

## Eigenvalue Problems

Let  $\underline{A} \in \mathbb{R}^{m \times m}$ ,  $\underline{x} \neq \underline{0} \in \mathbb{C}^m$  ( $\mathbb{C}$  is set of complex no.)  
then  $\underline{x}$  is an eigenvector of  $\underline{A}$  and  
 $\lambda \in \mathbb{C}$  is its corresponding eigenvalue

if  $\boxed{\underline{A}\underline{x} = \lambda\underline{x}}$

\* The set of all eigenvalues of a matrix  $\underline{A}$  is called the spectrum of  $\underline{A}$  denoted by  $\Lambda(\underline{A})$

Application areas:-

- \* Insights into evolution of system
  - vibration analysis
  - Study of resonance
  - Stability of structure
  - fluid flows subjected to small perturbations

- \* Quantum mechanical modeling  
(Solving Schrodinger equation)
- \* Principal stresses in solid mechanics
- \* PCA in data driven modeling
- \* Page rank algorithm used in search engines is an eigenvalue problem
- \* Eigenvectors of graph Laplacian matrix actually help in construction of efficient filters Graph Convolution Neural Network!

Eigenvalue decomposition!

An eigenvalue decomposition of

$A \in \mathbb{R}^{m \times m}$  is a factorization

$$A = \underline{X} \underline{\Lambda} \underline{X}^{-1} \text{ where}$$

$\underline{X}$  is non singular and  $\underline{\Lambda}$  is diagonal with  $\underline{X}$  comprising of eigenvectors of  $A$  as columns!

Note: Such decomposition may not always exist!

$$\underline{A} \underline{X} = \underline{X} \underline{\Lambda}$$

$$[A] \begin{bmatrix} | & | & | & \dots & | \\ \underline{x}_1 & \underline{x}_2 & \underline{x}_3 & \dots & \underline{x}_m \\ | & | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & | & \dots & | \\ \underline{x}_1 & \underline{x}_2 & \underline{x}_3 & \dots & \\ | & | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix}$$

$$\underline{A} \underline{x}_j = \lambda_j \underline{x}_j$$

i.e  $j^{\text{th}}$  column of  $\underline{X}$  is  $j^{\text{th}}$  eigen vector

and  $(j,j)$  entry of  $A$  is corresponding eigenvalue!

→ Geometric multiplicity :- The geometric multiplicity of an eigenvalue  $\lambda$  is the number of linearly independent eigenvectors associated with that eigenvalue  $\lambda$ . If  $\lambda \in \Lambda(A)$ , eigenspace  $E_\lambda$

$E_\lambda$  is an invariant subspace of  $A$

$$\text{i.e. } A E_\lambda \subseteq E_\lambda$$

The dimension of  $E_\lambda$  is the geometric multiplicity of  $\lambda$  i.e. maximum number of linearly independent eigenvectors that can be found for a given  $\lambda$

$$\begin{aligned} \underline{x} &= \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 \quad \textcircled{\lambda} \\ A \underline{x} &= A(\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2) \\ &= \alpha_1 \lambda \underline{v}_1 + \alpha_2 \lambda \underline{v}_2 \\ &= \lambda [\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2] \\ &= \lambda \underline{x} \end{aligned}$$

## \* Characteristic Polynomial:-

The characteristic polynomial  $p_A$  of  $A \in \mathbb{R}^{n \times n}$  is the  $n^{\text{th}}$  degree monic polynomial  $\boxed{p_A(z) = \det(zI - A)}$  ✓  
(coefficient of  $z^n$  is 1  $\rightarrow$  monic polynomial)

Thm:-  $\lambda$  is eigenvalue of  $A$  if and only if  $p_A(\lambda) = 0$

Note:-  $A \in \mathbb{R}^{n \times n}$   
 $\lambda$  can be complex  
any complex  $\lambda$  must appear in complex conjugate pairs  
Eigenvector of  $A$  lies in the null space of  $(A - \lambda I)$   
 $Ax = \lambda x$   
 $(A - \lambda I)x = 0$   
i.e.  $\lambda = a + ib$  is an eigenvalue  
 $\lambda^* = a - ib$  is an eigenvalue

## \* Algebraic multiplicity:-

Since  $p_A(z)$  is monic  $m$ -degree polynomial, it can be written as

$$p_A(z) = (z - \lambda_1)(z - \lambda_2) \dots (z - \lambda_m)$$

for some  $\lambda_j \in \mathbb{C}$  (roots of  $p_A(z)$ )

Each  $\lambda_j$  is an eigenvalue and in general may be repeated

"The multiplicity of  $\lambda$  as a root of  $p_A(z)$  is the algebraic multiplicity of an eigenvalue  $\lambda$ "

Remark:- (a) If  $A \in \mathbb{R}^{m \times m}$  then  $A$  has  $m$  eigenvalues counting algebraic multiplicity. In particular if roots of  $p_A(z)$  are simple, then  $A$  has  $m$  distinct eigenvalues.

