# DS 288: NUMERICAL METHODS

Sep - 2 - 2021

## Ch #3 INTERPOLATION METHODS

- DOWNLOAD HANDOUTS: 4 - 7
  - STUDY QUESTIONS: Set 1 & 2
  - MODEL MIDTERM POSTED
  - POLYNOMIAL INTERPOLATION.

TWO USE CASES:

- APPROXIMATE BETWEEN "TABULAR" VALUES (DISCRETE POINTS & MISSING VALUES IN BETWEEN)
- APPROXIMATE COMPLEX FUNCTION WITH SIMPLER ONES

↳ QUICK EXAMPLE:

SECANT METHOD (APPROXIMATED THE $f'$)

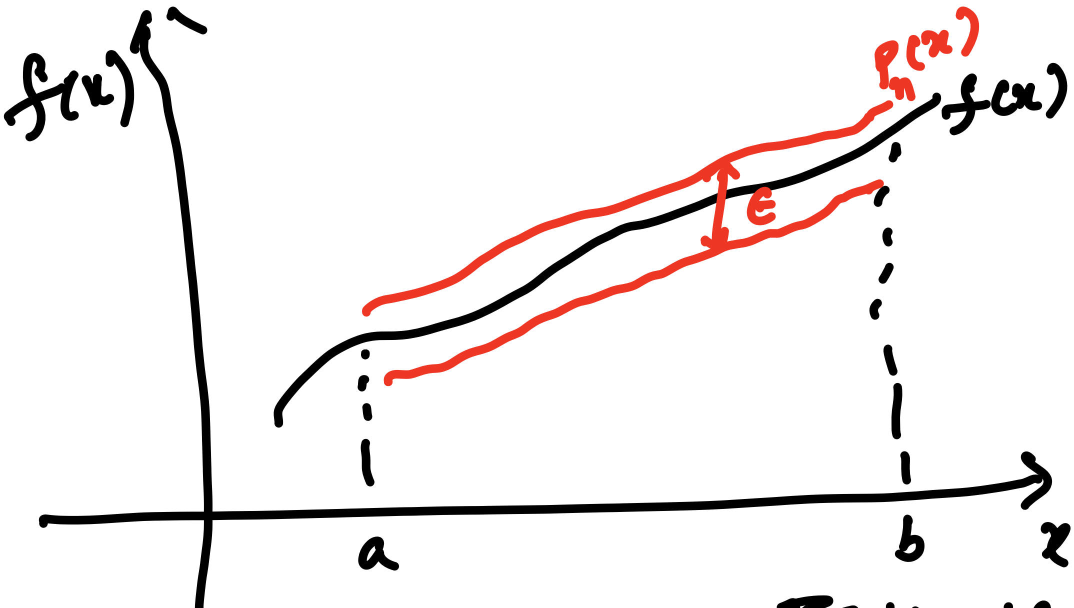
⌈Ch#2 : MULLER'S METHOD:
APPROXIMATE $f(x)$ WITH A
QUADRATIC POLYNOMIAL⌋

- WHY POLYNOMIALS?
  - EASY TO EVALUATE
  - EASY TO INTEGRATE / DIFFERENTIE

* APPROXIMATION PRINCIPLE IS BASED ON WEIERSTRASS APPROXIMATION THEOREM (3.1 IN TEXT)

If $f$ IS CONTINUOUS ON $[a,b]$ & $\epsilon > 0$, THEN A POLYNOMIAL $P$ EXISTS ON $[a,b]$ ∋

$$|f(x) - P| < \epsilon$$




HOWEVER, DOES NOT TELL US HOW TO FIND $P_n(x)$ OR EVEN THE ORDER (n) OF $P_n(x)$

- ONE TYPE OF POLYNOMIAL IS A TAYLOR POLYNOMIAL

(i) GREAT FOR ANALYSIS SINCE TRUNCATION ERROR IS EXPLICIT

$$\epsilon \cong \frac{f^{n+1}(\xi)\,(x-x_0)^{n+1}}{(n+1)!} \qquad \xi \in [x, x_0]$$

