NLA Final PYQs

2021

1) a) TRUE. If $\underset{\sim}{A}+\underset{\sim}{A}^T$ is positive definite, $\underset{\sim}{A}$ is non-singular.

Proof: $\underset{\sim}{A}+\underset{\sim}{A}^T$ is symmetric: $(\underset{\sim}{A}+\underset{\sim}{A}^T)^T = \underset{\sim}{A}^T + (\underset{\sim}{A}^T)^T$

$$= \underset{\sim}{A}^T + \underset{\sim}{A}$$
$$= A + A^T \quad \text{(proven)}$$

Hence $\underset{\sim}{A}+\underset{\sim}{A}^T$ is S.P.D. Therefore, we know that $\underset{\sim}{x}^T(\underset{\sim}{A}+\underset{\sim}{A}^T)\underset{\sim}{x} > 0 \quad \forall\, x \in \mathbb{R}^m$

$$\underset{\sim}{x}^T\underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{x}^T\underset{\sim}{A}^T\underset{\sim}{x} > 0$$

$$\underset{\sim}{x}^T\underset{\sim}{A}\underset{\sim}{x} + (\underset{\sim}{x}^T\underset{\sim}{A}^T\underset{\sim}{x})^T > 0$$

[$\underset{\sim}{x}^T\underset{\sim}{A}^T\underset{\sim}{x}$ is a number, and hence its own transpose]

$$\underset{\sim}{x}^T\underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{x}^T(\underset{\sim}{A}^T)^T(\underset{\sim}{x}^T)^T > 0$$

$$\underset{\sim}{x}^T\underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{x}^T\underset{\sim}{A}\underset{\sim}{x} > 0$$

$$2\underset{\sim}{x}^T\underset{\sim}{A}\underset{\sim}{x} > 0$$

$$\underset{\sim}{x}^T\underset{\sim}{A}\underset{\sim}{x} > 0$$

Hence, $\underset{\sim}{A}$ is also an S.P.D matrix. Therefore $\underset{\sim}{A}$ has to be non-singular.

b) <not part of syllabus>

c) FALSE.

$\underset{\sim}{F}$ is a Householder matrix. Hence, it will have a determinant of -1.

d) FALSE.

If a matrix $\underset{\sim}{Q}$ has n orthogonal columns, we can have m rows. For example, if $m = 3$, and $n = 1$, we can get a 3×1 valid matrix:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

e) FALSE.

$Ax = b$ will be an overdetermined system, due to it being full rank and $m > n$. In other words, there are more equations than unknowns. Therefore, there is no solution to $\hat{A}\hat{x} = \hat{b}$. $\hat{x}$ will be the least squares solution, which aims to minimise the residual, $\hat{r}$ in the 2-norm sense.

f) FALSE.

The n-dimensional Krylov subspace is a subspace within the eigenvectors of $\hat{A}$. The QR decomposition of K_n need not give the eigenvectors of $\hat{A}$. The columns of $\hat{Q}_n$ will just span the Krylov subspace.

2) a) $$\underset{\sim}{U} = \begin{bmatrix} -1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \\ -1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \end{bmatrix} \quad \underset{\sim}{\Sigma} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1/2 \end{bmatrix} \quad V = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\underset{\sim}{X} = \underset{\sim}{V}\underset{\sim}{\Sigma}\underset{\sim}{V}^T$$

$$= \begin{bmatrix} -1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \\ -1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} -1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \\ -1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \\ -1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \end{bmatrix}_{4\times 2} \begin{bmatrix} -\sqrt{2} & 0 \\ 0 & -1/2 \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1/2\sqrt{2} \\ 1 & 0 \\ 0 & 1/2\sqrt{2} \end{bmatrix} //$$

$$\|X\|_F = \sqrt{\sum_{i=0}^{m} \sum_{j=0}^{n} |a_{ij}|^2}$$

$$= \sqrt{1^2 + \left(\frac{1}{2\sqrt{2}}\right)^2 + 1^2 + \left(\frac{1}{2\sqrt{2}}\right)^2}$$

$$= \sqrt{2\,1/4}$$

$$= 3/2$$

$$= 1.5 //$$

We know that the maximum rank of a matrix of dimension $m \times n$ is $\min(m,n)$. Hence, the maximum rank of X can be 2. Because X has two linearly independent columns, the rank of X is 2.

b) Let X_1 be the rank-1 approximation of $\underset{\sim}{X}$:

$$X_1 = \sigma_1 \underset{\sim}{u_1} \underset{\sim}{v_1}^T$$

$$= \sqrt{2} \cdot \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{bmatrix}_{4\times1} \begin{bmatrix} -1 & 0 \end{bmatrix}_{1\times2}$$

$$= \sqrt{2} \cdot \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 0 \\ 1/\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

c) $$\underset{\sim}{C} = \underline{X}^T \hat{X} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \frac{1}{\sqrt{8}} & 0 & \frac{1}{\sqrt{8}} \end{bmatrix}_{2\times4} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{8}} \\ 1 & 0 \\ 0 & \frac{1}{\sqrt{8}} \end{bmatrix}_{4\times2}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 1/4 \end{bmatrix}_{2\times2}$$

Let us obtain eigenvalues of $\underset{\sim}{C}$. Because $\hat{C}$ is a diagonal matrix, the eigenvalues of $\underset{\sim}{C}$ are the diagonal values itself.

