Notations :-

* Vectors are denoted by lower case letters with a tilde below.
* Matrices are denoted by upper case with a tilde below
* Scalars are denoted using lower case alphabets

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→ $\underset{\sim}{x}$ is a n-dimensional vector :-

$$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

eg:- ① Sequence of number can represent 'n' measurements of a feature like % Urbanization of various countries, heights of students in class

② These vectors can also represent displacements, velocity fields, electron density, pressure fields on different grid pts in a domain

→ $\underset{\sim}{A}$ (m rows, n columns)

$$\underset{\sim}{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & \cdots & \cdot & a_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

Eg:- ① In data science, these can be collection of a different features

② Discretization of PDEs in a chosen basis can give rise to matrices

Eg:- Finite Element method, Finite difference method, Finite volume etc.

<u>Matrix - vector multiplication</u>:-

* $\underset{\sim}{b} = \underset{\sim}{A}\,\underset{\sim}{x}$; $\underset{\sim}{A} \in \mathbb{R}^{m \times n}$, $\underset{\sim}{x} \in \mathbb{R}^{n \times 1}$

$\underset{\sim}{b} \in \mathbb{R}^{m \times 1}$

$$b_i = \sum_{j=1}^{n} A_{ij}\, x_j$$

Eg:- $\underset{\sim}{A} = \begin{pmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{pmatrix}$ $\underset{\sim}{x} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$

$$\underset{\sim}{A}\,\underset{\sim}{x} = \begin{pmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 2\times 8 + 3\times 9 \\ 2\times 9 + 4\times 9 \\ 3\times 8 + 7\times 9 \end{pmatrix}$$

$$= \begin{pmatrix} \ \end{pmatrix}$$

<u>By rows</u>:-

Inner product of rows of $\underset{\sim}{A}$ with vector $\underset{\sim}{x}$

Defn:- $\quad \underset{\sim}{x} \to \underset{\sim}{A}\underset{\sim}{x}$ is a linear map

i.e for any vectors $\underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^n$

and any scalar $\alpha \in \mathbb{R}$

$$\underset{\sim}{A}(\underset{\sim}{x} + \underset{\sim}{y}) = \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{A}\underset{\sim}{y}$$

$$\underset{\sim}{A}(\alpha \underset{\sim}{x}) = \alpha \underset{\sim}{A}\underset{\sim}{x}$$

By columns:-

$$\underset{\sim}{A}\underset{\sim}{x} = \overset{\underset{\sim}{A}}{\begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix}} \overset{\underset{\sim}{x}}{\begin{pmatrix} 8 \\ 9 \end{pmatrix}} = 8\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} + 9\begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}$$

You are viewing $\underset{\sim}{A}\underset{\sim}{x}$ as a linear combination of $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_2$, the two columns of $\underset{\sim}{A}$

$$\underset{\sim}{A}\underset{\sim}{x} = \begin{bmatrix} | & | \\ \underset{\sim}{a}_1 & \underset{\sim}{a}_2 \\ | & | \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \underset{\sim}{a}_1 + x_2 \underset{\sim}{a}_2$$

Fundamental operation of Linear Algebra

$$\underset{\sim}{b} = \underset{\sim}{A}\underset{\sim}{x} = \sum_{j=1}^{n} x_j \underset{\sim}{a}_j$$

This will allow us to define range(A)

Range of a matrix A

Defn:- $\text{range}(A) = \{ \underset{\sim}{y} \mid \underset{\sim}{y} = \underset{\sim}{A}\underset{\sim}{x} \text{ for some } \underset{\sim}{x} \}$

Key idea is to take all combinations of columns of $\underset{\sim}{A}$ and the set of infinitely many output vectors $\underset{\sim}{A}\underset{\sim}{x}$ forms the range of (A).

