DS - 284

Assignment - 1

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Scalar	x
vector	$\underset{\sim}{x}$ (Notation)
Matrix	X

1. $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$

(a) $\dim [R(AB)] \leq \dim [R(A)]$

We know range of a linear transformation / matrix is a vectorspace. (B/c it satisfies closure property & 6 other associated properties of vector space.)

Hence dimension exists for range of a matrix.

Now,

$$R(A) = \{ \underset{\sim}{y} \mid \underset{\sim}{y} = A\underset{\sim}{x} \quad \forall \underset{\sim}{x} \in \mathbb{R}^n \}, \underset{\sim}{y} \in \mathbb{R}^m$$

$$R(AB) = \{ \underset{\sim}{z} \mid \underset{\sim}{z} = AB\underset{\sim}{x}, \forall \underset{\sim}{x} \in \mathbb{R}^p \}, \underset{\sim}{z} \in \mathbb{R}^m.$$

Now, let, $\underset{\sim}{z} \in R(AB)$

$$\Rightarrow \underset{\sim}{z} = (AB)\underset{\sim}{x}, \quad \underset{\sim}{x} \in \mathbb{R}^p$$

$$= A(B\underset{\sim}{x}) \quad [\because \text{matrix product is associative}]$$

$$= A\underset{\sim}{d}, \quad \underset{\sim}{d} \in \mathbb{R}^n.$$

By definition of $R(A)$,

$$\underset{\sim}{z} = A\underset{\sim}{d} \in R(A)$$

We started with $\underset{\sim}{z} \in R(AB)$ & showed $\underset{\sim}{z} \in R(A)$.

Hence $R(AB) \subseteq R(A)$.

So $R(AB)$ is a subspace of $R(A)$.

Hence $\dim [R(AB)] \leq \dim [R(A)]$

— (proved)

(b) If the matrix B is non-singular, $\dim[R(AB)] = \dim[R(A)]$

We've already proved $\dim[R(AB)] \leq \dim[R(A)]$. — ①

If we can show that, when B is non-singular, $\dim[R(A)] \leq \dim[R(AB)]$, then proof is done.

Now, B is non-singular. So B^{-1} exists.

(We also observe, both A & B will be square matrix of size '$m \times m$' as concept of invertibility is valid only for square matrix)

Now in eqn. ①,

replace $A \longrightarrow AB$.

replace $B \longrightarrow B^{-1}$.

We get,

$$\dim[R(AB \cdot B^{-1})] \leq \dim[R(AB)]$$

$$\Rightarrow \dim[R(A)] \leq \dim[R(AB)]$$ — ②

From ① & ②, we conclude

$$\dim[R(AB)] = \dim[R(A)]$$

when B is non-singular matrix.

— (proved)

(c) $N(M) = \{ \underset{\sim}{x} \mid M\underset{\sim}{x} = \underset{\sim}{0} \}$ — (Definition of Null space)

let, $\underset{\sim}{x} \in N(B)$

$\Rightarrow B\underset{\sim}{x} = \underset{\sim}{0}$ where $B \in R^{n \times p}$, $\underset{\sim}{x} \in R^{p}$, $\underset{\sim}{0} \in R^{n}$

$\Rightarrow AB\underset{\sim}{x} = A\underset{\sim}{0} = \underset{\sim}{0}$, where $A \in R^{m \times n}$

$\Rightarrow \underset{\sim}{x} \in N(AB)$

So ~~N(A~~ $N(B) \subseteq N(AB)$

$\Rightarrow \dim[N(B)] \leq \dim[N(AB)]$

let, $\{ \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_k \}$ be the basis of $N(B)$.

B/c $N(B)$ is a subspace of $N(AB)$, $\{ \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_k \}$ is also lin. independent in $N(AB)$.

If $\dim[N(AB)] = r$, we can extend the basis set of $N(B)$ to find basis of $N(AB)$.

So basis set of $N(AB) = \{ \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_k, \underset{\sim}{a}_{k+1}, \ldots, \underset{\sim}{a}_r \}$

B/c, $\{ \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_k, \underset{\sim}{a}_{k+1}, \ldots \underset{\sim}{a}_r \} \in N(AB)$,

$AB\underset{\sim}{x} = \underset{\sim}{0}$ where $\underset{\sim}{x} \in N(AB)$.

