DS 288: NUMERICAL METHODS

NOV-18-2021

Topics:

- ODE Solution Methods to IVPs
- Eulers Method
- Runge-Kutta 2^{nd} and 4^{th} order
- Adams-Bashforth (explicit) and Adams-Moulton (Implicit) Methods
- AB/AM Predictor/Corrector
- Systems and higher order ODES
- Stability
- Variable step-size
- Stiff ODEs

Sections in text: 5.1–5.11 (except 5.8)

Study Questions–IV [1]

1. Is it possible for ODE solution families to have regions of stability and instability?. What conditions ensure stability?.

   STABLE: $J \equiv \frac{\partial f}{\partial y} < 0$ UNSTABLE $J > 0$

2. In class we have argued that the global error of a particular method for solving ODEs is one order less than the order of its local truncation error which results from the Taylor series. Show that this is true by considering Euler's method and examining the order of truncation error after it has been accumulated over N steps.

3. Show graphically where w_1 would be for the Runge-Kutta method (Heun's method)

$$w_{i+1} = w_i + \frac{h}{4}\left[f_i + 3f\left(w_i + \frac{2}{3}hf_i,\ t_i + \frac{2}{3}h\right)\right] \tag{1}$$

   given $w_0 = y_0$ as the initial value.

4. In class, we showed that the Runge-Kutta 2^{nd} order (RK2) have the general form

$$w_{i+1} = w_i + h\left[f_i + \frac{h}{2}\left(\frac{\partial f_i}{\partial t} + \frac{\partial f_i}{\partial y}f_i\right)\right] \tag{2}$$

   An entire set of RK2 methods can be created by approximating

$$\left[f_i + \frac{h}{2}\left(\frac{\partial f_i}{\partial t} + \frac{\partial f_i}{\partial y}f_i\right)\right] \tag{3}$$

   as a weighted average of the form

$$a_1 f_i + a_2 f(w_i + \beta, t_i + \alpha) \tag{4}$$

   that is a weighted average of the slope at (w_i, t_i) and the slope at $(w_i + \beta, t_i + \alpha)$. Determine the relationships between a_1, a_2, α, and β that must hold by substituting the *zeroth* and *first* derivative terms in the 2D Taylor series for $f(w_i + \beta, t_i + \alpha)$ and equating Eq. 3 to Eq. 4.

5. For the fourth order Adams-Bashforth (explicit) and Adams-Moulton (Implicit) methods, show what derivatives of the rate function need to be approximated and to what order of accuracy they need to be realized in order to obtain 4^{th} order behavior. Indicate what function evaluations are needed for each derivative (i.e, f_{i+1}, f_i, f_{i-1}, etc.)

6. Define *consistency*, *stability*, and *convergence* in terms of ODE solution methods.

7. For a stiff ODE which of the following methods would be the most desirable

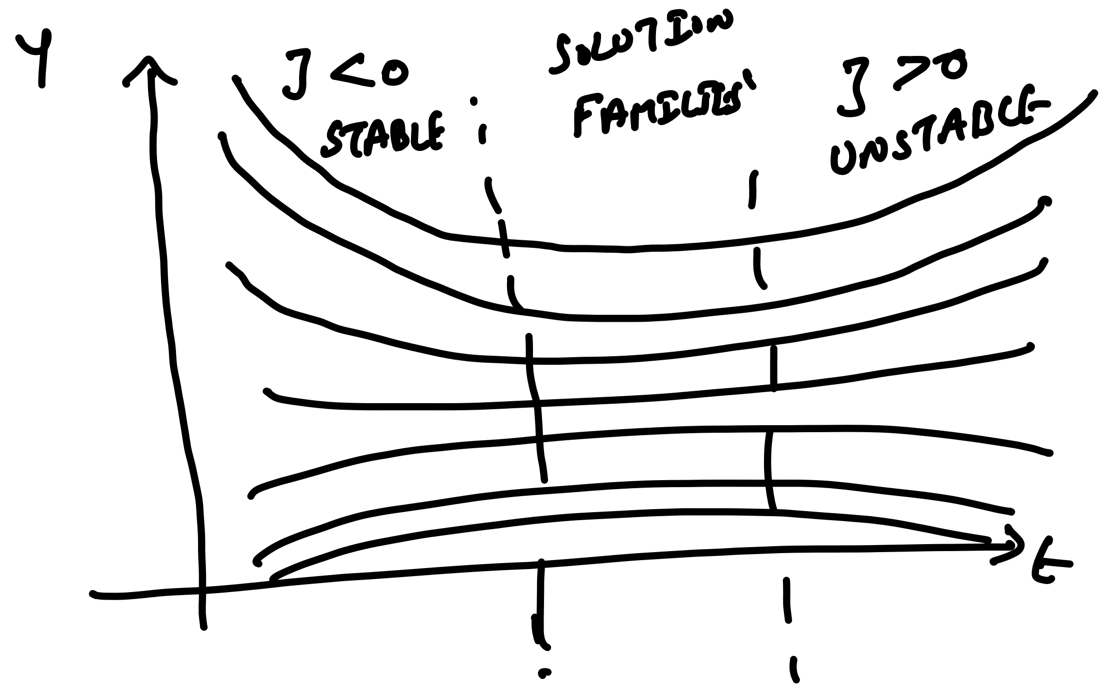
$$\epsilon_{i+1} = (1 + hJ)\epsilon_i + O(h^2) \tag{5}$$

$$\epsilon_{i+1} = \frac{1 + \frac{h}{2}J}{1 - \frac{h}{2}J}\epsilon_i + O(h^3) \tag{6}$$

$$\epsilon_{i+1} = (1 - hJ)^{-1}\epsilon_i + O(h^2) \tag{7}$$

[1] Posted on: October 27, 2021.

