Numerical Methods

DS288 and UMC201

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Chapter - 10

Numerical Solutions of Nonlinear System of Equations

System of nonlinear equations

A system of nonlinear equations has the form

$$\begin{aligned} f_1(x_1, x_2, \cdots, x_n) &= 0 \\ f_2(x_1, x_2, \cdots, x_n) &= 0 \\ \vdots \quad & \quad \vdots \\ f_n(x_1, x_2, \cdots, x_n) &= 0 \end{aligned} \tag{1}$$

where each function f_i can be thought of as mapping a vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)^t$ of the n-dimensional space $\mathbb{R}^n$ into the real line $\mathbb{R}$. This system of n nonlinear equations in n unknowns can also be represented by defining a function $\mathbf{F}$ mapping $\mathbb{R}^n$ into $\mathbb{R}^n$ as

$$\mathbf{F}(x_1, \cdots x_n) = [f_1(x_1, \cdots x_n), f_2(x_1, \cdots x_n), \cdots, f_n(x_1, \cdots x_n)]^t.$$

If vector notation is used to represent the variables $x_1, x_2, \cdots, x_n)$, then system (1) assumes the form

$$\mathbf{F}(x) = 0.$$

The functions $f_1, f_2, \cdots, f_n$ are called the coordinate functions of $\mathbf{F}$.

Definition

- Let f be a function defined on a set $D \subset \mathbb{R}^n$ and mapping into $\mathbb{R}$. The function f is said to have the **limit** L at $\mathbf{x}_0$, written

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = L, \tag{2}$$

if, given any number $\varepsilon > 0$, a number $\delta > 0$ exists with

$$|f(\mathbf{x}) - L| < \varepsilon, \quad \text{whenever } \mathbf{x} \in D \text{ and } \; 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta.$$

- Let f be a function from a set $D \subset \mathbb{R}^n$ into $\mathbb{R}$. The function f is **continuous** at $\mathbf{x}_0 \in D$ provided $\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x})$ exists and

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0). \tag{3}$$

Moreover, f is **continuous on a set** D if f is continuous at every point of D. This concept is expressed by writing $f \in C(D)$.

Definition

- We can now define the limit and continuity concepts for functions from $\mathbb{R}^n$ into $\mathbb{R}^n$ by considering the coordinate functions from $\mathbb{R}^n$ into $\mathbb{R}$.

Vector-valued functions and their limits/continuity

Let $\mathbf{F}$ be a function from $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$ of the form

$$\mathbf{F}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))^t, \tag{4}$$

where f_i is a mapping from $\mathbb{R}^n$ into $\mathbb{R}$ for each i. We define

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{F}(\mathbf{x}) = \mathbf{L} = (L_1, L_2, \ldots, L_n)^t, \tag{5}$$

if and only if $\lim_{\mathbf{x}\to\mathbf{x}_0} f_i(\mathbf{x}) = L_i$, for each $i = 1, 2, \ldots, n$.

The function $\mathbf{F}$ is **continuous** at $\mathbf{x}_0 \in D$ provided $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{F}(\mathbf{x})$ exists and $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{F}(\mathbf{x}) = \mathbf{F}(\mathbf{x}_0)$. In addition, $\mathbf{F}$ is continuous on the set D if $\mathbf{F}$ is continuous at each $\mathbf{x}$ in D. This concept is expressed by writing $\mathbf{F} \in C(D)$.

Definition

Theorem-1

Let f be a function from $D \subset \mathbb{R}^n$ into $\mathbb{R}$ and $\mathbf{x}_0 \in D$. Suppose that all the partial derivatives of f exist and constants $\delta > 0$ and $K > 0$ exist so that whenever $\|\mathbf{x} - \mathbf{x}_0\| < \delta$ and $\mathbf{x} \in D$, we have

$$\left| \frac{\partial f(\mathbf{x})}{\partial x_j} \right| \leq K, \quad \text{for each } j = 1, 2, \ldots, n. \tag{6}$$

Then f is continuous at $\mathbf{x}_0$.

