Root Finding

→ Regula Falsi $x_{n+1} = b - f(b)\left(\frac{b-a}{f(b)-f(a)}\right)$

→ Secant $x_{n+1} = x_n - f(x_n)\left(\frac{x_n - x_{n-1}}{f(x_n)-f(x_{n-1})}\right)$

→ Modified Newton : $x_{n+1} = x_n - \frac{f(x_n) f'(x_n)}{(f'(x_n))^2 - f(x_n) f''(x_n)}$

→ Muller : $d_1 = \frac{f_1 - f_0}{x_1 - x_0}$ $d_2 = \frac{f_2 - f_1}{x_2 - x_1}$ $a = \frac{d_2 - d_1}{x_2 - x_0}$

$b = d_2 + a(x_2 - x_1)$ $x_3 = x_2 - \frac{2c}{b + \text{sgn}(b)\sqrt{b^2 - 4ac}}$

$c = f_2$

→ System of non-linear eqns.

$x_1 = g_1(\underset{\sim}{x}) \cdots x_n = g_n(\underset{\sim}{x})$

Necessary condn. : $\left[\left|\frac{\partial g_j}{\partial x_1}\right| + \cdots + \left|\frac{\partial g_j}{\partial x_n}\right|\right] \le 1 \ \forall\ 1 \le j \le n$

→ Newton's method :

$\underset{\sim}{x}^{(i+1)} = \underset{\sim}{x}^{(i)} - \underset{\sim}{J}^{-1}_{\underset{\sim}{x}^{(i)}} f(\underset{\sim}{x}^{(i)})$

→ Richardson's Extrapolation : $I_{true} = \frac{2^{n-1} I(h/2) - I(h)}{2^{n-1} - 1}$

→ Romberg integration: Use Richardson multiple times

→ Gaussian Quadrature with undetermined coefficients : $\int_{-1}^{1} f(x)\,dx = f\left(-\sqrt{\tfrac{1}{3}}\right) + f\left(\sqrt{\tfrac{1}{3}}\right)$

ODEs - IVP

→ If $f(x,y)$ & $\frac{\partial f}{\partial y}$ are continuous on $R = \{|x - x_0| \le a, |y - y_0| \le b\}$ the IVP will have a unique soln. in $|x - x_0| \le h$ where $h = \min(a, b/M)$,

$M = \max_{(x,y)\in R} (f(x,y))$

→ Picard iteration : $y_n(x) = y(x_0) + \int_{x_0}^{x} f(t, y_{n-1}(t))\,dt$

Least Squares for Functions

→ System of eqns. : $\sum_{k=0}^{n} a_k \int_a^b x^{j+k}\,dx = \int_a^b x^j f(x)\,dx$

→ Orthogonal functions :

$\int_a^b w(x)\,\phi_k(x)\,\phi_j(x)\,dx = \begin{cases} 0, & j \ne k \\ \alpha_j > 0, & j = k \end{cases}$

$w(x)$ is a weight function which is integrable on interval I if $w(x) > 0\ \forall x \in I$

→ Soln. to large lst. sqs. w/o mat. multiplication:

$a_i \int_a^b w(x)(f_i(x))^2\,dx = \int_a^b w(x)\,y(x)\,f_i(x)\,dx, \quad i = 0, \dots, n$

Numerical Calculus

→ 3-pt forward difference : $f_j' \simeq \frac{-3f_j + 4f_{j+1} - f_{j+2}}{2h}$

→ " backward " : $f_j' \simeq \frac{3f_j - 4f_{j-1} + f_{j-2}}{2h}$

→ 2^{nd} derivative : $f_j'' \simeq \frac{f_{j+2} - 2f_{j+1} + f_j}{h^2} + O(h)$

→ Trapezoidal Rule $(n=1)$: $\int_a^b f(x)\,dx \simeq \frac{h}{2}(f(a) + f(b)) - \frac{h^3}{12} f''(\mu)$

→ Simpson's 1/3 " $(n=2)$: $\int_{x_0}^{x_2} f(x)\,dx \simeq \frac{h}{3}(f_0 + 4f_1 + f_2) - \frac{h^5}{90} f^{(4)}(\mu)$

→ " 3/8 " $(n=3)$: $\int_{x_0}^{x_3} f(x)\,dx \simeq \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3) - \frac{3h^5}{80} f^{(4)}(\mu)$

→ Boole's rule $(n=4)$: $\int_{x_0}^{x_4} f(x)\,dx \simeq \frac{2h}{45}(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4) + \frac{8h^7}{945} f^{(6)}(\mu)$

→ Midpoint rule $(n=0)$: $\int_{x_0}^{x_1} f(x)\,dx = 2hf_0 + \frac{h^3}{3} f''(\mu)$

