DS 288: NUMERICAL METHODS

OCT-14-2024 :

CH#5 SOLUTIONS TO ORDINARY DIFFERENTIAL EQUATIONS

ORDINARY — ONE INDEPENDENT VARIABLE.

PARTIAL - SEVERAL INDEPENDENT VARIABLES (≥ 2)

[PDE: DS 289: NUMERICAL SOLUTION TO DIFF. EQNS]

— THE CLASSIC INITIAL VALUE PROBLEM (IVP)

(PROTOTYPE PROBLEM)

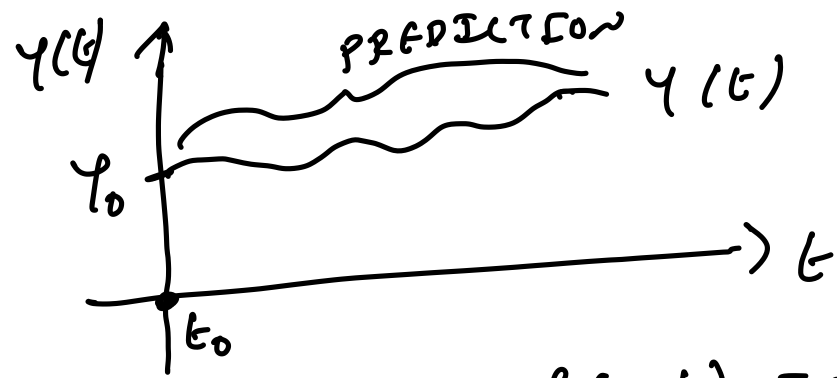
INITIAL VALUE

$$y'(t) = \underbrace{f(y, t)}_{\text{RATE FUNCTION}} \; ; \; y(t_0) = y_0$$

$\left[\frac{dy}{dt}\right]$

(TELLS HOW FAST y IS CHANGING)

- WE WANT TO FIND $y(t)$ GIVEN THE RATE FUNCTION $f(y, t)$ & INITIAL VALUE CONDITION y_0

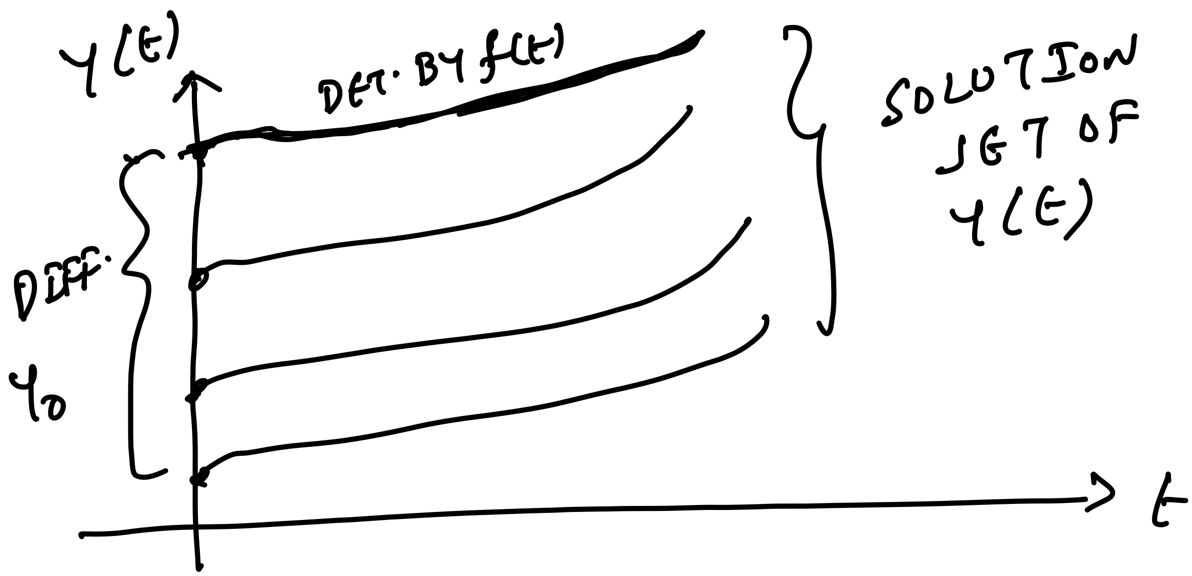


SIMPLEST CASE: $f(y,t) = f(t)$.
ONLY FUNCTION OF 't'

