Numerical Methods

DS288 and UMC201

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Upper Bound on Piecewise Linear interpolation

- Let us assume that the tabular points $x_0, x_1, \cdots, x_n$ are equispaced. Then

  $$x_{i+1} - x_i = h, \quad i = 1, 2, \cdots, n-1, \quad \text{and} \quad x_i - x_0 = ih, \quad i = 1, 2, \cdots, n$$

- Then the upper bound on piecewise linear interpolation is

  $$E([x_0, x_n]) = \frac{Mh^2}{8},$$

  where h is width of each sub interval and $M = \max_{x \in I} |f^{''}(x)|$

Disadvantage of linear function approximation

- No differentiability at the end points of the subintervals, i.e. the interpolating function is not smooth.
- From the physical conditions it is required that the approximation function must be continuously differentiable.

Piecewise Quadratic interpolation

- To obtain a somewhat smoother graph, consider using piecewise quadratic interpolation.
- Divided the interval [a, b] into 2n (even number) equal (not necessary) intervals, i.e.

$$\frac{b-a}{2n} = h, \quad \text{such that} \quad a = x_0, x_1, \cdots x_{2n-2}, x_{2n-1}, x_{2n} = b$$

- Begin by constructing the quadratic polynomial that interpolates $\{(x_{2i-1}, f_{2i-2}), (x_{2i-1}, f_{2i-1}), (x_{2i}, f_{2i})\}$
- Then, the quadratic interpolating polynomial is

$$\begin{aligned} P_{2i} =& f_{2i-2} + (x - x_{2i-2)} f[x_{2i-2}, x_{2i-1}] \\ & + (x - x_{2i-2)}(x - x_{2i-1)} f[x_{2i-2}, x_{2i-1}, x_{2i}], \\ & \text{for the interval } [x_{2i-2}, x_{2i}] \quad i = 1, 2, 3, \cdots, n \end{aligned}$$

- These polynomials are n in numbers.

Upper Bound on Piecewise Quadratic interpolation

Theorem (Basic Upper Bound)

If $f \in C^3[a, b]$, then for piecewise quadratic interpolation:

$$|f(x) - Q(x)| \leq \frac{\sqrt{3}}{9} h^3 \|f^{(3)}\|_\infty, \quad \textit{where:} \tag{1}$$

- $h =$ *maximum subinterval length*
- $\|f^{(3)}\|_\infty = \max\{|f^{(3)}(x)| : x \in [a, b]\}$

Example

Consider the data

n	0	1	2	2.5	3	3.5	4
x_n	2.5	0.5	0.5	1.5	1.5	1.125	0

Upper Bound on Piecewise Quadratic interpolation

- On each subinterval $[x_i, x_{i+2}]$, we have three points: x_i, x_{i+1}, x_{i+2}.
- The quadratic polynomial $Q_i(x)$ interpolates f at these three points.
- For quadratic interpolation on $[x_i, x_{i+2}]$, the error is

$$f(x) - Q_i(x) = \frac{f^{(3)}(\xi)}{3!} \cdot (x - x_i)(x - x_{i+1})(x - x_{i+2}) \quad \text{where } \xi \in [x_i, x_{i+2}].$$

- Maximize the Error Term. Here we need to find:

$$\max\{|(x - x_i)(x - x_{i+1})(x - x_{i+2})| : x \in [x_i, x_{i+2}]\}$$

  Let us transform to the standard interval. Set:
  - $t = \frac{x - x_{i+1}}{h}$, where $h = \frac{x_{i+2} - x_i}{2}$
  - The three points become: $t = -1, 0, 1$
  - The product becomes: $h^3 t(t^2 - 1) = h^3(t^3 - t)$

Upper Bound on Piecewise Quadratic interpolation

- Find Maximum of $|t^3 - t|$, by taking derivative: $\frac{d}{dt}(t^3 - t) = 3t^2 - 1 = 0$

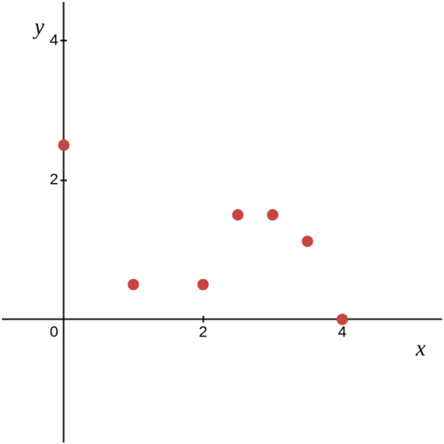
  Critical points: $t = \pm\frac{1}{\sqrt{3}}$. Evaluating $|t^3 - t|$:

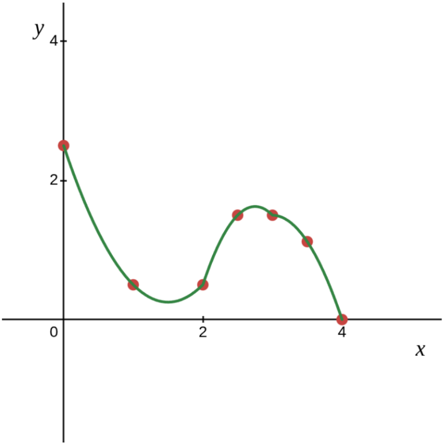
  - At $t = \frac{1}{\sqrt{3}}$: $\left|\frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}}\right| = \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9}$
  - At $t = -\frac{1}{\sqrt{3}}$: same value by symmetry
  - At boundaries $t = \pm 1$: $|\pm 1 \mp 1| = 0$ or 2

  Maximum value: $\frac{2\sqrt{3}}{9}$

- The final bound $|f(x) - Q_i(x)| \leq \frac{\|f^{(3)}\|_\infty}{6} \cdot h^3 \cdot \frac{2\sqrt{3}}{9} = \frac{\sqrt{3}}{9} h^3 \|f^{(3)}\|_\infty$

Piecewise Quadratic Interpolation





Piecewise Cubic interpolation
or
(Cubic Spline Interpolation)

Cubic Spline Interpolation

- The most common piecewise polynomial the approximation uses cubic polynomial between each successive pair of nodes called Cubic spline interpolation.
- These polynomials have the same slop and the curvature at the nodes where they join.
- Do not require the intervals to be of the same width

Cubic Spline Interpolation

Definition: Given a function f defined on $[a, b]$ and a set of nodes $a = x_0 < x_1 < \cdots < x_n = b$, a cubic spline interpolant S for f is a function that satisfies the following conditions:

(a) $S(x)$ is a cubic polynomial, denoted $S_j(x)$, on the subinterval $[x_j, x_{j+1}]$ for each $j = 0, 1, \cdots, n-1$;

(b) $S_j(x_j) = f(x_j)$ for each $j = 0, 1, \cdots, n-1$;

(c) $S_j(x_{j+1}) = f(x_{j+1})$ for each $j = 0, 1, \cdots, n-1$;.

(c1) $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ for each $j = 0, 1, \cdots, n-2$;

(d) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for each $j = 0, 1, \cdots, n-2$;

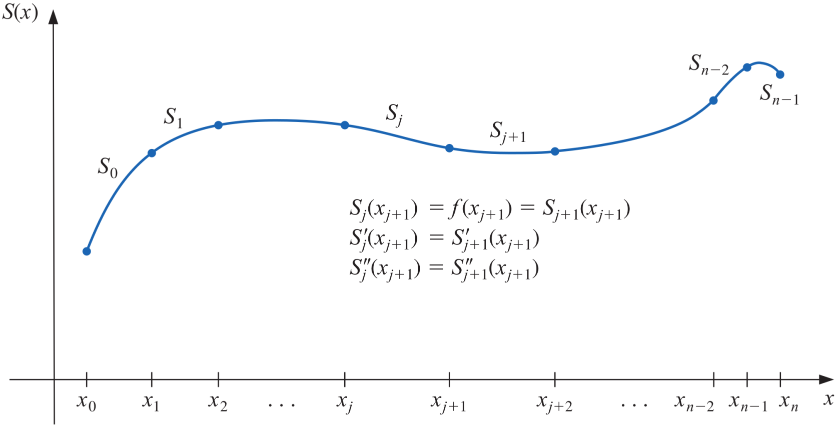
(e) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for each $j = 0, 1, \cdots, n-2$;

(f) One of the following sets of boundary conditions is satisfied:

(i) $S''(x_0) = S''(x_n) = 0$ (Natural (or free) boundary)

(ii) $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$ (Clamped boundary)

Cubic spline interpolation



Cubic spline interpolation

- To construct the cubic spline interpolant $S_j(x)$ for a given function f in the interval $[x_j, x_{j+1}]$, above conditions are applied to the cubic polynomial. For each $j = 0, 1, \cdots, n-1$;

