DS 288: NUMERICAL METHODS

AUG-24-2021

⟦ RECAP:
ERROR GROWTH: EXPONENTIAL & LINEAR ⟧

EXAMPLES OF RECURSIVE EQUATIONS (OR DIFFERENCE EQUATIONS)
⟦ § 1.3 ⟧

Ex:-

(a) $P_n = \frac{1}{3} P_{n-1}$

(b) $P_n = \frac{10}{3} P_{n-1} - P_{n-2}$

CAN WE EXPLAIN THE ERROR GROWTH BASED ON THESE RELATIONS

(ASSUME CONSTANTS ARE EXACT)

(a) $p_n = \frac{1}{3} p_{n-1}$ (exact)

$$- \quad \hat{p}_n = \frac{1}{3} \hat{p}_{n-1} \quad \text{(COMPUTER)}$$

$$\underbrace{p_n - \hat{p}_n}_{} = \frac{1}{3}(p_{n-1} - \hat{p}_{n-1})$$

$\epsilon_n \rightarrow$ ERROR AT STEP 'n'

$$\epsilon_n = \frac{1}{3} \epsilon_{n-1}$$

$$\epsilon_1 = \frac{1}{3} \epsilon_0$$

$$\epsilon_2 = \frac{1}{3} \epsilon_1 = \left(\frac{1}{3}\right)\left(\frac{1}{3} \epsilon_0\right) = \left(\frac{1}{3}\right)^2 \epsilon_0$$

$$\epsilon_3 = \frac{1}{3} \epsilon_2 = \frac{1}{3}\left(\frac{1}{3}\right)^2 \epsilon_0 = \left(\frac{1}{3}\right)^3 \epsilon_0$$

$$\vdots$$

$$\epsilon_n = \left(\frac{1}{3}\right)^n \epsilon_0$$

AS $n \rightarrow \infty$ $\left(\frac{1}{3}\right)^n \rightarrow 0$

ERRORS INTRODUCED WILL DECAY

NOW RELAX CONDITION THAT CONSTANTS ARE EXACT

$$\widehat{\frac{1}{3}} = \frac{1}{3} - \delta$$

$\delta \to$ ROUND OFF ERROR IN REPRESENTING $0 \cdot 3333 \cdots\cdots$

$$P_n = \frac{1}{3} P_{n-1} \quad \text{(EXACT)}$$

$$\hat{P}_n = \left(\frac{1}{3} - \delta\right) \hat{P}_{n-1} \quad \text{(COMPUTER)}$$

$$P_n - \hat{P}_n = \frac{1}{3}\left(P_{n-1} - \hat{P}_{n-1}\right) + \delta \hat{P}_{n-1}$$

$$\epsilon_n = \frac{1}{3} \epsilon_{n-1} + \delta \hat{P}_{n-1}$$

$$\epsilon_1 = \frac{1}{3} \epsilon_0 + \delta \hat{P}_0$$

$$\epsilon_2 = \left(\frac{1}{3}\right)^2 \epsilon_0 + \frac{1}{3} \delta \hat{P}_0 + \delta \hat{P}_1$$

$$\epsilon_3 = \left(\frac{1}{3}\right)^3 \epsilon_0 + \left(\frac{1}{3}\right)^2 \delta \hat{P}_0 + \frac{1}{3} \delta \hat{P}_1 + \delta \hat{P}_2$$

$$\vdots$$

$$\epsilon_n = \underline{\left(\tfrac{1}{3}\right)^n \epsilon_0} + \underbrace{\left(\tfrac{1}{3}\right)^{n-1} \delta \hat{P}_0 + \cdots + \delta \hat{P}_{n-1}}$$

n TERMS EACH ONE BEING SMALLER THAN $\delta \hat{P}_0$

$$\epsilon_n \le \left(\tfrac{1}{3}\right)^n \epsilon_0 + n\,\delta \hat{P}_0$$

$$\lim_{n\to\infty} \epsilon_n \le \underbrace{\left(\tfrac{1}{3}\right)^n \epsilon_0}_{\to 0} + \underbrace{n\,\delta \hat{P}_0}$$