Hence, the eigenvalues of $\underset{\sim}{C}$ are 2 and $1/4$. Taking the square rt. of 2 & $1/4$, we get $\sqrt{2}$ & $1/2$, which are the singular values of $\underset{\sim}{X}$. (proven)

3) a) Because $\underset{\sim}{A}$ is symmetric, we know that $\underset{\sim}{A}$ is non-defective, and hence possesses a full set of linearly independent eigenvectors. Because no. of eigenvalues = no. of eigenvectors for a non-defective matrix, each eigenvalue will correspond to a particular eigenvalue.

Hence, two eigenvectors corresponding to different eigenvalues will be linearly independent.

b) —

c) We know that if λ is an eigenvalue of $\underset{\sim}{A}$, λ^k will be the eigenvalue of $\underset{\sim}{A}^k$.

Hence, $\rho(\underset{\sim}{A}^k) = \max(\lambda^k)$, where λ is an eigenvalue of $\underset{\sim}{A}$

$$= (\rho(A))^k$$

Taking the definition of vector induced matrix norm,

$$\|\underset{\sim}{A}\|_1 = \max\left(\frac{\|\underset{\sim}{A}\underset{\sim}{x}\|_1}{\|x\|_1}\right)$$

Hence, $\|A\|_1 \geq \dfrac{\|\underset{\sim}{A}\underset{\sim}{x}\|}{\|x\|_1}$

Let $\underset{\sim}{x}$ be an eigenvector (normalised) of A.

Hence, $\|\underset{\sim}{A}\|_1 \geq \|\underset{\sim}{A}\underset{\sim}{x}\|$

$\geq \|\lambda \underset{\sim}{x}\|$ $\quad [\underset{\sim}{A}\underset{\sim}{x} = \lambda \underset{\sim}{x}]$

$\geq \lambda \|\underset{\sim}{x}\|$ [λ is a constant, $\|\alpha \underset{\sim}{x}\| = \alpha\|\underset{\sim}{x}\|$]

$\geq \lambda$ [$\|x\| = 1$]

Hence, we know that $\|\underset{\sim}{A}\|_1 >$ all values of λ.

Therefore $\|A\|_1$ must be greater than the greatest value of λ.

$\therefore \|\underset{\sim}{A}\|_1 \geq \rho(\underset{\sim}{A})$

4) a) If μ is an eigenvalue of $\underset{\sim}{A}$, then $\det(\underset{\sim}{A} - \mu \underset{\sim}{I})$ is 0. In other words, if μ is an eigenvalue of $\underset{\sim}{A}$, then $(\underset{\sim}{A} - \mu \underset{\sim}{I})$ will have a non-trivial nullspace wherein it will not have a solution.

Hence, to rephrase $(\underset{\sim}{A} - \mu \underset{\sim}{I})\underset{\sim}{x} = \underset{\sim}{b}$ will have a solution iff μ is not an eigenvalue of $\underset{\sim}{A}$.

b) Because $\underset{\sim}{A}$ is symmetric, $\underset{\sim}{A}$ is non-defective and hence has an eigen decomposition, i.e.

$$\underset{\sim}{A} = \underset{\sim}{X} \underset{\sim}{\Lambda} \underset{\sim}{X}^T$$
$$= \sum_{j=1}^{M} \lambda_j \cdot \underset{\sim}{u}_j \underset{\sim}{u}_j^T$$

, where the colms. of X are the eigenvectors, and Λ is a diagonal matrix with eigenvalues as the diagonals

Substituting this, we get:

$$(\underset{\sim}{A} - \mu \underset{\sim}{I})x = b$$

$$\left(\sum_{j=1}^{M} \lambda_j \cdot \underset{\sim}{u}_j \underset{\sim}{u}_j^T - \mu \underset{\sim}{I}\right) x = b$$

$$x = b \left(\sum_{j=1}^{M} \lambda_j \cdot \underset{\sim}{u}_j \underset{\sim}{u}_j^T - \mu \underset{\sim}{I}\right)^{-1}$$

5)

6) a) We know that the singular values of a matrix $\underset{\sim}{A}$ are the square roots of the eigenvalues of the matrix $\underset{\sim}{A}\underset{\sim}{A}^T$.

Hence $\sigma_i = \sqrt{\lambda_i}$, where λ_i are the eigenvalues of $\underset{\sim}{A}\underset{\sim}{A}^T$. Because $\underset{\sim}{A}$ is symmetric $\underset{\sim}{A}^T = \underset{\sim}{A}$. Hence, $\underset{\sim}{A}\underset{\sim}{A}^T = A^2$. We also know that the eigenvalues of A^2 are the squares of the eigenvalues of A. Hence, their sq. roots are the eigenvalues of $\underset{\sim}{A}$ as well as the singular values of $\underset{\sim}{A}$.

