Numerical Methods

DS288 and UMC201

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Chapter - 11

Boundary-Value Problems for Ordinary Differential Equations

The Shooting Method

Boundary-Value Problem (Second-order ODE)

Consider a second-order differential equation of the form

$$y'' = f(x, y, y'), \text{ for } a \le x \le b, \quad y(a) = \alpha \text{ and } y(b) = \beta$$

- The shooting method reduces a BVP to an IVP.
- Since this method is a two-point BVP, then it has two initial values.

  $$y'' = f(x, y, y'), \text{ for } a \le x \le b, \quad y(a) = \alpha \text{ and } y'(a) = t$$

  where t must be chosen so that the solution satisfies the remaining boundary condition, $y(b) = \beta$
- Since t, being the first derivative of $y(x)$ at $x = a$, is the initial slope of the solution, this approach requires selecting the proper slope, so that the solution will hit the target of $y(x) = \beta$ at $x = b$.
- Here value of t is guessed and compared to the known solution at the boundary conditions until the target thus shooting method.

Boundary-Value Problem (Second-order ODE)

Theorem: Suppose the function f in the boundary-value problem

$$y'' = f(x, y, y'), \text{ for } a \leq x \leq b, \text{ with } y(a) = \alpha \text{ and } y(b) = \beta$$

is continuous on the set

$$D = \{(x, y, y')| \text{ for } a \leq x \leq b, \text{ with } -\infty < y < \infty \text{ and } -\infty < y' < \infty\}$$

and that the partial derivatives f_y and $f_{y'}$ are also continuous on D. If

1. $f_y(x, y, y') > 0$, for all$(x, y, y') \in D$, and
2. a constant M exists, with

$$|f_{y'}(x, y, y')| \leq M, \text{ for all } (x, y, y') \in D,$$

then the boundary-value problem has a unique solution.

Example

Example

Show that following boundary-value problem has a unique solution.

$$y'' + e^{-xy} + \sin y' = 0, \text{ for } 1 \le x \le 2, \text{ with } y(1) = y(2) = 0$$

- We have $f(x, y, y') = -e^{-xy} - \sin y'$ and
- $f_y(x, y, y') = xe^{-xy} > 0$ for all $x \in [1, 2]$
- $|f_{y'}(x, y, y')| = |-\cos y'| \le 1$ for all $x \in [1, 2]$.
- Thus the problem has a unique solution.

Linear Boundary-Value Problems

The differential equation

$$y'' = f(x, y, y')$$

is linear when functions $p(x), q(x), r(x)$ exist with

$$f(x, y, y') = p(x)y' + q(x)y + r(x).$$

Corollary

Suppose the linear boundary-value problem

$$y'' = p(x)y' + q(x)y + r(x), \text{ for } a \leq x \leq b, \text{ with}$$

$y(a) = \alpha$ and $y(b) = \beta$, satisfies

1. $p(x), q(x), r(x)$ are continuous on $[a, b]$,
2. $q(x) > 0$ on $[a, b]$

Then the boundary-value problem has a unique solution.

Linear Boundary-Value Problems

To approximate the unique solution to this linear problem, we first consider the initial-value problems

- $y'' = p(x)y' + q(x)y + r(x),$ with $a \leq x \leq b,$ with $y(a) = \alpha$ and $y'(a) = 0$ and
- $y'' = p(x)y' + q(x)y,$ with $a \leq x \leq b,$ with $y(a) = 0$ and $y'(a) = 1$

Let $y_1(x), y_2(x)$ denote the solution for above equations with $y_2(b) \neq 0.$ Now we define

$$y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2(x) \tag{1}$$

$$\text{Then} \quad y'(x) = y_1'(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2'(x) \tag{2}$$

$$\text{and} \quad y''(x) = y_1''(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2''(x) \tag{3}$$

Linear Boundary-Value Problems

Substituting for $y_1''(x)$ and $y_2''(x)$ in this equation gives

$$y'' = p(x)y_1' + q(x)y_1 + r(x) + \frac{\beta - y_1(b)}{y_2(b)}(p(x)y_2' + q(x)y_2) \tag{4}$$

$$= p(x)y' + q(x)y + r(x), \tag{5}$$

Moreover,

$$y(a) = y_1(a) + \frac{\beta - y_1(b)}{y_2(b)}y_2(a) = \alpha + \frac{\beta - y_1(b)}{y_2(b)} \cdot 0 = \alpha \tag{6}$$

$$y(b) = y_1(b) + \frac{\beta - y_1(b)}{y_2(b)}y_2(b) = y_1(b) + \beta - y_1(b) = \beta \tag{7}$$

