DS 288: NUMERICAL METHODS

AUG-31-2021

ROOT MULTIPLICITY

$$f(x) = (x-p)^m q(x)$$

$$\lim_{x \to p} q(x) \neq 0 \qquad m \geq 2$$

$$g'(p) = \frac{m-1}{m} = \lambda$$

MODIFIED NEWTON'S METHOD

- $m \geq 2$ (USED FOR)

FOR $m = 2$ $\quad f(x) = (x-p)^2 q(x)$

$f'(p) = 0 \quad f''(p) \neq 0$

↳ WE CAN NOT USE NEWTON'S METHOD

STRATEGY

(i) DEFINE NEW FUNCTION:

$$u = \frac{f}{f'}$$

$$u(p) = \frac{f(p)}{f'(p)} = \frac{0}{0}$$

L'HOSPITAL RULE $= \frac{f'(p)}{f''(p)} = \frac{0}{(\neq 0)} = 0$

SO $u(p) = 0$ SAME ROOT AS $f(x)$

EXAMINE $u'(p) = 1 - \left.\frac{f f''}{f'^2}\right|_{x=p}$

L'HOSPITAL RULE

$u'(p) = 1 - 1/2 = 1/2$

SO $u'(p) \neq 0$ (UNLIKE $f'(p) = 0$)

p IS THE SIMPLE ROOT OF $f(x)$

NOW APPLY THE NEWTON'S METHOD

$$\boxed{p_{n+1} = p_n - \frac{u(p_n)}{u'(p_n)}}$$

MODIFIED NEWTON'S METHOD

WHERE $u(x) = f(x)/f'(x)$

NOTICE $g(x) = x - \dfrac{u(x)}{u'(x)}$

$$g'(x) = \frac{u(x)\,u''(x)}{u'(x)^2}$$

$g'(p) = 0 \Rightarrow \alpha = 2$ <u>QUADRATIC</u>

RECALL $\lambda = \dfrac{g''(p)}{2!}$

$$g''(p) = \frac{u''(p)}{u'(p)} \qquad \left[u(x) = \frac{f(x)}{f'(x)}\right]$$

$$u''(p) = \frac{-f'''(p)}{6f''(p)}$$

$$\lambda = u''(p) = \frac{-f'''(p)}{6f''(p)} \neq 0$$

[IN TEXT,
MODIFIED NEWTON'S METHOD
WORKS FOR ANY M ($m \geq 2$)

$$u'(p) = \frac{1}{m}$$

$\underset{m \to \infty}{Lt}\ u'(P) \to 0$ & METHOD BREAKS DOWN

NOTE: $g(x) = x - \dfrac{u(x)}{u'(x)}$

$$g(x) = x - \frac{f/f'}{\dfrac{f'(x)^2 - f(x)f''(x)}{f'(x)^2}}$$

$$g(x) = x - \frac{f(x)f'(x)}{\underbrace{f'(x)^2}_{0} - \underbrace{f(x)f''(x)}_{0}}$$

$x \to P$

ROUND OFF ERRORS IN COMPUTER CAN LEAD TO INSTABILITY.

SECANT METHOD

RECALL NEWTON'S METHOD

- $f'(x)$ SHOULD EXIST & FORM BE KNOWN

- MAKE APPROXIMATION TO DERIVATIVE $f'(x)$ IN NEWTON'S METHOD.

* $f'(x)$ CAN BE DIFFICULT TO COMPUTE OR MAY BE NOT KNOWN

- OFTEN KNOWN AS DISCRETE VERSION OF NEWTON'S METHOD

- BASED ON $f'(P_n) = \lim_{x \to P_n} \frac{f(x) - f(P_n)}{x - P_n}$

USE $x = P_{n-1}$

$$f'(P_n) \simeq \frac{f(P_n) - f(P_{n-1})}{P_n - P_{n-1}}$$

USE THIS IN NEWTONS METHOD

$$\boxed{p_{n+1} = p_n - \frac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}}$$

SECANT METHOD

* WE NEED TWO STARTING VALUES (p_0 & p_1). THEY NEED NOT BRACKET THE ROOT AS IN BISECTION METHOD

- SLOWER CONVERGENCE
  CAN SHOW $\alpha = 1.6$
  (SUPER LINEAR)

* ON YOUR OWN GRAPHICAL INTERPRENTATION*

METHOD OF FALSE POSITION

- SECANT METHOD BUT **ALWAYS** **BRACKET** THE ROOT.

$$f(P_n)\, f(P_{n-1}) < 0$$

[START $f(P_0)\, f(P_1) < 0$]

P_0, P_1 $\rightarrow P_2$

$f(P_0)\, f(P_2) < 0 \rightarrow$ [$P_1 = P_0$, P_2]

$f(P_1)\, f(P_2)$

- INITIAL GUESSES SHOULD BRACKET THE ROOT.

- CHOOSE SUBSEQUENT SECANT LINES SUCH THAT

$$f(P_n)\, f(P_{n-1}) < 0$$

* CAN BE SLOWER BUT GUARANTEES CONVERGENCE.

