Quiz 3 - 2023

1) a) FALSE.

Because $\underset{\sim}{Q}$ is a unitary matrix, we know that $\underset{\sim}{Q}^{\perp} = \underset{\sim}{Q}^{-1}$, where $\perp$ denotes the tranjugate (i.e. transpose + conjugate) of a matrix. Because $\underset{\sim}{Q} \in \mathbb{C}$, it is possible that $\underset{\sim}{Q}^{T} \neq \underset{\sim}{Q}^{\perp}$. Hence $\underset{\sim}{Q}\underset{\sim}{A}\underset{\sim}{Q}^{T}$ may not be a similarity transformation w.r.t. $\underset{\sim}{A}$. Therefore it cannot be conclusively said that $\underset{\sim}{A}$ & $\underset{\sim}{Q}\underset{\sim}{A}\underset{\sim}{Q}^{T}$ have the same eigenvalues.

b) FALSE.

We know that there is no direct algo. to calculate the eigenvalues of a matrix $\underset{\sim}{A} \in \mathbb{R}^{m \times m}$, where $m \geq 5$. We also know that if a matrix is in Schur form, i.e $\underset{\sim}{Q}\,\underset{\sim}{T}\,\underset{\sim}{Q}^{\perp}$, where $\underset{\sim}{T}$ is an upper triangular matrix, we can just read the eigenvalues from the diagonal of $\underset{\sim}{T}$. Hence, if there exists a direct method for Schur decomposition, we would essentially have a direct algo. for calculating the eigenvalues.

c) FALSE.

If a matrix $\underset{\sim}{A} \in \mathbb{R}^{m \times m}$ has m distinct eigenvalues, this implies that each eigenvalue has an algebraic multiplicity of 1. Because algebraic multiplicity $\geq$ geometric multiplicity, geometric multiplicity $= 1$. Therefore, $\underset{\sim}{A}$ has m linearly independent eigenvectors. Therefore the eigenvectors corresponding to different eigenvalues will be linearly independent.

d)

e) TRUE.

f)

The power iteration converges iff the starting vector $\underset{\sim}{x}^{(0)}$ is not linearly independent of the eigenvector corresponding to the largest eigenvalue. In this case, the largest eigenvector is:

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

$$\begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix} \underset{\sim}{x} = 7\underset{\sim}{x}$$

$$2x_1 + 5x_2 = 7x_1$$
$$-5x_1 + 5x_2 = 0$$
$$x_1 - x_2 = 0$$
$$x_1 = x_2$$

$$4x_1 + 3x_2 = 7x_2$$
$$4x_1 = 4x_2$$
$$x_1 = x_2$$

Hence, an eigenvector of $\underset{\sim}{A}$ corresponding to the largest eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

Hence, because the eigenvalue corresponding to the largest eigenvector of $\underset{\sim}{A}$ is orthogonal to $\underset{\sim}{x}^{(0)}$, the power iterations will not converge to the particular eigenvalue.

2) a) We can use power iteration with shift, i.e. inverse power iterations to obtain the eigenpair with eigenvalue closest to 2.0. The algorithm is as follows:

* Because $\underset{\sim}{A}$ is already tridiagonal, it can be considered as already being in upper Hessenberg format. Hence, we can apply inverse power iterations directly.

$\mu = 2.0$

$v^{(0)}$ = random vector

for $k = 1 \rightarrow \infty$:

$\underset{\sim}{w} = (\underset{\sim}{A} - \mu \underset{\sim}{I})^{-1} \; v^{(k-1)}$

$\underset{\sim}{v}^{(k)} = \frac{\underset{\sim}{w}}{\|\underset{\sim}{w}\|}$

$\underset{\sim}{\lambda}^{(k)} = (\underset{\sim}{v}^{(k)})^T \underset{\sim}{A} (\underset{\sim}{v}^{(k)})$

This algo shall converge to the eigenpair having an eigenvalue closest to 2.0.

b) ?

Quiz 3 (2024)

1) a) FALSE.

