Numerical methods - Questions from Notes

Interpolation

2)

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, \dots, x_{i+2}]$	$f[x_i, \dots x_{i+3}]$	$f[x_i, \dots x_{i+4}]$	
-2	1					
-1	4	3				
0	11	7	4			
1	16	5	-2	-6		
2	13	-3	-8	-6	0	
3	-4	-17	-14	-6	0	0

$\therefore P(x) = 1 + 3(x-(-2)) + 4(x-(-2))(x-(-1)) + (-6)(x-(-2))$
$(x-(-1))(x-0) + 0 \dots + 0 \dots$

$= 1 + 3(x+2) + 4(x+2)(x+1) - 6x(x+2)(x+1)$

3a) $P(x) = 3 - 2(x+1) + x(x+1)(x-1)$
$= 3 - 2x - 2 + x(x^2-1)$
$= 1 - 2x + x^3 - x$
$= 1 - 3x + x^3$

$Q(x) = -1 + 4(x+2) - 3(x+2)(x+1) + x(x+2)(x+1)$
$= -1 + 4x + 8 - 3(x^2+3x+2) + x(x^2+3x+2)$
$= 7 + 4x \cancel{-3x^2} - 9x - 6 + x^3 + \cancel{3x^2} + 2x$
$= 1 - 3x + x^3$

7) a)

t	$f[t_i]$	$f[t_i, t_{i+1}]$	$f[t_i, t_{i+2}]$	$f[t_i, t_{i+3}]$	$f[t_i, t_{i+4}]$	$f[t_i, t_{i+5}]$
0	0					
0	0	75				
3	225	75	0			
3	225	77	2	2		
5	383	79	2	0	2	
5	383	80	1	-1	-1	-3
8	623	80	0	-1	0	1
8	623	74	-6	-6	-5	-5
13	993	74	0	6	12	17 ...
13	993	72	-2	-2	-8	-20

Least Squares

Ex. 5.3.1

$$\underset{\sim}{A} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 5 \\ 1 & 7 \end{bmatrix}_{4\times 2} \qquad \underset{\sim}{x} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}_{2\times 1} \qquad \underset{\sim}{b} = \begin{bmatrix} 1 \\ 5 \\ 12 \\ 20 \end{bmatrix}$$

$$\underset{\sim}{A}^T\underset{\sim}{A} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 5 & 7 \end{bmatrix}_{2\times 4} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 5 \\ 1 & 7 \end{bmatrix}_{4\times 2} = \begin{bmatrix} 4 & 14 \\ 14 & 78 \end{bmatrix}$$

$$\underset{\sim}{A}^T\underset{\sim}{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 5 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 12 \\ 20 \end{bmatrix} = \begin{bmatrix} 38 \\ 210 \end{bmatrix}$$

$$\underset{\sim}{x} = (\underset{\sim}{A}^T\underset{\sim}{A})^{-1} \cdot \underset{\sim}{A}^T\underset{\sim}{b}$$

$$= \frac{1}{116} \begin{bmatrix} 78 & -14 \\ -14 & 4 \end{bmatrix} \begin{bmatrix} 38 \\ 210 \end{bmatrix}$$

$$= \frac{1}{116}\begin{bmatrix}24\\308\end{bmatrix}$$

$$\therefore y = \frac{24}{116} + \frac{308}{116}x = \frac{6+77x}{29}$$

$$\sum_{k=0}^{1} a_k \int_a^b x^{j+k}\,dx = \int_a^b x^j f(x)\,dx$$

3) a) Eqn 1: $a_0 \int_{-1}^{1} 1\,dx + a_1 \int_{-1}^{1} x\,dx = \int_{-1}^{1} (x^2 - 2x + 3)\,dx$

$$a_0(1-(-1)) + \frac{a_1}{2}\overset{0}{\cancel{(1^2-((-1)^2))}} = \left[\frac{x^3}{3} - x^2 + 3x\right]_{-1}^{1}$$

$$2a_0 = \left(\frac{1}{3} \cancel{-1} + 3\right) - \left(-\frac{1}{3} \cancel{-1} - 3\right)$$

$$2a_0 = 20/3$$

$$a_0 = 10/3$$

Eqn 2: $\frac{10}{3}\int_{-1}^{1} x\,dx + a_1 \int_{-1}^{1} x^2\,dx = \int_{-1}^{1} x(x^2-2x+3)\,dx$

$$\frac{10}{6}\overset{0}{\cancel{(1^2-(-1)^2)}} + \frac{a_1}{3}\left[x^3\right]_{-1}^{1} = \int_{-1}^{1} x^3 - 2x^2 + 3x\,dx$$

$$\frac{2a_1}{3} = \left[\frac{x^4}{4} - \frac{2x^3}{3} + \frac{3x^2}{2}\right]_{-1}^{1}$$

$$a_1 = \frac{3}{2}\left[\left(\cancel{\frac{1}{4}} - \frac{2}{3} + \cancel{\frac{3}{2}}\right) - \left(\cancel{\frac{1}{4}} + \frac{2}{3} + \cancel{\frac{3}{2}}\right)\right]$$

$$= \frac{\cancel{3}}{\cancel{2}}\left(\frac{\overset{2}{\cancel{-4}}}{\cancel{3}}\right)$$

$$= -2$$

$$\therefore f(x) = -2x + 10/3$$

b) Eqn 1: $a_0\int_{-1}^{1} 1\,dx + a_1\int_{-1}^{1} x\,dx = \int_{-1}^{1} x^3\,dx$

$2a_0 = \frac{1}{4}\left[x^4\right]_{-1}^{1}$

$a_0 = 0$

Eqn. 2: $\cancel{0\int_{-1}^{1} x\,dx}^{\,0} + a_1\int_{-1}^{1} x^2\,dx = \int_{-1}^{1} x^4\,dx$

$\frac{2a_1}{3} = \frac{1}{5}\left(1-(-1)\right)$

$a_1 = \frac{3}{\cancel{2}}\left(\frac{\cancel{2}}{5}\right) = 3/5$ $\quad\therefore f(x) = \frac{3}{5}x$

c) Eqn. 1: $a_0\int_{-1}^{1} 1\,dx + a_1\int_{-1}^{1} x\,dx = \int_{-1}^{1} \frac{1}{x+2}\,dx$