(ii) ALL INFORMATION ABOUT $f(x)$ IS CONCENTRATED AROUND $x_0$ (ALL DERIVATIVES ARE EVALUATED AT $x_0$)

(iii) LOCALLY VERY GOOD, BUT GETS WORSE AS $|x - x_0|$ GROWS ($\epsilon \sim (x - x_0)^{n+1}$)

* TRUNCATION ERROR $= 0$ WHEN $x = x_0$ (FUNCTION MATCHES ONLY AT $x_0$)

- A BETTER POLYNOMIAL (FOR INTERPOLATION) $P_n(x)$ MATCHES $f(x)$ AT A FINITE NUMBER OF VALUES.

* $N^{TH}$ ORDER POLYNOMIAL

$$P_N(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

THE NO. OF UNKNOWN'S ($a_i$'s)

$N+1$

* IF $f(x_i) = P_N(x_i)$ THEN WE NEED $N+1$ $x_i$'s TO FIND $a_i$'s ($N+1$ EQUATIONS & $N+1$ UNKWOWNS)

Ex :- $\underbrace{P_2(x)}_{\text{QUDRATIC.}} = a_2 x^2 + a_1 x + a_0$

$x_1, x_2, x_3$ AND $f(x_1), f(x_2), f(x_3)$

WANT $P_2(x_i) = f(x_i)$

$$a_2 x_1^2 + a_1 x_1 + a_0 = f(x_1)$$
$$a_2 x_2^2 + a_1 x_2 + a_0 = f(x_2)$$
$$a_2 x_3^2 + a_1 x_3 + a_0 = f(x_3)$$

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix} \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix}$$

UNKNOWN

$$\underline{\underline{X}} \cdot \underline{a} = \underline{f}$$

KNOWN

$$\underline{a} = \underline{\underline{X}}^{-1} \underline{f}$$

* CAN GET A SOLUTION $\underline{a}$

* SOLVING 'n' LINER SYSTEM OF EQUATIONS REQUIRES $O(n^3)$ OPERATIONS

* GIVEN $n+1$ DISTINCT SAMPLES OF $f(x)$ (along with $x_i$'s), ONE CAN ALWAYS FIND A UNIQUE POLYNOMIAL OF ORDER 'n' SUCH THAT $P_n(x_i) = f(x_i)$

- LINEAR SYSTEM OF EQUATIONS
- TEDIOUS. [$O(n^3)$ OPERATIONS]

WE WANT A QUICKER SOLUTION TO FIND $a_i$'s.

SOLUTION: USE A BASIS POLYNOMIAL

$$P_n(x) = \sum_{j=0}^{n} \alpha_j b_j(x)$$

WHERE $b_j(x) \equiv$ BASIS POLYNOMIAL (A KNOWN FUNCTION OF $x$)

*EX:- $b_j(x) = x^j$ THEN $\alpha_j = a_j$*

BY CLEVERALLY CHOOSING $\{b_j(x)\}$ WE CAN IDENTIFY $\{\alpha_j\}$ easily.

If $b_j(x_i) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

KRONECKER DELTA.

$$p_N(x_i) = \sum_{j=0}^{N} \alpha_j \delta_{ij} = \alpha_i$$

$$\parallel$$

$$f(x_i) = \alpha_i \quad \text{(KNOWN VALUE)}$$

$$\boxed{p_N(x) = \sum_{j=0}^{N} f(x_j)\, b_j(x)}$$

LEADS TO NUMBER OF POSSIBILITIES OF CHOOSING $b_j(x)$ WITH THE PROPERTY $b_j(x_i) = \delta_{ij}$

[REQUIRES $O(N)$ OPERATIONS]

ONE EXAMPLE :-

LAGRANGE POLYNOMIALS

$$L_{N,j}(x) = \prod_{\substack{k=0 \\ k \neq j}}^{N} \frac{(x - x_k)}{(x_j - x_k)}$$

ORDER CENTER

$$= \underbrace{\frac{x-x_0}{x_j-x_0} \cdot \frac{x-x_1}{x_j-x_1} \cdots \frac{x-x_N}{x_j-x_N}}_{N \text{ TERMS}}$$

