Numerical Methods

DS288 and UMC201

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Lagrange Interpolating Polynomial

Lagrange Polynomial

- We will define a function that passes through the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$. First, lets define

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} \quad \text{and} \quad l_1(x) = \frac{x - x_0}{x_1 - x_0}$$

- Then define the interpolating polynomial

$$p(x) = l_0(x) f(x_0) + l_1(x) f(x_1)$$

- We can verify that $p(x_0) = f(x_0)$, and $p(x_1) = f(x_1)$.
- $p(x)$ is the unique linear polynomial passing through the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

Lagrange interpolating polynomial

- Let $x_0, x_1, \cdots, x_n$ be $n+1$ distinct points in $[a, b]$.
- Consider a interpolating polynomial $p(x)$ of degree $\leq n$, i.e.,
- $p(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + \cdots + l_n(x)f(x_n)$ where $l_i(x),\ 0 \leq i \leq n$ are polynomial of degree n.
- The polynomial $p(x)$ will satisfy the interpolating conditions $p(x_i) = f(x_i)$ *if and only if*

$$l_i(x_j) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j \end{cases} \tag{1}$$

- The polynomial $l_i(x)$ satisfying the (1) can be written as $\ell_i(x) = \frac{(x - x_0)}{(x_i - x_0)} \cdots \frac{(x - x_{i-1})}{(x_i - x_{i-1})} \frac{(x - x_{i+1})}{(x_i - x_{i+1})} \cdots \frac{(x - x_n)}{(x_i - x_n)}$, where $0 \leq i \leq n$.

Lagrange interpolating polynomial

- Here $\ell_i(x) = \dfrac{(x-x_0)}{(x_i-x_0)} \cdots \dfrac{(x-x_{i-1})}{(x_i-x_{i-1})} \dfrac{(x-x_{i+1})}{(x_i-x_{i+1})} \cdots \dfrac{(x-x_n)}{(x_i-x_n)}$,
- The function $l_i(x)$ are called Lagrange coefficients and the polynomial $p(x)$ is called lagrange interpolating polynomial.

Theorem

If $x_0, x_1, \cdots, x_n$ are $n+1$ distinct numbers and f is a function whose values are given at these numbers, then a unique polynomial $p(x)$ of degree at most n exists with $f(x_k) = p(x_k)$, for each $k = 0, 1, \cdots, n$

$$p(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + \cdots + l_n(x)f(x_n) \tag{2}$$

where

$$L_k(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_0)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_0)} \tag{3}$$

Uniqueness of interpolating polynomial

Theorem

Let $x_0, x_1, \cdots, x_n$ be $n+1$ distinct points in $[a, b]$. There exists a unique polynomial $p(x)$ of degree n or less, that interpolates $f(x)$ at the points $\{x_i\}$, that is, $p(x_i) = f(x_i)$, for $0 \leq i \leq n$.

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- Let $q_n(x)$ be a polynomial of degree $\leq n$, which interpolates $f(x)$ at the points $x_0, x_1, \cdots, x_n$.

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- Let $q_n(x)$ be a polynomial of degree $\leq n$, which interpolates $f(x)$ at the points $x_0, x_1, \cdots, x_n$.
- Let $\phi(x) = p_n(x) - q_n(x)$.
- Then $\phi(x)$ is a polynomial of degree $\leq n$.
- However $\phi(x_j) = p_n(x_j) - q_n(x_j) = f_j - f_j = 0$ for $j = 0, 1, \cdots n$.
- Thus $\phi(x)$ which is a polynomial of degree $\leq n$ has $n+1$ zeros, i.e., $x_0, x_1, \cdots, x_n$.
- This can happen only if $\phi(x) \equiv 0$.
- Hence $p_n(x) = q_n(x)$

Particular case of Lagrange interpolation

- We have $p(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + \cdots + l_n(x)f(x_n)$.
- For $n = 1$ the corresponding interpolating polynomial is

$$p_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) \tag{4}$$

- For $n = 2$ the corresponding interpolating polynomial is

$$p_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

- For $n = 3$ the corresponding interpolating polynomial is

$$p_3(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0) + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1) + \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f(x_2) + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} f(x_3)$$

Example

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Find the interpolating polynomial for the following data
$f(0) = 1, f(-1) = 2, f(1) = 3$

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- The degree of the interpolating polynomial is at most two.
- $x_0 = 0, \quad x_1 = -1, \quad x_2 = 1$

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- $l_2(x) = \frac{1}{2}(x^2 + x)$

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- $l_2(x) = \frac{1}{2}(x^2 + x)$
- Thus $p_2(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2)$

