DS 288: NUMERICAL METHODS

OCT-12-2021

MID TERM EXAM SOLUTIONS

1) GIVEN 5 SAMPLE POINTS, THE HERMITE POLYNOMIAL WILL BE OF DEGREE 10 BECAUSE IT MATCHES BOTH THE FUNCTION AND IT'S DERIVATIVE AT THESE GIVEN POINTS.

ANS: FALSE

$n+1$ POINTS: $2n+1$ ORDER OF POLYNOMIAL

$n=4$ HERE :- $2n+1 = 9$ ORDER

($n+1$ DERIVATIVES & $n+1$ FUNCTION VALUES $2n+2$ EQNS)

(2) MODIFED NEWTON'S METHOD IS A ROOT FINDING TECHNIQUE WHICH IS QUADRATICALLY CONVERGING FOR ROOT OF ANY DEGREE OF MULTIPLICITY

AND DOES NOT REQUIRE ANY ADDITIONAL COMPUTATION COMPARED TO NEWTON'S METHOD.

ANS :- FALSE. IT REQUIRES $u = \frac{f}{f'}$ & u' (REQUIRES f'' AS WELL)

(3). AITKEN'S METHOD IS A TECHNIQUE FOR ACCELRATING A QUADRATICALLY CONVERGENT FIXED POINT ITERATION.

ANS: FALSE

AITKEN'S $\Rightarrow$ ACCELRATES LINEARLY CONVERGENT PROCESS.

(4). THE MAIN IDEA OF ORTHOGNAL BASIS FUNCTION FOR A POLYNOMIAL INTERPOLATION IS TO LEAD TO A SYSTEM OF EQUATIONS, WHICH IS EASILY SOLVABLE (INVOLVING ONLY ONE UNKNOWN IN ONE EQUATION).

ANS :- TRUE : ORTHOGONAL : δ_{ij}

LINEAR SYSTEM $\Rightarrow$ DIAGONAL ONE

(5) THE TRUNCATION ERROR IN NUMERICAL DIFFERENTIATION AND NUMERICAL INTEGRATION BEHAVE IN SIMILAR MANNER AS 'h' INCREASES THE TRUNCATION ERROR INCREASES.

ANS :- TRUE.

INT & DIFF $\Rightarrow$ E_{TRUNC} $\rightarrow$ 'h' IN NUMERATOR.

* 'h' $\rightarrow$ INCREASE $\Rightarrow$ E_{TRUNC} $\rightarrow$ INCREASE

(6). CONSTRUCT A TWO-DIMENSIONAL INTERPOLATING POLYNOMIAL, $P(x,y)$, WHICH APPROXIMATES $f(x,y)$ AS.

$$P(x,y) = A\_1(x) * B\_1(y)$$

WHERE A_1 & B_1 are LINEAR POLYNOMIALS IN x & y RESPECTIVELY. HOW MANY SAMPLES OF $f(x,y)$ ARE NEEDED TO UNIQULEY DETERMINE $P(x,y)$. WRITE THE RELATIONSHIPS THAT MUST BE SATISFIED.

ANS:-

$$A\_1(x) = a\_0 + a\_1(x)$$

$$B\_1(y) = b\_0 + b\_1(y)$$

$$P(x,y) = A\_1(x)\,B\_1(y)$$

$$P(x,y) = \underbrace{a\_0 b\_0}_{A} + \underbrace{a\_1 b\_0}_{B} x + \underbrace{a\_0 b\_1}_{C} y + \underbrace{a\_1 b\_1}_{D} xy$$

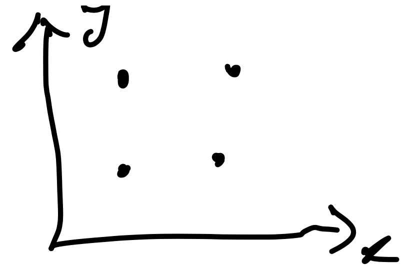
$$P(x,y) = A + Bx + Cy + Dxy$$

$A, B, C, D \rightarrow$ UNKNOWNS.

