Indian Institute of Science, Bangalore
Department of Computational and Data Sciences (CDS)

DS284: Numerical Linear Algebra

Assignment 0 [Posted Jul 29, 2025]

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This assignment helps you to refresh some of the concepts useful for this course

Notations: (i) Vectors and matrices are denoted by bold-faced lower case and upper case alphabets, respectively. (ii) Set of all real numbers is denoted by $\mathbb{R}$ (iii) Set of all n dimensional vectors is denoted by $\mathbb{R}^n$ and set of all $m \times n$ matrices is denoted by $\mathbb{R}^{m\times n}$

1. **Vector Space:** A non-empty set $\mathbb{V}$ is said to form a real vector space if it satisfies the following conditions:

    1. $\mathbf{u}+\mathbf{v} \in \mathbb{V},\ \forall\ \mathbf{u},\mathbf{v} \in \mathbb{V}$ [Closure under addition property]
    2. $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u},\ \forall\ \mathbf{u},\mathbf{v} \in \mathbb{V}$ [Commutative property]
    3. $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w},\ \forall\ \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{V}$ [Associative property]
    4. $\mathbf{0} \in \mathbb{V}$ such that $\mathbf{v}+\mathbf{0} = \mathbf{v},\ \forall\ \mathbf{v} \in \mathbb{V}$ [Zero vector]
    5. Each $\mathbf{v} \in \mathbb{V}$ has a $\mathbf{w} \in \mathbb{V}$ such that $\mathbf{v}+\mathbf{w} = \mathbf{0}$ [Additive inverse]
    6. $c\,\mathbf{u} \in \mathbb{V}, \forall \mathbf{u} \in \mathbb{V}$ and $c \in \mathbb{R}$ [Closure under scalar Multiplication]
    7. $(c+d)\,\mathbf{v} = c\,\mathbf{v} + d\,\mathbf{v}, \forall \mathbf{v} \in \mathbb{V}$ and $c, d \in \mathbb{R}$ [Distributive property for scalar addition]
    8. $c\,(\mathbf{v}+\mathbf{w}) = c\,\mathbf{v} + c\,\mathbf{w}, \forall \mathbf{v},\mathbf{w} \in \mathbb{V}$ and $c \in \mathbb{R}$ [Distributive property for vector addition]
    9. Multiplication by the scalar 1, $1\,\mathbf{v} = \mathbf{v}, \forall \mathbf{v} \in \mathbb{V}$ [Identity operation]
    10. $(cd)\mathbf{v} = c(d\mathbf{v}), \forall \mathbf{v} \in \mathbb{V}$ and $c, d \in \mathbb{R}$ [Compatibility of scalar multiplication with field multiplication]

    Now, verify if the following form a vector space.

    (a) $\mathbb{V} = \left\{ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \;\middle|\; w - x - y + z = 0;\ w, x, y, z \in \mathbb{R} \right\}$

    (b) $\mathbb{M}^{2\times 2} = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R}\} \right\}$

    (c) $\mathbb{N} = \{f : \mathbb{R} \to \mathbb{R} \mid \frac{\mathrm{d}f}{\mathrm{d}x} + 2f = 1\}$

2. **Subspace:** A non-empty subset $\mathbb{W}$ of a vector space $\mathbb{V}$ over a field F is a subspace of $\mathbb{V}$ if and only if

   1. $\mathbf{a} \in \mathbb{W},\ \mathbf{b} \in \mathbb{W} \Rightarrow \mathbf{a} + \mathbf{b} \in \mathbb{W}$
   2. $\mathbf{a} \in \mathbb{W},\ \alpha \in F \Rightarrow \alpha\mathbf{a} \in \mathbb{W}$

   Note: In the problems below we will consider $F = \mathbb{R}$.
   Verify the following for $\mathbb{V}$ defined as below:.

   (a) Does $\mathbb{V} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle|\ x, y, z \geq 0 \right\}$ form a subspace of $\mathbb{R}^3$?

