DS 288: NUMERICAL METHODS

Sep-30-2021

NEXT CLASS: MIDTERM EXAM

* ONE PAGE (BOTH SIDES): A4 OF WRITTEN NOTES IS ALLOWED IN THE EXAM.
* WHILE TYPING IN THE EXAM:

REPRESENT $a^i \rightarrow$ a^i

$a_i \rightarrow$ a_i

$a_{i,j} \rightarrow$ a_(i,j)

$a^{i,j} \rightarrow$ a^(i,j)

Topics:

- Interpolation
- Lagrange Polynomials
- Neville's Method
- Chebyshev Optimal Points
- Hermite Polynomials
- Piecewise Interpolation
- Splines
- Least Squares

Sections in text: 3.1–3.5, 8.1–8.3

Study Questions–II [1]

1. Given 8 samples of $f(x)$ at distinct values of x, what is the order of the unique interpolating polynomial that exactly reproduces the data samples provided?.

   7TH ORDER $P_7(x) = a_7 x^7 + a_6 x^6 + \dots + a_1 x + a_0$

2. How many roots (or zeros) does $L_{N,k}(x)$ have? Sketch $L_{5,0}(x)$, $L_{5,2}(x)$, and $L_{5,4}(x)$. What would $L'_{5,0}(x)$ and $L'_{5,2}(x)$ look like?.

   NTH ORDER POLYNOMIAL → N ROOTS

3. What is Neville's method? What is Divided Differences? If the data in Neville Table is rearranged at the start, will all the subsequent tabular entries be the same? Which will remain the same and which will be different if starting data x_0 through x_4 is changed such that x_2 and x_3 are interchanged?.

4. What is the advantage of using a basis function expansion to represent $P_N(x)$, an n^{th} order polynomial? What is the disadvantage?

   + $P_N = \sum_{i=0}^{N} \alpha_i b_i(x)$ $b_i(x_j) = \delta_{ij}$ THEN $\alpha_i = f(x_i)$ / − NEED N+1 NTH ORDER POLYNOMIAL $Cb_i(x)$

5. What is the value of using Chebyshev sampling of $f(x)$ to construct an interpolant? In this situation where would the sample points be located? Is the resulting polynomial representable as a Lagrange polynomial basis function expansion?

   - MINIMIZING MAXIMUM ERROR - AT THE ROOTS OF AN NTH ORDER Chebyshev POLYNOMIAL • YES

6. What special feature does a Hermite polynomial possess? What order Hermite polynomial results if 5 data samples (i.e., x_i with $i = 0, 1, 2, 3,$ and 4) are used? What advantages and disadvantages result from using a Hermite interpolant?.

   - $f(x_i) = H_{2N+1}(x_i)$ & $f'(x_i) = H'_{2N+1}(x_i)$

7. What is the motivation for using piecewise polynomial interpolation? Can such an interpolant be constructed from Lagrange polynomials? Hermite polynomials? What would be the advantages and disadvantages of each approach? Give an example of how this would be done in each case?

8. What is the concept behind a spline? Identify the constraints that could be imposed using a *quadratic* spline. Do any problems arise? Does this suggest why cubic splines are popular?

   - Piece wise continuous fn & REQUIRES CONTINUOUS DERIVATIVE

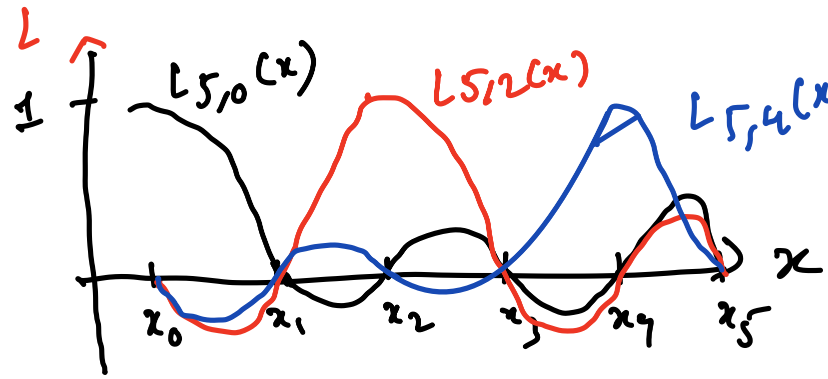
9. When would least squares curve fitting be appropriate? What is the difference between linear and nonlinear least squares? Give an example of a fit that results in a linear case and a nonlinear case.