MISSING $K=j$ TERM

$\Rightarrow N^{TH}$ ORDER POLYNOMIAL

If $i=j$ THEN ALL TERMS ARE UNITY

$(x=x_j)$

$$L_{N,j}(x_j)=1$$

If $i \neq j$ THEN ALL FACTORS BUT ONE ARE NON-ZERO, ONE NUMERATOR IS ZERO $= 0$.

$$L_{N,j}(x_k) = 0 \quad j \neq k.$$

$$\boxed{L_{N,j}(x_i) = \delta_{ij}}$$

RECALL $$P_N(x) = \sum_{j=0}^{N}{}' f(x_i)\, b_j(x)$$

$$b_j(x) \to L_{N,j}(x)$$

THERE ARE N+1 $b_j$'s FOR AN $N^{TH}$ ORDER POLYNOMIAL

<u>Ex</u> :- $$P_3(x) = \sum_{j=0}^{3}{}' f(x_j)\, L_{3,j}(x)$$

$$P_3(x) = f(x_0)\, L_{3,0}(x) + f(x_1)\, L_{3,1}(x) + f(x_2)\, L_{3,2}(x) + f(x_3)\, L_{3,3}(x)$$

$$L_{3,0}(x) = \underbrace{\frac{x-x_1}{x_0-x_1} \cdot \frac{x-x_2}{x_0-x_2} \cdot \frac{x-x_3}{x_0-x_3}}_{\text{CUBIC POLYNOMIAL}}$$

