DS 288 : NUMERICAL METHODS

Sep-23-2021

ROMBERG INTEGRATION

[§ 4.5 - BURDEN & FAIRES]

* CONCEPT: CAN BE USED WITH ANY NEWTON-COTES FORMULA

IDEA: COMPUTE QUADRATURE RULE WITH TWO DIFFERENT h INTERVALS & COMBINE THEM TO CANCEL THE ERROR TERM, THEREBY YIELDING A BETTER APPROXIMATION.

- SOMEWHAT ANALOGOUS TO NEVILLE'S METHOD

- SUCCESSIVE APPLICATIONS INCREASE THE ORDER OF ERROR TERM

FROM $O(h^{2N})$ TO $O(h^{2N+2})$

- SOMEWHAT INEFFICIENT, BUT EASY TO CODE AND ALLOWS ACCURACY TO ARBITRARY VALUE (AT LEAST TO WITHIN $\epsilon_{ROUNDOFF}$)

Ex:- TRPEZOIDAL RULE (COMPOSITE)

$$I(b) = \frac{h}{2}\left[f_0 + 2\sum_{i=1}^{N-1} f_i + f_N\right] + \underbrace{Ah^2 + Bh^4 + Ch^6 + \cdots}_{\text{ERROR TERMS}}$$

WITH A, B, C BEING CONSTANTS THAT ARE INDEPENDENT OF 'h' BUT THEY INVOLVE DERIVATIVES OF 'f' $\left[\underline{Ex}: A = \frac{f''(\xi)}{12}\right]$

DEFINE: $R_{\underline{N}} = \frac{h}{2}\left[f_0 + 2\sum_{i=1}^{N-1} f_i + f_N\right]$

TRAPEZOIDAL RULE ON N PANELS OF SIZE 'h'

NOW LOOK AT $I(b)$ FOR TWO DIFFERENT PANEL SIZES h_1, h_2

WITH $h_2 = \frac{h_1}{2}$

$2N$ PANELS OF h_2 & N PANELS OF h_1

$$4\times\left[\, I(b) = R_{2N} + Ah_2^2 + Bh_2^4 + Ch_2^6 + \cdots \right]$$

$$I(b) = R_N + Ah_1^2 + Bh_1^4 + Ch_1^6 + \cdots$$

$$= R_N + A(4h_2^2) + B(16h_2^4) + C(64h_2^6) + \cdots$$

$-$

$$3I(b) = 4R_{2N} - R_N - 12Bh_2^4 - 60h_2^6 - \cdots$$

[CANCELLED THE Ah_2^2 TERM]

$$I(b) = \frac{4R_{2N} - R_N}{3} - 4Bh_2^4 - 20Ch_2^6 - \cdots$$

$$R_{2N} \to O(h_2^2) = O\left(\frac{h_1^2}{4}\right) > O\left(\frac{h_1^4}{16}\right)$$

$$R_N \to O(h_1^2)$$

* EACH APPLICATION INCREASES ORDER OF ERROR TERM BY h_1^2

* COMPOSITE TRAPEZOIDAL RULE IS WHAT WE HAVE USED, THE SAME ANALOGY EXISTS FOR SINGLE APPLICATION.

[TEXT BOOK: COMPOSITE SIMPSON'S RULE]

GENERALIZE:

m = # OF PANEL DOUBLINGS ($h_2 = \frac{h_1}{2}$)

j = # OF ERROR REMOVALS (A, B, C, ETC.)

IN GENERAL:

$$\boxed{R_{m,j} = \frac{4^j R_{m,j-1} - R_{m-1,j-1}}{4^j - 1}} + O\left(\frac{h}{2^m}\right)^{2j+2}$$

ERROR

PANEL DOUBLING

ROMBERG TABLE

ERROR REMOVAL

INCREASING ACCURACY

	$m \backslash j$	0	1	2	3	4
h	0	$R_{0,0}$				
$h/2$	1	$R_{1,0}$	$R_{1,1}$			
$h/4$	2	$R_{2,0}$	$R_{2,1}$	$R_{2,2}$		
$h/8$	3	$R_{3,0}$	$R_{3,1}$	$R_{3,2}$	$R_{3,3}$	
$h/16$	4	$R_{4,0}$	$R_{4,1}$	$R_{4,2}$	$R_{4,3}$	$R_{4,4}$

$$\text{ACCURACY } O\left(\frac{h}{2^4}\right)^{2\times 4+2}$$

$$= O\left(\frac{h^{10}}{16^{10}}\right) \rightarrow \text{VERY ACCURATE}$$

NOTE THAT:

$R_{m,0} \rightarrow$ TRAPEZOIDAL RULE FOR DIFFERENT PANEL #'S

$R_{m,1} \rightarrow$ SIMPSON'S RULE FOR DIFFERENT PANEL #'S

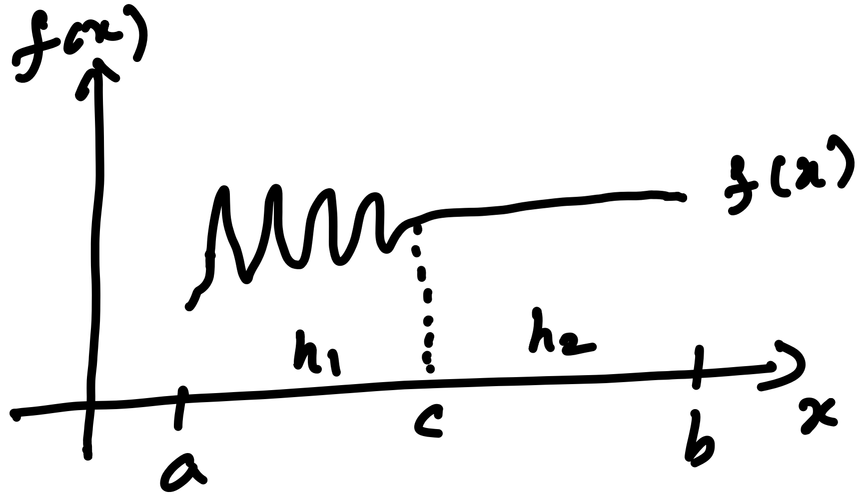
$R_{m,2} \rightarrow$ BOOLE'S RULE FOR DIFFERENT PANEL #'S

$R_{m,m}$ CONVERGES FASTER THAN $R_{m,0}$ (OR) $R_{m,1}$ (OR) $R_{m,2}$ AS LONG AS $m > 2$

CALCULATION OF $R_{m,m}$ REQUIRES $R_{m,0}$ AT LEAST. AND FOLLOWS REST OF PREVIOUSLY CALCULATED VALUES

ADAPTIVE QUADRATURE [§4.6 IN TEXT]

- RULES SO FAR HAVE BEEN BASED ON UNIFORM SAMPLING (CONSTANT h).



POSSIBLY $h_2 > h_1$

- WHAT WE REALLY WANT IS UNIFORMLY DISTRIBUTED ERROR, NOT FUNCTION SAMPLING.

* SOME REGION OF INTERVAL REQUIRE SMALLER (MORE) SAMPLING TO ACHEIVE SAME ERROR.

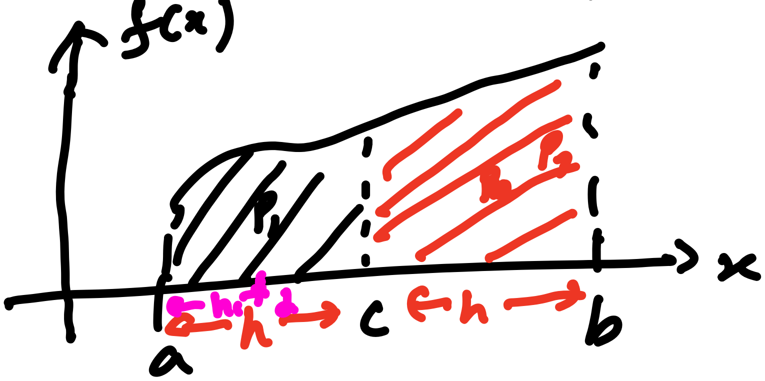
- WE DO NOT ERROR. BUT WE CAN ESTIMATE ε_{TRUNC}. THEN ADAPT SAMPLE SPACING ACCORDINLY.

