Q) 2) a) FALSE. We know of only one definite method of solving for eigenvalues: solving characteristic equations. However, solving characteristic equations is unstable. Even if we have an infinite precision computer, we do not have a method to solve characteristic equations above degree 5. To conclude, we do not have a numerically stable direct method to calculate eigenvalues for $m \geq 5$ even if we possess an infinite precision computer.

Q) 2) b) FALSE. Most eigensolvers employ a two phase method: the first phase converts the matrix to an upper Hessenberg matrix. An upper Hessenberg matrix is a upper triangular matrix with an addition 'line' of non-zero elements parallel to the diagonal. This phase has a time complexity of $O(m^3)$. The second phase further reduces the upper Hessenberg matrix to an upper triangular matrix using $O(m)$ iterations, each with $O(m^2)$ iterations, thereby bringing the time complexity of phase 2 to $O(m^3)$.

If phase 1 is not performed, we shall have to reduce a full (dense) matrix in $O(m)$ iterations, with each iteration requiring $O(m^3)$ operations. This shall raise the time complexity to $O(m^4)$.

To conclude, if a matrix does not go through phase-1, we will not be able to get the eigenvalue in $O(m^3)$ time. We shall require $O(m^4)$ time.

2) c) FALSE. In power iteration, we need to provide an initial "guess" vector denoted by $\underset{\sim}{v}^{(0)}$. The entire power iteration depends on the premise that $\underset{\sim}{v}^{(0)}$ is not orthogonal to the eigenvector corresponding to the largest eigenvalue. In other words, if the eigenvector corresponding to the largest eigenvalue is denoted by $\underset{\sim}{x}_1$, $\underset{\sim}{x}_1^T \underset{\sim}{v}^{(0)} \neq 0$ for $v^{(k)}$ to converge to $\underset{\sim}{x}_1$. Hence, not all vectors which are non-zero can qualify as an initial guess.

Q2) d) TRUE

We know that a householder matrix $\underset{\sim}{F}$ can be constructed using the following formula:

$$\underset{\sim}{F} = \underset{\sim}{I} - \frac{2\underset{\sim}{u}\underset{\sim}{u}^T}{\underset{\sim}{u}^T\underset{\sim}{u}}$$

Let us check whether $\underset{\sim}{F}$ is orthogonal. For $\underset{\sim}{F}$ to be orthogonal, it should satisfy the following property:

$$\underset{\sim}{F} \cdot \underset{\sim}{F}^T = \underset{\sim}{I}$$

Let us first find $\underset{\sim}{F}^T$:

$$\underset{\sim}{F} = \underset{\sim}{I} - \frac{2\underset{\sim}{u}\underset{\sim}{u}^T}{\underset{\sim}{u}^T\underset{\sim}{u}}$$

$$\underset{\sim}{F}^T = \left(\underset{\sim}{I} - \frac{2\underset{\sim}{u}\underset{\sim}{u}^T}{\underset{\sim}{u}^T\underset{\sim}{u}}\right)^T$$

$$= \underset{\sim}{I}^T - \left(\frac{2}{\underset{\sim}{u}^T\underset{\sim}{u}} \cdot \underset{\sim}{u}\underset{\sim}{u}^T\right)^T$$ [Transpose of sum is sum of transpose]

$$= \underset{\sim}{I} - (\underset{\sim}{u}^T)^T \cdot \underset{\sim}{u}^T \cdot \left(\frac{2}{\underset{\sim}{u}^T\underset{\sim}{u}}\right)^T$$ [Reversal law] [$\underset{\sim}{I}^T = \underset{\sim}{I}$]

$$= \underset{\sim}{I} - \underset{\sim}{u} \cdot \underset{\sim}{u}^T \cdot \frac{2}{\underset{\sim}{u}^T\underset{\sim}{u}}$$ [$\frac{2}{\underset{\sim}{u}^T\underset{\sim}{u}}$ is a constant]

$$= \underset{\sim}{I} - \frac{2}{\underset{\sim}{u}^T\underset{\sim}{u}}\underset{\sim}{u}\underset{\sim}{u}^T$$

$$= \underset{\sim}{F}$$

(Hence proven, $\underset{\sim}{F}$ is symmetric, i.e. $\underset{\sim}{F} = \underset{\sim}{F}^T$)

Hence, let us calculate $\underset{\sim}{F} \cdot \underset{\sim}{F}^T$:

$$\underset{\sim}{F} \cdot \underset{\sim}{F}^T = \underset{\sim}{F}^2 \qquad [\underset{\sim}{F} = \underset{\sim}{F}^T]$$

$$= \left(\underset{\sim}{I} - \frac{2}{\underset{\sim}{u}^T \underset{\sim}{u}} \underset{\sim}{u}\,\underset{\sim}{u}^T\right)\left(\underset{\sim}{I} - \frac{2}{\underset{\sim}{u}^T \underset{\sim}{u}} \underset{\sim}{u}\,\underset{\sim}{u}^T\right)$$