$$\Rightarrow y'(t) = f(t)$$

$$y(t) = \underbrace{\int_{t_0}^{t} f(t')\,dt'}_{\text{SOLVED BY NUMERICAL INTEGRATION METHOD.}} + y_0$$

INITIAL CONDITION



* THE SOLUTION SET IS A FAMILY OF CURVES WITH A CONSTANT SLOPE (RATE FUNCTION) AT A FIXED VALUE OF 't' (i.e. $y'(t) = f(t)$)

- MORE GENERAL CASE: $f(y,t)$.
FUNCTION OF 'y' & 't'

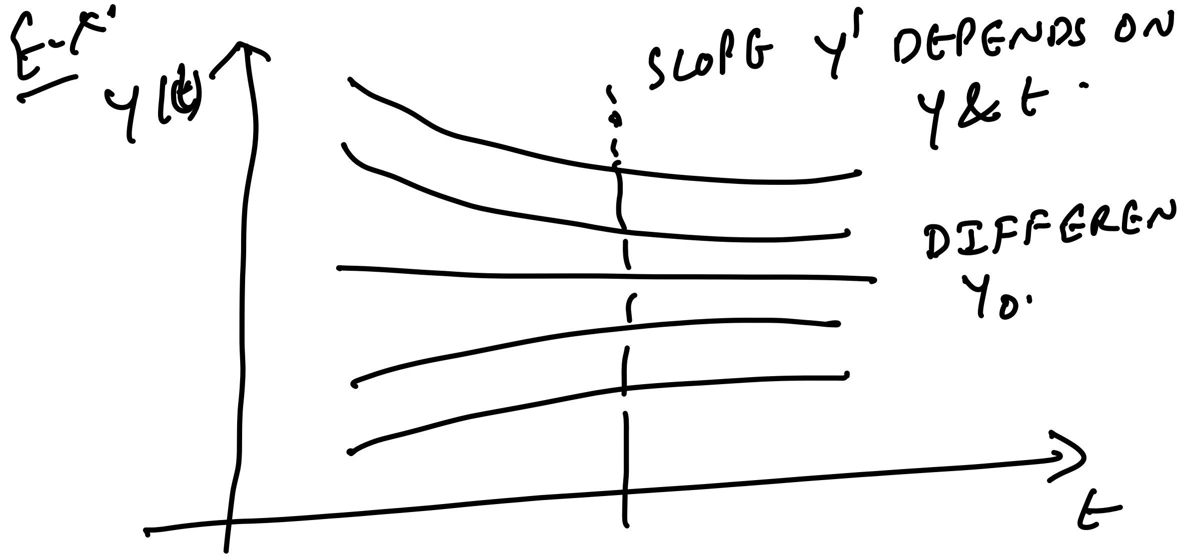
$$y'(t) = f(y,t) \Rightarrow y(t) = \int_{t_0}^{t} f(t', y)\,dt' + y_0$$

UNKNOWN

* SOLUTION FAMILY OF CURVES NOW HAS A VARIABLE SLOPE (RATE FUNCTION) AT A FIXED VALUE OF 't'
i.e. $y'(t) = f(y,t)$ ← DEPENDS ON y.

— SOLUTION FAMILY IS AN INTRINSIC PROPERTY OF THE PARTICULAR ODE.

E.g.



EULER'S METHOD [§5.2]

- RARELY USED IN PRACTISE.
  (NOT VERY ACCURATE).

BUT GOOD TO DEMONSTRATE PRINCIPLES USED TO SOLVE IVPs.

* 3 DIFFERENT WAYS WE CAN DERIVE APPROXIMATE SOLUTIONS TO ODE.

