

# DS 288: NUMERICAL METHODS

NOV-16-2021

VARIABLE STEP SIZE METHODS for ODES

## Variable Step Size Methods for ODEs

- Up to this point all our methods have been based on a fixed step size  $h$ .  
→ convenient and efficient in most cases.
- In some cases it is desirable to vary the step size  $h$  in order to control errors, especially when the error is nonuniform over the range of interest  $t \in [t_0, t_{end}]$ .  
→ analogous to adaptive quadrature.
- \* The key is to try and estimate the local truncation error (LTE) based on simple calculations.
- Two common strategies:

§5.7 Use two different methods of the same order (whose LTEs have known forms) to calculate  $w_{i+1}$  and relate their differences. ✓

§5.5 Use the same method with two different orders of LTE to calculate  $w_{i+1}$  and relate their differences.

### Two Different Methods of Same Order (§5.7)

Consider two different  $N^{th}$  order (GTE) methods which produce  $w_{i+1}$  and  $\tilde{w}_{i+1}$ :

- Assumptions (Big):
- a)  $y_k = w_k = \tilde{w}_k \quad k = 0, 1, \dots, i$
  - b)  $\xi$ -factors in  $E_{trunc}$  are same

Then

$$\begin{aligned} y_{i+1} - w_{i+1} &= Af^{(N)}(\xi_i) h^{N+1} \\ y_{i+1} - \tilde{w}_{i+1} &= Bf^{(N)}(\xi_i) h^{N+1} \\ \hline \tilde{w}_{i+1} - w_{i+1} &= (A - B)h^{N+1}f^{(N)}(\xi_i) \end{aligned}$$

and

$$f^{(N)} = \frac{\tilde{w}_{i+1} - w_{i+1}}{(A - B)h^{N+1}}$$

Now we want the global truncation error (GTE) to be less than some prescribed  $\epsilon$  and we consider taking a step size  $qh$ . What should the factor  $q$  be in order to maintain that GTE is bounded by some value  $\epsilon$ ? (Note that method A is our favored approach.)

$$\begin{aligned} |Af^{(N)}(\xi) (qh)^N| &< \epsilon \\ \frac{|A| (qh)^N |\tilde{w}_{i+1} - w_{i+1}|}{|A - B| h^{N+1}} &< \epsilon \end{aligned}$$

$$q < \left( \frac{\epsilon h |A - B|}{|A| |\tilde{w}_{i+1} - w_{i+1}|} \right)^{\frac{1}{N}}$$

← FORM FOR  
 $q$

Tracking  $q$  shows us how to alter the step-size in order to maintain the GTE below a prescribed amount, i.e.,  $\epsilon$ .

Typical Example: 4<sup>th</sup> Order Adams P/C

$$\begin{aligned} \text{A/B LTE : } & \underbrace{\frac{251}{720} h^5 f^{(4)}(\xi)}_B \\ \text{A/M LTE : } & \underbrace{\frac{-19}{720} h^5 f^{(4)}(\xi)}_A \\ q < & \left( \frac{\epsilon \frac{270}{720} h}{\frac{19}{720} |w_{i+1}^{A/M} - w_{i+1}^{A/B}|} \right)^{\frac{1}{4}} = \left( \frac{270}{19} \frac{\epsilon h}{|w_{i+1}^{A/M} - w_{i+1}^{A/B}|} \right)^{\frac{1}{4}} \end{aligned}$$

*INVOLVES  
TWO NUMERICAL  
SOLUTIONS*



### Same Method Two Different Orders (§5.5)

Consider using a method with  $\mathcal{O}(h^N)$  globally and the same method with  $\mathcal{O}(h^{N+1})$  which produce  $w_{i+1}$  and  $\tilde{w}_{i+1}$ :

$$\begin{aligned} y_{i+1} - w_{i+1} &= Ah^{N+1} f^{(N)}(\xi_i) \quad \xrightarrow{N+1} \mathcal{O}(h^{N+1}) \\ y_{i+1} - \tilde{w}_{i+1} &= Bh^{N+2} f^{(N)}(\xi_i) \quad \xrightarrow{N+2} \mathcal{O}(h^{N+2}) \\ \hline \tilde{w}_{i+1} - w_{i+1} &= Ah^{N+1} f^{(N)} - \underbrace{Bh^{N+2} f^{(N+1)}(\xi_i)}_0 \end{aligned}$$

assume small relative to first term, i.e.,  $h^{N+2} \ll h^{N+1}$  for small  $h$

So

$$f^{(N)} = \frac{\tilde{w}_{i+1} - w_{i+1}}{Ah^{N+1}}$$

... same as before with  $B = 0$ . Then it follows that

$$q < \left( \frac{\epsilon h}{|\tilde{w}_{i+1} - w_{i+1}|} \right)^{\frac{1}{N}}$$

*INVOLVES  
TWO NUMERICAL  
SOLUTIONS*

$\xrightarrow{\text{next step}} h \rightarrow q \sqrt{h}$

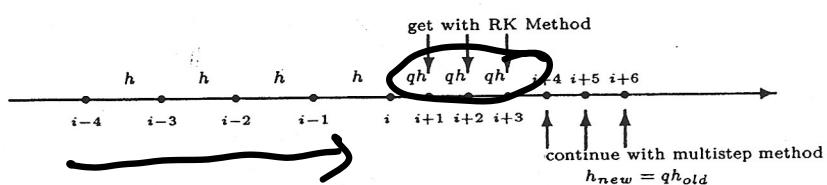
What  $q$  you will choose?  
 $0.1 \quad 0.01 \quad 0.0001 \quad \underline{0.09}$

