Numerical Methods

DS288 and UMC201

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Error Analysis for bisection method

Convergence of Bisection Method

Suppose that $f \in C[a, b]$ and $f(a) \cdot f(b) < 0$. Let $[a_0, b_0] = [a, b]$ be the initial interval, then the bisection method generates a sequence $\{x_n\}$ converges to a root α of f with

$$|x_n - \alpha| \leq \frac{b-a}{2^n}, \quad \text{when } n \geq 1.$$

- We observe

$$b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1}) = \frac{1}{2^2}(b_{n-2} - a_{n-2}) = \frac{1}{2^n}(b_0 - a_0)$$

- Since approximate root $x_n = \dfrac{(a_{n-1} + b_{n-1})}{2}$
- Therefore,

$$|x_n - \alpha| \leq \frac{1}{2}(b_{n-1} - a_{n-1}) = \frac{1}{2^n}(b_0 - a_0) = \frac{1}{2^n}(b - a)$$

Order of converges for bisection method

- First step of approximation,

$$\alpha = x_1 \pm e_1 \quad \text{and} \quad |e_1| < \frac{1}{2}(b_0 - a_0)$$

- Second step of the approximation

$$|e_2| \leq \frac{(b_0 - a_0)}{2^2} = \frac{(b-a)}{2^2}$$

- Error e_n at nth step $|e_n| \leq \dfrac{(b-a)}{2^n}$
- Therefore the bisection method converges linearly, as

$$\frac{e_{n+1}}{e_n} \leq \frac{1}{2} \quad \text{for} \quad n = 1, 2, 3, \cdots \qquad \lim_{n \to \infty} \frac{|e_{n+1}|}{|e_n|} = k \leq \frac{1}{2}$$

Remark

- Condition $f(a)f(b) < 0$ guarantees the existence of a root, but not the uniqueness i.e., there may be more than one root on the interval, and there is no way to know, theoretically, to which root the sequence will converge, but it will converge to one of them.
- Stopping Criteria: When the exact root/solution is not available, we can use the following condition (according to requirement)
  - Successive iterates $|x_n - x_{n-1}| < \epsilon$
  - Relative terms $\dfrac{|x_n - x_{n-1}|}{|x_n|} < \epsilon$
  - The function value is small enough $|f(x_n)| < \epsilon$
- However, no choice is perfect. In general, where no additional information about f is known, the second criterion is the preferred one (since it comes the closest to testing the relative error)

Advantage and disadvantages on bisection method

- **Advantage:**
  - This method definitely converges to the root of any type of non-linear equation since the error at any intermediate step is less than half of the error in the previous step.
- **Disadvantage:**
  - It fails to determine complex roots.
  - This method is too slow
  - Choosing one guess close to root has no advantage (i.e., required many iterations to converge.)
  - Order of the convergence is 1(or converge linearly).
  - We need two initial guesses for this method.
  - It can not be applied over an interval where the function takes values of the same sign.
  - Can not find root of some equations. For example: $f(x) = x^2$ as there are no bracketing values.

Regula Falsi Method

Regula Falsi Method

➢ Regula falsi method iteratively determines a sequence of root enclosing intervals (a_n, b_n) with a sequence of approximations p_n

- $f(x) = 0$ continuous on $[a, b]$ with $f(a) \cdot f(b) < 0$.
- Compute x-intercept x_1 of the line through $(a, f(a)), (b, f(b))$

$$x_1 = a - f(a)\left(\frac{b-a}{f(b)-f(a)}\right)$$

  - $f(x_1) = 0$; then x_1 is the root.
  - $f(a)f(x_1) < 0$; then the root lies in (a, x_1).
  - $f(x_1)f(b) < 0$; then the root lies in (x_1, b).

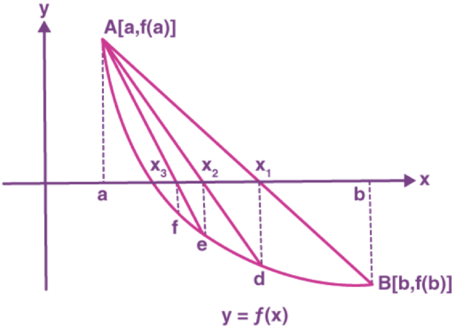
$$x_{n+1} = x_n - f(x_n)\left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right), \quad n = 1, 2, 3, \cdots$$

Regula Falsi (False Position) Method

- Curve $y = f(x)$ will meet x-axis at a certain point between A and B.
- The equation of the chord joining A and B

$$y - f(a) = \left(\frac{f(b) - f(a)}{b - a} \right) (x - a)$$

- Here $y = 0$, as the point of intersection of the chord with x-axis.



Regula Falsi Method

Example

Find an approximation to $\sqrt{3}$ correct to two decimal places

Here $f(x) = x^2 - 3$ and $f(1) < 0$ and $f(2) > 0$ root lies in $[1, 2]$.