(i) SUBSTITUTE FINITE DIFFERENCE APPROXIMATION TO y'.

$$y' = \frac{y_{i+1} - y_i}{h} - \frac{h}{2} y''(y(\xi_i), \xi_i)$$
$$= f(y_i, t_i)$$

HERE $h = t_{i+1} - t_i$

$\xi_i \in [t_i, t_{i+1}]$

y' HAS BEEN APPROXIMATE VIA FIRST FORWARD DIFFERENCE.

$$\boxed{y_{i+1} = y_i + h f_i + \frac{h^2}{2} f'(y(\xi_i), \xi_i)}$$

EXACT

$[y' = f]$

HERE $f_i \equiv f(y_i, t_i)$

WITH $y_i = y(t_i)$

SWITCH TO COMPUTED VALUES

$y_i \simeq w_i$

$$\boxed{w_{i+1} = w_i + h f_i}$$

USED IN COMPUTING.

(ii) INTEGRATE THE ODE

$$\int_{t_i}^{t_{i+1}} y' \, d\tau = \int_{t_i}^{t_{i+1}} f(y_i, \tau) \, d\tau$$

$$y_{i+1} - y_i = \underbrace{\int_{t_i}^{t_{i+1}} f(y_i, \tau)\, d\tau}_{I(t_{i+1})}$$

TAKE $I(t_{i+1})$ TAYLOR EXPAND AROUND t_i

$$I(t_{i+1}) = \cancelto{0}{I(t_i)} + h f_i + \frac{h^2 f'(\xi_i)}{2!}$$

$$\cancelto{0}{\int_{t_i}^{t_i} f(y_i, \tau)\, d\tau} \qquad t_{i+1} - t_i \qquad f(y_i, t_i)$$

$$\boxed{y_{i+1} = y_i + h f_i + \frac{h}{2} f'(\xi_i)}$$

SAME AS ABOVE.

