

3) a) For a matrix to be orthogonal, it should satisfy the following:

$$\underline{Q}^{-1} = \underline{Q}^T, \text{ i.e. inverse of } \underline{Q} = \text{transpose of } \underline{Q}$$

First let us calculate \underline{Q}^T :- \underline{Q} is given as $(\underline{I} - \underline{S})^{-1}(\underline{I} + \underline{S})$

$$\begin{aligned} \underline{Q}^T &= [(\underline{I} + \underline{S})^{-1}(\underline{I} + \underline{S})]^T \\ &= (\underline{I} + \underline{S})^T [(\underline{I} - \underline{S})^{-1}]^T \quad (\text{reversal law, i.e. } (\underline{AB})^T = \underline{B}^T \underline{A}^T) \\ &= (\underline{I}^T + \underline{S}^T) [(\underline{I} - \underline{S})^{-1}]^T \quad (\underline{A+B}^T = \underline{A}^T + \underline{B}^T) \\ &= (\underline{I}^T + \underline{S}^T) [(\underline{I} - \underline{S})^T]^{-1} \quad ((\underline{A}^{-1})^T = (\underline{A}^T)^{-1}) \\ &= (\underline{I} - \underline{S}) (\underline{I}^T - \underline{S}^T)^{-1} \quad (\underline{I}^T = \underline{I} \text{ and } \underline{S}^T = -\underline{S}) \\ &= (\underline{I} - \underline{S}) (\underline{I} + \underline{S})^{-1} \end{aligned}$$

To prove \underline{Q} 's orthogonality, we should prove $\underline{Q} \cdot \underline{Q}^T = \underline{I}$. For this we need to rearrange ~~\underline{Q} & \underline{Q}^T~~ $\underline{Q} \cdot \underline{Q}^T$

Proof of commutability:

In this proof, I want to prove that $(\underline{I} + \underline{S})(\underline{I} - \underline{S}) = (\underline{I} - \underline{S})(\underline{I} + \underline{S})$

$$\begin{aligned} (\underline{I} + \underline{S})(\underline{I} - \underline{S}) &= \underline{I}^2 + \underline{S} - \underline{S} - \underline{S}^2 = \underline{I} - \underline{S}^2 \\ (\underline{I} - \underline{S})(\underline{I} + \underline{S}) &= \underline{I}^2 - \underline{S} + \underline{S} - \underline{S}^2 = \underline{I} - \underline{S}^2 \end{aligned}$$

∴ Hence, the commutability is proved.

This proof shall help us to rearrange \underline{Q} & \underline{Q}^T .

Rearranging Q^T :

For this, we have to ~~prove~~ ^{use the} proven commutability:

$$(\underline{I} - \underline{S})(\underline{I} + \underline{S}) = (\underline{I} + \underline{S})(\underline{I} - \underline{S})$$

$$(\underline{I} - \underline{S})^{-1}(\underline{I} - \underline{S})(\underline{I} + \underline{S}) = (\underline{I} - \underline{S})^{-1}(\underline{I} + \underline{S})(\underline{I} - \underline{S}) \quad (\text{Left multiply by } (\underline{I} - \underline{S})^{-1})$$

$$\underline{I}(\underline{I} + \underline{S}) = (\underline{I} - \underline{S})^{-1}(\underline{I} + \underline{S})(\underline{I} - \underline{S})$$

$$(\underline{I} + \underline{S})(\underline{I} - \underline{S})^{-1} = (\underline{I} - \underline{S})^{-1}(\underline{I} + \underline{S})(\underline{I} - \underline{S})(\underline{I} - \underline{S})^{-1} \quad (\text{Right multiply by } (\underline{I} - \underline{S})^{-1})$$

$$(\underline{I} + \underline{S})(\underline{I} - \underline{S})^{-1} = (\underline{I} - \underline{S})^{-1}(\underline{I} + \underline{S})$$

Hence, \underline{Q} can be written as ~~$(\underline{I} - \underline{S})^{-1}(\underline{I} + \underline{S})$~~
 $(\underline{I} + \underline{S})(\underline{I} - \underline{S})^{-1}$.