$2a_0 = \left[\ln|x+2|\right]_{-1}^{1}$

$2a_0 = \ln 3 - \cancel{\ln 1}^{\,0}$

$a_0 = \frac{\ln 3}{2}$

Eqn 2: $\frac{\ln 3}{2}\cancel{\int_{-1}^{1} x\,dx}^{\,0} + a_1\cancel{\int_{-1}^{1} x^2\,dx}^{\,2/3} = \int_{-1}^{1} \frac{x}{x+2}\,dx$

$\frac{2a_1}{3} = \int_{-1}^{1} \cancel{\frac{x+2}{x+2}}^{\,1} - \frac{2}{x+2}\,dx$

$= \cancel{\int_{-1}^{1} 1\,dx}^{\,2} - 2\cancel{\int_{-1}^{1} \frac{1}{x+2}\,dx}^{\,\ln 3}$

$\frac{2a_1}{3} = 2 - 2\ln 3$

$a_1 = 3 - 3\ln 3$ $\quad\therefore f(x) = \frac{\ln 3}{2} + (3-\ln 3)x$

3) d) $a_0 \cancel{\int_{-1}^{1} 1\,dx}^{\,2} + a_1 \cancel{\int_{-1}^{1} x\,dx}^{\,0} = \int_{-1}^{1} e^x\,dx$

$2a_0 = \left[e^x\right]_{-1}^{1}$

$a_0 = \dfrac{e - e^{-1}}{2}$

$a_0 \cancel{\int_{-1}^{1} x\,dx}^{\,0} + a_1 \cancel{\int_{-1}^{1} x^2\,dx}^{\,2/3} = \int_{-1}^{1} x e^x\,dx$

$\int u\,dv = uv - \int v\,du$

$u = x \quad dv = e^x$

$du = 1 \quad v = e^x$

$\frac{2}{3} a_1 = \left[x \cdot e^x - \int e^x\,dx\right]_{-1}^{1}$

$\frac{2}{3} a_1 = \left[x e^x - e^x\right]_{-1}^{1}$

$\frac{2}{3} a_1 = \cancel{(e^1 - e^1)}^{\,0} - (-1 \cdot e^{-1} - e^{-1})$

$\frac{2}{3} a_1 = \frac{2}{e}$

$a_1 = \frac{3}{\cancel{2}} \cdot \frac{\cancel{2}}{e} = \frac{3}{e} \quad \therefore f(x) = \frac{e - 1/e}{2} + \frac{3}{e} x$

e) $2a_0 = \int_{-1}^{1} \frac{1}{2}\cos x + \frac{1}{3}\sin 2x\,dx$

$= \left[\frac{1}{2}\sin x - \frac{1}{6}\cos 2x\right]_{-1}^{1}$

$= \left(\frac{1}{2}\sin 1 - \frac{1}{6}\cos 2\right) - \left(\frac{1}{2}\cancel{\sin(-1)}^{\,-\sin 1} - \frac{1}{6}\cancel{\cos(-2)}^{\,\cos 2}\right)$

$= \frac{1}{2}\sin 1 + \frac{1}{2}\sin 1 \cancel{- \frac{1}{6}\cos 2} \cancel{+ \frac{1}{6}\cos 2}$

$= \sin 1$

Ex 7.2.1. $\int_1^2 x^2 = \frac{2-1}{2}(2^2+1^2) = 5/2$

2) $f'(0.5) = \frac{f(0.6)-f(0.5)}{0.1} = 0.852$ | Actual 0.878

$f'(0.6) = \frac{f(0.7)-f(0.5)}{0.2} = 0.824$ | 0.825

$f'(0.7) = \frac{f(0.7)-f(0.6)}{0.1} = 0.796$ | 0.765

Actual

3) $f'(1.1) = \frac{-3f(1.1)+4f(1.2)-f(1.3)}{2(0.1)} = 17.77$

$f'(1.2) = \frac{-3f(1.2)+4f(1.3)-f(1.4)}{0.2} = 21.70$

$f'(1.3) = \frac{3f(1.3)-4f(1.2)+f(1.1)}{0.2} = 26.62$

$f'(1.4) = \frac{3f(1.4)-4f(1.3)+f(1.2)}{0.2} = 32.51$

$\frac{d}{dx}(xe^x) = (x+1)e^x$

4) $f''(1) = \frac{f(0.9)-2f(1)+f(1.1)}{0.1^2}$

$= \frac{e^{1.8}-2e^2+e^{2.2}}{0.01}$

$= 24.7$

$f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

$f''(x) = 4e^{2x}$

$f''(1) = 4e^2$

$\simeq 24.56$

9) $f'(3) = \dfrac{f(1) - 8f(2) + 8f(4) - f(5)}{12 \cdot 1} = 0.211$

10) $v(0) = \dfrac{0.12 - 0}{0.2} = 0.6$

$v(0.2) = \dfrac{0.48 - 0}{0.4} = 1.2$

$v(0.4) = \dfrac{0 - 8(0.12) + 8(1.08) - 1.92}{12 \cdot 0.2} = 2.4$

$v(0.6) = \dfrac{0.12 - 8(0.48) + 8(1.92) - 3}{12 \cdot 0.2} = 3.6$

$v(0.8) = \dfrac{3 - 1.08}{0.4} = 4.8$

$v(1) = \dfrac{3 - 1.92}{0.2} = 5.4$

12) $\displaystyle\int_0^1 \frac{2}{1+x^2}\,dx \approx \frac{1-0}{2}\left(f(1) + f(0)\right) = \frac{1}{2}(1+2) = 3/2 = 1.5$

$\displaystyle\int_0^1 \frac{2}{1+x^2}\,dx = \frac{1-0}{6}\left(f(0) + 4f(0.5) + f(1)\right)$

