

## Assignment - 4

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1.  $f(t) = \sin(10t)$ ,  $t \in [0, 1]$

Code is attached at the end.

### Observation

- ① plot obtained by modified gram schmidt QR decomposition and Householder triangulization produced best result to the original curve.
- ② plot based on SVD is pretty skewed and looks like damped oscillation curve. Also python numpy default least squares method is also based on SVD. In this case also, we obtain similar curve. Hence the result was bad.
- ③ Further, there's no ~~any~~ default function for Normal equation method in python. When I tried to solve it using Cholesky decomposition, computationally it couldn't calculate that and showed error that  $\text{rank}(ATA) = 12$ , although  $\text{rank}(A) = 15$ . I'm not sure why it showed 8th <sup>(i.e. full rank)</sup> like this. One reason may be b/c  $t \in [0, 1]$  in  $At^T A$ , there's lots of entries with very low value which results in

SVD insinuates low values in  $\Sigma$ . Hence

Python adjudges  $A^T A$  as rank deficient and

could not calculate Cholesky decomposition.

However, if you directly calculate  $(A^T A)^{-1}$ , then it gives a result which looks like that obtained in SVD. Also the bad performance of SVD method might be due to this (though I'm not sure).

On the otherhand, modified Gram-Schmidt and Householder method performed as usual and gave expected result.

Plots and link to code are given at the end.

$$2. \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \\ 2 & 8 & 2 \end{bmatrix} \quad \tilde{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Least squares soln :  $\tilde{x}^* = [-1.33 \quad 0.67 \quad 0.67]$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \\ 5 & 7 & 9 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Least squares soln :  $\tilde{x}^* = [-1 \quad 10^{-16} \quad 1]$

We obtained this soln. using numpy default / SVD based method.

We also observe least squares problem always has a soln. Here the idea is just to minimize  $\|A\tilde{x} - b\|^2$ .

We know that, an orthogonal projector to  $\text{range}(A)$  can be expressed as

$$P = A(A^T A)^{-1} A^T$$

where  $A$  is full rank.

In this case  $P\tilde{b} \in \text{Range}(A)$  and solution will be unique.

If  $A$  is not full rank, even then we can have a projection onto  $\text{range}(A)$  let's say  $\tilde{c}$ .

In that case,  $A\tilde{x}^* = \tilde{c}$

So there always exists  $\tilde{x}^*$  b/c there's a projection onto  $\text{range}(A)$ .

And soln. is unique when  $A$  is full rank.

3. Let,  $A$  be a general square matrix.

Let,  $A \in \mathbb{R}^{m \times m}$

So

$$L_{m-1} P_{m-1} \cdots \cdots L_2 P_2 L_1 P_1 A = U$$

— (Gauss elimination with partial pivoting)

$L_i$ 's are lower triangular matrices with 1 at diagonal.

So  $\det(L_i) = 1$ .

$P_i$ 's are identity matrix with row interchange.

So  $\det(P_i) = \begin{cases} 1, & \text{if no row interchange} \\ -1, & \text{if row interchange} \end{cases}$

So  $\prod_{i=1}^{m-1} \det(P_i) \det(L_i) \cancel{\det A} \det A = \det U$

$$\Rightarrow (-1)^k \det A = \det U$$

where  $k = \text{Number of time row interchange has occurred.}$

$$\Rightarrow \det(A) = (-1)^{-k} \det(U).$$

4. Code is attached at the end and plots were presented. From the plot, it's clear that  $\|LU-A\|_F$  is much higher in case of without pivoting than with partial pivot.

As studied, LU factorization without pivoting is stable but not backward stable. Hence  $\|LU-A\|_F$  is pretty high in this case.

However, pivoting brings more numerical stability and  $\|LU - A\|_F$  is about three orders of magnitude lower. Actually with partial pivoting, it's backward stable. And in practice it also works well.

### ③ Time complexity Part

$$\det(A) = (-1)^{-k} \det(U)$$

$$\begin{aligned}\det U &= \text{product of diagonal elements} \\ &= \cancel{\text{m}} (m-1) \text{ operations}\end{aligned}$$

To check whether row interchange occurred or not in  $P_i$ , we need to check  $(m-1)$  diagonal elements.

We have to do this for  $(m-1)$   $P_i$ 's.

So this checking takes approx.  $(m-1)^2$  operations.

So total number of operations =  $(m-1)^2 + (m-1) + O(1)$

where  $O(1) \rightarrow$  Other small number of products or addition.