ERRORS INTRODUCED BOUNDED BY LINEAR GROWTH

(b) ASSUME CONSTANTS ARE EXACT.

$$P_n = \frac{10}{3} P_{n-1} - P_{n-2} \rightarrow \text{EXACT}$$

$$\hat{P}_n = \frac{10}{3} \hat{P}_{n-1} - \hat{P}_{n-2} \rightarrow \text{COMPUTER}$$

$$\epsilon_n = \frac{10}{3} \epsilon_{n-1} - \epsilon_{n-2} \rightarrow \text{DIFFERENCE EQUATION}$$

$\epsilon_n \propto \epsilon_0, \epsilon_1$

HANDOUT ON DIFFERENCE EQUATIONS

Solutions to Difference Equations: Review*

First let us look at the second-order homogeneous ordinary differential equation:

$$\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt} + 2\theta \;=\; 0 \qquad (1)$$

$\to \theta(t)$

with I.C.s

$$\begin{aligned} \theta(0) &= 1 \\ \frac{d\theta}{dt}(0) &= 0 \end{aligned}$$

We seek a solution of the form $\theta = Ce^{\lambda t}$. Plugging into equation (1)

$$\begin{aligned} C\left[\lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} + 2e^{\lambda t}\right] &= 0 \\ Ce^{\lambda t}\left[\lambda^2 + 2\lambda + 2\right] &= 0 \\ \theta\left[\lambda^2 + 2\lambda + 2\right] &= 0 \qquad (2) \end{aligned}$$

Equation (2) requires that either $\theta(t) = 0$ for all t, which is not very interesting as a solution, or that $\lambda^2 + 2\lambda + 2 = 0$. This quadratic equation is known as the *auxiliary* or *characteristic equation* and has two solutions or roots, $\lambda_1 = -1 + j$ and $\lambda_2 = -1 - j$. So the general solution to equation (1) is

$$\theta(t) \;=\; C_1\, e^{\lambda_1 t} + C_2\, e^{\lambda_2 t} \qquad (3)$$

Now we determine C_1 and C_2 from the I.C.s

$$\begin{aligned} 1 &= C_1 + C_2 \\ 0 &= \lambda_1 C_1 + \lambda_2 C_2 \end{aligned} \quad \overset{\text{solve}}{\Longrightarrow} \quad \begin{aligned} C_1 &= \frac{\lambda_2}{\lambda_2 - \lambda_1} \\ C_2 &= \frac{\lambda_1}{\lambda_1 - \lambda_2} \end{aligned}$$

The solution to equation (1) is, therefore,

$$\theta(t) \;=\; \frac{\lambda_2}{\lambda_2 - \lambda_1}\, e^{\lambda_1 t} \;+\; \frac{\lambda_1}{\lambda_1 - \lambda_2}\, e^{\lambda_2 t} \qquad (4)$$

Now let us look at a difference equation

$$y_{n+2} - \frac{5}{6}\, y_{n+1} + \frac{1}{6}\, y_n \;=\; 0 \qquad (5)$$

$\to y_n = \frac{5}{6} y_{n-1} - \frac{1}{6} y_{n-2}$

with I.C.s

$$\begin{aligned} y_0 &= 0 \\ y_1 &= 1 \end{aligned}$$

* Credit: Prof. Simon Shepard, Dartmouth College

Analogously, we seek a solution of the form $y_n = C\lambda^n$. Plugging into equation (5)

$$\begin{aligned} C\,\lambda^{n+2} - \frac{5}{6} C\,\lambda^{n+1} + \frac{1}{6} C\,\lambda^n &= 0 \\ C\,\lambda^n \left[\lambda^2 - \frac{5}{6}\,\lambda + \frac{1}{6}\right] &= 0 \\ y_n \left[\lambda^2 - \frac{5}{6}\,\lambda + \frac{1}{6}\right] &= 0 \end{aligned}$$

