Numerical Methods

DS288 and UMC201

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August-December 2025

Hermite's Interpolation

Hermite's Interpolation

Let $x_0, x_1, \cdots x_n$ be n distinct points in the interval $[a, b]$. Then $p_n(x)$ is a Hermite interpolating polynomial for $f(x)$ if

$$P(x_i) = f(x_i), \quad P'(x_i) = f'(x_i), \quad 0 \leq i \leq n$$

Remark:

- *$p(x)$ and $f(x)$ agree not only function values but also first derivative values at $x_i = 0, 1, \cdots n$.*
- *Since their first derivatives agree with those of f, they have the same shape as the function at $(x_i, f(x_i))$ in the sense that the tangent lines to the polynomial and the function agree.*

Hermite's Interpolation

Definition: If $f \in \mathbb{C}^1[a, b]$ and $x_0, x_1, \cdots, x_n$ are distinct numbers in $[a, b]$, the unique polynomial of least degree agreeing with f and f' at $x_0, \cdots x_n$ is the Hermite polynomial of degree at most $2n + 1$ given by

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j) H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j) \tilde{H}_{n,j}(x)$$

where, $L_{n,j}(x)$ is the jth Lagrange coefficient polynomial of degree n

$$H_{n,j}(x) = [1 - 2(x - x_j) L'_{n,j}] L^2_{n,j}(x) \quad \text{and} \quad \tilde{H}_{n,j}(x) = (x - x_j) L^2_{n,j}(x)$$

Further, if $f \in C^{2n+2}[a, b]$, then

$$f(x) = H_{2n+1}(x) + \frac{(x - x_0)^2 \cdots (x - x_n)^2}{(2n + 2)!} f^{(2n+2)}(\zeta(x))$$

for some $\zeta(x)$ in the interval (a, b).

- *$H_{2n+1}(x)$ is polynomial of degree at most $2n + 1$*

Proof

From Lagrange's interpolation formula $L_{n,j}(x_i) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j \end{cases}$

- when $i \neq j$: $H_{n,j}(x) = 0$; $\tilde{H}_{n,j}(x_i) = 0$
- when $i = j$:
  - $H_{n,j}(x_j) = [1 - 2(x_j - x_j)L'_{n,j}]L^2_{n,j}(x_j) = 1$
  - $\tilde{H}_{n,j}(x_j) = (x_j - x_j)L^2_{n,j}(x_j) = 0$

$\Rightarrow H_{2n+1}(x_j) = f(x_j)$. Thus $H_{2n+1}(x_j)$ agree with f at $x_0, \cdots, x_n$.

- $H'_{n,j}(x) = -2L'_{n,j}(x)L^2_{n,j}(x_j) + (1 - 2(x - x_j)L'_{n,j}(x))]2L_{n,j}(x)L'_{n,j}(x)$,
  - When $i \neq j$: $H'_{n,j}(x_i) = 0$;
  - when $i = j$: $H'_{n,j}(x_i) = 0$
- $\tilde{H}'_{n,j}(x) = L^2_{n,j}(x) + 2(x - x_j)L_{n,j}(x)L'_{n,j}(x)$,
  - When $i \neq j$: $\tilde{H}'_{n,j}(x_i) = 0$;
  - when $i = j$: $\tilde{H}'_{n,j}(x_i) = 1$

$\Rightarrow H'_{2n+1}(x_j) = f'(x_j)$. Thus $H'_{2n+1}(x_j)$ agree with f' at $x_0, \cdots, x_n$.

Example

Example

Use the Hermite polynomial that agrees with the data listed in the following Table to find an approximation of $f(1.5)$.

k	x_k	$f(x_k)$	$f'(x_k)$
0	1.3	0.6200860	-0.5220232
1	1.6	0.4554022	-0.5698959
2	1.9	0.2818186	-0.5811571

Example

Solution We first compute the Lagrange polynomials and their derivatives.

$$
\begin{aligned}
L_{2,0}(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}, \\
L'_{2,0}(x) &= \frac{100}{9}x - \frac{175}{9}, \\
L_{2,1}(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{-100}{9}x^2 + \frac{320}{9}x - \frac{247}{9}, \\
L'_{2,1}(x) &= \frac{-200}{9}x + \frac{320}{9}, \\
L_{2,2}(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{50}{9}x^2 - \frac{145}{9}x + \frac{104}{9}, \\
L'_{2,2}(x) &= \frac{100}{9}x - \frac{145}{9}.
\end{aligned}
$$

The polynomials $H_{2,j}(x)$ and $\hat{H}_{2,j}(x)$ are then

Example

$$
\begin{aligned}
H_{2,0}(x) &= [1-2(x-1.3)(-5)]\left(\frac{50}{9}x^2-\frac{175}{9}x+\frac{152}{9}\right)^2 \\
&= (10x-12)\left(\frac{50}{9}x^2-\frac{175}{9}x+\frac{152}{9}\right)^2, \\
H_{2,1}(x) &= 1\cdot\left(\frac{-100}{9}x^2+\frac{320}{9}x-\frac{247}{9}\right)^2, \\
H_{2,2}(x) &= 10(2-x)\left(\frac{50}{9}x^2-\frac{145}{9}x+\frac{104}{9}\right)^2, \\
\hat{H}_{2,0}(x) &= (x-1.3)\left(\frac{50}{9}x^2-\frac{175}{9}x+\frac{152}{9}\right)^2, \\
\hat{H}_{2,1}(x) &= (x-1.6)\left(\frac{-100}{9}x^2+\frac{320}{9}x-\frac{247}{9}\right)^2, \\
\hat{H}_{2,2}(x) &= (x-1.9)\left(\frac{50}{9}x^2-\frac{145}{9}x+\frac{104}{9}\right)^2.
\end{aligned}
$$

