Indian Institute of Science, Banglore
Department of Computational and Data Sciences (CDS)

DS284: Numerical Linear Algebra

Quiz 2

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Date: Sept 2, 2021 Duration 45 mins — Max Points: 20

Notations: (i) Vectors $\mathbf{v}$ and matrices $\mathbf{M}$ are denoted by bold faced lower case and upper case alphabets respectively. (ii) Set of all real numbers is denoted by $\mathbb{R}$ (iii) Set of all n dimensional vectors is denoted by $\mathbb{R}^n$ and set of all $m \times n$ matrices is denoted by $\mathbb{R}^{m\times n}$

1. Which of the following is/are true for matrix $\mathbf{H} \in \mathbb{R}^{m\times m}$, $\mathbf{H} = \mathbf{I} - 2\mathbf{q}\mathbf{q}^T$. where $\mathbf{q} \in \mathbb{R}^m$ is a unit vector, i.e, $||\mathbf{q}||_2 = 1$. (2 Point)

   (a) $\mathbf{H}$ always has a non-trivial null space.

   (b) $\mathbf{H}$ is always a symmetric matrix.

   (c) The action of $\mathbf{H}$ on a vector $\mathbf{p} \in \mathbb{R}^m$ orthogonal to $\mathbf{q}$ returns a vector with same magnitude but in the opposite direction of $\mathbf{p}$.

   (d) $\mathbf{H}^T\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^m$ is the vector of coefficients of expansion of $\mathbf{x}$ in the basis of columns of $\mathbf{H}$

2. Which of the following is/are NOT a valid norm for $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$? (1 Point)

   (a) $||\mathbf{x}||$ = Number of non-zero elements in $\mathbf{x}$

   (b) $||\mathbf{x}|| = (x_1 + x_2 + x_3)^2$

   (c) $||\mathbf{x}|| = |x_1^2 + x_2^2 + x_3^2|$

   (d) $||\mathbf{x}|| = (|x_1|^p + |x_2|^p + |x_3|^p)^{\frac{1}{p}}$ where $p = 0.5$

3. A lab scientist Mr. X has recorded some data from his experiment and it has 100 entries which are real numbers. He evaluated 3 properties T_1, T_2 and T_3 associated with this data. First, he computed the sum of the absolute values (T_1) and noted it to be 2.5, next he evaluated the square root of the sum of square of the entries (T_2) and found it to be 2.891, and then he computed the max entry in the data (T_3) and recorded it to be 1.2.
Mr. X has reported T_1, T_2, T_3 to his boss. If you were his boss, which of the following statements would you make on the data ? (2 Points)

   (a) Mr. X! Your recorded data is error free, since your data shows "T_1 to be less than $10T_2$" and "T_3 to be less than T_1 and T_2".

   (b) Mr. X! There is some problem with your recorded data , since your data shows "T_2 is greater than T_1".

   (c) Mr. X! There is some problem with your recorded data , since your data shows "T_3 is less than T_1 and T_2".

   (d) Mr. X! There is some problem with your recorded data , since your data shows "T_1 to be less than $10T_2$".

4. If set $\mathbb{Y}$ is a union of two sets $\mathbb{P}$ and $\mathbb{Q}$ then it contains elements of $\mathbb{P}$, elements of $\mathbb{Q}$, or elements from both $\mathbb{P}$ and $\mathbb{Q}$.
Now, $\mathbf{A} \in \mathbb{R}^{m\times m}$. Let $C(\mathbf{A})$ = Range($\mathbf{A}$), and $N(\mathbf{A})$ = Null($\mathbf{A}$)[non-trivial]. Let $\mathbb{S}$ be Union of $C(\mathbf{A})$ and $N(\mathbf{A})$. Which of the following is/are true? (2 Points)

   (a) There exists a non-zero vector in $\mathbb{R}^m$ that belongs to $\mathbb{S}$

   (b) There exists a non-zero vector in $\mathbb{R}^m$ that does not belong to $\mathbb{S}$

   (c) There can exist a non-zero vector in $C(\mathbf{A}^T)$ which belongs to $\mathbb{S}$

   (d) A non-zero vector in $C(\mathbf{A})$ can belong to $N(\mathbf{A})$

5. Let a unit vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$, $||v||_2 = 1$ be such that $\mathbf{Av} = \mathbf{0}$, where $\begin{bmatrix} \frac{5}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{bmatrix}$ then, $\sqrt{6}(|v_1| + v_2 + |v_3|)$ can be. (2 Points)

   (a) 4

   (b) 0

   (c) $4\sqrt{6}$

   (d) $\sqrt{6}$

6. Let $m, n \in \mathbb{N}, m < n$, $\mathbf{P} \in \mathbb{R}^{n\times m}$ and $\mathbf{Q} \in \mathbb{R}^{m\times n}$. Then which of the following is /are false? (1 Point)