Now, let's see if $\{ B\underset{\sim}{a}_{k+1}, \ldots, B\underset{\sim}{a}_r \}$ are lin. independent

$\alpha_1 B\underset{\sim}{a}_{k+1} + \ldots + \alpha_{r-k} B\underset{\sim}{a}_r = 0$ — (1)

$\Rightarrow B(\alpha_1 \underset{\sim}{a}_{k+1} + \ldots + \alpha_{r-k} \underset{\sim}{a}_r) = 0$ ($\because$ Every matrix is a lin. trans)

$\Rightarrow \alpha_1 \underset{\sim}{a}_{k+1} + \ldots + \alpha_{r-k} \underset{\sim}{a}_r \in N(B)$ $[T(\alpha \underset{\sim}{u} + \underset{\sim}{v}) = \alpha T(\underset{\sim}{u}) + T(\underset{\sim}{v})]$

$\Rightarrow \alpha_1 \underset{\sim}{a}_{k+1} + \cdots + \alpha_{r-k} \underset{\sim}{a}_r = \beta_1 \underset{\sim}{a}_1 + \cdots + \beta_k \underset{\sim}{a}_k$

$\Rightarrow \beta_1 \underset{\sim}{a}_1 + \cdots + \beta_k \underset{\sim}{a}_k - \alpha_1 \underset{\sim}{a}_{k+1} - \cdots - \alpha_{r-k} \underset{\sim}{a}_r = 0$

B/c $\{\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_r\}$ are basis of $N(AB)$, they are lin. independent.

Hence $\beta_1 = \cdots = \beta_k = 0$

$\alpha_1 = \cdots = \alpha_{r-k} = 0$

Hence from ①, we see $\{B \underset{\sim}{a}_{k+1}, \ldots, B \underset{\sim}{a}_r\}$ are lin. independent.

Also we see, all these elements belong to $N(A)$. ($\because \underset{\sim}{a}_{k+1}, \ldots, \underset{\sim}{a}_r \in N(AB)$).

Hence $N(A)$ has at least $(r-k)$ lin. independent vectors.

We know, No. of lin. ind. vector in a vector space $\leq$ Dimension of vector space.

Hence $\dim[N(A)] \geq r-k$.

So, we have, $\dim[N(AB)] = r$, $\dim[N(B)] = k$ — (From last page)

$\dim[N(A)] + \dim[N(B)] \geq r-k+k = r = \dim[N(AB)]$.

Hence

$\dim[N(AB)] \leq \dim[N(A)] + \dim[N(B)]$.

— (proved)

(d) $\dim[R(A)] + \dim[N(A)] = n$.

$A \in R^{m \times n}$. $\quad A : R^n \longrightarrow R^m$.

$N(A) = \{ \underset{\sim}{x} \in R^n \mid A\underset{\sim}{x} = \underset{\sim}{0} \}$

B/c $N(A) \subseteq R^n \Rightarrow \dim[N(A)] \leq n$.

let, $\dim[N(A)] = k$, $(k \leq n)$

let, basis of $N(A) = \{ \underset{\sim}{a_1}, \ldots, \underset{\sim}{a_k} \}$.

B/c $\{ \underset{\sim}{a_1}, \ldots, \underset{\sim}{a_k} \}$ are also lin. ind. in R^n,

we can extend it to form basis of R^n.

let, $\{ \underset{\sim}{a_1}, \ldots, \underset{\sim}{a_k}, \underset{\sim}{a_{k+1}}, \ldots, \underset{\sim}{a_n} \}$ is the basis of R^n.

Now, $\quad R(A) = \{ \underset{\sim}{y} \mid \underset{\sim}{y} = A\underset{\sim}{x}, \ \forall \underset{\sim}{x} \in R^n \}$

Now, any $\underset{\sim}{x} \in R^n$, can be written as

$$\underset{\sim}{x} = \alpha_1 \underset{\sim}{a_1} + \cdots + \alpha_k \underset{\sim}{a_k} + \alpha_{k+1} \underset{\sim}{a_{k+1}} + \cdots + \alpha_n \underset{\sim}{a_n}$$

$$\begin{aligned}
R(A) &= A\underset{\sim}{x} \quad \forall \underset{\sim}{x} \in R^n \\
&= A(\alpha_1 \underset{\sim}{a_1} + \cdots + \alpha_k \underset{\sim}{a_k} + \alpha_{k+1} \underset{\sim}{a_{k+1}} + \cdots + \alpha_n \underset{\sim}{a_n}) \\
&= \underbrace{\alpha_1 A\underset{\sim}{a_1} + \cdots + \alpha_k A\underset{\sim}{a_k}} + \alpha_{k+1} A\underset{\sim}{a_{k+1}} + \cdots + \alpha_n A\underset{\sim}{a_n} \\
&= 0 + \alpha_{k+1} A\underset{\sim}{a_{k+1}} + \cdots + \alpha_n A\underset{\sim}{a_n}
\end{aligned}$$

$(\because \{ \underset{\sim}{a_1}, \ldots, \underset{\sim}{a_k} \} \in N(A))$

$\{ A\underset{\sim}{a_{k+1}}, \ldots, A\underset{\sim}{a_n} \}$ span $R(A)$ from the above expression.