$= \frac{1}{6}(2 + 6.4 + 1)$

$= 1\frac{17}{30}$

≈ 1.567

13) a) $\int_0^2 x^2\ln(x^2+1)\,dx \approx \overset{1}{\cancel{\frac{2-0}{2}}}\left(f(0)+f(2)\right) = 0 + 4\ln 5 \approx 6.43$

$$\int_0^2 x^2\ln(x^2+1)\,dx \approx \frac{\frac{2-0}{2}}{3}\left(f(0)+4f(1)+f(2)\right)$$

$$= \frac{1}{3}\left(0+4\ln 2+4\ln 5\right)$$

$$= \frac{4\ln 10}{3}$$

$$\approx 3.07$$

b) $\int_{-1}^{1} \frac{x^2}{e^x}\,dx \approx \overset{1}{\cancel{\frac{1-(-1)}{2}}}\left(f(-1)+f(1)\right) = \frac{1}{e^{-1}} + \frac{1}{e} = e+\frac{1}{e} \approx 3.08$

$$\int_{-1}^{1} \frac{x^2}{e^x}\,dx \approx \frac{\overset{1}{\cancel{\frac{1-(-1)}{2}}}}{3}\left(\overset{e}{\cancel{f(-1)}}+4\overset{0}{\cancel{f(0)}}+\overset{1/e}{\cancel{f(1)}}\right) = \frac{e+1/e}{3} \approx 1.03$$

c) $\int_0^1 \frac{x}{x^2+4}\,dx \approx \frac{1-0}{2}\left(\overset{0}{\cancel{f(0)}}+\overset{1/5}{\cancel{f(1)}}\right) = 1/10 = 0.1$

$$\int_0^1 \frac{x}{x^2+4} \approx \frac{\frac{1-0}{2}}{3}\left(\overset{0}{\cancel{f(0)}}+4\overset{2/17}{\cancel{f(0.5)}}+\overset{1/5}{\cancel{f(1)}}\right) = \frac{1}{6}\left(\frac{57}{85}\right) = \frac{19}{170} \approx 0.112$$

d) $\int_1^3 \frac{1}{x^2} \approx \overset{0}{\cancel{\frac{3-1}{2}}}\left(\overset{1}{\cancel{f(1)}}+\overset{1/9}{\cancel{f(3)}}\right) = 1\tfrac{1}{9} \approx 1.11$

$$\int_1^3 \frac{1}{x^2} \approx \frac{\overset{1}{\cancel{\frac{3-1}{2}}}}{3}\left(\overset{1}{\cancel{f(1)}}+4\overset{1/4}{\cancel{f(2)}}+\overset{1/9}{\cancel{f(3)}}\right) = \frac{1}{3}\left(\frac{19}{9}\right) = \frac{19}{27} \approx 0.704$$

Convert -56 to binary (2's complement)

1) Convert 56 to binary : $(56)_{10} = (00111000)_2$

2) Convert to 2's complement

0	0	1	1	1	0	0	0	(Original)
↓	↓	↓	↓	↓	↓	↓	↓	
1	1	0	0	1	0	0	0	

* Blue represents direct copy * Red is inversion

Convert $(11001000)_2$ to decimal:

$= -128 + 64 + 8 = -56$

L20, S8 Ex

$\frac{dy}{dx} = \frac{y-x}{y+x} \quad y(0) = 1$

$y_1 = 1 + 0.02 \cdot \frac{1-0}{1+0}$

$y_2 = 1.02 + 0.02 \cdot \frac{\cancel{1.02-0.02}\ 1}{1.02+0.02}$

$y_3 = 1.039 + 0.02 \cdot \frac{1.039-0.04}{1.039+0.04}$

	x	y
0	0	1
1	0.02	1.02
2	0.04	1.039
3	0.06	1.0577
4	0.08	1.0756

L20 S12 Ex

$\frac{dy}{dx} = x+y \quad y(0) = 1 \quad h = 0.1$

$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))\right]$

	x	y
0	0	1
1	0.1	1.11
2	0.2	
3	0.3	
4	0.4	
5	0.5	

$f(x_0, y_0) = 1$

$\therefore y_1 = 1 + 0.05(1 + 1.2) = 1.11$

14) Composite Trapezoidal : $\int_a^b f(x)\,dx = \frac{h}{2}\left(f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right)$

a) $h = \frac{3-1}{4} = \frac{1}{2}$

$\therefore \int_1^3 \frac{1}{x}\,dx \approx \frac{1/2}{2}\left(f(1) + 2\sum_{j=1}^{n-1} f(1+jh) + f(3)\right)$

$= \frac{1}{4}\left[\frac{4}{3} + 2\left(\frac{1}{3/2} + \frac{1}{2} + \frac{1}{5/2}\right)\right]$

$= \frac{1}{4}\left(\frac{4}{3} + 2\left(\frac{2}{3} + \frac{1}{2} + \frac{2}{5}\right)\right)$

$= 1.02$

b) $h = \frac{3-1}{8} = 0.25 = \frac{1}{4}$

$\therefore \int_1^3 \frac{1}{x}\,dx \approx \frac{1/4}{2}\left(f(1) + f(3) + 2\sum_{j=1}^{n-1} f(x_j)\right)$

$= \frac{1}{8}\left(\frac{4}{3} + 2(3.75)\right)$

$= 1.10$

15) $\int_a^b f(x)\,dx = \frac{h}{3}\left(f(a) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right)$

a) $h = \frac{\pi - 0}{4} = \frac{\pi}{4}$

$\int_0^\pi x\sin x\,dx \approx \frac{\frac{\pi}{4}}{3}\left(\cancel{f(0)}^{\,0} + 2\sum_{j=1}^{n/2-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + \cancel{f(\pi)}^{\,0}\right)$

$$= \frac{\pi}{\cancel{18}\,6}\,\cancel{(2)}\left(\sum_{j=1}^{n/2-1} f(x_{2j}) + 2\sum_{j=1}^{n/2} f(x_{2j-1})\right)$$

$$= \frac{\pi}{6}\left(2f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right) + 2f\left(\frac{3\pi}{4}\right)\right)$$

