DS 288: NUMERICAL METHODS

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STIFF - ODE

$y' = -ky \Rightarrow y = y_0 e^{-kt}$

WHERE $\frac{\partial f}{\partial y} = -k$

LARGE $k \Rightarrow J$ IS LARGE

THESE TYPE ODE SOLUTIONS WILL BENEFIT FROM MORE THAN ONCE CORRECTION.

SYSTEM OF ODE'S :

* M FIRST ORDER ODE'S IN M VARIA-BLES & t.

$\frac{d\underline{z}}{dt} = \underline{f}(\underline{z}, t)$ WHERE $\underline{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix}$

& $\underline{f} = \begin{bmatrix} f_1(z_1, z_2, \dots z_m, t) \\ \vdots \\ f_m(z_1, z_2, \dots z_m, t) \end{bmatrix}$ M ELEMENT COLUMN VECTORS

System of ODEs: Example

Predictor/Corrector example:

$m=3$

$$\begin{aligned} y_1' &= y_2 y_3 - y_1 & \qquad y_1(0) &= \alpha \\ y_2' &= y_1 y_3 - y_2 & \qquad y_2(0) &= \beta \\ y_3' &= y_1 y_2 - y_3 & \qquad y_3(0) &= \gamma \end{aligned}$$

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Define: $Z_i^{P_n} =$ Predicted value of the n^{th} variable at time t_i.

$Z_i^{C_n} \equiv$ Corrected value of the n^{th} variable at time t_i.

Look at a two-step Adams Bashforth/Adams Moulton (A-B/A-M) method with **one** correction:

prediction

$$Z_{i+1}^{P_1} = Z_i^{C_1} + \frac{h}{2}\left[3\left(Z_i^{C_2} Z_i^{C_3} - Z_i^{C_1}\right) - \left(Z_{i-1}^{C_2} Z_{i-1}^{C_3} - Z_{i-1}^{C_1}\right)\right] \quad (1)$$

$$Z_{i+1}^{P_2} = Z_i^{C_2} + \frac{h}{2}\left[3\left(Z_i^{C_1} Z_i^{C_3} - Z_i^{C_2}\right) - \left(Z_{i-1}^{C_1} Z_{i-1}^{C_3} - Z_{i-1}^{C_2}\right)\right] \quad (2)$$

$$Z_{i+1}^{P_3} = Z_i^{C_3} + \frac{h}{2}\left[3\left(Z_i^{C_1} Z_i^{C_2} - Z_i^{C_3}\right) - \left(Z_{i-1}^{C_1} Z_{i-1}^{C_2} - Z_{i-1}^{C_3}\right)\right] \quad (3)$$

$\rightarrow$ COMPUTED IN ANY ORDER

correction

COMPUTED IN ANY ORDER

$$Z_{i+1}^{C_1} = Z_i^{C_1} + \frac{h}{12}\left[5\left(Z_{i+1}^{P_2} Z_{i+1}^{P_3} - Z_{i+1}^{P_1}\right) + 8\left(Z_i^{C_2} Z_i^{C_3} - Z_i^{C_1}\right) - \left(Z_{i-1}^{C_2} Z_{i-1}^{C_3} - Z_{i-1}^{C_1}\right)\right] \quad (4)$$

$$Z_{i+1}^{C_2} = Z_i^{C_2} + \frac{h}{12}\left[5\left(Z_{i+1}^{P_1} Z_{i+1}^{P_3} - Z_{i+1}^{P_2}\right) + 8\left(Z_i^{C_1} Z_i^{C_3} - Z_i^{C_2}\right) - \left(Z_{i-1}^{C_1} Z_{i-1}^{C_3} - Z_{i-1}^{C_2}\right)\right] \quad (5)$$

$$Z_{i+1}^{C_3} = Z_i^{C_3} + \frac{h}{12}\left[5\left(Z_{i+1}^{P_1} Z_{i+1}^{P_2} - Z_{i+1}^{P_3}\right) + 8\left(Z_i^{C_1} Z_i^{C_2} - Z_i^{C_3}\right) - \left(Z_{i-1}^{C_1} Z_{i-1}^{C_2} - Z_{i-1}^{C_3}\right)\right] \quad (6)$$

The main point here is that the *predictor* equations (1), (2), and (3) can be computed in **any** order, but they must **all** be computed **before** the *corrector* equations (4), (5), and (6). Once the *predictor* equations (1), (2), and (3) are computed, then the *corrector* equations (4), (5), and (6) can also be computed in **any** order.

- MUST PREDICT ALL EQUATIONS BEFORE COMING TO CORRECTION.

HIGHER ORDER ODE'S

Ex: $y'' = f(y', y, t)$ WHERE INITIAL CONDITIONS:

WITH $y'(0) = y_0'$

$y(0) = y_0$

* WRITE AS A SYSTEM OF FIRST ORER ODE'S, THEN SOLVE IT AS A SYSTEM OF ODES.

