DS 288: NUMERICAL METHODS

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* EULER'S METHOD
    - NOT VERY ACCURATE
    - HAS STABILITY LIMITATIONS
* WE WANT METHODS WITH HIGHER ORDER LTE.

EXTEND TAYLOR'S SERIES AN ADDITIONAL TERM

$$\underbrace{\underbrace{y_{i+1} = y_i + h f_i}_{\text{EULER'S METHOD: } O(h^2) \text{ LOCALLY (STEP)},\ O(h) \text{ GLOBALLY}} + \frac{h^2}{2} f_i'}_{O(h^3) \text{ LOCALLY},\ O(h^2) \text{ GLOBALLY}} + \frac{h^3}{3!} f_i'' + \frac{h^4}{4!} f_i''' + \cdots$$

EULER'S METHOD: $O(h^2)$ LOCALLY (STEP)

$O(h)$ GLOBALLY

$O(h^3)$ LOCALLY

$O(h^2)$ GLOBALLY

$$f = f(y, t) = y'(t)$$

$$f' = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \underline{\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f}$$

$O(h^4)$ LOCALLY

$O(h^3)$ GLOBALLY

$$f'' = \frac{\partial f'}{\partial t} + \frac{\partial f'}{\partial y}\frac{dy}{dt} = \frac{\partial f'}{\partial t} + \frac{\partial f'}{\partial y}\cdot f$$

$$= \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\cdot f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\cdot f\right)\cdot f$$

HIGHER ORDER TAYLOR SERIES METHODS QUICKLY BECOME TEDIOUS

* WHY NOT APPROXIMATE THE DERIVATIVES USING WEIGHTED FUNCTION EVALUATIONS OF 'f'
* RESTRICT EVALUATIONS TO $[t_i, t_{i+1}]$

  - RUNGE-KUTTA METHODS

    "SINGLE STEP" METHODS

RUNGE-KUTTA (RK-2) SECOND ORDER METHOD

$$w_{i+1} = w_i + \left(h f_i + \frac{h^2}{2}\left(\frac{\partial f_i}{\partial t} + \frac{\partial f_i}{\partial y} f_i\right)\right) \text{ — } (*)$$

REPLACE WITH FUNCTION EVALUATIONS ON $[t_i, t_{i+1}]$

RECALL IVP: $y' = f(y,t)$; $y(t_0) = y_0$

NEED 2D TAYLOR EXPANSION f.

EXPAND $f(w,t)$ AROUND w_i, t_i

$$f(w,t) = f_i + \frac{\partial f_i}{\partial t}(t - t_i) + \frac{\partial f_i}{\partial y}(w - w_i)$$

$$+ \frac{\partial^2 f}{\partial t^2}\frac{(t-t_i)^2}{2} + \frac{\partial^2 f_i}{\partial y^2}(w - w_i)^2$$

$$+ \frac{\partial^2 f}{\partial t \partial y}(t - t_i)(w - w_i) + \ldots$$

NOW EVALUATE AT $w = w_i + \alpha$
$t = t_i + \beta$

$$f(w_i+\alpha, t_i+\beta) = f_i + \beta \frac{\partial f_i}{\partial t} + \alpha \frac{\partial f_i}{\partial y} + \frac{\beta^2}{2} \frac{\partial^2 f_i}{\partial t^2} + \frac{\alpha^2}{2} \frac{\partial^2 f_i}{\partial y^2} + \alpha\beta \frac{\partial^2 f_i}{\partial t \partial y} + \cdots$$

FIRST ORDER IN (α, β)

$$f(w_i+\alpha, t_i+\beta) \cong f_i + \beta \frac{\partial f_i}{\partial t} + \alpha \frac{\partial f_i}{\partial y} \quad (+)$$

$$\alpha^2, \beta^2, \alpha\beta \to 0.$$

REWRITE ORIGINAL EXPRESSION (x)

