भारतीय विज्ञान संस्थान

Indian Institute of Science, Banglore
Department of Computational and Data Sciences (CDS)

DS284: Numerical Linear Algebra

Quiz 1 [Aug 30, 2024]

Faculty Instructor: Dr. Phani Motamarri

TAs: Gourab Panigrahi, Srinibas Nandi, Rushikesh Pawar, Nihar Shah, Surya Neelakandan

Exam duration: 5:45 PM - 6:45 PM — Max Points: 30

Important Information: (i) Vectors $\mathbf{x}$ and matrices $\mathbf{A}$ are denoted by bold faced lower case and upper case alphabets respectively. (ii) Set of all real numbers is denoted by $\mathbb{R}$. (iii) Set of all n dimensional vectors is denoted by $\mathbb{R}^n$ and set of all $m \times n$ matrices are denoted by $\mathbb{R}^{m\times n}$. (iv) $\mathcal{N}(\mathbf{A})$ denotes null-space of a matrix $\mathbf{A}$. (v) The canonical (standard) basis of $\mathbb{R}^n$ is $\{\mathbf{e}_i\}_{i=1}^n$ where $\mathbf{e}_i \in \mathbb{R}^n$ is a column vector with 1 as the i^{th} element and 0 everywhere else.

1. Answer whether the following statements are True or False (Yes or No) with appropriate justifications. **Marks will be awarded only for justification.** **$5 \times 3 = 15$ points**

   (a) In a friendly data science competition, Rushikesh and Nihar enthusiastically defined two functions $r\colon \mathbb{R}^m \to \mathbb{R}$ and $n\colon \mathbb{R}^m \to \mathbb{R}$ to analyze their data, as follows:

   $$r(\mathbf{x}) := \min\{\|\mathbf{x}\|_1, \|\mathbf{x}\|_2, \|\mathbf{x}\|_\infty\}, \quad \text{for any given } \mathbf{x} \in \mathbb{R}^m,$$
   $$n(\mathbf{x}) := \max\{\|\mathbf{x}\|_1, \|\mathbf{x}\|_2, \|\mathbf{x}\|_\infty\}, \quad \text{for any given } \mathbf{x} \in \mathbb{R}^m.$$

   Not to be outdone, Gourab combines best of both the worlds and claims to have defined a new norm as:

   $$\|\mathbf{x}\|_G := \lambda_1 r(\mathbf{x}) + \lambda_2 n(\mathbf{x}), \quad \text{for any given } \mathbf{x} \in \mathbb{R}^m \text{ and } \lambda_1, \lambda_2 > 0.$$

   Along comes Prof. Phani, who sceptically states, "This $\|\cdot\|_G$ can't possibly be a valid norm!". Do you agree with him?

   (b) Spectral radius $\rho(\mathbf{A})$ of a square matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is defined as the maximum of the absolute values of its eigenvalues i.e., $\rho(\mathbf{A}) = \max_i |\lambda_i|$ where λ_i is an eigenvalue of $\mathbf{A}$ (note that $|\cdot|$ here denotes a complex modulus if the eigenvalue is complex). Do you agree that the matrix norm induced by vector p-norm for any square matrix $\|\mathbf{A}\|_p$ is always greater than or equal to $\rho(\mathbf{A})$.

   (c) Srinibas is designing a numerical algorithm to solve a large-scale eigenvalue problem using a subspace projection technique. To this end, he constructs a subspace spanned by n orthonormal vectors and if these vectors form the columns of the matrix $\mathbf{Q} \in \mathbb{R}^{m\times n}(m \neq n)$, can Srinibas confidently conclude $\mathbf{Q}\mathbf{Q}^{\mathbf{T}} = \mathbf{I}$ where $\mathbf{I}$ is the identity matrix?

   (d) Curious Surya conducts an experiment, gathering n features for each of the m samples, and arranges them into a data matrix $\mathbf{X} \in \mathbb{R}^{m\times n}$. If he observes $\mathbf{X}$ to be a full-rank matrix, then he can always conclude that $dim[\mathcal{N}(\mathbf{X})] = 0$.

   (e) Let $\mathbb{W}$ be a subspace of $\mathbb{V}$ with $\{\mathbf{w_j}\}_{j=1}^n$ forming an orthonormal basis of $\mathbb{W}$. If $\mathbf{P} = \sum_{j=1}^n \mathbf{w_j}\mathbf{w_j}^T$, then $\mathbf{P^2} = \mathbf{P}$ is always true.

2. Answer the following subjective questions: **2+2+5+6 = 15 points**

(a) The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} \sqrt{2}/2 & \sqrt{6}/6 & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & \sqrt{3}/3 \end{pmatrix}$$

and the vector $\mathbf{b}$ represented in the standard basis (or the canonical basis) has the following components

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Write down the components of this vector $\mathbf{b}$ in the basis formed by the column vectors of the matrix $\mathbf{A}$.

(b) Consider a full rank matrix $\mathbf{H} \in \mathbb{R}^{n\times n}$ and let $k(\leq n)$ non-zero vectors $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots, \boldsymbol{d}_k\}$ belonging to $\mathbb{R}^n$ be such that they are $\mathbf{H}$-orthogonal i.e., they satisfy the property $\boldsymbol{d}_j^T \mathbf{H} \boldsymbol{d}_i = 0$ for all $i \neq j$. Verify whether the vectors $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots, \boldsymbol{d}_k\}$ are linearly dependent or independent.

(c) Let $\mathbf{B} \in \mathbb{R}^{m\times m}$ be a symmetric matrix i.e. $\mathbf{B} = \mathbf{B}^{\mathbf{T}}$. If $\mathbf{x}^{\mathbf{T}}\mathbf{B}\mathbf{x} = 0$ for all $\mathbf{x} \in \mathbb{R}^m$, then show that the matrix $\mathbf{B}$ has to be a zero matrix, i.e., $\mathbf{B} = \mathbf{0}$.

Hint: Since the statement $\mathbf{x}^{\mathbf{T}}\mathbf{B}\mathbf{x} = 0$ for all $\mathbf{x} \in \mathbb{R}^m$ is given to you, one can use appropriate choices of $\mathbf{x}$ to show that all the matrix entries of $\mathbf{B}$ are zeroes. In particular, first show that the diagonal entries of $\mathbf{B}$ are zeros by choosing $\mathbf{x}$ appropriately and then subsequently show that off-diagonal entries of $\mathbf{B}$ are also zero by making another choice of $\mathbf{x}$. The standard (canonical) basis $\{\mathbf{e}_j\}$ or a linear combination of them will come handy to you in making these choices. Of course there are other ways of solving the above problem as well.

(d) Let the matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ be such that the action of this $\mathbf{A}$ on any vector $\mathbf{x} \in \mathbb{R}^m$ preserves the Euclidean length of the vector. Does the action of such a matrix $\mathbf{A}$ on two different vectors preserve the angle between the vectors as well? In other words, you need to verify mathematically that if $\|\mathbf{A}\mathbf{x}\|_2 = \|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{R}^m$, then $(\mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{y}) = (\mathbf{x}, \mathbf{y})$ for any two vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$. Note that $(\cdot, \cdot)$ here denotes the inner product (dot product) between two vectors.