IN GENERAL

$$y^{(m)}(t) = f\left(y^{(m-1)}, y^{(m-2)}, \ldots y', y, t\right)$$

WITH INITIAL CONDITIONS

$$y_0, y_0', y_0'', \ldots y_0^{(m-1)}$$

NOW DEFINE

$$z_1(t) = y(t) \leftarrow \text{WANT TO FIND.}$$

$$z_2(t) = y'(t) = z_1'(t)$$

$$z_3(t) = y''(t) = z_2'(t)$$

$$\vdots \qquad \vdots$$

$$z_m(t) = y^{(m-1)}(t) = z_{m-1}'(t)$$

INITIAL CONDITIONS

$$z_1(0) = y_0$$
$$z_2(0) = y_0'$$
$$\vdots \quad \vdots \quad \vdots$$
$$z_m(0) = y_0^{(m-1)}$$

$$y^{(m)}(t) = z_m'(t)$$
$$= f(y^{m-1}, y^{m-2}, \dots y, t)$$

IN VECTOR NOTATION.

$$\begin{bmatrix} z_1'(t) \\ z_2'(t) \\ \vdots \\ z_m'(t) \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ f(z_m, z_{m-1}, \dots z_1, t) \end{bmatrix}$$

WITH $\underline{z}_0 = \begin{bmatrix} z_1(0) \\ z_2(0) \\ \vdots \\ z_m(0) \end{bmatrix}$

$$\underline{z}'(t) = \underline{f}(\underline{z}, t).$$

SYSTEM OF ODE'S.

EX:

$$y''(t) + 4y'(t) + 5y(t) = 0$$

I.C's: $y(0) = 3$, $y'(0) = 5$

$$y''(t) = \underbrace{-4y'(t) - 5y(t)}_{f(y, y', t)}$$

DEFINE: $z_1(t) \equiv y(t)$ ← WANT TO FIND

$z_2(t) = z_1'(t) = y'(t)$.

$z_2'(t) = y''(t) = f(y, y', t)$

$$\begin{cases} z_1'(t) = z_2(t) \to f_1 \\ z_2'(t) = -4z_2(t) - 5z_1(t) \to f_2 \end{cases}$$

WITH $z_1(0) = 3$ & $z_2(0) = 5$

→ SOLVE IT AS A SYSTEM OF FIRST ORDER ODES.

USING EULER'S METHOD

$$z_1^{i+1} = z_1^i + h f_1^i$$ $i \to$ ITERATION (TIMESTEP)

$$z_1^{i+1} = z_1^i + h z_2^i$$

$$z_2^{i+1} = z_2^i + h f_2^i$$

$$z_2^{i+1} = z_2^i + h(-4 z_2^i - 5 z_2^i)$$

$$\boxed{\begin{aligned} z_1^{i+1} &= z_1^i + h z_2^i \\ z_2^{i+1} &= (1-4h) z_2^i - 5h z_1^i \end{aligned}}$$

SOLVE SYSTEM AT EACH TIME STEP (i)

SOLUTION: $z_1(t_i) \Rightarrow y(t_i)$.

STABILITY (§5.10)

RECALL: EULER'S METHOD

$$\epsilon_{i+1} = \underbrace{(1+hJ_i)}_{\text{AMPLIFICATION FACTOR}}\epsilon_i + \underbrace{\frac{h^2}{2} f'(\xi_i)}_{LTE}$$

NEED TO CONSTRIN THIS TERM

* ERROR HAS BOTH A TRUNCATION COMPONENT AND A PROPAGATION COMPONENT

- ASSUME THAT ODE IS ANALYTICALLY STABLE

"WELL-POSED PROBLEM"

3 IDEAS ABOUT ODE SOLUTIONS

(i) CONSISTENCY :-

DOES THE DIFFERENCE EQUATION APPROACH THE DIFFERENTIAL EQUATIONS AS $h \to 0$

$$\lim_{h \to 0} LTE = 0$$

FOR ODE METHODS, WE HAVE STUDIED TILL NOW, WERE BASED ON TAYLOR SERIES EXPANSIONS $\Rightarrow$ h IS IN THE NUMERATOR OF LTE.

$$\Rightarrow \lim_{h \to 0} LTE = 0.$$

(ii) STABILITY :-

DO ERRORS, ONCE INTRODUCED REMAIN BOUNDED OR DO THE GROW EXPONENTIALLY?

* DIFFERENCE EQUATIONS
[RECALLING FROM WK # 1, HW # 1]

$|\lambda| \leq 1$.

- MORE ON STABILITY OF ODE SOLUTIONS LATER.

