

Numerical Linear Algebra

L21 - Eigenvalues & Eigenvectors - Gilbert Strang - MIT

Eigenvector: Vector which maintains direction even after being acted upon by the matrix.

- If $\lambda \neq 0$, \underline{A} is non-singular. The corresponding eigenvector will be part of \underline{A} 's null space.
- Consider a projection matrix. Only vectors in the column space of \underline{P} . This is because \underline{P} will bring a vector into its plane (and therefore changing its direction, if it does not exist in the column space of \underline{P} already). Hence, \underline{P} has only two eigenvalues: 0 or 1.
- Why is $\det(\underline{A} - \lambda \underline{I}) = 0$?

$$\underline{A}\underline{x} = \lambda \underline{x}$$

$$(\underline{A} - \lambda \underline{I})\underline{x} = 0$$

Because we know $\underline{x} \neq 0$, we know that $(\underline{A} - \lambda \underline{I})$ has a non-trivial nullspace.
Hence, $\det(\underline{A} - \lambda \underline{I}) = 0$.

→ If \underline{A} has eigenvalues $\lambda_1, \dots, \lambda_n$ & eigenvectors $\underline{x}_1, \dots, \underline{x}_n$, $\underline{A} + n \underline{I}$ will have eigenvalues $(\lambda_1 + n), \dots, (\lambda_n + n)$, and eigenvectors $\underline{x}_1, \dots, \underline{x}_n$. (Eigenvalues change, but eigenvectors remain the same.)

What if \underline{A} has eigenvalues $\lambda_1, \dots, \lambda_n$ & \underline{B} has eigenvalues $\alpha_1, \dots, \alpha_n$, does $(\underline{A} + \underline{B})$ have eigenvalues $(\lambda_1 + \alpha_1), \dots, (\lambda_n + \alpha_n)$?

$$\begin{aligned} \underline{A}\underline{x} &= \lambda \underline{x} \\ \underline{B}\underline{x} &= \alpha \underline{x} \\ (\underline{A} + \underline{B})\underline{x} &= (\lambda + \alpha) \underline{x} \end{aligned}$$

we are making an assumption here! we are thinking that \underline{x} is also an eigenvector of \underline{B} , and that may not be true!

rotate a vector by 90°

- What if \underline{A} is a rotation matrix? E.g. $\underline{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

→ Because no vector \underline{x} can be in the same direction after this rotation, no real

eigen vectors exist for such a matrix.

→ Hence, even real matrices can have complex eigenvalues.

→ If a matrix is anti-symmetric (i.e. $\tilde{A}^T = -\tilde{A}$), eigenvalues shall be purely complex.

→ For a triangular matrix, eigenvalues are the diagonal elements. Because $\det(\text{triangular matrix}) = \text{product of diagonal elements}$.

→ If \tilde{A} has eigenvalues $\lambda_1, \dots, \lambda_n$ & eigenvectors $\underline{x}_1, \dots, \underline{x}_n$,

→ \tilde{A}^2 will have eigenvalues $\lambda_1^2, \dots, \lambda_n^2$ & eigenvectors $\underline{x}_1, \dots, \underline{x}_n$:

$$\text{Pf: } \tilde{A}^2 \underline{x} = \tilde{A}(\tilde{A} \underline{x}) = \tilde{A}(\lambda \underline{x}) = \lambda (\tilde{A} \underline{x}) = \lambda (\lambda \underline{x}) = \lambda^2 \underline{x} \quad (\text{shown})$$

→ \tilde{A}^{-1} will have eigenvalues $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}$ & eigenvectors $\underline{x}_1, \dots, \underline{x}_n$:

$$\text{Pf: } \tilde{A}^{-1} \underline{v} = \tilde{A} \left(\frac{\underline{A} \underline{v}}{\lambda} \right) = \frac{\underline{v}}{\lambda} = \frac{1}{\lambda} \cdot \underline{v}$$
$$\begin{bmatrix} \underline{A} \underline{v} = \lambda \underline{v} \\ \underline{v} = \frac{\underline{A} \underline{v}}{\lambda} \end{bmatrix}$$

Defective Eigenvalues & Matrices

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$$\text{Let } \tilde{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

\tilde{A} & \tilde{B} have $\lambda=2$, with algebraic multiplicity 3.

For \tilde{A} , we can have 3 L.I. eigenvectors, i.e. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and geometric multiplicity = 3.

For \tilde{B} , we have only one eigenvector, i.e. \underline{e}_1 . Hence, the geometric multiplicity = 1.

→ An eigenvalue, whose algebraic multiplicity = geometric multiplicity is called

a defective eigenvalue.

→ A matrix that has at least one defective eigenvalue is a defective matrix, i.e. it does not possess a full set of m L.I. eigenvectors.

No eigen decomposition

→ Diagonal matrix is not defective.

→ Diagonalsability:

If $\tilde{A} \in \mathbb{R}^{m \times n}$ is not defective iff it has eigenvalue decomposition:

$$\tilde{A} = \tilde{X} \tilde{\Delta} \tilde{X}^{-1}, \text{ where } \tilde{X} \in \mathbb{R}^{n \times m}, \det(\tilde{X}) \neq 0.$$

If a): Given a diagonalsable matrix ($\tilde{A} = \tilde{X} \tilde{\Delta} \tilde{X}^{-1}$), \tilde{A} is non-defective

$\tilde{\Delta}$ & \tilde{I} are similar, hence they must have same eigenvalues (and corresponding geometric/algebraic multiplicity). Because $\tilde{\Delta}$ is not defective, \tilde{A} is also not defective.

$\tilde{\Delta}$ is a
diagonal
matrix

If b): Given \tilde{A} is not defective, \tilde{A} can be diagonalsed s.t $\tilde{A} = \tilde{X} \tilde{\Delta} \tilde{X}^{-1}$.

Let $\tilde{X} = \begin{bmatrix} \tilde{x}_1 & \dots & \tilde{x}_n \end{bmatrix}$ eigen vectors of \tilde{A}

$$\text{Then } \tilde{A} \tilde{X} = \begin{bmatrix} \tilde{x}_1 & \dots & \tilde{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \tilde{X} \tilde{\Delta}$$

$$\therefore \tilde{A} = \tilde{X} \tilde{\Delta} \tilde{X}^{-1}$$

Unitary (Orthogonal) Diagonalizability

→ Let $\tilde{Q} \in \mathbb{C}^{m \times m}$ is a unitary matrix if $\tilde{Q}^\perp \tilde{Q} = \tilde{Q} \tilde{Q}^\perp = I$

\tilde{Q}^\perp represents conjugate transpose of \tilde{Q} conjugate

If for a non-defective matrix $A \in \mathbb{R}^{m \times m}$, we define unitary diagonalizability if there a unitary matrix \tilde{Q} s.t $A = \tilde{Q} \tilde{\Delta} \tilde{Q}^{-1} = \tilde{Q} \tilde{\Delta} \tilde{Q}^\perp$.
* not all non-defective matrix are unitary diagonalizable

Symmetric Matrix & Eigenvalues & Eigenvectors

→ Asymmetric matrix has real eigenvalues & real eigenvectors.