Now, we can prove $\underline{Q}^T \cdot \underline{Q} = \underline{I}$

$$\underline{Q}^T \underline{Q} = [(\underline{I} - \underline{S})(\underline{I} + \underline{S})^{-1}] [(\underline{I} + \underline{S})(\underline{I} - \underline{S})^{-1}]$$

$$= (\underline{I} - \underline{S}) \left[(\underline{I} + \underline{S})^{-1} (\underline{I} + \underline{S}) \right] (\underline{I} - \underline{S})^{-1} \quad (\text{distributive property of matrix multiplication})$$

$$= (\underline{I} - \underline{S})(\underline{I})(\underline{I} - \underline{S})^{-1}$$

$$= (\underline{I} - \underline{S})(\underline{I} - \underline{S})^{-1}$$

$$= \underline{I}$$

\therefore Hence proven, $\underline{Q}^T = \underline{Q}$, or long as \underline{S} is skew-symmetric

3.b) Since this is an "if & only if" statement, we shall have to prove both ways.

Proof #1: If $\underline{\underline{S}} = \underline{\underline{O}}$, then $\underline{\underline{u}}^T \underline{\underline{S}} \underline{\underline{u}} = \underline{\underline{0}}$.

This is easy to prove. If $\underline{\underline{S}} = \underline{\underline{O}}$, then

$$\begin{aligned} \underline{\underline{u}}^T \underline{\underline{S}} \underline{\underline{u}} &= (\underline{\underline{u}}^T \underline{\underline{S}}) \underline{\underline{u}} && \text{(Distributive property of} \\ &= (\underline{\underline{u}}^T \underline{\underline{O}}) \underline{\underline{u}} && \text{matrix vector multiplication)} \\ &= \underline{\underline{0}} \cdot \underline{\underline{u}} \\ &= \underline{\underline{0}} && \text{(proven)} \end{aligned}$$

Proof #2: If $\underline{\underline{u}}^T \underline{\underline{S}} \underline{\underline{u}} = \underline{\underline{0}}$, then $\underline{\underline{S}} = \underline{\underline{O}}$

This proof has to be done in two stages. In the first stage, we need to prove that all diagonal elements of $\underline{\underline{S}}$ are 0. In the second phase, we prove all non-diagonal elements of $\underline{\underline{S}}$ to be 0. This shall hence prove $\underline{\underline{S}} = \underline{\underline{O}}$.

Phase #1: Prove $\{S_{ii} = 0 \mid \forall i=j\}$

Assume $\underline{\underline{u}} = \underline{\underline{e}}_i$, then ($\underline{\underline{e}}_i$ represents a canonical basis vector)

$$\underline{\underline{e}}_i^T \underline{\underline{S}} \underline{\underline{e}}_i = 0$$

$$S_{ii} = 0 \quad (S_{ii} = \underline{\underline{e}}_i^T \underline{\underline{S}} \underline{\underline{e}}_i)$$

Since i can be any index $1 \leq i \leq m$, all diagonal elements of $\underline{\underline{S}}$ are hence proven to be 0.

Hence, $S_{ii} = 0$.

Phase #2: Prove all non-diagonal elements of S are 0.

Assume $\underline{u} = \underline{e}_i + \underline{e}_j$ & $i \neq j$

$$\text{Hence } \underline{u}^T S \underline{u} = (\underline{e}_i + \underline{e}_j)^T S (\underline{e}_i + \underline{e}_j)$$

$$0 = \underline{e}_i^T S \underline{e}_i + \underline{e}_j^T S \underline{e}_j + \underline{e}_i^T S \underline{e}_j + \underline{e}_j^T S \underline{e}_i$$

$$(\underline{e}_i \& \underline{e}_j \text{ are } 0, \text{ as proven in phase \#1}) \quad 0 = S_{ii} + S_{jj} + S_{ij} + S_{ji}$$

0, as proven

in phase #1)

$$0 = S_{ij} + S_{ji}$$

$$(S_{ij} = S_{ji}, \quad 0 = 2S_{ij})$$

because S

$$S_{ij} = 0$$

is given to be

symmetric)

Hence, $S_{ij} = 0$ for $i \neq j$.

Combining results of both phases, it can be stated that $S = \underline{\underline{0}}$.

- c) ~~Strong~~ Because this statement is an "if & only if" statement, both directions need to be proven.

Proof #1: If S is skew symmetric, $\underline{u}^T S \underline{u} = 0$.

~~Proof~~ We know that $(\underline{u}^T S \underline{u}) = (\underline{u}^T S \underline{u})^T$ because $\underline{u}^T S \underline{u}$ is a 1×1 matrix, i.e. a scalar value, which has no effect on transpose.

