Indian Institute of Science Banglore
Department of Computational and Data Sciences (CDS)

DS284: Numerical Linear Algebra
Mid-semester Exam 2022

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Duration: 10:05 AM to 11:20 AM   Max Points: 50

Notations: Vectors and matrices are denoted below by bold faced lower case and upper case alphabets respectively. A matrix is said to be rank deficient if it is not full rank.

- Additional sheets beyond the answer booklet, if required, will be provided.
- You must return the question paper along with the exam booklet to the TAs proctoring the exam.
- The exam is an open book and open internet exam. You will not be allowed to use phones during the exam.

Problem 1 [5x3=15 points]

Assert if the following statements are True or False. Give detailed reasoning for your assertion. Marks will be awarded only for your reasoning.

(a) If $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^n$ $(n > 2)$ are two non-zero, non-parallel vectors, the determinant of the matrix $\mathbf{A} = \mathbf{u}\mathbf{v}^T + \mathbf{v}\mathbf{u}^T$ is always zero.

(b) Consider the system of equations $\mathbf{P}\mathbf{x} = \mathbf{b}$ and $\mathbf{P}^T\mathbf{P}\mathbf{x} = \mathbf{P}^T\mathbf{b}$, where $\mathbf{P} \in \mathbb{R}^{m\times n}$ with $m > n$ and $\mathbf{b} \in \mathbb{R}^m$. Assert the statements: (a) Both systems of equations do not always have a solution. (b) Both systems of equations have a unique solution if at all they have a solution. (c) Both systems of equations always have the same solution if they have a solution.

(c) Given a matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ one can always find a matrix $\mathbf{B} \in \mathbb{R}^{m\times m}$ such that $\mathbf{AB} - \mathbf{BA} = \mathbf{I}$.

(d) For a matrix $\mathbf{F} = \mathbf{I}_m - 2\mathbf{q}\mathbf{q}^T$, where $\mathbf{q} \in \mathbb{R}^m$ and m is an odd integer. Then the determinant of $\mathbf{F}$ is 1.

(e) Let $fl : \mathbb{R}^+ \to \mathbb{F}$ be a function that takes positive real numbers $x \in \mathbb{R}^+$ as input and returns the **nearest** floating point number approximation $x' \in \mathbb{F}$ (Assume that all inputs are representable as normalized floating point numbers and there is no risk of overflow/underflow in any of the operations). Then for $a, b, c \in \mathbb{R}^+$, $a + b = c \implies fl(fl(a) + fl(b)) = fl(c)$.

Problem 2 [2+1+4 =7 points]

Consider three vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

(a) Construct two orthonormal vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ such that they span the successive column subspaces of the matrix $\mathbf{A} = [\mathbf{a}_1\ \mathbf{a}_2]$

(b) Does the vector $\mathbf{v}$ lie in the space spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$? Explain.

(c) If the answer for (b) is yes, then solve for the linear combination coefficients of $\mathbf{a}_1$ and $\mathbf{a}_2$, when $\mathbf{v}$ is expressed as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. If the answer for (b) is no, then compute the projection of $\mathbf{v}$ onto the space spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$ and thereby compute the solution $\mathbf{x}$ which minimizes $||\mathbf{Ax} - \mathbf{v}||_2$

Problem 3 [5+3=8 points]

Suppose you are confronted with solving a linear system of equations $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is a symmetric non-singular matrix. Further, you would like to use an iterative solver to solve this system of equations as the matrix size is huge.

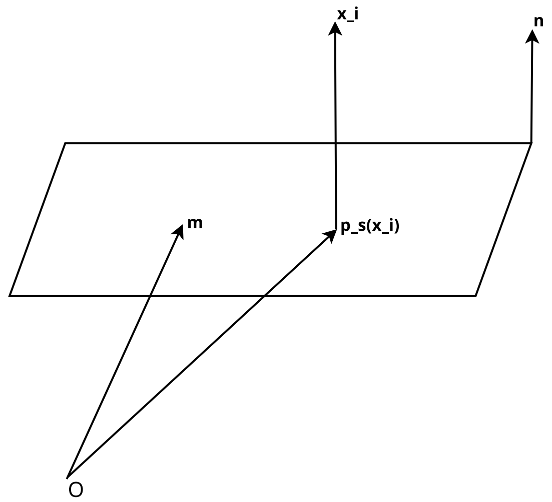
Remember that an iterative solver starts with some initial guess $\mathbf{x}_0$ and tries to seek sequence of approximations to $\mathbf{x}^*$ (the exact solution of $\mathbf{Ax} = \mathbf{b}$), in an iterative fashion so that the sequence of iterates $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n, \ldots)$ converge to $\mathbf{x}^*$, within a given tolerance for large n. You usually specify the tolerance ε_{tol} on the norm of the relative residual $\varepsilon = \frac{||\mathbf{r}||}{||\mathbf{b}||} = \frac{||\mathbf{b}-\mathbf{Ax}||}{||\mathbf{b}||}$ for convergence hoping that $\mathbf{x}_i$ at i^{th} iteration will be close enough to $\mathbf{x}^*$ if ε_{tol} is small enough. For the problem at hand, you would like to achieve a norm of relative error in the solution $\eta = \frac{||\mathbf{x}_i-\mathbf{x}^*||}{||\mathbf{x}^*||}$ to be $\eta_{ach} = 10^{-6}$ and for this you specify ε_{tol} in your iterative solver to be 10^{-12}. Another piece of information you know about the matrix $\mathbf{A}$ is that its 2-norm is 10 and its smallest eigenvalue is 10^{-7}. Now answer the following