If : A symmetric matrix has real eigenvalues :

$$\sum_{\tilde{x}} \tilde{x} = \lambda \tilde{x} \quad ①$$

$$\sum_{\tilde{x}} \tilde{x}^* = \lambda^* \tilde{x}^* \quad [\text{conjugate both sides}]$$

$$\sum_{\tilde{x}} \sum_{\tilde{x}} \tilde{x}^* = \lambda^* \sum_{\tilde{x}} \tilde{x}^* \tilde{x} \quad ② \quad [\text{pre-multiply } \tilde{x}^T \text{ on both sides}]$$

$$(\tilde{x}^*)^T \sum_{\tilde{x}} \tilde{x} = \lambda \tilde{x}^T \tilde{x} \quad ③$$

$$③ \Rightarrow (\sum_{\tilde{x}})^T \tilde{x}^* = \lambda \tilde{x}^T \tilde{x}$$

$$④ \Rightarrow (\sum_{\tilde{x}})^T \tilde{x}^* = \lambda^* \tilde{x}^T \tilde{x}$$

$$\mathcal{O} = (\lambda - \lambda^*) \underline{x}^* \underline{x}$$

$\lambda = \lambda^*$ $\Rightarrow \lambda$ is real valued

$(\lambda_1, \underline{x}_1)$ & $(\lambda_2, \underline{x}_2)$ are two eigenpairs, where $\lambda_1 \neq \lambda_2$

$$\underline{x}_1 = \lambda_1 \underline{x}_1$$

$$\underline{x}_2 = \lambda_2 \underline{x}_2$$

$$\underline{x}_1^T \underline{x}_2 = \lambda_1 \underline{x}_1^T \underline{x}_1$$

$$\underline{x}_1^T \underline{x}_2 = \lambda_2 \underline{x}_1^T \underline{x}_2$$

$$(\delta \underline{x}_2)^T \underline{x}_1 = \lambda_1 \underline{x}_2^T \underline{x}_1 \quad \textcircled{1}$$

$$(\delta \underline{x}_2)^T \underline{x}_1 = \lambda_2 \underline{x}_2^T \underline{x}_1 \quad \textcircled{2}$$

$$\textcircled{1} - \textcircled{2} \Rightarrow (\lambda_1 - \lambda_2) \underline{x}_1^T \underline{x}_2 = 0$$

$$\underline{x}_1^T \underline{x}_2 = 0 \quad [\lambda_1 \neq \lambda_2]$$

What happens if eigenvalues of $\underline{\Sigma}$ are degenerate? repeated eigenvalues

Let eigenvalue λ be repeated for $\underline{\Sigma} \in \mathbb{R}^{m \times m}$, where the geometric multiplicity is $r < m$. To prove that $\underline{\Sigma}$ is not defective, we need to show that algebraic multiplicity of $\underline{\Sigma}$ is also r .

Because geometric multiplicity of $\lambda = r$, we can find r L.I. vectors. We can use these to construct a r -dimensional subspace V^E .

$V^E = \{\underline{v}_1, \dots, \underline{v}_r\}$, and let $\underline{v}_1 \dots \underline{v}_r$ be orthogonal to each other.

Let us construct orthogonal vectors: $V^\perp = \{v_{r+1}, \dots, v_m\}$. v_{r+1}, \dots, v_m are orthogonal to each other, and V^\perp is orthogonal to V^F .

Let $\tilde{V} = \begin{bmatrix} \tilde{v}_1 & \dots & \tilde{v}_r & \tilde{v}_{r+1} & \dots & \tilde{v}_m \end{bmatrix}$, then:

$$\sum \tilde{v}_i = \sum_{j=1}^m c_{ji} \tilde{v}_j \quad [\sum \tilde{v}_i \text{ can be expressed as a linear combination}]$$

$$\therefore c_{ji} = \lambda \delta_{ji} + 1 \leq i \leq r$$

Also:

$$S \begin{bmatrix} \tilde{v}_1 & \dots & \tilde{v}_r & \tilde{v}_{r+1} & \dots & \tilde{v}_m \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{v}_1 & \dots & \tilde{v}_r & \tilde{v}_{r+1} & \dots & \tilde{v}_m \end{bmatrix} \left[\begin{array}{cccc|cc} \lambda & & & & c_{1,r+1} & c_{1,r+2} & \dots & c_{1,m} \\ & \ddots & & & \vdots & & & \vdots \\ 0 & & \lambda & & c_{r,r+1} & & & c_{r,m} \\ & & & \ddots & \vdots & & & \vdots \\ & & & & c_{m,r+1} & & & c_{m,m} \end{array} \right]$$



\tilde{M}

\tilde{M} will be symmetric

$$\text{Hence, } \tilde{V} = \tilde{V} \tilde{M} \Rightarrow \tilde{V}^T \tilde{V} = \tilde{M}$$

Hence:

$$\underbrace{V^T S V}_{\sim \sim \sim} = \left[\begin{array}{ccc|c} \lambda & \dots & 0 & 0 \\ 0 & \dots & \lambda & 0 \\ \hline 0 & & 0 & \zeta \end{array} \right], \text{ where } \zeta \text{ is a matrix with dimensions } (m-r) \times (m-r)$$

represent "characteristic polynomial of"

Hence $P_{\underbrace{V^T S V}_{\sim \sim \sim}}(z) = P_M(z) = P_\zeta(z)$ Because M is similar to ζ

$$\det(\zeta - z I_m) = (\lambda - z)^r \det(\zeta - z I_r)$$

We have to show that λ is not an eigenvalue of ζ . If λ is an eigenvalue of ζ , then:

$$\zeta u = \lambda u, \quad u \in \mathbb{R}^{m-r} \text{ eigenvector of } \zeta$$

$$\hat{u} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ u \end{bmatrix}$$

$$\underbrace{V^T S V}_{\sim \sim \sim} \hat{u} = \underbrace{M}_{\sim \sim} \hat{u} = \left[\begin{array}{ccc|c} \lambda & \dots & 0 & 0 \\ 0 & \dots & \lambda & 0 \\ \hline 0 & & 0 & \zeta \end{array} \right] \begin{bmatrix} 0 \\ \vdots \\ 0 \\ u \end{bmatrix} = \lambda \hat{u}$$

Hence λ is an eigenvalue of $\underbrace{V^T S V}_{\sim \sim \sim}$: $\underbrace{V^T S V}_{\sim \sim \sim} \hat{u} = \lambda \hat{u}$

$$\underbrace{S V}_{\sim \sim} \hat{u} = \lambda \underbrace{V^T}_{\sim} \hat{u}$$

By construction, $\underbrace{V^T}_{\sim} \hat{u}$ is an eigenvector of ζ corresponding to eigenvalue λ . It is a vector that lies in the space spanned by $\{v_{l+1}, \dots, v_m\}$. This means we are able to find another L.I. eigenvector $\underbrace{V^T}_{\sim} \hat{u}$ in addition to

what we had, i.e. $\{\underbrace{v_1 \dots v_r}\}$. This leads to a contradiction. Hence, λ cannot be an eigenvalue of $\underbrace{A - zI}_n$.

\Rightarrow Proven: The algebraic multiplicity = geometric multiplicity for a symmetric matrix. Hence, non-symmetric matrices are not defective.

Summary

\rightarrow A symmetric matrix $\underbrace{S} \in \mathbb{R}^{m \times m}$ is always non-defective and is orthogonally diagonalizable with real eigenvalues, i.e. $\underbrace{S = Q \Delta Q^T}$, where \underbrace{Q} is an orthogonal matrix & Δ is a diagonal matrix.

