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Question 2

Theory

In this question, we have to compress an image using Single Value Decomposition. The original (full-resolution) image is as follows:

This can be done using a concept called a low rank approximation. We know that the SVD of a matrix $\mathbf{A}$ is as follows:

$$\mathbf{A} = \mathbf{U\Sigma V^T}$$

In this decomposed format, the components are as follows:

- $\mathbf{U}$: The right singular vectors of the matrix.
- $\mathbf{V^T}$: The left singular vectors of the matrix.
- $\mathbf{\Sigma}$: The singular values of the matrix, in a diagonal format:

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_n \end{bmatrix}$$

We also know that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$.

The singular values matrix $\mathbf{\Sigma}$ can also be written as:

$$\mathbf{\Sigma} = \text{sum}_{j=1}^{r} \mathbf{\Sigma_j}$$

Where:

- $r =$ Rank of the matrix $\mathbf{A}$
- $\boldsymbol{\Sigma_j}[a][b] = \begin{cases} \sigma_j, & a = b = j \\ 0, & \text{otherwise} \end{cases}$, where $[a][b]$ represents the index of the matrix.

We can reconstruct the original matrix $\mathbf{A}$ by multiplying the components of the SVD:

$$\boldsymbol{A} = \boldsymbol{U\Sigma V^T}$$
$$\boldsymbol{A} = \boldsymbol{U} \left\{ \text{sum}_{j=1}^{r} \boldsymbol{\Sigma}_j \right\} \boldsymbol{V^T}$$
$$\boldsymbol{A} = \text{sum}_{j=1}^{r} \left(\boldsymbol{U\Sigma}_j \boldsymbol{V} \right)$$
$$\boldsymbol{A} = \text{sum}_{j=1}^{r} \left(\sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^{\boldsymbol{T}} \right)$$

Because we know that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$, we can see that the later σ values (”tail” values) have lesser of an effect on the overall array. Hence, we can ignore the later singular values, and still end up with a decent reconstruction of the image. This can be done by taking a partial sum. The formula for the k^{th} partial sum, $\mathbf{A_k}$ is as follows:

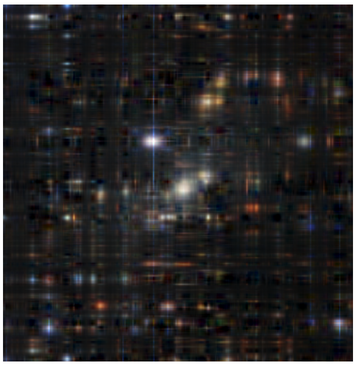
$$\boldsymbol{A_k} = \mathrm{sum}_{j=1}^{k}\left(\sigma_j \boldsymbol{u_j} \boldsymbol{v_j^T}\right)$$

Part (a)

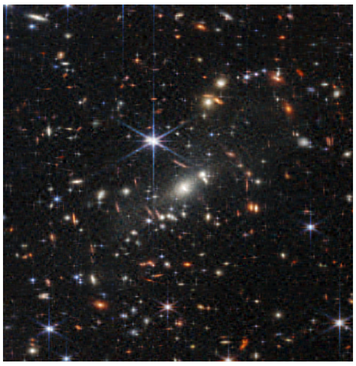
In this section, we have to obtain a value of k where the compressed image will be indistinguishable from the original image. We first split the image into the red, blue and green (RGB) components. The maximum rank of these three arrays is 1960. Hence, we know that the value of k can range from 1 to 1960.

Let us start with $k = 1$. Although this is very less, and will most definitely not give us an indistinguishable image, we can still have a starting point. The image generated at $k = 1$ is as follows:

Let us now slightly increase the value of k. At $k = 10$, we get this image:



We can see that we are getting closer to the original image. Let us try $k = 50$:

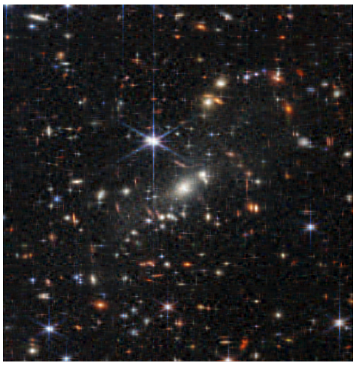


This image is quite close to the original image.

All the comparisons so far was done using "visual inspection". However, this is not an accurate measure of "indistinguishability". We should be using a more "numeric" system. We can do this by calculating the retained "energy" of the image, and ensuring that this is above an particular ϵ.

$$\text{Energy Retained} = \frac{\sum_{j=1}^{k} \sigma_j^2}{\sum_{j=1}^{r} \sigma_j^2} \geq \epsilon$$

If we take $\epsilon = 90\%$, the minimum value of k is 37. The image generated is as follows:



I believe that a retained energy of 90% is enough for this experiment. However, the code is very flexible and a higher percentage can be chosen at any time.

Part (b)

The total entries in the initial image is as follows:

$$Entries_{initial} = m \times n = 1960 \times 2000 = 3.92 \times 10^6$$

The number of entries in the low-rank approximation of the image is:

$$Entries_{low\_rank} = (m + n + 1) \times k = (1960 + 2000 + 1) \times 37 = 146557 \approx 1.46 \times 10^5$$

Hence, the number of entries that need to be sent back to Earth is $\approx 1.46 \times 10^5$. The number of entries has been reduced by a factor of $\frac{Entries_{initial}}{Entries_{low\_rank}} \approx 27$.

Part (c)

The error for each component is as follows:

Colour	$\|\mathbf{A} - \mathbf{A_k}\|_2$	$\|\mathbf{A} - \mathbf{A_k}\|_F$
Red	3666.71	26069.16
Green	3624.44	24762.16
Blue	4314.38	28680.76

Let us calculate σ_{k+1} and $\sqrt{\sigma_{k+1}^2 + \sigma_{k+2}^2 + \cdots \sigma_r^2}$ for each colour, so that we can verify the theorems given in the question.

Colour	σ_{k+1}	$\sqrt{\sigma_{k+1}^2 + \sigma_{k+2}^2 + \cdots \sigma_r^2}$
Red	3666.71	26069.16
Green	3624.44	24762.16
Blue	4314.38	28680.76

As can be seen, both theorems hold for the errors in the question.