(1)



(2) EULER'S METHOD

$$Y_{i+1} = Y_i + h f_i + \underbrace{\frac{h^2}{2} f'(\xi_i)}_{}\quad \text{EXACT}$$

$$w_{i+1} = w_i + h f_i$$

$$-$$

$$\epsilon_{i+1} = \epsilon_i + O(h^2) \quad [\text{ASSUMPTION } \underline{f(w_i, t_i) = f(Y_i, t_i)}]$$

LOOK AT ERROR ACCUMLATION

$$\epsilon_0 = 0$$

$$\epsilon_1 = O(h^2) = \epsilon$$

$$\epsilon_2 = \epsilon_1 + O(h^2) \cong 2\epsilon \quad [\xi_{i+1} = \xi_i]$$

$$\epsilon_3 = \epsilon_2 + O(h^2) \cong 3\epsilon$$

$\vdots \quad \vdots \qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\epsilon_N = \epsilon_{N-1} + O(h^2) \approx N\epsilon$$

$$N = \frac{b-a}{h} = \frac{t_{end} - t_0}{h}$$

$$\epsilon_N = \frac{t_{end} - t_0}{h}\epsilon = \frac{(t_{end} - t_0)}{h} O(h^2)$$

$$\epsilon_N = \bar{t}\, O(h)$$

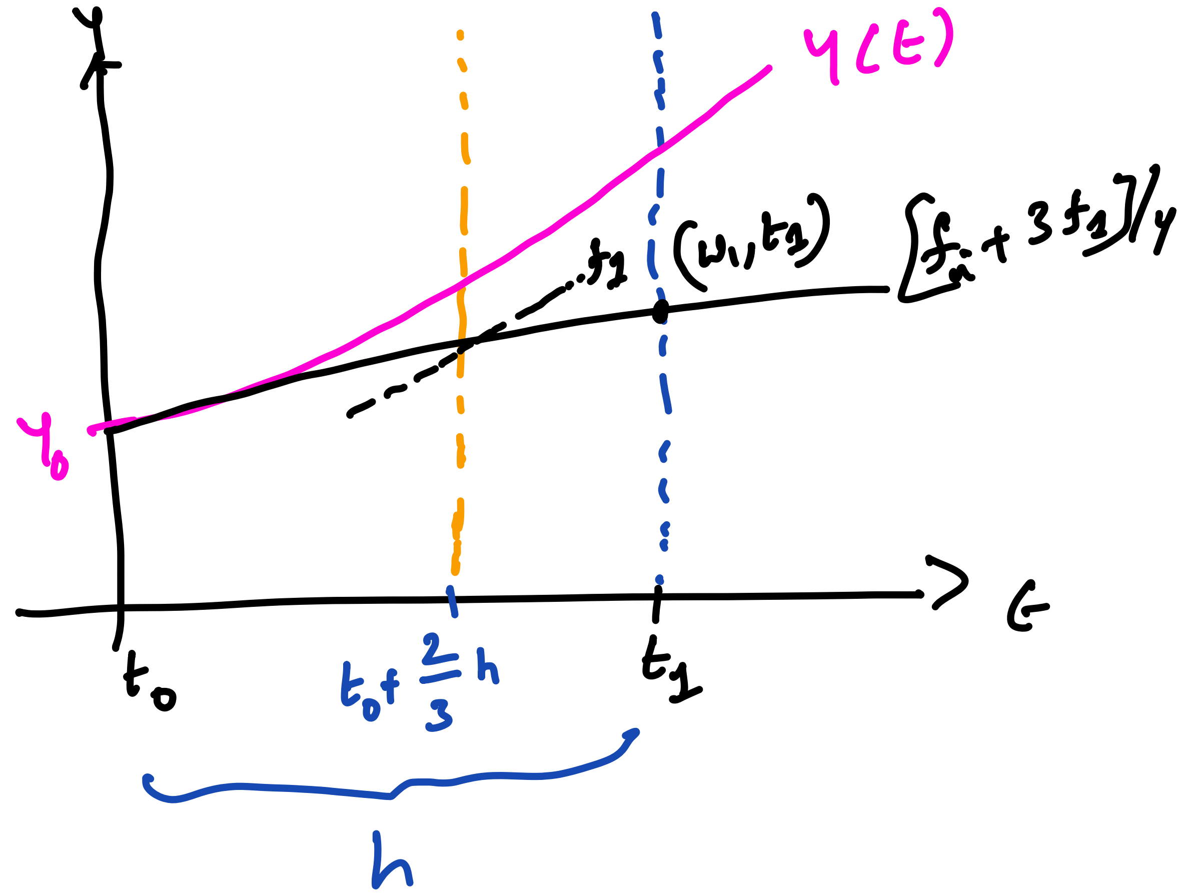
$$\boxed{\begin{array}{c}\epsilon_N = O(h) \\ \text{GLOBALLY}\end{array}}$$

