DS-288 Numerical Methods
UMC-202 Introduction to Scientific Computing
Mid Sem Exam 2025

Time: 2 hr

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Weight 30% (90 Points)

Part-I (Multiple Choice Questions) [Total: $4 \times 5 = 20$]

- *There is no penalty for incorrect answers.*
- ***More than one option may be correct. Credits will be given only if the correct option(s) and no wrong option(s) are marked. Partial credit(s) will be given.***

1. Richardson extrapolation improves accuracy by:

   (a) Reducing the effective step size

   (b) Cancelling the leading error term

   (c) Using higher derivatives

   (d) Combining results from different step sizes

2. Solving the BVP

$$-y'' = f(x), \quad y(0) = y(1) = 0$$

   with central differences yields a linear system that is:

   (a) Toeplitz and tridiagonal

   (b) Dense and symmetric

   (c) Upper triangular

   (d) Diagonal

3. The local truncation error of the explicit Euler method is:

   (a) $O(h)$

(b) $O(h^2)$

(c) $O(h^3)$

(d) $O(h^4)$

4. Consider the forward difference approximation for the second derivative:

$$f''(x_i) \approx \frac{af_i + bf_{i+1} + cf_{i+2}}{h^2}.$$

What are the values of a, b, and c that make this approximation *second-order accurate*?

(a) $a = 1, b = -2, c = 1$

(b) $a = 2, b = -5, c = 4$

(c) $a = -2, b = 5, c = -4$

(d) $a = 1, b = -3, c = 3$

Part-II (Assertion type question) [Total: $4 \times 5(1+4) = 20$]

- *Write True/False in the empty box and explain the reasoning for each question.*

1. An $n-$point Gaussian quadrature rule integrates exactly any polynomial of degree up to $2n-1$.

TRUE

Explanation:

A n-point Gaussian quadrature rule has 2n degrees of freedom. These 2n degrees can be used to satisfy 2n equations, thereby allowing the rule to integrate exactly for the first 2n monomials, i.e $1, x, x^2, \cdots, x^{2n-1}$. Hence, the highest degree polynomial will be of order $2n-1$.

2. Is $f(x,y) = 2x+3y$ is a Lipschitz function on the domain $R = \{(x,y) : 0 \le x \le 1, -\infty < y < \infty\}$.

TRUE.

Explanation:

$$|f(x,y_1) - f(x,y_2)|$$
$$= |2x + 3y_1 - 2x - 3y_2|$$
$$= 3|y_1 - y_2|$$
$$\therefore L = 3$$

3. The central difference second-derivative formula is exact for all polynomials of degree ≤ 3.

TRUE

Explanation:

The error term is $\frac{h^2}{12} f^{(4)}(\mu)$. Error $= 0$ $\forall$ polynomials with order ≤ 3.

4. For the linear system $Ax \approx b$ (overdetermined), the normal equations $A^\top Ax = A^\top b$ yield a least-squares solution only if A has full column rank.

TRUE.

Explanation:

To have a solution to $A^T A x = A^T b$, $A^T A$ needs be square and full rank. $A^T A$ will always be square, but will be full-rank iff A has full column rank..

Part-III (Fill in the numerical values) [Total: $4 \times 5 = 20$]

- *Write your answers in the missing place. Credits will be awarded only for the final answer.*

1. Using Simpson's rule with $n = 2$ subintervals, the value of

$$\int_0^2 (1 + x^2)dx$$

is approximated as $14/3 \simeq 4.6667$.

$$\int_0^2 (1+x^2) = \frac{\frac{2-0}{2}}{3}\left(f(0)+4f(1)+f(2)\right) = \frac{1+8+5}{3} = 14/3$$

2. When solving $f(x) = e^{-x} - x$ using Newton-Raphson with $x_0 = 0$, the absolute error $|x_1 - x^*|$ after the first iteration (where x^* is the true root ≈ 0.567143) is 0.067143 (round to two decimals). $x_0 = 0$ $\quad f'(x) = -e^{-x} - 1$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{1}{-2} = \frac{1}{2}$$

$$\therefore \text{Error} = |x_1 - x^*| = 0.067143$$

3. Using the 3rd-order Taylor polynomial of e^x around $x = 1$, the estimated value of $e^{1.5}$ is 4.474.

$$e^x \simeq e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 = e\left(1\tfrac{1}{2} + \tfrac{1}{8} + \tfrac{1}{48}\right) = \frac{31e}{48}$$

