

Root finding

$$\rightarrow \text{Regula Falsi} : x_{n+1} = b - f(b) \left(\frac{b-a}{f(b)-f(a)} \right)$$

$$\rightarrow \text{Secant} : x_{n+1} = x_n - f(x_n) \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \right)$$

$$\rightarrow \text{Modified Newton} : x_{n+1} = x_n - \frac{f(x_n) \cdot f'(x_n)}{(f'(x_n))^2 - f(x_n) \cdot f''(x_n)}$$

\rightarrow Müller:

$$d_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad d_2 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad a = \frac{d_2 - d_1}{x_2 - x_0}$$

$$b = d_2 + a(x_2 - x_1) \quad x_3 = x_2 - \frac{2c}{b + \text{sgn}(b) \sqrt{b^2 - 4ac}}$$

$$(= f(x_n))$$

System of Non-Linear Equations

\rightarrow Fixed point method:

$$\text{Rewrite eqns. as } x_1 = g_1(\tilde{x}) \dots x_n = g_n(\tilde{x})$$

$$\text{Necessary condn: } \left[\left| \frac{\partial g_i}{\partial x_j} \right| + \dots + \left| \frac{\partial g_i}{\partial x_n} \right| \right] \leq 1, \forall 1 \leq i \leq n$$

\rightarrow Newton's method: $\tilde{x}^{(i+1)} = \tilde{x}^{(i)} - J_{\tilde{x}^{(i)}}^{-1} \cdot f(\tilde{x}^{(i)})$, Where $J_{\tilde{x}^{(i)}}^{-1}$ is the Jacobian inverse evaluated at $\tilde{x}^{(i)}$.

Least Squares for functions

$$\rightarrow \text{System of equations: } \sum_{k=0}^1 a_k \int_a^b x^{j+k} dx = \int_a^b x^j f(x) dx$$

\rightarrow Orthogonal functions: (Alternative for least squares)

\rightarrow Weight function: An integrable function w is called a weight function on the interval I if $w(x) > 0 \quad \forall x \in I$.

\rightarrow Used to assign varying degrees of importance to approximations on certain portions of the interval.

\rightarrow Two functions are orthogonal w.r.t w if:

$$\int_a^b w(x) \phi_k(x) \phi_j(x) dx = \begin{cases} 0, & j \neq k \\ \omega_j > 0, & j = k \end{cases}$$

→ We can solve large least squares w/o matrix multiplication. We can reduce the system to:

$$a_i \int_a^b W(x) (f_i(x))^2 dx = \int_a^b W(x) y(x) f_i(x) dx, \quad i = 0, \dots, n$$

Numerical Differentiation & Integration

$$\rightarrow 3\text{-point forward difference: } f'_j \approx \frac{-3f_j + 4f_{j+1} - f_{j+2}}{2h}$$

$$\rightarrow "backward": f'_j \approx \frac{3f_j - 4f_{j-1} + f_{j-2}}{2h}$$

$$\rightarrow \text{second derivative: } f''_j = \frac{f_{j+2} - 2f_{j+1} + f_j}{h^2} + O(h)$$

→ Newton-Cotes formulae: Use equally spaced nodes

→ Open: exclude end points → Closed: include endpoints

$$\rightarrow \text{Trapezoidal Rule: } \int_a^b f(x) dx \approx \frac{h}{2} (f(a) + f(b)) - \frac{h^3}{12} f''(\xi) \quad (n=1)$$

$$\rightarrow \text{Simpson's } \frac{1}{3} \text{ Rule: } \int_{x_0}^{x_2} f(x) dx \approx \frac{h}{3} (f_0 + 4f_1 + f_2) - \frac{h^5}{90} f^{(4)}(\xi) \quad (n=2)$$

$$\rightarrow \text{Simpson's } \frac{3}{8} \text{ Rule: } \int_{x_0}^{x_3} f(x) dx \approx \frac{3h}{8} (f_0 + 3f_1 + 3f_2 + f_3) - \frac{3h^5}{80} f^{(4)}(\xi) \quad (n=3)$$

$$\rightarrow \text{Boole's Rule: } \int_{x_0}^{x_4} f(x) dx \approx \frac{2h}{45} (7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4) + \frac{8h^7}{945} f^{(6)}(\xi) \quad (n=4)$$

$$\rightarrow \text{Open formulae: } \int_{x_0}^{x_0} f(x) dx = 2hf_0 + \frac{h^3}{3} f''(\xi)$$

$$\rightarrow \text{Midpoint rule: } \int_a^{x_1} f(x) dx = 2hf_0 + \frac{h^3}{3} f''(\xi) \quad (n=0)$$

$$\rightarrow \text{Two-point open rule: } \int_{x_0}^{x_1} f(x) dx = \frac{3h}{2} (f_0 + f_1) + \frac{3h^3}{4} f''(\xi) \quad (n=1)$$

$$\rightarrow \text{Three-point open rule: } \int_{x_0}^{x_3} f(x) dx = \frac{4h}{3} (2f_0 + f_1 + 2f_2) + \frac{14h^5}{45} f^{(4)}(\xi)$$

$$\rightarrow \text{Four point open rule: } \int_{x_0}^{x_4} f(x) dx = \frac{5h}{24} (11f_0 + f_1 + f_2 + 11f_3) + \frac{95h^5}{144} f^{(4)}(\xi)$$

$$\rightarrow \text{Composite Trapezoidal: } \int_a^b f(x) dx = \frac{h}{2} \left(f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right) + \frac{b-a}{12} h^2 f''(\xi)$$