— (1)

If we can show that $\{A\underset{\sim}{a}_{k+1}, \dots, A\underset{\sim}{a}_n\}$ are lin. independent, then this set will be a basis of $R(A)$.

Now, $\beta_1 A\underset{\sim}{a}_{k+1} + \dots + \beta_{n-k} A\underset{\sim}{a}_n = 0$ — (2)

$\Rightarrow A(\beta_1 \underset{\sim}{a}_{k+1} + \dots + \beta_{n-k}\underset{\sim}{a}_n) = 0$

— (property of linear transformation)

So $\beta_1 \underset{\sim}{a}_{k+1} + \dots + \beta_{n-k}\underset{\sim}{a}_n \in N(A)$.

Hence

$$\beta_1 \underset{\sim}{a}_{k+1} + \dots + \beta_{n-k}\underset{\sim}{a}_n = \alpha_1 \underset{\sim}{a}_1 + \dots + \alpha_k \underset{\sim}{a}_k$$

B/c $\{\underset{\sim}{a}_1, \dots, \underset{\sim}{a}_n\}$ is basis of R^n, they're lin. independent

Hence $\beta_1 = \dots = \beta_{n-k} = 0$

$\alpha_1 = \dots = \alpha_k = 0$ — (3)

From (2) & (3), we see

$\{A\underset{\sim}{a}_{k+1}, \dots, A\underset{\sim}{a}_n\}$ are lin. independent.

From (1), we showed $\{A\underset{\sim}{a}_{k+1}, \dots, A\underset{\sim}{a}_n\}$ span $R(A)$.

Hence $\{A\underset{\sim}{a}_{k+1}, \dots, A\underset{\sim}{a}_n\}$ is a basis of $R(A)$.

Hence $\dim[R(A)] = n-k$.

So we see,

$$\dim[N(A)] + \dim[R(A)] = k + (n-k) = n$$

— (proved)

Before proving 1.(e), I want to prove 1.(g) & use that in proving 1.(e).

1.(g) Prove that row rank is equal to column rank.

let, $A \in R^{m \times n}$.

let, row rank of A is k.

$\dim[\text{rowspace}(A)] = k$, where rowspace(A) = lin. Combination of rows of A.

let $\{\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_k\}$ be basis of rowspace(A). where $\underset{\sim}{a}_i \in R^n$.

Let's check whether $\{A\underset{\sim}{a}_1, \ldots, A\underset{\sim}{a}_k\}$ are lin. ind.

$\alpha_1 A\underset{\sim}{a}_1 + \ldots + \alpha_k A\underset{\sim}{a}_k = \underset{\sim}{0}$ — ⓐ

$\Rightarrow A(\alpha_1 \underset{\sim}{a}_1 + \ldots + \alpha_k \underset{\sim}{a}_k) = \underset{\sim}{0}$ — ($\because$ property of linear transformation)

$\Rightarrow A\underset{\sim}{x} = \underset{\sim}{0}$ where $\underset{\sim}{x} = \alpha_1 \underset{\sim}{a}_1 + \ldots + \alpha_k \underset{\sim}{a}_k$.

So we observe, $\underset{\sim}{x} \in$ rowspace(A) — ①

We also see $A\underset{\sim}{x} = \underset{\sim}{0}$ imply $\underset{\sim}{x}$ is orthogonal to every row of A.

$$\begin{bmatrix} \text{— row } 1 \text{ —} \\ \vdots \\ \text{— row } m \text{ —} \end{bmatrix} \begin{bmatrix} \\ \underset{\sim}{x} \\ \\ \end{bmatrix} = \begin{bmatrix} \langle \text{row } 1, \underset{\sim}{x} \rangle \\ \vdots \\ \langle \text{row } m, \underset{\sim}{x} \rangle \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Hence $\underset{\sim}{x}$ is also $\perp$ to any lin. comb. of rows of A.

So $\underset{\sim}{x}$ is orthogonal to rowspace(A). — ②

From ① & ②, it's evident that

$$\underset{\sim}{x} = \underset{\sim}{0}$$

$$\Rightarrow \alpha_1 \underset{\sim}{a_1} + \dots + \alpha_k \underset{\sim}{a_k} = \underset{\sim}{0}$$

B/c $\{\underset{\sim}{a_1}, \dots, \underset{\sim}{a_k}\}$ are basis of rowspace (A), they're lin. independent.

