

3) a) For a matrix to be orthogonal, it should satisfy the following:

$$\underset{\sim}{Q}^{-1} = \underset{\sim}{Q}^T, \text{ i.e. inverse of } \underset{\sim}{Q} = \text{transpose of } \underset{\sim}{Q}$$

First let us calculate $\underset{\sim}{Q}^T$: - $\underset{\sim}{Q}$ is given as $\left(\underset{\sim}{I} - \underset{\sim}{S}\right)^{-1} \left(\underset{\sim}{I} + \underset{\sim}{S}\right)$

$$\begin{aligned} \underset{\sim}{Q}^T &= \left[\left(\underset{\sim}{I} - \underset{\sim}{S} \right)^{-1} \left(\underset{\sim}{I} + \underset{\sim}{S} \right) \right]^T \\ &= \left(\underset{\sim}{I} + \underset{\sim}{S} \right)^T \left[\left(\underset{\sim}{I} - \underset{\sim}{S} \right)^{-1} \right]^T \quad (\text{reversal law, i.e. } (AB)^T = B^T A^T) \\ &= \left(\underset{\sim}{I}^T + \underset{\sim}{S}^T \right) \left[\left(\underset{\sim}{I} - \underset{\sim}{S} \right)^T \right]^T \quad ((A+B)^T = A^T + B^T) \\ &= \left(\underset{\sim}{I}^T + \underset{\sim}{S}^T \right) \left[\left(\underset{\sim}{I} - \underset{\sim}{S} \right)^T \right]^{-1} \quad ((\tilde{A})^T = (\tilde{A}^T)^{-1}) \\ &= \left(\underset{\sim}{I} - \underset{\sim}{S} \right) \left(\underset{\sim}{I}^T - \underset{\sim}{S}^T \right)^{-1} \quad (\underset{\sim}{I}^T = \underset{\sim}{I} \text{ and } \underset{\sim}{S}^T = -\underset{\sim}{S}) \\ &= \left(\underset{\sim}{I} - \underset{\sim}{S} \right) \left(\underset{\sim}{I} + \underset{\sim}{S} \right)^{-1} \end{aligned}$$

To prove $\underset{\sim}{Q}$'s orthogonality, we should prove $\underset{\sim}{Q} \cdot \underset{\sim}{Q}^T = \underset{\sim}{I}$. For this we need to rearrange ~~$\underset{\sim}{Q} \cdot \underset{\sim}{Q}^T$~~

Proof of commutability :

In this proof, I want to prove that $(\underset{\sim}{I} + \underset{\sim}{S})(\underset{\sim}{I} - \underset{\sim}{S}) = (\underset{\sim}{I} - \underset{\sim}{S})(\underset{\sim}{I} + \underset{\sim}{S})$

$$\left(\underset{\sim}{I} + \underset{\sim}{S} \right) \left(\underset{\sim}{I} - \underset{\sim}{S} \right) = \underset{\sim}{I}^2 + \underset{\sim}{S} - \underset{\sim}{S} - \underset{\sim}{S}^2 = \underset{\sim}{I} - \underset{\sim}{S}^2$$

$$\left(\underset{\sim}{I} - \underset{\sim}{S} \right) \left(\underset{\sim}{I} + \underset{\sim}{S} \right) = \underset{\sim}{I}^2 - \underset{\sim}{S} + \underset{\sim}{S} - \underset{\sim}{S}^2 = \underset{\sim}{I} - \underset{\sim}{S}^2$$

∴ Hence, the commutability is proved.

This proof shall help us to rearrange $\underset{\sim}{Q}$ & $\underset{\sim}{Q}^T$.