So number floating point operation :  $O(m^2)$

and dominated by checking whether row interchange has occurred or not.

6.  $A \in \mathbb{C}^{m \times m}$

(a)  $A\tilde{x} = \lambda\tilde{x}$

$$\begin{aligned}(A - \mu I)\tilde{x} &= A\tilde{x} - \mu\tilde{x} \\ &= (\lambda - \mu)\tilde{x}\end{aligned}$$

So  $(\lambda - \mu)$  is eigenvalue of  $A - \mu I$ .

(b) False.

e.g.  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = 0 \Rightarrow \lambda = 1$$

-1 is not there in eigen value.

(c)  $A\tilde{x} = \lambda\tilde{x}$

$$\Rightarrow \overline{A\tilde{x}} = \overline{\lambda\tilde{x}}$$

$$\Rightarrow \bar{A}\bar{\tilde{x}} = \bar{\lambda}\bar{\tilde{x}}$$

B/c  $A$  is real,  $\bar{A} = A$

$$\Rightarrow A\bar{\tilde{x}} = \lambda^*\bar{\tilde{x}}$$

Hence  $\lambda^*$  is also eigen value of  $A$ .

(d)  $A\tilde{x} = \lambda\tilde{x}$

$$\Rightarrow A^{-1}A\tilde{x} = \lambda A^{-1}\tilde{x} \quad \text{so } \frac{1}{\lambda} \text{ is the eigenvalue}$$

$$\Rightarrow A^{-1}\tilde{x} = \frac{1}{\lambda}\tilde{x} \quad \text{or } A^{-1}$$

(e) False.

e.g.  $A = \begin{bmatrix} 0 & 2 & 3 \\ 5 & 0 & 4 \\ 6 & 7 & 0 \end{bmatrix}$

Here  $\det(A - \lambda I) = 0 \Rightarrow \lambda = 0$

But  $A \neq 0$

(f) A is diagonalisable.

So  $A = XDX^{-1}$  where  $D = \text{diagonal matrix}$   
 $X = \text{Invertible matrix}$

B/c all eigen values are equal,  $D = \lambda I$ .

So  $A = \lambda X X^{-1} = \lambda I = \text{diagonal matrix.}$

7. (a)  $A\tilde{x} = \lambda\tilde{x}, \quad \tilde{x} \in \mathbb{C}^n$ .

Let,  $\tilde{x}_i$  be the largest entry in  $\tilde{x}$ .  
in terms of modulus | . |.

So  $A\tilde{x} = \lambda\tilde{x}$

$$\Rightarrow \sum_{j=1}^n a_{ij}x_j = \lambda x_i$$

$$\Rightarrow \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij}x_j = (\lambda - a_{ii})x_i$$

$$\Rightarrow (\lambda - a_{ii}) = \frac{\sum_{j \neq i} a_{ij} x_j}{x_i}$$

$$\Rightarrow |\lambda - a_{ii}| \leq \sum_{j \neq i} \frac{|a_{ij}| |x_j|}{|x_i|} \leq \sum_{j \neq i} |a_{ij}|$$

(As  $|x_i| > |x_j|$  as per assumption)

So  $\lambda$  is present within  $\sum_{j \neq i} |a_{ij}|$  distance from one of the diagonal point  $a_{ii}$ . Hence it is present inside disc.

(b) Suppose,  $\exists \underline{x} \neq 0$  such that  $A\underline{x} = 0$ .

$$\Rightarrow \sum_{j=1}^n a_{ij} x_j = 0 \text{ for all } i.$$

Let,  $x_k$  be the largest entry in  $\underline{x}$  in terms of modulus.

$$\text{So } \sum_{j=1}^n a_{kj} x_j = 0$$

$$\Rightarrow a_{kk} x_k = - \sum_{j \neq k} a_{kj} x_j$$

$$\Rightarrow a_{kk} = - \frac{\sum_{j \neq k} a_{kj} x_j}{\otimes x_k}$$

$$|a_{kk}| \leq \sum_{j \neq k} \frac{|a_{kj}| |x_j|}{|x_k|} \leq \sum_{j \neq k} |a_{kj}|$$

$(\because |x_k| > |x_j|$   
by assumption)

$$|a_{kk}| \leq \sum_{j \neq k} |a_{kj}|$$

This violates diagonally dominant condition.