$$= \underset{\sim}{I}^2 - \frac{2}{\underset{\sim}{u}^T \underset{\sim}{u}} \underset{\sim}{u}\,\underset{\sim}{u}^T \cdot \underset{\sim}{I} - \underset{\sim}{I} \cdot \frac{2}{\underset{\sim}{u}^T \underset{\sim}{u}} \underset{\sim}{u}\,\underset{\sim}{u}^T + \frac{2}{\underset{\sim}{u}^T \underset{\sim}{u}} \underset{\sim}{u}\,\underset{\sim}{u}^T \cdot \frac{2}{\underset{\sim}{u}^T \underset{\sim}{u}} \underset{\sim}{u}\,\underset{\sim}{u}^T$$

$$= \underset{\sim}{I} - \frac{4}{\underset{\sim}{u}^T \underset{\sim}{u}} \underset{\sim}{u}\,\underset{\sim}{u}^T + \frac{4}{(\underset{\sim}{u}^T \underset{\sim}{u})^2} \cdot \underset{\sim}{u}\,\underset{\sim}{u}^T \cdot \underset{\sim}{u}\,\underset{\sim}{u}^T$$

$$= \underset{\sim}{I} - \frac{4}{\underset{\sim}{u}^T \underset{\sim}{u}} \underset{\sim}{u}\,\underset{\sim}{u}^T + \frac{4}{(\underset{\sim}{u}^T \underset{\sim}{u})^{\cancel{2}}} \cdot \underset{\sim}{u}\cancel{(\underset{\sim}{u}^T \underset{\sim}{u})}\underset{\sim}{u}^T$$

$$= \underset{\sim}{I} - \frac{4}{\underset{\sim}{u}^T \underset{\sim}{u}} \underset{\sim}{u}\,\underset{\sim}{u}^T + \frac{4}{\underset{\sim}{u}^T \underset{\sim}{u}} \underset{\sim}{u}\,\underset{\sim}{u}^T$$

$$= \underset{\sim}{I}$$

(Hence proven, $\underset{\sim}{F}$ is orthogonal)

Hence, because $\underset{\sim}{F}$ is orthogonal, we know that $\underset{\sim}{F}^T = \underset{\sim}{F}^{-1}$.

Let $\underset{\sim}{F}\underset{\sim}{A}\underset{\sim}{F}^T = \underset{\sim}{F}\underset{\sim}{A}\underset{\sim}{F}^{-1} = \underset{\sim}{B}$

Then $\underset{\sim}{A} = \underset{\sim}{F}^{-1}\underset{\sim}{B}\underset{\sim}{F}$

$$= \underset{\sim}{F}^{-1}\underset{\sim}{B}(\underset{\sim}{F}^{-1})^{-1}$$

Proof of $\underset{\sim}{A} = \underset{\sim}{F}^{-1}\underset{\sim}{B}\underset{\sim}{F}$:

$$\underset{\sim}{A} = \underset{\sim}{F}^{-1} \cdot \underset{\sim}{F}\underset{\sim}{A}\underset{\sim}{F}^{-1} \cdot \underset{\sim}{F}$$

$$= \underset{\sim}{I} \cdot \underset{\sim}{A} \cdot \underset{\sim}{I}$$

$$= \underset{\sim}{A}$$

Hence, because a similarity transformation exists between $\underset{\sim}{A}$ and $\underset{\sim}{B} = \underset{\sim}{F}\underset{\sim}{A}\underset{\sim}{F}^T$, $\underset{\sim}{A}$ and $\underset{\sim}{F}\underset{\sim}{A}\underset{\sim}{F}^T$ have the same eigenvalues. However, they may differ in eigenvectors.

3) a) We are given that $\underset{\sim}{A}\underset{\sim}{u}_i = \lambda_i \underset{\sim}{B}\underset{\sim}{u}_i$.

Because $\underset{\sim}{B}$ is a symmetric positive definite matrix, we can perform a Cholesky decomposition on $\underset{\sim}{B}$ such that $\underset{\sim}{B} = \underset{\sim}{R}\underset{\sim}{R}^T$.

Hence, the original equation can be re-written as:

$$\underset{\sim}{A}\underset{\sim}{u}_i = \lambda_i (\underset{\sim}{R}\underset{\sim}{R}^T) u_i$$ [Substitute $\underset{\sim}{B} = \underset{\sim}{R}\underset{\sim}{R}^T$]

$$\underset{\sim}{R}^{-1}\underset{\sim}{A}\underset{\sim}{u}_i = \lambda_i (\underset{\sim}{R}^{-1}\underset{\sim}{R}\underset{\sim}{R}^T\underset{\sim}{u}_i)$$ [Pre-multiply with $\underset{\sim}{R}^{-1}$, we know $\underset{\sim}{R}$ is non-singular due to properties of Cholesky decomposition]

$$\underset{\sim}{R}^{-1}\underset{\sim}{A}\underset{\sim}{u}_i = \lambda_i \underset{\sim}{R}^T\underset{\sim}{u}_i$$

$$\underset{\sim}{R}^{-1}\underset{\sim}{A} \cdot (\underset{\sim}{R}^T)^{-1}\underset{\sim}{v}_i = \lambda_i \underset{\sim}{v}_i$$ [Let $\underset{\sim}{R}^T\underset{\sim}{u}_i = \underset{\sim}{v}_i$, then $\underset{\sim}{u}_i = (\underset{\sim}{R}^T)^{-1}\underset{\sim}{v}_i$]

$$\therefore \underset{\sim}{H}\underset{\sim}{v}_i = \lambda_i \underset{\sim}{v}_i$$ [Let $\underset{\sim}{H} = \underset{\sim}{R}^{-1}\underset{\sim}{A}(\underset{\sim}{R}^T)^{-1}$]

