DS 288: NUMERICAL METHODS

OCT-26-2021

ADAMS METHODS

– CLASS OF MULTI-STEP METHODS

* ADAMS-BASHFORTH METHODS:
  "OPEN" OR EXPLICIT METHODS
  i.e. DO NOT INVOLVE f_{i+1}

* ADAMS-MOULTON METHODS:
  "CLOSED" OR IMPLICIT METHODS
  i.e. DO INVOLVE f_{i+1}

IDEA

$$y_{i+1} = y_i + hf_i + \frac{h^2}{2!} f_i' + \frac{h^3}{3!} f_i'' + \cdots$$

APPROXIMATE DERIVATIVES

WITH $f_{i+1}, f_i, f_{i-1}, f_{i-2}, \ldots$

EXPLICIT METHODS (ADAMS-BASHFORTH)

IMPLICIT METHODS (ADAMS-MOULTON)

ADAMS-BASHFORTH 2 STEP METHOD

$$Y_{i+1} = Y_i + hf_i + \underbrace{\frac{h^2}{2!} f_i'}_{} + \frac{h^3}{3!} f_i'' + \cdots \quad \circledast$$

FIRST BACKWARD DIFFERENCE APPROXIMATION $f_i' = \dfrac{f_i - f_{i-1}}{h} + O(h)$

$$w_{i+1} = w_i + hf_i + \frac{h}{2}\left[f_i - f_{i-1}\right] + O(h^3)$$

$$\boxed{w_{i+1} = w_i + h\left[\frac{3}{2} f_i - \frac{1}{2} f_{i-1}\right]} + O(h^3)$$

LOCALLY

GLOBALLY $O(h^2)$.

SAME ORDER AS RK-2, BUT NO HALF. INTERVALS

- KEEPS TRACK OF PREVIOUS FUNCTION EVALUATION AND ONLY ONE NEW FUNCTION EVALUATION IS REQUIRED AT EACH STEP.

ADAMS-BASHFORTH 3 STEP METHOD

USE 3 POINTS TO APPROXIMATE

$f_i' \to O(h^2)$
$f_i'' \to O(h)$
USE ONLY BACKWARD DIFFERENCE

$\Rightarrow O(h^4)$/STEP $\Rightarrow O(h^3)$/GLOBALLY

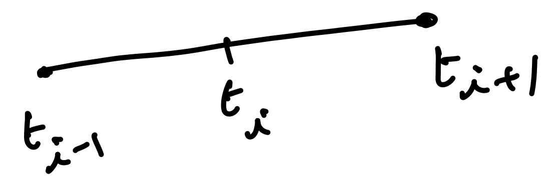
$$\boxed{w_{i+1} = w_i + \frac{h}{12}\left[23 f_i - 16 f_{i-1} + 5 f_{i-2}\right]}$$

A-B 3STEP METHOD

* CAN CONTINUE IN THIS FASHION.
  A-B 2-5 STEP METHODS IN TEXTBOOK
  [HANDOUT TO BE MADE]

* ADAMS-MOULTON METHODS
  INVOLVE f_{i+1}, SO WE CAN USE CENTERED DIFFERENCE APPROXIMATIONS
  i.e. f' TO $O(h^2)$ WITH ONLY 2 STEPS

ADAMS- MOULTON 2 STEP METHOD



USE CENTERED DIFFERENCE SCHEME.
TO APPROXIMATE f' & f''
TERMS FROM TAYLOR SERIES (REF. ⊛)

$$\left.\begin{aligned}
\frac{h^2}{2!} f_i' &\cong \frac{h^2}{2}\left[\frac{f_{i+1}-f_{i-1}}{2h} + O(h^2)\right] \\
\frac{h^3}{3!} f_i'' &\cong \frac{h^3}{6}\left[\frac{f_{i+1}-2f_i+f_{i-1}}{h^2} + O(h^2)\right]
\end{aligned}\right\} O(h^4) \text{ LOCALLY}$$

