DS 288: NUMERICAL METHODS

NOV - 9 - 2021

[[STABILITY OF ODE NUMERICAL SOLUTION

→ CONSISTENT
→ STABLE
→ CONVERGENT

BACKWARD EULER ⇒ UNCONDITIONALLY STABLE

IMPLICIT METHODS → MORE STABLE COMPARED TO EXPLICIT METHODS]]

STABILITY ANALYSIS FOR TRAPEZOIDAL RULE

$$\int_{t_i}^{t_{i+1}} y' \, d\tau = \int_{t_i}^{t_{i+1}} f(y_i, \tau) \, d\tau$$

APPROX. USING TRAPEZOIDAL RULE

$$\frac{h}{2}\left[f_i + f_{i+1}\right] - \frac{h^3}{12} f'(\xi)$$

$$Y_{i+1} - Y_i = \frac{h}{2}\left[f_i + f_{i+1}\right] + O(h^3)$$

$$\underline{- \quad w_{i+1} - w_i = \frac{h}{2}\left[\tilde{f}_i + \tilde{f}_{i+1}\right]} \rightarrow \text{COMPUTER}$$

$$\epsilon_{i+1} - \epsilon_i = \frac{h}{2}\left[f_i - \tilde{f}_i\right] + \frac{h}{2}\left[f_{i+1} - \tilde{f}_{i+1}\right]$$

WHERE $\tilde{f}_i = f(w_i, t_i)$

$f_i = f(Y_i, t_i)$

$$\rightarrow \epsilon_{i+1} = \epsilon_i + \frac{h}{2}\underbrace{\frac{\left[f_i - \tilde{f}_i\right]}{(Y_i - w_i)}}_{\equiv J_i}\epsilon_i + \frac{h}{2}\underbrace{\frac{\left[f_{i+1} - \tilde{f}_{i+1}\right]}{Y_{i+1} - w_{i+1}}}_{\equiv J_{i+1}}\epsilon_{i+1}$$

$$\epsilon_{i+1} = \epsilon_i + \frac{h}{2} J_i \epsilon_i + \frac{h}{2} J_{i+1} \epsilon_{i+1}$$

$$\boxed{\epsilon_{i+1} = \underbrace{\frac{\left(1 + \frac{h}{2} J_i\right)}{\left(1 - \frac{h}{2} J_{i+1}\right)}}_{\text{AMPLIFICATION FACTOR}} \epsilon_i}$$

$$\frac{e_{i+1}}{e_i} = \frac{(1 - \frac{h}{2}|J_i|)}{(1 + \frac{h}{2}|J_{i+1}|)} < 1 \quad \forall h > 0$$

REMEMBER $\underline{J<0}$ ANALYTICALLY STABLE

$\boxed{\text{UNCONDITIONALLY STABLE}}$

SO BACKWARD EULER & TRAPEZOIDAL RULE ARE STABLE : ← IMPLICIT METHODS.

- ORDER (LTE) FOR TRAPEZOIDAL RULE IS BETTER THAN BACKWARD EULER

$O(h^2)$ $O(h^3)$ LOCALLY

* FOR STIFF PROBLEMS AMP. FACTOR → 1. COUNTERS GAIN IN LTE RELATIVE TO BACKWARD EULER.

* FOR OTHER METHODS, IT GETS MORE DIFFICULT TO EXPRESS THE GLOBAL

ERROR AS A FUNCTION OF J.

- INSTEAD, CHOOSE A PROTOTYPE PROBLEM THAT CAN BE ANALYZED AND EXTRAPOLATE THE FINDINGS TO THE GENERAL CASE.

FOR STIFF PROBLEMS, PROTOTYPE

$y' = -Ky$; $y(0) = y_0$

ANALYTICAL SOLUTION: $y(t) = y_0 e^{-Kt}$ ←

K IS LARGE (STIFF)

$$\frac{\partial f}{\partial y} = -K = J$$

* LOOK AT MODIFIED EULER'S METHOD (RK-2) USING THIS PROTOTYPE

PROBLEM: $y' = f(y) = -Ky$.