The space spanned by the columns of A is range (A)
↳ column space (A)

Null-space :-

$\text{Null}(A) = \{ \underset{\sim}{x} \mid \underset{\sim}{A}\underset{\sim}{x} = 0 \}$

i.e set of all vectors $\underset{\sim}{x}$ that maps to a zero vector via $\underset{\sim}{A}$

→ If $\underset{\sim}{x} \in \text{Null}(A)$

$$\underset{\sim}{A}\underset{\sim}{x} = 0 \Rightarrow x_1 \underset{\sim}{a_1} + x_2 \underset{\sim}{a_2} + \dots x_n \underset{\sim}{a_n} = 0$$

If $\underset{\sim}{a_1}, \underset{\sim}{a_2}, .. \underset{\sim}{a_n}$ are linearly independent then $x_i = 0$

Rank of a matrix

Column Rank of $\underset{\sim}{A}$:→ Maximum number of linearly indepent columns of $\underset{\sim}{A}$

[Dimension of space spanned by its columns]

Row rank of $\underset{\sim}{A}$:→ Maximum number of linearly independent rows of $\underset{\sim}{A}$

[Dim of space spanned by rows]

Result :- Row rank of a matrix equals Column rank of a matrix and is called rank of a matrix.

$\underset{\sim}{A} \in \mathbb{R}^{m \times n}$ is said to have a full rank if it has the maximal rank

i.e rank $(A) = \min(m, n)$

Eg :- $\underset{\sim}{A} = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}_{m \times n}$ $\quad m \geq n$

If such a matrix $\underset{\sim}{A}$ has a full rank it must have 'n' linearly independent columns, rank $(A) = n$

<u>Thm 1:</u> A matrix $\underset{\sim}{A} \in \mathbb{R}^{m \times n}$ with $m \geq n$ maps no two distinct vectors to the same vector if and only if $\underset{\sim}{A}$ is a full rank.

$\rightarrow$ $\underset{\sim}{A}\,\underset{\sim}{x}_1 = \underset{\sim}{y}_1$; $\underset{\sim}{A}\,\underset{\sim}{x}_2 = \underset{\sim}{y}_2$ $(\underset{\sim}{x}_1 \neq \underset{\sim}{x}_2)$

then $\underset{\sim}{y}_1 \neq \underset{\sim}{y}_2$ if and only if $\underset{\sim}{A}$ is full rank

Pf:- Given $\underset{\sim}{A} \in \mathbb{R}^{m \times n}$ maps no two distinct vector to the same vector.

$S_1 \Rightarrow S_2$

$\boxed{\sim S_2 \Rightarrow \sim S_1}$

$\underset{\sim}{x}_1 \in \mathbb{R}^n$, $\underset{\sim}{x}_2 \in \mathbb{R}^n$ and $\underset{\sim}{x}_1 \neq \underset{\sim}{x}_2$

$\underline{\underset{\sim}{A}\,\underset{\sim}{x}_1} = \underset{\sim}{y}_1$; $\underset{\sim}{A}\,\underset{\sim}{x}_2 = \underset{\sim}{y}_2 \Rightarrow \underset{\sim}{y}_1 \neq \underset{\sim}{y}_2$

Now we need to show $\underset{\sim}{A}$ is full rank.

$\rightarrow$ Let us assume $\underset{\sim}{A}$ is not full rank it has a non-trivial null space, let that vector be $\underset{\sim}{c} \neq 0$ i.e $\underset{\sim}{A}\,\underset{\sim}{c} = \underset{\sim}{0}$

$\underset{\sim}{A}\,\underset{\sim}{x}_1 = \underset{\sim}{y}_1$; $\underset{\sim}{x}_2 = \underset{\sim}{x}_1 + \underset{\sim}{c}$

$$\underset{\sim}{y}_2 = \underset{\sim}{A}\underset{\sim}{x}_2 = \underset{\sim}{A}(\underset{\sim}{x}_1 + \underset{\sim}{c}) = \underset{\sim}{A}\underset{\sim}{x}_1 = \underset{\sim}{y}_1$$

i.e a contradiction, $\underset{\sim}{A}$ is full rank.