So either $y_n = 0$ or, more interestingly, $\lambda^2 - \frac{5}{6}\lambda + \frac{1}{6} = 0$, whose roots are $\lambda_1 = \frac{1}{2}$; $\lambda_2 = \frac{1}{3}$. So the general solution to equation (5) is

$$\boxed{y_n = C_1 \left(\frac{1}{2}\right)^n + C_2 \left(\frac{1}{3}\right)^n} \tag{6}$$

Now we determine C_1 and C_2 from the I.C.s

$$\begin{aligned} 0 &= C_1 + C_2 \\ 1 &= C_1 \left(\frac{1}{2}\right) + C_2 \left(\frac{1}{3}\right) \end{aligned} \quad \overset{\text{solve}}{\Longrightarrow} \quad \begin{aligned} C_1 &= 6 \\ C_2 &= -6 \end{aligned}$$

The solution to the difference equation (5) is, therefore,

$$y_n = 6 \left(\frac{1}{2}\right)^n - 6 \left(\frac{1}{3}\right)^n \tag{7}$$

Compare equation (7) with the difference equation (5)

$y_n = 6\left(\frac{1}{2}\right)^n - 6\left(\frac{1}{3}\right)^n$	$y_{n+2} - \frac{5}{6}\, y_{n+1} + \frac{1}{6}\, y_n = 0$
$y_0 = 0$	$y_0 = 0$
$y_1 = 1$	$y_1 = 1$
$y_2 = \frac{5}{6}$	$y_2 = \frac{5}{6}$
$y_3 = \frac{19}{36}$	$y_3 = \frac{19}{36}$
$y_4 = \frac{65}{216}$	$y_4 = \frac{65}{216}$
$\vdots$	$\vdots$

⋆ same ⋆

For our purposes, we do not need to get solutions to specific initial conditions, only the general behavior of the difference equation given by equation (6).

SOLVING DIFFERENCE EQUATION

$\epsilon_n \propto \lambda^n \quad [\epsilon_n = c\lambda^n]$

$\epsilon_n = \frac{10}{3}\epsilon_{n-1} - \epsilon_{n-2}$

$\lambda^n = \frac{10}{3}\lambda^{n-1} - \lambda^{n-2}$

DIVIDE λ^{n-2}

$\lambda^2 - \frac{10}{3}\lambda + 1 = 0 \rightarrow$ CHARACTERSTIC EQUATION

$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\Rightarrow \lambda_1 = \frac{1}{3}$ & $\lambda_2 = 3$

$\epsilon_n = c_1\left(\frac{1}{3}\right)^n + c_2(3)^n$

$$\lim_{n\to\infty} E_n = c_1 \cancelto{0}{\left(\frac{1}{3}\right)^n} + c_2 \cancelto{\infty}{(3)^n}$$

ONCE ERRORS INTRODUCE THEY GROW EXPONENTIALLY

「「WHY NOT $\widehat{\frac{10}{3}} = \frac{10}{3} - \delta$

BECAUSE ASSUMING $\frac{10}{3}$ BEING EXACT, WE ALREADY GOT EXPONENTIAL ERROR BEHAVIOR」」

FURTHER ON ERRORS:

$P_n \rightarrow$ SOLUTION AT n^{th} STEP

$P \rightarrow$ EXACT SOLUTION.