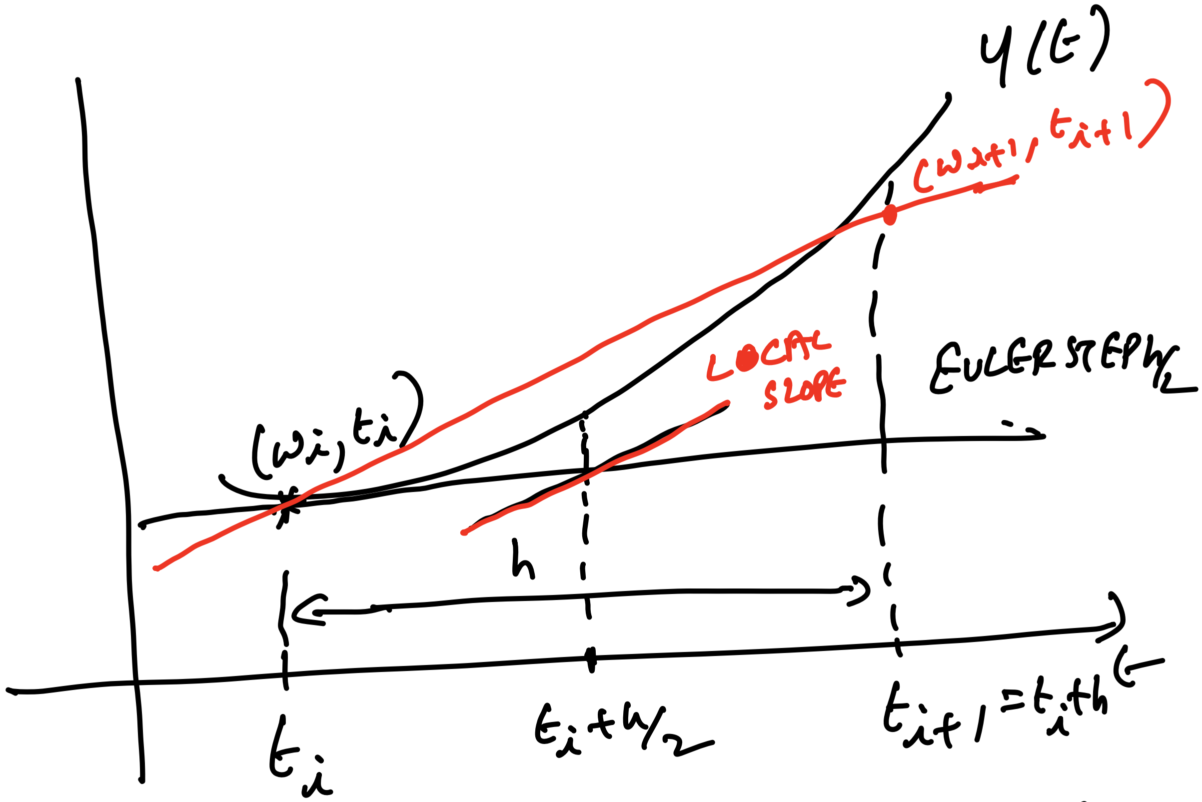
$$w_{i+1} = w_i + h\left[f_i + \frac{h}{2} \frac{\partial f_i}{\partial t} + \frac{h}{2} \frac{\partial f_i}{\partial y} f_i\right] \quad (x)$$

COMPARE $(+)$ WITH (x)

IF WE LET $\beta = \frac{h}{2}$ & $\alpha = \frac{h}{2} f_i$

$$\boxed{w_{i+1} = w_i + h f\left(w_i + \frac{h}{2} f_i, t_i + \frac{h}{2}\right)}$$

MIDPOINT RULE (RK-2)



NOTE THAT MIDPOINT RULE IS A PARTICULAR CHOICE OF A GENERAL APPROXIMATION TO

$$f_i + \frac{h}{2}\left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} y\right] = a_1 f(w_i, t_i) + a_2 f(w_i + \alpha, t_i + \beta)$$

SINGLE STEP $\{(t_i, w_i), (t_{i+1}, w_{i+1})\}$

MIDPOINT RULE: $a_1 = 0 \quad a_2 = 1$

$$\alpha = \frac{h}{2} f_i, \quad \beta = \frac{h}{2}$$

ANOTHER CHOICE LEADS US TO THE "MODIFIED EULER'S METHOD" OR "TRAPEZOIDAL RULE METHOD"

CHOICE: $a_1 = a_2 = 1/2$

$$\beta = h$$

$$w_{i+1} = w_i + \frac{h}{2}\left[f_i + f(w_i + hf_i, t_{i+1})\right]$$

SHOWING $\displaystyle\int_{t_i}^{t_{i+1}} y' \, d\tau = \underbrace{\int_{t_i}^{t_{i+1}} f(y, \tau)\, d\tau}_{\substack{\text{USE TRPEZOIDAL}\\ \text{APPROXIMATION}}}$

$$\frac{h}{2}\left[f_i + f_{i+1}\right] - \frac{h^3}{12} f'(\xi)$$

$$y_{i+1} - y_i = \frac{h}{2}\left[f_i + f_{i+1}\right] - \frac{h^3}{12} f'(\xi)$$

$$w_{i+1} = w_i + \frac{h}{2}\left[f_i + f(w_{i+1}, t_{i+1})\right]$$

HOW TO GET w_{i+1}

TAKE EULER STEP: $w_{i+1} = w_i + hf_i$

$$\boxed{w_{i+1} = w_i + \frac{h}{2}\left[f_i + f(w_i + hf_i, t_{i+1})\right]}$$

MODIFIED EULER'S METHOD

* MODIFIED EULER'S METHOD IS A PARTICULAR CHOICE/CASE OF THE TRAPEZOIDAL RULE.

