DS 288: NUMERICAL METHODS

AUG-26-2021

FIXED POINT

$$g(p) = p$$

THREE CONDITIONS FOR EXISTENCE OF UNIQUE p

(i) $g(x) \in C^1[a,b]$

(ii) $g(x)$ BOUNDED BY $[a,b]$

(iii) $|g'(x)| < 1 \ \forall x \in [a,b]$

FIXED POINT ITERATION

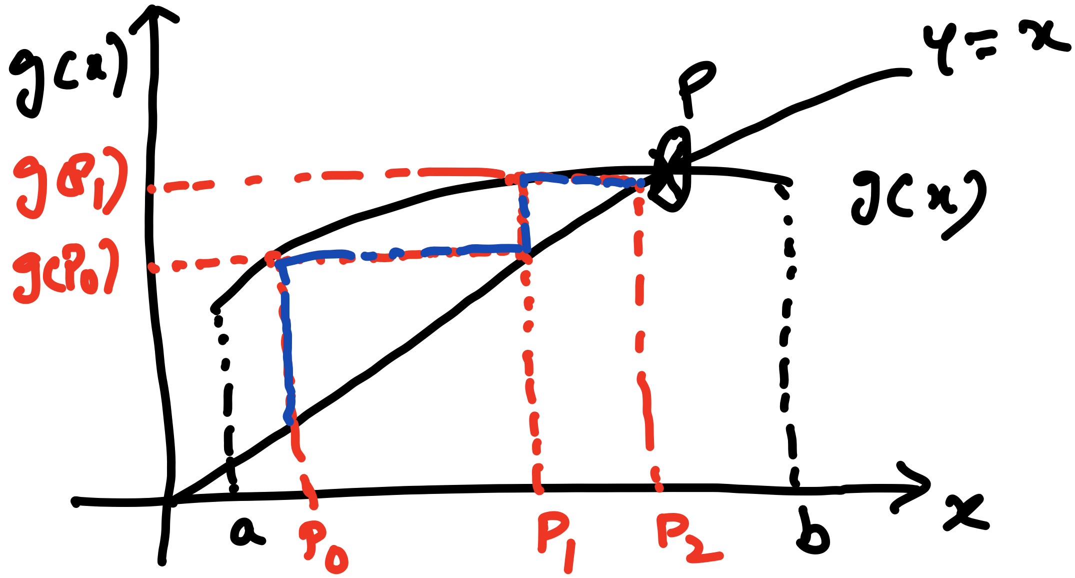
$p_n = g(p_{n-1})$ WHERE g IS THE ITERATION FUNCTION

START WITH p_0

$$p_1 = g(p_0)$$

$$p_2 = g(p_1)$$

$$\vdots \qquad \vdots \quad \vdots \quad \vdots$$



* NEED A STARTING VALUE
$p_0 \in [a, b]$ &
WANT $\lim_{n \to \infty} p_n = p$

ESTIMATE ERROR

ASSUME FIXED POINT CONDITIONS (i), (ii) & (iii) HOLD

$$p_n = g(p_{n-1}) \quad \text{COMPUTER}$$
$$p = g(p) \quad \text{EXACT}$$

$$\underbrace{|P_n - P|}_{\epsilon_n} = |g(P_{n-1}) - g(P)|$$

$\epsilon_n \rightarrow$ ERROR AT n^{th} step

MVT $\rightarrow$ MEAN VALUE THEOREM

$$\epsilon_n = |g'(c)|\,|P_{n-1} - P|$$

$$c \in [P_{n-1}, P]$$

$$g'(c) = \frac{g(P_{n-1}) - g(P)}{P_{n-1} - P}$$

$$\epsilon_n = |g'(c)|\,\epsilon_{n-1}$$

(iii) FIXED POINT CONDITION

$$|g'(x)| < 1.$$

$$\lim_{n \to \infty} \epsilon_n = 0$$

* GUARANTEED CONVERGENCE

$$\epsilon_n = K\,\epsilon_{n-1} \qquad K = g'(c)$$

EXPAND IT

$\epsilon_n = K^n \epsilon_0$ WHERE $K < 1$

$c \in [P_{n-1}, P]$

A FEW OBSERVATIONS:

