

DS 288: NUMERICAL METHODS

NOV-16-2021

VARIABLE STEP SIZE METHODS for ODEs

## Variable Step Size Methods for ODEs

- Up to this point all our methods have been based on a fixed step size  $h$ .  
→ convenient and efficient in most cases.
- In some cases it is desirable to vary the step size  $h$  in order to control errors, especially when the error is nonuniform over the range of interest  $t \in [t_0, t_{end}]$ .  
→ analogous to adaptive quadrature.
- ★ The key is to try and estimate the local truncation error (LTE) based on simple calculations.
- Two common strategies:
  - §5.7 Use two different methods of the same order (whose LTEs have known forms) to calculate  $w_{i+1}$  and relate their differences.
  - §5.5 Use the same method with two different orders of LTE to calculate  $w_{i+1}$  and relate their differences.

### Two Different Methods of Same Order (§5.7)

Consider two different  $N^{th}$  order (GTE) methods which produce  $w_{i+1}$  and  $\tilde{w}_{i+1}$ :

Assumptions (Big): a)  $y_k = w_k = \tilde{w}_k$   $k = 0, 1, \dots, i$   
b)  $\xi$ -factors in  $E_{trunc}$  are same

Then

$$\begin{aligned} y_{i+1} - w_{i+1} &= A f^{(N)}(\xi_i) h^{N+1} \\ y_{i+1} - \tilde{w}_{i+1} &= B f^{(N)}(\xi_i) h^{N+1} \end{aligned}$$

$$\tilde{w}_{i+1} - w_{i+1} = (A - B) h^{N+1} f^{(N)}(\xi_i)$$

and

$$f^{(N)} = \frac{\tilde{w}_{i+1} - w_{i+1}}{(A - B) h^{N+1}}$$

Now we want the global truncation error (GTE) to be less than some prescribed  $\epsilon$  and we consider taking a step size  $qh$ . What should the factor  $q$  be in order to maintain that GTE is bounded by some value  $\epsilon$ ? (Note that method A is our favored approach.)

$$\begin{aligned} |A f^{(N)}(\xi) (qh)^N| &< \epsilon \\ \frac{|A| (qh)^N |\tilde{w}_{i+1} - w_{i+1}|}{|A - B| h^{N+1}} &< \epsilon \end{aligned}$$

$$q < \left( \frac{\epsilon h |A - B|}{|A| |\tilde{w}_{i+1} - w_{i+1}|} \right)^{\frac{1}{N}}$$

Tracking  $q$  shows us how to alter the step-size in order to maintain the GTE below a prescribed amount, i.e.,  $\epsilon$ .

Typical Example: 4<sup>th</sup> Order Adams P/C

$$A/B \text{ LTE} : \underbrace{\frac{251}{720}}_B h^5 f^{(4)}(\xi)$$

$$A/M \text{ LTE} : \underbrace{\frac{-19}{720}}_A h^5 f^{(4)}(\xi)$$

$\sim \epsilon_0(h^5)$

INVOLVES TWO NUMERICAL SOLUTIONS

$$q < \left( \frac{\epsilon \frac{270}{720} h}{\frac{19}{720} |w_{i+1}^{A/M} - w_{i+1}^{A/B}|} \right)^{\frac{1}{4}} = \left( \frac{270}{19} \frac{\epsilon h}{|w_{i+1}^{A/M} - w_{i+1}^{A/B}|} \right)^{\frac{1}{4}}$$

### Same Method Two Different Orders (§5.5)

\*

Consider using a method with  $\mathcal{O}(h^N)$  globally and the same method with  $\mathcal{O}(h^{N+1})$  which produce  $w_{i+1}$  and  $\tilde{w}_{i+1}$ :

$$\begin{aligned} y_{i+1} - w_{i+1} &= Ah^{N+1} f^{(N)}(\xi_i) \quad \rightarrow n+1 \quad \mathcal{O}(h^{n+1}) \\ y_{i+1} - \tilde{w}_{i+1} &= Bh^{N+2} f^{(N)}(\xi_i) \quad \rightarrow n+2 \quad \mathcal{O}(h^{n+2}) \\ \hline \tilde{w}_{i+1} - w_{i+1} &= Ah^{N+1} f^{(N)} - Bh^{N+2} f^{(N+1)}(\xi_i) \quad \rightarrow 0 \end{aligned}$$

assume small relative to first term, i.e.,  $h^{N+2} \ll h^{N+1}$  for small  $h$

So

$$f^{(N)} = \frac{\tilde{w}_{i+1} - w_{i+1}}{Ah^{N+1}}$$

... same as before with  $B = 0$ . Then it follows that

$$q < \left( \frac{\epsilon h}{|\tilde{w}_{i+1} - w_{i+1}|} \right)^{\frac{1}{N}}$$

INVOLVES TWO NUMERICAL SOLUTIONS

$\rightarrow$  it's step  $\rightarrow h \rightarrow qh$

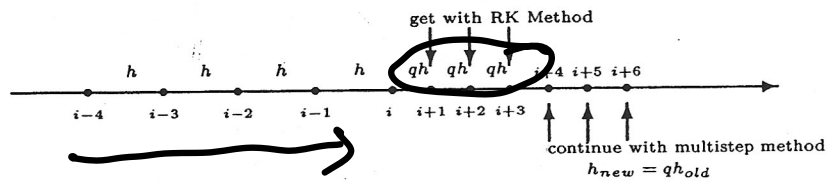
What  $q$  you will choose?  
0.1 0.01 0.0001 0.09

