

Numerical Methods

DS288 and UMC201

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Finite Difference Method

Consider a first-order ordinary differential equation:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \quad (1)$$

- The forward difference formula approximates the first derivative as:

$$\left. \frac{dy}{dx} \right|_{x=x_i} \approx \frac{y_{i+1} - y_i}{h} \quad (2)$$

where h is the step size and $x_i = x_0 + ih$. The truncation error is $O(h)$.

- The backward difference formula is:

$$\left. \frac{dy}{dx} \right|_{x=x_i} \approx \frac{y_i - y_{i-1}}{h}, \text{ The truncation error is } O(h). \quad (3)$$

- The central difference formula provides better accuracy:

$$\left. \frac{dy}{dx} \right|_{x=x_i} \approx \frac{y_{i+1} - y_{i-1}}{2h} \quad (4)$$

The truncation error is $O(h^2)$, making it more accurate than forward or backward differences

Finite Difference Method

Consider a second-order ordinary differential equation:

$$\frac{d^2y}{dx^2} = f \left(x, y, \frac{dy}{dx} \right), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0 \quad (5)$$

The second derivative can be approximated using:

$$\left. \frac{d^2y}{dx^2} \right|_{x=x_i} \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}, \text{ the truncation error is } O(h^2). \quad (6)$$

Using Taylor series expansion around x_i :

$$y_{i+1} = y_i + hy'_i + \frac{h^2}{2}y''_i + \frac{h^3}{6}y'''_i + O(h^4) \quad (7)$$

$$y_{i-1} = y_i - hy'_i + \frac{h^2}{2}y''_i - \frac{h^3}{6}y'''_i + O(h^4) \quad (8)$$

Adding equations (7) and (8): $y_{i+1} + y_{i-1} = 2y_i + h^2y''_i + O(h^4)$

Solving for y''_i :

$$y''_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + O(h^2)$$

Finite Difference Method

Consider Linear Second-Order Boundary Value Problem:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x), \quad a \leq x \leq b \quad (10)$$

with boundary conditions:

$$y(a) = \alpha, \quad y(b) = \beta \quad (11)$$

- Divide the interval $[a, b]$ into n subintervals with step size $h = \frac{b-a}{n}$.
- Let $x_i = a + ih$ for $i = 0, 1, 2, \dots, n$.
- At interior points x_i where $i = 1, 2, \dots, n - 1$, apply finite differences:

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + p(x_i)\frac{y_{i+1} - y_{i-1}}{2h} + q(x_i)y_i = r(x_i) \quad (12)$$

Finite Difference Method

Rearranging equation (12):

$$Ay_{i-1} + By_i + Cy_{i+1} = D_i \quad (13)$$

where:

$$A = \frac{1}{h^2} - \frac{p(x_i)}{2h} \quad (14)$$

$$B = -\frac{2}{h^2} + q(x_i) \quad (15)$$

$$C = \frac{1}{h^2} + \frac{p(x_i)}{2h} \quad (16)$$

$$D_i = r(x_i) \quad (17)$$

This results in a tridiagonal matrix system:

$$\begin{bmatrix} B_1 & C_1 & 0 & \cdots & 0 \\ A_2 & B_2 & C_2 & \cdots & 0 \\ 0 & A_3 & B_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & A_{n-1} & B_{n-1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} D_1 - A_1\alpha \\ D_2 \\ D_3 \\ \vdots \\ D_{n-1} - C_{n-1}\beta \end{bmatrix} \quad (18)$$



Section - 11.3

Finite-Difference Methods for Linear Problems



Discrete Approximation

- Consider the linear second-order boundary-value problem

$$\frac{d^2y}{dx^2} = p(x) \frac{dy}{dx} + q(x)y + r(x)$$

for $a \leq x \leq b$, and $y(a) = \alpha, y(b) = \beta$.