$$= \frac{\pi}{6}\left(\cancel{2}\left(\frac{\pi}{\cancel{4}_{2}}\,\cancel{\sin\frac{\pi}{4}}^{\sqrt{2}/2}\right) + \frac{\pi}{2}\,\cancel{\sin\frac{\pi}{2}}^{1} + \cancel{2}\left(\frac{3\pi}{\cancel{4}_{2}}\,\cancel{\sin\frac{3\pi}{4}}^{\sqrt{2}/2}\right)\right)$$

$$= \frac{\pi}{6}\left(\frac{\sqrt{2}\pi}{4} + \frac{\pi}{2} + \frac{3\sqrt{2}\pi}{4}\right)$$

$$= \frac{\pi}{6}\left(\sqrt{2}\pi + \frac{\pi}{2}\right)$$

$$= \frac{\pi^2\left(\sqrt{2}+\frac{1}{2}\right)}{6}$$

$$\simeq 3.149$$

16) a) $f(x) = xe^x$

$f'(x) = xe^x + e^x$

$= (x+1)e^x$

$f''(x) = (x+1)e^x + e^x$

$= (x+2)e^x$

$f^{(3)}(x) = (x+3)e^x$

$f^{(4)}(x) = (x+4)e^x$

b) Max value of $f^{(4)}(x)$ in $[0,2]$

$= f^{(4)}(2)$

$= 6e^2$

$\therefore \frac{\cancel{2}^{1}}{\cancel{180}_{90}} \cdot h^4 \cdot 6e^2 < 10^{-4}$

$h^4 < \frac{\cancel{90}^{15} \cdot 10^{-4}}{\cancel{6}_{1}e^2}$

$h^4 < 2.03 \times 10^{-4}$

$h < 0.119$

$\therefore \frac{b-a}{n} < 0.119$

$n > \frac{2}{0.119}$

$n > 16.8$

$\therefore n = 18$

17) $f(x) = e^{2x}\sin 3x$

$f'(x) = e^{2x}(3\cos 3x) + \sin 3x\,(2e^{2x})$
$= e^{2x}(3\cos 3x + 2\sin 3x)$

$f''(x) = 2e^{2x}(3\cos 3x + 2\sin 3x) + e^{2x}(6\cos 3x - 9\sin 3x)$
$= e^{2x}(6\cos 3x + 4\sin 3x + 6\cos 3x - 9\sin 3x)$
$= e^{2x}(12\cos 3x - 5\sin 3x)$

$f^{(3)}(x) = e^{2x}(24\cos 3x - 10\sin 3x) + e^{2x}(-36\sin 3x - 15\cos 3x)$

$= e^{2x}(9\cos 3x - 46\sin 3x)$

$f^{(4)}(x) = e^{2x}(18\cos 3x - 92\sin 3x) + e^{2x}(-27\sin 3x - 138\cos 3x)$

$= -e^{2x}(120\cos 3x + 119\sin 3x)$

$f^{(5)}(x) = -e^{2x}(240\cos 3x + 238\sin 3x) - e^{2x}(357\cos 3x - 360\sin 3x)$

$= -e^{2x}(597\cos 3x - 122\sin 3x)$
$= e^{2x}(122\sin 3x - 597\cos 3x)$

d) $\frac{b-a}{12} h^2 f''(\mu) \le 10^{-4}$

$f^{(3)}(x) = 0$

$\cancel{e^{2x}}(9\cos 3x - 46\sin 3x) = 0$

$46\sin 3x = 9\cos 3x$
$\tan 3x = 9/46$

$3x = 0.193, 3.33,$

$x = 0.0644, 1.11$

$f''(0) = 12$
$f''(0.0644) = 12.3$
$f''(1.11) = -99.91$
$f''(2) = 705$

$$\frac{\cancel{2-0}^{1}}{\cancel{12}_{6}} h^2 (705) \le 10^{-4}$$

$$h^2 \le \frac{6 \times 10^{-4}}{705}$$

$$h^2 \le 8.51 \times 10^{-7}$$

$$h \le 9.22 \times 10^{-4}$$

$$\frac{\cancel{b-a}^{2}}{n} \le 9.22 \times 10^{-4}$$

$$n \ge 2168.5$$

$$\therefore n = 2169$$

$$h = 9.22 \times 10^{-4}$$

b) $\frac{b-a}{180} h^4 f^{(4)}(\mu) \le 10^{-4}$

$$f^{(5)}(x) = 0$$

$$e^{2x}(122 \sin 3x - 597 \cos 3x) = 0$$

$$\tan 3x = \frac{597}{122}$$

$$3x = 1.36, 4.51$$

$$x = 0.456, 1.50$$

$$f^{(4)}(0) = -120$$

$$f^{(4)}(0.456) = -350$$

$$f^{(4)}(1.50) = 2845$$

$$f^{(4)}(2) = -4475$$

$$\therefore \frac{\cancel{2}}{\cancel{180}_{90}} h^4 (4475) \le 10^{-4}$$

$$h^4 \le \frac{10^{-4} \cdot 90}{4475}$$

$$h^4 \le 2.01 \times 10^{-6}$$

$$h \le 0.0377$$

$$\frac{\cancel{b-a}^{2}}{n} \le 0.0377$$

$$n \ge \frac{2}{0.0377}$$

$$n \ge 53.1$$

$$\therefore n = 54, \; h = 0.0370$$

ODEs

1) a) $f(x)$ is continuous on $[0, b]$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \cos t) = \cos t$ is also continuous on $[0, b]$

$$R = \{|t - t_0| \le 0, \; |y - y_0| \le 1\}$$

$$= \{|t - 0| \le 0, \; |y - 1| \le 1\}$$

$$= \{t = 0, \; |y - 1| \le 1\}$$

$$M = \max_{(x,y)\in R}\{f(x,y)\} = \max_{(t,y)\in R}\{y\cos t\} = \max_{(t,y)\in R}\{y\} = 2$$

$\therefore h = \min\{0, \frac{1}{2}\} = 0$

$\therefore$ There is a unique soln.