Remark: The continuity of a function of n variables at a point to the partial derivatives of the function at the point.

Definition (Fixed point)

A function $\mathbf{G}$ from $\mathbb{D} \subset \mathbb{R}^n$ into $\mathbb{R}^n$ has a fixed point at $\mathbf{p} \in \mathbb{D}$ if $\mathbf{G}(\mathbf{p}) = \mathbf{p}$.

Fixed-Point Theorem

Let $D = \{(x_1, x_2, \ldots, x_n)^T \mid a_i \leq x_i \leq b_i, \text{ for each } i = 1, 2, \ldots, n\}$ for some collection of constants $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$.

- Suppose $\mathbf{G}$ is a continuous function from $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$ with the property that $\mathbf{G}(\mathbf{x}) \in D$ whenever $\mathbf{x} \in D$. Then $\mathbf{G}$ has a fixed point in D.
- Suppose that all the component functions of $\mathbf{G}$ have continuous partial derivatives and a constant $K < 1$ exists with

$$\left| \frac{\partial g_i(\mathbf{x})}{\partial x_j} \right| \leq \frac{K}{n}, \quad \text{whenever } \mathbf{x} \in D, \tag{7}$$

  for each $j = 1, 2, \ldots, n$ and each component function g_i. Then the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by an arbitrarily selected $\mathbf{x}^{(0)}$ in D and generated by

$$\mathbf{x}^{(k)} = \mathbf{G}(\mathbf{x}^{(k-1)}), \quad \text{for each } k \geq 1 \tag{8}$$

  converges to the unique fixed point $\alpha \in D$
- Converges

$$\|\mathbf{x}^{(k)} - \alpha\|_\infty \leq \frac{K^k}{1-K} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|_\infty \tag{9}$$

Fixed-Point Theorem

Example

Place the nonlinear system

$$3x_1 - \cos(x_2 x_3) - \frac{1}{2} = 0, \tag{10}$$

$$x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 = 0, \tag{11}$$

$$e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0. \tag{12}$$

in a fixed-point form $\mathbf{x} = \mathbf{G}(\mathbf{x})$ by solving the ith equation for x_i, show that there is a unique solution on

$$D = \{(x_1, x_2, x_3)^t \mid -1 \leq x_i \leq 1, \quad \text{for each } i = 1, 2, 3\}.$$

and iterate starting with $\mathbf{x}^{(0)} = (0.1, 0.1, -0.1)^t$ until accuracy within 10^{-5} in the l_∞ norm is obtained.

Fixed-Point Theorem

Solution Solving the ith equation for x_i gives the fixed-point problem

$$\begin{aligned} x_1 &= \frac{1}{3}\cos(x_2 x_3) + \frac{1}{6}, \\ x_2 &= \frac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1, \\ x_3 &= -\frac{1}{20}e^{-x_1 x_2} - \frac{10\pi - 3}{60}. \end{aligned} \tag{13}$$

Let $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\mathbf{G}(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}), g_3(\mathbf{x}))^t$, where

$$g_1(x_1, x_2, x_3) = \frac{1}{3}\cos(x_2 x_3) + \frac{1}{6}, \tag{14}$$

$$g_2(x_1, x_2, x_3) = \frac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1, \tag{15}$$

$$g_3(x_1, x_2, x_3) = -\frac{1}{20}e^{-x_1 x_2} - \frac{10\pi - 3}{60}. \tag{16}$$

Above two Theorems will be used to show that $\mathbf{G}$ has a unique fixed point in

$$D = \{(x_1, x_2, x_3)^t \mid -1 \le x_i \le 1, \quad \text{for each } i = 1, 2, 3\}.$$