NUMBER EQUATIONS :- 4 ~~ARE~~ REQUIRED

(x_0, y_0) (x_0, y_1)

(x_1, y_0) (x_1, y_1)



$$P(x_0, y_0) = A + Bx_0 + Cy_0 + Dx_0y_0$$

$$P(x_0, y_1) = A + Bx_0 + Cy_1 + Dx_0y_1$$

$$P(x_1, y_0) = A + Bx_1 + Cy_0 + Dx_1y_0$$

$$P(x_1, y_1) = A + Bx_1 + Cy_1 + Dx_1y_1$$

(7). A DISPUTE HAS BROKEN OUT OVER BEHAVIOR OF THE ITERATION

x_(n+1) = c*(x_n)^2*(1- x_n)^2

WHERE c >= 1 & CONSTANT. YOU HAVE BEEN CALLED TO SETTLE THE DISPUTE. ANALYST A SAYS THAT THE CONSTANT C PLAYS NO ROLE IN DETERMINING THE ASYMPTOTIC RATE OF CONVERGENCE FOR THIS ITERATION TO x=0

FOR SOME STARTING VALUES OF x_0 ANALYST B SAYS THAT C IS CRITICAL TO THE ASYMPTOTIC CONVERGENCE RATE & THAT THE ITERATION WILL NOT EVEN CONVERGE FOR ANY STARTING VALUES OF x_0 GIVEN CERTAIN VALUES OF $c >= 1$.

IS EITHER ANLYSTS ARE CORRECT? IF SO, WHICH ONE & WHY? IF NOT, WHY ARE BOTH WRONG?

ANS:- $g(x) = c\,x^2(1-x)^2$

$g'(x) = 2cx(1-x)^2 + 2cx^2(1-x)(-1)$

$x=0 \quad g(0) = 0$

$g'(0) = 0$

$g''(x) = 2c(1-x)^2 + 4cx(1-x)(-1) + 4cx(x-1) + 2cx^2$

$g''(0) = 2C$

SINCE $g'(0)=0$ & $g''(0) \neq 0$

$\Rightarrow$ ORDER OF CONVERGENCE $=2$

$\alpha = 2$.

& $\lambda = \dfrac{|g''(0)|}{2!} = \dfrac{2C}{2} = C$

$$\lim_{n\to\infty} \frac{\epsilon_{n+1}}{\epsilon_n^2} = C$$ (ASYMPTOTIC ERROR BEHAVIOR)

$|g'(x)| < 1$ FOR ALL STARTING VALUES $x_0 \in [-\infty, \infty]$

FOR SOME $x_0 \Rightarrow |g'(x)| > 1$

CONVERGENCE IS NOT POSSIBLE FOR SOME x_0.

ANALYST B IS WRONG SAYING FOR ANY STARTING VALUE x_0 $g \rightarrow$ CONVERGES.

DOES NOT CONVERGE.

WHERE $|g'(x_0)| < 1 \rightarrow$ CONVERGES

IN THIS CASE $\lambda = c$ $c >= 1$

'c' PLAYS A ROLE OF ASYMPTOTIC CONVERGENCE RATE CONSTAN 'λ'.

THIS HAS SECOND-ORDER EFFECT.

ANALYST A IS WRONG.

(8). IF YOU WRITE DOWN STEPS OF IMPLEMENTING ADAPTIVE QUADRATURE ON A COMPUTER, WRITE DOWN THE MOST CRITICAL STEP? WILL THIS CRITICAL STEP BE SAME IRRESPECTIVE OF WHICH NEWTON-COTES FORMULA IS USED?

ANS:. PANEL SIZE

$$h \to h/_2$$

$$h = b-a \;;\quad h = \left(\frac{a+b}{2}\right) - a$$

$$\left| S_{10}' - S_{11}' - S_{21} \right| \leq \underline{15}\,\epsilon$$

$\hookrightarrow$ SIMPSON'S RULE

SECTION → S_{10} → REFINEMENT

*DETRMING SUCCESSIVE APPLICATION OF ANY NEWTON'S COTES FORMULA ON AN INTERVAL [a, b] WITH $h = h/2$ SHOULD BE ROUGHLY $\underline{\beta\epsilon}$ ACCURATE.

IF THIS IS SATISFIED $\Rightarrow$ STOP

$\beta \rightarrow$ FACTOR ARISING OUT OF ERROR IN NEWTON-COTES FORMULA

$\beta = 15 \rightarrow$ SIMPSON'S RULE

IT WILL BE SAME EXCEPT 'β' WILL CHANGE