Modified Euler:

$f(x_{n+1}, y_n + hf(x_n, y_n))$

$= (y_n + hf(x_n, y_n))\cos(x_{n+1})$

t	y
0	1
0.2	1.218
0.4	1.471
0.6	1.756
0.8	2.097
1	2.504

$\frac{dy}{dt} = y\cos t$

$\int \frac{1}{y}\,dy = \int \cos t\,dt$

$\ln|y| = \sin t + c$

$c = \ln|1| - \sin 0 = 0$

$\therefore \ln|y| = \sin t$

$y = e^{\sin t}$

b) $y' = \frac{2}{t}y + t^2e^t, \quad 1 \le t \le 2, \quad y(1) = 0$

$R = \{|t - t_0| \le a, \ |y - y_0| \le b\}$

$= \{|t-1| \le 1, \ |y-0| \le 2\}$

$= \{0 \le t \le 2, \ -2 \le y \le 2\}$

$y_{n+1} = y_n + hf(x_n, y_n)$

t	y
1	0
1.2	0.5437
1.4	1.681
1.6	3.751
1.8	7.225
2	12.75

c) $f(x,y) = t^2e^t - \frac{2}{t}y$ is continuous on $[1,2]$. Hence, existence.

$\frac{\delta f(x,y)}{\delta y} = -\frac{2}{t}$ is continuous on $t \in [1,2]$. Unique soln.

x	y	$f(x,y)$
1	$\sqrt{2e} \approx 2.332$	-1.945
1.25	1.845	2.501
1.5	2.471	6.190
1.75	4.168	12.860
2	7.383	–

d) $f(t,y) = \frac{4t^3y}{1+t^4}$ is continuous on $t \in [0,1]$,

$\frac{\delta f(t,y)}{\delta y} = \frac{4t^3}{1+t^4}$ is continuous on $t \in [0,1]$

t	y	$f(t,y)$
0	1	0
0.25	1	0.06226
0.5	1.016	0.4780
0.75	1.135	1.455
1	1.449	–

$\frac{dy}{dt} = \frac{4t^3y}{1+t^4}$

$\int \frac{1}{y}\,dy = \int \frac{4t^3}{1+t^4}\,dt$

$\ln|y| = \ln|1+t^4| + c$

$c = \ln|1| - \ln|1| = 0 \quad \therefore y = 1+t^4$

2) a) $f(t,y) = \frac{2-2ty}{t^2+1}$ is continuous

$\frac{\delta f(t,y)}{\delta y} = \frac{\delta}{\delta y}\left(\frac{2}{t^2-1}\right) - \frac{\delta}{\delta y}\left(\frac{2t}{t^2+1}\cdot y\right) = \frac{2t}{t^2+1}$ is also continuous.

t	y	$f(t,y)$
1	2	-1
1.25	1.75	-0.9268
1.5	1.568	-0.7861
1.75	1.322	-0.6464
2	1.160	✓

2b) $\frac{\delta f}{\delta y} = \frac{2y+1}{t}$

t	y	$f(t,y)$
1	-2	2
1.25	-1.5	0.6
1.5	-1.35	0.315
1.75	-1.271	0.1970
2	-1.222	0.1356
2.25	-1.188	0.09931
2.5	-1.163	0.07596
2.75	-1.144	0.06003
3	-1.129	✓

2c) $f(t,y) = t^2 - y$ ✓

$\frac{\delta f}{\delta y} = -1$ ✓

3)a) $y' = te^{3t} - 2y$

t	y	$f(t,y)$
0	0	0
0.5	0	2.241
1	1.120	✓

b) $y' = 1 + (t-y)^2$

t	y	$f(t,y)$
2	1	2
2.5	2	1.25
3	2.625	✓

c) $y' = 1 + y/t$

t	y	$f(t,y)$
1	2	3
1.25	2.75	3.2
1.5	3.55	3.367
1.75	4.392	3.510
2	5.269	—

d) $y' = \cos 2t + \sin 3t$

t	y	$f(t,y)$
0	1	1
0.25	1.25	1.560
0.5	1.640	1.538
0.75	2.024	0.8488
1	2.236	—

4) a) $y' = \dfrac{y^2 + y}{t}$

t	y
1	-2
1.5	-1
2	-1
2.5	-1
3	-1

b) $y' = \sin t + e^{-t}$

t	y
0	0
0.25	0.25
0.5	0.5066
0.75	0.7780
1	1.067

c) $y' = (t + 2t^3)y^3 - ty$

t	y
0	1
0.2	1
0.4	1.003
0.6	1.030
0.8	1.131
1	1.478
1.2	3.121
1.4	30.69
1.6	39862
1.8	1.240×10^{14}
2	5.140×10^{42}

5) a)

t	y	real	t	y	real	t	y	real
1	-1	-1	1.4	-0.6995	-0.7143	1.8	-0.5309	-0.5556
1.05	-0.95	-0.9524	1.45	-0.6735	-0.6897	1.85	-0.5148	-0.5405
1.1	-0.9045	-0.9091	1.5	-0.6491	-0.6667	1.9	-0.4996	-0.5263
1.15	-0.8630	-0.8646	1.55	-0.6264	-0.6452	1.95	-0.4850	-0.5128
1.2	-0.8249	-0.8333	1.6	-0.6049	-0.625	2	-0.4712	-0.5
1.25	-0.7898	-0.8	1.65	-0.5848	-0.6061			
1.3	-0.7574	-0.7692	1.7	-0.5658	-0.5882			
1.35	-0.7274	-0.7407	1.75	-0.5479	-0.5714			

b) $y(1.052) = y(1.05) + 0.002\left(\frac{y(1.1) - y(1.05)}{0.05}\right) = -0.9482$

$y(1.555) = y(1.55) + \frac{\cancel{0.005}\ 1}{\cancel{0.05}\ 10}\left(y(1.6) - y(1.55)\right) = -0.6243$

$y(1.978) = y(1.95) + \frac{0.028}{0.05}\left(y(2) - y(1.95)\right) = -0.4773$

real values:

$y(1.052) = -0.9506$ $y(1.555) = -0.6431$ $y(1.978) = -0.5056$

6) Modified Euler:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))\right]$$

a) $y' = \frac{y}{t} - \left(\frac{y}{t}\right)^2 = \frac{y}{t}\left(1 - \frac{y}{t}\right)$

t	y	$f(t_n, y_n)$	$f(t_{n+1}, y_n + hf(x_n, y_n))$	Real
1	1	0	0.08264	1
1.1	1.004	0.07956	0.1343	1.004
1.2	1.015	0.1306	0.1685	1.015
1.3				
1.4				
1.5				
1.6				
1.7				
1.8				
1.9				
2				

7)

$$\frac{dy}{dt} = 1 + t\sin(ty) \Rightarrow f = 1 + t\sin ty \qquad y(0) = 0$$

$$\frac{d^2y}{dt^2} = \frac{\delta f}{\delta t} + \frac{\delta f}{\delta y}\cdot\frac{dy}{dt}$$

$$= t\left(\frac{\delta}{\delta t}(\sin ty)\right) + \sin ty + t^2\cos ty \cdot t\sin ty$$

$$= ty\cos ty + \sin ty + t^3\cos ty\sin ty$$

$$\therefore y(x_1) = \sum_{i=0}^{2} \frac{h^i}{i!} y^{(i)}(x_0)$$

$$= \overset{1}{\cancel{\frac{0.1^0}{0!}}} y^{(0)}(x_0) + \overset{0.1}{\cancel{\frac{0.1^1}{1!}}} y^{(1)}(x_0) + \overset{0.005}{\cancel{\frac{0.1^2}{2!}}} y^{(2)}(x_0)$$

$y'(x_0) = 1 + t\sin ty$

$= 1 + 0\sin 0.0$

$= 1$

$y''(x_0) = 0$

$$= \overset{0}{\cancel{y(x_0)}} + 0.1\,\overset{1}{\cancel{y'(x_0)}} + 0.005\,\overset{0}{\cancel{y''(x_0)}}$$

$$= 0.1$$

General: $y_j = y_{j-1} + 0.1\,y'_{j-1} + 0.005\,y''_{j-1}$

t	y	t	y	t	y	t	y
0	0	0.6	0.6285	1.2	1.654	1.8	2.240
0.1	0.1	0.7	0.7545	1.3	1.862	1.9	2.197
0.2	0.2002	0.8	0.8954	1.4	2.036	2	2.134
0.3	0.3014	0.9	1.0551	1.5	2.160		
0.4	0.4050	1	1.237	1.6	2.230		
0.5	0.5131	1.1	1.440	1.7	2.254		

8) a) $\dfrac{dy}{dt} = 2yt^{-1} + t^2e^t \qquad f = 2yt^{-1} + t^2e^t$

$$\frac{d^2y}{dt^2} = \frac{\delta f}{\delta t} + \frac{\delta f}{\delta y}\cdot f$$

$$= -2yt^{-2} + t^2e^t + e^t(2t) + \frac{2}{t}\left(\frac{2y}{t} + t^2e^t\right)$$

$$= \frac{-2y}{t^2} + t^2e^t + 2te^t + \frac{4y^2}{t^2} + 2te^t$$

$$= \frac{4y^2 - 2y}{t^2} + 4te^t + t^2e^t$$

t	y	real
1	0	0
1.1	0.3398	0.3459
1.2	0.8468	0.8666
1.3	1.570	1.607
1.4	2.570	2.620

t	y	Real
1.5	3.929	3.968
1.6	5.749	5.721
1.7	8.169	7.964
1.8	11.38	10.79
1.9	15.64	14.32
2	21.33	18.68

b) Real

$y(1.04) = \cancel{y(1)} + \dfrac{\cancel{0.04}}{\cancel{0.1}}\left(y(1.1) - \cancel{y(1)}\right) \simeq 0.1359$ 0.1200

(0, 0.4, 0)

$y(1.55) = y(1.5) + \dfrac{0.05}{0.1}\left(y(1.6) - y(1.5)\right) \simeq 4.839$ 4.789

$y(1.97) = y(1.9) + \dfrac{0.07}{0.1}\left(y(2) - y(1.9)\right) \simeq 19.62$ 17.28

9) a)

x	y	k_1	k_2	k_3	k_4	real
1	5	-2.4	-1.64	-1.868	-1.199	5.889
1.2	3.231	-1.259	-0.8410	-0.9662	-0.5988	
1.4	2.319	-0.6314	-0.4020	-0.4708	-0.2690	
1.6	1.878	-0.2868	-0.1608	-0.1986	-0.08766	
1.8	1.696	-0.09747	-0.02823	-0.04980	0.01193	
2	1.656					

$k_1 = hf(x_n, y_n)$

$k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$

$k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$

$k_4 = hf(x_n + h, y_n + k_3)$

$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

14) b) $y_0(t) = 1$

$f(t, y_0(t)) = f(t, 1) = -1 + t + 1 = t$

$\therefore y_1(t) = 1 + \int_0^t t\,dt = 1 + t^2/2$

$f(t, y_1(t)) = f(t, 1 + t^2/2) = -1 - t^2/2 + t + 1 = t - t^2/2$

$y_2(t) = 1 + \int_0^t (t - t^2/2)\,dt = 1 + \frac{t^2}{2} - \frac{t^3}{6}$

$f(t, y_2(t)) = -1 - \frac{t^2}{2} + \frac{t^3}{6} + t + 1 = \frac{t^3}{6} - \frac{t^2}{2} + t$

$y_3(t) = 1 + \int_0^t \left(\frac{t^3}{6} - \frac{t^2}{2} + t\right)dt = 1 + \frac{t^4}{24} - \frac{t^3}{6} + \frac{t^2}{2}$

c) $y(t) \simeq \frac{y(0) \cdot t}{0!} + \frac{y'(0) \cdot t^1}{1!} + \frac{y''(0) \cdot t^2}{2!} + \frac{y^{(3)}(0) \cdot t^3}{3!}$

