Conditioning and Stability

→ Conditioning pertains to sensitivity of a mathematical problem to perturbations in inputs.

→ Stability pertains to perturbation behavior of an algorithm used to solve the mathematical problem on a computer.

(*) Conditioning of a problem :-

Solving a problem is like evaluating a function

$$\underset{\sim}{y} = f(\underset{\sim}{x})$$

Here $\underset{\sim}{x}$ represents input to the problem (the data), $\underset{\sim}{f}$ represents problem itself and $\underset{\sim}{y}$ represents its solution

* What happens to $\underset{\sim}{y}$ when given $\underset{\sim}{x}$ is perturbed slightly?

If small changes in $\underset{\sim}{x}$ leads to large changes in $\underset{\sim}{y}$, we say the problem is ill-conditioned and usually we are interested in solving well-conditioned problem.

* Absolute condition number:-

If a small perturbation of $\underset{\sim}{x}$ is denoted by $\delta \underset{\sim}{x}$, then let the resulting perturbation in the solution be represented as δf

i.e $\delta f = f(\underset{\sim}{x} + \delta \underset{\sim}{x}) - f(\underset{\sim}{x})$

then the absolute condition number $\hat{k} = K(x)$ of the problem f at x is given by

$$K(x) = \max_{\delta x} \left(\frac{\|\delta f\|}{\|\delta x\|} \right) \quad - (1)$$

for infinitesimally small $\underset{\sim}{\delta f}$ and $\underset{\sim}{\delta x}$

If $\underset{\sim}{f}$ has a derivative, we can evaluate the Jacobian matrix $\underset{\sim}{J}(x)$ as

$$J_{ij} = \frac{\partial f_i}{\partial x_j}$$

We have $\underset{\sim}{\delta f} \approx \underset{\sim}{J}(x)\, \underset{\sim}{\delta}(x)$ with equality as $\|\delta x\| \to 0$

$$K(x) = \max_{\delta x} \frac{\|J(x)\,\delta x\|}{\|\delta x\|}$$

$$\hat{K} = K(x) = \|J(x)\|$$

<u>Relative condition number</u>:-

Assume $\underset{\sim}{\delta f}$ and $\underset{\sim}{\delta x}$ are infinitesimal

$$\hat{K}^R = \max_{\underset{\sim}{\delta x}} \left[\frac{\frac{\|\delta f\|}{\|f(x)\|}}{\frac{\|\delta x\|}{\|x\|}} \right]$$

$$= \max_{\underset{\sim}{\delta x}} \left(\frac{\frac{\|\delta f\|}{\|\delta x\|}}{\frac{\|f(x)\|}{\|x\|}} \right)$$

$$\boxed{\hat{K}^R = \frac{\|J(x)\|}{\|f(x)\| / \|x\|}} \checkmark$$

A problem is well-conditioned if $\hat{K}$ is small (eg:- $1, 10, 10^2$). and ill-conditioned if $\hat{K}$ is large (eg: $10^6, 10^{16}$ ..)

We talk about relative condition number below and drop the superscript R in $\hat{K}^R$ below!

Examples :-

(1) $f(x) = \frac{x}{2}$ $\quad x \in \mathbb{R}$

Input : x $\qquad J = \frac{df}{dx} = \frac{1}{2}$

output : $\frac{x}{2}$

$$\hat{K} = \frac{\|J\|}{\frac{\|f(x)\|}{\|x\|}} = \frac{1/2}{\frac{(x/2)}{x}} = 1$$

well conditioned problem.

(2) $f(\underset{\sim}{x}) = x_1 - x_2$ where $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$J = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} = [1 \ \ -1]_{1\times 2}$$

$$\hat{K} = \frac{\|J\|_\infty}{\|f(\underset{\sim}{x})\|_\infty / \|\underset{\sim}{x}\|_\infty}$$

$\|J\|_\infty$ is max row sum $= 2$

$$\hat{K} = \frac{2}{\frac{|x_1 - x_2|}{\max\{|x_1|, |x_2|\}}}$$

$$= \frac{2\max\{|x_1|, |x_2|\}}{|x_1 - x_2|}$$

if $|x_1 - x_2|$ is small ≈ 0, K is large and is not well conditioned when you are subtracting two numbers which are very close by.

$$\hat{K} = \frac{\|J\|_1}{\frac{\|f(x)\|_1}{\|x\|_1}} = \frac{1}{\frac{|x_1 - x_2|}{(|x_1|+|x_2|)}} = \frac{(|x_1|+|x_2|)}{|x_1 - x_2|}$$

<u>Root finding of quadratic equation</u> :-

$$x^2 - 2xp + 1 = 0$$

$$x_1 = p - \sqrt{p^2-1}$$
$$x_2 = p + \sqrt{p^2-1}$$

<u>Input</u> :- value of p

<u>output</u> :- x_1, x_2 which are the roots of quadratic equation.

