DS284: Numerical Linear Algebra
Assignment 3

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Problem 1 [18 marks]

Recall the following from class:

(i) For all $x \in \mathbb{R}$ there exists $|\epsilon| \leq \epsilon_{machine}$ such that $fl(x) = x(1+\epsilon)$, where $fl(x)$ denotes floating point representation of x.

(ii) For all $x, y \in \mathbb{F}$ there exists $|\epsilon| \leq \epsilon_{machine}$ such that $x \circledast y = x * y(1+\epsilon)$ where $*$ denotes one of the operators $+, -, \times, \div$ and let $\circledast$ be its floating point analogue. Note $\mathbb{F}$ is a discrete subset of $\mathbb{R}$ which denote floating point representation of the real numbers.

Each of the following describes an algorithm implemented on a computer satisfying the properties (i) and (ii) described above. State with proper arguments whether the following algorithms are backward stable, stable but not backward stable, or unstable?

(a) Input data, $x \in \mathbb{R}$, computation of $2x$ as $x \oplus x$.

(b) Input data, $x \in \mathbb{R}$, computation of x^2 as $x \otimes x$.

(c) Input data, $x \in \mathbb{R} \setminus \{0\}$, computation of 1 as $x \mathbin{\bigcirc\!\!\!\!\div}\, x$.

(d) Input data, $\mathbf{A} \in \mathbb{R}^{m \times m}$, computation of full SVD of $\mathbf{A}$ i.e., $\mathbf{A} = \mathbf{U\Sigma V}^T$ with $\mathbf{U}$ and $\mathbf{V}$ being orthogonal matrices and $\mathbf{\Sigma}$ being diagonal.

- In the above, what would it mean for this algorithm to be backward stable. Verify that this algorithm can not be backward stable.
- Standard algorithms for computing SVD are stable. What does stability mean for such algorithms which compute SVD?

(e) Input data, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, computation of the inner product $\mathbf{x}^T\mathbf{y}$ as $(x_1 \otimes y_1) \oplus (x_2 \otimes y_2) \oplus (x_3 \otimes y_3) \oplus \ldots(x_m \otimes y_m)$.

(f) Input data $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, computation of eigen-values of $\mathbf{A}$ by evaluating the roots of characteristic polynomial.

[Hint: You need to examine the stability by looking at how the eigen-values of perturbed matrix $\mathbf{A}+\delta\mathbf{A}$ can be computed by finding the roots of the corresponding characteristic polynomial]

Problem 2 [8 marks]

This exercise will walk you through the steps in proving the existence of SVD of any rectangular matrix $\mathbf{A}$ of size $m \times n$ with rank r.

(a) Matrices of the form $\mathbf{G} = \mathbf{A}^T\mathbf{A}$ are called Gram matrices where $\mathbf{A} \in \mathbb{R}^{m\times n}$.

Show that $\mathbf{x}^T\mathbf{G}\mathbf{x} \geq 0 \; \forall \; \mathbf{x} \in \mathbb{R}^n$ and hence show that all eigen-values of $\mathbf{G}$ are non-negative.

(b) Show that $\mathbf{A}$ and $\mathbf{A}^T\mathbf{A}$ have the same rank.

(c) Show that a vector $\mathbf{u}$ of the form $\mathbf{A}\mathbf{v}/\sigma$ $(\sigma > 0)$ is a unit eigen-vector of $\mathbf{A}\mathbf{A}^T$ where $\mathbf{v}$ and σ^2 form the eigen-vector, eigen-value pair of $\mathbf{A}^T\mathbf{A}$.

(d) Note that i^{th} eigen-vector, eigen-value pair of $\mathbf{A}^T\mathbf{A}$ can be written as $\mathbf{A}^T\mathbf{A}\mathbf{v}_i = (\sigma_i^2)\mathbf{v}_i$.