- Need to approximate both y' and y''
- Divide the interval $[a, b]$ into $(N + 1)$ subintervals $[x_i, x_{i+1}]$, $i = 0, 1, \dots, N + 1$.
- $x_i = a + ih$, $h = (b - a)/(N + 1)$
- For the interior mesh points $x_i, i = 1, 2, \dots, N$

$$y''(x_i) = p(x_i)y'(x_i) + q(x_i)y(x_i) + r(x_i)$$

Discrete Approximation

- Assume that $y \in C^4[x_{i-1}, x_{i+1}]$, such that

$$y(x_{i+1}) = y(x_i + h)$$

$$= y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{6}y'''(x_i) + \frac{h^4}{24}y^{(4)}(\xi_i^+)$$

$$y(x_{i-1}) = y(x_i - h)$$

$$= y(x_i) - hy'(x_i) + \frac{h^2}{2}y''(x_i) - \frac{h^3}{6}y'''(x_i) + \frac{h^4}{24}y^{(4)}(\xi_i^-)$$

where $\xi_i^+ \in (x_i, x_{i+1})$ and $\xi_i^- \in (x_{i-1}, x_i)$

- We can derive the **centered-difference formula** for $y''(x_i)$

$$\begin{aligned}y''(x_i) &= \frac{1}{h^2}[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})] - \frac{h^2}{24}[y^{(4)}(\xi_i^+) + y^{(4)}(\xi_i^-)] \\&= \frac{1}{h^2}[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})] - \frac{h^2}{12}y^{(4)}(\xi_i)\end{aligned}$$

- Uses Intermediate value theorem, where $\xi_i \in (x_{i-1}, x_{i+1})$



Centered-difference Formula

- The centered-difference formulas for $y''(x_i)$ and $y'(x_i)$ are

$$y''(x_i) = \frac{1}{h^2} [y(x_{i+1}) - 2y(x_i) + y(x_{i-1})] - \frac{h^2}{12} y^{(4)}(\xi_i)$$

$$y'(x_i) = \frac{1}{2h} [y(x_{i+1}) - y(x_{i-1})] - \frac{h^2}{6} y'''(\eta_i)$$

where $\eta_i \in (x_{i-1}, x_{i+1})$

- Substituting these formulas in the given ODE

$$\begin{aligned} \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} &= p(x_i) \left[\frac{y(x_{i+1}) - y(x_{i-1})}{2h} \right] + q(x_i)y(x_i) \\ &\quad + r(x_i) - \frac{h^2}{12} [2p(x_i)y'''(\eta_i) - y^{(4)}(\xi_i)] \end{aligned}$$

- Consider $y_i = y(x_i)$, such that $y_0 = \alpha$ and $y_{N+1} = \beta$, then for truncation error $\mathcal{O}(h^2)$

$$\left(\frac{-y_{i+1} + 2y_i - y_{i-1}}{h^2} \right) + p(x_i) \left(\frac{y_{i+1} - y_{i-1}}{2h} \right) + q(x_i)y_i = -r(x_i)$$

Centered-Difference Formula - Matrix form

$$-\left(1 + \frac{h}{2}p(x_i)\right)y_{i-1} + \left(2 + h^2q(x_i)\right)y_i - \left(1 - \frac{h}{2}p(x_i)\right)y_{i+1} = -h^2r(x_i) \Rightarrow \mathbf{Ay} = \mathbf{b}$$

$$\mathbf{A} = \begin{bmatrix} 2 + h^2q(x_1) & -1 + \frac{h}{2}p(x_1) & \cdots & 0 \\ -1 - \frac{h}{2}p(x_2) & 2 + h^2q(x_2) & -1 + \frac{h}{2}p(x_2) & \cdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 + \frac{h}{2}p(x_{N-1}) \\ 0 & \cdots & -1 - \frac{h}{2}p(x_N) & 2 + h^2q(x_N) \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-1} \\ y_N \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -h^2r(x_1) + \left(1 + \frac{h}{2}p(x_1)\right)y_0 \\ -h^2r(x_2) \\ \vdots \\ -h^2r(x_{N_1}) \\ -h^2r(x_N) + \left(1 - \frac{h}{2}p(x_N)\right)y_{N+1} \end{bmatrix}$$



Finite-Difference Methods for Linear Problems

Theorem

Consider the following second-order boundary value problem

$$y'' = p(x) \frac{dy}{dx} + q(x)y + r(x), \quad x \in [a, b], \quad \text{with } y(a) = \alpha \text{ and } y(b) = \beta$$

Suppose that $p(x)$, $q(x)$, and $r(x)$ are continuous on $[a, b]$.