Because $\underset{\sim}{A}$ is a real matrix with complex eigenvalues, the eigenvalue's conjugate must also be an eigenvalue of A. Because magnitude of both an eigenvalue and its conjugate are equal, and power iteration converges to the eigenvalue with the largest magnitude (only if this magnitude is unique), power iterations do not converge in this case.

b) FALSE.

$$\underset{\sim}{B} \cdot \begin{bmatrix} \underset{\sim}{v}_{n+1} \\ 1 \end{bmatrix} = \lambda_{n+1} \begin{bmatrix} v_{n+1} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{x} \\ \underset{\sim}{x}^T & 0 \end{bmatrix}_{2\times 2} \begin{bmatrix} \underset{\sim}{v}_{n+1} \\ 1 \end{bmatrix}_{2\times 1} = \lambda_{n+1} \begin{bmatrix} \underset{\sim}{v}_{n+1} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \underset{\sim}{A} \cdot \underset{\sim}{v}_{n+1} + \underset{\sim}{x} \\ \underset{\sim}{x}^T \cdot \underset{\sim}{v}_{n+1} \end{bmatrix} = \begin{bmatrix} \lambda_{n+1} \cdot \underset{\sim}{v}_{n+1} \\ \lambda_{n+1} \end{bmatrix}$$

$\underset{\sim}{A} \cdot \underset{\sim}{v}_{n+1} + \underset{\sim}{x} = \lambda_{n+1} \cdot \underset{\sim}{v}_{n+1}$ and $\underset{\sim}{x}^T \cdot \underset{\sim}{v}_{n+1} = \lambda_{n+1}$

c) TRUE.

The Krylov subspace is actually defined as the subspace spanned by the vectors $\langle \underset{\sim}{b}, A\underset{\sim}{b}, \ldots, A^{n-1}\underset{\sim}{b} \rangle$. We do not use this method due to it being numerically unstable. Hence, we use the Arnoldi iteration to obtain this subspace.

d) TRUE.

Power Iteration works as follows:

Let $\underset{\sim}{v}^{(0)}$ = random unit vector

for $k = 1 \to \infty$:

$\underset{\sim}{w} = \underline{A}\, \underset{\sim}{v}^{(k-1)}$

$\underset{\sim}{v}^{(k)} = \dfrac{\underset{\sim}{w}}{\|\underset{\sim}{w}\|}$ // normalise

$\lambda^{(k)} = (\underset{\sim}{v}^{(k)})^T \underset{\sim}{A} (\underset{\sim}{v}^{(k)})$ // Rayleigh quotient

Due to the normalisation step, the largest eigenvalue becomes dominant, and all other smaller eigenvalues will become "overshadowed". We also know that the convergence for power iteration will be:

$$|\lambda^{(k)} - \lambda_1| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right)$$

$$= O(1^k) \qquad (\lambda_2 = \lambda_1)$$

$$= O(1)$$

$\therefore \lambda^{(k)} = c + \lambda_1$, where some c is some constant. As $k \to \infty$, $\lambda^{(k)} = c + \lambda_1 \neq \lambda_1$. Hence proven, no convergence.

e) FALSE.

Because $\underset{\sim}{Q}$ is a unitary matrix, we know that $Q^{\perp} = Q^{-1}$, where $\perp$ denotes the tranjugate (i.e. transpose + conjugate) of a matrix. Because $\underset{\sim}{Q} \in \mathbb{C}$, it is possible that $Q^{T} \neq Q^{\perp}$. Hence QAQ^{T} may not be a similarity transformation w.r.t. $\underset{\sim}{A}$. Therefore it cannot be conclusively said that A & QAQ^{T} have the same eigenvalues.