Hence $\alpha_1 = \dots = \alpha_k = 0$

Hence from eqn. ⓐ, we see

$\{A\underset{\sim}{a_1}, \dots, A\underset{\sim}{a_k}\}$ are lin. independent.

By definition of matrix vector multiplication,

$$A\underset{\sim}{x} \in \text{Columnspace}(A) \text{ b/c } \begin{bmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1 \underset{\sim}{c_1} + \dots + x_n \underset{\sim}{c_n}.$$

Hence $\{A\underset{\sim}{a_1}, \dots, A\underset{\sim}{a_k}\} \in$ Columnspace (A)

& this set is lin. independent.

Hence $\dim[\text{Columnspace}(A)] \geq k = \text{rowrank}(A)$

$\Rightarrow$ Columnrank (A) $\geq$ rowrank (A). — ③

We repeat the same operation with A^T.

Column rank $(A^T) \geq$ row rank (A^T)

$\Rightarrow$ Rowrank (A) $\geq$ Columnrank (A) — ④

$\left[\because \text{Rowspace \& column space are interchanged in } A \text{ \& } A^T\right]$

Hence from ③ & ④,

Rowrank (A) = Columnrank (A) — (proved)

1.(e) Prove that rank (A) + rank $(B) - n \leq$ rank (AB) &

rank $(AB) \leq \min[\text{rank}(A), \text{rank}(B)]$

→ We have shown in 1.(c) that

$$\dim[N(AB)] \leq \dim[N(A)] + \dim[N(B)]$$

From 1.(d) we can apply Rank-Nullity theorem.

$\Rightarrow$ $p - \text{rank}(AB) \leq n - \text{rank}(A) + p - \text{rank}(B)$

$\Rightarrow$ $\text{rank}(AB) \geq \text{rank}(A) + \text{rank}(B) - n$

— (proved)

In 1.(a), we showed, $\dim[R(AB)] \leq \dim[R(A)]$

$\Rightarrow$ $\text{rank}(AB) \leq \text{rank}(A)$. — ①

We know row rank = Column rank.

Let, Q has rank m. (rank = Row rank = Column rank)

Q^T ~~has~~ Column rank = Q's row rank $= m$

Hence rank (Q^T) = rank (Q). ($\because$ Col. space of Q^T = Row space of Q)

Using this,

$$\text{rank}(AB) = \text{rank}[(AB)^T]$$
$$= \text{rank}(B^T A^T) \leq \text{rank}(B^T)$$

— ($\because$ by Ques. 1.(a))

$\Rightarrow$ $\text{Rank}(AB) \leq \text{Rank}(B)$. — ②

From eq. ① & ②

$$\text{Rank}(AB) \leq \min\{\text{Rank}(A), \text{Rank}(B)\}$$ — (proved)

(f) $\underset{\sim}{u} \in \mathbb{R}^n$. Prove that Rank $(uu^T) = 1$

$\longrightarrow$ Rank = Row rank = column rank —— ($\because$ From 1.(g))

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} = \underset{\sim}{u}\,\underset{\sim}{u}^T$$

$$\Rightarrow \underset{\substack{\sim \\ c_1}}{} \begin{bmatrix} u_1 \cdot u_1 & \cdots & u_1 \cdot u_n \\ \vdots & & \vdots \\ u_n \cdot u_1 & \cdots & u_n \cdot u_n \end{bmatrix}_{n \times n} = \underset{\sim}{u}\,\underset{\sim}{u}^T$$

$$\underbrace{\text{(column 1)}}_{\underset{\sim}{c}_1} \quad \cdots \quad \underbrace{\text{(column n)}}_{\underset{\sim}{c}_n}$$

We observe that

$$\underset{\sim}{c}_i = u_i \underset{\sim}{u} = \frac{u_i}{u_1} \cdot u_1 \underset{\sim}{u} = \left(\frac{u_i}{u_1}\right) \underset{\sim}{c}_1$$

$\frac{u_i}{u_1} \longrightarrow$ Scalar (Assuming $u_1 \neq 0$)

$\boxed{\text{If } u_1 = 0 \text{, choose } u_j \text{ such that } u_j \neq 0.}$ $\longrightarrow$ $\boxed{\text{Then } \underset{\sim}{c}_i = \frac{u_i}{u_j} \underset{\sim}{c}_j}$

So we see, any column of $\underset{\sim}{u}\,\underset{\sim}{u}^T$ is a Scalar multiple of First column / any one non-zero column.

Hence Column rank = No. of lin. independent columns in matrix

$= \dim\left[\text{colspace}(\underset{\sim}{u}\,\underset{\sim}{u}^T)\right] = 1.$

B/c any lin. combination of columns of $\underset{\sim}{u}\,\underset{\sim}{u}^T$ can be expressed as a scalar multiple of First column.

