Numerical Methods

DS288 and UMC201

Ratikanta Behera

Department of Computational and Data Sciences,
Indian Institute of Science Bangalore

August-December 2025

Chapter - 12

Numerical Solutions to Partial Differential Equations

Classification of Partial Differential Equations

Second order PDEs

The second-order linear partial differential equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fy = 0$$

where A, B, C, D, E, F are functions of x, y or are real constants, is said to be

- Parabolic equation if $B^2 - 4AC = 0$
- Hyperbolic equation if $B^2 - 4AC > 0$
- Elliptic equation if $B^2 - 4AC < 0$

The lower order (first order) terms do not play any role in the classification of the equation.

Example

Example

Classify the following PDE

$$yU_{xx} + U_{yy} = 0$$

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fy = 0$$

$$B^2 - 4AC = -4y$$

- $y > 0, \quad \longrightarrow$ Elliptic
- $y = 0, \quad \longrightarrow$ Parabolic
- $y < 0, \quad \longrightarrow$ Hyparabolic

Simple example of PDEs

- Parabolic equation: (Heat equation)

$$\begin{aligned}\frac{\partial u}{\partial t} &= c^2 \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial u}{\partial t} &= c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)\end{aligned}$$

- Hyperbolic equation : (Wave equation)

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial^2 u}{\partial t^2} &= c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)\end{aligned}$$

- Elliptic equation : (Laplace equation)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Section - 12.1

Elliptic Partial Differential Equations

Elliptic Partial Differential Equations

Elliptic PDEs

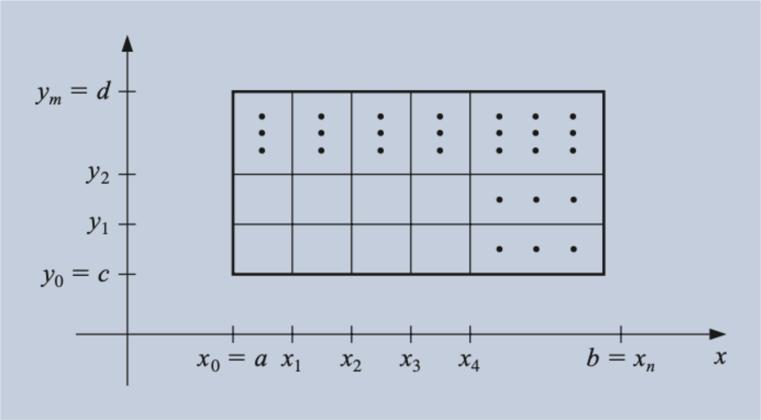
Consider the Poisson equation

$$\nabla^2 u(x,y) \equiv \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = f(x,y)$$

on $R = \{(x,y) | a < x < b, c < y < d\}$, with $u(x,y) = g(x,y)$ for $(x,y) \in S$, where S is the boundary of R. If f and g are continuous on their domains, then there is a unique solution to this equation.

- Partition the intervals $[a,b]$ and $[c,d]$ into n and m equal parts with step sizes $h = (b-a)/m$ and $k = (d-c)/m$

Grid Selection



Domain discretization

- The domain has equidistant mesh points (x_i, y_j), where $x_i = a + ih$, $i = 0, 1, \ldots, n$ and $y_j = c + jk$, $j = 0, 1, \ldots, m$
- For the mesh points in the interior of the grid (x_i, y_j), $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, m-1$, we use the centered-difference formula

$$\frac{\partial^2 u}{\partial x^2}(x_i, y_j) \approx \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2}$$
$$\frac{\partial^2 u}{\partial y^2}(x_i, y_j) \approx \frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})}{k^2}$$

with respective truncation errors $-\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, y_j)$ and $-\frac{k^2}{12}\frac{\partial^4 u}{\partial y^4}(x_i, \eta_j)$, $\xi \in (x_{i-1}, x_{i+1}), \eta_j \in (y_{j-1}, y_{j+1})$

Finite-Difference Method

- Substituting the above formulas in the Poisson equation we get

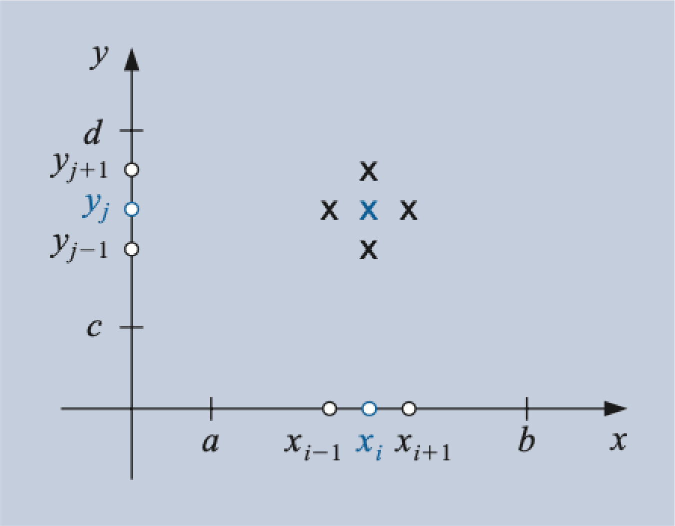
$$\frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} + \frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})}{k^2}$$
$$= f(x_i, y_j) + \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, y_j) + \frac{k^2}{12}\frac{\partial^4 u}{\partial y^4}(x_i, \eta_j)$$

for $i = 1, 2, \ldots, n-1$ and $j = 1, 2, \ldots, m-1$

- Use the boundary conditions as

$$u(x_0, y_j) = g(x_0, y_j), u(x_n, y_j) = g(x_n, y_j), j = 0, 1, \ldots, m$$
$$u(x_i, y_0) = g(x_i, y_0), u(x_i, y_m) = g(x_i, y_m), i = 1, 2, \ldots, n-1$$