Example 3

Example

Solve the following boundary-value problem with $N = 10$

$$y'' = -\frac{2}{x}y' + \frac{2}{x^2}y + \frac{\sin \ln x}{x^2}, \text{ for } 1 \le x \le 2, \text{ with } y(1) = 1,\ y(2) = 2,$$

and compare the results to those of the exact solution

$$y = c_1 x + \frac{c_2}{x^2} - \frac{3}{10}\sin \ln x - \frac{1}{10}\cos \ln x$$

where $c_2 = -0.03920701320$ and $c_1 = 1.1392070132$.

The solutions to the initial-value problems:

$y'' = -\frac{2}{x}y' + \frac{2}{x^2}y + \frac{\sin \ln x}{x^2}$, for $1 \le x \le 2$, with $y_1(1) = 1,\ y_1'(1) = 0$

and

$y'' = -\frac{2}{x}y' + \frac{2}{x^2}y + \frac{\sin \ln x}{x^2}$, for $1 \le x \le 2$, with $y_2(1) = 0,\ y_2'(1) = 1$

Linear Boundary-Value Problems

- Here $N = 10$ and $h = 0.1$
- Consider $u_{1,i}$ approximates to the value of $y_1(x_i)$,
- Similarly the value $v_{1,i}$ approximates $y_2(x_i)$ to the value of w_i approximates
- Now $y(x_i) = y_1(x_i) + \dfrac{2 - y_1(2)}{y_2(2)} y_2(x_i)$.

Algorithm for Linear Boundary-Value Problems

To approximate the solution of the boundary-value problem:

$$-y'' + p(x)y' + q(x)y + r(x) = 0, \quad \text{for } a \leq x \leq b, \quad \text{with } y(a) = \alpha, \quad y(b) = \beta$$

Approximation Algorithm

- **Step 1:** Set initial conditions

$$h = \frac{b-a}{N}$$
$$u_{1,0} = \alpha$$
$$u_{2,0} = 0$$
$$v_{1,0} = 0$$
$$v_{2,0} = 1$$

- **Step 2:** For $i = 0, \dots, N-1$ do Steps 3 and 4.
  (The Runge-Kutta method for systems is used in Steps 3 and 4.)
- **Step 3:** Set $x = a + ih$.

Algorithm for Linear Boundary-Value Problems

Step 4: Compute Runge-Kutta coefficients and update solutions:

- First set of coefficients:

$$
\begin{aligned}
k_{1,1} &= hu_{2,i}\\
k_{1,2} &= h[p(x)u_{2,i} + q(x)u_{1,i} + r(x)]\\
k_{2,1} &= h[u_{2,i} + \frac{1}{2}k_{1,2}]\\
k_{2,2} &= h\left[p(x+h/2)\left(u_{2,i} + \frac{1}{2}k_{1,2}\right) + q(x+h/2)\left(u_{1,i} + \frac{1}{2}k_{1,1}\right) + r(x+h/2)\right]
\end{aligned}
$$

- Continuing with more coefficients:

$$
\begin{aligned}
k_{3,1} &= h[u_{2,i} + \frac{1}{2}k_{2,2}]\\
k_{3,2} &= h\left[p(x+h/2)\left(u_{2,i} + \frac{1}{2}k_{2,2}\right) + q(x+h/2)\left(u_{1,i} + \frac{1}{2}k_{2,1}\right) + r(x+h/2)\right]\\
k_{4,1} &= h[u_{2,i} + k_{3,2}]\\
k_{4,2} &= h\left[p(x+h)(u_{2,i} + k_{3,2}) + q(x+h)(u_{1,i} + k_{3,1}) + r(x+h)\right]
\end{aligned}
$$

- Update solutions:

$$
\begin{aligned}
u_{1,i+1} &= u_{1,i} + \frac{1}{6}\left[k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}\right]\\
u_{2,i+1} &= u_{2,i} + \frac{1}{6}\left[k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}\right]
\end{aligned}
$$