$q$  to be lesser at the same time  
 If  $qh$  too small  $\rightarrow$  more steps  
 $\rightarrow$  more computation

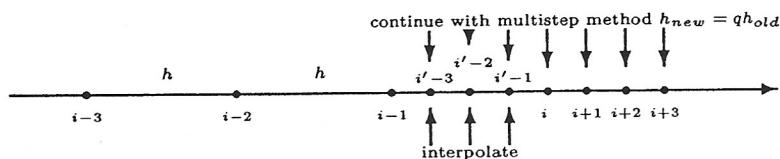
- \* Note that multi-step methods are derived assuming constant  $h$ ! If the step size changes along the way an adjustment must be made.

- Two common strategies:

- use a single-step method (i.e., self-starting) to calculate enough new values to proceed with the new step size



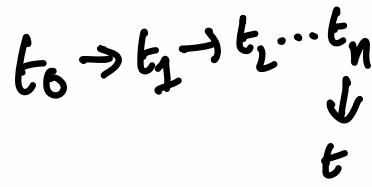
- interpolate back in time with the new step size



- Many methods do not change  $h$  continuously (which could potentially be quite costly) but rather monitor  $q$  and change only if it is significantly different than 1.

$$\begin{aligned}
 h_4 &= 0.9 h_3 \\
 h_5 &= 0.95 h_4 \\
 &\vdots \quad \text{(Significant)} \\
 h_i &\rightarrow 0.5 h_{i-1}
 \end{aligned}$$

# Summary of ODE Methods



Prototype problem:

$$\begin{array}{ll} y'(t) = f(y, t) & \text{rate function} \\ y(t_0) = y_0 & \text{initial value} \end{array}$$



Method	Approximate Solution to ODE	LTE	GTE	Ex/Im plicit
<i>Single-step</i>				
Euler's Method	$w_{i+1} = w_i + h f_i$	$O(h^2)$	$O(h)$	Explicit
Midpoint Rule	$w_{i+1} = w_i + h f\left(w_i + \frac{h}{2} f_i, t_i + \frac{h}{2}\right)$	$O(h^3)$	$O(h^2)$	Explicit
Trapezoidal Rule	$w_{i+1} = w_i + \frac{h}{2} [f_i + f(w_{i+1}, t_{i+1})]$	$O(h^3)$	$O(h^2)$	Implicit
Modified Euler's	$w_{i+1} = w_i + \frac{h}{2} [f_i + f(w_i + h f_i, t_{i+1})]$	$O(h^3)$	$O(h^2)$	Explicit
Heun's Method	$w_{i+1} = w_i + \frac{h}{4} [f_i + 3f(w_i + \frac{2}{3} h f_i, t_i + \frac{2}{3} h)]$	$O(h^3)$	$O(h^2)$	Explicit
Runge-Kutta 4 <sup>th</sup> Order	$w_{i+1} = w_i + \frac{h}{6} [f_i + 4f_{i+\frac{1}{2}} + f_{i+1}]$	$O(h^5)$	$O(h^4)$	Implicit
<u>most popular RK4</u>	$w_{i+1} = w_i + \frac{h}{6} [f_1 + 2(f_2 + f_3) + f_4]$ where $f_1 = f_i$ $f_2 = f(w_i + \frac{h}{2} f_1, t_{i+\frac{1}{2}})$ $f_3 = f(w_i + \frac{h}{2} f_2, t_{i+\frac{1}{2}})$ $f_4 = f(w_i + h f_3, t_i + 1)$	$O(h^5)$	$O(h^4)$	Explicit

*Multi-step*

Adams-Basforth

2 Step	$w_{i+1} = w_i + \frac{h}{2} [3f_i - f_{i-1}]$	$O(h^3)$	$O(h^2)$	Explicit
3 Step	$w_{i+1} = w_i + \frac{h}{12} [23f_i - 16f_{i-1} + 5f_{i-2}]$	$O(h^4)$	$O(h^3)$	Explicit
4 Step	$w_{i+1} = w_i + \frac{h}{24} [55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}]$	$O(h^5)$	$O(h^4)$	Explicit
5 Step	$w_{i+1} = w_i + \frac{h}{720} [1901f_i - 277f_{i-1} + 2616f_{i-2} - 1274f_{i-3} - 251f_{i-4}]$	$O(h^6)$	$O(h^5)$	Explicit

Adams-Moulton

2 Step	$w_{i+1} = w_i + \frac{h}{12} [5f_{i+1} + 8f_i - f_{i-1}]$	$O(h^4)$	$O(h^3)$	Implicit
3 Step	$w_{i+1} = w_i + \frac{h}{24} [9f_{i+1} + 19f_i - 5f_{i-1} + f_{i-2}]$	$O(h^5)$	$O(h^4)$	Implicit
4 Step	$w_{i+1} = w_i + \frac{h}{720} [251f_{i+1} + 646f_i - 264f_{i-1} + 106f_{i-2} - 19f_{i-3}]$	$O(h^6)$	$O(h^5)$	Implicit

- Multi-step methods are not self-starting; need to start with a single-step method
- Implicit methods are not self-starting; need to start (*predict*) with an explicit method