\rightarrow A skew-symmetric matrix $\underbrace{W} \in \mathbb{R}^{m \times m}$ is also non-defective and has purely imaginary eigenvalues. \underbrace{W} is also unitary diagonalizable: $\underbrace{W = Q \Delta Q^{-1}}$

\rightarrow Normal matrices: Any matrix $\underbrace{A} \in \mathbb{C}^{m \times m}$, s.t. $\underbrace{A^\perp A = A A^\perp}$

\rightarrow A matrix $\underbrace{A} \in \mathbb{C}^{m \times m}$ is unitary diagonalizable iff it is normal.



Schur Factorization

\downarrow SVD of $T =$ SVD of A^\dagger

\rightarrow A factorization of $\underbrace{A} \in \mathbb{C}^{m \times m}$ of the form: $\underbrace{A = Q T Q^\perp}$, where \underbrace{Q} is unitary, i.e. $\underbrace{Q^\perp Q = I}_n$, and \underbrace{T} is upper triangular.

\rightarrow Every square matrix admits for every square matrix (even for those of which EVD does not exist). This makes it useful in computations.

\downarrow orthogonal)

\rightarrow If $\underbrace{A} \in \mathbb{R}^{m \times m}$, a factorization with real matrices exists, where $\underbrace{A = V T V^T}$,

Where \tilde{L} is quasi upper triangular!

Either 1×1 blocks in the diagonal (normal UTM) or 2×2 blocks in the diagonal.

Usually occurs with complex eigenvalues!

→ Schur factorization need not be unique.

→ We can guarantee pure upper triangular for \tilde{L} , if we allow for complex arithmetic.

Summary of Decompositions

→ A diagonalization $A = \tilde{X} \tilde{\Delta} \tilde{X}^{-1}$ exists iff \tilde{A} is non-defective

$$A = Q \tilde{\Delta} Q^{-1}$$

→ " unitary triangularization called Schur factorization always exists for a q. matrix \tilde{L} of the form $A = Q \tilde{T} Q^{-1}$

→ All three of the above are basically similarity transformations. Hence, we know that A and $\tilde{\Delta}$ / \tilde{T} will have same eigenvalues, which is why we use these to get the eigenvalues of A .

Eigenvalue Algorithms

→ Solving very characteristic eqn. is unstable!

→ Two classes:

→ Direct: We can get ∞ precision in finite no. of steps

→ Iterative: $\infty \infty \infty \infty \infty \infty \infty$ iterations

→ For $m \leq 4$, we can have a direct method i.e. solving characteristic

eqns). It may be inaccurate, but it's direct.

→ For $m \geq 5$, we have no way to deterministically solve a degree- 5 polynomial eqn. Hence, we shall have to use iterative methods.

→ Monic polynomial: $p(z) = z^m + a_{m-1}z^{m-1} + \dots + a_1z + a_0$.

Roots of $p(z)$ are equal to eigenvalues of this matrix:

$$\tilde{A} = \begin{bmatrix} 0 & 0 & & -a_0 \\ 1 & 0 & & -a_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -a_{m-2} \\ 1 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & -a_{m-1} \end{bmatrix}$$

\tilde{A} is the " companion" matrix of $p(z)$

No algorithm can exactly produce all the roots of an arbitrary polynomial in a finite no. of steps.

→ Most eigen solvers reduce the matrix to Schur factorisation, where they can just read the eigenvalues from the diagonal of \tilde{T} , because of the similarity transformation that exists between \tilde{A} & \tilde{T} .

Schur triangulization: similar to householder triangularisation

$$Q_1^\perp \cdots \tilde{Q}_j^\perp \tilde{A} \tilde{Q}_1 \cdots \tilde{Q}_j = \tilde{T}$$

otherwise it will be a direct method!!!

We have to pre & post multiply to maintain the eigenvalues (similarity transformation)

If \tilde{A} is real & symmetric, \tilde{T} will be a diagonal matrix

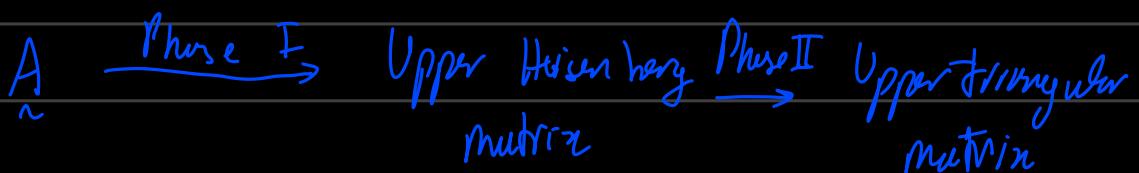
→ Two phases of eigenvalue computation:

1	4	2	3
3	4	1	7
0	2	3	4
0	0	1	3

→ Direct method to reduce \underline{A} to upper Hessenberg matrix

↳ UTM , last with m
addl. line of non-zero
entries parallel to
diagonal

→ An iterative method to produce a sequence of similarity transformations
that reduce \underline{A} to converge to UTM .



→ Phase I : $O(m^3)$ flops

→ " II : $O(m^3)$ "

→ $O(m)$ no. of iterations are reqd.

→ Each iteration requires $O(m^2)$

to reduce a Hessenberg matrix. Otherwise, we may have to reduce a full matrix, requiring $O(m^3)$ flops per iteration, thereby bringing the total TC to $O(m^4)$. This is why 2-phased soln. is employed.

Reduction to upper Hessenberg form using Householder

→ Householder transformation = $Q_1 A$

→ To preserve the eigenvalues, we need to post-multiply Q_1^T to $\underline{Q}_1 \underline{A}$ so
that there is a similarity transformation b/t $\underline{A} \Leftarrow \underline{Q}_1 \underline{A} \underline{Q}_1^T$. However,
doing this will result in $\underline{Q}_1 \underline{A} \underline{Q}_1^T$ becoming a dense matrix

again $\textcircled{2}$.

again this will result in determinate algo

→ Hence we cannot reduce \tilde{A} to a UTM / Diagonal matrix using this method.
Therefore, we use upper Hessenberg matrix:

$$\begin{bmatrix} - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{bmatrix} \xrightarrow{\text{(pre-multiply) } Q_1^\perp} \begin{bmatrix} - & - & - & - & - \\ - & - & - & - & - \\ 0 & - & - & - & - \\ 0 & - & - & - & - \\ 0 & - & - & - & - \end{bmatrix}$$

$$\begin{array}{c} \text{post-multiply} \\ \downarrow \end{array} \quad \begin{array}{c} \text{pre-multiply} \\ \downarrow \end{array} \quad \begin{array}{c} \text{Q}_2 \text{ (pre-multiply)} \\ \downarrow \end{array}$$

$$\begin{bmatrix} - & - & - & - & - \\ - & - & - & - & - \\ 0 & - & - & - & - \\ 0 & 0 & - & - & - \\ 0 & 0 & - & - & - \end{bmatrix} \xleftarrow{Q_2^\perp Q_2} \begin{bmatrix} - & - & - & - & - \\ - & - & - & - & - \\ 0 & - & - & - & - \\ 0 & - & - & - & - \\ 0 & - & - & - & - \end{bmatrix}$$

We repeat this process $(m-2)$ times, after which \tilde{A} shall be reduced to upper Hessenberg form:

$$\underbrace{Q_{m-2}^\perp \cdots Q_1^\perp}_{Q_\sim^\perp} \underbrace{A}_{Q_\sim} \underbrace{Q_1 \cdots Q_{m-2}}_{Q_\sim} = \tilde{H} \Rightarrow \tilde{A} = \underbrace{Q_\sim}_{\sim} \tilde{H} \underbrace{Q_\sim^\perp}_{\sim}$$

For a real symmetric matrix with $\lambda_1, \dots, \lambda_m$ real eigenvalues, and $\tilde{q}_1, \dots, \tilde{q}_m$ orthogonal eigenvectors.