Hence rank $(\underset{\sim}{u}\,\underset{\sim}{u}^T) = 1$, where $\underset{\sim}{u} \neq \underset{\sim}{0}$.

2. Suppose $\exists$ a set of real coefficients $c_1, \dots, c_{10}$ for anyset of real numbers $d_1, \dots, d_{10}$ such that

$$\sum_{j=1}^{10} c_j f_j(i) = d_i \quad \text{for } i \in \{1, \dots, 10\}$$

(a) We can express the above set of eqn. as

$$M \underset{\sim}{c} = \underset{\sim}{d}$$

where $\underset{\sim}{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_{10} \end{bmatrix}$ $\underset{\sim}{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_{10} \end{bmatrix}$ $\underset{\sim}{c} \in R^{10}$, $\underset{\sim}{d} \in R^{10}$

$M_{i,j} = f_j(i)$, $M \in R^{10 \times 10}$

Given condition is for any $\underset{\sim}{d}$, we can find $\underset{\sim}{c} \in R^{10}$.

Hence Range $(M) = R^{10}$ b/c $\underset{\sim}{d} \in R^{10}$.

So Nullity $(M) = \dim[R^{10}] - \dim[R(M)]$
$= 10 - 10 = 0$

As discussed in class, square matrix with nullity 0 has full rank & invertible.

So M is invertible and M^{-1} exists & it's unique $(M \cdot M^{-1} = M^{-1} \cdot M = I)$ (Discussed in class)

$M \underset{\sim}{c} = \underset{\sim}{d} \Rightarrow M^{-1}(M \underset{\sim}{c}) = M^{-1} \underset{\sim}{d}$

$\Rightarrow \underset{\sim}{c} = M^{-1} \underset{\sim}{d}$.

Hence we have proved that given $\underset{\sim}{d} \in R^{10}$, $\underset{\sim}{c}$ is determined uniquely by $M^{-1} \underset{\sim}{d}$.

— (proved)

(b) Give $\quad A\underset{\sim}{d} = \underset{\sim}{c}$, $A \in R^{10}$, $\underset{\sim}{d} \in R^{10}$

for any $\underset{\sim}{c}$. $\quad \underset{\sim}{c} \in R^{10}$

We've shown in 2.(a) that $M\underset{\sim}{c} = \underset{\sim}{d}$ for any $\underset{\sim}{d}$.

Hence

$$A M \underset{\sim}{c} = \underset{\sim}{c} \qquad \forall\, \underset{\sim}{c} \in R^{10}.$$

Hence $\quad AM = I \qquad \Rightarrow A^{-1}(AM) = A^{-1}I$

$$\Rightarrow M = A^{-1}$$

So $\quad (A^{-1})_{i,j} = (M)_{ij} = f_j(i)$.

Hence (i,j)th entry of A^{-1} is $f_j(i)$.

3. $A \in R^{m \times m}$. $\quad A\underset{\sim}{x} = \lambda \underset{\sim}{x}$, $\underset{\sim}{x} \neq \underset{\sim}{0}$, λ is a scalar.

(a) If A is symmetric, eigen value is real.

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x} \qquad , \quad A^* = (\bar{A})^T = \text{conjugate transpose}$$

$$\Rightarrow \underset{\sim}{x}^* A \underset{\sim}{x} = \lambda \underset{\sim}{x}^* \underset{\sim}{x} \qquad , \quad \underset{\sim}{x} \in R^m.$$

$$\Rightarrow \lambda = \frac{\underset{\sim}{x}^* A \underset{\sim}{x}}{\underset{\sim}{x}^* \underset{\sim}{x}}$$

$$\lambda^* = \frac{(\underset{\sim}{x}^* A \underset{\sim}{x})^*}{(\underset{\sim}{x}^* \underset{\sim}{x})^*} = \frac{\underset{\sim}{x}^* A^* \underset{\sim}{x}}{\underset{\sim}{x}^* \underset{\sim}{x}}$$

Now $\lambda^* = \dfrac{\underset{\sim}{x}^* A^* \underset{\sim}{x}}{\underset{\sim}{x}^* \underset{\sim}{x}} = \dfrac{\underset{\sim}{x}^* A \underset{\sim}{x}}{\underset{\sim}{x}^* \underset{\sim}{x}}$

($\because$ A is symmetric, $A = A^T$.
Also A is real $\in R^{m \times m}$.
Hence $A^* = A$)

So we observe, $\lambda = \lambda^*$

This is possible only when λ is real.
Hence eigenvalues are real. — (proved).