2) $r(x) = x^T A x$

$$= \left(\sum_{j=1}^{m} c_j q_j^T\right) A \left(\sum_{j=1}^{m} c_j q_j\right)$$

$$= \sum_{j=1}^{m} c_j^2 q_j^T A q_j$$

$$= \sum_{j=1, j\neq J}^{m} c_j^2 \lambda_j + c_J^2 \lambda_J$$

$$\leq \sum_{j=1, j\neq J}^{m} \varepsilon^2 |c_J|^2 \lambda_j + c_J^2 \lambda_J$$

$$\leq \varepsilon^2 |c_J|^2 \sum_{j=1}^{m} \lambda_j$$

$r(q_J) = q_J^T A q_J$

$= \lambda_J q_J^T q_J$

$= \lambda_J$

$A q_J = \lambda_J q_J$

$$\frac{|c_j|}{|c_J|} \leq \varepsilon$$

$|c_j| \leq \varepsilon \cdot |c_J|$

$|c_j|^2 \leq \varepsilon^2 |c_J|^2$

Quiz 4 (2021)

1) ii)

Because $\underset{\sim}{M}^2 = \underset{\sim}{0}$, $\lambda^2 = 0$, where λ is an eigenvalue of $\underset{\sim}{M}$.
Hence, $\lambda = 0$ is an eigenvalue of $\underset{\sim}{M}$

Proof against ii):

α is an eigenvalue of $\underset{\sim}{M} + \alpha \underset{\sim}{I}$?

$$(\underset{\sim}{M} + \alpha \underset{\sim}{I})\underset{\sim}{x} = \alpha \underset{\sim}{x}$$
$$\underset{\sim}{M}\underset{\sim}{x} + \alpha \underset{\sim}{x} = \alpha \underset{\sim}{x}$$
$$\underset{\sim}{M}\underset{\sim}{x} = \underset{\sim}{0}$$
$$\lambda \underset{\sim}{x} = 0$$
$$0 \underset{\sim}{x} = 0 \checkmark$$

$$(\underset{\sim}{M} - \alpha \underset{\sim}{I})\underset{\sim}{x} = \alpha \underset{\sim}{x}$$
$$\underset{\sim}{M}\underset{\sim}{x} - \alpha \underset{\sim}{x} = \alpha x$$
$$\underset{\sim}{M}\underset{\sim}{x} = 2\alpha \underset{\sim}{x}$$
$$\lambda \underset{\sim}{x} = 2\alpha \underset{\sim}{x}$$

$$\lambda = 2\alpha$$
$$0 = 2\alpha$$
$\alpha = 0$ (not valid because $\alpha \in \mathbb{R} - \{0\}$)

Proof against b):

$$(\alpha \underset{\sim}{I} - \underset{\sim}{M})\underset{\sim}{x} = \alpha \underset{\sim}{x}$$

$$\alpha \underset{\sim}{x} - \underset{\sim}{M}\underset{\sim}{x} = \alpha x$$

$$-\underset{\sim}{M}\underset{\sim}{x} = \underset{\sim}{0}$$

$$-\lambda x = \underset{\sim}{0}$$

$$\underset{\sim}{0} = \underset{\sim}{0}$$

$\therefore$ α is an eigenvalue of $(\alpha \underset{\sim}{I} - \underset{\sim}{M})$. Its algebraic multiplicity need not be n. Hence, b is also false.

Proof against c):

$$(\underset{\sim}{M} - \alpha \underset{\sim}{I})\underset{\sim}{x} = -\alpha \underset{\sim}{x}$$

$$\underset{\sim}{M}x - \alpha \underset{\sim}{x} = -\alpha x$$

$$\underset{\sim}{M}\underset{\sim}{x} = 0$$

$$\lambda \underset{\sim}{x} = 0$$

$$0 = 0$$

However, we do not know the no. of distinct eigenvalues of $\underset{\sim}{M} - \alpha \underset{\sim}{I}$. For all we know, α may be the only eigenvalue of $M - \alpha I$.

2) a)

There exists a similarity transformation b/t $\underset{\sim}{A}$ & $\underset{\sim}{B}$. Hence, $\underset{\sim}{A}$ & $\underset{\sim}{B}$ will have same eigenvalue.