→ 2-point open $(n=1)$: $\int_{x_0}^{x_2} f(x)\,dx = \frac{3h}{2}(f_0 + f_1) + \frac{3h^3}{4} f''(\mu)$

→ 3-point open $(n=2)$: $\int_{x_0}^{x_3} f(x)\,dx = \frac{4h}{3}(2f_0 + f_1 + 2f_2) + \frac{14h^5}{45} f^{(4)}(\mu)$

→ 4-point open $(n=3)$: $\int_{x_0}^{x_4} f(x)\,dx = \frac{5h}{24}(11f_0 + f_1 + f_2 + 11f_3) + \frac{95h^5}{144} f^{(4)}(\mu)$

→ Composite Simpsons : $\int_a^b f(x)\,dx = \frac{h}{3}\left(f(a) + 2\sum_{j=1}^{n/2 - 1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right) + \frac{b-a}{180} h^4 f^{(4)}(\mu)$

→ Modified Euler (Predictor Corrector): $y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))\right]$

→ Lipschitz condn: A function satisfies Lipschitz condition in y in R if a constant $L \geq 0$ exists, s.t. $|f(x,y_1) - f(x,y_2)| \leq L|y_2 - y_1|$

→ Shooting method (convert to IVP from BVP)

→ Existence: If f, f_y & $f_{y'}$ is continuous on $a \leq x \leq b$, and

→ $f_y(x,y,y') \geq 0$, and

→ $f_{y'}(x,y,y') \leq M$

→ RK-2:

$$y_{n+1} = y_n + \tfrac{1}{2}(k_1 + k_2)$$
$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf(x_n + h, y_n + k_1)$$

→ RK-4:

$$y_{n+1} = y_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf(x_n + h/2, y_n + k_1/2)$$
$$k_3 = hf(x_n + h/2, y_n + k_2/2)$$
$$k_4 = hf(x_n + h, y_n + k_3)$$

n	Adam-Bashforth	Adam-Moulton
2	$0, 3/2, -5/2$	$5/2, 8/2, -1/2$
3	$0, \frac{23}{12}, -\frac{16}{12}, 5/12$	$9/24, 19/24, -5/24, 1/24$
4	$0, \frac{55}{24}, -\frac{59}{24}, \frac{37}{24}, -9/24$	$\frac{251}{720}, \frac{646}{720}, -\frac{264}{720}, \frac{106}{720}, -\frac{19}{720}$
5	$0, \frac{1901}{720}, -\frac{2774}{720}, \frac{2616}{720}, -\frac{1274}{720}, \frac{251}{720}$	

→ Higher order differential eqns.: $y^{(m)}(x) = f(x, y, y', \ldots, y^{(m-1)})$ $a \leq x \leq b$, $y(a) = \alpha_1, \ldots y^{(m-1)}(a) = \alpha_{m-1}$

→ Taylor series method: $\frac{d^2y}{dx^2} = \frac{\delta f}{\delta x} + \frac{\delta f}{\delta y} \cdot y' \ldots$ Using Taylor series, $y(x_1) = \sum_{i=0}^{\infty} \frac{h^i}{i!} y^{(i)}(x_0)$

→ Linear BVP: $y'' = f(x, y, y') = p(x)y' + q(x)y + r(x)$

→ If p, q & r are continuous, and $q > 0$ on $a \leq x \leq b$, then the BVP has a unique soln. Split into two IVPs:

$$y'' = p(x)y' + q(x)y + r(x) \qquad y(a) = \alpha, y'(a) = 0$$
$$y'' = p(x)y' + q(x)y \qquad y(a) = 0, \; y'(a) = 1$$

→ Finite difference method for non-linear problems (BVP)

If i) f, f_y & $f_{y'}$ are continuous on $a \leq x \leq b$

2) $f_y \geq \delta \geq 0$ on $a \leq x \leq b$

3) There exist K & L s.t.:

$$K = \max_{x \in [a,b]} |f_y| \; \& \; L = \max_{x \in [a,b]} |f_{y'}|$$

Then there exists a unique soln to the BVP.

Divide the interval to (N+1) equal sub-intervals, and replace $y''(x_i)$ & $y'(x_i)$ with the central difference formulae: $\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = f\left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right)$

There shall be a unique soln. for $h < 2/L$

Second order P.D.E.: $A\frac{\delta^2 u}{\delta x^2} + B\frac{\delta u^2}{\delta y^2} + C\frac{\delta u^2}{\delta x \delta y} + D\frac{\delta u}{\delta x} + E\frac{\delta u}{\delta y} + Fu = 0$

→ A, B, … F can be functions of x.

→ If $B^2 - 4AC = 0$, parabolic eqn.

→ > 0, hyperbolic "

→ < 0, elliptic "