If $q(x) \geq 0$ on $[a, b]$, then the tridiagonal linear system has a unique solution provided that $h < 2/L$, where $L = \max_{a \leq x \leq b} |p(x)|$.

Example

Solve the following linear BVP by using $N = 9$ and $h = 0.1$

$$y'' = -\frac{2}{x}y' + \frac{\sin(\ln x)}{x^2}, \quad x \in [1, 2], \quad y(1) = 1, \quad y(2) = 2$$

Example

Using the central-difference formula we obtain

x_i	w_i	$y(x_i)$	$ w_i - y(x_i) $
1.0	1.0000000	1.0000000	-
1.1	1.09260052	1.09262930	2.88×10^{-5}
1.2	1.18704313	1.18708484	4.17×10^{-5}
1.3	1.28333687	1.28338236	4.55×10^{-5}
1.4	1.38140205	1.38144595	4.39×10^{-5}
1.5	1.48112026	1.48115942	3.92×10^{-5}
1.6	1.58235990	1.58239246	3.26×10^{-5}
1.7	1.68498902	1.68501396	2.49×10^{-5}
1.8	1.78888175	1.78889853	1.68×10^{-5}
1.9	1.89392110	1.89392951	8.41×10^{-6}
2.0	2.0000000	2.0000000	-

Section - 11.4

Finite-Difference Methods for Nonlinear Problems



Finite-Difference Methods for Nonlinear Problems

- Consider the general nonlinear boundary-value problem

$$y'' = f(x, y, y'), \quad x \in [a, b], \text{ with } y(a) = \alpha, y(b) = \beta$$

- Assume that f satisfies the following conditions

- $f, f_y, f_{y'}$ are all continuous on

$$D = \{(x, y, y') | x \in [a, b], y \in (-\infty, \infty), y' \in (-\infty, \infty)\}$$

- $f_y(x, y, y') \geq \delta$ on D , for some $\delta > 0$
- There exist constants k and L , where

$$k = \max_{(x,y,y') \in D} |f_y(x, y, y')| \text{ and } L = \max_{(x,y,y') \in D} |f_{y'}(x, y, y')|$$

- Then there exists a unique solution to the given nonlinear BVP



Finite-Difference Methods for Nonlinear Problems

- Divide the interval $[a, b]$ into $(N + 1)$ equal subintervals $[x_i, x_{i+1}]$, $x_i = a + ih$, $i = 0, 1, \dots, N + 1$, such that

$$y''(x_i) = f(x_i, y(x_i), y'(x_i))$$

- Replace $y''(x_i)$ and $y'(x_i)$ by the appropriate central-difference formulas

$$\begin{aligned} \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} &= f\left(x_i, y(x_i), \frac{y(x_{i+1}) - y(x_{i-1})}{2h} - \frac{h^2}{6}y'''(\eta_i)\right) \\ &\quad + \frac{h^2}{12}y^{(4)}(\xi_i) \end{aligned}$$

for some ξ_i and η_i in (x_{i-1}, x_{i+1})

- Removing the error terms, we obtain the finite-difference methods

$$-\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + f\left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right) = 0, i = 1, 2, \dots, N$$

where $y_i = y(x_i)$, and $y_0 = \alpha$, $y_{N+1} = \beta$

Finite-Difference Methods for Nonlinear Problems

- We can obtain the $N \times N$ nonlinear system as

$$2y_1 - y_2 + h^2 f \left(x_1, y_1, \frac{y_2 - \alpha}{2h} \right) - \alpha = 0,$$

$$-y_1 + 2y_2 - y_3 + h^2 f \left(x_2, y_2, \frac{y_3 - y_1}{2h} \right) = 0,$$

⋮

$$-y_{N-2} + 2y_{N-1} - y_N + h^2 f \left(x_{N-1}, y_{N-1}, \frac{y_N - y_{N-2}}{2h} \right) = 0,$$

$$-y_{N-1} + 2y_N + h^2 f \left(x_N, y_N, \frac{\beta - y_{N-1}}{2h} \right) - \beta = 0$$