(contd.)

$$\underline{\underline{u}}^T \underline{\underline{S}} \underline{\underline{u}} = (\underline{\underline{u}}^T \underline{\underline{S}} \underline{\underline{u}})^T$$

$$\underline{\underline{u}}^T \underline{\underline{S}} \underline{\underline{u}} = \underline{\underline{u}}^T \underline{\underline{S}}^T (\underline{\underline{u}}^T)^T \quad (\text{Law of reversal, i.e. } (ABC)^T = C^T B^T A^T)$$

$$\underline{\underline{u}}^T \underline{\underline{S}} \underline{\underline{u}} = \underline{\underline{u}}^T \underline{\underline{S}}^T \underline{\underline{u}} \quad ((\underline{\underline{u}}^T)^T = \underline{\underline{u}})$$

$$\underline{\underline{u}}^T \underline{\underline{S}} \underline{\underline{u}} = \underline{\underline{u}}^T (-\underline{\underline{S}}) \underline{\underline{u}} \quad (\text{Since } \underline{\underline{S}} \text{ is skew-symmetric, } \underline{\underline{S}}^T = -\underline{\underline{S}})$$

$$\underline{\underline{u}}^T \underline{\underline{S}} \underline{\underline{u}} = -\underline{\underline{u}}^T \underline{\underline{S}} \underline{\underline{u}}$$

$$x = -x \quad (\text{Assume } x = \underline{\underline{u}}^T \underline{\underline{S}} \underline{\underline{u}})$$

$$2x = 0$$

$$x = 0$$

(Hence proven)

\therefore If $\underline{\underline{S}}$ is skew-symmetric, then $x = \underline{\underline{u}}^T \underline{\underline{S}} \underline{\underline{u}} = 0$.

Proof #2: If $\underline{\underline{u}}^T \underline{\underline{S}} \underline{\underline{u}} = 0$, then $\underline{\underline{S}}$ is skew-symmetric.

Any square matrix can be split into a symmetric & skew symmetric portion as follows:

$$\underline{\underline{A}} = \underbrace{\frac{1}{2}(\underline{\underline{A}} + \underline{\underline{A}}^T)}_{\text{symmetric part}} + \underbrace{\frac{1}{2}(\underline{\underline{A}} - \underline{\underline{A}}^T)}_{\text{skew-symmetric part}}$$

Hence, $\underline{\underline{S}}$ can be represented as $\frac{1}{2}(\underline{\underline{S}} + \underline{\underline{S}}^T) + \frac{1}{2}(\underline{\underline{S}} - \underline{\underline{S}}^T)$.

If we assume $\underline{\underline{S}}_{\text{symm}} = \frac{1}{2}(\underline{\underline{S}} + \underline{\underline{S}}^T)$ and $\underline{\underline{S}}_{\text{skew}} = \frac{1}{2}(\underline{\underline{S}} - \underline{\underline{S}}^T)$.

$\underline{\underline{S}}$ can be re-written as $\underline{\underline{S}}_{\text{symm}} + \underline{\underline{S}}_{\text{skew}}$.

Hence, $\underline{u}^T \underline{S} \underline{u} = 0$

$$\underline{u}^T (\underline{S}_{\text{symm}} + \underline{S}_{\text{skew}}) \underline{u} = 0$$

$$\underline{u}^T \underline{S}_{\text{symm}} \underline{u} + \underline{u}^T \underline{S}_{\text{skew}} \underline{u} = 0$$

$$\underline{u}^T \underline{S}_{\text{symm}} \underline{u} = 0$$

(Because $\underline{u}^T \underline{S}_{\text{skew}} \underline{u} = 0$ as proven in proof #1 of Question 3) a))

$$\therefore \underline{S}_{\text{symm}} = \underline{0}$$

(This has been proven in proof #2 of Question 3) b))

$$\frac{1}{2} (\underline{S} + \underline{S}^T) = \underline{0} \quad (\text{Substituting definition of } \underline{S}_{\text{symm}})$$

$$\underline{S} + \underline{S}^T = \underline{0}$$

$$\underline{S}^T = -\underline{S}$$

Since this is the fundamental property of a skew-symmetric matrix, we can henceforth prove that \underline{S} is a skew-symmetric matrix.