Example

$$
\begin{aligned}
H_5(x) &= 0.6200860 H_{2,0}(x) + 0.4554022 H_{2,1}(x) + 0.2818186 H_{2,2}(x) \\
&\quad -0.5220232 \hat{H}_{2,0}(x) - 0.5698959 \hat{H}_{2,1}(x) - 0.5811571 \hat{H}_{2,2}(x) \\
H_5(1.5) &= f[1.3] + f'(1.3)(1.5-1.3) + f[1.3,1.3,1.6](1.5-1.3)^2 \\
&\quad + f[1.3,1.3,1.6,1.6](1.5-1.3)^2(1.5-1.6) \\
&\quad + f[1.3,1.3,1.6,1.6,1.9](1.5-1.3)^2(1.5-1.6)^2 \\
&\quad + f[1.3,1.3,1.6,1.6,1.9,1.9](1.5-1.3)^2(1.5-1.6)^2(1.5-1.9) \\
&= 0.6200860 + (-0.5220232)(0.2) + (-0.0897427)(0.2)^2 \\
&\quad + 0.0663657(0.2)^2(-0.1) + 0.0026663(0.2)^2(-0.1)^2 \\
&\quad + (-0.0027738)(0.2)^2(-0.1)^2(-0.4) \\
&= 0.5118277.
\end{aligned}
$$

Hermite Polynomial by divided Difference

- Suppose $x_0, \cdots x_n$ and f, f' are given at these numbers.
- Define $z_0, \cdots z_{2n+1}$ by $z_{2i} = z_{2i+1} = x_i$ for $i = 0, \cdots n$
- Construct divided difference table, but use

$$f'(x_0), f'(x_1), \cdots f'(x_n)$$

- To set the following the divided difference

$$f[z_0, z_1], f[z_1, z_2], f[z_3, z_4], \cdots, f[z_{2n}, z_{2n+1}]$$

- $f[z_0, z_1] = f'(x_0)$, $f[z_2, z_3] = f'(x_1), \cdots, f[z_{2n}, z_{2n+1}] = f'(x_n)$.
- The Hermite Polynomial

$$H_{2n+1}(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0, \cdots z_k](x - z_0) \cdots (x - z_{k-1})$$

Hermite Polynomial via Divided Difference

z	f(z)	First divided differences	Second divided differences
$z_0 = x_0$	$f[z_0] = f(x_0)$		
		$f[z_0, z_1] = f'(x_0)$	
$z_1 = x_0$	$f[z_1] = f(x_0)$		$f[z_0, z_1, z_2] = \frac{f[z_1,z_2]-f[z_0,z_1]}{z_2-z_0}$
		$f[z_1, z_2] = \frac{f[z_2]-f[z_1]}{z_2-z_1}$	
$z_2 = x_1$	$f[z_2] = f(x_1)$		$f[z_1, z_2, z_3] = \frac{f[z_2,z_3]-f[z_1,z_2]}{z_3-z_1}$
		$f[z_2, z_3] = f'(x_1)$	
$z_3 = x_1$	$f[z_3] = f(x_1)$		$f[z_2, z_3, z_4] = \frac{f[z_3,z_4]-f[z_2,z_3]}{z_4-z_2}$
		$f[z_3, z_4] = \frac{f[z_4]-f[z_3]}{z_4-z_3}$	
$z_4 = x_2$	$f[z_4] = f(x_2)$		$f[z_3, z_4, z_5] = \frac{f[z_4,z_5]-f[z_3,z_4]}{z_5-z_3}$
		$f[z_4, z_5] = f'(x_2)$	
$z_5 = x_2$	$f[z_5] = f(x_2)$		

Example

Example

Use the Hermite polynomial (Newton's divided difference) that agrees with the data listed in the following Table to find an approximation of $f(1.5)$.

k	x_k	$f(x_k)$	$f'(x_k)$
0	1.3	0.6200860	-0.5220232
1	1.6	0.4554022	-0.5698959
2	1.9	0.2818186	-0.5811571

Hermite Polynomial by Divided Difference

1.3	0.6200860					
		-0.5220232				
1.3	0.6200860		-0.0897427			
		-0.5489460		0.0663657		
1.6	0.4554022		-0.0698330		0.0026663	
		−0.5698959		0.0679655		-0.0027738
1.6	0.4554022		-0.0290537		0.0010020	
		-0.5786120		0.0685667		
1.9	0.2818186		-0.0084837			
		-0.5811571				
1.9	0.2818186					

Hermite Polynomial by Divided Difference

$$
\begin{aligned}
H_5(1.5) \quad &= \quad f[1.3] + f'(1.3)(1.5-1.3) + f[1.3, 1.3, 1.6](1.5-1.3)^2 \\
&\quad +f[1.3, 1.3, 1.6, 1.6](1.5-1.3)^2(1.5-1.6) \\
&\quad +f[1.3, 1.3, 1.6, 1.6, 1.9](1.5-1.3)^2(1.5-1.6)^2 \\
&\quad +f[1.3, 1.3, 1.6, 1.6, 1.9, 1.9](1.5-1.3)^2(1.5-1.6)^2(1.5-1.9) \\
&= \quad 0.6200860 + (-0.5220232)(0.2) + (-0.0897427)(0.2)^2 \\
&\quad +0.0663657(0.2)^2(-0.1) + 0.0026663(0.2)^2(-0.1)^2 \\
&\quad +(-0.0027738)(0.2)^2(-0.1)^2(-0.4) \\
&= \quad 0.5118277.
\end{aligned}
$$


ANY
QUESTIONS?