「「TRUTH $P_n \neq P$

WE DO NOT KNOW P」」

RELATIVE ERROR$_n$ = $\dfrac{|P_n - P|}{|P|}$

WHAT HAPPENS WHEN YOU DO NOT KNOW 'P'

$$\epsilon_n = \frac{|P_n - P_{n-1}|}{|P_n|}$$

BEST GUESS FOR $P \approx P_n$

IF 'P' IS MULTI DIMENSIONAL

$$\epsilon_n = \frac{\|\bar{P}_n - \bar{P}_{n-1}\|_m}{\|\bar{P}_n\|_m}$$

$\|\ \|_m \rightarrow$ IS THE l-m NORM

$\|\bar{x}\|_2 = \left(\sum x_i^2\right)^{1/2}$

MAGNITUDE

$\|\bar{x}\|_\infty = \text{MAX}(\bar{x})$

$|P_n - P| \rightarrow$ ABSOLUTE ERROR

WHEN P IS UNKNOWN

$|P_n - P_{n-1}| \rightarrow$ ABSOLUTE ERROR.

Example 2 Determine the absolute and relative errors when approximating p by p^* when

(a) $p = 0.3000 \times 10^1$ and $p^* = 0.3100 \times 10^1$;

(b) $p = 0.3000 \times 10^{-3}$ and $p^* = 0.3100 \times 10^{-3}$;

(c) $p = 0.3000 \times 10^4$ and $p^* = 0.3100 \times 10^4$.

Solution

(a) For $p = 0.3000 \times 10^1$ and $p^* = 0.3100 \times 10^1$ the absolute error is 0.1, and the relative error is $0.333\overline{3} \times 10^{-1}$.

(b) For $p = 0.3000 \times 10^{-3}$ and $p^* = 0.3100 \times 10^{-3}$ the absolute error is 0.1×10^{-4}, and the relative error is $0.333\overline{3} \times 10^{-1}$.

(c) For $p = 0.3000 \times 10^4$ and $p^* = 0.3100 \times 10^4$, the absolute error is 0.1×10^3, and the relative error is again $0.333\overline{3} \times 10^{-1}$.

This example shows that the same relative error, $0.333\overline{3} \times 10^{-1}$, occurs for widely varying absolute errors. As a measure of accuracy, the absolute error can be misleading and the relative error more meaningful, because the relative error takes into consideration the size of the value. ■

We often cannot find an accurate value for the true error in an approximation. Instead we find a bound for the error, which gives us a "worst-case" error.

The following definition uses relative error to give a measure of significant digits of accuracy for an approximation.

Definition 1.16 The number p^* is said to approximate p to t **significant digits** (or figures) if t is the largest nonnegative integer for which

$$\frac{|p - p^*|}{|p|} \leq 5 \times 10^{-t}.$$ ■

The term significant digits is often used to loosely describe the number of decimal digits that appear to be accurate. The definition is more precise, and provides a continuous concept.

Table 1.1 illustrates the continuous nature of significant digits by listing, for the various values of p, the least upper bound of $|p - p^*|$, denoted max $|p - p^*|$, when p^* agrees with p to four significant digits.

Table 1.1

p	0.1	0.5	100	1000	5000	9990	10000
max $\|p - p^*\|$	0.00005	0.00025	0.05	0.5	2.5	4.995	5.

RELATIVE ERRORS ARE MORE MEANINGFUL.

CASES WHERE WE HAVE TO USE ABSOLUTE ERROR

<u>EX</u> :- ROOT FINDING

FIND $P \ni f(P)=0$

ERROR GROWTH.

$E_n = c(K)^n E_0$ Exponential $|K|>1$

SO THE CRITICAL PART.

ANALYZE <u>MAGNITUDE</u> OF ROOTS OF CHARACTERSTIC EQUATION

ROOTS COULD BE COMPLEX.

