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Numerical Linear Algebra - Assignment 4

2 a)

Theory: The algorithm for computing the columns of $\underset{\sim}{Q}$ is as follows:

for $j = 1 \longrightarrow n$:
{
$\underset{\sim}{v_j} = \underset{\sim}{a_j}$
for $i = 1 \rightarrow (j-1)$:
{
$r_{ij} = \underset{\sim}{v_j}^T \underset{\sim}{a_i}$
$\underset{\sim}{v_j} = \underset{\sim}{v_j} - r_{ij} \underset{\sim}{q_i}$
}
$\underset{\sim}{q_j} = \dfrac{\underset{\sim}{v_j}}{\|\underset{\sim}{v_j}\|_2}$
}

Hence, let us now calculate $q_0(x) \cdots q_3(x)$ for $\underset{\sim}{A}$:

i) Calculating $q_0(x)$:

$$v_0(x) = a_0(x) = 1$$

$$\|v_0(x)\|_2 = \sqrt{(a_0(x), a_0(x))} = \sqrt{\int_{-1}^{1} 1^2 dx} = \sqrt{[x]_{-1}^{1}} = \sqrt{2}$$

$$q_0(x) = \frac{v_0(x)}{\|v_0(x)\|_2} = \frac{1}{\sqrt{2}}$$

ii) Calculating $q_1(x)$

$$v_1(x) = a_1(x) - (a_1(x), q_0(x))\, q_0(x)$$

$$(a_1(x), q_0(x)) = \int_{-1}^{1} x \cdot \frac{1}{\sqrt{2}} dx = \frac{1}{\sqrt{2}} \cdot \frac{1}{2} [x^2]_{-1}^{1} = 0$$

$$\therefore v_1(x) = a_1(x) - 0 \cdot q_0(x) = a_1(x) = x$$

$$\|v_1\|_2 = \sqrt{\int_{-1}^{1} x^2\,dx} = \sqrt{\left[\frac{x^3}{3}\right]_{-1}^{1}} = \sqrt{\frac{2}{3}}$$

$$\therefore q_1(x) = \frac{x}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}}\,x$$

iii) Calculating $q_2(x)$

$$q_2(x) = a_2(x) - (a_2(x), q_0(x))\,q_0(x) - (a_2(x), q_1(x))\,q_1(x)$$

$$(a_2(x), q_0(x)) = \int_{-1}^{1} x^2 \cdot \frac{1}{\sqrt{2}}\,dx = \frac{1}{\sqrt{2}}\,[x^3]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{3} \cdot 2$$

$$= \frac{\sqrt{2}}{3}$$

$$(a_2(x), q_1(x)) = \int_{-1}^{1} x^2 \cdot \sqrt{\frac{3}{2}}\,x = \sqrt{\frac{3}{2}} \int_{-1}^{1} x^3\,dx$$

$$= \sqrt{\frac{3}{2}} \cdot \frac{1}{4}\,[x^4]_{-1}^{1}$$

$$= \sqrt{\frac{3}{2}} \cdot \frac{1}{4} \cdot 0$$

$$= 0$$

$$\therefore v_2(x) = x^2 - \frac{\cancel{\sqrt{2}}}{3}\left(\frac{1}{\cancel{\sqrt{2}}}\right) - \cancel{0 \cdot q_1(x)}\ 0$$

$$= x^2 - \tfrac{1}{3}$$

$$\|v_2(x)\|_2 = \sqrt{\int_{-1}^{1} \left(x^2 - \tfrac{1}{3}\right)^2 dx}$$

$$= \sqrt{\int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) dx}$$

$$= \left[\frac{x^5}{5} - \frac{2x^3}{9} + \frac{1}{9}x\right]_{-1}^{1}$$

$$= \sqrt{\left(\frac{4}{45}\right) - \left(-\frac{4}{45}\right)}$$

$$= \sqrt{\frac{8}{45}}$$

$$= \frac{2}{3}\sqrt{\frac{2}{5}}$$

$$\therefore q_2(x) = \frac{x^2 - 1/3}{\frac{2}{3}\sqrt{\frac{2}{5}}} = \frac{3}{2}\sqrt{\frac{5}{2}}\left(x^2 - \frac{1}{3}\right)$$

$$= \sqrt{\frac{5}{8}}\left(3x^2 - 1\right)$$

iv) Calculating $q_3(x)$:

$$v_3(x) = a_3(x) - (a_3(x), q_0(x))\, q_0(x) - (a_3(x), q_1(x))\, q_1(x) - (a_3(x), q_2(x))\, q_2(x)$$