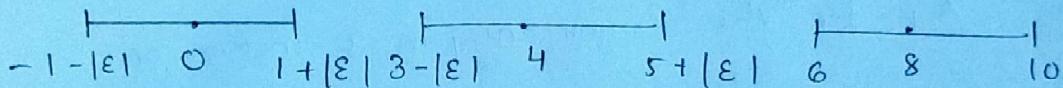
Hence our assumption of  $\underline{x} \neq 0$  and  $A\underline{x} = 0$  is wrong.

Only  $\underline{x} = 0$  can lead to  $A\underline{x} = 0$

Hence A is invertible.

③  $A = \begin{bmatrix} 8 & 2 & 0 \\ 1 & 4 & \varepsilon \\ 0 & \varepsilon & 1 \end{bmatrix}$   $|\varepsilon| < 1$

Gershgorin's Disks



$$|\varepsilon| < 1.$$

$$\text{So } \lambda \in (-2, 2) \cup (2, 6) \cup (6, 10)$$

But  $\lambda$  can't exactly take values like 2, 6, 10.

5. @ A is a non-singular square matrix.  $\in \mathbb{R}^{m \times m}$

$$A = QR, \quad A^T = R^T Q^T$$

$$A^T A = R^T Q^T Q R = R^T R \quad (\because Q^T Q = I) \\ (\because \text{orthogonal})$$

$$A^T A = U^T U \quad - \text{(given)}$$

Also

$$\begin{aligned} \underline{x}^T A^T A \underline{x} &\geq 0 \\ \Leftrightarrow \underline{x}^T \underline{x} &\geq 0 \end{aligned}$$

So  $A^T A$  is pos. semidefinite.

Now  $A^T A \underline{x} = 0 \Rightarrow \underline{x} = 0$  b/c A is nonsingular.  
Hence  $A^T A$  has inverse.

$$\text{So } \|A \underline{x}\|^2 > 0 \text{ for } \underline{x} \neq 0$$

So  $A^T A$  is positive definite.

Also ②  $(A^T A)^T = A^T A = \text{symmetric matrix.}$

We know for real symmetric positive definite matrix,  
there's unique Cholesky factorization.

$$\text{Hence } R^T R = U^T U$$

$$\Rightarrow R = U$$

(C) Case-1

$$A = B^T B$$

$$\underline{x}^T A \underline{x} = \underline{x}^T B^T B \underline{x} = \|B \underline{x}\|^2 \geq 0$$

$\|B \underline{x}\| = 0$  only if  $\underline{x} = 0$  ( $\because B$  is full rank)

Hence for all  $\underline{x} \neq 0$ ,  $\underline{x}^T A \underline{x} > 0$

So  $A$  is symmetric positive definite.

Case-2

$A \in \mathbb{R}^{n \times n}$  is symmetric positive definite.

So  $A = Q^T D Q$  where  $Q$  = orthogonal matrix  
 $D$  = Diagonal matrix

B/c  $A$  is pos. definite, all eigen values are positive.

So  $D^{1/2}$  is diagonal with all +ve elements.

$$\begin{aligned} \text{So } A &= Q^T D^{1/2 T} D^{1/2} Q = (D^{1/2} Q)^T (D^{1/2} Q) \\ &= B^T B \end{aligned}$$

Under some conditions, some part of these square  
 (rank limited)

$B$  will be zero and they can be  $m \times n$  matrix.

From this for this choice of  $B$  is not unique b/c  
 we know for sure there's another decomposition  
 (Cholesky) :  $A = R^T R$ .  $R \in$  upper triangular.

$$\textcircled{b} \quad M_{ij} = \int_1^1 \phi_i(x) \phi_j(x) dx \quad (\text{inner product})$$

$$\begin{aligned} \text{We observe, } M_{ii} &= \int_1^1 \phi_i(x) \phi_i(x) dx \\ &= \langle \phi_i(x), \phi_i(x) \rangle > 0 \\ &\quad (\because \phi_i(x) \neq 0) \end{aligned}$$

$M_{ij} = M_{ji}$  b/c  $\phi_j$  and  $\phi_i$  are interchangeable here.

Sylvester's criterion for positive definiteness : All the leading principal minors should be positive.

First principal minor =  $\langle \phi_1(x), \phi_1(x) \rangle > 0$

$$\begin{aligned} \text{Second principal minor} &= \langle \phi_1(x), \phi_1(x) \rangle \langle \phi_2(x), \phi_2(x) \rangle \\ &\quad - \langle \phi_1(x), \phi_2(x) \rangle^2 \end{aligned}$$