Numerical Methods

DS288 and UMC201

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Fixed-Point Method
or
Fixed-Point Iteration Method

Fixed-Point Method

- A point α is called a fixed point of a function $g(x)$ if $g(\alpha) = \alpha$.

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- $g(x) = (3x + 4)^{1/2}; \longrightarrow \alpha = 4$

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- Given an equation $f(x) = 0$, it is possible to rearrange this into the form $g(x) = x$ in several ways.
  - ➢ If $f(x) = 0$ if and only if $g(x) = x$.
  - ➢ α is a root of $f(x) = 0$ if and only if α is a fixed point of $g(x)$.

Fixed-Point Method

- The equation $x^3 + 4x^2 - 10 = 0$ has a unique root in $[1, 2]$.
- There are many ways to change the equation to the fixed point form $x = g(x)$ using simple algebraic manipulation.

Example

1. $x = g_1(x) = x - x^3 - 4x^2 + 10$
2. $x = g_2(x) = x - \dfrac{x^3 + 4x^2 - 10}{3x^2 + 8x}$
3. $x = g_3(x) = \left(\dfrac{10}{3} - 4x\right)^{1/2}$
4. $x = g_4(x) = \dfrac{1}{2}\left(10 - x^3\right)^{1/2}$
5. $x = g_5(x) = \left(\dfrac{10}{4 + x}\right)^{1/2}$

Fixed-Point Method

- It is possible to define a sequence of approximations as $x_n = g(x_{n-1})$ with an initial approximation x_0 .

Questions

(a) Does the sequence $\{x_n\}$ always converge to a root α ?

(b) If it does, will α is a root of $g(x) = x$?

(c) How should we chose $g(x)$ such that $\{x_n\}$ converges to the root α?

Fixed-Point Method

Theorem

(1) *If* $g \in C[a, b]$ *and* $a \leq g(x) \leq b$.

(2) $g'(x)$ *exists for all* $x \in [a, b]$ *and there exists a positive number* k *such that* $|g'(x)| \leq k < 1$ *for all* $x \in [a, b]$

Then

(a) $g(x)$ *has at least one fixed point* $\alpha \in [a, b]$.

(b) *The fixed point* α *is unique.*

(c) *The sequence* $\{x_n\}$ *generated by the rule* $x_n = g(x_{n-1})$, *starting with initial approximation* $x_0 \in [a, b]$, *converges to the fixed point* α

Example

Example

Find a suitable iteration function $g(x)$ and an interval to compute the smallest positive root of $x^3 - x - 1 = 0$ by fixed point method (FPM). Check the conditions of FPM.

➢ $f(x) = x^3 - x - 1$, thus $f(0) = -1$, $f(1) = -1$ and $f(2) = 5$
➢ By IVT the smallest positive root lies in $[1, 2]$.

- Since $x^3 - x - 1 = 0$, we design $x = x^3 - 1 = g(x)$ (say)
- Thus $g'(x) = 3x^2$ but $|g'(x)| > 1$ for $x \in [1, 2]$
- $g(x)$ is not suitable fixed point method for this problem .

Example

- Since $x^3 - x - 1 = 0$, we design $x = (1+x)^{1/3} = g(x)$ (say)
- Thus $g'(x) = \frac{1}{3}(1+x)^{-2/3}$ but $|g'(x)| > 0$ for $x \in [1,2]$
- $g(x)$ is increasing function. Further, $\max\limits_{x\in[1,2]} |g'(x)| < 1$
- Next, we have to test $a = 1 \leq g(x) \leq 2 = b$, that is

$$\min_{x\in[a,b]} |g(x)| = g(1) = 2^{1/3} > 1, \quad \max_{x\in[a,b]} |g(x)| = g(2) = 3^{1/3} < 2$$

- Thus $g(x) = (1+x)^{1/3}$ is the iteration function, which converges to the root $\alpha = 1.32472$

n	0	1	2	3	4	5	6	7
x_n	1.5	1.3572	1.3309	1.3259	1.3249	1.3248	1.3247	1.3247

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Fixed-Point Method

Proof (a) (Existence of Fixed Point)

- Given $g \in C[a, b]$ and $a \leq g(x) \leq b$ for all $x \in [a, b]$.
- If $g(a) = a$ or $g(b) = b$, then g has a fixed point.
- Without loss of generality, assume $a < g(a)$ and $g(b) < b$.
- Define $h(x) = g(x) - x$ for all $x \in [a, b]$. Then, $h(x)$ is a continuous function on $[a, b]$ with $h(a) > 0$ and $h(b) < 0$.
- By intermediate value theorem, there exists a $\alpha \in (a, b)$ such that $h(\alpha) = 0 \Rightarrow g(\alpha) = \alpha$. Thus, g has a fixed point $\alpha \in [a, b]$.

Theorem (Intermediate Value Theorem (IVT))

Let $f(x)$ be continuous function in $[a, b]$, where $a < b$ and let k be any number between $f(a)$ and $f(b)$. Then, there exists a number c in (a, b) such that $f(c) = k$.

Fixed-Point Method

Proof (b) (uniqueness)

- If not, let $g(x)$ has two distinct fixed points α_1 and α_2 in $[a, b]$.
- Thus, $g(\alpha_1) = \alpha_1$ and $g(\alpha_2) = \alpha_2$. Hence

$$|\alpha_1 - \alpha_2| = |g(\alpha_1) - g(\alpha_2)|$$

- By Mean value Theorem, $g(\alpha_1) - g(\alpha_2) = g'(\zeta)(\alpha_1 - \alpha_2)$, where ζ lies between α_1 and α_2. Hence

$$\begin{aligned} |\alpha_1 - \alpha_2| &= |g'(\zeta)(\alpha_1 - \alpha_2)| = |g'(\zeta)||\alpha_1 - \alpha_2| \\ &\leq k|\alpha_1 - \alpha_2| < |\alpha_1 - \alpha_2| \end{aligned}$$

- This is a contradiction. Thus $g(x) = x$ is unique.

Theorem (Mean Value Theorem (MVT))

If $f(x)$ be continuous function in $[a, b]$ and $f(x)$ is differentiable on (a, b), then, there exists a number c in (a, b) such that

$$f(b) - f(a) = f'(c)(b - a)$$


ANY
QUESTIONS?