$$L_{3,0}(x_0) = 1$$

$L_{3,0}(x_1) = 0$ $\quad L_{3,0}(x_2) = 0$

$L_{3,0}(x_3) = 0$

HANDOUT ON

LAGRANGE POLYNOMIAL

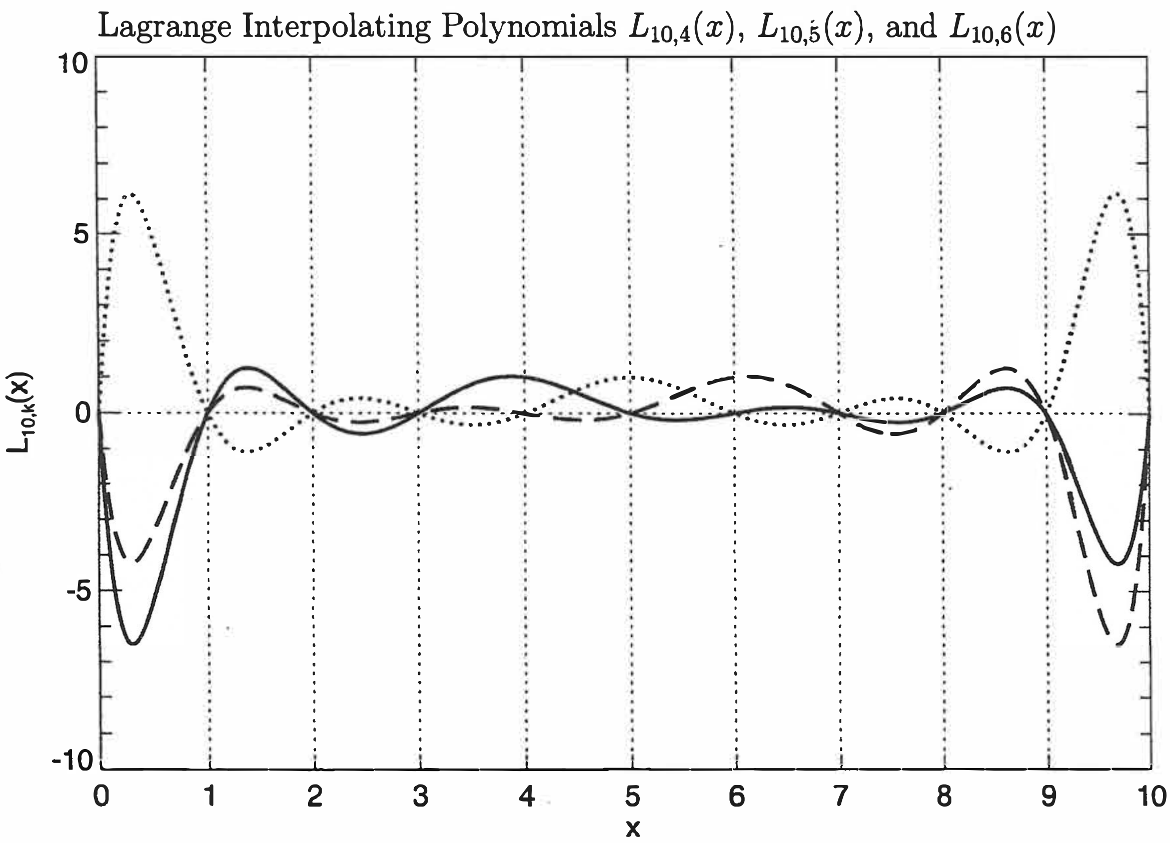
# Lagrange Interpolating Polynomials

Definition:

$$\boxed{L_{N,k}(x) = \prod_{\substack{i=0 \\ i \neq k}}^{N} \frac{(x - x_i)}{(x_k - x_i)}}$$

$N$ is the order of the polynomial
$k$ is the index of the center location of the polynomial (i.e., $x_k$)

Example:




Note the desirable behavior of Lagrange interpolating polynomials:

$$L_{N,k}(x_i) = \delta_{ik} \equiv \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

Error term:

$$\boxed{f(x) - P_N(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!} \prod_{i=0}^{N} (x - x_i) \qquad \xi \in [x_0, x_N]}$$

Courtesy: Prof. Simon Shepard

* EASY TO WRITE

BUT INEFFICIENT TO USE

$$P_N(x) = \sum_{j=0}^{N} f(x_j)\, \underline{L_{N,j}(x)}$$

USING A WEIGHTED SUM OF

* $N+1$ $N^{TH}$ ORDER POLYNOMIALS *

$\Longrightarrow$ $N^{TH}$ ORDER POLYNOMIAL

ERROR TERM:

$$f(x) - P_N(x) = \frac{f^{N+1}(\xi)}{(N+1)!} \prod_{i=0}^{N} (x - x_i)$$

$\xi \in [x_0, x_N]$ [EXCERCISE]

COMPARE WITH TAYLOR POLYNOMIAL

$$\frac{f^{N+1}(\xi)}{(N+1)!} (x - x_0)^{N+1}$$

INFORMATION CONCENTRATED AT $x_0$

LAGRANGE: INTERPOLATION $\frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\dots(x-x_n)$

## CHEBYSHEV OPTIMAL POINTS

[[§8.3]]

- BASED ON ROOTS OF CHEBYSHEV POLYNOMIALS

* HANDOUT OF CHEBYSHEV POLYNOMIALS *

# Chebyshev Polynomials

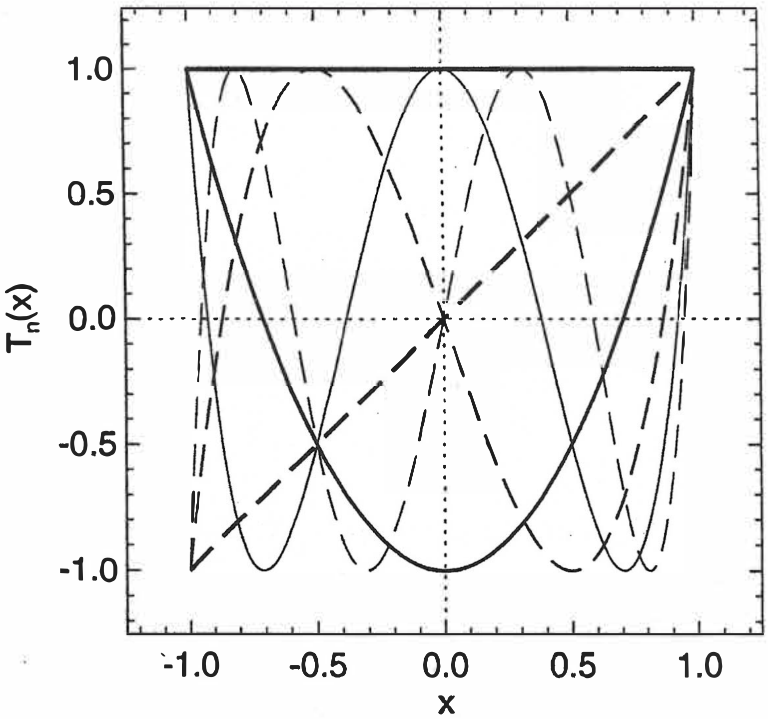
Definition:

$$\boxed{T_n(x) = \cos[n \arccos x] \quad n \geq 0}$$

Recursion relation:

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 12x^3 + 5x \\
&\vdots \quad \vdots \\
T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x) \quad n \geq 2
\end{aligned}$$

Chebyshev Polynomials $T_n(x)$ for $n = 0, 1, 2, 3, 4, 5$




Courtesy: Prof. Simon Shepard

MOTIVATION:

- TAYLOR AT $x_0$
- LAGRANGE $\{x_i\}$

DOES THERE EXIST A SET OF POINTS $\{x_i\}$ FOR WHICH THE "OPTIMAL" INTERPOLATION IS ACHIEVED FOR LAGRANGE POLYNOMIALS?

- OPTIMAL IS DEFINED AS MINIMIZING THE MAXIMUM ERROR ON $[a, b]$

$$f(x) - P_N(x) = \underbrace{\frac{f^{N+1}(\xi)}{(N+1)!}}_{\text{FIXED}} \underbrace{\prod_{i=0}^{N} (x - x_i)}_{\text{MINIMIZE}}$$

CAN PROVE (IN TEXT) A MINIMAL MAXIMUM EXISTS ON [-1, 1] IF WE USE THE ROOTS OF THE $N+1^{TH}$ ORDER CHEBYSHEV POLYNOMIAL FOR $\{x_i\}$

$$\bar{x}_i = \cos\left[\frac{(2i+1)\pi}{2(N+1)}\right]$$

$$i = 0, 1, \ldots N$$

FOR A GENERAL INTERVAL [a,b]

$$\boxed{x_i = \frac{a+b}{2} + \frac{b-a}{2}\bar{x}_i}$$

OPTIMAL SAMPLE POINTS FOR $f(x)$ ON [a, b]

* NOT ALWAYS POSSIBLE TO SAMPLE $f(x)$ AT THESE $\{x_i\}$'s [EXAMPLE IN TEXTBOOK]

# ITERATIVE INTERPOLATION (§3.1)

- PRODUCE HIGHER ORDER POLYNOMIALS FROM COMBINATIONS OF LOWER ORDER POLYNOMIALS.

- HOW DOES THIS WORK?

LOOK AT LAGRANGE POLYNOMIALS

$$P_N(x) = \sum_{j=0}^{N} f(x_j)\, b_j(x)$$

$$b_j(x) = L_{N,j}(x) = \prod_{\substack{i=0 \\ i \neq j}}^{N} \frac{(x - x_i)}{(x_j - x_i)}$$

LOOK AT SOME TERMS

$$P_0(x) = f(x_0)\, b_0(x) = f(x_0)$$

NOT ORDER OF POLYNOMIAL

1 TERM

A POINT AT WHICH $f(x_0) = P_0$

$$P_{0,1}(x) = f(x_0)\, b_0(x) + f(x_1)\, b_1(x)$$

$$= f(x_0) \frac{(x - x_1)}{x_0 - x_1} + f(x_1) \frac{(x - x_0)}{(x_1 - x_0)}$$

2 TERMS

$$= \frac{P_1(x)(x-x_0) - P_0(x)(x-x_1)}{(x_1-x_0)}$$

WHERE $P_1(x) = f(x_1)\, b_1(x) = f(x_1)$

NOT ORDER OF POLYNOMIAL

NEVILLE'S METHOD

— END OF CLASS —