Let us also check if $\underset{\sim}{H}$ is symmetric:

$$\underset{\sim}{H}^T = (\underset{\sim}{R}^{-1}\underset{\sim}{A}(\underset{\sim}{R}^T)^{-1})^T = ((\underset{\sim}{R}^T)^{-1})^T \underset{\sim}{A}^T (\underset{\sim}{R}^{-1})^T$$

$$= ((\underset{\sim}{R}^T)^T)^{-1} \underset{\sim}{A}^T (\underset{\sim}{R}^T)^{-1}$$ [$(\underset{\sim}{R}^T)^{-1} = (\underset{\sim}{R}^{-1})^T$]

$$= R^{-1}\underset{\sim}{A}^T(R^T)^{-1}$$ [$(\underset{\sim}{R}^T)^T = \underset{\sim}{R}$]

$$= \underset{\sim}{H}$$ (shown)

Q3) b) Because we are given a value 2.0 and are told that this value is close to an eigenvalue λ_i, we can use inverse power iterations. The algorithm for this iteration is as follows:

1 Initialise $\mu = 2.0$ // value close to eigenvalue

2 Initialise $\underset{\sim}{v}^{(0)}$ = some random vector

3 For $k = 1 \longrightarrow \infty$:

4 $\quad \underset{\sim}{w} = (\underset{\sim}{H} - \mu \underset{\sim}{I})^{-1} \underset{\sim}{v}^{(k-1)}$ // We can solve for $\underset{\sim}{w}$ in $(\underset{\sim}{H} - \mu \underset{\sim}{I})\underset{\sim}{w} = \underset{\sim}{v}^{(k-1)}$ to get $\underset{\sim}{w}$. This avoids the calculation of matrix inverse.

5 $\quad \underset{\sim}{v}^{(k)} = \dfrac{\underset{\sim}{w}}{\|\underset{\sim}{w}\|}$ // Normalise $\underset{\sim}{w}$

6 $\quad \lambda^{(k)} = (\underset{\sim}{v}^{(k)})^T \underset{\sim}{H} (\underset{\sim}{v}^{(k)})$

$\underset{\sim}{u}_i = (\underset{\sim}{L}^T)^{-1} \underset{\sim}{v}^{(k)}$ // $\underset{\sim}{v}^{(k)}$ is an eigenvector of $\underset{\sim}{H}$, we need to transform it back to the eigenvector in the original problem.

For this algorithm to converge to λ_i as $k \to \infty$, the following conditions must be fulfilled:

i) There must be a unique eigenvalue, i.e. an eigenvalue with unitary algebraic multiplicity, close to the initial value of 2.0.

ii) The random vector chosen in the second step must not be orthogonal to the eigenvector corresponding to the eigenvalue λ_i.

Q3) c) The computationally dominant step is step 4, which is obtaining the solution to the linear system of equations $(\underset{\sim}{H} - \mu \underset{\sim}{I})\underset{\sim}{w} = \underset{\sim}{v}^{(k-1)}$.

Because $\underset{\sim}{A}$ & $\underset{\sim}{B}$ are large sparse matrices, it will be easier for us to work with them.

Let us convert this linear system of equations to use the original sparse matrices:

$$(\underset{\sim}{H} - \mu \underset{\sim}{I})\underset{\sim}{w} = \underset{\sim}{v}^{(k-1)}$$

$$(\underset{\sim}{R}^{-1}\underset{\sim}{A}(\underset{\sim}{R}^{T})^{-1} - \mu \underset{\sim}{I})\underset{\sim}{w} = \underset{\sim}{v}^{(k-1)} \quad \left[\text{Substitute } \underset{\sim}{H} = \underset{\sim}{R}^{-1}\underset{\sim}{A}(\underset{\sim}{R}^{T})^{-1}\right]$$

$$(\underset{\sim}{R}\,\underset{\sim}{R}^{-1}\underset{\sim}{A}(\underset{\sim}{R}^{T})^{-1} - \underset{\sim}{R}(\mu \underset{\sim}{I}))\underset{\sim}{w} = \underset{\sim}{R}\underset{\sim}{v}^{(k-1)} \quad \left[\text{Left multiply } \underset{\sim}{R}\right]$$

$$(\underset{\sim}{A}(\underset{\sim}{R}^{T})^{-1} - \mu \underset{\sim}{R})\underset{\sim}{w} = \underset{\sim}{R}\underset{\sim}{v}^{(k-1)}$$

$$(\underset{\sim}{A} - \mu \underset{\sim}{R}\underset{\sim}{R}^{T})(\underset{\sim}{R}^{T})^{-1}\underset{\sim}{w} = \underset{\sim}{R}\underset{\sim}{v}^{(k-1)} \quad \left[\text{Factorise } (\underset{\sim}{R}^{T})^{-1}\right]$$

