

Root Finding

→ Regular Falsi: $x_{n+1} = b - f(b) \left(\frac{b-a}{f(b)-f(a)} \right)$

→ Secant: $x_{n+1} = x_n - f(x_n) \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \right)$

→ Modified Newton: $x_{n+1} = x_n - \frac{f(x_n) \cdot f'(x_n)}{(f'(x_n))^2 - f(x_n) \cdot f''(x_n)}$

→ Müller:

$$d_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad d_2 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad u = \frac{d_2 - d_1}{x_2 - x_1}$$

$$b = d_2 + u(x_2 - x_1) \quad x_3 = x_2 - \frac{2c}{b + \operatorname{sgn}(b) \sqrt{b^2 - 4ac}}$$

($c = f(x_2)$)

System of Non-linear equations

→ Fixed point method:

Rewrite eqns. as $x_1 = g_1(x) \dots x_n = g_n(x)$

Necessary condn: $\left| \frac{\partial g_i}{\partial x_1} \right| + \dots + \left| \frac{\partial g_i}{\partial x_n} \right| \leq 1, \forall 1 \leq i \leq n$

→ Newton's method: $x_{n+1}^{(i)} = x_n^{(i)} - J_{x_n^{(i)}}^{-1} \cdot f(x_n^{(i)})$, where $J_{x_n^{(i)}}$ is the Jacobian inverse evaluated at $x_n^{(i)}$.

Least Squares for function

→ System of equations: $\sum_{k=0}^n a_k \int_a^b x^{j+k} dx = \int_a^b x^j f(x) dx$

→ Orthogonal functions: (Alternative for least squares)

→ Weight function: An integrable function w is called a weight function on the interval I if $w(x) > 0 \forall x \in I$.

→ Used to assign varying degrees of importance to approximations on certain portions of the interval.

→ Two functions are orthogonal w.r.t w if:

$$\int_a^b w(x) \phi_k(x) \phi_j(x) dx = \begin{cases} 0, & j \neq k \\ \Delta_j > 0, & j = k \end{cases}$$

→ We can solve large least squares w/o matrix multiplication. We can reduce the system to:

$$a_i \int_a^b w(x) (f_i(x))^2 dx = \int_a^b w(x) y(x) f_i(x) dx, i=0, \dots, n$$

Numerical Differentiation & Integration

→ 3-point forward difference: $f'_i \approx \frac{-3f_i + 4f_{i+1} - f_{i+2}}{2h}$

→ "backward" : $f'_i \approx \frac{3f_i - 4f_{i-1} + f_{i-2}}{2h}$

→ Second derivative: $f''_i \approx \frac{f_{i+2} - 2f_{i+1} + f_i}{h^2} + O(h)$

→ Newton-Cotes formulae: Unequally spaced nodes

→ Open: exclude end points → Closed: include end points

→ Trapezoidal Rule: $\int_a^b f(x) dx \approx \frac{h}{2} (f(a) + f(b)) - \frac{h^3}{12} f''(\xi)$
(n=1)

→ Simpson's 1/3 Rule: $\int_{x_0}^{x_2} f(x) dx \approx \frac{h}{3} (f_0 + 4f_1 + f_2) - \frac{h^5}{90} f^{(4)}(\xi)$
(n=2)

→ Simpson's 3/8 Rule: $\int_{x_0}^{x_3} f(x) dx \approx \frac{3h}{8} (f_0 + 3f_1 + 3f_2 + f_3) - \frac{3h^5}{80} f^{(4)}(\xi)$
(n=3)

→ Boole's Rule: $\int_{x_0}^{x_4} f(x) dx \approx \frac{2h}{45} (7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4) + \frac{8h^7}{945} f^{(6)}(\xi)$
(n=4)

→ Open formulae:

→ Midpoint rule: $\int_x^{x_1} f(x) dx = 2hf_0 + \frac{h^3}{3} f''(\xi)$
(n=0)

→ Two-point open rule: $\int_{x_0}^{x_2} f(x) dx = \frac{3h}{2} (f_0 + f_1) + \frac{3h^3}{4} f''(\xi)$
(n=1)

→ Three-point open rule: $\int_{x_0}^{x_3} f(x) dx = \frac{4h}{3} (2f_0 + f_1 + 2f_2) + \frac{14h^5}{45} f^{(4)}(\xi)$

