

Numerical Methods

DS288 and UMC201

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Runge-Kutta Methods

Why RK method

- From application point of view, the Taylor series method has a major disadvantage.
- It required evaluation of partial derivatives each time.
- This is not possible in any practical application.
- We need to discuss a method for higher order accuracy and not needed for evaluating of partial derivatives.

Taylor Polynomial in two variable

Suppose that $f(t, y)$ and all its partial derivatives of order less than or equal to $n + 1$ are continuous on $D = \{(t, y) \mid a \leq t \leq b, c \leq y \leq d\}$, and let $(t_0, y_0) \in D$. For every $(t, y) \in D$, there exists ξ between t and t_0 and μ between y and y_0 with $f(t, y) = P_n(t, y) + R_n(t, y)$, where

$$\begin{aligned}P_n(t, y) &= f(t_0, y_0) + \left[(t - t_0) \frac{\partial f}{\partial t}(t_0, y_0) + (y - y_0) \frac{\partial f}{\partial y}(t_0, y_0) \right] \\&\quad + \left[\frac{(t - t_0)^2}{2} \frac{\partial^2 f}{\partial t^2}(t_0, y_0) + (t - t_0)(y - y_0) \frac{\partial^2 f}{\partial t \partial y}(t_0, y_0) \right. \\&\quad \left. + \frac{(y - y_0)^2}{2} \frac{\partial^2 f}{\partial y^2}(t_0, y_0) \right] + \cdots + \left[\frac{1}{n!} \sum_{j=0}^n \binom{n}{j} (t - t_0)^{n-j} (y - y_0)^j \frac{\partial^n f}{\partial t^{n-j} \partial y^j}(t_0, y_0) \right]\end{aligned}$$

and

$$R_n(t, y) = \frac{1}{(n+1)!} \sum_{j=0}^{n+1} \binom{n+1}{j} (t - t_0)^{n+1-j} (y - y_0)^j \frac{\partial^{n+1} f}{\partial t^{n+1-j} \partial y^j}(\xi, \mu).$$

The function $P_n(t, y)$ is called the **nth Taylor polynomial in two variables** for the function f about (t_0, y_0) , and $R_n(t, y)$ is the remainder term associated with $P_n(t, y)$.



Runge-Kutta Methods

2nd order RK method

- We have IVP

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0, \quad a \leq x \leq b \quad (1)$$

- The second order Runge-Kutta method

$$\begin{aligned} y_{n+1} &= y_n + \text{Average Slope} * (x_{n+1} - x_n) \\ &= y_n + h \frac{f(x_n, y_n) + f(x_{n+1}, y_{n+1})}{2} \\ &= y_n + \frac{1}{2}(k_1 + k_2) \end{aligned}$$

$$\text{where } k_1 = hf(x_n, y_n),$$

$$k_2 = hf(x_n + h, y_n + k_1),$$

Runge-Kutta Methods

4th order RK method

- We have IVP

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0, \quad a \leq x \leq b \quad (2)$$

$$y_{n+1} = y_n + \omega_1 k_1 + \omega_2 k_2 + \omega_3 k_3 + \omega_4 k_4 \quad n = 0, 1, 2, \dots$$

where

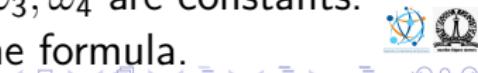
$$k_1 = hf(x_n, y_n),$$

$$k_2 = hf(x_n + \alpha h, y_n + \beta k_1),$$

$$k_3 = hf(x_n + \gamma h, y_n + \delta_1 k_1 + \delta_2 k_2),$$

$$k_4 = hf(x_n + \mu h, y_n + \zeta_1 k_1 + \zeta_2 k_2 + \zeta_3 k_3),$$

where $\alpha, \beta, \gamma, \delta_1, \delta_2, \mu, \zeta_1, \zeta_2, \zeta_3, \omega_1, \omega_2, \omega_3, \omega_4$ are constants.
which will be chosen that accuracy of the formula.