$$(\underset{\sim}{A} - \mu \underset{\sim}{B})(\underset{\sim}{R}^{T})^{-1}\underset{\sim}{w} = \underset{\sim}{R}\underset{\sim}{v}^{(k-1)} \quad \left[\text{Substitute } \underset{\sim}{R}\underset{\sim}{R}^{T} = \underset{\sim}{B}\right]$$

$$(\underset{\sim}{A} - \mu \underset{\sim}{B})\underset{\sim}{t} = \underset{\sim}{R}\underset{\sim}{v}^{(k-1)} \quad \left[\text{Let } \underset{\sim}{t} = (\underset{\sim}{R}^{T})^{-1}\underset{\sim}{w}\right]$$

We can solve this equation in a two-step process:

1) Solve for $\underset{\sim}{t}$ in the linear system $(\underset{\sim}{A} - \mu \underset{\sim}{B})\underset{\sim}{t} = \underset{\sim}{R}\underset{\sim}{v}^{(k-1)}$

2) Solve for $\underset{\sim}{w}$ in the linear system $(\underset{\sim}{R}^{T})^{-1}\underset{\sim}{w} = \underset{\sim}{t}$

Because the matrices $\underset{\sim}{A}$ & $\underset{\sim}{B}$ are sparse, this method shall be more computationally efficient than solving directly for $\underset{\sim}{w}$ in the linear system $(\underset{\sim}{H} - \mu \underset{\sim}{I})\underset{\sim}{w} = \underset{\sim}{v}^{(k-1)}$.

Q3) d) Because we are given 2.0 is close to λ_i, and λ_i is the smallest eigenvalue (by magnitude), we can further optimise by tuning a value of μ such that $\mu < \lambda_i$. In this case, we can choose $\mu = 1.5$.

When we tune a value of $\mu < \lambda_i$, we can guarantee that $(\underset{\sim}{A} - \mu \underset{\sim}{B})$ is symmetric, positive & definite (S.P.D.)

This can be easily proven as follows:

S.P.D.'s must follow $\underset{\sim}{x}^T \underset{\sim}{M} \underset{\sim}{x} > 0 \quad \forall \underset{\sim}{x} \in \mathbb{R}^m$

$$\underset{\sim}{x}^T (\underset{\sim}{A} - \mu \underset{\sim}{B}) \underset{\sim}{x} = \underset{\sim}{x}^T \underset{\sim}{A} \underset{\sim}{x} - \mu \underset{\sim}{x}^T \underset{\sim}{B} \underset{\sim}{x}$$
$$\geq (\lambda_{min} - \sigma) \underset{\sim}{x}^T \underset{\sim}{B} \underset{\sim}{x} \quad [\underset{\sim}{A} \text{ is given as S.P.D.}]$$
$$\geq 0 \quad [\underset{\sim}{B} \text{ " " " " }]$$

Hence, because we know $(\underset{\sim}{A} - \mu \underset{\sim}{B})$ is S.P.D., we can perform an L.U. decomposition on this and hence solve the first linear system of equations faster.

Q4) a) From the definition of backward stability, we can state that:

$$(\underset{\sim}{A} + \underset{\sim}{\delta A})\underset{\sim}{\tilde{w}} = \underset{\sim}{v}, \text{ where } \|\underset{\sim}{\delta A}\| = O(\varepsilon_m) \cdot \|\underset{\sim}{A}\|$$

$$(\underset{\sim}{A} + \underset{\sim}{\delta A})(\underset{\sim}{w} + \underset{\sim}{\delta w}) = \underset{\sim}{v} \quad [\underset{\sim}{\tilde{w}} = \underset{\sim}{w} + \underset{\sim}{\delta w}]$$

$$\cancel{\underset{\sim}{A}\underset{\sim}{w}} + \underset{\sim}{\delta A}\,\underset{\sim}{w} + \underset{\sim}{A}\,\underset{\sim}{\delta w} + \underset{\sim}{\delta A}\,\underset{\sim}{\delta w} = \cancel{\underset{\sim}{A}\underset{\sim}{w}} \quad [\underset{\sim}{A}\,\underset{\sim}{w} = \underset{\sim}{v}]$$

$$\underset{\sim}{\delta A}\,\underset{\sim}{w} + \underset{\sim}{A}\,\underset{\sim}{\delta w} + \underset{\sim}{\delta A}\,\underset{\sim}{\delta w} = 0$$

$$(\underset{\sim}{A} + \underset{\sim}{\delta A})\underset{\sim}{\delta w} = -\underset{\sim}{\delta A}\,\underset{\sim}{w}$$

$$\underset{\sim}{\delta w} = (\underset{\sim}{A} + \underset{\sim}{\delta A})^{-1}(-\underset{\sim}{\delta A}\,\underset{\sim}{w})$$

$$= -(\underset{\sim}{A} + \underset{\sim}{\delta A})^{-1}\underset{\sim}{\delta A}\,\underset{\sim}{w} \quad \text{(shown)}$$

Q4) b) Because $\underset{\sim}{A}$ is symmetric, we know that $\underset{\sim}{A}$ is non-defective and hence has an eigen-decomposition. Furthermore, we also know that the set of vectors $\{q_1, \ldots, q_m\}$ forms a basis for $\mathbb{R}^m$.