→ Four point open rule: $\int_{x_0}^{x_4} f(x) dx = \frac{5h}{24} (11f_0 + f_1 + f_2 + 11f_3) + \frac{95h^5}{144} f^{(4)}(\xi)$

→ Composite Trapezoidal: $\int_a^b f(x) dx = \frac{h}{2} (f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b)) + \frac{b-a}{12} h^2 f''(\xi)$

→ Composite Simpson's:

$$\int_a^b f(x) dx = \frac{h}{3} (f(a) + 2 \sum_{j=1}^{n/2-1} f(x_{2j}) + 4 \sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)) + \frac{b-a}{180} h^4 f^{(4)}(\xi)$$

→ Richardson's Extrapolation

Let $I_{true} = \int_a^b f(x) dx$, $I(h)$ be trapezium rule with h as length of interval, once
 $E(h)$ be error when using trapezium rule

We know: $I(h) + E(h) = I(h/2) + E(h/2) = I_{true}$

Let us take $c = \frac{-f''(h)}{12}$

$$I_{true} = I(h) + ch^3 \quad (1)$$

$$I_{true} = I(h/2) + 2c(h/2)^3 = I(h/2) + \frac{ch^3}{2^2} \quad (2)$$

Final: $I_{true} = \frac{2^{n-1} I(h/2) - I(h)}{2^{n-1} - 1} + \frac{ch^{n+1}}{2^{n+2} - 2}$ ← Error

→ Increases level of accuracy by 1.

→ Romberg Integration

→ Use Richardson's extrapolation multiple times

$$\text{i.e. } I(h, h/2) = I(h/2) + \frac{I(h/2) - I(h)}{3}$$

$$I(h/2, h/4) = I(h/4) + \frac{I(h/4) - I(h/2)}{3}$$

$$I(h, h/2, h/4) = I(h/2, h/4) - I(h, h/2)$$

Eventually, $I(h, \dots, h/2^n)$ will get more accurate.

→ Gaussian Quadrature

→ Rather than use points at equally spaced points, we choose the points in a optimal way.

→ Used to calculate: $\int_{-1}^1 f(x) dx$ ↖ only integral can be brought to this form

→ Applying method of undetermined coefficients (upto x^3), we get this to be the final result:

$$\int_{-1}^1 f(x) dx = f(-\sqrt{1/3}) + f(\sqrt{1/3})$$

ODEs - IVPs

→ If $f(x, y)$ & $\frac{df}{dx}$ are both continuous $R = \{ |x - x_0| \leq a, |y - y_0| \leq b \}$, the IVP will have unique soln. in $|x - x_0| \leq h$, where:

$$h = \min(a, b/M), \quad M = \max_{(x,y) \in R} (f(x,y))$$

→ Picard iteration: $y_n(x) = y(x_0) + \int_{x_0}^x f(t, y_{n-1}(t)) dt$

→ Euler's Method: $y_{n+1} = y_n + hf'(x_n, y_n)$

→ Modified Euler (Predictor-Corrector) method:

$$y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))]$$

→ Lip-shit condition:

A function satisfies Lip-shit condn. in the variable y on \mathbb{R} if a constant $L \geq 0$, exists s.t. $|f(x, y_1) - f(x, y_2)| \leq L |y_2 - y_1|$

Lipshitz constant ↑

→ RK methods

→ Taylor series method requires partial derivatives every time

→ RK-2: $y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$, where $k_1 = hf(x_n, y_n)$

$k_2 = hf(x_n + h, y_n + k_1)$

→ RK-4

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

, where $k_1 = hf(x_n, y_n)$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

→ Higher order differential eqns.:

$$y^{(m)}(x) = f(x, y, y', \dots, y^{(m-1)}) \quad , \quad a \leq x \leq b$$

$$y(a) = \alpha_1, \dots, y^{(m-1)}(a) = \alpha_{m-1}$$

→ single step methods

→ process: $y(x_{n+1}) = f(x_n, y_n, y'_n, h)$

* process is demanding only one past value

→ Taylor series method

Assumption: The differential eqn. has a unique soln., and continuous partial derivatives of order $(p+1)$

$$\frac{dy}{dx} = f(x, y) \quad , \quad y(x_0) = y_0 \quad a \leq x \leq b$$

Differentiate:

$$\frac{d^2 y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} \Rightarrow y'' = f_x + f_y f$$

Similarly, we can get $y^{(2)}, y^{(4)} \dots$

Using Taylor series: $y(x_1) = \sum_{i=0}^{\infty} \frac{h^i}{i!} \cdot y^{(i)}(x_0)$

→ Multistep method

→ general form: $y_{j+1} = \sum_{k=1}^m a_k y_{j-k+1} + h \sum_{k=0}^m b_k f_{j-k+1}$

→ " " of Adams' multistep methods:

$$y_{j+1} = y_j + h \sum_{k=0}^m b_k f_{j-k+1}$$

→ Adams-Bashforth technique (when $b_0 = 0$)

→ Because $b_0 = 0$, the method is explicit, i.e. does not require calculation of f_{j+1} .