Examine sensitivity of x_2 w.r.t p

$$\hat{K} = K(p) = \frac{\|J\|}{\frac{\|f(p)\|}{\|p\|}} \quad ; \quad J = \frac{dx_2}{dp}$$

$$K(p) = \left|\frac{dx_2}{dp}\right| \times \frac{|p|}{|x_2|}$$

$$= \left|1 + \frac{p}{\sqrt{p^2-1}}\right| \times \frac{|p|}{|p + \sqrt{p^2-1}|}$$

$$= \left| \frac{p+\sqrt{p^2-1}}{\sqrt{p^2-1}} \right| \times \frac{|p|}{|p+\sqrt{p^2-1}|}$$

$$= \frac{|p|}{\sqrt{p^2-1}}$$

The root x_2 is sensitive for $|p|$ close to 1 with $K(p) \to \infty$ and insensitive for large p with $K \simeq 1$

<u>Eigenvalues of matrix</u> :-

$$\underset{\sim}{A}\, \underset{\sim}{x} = \lambda \underset{\sim}{x}$$

$$\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = 0$$

<u>Input</u> :- $\underset{\sim}{A}$

<u>output</u> : eigenvalues λ

$\underset{\sim}{A} = \begin{pmatrix} 1 & 1000 \\ 0 & 1 \end{pmatrix}$ eigenvalues of $A = \{1, 1\}$

$\hat{\underset{\sim}{A}} = \begin{pmatrix} 1 & 1000 \\ 0.001 & 1 \end{pmatrix}$ eigenvalues of A after perturbation $\{0, 2\}$

Consider a symmetric matrix $\underset{\sim}{A} = \underset{\sim}{A}^T$

λ and $\lambda + \delta\lambda$ are corresponding eigenvalues $\underset{\sim}{A}$ and $\underset{\sim}{A} + \delta\underset{\sim}{A}$, then $\underbrace{|\delta\lambda| \le \|\delta A\|_2}$ — (1)

Relative condition number $= \max_{\|\delta A\|} \dfrac{\dfrac{|\delta\lambda|}{|\lambda|}}{\dfrac{\|\delta A\|_2}{\|A\|_2}}$

use ① $\rightarrow$ $\boxed{\text{Relative condition number } \hat{k} = \dfrac{\|A\|_2}{|\lambda|}}$

Conditioning of matrix-vector multiplication :-

$y = Ax$

Fixed A

Input :- x

Output :- Ax

Consider the problem of computing Ax for fixed A and input x

$$\boxed{k = \max_{\delta x} \dfrac{\dfrac{\|\delta f\|}{\|f\|}}{\dfrac{\|\delta x\|}{\|x\|}}}$$

$$k = \max_{\delta x} \dfrac{\dfrac{\|A(x+\delta x) - Ax\|}{\|Ax\|}}{\dfrac{\|\delta x\|}{\|x\|}}$$

$$= \max_{\delta x} \dfrac{\dfrac{\|A\delta x\|}{\|Ax\|}}{\dfrac{\|\delta x\|}{\|x\|}} = \underbrace{\left(\max_{\delta x} \dfrac{\|A\delta x\|}{\|\delta x\|}\right) \times \dfrac{\|x\|}{\|Ax\|}}$$

$$\hat{k} = \dfrac{\|A\| \, \|x\|}{\|Ax\|} \quad — ②$$

$\hat{k}$ for a give A and at x

To loosen above equality to get a bound of $\underset{\sim}{x}$, let us assume $\underset{\sim}{A}$ is square and non-singular

$$\underset{\sim}{x} = \underset{\sim}{A}^{-1} \underset{\sim}{A} \underset{\sim}{x}$$

$$\|\underset{\sim}{x}\| = \|\underset{\sim}{A}^{-1} \underset{\sim}{A} \underset{\sim}{x}\| \leq \|\underset{\sim}{A}^{-1}\| \|\underset{\sim}{A} \underset{\sim}{x}\|$$

$$\frac{\|\underset{\sim}{x}\|}{\|\underset{\sim}{A}\underset{\sim}{x}\|} \leq \|\underset{\sim}{A}^{-1}\| \quad - \text{③}$$

From eqn ② $\hat{K} = \dfrac{\|\underset{\sim}{A}\| \|\underset{\sim}{x}\|}{\|\underset{\sim}{A}\underset{\sim}{x}\|}$ ✓

using ③ $\hat{K} \leq \|\underset{\sim}{A}\| \|\underset{\sim}{A}^{-1}\| \quad - \text{④}$

What about for a give $\underset{\sim}{A}$, if you want to compute $\underset{\sim}{A}^{-1}\underset{\sim}{b}$ from a given input $\underset{\sim}{b}$?

