Indian Institute of Science, Bangalore
Department of Computational and Data Sciences (CDS)

DS284: Numerical Linear Algebra

Quiz 3 [Nov 4, 2023]

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Exam duration: 3:00 PM - 4:15 PM Max Points: 25

Notations: (i) Vectors and matrices are denoted by bold faced lower case and upper case alphabets respectively. (ii) Set of all real numbers is denoted by $\mathbb{R}$ (iii) Set of all n dimensional vectors is denoted by $\mathbb{R}^n$ (iv) Set of all $m \times n$ real matrices is denoted by $\mathbb{R}^{m\times n}$. (v) Set of all $m \times n$ complex matrices is denoted by $\mathbb{C}^{m\times n}$

1. Assert if the following statements are True or False. Give a short reasoning for your assertion. Marks will be awarded only for your reasoning. **$6 \times 2.5 = 15.0$ points**

   (a) Let $\mathbf{A} \in \mathbb{C}^{m\times m}$ and $\mathbf{Q} \in \mathbb{C}^{m\times m}$, with $\mathbf{Q}$ being a unitary matrix. Then $\mathbf{A}$ and $\mathbf{QAQ}^T$ have the same eigenvalues.

   (b) One can always design an algorithm to compute Schur decomposition of a given matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ in a finite number of steps.

   (c) If matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ has m distinct eigenvalues, then the eigenvectors of $\mathbf{A}$ corresponding to the eigenvalues λ_i and λ_j $(i \neq j)$ may be linearly dependent.

   (d) A Householder reflector $\mathbf{F} = \mathbf{I} - 2\mathbf{q}\mathbf{q}^T$ with $\|\mathbf{q}\|_2 = 1$, $\mathbf{F} \in \mathbb{R}^{m\times m}$, has the eigenvalue -1 with algebraic multiplicity $m-1$ whenever m is even.

   (e) Rayleigh quotient iteration produces a sequence of vectors $\mathbf{v}^{(k)}$ that converges to the eigenvector corresponding to the largest eigenvalue of symmetric matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ as $k \to \infty$ starting with any initial guess vector $\mathbf{v}^{(0)}$.

   (f) Let $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$ and $\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The matrix $\mathbf{A}$ has 7, -2 as the eigenvalues. If you devise a power iteration using $\mathbf{A}$ with $\mathbf{x}^{(0)}$ as the starting guess vector, you will converge to the eigenvector corresponding to the eigenvalue 7.

2. (a) Answer the following question: Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be large symmetric positive definite matrix which is tridiagonal and you have the problem of finding eigenvector, eigenvalue pairs $(\lambda_i, \mathbf{u}_i)$ satisfying the equation $\mathbf{A}\mathbf{u}_i = \lambda_i \mathbf{u}_i$. Devise an iterative algorithm to find the eigenvalue λ_i closest to 2.0 (assume 2.0 is not an eigenvalue of $\mathbf{A}$) and a corresponding eigenvector. Will your algorithm always converge, or is there any condition that needs to be satisfied for convergence? **[5 marks]**

   (b) Recall that $\mathbf{A}$ is a tridiagonal matrix. Assume that you already obtained max eigenvalue λ_{max} of $\mathbf{A}$ using a power iteration. Now you are also told that the eigenvalue closest to 2.0 is, in fact, the lowest eigenvalue. Can you think of devising a power iteration based approach different from what you used in (a) and further does not involve any explicit inversion of $\mathbf{A}$ matrix to compute this lowest eigenvalue? Do you think the new algorithm can be better in terms of computational cost than what you used in (a) to compute the eigenvalue closest to 2.0? **[5 marks]**
   (**Hint:** Shift the given matrix using a scalar multiple of identity matrix!)