\rightarrow Composite Simpsons:

$$\int_a^b f(x) dx = \frac{h}{3} \left(f(a) + 2 \sum_{j=1}^{\frac{n-1}{2}} f(x_{2j}) + 4 \sum_{j=1}^{\frac{n-1}{2}} f(x_{2j-1}) + f(b) \right) + \frac{b-a}{180} h^4 f^{(4)}(\xi)$$

\rightarrow Richardson's Extrapolation

Let $I_{\text{true}} = \int_a^b f(x) dx$, $I(h)$ be trapezium rule with h as length of interval, and $E(h)$ be error when using trapezium rule

$$\text{We know: } I(h) + E(h) = I\left(\frac{h}{2}\right) + E\left(\frac{h}{2}\right) = I_{\text{true}}$$

$$\text{Let us take } c = -\frac{f''(n)}{12}$$

$$I_{\text{true}} = I(h) + ch^3 \quad (1)$$

$$I_{\text{true}} = I\left(\frac{h}{2}\right) + 2c\left(\frac{h}{2}\right)^3 = I\left(\frac{h}{2}\right) + \frac{ch^3}{2^2} \quad (2)$$

$$\text{Final: } I_{\text{true}} = \frac{2^{n-1}}{2^{n-1}-1} I\left(\frac{h}{2}\right) - I(h) + \frac{(ch)^{n+1}}{2^{n+2}-2}$$

\rightarrow Increases level of accuracy by 1.

→ Romberg Integration

→ Use Richardson's extrapolation multiple times

$$\text{I.e } I(h, h/2) = I(h/2) + \frac{I(h/2) - I(h)}{3}$$

$$I(h/2, h/4) = I(h/4) + \frac{I(h/4) - I(h/2)}{3}$$

$$I(h, h/2, h/4) = I(h/2, h/4) - I(h, h/2)$$

Eventually, $I(h, \dots, h/2^n)$ will get more accurate.

→ Gaussian Quadrature

→ Rather than use points at equally spaced points, we choose the points in an optimal way.

→ Used to calculate:

$$\int_{-1}^1 f(x) dx$$

any integral can be brought to this form

→ Applying method of undetermined coefficients (upto x^3), we get this to be the final result:

$$\int_{-1}^1 f(x) dx = f(-\sqrt{3}) + f(\sqrt{3})$$

→ If $f(x, y)$ & $\frac{\partial f}{\partial y}$ are both continuous $R = \{(x-x_0) \leq a, |y-y_0| \leq b\}$, the IVP will have unique soln. in $|x-x_0| \leq h$, where:

$$h = \min(a, b/M), \quad M = \max_{(x,y) \in R} (f(x,y))$$

→ Picard iteration: $y_n(x) = y(x_0) + \int_{x_0}^x f(t, y_{n-1}(t)) dt$

→ Euler's Method: $y_{n+1} = y_n + hf(x_n, y_n)$

→ Modified Euler (Predictor-Corrector) method:

$$y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))]$$

→ Lip-shit condition:

A function satisfies Lip-shit condn. in the variable y on \mathbb{R} if a constant $L \geq 0$, exist s.t. $|f(x, y_1) - f(x, y_2)| \leq L |y_2 - y_1|$

Lipshitz constant \nearrow

→ RK methods

→ Taylor series method requires partial derivatives every time

→ RK-2: $y_{n+1} = y_n + \frac{h}{2}(u_1 + u_2)$, where $u_1 = hf(x_n, y_n)$
 $u_2 = hf(x_n + h, y_n + u_1)$

→ RK-4

$$y_{n+1} = y_n + \frac{h}{6}(u_1 + 2u_2 + 2u_3 + u_4)$$

, where $u_1 = hf(x_n, y_n)$

$$u_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{u_1}{2}\right)$$

$$u_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{u_2}{2}\right)$$

$$u_4 = hf(x_n + h, y_n + u_3)$$

→ Higher order differential eqns.:

$$y^{(m)}(x) = f(x, y, y', \dots, y^{(m-1)}), a \leq x \leq b$$

$$y(a) = \alpha_1, \dots, y^{(m-1)}(a) = \alpha_{m-1}$$

→ Single step methods

$$\rightarrow \text{Process: } y(x_{n+1}) = f(x_n, y_n, y'_n, h)$$

* Process is demanding only one past value

→ Taylor series method

Assumption: The differential eqn. has a unique soln., and continuous partial derivatives of order $(p+1)$

$$\frac{dy}{dx} = f(x, y) \quad , \quad y(x_0) = y_0 \quad a \leq x \leq b$$

Differentiate:

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} \Rightarrow y'' = f_x + f_y f$$

Similarly, we can get $y^{(3)}, y^{(4)} \dots$

Why Taylor series: $y(x_1) = \sum_{i=0}^{\infty} \frac{h^i}{i!} \cdot y^{(i)}(x_0)$

→ Multistep method

→ general form: $y_{j+1} = \sum_{n=1}^m a_n y_{j-n+1} + h \sum_{n=0}^m b_n f_{j-n+1}$

→ " of Adams' multistep methods:

$$y_{j+1} = y_j + h \sum_{n=0}^m b_n f_{j-n+1}$$

→ Adams Bashforth technique (when $b_0 = 0$)

→ Because $b_0 = 0$, the method is explicit, i.e. does not require calculation of f_{j+1} .

→ E.g. for a 3-step Adams Bashforth method,

$$y_{j+1} = y_j + h(b_1 f_j + b_2 f_{j-1} + b_3 f_{j-2})$$

How do we get the b -values? Construct an interpolating polynomial for f on x_j, x_{j-1}, x_{j-2} , and integrate it

Order	Adams-Bashforth	Adams-Moulton
2	$0, \frac{3}{2}, -\frac{5}{2}$	$\frac{9}{2}, \frac{8}{2}, -\frac{1}{2}$
3	$0, \frac{23}{12}, -\frac{16}{12}, \frac{5}{12}$	$\frac{9}{24}, \frac{19}{24}, -\frac{5}{24}, \frac{1}{24}$
4	$0, \frac{55}{24}, -\frac{59}{24}, \frac{37}{24}, \frac{9}{24}$	$\frac{251}{720}, \frac{646}{720}, -\frac{264}{720}, \frac{106}{720}, -\frac{19}{720}$
5	$0, \frac{1901}{720}, -\frac{2774}{720}, \frac{2616}{720}, -\frac{1274}{720}, \frac{211}{720}$	/

ODEs - BVP

→ Consider a BVP: $y'' = f(x, y, y')$, $a \leq x \leq b$, $y(a) = \alpha$, $y(b) = \beta$

→ Shooting method (reduce BVP to IVP)

→ Existence theorem:

If $f, f_y \in F_Y$, it continues on $a \leq x \leq b$, and:

i) $f_y(x, y, y') > 0$ in $a \leq x \leq b$

ii) $|f_y(x, y, y')| \leq M$ in $a \leq x \leq b$

→ Linear BVP: $y'' = f(x, y, y')$

$$= p(x) y' + q(x) y + r(x)$$

→ If p, q , & r are continuous, and $q > 0$ on $a \leq x \leq b$,
then the BVP has a unique soln.

→ Split into two IVPs:

$$\textcircled{1} \quad y'' = p(x) y' + q(x) y + r(x) \quad y(a) = \alpha, \quad y'(a) = 0$$

$$\textcircled{2} \quad y'' = p(x) y' + q(x) y \quad y(a) = 0, \quad y'(a) = 1$$

If $\textcircled{1}$ has soln. y_1 & $\textcircled{2}$ has soln y_2 ,

→ Finite-difference methods for linear problems:

→ Consider the 2nd order BVP:

$$y'' = p(x) \frac{dy}{dx} + q(x) y + r(x), \quad x \in [a, b], \quad y(a) = \alpha, \\ y(b) = \beta$$

If $p(x), q(x), r(x)$ are continuous on $[a, b]$, and $q(x) > 0$

on $[a, b]$, then the tridiagonal linear system has a unique

sln if $h < \frac{2}{\max_{[a,b]} |p(x)|}$

→ Finite difference methods for Non-linear problems

Consider the BVP:

$$y'' = f(x, y, y'), \quad x \in [a, b], \quad y(a) = \alpha, \quad y(b) = \beta$$

If: 1) f, f_y , & f_{yy} are continuous on $x \in [a, b]$

2) $f_y \geq s > 0$ on $x \in [a, b]$

3) There exists constants K & L s.t:

$$K = \max_{x \in [a, b]} |f_y| \quad \& \quad L = \max_{x \in [a, b]} |f_{yy}|$$

Then there exists a unique soln. to the BVP.

Divide the interval into $(N+1)$ equal sub-intervals, and replace
 $y''(x_i)$ & $y'(x_i)$ with the central difference formulae:

$$y'' = f(x, y, y') \quad \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = f(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h})$$

There will be a unique soln. if $h < \frac{2}{L}$

PDEs

Second order P.D.E: $A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F_u = 0$

- A, B, ..., F can be functions of x, y
- If $B^2 - 4AC = 0$, parabolic egn.
- " " > 0 , hyperbolic "
- " " < 0 , elliptic "