Now we need to show if $\underset{\sim}{A}$ is full rank

$\underset{\sim}{A}\underset{\sim}{x}_1 = \underset{\sim}{y}_1$; $\underset{\sim}{A}\underset{\sim}{x}_2 = \underset{\sim}{y}_2$ where $\underset{\sim}{x}_1 \neq \underset{\sim}{x}_2 \Rightarrow \underset{\sim}{y}_1 \neq \underset{\sim}{y}_2$

→ Consider $\underset{\sim}{x}_1 \neq \underset{\sim}{x}_2$ i.e $\underset{\sim}{A}\underset{\sim}{x}_1 = \underset{\sim}{y}$ and $\underset{\sim}{A}\underset{\sim}{x}_2 = \underset{\sim}{y}$

i.e $\underset{\sim}{A}(\underset{\sim}{x}_1 - \underset{\sim}{x}_2) = \underset{\sim}{0}$

$\underset{\sim}{x}_1 - \underset{\sim}{x}_2 = 0$ because $\underset{\sim}{A}$ is full rank.

i.e $\underset{\sim}{x}_1 = \underset{\sim}{x}_2$

(again a contradiction)

<u>Matrix - matrix multiplication</u> :-

$$\underset{\sim}{A} \in \mathbb{R}^{l \times m} \qquad \underset{\sim}{C} \in \mathbb{R}^{m \times n}$$

$$\underset{\sim}{B} = \underset{\sim}{A}\underset{\sim}{C} \quad \text{i.e } B \in \mathbb{R}^{l \times n}$$

$$B_{ij} = \sum_{k=1}^{m} A_{ik} C_{kj}$$

<u>Inner product form</u> :-

$$B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a+3c & 2b+3d \\ 3a+4c & 3b+4d \\ 4a+7c & 4b+7d \end{bmatrix}$$

$$\qquad A \qquad\qquad C$$

Linear combination of columns

$$\underbrace{\begin{bmatrix} | & | & & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \\ | & | & & | \end{bmatrix}}_{l\times n} = \underbrace{\begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m \\ | & | & & | \end{bmatrix}}_{l\times m} \underbrace{\begin{bmatrix} | & | & & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \\ | & | & & | \end{bmatrix}}_{m\times n}$$

$$B = \mathbf{A} \quad \mathbf{C}$$

Columns of $\mathbf{B}$ can be written as linear combination of columns of $\mathbf{A}$ with column vectors of $\mathbf{C}$ as linear combination coefficients

$$\boxed{\mathbf{b}_j = \sum_{k=1}^{m} c_{kj}\, \mathbf{a}_k}$$

Outer-product form:-

$\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ outer product of two vectors $\mathbf{u}$ and $\mathbf{v}$ as a rank 1 matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ and $\mathbf{A} := \mathbf{u}\mathbf{v}^T$

$$\underset{l\times m}{\mathbf{B}} = \mathbf{A}\underset{m\times n}{\mathbf{C}} = \underbrace{\begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m \\ | & | & & | \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} - & \mathbf{r}_1^T & - \\ - & \mathbf{r}_2^T & - \\ & \vdots & \\ - & \mathbf{r}_m^T & - \end{bmatrix}}_{\mathbf{C}}$$

i^{th} row of $\mathbf{C}$ matrix is denoted as a column vector $\mathbf{r}_i \in \mathbb{R}^n$

$$\mathbf{B} = \mathbf{A}\mathbf{C} = \mathbf{a}_1\mathbf{r}_1^T + \mathbf{a}_2\mathbf{r}_2^T + \mathbf{a}_3\mathbf{r}_3^T + \cdots \mathbf{a}_m\mathbf{r}_m^T$$

Sum of 'm' rank 1 matrices

Four fundamental Subspaces of $\underset{\sim}{A}$

Column space :- $C(\underset{\sim}{A}) \rightarrow$ combination of columns of $\underset{\sim}{A}$

Row space :- $C(\underset{\sim}{A}^T) \rightarrow$ combination of columns of $\underset{\sim}{A}^T$

Null space :- $N(A)$ $\rightarrow$ all solutions $\underset{\sim}{x}$ to $\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{0}$

Left Null space :- $N(A^T)$ $\rightarrow$ all solutions $\underset{\sim}{y}$ to $\underset{\sim}{A}^T\underset{\sim}{y} = \underset{\sim}{0}$

