Numerical Methods

DS288 and UMC201

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Chapter-3

Interpolation and Polynomial Approximation

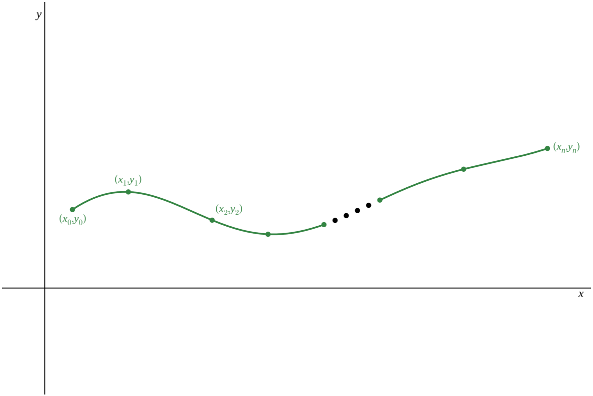
Interpolation

- List of students taking admission in different years

Year	2006	2009	2010	2012	2015	2016	2018
Student	600	800	950	1000	1050	1100	1200

- One may ask whether these data could be used to provide a reasonable estimate of students, say in 2008 or 2011 or 2017.
- Predictions of these types can be obtained using a function fitting the given data.
- This process is called interpolation.

Interpolation



- If data $(x_i, y_i), i = 0, 1, 2, ..., n$ are available from an experiment or otherwise, such that y_i depends on x_i.
- Then, we want to find the nature of the relationship of y on x.

Interpolation

- Approximate the value of y at some value of x not listed among the x_i, or determine a function that in some sense approximates the data.
- The points where the values of polynomial and the function coincide are called interpolating points or nodes or tabular points. The polynomial is known as the interpolating polynomial.

The function $f(x)$ generally replace by a polynomial $p_n(x)$

$$p_n(x) = a_0 + a_1 x + \cdots + a_n x^n$$

Weisestrass Approximation Theorem

- The following theorem is the basis for polynomial approximation:

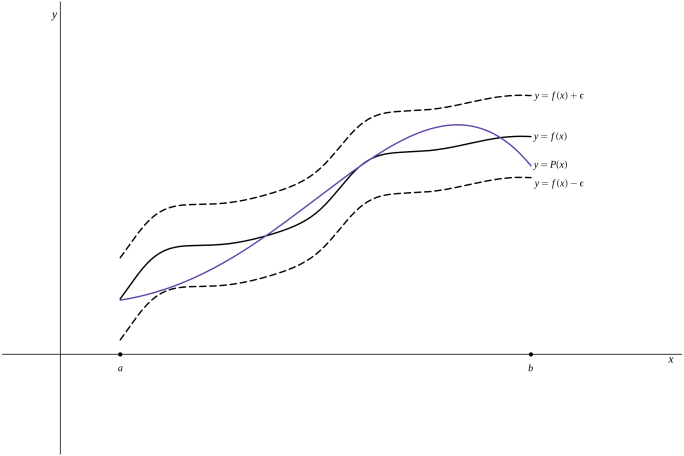
Theorem

Suppose that $f \in C[a, b]$*. For each* $\epsilon > 0$ *there exists a polynomial* $P(x)$*, such that*

$$\|f(x) - P(x)\| < \epsilon, \quad \textit{for all} \;\; x \in [a, b]$$

- The theorem says nothing about finding polynomial or its order.

Weisestrass Approximation Theorem



Weierstrass approximation Theorem guarantees that we (maybe with substantial work) can find a polynomial that fits into the tube around the function f, no matter how thin we make the tube.

Interpolations

Definition

Let $x_0, x_1, \cdots x_n$ be $n+1$ distinct points in the interval $[a, b]$. Then $p_n(x)$ is an interpolating polynomial to $f(x)$ if

- $p_n(x_i) = f(x_i)$ for $i = 1, 2, 3 \cdots, n$

or

- $p_n(x_i) = f(x_i)$
- $p_n'(x_i) = f'(x_i)$ for $i = 1, 2, 3 \cdots, n$

The derivative conditions may be replaced by more general conditions involving higher-order derivatives.

Note: The Taylor expansion works very hard to be accurate in the neighborhood of one point. But we want to fit data at many points (in an extended interval).

Discussion

Let $x_0, x_1, \cdots x_n$ be $n+1$ distinct numbers, and let $f(x_0), f(x_1), \cdots f(x_n)$ be associated function values. We now study the problem of finding a polynomial $p(x)$ that interpolates the given data

$$p_n(x_i) = f(x_i) \quad \text{for } i = 1, 2, 3 \cdots, n$$

Questions

- Does such polynomial exist, and if so, what is the degree?
- Is the polynomial unique?
- What is a formula for producing $p(x)$ from the given data?

- Suppose we have $n+1$ distinct points $x_0 < x_1 < \cdots < x_n$
- $p_n(x)$ is polynomial interpolating $f(x)$ a set of $n+1$ points.
- Then

$$p_n(x_i) = f(x_i) \quad \text{for } i = 1, 2, 3 \cdots, n$$

- Now we can write

$$\begin{aligned}
& a_0 + a_1 x_0 + \cdots + a_n x_0^n = f(x_0) = f_0 \\
& a_0 + a_1 x_1 + \cdots + a_n x_1^n = f(x_1) = f_1 \\
& \vdots \\
& a_0 + a_1 x_n + \cdots + a_n x_n^n = f(x_n) = f_n
\end{aligned}$$

- This is a system of $n+1$ linear equations in $n+1$ unknowns; $a_0, a_1, \cdots a_n$

- This above system of $n+1$ linear equation in $n+1$ unknowns.
- This system will have a unique solution if the determinant

$$\Delta = \begin{vmatrix} 1 & x_0 & x_0^2 \cdots x_0^n \\ 1 & x_1 & x_1^2 \cdots x_1^n \\ \vdots & & \\ 1 & x_n & x_n^2 \cdots x_n^n \end{vmatrix} \neq 0$$

- Indeed, the value of the determinant is not zero.
- $\Delta = \prod_{0 \leq j < i \leq n} x_i - x_j.$
- therefore unique interpolating polynomial exists.

Example

Example

Find the interpolating polynomial for the following data
$f(-1) = 0, f(0) = 1, f(1) = 2$

- Consider the interpolating polynomial $p(x) = a_0 + a_1 x + a_2 x^2$
- $a_0 = 1,\ a_1 = 1,\ a_2 = 0.$ Thus $p(x) = 1 + x$


ANY
QUESTIONS?