Runge-Kutta Methods

4th order RK method

- We have IVP

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0, \quad a \leq x \leq b \quad (3)$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad n = 0, 1, 2, \dots$$

where

$$k_1 = hf(x_n, y_n),$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right),$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right),$$

$$k_4 = hf(x_n + h, y_n + k_3),$$



Example of RK4 method

Example

consider the IVP

$$\frac{dy}{dx} = x + y, \quad y(0) = 1$$

Find the approximate value of y when $x = 0.2$

- Here $f(x, y) = x + y$, $x_0 = 0$, $y(x_0) = 1$, $h = 0.2$
- $k_1 = hf(x_0, y_0) = 0.2(1) = 0.2$
- $k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$
- $k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$
- $k_4 = hf(x_0 + h, y_0 + k_3)$
- $y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.2428$

Higher-Order Equations and Systems of DEs

System of first-order initial-value problems

An m th-order system of first-order initial-value problems has the form:

$$\begin{aligned}\frac{dy_1}{dx} &= f_1(x, y_1, y_2, \dots, y_m) \\ \frac{dy_2}{dx} &= f_2(x, y_1, y_2, \dots, y_m) \\ &\vdots \\ \frac{dy_m}{dx} &= f_m(x, y_1, y_2, \dots, y_m)\end{aligned}\tag{4}$$

for $a \leq x \leq b$ with initial conditions

$$y_1(a) = \alpha_1, y_2(a) = \alpha_2 \dots y_m(a) = \alpha_m\tag{5}$$

The objective is to find m functions $y_1(x), y_2(x), \dots, y_m(x)$ that satisfy each of the differential equations together with all the initial conditions.

System of first-order initial-value problems

Definition (Lipschitz condition) The function $f(x, y_1, \dots, y_m)$, defined on the set

$D = \{(x, y_1, \dots, y_m) \mid a \leq x \leq b \text{ and } -\infty < y_i < \infty, \text{ for each } i = 1, 2 \dots m\}$, is said to satisfy Lipschitz condition on D in the variables y_1, y_2, \dots, y_m if a constant $L > 0$ exists with

$$|f(x, y_1, \dots, y_m) - f(x, z_1, \dots, z_m)| \leq L \sum_{j=1}^m |y_j - z_j|, \quad (6)$$

for all (x, y_1, \dots, y_m) and (x, z_1, \dots, z_m) in D .

Theorem: Suppose

$D = \{(x, y_1, \dots, y_m) \mid a \leq x \leq b \text{ and } -\infty < y_i < \infty, \text{ for each } i = 1, 2 \dots m\}$ and let $f(x, y_1, \dots, y_m)$, for each $i = 1, 2 \dots m\}$, be continuous and satisfy a Lipschitz condition on D . The system of first-order equations given in (4), subject to initial conditions (5) has a unique solution $y_1(t), y_2(t), \dots, y_m(t)$ for $a \leq x \leq b$.

Runge-Kutta method, fourth order

Consider the first-order initial value problem

$$y' = f(x, y), \quad a \leq x \leq b, \quad y(a) = \alpha$$

$$w_0 = \alpha,$$

$$k_1 = hf(x_i, w_i),$$

$$k_2 = hf\left(x_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right),$$

$$k_3 = hf\left(x_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right),$$

$$k_4 = hf(x_{i+1}, w_i + k_3),$$

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad \forall i = 0, 1, \dots, N-1$$

Which we would like to generalized

Fourth order Runge-Kutta method

- Let an integer $N > 0$ be chosen and $h = (b - a)/N$.
- Partition the interval $[a, b]$ into N sub-intervals with mesh points $x_j = a + jh$, $j = 0, 1, \dots, N$.
- Let $w_{i,j}$ be the approximation to $u_i(x_j)$, for $i = 1, 2, \dots, m$, and $j = 0, 1, 2, \dots, N$

$$w_{1,0} = \alpha_1, \quad w_{2,0} = \alpha_2, \quad \dots \quad w_{m,0} = \alpha_m$$

$$k_{1,i} = hf_i(x_j, w_{1,j}, w_{2,j}, \dots, w_{m,j})$$

$$k_{2,i} = hf_i\left(x_j + \frac{h}{2}, w_{1,j} + \frac{1}{2}k_{1,1}, w_{2,j} + \frac{1}{2}k_{1,2}, \dots, w_{m,j} + \frac{1}{2}k_{1,m}\right)$$

$$k_{3,i} = hf_i\left(x_j + \frac{h}{2}, w_{1,j} + \frac{1}{2}k_{2,1}, w_{2,j} + \frac{1}{2}k_{2,2}, \dots, w_{m,j} + \frac{1}{2}k_{2,m}\right)$$

$$k_{4,i} = hf_i\left(x_j + h, w_{1,j} + k_{3,1}, w_{2,j} + k_{3,2}, \dots, w_{m,j} + k_{3,m}\right)$$