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Question 5

Calculated Quantities

Since the matrix \mathbf{A} and the vectors are random, the output of each code run will vary. However, the quantities calculated by one code run is as follows (in tabular format):

p-value	Maximum value of $\ \hat{A}\hat{x}\ _p$	Actual value of $\ \mathbf{A}\ _p$	Order of Relative Error
1	83.6969	83.6971	10^{-6}
2	10.5423	10.5423	10^{-10}
∞	4.3310	4.3301	10^{-4}

Although the program has output numbers to up to 14 decimal places, I have rounded them down to 4 decimal places, for better readability. Furthermore, I have not added the other norms of $\mathbf{A}\hat{x}$, because they do not have a corresponding norm of $\|\mathbf{A}\|$. The other norms, which were calculated, are as follows:

p-value	Maximum value of $\ \hat{A}\hat{x}\ _p$
3	6.1872
4	5.0428
5	4.5759
6	4.3540

Code and Algorithm

The Python code for the calculation is as follows:

```
import numpy as np

#generate random matrix A
A = np.random.randn(100,2)

#this array keeps track of maximum 1-norm, 2-norm etc. of the vector Ax
#max_norm[0] represents maximum 1-norm of Ax,
#max_norm[1] represents maximum 2-norm of Ax, etc
max_norm = [-1,-1,-1,-1,-1,-1,-1]

for _ in range(1000):
    #Generate random x vector
    #The np.random.randn() method returns a matrix of 2x1 instead of a vector
    #the squeeze function is used to convert the 2x1 matrix to a vector
    x = np.squeeze(np.random.randn(2,1))

    #take 1-norm to 6-norm
    #x is converted to unit vector for the particular norm
    #i.e if 1-norm of Ax is to be calculated, x is normalised by its 1-norm, and so on
    for p in range(1,7):
        #Calculate the relevant norm of x and normalise x accordingly
        pnorm_x = np.linalg.norm(x,ord=p)
        unit_x = [element/pnorm_x for element in x]

        #Calculate Ax
        #Again the @ operator produces a matrix, which needs to be squeezed to a vector
        Ax = np.squeeze(A @ unit_x)

        #Update the max_norm
        max_norm[p-1]=max(max_norm[p-1], np.linalg.norm(Ax, ord=p))
```

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```
#take infinity norm
#This section is outside the loop because p cannot
#be suddenly assigned to infinity
pnorm_x = np.linalg.norm(x,ord=np.inf)
unit_x = [element/pnorm_x for element in x]
Ax = np.squeeze(A @ unit_x)
max_norm[6]=max(max_norm[6], np.linalg.norm(Ax, ord=np.inf))

print("1-norm of A =", np.linalg.norm(A, ord=1))
print("Max 1-norm of Ax =", max_norm[0], end="\n")

print("2-norm of A =", np.linalg.norm(A, ord=2))
print("Max 2-norm of Ax =", max_norm[1], end="\n")

print("infinity-norm of A =", np.linalg.norm(A, ord=np.inf))
print("Max infinity-norm of Ax =", max_norm[6])

print("Remaining norms of x: ", max_norm[2:6])
```

The program steps are as follows:

1. A random matrix with dimensions 100×2 is created. This is the **A** matrix.
2. A list called `max_norm` is created with 7 elements. This list shall store the maximum of the norms. The first element of the list, i.e. `max_norm[0]`, contains the maximum 1-norm of **Ax**. The second element stores the maximum 2-norm, and so on. The last element shall store the ∞ -norm of **Ax**.
3. A for loop is present, which shall generate 1000 random **x** vectors, such that $\mathbf{x} \in \mathbb{R}^{100}$. This vector is then normalised. Its norm is calculated and updated to the `max_norm` list (if it is the maximum).
4. Lastly, the 1-norm, 2-norm and ∞ -norm of **A** is calculated, to compare with the maximum values obtained from the iterations above.

Sample Output

Since the matrix **A** and the vectors are random, the output of each code run will vary. However, one such output is as follows:

```
1-norm of A = 77.14076257113005
Max 1-norm of Ax = 77.12304412473856
2-norm of A = 9.568208854655976
Max 2-norm of Ax = 9.568208654815267
infinity-norm of A = 4.49453960570396
Max infinity-norm of Ax = 4.487772423831027
Remaining norms of x: [np.float64(5.627964886251967), np.float64(4.679666918956469),
np.float64(4.341368483515785), np.float64(4.219939313675693)]
```

Observations

From the above table, we can devise the following:

$$\|A\hat{x}\|_p^m \leq \|A\|_p^{(m,n)}, \forall x \in \mathbb{R}^n, \|\hat{x}\|_p = 1$$

From this, we can also infer that there is one $\widehat{x_{max}}$ such that the above inequality is converted to equality, i.e.

$$\|A\widehat{x_{max}}\|_p^m = \|A\|_p^{(m,n)}, \|\widehat{x_{max}}\|_p = 1$$