$$\simeq 4.474$$

4. For $f(x) = x(|x| + |x - 1|)$ the smallest Lipschitz constant L in the interval $[-2, 2]$ is 9.

$$f(x) = \begin{cases} x(1-2x) & -2 \le x < 0 \\ x & 0 \le x \le 1 \\ x(2x-1) & 1 < x \le 2 \end{cases} \qquad f'(x) = \begin{cases} 1-4x, & -2 \le x \le 0 \\ 1, & 0 \le x \le 1 \\ 4x-1, & 1 < x \le 2 \end{cases} \qquad \max(f'(x)) = \begin{cases} 9 \\ 1 \\ 7 \end{cases}$$

$$\therefore L = 9$$

...

Part-IV (Subjective type question) [Total: $2 \times 15 = 30$]

- *Write your answers in the space provided.*

1. (a) Since set of Legendre polynomials, $\{P_n(x)\}$, is orthogonal on $[-1, 1]$ with respect to the weight function $w(x) \equiv 1$. Use first three Legendre polynomials, $P_0(x) = 1, P_1(x) = x$, and $P_2(x) = \frac{3x^2-1}{2}$ to approximate $f(x) = x^3 + x^2 + x$ as $f(x) \approx \alpha P_0 + \beta P_1 + \gamma P_2$. [1.5+1.5+1.5]

   (b) Determine a, b, c such that the following formula

$$\int_0^h f(x)dx = h\left\{af(0) + bf(\frac{h}{3}) + cf(h)\right\}$$

   is exact for the polynomial of as high order as possible.

a) $\alpha \int_{-1}^{1} w(x)\,(P_0(x))^2 dx = \int_{-1}^{1} w(x)\,(f(x))(P_0(x))\,dx$

$\alpha \cdot 2 = 2/3$

$\alpha = 1/3$

$\beta \cdot \int_{-1}^{1} w(x)\,(P_1(x))^2 dx = \int_{-1}^{1} w(x)\,(f(x))(P_1(x))\,dx$

$\beta \cdot 2/3 = \frac{16}{15}$

$\beta = 8/5$

$\gamma \int_{-1}^{1} w(x)(P_2(x))^2 dx = \int_{-1}^{1} w(x)(f(x))(P_2(x))dx$

$\gamma \cdot 2/5 = 4/15$

$\gamma = 2/3$

b) Method of undetermined coefficients!

$$\int_0^h f(x)\,dx = h\{af(0) + bf(h/3) + cf(h)\}$$

Let $f(x)=1$: $\int_0^h 1\,dx = h$

$h\{a\cdot 1 + b + c\}$ $\therefore (a+b+c)h = h \Rightarrow a+b+c=1$ (1)

Let $f(x)=x$ $\int_0^h f(x)\,dx = \frac{h^2}{2}$

$h(\cancel{af(0)}^{0} + bf(h/3) + cf(h)) = b\cdot\frac{h^2}{3} + c\cdot h^2$

$b\cdot\frac{h^2}{3} + ch^2 = \frac{h^2}{2}$

$\frac{b}{3} + c = 1/2$ (2)

Let $f(x) = x^2$

$\int_0^h f(x)\,dx = \frac{h^3}{3}$

$h(\cancel{af(0)}^{0} + bf(h/3) + cf(h))$

$b\cdot\frac{h^3}{9} + ch^3$

$h^3\left(\frac{b}{9} + c\right) = h^3/3$

$b/9 + c = 1/3$ (3)

Solving, $a=0, b=3/4, c=1/4$

$\therefore \int_0^h f(x)\,dx = h\{3/4 f(h/3) + \frac{f(h)}{4}\}$

is accurate upto degree 2.

2. Approximate the integral below using Composite Simpson's rule with $n = m = 4$. In every intermediate arithmetic step, round to 3 decimal places.

$$\int_1^3 \frac{1}{x}\, dx$$

Compare with the exact value $\ln 3 = 1.099$.

$$\int_1^3 \frac{1}{x}\, dx \simeq \frac{\frac{3-1}{4}}{3}\left(\frac{1}{1} + 4\left(\frac{1}{1.5}\right) + 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{2.5}\right) + \frac{1}{3}\right)$$

$$\simeq \frac{1}{6}\left(2 + \frac{8}{3} + \frac{8}{5} + \frac{1}{3}\right)$$

$$\simeq 1.1$$

$$\text{Error} = |1.1 - \ln 3| = 1.388 \times 10^{-3}$$