Because it is given that $\underset{\sim}{v}$ has components from all eigenvectors of $\underset{\sim}{A}$, we can state:

$$\underset{\sim}{v} = \sum_{i=1}^{m} c_i \underset{\sim}{q_i}$$

Let us try to solve for $\underset{\sim}{A}\underset{\sim}{w} = \underset{\sim}{v}$:

$$\underset{\sim}{A}\,\underset{\sim}{w} = \underset{\sim}{v}$$

$$\underset{\sim}{w} = \underset{\sim}{A}^{-1}\underset{\sim}{v} \quad [A \text{ is given to be invertible}]$$

$$\underset{\sim}{w} = \underset{\sim}{A}^{-1}\left(\sum_{i=1}^{m} c_i \underset{\sim}{q_i}\right) \quad \left[\text{Substitute } \underset{\sim}{v} = \sum_{i=1}^{m} c_i \underset{\sim}{q_i}\right]$$

$$= \sum_{i=1}^{m} c_i \underset{\sim}{A}^{-1} \underset{\sim}{q_i}$$

$$= \sum_{i=1}^{m} c_i \cdot \frac{\underset{\sim}{q_i}}{\lambda_i} \quad \left[\text{Because } \underset{\sim}{A}\underset{\sim}{q_i} = \lambda_i \underset{\sim}{q_i},\ \underset{\sim}{A}^{-1}\underset{\sim}{q_i} = \frac{\underset{\sim}{q_i}}{\lambda_i}\right]$$

Because $|\lambda_1| << |\lambda_2| \leq |\lambda_3| \leq \cdots \leq |\lambda_m|$, we constitute that:

$$\left|\frac{1}{\lambda_1}\right| >> \left|\frac{1}{\lambda_2}\right| \geq \left|\frac{1}{\lambda_3}\right| \geq \cdots \geq \left|\frac{1}{\lambda_m}\right|$$

Hence the coefficient of $\underset{\sim}{q_1}$, i.e $\frac{c_1}{\lambda_1}$ shall 'dominate' over all of the other eigenvectors' coefficients.

Therefore, $\underset{\sim}{w} \simeq \frac{c_1}{\lambda_1} \underset{\sim}{q_1}$

Normalising $\underset{\sim}{w}$, we get:

$$\frac{\underset{\sim}{w}}{\|\underset{\sim}{w}\|} = \frac{\frac{c_1}{\lambda_1} \underset{\sim}{q_1}}{\left\|\frac{c_1}{\lambda_1} \underset{\sim}{q_1}\right\|}$$

$$= \frac{\frac{c_1}{\lambda_1} \underset{\sim}{q_1}}{\left|\frac{c_1}{\lambda_1}\right| \|\underset{\sim}{q_1}\|_2} \quad \left[\|\alpha \underset{\sim}{x}\| = \alpha \|\underset{\sim}{x}\|\right]$$

$$= \pm 1 \cdot \frac{\underset{\sim}{q_1}}{\|\underset{\sim}{q_1}\|_2} \quad \left[\frac{a}{|a|} = \pm 1\right]$$

$$= \pm \underset{\sim}{q_1} \quad (\text{shown}) \quad \left[\|\underset{\sim}{q_1}\| = 1\right]$$

Q4) c) We are given that $(\underset{\sim}{A}+\delta\underset{\sim}{A})^{-1} \simeq \underset{\sim}{A}^{-1} + \underset{\sim}{A}^{-1}(\delta\underset{\sim}{A})\underset{\sim}{A}^{-1}$

From a), we know that $\delta\underset{\sim}{w} = -(\underset{\sim}{A}+\delta\underset{\sim}{A})^{-1}(\delta\underset{\sim}{A})\underset{\sim}{w}$

Substituting the Taylor series expansion for $(\underset{\sim}{A}+\delta\underset{\sim}{A})^{-1}$, we get:

$$\delta\underset{\sim}{w} = -(\underset{\sim}{A}+\delta\underset{\sim}{A})^{-1}(\delta\underset{\sim}{A})\underset{\sim}{w}$$

$$\delta\underset{\sim}{w} \simeq -\left(\underset{\sim}{A}^{-1} + \underset{\sim}{A}^{-1}(\delta\underset{\sim}{A})\underset{\sim}{A}^{-1}\right)(\delta\underset{\sim}{A})\underset{\sim}{w}$$

$$\simeq \underset{\sim}{A}^{-1}(\delta\underset{\sim}{A})\underset{\sim}{A}^{-1}(\delta\underset{\sim}{A})\underset{\sim}{w} - \underset{\sim}{A}^{-1}(\delta\underset{\sim}{A})\underset{\sim}{w}$$

Because $\|\delta A\| = O(\epsilon_m)$, $\underset{\sim}{A}^{-1}(\delta\underset{\sim}{A})\underset{\sim}{A}^{-1}(\delta\underset{\sim}{A})\underset{\sim}{w} \simeq 0$