Algorithm for Linear Boundary-Value Problems

- Primed coefficients (similar calculations):

$$k'_{1,1} = hv_{2,i}$$
$$k'_{1,2} = h[p(x)v_{2,i} + q(x)v_{1,i}]$$

$$k'_{2,1} = h\left[v_{2,i} + \frac{1}{2}k'_{1,2}\right]$$
$$k'_{2,2} = h\left[p(x+\frac{h}{2})\left(v_{2,i} + \frac{1}{2}k'_{1,2}\right) + q(x+\frac{h}{2})\left(v_{1,i} + \frac{1}{2}k'_{1,1}\right)\right]$$

$$k'_{3,1} = h\left[v_{2,i} + \frac{1}{2}k'_{2,2}\right]$$
$$k'_{3,2} = h\left[p(x+\frac{h}{2})\left(v_{2,i} + \frac{1}{2}k'_{2,2}\right) + q(x+\frac{h}{2})\left(v_{1,i} + \frac{1}{2}k'_{2,1}\right)\right]$$

$$k'_{4,1} = h[v_{2,i} + k'_{3,2}]$$
$$k'_{4,2} = h[p(x+h)(v_{2,i} + k'_{3,2}) + q(x+h)(v_{1,i} + k'_{3,1})]$$

$$v_{1,i+1} = v_{1,i} + \frac{1}{6}\left[k'_{1,1} + 2k'_{2,1} + 2k'_{3,1} + k'_{4,1}\right]$$
$$v_{2,i+1} = v_{2,i} + \frac{1}{6}\left[k'_{1,2} + 2k'_{2,2} + 2k'_{3,2} + k'_{4,2}\right]$$

Chapter - 11.2

The Shooting Method for Nonlinear Problems

The Shooting Method for Nonlinear Problems

- Consider the nonlinear second-order boundary value problem

$$y'' = f(x, y, y'), \ x \in [a, b], \text{ with } y(a) = \alpha \text{ and } y(b) = \beta \tag{8}$$

- Approximate the solution to BVP by using the solutions to a sequence of initial-value problems involving a parameter t
- Choose $t = t_k$, such that

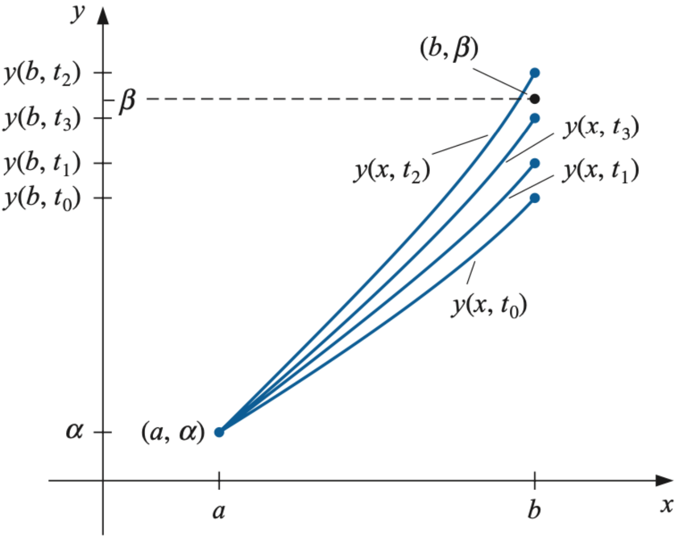
$$\lim_{k \to \infty} y(b, t_k) = y(b) = \beta$$

where $y(x, t_k)$ denotes the solution to the IVP

$$y'' = f(x, y, y'), \ x \in [a, b], \text{ with } y(a) = \alpha \text{ and } y'(a) = t_k$$

- And $y(x)$ denotes the solution to the BVP (8)
- We start with a parameter t_0, such that $y'(a) = t_0$

The Shooting Method for Nonlinear Problems



The Shooting Method for Nonlinear Problems

- If $y(b, t_0)$ is not sufficiently close to β, we correct our approximation by choosing elevations t_1, t_2, and soon, until $y(b, t_k)$ is sufficiently close to hitting β.

- Since $y(x, t)$ denotes the solution to the IVP, we next determine t in the following nonlinear equation in the variable t

$$y(b, t) - \beta = 0$$

- To use the Secant method to solve the problem but we need to choose two initial approximations.

- But Newton's method to generate the sequence $\{t_k\}$, only one initial approximation, t_0, is needed.