⑥ The algebraic multiplicity of an eigenvalue  $\lambda$  is always at least as large as its geometric multiplicity.

\* Similarity transformation:-

If  $\underline{X} \in \mathbb{R}^{n \times n}$  is non singular, then

$$\underline{A} \rightarrow \underline{X}^{-1} \underline{A} \underline{X} \text{ is called a}$$

similarity transformation of  $\underline{A} \in \mathbb{R}^{n \times n}$

We say that two matrices  $\underline{A}$  and  $\underline{B}$  are similar if there is a similarity transformation of one to another

i.e. if there is a nonsingular  $\underline{X} \in \mathbb{R}^{n \times n}$  such that  $\underline{B} = \underline{X}^{-1} \underline{A} \underline{X}$   
then  $\underline{A}$  and  $\underline{B}$  are said to be similar!

Thm:- If  $\underline{X}$  is nonsingular, then  $\underline{A}$  and  $\underline{X}^{-1}\underline{A}\underline{X}$  have the same characteristic polynomial, eigenvalues and algebraic multiplicity and geometric multiplicity!

Pf:-

$$\begin{aligned}
 p(\underline{z}) &= \det(\underline{z}\underline{I} - \underline{X}^{-1}\underline{A}\underline{X}) \\
 \underline{X}^{-1}\underline{A}\underline{X} &= \det(\underline{z}\underline{X}\underline{X}^{-1} - \underline{X}^{-1}\underline{A}\underline{X}) \\
 &= \det(\underline{X}^{-1}(\underline{z}\underline{I} - \underline{A})\underline{X}) \\
 &= (\det \underline{X}^{-1})(\det(\underline{z}\underline{I} - \underline{A}))(\det \underline{X}) \\
 &= (\det \underline{X})^{-1}(\det(\underline{z}\underline{I} - \underline{A}))(\det \underline{X}) \\
 &= \det(\underline{z}\underline{I} - \underline{A}) = p_{\underline{A}}(\underline{z})
 \end{aligned}$$

Hence  $\underline{A}$ ,  $\underline{X}^{-1}\underline{A}\underline{X}$  have the same eigenvalues (same roots of  $p_{\underline{A}}(\underline{z})$ ) and

hence same algebraic multiplicity

Build a matrix  $E_\lambda$  whose column vectors span the eigenspace for the matrix  $A$ , corresponding to the eigenvalue  $\lambda$

$$\begin{aligned}(\underbrace{X^{-1}}_{\sim} \underbrace{A}_{\sim} \underbrace{X}_{\sim}) (\underbrace{X^{-1}}_{\sim} \underbrace{E_\lambda}_{\sim}) &= \underbrace{X^{-1}}_{\sim} \underbrace{A}_{\sim} \underbrace{E_\lambda}_{\sim} \\ &= \underbrace{X^{-1}}_{\sim} \underbrace{E_\lambda}_{\sim} \lambda\end{aligned}$$

$\underbrace{X^{-1}}_{\sim} \underbrace{E_\lambda}_{\sim}$  is the eigenspace for  $\underbrace{X^{-1}}_{\sim} \underbrace{A}_{\sim} \underbrace{X}_{\sim}$  corresponding to the eigenvalue  $\lambda$ .

Hence geometric multiplicity of

$A$  and  $\underbrace{X^{-1}}_{\sim} \underbrace{A}_{\sim} \underbrace{X}_{\sim}$  are the same.

( $\because E_\lambda$  and  $\underbrace{X^{-1}}_{\sim} \underbrace{E_\lambda}_{\sim}$  has the same rank)

Defective eigenvalues  
and matrices

\* A generic matrix need not have distinct eigenvalues  
 i.e algebraic multiplicity need not be 1 and geometric multiplicity need not be 1 as well and need not be equal to algebraic multiplicity as well)

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}; B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Both  $A$  and  $B$  eigenvalue

$$\lambda = 2.$$

Algebraic multiplicity of  $\lambda = 2$  for  $A$ ?

→ 3

Algebraic multiplicity of  $\lambda = 2$  for  $B$ ?

→ 3

For  $A$ , we can choose 3  
linearly independent eigenvectors  
 $\underline{e}_1, \underline{e}_2, \underline{e}_3$  and geometric  
multiplicity is also 3.

For  $B$ , we can only have  
only linearly independent eigenvectors  
 $\underline{e}_1$ , the geometric multiplicity  
of  $\underline{B}$  is 1

\* An eigenvalue whose algebraic  
multiplicity is greater than its  
geometric multiplicity is called a  
defective eigenvalue!