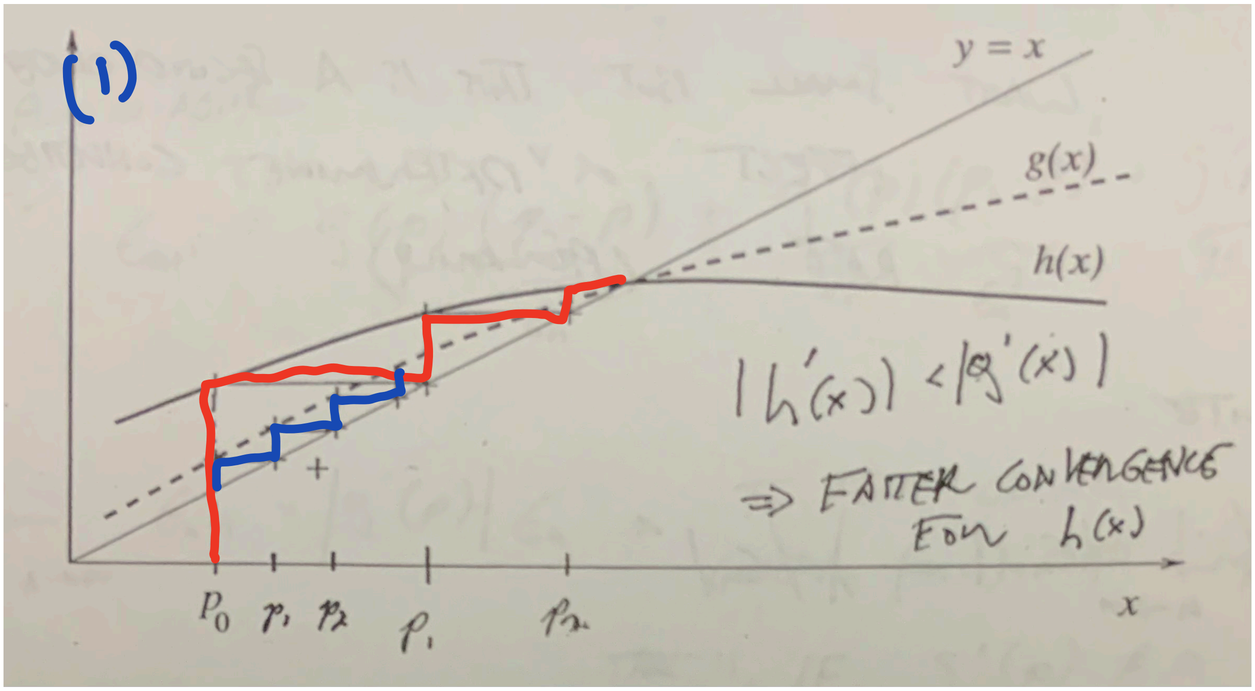
(a) THE SMALLER $|g'(x)| = K$ THE FASTER IS CONVERGENCE