Fixed-Point Theorem

For $\mathbf{x} = (x_1, x_2, x_3)^t$ in D,

$$|g_1(x_1, x_2, x_3)| \le \frac{1}{3}|\cos(x_2 x_3)| + \frac{1}{6} \le 0.50, \tag{17}$$

$$|g_2(x_1, x_2, x_3)| = \left|\frac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1\right| \le \frac{1}{9}\sqrt{1 + \sin 1 + 1.06} - 0.1 < 0.09, \tag{18}$$

and

$$|g_3(x_1, x_2, x_3)| = \left|\frac{1}{20}e^{-x_1 x_2} + \frac{10\pi - 3}{60}\right| \le \frac{1}{20}e + \frac{10\pi - 3}{60} < 0.61.$$

So we have, for each $i = 1, 2, 3,$

$$-1 \le g_i(x_1, x_2, x_3) \le 1.$$

Thus $\mathbf{G}(\mathbf{x}) \in D$ whenever $\mathbf{x} \in D$.
Finding bounds for the partial derivatives on D gives

$$\left|\frac{\partial g_1}{\partial x_1}\right| = 0, \quad \left|\frac{\partial g_2}{\partial x_2}\right| = 0, \quad \text{and} \quad \left|\frac{\partial g_3}{\partial x_3}\right| = 0,$$

Fixed-Point Theorem

$$\left|\frac{\partial g_1}{\partial x_2}\right| \leq \frac{1}{3}|x_3| \cdot |\sin x_2 x_3| \leq \frac{1}{3}\sin 1 < 0.281,$$

$$\left|\frac{\partial g_1}{\partial x_3}\right| \leq \frac{1}{3}|x_2| \cdot |\sin x_2 x_3| \leq \frac{1}{3}\sin 1 < 0.281,$$

$$\left|\frac{\partial g_2}{\partial x_1}\right| = \frac{|x_1|}{9\sqrt{x_1^2 + \sin x_3 + 1.06}} < \frac{1}{9\sqrt{0.218}} < 0.238,$$

$$\left|\frac{\partial g_2}{\partial x_3}\right| = \frac{|\cos x_3|}{18\sqrt{x_1^2 + \sin x_3 + 1.06}} < \frac{1}{18\sqrt{0.218}} < 0.119,$$

$$\left|\frac{\partial g_3}{\partial x_1}\right| = \frac{|x_2|}{20}e^{-x_1 x_2} \leq \frac{1}{20}e < 0.14,$$

$$\left|\frac{\partial g_3}{\partial x_2}\right| = \frac{|x_1|}{20}e^{-x_1 x_2} \leq \frac{1}{20}e < 0.14.$$

Solution

- The partial derivatives of g_1, g_2, and g_3 are all bounded on $\mathbb{D}$,
- Following fixed-point Theorem, **G** is continuous on $\mathbb{D}$.

$$\left|\frac{\partial g_i(\mathbf{X})}{\partial x_j}\right| \leq 0.281 \text{ for each } i = 1, 2, 3 \text{ and } j = 1, 2, 3,$$

  and the condition in the second part of n-dimensional fixed-point theorem holds with $K = 3(0.281) = 0.843$.

- Similarly, it can also be shown that $\dfrac{\partial g_i}{\partial x_j}$ is continuous on D for each i = 1, 2, 3 and j = 1, 2, 3.
- Consequently, **G** has a unique fixed point in $\mathbb{D}$, and the nonlinear system has a solution in $\mathbb{D}$.

Remark: **G** having a unique fixed point in $\mathbb{D}$ does not imply that the solution to the original system is unique in this domain, because the solution for x_2 in (13) involved the choice of the principal square root.

