Numerical Methods

DS288 and UMC201

Ratikanta Behera

Department of Computational and Data Sciences,
Indian Institute of Science Bangalore

August-December 2025

Solving Initial Value Problem

- Sometimes the function is very complex and the derivative or integration of the function can't be evaluated analytically.
- Need to use numerical methods.
- Which is based on Taylor's series expansion.
- Their accuracy depends on the number of terms involved in the calculation of the derivative.

➢ The Taylor series of a function $f(x)$ at a point x_0

$$\begin{aligned} f(x) &= f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 \\ &+ \cdots + \frac{f^{(n+1)}(\zeta_n)}{(n+1)!}(x - x_0)^{(n+1)} \quad \text{for } x_0 < \zeta_n < x \end{aligned}$$

where $f(x)$ infinitely differentiable at x_0.

Solving Initial Value Problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0, \quad a \leq x \leq b \tag{1}$$

- Finite number of subintervals by points (grid points)

$$a = x_0 \leq x_1 \leq x_2 \leq \cdots\cdots \leq x_{n-1} \leq x_n = b$$

- Step length: The spacing between the points $h_i = x_i - x_{i-1}$
- Uniform grid points: $x_i = x_0 + ih$

➢ Let y_i is the approximation solution of $y(x)$ at the points x_i.
➢ Estimates the solution at different grid point $y_{x_0+h}, y_{x_0+2h}, \cdots$ using the truncated Taylor series expansion.
➢ The set of numbers $\{y_i\}$ is called numerical solution of IVP.

Euler's Method

- Euler's method define by $y_{n+1} = y_n + hf(x_n, y_n)$.
- Using Taylor's series

$$\begin{aligned} y(x_{n+1}) &= y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(\zeta_n) \text{ for } x_n < \zeta_n < x \\ y(x_{n+1}) &\approx y(x_n) + hy'(x_n) \\ y_{n+1} &= y_n + hf(x_n, y_n) \end{aligned}$$

- By dropping the error term in the Euler method. The term

$$T_n = \frac{h^2}{2}y''(\zeta_n)$$

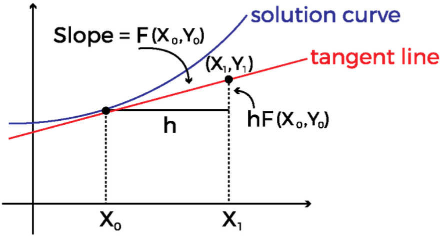
is called the truncation error or discretization error at x_{n+1}.

Geometrical Interpretation

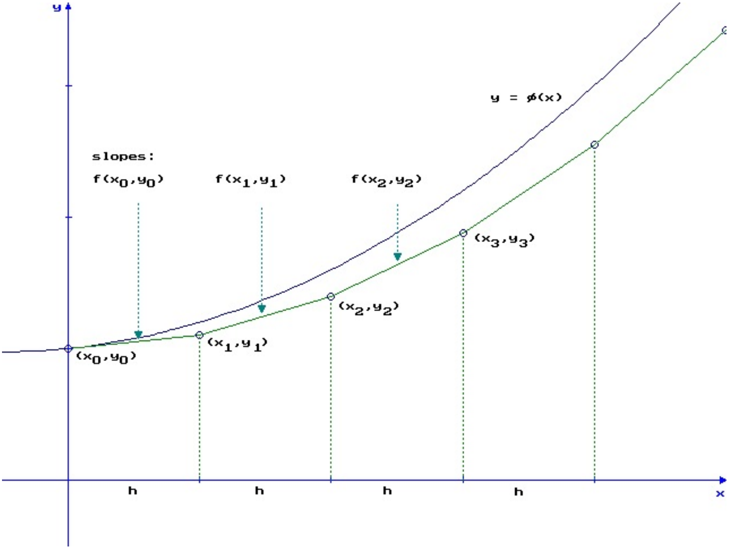
We have first order initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \tag{2}$$

- $x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n$
- Let $h = x_1 - x_0$
- Slope$= \frac{y_1 - y_0}{x_1 - x_0} = f(x_0, y_0)$
- $y_1 = y_0 + hf(x_0, y_0)$
- $y_2 = y_1 + hf(x_1, y_1)$
- $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$