Newton Iteration

- The iteration formula

$$t_k = t_{k-1} - \frac{y(b, t_{k-1}) - \beta}{\dfrac{d}{dt} y(b, t_{k-1})}$$

  requires the knowledge of $\frac{d}{dt} y(b, t_{k-1})$
- Rewrite the IVP such that the solution depends on both x and t

$$y''(x,t) = f(x, y, y'), x \in [a, b], \text{ with } y(a,t) = \alpha, y'(a,t) = t$$

- Consider the partial derivative

$$\begin{aligned} \frac{\partial y''}{\partial t}(x,t) &= \frac{\partial f}{\partial t}(x, y(x,t), y'(x,t)) \\ &= \frac{\partial f}{\partial x}(x,y,y')\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}(x,y,y')\frac{\partial y}{\partial t} + \frac{\partial f}{\partial y'}(x,y,y')\frac{\partial y'}{\partial t} \end{aligned}$$

Newton Iteration

- Since x and t are independent, $\partial x/\partial t = 0$, such that

$$\frac{\partial y''}{\partial t}(x,t) = \frac{\partial f}{\partial y}(x,y,y')\frac{\partial y}{\partial t} + \frac{\partial f}{\partial y'}(x,y,y')\frac{\partial y'}{\partial t}, x \in [a,b]$$

- The initial condition gives

$$\frac{\partial y}{\partial t}(a,t) = 0, \quad \text{and} \quad \frac{\partial y'}{\partial t}(a,t) = 1$$

- Assume $z(x,t) = (\partial y/\partial t)(x,t)$, and that order of differentiation of x and t can be reversed, we have the IVP

$$z''(x,t) = \frac{\partial f}{\partial y}(x,y,y')z(x,t) + \frac{\partial f}{\partial y'}(x,y,y')z'(x,t), x \in [a,b]$$

with $z(a,t) = 0$ and $z'(a,t) = 1$

- Therefore $t_k = t_{k-1} - \dfrac{y(b,t_{k-1}) - \beta}{z(b,t_{k-1})}$

Example

Example

Solve the following BVP using Shooting method with Newton's Method

$$y'' = \frac{1}{8}(32 + 2x^3 - yy'), x \in [1, 3], \text{ with } y(1) = 17, \ y(3) = \frac{43}{3}$$

Use $N = 20, M = 10, \epsilon = 10^{-5}$, and compare the results with the exact solution $y(x) = x^2 + 16/x$

Need approximate solutions to the IVPs

$$y'' = \frac{1}{8}(32 + 2x^3 - yy'), x \in [1, 3], \text{ with } y(1) = 17, y'(1) = t_k, \quad \text{and}$$

$$z'' = \frac{\partial f}{\partial y} z + \frac{\partial f}{\partial y'} z' = -\frac{1}{8}(y'z + yz'), x \in [1, 3] \text{ with } z(1) = 0, z'(1) = 1$$

at each step in the iteration For the tolerance value 10^{-5}, four iterations are required,

$$|w_{1,N}(t_k) - y(3)| \leq 10^{-5}$$

such that $t_4 = -14.000203$. Results obtained for t

Example

x_i	$w_{1,i}$	$y(x_i)$	$\lvert w_{1,i} - y(x_i)\rvert$
1.0	17.000000	17.000000	-
1.1	15.755495	15.755455	4.06×10^{-5}
1.2	14.733389	14.733333	5.60×10^{-5}
1.3	13.997752	13.997692	5.94×10^{-5}
1.4	13.388629	13.388571	5.71×10^{-5}
1.5	12.916719	12.916667	5.23×10^{-5}
1.6	12.560046	12.560000	4.64×10^{-5}
1.7	12.301805	12.301765	4.02×10^{-5}
1.8	12.128923	12.128889	3.14×10^{-5}
1.9	12.031081	12.031053	2.84×10^{-5}
2.0	12.000023	12.000000	2.32×10^{-5}
2.1	12.029066	12.029048	1.84×10^{-5}
2.2	12.112741	12.112727	1.40×10^{-5}
2.3	12.246532	12.246522	1.01×10^{-5}
2.4	12.426673	12.426667	6.68×10^{-6}
2.5	12.650004	12.650000	3.61×10^{-6}
2.6	12.913847	12.913845	9.17×10^{-7}
2.7	13.215924	13.215926	1.43×10^{-6}
2.8	13.554282	13.554286	3.46×10^{-6}
2.9	13.927236	13.927241	5.21×10^{-6}
3.0	14.333327	14.333333	6.69×10^{-6}


ANY
QUESTIONS?