TRAP ~~RULE~~ : $w_{i+1} = w_i + \frac{h}{2}[f_i + f_{i+1}]$

$$f_{i+1} = f(w_i + hf_i, t_{i+1})$$

$$w_{i+1} = w_i + \frac{h}{2}\left[-kw_i + (-k)(w_i + hf_i)\right]$$

WHERE $f(w_i) = -kw_i \leftarrow f(y) = -ky$

$$w_{i+1} = \frac{h}{2}\left(-kw_i - kw_i - kh(-kw_i)\right)$$

$$w_{i+1} = w_i \underbrace{\left[1 - hk + \frac{(hk)^2}{2}\right]}_{\text{AMPLIFICATION FACTOR} < 1}$$

AS $t \to \infty$: FOR STIFF PROBLEMS $y \to 0$

$t_{i+1} \to \infty \quad w_{i+1} \to 0$

TWO OBSERVATIONS

(A). THE ERROR PROPAGATES IN EXACTLY THE SAME WAY AS SOLUTION DOES (WILL SHOW SHORTLY)

(B). THE FIRST 3 TERMS IN THE TAYLOR SERIES EXPANSION OF THE EXACT SOLUTION. FORMS THE AMPLIFICATION FACTOR IN THIS PROTO-

TYPE PROBLEM. (RK-4 PRODUCES FIRST 5 TERMS).

SHOW (B) FIRST.

$$y_{i+1} = y_0 e^{-k t_{i+1}} = y_0 e^{-k(t_i + h)}$$

$$y_i = y_0 e^{-k t_i}$$

$$\frac{y_{i+1}}{y_i} = e^{-hk} = \overbrace{\underbrace{1 - hk + \frac{(hk)^2}{2!}}_{\text{RK-2}} - \frac{(hk)^3}{3!} + \frac{(hk)^4}{4!}}^{\text{RK-4}} - \cdots$$

SHOW (A) BY USING PROTOTYPE PROBLEM:

$$y_{i+1} = y_i + h f_i + \frac{h^2}{2!} f_i' + \frac{h^3}{3!} f_i''(\xi_i)$$

$$= y_i + h f_i + \frac{h^2}{2!}\left[\frac{\partial f_i}{\partial t} + \frac{\partial f_i}{\partial y} f_i\right] + \frac{h^3}{3!} f_i''(\xi_i)$$

NOW WE USE TEST PROBLEM

$$\frac{\partial f_i}{\partial t} = 0 \quad \frac{\partial f}{\partial y} = -k \; ; \; f_i = -k y_i$$

$$y_{i+1} = y_i - hk y_i + \frac{(hk)^2}{2!} y_i + \frac{h^3}{3!} f_i''(\xi_i)$$

$$w_{i+1} = w_i - hk w_i + \frac{(hk)^2}{2!} w_i$$ ← COMPUTER

$$\epsilon_{i+1} = \epsilon_i \underbrace{\left(1 - hk + \frac{(hk)^2}{2}\right)}_{\text{AMPLIFICATION FACTOR}} + O(h^3)$$

* ERROR BEHAVES IN EXACTLY THE SAME WAY AS SOLUTION DOES.

MULTI STEP METHOD STABILITY

- APPLY METHOD TO A PROTOTYPE PROBLEM.

$\Rightarrow$ DIFFERENCE EQUATION WITH CONSTANT COEFFICIENTS (ATLEAST FOR PROTOTYPE PROBLEM)

Ex: 4^{TH} ORDER ADAMS-BASHFORTH METHOD

$$w_{i+1} = w_i + \frac{h}{24}\left[55 f_i - 59 f_{i-1} + 37 f_{i-2} + 9 f_{i-3}\right]$$

TEST PROBLEM: $f_i = -Kw_i$; $f_{i-1} = -Kw_{i-1}$ . .

GROUP TERMS

$$0 = w_{i+1} - \left(1 - \frac{55hk}{24}\right) w_i - \frac{59}{24} hk\, w_{i-1} + \frac{37}{24} hk\, w_{i-2} + \frac{9hk}{24} w_{i-3}$$

SINCE ϵ BEHAVES AS $w \Rightarrow w_i \sim \lambda^i$

$$\lambda^4 - \left(1 - \frac{55hk}{24}\right)\lambda^3 - \frac{59}{24} hk\, \lambda^2 + \frac{37}{24} hk\, \lambda + \frac{9hk}{24} = 0$$

FOURTH ORDER POLYNOMIAL

* NEED ALL ROOTS ≤ 1 IN MAGNITUDE FOR STABILITY.

- FROM HERE WE CAN DERIVE 'h' LIMITS FOR STABILITY

[READ HANDOUT ON VARIABLE STEP SIZE METHODS FOR ODES FOR NEXT CLASS]