$+ \frac{y^{(4)}(0) \cdot t^4}{4!} + \dots$

$y(0) = 0 + e^0 = 1$	$y'(0) = 1 - 1 = 0$	$y''(0) = 1$	$y^{(3)}(0) = -1$
$y'(t) = 1 - e^{-t}$	$y''(t) = e^{-t}$	$y^{(3)}(t) = -e^{-t}$	$y^{(4)}(t) = e^{-t}$
			$y^{(4)}(0) = 1$

$\therefore\ y(t) \approx t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24}$

15) a) $v' = -g - \frac{kv|v|}{m}$

$v' = -9.8 - \frac{0.002}{0.11} v|v|$

$v_{n+1} = v_n + hf(t_n, v_n)$

t	v
0	8
0.1	6.904
0.2	5.837
0.3	4.795
0.4	3.773
0.5	2.767
0.6	1.773
0.7	0.7877
0.8	-0.1934
0.9	-1.173
1	-2.151

b) The max. height is reached when $v = 0$. We know $v = 0$ between 0.7 & 0.8 s. Let us try linear interpolation:

$= 0.7 + \frac{0 - 0.787}{-0.194 - 0.787}(0.1) = 0.78\,s = 0.8\,s$ (round to 0.1 s)

18) a)

x	$f(x) = \sin x$	$f'(x)$		$g(x) = \sqrt{1 + (f'(x))^2}$
0	0	3-pt forward:	1.049	1.449
$\pi/8$	0.3827	2-pt central:	0.9003	1.346
$\pi/4$	0.7071	"	0.6891	1.214
$3\pi/8$	0.9239	"	0.3729	1.067
$\pi/2$	1	"	0	1
$5\pi/8$	0.9239	"	-0.3729	1.067
$3\pi/4$	0.7071	"	-0.6891	1.214
$7\pi/8$	0.3827	"	-0.9003	1.346
π	0	3-pt backward	-1.049	1.449

b) $L \approx \frac{\pi/8}{3}\left(g(0) + g(\pi) + 2\left(g\left(\frac{\pi}{8}\right) + g\left(\frac{3\pi}{8}\right) + g\left(\frac{5\pi}{8}\right) + g\left(\frac{7\pi}{8}\right)\right) + 4\left(g\left(\frac{\pi}{4}\right) + g\left(\frac{\pi}{2}\right) + g\left(\frac{3\pi}{4}\right)\right)\right)$

$= \frac{7}{24}(26.26)$

$= 3.438$

Error $= |3.8202 - 3.438| = 0.3825$

NM- Sample Final

Part - I 1) d) 2) c) 3) b) 4) a)

Part - II 1) TRUE.

2) FALSE.

$f(x) = 2x + 3y$ is an increasing function, hence

$\max |f(x, y_1) - f(x, y_2)| = |f(1, \infty) - f(1, -\infty)| =$

Because no L exists s.t. $|f(1, \infty) - f(1, -\infty)| \leq L|y_2 - y_1|$, the Lipshit condition is false

3) TRUE.

The error is of order $f^{(4)}(\mu)$, which shall be 0 for all polynomials with degree ≤ 3

4) TRUE.

When we obtain the Vandermonde-like matrix, i.e. $A^T A$, it will be full-rank iff A has full column rank.

E.g. Let $A \in \mathbb{R}^{m \times n}$, $m >> n$, $rank(A) = n$

Then, $A^T A \in \mathbb{R}^{n \times n}$, $rank(A^T A) = \min(rank(A^T), rank(A^T)) = n$

$\therefore$ Because $\underset{\sim}{A}^T \underset{\sim}{A}$ shall be a $n \times n$ matrix, with rank $= n$, it will be full rank. This is only possible if $\underset{\sim}{A}$ is full column rank.

Part III

1) $$\int_0^2 (1+x^2)\, dx = \frac{\frac{2-0}{2}}{3} \left(f(0) + 4f(1) + f(2) \right)$$
$$= \tfrac{1}{3} (1 + 8 + 5)$$
$$= 14/3$$

2) $f(x) = e^{-x} - x$ $\qquad x_0 = 0$

$f'(x) = -e^{-x} - 1$ $\qquad x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$
$$= 0 - \frac{e^0 - 0}{-e^0 - 1}$$
$$= -\frac{1}{-2}$$
$$= 1/2$$

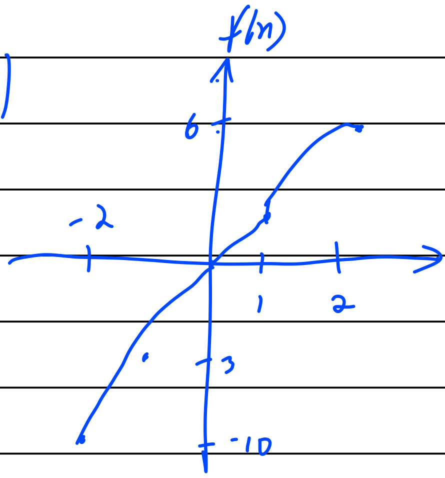
$\therefore$ Error $= |x_1 - x^*| = |1/2 - 0.567143| = 0.067143$

3) $$f(x) \simeq \sum_{n=0}^{\infty} \frac{f^{(n)}(a)\,(x-a)^n}{n!}$$

$$\therefore e^x \simeq \frac{f(1)}{0!} + \frac{f'(1)\cdot(x-1)}{1!} + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!}$$
$$\simeq e + e(x-1) + \frac{e(x-1)^2}{2} + \frac{e(x-1)^3}{6}$$
$$\simeq e\left[x + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} \right]$$

$$\therefore e^{1.5} \simeq e\left(1.5 + \frac{0.5^2}{2} + \frac{0.5^3}{6}\right)$$
$$\simeq e\left(\frac{79}{48}\right)$$
$$\simeq 4.4738$$

4) $f(x) = \begin{cases} x(1-2x) & -2 \le x < 0 \\ x & 0 \le x \le 1 \\ x(2x-1) & 1 < x \le 2 \end{cases}$ $\qquad x(x+1-x)$