(ii) TAYLOR SERIES EXPANSION OF $y(t)$ AROUND t_i EVALUATED AT t_{i+1}

$$y(t) = y(t_i) + y'(t_i)(t - t_i) + \frac{y''(\xi_i)(t - t_i)^2}{2}$$

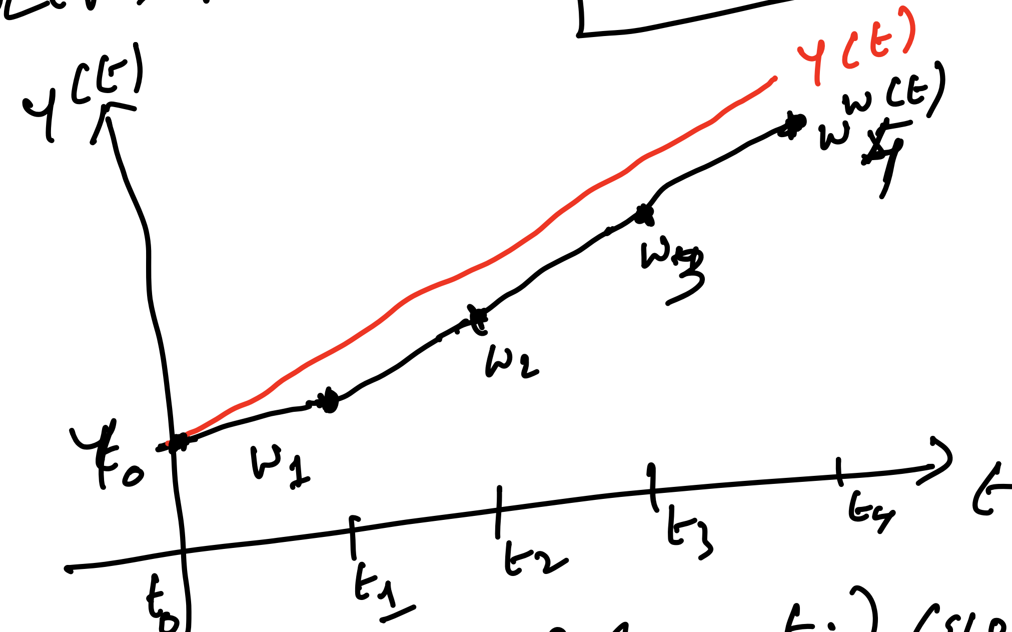
$$t = t_{i+1}$$

$$y(t_{i+1}) = y(t_i) + \underbrace{y'(t_i)}_{f}(t_{i+1}-t_i) + y''(\xi_i)\frac{(t_{i+1}-t_i)^2}{2}$$

$$\boxed{y_{i+1} = y_i + h f_i + h^2 \frac{f'(\xi_i)}{2}}$$

SAME AS ABOVE.

EULER'S METHOD: $\boxed{w_{i+1} = w_i + h f_i}$



* REMEMBER $f_i = f(w_i, t_i)$ (SLOPE)

NOT $f(y_i, t_i)$

THIS WILL BE EXACT $f(w_i, t_i) = f(y_i, t_i)$

$y_i = y_0$

OR $f(y,t) = f(t)$ [SAME SLOPE]

NOTES ON EULER'S METHOD

* EULER'S METHOD IS AN EXPLICIT METHOD WHERE NEW SOLUTION IS A FUNCTION OF THE OLD SOLUTION.

- HOW GOOD IS EULER'S METHOD?
  * LOCAL TRUNCATION ERROR IS $O(h^2)$
  * $y_{i+1} - y_i$ IS $O(h)$ AS "t" ADVANCES.
  * ADDITIONAL ERROR COMPONENT DUE TO EVALUATION OF RATE FUNCTION IN WRONG PLACE $f(w_i, t_i)$ NOT $f(y_i, t_i)$.

- LET US LOOK AT ERROR BEHAVIOR MORE QUANTITATIVELY.

TOTAL ERROR $\equiv \epsilon_{i+1} = |y_{i+1} - w_{i+1}|$

STANDARD ANALYSIS

$$y_{i+1} = y_i + h f_i + \frac{h^2}{2} f'(y(\xi_i), \xi_i) \quad \text{EXACT}$$

$f_i = f(y_i, t_i)$

$$w_{i+1} = w_i + h \bar{f}_i \qquad \bar{f}_i = f(w_i, t_i)$$

$$\underbrace{y_{i+1} - w_{i+1}}_{\epsilon_{i+1}} = \underbrace{y_i - w_i}_{\epsilon_i} + h(f_i - \bar{f}_i) + \frac{h^2}{2} f'(y, \xi_i)$$

$$\epsilon_{i+1} = \epsilon_i \left[1 + h \underbrace{\frac{(f_i - \bar{f}_i)}{y_i - w_i}}_{\substack{= \frac{\partial f}{\partial y}(\bar{y}_i, t_i) \\ \bar{y}_i \in [y_i, w_i]}} \right] + \frac{h^2}{2} f'(y, \xi_i)$$

BY MEAN VALUE THEOREM

$J_i \rightarrow$ JACOBIAN OF THE EQUATION AT $t = t_i$

$$\boxed{\epsilon_{i+1} = \epsilon_i (1 + h J_i) + \underbrace{\frac{h^2}{2} f'(y(\xi_i), \xi_i)}_{\text{TRUNCATION ERROR}}}$$

$(1+hJ_i) \rightarrow$ AMPLIFICATION FACTOR

GE $\rightarrow$ GLOBAL ERROR

LTE $\rightarrow$ LOCAL TRUNCATION ERROR

$$\boxed{GE_{i+1} = (1+hJ_i)\,GE_i + LTE_{i+1}}$$

FOR EULER'S METHOD TO BE STABLE (ERROR DOES NOT BLOWUP WITH ITERATIONS).

$$|1+hJ_i| < 1$$

OR $-2 < hJ_i < 0$.

