Stability, Consistency, and Convergence Analysis of Euler's Method

1 Introduction

We consider the initial value problem (IVP) for ordinary differential equations:

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0, \tag{1}$$

where $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a given function satisfying appropriate smoothness conditions.

The **explicit Euler method** (or forward Euler method) is given by:

$$y_{n+1} = y_n + h \cdot f(t_n, y_n), \quad n = 0, 1, 2, \dots, \tag{2}$$

where $h > 0$ is the step size and $t_n = t_0 + nh$.

2 Consistency Analysis

Definition 2.1 (Consistency)**.** A numerical method is **consistent** if the local truncation error tends to zero as the step size approaches zero.

2.1 Local Truncation Error

The local truncation error (LTE) at step $n + 1$ is defined as:

$$\tau_{n+1} = \frac{y(t_{n+1}) - y(t_n)}{h} - f(t_n, y(t_n)), \tag{3}$$

where $y(t)$ is the exact solution of (1).

Using Taylor's expansion around t_n:

$$\begin{aligned} y(t_{n+1}) &= y(t_n) + h \cdot y'(t_n) + \frac{h^2}{2} y''(t_n) + \mathcal{O}(h^3) \\ &= y(t_n) + h \cdot f(t_n, y(t_n)) + \frac{h^2}{2} y''(t_n) + \mathcal{O}(h^3). \end{aligned} \tag{4}$$

Therefore, the local truncation error is:

$$\tau_{n+1} = \frac{h}{2} y''(t_n) + \mathcal{O}(h^2) = \mathcal{O}(h). \tag{5}$$

Theorem 2.1 (Consistency of Euler's Method)**.** Euler's method is consistent with order $p = 1$, i.e., $\tau_{n+1} = \mathcal{O}(h)$ as $h \to 0$.

Remark 2.1. The dominant term in the local truncation error is $\frac{h}{2} y''(t_n)$. This makes Euler's method a **first-order method**.

3 Stability Analysis

Stability analysis examines how errors propagate through the numerical solution. We consider several types of stability.

3.1 Absolute Stability

To analyze absolute stability, we apply Euler's method to the scalar test equation:

$$y' = \lambda y, \quad y(0) = y_0, \quad \lambda \in \mathbb{C}. \tag{6}$$

The exact solution is $y(t) = y_0 e^{\lambda t}$.

Applying Euler's method (2):

$$y_{n+1} = y_n + h\lambda y_n = (1 + h\lambda) y_n. \tag{7}$$

By induction:

$$y_n = (1 + h\lambda)^n y_0. \tag{8}$$

Definition 3.1 (Absolute Stability)**.** Euler's method is **absolutely stable** for a given $h\lambda$ if

$$|1 + h\lambda| \leq 1. \tag{9}$$

Definition 3.2 (Region of Absolute Stability)**.** The **region of absolute stability** is the set

$$S = \{z \in \mathbb{C} : |1 + z| \leq 1\}, \tag{10}$$

where $z = h\lambda$.

Theorem 3.1 (Stability Region for Euler's Method)**.** The region of absolute stability for Euler's method is a closed disk in the complex plane with:

- Center: $(-1, 0)$
- Radius: 1

This region is entirely contained in the left half-plane $\{\mathrm{Re}(z) \leq 0\}$.

Proof. The condition $|1 + z| \leq 1$ describes all complex numbers z whose distance from -1 is at most 1. Setting $z = x + iy$:

$$|1 + z|^2 = (1 + x)^2 + y^2 \leq 1, \tag{11}$$
$$x^2 + 2x + 1 + y^2 \leq 1, \tag{12}$$
$$x^2 + 2x + y^2 \leq 0, \tag{13}$$
$$(x + 1)^2 + y^2 \leq 1. \tag{14}$$

This is a disk centered at $(-1, 0)$ with radius 1. □

3.1.1 Implications for Stability

- For real $\lambda < 0$, the stability condition becomes:

$$-1 \leq 1 + h\lambda \leq 1 \quad \Rightarrow \quad h \leq \frac{2}{|\lambda|}. \tag{15}$$

- For stiff problems with large $|\lambda|$, extremely small step sizes are required.
- Euler's method is **conditionally stable**.

4 Convergence Analysis

Definition 4.1 (Convergence)**.** A numerical method is **convergent** if the global error approaches zero as the step size approaches zero:

$$\max_{0 \le n \le N} |y(t_n) - y_n| \to 0 \quad \text{as} \quad h \to 0, \tag{16}$$

where $Nh = T - t_0$ is fixed.

4.1 Lax Equivalence Theorem

Theorem 4.1 (Lax Equivalence Theorem)**.** For a consistent, zero-stable linear multistep method:

$$\text{Consistency} + \text{Zero-Stability} \iff \text{Convergence}. \tag{17}$$

Since Euler's method is both consistent and zero-stable, it is convergent.

Theorem 4.2 (Convergence Order of Euler's Method)**.** The global error of Euler's method satisfies:

$$|e_n| = \mathcal{O}(h), \quad \text{as} \quad h \to 0. \tag{18}$$

Euler's method has **first-order convergence**.

Remark 4.1. Although the local truncation error is $\mathcal{O}(h^2)$, the global error accumulates over $n = \mathcal{O}(1/h)$ steps, resulting in an overall $\mathcal{O}(h)$ global error.

5 Practical Considerations

5.1 Step Size Selection

The choice of step size h must balance:

- **Accuracy**: h should be small enough that the $\mathcal{O}(h)$ error is acceptable.
- **Stability**: For the test equation $y' = \lambda y$ with $\lambda < 0$, we need $h \le \frac{2}{|\lambda|}$.
- **Computational cost**: Smaller h requires more steps and increased computation time.

5.2 Advantages and Disadvantages

Advantages:

- Simple to implement
- Computationally inexpensive per step (one function evaluation)
- Good for non-stiff problems with moderate accuracy requirements
- Provides intuitive geometric interpretation

Disadvantages:

- Low order accuracy (many steps needed for high precision)
- Conditional stability (restrictive for stiff problems)
- Error accumulation over long time integrations
- Not A-stable

5.3 Improvements and Alternatives

- **Implicit Euler method**: Unconditionally stable, better for stiff problems

$$y_{n+1} = y_n + h \cdot f(t_{n+1}, y_{n+1}). \tag{19}$$

- **Higher-order Runge-Kutta methods**: RK2, RK4 for better accuracy with the same stability properties.
- **Adaptive step size methods**: Adjust h dynamically based on local error estimates (e.g., Runge-Kutta-Fehlberg).
- **Multistep methods**: Adams-Bashforth, Adams-Moulton methods for improved efficiency.

6 Summary

Property	**Euler's Method**
Consistency	Yes, order $p = 1$
Local Truncation Error	$\mathcal{O}(h)$
Global Error	$\mathcal{O}(h)$
Convergence	Yes, first-order
Stability Region	Disk: center $(-1, 0)$, radius 1
Function Evaluations per Step	1

Table 1: Summary of properties of Euler's method

7 Conclusion

Euler's method is a fundamental first-order numerical method for solving ordinary differential equations. Its key theoretical properties are:

1. **Consistency**: The local truncation error is $\mathcal{O}(h)$, ensuring consistency.
2. **Stability**: The method is zero-stable and conditionally stable with a stability region that is a disk of radius 1 centered at -1 in the complex plane.
3. **Convergence**: By the Lax Equivalence Theorem, consistency and zero-stability imply convergence, with global error $\mathcal{O}(h)$.