Suhas Kumath
25945

Numerical Linear Algebra - Assignment 4

7) a)

True. The proof is as follows:

Let λ be an eigenvalue of $\underset{\sim}{A}$. We hence know there exists an $\underset{\sim}{x}$ s.t.

$$\underset{\sim}{A}\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

We want to show that $(\underset{\sim}{A} - \mu \underset{\sim}{I})\hat{\underset{\sim}{x}} = (\lambda - \mu)\hat{\underset{\sim}{x}}$

$$\begin{aligned} \text{L.H.S.} = (\underset{\sim}{A} - \mu \underset{\sim}{I})\underset{\sim}{x} &= \underset{\sim}{A}\underset{\sim}{x} - \mu \underset{\sim}{I}\underset{\sim}{x} && [\text{Opening the bracket}] \\ &= \underset{\sim}{A}\underset{\sim}{x} - \mu \underset{\sim}{x} && [\underset{\sim}{I}\hat{\underset{\sim}{x}} = \hat{\underset{\sim}{x}}] \\ &= \lambda \underset{\sim}{x} - \mu \underset{\sim}{x} && [\underset{\sim}{A}\underset{\sim}{x} = \lambda \underset{\sim}{x}] \\ &= (\lambda - \mu)\underset{\sim}{x} && [\text{Factorise w.r.t. } \underset{\sim}{x}] \end{aligned}$$

Hence proven, L.H.S. = R.H.S..

7) b)

False. The counter-example is as follows:

Consider a 2x2 matrix $\underset{\sim}{A} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

Let us calculate the eigen values of $\underset{\sim}{A}$:

$$|\underset{\sim}{A} - \lambda \underset{\sim}{I}| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda) = 0$$

$$\lambda = 1, 2$$

If the statement is true, $\lambda = -1$ & $\lambda = -2$ must also be an eigenvalue. However, we just calculated both eigenvalues of $\underset{\sim}{A}$ and neither -1 nor -2 belong to the calculated values. Hence neither -1 nor -2 can be eigenvalues of $\underset{\sim}{A}$. This itself proves the statement to be false.

7) c) True. The proof is as follows:

Let $\underset{\sim}{A}$ be a real matrix, and λ be its complex-valued eigenvalue.

Then, $\underset{\sim}{A}\,\underset{\sim}{x} = \lambda \underset{\sim}{x}$

$\overline{(\underset{\sim}{A}\,\underset{\sim}{x})} = \overline{(\lambda \underset{\sim}{x})}$ [Take conjugate of both sides]

$\overline{\underset{\sim}{A}}\,\overline{\underset{\sim}{x}} = \overline{\lambda}\,\overline{\underset{\sim}{x}}$ [Property of conjugate : $\overline{(MN)} = \overline{M} \cdot \overline{N}$]

$\underset{\sim}{A}\,\overline{\underset{\sim}{x}} = \lambda^{*}\,\overline{\underset{\sim}{x}}$ [$\overline{A} = A$ because A is a real matrix]

Because $\underset{\sim}{x}$ is an eigenvector $\underset{\sim}{x} \neq \underset{\sim}{0}$. Therefore $\overline{\underset{\sim}{x}}$ also cannot be zero vector.

Hence proven, if λ is an eigenvalue of $\underset{\sim}{A}$, λ^{*} will also be an eigenvalue of $\underset{\sim}{A}$.

7) d) True. The proof is as follows:

We need to first prove that $\lambda^{-1} = 1/\lambda$ exists. For this, λ cannot be equal to 0.

We know that $\lambda \neq 0$. This is because $\underset{\sim}{A}$ is non-singular, i.e full rank. We also know that no. of non-zero eigenvalues of $\underset{\sim}{A}$ = rank of $\underset{\sim}{A}$.