— END OF CHAPER 1 —

CHAPTER-2 :

SOLNS TO EQUATIONS IN ONE VARIABLE

"ROOT FINDING" PROBLEM

FIND $P \ni f(P) = 0$

P IS THE ROOT OR ZERO OF $f(x)$

Ex:- $x^2 e^x + e^{-2x} = 4$

$\Rightarrow f(x) = x^2 e^x + e^{-2x} - 4$

ITERATIVE METHODS

- SUCCESSIVELY IMPROVE ESTIMATION OF 'P'

(0) SCAN x UNTIL $f(x)=0$

$x_0 \rightarrow \quad x_1 = x_0 + \underset{\swarrow}{s} \quad x \in [a,b]$

STEP SIZE

$f(x_0) \neq 0$

$f(x_1) \neq 0 \rightarrow x_2 = x_1 + s$

WHAT STEP SIZE?

s TOO SMALL $\rightarrow$ ∞ STEPS

TOO BIG $\rightarrow$ MISS 'p'.

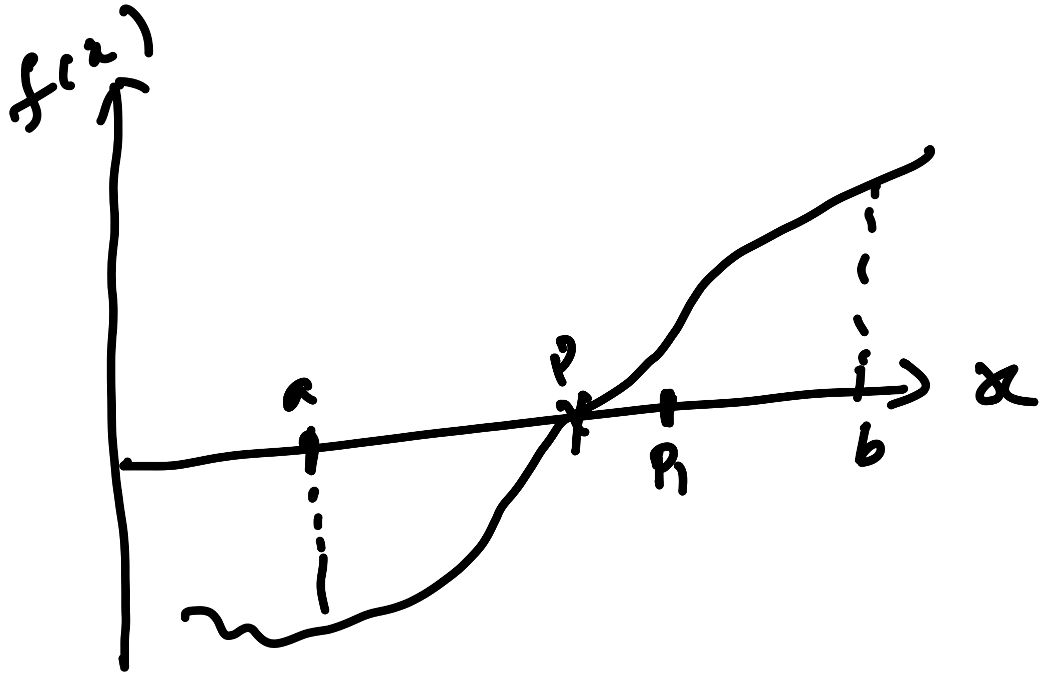
"NO GUARANTEES"

1) BISECTION METHOD

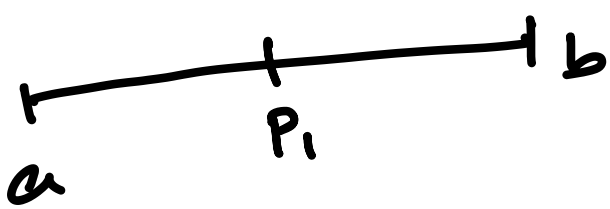
(§2.1)

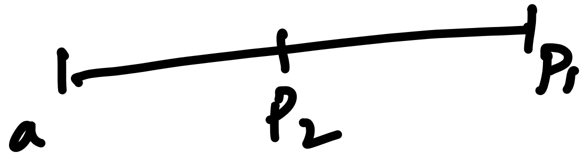
If $f(a)f(b) < 0$ THEN IVT GUARANTEES A ROOT $p \in [a,b]$

