

Numerical Methods

DS288 and UMC201

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August-December 2025



Newton's Method or Newton-Raphson method



Newton's Method (or Newton-Raphson method)

- Best known method for approximating the root of a differentiable function.
- It can be generalized many ways for the solution of others,
 - system of nonlinear equations
 - nonlinear integral equations
 - differential equations.
- Newton's method can often converge remarkably quickly, if the iteration begins sufficiently near to the desired root.



Newton's Method (or Newton-Raphson method)

- Choose a value x_0 (called initial approximation) that is reasonably close to the root of the equation.
- Find the tangent line at $(x_0, f(x_0))$

$$y - f(x_0) = f'(x_0)(x - x_0)$$

- Compute the x -intercept of this tangent.

$$0 - f(x_0) = f'(x_0)(x - x_0) \Rightarrow x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

- The x -intercept of this tangent will be the next approximation

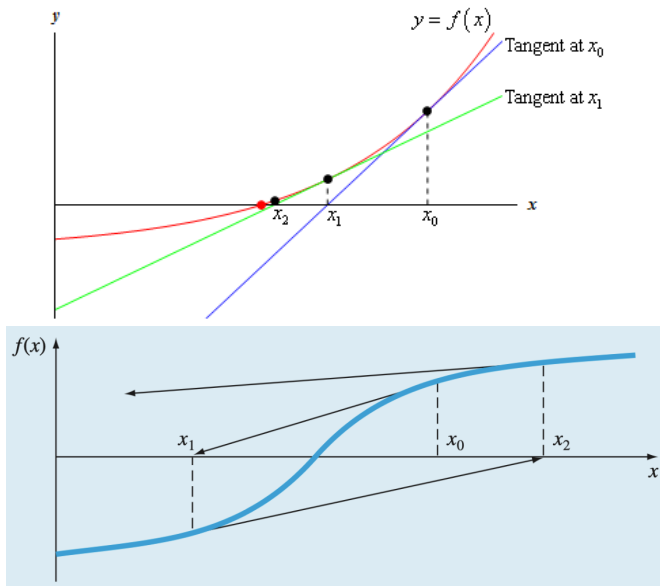
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

- Continuing this process and generate the sequence

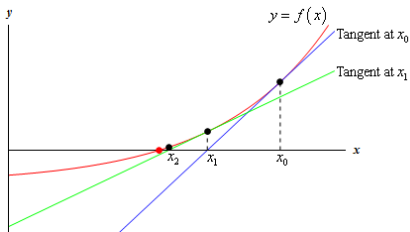
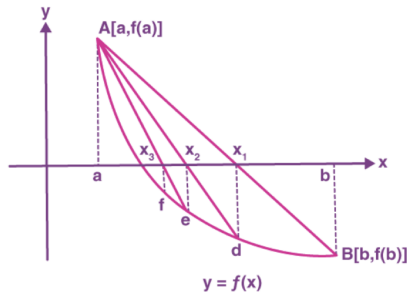
$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$



Newton's Method



Regula Falsi and Newton's Method



Taylor's Theorem

Suppose $f \in C^n[a, b]$, that $f^{(n+1)}$ exists on $[a, b]$, and $x_0 \in [a, b]$. For every $x \in [a, b]$, there exists a number $\xi(x)$ between x_0 and x with

$$f(x) = P_n(x) + R_n(x),$$

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1} \\ &= \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1} \end{aligned}$$

- Here $P_n(x)$ is called the n th Taylor polynomial for f about x_0 , and $R_n(x)$ is called the remainder term or truncation error associated with $P_n(x)$.
- The number $\xi(x)$ in the truncation error $R_n(x)$ depends on the value of x at which polynomial $P_n(x)$ is being evaluated, it is a function of the variable x .
- However, we should not expect to be able to explicitly determine the function $\xi(x)$. Taylor's Theorem simply ensures that such a function exists, and that its value lies between x and x_0 .
- In fact, one of the common problems in numerical methods is to try to determine a realistic bound for the value of $f^{(n+1)}(\xi(x))$ when x is in some specified interval.