If \underline{x} is some vector close to an eigenvector, we have to find α s.t. $\|\underline{A}\underline{x} - \alpha \underline{x}\|_2$ is minimised. This is a least sq. problem.

Hence,

$$\begin{aligned} & \underline{x}\alpha = \underline{A}\underline{x} \\ \underline{A}\rightarrow \underline{x} & \quad \underline{x}^T \underline{x} \alpha = \underline{x}^T \underline{A}\underline{x} \\ \underline{x} \rightarrow \alpha & \quad \therefore \boxed{\alpha = \frac{\underline{x}^T \underline{A}\underline{x}}{\underline{x}^T \underline{x}}} \\ \underline{b} \rightarrow \underline{A}\underline{x} & \quad \text{reduces quotient, very useful} \end{aligned}$$

→ This α is the natural eigenvalue estimate to consider if \underline{x} is approximately equal to an eigenvector.

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Given $\underline{x} \Rightarrow$ we can say $\underline{x} = \sum_{j=1}^m c_j \underline{v}_j$, where c_j 's are basically eigenvalues of \underline{A} .

→ If \underline{x} is close to one of the eigenvectors \underline{v}_j , then $\frac{|c_j|}{|c_i|} \leq \epsilon_j < 1$

Let $\epsilon = \max_j (\epsilon_j) + \epsilon_i$,

then

$$\frac{|c_j|}{|c_i|} < \epsilon \quad \forall j \neq i$$

estimated eigenvector
actual eigenvector

We can show that $|r(\underline{x}) - r(\underline{v}_j)| = O(\epsilon^2)$

(Exam question possibility !!)

H.W.: Prove $|r(\underline{x}) - r(\underline{v}_j)| = O(\epsilon^2)$ using the following:

$$r(\underline{x}) = \frac{\underline{x}^T \underline{A} \underline{x}}{\underline{x}^T \underline{x}} = \underline{x}^T \underline{A} \underline{x} = \sum_{j=1}^m c_j^2 \lambda_j^2$$

Hint: Estimate O.B. for $\left| \sum_{j=1}^m \zeta_j^2 \lambda_j^2 - \text{r}(A_S) \right| = O(\varepsilon^2)$

" λ : Use triangular inequality"

Eigenvalue Algorithms

Power iteration

by magnitude

→ Objective: Find the largest eigenvalue

→ Method:

Let $A \in \mathbb{R}^{m \times m}$, $v^{(0)}$ is a random vector

1) Get $v^{(0)}$

2) Get $\frac{\tilde{A} \tilde{v}^{(0)}}{\|\tilde{A} \tilde{v}^{(0)}\|}$

3) Get $\frac{\tilde{A} \tilde{v}^{(0)}}{\|\tilde{A} \tilde{v}^{(0)}\|}$ and then normalize it

Repeat this n times. Eventually we get $\frac{\tilde{A}^n \cdot \tilde{v}^{(0)}}{\|\tilde{A}^n \cdot \tilde{v}^{(0)}\|}$.

This approaches the longest eigenvector.

by magnitude

Proof: $\tilde{v}^{(0)} = \underbrace{a_1}_{\sim} \underbrace{q_1}_{\sim} + \dots + \underbrace{a_m}_{\sim} \underbrace{q_m}_{\sim}$

(Because $\tilde{v}^{(0)}$ is not an eigenvector, it should be having some components of q_1, q_2, \dots, q_m)

$$\tilde{A}^n \tilde{v}^{(0)} = a_1 \lambda_1^n \underbrace{q_1}_{\sim} + \dots + a_m \lambda_m^n \underbrace{q_m}_{\sim}$$

Let $|\lambda_1| > |\lambda_2| > \dots > |\lambda_m|$, then :

$$\underset{n}{\underbrace{A^n}} \underset{\sim}{v}^{(0)} = \alpha_1 \lambda_1^n \left(\underset{\sim}{q}_1 + \frac{\alpha_2}{\alpha_1} \cdot \left(\frac{\lambda_2}{\lambda_1} \right)^n \underset{\sim}{q}_2 + \dots + \frac{\alpha_m}{\alpha_1} \cdot \left(\frac{\lambda_m}{\lambda_1} \right)^n \underset{\sim}{q}_m \right)$$

Because $\left| \frac{\lambda_j}{\lambda_1} \right| < 1 \quad \forall j=1, \dots, m$, all these terms tend to 0 as $n \rightarrow \infty$.

$$\therefore \underset{n}{\underbrace{A^n}} \underset{\sim}{v}^{(0)} = \alpha_1 \lambda_1^n \underset{\sim}{q}_1$$

Hence $\underset{n}{\underbrace{v}}^{(n)} = \frac{\underset{n}{\underbrace{A^n}} \underset{\sim}{v}^{(0)}}{\| \underset{n}{\underbrace{A^n}} \underset{\sim}{v}^{(0)} \|} \rightarrow \underset{\sim}{q}_1 \text{ as } n \rightarrow \infty$

→ This method converges very slowly for repeated eigenvalues.

→ For this method $\underset{\sim}{v}^{(0)}$ must not be orthogonal to $\underset{\sim}{q}_j$. However, the chance of this is very low (because we are using a random vector).

Algo :

$$\text{Initialise } \underset{\sim}{v}^{(0)}, \text{ s.t. } \| \underset{\sim}{v}^{(0)} \| = 1$$

for $n=1 \rightarrow \infty$

$$\underset{\sim}{w} = \underset{n}{\underbrace{A}} \underset{\sim}{v}^{(n-1)}$$

$$\underset{\sim}{v}^{(n)} = \frac{\underset{\sim}{w}}{\| \underset{\sim}{w} \|}$$

$$\lambda^{(n)} = (\underset{\sim}{v}^{(n)})^T \underset{n}{\underbrace{A}} \underset{\sim}{v}^{(n)}$$

Result: Let $|\lambda_1| \geq |\lambda_2| \cdots \geq |\lambda_m|$ and $\tilde{q}_1^\top \tilde{v}^{(0)} \neq 0$, then iteration of the power iteration shall satisfy:

$$\|\tilde{v}^{(n)} - (\pm \tilde{q}_1)\| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^n\right)$$

$\tilde{v}^{(n)}$ tends to \tilde{q}_1 or $-\tilde{q}_1$ and

$$\text{ratio of } \left|\lambda^{(n)} - \lambda_1\right| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^{2n}\right)$$

Because of ratio of eigenvalues, error in eigenvalue is square of error in eigenvector

If n is even, $\tilde{v}^{(n)} \rightarrow \tilde{q}_1$
Else, $\tilde{v}^{(n)} \rightarrow -\tilde{q}_1$

Shortcomings of this method:

1) We are finding eigenvector corresponding to longest eigenvalue only. What about the remaining eigenvalues / eigenvectors?

2) Convergence is linear with error being reduced by $\left|\frac{\lambda_2}{\lambda_1}\right|$ at every iteration

\rightarrow if $\lambda_2 \approx \lambda_1$, convergence can be very slow.