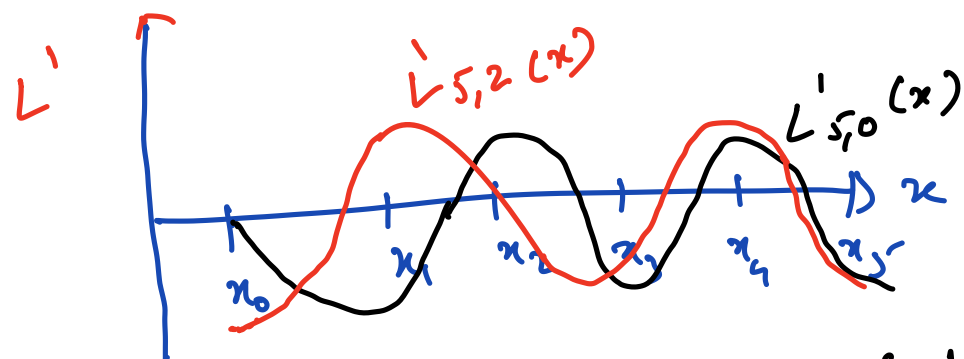
   * WHEN ERROR IS PRESENT IN DATA LINEAR: $y = ax + b$ NON-LINEAR: $y = be^{ax}$

10. What is the difference between discrete and continuous least squares? Why are orthogonal basis function expansions so useful in the continuous case?

[1] Posted on: September 1, 2021.

(ANS 2)





(ANS 3) NEVILLE'S METHOD: TEHNIQUE USED TO CONSTRUCT HIGHER ORDER LAGRANGE'S POLYNOMIAL FROM PAIRS OF LOWER ORDER.

DIVIDED DIFFERENCE: NOTATION THAT ALLOWS US TO WRITE DOWN COEFFICIENTS OF INTERPOLATING POLYNOMIAL.

NEVILLE'S METHOD.

$f(x_0) = P_0$				
	$P_{0,1}$			
$f(x_1) = P_1$		$P_{0,1,2}$		
	$P_{1,2}$		$P_{0,1,2,3}$	
$f(x_2) = P_2$		$P_{1,2,3}$		$P_{0,1,2,3,4}$
	$P_{2,3}$		$P_{1,2,3,4}$	ORDER: 4
$f(x_3) = P_3$		$P_{2,3,4}$		
	$P_{3,4}$			
$f(x_4) = P_4$				

INTERCHANGING x_2 & x_3

BOXED VALUES CHANGE.

P_0				
	$P_{0,1}$			
P_1		$\boxed{P_{0,1,3}}$		
	$\boxed{P_{1,3}}$		$P_{0,1,2,3}$	
$\boxed{P_3}$		$P_{1,2,3}$		$P_{0,1,2,3,4}$
	$P_{3,2}$		$P_{1,2,3,4}$	
$\boxed{P_2}$		$P_{2,3,4}$		
	$\boxed{P_{2,4}}$			
P_4				

LAGRANGE'S ERROR TERM

n^{th} ORDER POLYNOMIAL

$$\frac{f^{n+1}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

$$\xi \in [x_0, x_4]$$

(ANS 6) FUNCTION & DERIVATIVE MATCHES AT N+1 POINTS

$x_i, \ i = 0, 1, 2, \ldots 4 \quad N = 4$

$2N+1 \Rightarrow 9^{TH}$ ORDER HERMITE POLYNOMIAL

+ MATCHES VALUE & FIRST DERIVATIVE
⇒ MORE ACCURATE

− NEED TO KNOW $f'(x_i)$ VALUES ALSO.
[MAY NOT BE PRACTICAL OR POSSIBLE]

(ANS 7) ELIMANTES WIGGLES/OSCILLATIONS ESPECIALLY AT END POINTS CAUSED BY HIGHER ORDER POLYNOMIALS

* YES * YES

LAGRANGE'S : LINEAR (STRAIGHT LINE)
HERMITE'S : CUBIC (CUBIC-SPLINE)
→ +SIMPLE − NOT VERY ACCURATE
→ +ACCURATE − MUST KNOW $f'(x_i)$

(ANS 8) QUDRATIC SPLINE CONDITIONS

(i) $S_j(x_j) = f(x_j)$ $j=0,1,\dots N-1$

(ii) $S_j(x_{j+1}) = f(x_{j+1})$ $j=0,1,\dots N-1$

(iii) $S_j'(x_{j+1}) = S_{j+1}'(x_{j+1})$ $j=0,1,\dots N-2$

NUMBER OF CONDITIONS: (i) $\rightarrow N$

(ii) $\rightarrow N$

(iii) $\rightarrow N-1$

$3N-1$ CONDITIONS

N SEGMENTS :- FOR EACH SEGMENTS: 3

TOTAL UNKNOWNS: $3N$

$\therefore$ LOPSIDED BY ONE BOUNDARY CONDITION.