SO J_i MUST BE < 0 $(h \not< 0)$

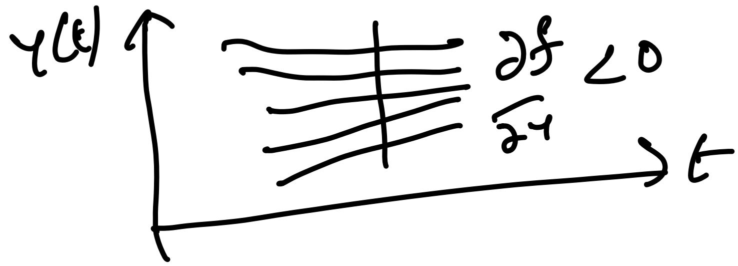
OR $\boxed{h < \frac{2}{|J|}}$ STABILITY FOR EULER'S METHOD

LIMITATION ON HOW LARGE SHOULD BE 'h'

A STABLE ODE WILL HAVE $J < 0$.

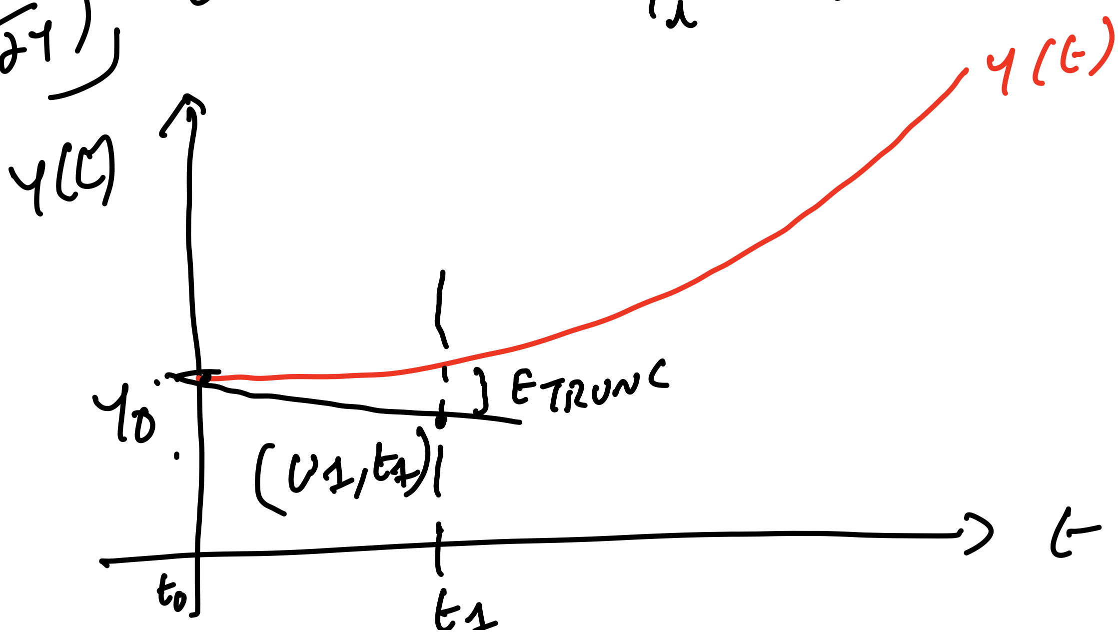
$$\Rightarrow \frac{\partial f}{\partial y} < 0$$

SOL: FAMILY OF CURVES CONVERGE AS t INCREASES



RECALL

$$\left(\frac{\partial f}{\partial y}\right) \quad J_i = \frac{f(\overbrace{y(t_i)}^{\text{UNKNOWN}}, t_i) - f(\overbrace{w_i}^{\text{COMPUTING}}, t_i)}{y_i - w_i}$$

