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Assignment - 1

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3) a) For a matrix to be orthogonal, it should satisfy the following:

$\tilde{Q}^{-1} = \tilde{Q}^T$, i.e. inverse of $\tilde{Q}$ = transpose of $\tilde{Q}$

First let us calculate $\tilde{Q}^T$:- $\tilde{Q}$ is given as $(\tilde{I}-\tilde{S})^{-1}(\tilde{I}+\tilde{S})$

$$\tilde{Q}^T = \left[(\tilde{I}-\tilde{S})^{-1}(\tilde{I}+\tilde{S})\right]^T$$

$$= (\tilde{I}+\tilde{S})^T\left[(\tilde{I}-\tilde{S})^{-1}\right]^T \quad (\text{reversal law, i.e. } (\tilde{A}\tilde{B})^T = \tilde{B}^T\tilde{A}^T)$$

$$= (\tilde{I}^T+\tilde{S}^T)\left[(\tilde{I}-\tilde{S})^{-1}\right]^T \quad ((\tilde{A}+\tilde{B})^T = \tilde{A}^T+\tilde{B}^T)$$

$$= (\tilde{I}^T+\tilde{S}^T)\left[(\tilde{I}-\tilde{S})^T\right]^{-1} \quad ((\tilde{A}^{-1})^T = (\tilde{A}^T)^{-1})$$

$$= (\tilde{I}-\tilde{S})(\tilde{I}^T-\tilde{S}^T)^{-1} \quad (\tilde{I}^T = \tilde{I} \text{ and } \tilde{S}^T = -\tilde{S})$$

$$= (\tilde{I}-\tilde{S})(\tilde{I}+\tilde{S})^{-1}$$

To prove $\tilde{Q}$'s orthogonality, we should prove $\tilde{Q}\cdot\tilde{Q}^T = \tilde{I}$. For this we need to rearrange ~~both~~ $\tilde{Q}$ ~~& $\tilde{Q}^T$~~

Proof of commutability:

In this proof, I want to prove that $(I+S)(I-S) = (I-S)(I+S)$

$$(\tilde{I}+\tilde{S})(\tilde{I}-\tilde{S}) = \tilde{I}^2+\tilde{S}-\tilde{S}-\tilde{S}^2 = \tilde{I}-\tilde{S}^2$$
$$(\tilde{I}-\tilde{S})(\tilde{I}+\tilde{S}) = \tilde{I}^2-\tilde{S}+\tilde{S}-\tilde{S}^2 = \tilde{I}-\tilde{S}^2$$

$\therefore$ Hence, the commutability is proven.

This proof shall help us to rearrange $\tilde{Q}$ & $\tilde{Q}^T$.

Rearranging Q^T:

For this, we have to use the proven commutability:

$$(I-S)(I+S) = (I+S)(I-S)$$

$$(I-S)^{-1}(I-S)(I+S) = (I-S)^{-1}(I+S)(I-S) \quad \text{(left multiply by } (I-S)^{-1}\text{)}$$

$$I(I+S) = (I-S)^{-1}(I+S)(I-S)$$

$$(I+S)(I-S)^{-1} = (I-S)^{-1}(I+S)(I-S)(I-S)^{-1} \quad \text{(right multiply } (I-S)^{-1}\text{)}$$

$$(I+S)(I-S)^{-1} = (I-S)^{-1}(I+S)$$

Hence, Q can be written as $\cancel{(I-S)(I+S)}$ $(I+S)(I-S)^{-1}$.

Now, we can prove $Q^T \cdot Q = I$

$$Q^T Q = \left[(I-S)(I+S)^{-1}\right]\left[(I+S)(I-S)^{-1}\right]$$

$$= (I-S)\left[(I+S)^{-1}(I+S)\right](I-S)^{-1} \quad \text{(distributive property of matrix multiplication)}$$

$$= (I-S)(I)(I-S)^{-1}$$

$$= (I-S)(I-S)^{-1}$$

$$= I$$

$\therefore$ Hence proven, $Q^{-1} = Q^T$, as long as S is skew-symmetric

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3.b) Since this is an "if & only if" statement, we shall have to prove both ways.