We know that the two-norm of $\underset{\sim}{A}$ is the largest singular value of $\underset{\sim}{A}$. In this case, the two-norm of $\underset{\sim}{A}$ will be largest eigenvalue of A, or the spectral radius, $\rho(\underset{\sim}{A})$ of $\underset{\sim}{A}$.

b) $$|\underset{\sim}{x}^T \underset{\sim}{A}\underset{\sim}{x}| = \|\underset{\sim}{x}^T\|_2 \, \|\underset{\sim}{A}\underset{\sim}{x}\|_2 \qquad \|A\| \geq \frac{\|Ax\|}{\|x\|}$$

$$\leq \|\underset{\sim}{x}\|_2 \cdot \|\underset{\sim}{A}\underset{\sim}{x}\|_2$$

$$\leq \|\underset{\sim}{x}\|_2 \cdot \|\underset{\sim}{A}\|_2 \cdot \|\underset{\sim}{x}\|_2 \qquad \|\underset{\sim}{A}\underset{\sim}{x}\|_2 \leq \|\underset{\sim}{A}\|_2 \|\underset{\sim}{x}\|$$

$$\leq \|\underset{\sim}{A}\|_2 \, \|\underset{\sim}{x}\|_2^2$$

$$\leq \|\underset{\sim}{A}\|_2 \qquad (\underset{\sim}{x} \text{ is given to be a unit vector, } \|\underset{\sim}{x}\|_2 = 1)$$

c) Because $(\tilde{\lambda}, \tilde{\underset{\sim}{u}})$ is an eigenpair for $\tilde{\underset{\sim}{A}}$:

$$\tilde{\underset{\sim}{A}}\tilde{\underset{\sim}{u}} = \tilde{\lambda}\tilde{\underset{\sim}{u}}$$

$$(\underset{\sim}{A} + \delta\underset{\sim}{A})(\underset{\sim}{u} + \delta\underset{\sim}{u}) = (\lambda + \delta\lambda)(\underset{\sim}{u} + \delta\underset{\sim}{u})$$

$$\cancel{\underset{\sim}{A}\underset{\sim}{u}} + \delta\underset{\sim}{A}\underset{\sim}{u} + \underset{\sim}{A}\delta\underset{\sim}{u} + \delta\underset{\sim}{A}\,\delta\underset{\sim}{u} = \cancel{\lambda\underset{\sim}{u}} + \delta\lambda\underset{\sim}{u} + \lambda\,\delta\underset{\sim}{u} + \delta\lambda\,\delta\underset{\sim}{u}$$

$$\delta\underset{\sim}{A}\underset{\sim}{u} + \underset{\sim}{A}\delta\underset{\sim}{u} + \delta\underset{\sim}{A}\,\delta\underset{\sim}{u} = \delta\lambda\underset{\sim}{u} + \lambda\,\delta\underset{\sim}{u} + \delta\lambda\,\delta\underset{\sim}{u}$$

$$\underset{\sim}{u}^T(\delta A \underset{\sim}{u} + A\,\delta\underset{\sim}{u} + \delta A\,\delta\underset{\sim}{u}) = \underset{\sim}{u}^T(\delta\lambda\underset{\sim}{u} + \lambda\,\delta\underset{\sim}{u} + \delta\lambda\,\delta\underset{\sim}{u})$$

$$\underset{\sim}{u}^T\delta A\underset{\sim}{u} + \cancel{\underset{\sim}{u}^T A\,\delta\underset{\sim}{u}} + \cancel{\underset{\sim}{u}^T\delta A\,\delta\underset{\sim}{u}} = \delta\lambda\,\underbrace{\underset{\sim}{u}^T\underset{\sim}{u}}_{1} + \underbrace{\cancel{\lambda\underset{\sim}{u}^T\delta\underset{\sim}{u}}}_{0} + \underbrace{\cancel{\delta\lambda\,\underset{\sim}{u}^T\delta\underset{\sim}{u}}}_{0}$$

(Because $\underset{\sim}{u}$ is orthogonal to $\delta\underset{\sim}{u}$) (Because we assume $\underset{\sim}{u}$ to be unit eigenvector)

$$\therefore\ \underset{\sim}{u}^T\delta A\underset{\sim}{u} = \delta\lambda$$

Taking 2-norm on both sides: $\|\delta\lambda\|_2 = \|\underset{\sim}{u}^T\delta A\underset{\sim}{u}\|_2$

($\delta\lambda$ is a number) $|\delta\lambda| \leq \underbrace{\cancel{\|\underset{\sim}{u}^T\|_2}}_{1}\|\delta A\|_2\underbrace{\cancel{\|\underset{\sim}{u}\|_2}}_{1}$