Convergence of Newton's method

$$e_{n+1} = \alpha - x_{n+1} = \alpha - \left(x_n - \frac{f(x_n)}{f'(x_n)} \right) = e_n + \frac{f(x_n)}{f'(x_n)}$$

Now using Taylor's theorem, we have.

$$\begin{aligned} 0 &= f(\alpha) = f(\alpha - x_n + x_n) = f(e_n + x_n) \\ &= f(x_n) + e_n f'(x_n) + \frac{e_n^2}{2} f''(\zeta_n), \\ \Rightarrow e_n + \frac{f(x_n)}{f'(x_n)} &= -\frac{e_n^2}{2} \frac{f''(\zeta_n)}{f'(x_n)} \quad (\text{dividing } f'(x_n) \text{ both side}) \\ \Rightarrow e_{n+1} &= -\frac{e_n^2}{2} \frac{f''(\zeta_n)}{f'(x_n)} \\ \Rightarrow |e_{n+1}| &\leq \frac{1}{2} \frac{f''(\zeta_n)}{f'(x_n)} |e_n|^2 \quad (\text{converges quadratically}) \end{aligned}$$



Newton's Method

Theorem

Let $f(x) \in \mathbb{C}^2[a, b]$. If $x \in [a, b]$ such that $f(x) = 0$ and $f'(x) \neq 0$, then there exists a $\delta > 0$ such that the sequence $\{x_n\}_{n=1}^{\infty}$ defined by the iteration

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}, \text{ for } n = 1, 2, 3, \dots$$

will converge to α for any initial approximation $x_0 \in [\alpha - \delta, \alpha + \delta]$



Newton's Method

➤ We have $|e_{n+1}| \leq e_n^2 \frac{1}{2} \left| \frac{f''(\zeta_n)}{f'(\zeta_n)} \right|$, Let $M = \frac{1}{2} \frac{\max_{x \in [x-\delta, x+\delta]} |f''(x)|}{\min_{x \in [x-\delta, x+\delta]} |f'(x)|}$

$$\begin{aligned} |e_{n+1}| &\leq M e_n^2 \text{ multiply both sides by } M \text{ to get} \\ \Rightarrow M |e_{n+1}| &\leq M^2 e_n^2 = (M e_n)^2 \leq \dots \leq (M e_0)^{2^n} \end{aligned}$$

$$M(\alpha - x_{n+1}) \leq [M(\alpha - x_0)]^{2^n}$$

- Since we want $(\alpha - x_{n+1})$ to converge to zero, this says, if x_0 is chosen such that $M|\alpha - x_0| = k < 1$
- In addition to $|\alpha - x_0| < \delta$ then $\lim_{n \rightarrow \infty} (M e_0)^{2^n} = \lim_{n \rightarrow \infty} (k)^{2^n} = 0$, i.e., $e_n \rightarrow 0$ as $n \rightarrow \infty$
- If we choose $\delta < \frac{1}{M}$, then NR-Method converges for any initial approximation in $[\alpha - \delta, \alpha + \delta]$.



Newton's Method

Example

Find the root of $x^2 - 3 = 0$ using Newton-Raphson method

➤ Here $f(1) < 0$ and $f(2) > 0$ root lies in $[1, 2]$.

Consider the following cases to find approximations

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

| Iteration | $x_0 = 1$ | $x_0 = 1.5$ | $x_0 = 1.7$ |
|-----------|-----------|-------------|-------------|
| n | x_n | x_n | x_n |
| 1 | 2 | 1.75 | 1.7323529 |
| 2 | 1.75 | 1.7321429 | 1.7320508 |
| 3 | 1.7321429 | 1.7320508 | |
| 4 | 1.7320508 | | |

Advantage and disadvantages

Advantage of Newtons method

- Fast convergence: It converges fast (if it converges) (since $p = 2$ in most cases we get root in less number of steps.) It requires only one guess.
- Simple in formulation for computing and converting to multiple dimensions.
- Can be used to polish a root found by other methods. (Near a root, the number of significant digits approximately doubles with each step)
- Derivation is more intuitive, it is easier to understand its behaviour, when it is likely to converge or diverge.