Consider the case of a full rank matrix $\mathbf{A}$ ie. $(\sigma_i > 0 \;\; \forall i)$, if we define a new vector $\mathbf{u}_i = \frac{\mathbf{A}\mathbf{v}_i}{\sigma_i}$, show that $\mathbf{A}$ can be written as $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$, where $\mathbf{U} \in \mathbb{R}^{m\times m}$ is an orthogonal matrix, $\mathbf{\Sigma} \in \mathbb{R}^{m\times n}$ is diagonal matrix and $\mathbf{V} \in \mathbb{R}^{n\times n}$ is again an orthogonal matrix, $\mathbf{u}_i$ is i^{th} column of $\mathbf{U}$, and $\mathbf{v}_i$ is i^{th} column of $\mathbf{V}$

[Note: In low rank scenario some $\sigma_i = 0$ if other non-zero σ_i are sorted, we can compute $\mathbf{U}$ by adding additional column vectors that span $\mathbb{R}^m$ and add rows of 0-vector to $\mathbf{\Sigma}$.]

Problem 3 [15 marks]

A satellite revolving around one of the Juipiter's moon **Europa** has unexpectedly found alien life on its surface, and the camera harbored in the satellite has taken a photograph of it and stored it as .bmp file. However, due to some technical difficulties it has not been able to transmit full resolution image in bmp format to earth. However, a simple onboard computer can be programmed from Earth remotely and the control station on Earth decided to use SVD to compress the image. This image is as below and also downloadable from Teams class page as a bmp file. As a scientist working to program the onboard computer, think about the following:

(a) How many singular values are required to make the approximated image indistinguishable from the original image? (Hint: Load the image in Python/Matlab/ Octave and the matrix representation of the image will be accessible to you. For $r \times r$ pixel image, the image will have $r \times r \times 3$ matrix entries with the number 3 corresponding to color depth of the image representing Red, Blue, Green.)

(b) Based on your observation in (a), how many entries need to be transmitted to earth to reconstruct the approximate image as opposed to sending the original image?

[Perform the tasks in a programming environment comfortable to you like Matlab/ Octave/ Python. You can use inbuilt functions for computing SVD.]

(c) What is the 2-Norm and Frobenius-Norm error between the matrix representation of the original image and the approximate image obtained for different number of singular values. Check if the following theorems hold for these errors:

For the matrix $\mathbf{A}$ of rank r, with singular values $\sigma_1, \sigma_2, \sigma_3, ...\sigma_r$, $\mathbf{A}_v$ is the v rank approximation of $\mathbf{A}$. ($\mathbf{A}_v = \sum_{i=1}^{v} \sigma_i \mathbf{u}_i \mathbf{v}_i$) such that $1 < v < r$, then:

$$\|\mathbf{A} - \mathbf{A}_v\|_2 = \sigma_{v+1}$$

$$\|\mathbf{A} - \mathbf{A}_v\|_F = \sqrt{\sigma_{v+1}^2 + \sigma_{v+2}^2 + ... + \sigma_r^2}$$

Problem 4 [15 marks]

(a) Geometrically, the orthogonal matrix is a matrix transformation that preserves 2-Norm of a matrix and causes rotation / reflection.

Can you justify $\mathbf{I} - 2\mathbf{P}$ is orthogonal matrix if $\mathbf{P}$ is orthogonal projector?

Prove the same algebraically as well.

(b) Given a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and its projector $\mathbf{P}$ which projects all vectors orthogonally on to column space of $\mathbf{A}$, then answer the following questions:

- If $\mathbf{A}$ is full rank, what is $\mathbf{P}$?
- Given $\mathbf{P}$ is there any way to find out the null space of $\mathbf{A}$?
- What can you say about the eigen-values of $\mathbf{P}$?

(c) If $\mathbf{P} \in \mathbb{R}^{m \times m}$ be a non-zero projection. Show that $\|\mathbf{P}\|_2 \geq 1$ with equality, if and only if $\mathbf{P}$ is orthogonal projector.

Problem 5 [8 marks]

Given below is a magic matrix of size 3.

$$\mathbf{A} = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}$$

Find $\mathbf{Q}$ and $\mathbf{R}$ from QR factorization of given matrix by hand. Now do the following in Matlab/ Octave/ Python. For Matlab the command is:

[$\mathbf{Q}$, $\mathbf{R}$] = qr(magic(3))

Do these $\mathbf{Q}$ and $\mathbf{R}$ match your $\mathbf{Q}$ and $\mathbf{R}$? Is the QR factorization unique? If not unique, can you impose a condition on $\mathbf{R}$ to make the factorization unique?

