Indian Institute of Science, Banglore
Department of Computational and Data Sciences (CDS)

DS284: Numerical Linear Algebra

Quiz 1 [Aug 31, 2023]

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Submission Deadline: Aug 31, 2023, 11:15 AM — Max Points: 30

Important Information: (i) Vectors $\mathbf{x}$ and matrices $\mathbf{A}$ are denoted by bold faced lower case and upper case alphabets respectively. (ii) Set of all real numbers is denoted by $\mathbb{R}$. (iii) Set of all rational numbers is denoted by $\mathbb{Q}$. (iv) Set of all n dimensional vectors is denoted by $\mathbb{R}^n$ and set of all $m \times n$ matrices are denoted by $\mathbb{R}^{m\times n}$ (v) Matrices $\mathbf{P}$ and $\mathbf{Q}$ are said to commute with each other if $\mathbf{PQ} = \mathbf{QP}$

1. Assert if the following statements are True or False. Give a short reasoning for your assertion. Marks will be awarded only for your reasoning. **$7 \times 3 = 21$ points**

   (a) Suppose we know that for a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{A}^T\mathbf{A} = \mathbf{I}$, where $\mathbf{I}$ is the $n \times n$ identity matrix. Then both the rows and columns of $\mathbf{A}$ are linearly independent

   (b) Suppose $\mathbf{A} \in \mathbb{R}^{n\times n}$ be such that $rank(\mathbf{A}) = rank(\mathbf{A}^2)$. Then $null(\mathbf{A}) \cap range(\mathbf{A}) = \{0\}$.

   (c) If $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a skew-symmetric matrix ($\mathbf{A} = -\mathbf{A}^T$) and n is odd, then $\mathbf{A}$ may be invertible.

   (d) Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\|\mathbf{A}\|$ denotes induced matrix norm discussed in class. If λ is an eigen value of $\mathbf{A}$, then $|\lambda| \leq \|\mathbf{A}\|$. (**Hint:** Recall the definition of eigenvalue problem to be $\mathbf{Ax} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq 0$.)

   (e) We have a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, and further we also know that for a matrix $\mathbf{C} \in \mathbb{R}^{n\times n}$, $\mathbf{AC} = \mathbf{I}$. Then the matrices $\mathbf{A}$ and $\mathbf{C}$ should always commute with each other.

   (f) A real-valued function $\|\mathbf{v}\| : \mathbb{R}^n \to \mathbb{R}$ is defined on a vector space as the number of non-zero entries in a given vector. The above $\|\mathbf{v}\|$ is a valid definition of norm.

   (g) If $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a skew-symmetric matrix ($\mathbf{A} = -\mathbf{A}^T$) and n is odd, then $\mathbf{A}$ may be invertible.

2. Starting with an initial guess $\mathbf{x}^{(0)}$, a fixed point iteration of the form $\mathbf{x}^{(k+1)} = \mathbf{T}\mathbf{x}^{(k)} + \mathbf{v}$ can be devised to solve a linear system of equations $\mathbf{Ax} = \mathbf{b}$ ($\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{x}, \mathbf{b} \in \mathbb{R}^n$) involving the matrix $\mathbf{T}(\in \mathbb{R}^{n\times n})$ and a vector $\mathbf{v}(\in \mathbb{R}^n)$. The iterations are repeated till $\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\| \leq \epsilon$, where ϵ is a specified tolerance value.

   Answer the following: **3+4+2 = 9 points**

   (a) Under what conditions (of $\mathbf{T}, \mathbf{v}$) does the fixed point problem $\mathbf{x} = \mathbf{Tx} + \mathbf{v}$ have a solution. Note that a fixed-point solution, $\mathbf{x}^*$ should satisfy $\mathbf{x}^* = \mathbf{Tx}^* + \mathbf{v}$.

   (b) Assume that the above condition is satisfied, i.e. a fixed point solution exists. Under what condition on $\mathbf{T}$ does the fixed point iteration $\mathbf{x}^{(k+1)} = \mathbf{T}\mathbf{x}^{(k)} + \mathbf{v}$ converge to the fixed point solution for any initial value $\mathbf{x}^{(0)}$? [**Hint:** Recall that $\mathbf{x}^*$ is a fixed point solution and for convergence, one has to drive the error vector as close to zero as possible at every k^{th} iteration]

(c) Under what conditions (of $\mathbf{T}, \mathbf{v}$) does the fixed point problem converge to a unique solution for any initial value?