$\hookrightarrow$ BRACKETING THE ROOT.



$$p_1 = a + \frac{b-a}{2} = \frac{b+a}{2}$$





EVALUATE $f(a_n), f(b_n), f(p_n)$

$a_0 = a$; $b_0 = b$

DETERMINE NEW INTERVAL

SUCH THAT $f(a_{n+1}) f(b_{n+1}) < 0$

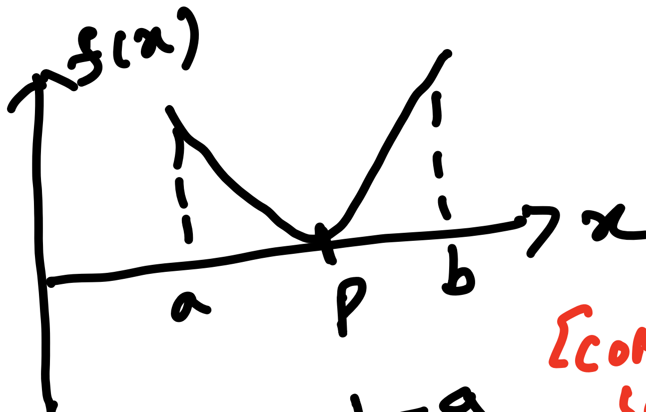
$$a_{n+1} = \begin{cases} a_n & f(a_n)f(p_n) < 0 \\ p_n & f(a_n)f(p_n) > 0 \end{cases}$$

$$b_{n+1} = \begin{cases} b_n & f(b_n)f(p_n) < 0 \\ p_n & f(b_n)f(p_n) > 0 \end{cases}$$

+ SIMPLE, ALWAYS FINDS ROOT.

− NEED $[a, b] \ni f(a)f(b) < 0$

Ex:-



[COMPUER SCIENE:- BINARY SEARCH PROC]]

* SLOW $\quad \epsilon_n \le \dfrac{b-a}{2^n}$

ERROR BEHAVIOR

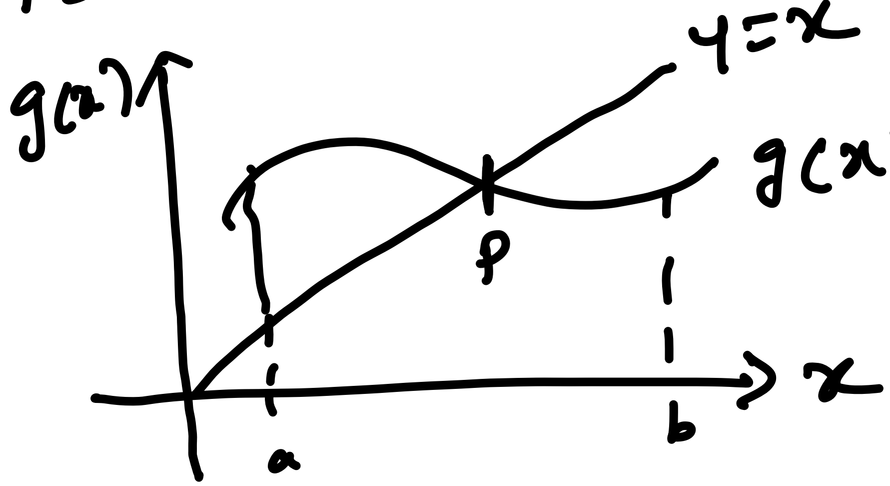
(PROOF IN TEXT THEOREM 2.1)