Inverse Power Iteration

$\rightarrow \mu \in \mathbb{R}$ that is not eigenvalue of \tilde{A} , the eigenvectors of $(\tilde{A} - \mu \tilde{I})^{-1}$ are some of \tilde{A} . The corresponding eigenvalues are $\left\{\frac{1}{\lambda_j - \mu}\right\}_{j=1}^m$, where $\{\lambda_j\}$ are eigenvalues of \tilde{A} .

→ Suppose μ is close to λ_3 of \tilde{A} , then $\frac{1}{\lambda_3 - \mu}$ will be much larger
 then $\frac{1}{\lambda_j - \mu}$ if $j \neq 3$

→ Apply power iteration to $(\tilde{A} - \mu \tilde{I})^{-1}$, then the process converges to q_3

Algo:

- Initialise $\mu = \text{some value near } \lambda_3$,
- $\tilde{v}^{(0)} = \text{"random vector"}$
- for $n = 1 \rightarrow \infty$:
- $\tilde{w} = (\tilde{A} - \mu \tilde{I})^{-1} \tilde{v}^{(n-1)}$

We consider for \tilde{w} in $\tilde{v}^{(n)} = \frac{\tilde{w}}{\|\tilde{w}\|}$

$$(\tilde{A} - \mu \tilde{I}) \tilde{w} = \tilde{v}^{(n-1)} \quad \lambda^{(n)} = (\tilde{v}^{(n)})^T \tilde{A} \tilde{v}^{(n)}$$

Although this is badly conditioned, this does not matter so much due to the normalisation.

Suppose λ_3 is the closest eigenvalue to μ and λ_n is the second closest:

$$|\mu - \lambda_3| < |\mu - \lambda_n| \leq \dots \leq |\mu - \lambda_i|,$$

$\forall j \neq 3$ and $\tilde{q}_j^T \tilde{v}^{(0)} = 0$, then iterates of inverse iterations $\tilde{v}^{(n)}$ satisfy:

$$\|\tilde{v}^{(n)} - (\pm q_3)\| = O\left(\left|\frac{\mu - \lambda_3}{\mu - \lambda_n}\right|^n\right)$$

and

$$|\lambda^{(n)} - \lambda_3| = O\left(\left|\frac{\mu - \lambda_3}{\mu - \lambda_n}\right|^{2n}\right)$$

Summary:

- Power iteration: eigen vector estimate → eigen value estimate
 → Inverse power iteration: eigen value estimate → eigen vector estimate

Rayleigh quotient iteration

Algo:
 Initialize $\tilde{v}^{(0)}$ to some vector s.t $\|\tilde{v}^{(0)}\|_2 = 1$
 $\lambda^{(0)} = (\tilde{v}^{(0)})^\top \tilde{A} \tilde{v}^{(0)}$

for $n=1 \rightarrow \infty$

$$(\tilde{A} - \lambda^{(n-1)} \tilde{I}) \tilde{w} = \tilde{v}^{(n-1)}$$

$$\tilde{q}^{(n)} = \frac{\tilde{w}}{\|\tilde{w}\|}$$

$$\lambda^{(n)} = (\tilde{q}^{(n)})^\top \tilde{A} \tilde{q}^{(n)}$$

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Convergence is very fast! We can get eigenvalues accurate to 10 d.p. in 3-4 iterations.

Hence $\|\tilde{v}^{(n+1)} - (\pm q_j)\| = O(\|\tilde{v}^{(n)} - (\pm q_j)\|^3)$

and $|\lambda^{(n+1)} - \lambda_j| = O(|\lambda^{(n)} - \lambda_j|^3)$ Because we are relating the current & prev. iteration of the eigenvalue.

Operation counts

Let $A \in \mathbb{R}^{m \times m}$ be dense.

→ Each step of power iteration has one matrix-vector multiplication, thereby requiring

$O(m^2)$ flops.

- Each step of inverse power iteration requires a soln. of linear system of eqns, which has a cost of $O(m^3)$ flops.
 - We can solve for $(\tilde{A} - \mu \tilde{I})^{-1}$ once (outside the loop) because $(\tilde{A} - \mu \tilde{I})^{-1}$ is a constant. We can reduce the per-iteration time complexity to $O(m^2)$
- Rayleigh quotient iteration requires $O(m^5)$ time per operation, but it is still worth it because lesser iterations are reqd.
- * If \tilde{A} has been reduced to tridiagonal / upper Hessenberg, time complexity per-iterations shall be reduced to $O(m^2)$

Multiple Eigenvectors

Subspace iterations (Simultaneous iterations)

- Instead of one vector, we have multiple vectors which are linearly independent to each other.
- * If we have an ∞ precision computer, all of these vectors shall converge to different eigen vectors. Otherwise, they shall all converge to the "largest" eigenvector.

$$\text{Let } \tilde{V}^{(0)} = \begin{bmatrix} | & & | \\ V_1^{(0)} & \dots & V_n^{(0)} \\ | & & | \end{bmatrix}, \quad \tilde{V}^{(n)} = \begin{bmatrix} | & & | \\ \tilde{A}^{(n)} V^{(0)} & & V^{(0)} \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1^{(n)} & \dots & V_n^{(n)} \\ | & & | \end{bmatrix}$$

Let the eigenvectors of \tilde{A} be denoted by $\tilde{q}_1, \dots, \tilde{q}_n$, and a matrix \tilde{Q} is

$$\tilde{Q} = \begin{bmatrix} 1 & 1 \\ \tilde{q}_1 & \cdots & \tilde{q}_n \\ \tilde{1} & & \tilde{1} \end{bmatrix}$$

Proof: $V^{(0)} = \begin{bmatrix} 1 \\ \tilde{1} \end{bmatrix}$

$$= \tilde{Q} \tilde{Q}^T V^{(0)} \quad \left[\tilde{C} = \tilde{Q}^T \tilde{V}^{(0)} \right]$$

$$= \tilde{V}^{(0)} \quad (\text{shown})$$

Hence $V^{(k)} = \tilde{A}^k \cdot V^{(0)}$

$$= (\tilde{Q} \tilde{\Delta} \tilde{Q}^T)^k \cdot V^{(0)}$$

$$= \tilde{Q} \tilde{\Delta}^k \tilde{Q}^T \cdot V^{(0)} \quad [\text{Q \& Q}^T \text{ are orthogonal}]$$

$$= \tilde{Q} \tilde{\Delta}^k \tilde{Q}^T \cdot \tilde{Q} \tilde{C} \quad \left[V^{(0)} = \tilde{Q} \tilde{C} \right]$$

$$= \tilde{Q} \tilde{\Delta}^k \tilde{C}$$

"unwanted eigenvalues"

To extract first n eigenvalues:

$$\text{Let } \tilde{Q} = \begin{bmatrix} \tilde{Q}_1 & \tilde{Q}_2 \end{bmatrix}, \quad \tilde{Q}_1 = \begin{bmatrix} \tilde{q}_1 & \cdots & \tilde{q}_n \end{bmatrix}, \quad \tilde{Q}_2 = \begin{bmatrix} \tilde{q}_{n+1} & \cdots & \tilde{q}_m \end{bmatrix}$$

$$\text{Let } \tilde{\Delta} = \begin{bmatrix} \tilde{\Delta}_1 & 0 \\ 0 & \tilde{\Delta}_2 \end{bmatrix} \quad \text{and} \quad \tilde{C} = \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix}$$

$$\text{Hence, } \underline{V}_{\sim}^{(n)} = \underline{Q}_{\sim} \underbrace{\Delta_{\sim 1}^n}_{\sim} \zeta_1 + \underline{Q}_{\sim} \underbrace{\Delta_{\sim 2}^n}_{\sim} \zeta_2$$

$$= \left(\underline{Q}_{\sim} \underbrace{\Delta_{\sim 1}^n}_{\sim} + \underline{Q}_{\sim} \underbrace{\Delta_{\sim 2}^n}_{\sim} \zeta_2 \zeta_1^{-1} \right) \zeta_1 \quad [\text{Factor out } \zeta_1]$$

* Assumption: ζ_1 is invertible

$$= \left(\underbrace{\underline{Q}_{\sim} \Delta_{\sim 1}^n}_{\text{dominant spectrum}} + O\left(\|\lambda_{n+1}\|^n\right) \right) \zeta_1$$

\nearrow "unwanted" spectrum

As $n \rightarrow \infty$, we are magnifying the first term.