Problem 6 [8 marks]

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 \mid & x \mid & x^2 \mid & ... \mid & x^{n-1} \end{bmatrix}$$

Each column is a function in $L^2[-1, 1]$ i.e., a vector space of real-valued function on $[-1, 1]$ which has inner-product of two functions f and g defined as:

$$(f, g) = \int_{-1}^{1} f(x)g(x)dx \tag{1}$$

If the QR factorization of $\mathbf{A}$ using the above definition of inner product can be written as

$$\mathbf{A} = \mathbf{QR} = \begin{bmatrix} q_0(x) \mid & q_1(x) \mid & q_2(x) \mid & ... \mid & q_{n-1}(x) \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & ... & r_{1n} \\ 0 & r_{22} & ... & r_{2n} \\ 0 & 0 & ... & r_{3n} \\ . & & & \\ . & & & \\ 0 & 0 & ... & r_{mn} \end{bmatrix}$$

where columns of $\mathbf{Q}$ are functions of x, and are orthonormal with respect to inner product defined in equation (1). Now answer the following:

(a) Consider $n = 4$, derive expressions of $q_0(x), q_1(x), q_2(x), q_3(x)$ by using Gram Schmidt orthogonalization procedure.

(b) Show that $\int_{-1}^{1} q_{n-1}(x)dx = 0$ for $n \geq 2$.

Note that these $q_{n-1}(x)$ are called Legendre polynomials and the roots of these polynomials are called Gauss Legendre points and play a very important role in numerical integration as quadrature rules.

Problem 7 [16 marks]

Let matrix:

$$\mathbf{A} = \begin{bmatrix} 0.70000 & 0.70711 \\ 0.70001 & 0.70711 \end{bmatrix}$$

(a) Consider a computer which rounds all computed results to five digits of relative accuracy. Using CGS or MGS, what will be the matrix $\mathbf{Q}$ associated with QR decomposition of $\mathbf{A}$ assuming that you are working on such a computer.

(b) Apply Householder's method to compute QR factorization of $\mathbf{A}$ using the same 5 digit arithmetic.

Compare the $\mathbf{Q'}s$ obtained in (a) and (b) and comment on orthogonal nature of the $\mathbf{Q}$ matrix.

Problem 8 [12 marks]

Assuming that floating point properties (i) and (ii) described in Problem 1 hold good, let us analytically find out the loss of orthogonality between two linearly independent unit normalized vectors $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^m$, during Gram-Schmidt Orthogonalization.

(a) If $\mathbf{q}_1 = \mathbf{a}_1$, then mathematically we know that the vector orthogonal to $\mathbf{q}_1$ can be computed as $\mathbf{w}_2 = \mathbf{a}_2 - (\mathbf{q}_1^T \mathbf{a}_2)\mathbf{q}_1$. Denoting $\overline{r}_{12} = fl(\mathbf{q}_1^T \mathbf{a}_2)$, show that

$$|\overline{r}_{12} - r_{12}| \leq m\, \epsilon_{machine} + O(\epsilon_{machine}^2)$$

(b) If $\overline{\mathbf{w}}_2 = fl(\mathbf{a}_2 - fl(\overline{r}_{12}\mathbf{q}_1))$, Show that error in $\overline{\mathbf{w}}_2$ i.e.

$$|\overline{\mathbf{w}}_2 - \mathbf{w}_2| \leq (m+2)\epsilon_{machine} + O(\epsilon_{machine}^2)$$

(c) Assuming that normalization $\overline{\mathbf{q}}_2 = \frac{\overline{w}_2}{\overline{r}_{22}}$ and $\overline{r}_{22} = \|\overline{w}_2\|_2$ is carried out without error, show that

$$|\mathbf{q}_1^T \overline{\mathbf{q}}_2| \leq \frac{(m+2)\epsilon_{machine}}{\overline{r}_{22}} \tag{2}$$

[Note: If $\overline{r}_{22}$ is small, we can see from equation (2) that we incur considerable loss of orthogonality.]