$\max |f(x_1) - f(x_2)| = |f(2) - f(-2)|$

$= |6 - (-10)|$

$= 16$

$\therefore L = \frac{16}{4} = 4$

Part IV 1) a) $P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{3x^2 - 1}{2}$

$a_i \int_a^b w(x) (P_i(x))^2 dx = \int_a^b w(x) y(x) (P_i(x)) dx$

$\alpha \int_{-1}^{1} (1)^2 dx = \int_{-1}^{1} (x^3 + x^2 + x)(1) dx$

$2\alpha = \left[\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{1}$

$\alpha = \frac{1}{2}\left[\frac{1}{4} + \frac{1}{3} + \frac{1}{2} - \left(\frac{1}{4} - \frac{1}{3} + \frac{1}{2}\right)\right]$

$= \frac{1}{2}\left(\frac{2}{3}\right)$

$= \frac{1}{3}$

$\beta \int_{-1}^{1} x^2 dx = \int_{-1}^{1} (x^3 + x^2 + x)(x) dx$

$\beta \left(\frac{x^2}{3}\right)_{-1}^{1} = \left(\frac{x^5}{5} + \frac{x^4}{4} + \frac{x^3}{3}\right)_{-1}^{1}$

$\frac{2}{3}\beta = \frac{1}{5} + \frac{1}{4} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{3}$

$\frac{2}{3}\beta = \frac{2}{5} + \frac{2}{3}$

$\beta = 3\left(\frac{1}{5} + \frac{1}{3}\right) = 1.6$

$\gamma \int_{-1}^{1} \left(\frac{3x^2-1}{2}\right)^2 dx = \int (x^3+x^2+x)\left(\frac{3x^2-1}{2}\right) dx$

$\frac{\gamma}{4} \int_{-1}^{1} (9x^4 - 6x^2 + 1) dx = \frac{1}{2} \int_{-1}^{1} (3x^5 + 3x^4 + 3x^3 - x^3 - x^2 - x) dx$

$\frac{\gamma}{2} \left[\frac{9x^5}{5} - 2x^3 + x\right]_{-1}^{1} = \int_{-1}^{1} (3x^5 + 3x^4 + 2x^3 - x^2 - x) dx$

$\frac{\gamma}{2} \left(\left(\frac{9}{5} - 2 + 1\right) - \left(-\frac{9}{5} + 2 - 1\right)\right) = \left[\frac{x^6}{2} + \frac{3x^5}{5} + \frac{x^4}{2} - \frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{1}$

$\frac{\gamma}{2} \left(\frac{9}{5} - 1 + \frac{9}{5} - 1\right) = \left(\frac{1}{2} + \frac{3}{5} + \cancel{\frac{1}{2}} - \frac{1}{3} - \cancel{\frac{1}{2}}\right) - \left(\frac{1}{2} - \frac{3}{5} + \cancel{\frac{1}{2}} + \frac{1}{3} - \cancel{\frac{1}{2}}\right)$

$\gamma\left(\frac{4}{5}\right) = \cancel{\frac{1}{2}} + \frac{3}{5} - \frac{1}{3} - \cancel{\frac{1}{2}} + \frac{3}{5} - \frac{1}{3}$

$\gamma = \frac{5}{4}\left(\frac{6}{5} - \frac{2}{3}\right)$

$= \frac{\cancel{5}^{1}(18-10)}{4(\cancel{15})3}$

$= \frac{8}{12}$

$= \frac{2}{3}$

b) Method of undetermined coefficients:

Let $f(x) = 1$

$$\therefore \int_0^h 1\,dx = h(a+b+c)$$

$$h = h(a+b+c)$$

$$a+b+c = 1 \quad (1)$$

Let $f(x) = x$

$$\int_0^h x\,dx = h\left(a(0)+b\left(\frac{h}{3}\right)+c\cdot h\right)$$

$$\frac{\cancel{h^2}}{2} = \cancel{h}\left(\frac{b\cancel{h}}{3}+c\cdot\cancel{h}\right)$$

$$\frac{b}{3}+c = \frac{1}{2} \quad (2)$$

Let $f(x) = x^2$

$$\int_0^h x^2\,dx = h\left(a(0)+b\left(\frac{h^2}{9}\right)+ch^2\right)$$

$$\frac{\cancel{h^3}}{3} = \cancel{h^3}\left(\frac{b}{9}+c\right)$$

$$\frac{b}{9}+c = \frac{1}{3} \quad (3)$$

$(2)-(3)$ $$\frac{b}{3}+\cancel{c}-\frac{b}{9}-\cancel{c} = \frac{1}{2}-\frac{1}{3}$$

$$\frac{2b}{9} = \frac{1}{6}$$

$$b = \frac{9}{12} = \frac{3}{4}$$

$$\therefore c = \frac{1}{2}-\frac{b}{3} = \frac{1}{2}-\frac{1}{4} = \frac{1}{4}$$

$$\therefore a = 1-\frac{3}{4}-\frac{1}{4} = 0 \qquad \therefore a=0,\ b=\frac{3}{4},\ c=\frac{1}{4}$$

$$\therefore \int_0^h f(x)\,dx = h\left(\frac{3}{4}f\left(\frac{h}{3}\right)+\frac{1}{4}f(h)\right) \quad \forall\ f(x) \text{ of degree } \leq 2$$

2) $$\int_1^3 \frac{1}{x}\,dx \simeq \frac{\frac{3-1}{4}}{3}\left(f(1)+4f(1.5)+2f(2)+4f(2.5)+f(3)\right)$$

$$\simeq \frac{1}{6}\left(1+4\left(\frac{2}{3}\right)+\frac{2}{2}+4\left(\frac{2}{5}\right)+\frac{1}{3}\right)$$

$$\simeq \frac{1}{6}\left(1+\frac{8}{3}+1+\frac{8}{5}+\frac{1}{3}\right)$$

$$\simeq \frac{1}{6}\left(2+\frac{40+24+5}{15}\right)$$

$$= \frac{1}{6}\left(2+\frac{23}{5}\right)$$

$$= \frac{33}{30}$$

$$\simeq 1.1$$