Rearranging \underline{Q}^T :

For this, we have to ~~use the~~ prove commutativity:

$$\underline{(I-S)(I+S)} = \underline{(I+S)(I-S)}$$

$$\underline{(I-S)^{-1}(I-S)(I+S)} = \underline{(I-S)^{-1}(I+S)(I-S)} \quad \begin{matrix} \text{(left multiply} \\ \text{by } (I-S)^{-1}) \end{matrix}$$

$$\underline{I} = \underline{(I+S)^{-1}(I+S)(I-S)}$$

$$\underline{(I+S)(I-S)^{-1}} = \underline{(I-S)^{-1}(I+S)(I-S)(I-S)^{-1}} \quad \begin{matrix} \text{(right} \\ \text{multiply} \\ \text{by } (I-S)^{-1}) \end{matrix}$$

$$\underline{(I+S)(I-S)^{-1}} = \underline{(I-S)^{-1}(I+S)}$$

Hence, \underline{Q} can be written as ~~$(I+S)(I-S)^{-1}$~~

$$\underline{(I+S)(I-S)^{-1}}$$

Now, we can prove $\underline{Q}^T \cdot \underline{Q} = \underline{I}$

$$Q^T Q = [(I-S)(I+S)^{-1}] [(I+S)(I-S)^{-1}]$$

$$= (I-S) [(I+S)^{-1} (I+S)] (I-S)^{-1} \quad \begin{matrix} \text{(distributive} \\ \text{property} \\ \text{of matrix} \\ \text{multiplication}) \end{matrix}$$

$$= (I-S) I (I-S)^{-1}$$

$$= (I-S) (I-S)^{-1}$$

$$= \underline{I}$$

\therefore Hence proven, $\underline{Q} = \underline{Q}^T$, or long as S is skew-symmetric

3-b) Since this is an "if and only if" statement, we shall have to prove both ways.

~~Proof #1:~~ If $\underline{\underline{S}} = \underline{\underline{0}}$, then $\underline{\underline{u}}^T \underline{\underline{S}} \underline{\underline{u}} = \underline{\underline{0}}$.

This is easy to prove. If $\underline{\underline{S}} = \underline{\underline{0}}$, then

$$\begin{aligned} \underline{\underline{u}}^T \underline{\underline{S}} \underline{\underline{u}} &= (\underline{\underline{u}}^T \underline{\underline{S}}) \underline{\underline{u}} && \text{(Distributive property of} \\ &= (\underline{\underline{u}}^T \underline{\underline{0}}) \underline{\underline{u}} && \text{matrix-vector multiplication)} \\ &= \underline{\underline{0}} \cdot \underline{\underline{u}} \\ &= \underline{\underline{0}} \quad (\text{proven}) \end{aligned}$$

~~Proof #2:~~ If $\underline{\underline{u}}^T \underline{\underline{S}} \underline{\underline{u}} = \underline{\underline{0}}$, then $\underline{\underline{S}} = \underline{\underline{0}}$

This proof has to be done in two stages. In the first stage, we need to prove that all diagonal elements of $\underline{\underline{S}}$ are 0. In the second phase, we prove all non-diagonal elements of $\underline{\underline{S}}$ to be 0. This shall hence prove $\underline{\underline{S}} = \underline{\underline{0}}$.

Phase #1: Prove $\{ \forall S_{ii} = 0 \mid i = j \}$

Assume $\underline{\underline{u}} = \underline{\underline{e}}_i$, then ($\underline{\underline{e}}_i$ represents a canonical basis vector)

$$\underline{\underline{e}}_i^T \underline{\underline{S}} \underline{\underline{e}}_i = 0$$

$$S_{ii} = 0 \quad (S_{ii} = \underline{\underline{e}}_i^T \underline{\underline{S}} \underline{\underline{e}}_i)$$

Since i can be any index $1 \leq i \leq m$, all diagonal elements of $\underline{\underline{S}}$ are hence proven to be 0.

Hence, $S_{ii} = 0$.

Phase #2: Prove all non-diagonal elements of \tilde{S} are 0.

