Numerical Methods

DS288 and UMC201

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Chapter-5
Initial-Value Problems for Ordinary Differential Equations

Differential Equations

- An equation involving one dependent variable and its derivatives with respect to one or more independent variables is called a differential equation.

- Ordinary differential equations: A differential equation involving only ordinary derivatives w.r.t only one independent variable.
- Partial differential equations: A differential equation involving partial derivatives w.r.t more than one independent variable.

- The order of a differential equation is the order of the highest derivative present.
- The general ordinary differential equation of the nth order is

$$F(x, y, y', y'', \cdots y^{(n)}) = 0$$

Well-posed problem

The initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0, \quad a \leq x \leq b \tag{1}$$

is said to be well-posed problem if.

- Unique solution, $y(x)$.
- The solution must exhibit continuous dependence on initial data, i.e., There exist constants $\epsilon_0 > 0$ and $k > 0$ such that for any ϵ, with $\epsilon_0 > \epsilon > 0$, whenever $\delta(x)$ is continuous with $|\delta(x)| < \epsilon$ for all $x \in [a, b]$, and when $|\delta_0| < \epsilon$, the initial value problem

  $$\frac{dz}{dx} = f(x, z) + \delta(x), \quad a \leq x \leq b, \quad z(x_0) = y_0 + \delta_0, \tag{2}$$

  has a unique solution $z(t)$ that satisfies

  $$|y(x) - z(x)| < k\epsilon \quad \forall x \in [a, b].$$

- The problem specified by (2) is called a perturbed problem.

Solving ODEs

Consider first order initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \tag{3}$$

- Has a solution
- Infinite number of solution
- No solution

Example

$$\frac{dy}{dx} = \frac{3y}{x}, \quad y(0) = 1 \tag{4}$$

- The general solution $y = cx^3$
- But the IVP has no solution.
- If we modify the IC as $y(0) = 0$ then the new IVP admits a one parameter family of solution $y = cx^3$ (infinite no. solutions)

Existence of initial value problem

We have first order initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \tag{5}$$

Now we are interested to know the following questions

- Under what conditions there exist a solution
- Under what conditions the solution of IVP is unique.

Theorem (Peano existence theorem)

Let $f(x, y)$ be continuous function in the closed rectangular domain

$$R = \{|x - x_0| \leq a,\ |y - y_0| \leq b\}.$$

Then, the IVP (5) has at least one solution in the interval, $I = |x - x_0| < h$, *where* $h = \min\left\{a, \frac{b}{M}\right\}$ *and* $M = \max_{(x,y)\in R} f(x, y)$.

Existence of initial value problem

➢ This theorem is a sufficient condition but not necessary, that is, even if the conditions are violated, the problem may have a solution.

Example

Consider the initial value problem

$$\frac{dy}{dx} = \frac{y}{x}, \qquad y(0) = 0 \tag{6}$$

- Here $f(x, y) = \frac{y}{x}$, $x_0 = 0$ and $y_0 = 0$.
- Rectangular domain $R = \{|x - 0| \leq 1, \ |y - 0| \leq 1\}$
- $f(x, y)$ is continuous everywhere in the R except at $x = 0$.

➢ Hence, the existence theorem cannot be applied, i.e., the theorem can not give any conclusion about the existence of the solution in the domain.

➢ However $y = x$ is a solution.

Uniqueness of the solution

We have first order initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \tag{7}$$

Theorem (Picard's theorem)

Let $f(x, y)$ and $\frac{\partial f}{dy}$ be continuous function in the closed rectangular domain

$$R = \{|x - x_0| \leq a, \ |y - y_0| \leq b\}.$$

Then, the IVP (7) has unique solution in the interval,
$I = |x - x_0| < h$, *where* $h = \min\left\{a, \frac{b}{M}\right\}$ *and* $M = \max_{(x,y)\in R} f(x, y)$.

➢ This theorem is also a sufficient condition but not necessary, that is, even if the conditions are violated, the IVP may have a unique solution.

Picard Iteration for IVP

- Picard Iteration gives approximate solution to the IVP

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \tag{8}$$

- The IVP (8) is equivalent to the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt \tag{9}$$

To solve the integral equation iteratively, we consider a constant function $y_0(x) = y(x_0) = y_0$ as rough approximation of a solution $y(x)$.

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0(t))dt = y_0 + \int_{x_0}^{x} f(t, y_0)dt \tag{10}$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))dt \tag{11}$$

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))dt \tag{12}$$

Example

Theorem

If the function $f(x, y)$ satisfy the existence and uniqueness theorem for the IVP

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \tag{13}$$

then the successive approximation $y_n(x)$ converges to the unique solution $y(x)$ of the IVP.

Example

Apply Picard iteration for IVP

$$\frac{dy}{dx} = 2x(1 - y), \quad y(0) = 2. \tag{14}$$

Example

Example

Here $f(x,y) = 2x(1-y)$, $y_0(x) = y(x_0) = y_0 = 2$, $x_0 = 0$. Now

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0(t))dt$$

$$y_1(x) = 2 + \int_0^x 2t(1-2)dt = 2 - x^2$$

$$y_2(x) = 2 + \int_0^x 2t(t^2-1) = 2 - x^2 + \frac{x^4}{2}$$

$$y_3(x) = 2 + \int_0^x 2t(t^2 - \frac{t^4}{2} - 1)dt = 2 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!}$$

$$\vdots$$

$$y_n(x) = 2 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!}$$

Exact Solution of the example

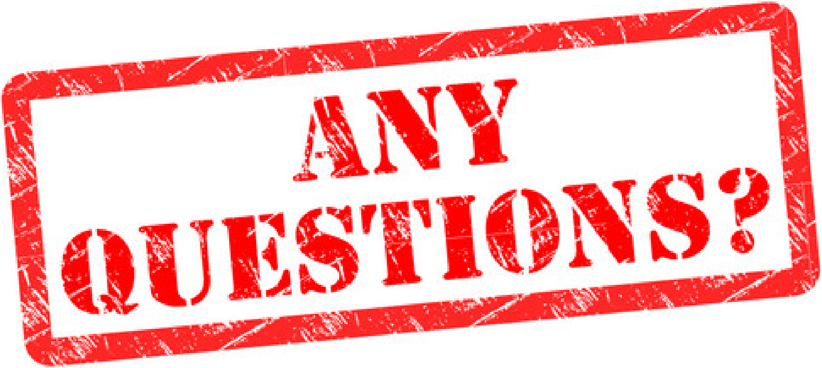
- Hence, $y_n \to 1 + e^{-x^2}$ as $n \to \infty$.
  Since we have the IVP

$$\frac{dy}{dx} = 2x(1-y), \quad y(0) = 2.$$

$$\frac{dy}{y-1} = -2xdx \Rightarrow \int_{x_0}^{x} \frac{dy}{y-1} = -\int_{x_0}^{x} 2x \ dx \tag{15}$$

$$y(x) = e^{-x^2} + c \tag{16}$$

- Hence the exact solution of the IVP is $y(x) = 1 + e^{-x^2}$.
- Thus, the Picard iterates converges to the unique solution


ANY
QUESTIONS?