Inverse of a matrix :-

A full rank square matrix is invertible

i.e., for $\underset{\sim}{A} \in \mathbb{R}^{m\times m}$, 'm' columns of $\underset{\sim}{A}$ form a basis for $\mathbb{R}^m$.

$$\longleftrightarrow \quad \underset{\sim}{A} = \begin{bmatrix} | & | & & | \\ \underset{\sim}{a}_1 & \underset{\sim}{a}_2 & \cdots & \underset{\sim}{a}_m \\ | & | & & | \end{bmatrix}_{m\times m}$$

$\underset{\sim}{A}$ is a full rank matrix

Hence,

Any vector in $\mathbb{R}^m$ can be expressed as a linear combination of columns of $\underset{\sim}{A}$

$$\underset{\sim}{e}_j = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1} \leftarrow j^{th}$$

$$\underset{\sim}{e}_j = \sum_{i=1}^{m} z_{ij}\,\underset{\sim}{a}_i$$

$$= \underset{\sim}{A}\,\underset{\sim}{z}_j \quad - (*)$$

$j = 1, 2, \dots m$

$$\begin{bmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \cdots & \underset{\sim}{e}_m \end{bmatrix} = \left[\begin{array}{c|c|c|c|c} 1 & 0 & 0 & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ & & & & 1 \end{array}\right] = I$$

$$\underset{\sim}{I} = \underset{\sim}{A}\,\underset{\sim}{Z} = \underset{\sim}{A}\begin{bmatrix} | & | & & | \\ \underset{\sim}{z}_1 & z_2 & \cdots & \underset{\sim}{z}_m \\ | & | & & | \end{bmatrix}$$

This matrix is called the inverse of $\underset{\sim}{A}$.

$\underset{\sim}{I} = \underset{\sim}{A}\,\underset{\sim}{Z} = \underset{\sim}{Z}\,\underset{\sim}{A}$ (Exercise)

And this $\underset{\sim}{Z}$ is called inverse of matrix $\underset{\sim}{A}$ and denote as $\underset{\sim}{A}^{-1}$

" Every non-singular square matrix $\underset{\sim}{A}$ has a unique inverse $\underset{\sim}{A}^{-1}$ that satisfies "

$$\boxed{\underset{\sim}{A}\,\underset{\sim}{A}^{-1} = \underset{\sim}{A}^{-1}\underset{\sim}{A} = I}$$

<u>Matrix</u> <u>inverse</u> <u>times</u> <u>vector</u>:-

$\underset{\sim}{A} \in \mathbb{R}^{m \times m}$ (Full rank) ; $\underset{\sim}{b} \in \mathbb{R}^m$; $\underset{\sim}{x} \in \mathbb{R}^m$

$$\boxed{\underset{\sim}{x} = \underset{\sim}{A}^{-1}\underset{\sim}{b}}$$

We can think $\underset{\sim}{x}$ as a unique vector satisfying $\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{b}$

$$\underset{\sim}{b} = \sum_{i=1}^{m} b_i \underset{\sim}{e_i} \quad \text{where } \underset{\sim}{e_i} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \rightarrow i$$

$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ are the coefficients of expansion of $\underset{\sim}{b}$ in $\{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \dots e_m\}$

$$\underset{\sim}{b} = \underset{\sim}{A}\underset{\sim}{x} = \sum_{j=1}^{m} x_j \underset{\sim}{a_j} \quad \text{where } \underset{\sim}{A} = \begin{bmatrix} | & | & & | \\ \underset{\sim}{a_1} & \underset{\sim}{a_2} & \cdots & a_m \\ | & | & & | \end{bmatrix}$$

$$\Downarrow$$

$\underset{\sim}{A}^{-1}\underset{\sim}{b} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$ are the coefficients of expansion of $\underset{\sim}{b}$ in $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \dots a_m\}$

This means we can think of $\underset{\sim}{A}^{-1}\underset{\sim}{b}$ as a change of basis operation from $\{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \dots e_m\}$ to $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \dots a_m\}$ i.e column-space of $\underset{\sim}{A}$

Transpose of a matrix ($\underset{\sim}{A}^T$) :-

If $\underset{\sim}{A} \in \mathbb{R}^{m \times n}$ then $\underset{\sim}{A}^T \in \mathbb{R}^{n \times m}$ where $A^T_{ij} = A_{ji}$