(b) If $\underset{\sim}{x}$ & $\underset{\sim}{y}$ are eigen vectors of symmetric matrix A, with distinct eigen values, Show that $\underset{\sim}{x}$ & $\underset{\sim}{y}$ are orthogonal.

$A\underset{\sim}{x} = \lambda_1 \underset{\sim}{x}$, $A\underset{\sim}{y} = \lambda_2 \underset{\sim}{y}$

$\Rightarrow \underset{\sim}{y}^* A \underset{\sim}{x} = \lambda_1 \underset{\sim}{y}^* \underset{\sim}{x}$

$\Rightarrow \underset{\sim}{x}^* A \underset{\sim}{y} = \lambda_2 \underset{\sim}{x}^* \underset{\sim}{y}$ — ②

$\Rightarrow (\underset{\sim}{y}^* A \underset{\sim}{x})^* = (\lambda_1 \underset{\sim}{y}^* \underset{\sim}{x})^*$

$\Rightarrow \underset{\sim}{x}^* A^* \underset{\sim}{y} = \lambda_1^* \underset{\sim}{x}^* \underset{\sim}{y} = \lambda_1 \underset{\sim}{x}^* \underset{\sim}{y}$ [$\because$ eigen values are real as per 3.(a)]

$\Rightarrow \underset{\sim}{x}^* A \underset{\sim}{y} = \lambda_1 \underset{\sim}{x}^* \underset{\sim}{y}$ — ① [$\because$ A is real & symmetric]

From eq. ① & ②,

$\lambda_1 \underset{\sim}{x}^* \underset{\sim}{y} = \lambda_2 \underset{\sim}{x}^* \underset{\sim}{y}$

$$\Rightarrow (\lambda_1 - \lambda_2)\, \underset{\sim}{x}^* \underset{\sim}{y} = 0$$

Given $\lambda_1 \neq \lambda_2$

Hence

$$\underset{\sim}{x}^* \underset{\sim}{y} = 0$$

$$\Rightarrow \langle \underset{\sim}{x}, \underset{\sim}{y} \rangle = 0$$

Hence $\underset{\sim}{x}$ & $\underset{\sim}{y}$ are orthogonal. —— (proved)

(c) S is skew-symmetric. $S^T = -S$.

$$S\underset{\sim}{x} = \lambda \underset{\sim}{x} \Rightarrow \underset{\sim}{x}^* S \underset{\sim}{x} = \lambda \underset{\sim}{x}^* \underset{\sim}{x}$$

$$\Rightarrow \lambda = \frac{\underset{\sim}{x}^* S \underset{\sim}{x}}{\underset{\sim}{x}^* \underset{\sim}{x}}$$

$$\lambda^* = \left[\frac{\underset{\sim}{x}^* S \underset{\sim}{x}}{\underset{\sim}{x}^* \underset{\sim}{x}}\right]^* = \frac{\underset{\sim}{x}^* S^* \underset{\sim}{x}}{\underset{\sim}{x}^* \underset{\sim}{x}} = -\frac{\underset{\sim}{x}^* S \underset{\sim}{x}}{\underset{\sim}{x}^* \underset{\sim}{x}} = -\lambda$$

$$\lambda^* = -\lambda$$

$\left[\begin{array}{l} \because S^* = -S \\ \text{b/c } S \in \mathbb{R}^{m \times m} \\ S^T = -S \end{array}\right]$

$\Rightarrow \lambda$ is 0 or purely imaginary.

—— (proved)

(d) S is skew-symmetric.

Eigen values are 0 or purely imaginary.

Eigen value is obtained by solving $\det(S-\lambda I)=0$, which gives rise to characteristic polynomial.

We know, if solution of polynomial eqn's are purely imaginary, then they come in conjugate pairs.

Hence eigen values of S are of form $\pm ib$ where $i=\sqrt{-1}$ & $b\in\mathbb{R}$.

So

$$S\underset{\sim}{x} = \lambda \underset{\sim}{x}, \quad \lambda = \pm ib$$

Now, for same eigen vectors,

$$(I-S)\underset{\sim}{x} = I\underset{\sim}{x} - S\underset{\sim}{x} = \underset{\sim}{x} - \lambda\underset{\sim}{x}$$

$$= (1-\lambda)\underset{\sim}{x}$$

Hence eigen values of $I-S$ is $1-\lambda$ where $\lambda = \pm ib$.

Now, we know,

$$\det(A) = \text{product of eigen values of eqn. } A\underset{\sim}{x} = \lambda\underset{\sim}{x}.$$

We observe, eigen values of $I-S$ are $1 \pm ib$.