Hence, $\delta\underset{\sim}{w} \simeq -\underset{\sim}{A}^{-1}(\delta\underset{\sim}{A})\underset{\sim}{w}$

$$\simeq -\underset{\sim}{A}^{-1}\underset{\sim}{z} \qquad \left[\text{Let } \underset{\sim}{z} = (\delta\underset{\sim}{A})\underset{\sim}{w}\right]$$

Because $\{q_1, \cdots, q_m\}$ form a basis for $\mathbb{R}^m$, we can express $\underset{\sim}{z}$ as a linear combination of these vectors:

$$\underset{\sim}{z} = \sum_{j=0}^{m} d_j \underset{\sim}{q_j}$$

Substituting this, we get:

$$\delta\underset{\sim}{w} \simeq -\underset{\sim}{A}^{-1}\sum_{j=0}^{m} d_j \underset{\sim}{q_j}$$

$$\simeq -\sum_{j=0}^{m} d_j \underset{\sim}{A}^{-1}\underset{\sim}{q_j}$$

$$\simeq -\sum_{j=0}^{m} \frac{d_j}{\lambda_j} q_j \qquad \left[\underset{\sim}{A}^{-1}\underset{\sim}{q_j} = \frac{1}{\lambda_j}\underset{\sim}{q_j}\right]$$

Because we know from part b) that $\left|\frac{1}{\lambda_1}\right| >> \left|\frac{1}{\lambda_2}\right| \geq \left|\frac{1}{\lambda_3}\right| \geq \cdots \geq \left|\frac{1}{\lambda_m}\right|$,

we can state that $\underset{\sim}{\delta w} \simeq -\frac{d_1}{\lambda_1} \underset{\sim}{q_1}$

Because $\underset{\sim}{\delta w} \simeq K \cdot \underset{\sim}{q_1}$, $\underset{\sim}{\delta w}$ is approximately in the direction of $\underset{\sim}{q_1}$.

(proven)

Q) 5) a) Because ε_i is said to be the exact eigenvalue of $\underset{\sim}{A}$, we can state:

$$\underset{\sim}{A}\,\underset{\sim}{x}_i = \varepsilon_i\,\underset{\sim}{x}_i$$

It is given that the exact eigenvector is the sum of the current approximation, i.e. $\underset{\sim}{\tilde{x}}_i^{(0)}$ and the corrector, i.e $\underset{\sim}{t}_i$. Hence we can state that:

$$\underset{\sim}{x}_i = \underset{\sim}{\tilde{x}}_i^{(0)} + \underset{\sim}{t}_i$$

Substituting this into the previous equation, we get:

$$\underset{\sim}{A}\left(\underset{\sim}{\tilde{x}}_i^{(0)} + \underset{\sim}{t}_i\right) = \varepsilon_i\left(\underset{\sim}{\tilde{x}}_i^{(0)} + \underset{\sim}{t}_i\right)$$

$$\underset{\sim}{A}\,\underset{\sim}{\tilde{x}}_i^{(0)} + \underset{\sim}{A}\,\underset{\sim}{t}_i = \varepsilon_i\,\underset{\sim}{\tilde{x}}_i^{(0)} + \varepsilon_i\,\underset{\sim}{t}_i$$

$$\underset{\sim}{A}\,\underset{\sim}{t}_i - \varepsilon_i\,\underset{\sim}{t}_i = \varepsilon_i\,\underset{\sim}{\tilde{x}}_i^{(0)} - \underset{\sim}{A}\,\underset{\sim}{\tilde{x}}_i^{(0)}$$

$$\left(\underset{\sim}{A} - \varepsilon_i\underset{\sim}{I}\right)\underset{\sim}{t}_i = \left(\varepsilon_i\underset{\sim}{I} - \underset{\sim}{A}\right)\underset{\sim}{\tilde{x}}_i^{(0)} \quad \text{(shown)}$$

From the above expression, we see that it is actually a linear system of equations. However, we know that $\det(\underset{\sim}{A} - \varepsilon_i \underset{\sim}{t}_i)$ is zero, which means $(\underset{\sim}{A} - \varepsilon_i \underset{\sim}{t}_i)$ singular. Hence, there does not exist a unique solution for this equation.

Q5) b) Given an approximate eigenvector, i.e $\underset{\sim}{\tilde{x}}_i^{(0)}$, the best possible estimate for its corresponding eigenvalue is the Rayleigh quotient. This quotient can be expressed as:

$$\tilde{\varepsilon}_i^{(0)} = \frac{\left(\underset{\sim}{\tilde{x}}_i^{(0)}\right)^T A \left(\underset{\sim}{\tilde{x}}_i^{(0)}\right)}{\left(\underset{\sim}{\tilde{x}}_i^{(0)}\right)^T \left(\underset{\sim}{\tilde{x}}_i^{(0)}\right)}$$

