Numerical Methods

DS288 and UMC201

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Global error for Euler's method

Theorem

1. $f(x,y)$ *is continuous and satisfies the Lipschitz condition with constant* L. *Let* $y = y(x)$ *is the unique solution of the IVP*

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

2. *There exists a constant* M *such that*

$$|y''(x)| \leq M, \text{ for all } x \in [x_0, b].$$

3. *Let* $\omega_0, \omega_1, \ldots, \omega_N$ *be the approximations generated by Euler method. Then*

$$|e_i| = |y(x_i) - \omega_i| \leq \frac{Mh}{2L}[e^{(x_i - x_0)L} - 1]; \quad i = 0, 1, \cdots N.$$

Global error for Euler's method

For $i = 0$ the result is true, $y(x_0) = \omega_0 = x_0$ By Taylor's theorem,

$$\begin{aligned} y(x_{i+1}) &= y(x_i) + hy'(x_i) + \frac{h^2}{2} y''(\zeta); \quad x_i < \zeta < x_{i+1} \\ &= y(x_i) + hf(x_i, y(x_i)) + \frac{h^2}{2} y''(\zeta). \qquad (1) \end{aligned}$$

We have Euler formula $\omega_{i+1} = \omega_i + hf(x_i, \omega_i)$
From (1) and the Euler formula, we get

$$y(x_{i+1}) - \omega_{i+1} = y(x_i) - \omega_i + h\{f(x_i, y_i) - f(x_i, \omega_i)\} + \frac{h^2}{2} y''(\zeta).$$

$$|y_{i+1} - \omega_{i+1}| \leq |y_i - \omega_i| + h|(f(x_i, y_i)) - f(x_i, \omega_i))| + \frac{h^2}{2} |y''\zeta|.$$

Global error for Euler's method

Applying Lipschitz condition with constant L and $|y''(x)| \leq M$,

$$|y_{i+1} - \omega_{i+1}| \leq (1 + hL)|y_i - \omega_i| + \frac{h^2 M}{2}. \tag{2}$$

Now we have the following lemma

If s and t are +ve real number, $\{a_i\}_{i=0}^{k}$ is a sequence satisfying $a_0 \geq -t/s$ and $a_{i+1} \leq (1+s)a_i + t$ for each $i = 0, 1, 2, \cdots k-1$

$$\text{then} \quad a_{i+1} \leq e^{(i+1)s}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}$$

Consider $s = hL,\; t = h^2M/2$ and $a_j = |y_j - \omega_j|$ for each $j = 0, \cdots N$

$$|y_{i+1} - \omega_{i+1}| \leq e^{(i+1)hL}\left(|y_0 - \omega_0| + \frac{h^2 M}{2hL}\right) - \frac{Mh^2}{2hL} \tag{3}$$

Global error for Euler's method

$$|y_{i+1} - \omega_{i+1}| \leq e^{(i+1)hL}\left(|y_0 - \omega_0| + \frac{h^2 M}{2hL}\right) - \frac{Mh^2}{2hL} \tag{4}$$

Since $|y_0 - \omega_0| = 0$ and $(i+1)h = x_{i+1} - x_0$. Then

$$|y_{i+1} - \omega_{i+1}| \leq \frac{Mh}{2L}[e^{(x_{i+1}-x_0)L} - 1]; \quad i = 0, 1, \cdots N. \tag{5}$$

- It is clear that $|y_i - \omega_i| \to 0$ as $h \to 0$ so Euler's method is convergent.
- One weakness of the theorem is that we don't know the M, the second derivative of the exact solution. If $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial x}$ exist, then we can get M. i.e.,

$$y'' = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f$$

Lipschitz condition

Lipschitz condition

The Lipschitz condition may be replaced by $|f_y(x, y)| \leq L$,

- $|\frac{\partial f}{\partial y}| \leq L \Rightarrow$ the Lipschitz condition.
- The converse is not true in general (i.e., the Lipschitz condition holds the derivative may not exist at certain points.)

By mean value theorem we obtained for $(x, y_1), (x, y_2) \in R$

$$|f(x, y_1) - f(x, y_2)| \leq f_y(x, \zeta)|y_1 - y_2| \quad \text{where} \quad y_1 \leq \zeta \leq y_2$$

Here the point (x, ζ) lies in R and $|f_y(x, \zeta)| \leq K$. Hence

$$|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2|$$

which is the Lipschitz condition.

Example

Example

Consider the function $f(x, y) = x|y|$ where R is the rectangle defined by $|x| \leq a, \ |y| \leq b$.

- For all $(x, y_1), (x, y_2) \in R$, we can write

$$|f(x, y_1) - f(x, y_2)| = |x| \, [|y_1 - y_2|] \leq a|y_1 - y_2|$$

  Thus f satisfies Lipschitz condition in R.

- But the partial derivatives $\frac{\partial f}{\partial y}$ does not exist at any point $(x, 0) \in R$ for which $x \neq 0$.

Example

Example

Consider the initial value problem

$$\frac{dy}{dx} = y - x^2 + 1, \quad y(0) = 0.5, \quad 0 \leq x \leq 2, \quad h = 0.2 \tag{6}$$

Find the approximation error bound.

- Here $f(x, y) = y - x^2 + 1, \quad \frac{\partial f}{\partial y} = 1$ for all y. So $L = 1$.
- Exact solution $y(x) = (x+1)^2 - 0.5e^x$. So $y''(x) = 2 - 0.5e^x$
- $|y''(x)| = 0.5e^2 - 2$ for all $x \in [0, \ 2]$
- Using Theorem-1 with $h = 0.2 \ \ L = 1$ and $M = 0.5e^2 - 2$ we get

$$|y_i - \omega_i| \leq 0.1(0.5e^2 - 2)(e^{x_i} - 1)$$

Taylor's series method

We have first order initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0, \quad a \leq x \leq b \tag{7}$$

Differentiating IVP we get

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \quad \Rightarrow \quad y'' = f_x + f_y f \tag{8}$$

Differentiating this successively, we get y''', $y^{iv}, \cdots$ etc.
Now Taylor's series

$$y(x_1) = y(x_0) + hy'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \cdots + \frac{h^{n+1}}{(n+1)!}y^{n+1}(\zeta) \tag{9}$$

➥ where $y_1 = y(x_1)$ is the approximate value of y at $x = x_1$

Taylor's series method

Example

Using Taylor series method find $y(0.2)$ for the following IVP with with step length 0.1

$$\frac{dy}{dx} = x^2y - 1, \; y(0) = 1$$

- Here $y' = x^2y - 1$, y(0)=1, h=0.1, y(0.2)= ?
- Differentiating successively, we get
  $y' = x^2y - 1, \; y'' = 2xy + x^2y', \; y''' = 2y + 4xy' + x^2y'', \; y^{iv} = ?,$
- Now substituting, we get
  $y'(0) = 0^2y(0) - 1 = -1, \; y''(0) = ?$
- Following Eq. (9) and above values, we get $y(0.1) = 0.90031,$
- Similarly, we can compute $y(0.2) = 0.80227$


ANY
QUESTIONS?