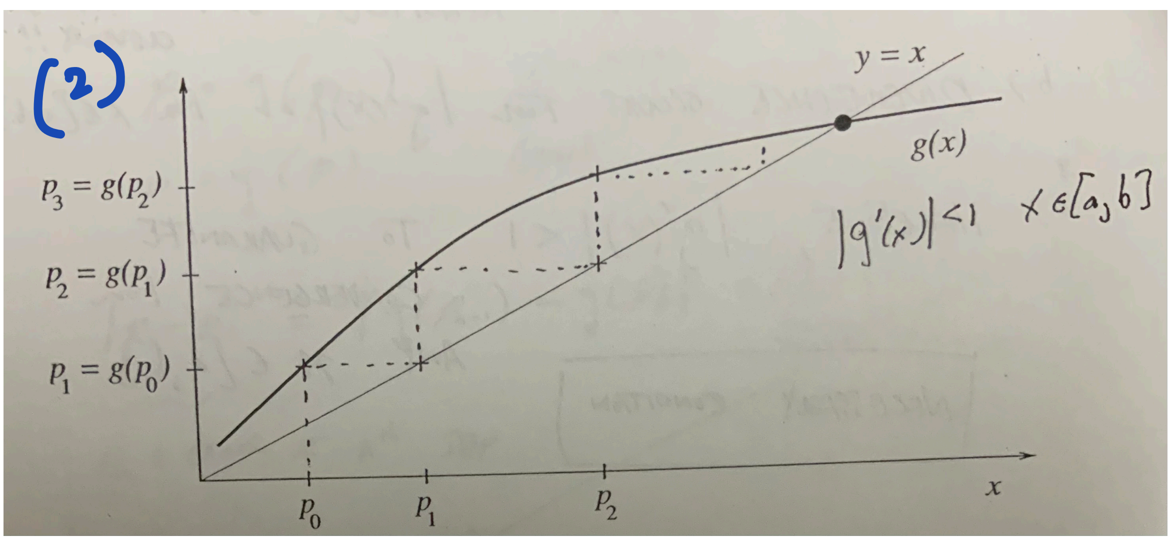
* PICK $g(x)$ FOR FASTER CONVERGENCE

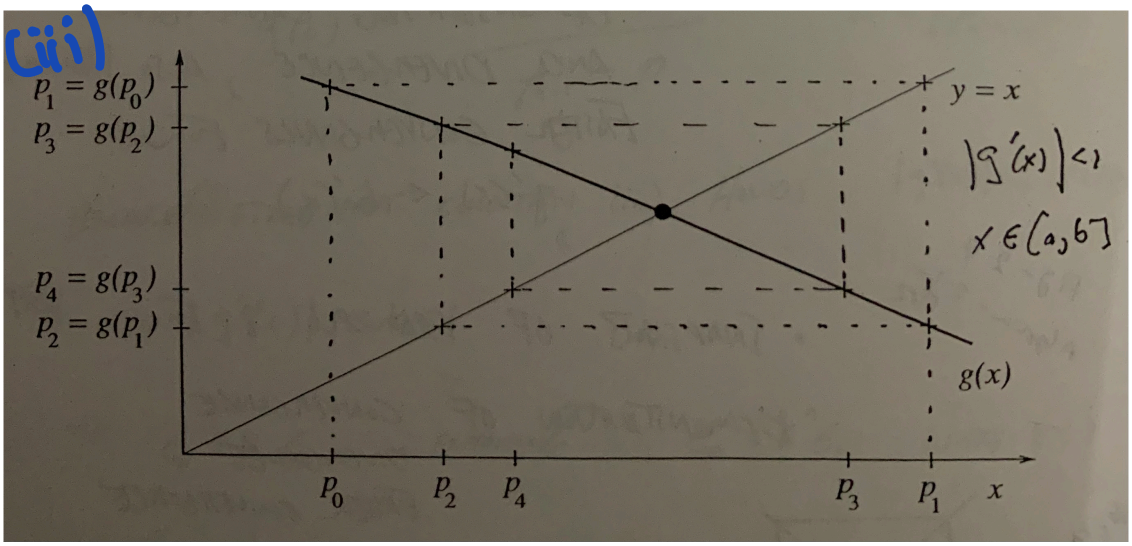
(b) DIVERGENCE HAPPENS FOR $|g'(x)| > 1$ FOR $x \in [a, b]$

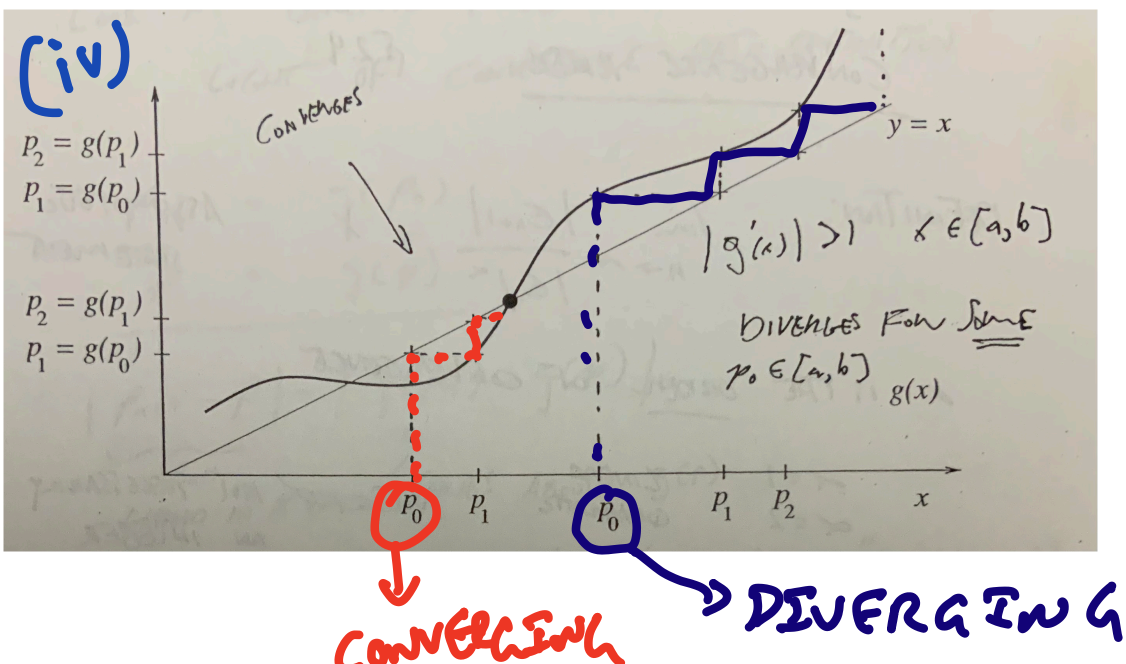
(c) THEREFORE, $|g'(x)| < 1$ FOR GUARATEED CONVERGENCE FOR ANY $P_0 \in [a, b]$