Substituting $\varepsilon_i \approx \tilde{\varepsilon}_i^{(0)}$ and replacing $\underset{\sim}{A}$ with $\underset{\sim}{D} = \text{diag}(\underset{\sim}{A})$, we get:

$$(\underset{\sim}{D} - \tilde{\varepsilon}_i^{(0)} \underset{\sim}{I}) \hat{\underset{\sim}{t}}_i = (\varepsilon_i^{(0)} \underset{\sim}{I} - \underset{\sim}{A}) \tilde{\underset{\sim}{x}}_i^{(0)}$$

We also know that $(\underset{\sim}{D} - \tilde{\varepsilon}_i^{(0)} \underset{\sim}{I})$ shall be a diagonal matrix, and shall be invertible, as long as $\varepsilon_i^{(0)} \neq D_{ij} \ \forall i=j$. Finally, we can derive an expression for $\underset{\sim}{t}_i$:

$$\hat{\underset{\sim}{t}}_i = (\underset{\sim}{D} - \tilde{\varepsilon}_i^{(0)} \underset{\sim}{I})^{-1} (\varepsilon_i^{(0)} \underset{\sim}{I} - \underset{\sim}{A}) \tilde{\underset{\sim}{x}}_i^{(0)}$$

Q5) c) In the current method, we are keeping track only of the approximate eigenvectors. The new space shall store the corrector values as well as the approximate eigenvectors.

If we do such a thing, we can write more efficient algorithms to essentially "point" the eigenvectors in the correct direction. We can also express the next iteration (or approximation) of eigenvalues as a linear combination of the current approximation as well as the correction factor. This shall speeden up the convergence greatly.

Q5) d) The Galerkin condition states that: for an approximate eigenpair $(\tilde{\underset{\sim}{x}}_i^{(1)}, \varepsilon_i^{(1)})$, the residual must be orthogonal to the entire subspace.

$$(\underset{\sim}{V}_{2n}^{(0)})^T (\underset{\sim}{A} \tilde{\underset{\sim}{x}}_i^{(1)} - \tilde{\varepsilon}_i^{(1)} \tilde{\underset{\sim}{x}}_i^{(1)}) = 0$$

Let us construct a matrix $\underset{\sim}{V}$ s.t. $\underset{\sim}{V}$ forms an orthonormal basis for $V_{2n}^{(0)}$. This can be done by any one of the orthogonalisation methods. In the interest of stability, we may like to use Householder orthogonalisation.

Using the Galerkin condition, we can yield:

$$\underset{\sim}{H}\,\underset{\sim}{y_i} = \tilde{\varepsilon}_i^{(1)}\,\underset{\sim}{y_i} \quad , \text{where } \underset{\sim}{H} = \underset{\sim}{V}^T \underset{\sim}{A}\,\underset{\sim}{V}$$

The Ritz pair will be $(\mu, \underset{\sim}{V}\underset{\sim}{y})$

The consequence of performing this orders term is that we reduce the problem from a large $m \times m$ eigenvalue problem to a relatively small $2n \times 2n$ eigenvalue problem.

The procedure to obtain the next approximation is as follows:

i) Construct the orthonormal basis for $V_{2n}^{(0)}$. Construct a matrix $\underset{\sim}{V}$ with these orthonormal basis vectors as its columns.

ii) Form the matrix $\underset{\sim}{H} = \underset{\sim}{V}^T \underset{\sim}{A}\,\underset{\sim}{V}$.

iii) Solve the "small" problem $\underset{\sim}{H}\,\underset{\sim}{y_i} = \tilde{\varepsilon}_i^{(1)}\,\underset{\sim}{y_i}$ to obtain $2n$ eigenpairs.

iv) Select the n smallest eigenvalues (and their corresponding eigenvectors). This shall represent the next iteration.

v) Map these eigenvectors back to the m-dimensional space, so that we can start the next iteration. This can be done by:

$$\underset{\sim}{x}_i^{(1)} = \underset{\sim}{Q}\,\underset{\sim}{y_i}$$

Q 6) a) The characteristic equation is given by $\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = 0$

$$\det\left[\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right] = 0$$

$$\det\begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 2 & 3 & 3-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)\cdot\det\begin{bmatrix} 2-\lambda & 0 \\ 3 & 3-\lambda \end{bmatrix} - 0\cdot\det\begin{bmatrix} 1 & 0 \\ 2 & 3-\lambda \end{bmatrix} + 0\cdot\det\begin{bmatrix} 1 & 2-\lambda \\ 2 & 3 \end{bmatrix} = 0$$

$$(1-\lambda)\cdot\det\begin{bmatrix} 2-\lambda & 0 \\ 3 & 3-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)\left[(2-\lambda)(3-\lambda) - 0(3)\right] = 0$$

$$(1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$$\lambda = 1, 2, 3_{//}$$

Q 6) b) For this question, we need to find an orthonormal basis $\{\underset{\sim}{q_1}, \underset{\sim}{q_2}\}$ for the 2-D Krylov subspace $K_2 = \langle \underset{\sim}{b}, \underset{\sim}{A}\underset{\sim}{b} \rangle$, where:

$$\underset{\sim}{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Computation of $\underset{\sim}{q}_1$: $\underset{\sim}{q}_1 = \frac{\underset{\sim}{b}}{\|\underset{\sim}{b}\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Computation of $\underset{\sim}{q}_2$:-