Solution

- To approximate the fixed point α, we choose $\mathbf{x}^{(0)} = (0.1, 0.1, -0.1)^t$.
- The sequence of vectors generated by

$$x_1^{(k)} = \frac{1}{3}\cos x_2^{(k-1)} x_3^{(k-1)} + \frac{1}{6}, \tag{19}$$

$$x_2^{(k)} = \frac{1}{9}\sqrt{\left(x_1^{(k-1)}\right)^2 + \sin x_3^{(k-1)} + 1.06} - 0.1, \tag{20}$$

$$x_3^{(k)} = -\frac{1}{20} e^{-x_1^{(k-1)} x_2^{(k-1)}} - \frac{10\pi - 3}{60} \tag{21}$$

converges to the unique solution of the system in (13). The results are generated until

$$\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty < 10^{-5}$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty$
0	0.10000000	0.10000000	−0.10000000	
1	0.49998333	0.00944115	−0.52310127	0.423
2	0.49999593	0.00002557	−0.52336331	9.4×10^{-3}
3	0.50000000	0.00001234	−0.52359814	2.3×10^{-4}
4	0.50000000	0.00000003	−0.52359847	1.2×10^{-5}
5	0.50000000	0.00000002	−0.52359877	3.1×10^{-7}

Error bound

- Following the error bound in the theorem, with $K = 0.843$. This gives
$$\|\mathbf{x}^{(5)} - \mathbf{p}\|_\infty \leq \frac{(0.843)^5}{1 - 0.843}(0.423) < 1.15,$$
- This does not indicate the true accuracy of $\mathbf{x}^{(5)}$.
- The actual solution is $\mathbf{p} = \left(0.5, 0, -\frac{\pi}{6}\right)^T \approx (0.5, 0, -0.5235987757)^T$,
- Thus $\|\mathbf{x}^{(5)} - \mathbf{p}\|_\infty \leq 2 \times 10^{-8}$.

Chapter - 10

Section 10:2 (Newton-Raphson method)

Newton-Raphson method via fixed point

- Consider the sequence $x_n = g(x_{n-1}), \quad \text{for } n \geq 1$
- For g in the form $g(x) = x - \phi(x)f(x)$, where ϕ is a differentiable function that will be chosen later.
- For the iterative procedure derived from g to be quadratically convergent, we need to have $g'(\alpha) = 0$ when $f(\alpha) = 0$. Because

$$g'(x) = 1 - \phi'(x)f(x) - f'(x)\phi(x), \text{ and } f(\alpha) = 0, \tag{22}$$

- We have

$$g'(\alpha) = 1 - \phi'(\alpha)f(\alpha) - f'(\alpha)\phi(\alpha) = 1 - \phi'(\alpha)\cdot 0 - f'(\alpha)\phi(\alpha) = 1 - f'(\alpha)\phi(\alpha),$$

and $g'(\alpha) = 0$ if and only if $\phi(\alpha) = 1/f'(\alpha)$.

- If we let $\phi(x) = 1/f'(x)$, then we will ensure that $\phi(\alpha) = 1/f'(\alpha)$ and produce the quadratically convergent procedure

$$x_n = g(x_{n-1}) = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}. \tag{23}$$

This, of course, is simply Newton's method.

Newton-Raphson method

- An appropriate fixed-point method in the one-dimensional case, we need a function ϕ with the property that $g(x) = x - \phi(x)f(x)$ gives quadratic convergence to fixed point α of the function g.
- From this condition Newton's method evolved by choosing

$$\phi(x) = \frac{1}{f'(x)}, \quad \text{assuming that} \quad f'(x) \neq 0.$$

- A similar approach in the n-dimensional case involves a matrix

$$A(\mathbf{x}) = \begin{pmatrix} a_{11}(\mathbf{x}) & a_{12}(\mathbf{x}) & \cdots & a_{1n}(\mathbf{x}) \\ a_{21}(\mathbf{x}) & a_{22}(\mathbf{x}) & \cdots & a_{2n}(\mathbf{x}) \\ \vdots & \vdots & & \vdots \\ a_{n1}(\mathbf{x}) & a_{n2}(\mathbf{x}) & \cdots & a_{nn}(\mathbf{x}) \end{pmatrix} \tag{24}$$

where each of the entries $a_{ij}(\mathbf{x})$ is a function from $\mathbb{R}^n$ into $\mathbb{R}$. This requires that $A(\mathbf{x})$ be found so that

$$\mathbf{G}(\mathbf{x}) = \mathbf{x} - A(\mathbf{x})^{-1}\mathbf{F}(\mathbf{x})$$

gives quadratic convergence to the solution of $\mathbf{F}(\mathbf{x}) = 0$, assuming that $A(\mathbf{x})$ is nonsingular at the fixed point α of $\mathbf{G}$.