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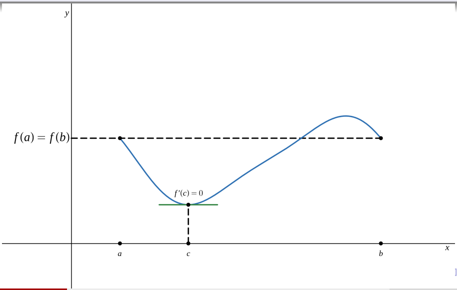
- $l_0(x) = -(x^2 - 1)$
- $l_1(x) = \frac{1}{2}(x^2 - x)$
- $l_2(x) = \frac{1}{2}(x^2 + x)$
- Thus $p_2(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2)$
- $p_2(x) = 1 + \frac{1}{2}x + \frac{3}{2}x^2$

Rolle's Theorem

Rolle's Theorem: If $f(x)$ is a real valued continuous function on $[a, b]$ and $f'(x)$ exists on (a, b). Also if $f(a) = f(b)$, then there exits at least one number $c \in (a, b)$ such that $f'(c) = 0$.

Generalized Rolle's Theorem: Suppose $f \in \mathbb{C}[a, b]$ is n-times differentiable on (a, b). If $f(x)$ is zero at the $n + 1$ distinct point $x_0, x_1, \cdots, x_n$ in $[a, b]$, then there exists a number $c \in (a, b)$ such that $f^{(n)}(c) = 0$.



Interpolation Error

Theorem

- Let $f(x) \in \mathbb{C}^{n+1}[a, b]$ and $x_i \in [a,\ b]$ for $0, 1, \cdots n$. Consider $p_n(x)$ is a polynomial interpolating $f(x)$ at $x_i = 0, 1, \cdots n$.
- Then
$$f(x) - p(x) = \frac{f^{(n+1)(x^*)}}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n),$$

  where x^* is an unknown point in the interval $(a,\ b)$.

- Consider $M = \max_{a \le x \le b} |f^{(n+1)}(x^*)|$, then
- $|E(x)| = |f(x) - p(x)| \le \frac{M}{(n+1)!}|(x - x_0)(x - x_1)\cdots(x - x_n)|$,
- Further, if $N = \max_{a \le x \le b} |(x - x_0)(x - x_1)\cdots(x - x_n)|$ then

$$|E(x)| \le \frac{MN}{(n+1)!}$$

Linear Interpolation

Example

Find a bound for the error in linear interpolation.

- Let the nodes be x_0, x_1 then the linear interpolation

$$p_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) \tag{5}$$

- The interpolation error formula is $|E(x)| \leq \frac{M}{2!} |(x - x_0)(x - x_1)|$.
- Define $g(x) = (x - x_0)(x - x_1)$, on $[x_0, x_1]$.
- Differentiate: $g'(x) = (x_1 - x) - (x - x_0) = x_1 + x_0 - 2x$.
- Setting $g'(x) = 0$ gives $x = \frac{x_0 + x_1}{2}$.
- At this midpoint, $g\left(\frac{x_0+x_1}{2}\right) = \left(\frac{x_1 - x_0}{2}\right)^2 = \frac{(x_1 - x_0)^2}{4}$.
- Hence,

$$|E(x)| \leq \frac{M}{2} \cdot \frac{(x_1 - x_0)^2}{4} = \frac{M}{8}(x_1 - x_0)^2.$$

Lagrange interpolating polynomial

Example

Determine the step size h that can be used in the tabulation of $f(x) = \sin x$ in the interval $[1, 3]$ so that the linear interpolation will be correct to four decimal places after rounding.

- Given $f(x) = \sin x$, then $f'(x) = \cos x$ and $f''(x) = -\sin x$
- We need to find: $\max_{x\in[1,3]} |f''(x)| = \max_{x\in[1,3]} |-\sin x| = \max_{x\in[1,3]} |\sin x|$
- Since $\sin x$ achieves its maximum value of 1 at $x = \frac{\pi}{2} \approx 1.5708$, and $\frac{\pi}{2} \in [1, 3]$, we have:
$$\max_{x\in[1,3]} |f''(x)| = 1$$
- For 4-decimal-place accuracy after rounding: $\varepsilon = 0.5 \times 10^{-4} = 0.00005$
- The error bound for linear interpolation is: $|E(x)| \leq \frac{h^2}{8} \max_{x\in[a,b]} |f''(x)| = \frac{h^2}{8}$
- The step size must satisfy: $\boxed{h \leq 0.02}$


ANY
QUESTIONS?