AS n INCREASES $p_n \to p$

FIXED POINT SCHEMES

- USEFUL FOR ROOT FINDING

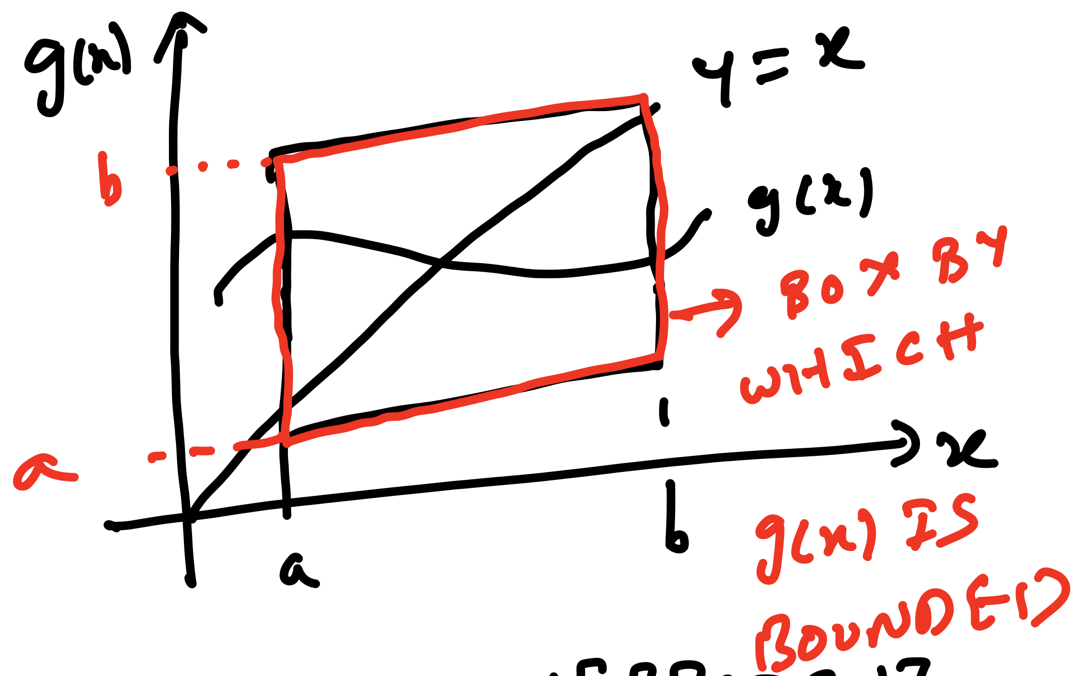
* FIXED POINT 'P' WHEN $g(P)=P$



P EXISTS ON $[a,b]$ IF

(i) $g(x)$ MUST BE CONTINUOUS ON $[a,b]$

(ii) $g(x)$ MUST BE "BOUNDED" ON $[a,b]$ BY $[a,b]$



(i) & (ii) ARE SUFFICIENT BUT NOT NECESSARY

↳ ALONE DOES NOT G.UARATEE UNIQUENESS.

SUFFICINCY FOR EXISTENCE OF UNIQUE FIXED POINT $\in [a,b]$

(i) $g(x) \in C[a,b]$

(ii) $g(x)$ IS BOUNDED BY $[a,b]$

(iii) $|g'(x)| < 1 \ \forall x \in [a,b]$

↳ ADDITIONAL CONSTRAINT

RECALL "ROOT FINDING" $f(P) = 0$

HOW TO GET $g(x)$ from $f(x)$

$\exists$ WHEN $g(P) = P$ THEN $f(P) = 0$

Ex: $g(x) = x - f(x)$

WHEN $g(P) = P$ THEN $f(P) = 0$

* FIND FIXED POINT P USING $g(x)$ WHICH CORRESPONDS TO ROOT IN $f(x)$ *

USE FIXED POINT METHODS
BECAUSE THEY HAVE DESIRABLE
& PROVABLE FEATURES SUCH AS
* SIMPLE ALOGRITHMS
* CONVERGENCE CONDITIONS
ARE PROVABLE
* ROUND-OFF ERRORS BEING
SMALLER.

$$\left[f(p)=0 \equiv g(p)=p \right.$$

↓

REPRSENTATION IS HARDER
IN COMPUTER