Proof #1: If $\underset{\sim}{S} = \underset{\sim}{O}$, then $\underset{\sim}{u}^T \underset{\sim}{S} \underset{\sim}{u} = \underset{\sim}{0}$.

This is easy to prove. If $\underset{\sim}{S} = \underset{\sim}{O}$, then

$$\underset{\sim}{u}^T \underset{\sim}{S} \underset{\sim}{u} = (\underset{\sim}{u}^T \underset{\sim}{S}) \underset{\sim}{u}$$ (Distributive property of matrix vector multiplication)

$$= (\underset{\sim}{u}^T \underset{\sim}{O}) \underset{\sim}{u}$$
$$= \underset{\sim}{0} \cdot \underset{\sim}{u}$$
$$= \underset{\sim}{0}$$ (proven)

Proof #2: If $\underset{\sim}{u}^T \underset{\sim}{S} \underset{\sim}{u} = \underset{\sim}{0}$, then $\underset{\sim}{S} = \underset{\sim}{O}$

This proof has to be done in two stages. In the first stage, we need to prove that all diagonal elements of $\underset{\sim}{S}$ are 0. In the second phase, we prove all non-diagonal elements of $\underset{\sim}{S}$ to be 0. This shall hence prove $\underset{\sim}{S} = \underset{\sim}{O}$.

Phase #1: Prove $\{ S_{ij} = 0 \mid \forall\, i = j \}$

Assume $\underset{\sim}{u} = \underset{\sim}{e_i}$, then (ei represents a canonical basis vector)

$$\underset{\sim}{e_i}^T \underset{\sim}{S} \underset{\sim}{e_i} = 0$$

$$S_{ii} = 0$$ $$\left(S_{ii} = \underset{\sim}{e_i}^T \underset{\sim}{S} \underset{\sim}{e_i}\right)$$

Since i can be any index $1 \le i \le m$, all diagonal elements of $\underset{\sim}{S}$ are hence proven to be 0.

Hence, $S_{ii} = 0$.

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Phase #2: Prove all non-diagonal elements of $\underset{\sim}{S}$ are 0.

Assume $\underset{\sim}{u} = \underset{\sim}{e_i} + \underset{\sim}{e_j}$ & $i \neq j$

Hence $\underset{\sim}{u}^T \underset{\sim}{S} \underset{\sim}{u} = (\underset{\sim}{e_i} + \underset{\sim}{e_j})^T \underset{\sim}{S} (\underset{\sim}{e_j} + \underset{\sim}{e_i})$

$0 = e_i S e_j + e_j S e_j + e_i S e_i + e_j S e_i$

(S_{ii} & S_{jj} are 0, as proven in phase #1) $\quad 0 = S_{ii} + S_{jj} + S_{ij} + S_{ji}$

$0 = S_{ij} + S_{ji}$

($S_{ij} = S_{ji}$, because $\underset{\sim}{S}$ is given to be symmetric) $\quad 0 = 2S_{ij}$

$S_{ij} = 0$

Hence, $S_{ij} = 0$ for $i \neq j$.

Combining results of both phases it can be ~~st~~ stated that $\underset{\sim}{S} = \underset{\sim}{0}$.

c) ~~Sing~~ Because this statement is an "if & only if" statement, both directions need to ~~p~~ be proven.

Proof #1: If S is skew symmetric, $\underset{\sim}{u}^T \underset{\sim}{S} \underset{\sim}{u} = 0$.

~~Lexx~~ We know that $(\underset{\sim}{u}^T \underset{\sim}{S} \underset{\sim}{u}) = (\underset{\sim}{u}^T \underset{\sim}{S} \underset{\sim}{u})^T$ because $\underset{\sim}{u}^T \underset{\sim}{S} \underset{\sim}{u}$ is a 1×1 matrix, i.e. a scalar value, which has no effect on transpose.