3) a) FALSE. $r(\underset{\sim}{x})$ minimises the expression $\|\underset{\sim}{A}\underset{\sim}{x} - r(\underset{\sim}{x}) \cdot \underset{\sim}{x}\|$, this minimisation may not be within the largest & smallest eigen values of A. It may be just less than the smallest or just more than the largest.

b) FALSE. Because $\hat{x}$ is a unit vector $\hat{x}^T \hat{A} \hat{x}$ is actually the Rayleigh quotient. As per rules of the Rayleigh quotient, if

$$\|\hat{x} - \underset{\sim}{q}_J\| = \Delta, \quad |r(\hat{x}) - \lambda_J| = O(\Delta^2) \neq O(\Delta)$$

c) We want to find a β s.t. $\|\underset{\sim}{A}\underset{\sim}{x} - \beta \hat{x}\|$ is reduced. This is the Rayleigh quotient corresponding to $\underset{\sim}{x}$.

$$\therefore \beta = \frac{\hat{x}^T \hat{A} \tilde{x}}{x^T x}$$

$$= \frac{\begin{bmatrix} 3 & 7 \end{bmatrix}_{1\times 2} \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}_{2\times 2} \begin{bmatrix} 3 \\ 7 \end{bmatrix}}{\begin{bmatrix} 3 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix}}$$

$$= \frac{\begin{bmatrix} 34 & 36 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix}}{58}$$

$$= \frac{\begin{bmatrix} 17 & 18 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix}}{29} \qquad \begin{array}{r} 51 \\ 126 \\ \hline 177 \end{array}$$

$$= \frac{177}{29} \approx 6 \, {}^{3}/_{29}$$

$$\neq 5$$

Hence, c) is not the answer.

d) FALSE.

4 a) FALSE. There is no direct, numerically stable algorithm to obtain the eigenvalues of a matrix.

b) FALSE. If the matrix is not reduced to upper Hessenberg, $O(m^4)$ work is required.

c) FALSE. For a "guess" vector $\underset{\sim}{v}^{(0)}$ to converge to q_1, where q_i is the eigenvector corresponding to the largest eigenvalue, $(\underset{\sim}{v}^{(0)})^T \cdot q_J \neq 0$. In other words $\underset{\sim}{v}^{(0)} \neq \underset{\sim}{0}$, AND, $\underset{\sim}{v}^{(0)}$ must not be orthogonal to $\underset{\sim}{q}_J$.

d) TRUE. A Householder reflector is usually orthogonal. Hence, $\underset{\sim}{F}^T = \underset{\sim}{F}^{-1}$

Therefore $\underset{\sim}{F}\underset{\sim}{A}\underset{\sim}{F}^T = \underset{\sim}{F}\underset{\sim}{A}\underset{\sim}{F}^{-1}$

This implies that there is a similarity transformation between $\underset{\sim}{A}$ & $\underset{\sim}{F}\underset{\sim}{A}\underset{\sim}{F}^{-1}$, which means that they have the same eigenvalue.

5) a) TRUE.

b) TRUE. As $k \to \infty$, $\underset{\sim}{A}^{(k)}$ approaches Schur form, which means that $Q^{(k)}$ will form an eigenvector matrix.

c) TRUE.

d) TRUE. Both are $O(m^3)$

6) a) TRUE. If a real matrix has a complex eigenvalue, its complement shall also be an eigenvalue.

$$\overline{\underset{\sim}{A}\underset{\sim}{x}} = \overline{\lambda \underset{\sim}{x}}$$

$$\bar{\underset{\sim}{A}}\, \bar{\underset{\sim}{x}} = \bar{\lambda}\, \bar{\underset{\sim}{x}} \qquad [\overline{ab} = \bar{a}\,\bar{b}]$$

$$\underset{\sim}{A}\, \bar{\underset{\sim}{x}} = \bar{\lambda}\, \bar{\underset{\sim}{x}} \qquad [\text{Because } \underset{\sim}{A} \text{ is real, } \bar{\underset{\sim}{A}} = \underset{\sim}{A}] \ **$$

$$\underset{\sim}{A}\, \underset{\sim}{y} = \bar{\lambda}\, \underset{\sim}{y} \qquad [\text{Let } \bar{\underset{\sim}{x}} = \underset{\sim}{y}]$$

(shown)

b) TRUE. Because $\underset{\sim}{A}$ is diagonalizable, $\underset{\sim}{A}$ is not defective. Hence, $\underset{\sim}{A}$ must have a full set of linearly independent eigenvectors. Therefore $\underset{\sim}{A}$ will be full-rank.

c) TRUE.