$q$  to be lesser at the same time  
 If  $qh$  too small  $\rightarrow$  MORE STEPS  
 $\rightarrow$  MORE COMPUTATION

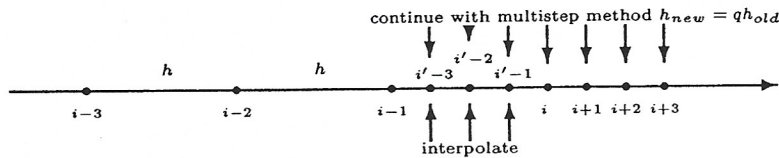
\* Note that multi-step methods are derived assuming constant  $h$ ! If the step size changes along the way an adjustment must be made.

– Two common strategies:

(i) use a single-step method (i.e., self-starting) to calculate enough new values to proceed with the new step size



(ii) interpolate back in time with the new step size



– Many methods do not change  $h$  continuously (which could potentially be quite costly) but rather monitor  $q$  and change only if it is significantly different than 1.

$$\begin{aligned}
 h_4 &= 0.9 h_3 \\
 h_5 &= 0.95 h_4 \leftarrow \\
 &\vdots \\
 h_i &\rightarrow 0.5 h_{i-1} \quad (\text{significant})
 \end{aligned}$$

## Summary of ODE Methods

$$t_0 \rightarrow t_1 \rightarrow t_2 \cdots t_n$$

↓  
t

Prototype problem:

$$y'(t) = f(y, t)$$

$$y(t_0) = y_0$$

rate function  
initial value

Method	Approximate Solution to ODE	LTE	GTE	Ex/Im plicit
<i>Single-step</i>				
Euler's Method	$w_{i+1} = w_i + hf_i$	$O(h^2)$	$O(h)$	Explicit
Midpoint Rule	$w_{i+1} = w_i + hf\left(w_i + \frac{h}{2}f_i, t_i + \frac{h}{2}\right)$	$O(h^3)$	$O(h^2)$	Explicit
Trapezoidal Rule	$w_{i+1} = w_i + \frac{h}{2}[f_i + f(w_{i+1}, t_{i+1})]$	$O(h^3)$	$O(h^2)$	Implicit
Modified Euler's	$w_{i+1} = w_i + \frac{h}{2}[f_i + f(w_i + hf_i, t_{i+1})]$	$O(h^3)$	$O(h^2)$	Explicit
Heun's Method	$w_{i+1} = w_i + \frac{h}{4}\left[f_i + 3f\left(w_i + \frac{2}{3}hf_i, t_i + \frac{2}{3}h\right)\right]$	$O(h^3)$	$O(h^2)$	Explicit
Runge-Kutta 4 <sup>th</sup> Order	$w_{i+1} = w_i + \frac{h}{6}\left[f_i + 4f_{i+\frac{1}{2}} + f_{i+1}\right]$	$O(h^5)$	$O(h^4)$	Implicit
<u>most popular RK4</u>	$w_{i+1} = w_i + \frac{h}{6}[f_1 + 2(f_2 + f_3) + f_4]$	$O(h^5)$	$O(h^4)$	Explicit
	where $f_1 = f_i$			
	$f_2 = f\left(w_i + \frac{h}{2}f_1, t_i + \frac{1}{2}h\right)$			
	$f_3 = f\left(w_i + \frac{h}{2}f_2, t_i + \frac{1}{2}h\right)$			
	$f_4 = f(w_i + hf_3, t_i + h)$			
<i>Multi-step</i>				
Adams-Bashforth				
2 Step	$w_{i+1} = w_i + \frac{h}{2}[3f_i - f_{i-1}]$	$O(h^3)$	$O(h^2)$	Explicit
3 Step	$w_{i+1} = w_i + \frac{h}{12}[23f_i - 16f_{i-1} + 5f_{i-2}]$	$O(h^4)$	$O(h^3)$	Explicit
4 Step	$w_{i+1} = w_i + \frac{h}{24}[55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}]$	$O(h^5)$	$O(h^4)$	Explicit
5 Step	$w_{i+1} = w_i + \frac{h}{720}[1901f_i - 277f_{i-1} + 2616f_{i-2} - 1274f_{i-3} - 251f_{i-4}]$	$O(h^6)$	$O(h^5)$	Explicit
Adams-Moulton				
2 Step	$w_{i+1} = w_i + \frac{h}{12}[5f_{i+1} + 8f_i - f_{i-1}]$	$O(h^4)$	$O(h^3)$	Implicit
3 Step	$w_{i+1} = w_i + \frac{h}{24}[9f_{i+1} + 19f_i - 5f_{i-1} + f_{i-2}]$	$O(h^5)$	$O(h^4)$	Implicit
4 Step	$w_{i+1} = w_i + \frac{h}{720}[251f_{i+1} + 646f_i - 264f_{i-1} + 106f_{i-2} - 19f_{i-3}]$	$O(h^6)$	$O(h^5)$	Implicit

- Multi-step methods are not self-starting; need to start with a single-step method
- Implicit methods are not self-starting; need to start (*predict*) with an explicit method