[SIMILAL COMPOSITE INTEGRATION RULE]

(3) HEUN'S METHOD:

$$w_{i+1} = w_i + \frac{h}{4}\left[f_i + 3f\left(w_i + \frac{2}{3}hf_i, t_i + \frac{2}{3}h\right)\right]$$

$$= w_i + h\left[f_i + 3f_1\right]/4$$



(4) 2D TAYLOR POLYNOMIAL

$$\underline{f(w_i+\beta, t_i+\alpha)} \cong \underbrace{f(w_i, t_i)}_{f_i} + \frac{\partial f_i}{\partial y}\beta + \frac{\partial f_i}{\partial t}\alpha$$

$$a_1 f_i + a_2\left[f_i + \frac{\partial f_i}{\partial y}\beta + \frac{\partial f_i}{\partial t}\alpha\right] = f_i + \frac{h}{2}\left[\frac{\partial f_i}{\partial y} f_i + \frac{\partial f_i}{\partial t}\right]$$

$$(a_1+a_2) f_i + a_2\beta\frac{\partial f_i}{\partial y} + a_2\alpha\frac{\partial f_i}{\partial t}$$

$$= f_i + \frac{h}{2} f_i \frac{\partial f_i}{\partial y} + \frac{h}{2}\frac{\partial f_i}{\partial t}$$

EQUATE THE TERMS

$$\boxed{\begin{aligned} & a_1 + a_2 = 1 \\ & \left.\begin{aligned} a_2\beta &= \frac{h}{2} f_i \\ a_2\alpha &= \frac{h}{2} \end{aligned}\right\} \quad \beta = f_i \alpha \end{aligned}}$$

PICK ANY ONE i.e. $a_1 = 0$

$$\Rightarrow a_2 = 1$$

$$\beta = \frac{h}{2} f_i \quad \alpha = \frac{h}{2}$$

$$\Rightarrow w_{i+1} = w_i + h\left[f\left(w_i + \frac{h}{2} f_i, t_i + \frac{h}{2}\right)\right]$$

MID POINT RULE

(5) $y' = f(y, t) \Rightarrow y_{i+1} - y_i = \int_{t_i}^{t_{i+1}} f \, d\tau$

TAYLOR SERIES

$$y_{i+1} = y_i + h f_i + \frac{h^2}{2} f_i' + \frac{h^3}{3!} f_i'' + \frac{h^4}{4!} f_i''' + \frac{h^5}{5!} f_i^{iv} + \ldots$$

WANT GTE : $O(h^4)$ LOCALLY : $O(h^5)$

$f_i' \to O(h^3)$

$f_i'' \to O(h^2)$

$f_i''' \to O(h)$

A-B : $f_i' \to f_i, f_{i-1}, f_{i-2}, f_{i-3}$

A-M : $f_i' \to f_{i+1}, f_i, f_{i-1}, f_{i-2}$

A-B :- $f_i'' \to f_i, f_{i-1}, f_{i-2}, f_{i-3}$

A-M :- $f_i'' \to f_{i+1}, f_i, f_{i-1}, f_{i-2}$

A-B : $f_i''' \to f_i, f_{i-1}, f_{i-2}, f_{i-3}$

A-M : $f_i''' \to f_{i+1}, f_i, f_{i-1}, f_{i-2}$

(6) REFER TO NOTES

ODE NUMERICAL SOLUTIONS

$\Rightarrow$ EQUIVALENCE THEOREM

(7) STIFF ODE

$J = \frac{\partial f}{\partial \epsilon}$ IS LARGE $\quad J \to \infty$

(i) $\epsilon_{i+1} = \epsilon_i (1 + hJ) + O(h^2)$

$\epsilon_{i+1} \to \infty \quad$ ERROR IS NOT BOUNDED

NOT STABLE

EQ. THEOREM $\Rightarrow$ NOT CONVERGENT

(ii) $\epsilon_{i+1} = \left(\dfrac{1 + \frac{h}{2} J}{1 - \frac{h}{2} J} \right) \epsilon_i + O(h^3)$

$J \to \infty \quad \epsilon_{i+1} \to \epsilon_i$

$\therefore$ STABLE $\quad$ "ERROR IS BOUNDED"

(iii) $\epsilon_{i+1} = \dfrac{1}{1-hJ}\,\epsilon_i + O(h^2)$

$J \to \infty \qquad \epsilon_{i+1} \to 0$

FOR FINITE 'h' → ERRORS GO TO ZERO

BEST METHOD : (iii)

SECOND DESIRABLE: (ii)

(i) UNSTABLE, NOT DESIRABLE.