Useful properties of A^T and A^{-1} :-

ⓐ $(AB)^T = B^T A^T$ ⓑ $(AB)^{-1} = B^{-1} A^{-1}$

ⓒ $(A^T)^{-1} = (A^{-1})^T$

If $A = A^T$ (Symmetric matrix)

If $A = A^+$ (Hermitian matrix)

A^+ is conjugate transpose of $A \in \mathbb{C}^{m \times n}$

Inner products :→

Let $x, y \in \mathbb{R}^m$, then inner product of x and y is a scalar $(x, y) = x^T y$

$$= \sum_{i=1}^{m} x_i y_i$$

Euclidean length of a vector x :-

$$\|x\| = \sqrt{x^T x} = \left(\sum_{i=1}^{m} x_i^2 \right)^{1/2}$$

Angle between vectors x and y is α then $\cos \alpha = \dfrac{x^T y}{\|x\| \|y\|}$

Useful properties of inner product :-

ⓐ $(x_1 + x_2, y) = (x_1, y) + (x_2, y)$

(b) $(\underset{\sim}{x}, \underset{\sim}{y}_1 + \underset{\sim}{y}_2) = (\underset{\sim}{x}, \underset{\sim}{y}_1) + (\underset{\sim}{x}, \underset{\sim}{y}_2)$

(c) $(\alpha \underset{\sim}{x}, \beta \underset{\sim}{y}) = \alpha\beta\, (\underset{\sim}{x}, \underset{\sim}{y})$

Orthogonality :-

(a) Orthogonal vectors $\underset{\sim}{x}$ and $\underset{\sim}{y}$:→ Angle b/w vectors is 90° i.e $\cos 90^\circ = 0 \Rightarrow (\underset{\sim}{x}, \underset{\sim}{y}) = 0$

or $\underset{\sim}{x}^T \underset{\sim}{y} = 0$

(b) Orthogonal basis for a subspace :-

$\underset{\sim}{S} = \{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \ldots v_n\}$ is said to be an orthogonal basis spanning a n-dimensional space if every pair of vectors in $\underset{\sim}{S}$ is orthogonal i.e $\underset{\sim}{v}_i^T v_j = 0$

orthonormal basis → In addition to orthogonality, $\underset{\sim}{v}_i^T \underset{\sim}{v}_i = 1$ (Length of each vector is 1)

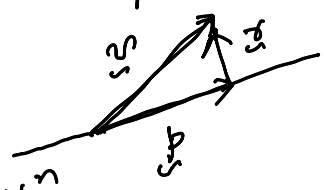
Result: Vectors in a orthogonal set $\underset{\sim}{S}$ are linearly independent

Pf :- exercise

Components of vector :- Consider $\mathbb{R}^m$ and n-dimensional space $\mathbb{V}^n \subseteq \mathbb{R}^m$ spanned by an orthonormal set $Q = \{q_1, q_2, \dots q_n\}$

(Note :- $n \le m$)

Let $v \in \mathbb{R}^m$ and let us decompose



$v = p + r$ where $p \in \mathbb{V}^n$ the space spanned by Q and r is orthogonal to $\mathbb{V}^n$

Since $p \in \mathbb{V}^n$, $p = \sum_{i=1}^{n} \alpha_i q_i$ — (*)

$$v = p + r \Rightarrow q_j^T v = q_j^T p + \cancelto{0}{q_j^T r}$$

$$\Rightarrow q_j^T v = q_j^T \left(\sum_{i=1}^{n} \alpha_i q_i \right) \quad — \square$$

$i \neq j$ will vanish

$$\Rightarrow \boxed{\alpha_j = q_j^T v}$$

$$\Rightarrow v = r + \sum_{i=1}^{n} (q_i^T v)\, q_i$$

If $m = n$ then $v = \sum_{i=1}^{m} (q_i^T v)\, q_i$

i.e Sum of coefficients of $q_i^T v$ times basis vectors q_i

Also $$\underset{\sim}{v} = \sum_{i=1}^{m} \underset{\sim}{q_i}(\underset{\sim}{q_i}^T \underset{\sim}{v}) = \sum_{i=1}^{m} \underbrace{(\underset{\sim}{q_i}\underset{\sim}{q_i}^T)}\underset{\sim}{v} = \sum_{i=1}^{m} \underset{\sim}{P_i}\underset{\sim}{v}$$

i.e Sum of orthogonal projections of $\underset{\sim}{v}$ onto various directions $\underset{\sim}{q_i}$