Result: Assumption #1: The first n eigenvalues are distinct and well-separated from others, i.e. $|\lambda_1| > |\lambda_2| \dots > |\lambda_n| > |\lambda_{n+1}| \geq \dots \geq |\lambda_m|$

#2: If $\underline{Q}_{\sim} = [\underline{q}_1 \dots \underline{q}_n] \in \mathbb{R}^{m \times n}$, $\underline{Q}_{\sim}^T \underline{V}_{\sim}^{(0)}$ is non-singular, but all principal submatrices of $\underline{Q}_{\sim}^T \underline{V}_{\sim}^{(0)}$ are also non-singular.

$$\underline{V}_{\sim}^{(n)} = \underline{A}_{\sim}^n \underline{V}_{\sim}^{(0)} \implies \underline{Q}_{\sim}^{(n)} \underline{R}_{\sim}^{(n)} = \underline{V}_{\sim}^{(n)}$$

QR factorization

Then as $n \rightarrow \infty$, the cols. of $\underline{Q}_{\sim}^{(n)}$ converges linearly to the eigenvectors of \underline{A}_{\sim} .

$$\left\| \underline{q}_{\sim j}^{(n)} - (\pm \underline{q}_{\sim j}) \right\| = O(C^n) \text{ for } j = 1 \rightarrow m, \text{ where}$$

$$C = \max_{1 \leq n \leq n} \left| \frac{\lambda_{n+1}}{\lambda_n} \right| < 1$$

This is unnormalised simultaneous Lanczos iterations, which is numerically unstable, due to loss of orthogonality.

Let $\tilde{V}_n^{(0)}$ be a set of random vectors, which are linearly independent.

$$\tilde{V}_n^{(0)} = \tilde{A} \tilde{V}_n^{(0)} = \begin{bmatrix} V_1^{(1)} & V_2^{(1)} & \dots & V_n^{(1)} \end{bmatrix}$$

$$= \tilde{Q}_n^{(1)} \tilde{R}_n^{(1)}$$

↑ This ensures that $\tilde{V}_1^{(1)}, \tilde{V}_2^{(1)}, \dots, \tilde{V}_n^{(1)}$ are orthonormal

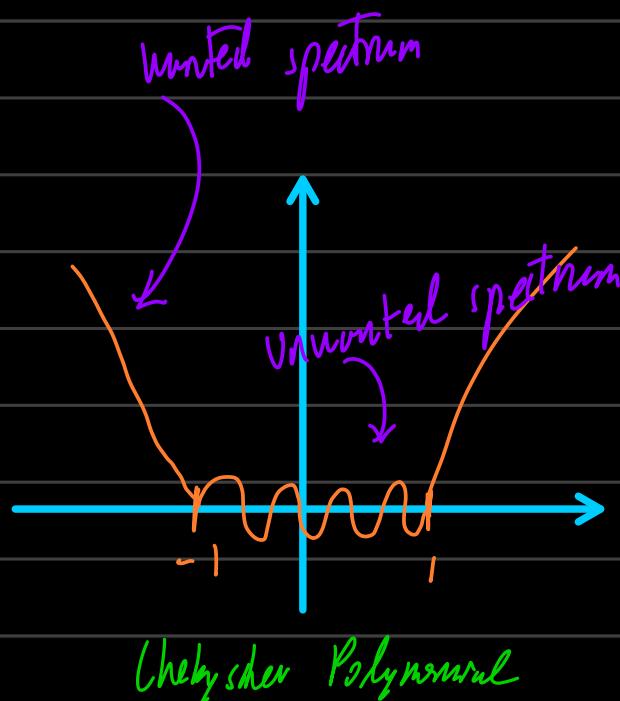
Hence, at every step we are orthonormalizing.

Alg: Choose $\tilde{Q}_n^{(0)} \in \mathbb{R}^{m \times n}$

for $n = 1 \rightarrow \infty$

$$\tilde{Z}_n^{(n)} = \tilde{A} \tilde{Q}_n^{(n-1)}$$

$$\tilde{Q}_n^{(n)} \tilde{R}_n^{(n)} = \tilde{Z}_n^{(n)}$$



We can construct \hat{A} from \tilde{A} such that \hat{A} has the wanted spectrum to the left of -1 , and the unwanted spectrum between -1 & 1 .

This is a practically applicable method for large sparse matrices.

QR Algorithm

→ Used mostly with dense matrices

Alg0: $\tilde{A}^{(0)} = A$ ↑ This is mathematically same as subspace iteration, but makes it better conditioned.

for $k = 1 \rightarrow \infty$

$$Q^{(n)} R^{(n)} = A^{(n-1)} \quad // \text{orthogonalise}$$

$$A^{(n)} = R^{(n)} Q^{(n)}$$

diagonalised

Eventually, for a real symmetric matrix, $A^{(n)}$ will approach Schur form, where eigenvalues can simply be read.

This shall also be useful in SVD, as SVD is essentially an eigenproblem.

Proof: $Q^{(n)} R^{(n)} = A^{(n-1)}$

$$R^{(n)} = (Q^{(n)})^T A^{(n-1)}$$

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Substituting, $A^{(n)} = R^{(n)} Q^{(n)}$
 $= (Q^{(n)})^T A^{(n-1)} Q^{(n)}$ (similarly transformation)

→ Simultaneous iteration is equivalent to QR algorithm.

If: Let $\tilde{Q}_R^{(n)} = \underbrace{Q^{(1)} \cdot Q^{(2)} \cdots Q^{(n)}}_{\sim} = \prod_{i=1}^n Q^{(i)} \sim$

" $\tilde{R}_R^{(n)} = \underbrace{R^{(n)} \cdot R^{(n-1)} \cdots R^{(1)}}_{\sim} = \prod_{i=k}^1 R^{(i)} \sim$

What happens to \tilde{A}^n ?