- ADAPTIVE QUDRATURE RULE FOR SIMPSON'S METHOD. (SINGLE APPLECATION)

$$I(b) = \int_a^b f(x)\,dx = \frac{h}{3}\left[f_a + 4f_c + f_b\right] - \frac{h^5 f^4(\xi)}{90}$$

DEFINE $S(a,b) \equiv$ SIMPSON'S RULE ON $[a,b]$

$$I(b) = S(a,b) - \frac{h^5 f^4(\xi)}{90} \quad \text{—— I}$$



APPLY SIMPSONS RULE ON PANEL P1 & P2

$$I(b) = S(a,c) - \frac{h_1^5 f^4(\xi_1)}{90} \qquad \xi_1 \in [a,c]$$

$$+ S(c,b) - \frac{h_1^5 f^4(\xi_2)}{90} \quad \xi_2 \in [c,b]$$

LET $\overline{f^4(\xi)} = \frac{f^4(\xi_1) + f^4(\xi_2)}{2}$ (MEAN)

$$I(b) = S(a,c) + S(c,b) - \frac{2\left(\frac{h}{2}\right)^5 \overline{f^4(\xi)}}{90}$$

$[h_1 = h/2]$ $\leftarrow$ II

EQUATE I & II

$$S(a,b) - \frac{h^5 f^4(\xi)}{90} = S(a,c) + S(c,b) - \frac{1}{16}\left[\frac{h^5 \overline{f_4(\xi)}}{90}\right]$$

ASSUME THAT $f^4(\xi) = \overline{f^4(\xi)}$

$$\frac{15}{16}\left[\frac{h^5 f^4(\xi)}{90}\right] = S(a,b) - S(a,c) - S(c,b)$$

$\longrightarrow$ APPROXIMATION TO ERROR

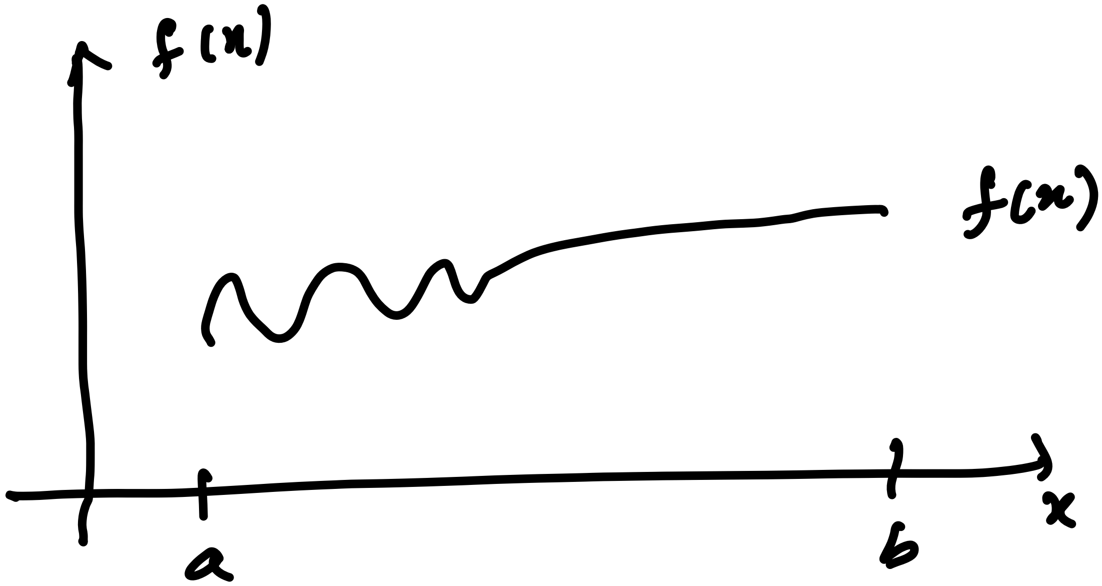
$$\left| \int_a^b f(x)\,dx - S(a,c) - S(c,b) \right| \cong \frac{1}{15} \left| S(a,b) - S(a,c) - S(c,b) \right|.$$

* $S(a,c) + S(c,b)$ APPROXIMATES $I(b)$ 15 TIMES BETTER THAN. $S(a,b)$

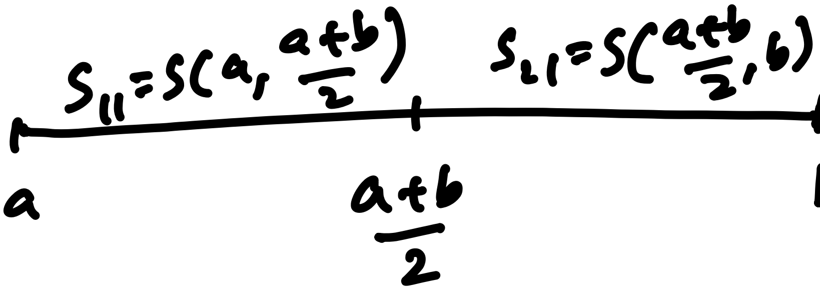
- SO EACH SUCCESSIVE APPLICATION OF SIMPSON'S RULE ON AN INTERVAL $[a,b]$ WITH $h = h/2$ IS ROUGHLY 15 TIMES MORE ACCURATE.