- It has a unique solution for $h < 2/L$

Newton's Method for Iterations

- Generate a sequence of iterates $\{(y_1^{(k)}, y_2^{(k)}, \dots, y_N^{(k)})^T\}$, that converges to the solution of the above system of nonlinear equations, provided that
 - The initial approximation $\{(y_1^{(0)}, y_2(0), \dots, y_N^{(0)})^T\}$ is very close to the true solution
 - And that the Jacobian matrix $J(y_1, y_2, \dots, y_N)$ for the system is nonsingular, where

$$J_{ij} = \begin{cases} -1 + \frac{h}{2} f_{y'} \left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h} \right), & \text{for } i = j-1, j = 2, \dots, N \\ -2 + h^2 f_y \left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h} \right), & \text{for } i = j, j = 1, \dots, N \\ -1 - \frac{h}{2} f_{y'} \left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h} \right), & \text{for } i = j+1, j = 2, \dots, N-1 \end{cases}$$

Newton's Method for Iterations

- Newton's method for nonlinear system requires that at each iteration the $N \times N$ linear system

$$\begin{aligned} J(y_1, \dots, y_N)(v_1, \dots, v_N)^T = & - \left(2y_1 - y_2 - \alpha + h^2 f \left(x_1, y_1, \frac{y_2 - \alpha}{2h} \right), \right. \\ & - y_1 + 2y_2 - y_3 + h^2 f \left(x_2, y_2, \frac{y_3 - y_1}{2h} \right), \dots, \\ & - y_{N-2} + 2y_{N-1} - y_N + h^2 f \left(x_{N-1}, y_{N-1}, \frac{y_N - y_{N-2}}{2h} \right), \\ & \left. - y_{N-1} + 2y_N + h^2 f \left(x_N, y_N, \frac{\beta - y_{N-1}}{2h} \right) - \beta \right)^T \end{aligned}$$

be solved for v_1, v_2, \dots, v_N , since

$$w_i^{(k)} = y_i^{(k-1)} + v_i, i = 1, 2, \dots, N$$

- J is tridiagonal

Example

Example

Solve the following nonlinear BVP using finite difference method

$$y'' = \frac{1}{8}(32 + 2x^3 - yy'), \quad x \in [1, 3], \quad y(1) = 17, \quad y(3) = \frac{43}{3}$$

- Use $h = 0.1$ and $\epsilon = 10^{-8}$, such that

$$|y_i^{(k)} - y_i^{(k-1)}| < 10^{-8}$$

Example

x_i	w_i	$y(x_i)$	$ w_i - y(x_i) $
1.0	17.000000	17.000000	-
1.1	15.754503	15.755455	9.520×10^{-4}
1.2	14.771740	14.733333	1.594×10^{-3}
1.3	13.995677	13.997692	2.015×10^{-3}
1.4	13.386297	13.388571	2.275×10^{-3}
1.5	12.914252	12.916667	2.414×10^{-3}
1.6	12.557538	12.560000	2.462×10^{-3}
1.7	12.299326	12.301765	2.438×10^{-3}
1.8	12.126529	12.128889	2.360×10^{-3}
1.9	12.028814	12.031053	2.239×10^{-3}
2.0	11.997915	12.000000	2.085×10^{-3}
2.1	12.027142	12.029048	1.905×10^{-3}
2.2	12.111020	12.112727	1.707×10^{-3}
2.3	12.245025	12.246522	1.497×10^{-3}
2.4	12.425388	12.426667	1.278×10^{-3}
2.5	12.648944	12.650000	1.056×10^{-3}
2.6	12.913013	12.913846	8.335×10^{-4}



**ANY
QUESTIONS?**