Another way to prove it is as follows: (Proof by contradiction)

Let $\underset{\sim}{A}\,\underset{\sim}{x} = \lambda \underset{\sim}{x}$, then

$\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = 0$

Let us try $\lambda = 0$:

$$\det(\underset{\sim}{A} - 0\underset{\sim}{I}) = 0$$

$$\det(\underset{\sim}{A}) = 0$$ $\Rightarrow$ We know this cannot be true because A is given to be non-singular.

Let λ be the eigen value of $\underset{\sim}{A}$. Then:

$$\underset{\sim}{A}\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

$$\underset{\sim}{A}^{-1}\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{A}^{-1}\lambda\underset{\sim}{x}$$ [Left-multiply both sides by $\underset{\sim}{A}^{-1}$]

$$\underset{\sim}{x} = \lambda \underset{\sim}{A}^{-1}\underset{\sim}{x}$$ [$\underset{\sim}{A}^{-1} \cdot \underset{\sim}{A} = \underset{\sim}{I}$ and $\underset{\sim}{I}\underset{\sim}{x} = \underset{\sim}{x}$]

$$\underset{\sim}{A}^{-1}\underset{\sim}{x} = \frac{1}{\lambda}\underset{\sim}{x}$$ [Divide both sides by λ]

$$\underset{\sim}{A}^{-1}\underset{\sim}{x} = \lambda^{-1}\underset{\sim}{x}$$

Hence proven, λ^{-1} is the eigen value of $\underset{\sim}{A}^{-1}$, as long as λ is an eigenvalue of $\underset{\sim}{A}$ and $\underset{\sim}{A}$ is non-singular.

7) e)

False. Counter-example is as follows:

Let $\underset{\sim}{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

The eigenvalues of $\underset{\sim}{A}$ are 0, 0, whereas not all entries of $\underset{\sim}{A}$ are 0.

$$\boxed{\begin{aligned} &|A - \lambda I| = 0 \\ &\begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = 0 \\ &(-\lambda)^2 = 0 \\ &\lambda = 0, 0 \end{aligned}}$$

7) f)

True. The proof is as follows:

Because $\underset{\sim}{A}$ is diagonalisable, $\underset{\sim}{A}$ can be expressed as:

$\underset{\sim}{A} = \underset{\sim}{P}\,\underset{\sim}{D}\,\underset{\sim}{P}^{-1}$, where $\underset{\sim}{D}$ is a diagonal matrix, and $\underset{\sim}{P}$ is an invertible matrix.

The diagonal entries of $\underset{\sim}{D}$ are the eigenvalues of $\underset{\sim}{A}$. Because all eigenvalues of $\underset{\sim}{A}$ are given to be λ, we can say that:

$$\underset{\sim}{D} = \begin{bmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \ddots \; \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix} = \lambda \underset{\sim}{I},$$

where $\underset{\sim}{I}$ is the identity matrix.

Substituting $\underset{\sim}{D} = \lambda \underset{\sim}{I}$ into the original equation,

$$\begin{aligned} \underset{\sim}{A} = \underset{\sim}{P}\,\underset{\sim}{D}\,\underset{\sim}{P}^{-1} &= \underset{\sim}{P}(\lambda \underset{\sim}{I})\underset{\sim}{P}^{-1} && [\underset{\sim}{D} = \lambda \underset{\sim}{I}] \\ &= \lambda(\underset{\sim}{P}\,\underset{\sim}{I}\,\underset{\sim}{P}^{-1}) && [\text{Re-arrange to take scalar out}] \\ &= \lambda(PP^{-1}) && [\underset{\sim}{P}\,\underset{\sim}{I} = \underset{\sim}{P}] \\ &= \lambda \underset{\sim}{I} && [\underset{\sim}{P}\,\underset{\sim}{P}^{-1} = \underset{\sim}{I}] \end{aligned}$$

Hence, because $\underset{\sim}{A} = \lambda \underset{\sim}{I}$, and I is diagonal, we know that $\underset{\sim}{A}$ is also diagonal.