When $n=1$: $\tilde{A} = \tilde{A}^{(0)} = \underbrace{Q^{(1)}}_{\sim} \underbrace{R^{(1)}}_{\sim} = \underbrace{Q^{(1)} R^{(1)}}_{\sim \sim}$

" $n=2$: $\tilde{A}^2 = \underbrace{\tilde{A}}_{\sim} \cdot \underbrace{\tilde{A}}_{\sim} = (\underbrace{Q_R^{(1)} R_R^{(1)}}_{\sim \sim}) (\underbrace{Q_R^{(1)} R_R^{(1)}}_{\sim \sim})$

Simultaneous subspace iteration: Let $\tilde{Q}_\sim^{(0)} = I$

for $n = 1 \rightarrow \infty$:

$$Z = \tilde{A} \tilde{Q}_\sim^{(n-1)}$$

$$Z = \tilde{Q}_\sim^{(n)} \tilde{R}_S^{(n)}$$

$$\tilde{A}_\sim^{(n)} = (\tilde{Q}_\sim^{(n)})^\top \tilde{A} \tilde{Q}_\sim^{(n)}$$

$$\tilde{R}_S^{(n)} = R_S^{(n)} \circ R_T^{(n-1)} \circ \dots \circ R_S^{(1)}$$

What is \tilde{A}_\sim^n in simultaneous iteration?

$$\text{When } n=1: \quad \tilde{A}_\sim^1 = \tilde{A} \tilde{Q}_\sim^{(0)} = \tilde{Q}_\sim^{(1)} \tilde{R}_S^{(1)}$$

$$\text{When } n=2: \quad \tilde{A}_\sim^2 = \tilde{A} \left(\tilde{Q}_\sim^{(1)} \tilde{R}_S^{(1)} \right)$$

$$= \tilde{A} \tilde{Q}_\sim^{(1)} \tilde{R}_S^{(1)}$$

$$= \tilde{Q}_\sim^{(2)} R_S^{(2)} \tilde{R}_S^{(1)}$$

$$= \tilde{Q}_\sim^{(2)} \tilde{R}_S^{(2)}$$

$$\text{Hence, } \tilde{A}_\sim^n = \tilde{Q}_\sim^{(n)} \tilde{R}_S^{(n)} \quad \textcircled{3}$$

 QR factorization is unique for full-rank matrices

From eqn ① & ③, and assuming \tilde{A}_\sim^n is full-rank, we know that $\tilde{Q}_\sim^{(n)} = \tilde{Q}_R^{(n)}$ and $\tilde{R}_S^{(n)} = R_R^{(n)}$.

$$\text{We can also state that } \tilde{A}_\sim^n = (\tilde{Q}_\sim^{(n)})^\top \tilde{A} \tilde{Q}_\sim^{(n)}$$

$$\text{Hence, } \underbrace{(A^{-1})^n}_{\sim} P = \underbrace{\left(\bar{Q}^{(n)}\right)}_{\sim} \underbrace{\left(P\left(\bar{R}^{(n)}\right)^{-T}P\right)}_{\sim}$$

Therefore, at the n^{th} iteration, we can do the following to speed up convergence.

Full algo (modified QR): (most used algo in eigenvalues)

Define $\underbrace{A^{(0)}}_{\sim}$ s.t. $\underbrace{(Q^{(0)})^T}_{\sim} \underbrace{A^{(0)}}_{\sim} \underbrace{Q^{(0)}}_{\sim} = A$ $\| A^{(0)} \|$ is tridiagonalization of \underbrace{A}_{\sim}

for $n = 1 \rightarrow \infty$:

Pick a shift $\mu^{(n)}$ // there are many methods, e.g. $\mu^{(n)} = A_{mm}^{(n-1)}$

$$\underbrace{Q^{(n)} R^{(n)}}_{\sim} = \underbrace{A^{(n-1)}}_{\sim} - \mu^{(n)} I \quad \text{// shifted QR factorization}$$

$$\underbrace{A^{(n)}}_{\sim} = \underbrace{R^{(n)}}_{\sim} \underbrace{Q^{(n)}}_{\sim} + \mu^{(n)} I \quad \text{// deflating strategy}$$

If any off-diagonal entries is sufficiently close to 0, set $A_{i,j+1} = A_{j+1,j} = 0$

Split $\underbrace{A^{(n)}}_{\sim}$ s.t. $\underbrace{A^{(n)}}_{\sim} = \begin{bmatrix} \underbrace{A_1}_{\sim} & 0 \\ 0 & \underbrace{A_2}_{\sim} \end{bmatrix}$ // decoupling the matrix
 $\| A^{(n)} \|$ need not be tridiagonal

Apply QR algo on $\underbrace{A_1}_{\sim}$ & $\underbrace{A_2}_{\sim}$ // start again from tri-diagonalisation

So far, the discussion on eigenvalue algorithms is half-direct, half-iterative. Let us now discuss purely iterative algorithms.

Fully Iterative Methods

Advantages of fully iterative methods :

- It can be faster
 - If a matrix exceeds a particular size, we can devise a much more efficient eigensolver; especially if the matrix is sparse.
 - Based on subspace methods
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- subspace must be rich in the soln. that we are seeking. In this case, the subspace must be rich in eigenvectors. Such a subspace is called a Krylov subspace.
- Krylov subspace : $\tilde{A} \in \mathbb{R}^{m \times m}$, $b \in \mathbb{R}^m$. The Krylov sequence is the set of vectors $\tilde{b}, \tilde{A}\tilde{b}, \tilde{A}^2\tilde{b}, \dots$. A subspace generated in this sequence is a Krylov subspace.   Similar to power iterations

for this to be an actual subspace, $\tilde{b}, \tilde{A}\tilde{b}, \tilde{A}^2\tilde{b}$ must be L.I.. This is true when \tilde{A} is full rank, but may break down when \tilde{A} is not full rank

Constructing the subspace in this method is computationally unstable, because $\tilde{A}^k\tilde{b}$ approaches the largest eigenvector of \tilde{A} as $k \rightarrow \infty$.

- Arnoldi Iteration : A procedure to construct a Krylov subspace.

Phase I : $\tilde{A} = \tilde{Q} \tilde{H} \tilde{Q}^T$ ① // reduce \tilde{A} to upper Hessenberg form

$$\tilde{A} \tilde{Q} = \tilde{Q} \tilde{H} \quad ②$$

We need to break this into a iterative problem from a direct problem.

useful fashion.

Let $\theta_i = \{\text{eigenvalues of } \tilde{A}_n\}$ || these are called Arnoldi eigenvalue estimators (at step n)

|| these are also called or Ritz values



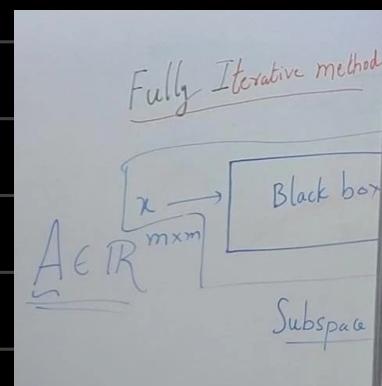
Arnoldi iterations can be viewed as a polynomial approximation

Let $x \in K_n$, where $K_n = \langle b, \tilde{A}b, \dots, \tilde{A}^{n-1}b \rangle$, then:

$$x = c_0 b + \dots + c_{n-1} \tilde{A}^{n-1} b$$

$$\text{Let } q(z) = c_0 + c_1 z + c_2 z^2 + \dots + c_{n-1} z^{n-1}.$$

$$\text{Hence } x = (q(A))b$$



Phase I

Arnoldi iteration

$$\tilde{A} = Q \tilde{H} Q^T \quad \text{--- ①}$$

$$\tilde{A}Q = Q \tilde{H} \quad \text{--- ②}$$

$$\tilde{A} \hat{Q}_n = \hat{Q}_{n+1} \tilde{H}_n \quad \boxed{\text{--- ③}}$$

$$\tilde{H}_n = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_m \\ 0 & h_{22} & \cdots & 0 \\ \vdots & \ddots & \ddots & h_{nn} \\ 0 & 0 & \cdots & h_{(n+1)(n+1)} \end{bmatrix}_{(n+1) \times n}$$

$$\hat{Q}_n = \begin{bmatrix} 1 & & & \\ q_1 & q_2 & \cdots & q_n \end{bmatrix}$$

$$\hat{Q}_{n+1} = \begin{bmatrix} 1 & & & \\ q_1 & q_2 & \cdots & q_{n+1} \end{bmatrix}$$

$$\begin{aligned} \tilde{A}q_1 &= h_{11}q_1 + h_{21}q_2 \\ \tilde{A}q_2 &= h_{12}q_1 + h_{22}q_2 + h_{32}q_3 \\ &\vdots \\ \tilde{A}q_n &= h_{1n}q_1 + h_{2n}q_2 + \cdots + h_{nn}q_n \end{aligned} \quad \text{--- ③}$$