Assume $u = \underbrace{e_i}_{\sim} + \underbrace{e_j}_{\sim} \quad \& \quad i \neq j$

$$\text{Hence } u^T \underbrace{\tilde{S} u}_{\sim} = (\underbrace{e_i + e_j}_{\sim})^T \underbrace{\tilde{S} (e_i + e_j)}_{\sim}$$

$$0 = \underbrace{e_i^T e_j}_{\sim} + \underbrace{e_j^T e_j}_{\sim} + \underbrace{e_i^T e_j}_{\sim} + \underbrace{e_j^T e_i}_{\sim}$$

$(S_{ii} \& S_{jj} \text{ are } 0, \text{ as proven in phase #1}) \quad 0 = S_{ii} + S_{jj} + S_{ij} + S_{ji}$

$$(S_{ij} = S_{ji}, \quad 0 = 2S_{ij}; \quad S_{ij} = 0)$$

is given to be symmetric) Hence, $S_{ij} = 0 \text{ for } i \neq j.$

Combining results of both phases, it can be stated that

$$\underbrace{S}_{\sim} = \underbrace{0}_{\sim}.$$

- c) Why Because this statement is an "if and only if" statement, both directions need to be proven.

Proof #1: If S is skew-symmetric, $\underbrace{u^T S u}_{\sim} = 0.$

Now we know that $(\underbrace{u^T S u}_{\sim}) = (\underbrace{u^T S u}_{\sim})^T$ because $\underbrace{u^T S u}_{\sim}$ is a 1×1 matrix, i.e. a scalar value, which has no effect on transpose.

(contd.)

$$\underbrace{u^T}_{\sim} \underbrace{S}_{\sim} \underbrace{u}_{\sim} = (\underbrace{u^T}_{\sim} \underbrace{S}_{\sim} \underbrace{u}_{\sim})^T$$

$$\underbrace{u^T}_{\sim} \underbrace{S}_{\sim} \underbrace{u}_{\sim} = \underbrace{u^T}_{\sim} \underbrace{S^T}_{\sim} (\underbrace{u^T}_{\sim})^T \quad (\text{Law of reversal, i.e. } (ABC)^T = C^T B^T A^T)$$

$$\underbrace{u^T}_{\sim} \underbrace{S}_{\sim} \underbrace{u}_{\sim} = \underbrace{u^T}_{\sim} \underbrace{S^T}_{\sim} \underbrace{u}_{\sim} \quad ((\underbrace{u^T}_{\sim})^T = \underbrace{u}_{\sim})$$

$$\underbrace{u^T}_{\sim} \underbrace{S}_{\sim} \underbrace{u}_{\sim} = \underbrace{u^T}_{\sim} (-\underbrace{S}_{\sim}) \underbrace{u}_{\sim} \quad (\text{Since } S \text{ is skew-symmetric, } \underbrace{S^T}_{\sim} = -\underbrace{S}_{\sim})$$

$$\underbrace{u^T}_{\sim} \underbrace{S}_{\sim} \underbrace{u}_{\sim} = -\underbrace{u^T}_{\sim} S \underbrace{u}_{\sim}$$

$$n = -x \quad (\text{Assume } n = \underbrace{u^T}_{\sim} \underbrace{S}_{\sim} \underbrace{u}_{\sim})$$

$$2n = 0$$

$$n = 0 \quad (\text{Hence proven})$$

\therefore If \underbrace{S}_{\sim} is skew-symmetric, $n = \underbrace{u^T}_{\sim} \underbrace{S}_{\sim} \underbrace{u}_{\sim} = 0$.

Proof#2: If $\underbrace{u^T}_{\sim} \underbrace{S}_{\sim} \underbrace{u}_{\sim} = 0$, then S is skew-symmetric.

Any square matrix can be split into a symmetric & skew symmetric portion as follows:

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric part}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{skew-symmetric part}}$$

Hence, S can be represented as $\frac{1}{2}(\underbrace{S + S^T}_{\sim}) + \frac{1}{2}(\underbrace{S - S^T}_{\sim})$.