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$$\underset{\sim}{u}^T \underset{\sim}{S} \underset{\sim}{u} = (\underset{\sim}{u}^T \underset{\sim}{S} \underset{\sim}{u})^T$$

$$\underset{\sim}{u}^T \underset{\sim}{S} \underset{\sim}{u} = \underset{\sim}{u}^T \underset{\sim}{S}^T (\underset{\sim}{u}^T)^T$$ (Law of reversal, i.e. $(ABC)^T = C^T B^T A^T$)

$$\underset{\sim}{u}^T \underset{\sim}{S} \underset{\sim}{u} = \underset{\sim}{u}^T \underset{\sim}{S}^T \underset{\sim}{u}$$ $((\underset{\sim}{u}^T)^T = \underset{\sim}{u})$

$$\underset{\sim}{u}^T \underset{\sim}{S} \underset{\sim}{u} = \underset{\sim}{u}^T (-\underset{\sim}{S}) \underset{\sim}{u}$$ (Since $\underset{\sim}{S}$ is skew-symmetric, $\underset{\sim}{S}^T = -\underset{\sim}{S}$)

$$\underset{\sim}{u}^T \underset{\sim}{S} \underset{\sim}{u} = -\underset{\sim}{u}^T \underset{\sim}{S} \underset{\sim}{u}$$

$$x = -x$$ (Assume $x = \underset{\sim}{u}^T \underset{\sim}{S} \underset{\sim}{u}$)

$$2x = 0$$

$$x = 0$$ (Hence proven)

$\therefore$ If $\underset{\sim}{S}$ is skew-symmetric, then $x = \underset{\sim}{u}^T \underset{\sim}{S} \underset{\sim}{u} = 0$.

Proof #2: If $\underset{\sim}{u}^T \underset{\sim}{S} \underset{\sim}{u} = 0$, then $\underset{\sim}{S}$ is skew-symmetric.

Any square matrix can be split into a symmetric & skew symmetric portion as follows:

$$\underset{\sim}{A} = \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric part}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{skew-symmetric part}}$$

Hence, $\underset{\sim}{S}$ can be represented as $\frac{1}{2}(\underset{\sim}{S} + \underset{\sim}{S}^T) + \frac{1}{2}(\underset{\sim}{S} - \underset{\sim}{S}^T)$.

If we assume $\underset{\sim}{S}_{symm} = \frac{1}{2}(\underset{\sim}{S} + \underset{\sim}{S}^T)$ and $\underset{\sim}{S}_{skew} = \frac{1}{2}(\underset{\sim}{S} - \underset{\sim}{S}^T)$,

$\underset{\sim}{S}$ can be re-written as $\underset{\sim}{S}_{symm} + \underset{\sim}{S}_{skew}$.

Hence, $\underset{\sim}{u}^T \underset{\sim}{S} \underset{\sim}{u} = 0$

$$\underset{\sim}{u}^T (\underset{\sim}{S}_{symm} + \underset{\sim}{S}_{skew}) \underset{\sim}{u} = 0$$

$$\underset{\sim}{u}^T \underset{\sim}{S}_{symm} \underset{\sim}{u} + \underset{\sim}{u}^T \underset{\sim}{S}_{skew} \underset{\sim}{u} = \underset{\sim}{0}$$

$\underset{\sim}{u}^T \underset{\sim}{S}_{symm} \underset{\sim}{u} = \underset{\sim}{0}$ (Because $\underset{\sim}{u}^T \underset{\sim}{S}_{skew} \underset{\sim}{u} = 0$. as proven in proof #1 of Question 3) c))

$\therefore \underset{\sim}{S}_{symm} = \underset{\sim}{0}$ (This has be proven in proof #2 of Question 3) b))

$\frac{1}{2} (\underset{\sim}{S} + \underset{\sim}{S}^T) = \underset{\sim}{0}$ (Substituting definition of $\underset{\sim}{S}_{symm}$)

$$\underset{\sim}{S} + \underset{\sim}{S}^T = 0$$

$$\underset{\sim}{S}^T = -\underset{\sim}{S}$$

Since this is the fundamental property of a skew-symmetric matrix, we can henceforth prove that $\underset{\sim}{S}$ is a skew-symmetric matrix.