   (b) Does $\mathbb{V} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle|\ x, y, z \in \mathbb{R},\ x + z = 0 \right\}$ form a subspace of $\mathbb{R}^3$?

   (c) Does $\mathbb{V} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle|\ \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0 \right\}$ form a subspace of $\mathbb{R}^{2\times 2}$?

3. **Linear Dependence and Linear Independence:** A finite set vectors $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ of a vector space $\mathbb{V}$ over a field F (in our case it is $\mathbb{R}$ or $\mathbb{C}$) is said to be linearly dependent in $\mathbb{V}$ if there exist scalars $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ not all zero in F such that

   $$\alpha_1\mathbf{v_1} + \alpha_2\mathbf{v_2} + ... + \alpha_n\mathbf{v_n} = \mathbf{0} \tag{1}$$

   The set is said to be linearly independent in $\mathbb{V}$ if the equality (1) is satisfied only when $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$

   Basis: Let $\mathbb{V}$ be a vector space over a field F. A set $\mathbb{S}$ of vectors in $\mathbb{V}$ is said to be a basis of $\mathbb{V}$ if

   1. $\mathbb{S}$ is linearly independent in $\mathbb{V}$, and
   2. Any vector in $\mathbb{V}$ can be written as linear combination of the vectors in $\mathbb{S}$.

   Now answer the following questions:

   (a) Can $(0, -26, -9)$ and $(1, 3, 5)$ be expressed as a linear combinations of $(5, 3, 7)$ and $(2, -4, 1)$. Explain.

   (b) Find a basis of the given subspace.

   $$\mathbf{U} = \left\{ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4 \mid 3w + x - 7z = 0 \right\}$$

4. **Solving linear system of equations multiple ways:**

   (a) Solve the following set of linear system of equations graphically and comment on the solvability (no solution, infinite solutions or unique solution) of the systems.

   - Set 1
$$x + 2y = 3$$
$$4x + 5y = 6$$
   - Set 2
$$x + 2y = 3$$
$$4x + 8y = 6$$
   - Set 3
$$x + 2y = 3$$
$$4x + 8y = 12$$

   (b) Solve the same set of linear system of equations algebraically. Confirm your observations obtained from part (a).

   (c) The matrix vector product $\mathbf{Ax}$ can also be viewed as linear combination of column vectors of $\mathbf{A}$.
$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + y \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$
   Using the above view point and the definition of linear independence/dependence of vectors, analyse the solvability of the above system of equations without having to solve the system of equations completely.

5. **Review of Matrices:**

   Consider three vectors, $\mathbf{a} = \begin{pmatrix} -209/362 \\ -209/362 \\ 209/362 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ -408/577 \\ -408/577 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 396/485 \\ -198/485 \\ 198/485 \end{pmatrix}$

   (a) Evaluate the dot product/inner product $\mathbf{a.b}$, $\mathbf{a.c}$ and $\mathbf{b.c}$ The dot product is given by $\mathbf{a.b} = \sum_{i=1}^{n} a_i b_i$ where $\mathbf{a,b} \in \mathbb{R}^n$

   (b) Calculate the geometric length of the above vectors **a, b, c** and also of $\mathbf{x}$ where $\mathbf{x} = \begin{pmatrix} 2 \\ -40/57 \\ 8/77 \end{pmatrix}$. Note that the geometric length of the vector $\mathbf{a} \in \mathbb{R}^n$ is computed as $\sqrt{\mathbf{a.a}}$.

(c) Construct a matrix $\mathbf{A}$ with the above vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as the first, second and third columns of $\mathbf{A}$. Calculate the geometric length of $\mathbf{Ax}$ and compare with that of $\mathbf{x}$. What do you observe?

(d) Evaluate $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$

(e) Prove that for full rank square matrices with the property $\mathbf{A}\mathbf{A}^T = \mathbf{I}$, your observation in part (c) will always be true for any $\mathbf{x}$.

Reference textbooks for review:

1. Linear Algebra and its Applications by Gilbert Strang.
2. Linear Algebra by S. Friedberg, A. Insel and L. Spence. (Solved examples)