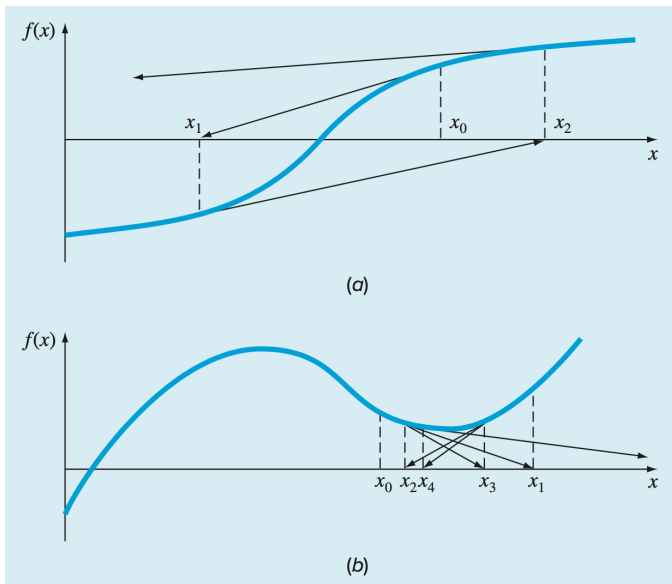


Disadvantages of Newtons Method

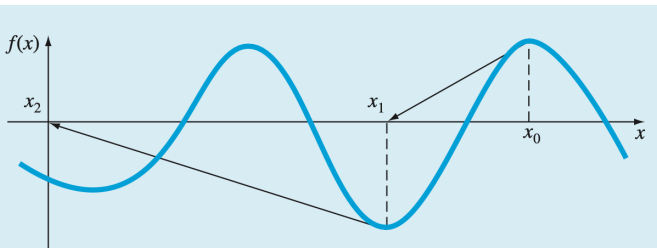
- Convergence is **not guaranteed**. So, sometimes, for given equation and for given guess we may not get solution.
- It may converge to a **root different** from the expected one or diverge if the starting value is **not close enough to the root**.
- It is likely to have difficulty if $f'(x_n) = 0$. That means the X-axis is **tangent** to the graph of $y = f(x)$ at $x = x_n$.
- This method can spin forever if there is **no real root** of the equation. (Example: $x^2 + 2 = 0$).
- It needs to know **both $f(x)$ and $f'(x)$** . On the other hand, bisection method requires only $f(x)$.
- In case of **multiple roots**, this method converges slowly. Further, near local maxima and local minima, due to oscillation, its convergence is slow.



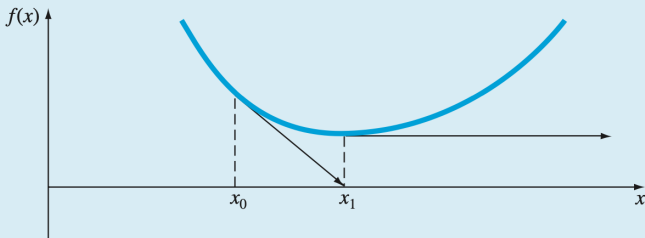
Case of failure of Newton's method



Case of failure of Newton's method



(c)



(d)

Multiple roots

Definition

The root α of $f(x)$ is said to be of multiplicity m if $f(x) = (x - \alpha)^m q(x)$ for some continuous function $q(x)$ with $q(\alpha) \neq 0$, m a positive integer.

Theorem

The function $f \in C^1[a, b]$ has a simple zero at $\alpha \in [a, b]$ if and only if $f(\alpha) = 0$, but $f'(\alpha) \neq 0$.