Compute $\underset{\sim}{A}\underset{\sim}{q}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

Orthogonalise $\underset{\sim}{A}\underset{\sim}{q}_1$ from $\underset{\sim}{q}_1$:

$$\underset{\sim}{v}_2 = \underset{\sim}{A}\underset{\sim}{q}_1 - \underset{\sim}{q}_1^T(\underset{\sim}{A}\underset{\sim}{q}_1)\underset{\sim}{q}_1$$

$$= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Normalise $\underset{\sim}{v}_2$ to get $\underset{\sim}{q}_2$:

$$\underset{\sim}{q}_2 = \frac{\underset{\sim}{v}_2}{\|\underset{\sim}{v}_2\|} = \frac{1}{\sqrt{1^2+2^2}}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

Hence, the basis vectors for K_2 are : $\underset{\sim}{q}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\underset{\sim}{q}_2 = \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

Q6) c) To project $\underset{\sim}{A}$ onto K, a 2×2 projection matrix $\underset{\sim}{H}$ shall be used.

During Q6) b), we calculated $h_{11} = 1$ and $h_{21} = \sqrt{5}$.

Let us calculate h_{12}:

$$h_{12} = \underset{\sim}{q_1}^T \underset{\sim}{A} \underset{\sim}{q_2} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 9 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} (0)$$

$$= 0 \,//$$

Let us calculate h_{22}:

$$h_{22} = \underset{\sim}{q_2}^T \underset{\sim}{A} \underset{\sim}{q_2} = \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} \cdot \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 9 \end{pmatrix}$$

$$= 20/5$$

$$= 4 \,//$$

Hence, $\underset{\sim}{H} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \sqrt{5} & 4 \end{pmatrix}$

Because $\underset{\sim}{H}$ is a lower triangular matrix, the eigenvalues of $\underset{\sim}{H}$ are 1 & 4. These are the "Ritz" values

The error in the largest eigenvalue $= |1-1| = 0$

The error in the smallest eigenvalue $= |4-3| = 1$

Q6) d) Let us find the solution to $\underset{\sim}{A}\underset{\sim}{x}^* = \underset{\sim}{b}$, $\underset{\sim}{x}^* \in \mathbb{R}^3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 3 \end{pmatrix} x^* = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow x^* = \begin{pmatrix} 1 \\ -1/2 \\ -1/6 \end{pmatrix}$$

Now, we want to find a vector $\hat{x}$ such that $\|\underset{\sim}{A}\hat{x} - \underset{\sim}{b}\|$ is minimal, where $\hat{x} \in K_2$. We cannot solve this as a simple least squares problem due to the restriction on $\underset{\sim}{x}$.

However, we can can get a solution: $\hat{x} = \underset{\sim}{Q}\underset{\sim}{y}$, where:

$$\underset{\sim}{Q} = \begin{bmatrix} | & | \\ \underset{\sim}{q}_1 & q_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{5} \\ 0 & 2/\sqrt{5} \end{bmatrix}$$

and $\underset{\sim}{y}$ is the solution to $\underset{\sim}{H}_2 \underset{\sim}{y} = \|b\|_2 e_1$.

Let us obtain $\underset{\sim}{y}$:

$$\begin{pmatrix} 1 & 0 \\ \sqrt{5} & 4 \end{pmatrix} \underset{\sim}{y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \underset{\sim}{y} = \begin{pmatrix} 1 \\ -\frac{\sqrt{5}}{4} \end{pmatrix}$$

Hence, let us obtain $\hat{x}$:

$$\hat{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{5} \\ 0 & 2/\sqrt{5} \end{bmatrix} \begin{pmatrix} 1 \\ -\frac{\sqrt{5}}{4} \end{pmatrix} = \begin{bmatrix} 1(1) + 0\left(-\frac{\sqrt{5}}{4}\right) \\ 0(1) + \frac{1}{\cancel{\sqrt{5}}}\left(-\frac{\cancel{\sqrt{5}}}{4}\right) \\ 0(1) + \frac{2}{\cancel{\sqrt{5}}}\left(-\frac{\cancel{\sqrt{5}}}{4}\right) \end{bmatrix} = \begin{bmatrix} 1 \\ -1/4 \\ -1/2 \end{bmatrix}$$

Finally, let us calculate the error in $\underset{\sim}{x}^*$ and $\underset{\sim}{\hat{x}}$:

$$\text{Error} = \| \underset{\sim}{x}^* - \underset{\sim}{\hat{x}} \|_2$$

$$= \left\| \begin{pmatrix} 1 \\ -1/2 \\ -1/6 \end{pmatrix} - \begin{pmatrix} 1 \\ -1/4 \\ -1/2 \end{pmatrix} \right\|_2$$

$$= \left\| \begin{bmatrix} 0 \\ -1/4 \\ 1/3 \end{bmatrix} \right\|$$

$$= \sqrt{\left(-1/4\right)^2 + \left(1/3\right)^2}$$

$$= \sqrt{\frac{1}{16} + \frac{1}{9}}$$

$$= \sqrt{\frac{25}{144}}$$

$$= 5/12$$