Indian Institute of Science, Banglore
Department of Computational and Data Sciences (CDS)

DS284: Numerical Linear Algebra

Assignment 4 [Posted Sep 27, 2025]

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Submissions required: Problems 2, 5 and 7. **Max Points: 50**

Notations: Vectors and matrices are denoted below by bold faced lower case and upper case alphabets respectively.

Problem 1

Given below is a magic matrix of size 3.

$$\mathbf{A} = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}$$

Find $\mathbf{Q}$ and $\mathbf{R}$ from QR factorization of given matrix by hand. Now do the following in Matlab/ Octave/ Python. For Matlab the command is:

$[\mathbf{Q}, \mathbf{R}] = \mathrm{qr}(\mathrm{magic}(3))$

Do these $\mathbf{Q}$ and $\mathbf{R}$ match your $\mathbf{Q}$ and $\mathbf{R}$? Is the QR factorization unique? If not unique, can you impose a condition on $\mathbf{R}$ to make the factorization unique?

Problem 2

Solution to this problem needs to be submitted by 11 OCT and will be graded.

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 \,|\, & x \,|\, & x^2 \,|\, & \dots \,|\, & x^{n-1} \end{bmatrix}$$

Each column is a function in $L^2[-1, 1]$ i.e., a vector space of real-valued function on $[-1, 1]$ which has inner-product of two functions f and g defined as:

$$(f, g) = \int_{-1}^{1} f(x)g(x)dx \tag{1}$$

If the QR factorization of $\mathbf{A}$ using the above definition of inner product can be written as

$$\mathbf{A} = \mathbf{QR} = \begin{bmatrix} q_0(x) \,|\, & q_1(x) \,|\, & q_2(x) \,|\, & \dots \,|\, & q_{n-1}(x) \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ 0 & 0 & \dots & r_{3n} \\ . & & & \\ . & & & \\ 0 & 0 & \dots & r_{mn} \end{bmatrix}$$

where columns of $\mathbf{Q}$ are functions of x, and are orthonormal with respect to inner product defined in equation (1). Now answer the following:

(a) Consider $n = 4$, derive expressions of $q_0(x), q_1(x), q_2(x), q_3(x)$ by using Gram Schmidt orthogonalization procedure.

(b) Show that $\int_{-1}^{1} q_{n-1}(x)dx = 0$ for $n \geqslant 2$.

Note that these $q_{n-1}(x)$ are called Legendre polynomials and the roots of these polynomials are called Gauss Legendre points and play a very important role in numerical integration as quadrature rules.

Problem 3

Let matrix:

$$\mathbf{A} = \begin{bmatrix} 0.70000 & 0.70711 \\ 0.70001 & 0.70711 \end{bmatrix}$$

(a) Consider a computer which rounds all computed results to five digits of relative accuracy. Using CGS or MGS, what will be the matrix $\mathbf{Q}$ associated with QR decomposition of $\mathbf{A}$ assuming that you are working on such a computer.

(b) Apply Householder's method to compute QR factorization of $\mathbf{A}$ using the same 5 digit arithmetic.

Compare the $\mathbf{Q}'s$ obtained in (a) and (b) and comment on the orthogonal nature of the $\mathbf{Q}$ matrix.

Problem 4

(a) Let $\mathbf{A}$ be a non-singular square matrix and let $\mathbf{A} = \mathbf{QR}$ be its QR factorization. Let also $\mathbf{A}^T\mathbf{A} = \mathbf{U}^T\mathbf{U}$ be the Cholesky factorization of $\mathbf{A}^T\mathbf{A}$. Can you conclude that $\mathbf{R} = \mathbf{U}$? If yes, prove it; if not, why not?

(b) Recall that by $\mathbf{A} \in \mathbb{R}^{m\times m}$, being symmetric and strictly positive definite, we mean $\mathbf{A} = \mathbf{A}^T$ and $\forall \mathbf{x} \in \mathbb{R}^m, \mathbf{x} \neq 0$, we have $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$. A symmetric matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ is positive semi-definite if $\forall \mathbf{x} \in \mathbb{R}^m, \mathbf{x} \neq 0$, we have $\mathbf{x}^T\mathbf{A}\mathbf{x} \geqslant 0$

If $\{\phi_i(x)\}_{i=1\ldots m}$ denote m linearly independent basis functions (non-zero) defined over $[-1, 1]$ in an m-dimensional vector space then show that the matrix $\mathbf{M} = \int_{-1}^{1} \phi_i(x)\phi_j(x)dx$ for $i, j = 1, 2\cdots m$ is a symmetric positive definite matrix.