Newton-Raphson method

Theorem

Let $\mathbf{p}$ be a solution of $\mathbf{G}(\mathbf{x}) = \mathbf{x}$. Suppose a number $\delta > 0$ exists with

(i) $\frac{\partial g_i}{\partial x_j}$ is continuous on $N_\delta = \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{p}\| < \delta\}$, for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$;

(ii) $\frac{\partial^2 g_i(\mathbf{x})}{\partial x_j \partial x_k}$ is continuous, and $\left| \frac{\partial^2 g_i(\mathbf{x})}{\partial x_j \partial x_k} \right| \leq M$ for some constant M, whenever $\mathbf{x} \in N_\delta$, for each $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$, and $k = 1, 2, \ldots, n$;

(iii) $\frac{\partial g_i(\mathbf{p})}{\partial x_k} = 0$, for each $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n$.

Then a number $\tilde{\delta} \leq \delta$ exists such that the sequence generated by $\mathbf{x}^{(k)} = \mathbf{G}(\mathbf{x}^{(k-1)})$ converges quadratically to $\mathbf{p}$ for any choice of $\mathbf{x}^{(0)}$, provided that $\|\mathbf{x}^{(0)} - \mathbf{p}\| < \tilde{\delta}$. Moreover,

$$\|\mathbf{x}^{(k)} - \mathbf{p}\|_\infty \leq \frac{n^2 M}{2} \|\mathbf{x}^{(k-1)} - \mathbf{p}\|_\infty^2, \quad \text{for each } k \geq 1.$$

Fixed-Point Iteration

- Suppose that $\mathbf{A}(\mathbf{x})$ is an $n \times n$ matrix of functions from $\mathbb{R}^n$ into $\mathbb{R}$ in the form of Eq. (24), where the specific entries will be chosen later.
- Assume $\mathbf{A}(\mathbf{x})$ is nonsingular near a solution $\mathbf{p}$ of $\mathbf{F}(\mathbf{x}) = \mathbf{0}$, and let $b_{ij}(\mathbf{x})$ denote the entry of $\mathbf{A}(\mathbf{x})^{-1}$ in the ith row and jth column.
- For $\mathbf{G}(\mathbf{x}) = \mathbf{x} - \mathbf{A}(\mathbf{x})^{-1}\mathbf{F}(\mathbf{x})$, we have $g_i(\mathbf{x}) = x_i - \sum_{j=1}^{n} b_{ij}(\mathbf{x}) f_j(\mathbf{x})$. So

$$\frac{\partial g_i}{\partial x_k}(\mathbf{x}) = \begin{cases} 1 - \sum_{j=1}^{n} \left(b_{ij}(\mathbf{x}) \frac{\partial f_j}{\partial x_k}(\mathbf{x}) + \frac{\partial b_{ij}}{\partial x_k}(\mathbf{x}) f_j(\mathbf{x}) \right), & \text{if } i = k, \\ -\sum_{j=1}^{n} \left(b_{ij}(\mathbf{x}) \frac{\partial f_j}{\partial x_k}(\mathbf{x}) + \frac{\partial b_{ij}}{\partial x_k}(\mathbf{x}) f_j(\mathbf{x}) \right), & \text{if } i \neq k. \end{cases}$$

- The above Theorem implies that we need $\frac{\partial g_i(\mathbf{p})}{\partial x_k} = 0$, for each $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n$. This means that for $i = k$,

$$0 = 1 - \sum_{j=1}^{n} b_{ij}(\mathbf{p}) \frac{\partial f_j}{\partial x_i}(\mathbf{p}),$$