[GRAPHICAL INTERPRETATION ON YOUR OWN]

ACCELERATING LINEARLY CONVERGENT PROCESS (§2.5)

(FOR FIXED POINT METHODS)

MOTIVATION: CONSTRUCT A NEW SET OF ITERATES $\{\tilde{p}_n\}$ THAT CONVERGES MORE RAPIDLY TO p THAN $\{p_n\}$

WE HAVE

$$\lim_{n\to\infty} E_{n+1} = g'(p)\, E_n$$

[LINEAR $\alpha = 1$, $\lambda = g'(p)$]

OR $\lim_{n \to \infty} P_{n+1} - P = g'(P)(P_n - P)$ ⊛

$\lim_{n \to \infty} P_{n+2} - P = g'(P)(P_{n+1} - P)$

FOR LARGE 'n' $n >> 1$ $\Big\}$ $P_{n+1} - P_{n+2} = g'(P)(P_n - P_{n+1})$

$$\Rightarrow g'(P) \cong \frac{P_{n+2} - P_{n+1}}{P_{n+1} - P_n}$$

USE THIS IN ⊛

$$P_{n+1} - P = \frac{P_{n+2} - P_{n+1}}{P_{n+1} - P_n}(P_n - P)$$

SOLVE FOR 'P'

$$(P_{n+1} - P)(P_{n+1} - P_n) = (P_{n+2} - P_{n+1})(P_n - P)$$

(DO ON YOUR OWN)

$$\boxed{p = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}}$$

NEW ITERATES

$$\boxed{\tilde{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}}$$

AITKENS Δ^2 METHOD

WITH $\{\tilde{p}_n\}$ NEW SET OF ITERATES

p_0

$p_1 = g(p_0)$

$p_2 = g(p_1)$

$p_3 = g(p_2)$

$p_4 = g(p_3)$

$\xrightarrow[\Delta^2 \text{ METHOD}]{\text{AITKENS}}$

$\tilde{p}_0 \Rightarrow \tilde{p}_1 = g(\tilde{p}_0)$

$\tilde{p}_1 \Rightarrow \tilde{p}_2 = g(\tilde{p}_1)$

$\tilde{p}_2 \Rightarrow \tilde{p}_3 = g(\tilde{p}_2)$

$$\lim_{n\to\infty} \tilde{\epsilon}_{n+1} = |g'(p)|^2 \tilde{\epsilon}_n$$

NOT QUADRATIC BUT FASTER

$$\tilde{\epsilon}_{n+1} < \epsilon_{n+1}$$

<u>AITKENSON'S METHOD</u>: WILL ALWAYS CONSTRUCT NEW ITERATES BASED ON OLD ITERATES

<u>STEFFENSON'S METHOD</u>

- USE $\tilde{p}_0$ TO COMPUTE $\underline{p_3 = g(\tilde{p}_0)}$ ASSUME THAT $\tilde{p}_0$ IS A BETTER GUESS THAN p_2
- METHOD IS <u>QUADRATICALLY</u> CONVERGENT

BUT MORE COMPUTATIONS
(FUNCTION EVALUATIONS) NECE-
SSARY TO GET A TERM IN THE
ITERATION.
(NOT SURE IF BETTER)

HAND OUT

Summary of Root Finding Methods

Method	Root Type	Asymptotic Convergence Rate	Bracketing Method
Bisection	simple	$\alpha = 1$ ✓	⋆
	multiple (odd)	$\alpha = 1$	⋆
	multiple (even)	NA	⋆
False Position	simple	$\alpha = 1$ ✓	⋆
	multiple (odd)	$\alpha = 1$	⋆
	multiple (even)	NA	
Secant	simple	$\alpha = 1.62$	
	multiple	$\alpha = 1$	
Newton's	simple	$\alpha = 2$	
	multiple	$\alpha = 1$	
Modified Newton's	simple	$\alpha = 2$	
	multiple	$\alpha = 2$	

BEST

Where the asymptotic convergence rate is defined as

$$\lim_{n\to\infty} |\epsilon_{n+1}| = \lambda |\epsilon_n|^\alpha$$

and

λ is the *asymptotic error constant*, and
α is the *order of convergence*.

CONVERGENCE WISE

* REQUIRES f, f', f'' (MORE COMPUTATION PER ITERATE)

*Courtesy: Prof. Simon Shepard Lecture Notes

* NO GUARANTEED CONVERGENCE.

[EXCERCISE:
WRITE DOWN ATLEAST
ONE ADVANTAGE & ONE
DISADVATAGE OF EACH OF
ROOT FINDING METHODS]

- NUMERICAL SOFTWARE READING (SURVEY OF)
  ON YOUR OWN

- TEXT BOOK HAS ALGORITHMS (PSEUDO CODE)
  (READ)

Ex:- STEFFENSON'S METHOD
ALGORITHM-2.6 IN PAGE-89