Indian Institute of Science, Bangalore
Department of Computational and Data Sciences (CDS)

DS284: Numerical Linear Algebra

Quiz 3 [Nov 13, 2024]

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Exam duration: 5:45 PM - 6:45 PM Max Points: 30

Notations: (i) Vectors and matrices are denoted by bold faced lower case and upper case alphabets respectively. (ii) Set of all real numbers is denoted by $\mathbb{R}$ (iii) Set of all n dimensional vectors is denoted by $\mathbb{R}^n$ (iv) Set of all $m \times n$ real matrices is denoted by $\mathbb{R}^{m\times n}$ (v) Set of all $m \times n$ complex matrices is denoted by $\mathbb{C}^{m\times n}$

1. Answer the following statements as True/False or Yes/No. Give a short reason for your answer. Marks will be awarded only for your reasoning. **$5 \times 3.5 = 17.5$ points**

   (a) Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be a non-normal non-defective real matrix with complex eigenvalues having non-zero imaginary part. Do you think "Power iteration" still converges to an eigenvector corresponding to a complex eigenvalue with the largest magnitude? (**Hint:** A complex number z can be written in polar form as $z = re^{\mathrm{i}\theta}$ where r is the magnitude of the complex number)

   (b) Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix with eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ with corresponding eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. Let $\mathbf{x} \in \mathbb{R}^n$ be a non-zero vector, and we define the matrix $\mathbf{B} = \begin{bmatrix} \mathbf{A} & \mathbf{x} \\ \mathbf{x}^T & 0 \end{bmatrix}$ having eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n+1}$ with corresponding eigenvectors $\begin{bmatrix} \mathbf{v}_1 \\ 1 \end{bmatrix}, \begin{bmatrix} \mathbf{v}_2 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{v}_{n+1} \\ 1 \end{bmatrix}$, where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n+1} \in \mathbb{R}^n$ are non-zero vectors. Then assert if the following statement is true or false: "Largest eigenvalue of $\mathbf{A}$ is strictly less than the largest eigenvalue of $\mathbf{B}$ i.e. $\mu_n < \lambda_{n+1}$". (**Hint:** For any symmetric matrix $\mathbf{A}$, the Rayleigh quotient $r(\mathbf{x})$ of any vector $\mathbf{x} \neq \mathbf{0}$, satisfies the inequality $\lambda_{min} \leq r(\mathbf{x}) \leq \lambda_{max}$, where λ_{min} and λ_{max} are the minimum and maximum eigenvalues, respectively of $\mathbf{A}$)

   (c) The following $(n+1)$ term Arnoldi recurrence relation is used to build the Krylov subspace $\mathcal{K}_n$ associated with a matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$

   $$\mathbf{A}\mathbf{q}_n = h_{1n}\mathbf{q}_1 + \cdots + h_{nn}\mathbf{q}_n + h_{n+1,n}\mathbf{q}_{n+1}$$

   using some starting vector $\mathbf{q}_1 = \mathbf{b}/||\mathbf{b}||$. Then the orthonormal column vectors $\{\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_n\}$ has to span the successive column subspaces of $\{\mathbf{b}, \mathbf{A}\mathbf{b}, \cdots, \mathbf{A}^{n-1}\mathbf{b}\}$.

   (d) Assume you have a computer with infinite precision. Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be a real positive definite symmetric matrix with an algebraic multiplicity of 2 corresponding to the largest eigenvalue λ. Power iteration for this matrix $\mathbf{A}$ fails to converge to an eigenvector corresponding to this largest eigenvalue.

   (e) Let $\mathbf{A} \in \mathbb{C}^{m\times m}$ and $\mathbf{Q} \in \mathbb{C}^{m\times m}$, with $\mathbf{Q}$ being a unitary matrix. Then $\mathbf{A}$ and $\mathbf{Q}\mathbf{A}\mathbf{Q}^T$ should have the same eigenvalues.

2. We have seen in the class that the Rayleigh quotient of a unit-vector $\mathbf{x} \in \mathbb{R}^m$ for a real symmetric matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ is defined as $\alpha = r(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ and this $r(\mathbf{x})$ acts like an eigenvalue of $\mathbf{A}$ if $\mathbf{x}$ acts like an approximate eigenvector. You will now show that this Rayleigh quotient is a quadratically accurate estimate of an eigenvalue. To this end, consider the case where $\mathbf{x}$ is close to one of the eigenvectors $\mathbf{q_J}$ i.e. if $\mathbf{x} = \sum_{j=1}^m c_j\mathbf{q_j}$ and $\frac{|c_j|}{|c_J|} = \epsilon_j < 1 \quad \forall \quad j \neq J$, we have $\frac{|c_j|}{|c_J|} \leq \epsilon$ where $\epsilon = \max(\epsilon_j)$, now argue that $|r(\mathbf{x}) - r(\mathbf{q_J})| = \mathcal{O}(\epsilon^2)$ **[6 points]**

   (**Hint:** Since $\mathbf{x}$ is a unit vector, you have $\sum_{j=1}^m {c_j}^2 = 1$. First, substitute $\mathbf{x} = \sum_{j=1}^m c_j\mathbf{q_j}$ in the expression of $r(\mathbf{x})$ and then try to use triangle inequalities to arrive at a bound for $|r(\mathbf{x}) - r(\mathbf{q_J})|$. You may also have to use the fact $\frac{|c_j|}{|c_J|} \leq \epsilon$)

3. Consider the eigenvalue problem corresponding to a very large sparse symmetric matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$. We are interested in solving $\mathbf{A}\mathbf{x}_i = \lambda_i\mathbf{x}_i$ for $i = 1 \cdots n$ largest eigenvalue and eigenvector pairs of $\mathbf{A}$ ($n << m$). To this end, let us say we have devised some procedure to construct a smaller r-dimensional ($n < r << m$) subspace rich in the desired eigenvectors of $\mathbf{A}$ spanned by some r orthonormal basis $\mathbb{V}^r = \{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_r\}$. We now wish to seek n vectors in this subspace $\mathbb{V}^r$, which can approximate the n largest eigenvectors of $\mathbf{A}$. One way to find these approximate eigenvectors in $\mathbb{V}^r$ is to make the eigenproblem residuals computed using these vectors orthogonal to the subspace $\mathbb{V}^r$. Deduce an algorithm to find these approximate eigenvector-eigenvalue pairs by imposing this orthogonality condition. **[6.5 points]**