Algo:-
 b = arbitrary
 $q_1 = \frac{b}{\|b\|}$
 for $n = 1, 2, 3, \dots$
 $v = Aq_n$
 for $j = 1 \dots n$
 $h_{jn} = q_j^T v$
 $v = v - h_{jn}q_j$
 $h_{n+1,n} = \|v\|$
 $q_{n+1} = \frac{v}{h_{n+1,n}}$

$$q_1 = \frac{b}{\|b\|} \quad K_n = \langle q_1, q_2, \dots, q_n \rangle = \langle b, Ab, \dots, A^{n-1}b \rangle$$

$$\begin{aligned} b &= \beta_1 q_1 + \beta_2 q_2 \\ &= \beta_1 \frac{b}{\|b\|} + \frac{\beta_2}{h_{21}} \left(Aq_1 - h_{11}q_1 \right) \\ &= \beta_1 \frac{b}{\|b\|} + \frac{\beta_2}{h_{21}} \left(\frac{Ab}{\|b\|} - h_{11} \frac{b}{\|b\|} \right) \\ &= b \underbrace{\left[\frac{\beta_1}{\|b\|} - \frac{\beta_2 h_{11}}{h_{21} \|b\|} \right]}_{d_1} + Ab \underbrace{\left[\frac{\beta_2}{h_{21}} \right]}_{d_2} \end{aligned}$$

Arnoldi approximation problem: Find $p^n \in P^n$
s.t. $\|p^n(\tilde{A})b\|_2$ is minimum.

* P^n is the set of all monic polynomials of degree n .

polynomials with coefficient of highest power = 1

Hence, we can restate the problem as: $\min_{P^n \in P^n} \|p^n(\tilde{A})b\|_2$

The solution to this problem is actually $P_{H_n}(z) = \det(zI - H_n)$

As we increase n , we shall essentially be trying to solve the characteristic polynomial of \tilde{A} .

$$q_1 = \frac{b}{\|b\|} \quad K_n = \langle q_1, q_2, \dots, q_n \rangle = \langle b, Ab, \dots, A^{n-1}b \rangle$$

$$\begin{aligned} b &= \beta_1 q_1 + \beta_2 q_2 \\ &= \beta_1 \frac{b}{\|b\|} + \frac{\beta_2}{h_{21}} \left(Aq_1 - h_{11}q_1 \right) \\ &= \beta_1 \frac{b}{\|b\|} + \frac{\beta_2}{h_{21}} \left(\frac{Ab}{\|b\|} - h_{11} \frac{b}{\|b\|} \right) \\ &= b \underbrace{\left[\frac{\beta_1}{\|b\|} - \frac{\beta_2 h_{11}}{h_{21} \|b\|} \right]}_{d_1} + Ab \underbrace{\left[\frac{\beta_2}{h_{21}} \right]}_{d_2} \end{aligned}$$

$$\begin{aligned} \tilde{A} \tilde{Q}_n &= \tilde{Q}_{n+1} \tilde{H}_n \\ \tilde{A}, \tilde{Q}_n &\in \langle q_1, q_2, \dots, q_n \rangle \\ \tilde{A}_p &= \tilde{Q}_n \tilde{Q}_n^T \tilde{A} \\ \tilde{A}_{p,j} &= q_j^T \tilde{A} q_j \\ \hat{A}_p &= \tilde{Q}_n^T (\tilde{Q}_n \tilde{Q}_n^T \tilde{A}) \tilde{Q}_n \\ &= \tilde{Q}_n^T \tilde{A} \tilde{Q}_n \end{aligned}$$

$$\begin{aligned} B_{ij} &= q_i^T B q_j \\ e_i^T B e_j & \end{aligned}$$

Projection of \tilde{A} onto $K_n = \langle q_1, \dots, q_n \rangle$

$$\hat{A}_p = \tilde{Q}_n^T (\tilde{A}) \tilde{Q}_n$$

$$\hat{A}_p = \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1n} \\ h_{21} & h_{22} & \dots & \vdots \\ \vdots & \ddots & \ddots & h_{nn} \end{bmatrix} = \boxed{H_n}$$

$$\begin{aligned} \tilde{A} \tilde{Q}_n &= \tilde{Q}_{n+1} \tilde{H}_n \\ \Rightarrow \tilde{Q}_n^T \tilde{A} \tilde{Q}_n &= \tilde{Q}_n^T \tilde{Q}_{n+1} \tilde{H}_n \\ &= \begin{bmatrix} 1 & & & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}_{n \times (n+1)} \tilde{H}_n \end{aligned}$$

$$K_n = \langle q_1, \dots, q_n \rangle$$

$$\underline{A} \hat{\underline{Q}}_n = \hat{\underline{Q}}_{n+1} \tilde{H}_n$$

$$\Rightarrow \hat{\underline{Q}}_n^T \underline{A} \hat{\underline{Q}}_n = \hat{\underline{Q}}_n^T \hat{\underline{Q}}_{n+1} \tilde{H}_n \\ = \begin{bmatrix} 1 & & & & 0 \\ 0 & \ddots & & & 0 \\ 0 & & \ddots & & 0 \\ \vdots & & & 1 & 0 \\ 0 & & & & \ddots \end{bmatrix}_{n \times (n+1)} \tilde{H}_n$$

Since $\tilde{H}_n (n \times n)$ is a projection of A onto K_n , there is a possibility that eigenvalues of \tilde{H}_n are related to eigenvalues of A in a useful fashion!

$\Omega_f = \{ \text{eigenvalues of } H_n \}$
are called Arnoldi eigenvalue estimates (at step n) or Ritz values

Arnoldi iteration \rightarrow Polynomial approximation!

$$x \in K_n, K_n = \langle b, Ab, \dots, A^{n-1}b \rangle$$

$$x = c_0 b + c_1 A b + c_2 A^2 b + \dots + c_{n-1} \underbrace{A^{n-1} b}_{\boxed{x = q(A) b}}$$

$$q(z) = c_0 + c_1 z + c_2 z^2 + \dots + c_{n-1} z^{n-1}$$

Algo:-

b = arbitrary

$$q_1 = \frac{b}{\|b\|}$$

for $n = 1, 2, 3, \dots$

$$v = A q_n$$

for $j = 1 \dots n$

$$h_{jn} = q_j^T v$$

$$v = v - h_{jn} q_j$$

$$h_{n+1,n} = \|v\|$$

$$q_{n+1} = \frac{v}{h_{n+1,n}}$$

Arnoldi approximation problem:-

Find $p^n \in P^n$ such that $\underbrace{\|p^n(A)b\|_2}$
is minimum!

P^n denotes the space of monic
of degree n \rightarrow polynomial

$$\min_{P^n \in P^n} \|p^n(A)b\|_2$$

$$p_{H^n}(z) = \det(zI - H_n)$$