(a) Derive a relationship between ε and η in terms of property of the matrix $\mathbf{A}$.

(b) Based on the result derived in (a) and the other information you have about $\mathbf{A}$, will you achieve $\eta_{ach} = 10^{-6}$ if you prescribe $\varepsilon_{set} = 10^{-12}$

Problem 4 [4+3+3+2+3+5 =20 points]

You have n data points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^3$ corresponding to some measurements obtained in an experiment. The following exercise seeks you to derive an expression for the best fit 2-D plane $\mathbb{S}$, which minimizes the sum of squares of orthogonal distances from each of the n data points to the best-fit plane. In other words, we are seeking to minimize the orthogonal fitting errors.



(a) Assume $\mathbf{m} \in \mathbb{R}^3$ be a point on the plane $\mathbb{S}$. Let $\mathbf{n}$ be the unit vector in the direction orthogonal to the candidate plane $\mathbb{S}$. If $P_S(\mathbf{x}_i)$ is the point on this plane which is the orthogonal projection of $\mathbf{x}_i$ onto the plane $\mathbb{S}$, derive an expression for $P_S(\mathbf{x}_i)$ in terms of $\mathbf{m}$, $\mathbf{x}_i$ and $\mathbf{n}$. Write this expression for $P_S(\mathbf{x}_i)$ in terms of the orthogonal projector $\mathbf{I} - \mathbf{n}\mathbf{n}^T$ (Hint: use the fact that any vector on the plane $\mathbb{S}$ is orthogonal to $\mathbf{n}$)

(b) Let $\mathbf{B} \in \mathbb{R}^{3\times 2}$ be a matrix whose column vectors $\mathbf{b}_1, \mathbf{b}_2$ span the 2-dimensional vector space orthogonal to $\mathbf{n}$. Derive the expression for $\mathbf{n}\mathbf{n}^T$ in terms of the matrix $\mathbf{B}$ and hence write the final expression for $P_S(\mathbf{x}_i)$ in terms of $\mathbf{x}_i$, $\mathbf{m}$ and the matrix $\mathbf{B}$.

(c) Recall that our objective is to find the best-fit plane that minimizes the sum of squares of the Euclidean distances between each data point and its corresponding orthogonal projection onto the candidate plane $\mathbb{S}$. Pose this problem mathematically using the expression of $P_S(\mathbf{x}_i)$ derived in (b).

(d) Note that the minimization problem posed in (c) has to be minimized with respect to the point $\mathbf{m} \in \mathbb{R}^3$ and the orthogonal matrix $\mathbf{B} \in \mathbb{R}^{3\times 2}$. Using the fact that optimal $\mathbf{m}$ for any given $\mathbf{B}$ is of the form $\mathbf{m}^* = \frac{1}{n}\sum \mathbf{x}_i$, rewrite the minimization problem in (c) in terms of $\widetilde{\mathbf{x}}_i = \mathbf{x}_i - \mathbf{m}^*$

(e) Rewrite the minimization problem in terms of the matrix $\widetilde{\mathbf{X}} \in \mathbb{R}^{n\times 3}$ such that $\widetilde{\mathbf{X}} = [\widetilde{\mathbf{x}}_1\ \widetilde{\mathbf{x}}_2\ \ldots\ \widetilde{\mathbf{x}}_n]^T$ (Hint: Use the definition of Frobenius norm of a matrix in terms of the columns of the matrix)

(f) Comment about the minimization problem obtained in (e) by exploring the connections to the low-rank approximation of the given data matrix $\widetilde{\mathbf{X}}$ and thereby deduce that the minimizer $\mathbf{B}^*$ has to be related to the singular vectors of $\widetilde{\mathbf{X}}$.