A normal matrix is one that satisfies $\underset{\sim}{A}\underset{\sim}{A}^{+} = \underset{\sim}{A}^{+}\underset{\sim}{A}$.

A real orthogonal matrix (an orthogonal matrix is always real) will follow this property because $\underset{\sim}{Q}^{T} = \underset{\sim}{Q}^{-1}$:

$$\underset{\sim}{Q}\underset{\sim}{Q}^{T} = \underset{\sim}{Q}\underset{\sim}{Q}^{-1} = \underset{\sim}{I}$$

$$\underset{\sim}{Q}^{T}\underset{\sim}{Q} = \underset{\sim}{Q}^{-1}\underset{\sim}{Q} = \underset{\sim}{I}$$

d) FALSE.

7) a) FALSE. Sum of eigenvalues of A = Trace of A = $2+1+3+4+4 = 14 \neq 16$

b) TRUE.

$$AA^T = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 4 & 0 & 3 & 0 & 0 \\ 0 & 8 & 0 & 4 & 2 \\ 0 & 0 & 0 & 8 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 8 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 8 & 0 & 0 \\ 0 & 1 & 0 & 8 & 0 \\ 8 & 0 & 25 & 0 & 0 \\ 0 & 8 & 0 & 84 & 40 \\ 0 & 0 & 0 & 40 & 80 \end{bmatrix}$$

Sum of eigenvalues of AA^T = Trace of AA^T

$= 4+1+25+84+80$

$= 30+164$

$= 194$ ✓

c) FALSE.

Product of eigenvalues of AA^T

$= \det(AA^T)$

$$= \det \begin{bmatrix} 4 & 0 & 8 & 0 & 0 \\ 0 & 1 & 0 & 8 & 0 \\ 8 & 0 & 25 & 0 & 0 \\ 0 & 8 & 0 & 84 & 40 \\ 0 & 0 & 0 & 40 & 80 \end{bmatrix}$$

$$= -40 \cdot \det \begin{bmatrix} 4 & 0 & 8 & 0 \\ 0 & 1 & 0 & 0 \\ 8 & 0 & 25 & 0 \\ 0 & 8 & 0 & 40 \end{bmatrix} + 80 \cdot \det \begin{bmatrix} 4 & 0 & 8 & 0 \\ 0 & 1 & 0 & 8 \\ 8 & 0 & 25 & 0 \\ 0 & 8 & 0 & 84 \end{bmatrix}$$

$$= 40(1440) + 80(720)$$

$$= 2(80)(720)$$

$$= 11{,}520$$

$$\neq 9216$$

$$\det \begin{bmatrix} 4 & 0 & 8 & 0 \\ 0 & 1 & 0 & 0 \\ 8 & 0 & 25 & 0 \\ 0 & 8 & 0 & 40 \end{bmatrix} = 40 \cdot \det \begin{bmatrix} 4 & 0 & 8 \\ 0 & 1 & 0 \\ 8 & 0 & 25 \end{bmatrix}$$

$$= 40 \det \begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix}$$

$$= 40(36)$$

$$= 1440$$

$$\det \begin{bmatrix} 4 & 0 & 8 & 0 \\ 0 & 1 & 0 & 8 \\ 8 & 0 & 25 & 0 \\ 0 & 8 & 0 & 84 \end{bmatrix} = 4 \cdot \det \begin{bmatrix} 1 & 0 & 8 \\ 0 & 25 & 0 \\ 8 & 0 & 84 \end{bmatrix} + 8 \cdot \det \begin{bmatrix} 0 & 1 & 8 \\ 8 & 0 & 0 \\ 0 & 8 & 84 \end{bmatrix}$$

$$= 4(500) + 8(-160)$$

$$= 2000 - 1280$$

$$= 720$$

$$\det \begin{bmatrix} 1 & 0 & 8 \\ 0 & 25 & 0 \\ 8 & 0 & 84 \end{bmatrix} = 25 \cdot \det \begin{bmatrix} 1 & 8 \\ 8 & 84 \end{bmatrix} = 25 \cdot 20 = 500$$