Their product $= (1+ib_1)(1-ib_1)(1+ib_2)(1-ib_2)\ldots\ldots$

$= (1+b_1^2)(1+b_2^2)\ldots\ldots$ (Finite number of times)

Each of these terms are +ve.

Hence product of eigen values = determinant of $I-S$ is +ve.

B/c $\det(I-S) \neq 0$, $I-S$ is non-singular & invertible

— (proved)

(e) prove that $Q = (I-S)^{-1}(I+S)$ is orthogonal for any skew symmetric matrix S.

→ A matrix P is orthogonal if ~~$QQ^T = Q^TQ = I$~~.
$PP^T = P^TP = I$.

$$Q^TQ = \left[(I-S)^{-1}(I+S)\right]^T (I-S)^{-1}(I+S)$$
$$= (I+S)^T\left((I-S)^{-1}\right)^T (I-S)^{-1}(I+S)$$
$$= (I+S^T)\left((I-S)^T\right)^{-1}(I-S)^{-1}(I+S)$$
$$= (I-S)(I+S)^{-1}(I-S)^{-1}(I+S)$$
$(\because S^T = -S)$

$$\left[\because \begin{array}{l} (A+B)^T = A^T + B^T \\ (AB)^T = B^T A^T \\ (A^T)^{-1} = (A^{-1})^T \end{array}\right]$$

$$= (I-S)\left[(I-S)(I+S)\right]^{-1}(I+S)$$

Now, $(I-S)(I+S) = I + S - S - S^2 = I - S^2$
$(I+S)(I-S) = I - S + S - S^2 = I - S^2$

So $(I-S)(I+S) = (I+S)(I-S)$. — (1)

Hence coming back to eqn.

$$Q^TQ = (I-S)\left[(I-S)(I+S)\right]^{-1}(I+S)$$

$$= (I-S)\left[(I+S)(I-S)\right]^{-1}(I+S)$$

($\because$ From eqn. (1))

$$= \underbrace{(I-S)(I-S)^{-1}}_{I}\underbrace{(I+S)^{-1}(I+S)}_{I} \quad (\because (AB)^{-1} = B^{-1}A^{-1})$$

$$= I \cdot I$$

$$= I$$

Similarly, we can show $QQ^T = I$.
Hence Q is orthogonal. — (proved)

(f) $A \in R^{m \times m}$, A is symmetric.

Show that $\underset{\sim}{u}^T A \underset{\sim}{u} = 0 \ \forall u \in R^m \iff A = 0$

$\longrightarrow$ One direction is trivial.

If $A = 0$, $A\underset{\sim}{u} = \underset{\sim}{0}$

Hence $\underset{\sim}{u}^T A \underset{\sim}{u} = \underset{\sim}{u}^T \underset{\sim}{0} = \underset{\sim}{0}$

Now show that $\underset{\sim}{u}^T A \underset{\sim}{u} = \underset{\sim}{0} \implies A = 0$

$\forall \underset{\sim}{u} \in R^m$

Now, A can be expanded as QDQ^T,

where Q = orthogonal matrix

D = Diagonal Matrix

B/c Q is an ~~orthonormal~~ orthogonal matrix in $\mathbb{R}^{m\times m}$, its columns form orthonormal basis in $\mathbb{R}^m$.

So any vector $\underset{\sim}{y} \in \mathbb{R}^m$ can be written as $Q\underset{\sim}{\omega}$.

$\underset{\sim}{y} = Q\underset{\sim}{\omega}$ b/c
$$\underbrace{\begin{bmatrix} | & & | \\ \underset{\sim}{c}_1 & \cdots & \underset{\sim}{c}_m \\ | & & | \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \omega_1 \\ \vdots \\ \omega_m \end{bmatrix}}_{\underset{\sim}{\omega}} = \underset{\sim}{y}$$

$$\Rightarrow \omega_1 \underset{\sim}{c}_1 + \cdots + \omega_m \underset{\sim}{c}_m = \underset{\sim}{y}$$

B/c $\{\underset{\sim}{c}_1, \ldots, \underset{\sim}{c}_m\}$ are basis of $\mathbb{R}^m$, $\underset{\sim}{y}$ can be expressed as lin. combination of them & in condensed form,

$\underset{\sim}{y} = Q\underset{\sim}{\omega}$. where $\underset{\sim}{\omega} \in \mathbb{R}^m$ & unique for a particular $\underset{\sim}{y}$.