Input :- $\underset{\sim}{b}$ Fixed $\underset{\sim}{A}$, $\underset{\sim}{A}^{-1}$

Output :- $\underset{\sim}{A}^{-1}\underset{\sim}{b} = \underset{\sim}{x}$

$$\hat{K} = \frac{\|\underset{\sim}{A}^{-1}\| \|\underset{\sim}{b}\|}{\|\underset{\sim}{A}^{-1}\underset{\sim}{b}\|}$$

$$\hat{K} = \frac{\|A^{-1}\| \|b\|}{\|\underset{\sim}{x}\|}$$

using ④, even for the problem of $\underset{\sim}{A}^{-1}\underset{\sim}{b}$

$$\hat{K} \leq \|\underset{\sim}{A}^{-1}\| \|\underset{\sim}{A}\|$$

Result:-

Let $\underset{\sim}{A} \in \mathbb{R}^{m \times m}$ be non-singular and consider the equation $\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{b}$, the problem of computing $\underset{\sim}{b}$ given $\underset{\sim}{x}$ has condition number

$$\hat{K} = \frac{\|\underset{\sim}{A}\| \|\underset{\sim}{x}\|}{\|\underset{\sim}{b}\|} \leq \|\underset{\sim}{A}\| \|\underset{\sim}{A}^{-1}\|$$

with respect to perturbations of $\underset{\sim}{x}$.

The problem of computing $\underset{\sim}{x}$ given $\underset{\sim}{b}$ has condition number

$$\hat{K} = \frac{\|\underset{\sim}{A}^{-1}\| \|\underset{\sim}{b}\|}{\|\underset{\sim}{x}\|} \leq \|\underset{\sim}{A}\| \|\underset{\sim}{A}^{-1}\|$$

Condition number of a matrix:-

The condition number of $\underset{\sim}{A}$ (relative to norm $\|\cdot\|$) denoted by $\hat{K}(\underset{\sim}{A}) = \|\underset{\sim}{A}\| \|\underset{\sim}{A}^{-1}\|$

If $\hat{K}(\underset{\sim}{A})$ is small, $\underset{\sim}{A}$ is said to be well-conditioned

if $\hat{K}(A)$ is large, $\underset{\sim}{A}$ is said to be ill conditioned.

If $\underset{\sim}{A}$ is singular, or close to singular $K(A) \to \infty$

In the 2-norm

$$\hat{K}(\underset{\sim}{A}) = \|\underset{\sim}{A}\|_2 \|\underset{\sim}{A}^{-1}\|_2$$

$\sigma_1 \rightarrow$ max singular value of $\underset{\sim}{A}$

$(1/\sigma_m) \rightarrow$ max singular value of $\underset{\sim}{A}^{-1}$

$\sigma_m \rightarrow$ min singular of $\underset{\sim}{A}$

$$\hat{K}(\underset{\sim}{A}) = \frac{\sigma_1}{\sigma_m}$$

~~For a square matrix~~ non zero singular values are square roots of non-zero eigenvalues of A^TA or AA^T.

$$K(A) = \frac{\sqrt{\lambda_{max}(A^TA)}}{\sqrt{\lambda_{min}(A^TA)}}$$

~~$K(A) = \frac{|\lambda_{max}(A)|}{|\lambda_{min}(A)|}$~~ ~~{ If A is symmetric}~~

If A is symmetric $K(A)$ is ratio of max of absolute value of λ and min of absolute value of λ

If $\underset{\sim}{A} \in \mathbb{R}^{m \times n}$ $(m > n)$, $K(\underset{\sim}{A})$ is defined in terms of pseudoinverse A^+.

$$K(A) = \|\underset{\sim}{A}\| \|\underset{\sim}{A}^+\|$$

$$\underset{\sim}{A}^+ = (\underset{\sim}{A}^T \underset{\sim}{A})^{-1} \underset{\sim}{A}^T$$

<u>Conditioning of system of equations</u>:-

So far we fixed $\underset{\sim}{A}$ and perturbed $\underset{\sim}{x}$ or $\underset{\sim}{b}$