$\leq \|\delta A\|_2$ (shown)

d) Relative condn. no. $= \dfrac{\text{Perturbation in output}}{\text{Perturbation in input}}$

$$= \frac{|\delta\lambda|}{|\delta\underset{\sim}{A}|}$$

$$\leq \frac{\|\underset{\sim}{A}\|}{\|\delta\underset{\sim}{A}\|}$$

$$\leq \frac{\|\underset{\sim}{A}\|}{\|\delta\underset{\sim}{A}\|}$$

2022

1) a) TRUE. If A & B have a similarity transformation between them, they shall have the same eigenvalues, but need not have the same eigenvectors. Hence, their row space need not be identical. However, in this case, because A&B are symmetric, they are not defective and hence possess a full set of linearly independent eigenvectors. These eigenvectors shall span $\mathbb{R}^n$. Hence, the row space of Q & $\bar{Q}$ will be the same.

b) FALSE. Because the range space & null space need not be orthogonal. It is true only for a special class of projectors, i.e. orthogonal projectors.

c) FALSE. It is $\hat{\kappa}^R \cdot O(\epsilon_{mach})$, where $\hat{\kappa}^R$ denotes the condn. no. of the problem.

d) TRUE. The similarity transformation b/t AB & BA is as follows:

$$AB = B^{-1}(BA)B$$ [B^{-1} exists because B is positive-definite]

e) FALSE. The value of the condn. no. of matrix $A = \frac{\sigma_1}{\sigma_n}$, where σ_1 = largest singular value of A and σ_n is the smallest eigenvalue of A.

Similarly, let $R = V\Sigma V^T$ be the s.V.D. of R.
Let the condn. no. of R be $\frac{\sigma_1}{\sigma_n}$.

$M = R^T R$

$= (V\Sigma V^T)^T (V\Sigma V^T)$

$= (V^T)^T \Sigma^T V^T V \Sigma V^T$

$= V \Sigma \cdot \Sigma \cdot V^T$

$= V \cdot \Sigma^2 \cdot V^T$

$= V \cdot \Sigma^2 V^{-1}$

The condition no. of a matrix can also be given as $\sqrt{\frac{\lambda_{max}}{\lambda_{min}}}$, where λ_{max} & λ_{min} denote the largest & smallest eigenvalues respectively.

Because $\underset{\sim}{M}$ & $\underset{\sim}{\Sigma}^2$ have a similarity transformation, their eigenvalues will the same. Because $\underset{\sim}{\Sigma}^2$ is a diagonal matrix, Σ^2 has eigenvalues equal to its diagonal values. Note that the diagonal values of Σ^2 are $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2$, where $\sigma_1, \dots, \sigma_n$ are singular values of $\underset{\sim}{R}$.

$$\therefore \kappa(\underset{\sim}{M}) = \frac{\lambda_{max}}{\lambda_{min}} = \frac{\sigma_1^2}{\sigma_n^2} = \left(\frac{\sigma_1}{\sigma_n}\right)^2 = (\kappa(\underset{\sim}{R}))^2$$

$$\therefore \kappa(\underset{\sim}{M}) \neq \kappa(\underset{\sim}{R})$$

f) FALSE. $\underset{\sim}{X}^T \underset{\sim}{A} \underset{\sim}{X}$ is S.P.D. iff X is full-rank.

If $\underset{\sim}{X}^T \underset{\sim}{A} \underset{\sim}{X}$ is S.P.D., it should follow, $\underset{\sim}{v}^T(\underset{\sim}{X}^T \underset{\sim}{A} \underset{\sim}{X})\underset{\sim}{v} > 0 \quad \forall\, \underset{\sim}{v} \neq \underset{\sim}{0}$.

$$\underset{\sim}{v}^T \underset{\sim}{X}^T \underset{\sim}{A} \underset{\sim}{X} \underset{\sim}{v} > 0$$

$$(\underset{\sim}{X}\underset{\sim}{v})^T \underset{\sim}{A} \underset{\sim}{X} \underset{\sim}{v} > 0$$

$$\underset{\sim}{y} \underset{\sim}{A} \underset{\sim}{y} > 0 \quad \text{which is true for all } \underset{\sim}{y} \neq \underset{\sim}{0}.$$

However, if X is rank-deficient, i.e. has a non-trivial nullspace, $\underset{\sim}{X}\underset{\sim}{v}$ may be $\underset{\sim}{0}$ even if $\underset{\sim}{v} \neq 0$.

g) TRUE. The matrix V_2 will span the null space of $\underset{\sim}{A}$. "All vector b satisfying $\underset{\sim}{A}\underset{\sim}{b} = \underset{\sim}{0}$" covers all vectors in the nullspace of $\underset{\sim}{A}$. Because V is orthogonal, V_2 will be orthogonal.

→ Skipped question 2: Absolutely stupid

3) a) $(\underset{\sim}{A} - \varepsilon_i \underset{\sim}{I})\underset{\sim}{t}_i = (\varepsilon_i \underset{\sim}{I} - \underset{\sim}{A})\underset{\sim}{x}_i^{(0)}$

$$\underset{\sim}{A}\underset{\sim}{t}_i - \varepsilon_i \underset{\sim}{t}_i = \varepsilon_i \underset{\sim}{x}_i^{(0)} - \underset{\sim}{A}\underset{\sim}{x}_i^{(0)}$$

$\underset{\sim}{A}(\underset{\sim}{t_i} + \underset{\sim}{x_i}^{(0)}) = \varepsilon_i(\underset{\sim}{x_i}^{(0)} + \underset{\sim}{t_i})$ $\quad x_i^{(0)} + t_i = x_i$

$\underset{\sim}{A}\underset{\sim}{x_i} = \varepsilon_i \underset{\sim}{x_i}$ (shown)

If ε_i is the exact eigenvalue, $(\underset{\sim}{A} - \varepsilon_i \underset{\sim}{I})$ shall have a non-trivial null space, hence the equation is not solvable.