$$\det \begin{bmatrix} 0 & 1 & 8 \\ 8 & 0 & 0 \\ 0 & 8 & 84 \end{bmatrix} = -8 \det \begin{bmatrix} 1 & 8 \\ 8 & 84 \end{bmatrix} = -8 \cdot 20 = -160$$

d) TRUE. Let the eigenvalues of $\underset{\sim}{A}$ be denoted by $\lambda_1, \dots \lambda_5$. $\underset{\sim}{A}^T$ will have the same eigenvalues as $\underset{\sim}{A}$. Furthermore the eigenvalues of both $\underset{\sim}{A}\underset{\sim}{A}^T$ & $\underset{\sim}{A}^T\underset{\sim}{A}$ will have the same eigenvalues, i.e $\lambda_1^2, \dots \lambda_5^2$.

Pf:

$$\underset{\sim}{A}\underset{\sim}{x} = \lambda \underset{\sim}{x} \quad (1)$$

$$\underset{\sim}{A}^T\underset{\sim}{x} = \lambda \underset{\sim}{x} \quad (2)$$

$$\underset{\sim}{A}^T\underset{\sim}{A}\underset{\sim}{x} = \mu \underset{\sim}{x}$$

$$\underset{\sim}{A}^T\lambda\underset{\sim}{x} = \mu x$$

$$\lambda \underset{\sim}{A}^T\underset{\sim}{x} = \mu \underset{\sim}{x}$$

$$\lambda \lambda \underset{\sim}{x} = \mu \underset{\sim}{x}$$

$$\lambda^2 \underset{\sim}{x} = \mu \underset{\sim}{x}$$

$$\therefore \mu = \lambda^2$$

Quiz 4 (2022)

1) a) FALSE. $\underset{\sim}{M}$ may have complex eigenvalue. Because $\underset{\sim}{M}^3 = \underset{\sim}{I}$, $\lambda^3 = 1$ $\forall$ eigenvalues λ of $\underset{\sim}{M}$. Therefore $\underset{\sim}{M}$ may have complex eigenvalues equal to $\sqrt[3]{1}$.

b) TRUE. If $\underset{\sim}{M}$ has a complex eigenvalue, M^{-1} will have the complement of that eigenvalue. When added together, the complex parts shall cancel out, leaving only the real part.

c) TRUE

d) FALSE.

2) a) TRUE. Because $\underset{\sim}{A}$ is skew-symmetric, it will have an eigen decomposition, due to it being non-defective. The Schur decomposition actually will be the same as the eigendecomposition

b) FALSE. When λ is an eigenvalue of $\underset{\sim}{A}$, the eigenvectors corresponding to λ are essentially the null space of $(\underset{\sim}{A} - \lambda \underset{\sim}{I})$.

Proof:

$$\underset{\sim}{A}\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

$$\underset{\sim}{A}\underset{\sim}{x} - \lambda \underset{\sim}{x} = \underset{\sim}{0}$$

$$(\underset{\sim}{A} - \lambda \underset{\sim}{I})\underset{\sim}{x} = \underset{\sim}{0} \Rightarrow \underset{\sim}{x} \text{ will be nullspace of } \underset{\sim}{A} - \lambda \underset{\sim}{I}.$$

Because the question states that the geometric multiplicity of λ is 3, it means that there are 3 linearly independent eigenvectors associated with λ. In other words, in the above equation, there exist 3 linearly independent vectors that can be substituted for $\underset{\sim}{x}$.

Therefore, the dimension of null space, i.e. $\dim(N(\underset{\sim}{A} - \lambda \underset{\sim}{I})) = 3$. 3 may not equal to $m-3$. Hence, the statement is false.

c) FALSE. $\hat{A} = \underline{Q}\,\underline{D}\,\underline{Q}^{-1}$ is called as unitary or orthogonal decomposition, and may not exist for all non-defective matrices.

d) Because $\underline{u}\underline{u}^T \in \mathbb{R}^{2\times 2}$ is rank-deficient, one eigenvalue of $\underset{\sim}{A}$ will be 0.

Another eigenvector of $\underset{\sim}{A}$ will be $\underset{\sim}{u}$ itself.