Grid Selection



Finite-Difference Method

- Can be expressed in the difference-equation form

$$2\left[\left(\frac{h}{k}\right)^2+1\right]u_{ij}-(u_{i+1,j}+u_{i-1,j})-\left(\frac{h}{k}\right)^2(u_{i,j+1}+u_{i,j-1})=-h^2f(x_i,y_j)$$

where $u_{ij}=u(x_i,y_j)$ and $i=1,2,\ldots,n-1$ and $j=1,2,\ldots m-1$ with the boundary conditions

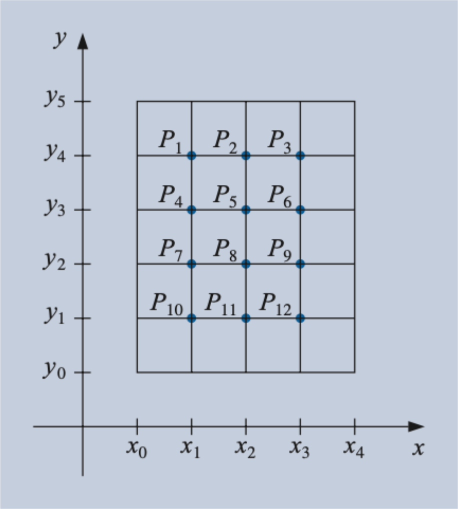
$$u_{0j}=g(x_0,y_j),u_{nj}=g(x_n,y_j),j=0,1,\ldots m$$
$$u_{i0}=g(x_i,y_0),u_{im}=g(x_i,y_m),i=1,2,\ldots,n-1$$

(1) It produce an $(n-1)(m-1)\times(n-1)(m-1)$ linear system with unknowns u_{ij}

(2) Relabel the grid points as $P_l=(x_i,y_j)$ and $u_l=u_{i,j}$, where $l=i+(m-1-j)(n-1)$, $i=1,2\ldots n-1$, $j=1,2\ldots,m-1$

- The relabeling provides a banded matrix system having bandwidth at most $2n-1$

Grid Selection



Example

- Consider a steady-state heat distribution in a thin square metal plate with dimensions $0.5m \times 0.5m$
- Two adjacent boundaries are held at $0^\circ C$ and the heat on the other two boundaries increases linearly from $0^\circ C$ at one corner to $100^\circ C$ where the sides meet
- Assume zero boundary conditions along the x- and y-axes and express the problem as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial u^2}{\partial y^2} = 0$$

for $(x, y) \in R = \{(x, y) \mid 0 < x < 0.5, 0 < y < 0.5\}$

- The boundary conditions are

$$u(0, y) = 0, u(x, 0) = 0, u(x, 0.5) = 200x, u(0.5, y) = 200y$$

- For $n = m = 4$, we have the difference equation

$$4u_{ij} - u_{i+1,j} - u_{i-1,j} - u_{i,j-1} - u_{i,j+1} = 0, i, j = 1, 2, 3$$

Example

- Use the relabeled grid points $u_l = u(P_l)$

$$
\begin{aligned}
P_1 &: 4u_1 - u_2 - u_4 = u_{0,3} + u_{1,4}\\
P_2 &: 4u_2 - u_3 - u_1 - u_5 = u_{2,4}\\
P_3 &: 4u_3 - u_2 - u_6 = u_{4,3} + u_{3,4}\\
P_4 &: 4u_4 - u_5 - u_1 - u_7 = u_{0,2}\\
P_5 &: 4u_5 - u_6 - u_4 - u_2 - u_8 = 0\\
P_6 &: 4u_6 - u_5 - u_3 - u_9 = u_{4,2}\\
P_7 &: 4u_7 - u_8 - u_4 = u_{0,1} + u_{1,0}\\
P_8 &: 4u_8 - u_9 - u_7 - u_5 = u_{2,0}\\
P_9 &: 4u_9 - u_8 - u_6 = u_{3,0} + u_{4,1}
\end{aligned}
$$

- With the boundary conditions

$$
\begin{aligned}
&u_{1,0} = u_{2,0} = u_{3,0} = u_{0,1} = u_{0,2} = u_{0,3} = 0\\
&u_{1,4} = u_{4,1} = 25,\ u_{2,4} = u_{4,2} = 50,\ u_{3,4} = u_{4,3} = 75
\end{aligned}
$$

Example

This gives us the linear system in the matrix form as

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix} = \begin{bmatrix} 20 \\ 50 \\ 150 \\ 0 \\ 0 \\ 50 \\ 0 \\ 0 \\ 25 \end{bmatrix}$$

- The obtained answers are exact, since the true solution $u(x, y) = 400xy$ satisfies $\dfrac{\partial^4 u}{\partial x^4} = \dfrac{\partial^4 u}{\partial y^4} \equiv 0$ and the truncation error is zero at each step


ANY
QUESTIONS?