Newton-Raphson method

- That is,

$$\sum_{j=1}^{n} b_{ij}(\mathbf{p}) \frac{\partial f_j}{\partial x_i}(\mathbf{p}) = 1. \tag{25}$$

When $k \neq i$,

$$0 = -\sum_{j=1}^{n} b_{ij}(\mathbf{p}) \frac{\partial f_j}{\partial x_k}(\mathbf{p}),$$

so

$$\sum_{j=1}^{n} b_{ij}(\mathbf{p}) \frac{\partial f_j}{\partial x_k}(\mathbf{p}) = 0. \tag{26}$$

Define the matrix $J(\mathbf{x})$ by

$$J(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(\mathbf{x}) & \frac{\partial f_n}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_n}{\partial x_n}(\mathbf{x}) \end{bmatrix}. \tag{27}$$

Then conditions (25) and (26) require that

$$\mathbf{A}(\mathbf{p})^{-1} J(\mathbf{p}) = I, \text{ the identity matrix,} \quad \text{so } \mathbf{A}(\mathbf{p}) = J(\mathbf{p}).$$

Fixed Point Iteration

- That is, An appropriate choice for $\mathbf{A}(\mathbf{x})$ is, consequently, $\mathbf{A}(\mathbf{x}) = J(\mathbf{x})$, since this satisfies condition (iii) in the above Theorem. The function $\mathbf{G}$ is defined by

$$\mathbf{G}(\mathbf{x}) = \mathbf{x} - J(\mathbf{x})^{-1}\mathbf{F}(\mathbf{x}),$$

  and the functional iteration procedure evolves from selecting $\mathbf{x}^{(0)}$ and generating, for $k \geq 1$,

$$\mathbf{x}^{(k)} = \mathbf{G}(\mathbf{x}^{(k-1)}) = \mathbf{x}^{(k-1)} - J(\mathbf{x}^{(k-1)})^{-1}\mathbf{F}(\mathbf{x}^{(k-1)}). \tag{10.9}$$

  This is called **Newton's method for nonlinear systems**.

Fixed Point Iteration

Example

$$3x_1 - \cos(x_2x_3) - \frac{1}{2} = 0, \tag{28}$$

$$x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 = 0, \tag{29}$$

$$e^{-x_1x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0 \tag{30}$$

was shown fixed-point method that the approximate solution $(0.5, 0, -0.5235877)^t$. Apply Newton's method to this problem with $\mathbf{x}^{(0)} = (0.1, 0.1, -0.1)^t$.

Solution Define

$$\mathbf{F}(x_1, x_2, x_3) = (f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3))^t, \tag{31}$$

where

$$f_1(x_1, x_2, x_3) = 3x_1 - \cos(x_2x_3) - \frac{1}{2}, \tag{32}$$

$$f_2(x_1, x_2, x_3) = x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06, \tag{33}$$

$$f_3(x_1, x_2, x_3) = e^{-x_1x_2} + 20x_3 + \frac{10\pi - 3}{3}. \tag{34}$$

Fixed Point Iteration

- The Jacobian matrix $J(\mathbf{x})$ for this system is

$$J(x_1, x_2, x_3) = \begin{bmatrix} 3 & x_1 \sin x_2 x_3 & x_2 \sin x_2 x_3 \\ 2x_1 & -162(x_2 + 0.1) & \cos x_3 \\ -x_2 e^{-x_1 x_2} & -x_1 e^{-x_1 x_2} & 20 \end{bmatrix}.$$