2023

1) a) FALSE. If $\underset{\frown}{A}$ is a full rank $m \times n$ matrix, with $m >> n$, $\underset{\frown}{A}\underset{\frown}{x} = \underset{\frown}{b}$ forms an overdetermined system, which has no solution. $\hat{\underset{\frown}{x}}$ is the least squares solution, which minimises $\| \underset{\frown}{A} \hat{\underset{\frown}{x}} - \underset{\frown}{b} \|_2$

b) TRUE.

$\underset{\sim}{P}$ & $\underset{\sim}{Q}$ are basically row vectors of dimension $(m+1)$. Because $A \in \mathbb{R}^{(m+1)\times m}$, A has a non-trivial null space of dimension 1. Hence, all the row vectors which satisfy $\underset{\frown}{X}\underset{\frown}{A} = 0$ will be linearly dependent. Therefore, $\underset{\frown}{P}$ & $\underset{\frown}{Q}$ must be linearly dependent. This implies that $\underset{\frown}{P}\underset{\frown}{Q}^T \neq 0$.

c) TRUE. Such a matrix is simply one whose column space spans that of $\mathbb{R}^n - W$. We simply take linearly independent columns from this space and use it as columns of $\underset{\frown}{A}$. The remaining columns can be simply $\underset{\sim}{0}_n$.

d) TRUE. Both properties of orthogonal projectors are followed:

$$\underset{\frown}{E}\underset{\frown}{x} = \frac{(\underset{\frown}{I} + \underset{\frown}{F})\underset{\frown}{x}}{2}$$

$$\therefore \underset{\frown}{E} = \frac{\underset{\frown}{I} + \underset{\frown}{E}}{2}$$

$\underset{\frown}{E}^2 = \underset{\frown}{E}$?

$$\underset{\frown}{E}^2 = \frac{(\underset{\frown}{I} + \underset{\frown}{F})^2}{2^2} = \frac{\underset{\frown}{I}^2 + 2\underset{\frown}{F} + \underset{\frown}{F}^2}{4} = \frac{\underset{\frown}{I} + 2\underset{\frown}{F} + \underset{\frown}{I}}{4} = \frac{\underset{\frown}{I} + \underset{\frown}{F}}{2} = \underset{\frown}{E} \checkmark$$

$$\underset{\frown}{E}^T = \left(\frac{\underset{\frown}{I} + \underset{\frown}{E}}{2}\right)^T = \tfrac{1}{2}(\underset{\frown}{I}^T + \underset{\frown}{F}^T) = \tfrac{1}{2}(\underset{\frown}{I} + \underset{\frown}{F})$$

e) FALSE. The columns of $Q_n \cdots Q_1$ will approach eigenvectors of $\underset{\sim}{A}$.

f) TRUE. $\mathcal{K}_1 = \{\underset{\sim}{b}\}$. Hence, any vector $\underset{\sim}{x} \in \mathcal{K}_1$ must be a scalar multiple of $\underset{\sim}{b}$.

2) a) The projector that projects a vector onto a column space of any matrix $\underset{\sim}{A}$ is:

$$\underset{\sim}{P} = A(A^T \underset{\sim}{A})^{-1} \underset{\sim}{A}^T$$

Let $\underset{\sim}{Q}_{j-1} = \begin{bmatrix} | & & | \\ \underset{\sim}{q}_1 & \cdots & \underset{\sim}{q}_{j-1} \\ | & & | \end{bmatrix}$

$$\underset{\sim}{P} = \underset{\sim}{Q}_{j-1} (\underbrace{\cancel{\underset{\sim}{Q}_{j-1}^T \underset{\sim}{Q}_{j-1}}}_{}\overset{\underset{\sim}{I}}{})^{-1} \underset{\sim}{Q}_{j-1}^T$$

[$\underset{\sim}{Q}_{j-1}^T \underset{\sim}{Q}_{j-1} = \underset{\sim}{I}$ because Q_{j-1} has orthonormal columns]

$$= \underset{\sim}{Q}_{j-1} \underset{\sim}{Q}_{j-1}^T$$

b) The complementary projector $\bar{\underset{\sim}{P}}$ is $\underset{\sim}{I} - \underset{\sim}{P}$

$$\underset{\sim}{I} - \underset{\sim}{P} = \underset{\sim}{I} - \underset{\sim}{Q}_{j-1} \underset{\sim}{Q}_{j-1}^T$$