FIXED POINT FUNCTIONS









CONVERGENCE RATES [§2-4]

DEFINITION $\displaystyle \lim_{n \to \infty} \frac{|\epsilon_{n+1}|}{|\epsilon_n|^{\alpha}} = \lambda$

WHERE $\alpha \rightarrow$ ORDER OF CONVERGENCE.

$\lambda \rightarrow$ ASYMPTOTIC ERROR CONSTANT

$\alpha = 1 \rightarrow$ LINEAR

$\alpha = 2 \rightarrow$ QUADRATIC

$\alpha = 3 \rightarrow$ CUBIC

[α NEED NOT BE INTEGER]

WANT SMALL, THIS IS A SECOND ORDER EFFECT

$\alpha \rightarrow$ { DETERMINES CONVERGENCE RATE PRIMARILY

$$\boxed{\lim_{n \to \infty} |\epsilon_{n+1}| = \lambda |\epsilon_n|^{\alpha}}$$

α CONVERGENCE RATE IS IN THE LIMIT, NOT NECESSARILY IN THE INITIAL STEPS.

LOOK AT GENERAL FIXED POINT ITERATION IN LIGHT OF CONVERGENCE RATE DEFINITION

$$p_{n+1} = g(p_n) \rightarrow \text{COMPUTER}$$

$$p = g(p) \rightarrow \text{EXACT}$$

$$\underbrace{|p_{n+1} - p|}_{\epsilon_{n+1}} = |g(p_n) - g(p)| \quad \text{—} \circledast$$

EXPAND IN TAYLOR SERIES AROUND 'p'

$$g(p_n) = g(p) + g'(p)(p_n - p) + \frac{g''(p)}{2!}(p_n - p)^2 + \frac{g'''(p)}{3!}(p_n - p)^3 + \ldots\ldots$$

PLUG INTO $\circledast$

$$\epsilon_{n+1} = g'(p)\underbrace{(p_n - p)}_{\epsilon_n} + \frac{g''(p)}{2!}(p_n - p)^2 + \frac{g'''(p)}{3!}(p_n - p)^3 + \cdots\cdots$$

$$\boxed{\epsilon_{n+1} = g'(p)\,\epsilon_n + \frac{g''(p)}{2!}\epsilon_n^2 + \frac{g'''(p)}{3!}\epsilon_n^3 + \cdots\cdots}$$

FIRST ORDER (TERM)

$$\lim_{n \to \infty} \epsilon_{n+1} = |g'(p)|\,\epsilon_n$$

$\alpha \to 1$ LINEAR CONVERGENCE

$$\lambda = |g'(p)|$$

$$\lim_{n \to \infty} \epsilon_{n+1} = \left|\frac{g''(p)}{2!}\right| \epsilon_n^2 \quad \text{IF } g'(p) = 0$$

$\alpha \to 2$ QUADRATIC CONVERGENCE

WANT $g(x)$ SUCH THAT

$g^{n}(x) = 0$ FOR AS MANY

$n = 1, 2, 3 \ldots$ FOR $n+1^{th}$ ORDER

CONVERGENCE.