Euler's Method



Example for Euler's Method

Example

Find an approximate value of y corresponding to $x = 0.5$ with $h = 0.1$

$$\frac{dy}{dx} = x + y, \quad y(0) = 1$$

➢ Here $x_0 = 0,\ y_0 = 1,\ h = 0.1,\ f(x, y) = x + y$

x_i	x	$y(x_i)$	$y_e(x) = 2e^x - x - 1$	$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$
x_0	0	1.0000	1.0000	1.00+0.1(1.00)=1.10
x_1	0.1	1.1000	1.1103	1.10+0.1(1.20)=1.22
x_2	0.2	1.2200	1.2428	1.22+0.1(1.42)=1.36
x_3	0.3	1.3620	1.3997	1.36+0.1(1.66)=1.53
x_4	0.4	1.5282	1.5836	1.53+0.1(1.93)=1.7210
x_5	0.5	1.7210	1.7974	

➢ Thus the required approximation value is $y(0.5) = 1.7210$

Example for Euler's Method

Example

- For given IVP, find the value of y for $x = 0.08$ with $h = 0.02$

$$\frac{dy}{dx} = \frac{y - x}{y + x}, \quad y(0) = 1$$

- Here $x_0 = 0$, $y_0 = 1$ $h = 0.02$, $f(x, y) = \frac{y-x}{y+x}$

x	y	$\frac{dy}{dx} = \frac{y-x}{y+x}$	$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$
0.00	1.0000	1.0000	1.0000+0.02(1.0000)=1.0200
0.02	1.0200	0.9615	1.0200+0.02(0.9615)=1.0392
0.04	1.0392	0.926	1.0392+0.02(0.926)=1.0577
0.06	1.0577	0.893	1.0577+0.02(0.893)=1.0756
0.08	1.0756		

- Thus the required approximation value is $y(0.08) = 1.0756$

Modified-Euler (Predictor-Corrector) Method

Why Modified Euler's Method

- The Euler scheme may be very easy to implement but it can't give accurate solutions.
- A very small step size is required for any meaningful result.
- The solution is correct only if the function is linear, as the starting point of each sub-interval is used to find the slope of the solution curve.

Improvement

- To take the arithmetic average of the slopes at x_n and x_{n+1}
- Average slope $\dfrac{y'(x_n)+y'(x_{n+1})}{2} = \dfrac{[f(x_n,y_n)+f(x_{n+1},y_{n+1})]}{2}$
- $y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$

Modified Euler's Method

(1) Find the predicted value: $y_{n+1,p} = y_n + h * f(x_n, y_n)$,

(2) Average slope $\frac{[f(x_n,y_n)+f(x_{n+1},y_{n+1})]}{2}$

(3) Then correct the predicted value (modified Euler's Method):

$$y_{n+1,c} = y_n + h\frac{f(x_n, y_n) + f(x_{n+1}, y_{n+1,p})}{2}$$

- We can write the modified Euler's Method

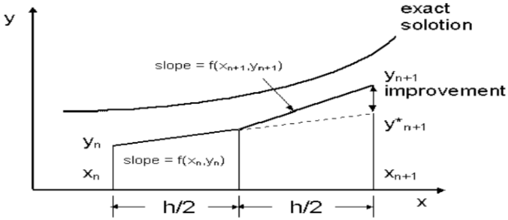
$$y_{n+1} = y_n + h\left[\frac{f(x_n, y_n) + f(x_{n+1}, y_n + h * f(x_n, y_n))}{2}\right]$$

Geometrical Interpretation

We have first order initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

- Slope at $(x_n, y_n) \quad \longrightarrow \quad f(x_n, y_n)$
- Slope at $(x_{n+1}, y_{n+1}) \quad \longrightarrow \quad f(x_{n+1}, y_{n+1})$
- $y_{n+1,c} = y_n + h\frac{f(x_n,y_n)+f(x_{n+1},y_{n+1,p})}{2}$



Example (Euler's Method)

Example

Find an approximate value of y corresponding to $x = 0.5$ with $h = 0.1$

$$\frac{dy}{dx} = x + y, \quad y(0) = 1$$

➢ Here $x_0 = 0,\ y_0 = 1,\ h = 0.1,\ f(x, y) = x + y$

x_i	x	$y(x_i)$	$y_e(x) = 2e^x - x - 1$	$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$
x_0	0	1.0000	1.0000	1.00+0.1(1.00)=1.10
x_1	0.1	1.1000	1.1103	1.10+0.1(1.20)=1.22
x_2	0.2	1.2200	1.2428	1.22+0.1(1.42)=1.36
x_3	0.3	1.3620	1.3997	1.36+0.1(1.66)=1.53
x_4	0.4	1.5282	1.5836	1.53+0.1(1.93)=1.7210
x_5	0.5	1.7210	1.7974	