Theorem

The function $f \in C^m[a, b]$ has a zero of multiplicity m at $\alpha \in (a, b)$ if and only if

$$0 = f(\alpha) = f'(\alpha) = f''(\alpha) = \dots = f^{(m-1)}(\alpha), \text{ but } f^{(m)}(\alpha) \neq 0.$$

Modified Newton's Method



Modified Newton's Method

- To handle multiple roots of a function f , we define $\mu(x) = \frac{f(x)}{f'(x)}$
- α is a zero of f of multiplicity m with $f(x) = (x - \alpha)^m q(x)$, then

$$\mu(x) = \frac{(x - \alpha)^m q(x)}{m(x - \alpha)^{m-1} q(x) + (x - \alpha)^m q'(x)} = (x - \alpha) \left[\frac{q(x)}{mq(x) + (x - \alpha)q'(x)} \right]$$

Hence, α is a simple zero of $\mu(x)$.

- But $q(\alpha) \neq 0$, and $\frac{q(\alpha)}{mq(\alpha) + (\alpha - \alpha)q'(\alpha)} = \frac{1}{m} \neq 0$
- Applying Newton's method to $\mu(x)$, i.e., $\mu(x_n)$,

$$x_{n+1} = x_n - \frac{\mu(x_n)}{\mu'(x_n)} \Rightarrow x_{n+1} = x_n - \frac{f(x_n)/f'(x_n)}{([f'(x_n)]^2 - [f(x_n)][f''(x_n)]) / [f'(x_n)]^2}$$

- Thus Modified Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{[f'(x_n)]^2 - f(x_n)f''(x_n)}$$



Modified Newton's Method

- Thus Modified Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{[f'(x_n)]^2 - f(x_n)f''(x_n)}$$

Example $f(x) = (x - 1.1)^3(x - 2.1)$. Here **1.1** is a root of the function $f(x)$ with multiplicity 3. Take $x_0 = 0.5$.

| n | Newton's Method | Modified Newton's Method |
|-----|-----------------|--------------------------|
| 1 | 0.67778 | 1.03333 |
| 2 | 0.80585 | 1.09864 |
| 3 | 0.89699 | |
| 4 | 0.96106 | |
| 5 | 1.00556 | |
| 6 | 1.03616 | |
| 7 | 1.05702 | |
| 8 | 1.07115 | |
| 9 | 1.08068 | |
| 10 | 1.08708 | |
| 11 | 1.09137 | |



Advantage and Disadvantage

- The Modified Newton's method converges quadratically regardless of the multiplicity of the zero of $f(x)$.
- The drawback of the method is additional calculation of $f''(x)$.
- This method can cause serious round-off problems since the denominator in the formula consists of the difference of two numbers that are both close to zero.

Remark

- The Newton's method converges quadratically, when α is a simple root of $f(x) = 0$ ($f'(\alpha) \neq 0$).
- If you use the same formula for **multiple roots** then your order of convergence will not be 2 (Verify yourself).



Secant Method



Secant Method

- It is a variation on either Newton's method or Regula Falsi method.
- Needed two initial points a and b , but not required the condition $f(a) \cdot f(b) < 0$.
- Find the straight line which passes through $(a, f(a))$ and $(b, f(b))$.
- Compute x -intercept x_1 of the line through $(a, f(a)), (b, f(b))$

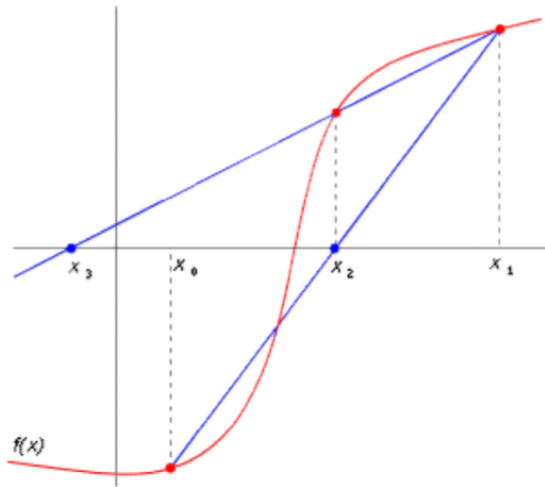
$$x_1 = a - f(a) \left(\frac{b - a}{f(b) - f(a)} \right)$$