$$(a_3(x), q_0(x)) = \int_{-1}^{1} x^3 \cdot \frac{1}{\sqrt{2}}\, dx = \frac{1}{4\sqrt{2}}\left[x^4\right]_{-1}^{1} = 0$$

$$(a_3(x), q_1(x)) = \int_{-1}^{1} x^3 \cdot \sqrt{\frac{3}{2}}\, x\, dx = \sqrt{\frac{3}{2}} \int_{-1}^{1} x^4\, dx$$

$$= \sqrt{\frac{3}{2}} \cdot \frac{1}{5} \cdot \left[x^5\right]_{-1}^{1}$$

$$= \frac{\sqrt{3}}{\cancel{\sqrt{2}}_{1}} \cdot \frac{1}{5} \cdot \cancel{x}^{\sqrt{2}}$$

$$= \frac{\sqrt{3} \cdot \sqrt{2}}{5}$$

$$= \frac{\sqrt{6}}{5}$$

$$(a_3(x), q_2(x)) = \int_{-1}^{1} x^3 \cdot \sqrt{\frac{5}{8}} (3x^2 - 1)\, dx$$

$$= \sqrt{\frac{5}{8}} \int_{-1}^{1} (3x^5 - x^3)\, dx$$

$$= \sqrt{\frac{5}{8}} \left[\frac{3x^6}{6} - \frac{x^4}{4}\right]_{-1}^{1}$$

$$= 0$$

$$\therefore v_3(x) = a_3(x) - \cancel{0 \cdot q_0(x)}^{0} - \frac{\sqrt{6}}{5} q_1(x) - \cancel{0 \cdot q_2(x)}^{0}$$

$$= x^3 - \frac{\sqrt{6}}{5} \cdot \frac{\sqrt{3}}{\sqrt{2}} x$$

$$= x^3 \; \frac{\sqrt{3} \cdot \cancel{\sqrt{2}}}{5} \cdot \frac{\sqrt{3}}{\cancel{\sqrt{2}}} x$$

$$= x^3 - 3/5\, x$$

Using property of even function:

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx,$$

if $f(-x) = f(x)$

$$\|v_3(x)\|_2 = \sqrt{\int_{-1}^{1}\left(x^3 - \tfrac{3}{5}x\right)^2 dx}$$

$$= \int_{-1}^{1}\left(x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2\right)dx$$

$$= \sqrt{2\int_{0}^{1}\left(x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2\right)dx}$$

$$= \sqrt{2\left[\frac{x^7}{7} - \frac{6x^5}{25} + \frac{3x^3}{25}\right]_0^1}$$

$$= \sqrt{2\left(\frac{1}{7} - \frac{6}{25} + \frac{3}{25}\right)}$$

$$= \sqrt{\frac{8}{175}}$$

$$= \frac{2}{5}\sqrt{\frac{2}{7}}$$

$$q_3 = \frac{x^3 - \tfrac{3}{5}x}{\tfrac{2}{5}\sqrt{\tfrac{2}{7}}} = \frac{5}{2}\sqrt{\frac{7}{2}}\left(x^3 - \tfrac{3}{5}x\right)$$

$$= \sqrt{\frac{7}{2}}\cdot\frac{1}{2}\left(5x^3 - 3x\right)$$

$$= \sqrt{\frac{7}{8}}\left(5x^3 - 3x\right)$$

Hence, the expressions for $q_0(x) \dots q_3(x)$ are:

$$q_0(x) = 1/\sqrt{2}$$

$$q_1(x) = \sqrt{\frac{3}{2}}\, x$$

$$q_2(x) = \sqrt{\frac{5}{8}}\,(3x^2 - 1)$$

$$q_3(x) = \sqrt{\frac{7}{8}}\,(5x^3 - 3x)$$

2) b)

The given integral can be expressed as the inner product:

$$\int_{-1}^{1} q_{n-1}(x)\, dx = \int_{-1}^{1} q_{n-1}(x) \cdot 1\, dx$$

$$= (q_{n-1}(x), 1)$$

Because we know from part (a) that $1 = q_0(x) \cdot \|1\|_2$,

$$\int_{-1}^{1} q_{n-1}(x)\, dx = (q_{n-1}(x), q_0(x) \cdot \|1\|_2)$$

$$= \|1\|_2 \, (q_{n-1}(x), q_0(x))$$

Because we know that $(q_i, q_j) = 0 \;\forall\, i \neq j$, and $n > 2$, we can say that:

$$\int_{-1}^{1} q_{n-1}(x)\, dx = \|1\|_2 \cdot 0 = 0 \quad \text{(proven)}$$