→ Eg. for a 3-step Adams-Bashforth method,

$$y_{j+1} = y_j + h(b_1 f_j + b_2 f_{j-1} + b_3 f_{j-2})$$

How do we get the b -values? Construct an interpolating polynomial for f on x_j, x_{j-1}, x_{j-2} , and integrate it

Order	Adams-Bashforth	Adams-Moulton
2	$0, \frac{3}{2}, -\frac{5}{2}$	$\frac{5}{2}, \frac{8}{2}, -\frac{1}{2}$
3	$0, \frac{23}{12}, -\frac{16}{12}, \frac{5}{12}$	$\frac{9}{24}, \frac{19}{24}, -\frac{5}{24}, \frac{1}{24}$
4	$0, \frac{55}{24}, -\frac{59}{24}, \frac{37}{24}, -\frac{9}{24}$	$\frac{251}{720}, \frac{646}{720}, -\frac{264}{720}, \frac{106}{720}, -\frac{19}{720}$
5	$0, \frac{1901}{720}, -\frac{2774}{720}, \frac{2616}{720}, -\frac{1274}{720}, \frac{251}{720}$	/

ODEs - BVP

→ Consider a BVP: $y'' = f(x, y, y')$, $a \leq x \leq b$, $y(a) = \alpha$, $y(b) = \beta$

→ Shooting method (reduce BVP to IVP)

→ Existence theorem:

If $f, f_y \in C$, it continues on $a \leq x \leq b$, and:

i) $f_y(x, y, y') \geq 0$ in $a \leq x \leq b$

ii) $|f_{y'}(x, y, y')| \leq M$ in $a \leq x \leq b$

→ Linear BVP: $y'' = f(x, y, y')$
 $= p(x) y' + q(x) y + r(x)$

→ If p, q, r are continuous, and $q > 0$ on $a \leq x \leq b$, then the BVP has a unique soln.

→ Split into two IVPs:

① $y'' = p(x) y' + q(x) y + r(x)$ $y(a) = \alpha, y'(a) = 0$

② $y'' = p(x) y' + q(x) y$ $y(a) = 0, y'(a) = 1$

If ① has soln. y_1 & ② has soln. y_2 ,

→ Finite difference methods for linear problems:

→ Consider the 2nd order BVP:

$$y'' = p(x) \frac{dy}{dx} + q(x) y + r(x), \quad x \in [a, b], \quad y(a) = \alpha, \quad y(b) = \beta$$

If $p(x), q(x), r(x)$ are continuous on $[a, b]$, and $q(x) > 0$ on $[a, b]$, then the tridiagonal linear system has a unique soln if $h < \frac{2}{\max_{x \in [a, b]} |p(x)|}$

→ Finite difference methods for Non-linear problems

Consider the BVP:

$$y'' = f(x, y, y'), \quad x \in [a, b], \quad y(a) = \alpha, \quad y(b) = \beta$$

If: 1) f, f_y & $f_{y'}$ are continuous on $x \in [a, b]$

2) $f_y \geq \delta > 0$ on $x \in [a, b]$

3) There exists constants K & L s.t.:

$$K = \max_{x \in [a, b]} |f_y| \quad \& \quad L = \max_{x \in [a, b]} |f_{y'}|$$

Then there exists a unique soln. to the BVP.

Divide the interval into $(N+1)$ equal sub-intervals, and replace $y''(x_i)$ & $y'(x_i)$ with the central difference formulae:

$$y'' = f(x, y, y') \quad \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = f\left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right)$$

There will be a unique soln. if $h < \frac{3}{L}$

PDEs

Second order P.D.E: $A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F u = 0$

- A, B, \dots, F can be functions of x, y
- If $B^2 - 4AC = 0$, parabolic eqn.
- " " > 0 , hyperbolic "
- " " < 0 , elliptic "