x.$$

and hence (1) becomes

$$a_j^T (A x - v) = 0 \quad \forall j = 1 \ldots n$$

$$A^T [A x - v] = 0$$

$$\Rightarrow A^T A x = A^T v$$

$$x = (A^T A)^{-1} A^T v$$

(show $(A^T A)^{-1}$ exists?)

$$y = P v = A x$$

$$\Rightarrow P v = A (A^T A)^{-1} A^T v \quad \forall v \in \mathbb{R}^m$$

$$\Rightarrow \boxed{\underset{\sim}{P} = \underset{\sim}{A}(\underset{\sim}{A}^T\underset{\sim}{A})^{-1}\underset{\sim}{A}^T}$$

"Show that this $\underset{\sim}{P}$ is same as $\underset{\sim}{P}$ obtained by $\hat{Q}\hat{Q}^T$ before?"

Exercise!