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$$

- These constraints total $4n - 2$ conditions. But we need $4n$ to get a unique solution.
- Solution is to impose one condition at each endpoint, x_0 and x_n

Cubic spline interpolation

1. $S_j(x_j) = f(x_j) = a_j, \; j = 0, 1, 2, \cdots n-1$
2. $S_j(x_{j+1}) = f(x_{j+1}) = a_j + b_j h_j + c_j h_j^2 + d_j h_j^3 = a_{j+1}, \; j = 0, \cdots n-1,$ where $h_j = x_{j+1} - x_j$
3. $S'_j(x_{j+1}) = S'_{j+1}(x_{j+1})$, where $S'_j(x) = b_j + 2c_j(x - x_j) + 3d_j(x - x_j)^2$
$$b_j + 2c_j h_j + 3d_j h_j^2 = b_{j+1}, \quad \text{for } j = 0, 1, 2, \cdots n-2$$
4. $S''_j(x_{j+1}) = S''_{j+1}(x_{j+1})$, where $S''_j(x) = 2c_j + 6d_j(x - x_j)$
$$2c_j + 6d_j h_j = 2c_{j+1}, \quad \text{for } j = 0, 1, 2, \cdots n-2$$

- Combine these four equation into a single equation involving on the set of $\{c_j\}$ and known quantity by various substitutions

➢ Solve (4) for d_j and plug into (2),

➢ Solve (2) for b_j and plug into (3),

Cubic spline interpolation

We get

$$h_{j-1}c_{j-1} + 2(h_{j-1} + h_j)c_j + h_j c_{j+1} = \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1})$$

holds for $j = 0, 1, 2, \cdots n-1$, that is the system of $n-1$ equation with $n+1$ variable, $\{c_j\}_{j=0}^{n}$. To get unique solution
For example (Natural (or free) boundary), i.e., $c_0 = c_n = 0$, see details

- We need $S_0''(x_0) = S_n''(x_n) = 0$
  Since $S_j''(x) = 2c_j + 6d_j(x - x_j)$ we get

$$S_0''(x_0) = 2c_0 = 0 \quad (2)$$
$$S_n''(x_n) = 2c_n = 0 \quad (3)$$

Cubic spline interpolation

Now we write a matrix system $Ac = b$ (with each j represents as a row in the matrix), where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 & \cdots & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ h_0 & 0 & \cdots & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ h_0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$c = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \\ c_n \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \frac{3}{h_2}(a_3 - a_2) - \frac{3}{h_1}(a_2 - a_1) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - h\frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 0 \end{pmatrix}$$

Cubic spline interpolation

Example

Construct a natural cubic spline that passes through the points $(1, 2), (2, 3)$, and $(3, 5)$.

This spline consists of two cubics.

$$\begin{aligned} S_0(x) &= a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3, \text{ on the interval } [1,2] \\ S_1(x) &= a_1 + b_1(x-2) + c_1(x-2)^2 + d_1(x-2)^3 \text{ on the interval } [2,3]. \end{aligned}$$

- 8 constants to be determined, which requires 8 conditions.
- 4 conditions must agree with the data at the nodes. Hence

$$\begin{aligned} 2 = f(1) &= a_0, \\ 3 = f(2) &= a_0 + b_0 + c_0 + d_0, \\ 3 = f(2) &= a_1, \\ 5 = f(3) &= a_1 + b_1 + c_1 + d_1. \end{aligned}$$

Cubic spline interpolation

Two more come from the fact that $S_0'(2) = S_1'(2)$ and $S_0''(2) = S_1''(2)$. These are

$$\begin{aligned} b_0 + 2c_0 + 3d_0 &= b_1 \qquad S_0'(2) = S_1'(2), \\ 2c_0 + 6d_0 &= 2c_1 \qquad S_0''(2) = S_1''(2) \end{aligned}$$

The final two come from the natural boundary conditions:

$$\begin{aligned} 2c_0 &= 0 \qquad S_0''(1) = 0, \\ 2c_1 + 6d_1 &= 0 \qquad S_1''(3) = 0 \end{aligned}$$

Solving this system of equations gives the spline

$$S(x) = \begin{cases} 2 + \frac{3}{4}(x-1) + \frac{1}{4}(x-1)^3, & \text{for } x \in [1,2] \\ 3 + \frac{3}{2}(x-2) + \frac{3}{4}(x-2)^2 - \frac{1}{4}(x-2)^3, & \text{for } x \in [2,3] \end{cases}$$


ANY
QUESTIONS?