- Continuing this process and generating the sequence

$$x_{n+1} = x_n - f(x_n) \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \right)$$

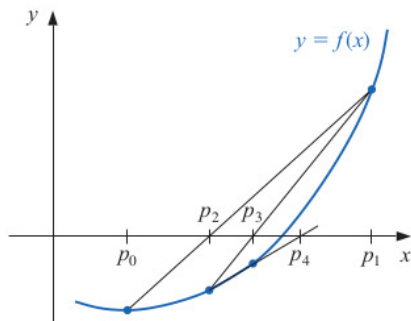


Secant method

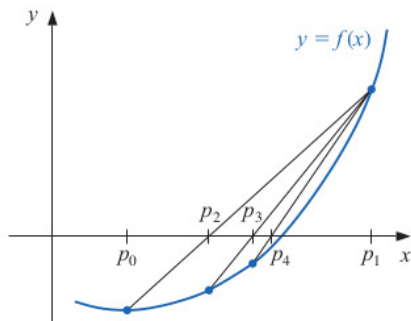


Secant method and Regula Falsi method

Secant Method



Method of False Position



- First three approximations are same, but the fourth approximations differ.

Newton's formula to Secant formula

➤ Get by replacing the derivative term in Newton's formula

$$f'(x_{n-1}) = \lim_{h \rightarrow 0} \frac{f(x_{n-1} + h) - f(x_{n-1})}{h} \quad (1)$$

$$f'(x_{n-1}) \approx \frac{f(x_{n-1} + h) - f(x_{n-1})}{h} \quad \text{for small } h. \quad (2)$$

Consider $h = x_{n-2} - x_{n-1}$, then $x_{n-2} = x_{n-1} + h$, this implies

$$f'(x_{n-1}) \approx \frac{f(x_{n-2}) - f(x_{n-1})}{x_{n-2} - x_{n-1}} \quad (3)$$

We have (Newton's method) $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$

$$x_n = x_{n-1} - f(x_{n-1}) \left(\frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})} \right)$$



Example

Example

Find the real root of the equation $x^3 - 2x - 5 = 0$ (secant method)

- Let two initial approximation $x_{-1} = 2$ and $x_0 = 3$
- Then $f(x_{-1}) = -1$ and $f(x_0) = 16$. Now we have

$$x_n = x_{n-1} - f(x_{n-1}) \left(\frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})} \right) \quad (5)$$

- $x_1 = 35/17 = 2.05882529$ and $f(x_1) = -0.390799923$
- $x_2 = 2.08126366$ and $f(x_1) = -0.147204057$
- $x_3 = 2.094824145$

➤ Find the real root of the equation $x^3 + 2x^2 - 3x - 1 = 0$ on the interval $(1,2)$ using secant method.



Advantage and disadvantage of secant method

- **Advantage:**

- Converges fast (if it converges) in comparison with Bisection method and Regula Falsi method
- Need two guesses that do not need to bracket the root.
- Need only one function evaluation per iteration.
- It does not necessitate the usage of the function's derivative, which is not available in a number of applications.
- Order of the convergences is 1.618. (Homework)

- **Disadvantage:**

- This method can cause serious round-off problems since the denominator in the formula consists of the difference between two numbers that are both close to zero.
- The computed iterates have no guaranteed error bounds.
- It is not necessary to have two initial guesses bracketing the root, but on the other hand, convergence is not guaranteed.



**ANY
QUESTIONS?**