(c) <u>Orthogonal matrices</u> :-

A square matrix $\underset{\sim}{Q} \in \mathbb{R}^{m \times m}$ is called orthogonal matrix if $\underset{\sim}{Q}^T = \underset{\sim}{Q}^{-1}$

i.e $\boxed{\underset{\sim}{Q}^T \underset{\sim}{Q} = \underset{\sim}{Q}\underset{\sim}{Q}^T = I}$ $(\underset{\sim}{A}\underset{\sim}{Z} = \underset{\sim}{Z}\underset{\sim}{A} = I)$

$$\underset{\sim}{Q} = \left[\underset{\sim}{q_1} \middle| \underset{\sim}{q_2} \middle| \cdots \middle| \underset{\sim}{q_m} \right]$$

$$\underset{\sim}{Q}^T \underset{\sim}{Q} = \begin{bmatrix} \underset{\sim}{q_1}^T \\ \underset{\sim}{q_2}^T \\ \vdots \\ q_m^T \end{bmatrix} \left[\underset{\sim}{q_1} \middle| \underset{\sim}{q_2} \cdots \middle| q_m \right] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \cdots & & \end{bmatrix} = I$$

✓

This means $\underset{\sim}{q}_i^T \underset{\sim}{q}_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

If $\underset{\sim}{Q} \in \mathbb{C}^{m \times m}$ then $\underset{\sim}{Q}^{\dagger} \underset{\sim}{Q} = \underset{\sim}{Q} \underset{\sim}{Q}^{\dagger} = I$

then $\underset{\sim}{Q}$ is called unitary matrix

We saw

$\underset{\sim}{A} \underset{\sim}{x}$ and $\underset{\sim}{A}^{-1} \underset{\sim}{b}$ interpretations

$$\underset{\sim}{A} \rightarrow \underset{\sim}{Q}$$

i.e $\underset{\sim}{Q} \underset{\sim}{x}$ and $\underset{\sim}{Q}^{-1} \underset{\sim}{b} \Rightarrow \underset{\sim}{Q}^T \underset{\sim}{b}$

$\underset{\sim}{Q}^T \underset{\sim}{b}$ can be thought of change of basis from $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, .. e_m\}$ to orthogonal basis $\{\underset{\sim}{q}_1, \underset{\sim}{q}_2, .. q_m\}$

i.e columns of $\underset{\sim}{Q}$

(ii) Orthogonal matrices preserves inner products

$$(\underset{\sim}{x}, \underset{\sim}{y}) := \underset{\sim}{x}^T \underset{\sim}{y}$$

$$(\underset{\sim}{Q} \underset{\sim}{x}, \underset{\sim}{Q} \underset{\sim}{y}) = (\underset{\sim}{Q} \underset{\sim}{x})^T (\underset{\sim}{Q} \underset{\sim}{y}) = \underset{\sim}{x}^T \underset{\sim}{Q}^T \underset{\sim}{Q} \underset{\sim}{y}$$

$$= \underset{\sim}{x}^T \underset{\sim}{y} = (\underset{\sim}{x}, \underset{\sim}{y})$$

(ii) <u>Orthogonal matrices preserves lengths</u>

$$\|\underset{\sim}{Q}\,\underset{\sim}{x}\| = \|x\| \quad \text{(verify?)}$$

(iii) $\det \underset{\sim}{Q} = \pm 1 \qquad [\underset{\sim}{Q}^T \underset{\sim}{Q} = I]$

(iv) Action of $\underset{\sim}{Q} \in \mathbb{R}^{m \times m}$ on a vector is a rigid rotation (if $\det \underset{\sim}{Q} = 1$) or reflection (if $\det \underset{\sim}{Q} = -1$)