If we assume $\underbrace{S}_{\sim} \text{symm} = \frac{1}{2}(\underbrace{S + S^T}_{\sim})$ and $\underbrace{S}_{\sim} \text{skew} = \frac{1}{2}(\underbrace{S - S^T}_{\sim})$,

\underbrace{S}_{\sim} can be re-written as $\underbrace{S}_{\sim} \text{symm} + \underbrace{S}_{\sim} \text{skew}$.

$$\text{Hence, } \underbrace{\mathbf{u}^T}_{\sim} \underbrace{\mathbf{S}}_{\sim} \underbrace{\mathbf{u}}_{\sim} = \mathbf{0}$$

$$\underbrace{\mathbf{u}^T}_{\sim} \left(\underbrace{\mathbf{S}_{\text{symm}}}_{\sim} + \underbrace{\mathbf{S}_{\text{skew}}}_{\sim} \right) \underbrace{\mathbf{u}}_{\sim} = \mathbf{0}$$

$$\underbrace{\mathbf{u}^T}_{\sim} \underbrace{\mathbf{S}_{\text{symm}}}_{\sim} \underbrace{\mathbf{u}}_{\sim} + \underbrace{\mathbf{u}^T}_{\sim} \underbrace{\mathbf{S}_{\text{skew}}}_{\sim} \underbrace{\mathbf{u}}_{\sim} = \mathbf{0}$$

$$\underbrace{\mathbf{u}^T}_{\sim} \underbrace{\mathbf{S}_{\text{symm}}}_{\sim} \underbrace{\mathbf{u}}_{\sim} = \mathbf{0} \quad (\text{Because } \underbrace{\mathbf{u}^T}_{\sim} \underbrace{\mathbf{S}_{\text{skew}}}_{\sim} \underbrace{\mathbf{u}}_{\sim} = \mathbf{0}, \\ \text{as proven in proof \#1 of Question 3) b(c)})$$

$$\therefore \underbrace{\mathbf{S}_{\text{symm}}}_{\sim} = \mathbf{0} \quad (\text{This has been proven in proof \#2 of Question 3) b)})$$

$$\frac{1}{2} \left(\underbrace{\mathbf{S}}_{\sim} + \underbrace{\mathbf{S}^T}_{\sim} \right) = \mathbf{0} \quad (\text{Substituting definition of } \underbrace{\mathbf{S}_{\text{symm}}}_{\sim})$$

$$\underbrace{\mathbf{S}}_{\sim} + \underbrace{\mathbf{S}^T}_{\sim} = \mathbf{0}$$

$$\underbrace{\mathbf{S}^T}_{\sim} = -\underbrace{\mathbf{S}}_{\sim}$$

Since this is the fundamental property of a skew-symmetric matrix, we can henceforth prove that \mathbf{S} is a skew-symmetric matrix.

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Course Code: DS284
Course Name: Numerical Linear Algebra
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Question 5

Calculated Quantities

Since the matrix \mathbf{A} and the vectors are random, the output of each code run will vary. However, the quantities calculated by one code run is as follows (in tabular format):

p-value	Maximum value of $\ \mathbf{A}\hat{\mathbf{x}}\ _p$	Actual value of $\ \mathbf{A}\ _p$	Order of Relative Error
1	83.6969	83.6971	10^{-6}
2	10.5423	10.5423	10^{-10}
∞	4.3310	4.3301	10^{-4}

Although the program has output numbers to up to 14 decimal places, I have rounded them down to 4 decimal places, for better readability. Furthermore, I have not added the other norms of $\mathbf{A}\hat{\mathbf{x}}$, because they do not have a corresponding norm of $\|\mathbf{A}\|$. The other norms, which were calculated, are as follows:

p-value	Maximum value of $\ \mathbf{A}\hat{\mathbf{x}}\ _p$
3	6.1872
4	5.0428
5	4.5759
6	4.3540