Proof of $(I - P)$ being symmetric:

$$(\underset{\sim}{I} - \underset{\sim}{P})^T = \underset{\sim}{I}^T - P^T$$
$$= \underset{\sim}{I} - (\underset{\sim}{Q}_{j-1} \underset{\sim}{Q}_{j-1}^T)^T$$
$$= \underset{\sim}{I} - \underset{\sim}{Q}_{j-1} \underset{\sim}{Q}_{j-1}^T$$
$$= \underset{\sim}{I} - \underset{\sim}{P} \quad \text{(shown)}$$
$$= \bar{\underset{\sim}{P}}$$

$$\therefore \tilde{\underset{\sim}{a}}_j = (\underset{\sim}{I} - \underset{\sim}{Q}_{j-1} \underset{\sim}{Q}_{j-1}^T)\, \underset{\sim}{a}_j$$

$$= \underset{\sim}{a}_j - \left[\sum_{i=0}^{j-1} \underset{\sim}{q}_i \underset{\sim}{q}_i^T\right] \cdot \underset{\sim}{a}_j$$

$$= \underset{\sim}{a_j} - \sum_{i=0}^{j-1} \underset{\sim}{q_i} \underset{\sim}{q_i}^T \underset{\sim}{a_j}$$

Because $\bar{a}_j$ is orthogonal to the space spanned by $\langle q_1, \dots, q_{j-1} \rangle$, but $\tilde{a}_j$ must be in the space spanned by $\langle q_1, \dots, q_j \rangle$, $\hat{a}_j$ must be in the vector space spanned by q_j. In other words, $\hat{a}_j$ is a scalar multiple of q_j.

3) a) If A has 1 as an eigenvalue,
$\det(A - 1 \cdot I)$ must be 0.

$\det(A - \lambda I)$
$\det(A - I)$

$\det(I + uv^T - I)$
$\det(uv^T)$
$\det(uv^T) = 0 \quad \forall\, m > 1$

If the eigenvalue 1 has multiplicity $m-1$, $\dim(\text{null}(A - I)) = m-1$

We know that $\dim(\text{range}(A - I)) = \dim(\text{range}(uv^T)) = 1$, and
$\dim(\text{range}(A - I)) + \dim(\text{null}(A - I)) = m$.

Hence, $\dim(\text{null}(A - I)) = m - \dim(\text{range}(A - I)) = m-1$ (shown)

b) u is an eigenvector of A:

$Au = \lambda u$

$(I + uv^T)u = \lambda u$

$u + uv^Tu = \lambda u$

$u(1 + v^Tu) = \lambda u$

$\lambda = 1 + v^Tu$
$= 1 + (v, u)$, where $(\cdot, \cdot)$ denotes inner prod.

c) $A A^{-1} = I$

$(I + uv^T)(I + \kappa uv^T) = I$

$\cancel{I} + \kappa uv^T + uv^T + \kappa uv^T uv^T = \cancel{I}$

$\kappa uv^T + uv^T + \kappa uv^T uv^T = 0$

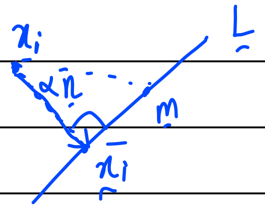
$$\kappa = -\frac{1}{1+v^Tu}$$

$$\therefore A^{-1} = I - \frac{1}{1+v^Tu} uv^T$$

d) If $v^Tu = -1$, $A^{-1} = I - \frac{uv^T}{1-1} = I - \frac{uv^T}{0}$ = D.N.E.

$\therefore$ A^{-1} is singular if $v^Tu = -1$.

4) a)



$\bar{x}_i = x_i + \alpha n$, $\alpha = -(x_i - m)^T n$

$\therefore \bar{x}_i = x_i - (x_i - m)^T n n$

b) $r = \|x_i - \bar{x}_i\|_2^2$

$= \|x_i - x_i + (x_i - m)^T nn\|_2^2$

$= \|(x_i - m)^T n n\|_2^2$

$= ((x_i - m) nn)^T ((x_i - m) nn)$

$= n^T n^T \underbrace{(x_i - m)^T (x_i - m)}_{\text{number}} nn$

$= (x_i - m)^T (x_i - m) \cancel{\overset{1}{n^T n^T}\, \overset{1}{nn}}$

$= (x_i^T - m^T)(x_i$

i) a) Backward stable : $\|\hat{f}(x) - f(x)\| = O(\varepsilon_m)$

$(A+\delta A)(w+\delta w) = v$

$\cancel{Aw} + \delta A w + \delta A \delta w + A\delta w = \cancel{v}$

$\delta A w + \delta A \delta w + A \delta w = 0$

$(\delta A + A)\delta w = -\delta A w$

$\delta w = -(\delta A + A)^{-1} \delta A w$ (shown)

b) $v = c_1 q_1 + \cdots + c_m q_m$

$w = A^{-1} v$

$= c_1 A^{-1} q_1 + \cdots + c_m A^{-1} q_m$

$= \dfrac{c_1 q_1}{\lambda_1} + \cdots + \dfrac{c_m q_m}{\lambda_m}$

$A q_i = \lambda_i q_i$

$q_i = \lambda_i A^{-1} q_i$

$A^{-1} q_i = \dfrac{q_i}{\lambda_i}$

Because $|\lambda_1| << |\lambda_2| \le |\lambda_3| \le \cdots \le |\lambda_m|$,

$\left\|\dfrac{c_1 q_1}{\lambda_1}\right\| >> \left\|\dfrac{c_2 q_2}{\lambda_2}\right\| \ge \left\|\dfrac{c_3 q_3}{\lambda_3}\right\| \ge \cdots \ge \left\|\dfrac{c_m q_m}{\lambda_m}\right\|$