$$w_{i,j+1} = w_{i,j} + \frac{1}{6}(k_{1,i} + 2k_{2,i} + 2k_{3,i} + k_{4,i})$$



RK4 Example

Now the initial condition $w_{1,0} = -0.4$, $w_{2,0} = -0.6$

$$k_{1,1} = hf_1(x_0, w_{1,0}, w_{2,0}) = hw_{2,0} - 0.06 \quad (7)$$

$$k_{1,2} = hf_2(x_0, w_{1,0}, w_{2,0}) = h[e^{2x_0} \sin x_0 - 2w_{1,0}] = -0.04$$

$$k_{2,1} = hf_1\left(x_0 + \frac{h}{2}, w_{1,0} + \frac{1}{2}k_{1,1}, w_{2,0} + \frac{1}{2}k_{1,2}\right) = -0.062$$

$$k_{2,2} = hf_2\left(x_0 + \frac{h}{2}, w_{1,0} + \frac{1}{2}k_{1,1}, w_{2,0} + \frac{1}{2}k_{1,2}\right) = -0.003247644757$$

$$k_{3,1} = hf_1\left(t_0 + \frac{h}{2}, w_{1,0} + \frac{1}{2}k_{2,1}, w_{2,0} + \frac{1}{2}k_{2,2}\right) = -0.06162832238$$

$$k_{3,2} = hf_2\left(x_0 + \frac{h}{2}, w_{1,0} + \frac{1}{2}k_{2,1}, w_{2,0} + \frac{1}{2}k_{2,2}\right) = -0.03152409237$$

$$k_{4,1} = hf_1(x_0 + h, w_{1,0} + k_{3,1}, w_{2,0} + k_{3,2}) = -0.06315240924$$

$$k_{4,2} = hf_2(x_0 + h, w_{1,0} + k_{3,1}, w_{2,0} + k_{3,2}) = -0.02178637298$$



RK4 Example

$$w_{1,1} = w_{1,0} + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}) = -0.4617333423$$

$$w_{2,1} = w_{2,0} + \frac{1}{6}(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}) = -0.63163112421$$

The values $w_{1,1} \approx u_1(0.1) = y(0.1)$, $w_{2,1} \approx u_2(0.1) = y'(0.1)$ similarly proceed for $j = 1, 2, \dots, 10$ that is $x = 0.2, 0.3, \dots, 1.0$.

Higher Order Differential Equations

A general m th-order initial-value problem

$$y^{(m)}(x) = f(x, y, y', y'', \dots, y^{(m-1)}), a \leq x \leq b$$

with the ICs: $y(a) = \alpha_1, y'(a) = \alpha_2, \dots, y^{(m-1)}(a) = \alpha_m$

- Consider $y_1(x) = y(x), y_2(x) = y'(x), \dots, y_m(x) = y^{(m-1)}(x)$
- This produces the first order system

$$\frac{dy_1}{dx} = \frac{dy}{dx} = y_2, \quad \frac{dy_2}{dx} = \frac{dy'}{dx} = y_3, \quad \frac{dy_{m-1}}{dx} = \frac{dy^{(m-2)}}{dx} = y_m$$

$$\frac{dy_m}{dx} = \frac{dy^{(m-1)}}{dx} = y^{(m)} = f(x, y, y', \dots, y^{(m-1)}) = f(x, y_1, y_2, \dots, y^{(m-1)})$$

- with initial conditions (ICs)

$$y_1(a) = y(a) = \alpha_1, \quad y_2(a) = y'(a) = \alpha_2, \quad y_m(a) = y^{(m-1)}(a) = \alpha_m$$

RK4 Example

Example

Transform the second order initial-value problem

$$y'' = 2y' + 2y = e^{2x} \sin x, \quad y(0) = -0.4, \quad y'(0) = -0.6 \quad \text{for } 0 \leq x \leq 1,$$

into a system of first-order initial-value problems, and use the Runge-Kutta method with $h = 0.1$ to approximate the solution.

Let $y_1(x) = y(x)$ and $y_2(x) = y'(x)$

$$y'_1(x) = y_2(x)$$

$$y'_2(x) = e^{2x} \sin x = 2y_1(x) + 2y_2(x)$$

using initial conditions $y_1(0) = -0.4$ and $y_2(0) = -0.6$

**ANY
QUESTIONS?**