➢ Thus the required approximation value is $y(0.5) = 1.7210$

Example (Modified Euler's Method)

Example

Find an approximate value of y corresponding to $x = 0.5$ with $h = 0.1$

$$\frac{dy}{dx} = x + y, \quad y(0) = 1$$

➢ Here $x_0 = 0,\ y_0 = 1,\ h = 0.1,\ f(x, y) = x + y$

x	y	$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_n + h * f(x_n, y_n))\right]$
0	1.00	$1.00 + \frac{0.1}{2}(0 + 1 + 0.1 + 1.1) = 1.1100$
0.1	1.11	$1.11 + \frac{0.1}{2}(0.1 + 1.11 + 0.2 + 1.2200) = 1.2415$
0.2	1.2415	$1.2415 + \frac{0.1}{2}(0.2 + 1.2415 + 0.3 + 1.362) = 1.3967$
0.3	1.3967	1.5779
0.4	1.5779	1.7878
0.5	1.7878	

➢ Thus the required approximation value is $y(0.5) = 1.7878$

Example for Modified Euler's Method

Example

Find an approximate value of y corresponding to $x = 0.5$ with $h = 0.1$

$$\frac{dy}{dx} = log(x + y), \quad y(0) = 2$$

Analysis of Euler's method

Lemma-1

For all $x \geq 1$ and any positive m, then $0 \leq (1+x)^m \leq e^{mx}$

Apply Taylor's theorem with $f(x) = e^x$, $x_0 = 1$ and $n = 1$.

$$e^x = 1 + x + \frac{1}{2}x^2 e^{\zeta}, \qquad 0 < \zeta < x$$

since $0 \leq 1 + x \leq 1 + x + \frac{1}{2}x^2 e^{\zeta} = e^x$
Further, since $1 + x \geq 0$

$$0 \leq (1+x)^m \leq (e^x)^m$$

Analysis of Euler's method

Lemma-2

If s and t are +ve real number, $\{a_i\}_{i=0}^{k}$ is a sequence satisfying $a_0 \geq -t/s$ and

$$a_{i+1} \leq (1+s)a_i + t \text{ for each } i = 0, 1, 2, \cdots k-1$$

$$\text{then} \quad a_{i+1} \leq e^{(i+1)s}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}$$

For a fixed integer i,

$$\begin{aligned} a_{i+1} &\leq (1+s)a_i + t \\ &\leq (1+s)[(1+s)a_{i-1} + t] + t \\ &= (1+s)^2 a_{i-1} + [1 + (1+s)]t \\ &= (1+s)^3 a_{i-2} + [1 + (1+s) + (1+s)^2]t \\ &\vdots \end{aligned}$$

Analysis of Euler's method

$$a_{i+1} \leq (1+s)^{i+1} a_0 + [1 + (1+s) + (1+s)^2 + \cdots + (1+s)^i] t$$

The geometric series $\rightarrow 1 + (1+s) + \cdots + (1+s)^i = \sum_{j=0}^{i} (1+s)^j$

This sum: $\longrightarrow \dfrac{1 - (1+s)^{i+1}}{1 - (1+s)} = \dfrac{1}{s}[(1+s)^{i+1} - 1]$. Thus

$$\begin{aligned} a_{i+1} &\leq (1+s)^{i+1} a_0 + \frac{t}{s}[(1+s)^{i+1} - 1] \\ &= (1+s)^{i+1} \left(a_0 + \frac{t}{s} \right) - \frac{t}{s} \end{aligned}$$

with Lemma-1 we can write for $x = 1 + s$

$$a_{i+1} \leq e^{(i+1)s} \left(a_0 + \frac{t}{s} \right) - \frac{t}{s}$$

Lipschitz condition

Definition

A function $f(x, y)$ is said to satisfy a Lipschitz condition in the variable y on R if a constant $L \geq 0$, exists with

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$$

where $\quad (x, y_1), (x, y_2) \in R = \{(x, y): \ a \leq x \leq b, -\infty < y < \infty\}$

The constant L is called Lipschitz constant for f.


ANY
QUESTIONS?