hence, $w \simeq \dfrac{c_1 q_1}{\lambda_1}$

because q_1 is a unit vector, $q_1 \simeq \dfrac{w \lambda_1}{c_1} \simeq \dfrac{w}{\left(\frac{c_1}{\lambda_1}\right)} \simeq \dfrac{w}{\|w\|}$

c) $\delta w = -(A+\delta A)^{-1}(\delta A) w$

$= -\left(A^{-1} - A^{-1}(\delta A)A^{-1} + O(\varepsilon_m^2)\right)(\delta A) w$

$\simeq -A^{-1} \delta A w$

$= -A^{-1} z$ (let $z = \delta A w$)

$\simeq \kappa q_1$

$$\therefore \ \tilde{\underset{\sim}{w}} = \underset{\sim}{w} + \delta \underset{\sim}{w} \,\hat{=}\, \|\underset{\sim}{w}\| \underset{\sim}{q}_1 + \kappa \underset{\sim}{q}_1 \,\hat{=}\, (\|\underset{\sim}{w}\| + \kappa)\, \underset{\sim}{q}_1 \,//$$

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1) a) YES.

$$\underset{\sim}{Q}\,\underset{\sim}{Q}^T = \begin{bmatrix} \underset{\sim}{q}_1^T \underset{\sim}{q}_1 & \cdots & \underset{\sim}{q}_1^T \cdot \underset{\sim}{q}_m \\ \vdots & \ddots & \vdots \\ \underset{\sim}{q}_m^T \underset{\sim}{q}_1 & \cdots & \underset{\sim}{q}_m^T \underset{\sim}{q}_m \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 1 \end{bmatrix} = \underset{\sim}{I}_m$$

b) YES. An orthogonal projector is unique for a subspace, even if it is calculated using different bases.

c) TRUE. Eventhough this is an underdetermined system (less observations than no. of variables), there will exist infinite solutions to the problem.

d) ✓

e) $$n(x) = \frac{\dfrac{\|\delta f\|}{\|f\|}}{\dfrac{\|\delta x\|}{\|x\|}}$$

$$= \frac{\|\delta f\|\,\|x\|}{\|f\|\,\|\delta x\|}$$

$$= \frac{\|J(\underset{\sim}{x})\|\,\|\underset{\sim}{x}\|}{\|f\|} \qquad J(\underset{\sim}{x}) = \begin{bmatrix} \dfrac{\delta f}{\delta x_1} & \dfrac{\delta f}{\delta x_2} \end{bmatrix} = \begin{bmatrix} x_2 & x_1 \end{bmatrix}$$

$$= \frac{(|x_2|+|x_1|)(|x_1|+|x_2|)}{|x_1|\,|x_2|}$$

$$= \frac{2\cancel{|x_1|}\cancel{|x_2|}}{\cancel{|x_1|}\cancel{|x_2|}} + \frac{|x_1|^{\cancel{2}}}{\cancel{|x_1|}\,|x_2|} + \frac{|x_2|^{\cancel{2}}}{|x_1|\,\cancel{|x_2|}}$$

$$= 2 + \left|\frac{x_1}{x_2}\right| + \left|\frac{x_2}{x_1}\right|$$

$\therefore$ If $x_1 >> x_2$, or $x_2 >> x_1$, $\hat{\kappa}^R(x)$ will be large

f) TRUE. The rate of convergence of power iteration is decided by $\left|\frac{\lambda_2}{\lambda_1}\right|$, where λ_1 = largest eigenvalue (by magnitude), and λ_2 = second largest eigenvalue by magnitude.

2) a) $\underset{\sim}{X} = \underset{\sim}{U}\,\underset{\sim}{\Sigma}\,V^T$

$$= \underbrace{\begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}}_{4\times 2} \underbrace{\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}}_{2\times 2} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -\frac{1}{2\sqrt{2}} \\ -1 & 0 \\ 0 & -\frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2\sqrt{2}} \\ 1 & 0 \\ 0 & \frac{1}{2\sqrt{2}} \end{bmatrix}$$

$$\|\underset{\sim}{X}\|_F = \sqrt{\sigma_1^2 + \sigma_2^2} = \sqrt{2 + \tfrac{1}{4}} = \sqrt{\tfrac{9}{4}} = \tfrac{3}{2}$$

rank of $\underline{X}$ = $\dim(R(\underset{\sim}{X}))$ = no. of L.I. columns.