   (a) $rank(\mathbf{PQ}) = n$

   (b) $rank(\mathbf{QP}) = m$

   (c) $rank(\mathbf{PQ}) = m$

   (d) $rank(\mathbf{QP}) = \lceil \frac{m+n}{2} \rceil$, where $\lceil x \rceil$ denotes the smallest integer $\geq x$

7. Let $\mathbb{W}_1$ and $\mathbb{W}_2$ be subspaces of the real vector space $\mathbb{R}^{100}$ where

   $\mathbb{W}_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_{100} \end{bmatrix} \mid x_i = 0 \text{ if } i \text{ is divisible by } 4 \right\}$ and

   $\mathbb{W}_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_{100} \end{bmatrix} \mid x_i = 0 \text{ if } i \text{ is divisible by } 5 \right\}$ then, dimension of $\mathbb{W}_1 \cap \mathbb{W}_2$ is

   (1 Point)

   (a) 75

   (b) 95

   (c) 60

   (d) 40

8. Let $\mathbf{M} \in \mathbb{R}^{4\times 3}$ be a non zero matrix, and let ($\mathbf{e}_1$,$\mathbf{e}_2$,$\mathbf{e}_3$) be the standard canonical basis of $\mathbb{R}^3$. Which of the following is/are true?

   Note: Standard canonical basis imply $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (1 Point)

   (a) if $rank(\mathbf{M}) = 1$, the vectors $\{\mathbf{Me}_1, \mathbf{Me}_2\}$ are always linearly independent.

   (b) if $rank(\mathbf{M}) = 2$, the vectors $\{\mathbf{Me}_1, \mathbf{Me}_2\}$ are always linearly independent.

   (c) if $rank(\mathbf{M}) = 2$, the vectors $\{\mathbf{Me}_1, \mathbf{Me}_3\}$ are always linearly independent.

   (d) if $rank(\mathbf{M}) = 3$, the vectors $\{\mathbf{Me}_1, \mathbf{Me}_3\}$ are always linearly independent.

9. For matrices $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times p}$, $\mathbf{C} \in \mathbb{R}^{p \times q}$ and vector $\mathbf{v} \in \mathbb{R}^n$. Which of the following is/are true? (2 Points)

   (a) $\mathbf{Av}$ has mn multiplications and $m(n-1)$ additions.

   (b) $\mathbf{Av}$ has mn multiplications and mn additions.

   (c) $\mathbf{AB}$ has mnp multiplications and $m(n-1)p$ additions.

   (d) $\mathbf{AB}$ has mnp multiplications and mnp additions.

10. We say a subspace $\mathbb{S}$ is an orthogonal subspace to $\mathbb{T}$, if every vector in $\mathbb{S}$ is orthogonal to every vector in $\mathbb{T}$.
    Which of the following are True? (2 Points)

    (a) Null($\mathbf{A}$) and Range($\mathbf{A^T}$) are orthogonal spaces

    (b) If $\mathbf{A} \in \mathbb{R}^{m \times n}, m > n$, there can exist a $\mathbf{B} \in \mathbb{R}^{n \times m}$ such that $\mathbf{AB} = \mathbf{I}$

    (c) If $\mathbf{A} \in \mathbb{R}^{m \times n}, m > n$ with rank $= n$, then $\mathbf{A}^T\mathbf{A}$ is a full rank matrix

    (d) $\mathbf{A} \in \mathbb{R}^{m \times m}$ satisfies $\mathbf{u}^T\mathbf{A}\mathbf{u} \neq 0 \ \forall \ \mathbf{u} \neq \mathbf{0}$, then $\mathbf{A}^2$ is a full rank matrix

11. In the below options, $f : \mathbb{R}^{m \times n} \longrightarrow \mathbb{R}^+$ is defined as the square root of sum of squares of all entries of a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$.
    $||.||_p$ for a vector represents p norm of a vector, and for a matrix, represents induced matrix p-norm.
    Then which of the following is/are true? (2 Points)

    (a) $||\mathbf{A}||_\infty = ||\mathbf{A}^T||_1$

    (b) $f(\mathbf{A} + \mathbf{B}) \leq f(\mathbf{A}) + f(\mathbf{B})$

    (c) $f(\mathbf{A})$ satisfies $f(\mathbf{AB}) \leq f(\mathbf{A})f(\mathbf{B})$

    (d) The function $g : \mathbb{R}^{m \times n} \longrightarrow \mathbb{R}^+$ that finds the absolute value of the largest element of $\mathbf{A} \in \mathbb{R}^{m \times n}$ is a valid matrix norm

12. Suppose we know for a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{A}^T\mathbf{A} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix, then which of the following is/are true for such a matrix $\mathbf{A}$? (2 Point)

    (a) $\mathbf{AA}^T$ is always an identity matrix

    (b) $\mathbf{A}^{-1}$ always exists

    (c) $\mathbf{Ax} = \mathbf{0}$ has only a trivial solution i.e., $\mathbf{x} = \mathbf{0}$

    (d) If angle between $\mathbf{x}$ and $\mathbf{y}$ is theta, then angle between $\mathbf{Ax}$ and $\mathbf{Ay}$ is also theta