NEWTON'S METHOD (§2.3)

If $g(x) = x - \underbrace{\phi(x)}_{\text{SCALING FUNCTION}} \underbrace{f(x)}_{\text{FIND ROOTS FOR}}$

$$g(P) = P - \phi(P) f(P)$$

IF P IS THE ROOT $f(P) = 0$

$\Rightarrow g(P) = P$

* $g(x)$ IS A FUNCTION WITH FIXED POINT AT 'P' $\rightarrow$

ROOT OF $f(x)$

$$g'(x) = \frac{d}{dx}(g(x)) = 1 - [\phi'(x) f(x) + \phi(x) f'(x)]$$

$x = P$

$$g'(P) = 1 - [\phi'(P) \underbrace{\cancel{f(P)}}_{0} + \phi(P) f'(P)]$$

$f(P) = 0$ ROOT

* IF $g'(P) = 0 \rightarrow$ [ENSURING $\alpha = 2$]

$$0 = 1 - \phi(P) f'(P)$$

$$\phi(P) = \frac{1}{f'(P)}$$ ASSUMING $f'(P) \neq 0$

SO $g(x) = x - \phi(x) f(x)$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

FOR NEWTON'S METHOD

$$\boxed{p_{n+1} = g(p_n) = p_n - \frac{f(p_n)}{f'(p_n)}}$$

NEWTON'S METHOD

- GUARANTEED QUADRATIC CONVERGENCE FOR 'p_0' PROVIDED $f'(p) \neq 0$

IF YOU ARE CLOSE TO THE ROOT: $p_n = g(p_{n-1})$

$|p_n - g(p_{n-1})| \sim 0$

$\to p = g(p)$

RECALL EQUATION FOR A LINE

$$y = \underset{\substack{\downarrow \\ \text{SLOPE}}}{m}\, x + b$$

$b \rightarrow$ Y-INTERCEPT

$Y(x=0)$

SOLVE FOR x

$$x = \frac{y-b}{m} = \frac{y}{m} - \frac{b}{m}$$

LET $-\frac{b}{m} = C$

$$x = \frac{y}{m} + C \rightarrow$$ X-INTERCEPT

$X(Y=0)$

LOOK AT NEWTON'S ITERATION

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$

$P_n \rightarrow X \quad f(P_n) \rightarrow Y$

$f'(P_n) \rightarrow m$

$$P_{n+1} = x - \frac{y}{m} = c \quad (x\text{-INTERCEPT})$$

FEW OBSERVATIONS FOR NEWTON'S METHOD:-

(i) FASTER CONVERGENCE ($\alpha = 2$)

(ii) TEXBOOK:- TAYLOR EXPANSION - FIRST THEOREM

(iii) PRACTICALLY: NEWTON'S METHOD IS SENSITIVE TO INITIAL GUESS.

* BAD GUESSES (P_0) MAY DIVERGE.

(iv) DEMANDS $f'(x)$ EXISTENCE & BEING KNOWN WITH $f'(P) \neq 0$

WHAT <u>IF $f'(p)=0$</u> ($\&$ $f''(p) \neq 0$)

RECALL $g(x) = x - \dfrac{f(x)}{f'(x)}$ NEWTON'S METHOD

DERIVED THIS BY $g'(p)=0$

$$g'(x) = \frac{d}{dx}\left(x - \frac{f}{f'}\right)$$

$$= 1 - f'(f')^{-1} + ff''(f')^{-2}$$

$$g'(x) = \frac{f(x)\,f''(x)}{[f'(x)]^2}$$

$$g'(p) = \frac{\cancel{f(p)}^{\,0 \text{ (ROOT)}}\,f''(p)}{[f'(p)]^2} = \frac{0}{0}$$

UNDEFINED

USE L'HOPITAL'S RULE. $\dfrac{f''^2}{2f''^2}$

(2 TIMES)

$$g'(p) = \frac{1}{2} \neq 0$$

$$\lim_{n \to \infty} \frac{|\epsilon_{n+1}|}{|\epsilon_n|} = |g'(p)|$$

IF $g'(p) \neq 0$

$\alpha = 1$ LINEAR WITH $\lambda = \frac{1}{2}$

If $f'(p) = 0$ USING NEWTON'S METHOD, WE GET ORDER OF CONVERGENCE TO BE <u>LINEAR</u>

GENERLIZE TO ROOT OF "MULTIPLICITY" m.

<u>DEFINITION</u> $f(x) = (x-p)^m q(x)$

$$\lim_{x \to p} q(x) \neq 0$$

↳ PART OF FUNCTION THAT DOES NOT CONTRIBUTE TO ROOT.

$$g'(p) = \left.\frac{f(x)f''(x)}{[f'(x)]^2}\right|_{\lim_{x\to p}} = \frac{m-1}{m} = \lambda$$

$$\lim_{m\to\infty} g'(p) = 1 \qquad \lambda = 1$$

NO CONVERGENCE

STILL $\alpha = 1 \rightarrow$ LINEAR CONVERGENCE.

MODIFIED NEWTON'S METHOD

- USE FOR $m \geq 2$