Proof. $A u = \lambda u$

$\underline{u}\underline{u}^T\underline{u} = \lambda u$

$(\underset{\sim}{u}^T\underset{\sim}{u})\,\underline{u} = \lambda \underset{\sim}{u}$ (shown)

$\lambda = u^T u = \|u\|_2^2$

Using this, let us obtain the respective eigenvalue of $\hat{A}$ using Rayleigh quotient:

$$r(\hat{u}) = \frac{\underline{u}^T\underline{A}\,\underline{u}}{\underset{\sim}{u}^T\hat{u}} = \frac{\underline{u}^T\underline{u}\,\underline{u}^T\underline{u}}{\underset{\sim}{u}^T\underline{u}} = \frac{\|u\|_2^2 \cdot \cancel{\|u\|_2^2}}{\cancel{\|u\|_2^2}} = \|u\|_2^2$$

Because $\|\hat{u}\|_2^2$ is always non-negative, and 0 is also non-negative, the statement is true.

3)a) TRUE. Because $\hat{A}$ is symmetric, and has all positive eigenvalues, $\underset{\sim}{A}$ is actually S.P.D.. Hence, $\underset{\sim}{x}^T\underline{A}\underline{x} \geq 0 \quad \forall\, x \in \mathbb{R}^m$.

The Rayleigh quotient, $r(x)$ is calculated as follows:

$$r(\underset{\sim}{x}) = \frac{x^T A x}{x^T x} = \frac{\hat{x}^T\hat{A}\,x}{\|\hat{x}\|_2^2}$$

Because the numerator & denominator are positive, $r(\hat{x})$ will also be positive.

c) TRUE. The eigenvalues of a skew symmetric matrix are purely imaginary or zero. Because we know that the matrix is non-singular, zero cannot be an eigenvalue. Furthermore, we also know that eigenvalues appear are complex conjugates for a real matrix. Hence, the product of eigenvalues will always be positive.

d) TRUE. After the completion of phase 1, there exists a similarity transformation between $\hat{A}$ & $\hat{M}$. Therefore, the eigenvalues of $\hat{A}$ & $\underline{M}$ will be the same, and have the same algebraic / geometric multiplicity.

4) a) Computation of $\hat{A}^T\hat{A} = n \times m \times n = n^2 m$

" " Cholesky decomposition : $n^3/3$

Total work $= mn^2 + n^3/3 = n^2(m + n/3)$

b) $2mn^2 - \frac{2n^3}{3} = 2n^2(m - n/3)$

Comparison b/t a) & b): $\cancel{n^2}(m+n/3) > 2\cancel{n^2}(m-n/3)$

$m + n/3 > 2m - 2n/3$

$-m > -3n/3$

$-m > -n$

$m < n$ (FALSE)

$\therefore$ a) is better than b)

c) MGS : $2mn^2$

$2m\cancel{n^2} > \cancel{n^2}(m+n/3)$

$2m > m + n/3$

$m > n/3$

$3m > n$ (TRUE) $\Rightarrow$ a) is better than c)

d) $2mn^2 + 11n^3 = n^2(2m + 11n)$

$\cancel{n^2}(m + n/3) > \cancel{n^2}(2m + 11n)$

$m + n/3 > 2m + 11n$

$-m > \frac{32n}{3}$

$m < -\frac{32n}{3}$ (FALSE) $\Rightarrow$ a) is better than d)

$\therefore$ a) is the best.

5) a)

Trefethen & Bau - Ch 24

24.1 a) $(A-\mu I)x = \beta x$

$Ax - \mu x = \beta x$

$\lambda x - \mu x = \beta x$

$(\lambda-\mu)x = \beta x$

$\therefore \beta = \lambda - \mu \Rightarrow$ TRUE

b) FALSE.

c) We know: $\overline{Ax} = \overline{\lambda x}$

$\bar{A}\bar{x} = \bar{\lambda}\bar{x}$

$A\bar{x} = \bar{\lambda}\bar{x}$

TRUE

d) TRUE.

$Ax = \lambda x$

$A^{-1} \cdot A^{-1} \cdot Ax = A^{-1} \cdot A^{-1} \cdot \lambda x$

$\cdot\ A^{-1}x = \lambda \cdot A^{-1}A^{-1}x$