Hence
$$\begin{aligned}
\underset{\sim}{y}^T A \underset{\sim}{y} &= (Q\underset{\sim}{\omega})^T Q D Q^T Q \underset{\sim}{\omega} \\
&= \underset{\sim}{\omega}^T Q^T Q D Q^T Q \underset{\sim}{\omega} \\
&= \underset{\sim}{\omega}^T D \underset{\sim}{\omega} \qquad (\because Q^T Q = Q Q^T = I) \\
&= \lambda_1 |\omega_1|^2 + \cdots + \lambda_m |\omega_m|^2
\end{aligned}$$

where D = diagonal matrix = $\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{bmatrix}_{m\times m}$

If $\underset{\sim}{y}^T A \underset{\sim}{y} = 0 \quad \forall \underset{\sim}{y} \in \mathbb{R}^m$

$\Rightarrow \lambda_1 |\omega_1|^2 + \cdots + \lambda_m |\omega_m|^2 = 0$ for any $\underset{\sim}{\omega} \in \mathbb{R}^m$.

Now if $\lambda_1 |\omega_1|^2 + \dots + \lambda_m |\omega_m|^2 = 0$ for any $\underset{\sim}{\omega} \in R^m$

$$\Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_m = 0.$$

B/c $\lambda_1 = \dots = \lambda_m = 0$, $D = 0$

Hence $A = QDQ^T = 0$.

Hence

$$\underset{\sim}{u}^T A \underset{\sim}{u} = 0 \quad \forall \underset{\sim}{u} \in R^m \Rightarrow A = 0 \qquad \text{— (proved)}$$

So overall, we proved $\underset{\sim}{u}^T A \underset{\sim}{u} = 0 \;\; \forall \underset{\sim}{u} \in R^m \iff A = 0$

(9) Show that $\underset{\sim}{u}^T S \underset{\sim}{u} = 0 \;\; \forall \underset{\sim}{u} \in R^m \iff S$ is skew-symmetric.

Case - 1 (Right $\longrightarrow$ Left proof)

S is skew-symmetric. $S^T = -S$.

Now, let, $\underset{\sim}{u}^T S \cancel{A} \underset{\sim}{u} = x$

$(\underset{\sim}{u}^T S \cancel{A} \underset{\sim}{u})^T = x^T = x$ $\quad (\because x$ is a scalar$)$

$\Rightarrow \underset{\sim}{u}^T S \cancel{A}^T \underset{\sim}{u} = x$

$\Rightarrow \underset{\sim}{u}^T S \underset{\sim}{u} = -x$ $\quad (\because S^T = -S)$

So we observe, $\underset{\sim}{u}^T S \underset{\sim}{u} = x = -x$

$$\Rightarrow x = 0.$$

Hence $\underset{\sim}{u}^T S \underset{\sim}{u} = 0$, when S is skew-symmetric.

— (proved)

Case - 2 (Left $\longrightarrow$ Right proof)

$$\underset{\sim}{u}^T S \underset{\sim}{u} = 0 \quad \forall \underset{\sim}{u} \in R^m$$

$$\Rightarrow (\underset{\sim}{u}^T S \underset{\sim}{u})^T = 0 \quad \Rightarrow \underset{\sim}{u}^T S^T \underset{\sim}{u} = 0$$

So we Observe $\underset{\sim}{u}^T S \underset{\sim}{u} = 0$ & $\underset{\sim}{u}^T S^T \underset{\sim}{u} = 0$

$$\Rightarrow \langle \underset{\sim}{u}, S\underset{\sim}{u} \rangle = 0 \ \& \ \langle \underset{\sim}{u}, S^T \underset{\sim}{u} \rangle = 0$$

$$\Rightarrow \langle \underset{\sim}{u}, (S+S^T)\underset{\sim}{u} \rangle = 0$$ (Adding both or them)

$$\Rightarrow \underset{\sim}{u}^T (S+S^T) \underset{\sim}{u} = 0 \quad \text{--- (1)}$$

For any matrix A, $C = A + A^T$ is symmetric.

B/c $C_{ij} = a_{ij} + a_{ji}$ & $C_{ji} = a_{ji} + a_{ij}$

Hence $C_{ij} = C_{ji} \Rightarrow C$ is symmetric.

Hence $S + S^T$ is symmetric matrix.

In 3.(f), we showed $\underset{\sim}{u}^T A \underset{\sim}{u} = 0 \ \forall \underset{\sim}{u} \in R^m \Rightarrow A = 0$
where A is symmetric. --- (2).

Hence from (1) & (2), $S + S^T = 0$

$$\Rightarrow S^T = -S. \Rightarrow S \text{ is skew-symmetric.}$$

Hence we proved

$\underset{\sim}{u}^T S \underset{\sim}{u} = 0 \ \forall \underset{\sim}{u} \in R^m \iff S$ is skew-symmetric. (proved)