(iii) CONVERGENCE :-

DOES THE NUMERICAL SOLUTION APPROACH THE EXACT SOLUTION

. AS $h \to 0$

$$\lim_{h \to 0} E_{i+1} \to 0 \quad i \to \infty$$

* HARD TO PROVE THEORETICALLY

EQUIVALENCE THEOREM :-

A NECESSARY AND SUFFICIENT CONDITION FOR CONVERGENCE

(iii) IS TO HAVE CONSISTENCY (i) AND STABILITY (ii)

- NOTE THAT STABILITY DOES NOT GUARANTEE ACCURACY. (h IS A NON-ZERO)

* LOOK AT STABILITY OF VARIOUS METHODS:

STABILITY DEPENDS ON
- STEP SIZE 'h'
- ODE SOLUTION FAMILY ($f \to J$)
- NUMERICAL METHOD (LTE)

$\therefore$ LET US START LOOKING AT STABILITY CRITERIA FOR FEW METHODS.

BACKWARD EULER

$$w_{i+1} = w_i + h f_{i+1}$$ ← IMPLICIT METHOD

[COMES FROM EXPANDING $I(t) = \int_t^{t_{i+1}} f \, d\tau$
AROUND t_{i+1} & EVALATING AT t_i]

$$\epsilon_{i+1} = (1 - h J_{i+1})^{-1} \epsilon_i + O(h^2)$$

[EULER $\epsilon_{i+1} = (1 + h J_i) \epsilon_i + O(h^2)$]

∴ GIVEN ANALYTICALLY STABLE ODE $J \leq 0$

AMPLIFICATION FACTOR.

$$(1 - h J_{i+1})^{-1} \Rightarrow \frac{1}{(1 + \underbrace{h J_{i+1}}_{>0})} \leq 1$$

FOR ANY $h > 0$

$\boxed{\text{UNCONDITIONALLY STABLE METHOD}}$

* STIFF PROBLEMS: J IS LARGE.

AMPLIFICATION FACTOR $\to 0$

VERY GOOD CHOICE OF METHOD FOR STIFF PROBLEMS

DERIVATION OF STABILITY CRITERION FOR BACKWARD EULER METHOD

$$\int_{t_i}^{t_{i+1}} y' = \underbrace{\int_{t_i}^{t_{i+1}} f(y(\tau), \tau)\, d\tau}_{\text{EXPAND AROUND } t_{i+1} \text{ \& EVALUATE AT } t_i}$$

FORWARD CASE:

$$I(t) = \int_{t_i}^{t} f(y(\tau), \tau)\, d\tau$$

$$= \cancel{I(t_i)} + f_i (t - t_i) + \frac{f_i' (t - t_i)^2}{2!} + \ldots$$

$$I(t_{i+1}) = h f_i + \frac{h^2}{2!} f_i' + \frac{h^3}{3!} f_i'' + \cdots$$

$$h = t_{i+1} - t_i$$

BACKWARD CASE:

$$I(t) = \int_{t}^{t_{i+1}} f(y(\tau), \tau)\, d\tau$$

$$= \cancel{I(t_{i+1})}^{\,0} - (t - t_{i+1}) f_{i+1} - \frac{(t-t_{i+1})^2}{2!} f'_{i+1} - \frac{(t-t_{i+1})^3}{3!} f''_{i+1} - \cdots$$

$$\int_{t}^{t_{i+1}} d\tau = -\int_{t_{i+1}}^{t} d\tau$$

$$I(t_i) = \int_{t_i}^{t_{i+1}} f\, d\tau = h f_{i+1} + \frac{h^2}{2} f'_{i+1} + \cdots$$

$$h = t_{i+1} - t_i$$

$$\Rightarrow y_{i+1} - y_i = h f_{i+1} + \frac{h^2}{2} f'_{i+1} + O(h^3)$$

$$w_{i+1} - w_i = h f(w_{i+1}, t_{i+1}) \leftarrow \text{APPROXIMATION}$$

$$\epsilon_{i+1} - \epsilon_i = h\, f(y_{i+1}, t_{i+1}) - f(w_{i+1}, t_{i+1}) + O(h^2)$$

$$\epsilon_{i+1} = \epsilon_i + h \overbrace{\frac{f(y_{i+1}, t_{i+1}) - f(w_{i+1}, t_{i+1})}{y_{i+1} - w_{i+1}}}^{J_{i+1}} \overbrace{(y_{i+1} - w_{i+1})}^{\epsilon_{i+1}} + O(h^2)$$

$$\epsilon_{i+1} = \epsilon_i + h \underbrace{\left.\frac{\partial f}{\partial y}\right|_{t_{i+1}}}_{J_{i+1}} \epsilon_{i+1} + O(h^2)$$

$$\epsilon_{i+1} = (1 - h J_{i+1})^{-1} \epsilon_i + O(h^2)$$

$$\frac{\epsilon_{i+1}}{\epsilon_i} = \frac{1}{1 - hJ_{i+1}} = \frac{1}{1 + h|J_{i+1}|} < 1.$$

ALWAYS HAVE $h > 0$

AS LONG AS $J \le 0$

ODE BEING ANALYTICALLY STABLE

$$\frac{\epsilon_{i+1}}{\epsilon_i} < 1 \Rightarrow$$ UNCONDITIONAL STABILITY.