SUBSTITUTE THESE IN ⊛

$$w_{i+1} = w_i + h f_i + \frac{h^2}{2}\left[\frac{f_{i+1}-f_{i-1}}{2h}\right] + \frac{h^3}{3!}\left[\frac{f_{i+1}-2f_i+f_{i-1}}{h^2}\right]$$

COLLECT TERMS

$$\boxed{w_{i+1} = w_i + \frac{h}{12}\left[5f_{i+1} + 8f_i - f_{i-1}\right]}$$ A-M 2 STEP METHOD

$O(h^4)$ LOCALLY

$O(h^3)$ GLOBALLY

$$LTE = \frac{-h^4}{24} f'''(y_i, \xi_i) \qquad \xi_i \in [t_{i-1}, t_{i+1}]$$

[TEXT BOOK: GTE IN CLASS IS BEING CALLED AS LTE]

$$f_{i+1} \equiv f(w_{i+1}, t_{i+1})$$

TWO SOLUTIONS:

(1) y APPEARS LINEARLLY IN $f(y,t)$

i.e. $y' = \underline{\underline{-y+t-1}}$

THEN f_{i+1} EVALUATION PRESENTS NO PROBLEM

<u>Ex:</u> $w_{i+1} = w_i + \frac{h}{12}\left[5(-w_{i+1} + t_{i+1} + 1) + \cdots\right.$

SOLVE FOR w_{i+1}

$$w_{i+1}\left(1+\frac{5h}{12}\right) = w_i + \frac{h}{12}\left[5\left(t_{i+1}\right)\right. + \cdots$$

CALCULATE w_{i+1}

(ii) y APPEARS NON-LINEARALLY IN $f(y,t)$.

i.e. $y' = \cos(y) + t$

THEN f_{i+1} EVALUATION PRESENTS A PROBELM

* USING 2 STEP A-M

$$w_{i+1} = w_i + \frac{h}{12}\left[5\left(\underbrace{\cos(w_{i+1}) + t_{i+1}}_{f_{i+1}}\right) + \cdots\right.$$

* NOT POSSIBLE TO COLLECT w_{i+1} TERMS

* INSTEAD WE MUST PREDICT w_{i+1} AND CORRECT USING A-M METHOD

PREDICTOR / CORRECTOR SCHEME

* USE AN EXPLICIT METHOD TO PREDICT w_{i+1}
THEN USE IMPLICIT METHOD TO IMPROVE/CORRECT w_{i+1}

* WE DID THIS TO GET EXPLICIT FORM OF THE TRAPEZOIDAL METHOD IN THE FORM OF MODIFIED EULER'S METHOD

$$w_{i+1} = w_i + \frac{h}{2}\left[f_i + f_{i+1}\right]$$ TRAPEZOIDAL METHOD

$f_{i+1} = f(w_{i+1}, t_{i+1})$ USE

$$w_{i+1} = w_i + h f_i \longleftarrow$$ EULER STEP

<u>E.X</u>: - A-B/A-M PREDICTOR/CORRECTOR

$$\hat{w}_{i+1} = w_i + \frac{h}{2}(3f_i - f_{i-1})$$ A-B 2STEP PREDICTOR $O(h^2)$/GLOBALLY

$$w_{i+1} = w_i + \frac{h}{12}\left[5f(\hat{w}_{i+1}, t_{i+1}) + 8f_i - f_{i-1}\right]$$

A-M 2STEP CORRECTOR

$O(h^3)$/ GLOBALLY

PREDICTOR OUTPUT.

* THINK OF THIS AS A FIXED POINT ITERATION

$$w_{i+1} = g(\hat{w}_{i+1})$$

PREDICTOR (EXPLICIT METHOD)

*MULTIPLE STEPS OF CORRECTION?

- IN MOST CASES THERE IS NOT MUCH ADVANTAGE OF "CORRECTING" MORE THAN ONCE.
- IF WE DO ITERATE (CORRECT

MUTIPLE TIMES) THEN FOR CONVE-RGENCE TO w_{i+1} WE REQUIRE

$$|g'(w_{i+1})| < 1 \text{. IS REQUIRED}$$

2-STEP A-B/A-M P/C:

$$|g'(w_{i+1})| = \left| \frac{5h}{12} \underbrace{\frac{\partial f}{\partial y}\Big|_{w_{i+1}}}_{J = \text{JACOBIAN.}} \right| < 1.$$

*RECALL J FORMS THE AMPLIFICATION FACTOR WHEN DERIVING THE ERROR BEHAVIOR OF EULER'S METHOD.