- ANOTHER CHOICE IS <u>HEUN'S METHOD</u>

{TEXT BOOK, FOR EXCERCISE}

{DERIVE THIS}

* ALL CASES OF RK2 METHOD - $O(h^2)$

* 4^{TH} ORDER RUNGE KUTTA METHODS LEADING HIGHER ACCURACIES.

* HIGHER ORDER BEYOND 4^{TH} ORDER HAVE TRUNCATION ERRORS AROUND THE ROUND OFF ERRORS.

RUNGE-KUTTA 4^{TH} ORDER METHOD

RK-4

* VERY POPULAR METHOD FOR SOLVING ODE'S (IVP)

- GOOD TRADE OFF BETWEEN TRUNCATION ERROR REDUCTION AND RELATIVE EFFORT. (BEST IN THE RK-CLASS)

* DERIVED FROM INTEGRATION OF THE ODE USING SIMPSON'S RULE
[RK-2 $\Rightarrow$ TRAPEZOIDAL RULE]

$$\int_{t_i}^{t_{i+1}} y' \, d\tau = \underbrace{\int_{t_i}^{t_{i+1}} f(y, \tau) \, d\tau}_{\text{SIMPSON'S RULE}}$$

$$= \frac{h}{6} \left[f_i + 4 f_{i+1/2} + f_{i+1} \right] + O(h^5)$$

$$\boxed{w_{i+1} = w_i + \frac{h}{6} \left[f_i + 4 f_{i+1/2} + f_{i+1} \right]}$$

RK-4 $\quad O(h^4) \to$ GLOBALY

* GENERAL FORMULA BUT WE NEED TO EVALUATE $f_{i+1/2}$ & f_{i+1}

THERE ARE MANY POSSOBILITIES

MOST FAMOUS. (AVERAGING SLOPES).

$$\boxed{\begin{aligned} w_{i+1} &= w_i + \frac{h\,[f_1 + 2(f_2+f_3) + f_4]}{6} \\ \text{WHERE } f_1 &= f_i \quad \text{(SLOPE AT } f_i) \\ f_2 &= f(w_i + \tfrac{h}{2} f_1,\ t_{i+1/2}) \quad \text{SLOPE AT MIDPOINT } f_1 \\ f_3 &= f(w_i + \tfrac{h f_2}{2},\ t_{i+1/2}) \quad \text{SLOPE AT MIDPOINT USING } f_2 \\ f_4 &= f(w_i + h f_3,\ t_{i+1}) \quad \text{MODIFIED EULER'S WITH } f_3 \end{aligned}}$$

RK-4 METHOD $O(h^4) \rightarrow$ GLOBALY

* SAFE & STABLE METHOD *

- SINGLE-STEP METHOD TILL NOW

$(w_i, t_i) \rightarrow (w_{i+1}, t_{i+1})$

* WE DON'T USE ANY INFORMATION ABOUT RATE FUNCTION 'f' LEARNED ON INTERVAL $[t_i, t_{i+1}]$ TO SOLVE

FOR INTERVAL $[t_{i+1}, t_{i+2}]$

* MULTI - STEP METHODS

- USE PAST INFORMATION ABOUT RATE FUNCTION
- MORE RISKY & LIKELY TO BE UNSTABLE.

* WE WILL NOT RESTRICT APPROXIMATIONS TO DERIVATIVE TO SAMPLES ON ONE INTERVAL

- WE CAN USE $f_i, f_{i-1}, f_{i-2}, \ldots$ TO APPROXIMATE $f', f'', \ldots$
  - IF WE USE f_{i+1} [EX: TRAPEZOIDAL RULE) THEN WE MUST FIGURE OUT WHAT WILL BE w_{i+1}
- NOT SELF STARTING. FOR N STEP METHOD MUST TAKE N-1 STEPS USING

ONE OF THE SINGLE-STEP METHODS OF COMPARABLE ORDER TO GET ENOUGH POINTS IN ORDER TO EVALUATE FIRST STEP.

ADAMS METHODS → CLASS OF MULTI-STEP METHODS

* ADAMS-BASHFORTH METHODS:
  "OPEN" OR EXPLICIT METHODS
  i.e. DO NOT INVOLVE f_{i+1}

* ADAMS-MOULTON METHODS:
  "CLOSED" OR IMPLICIT METHODS
  i.e. DO INVOLVE f_{i+1}

IDEA:

$$y_{i+1} = y_i + h f_i + \frac{h^2}{2} f_i' + \frac{h^3}{3!} f_i'' + \cdots$$

APPROXIMATE DERIVATIVE WITH

$$\underbrace{f_{i+1}, \underbrace{f_i, f_{i-1}, f_{i-2}, \ldots}_{\text{EXPLICIT METHODS (A-B)}}}_{\text{IMPLICIT METHODS (A-M)}}$$