- HOW TO USE IT?

  * WANT $I(b)$ TO ACCURACY ε



$S(a,b) = S_{10}$ → SECTION
→ REFINEMENT

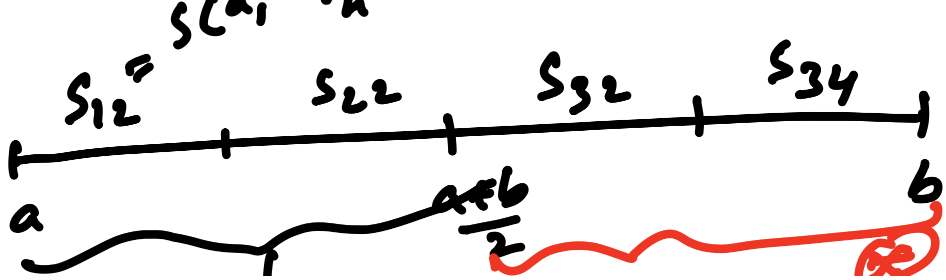


$$|S_{10} - S_{11} - S_{21}| \le 15\epsilon$$

STOP $\left| \int_a^b f(x)\,dx - S_{11} - S_{21} \right| < \epsilon$

ELSE

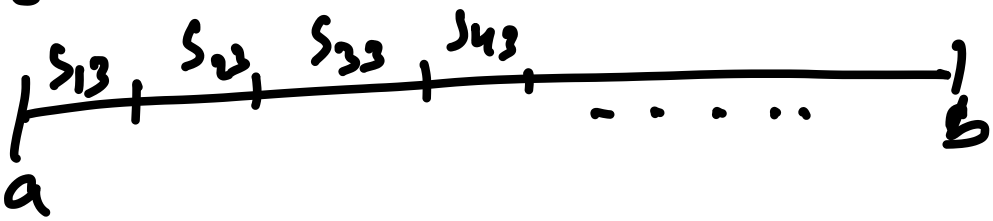
$S_{12} = S(a, \frac{a+b}{4})$



IF $|S_{11} - S_{12} - S_{22}| < 15(\frac{\epsilon}{2})$

STOP $|\int_{1,2} f(x)dx - S_{12} - S_{22}| < \epsilon/2$

ELSE



FOR (*)

IF $|S_{21} - S_{32} - S_{42}| < 15(\frac{\epsilon}{2})$

STOP $\Rightarrow |\int_{3,4} f(x)dx - S_{32} - S_{42}| < \frac{\epsilon}{2}$

ELSE $\rightarrow$ WE SUCCESSIVELY APPY SIMPSON'S RULE $(\frac{a+b}{8})$

NOTE THAT: 15 FACTOR WAS DUE TO SIMPSON'S RULE.

* ASSUMPTION $f^4(\xi) = f^4(\xi)$

IN PRACTISE THIS MAY NOT BE TRUE. FACTOR '15' MIGHT BE ON THE HIGHER SIDE. WE CAN USE A CONSERVATIVE FACTOR (< 15) IS USED IN PRACTISE.

* GAUSSIAN QUADRATURE

[§4.7 TEXT]

* MOTIVATION: IS THERE ANY OPTIMAL SET OF SAMPLE POINTS FOR BETTER APPROXIMATING

$$I(b) = \int_a^b f(x)\,dx$$

- OPTIMAL IN THE SENSE THAT WE CAN INTEGRATE EXACTLY THE MAXIMUM DEGREE OF POLYNOMIAL USING THE MIN. NUMBER OF SAMPLE POINTS.

QUADRATURE RULE $I(b) \approx \sum_{i=0}^{N} a_i f(x_i)$

THE SPEICAL POINTS :
GAUSS POINTS: WHICH ARE ROOTS OF LEGENDRE POLYNOMIALS.

* BY USING 'N' SUCH GAUSS POINTS ONE CAN EXACTLY INTEGRATE A POYNOMIAL OF DEGREE $\underline{\underline{2N-1}}$.

[NO DERIVATION]