[ONE SIDE B.C. CAN BE APPLIED]

(ANS 10) DISCRETE $E = \sum_{i=1}^{N} [y_i - P_N(x_i)]^2$

CONTINUOUS $E = \int_a^b w(x)\,[f(x) - P_N(x)]^2\,dx$

$P_N(x) = \sum_{i=0}^{N} a_i\,\phi_i(x)$

ORTHOGNAL $\Rightarrow a_i = \frac{1}{\alpha_i}\int_a^b w(x) f(x)\,\phi_i(x)\,dx$

Topics:

- Forward/Backward/Centered Differences
- Numerical Quadrature
- Composite Integration
- Newton-Cotes Open/Closed Formulas
- Romberg Integration
- Adaptive Quadrature
- Gauss Quadrature

Sections in text: 4.1–4.7 (except 4.2)

ONE ORDER 'h' LOWER THAN SINGLE APPLICATION

Study Questions–III [1]

1. Describe how to derive $f'''(x)$ to $O(h^3)$. How many function evaluations would be involved?. Could a centered difference formula be derived?.

fjH

2. How many more sample points are needed to increase a central difference formula from $O(h^m)$ to $O(h^{m+2})$?. Is this the same number that would be required for a backward difference formula?.

2 POINTS SAME FOR FORWARD/BACKWARD

3. How does one interpret the error term when using a *single* application of a given quadrature rule?. How about for a composite quadrature rule?.

-ORDER OF POLYNOMIAL THAT CAN BE EXACTLY INTEGRATED WITH RULE : $P_N^{(n+1)}(x) = 0$

4. What happens to the error term behavior when composite rule integration is used (i.e., relative to a single application of the quadrature rule)?

5. What problem arise when $h \to 0$ in difference formulas?. What is the best strategy to adopt to help to overcome these difficulties?.

$E_{ROUND} \to \infty \Rightarrow E_{TOTAL} \to \infty$; USE HIGHER ORDER ACCURACY FORMULA WITH LARGER h.

6. What is Romberg integration? and why is it useful?.

7. What is the rationale for adaptive quadrature?. What assumptions were made to develop the adaptive quadrature scheme studied in class?.

- UNIFORMLY DISTRIBUTE THE ERROR BASED FUNCTION ERROR

8. What is Gauss quadrature? Why is it more accurate than a quadrature rule with uniform sampling of the integrand with the same number of function evaluations?

-ALLOW SAMPLE POINTS TO BE FREE GIVING TWICE DEGREES OF FREEDOM. INTEGRATE $P_{2N-1}(x)$ EXACTLY

9. How are the sample points chosen in Gauss quadrature? Is it generally useful in any interval [a,b]?.

ROOTS OF N^{th} ORDER LEGENDRE POLYNOMIAL $L_N(x)$ $[-1,1] \to [a,b]$

10. How many error removals can be achieved in Romberg integration after 6 panel doublings?.

6 ERROR REMOVAL $O\left(\frac{h}{2^n}\right)^{2j+2} \Rightarrow O\left(\frac{h}{64}\right)^{14}$

11. What is a panel and how many are involved in Simpson's $\frac{3}{8}^{th}$ rule (single application)?

PANEL IS BASIC UNIT OF INTEGRATION : 3 PANELS (4 POINTS)

12. What are Newton-Cotes open formulas?. How do they differ from Newton-Cotes closed formulas?. In terms of the accuracy benchmark for a single application of these rules, is either more accurate than the other for same number of integrand evaluations?.

OPEN: DO NOT INCLUDE END PTS. CLOSED: DO INCLUDE END POINTS

13. Why is an optimal h theoretically possible to find for numerical differentiation? Is the same true for numerical integration?

DIFF: $E_{TRUNC} \to h^n$ $E_{ROUND}: \frac{1}{h}$; INT : $E_{ROUND} \sim \epsilon(b-a)$ CONSTANT

14. How do rounding errors behave in numerical differentiation relative to numerical integration?.

[1] Posted on: September 15, 2021.

(ANS 1) $f_{j+1} = f_j + h f_j' + \frac{h^2}{2} f_j'' + \frac{h^3}{3!} f_j''' + \frac{h^4}{4!} f^{4} + \frac{h^5}{5} f^{5} + \dots$

TAYLOR SERIES OF $f(x)$ ABOUT x_j

EVALUATED AT (x_{j+1})

KILL THESE TERMS

FORWARD/BACKWARD TAYLOR SERIES FOR EACH KILL (4) → FORWARD ONE BACKWARD

f''' TO $O(h^3) \to 6$ POINTS.

RECALL N^{TH} ORDER DERAVIATEVE TO $O(h^M)$

$N+M$ POINTS & $N+M-1$ TAYLOR SERIES

~~BA~~ $N=3$ 6 POINTS

$M=3$ 5 TAYLOR SERIES.

NO: CENTERED DIFFERENCE IS FOR EVEN M (h^M)

BEACAUSE $M=3$, IT IS NOT POSSIBLE

(ANS 6) METHOD WHICH ALLOWS MORE ACCURATE INTEGRATION BASED ON PREVIOUS VALUES

(USING LARGE PANEL SIZES)

(ANS 7) CONT... AVERAGE f(x) OVER TWO SEGMENTS (DERIVATIVES)

EQUALS VALUES OVER WHOLE SEGMENT.

(ANS 9) CONT..

$$y_i = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right) x_i$$

(ANS 12) BOTH INTEGRATE SAME ORDER POLYNOMIALS BUT OPEN FORMULAS SLIGHTLY MORE ACCURATE SINCE h IS SMALLER