- $x_1 = b - f(b)\left(\frac{b-a}{f(b)-f(a)}\right) = 1.6666667, \; f(x_1) = -222.2221 < 0$

$$f(x_1) < 0 \quad \text{and} \quad f(b) > 0 \text{ root lies in } [x_1, b]$$

- $x_2 = b - f(b)\left(\frac{b-x_1}{f(b)-f(x_1)}\right) = 1.727272, \; f(x_2) = -0.0165289 < 0$

$$f(x_2) < 0 \quad \text{and} \quad f(b) > 0 \text{ root lies in } [x_2, b]$$

- $x_3 = 1.7317073$ $x_4 = 1.7320262$, $x_5 = 1.7320491$, $x_6 = 1.7320507$ $x_8 = 1.7320508$

Regula Falsi Method

➡ Order of convergence of both bisection method and Regula Falsi method is one.

➡ Regular falsi method converges faster than bisection method.

➡ However, it is not true in general.

Example

Use bisection method and Regula Falsi method to locate the root of $f(x) = x^{10} - 1$ between 0 and 1.3

Regular Falsi method can cause serious round-off problems; since the denominator in the formula consists of the difference between two numbers that are both close to zero.

Newton's Method
or
Newton-Raphson method

Newton's Method (or Newton-Raphson method)

- Best known method for approximating the root of a differentiable function.
- It can be generalized many ways for the solution of others,
  - system of nonlinear equations
  - nonlinear integral equations
  - differential equations.
- Newton's method can often converge remarkably quickly, if the iteration begins sufficiently near to the desired root.

Newton's Method (or Newton-Raphson method)

- Choose a value x_0 (called initial approximation) that is reasonably close to the root of the equation.
- Find the tangent line at $(x_0, f(x_0))$

$$y - f(x_0) = f'(x_0)(x - x_0)$$

- Compute the x-intercept of this tangent.

$$0 - f(x_0) = f'(x_0)(x - x_0) \Rightarrow x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

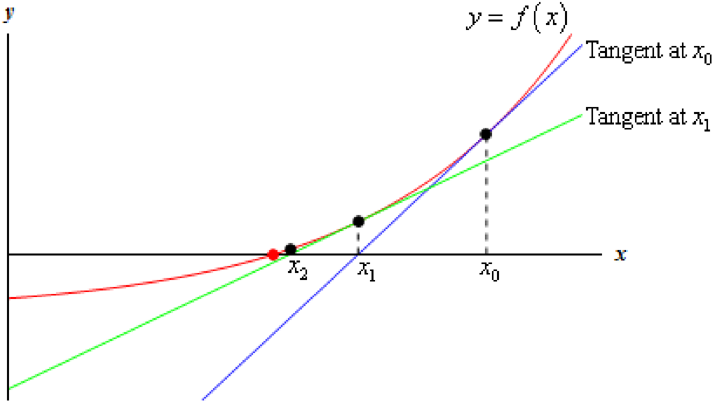
- The x-intercept of this tangent will be the next approximation

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

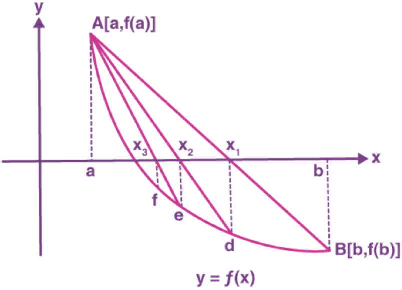
- Continuing this process and generate the sequence

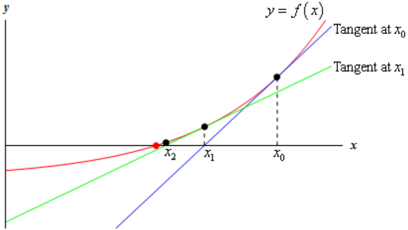
$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

Newton's Method



Regula Falsi and Newton's Method





Convergence of Newton's method

$$e_{n+1} = \alpha - x_{n+1} = \alpha - \left(x_n - \frac{f(x_n)}{f'(x_n)} \right) = e_n + \frac{f(x_n)}{f'(x_n)}$$

Now using Taylor's theorem, we have.

$$\begin{aligned}
0 &= f(\alpha) = f(\alpha - x_n + x_n) = f(e_n + x_n) \\
&= f(x_n) + e_n f'(x_n) + \frac{e_n^2}{2} f''(\zeta_n), \\
\Rightarrow \quad & e_n + \frac{f(x_n)}{f'(x_n)} = -\frac{e_n^2}{2} \frac{f''(\zeta_n)}{f'(x_n)} \quad \text{(dividing } f'(x_n) \text{ both side)} \\
\Rightarrow \quad & e_{n+1} = -\frac{e_n^2}{2} \frac{f''(\zeta_n)}{f'(x_n)} \\
\Rightarrow \quad & |e_{n+1}| \leq \frac{1}{2} \frac{f''(\zeta_n)}{f'(x_n)} |e_n|^2 \quad \text{(converges quadratically)}
\end{aligned}$$


ANY
QUESTIONS?