Non-zero singular values are square roots of non-zero eigenvalues of $\underset{\sim}{A}^T\underset{\sim}{A}$ or $\underset{\sim}{A}\underset{\sim}{A}^T$

$$K(\underset{\sim}{A}) = \frac{\sqrt{\lambda_{max}(\underset{\sim}{A}^T\underset{\sim}{A})}}{\sqrt{\lambda_{min}(\underset{\sim}{A}^T\underset{\sim}{A})}}$$

If $\underset{\sim}{A}$ is symmetric $K(A) = \dfrac{|\lambda_{max}(\underset{\sim}{A})|}{|\lambda_{min}(\underset{\sim}{A})|}$

If $\underset{\sim}{A} \in \mathbb{R}^{m \times n}$ $(m > n)$, $K(\underset{\sim}{A})$ is defined in terms of pseudoinverse A^+.

$$K(A) = \|A\| \, \|A^+\|$$

$$A^+ = (A^TA)^{-1}A^T$$

Conditioning of system of equations:-

So far we fixed $\underset{\sim}{A}$ and perturbed $\underset{\sim}{x}$ or $\underset{\sim}{b}$

$\rightarrow$ What about if we perturb A?

Fix $\underset{\sim}{b}$ and consider $f : A \rightarrow x = \underset{\sim}{A}^{-1}\underset{\sim}{b}$

inputs :- $\underset{\sim}{A}$

output :- $\underset{\sim}{x}$

$\underset{\sim}{A}$ is perturbed by $\delta\underset{\sim}{A}$ and let $\underset{\sim}{x}$, the output be perturbed by $\delta\underset{\sim}{x}$

$$(\underset{\sim}{A} + \delta\underset{\sim}{A})(\underset{\sim}{x} + \delta\underset{\sim}{x}) = b \quad - (1)$$

$$\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{b} \quad - (2)$$

using (1) and (2) above, we have

$$(\delta\underset{\sim}{A})\underset{\sim}{x} + \underset{\sim}{A}(\delta\underset{\sim}{x}) = 0$$

$$\delta\underset{\sim}{x} = -\underset{\sim}{A}^{-1}(\delta\underset{\sim}{A})\underset{\sim}{x}$$

$$\|\delta\underset{\sim}{x}\| = \|-\underset{\sim}{A}^{-1}(\delta\underset{\sim}{A})\underset{\sim}{x}\|$$

$$\leq \|\underset{\sim}{A}^{-1}\| \, \|(\delta\underset{\sim}{A})\underset{\sim}{x}\|$$

$$\leq \|\underset{\sim}{A}^{-1}\| \, \|\delta\underset{\sim}{A}\| \, \|\underset{\sim}{x}\| \quad - (3)$$

$$\hat{K} = \max_{\delta A} \frac{\dfrac{\|\delta\underset{\sim}{x}\|}{\|\underset{\sim}{x}\|}}{\dfrac{\|\delta\underset{\sim}{A}\|}{\|\underset{\sim}{A}\|}}$$

using eq 3

$$\underbrace{\left(\frac{\dfrac{\|\delta x\|}{\|x\|}}{\dfrac{\|\delta A\|}{\|A\|}} \right) \leq \|\underset{\sim}{A}^{-1}\| \, \|A\|}$$

If perturbation $\underset{\sim}{\delta A}$ exists which makes the above inequality an equality

then

$\hat{K}$ of solving system of equations

$$\hat{K} = \max_{\underset{\sim}{\delta A}} \left(\frac{\frac{\|\underset{\sim}{\delta x}\|}{\|\underset{\sim}{x}\|}}{\frac{\|\underset{\sim}{\delta A}\|}{\|\underset{\sim}{A}\|}} \right) = K(\underset{\sim}{A})$$

Result :-

For a fixed $\underset{\sim}{b}$, Input $\rightarrow \underset{\sim}{A}$, output $\underset{\sim}{x} = \underset{\sim}{A}^{-1}\underset{\sim}{b}$

condition number of this problem with respect to perturbations in $\underset{\sim}{A}$ is

$$\hat{K} = \|\underset{\sim}{A}\|\|\underset{\sim}{A}^{-1}\| = K(\underset{\sim}{A})$$

eg:-

$$K(\underset{\sim}{A}) = 10^{6}$$

$$\frac{\|\underset{\sim}{\delta x}\|}{\|\underset{\sim}{x}\|} \leq 10^{6} \times \frac{\|\underset{\sim}{\delta A}\|}{\|\underset{\sim}{A}\|} \qquad \varepsilon_M \approx 10^{-8}$$

$$\frac{\|\underset{\sim}{\delta A}\|}{\|\underset{\sim}{A}\|} \leq O(\varepsilon_M)$$

$$\leq 10^{6} \times 10^{-8}$$

$$\approx O(10^{-2})$$