= 2

b) $\underset{\sim}{X}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 0 \end{bmatrix} = \underset{4\times 1}{\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}} \begin{bmatrix} -1 & 0 \end{bmatrix}_{1\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$

c) $\underset{\sim}{C} = \underset{\sim}{X}^T \underset{\sim}{X}$

$= \underset{2\times 4}{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1/\sqrt{8} & 0 & 1/\sqrt{8} \end{bmatrix}} \underset{4\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{8} \\ 1 & 0 \\ 0 & 1/\sqrt{8} \end{bmatrix}}$

$= \begin{bmatrix} 2 & 0 \\ 0 & 1/4 \end{bmatrix}$

eigenvalues of $\underset{\sim}{C} = 2, 1/4$

sq. roots of eigenvalues of $C = \sqrt{2}, 1/2$ = singular values of $\underset{\sim}{X}$ (shown)

3) a) Because the basis vectors are orthogonal (due to them being canonical), the vector has to have unique coefficients.

Let $\underset{\sim}{v} = \sum_{i=1}^{m} \alpha_i \underset{\sim}{b}_i = \sum_{i=1}^{m} \beta_i \underset{\sim}{b}_i$

$\underset{\sim}{v} - \underset{\sim}{v} = \sum_{i=1}^{m} (\alpha_i - \beta_i) \underset{\sim}{b}_i$

$0 = \sum_{i=1}^{m} (\alpha_i - \beta_i) \underset{\sim}{b}_i$

b) $\underset{\sim}{v} = \sum_{i=1}^{m} \alpha_i \underset{\sim}{b}_i$

$= \underset{\sim}{B} \underset{\sim}{\alpha}$

$\underset{\sim}{B} \underset{\sim}{\alpha} = \underset{\sim}{v}$

$\therefore \underset{\sim}{\alpha} = \underset{\sim}{B}^{-1} \underset{\sim}{v}$,

c) $\underset{\sim}{w} = \sum_{i=1}^{p} \beta_i \underset{\sim}{n_i}$ $\therefore$ $\underset{\sim}{N}\underset{\sim}{\beta} = \underset{\sim}{w}$

$= \underset{\sim}{N} \cdot \underset{\sim}{\beta}$ $\quad \underset{\sim}{\beta} = \underset{\sim}{N}^{-1}\underset{\sim}{w}$ cannot be done because N is rank deficient (singular)

Hence, we will use pseudo-inverse: $\underset{\sim}{\beta} = (\underset{\sim}{N}^T\underset{\sim}{N})^{-1}\underset{\sim}{N}^T\underset{\sim}{w}$

d) ✓

e) ✓

4) a) Because $\underset{\sim}{x_n} \in \underset{\sim}{K_n}$, $x_n = c_0\underset{\sim}{b} + c_1\underset{\sim}{A}\underset{\sim}{b} + \cdots + c_{n-1}\underset{\sim}{A}^{n-1}\underset{\sim}{b}$

$= q_n(\underset{\sim}{A})\underset{\sim}{b}$

$\therefore \underset{\sim}{r_n} = \underset{\sim}{b} - \underset{\sim}{A}\underset{\sim}{x_n}$

$= \underset{\sim}{b} - A \cdot q_n(\underset{\sim}{A})\underset{\sim}{b}$

$= (\underset{\sim}{I} - \underset{\sim}{A} \cdot q_n(\underset{\sim}{A}))\underset{\sim}{b}$

$= p_n(\underset{\sim}{A})\underset{\sim}{b}$

p_n has degree at most n because q_n has degree at most n.

$p_n(\underset{\sim}{0}) = \underset{\sim}{I} - \underset{\sim}{0} \cdot q_n(\underset{\sim}{0}) = \underset{\sim}{I}$ //

b) $\|r_1\| = \|p_1(\underset{\sim}{A})\underset{\sim}{b}\|$

$= \|(\underset{\sim}{I} - q_1(\underset{\sim}{A}))\underset{\sim}{b}\|$

$= \|(\underset{\sim}{I} - \cancel{c_0 b} - c_1\underset{\sim}{A}\underset{\sim}{b})\underset{\sim}{b}\|$

c) $x^T A_s x = x^T\left(\frac{1}{2}(A + A^T)\right)x$

$= \frac{1}{2} x^T (A + A^T) x$

$= \frac{1}{2}(x^T A x + x^T A^T x) = x^T A x$

$x^T A x + x^T A^T x = 2x^T A x$

$x^T A x = x^T A^T x$

5) a) Algorithm 1: for loop: n floating pt. ops.

Algorithm 2: inner for loop: $= \frac{n}{2} + \frac{n}{4} + \cdots + \frac{n}{2^{\log n}}$

$= n\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{\log n}}\right)$

$= n\left(\frac{\cancel{\frac{1}{2}}\left(1 - \frac{1}{2}^{\log_2 n}\right)}{\cancel{1 - \frac{1}{2}}}\right)$

$= n\left(1 - \frac{1}{n}\right)$

$= n - 1$

b) Algo 1 has n flops, thereby having num. error $= O(n \cdot \varepsilon_m)$

Algo 2, because it is does a pairwise summation, and there are $\log_2 n$ pairs, results in $O(\log_2 n \cdot \varepsilon_m)$ num. error.

Hence, algo 2 is more accurate.