Indian Institute of Science Banglore
Department of Computational and Data Sciences (CDS)

DS284: Numerical Linear Algebra

Mid-semester Exam 2021

Faculty Instructor: Dr. Phani Motamarri

TAs: Kartick Ramakrishnan, Sameer Khadatkar, Soumalya Nandi

Duration: 15:00 hrs to 16:15 hrs Max Points: 50

Notations: Vectors and matrices are denoted below by bold faced lower case and upper case alphabets respectively. A matrix is said to be rank deficient if it is not full rank.

Start each problem in a new page.

Problem 1 [6x3=18 points]

Assert if the following statements are True or False. Give a detailed reasoning for your assertion. Marks will be awarded only for your reasoning.

(a) The accuracy of the solution $\mathbf{x}$ for the square system of equations $\mathbf{Ax} = \mathbf{b}$ only depends on (i) the condition number of $\mathbf{A}$ and (ii) the precision of arithmetic operations carried out.

(b) Let $\mathbf{P} \in \mathbb{R}^{n \times n}$ be an orthogonal matrix with $det(\mathbf{P}) = -1$. Then the matrix $\mathbf{P} + \mathbf{I}_n$ is not invertible (i.e a singular matrix).

(c) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be of $rank(\mathbf{A}) = r(< \min(m, n))$. If $\mathbf{A} = \mathbf{U\Sigma V^T}$ be the full SVD of $\mathbf{A}$, then $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r\}$, the first r columns of $\mathbf{V}$ form an orthogonal basis for $null(\mathbf{A}^T)$

(d) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix with orthogonal columns. Let $\mathbf{A} = \hat{\mathbf{Q}}\hat{\mathbf{R}}$ be its reduced $\mathbf{QR}$ factorization. Then $\hat{\mathbf{R}}$ is a diagonal matrix.

(e) If $\hat{\mathbf{x}} \in \mathbb{R}^n$ satisfies the system of equations $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$ where $\mathbf{A} \in \mathbb{R}^{m \times n}$ is a full rank matrix with $m > n$ and $\mathbf{b} \in \mathbb{R}^m$, then $\hat{\mathbf{x}}$ always satisfies the system of equations $\mathbf{Ax} = \mathbf{b}$

(f) The following matrix $\mathbf{P}$ is a projector matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Problem 2 [8 points]

Consider the three vectors $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$; $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; $\mathbf{a}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(a) Construct the matrix $\mathbf{A}$ whose columns are the three vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. Show that the *range*($\mathbf{A}$) spans $\mathbb{R}^3$ [3 points]

(b) Construct the successive orthogonal transformations $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3$ which transforms $\mathbf{A}$ to an upper triangle matrix $\mathbf{R}$ as shown below:

$$\mathbf{Q}_3\mathbf{Q}_2\mathbf{Q}_1\mathbf{A} = \mathbf{R}$$

(Write down explicitly the matrices $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3$ in your answer sheet) [5 points]

Problem 3 [24 points]

Consider a rank deficient matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ with $m > n$ and a vector $\mathbf{b} \in \mathbb{R}^m$. Let $rank(\mathbf{A}) = r(< n)$. Now answer the following questions:

(a) Recall in the class we derived $\mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$ minimizes the least square error $||\mathbf{b} - \mathbf{A}\mathbf{x}||_2$. Is this solution $\mathbf{x}$, a valid minimizer of $||\mathbf{b} - \mathbf{A}\mathbf{x}||_2$ for the problem at hand where $\mathbf{A}$ is rank deficient. Explain why/why not. [3 points]

(b) If $\mathbf{b}$ lies in the *range*($\mathbf{A}$), explain why the system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ should have infinite solutions? [3 points]

(c) Let $\mathbf{U}_1\mathbf{\Sigma_1}\mathbf{V}_1^T$ be the reduced SVD of our rank deficient matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ with $\mathbf{U}_1 \in \mathbb{R}^{m \times r}$ $\mathbf{\Sigma_1} \in \mathbb{R}^{r \times r}$ and $\mathbf{V}_1 \in \mathbb{R}^{n \times r}$, then show that $\mathbf{x} = \mathbf{V}_1\mathbf{\Sigma_1}^{-1}\mathbf{U}_1^T\mathbf{b}$ satisfies $\mathbf{A}\mathbf{x} = \mathbf{b}$ in the case where $\mathbf{b}$ lies in the *range*($\mathbf{A}$). [5 points]

(d) Show that above solution $\mathbf{x} = \mathbf{V}_1\mathbf{\Sigma_1}^{-1}\mathbf{U}_1^T\mathbf{b}$ is the minimum L^2 norm solution out of all possible solutions that satisfy $\mathbf{A}\mathbf{x} = \mathbf{b}$ in the case where $\mathbf{b}$ lies in the *range*($\mathbf{A}$) (Hint: Express $\mathbf{x} \in \mathbb{R}^n$ in the orthonormal basis of $\mathbf{V}$) [7 points]

(e) Let us consider the case where $\mathbf{b}$ does not lie in the *range*($\mathbf{A}$). Now, to find the least square solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$, we need to solve $\mathbf{A}\mathbf{x} = \mathbf{P}\mathbf{b}$, where $\mathbf{P}$ is the orthogonal projector onto the *range*($\mathbf{A}$). Comment on the solvability of the system $\mathbf{A}\mathbf{x} = \mathbf{P}\mathbf{b}$. Show that the solution $\mathbf{x} = \mathbf{V}_1\mathbf{\Sigma_1}^{-1}\mathbf{U}_1^T\mathbf{b}$ corresponds to the minimum L^2 norm least square solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ [6 points]

Note for Problem 3: Answer all the above questions keeping in mind $\mathbf{A} \in \mathbb{R}^{m \times n}$ is a rank deficient matrix with $m > n$. In data science, least squares problems of the above kind arise in the form of inverse problems relevant to various domain areas such as electromagnetic scattering, geophysics, image restoration, compressed sensing (reconstruction of sparse signal from small number of random measurements), inverse scattering, medical imaging and the study of atmospheres etc.,