Code and Algorithm

The Python code for the calculation is as follows:

```

import numpy as np

#generate random matrix A
A = np.random.randn(100,2)

#this array keeps track of maximum 1-norm, 2-norm etc. of the vector Ax
#max_norm[0] represents maximum 1-norm of Ax,
#max_norm[1] represents maximum 2-norm of Ax, etc
max_norm = [-1,-1,-1,-1,-1,-1,-1]

for _ in range(1000):
    #Generate random x vector
    #The np.random.randn() method returns a matrix of 2x1 instead of a vector
    #the squeeze function is used to convert the 2x1 matrix to a vector
    x = np.squeeze(np.random.randn(2,1))

    #take 1-norm to 6-norm
    #x is converted to unit vector for the particular norm
    #i.e if 1-norm of Ax is to be calculated, x is normalised by its 1-norm, and so on
    for p in range(1,7):
        #Calculate the relevant norm of x and normalise x accordingly
        pnorm_x = np.linalg.norm(x,ord=p)
        unit_x = [element/pnorm_x for element in x]

        #Calculate Ax
        #Again the @ operator produces a matrix, which needs to be squeezed to a vector
        Ax = np.squeeze(A @ unit_x)

        #Update the max_norm
        max_norm[p-1]=max(max_norm[p-1], np.linalg.norm(Ax, ord=p))
    
```

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```

#take infinity norm
#This section is outside the loop because p cannot
#be suddenly assigned to infinity
pnorm_x = np.linalg.norm(x, ord=np.inf)
unit_x = [element/pnorm_x for element in x]
Ax = np.squeeze(A @ unit_x)
max_norm[6]=max(max_norm[6], np.linalg.norm(Ax, ord=np.inf))

print("1-norm of A =", np.linalg.norm(A, ord=1))
print("Max 1-norm of Ax =", max_norm[0], end="\n")

print("2-norm of A =", np.linalg.norm(A, ord=2))
print("Max 2-norm of Ax =", max_norm[1], end="\n")

print("infinity-norm of A =", np.linalg.norm(A, ord=np.inf))
print("Max infinity-norm of Ax =", max_norm[6])

print("Remaining norms of x: ", max_norm[2:6])
    
```

The program steps are as follows:

1. A random matrix with dimensions 100×2 is created. This is the **A** matrix.
2. A list called `max_norm` is created with 7 elements. This list shall store the maximum of the norms. The first element of the list, i.e. `max_norm[0]`, contains the maximum 1-norm of **Ax**. The second element stores the maximum 2-norm, and so on. The last element shall store the ∞ -norm of **Ax**.
3. A for loop is present, which shall generate 1000 random **x** vectors, such that $\mathbf{x} \in \mathbb{R}^{100}$. This vector is then normalised. Its norm is calculated and updated to the `max_norm` list (if it is the maximum).
4. Lastly, the 1-norm, 2-norm and ∞ -norm of **A** is calculated, to compare with the maximum values obtained from the iterations above.

Sample Output

Since the matrix **A** and the vectors are random, the output of each code run will vary. However, one such output is as follows:

```

1-norm of A = 77.14076257113005
Max 1-norm of Ax = 77.12304412473856
2-norm of A = 9.568208854655976
Max 2-norm of Ax = 9.568208654815267
infinity-norm of A = 4.49453960570396
Max infinity-norm of Ax = 4.487772423831027
Remaining norms of x: [np.float64(5.627964886251967), np.float64(4.679666918956469),
np.float64(4.341368483515785), np.float64(4.219939313675693)]
    
```

Observations

From the above table, we can devise the following:

$$\|A\hat{x}\|_p^m \leq \|A\|_p^{(m,n)}, \forall x \in \mathbb{R}^n, \|\hat{x}\|_p = 1$$

From this, we can also infer that there is one $\widehat{x_{max}}$ such that the above inequality is converted to equality, i.e.

$$\|A\widehat{x_{max}}\|_p^m = \|A\|_p^{(m,n)}, \|\widehat{x_{max}}\|_p = 1$$