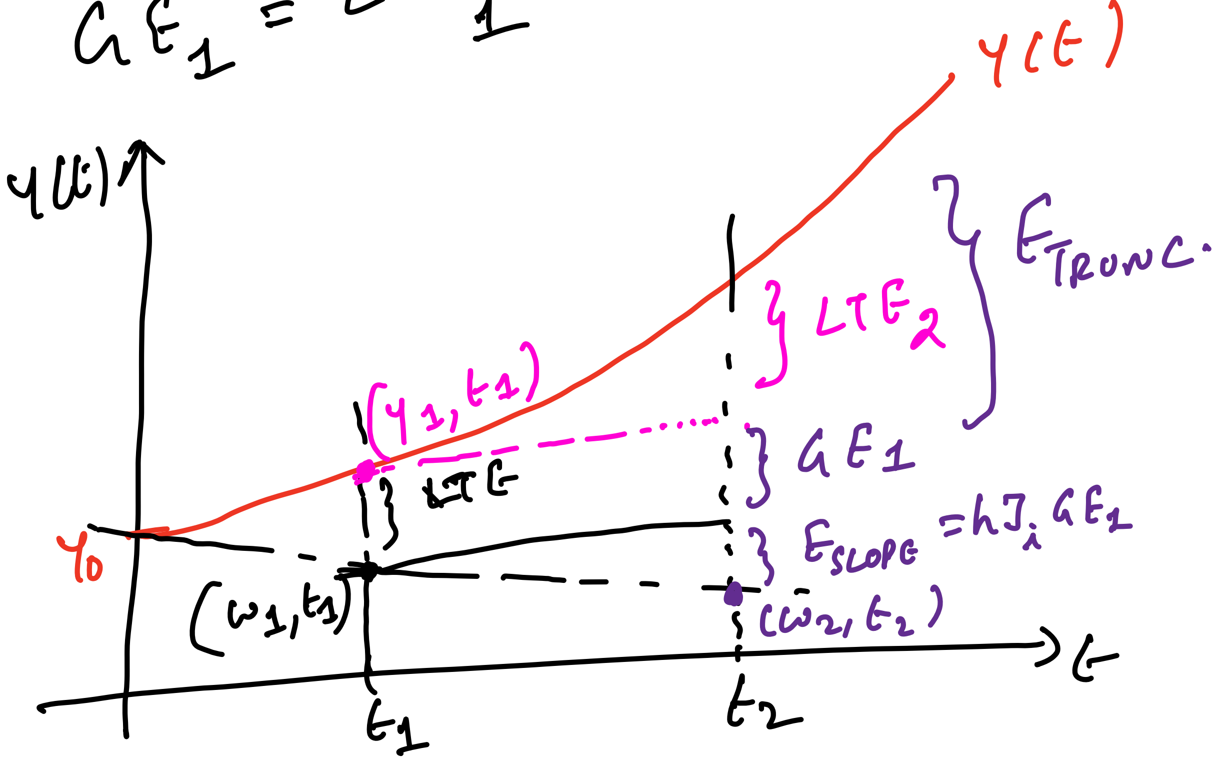


FIRST STEP

$$y_1 = y_0 + h f_0 + \underbrace{\frac{h^2}{2} f'(y_0)}_{E_{TRUNC}} \to \text{EXACT}$$

$$\begin{matrix} \| & & \| \\ w_1 = w_0 & + & h f_0 \end{matrix}$$

$$GE_1 = LTE_1 \quad (GE_0 = 0)$$



$$y_2 = y_1 + h f(y_1, t_1) + E_{TRUNC_2} \to \text{EXACT}$$

$$w_2 = w_1 + h f(w_1, t_1) \to \text{COMPUTIN}$$

AS t PROGRESSES

* E_{TRUNC} GROWS.

IF $J = 0$ $(f = f(t)) \Rightarrow E_{SLOPE} = 0$

$J < 0 \Rightarrow E_{SLOPE} < 0 \rightarrow$ CANCELS SOME OF THE ERRORS.

* EULER'S METHOD IS NOT VERY ACCURATE AND HAS STABILITY LIMITATIONS (CHOOSE 'h' WISELY)

  - SUSCEPTIBLE TO EXPONENTIAL ERROR AMPLICATION.

* WE WANT METHOD WITH HIGHER ORDER LTE.