Similarly show that the matrix $\mathbf{K} = \int_{-1}^{1} \frac{d\phi_i(x)}{dx}\frac{d\phi_j(x)}{dx}dx$ for $i, j = 1, 2\cdots m$ is a symmetric positive semi-definite matrix.

(c) Show that a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is symmetric, strictly positive definite if and only if there exists a matrix $\mathbf{B} \in \mathbb{R}^{m\times n}$ of rank n, where $n \leqslant m$, such that $\mathbf{A} = \mathbf{B}^T\mathbf{B}$. Assuming that $\mathbf{A}$ is of this form, is there a unique such $\mathbf{B}$?

Problem 5

Solution to this problem needs to be submitted by 11 OCT and will be graded.

In this problem you will test different algorithms for the least squares problem to approximate the function $f(t) = \sin(10t)$ for $t \in [0, 1]$ using a polynomial fit . To this end, first generate $m = 100$ data points using the above function which forms your given data i.e $(t_i, f(t_i)$ for

$i = 1 \cdots m$. Using this data, we would like to construct a 14th degree least squares polynomial fit to $f(t)$. Determine its least square it using the following methods.

(a) Use QR Factorization with your implementation of Modified Gram Schmidt. You should write your own back substitution code for solving the resulting triangular system.

(b) Using QR Factorization with your implementation of Householder factorization.

(c) Using SVD (Computed with any inbuilt libraries in MATLAB/Python/Octave)

(d) Using normal equations, you can use backslash command in MATLAB to solve this system.

Accept the MATLAB/Octave/Python least squares solution (given by backslash "\" in MATLAB) as the truth. Display and plot the approximation given by this "true" solution and compare it with $f(t)$. Compare with the solution given by 4 methods described above. Explain the results.

Problem 6

Given below is the matrix $\mathbf{A}$ and vector $\mathbf{b}$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \\ 2 & 8 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Find the least square solution for the above. Now do the same for the below matrix also.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \\ 5 & 7 & 9 \end{bmatrix}$$

Were you able to find the least square solution? If not explain why? Must there be a restriction on $\mathbf{A}$ for a least square solution to exist or will it always exist?

Problem 7

Solution to this problem needs to be submitted by 11 OCT and will be graded.

For each of the following statements prove that it is true or give an example to show that it is false. Assume $\mathbf{A} \in \mathbb{C}^{m \times m}$ unless otherwise indicated.

(a) If λ is an eigenvalue of $\mathbf{A}$ and $\mu \in \mathbb{C}$, then $\lambda - \mu$ is an eigenvalue of $\mathbf{A} - \mu\mathbf{I}$.

(b) If $\mathbf{A}$ is real and λ is an eigenvalue of $\mathbf{A}$ then so it $-\lambda$.

(c) If $\mathbf{A}$ is real and λ is an eigenvalue of $\mathbf{A}$, then so is λ^*. (λ^* is the complex conjugate of λ).

(d) If λ is an eigen value of $\mathbf{A}$ and $\mathbf{A}$ is non-singular, then λ^{-1} is the eigenvalue of $\mathbf{A}^{-1}$.

(e) If all the eigenvalues of $\mathbf{A}$ are zero, then $\mathbf{A} = 0$.

(f) If $\mathbf{A}$ is diagonalizable and all its eigenvalues are equal, then $\mathbf{A}$ is diagonal.

Problem 8

Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ with entries a_{ij} for $i, j = 1, 2, ..., n$ and define the closed disks $D(a_{ii}, r_i)$ centered at the diagonal entries a_{ii} of $\mathbf{A}$ of radius $r_i = \sum_{j=1}^{n}(1-\delta_{ij})|a_{ij}|$ for $i = 1, 2, ..., n$. Note that δ_{ij} represents Kronecker delta i.e $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. The above disks are called Greshgorin's disks.

(a) Prove that every eigenvalue of $\mathbf{A}$ lies in a Greshgorin disk.
(*Hint*: Let λ be any eigenvalue of $\mathbf{A}$ and $\mathbf{x}$ be the corresponding eigenvector with largest entry 1.)

(b) Suppose that $\mathbf{A}$ is diagonally dominant i.e. $|a_{ii}| > \sum_{j=1}^{n}(1-\delta_{ij})|a_{ij}|$ for all $i = 1, 2, ..., n$. Prove that $\mathbf{A}$ is invertible.

(c) Give estimates based on (a), for the eigenvalues of:

$$\mathbf{A} = \begin{bmatrix} 8 & 2 & 0 \\ 1 & 4 & \epsilon \\ 0 & \epsilon & 1 \end{bmatrix} \qquad \text{where } |\epsilon| < 1$$