- Let $\mathbf{x}^{(0)} = (0.1, 0.1, -0.1)^T$. Then $\mathbf{F}(\mathbf{x}^{(0)}) = (-0.199995, -2.269833417, 8.462025346)^T$ and

$$J(\mathbf{x}^{(0)}) = \begin{bmatrix} 3 & 9.999833334 \times 10^{-4} & 9.999833334 \times 10^{-4} \\ 0.2 & -32.4 & 0.995004165 \\ -0.0990049833 & -0.0990049833 & 20 \end{bmatrix}.$$

- Solving the linear system, $J(\mathbf{x}^{(0)})\mathbf{y}^{(0)} = -\mathbf{F}(\mathbf{x}^{(0)})$ gives

$$\mathbf{y}^{(0)} = \begin{bmatrix} 0.3998696728 \\ -0.08053315147 \\ -0.4215204718 \end{bmatrix} \quad \text{and} \quad \mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \mathbf{y}^{(0)} = \begin{bmatrix} 0.4998696782 \\ 0.01946684853 \\ -0.5215204718 \end{bmatrix}.$$

- Continuing for $k = 2, 3, \ldots$, we have

$$\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{bmatrix} = \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{bmatrix} + \begin{bmatrix} y_1^{(k-1)} \\ y_2^{(k-1)} \\ y_3^{(k-1)} \end{bmatrix},$$

Fixed Point Iteration

- where

$$\begin{bmatrix} y_1^{(k-1)} \\ y_2^{(k-1)} \\ y_3^{(k-1)} \end{bmatrix} = -\left(J\left(x_1^{(k-1)}, x_2^{(k-1)}, x_3^{(k-1)}\right)\right)^{-1} \mathbf{F}\left(x_1^{(k-1)}, x_2^{(k-1)}, x_3^{(k-1)}\right).$$

- Thus, at the kth step, the linear system $J\left(\mathbf{x}^{(k-1)}\right)\mathbf{y}^{(k-1)} = -\mathbf{F}\left(\mathbf{x}^{(k-1)}\right)$ must be solved, where

$$J\left(\mathbf{x}^{(k-1)}\right) = \begin{bmatrix} 3 & x_1^{(k-1)} \sin x_2^{(k-1)} x_3^{(k-1)} & x_2^{(k-1)} \sin x_2^{(k-1)} x_3^{(k-1)} \\ 2x_1^{(k-1)} & -162\left(x_2^{(k-1)} + 0.1\right) & \cos x_3^{(k-1)} \\ -x_2^{(k-1)} e^{-x_1^{(k-1)} x_2^{(k-1)}} & -x_1^{(k-1)} e^{-x_1^{(k-1)} x_2^{(k-1)}} & 20 \end{bmatrix},$$

$$\mathbf{y}^{(k-1)} = \begin{bmatrix} y_1^{(k-1)} \\ y_2^{(k-1)} \\ y_3^{(k-1)} \end{bmatrix},$$

and

$$\mathbf{F}\left(\mathbf{x}^{(k-1)}\right) = \begin{bmatrix} 3x_1^{(k-1)} - \cos x_2^{(k-1)} x_3^{(k-1)} - \frac{1}{2} \\ \left(x_1^{(k-1)}\right)^2 - 81\left(x_2^{(k-1)} + 0.1\right)^2 + \sin x_3^{(k-1)} + 1.06 \\ e^{-x_1^{(k-1)} x_2^{(k-1)}} + 20x_3^{(k-1)} + \frac{10\pi - 3}{3} \end{bmatrix}.$$

- The results using this iterative procedure are shown in Table.

Fixed Point Iteration

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty$
0	0.1000000000	0.1000000000	-0.1000000000	
1	0.4998696728	0.0194608485	-0.5215204718	0.4215204718
2	0.5000142403	0.0015885914	-0.5235569638	1.788×10^{-2}
3	0.5000000113	0.0000124448	-0.5235984500	1.576×10^{-3}
4	0.5000000000	8.516×10^{-10}	-0.5235987755	1.244×10^{-5}
5	0.5000000000	-1.375×10^{-11}	-0.5235987756	8.654×10^{-10}


ANY
QUESTIONS?