* IF J IS LARGE ⇒ WE REQUIRE SMALL 'h' ⇒ WE REQUIRE MORS STEPS ⇒ MORE COMPUTATIONAL COMPLEXITY.

* WE COULD USE OTHER TYPES OF FIXED POINT ITERATIONS (i.e. OTHER 'g' FUNCTIONS).

NEWTON'S METHOD.

<u>Ex:</u> - REWRITE $g(w_{i+1}) = w_{i+1}$ AS $f(w_{i+1}) = 0$

2 STEP A-B/A-M P/C EXAMPLE:

$$w_{i+1} - \left\{ w_i + \frac{h}{12}\left[5 f(w_{i+1}, t_{i+1}) + 8 f_i - f_{i-1}\right]\right\} = 0$$

$f(z) = 0$ WHERE $z = w_{i+1}$

$$z^{k+1} = z^k - \frac{f(z^k)}{f'(z^k)}$$ NEWTON'S METHOD

WRITE IT OUT

$$w_{i+1}^{k+1} = w_{i+1}^{k} - \frac{\left\{ w_{i+1}^{k} - w_i - \frac{h}{12}\left[5 f(w_{i+1}^{k}, t_{i+1}) + 8 f_i - f_{i-1}\right]\right\}}{1 - \frac{5h}{12} \frac{\partial f}{\partial y}\Big|_{w_{i+1}^{k}}}$$

* CONVERGENCE IS FASTER BUT REQUIRES A GOOD GUESS OF w_{i+1}^{0} (PREDICTOR)

- IN PRACTICAL PROBLEMS, IMPROVING THE CORRECTED VALUE IS <u>NOT</u> WORTH IT. MUCH BETTER TO CUT THE STEP SIZE ('h' AS SMALL).

* FOR "STIFF" EQUATIONS

$\hookrightarrow$ (LARGE J)

THERE IS A BENEFIT TO MULTIPLE CORRECTIONS (ITERATIVE CORRECTOR)

- DEFINETLY SHOULD USE NEWTON'S METHOD SINC $g'(w_{i+1})=0$ (AS OPPOSED TO $\frac{5h}{12}J$ FOR GENERAL

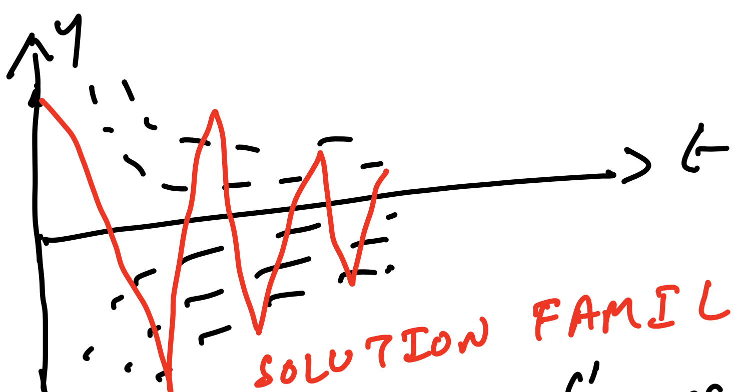
(FIXED POINT ITER)

EX: OF A STIFF EQUATION

$$y' = \underbrace{-Ky}_{f} \Rightarrow y = y_0 e^{-Kt}.$$

$$J = \frac{\partial f}{\partial y} = -K$$ FOR LARGE 'K' 'J' IS LARGE

SLOPE IS STEEP FOR LARGER y



"CAN OSCILATE"

SLOPE $\to$ 0 FOR $y = 0$

* MORE LATER ON "STIFF" EQUATIONS WHEN WE DISCUSS STABILITY.