NAME - SUHAS NAGARAJ
UID - 119505373

ENPM 667

PROBLEM SET - 2

PROBLEM - 1

a) $\begin{vmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ 3 & -3 & 4 & -2 \\ -2 & 1 & -2 & 1 \end{vmatrix}$ => Determinant of Matrix M

$$\rightarrow \begin{vmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ 3 & -3 & 4 & -2 \\ -2 & 1 & -2 & 1 \end{vmatrix} = M_{11}C_{11} + M_{12}C_{12} + M_{13}C_{13} + M_{14}C_{14} \quad - (1)$$

where $C_{11} = \begin{vmatrix} 1 & -2 & 1 \\ -3 & 4 & -2 \\ 1 & -2 & 1 \end{vmatrix} \times (-1)^{1+1}$

$$= 1\big((4\times1) - (-2)(-2)\big) - (-2)\big((-3)(1) - (-2)(1)\big) + 1\big((-3)(-2) - (4)(1)\big)$$

$$= (4-4) + 2(-3+2) + (6-4)$$

$$= 0 - 2 + 2$$

$$C_{11} = 0$$

$C_{12} = \begin{vmatrix} 0 & -2 & 1 \\ 3 & 4 & -2 \\ -2 & -2 & 1 \end{vmatrix} \times (-1)^{1+2}$

$$C_{12} = -1\left[0-(-2)\Big((3)(1)-(-2)(-2)\Big)+(1)\Big((3)(-2)-(4)(-2)\Big)\right]$$

$$= -1\Big(+2(3-4)+1(-6+8)\Big)$$

$$= -1(-2+2)$$

$$\underline{C_{12} = 0}$$

$$C_{13} = \begin{vmatrix} 0 & 1 & 1 \\ 3 & -3 & -2 \\ -2 & 1 & 1 \end{vmatrix} \times (-1)^{1+3}$$

$$= 0 - 1\Big((3)(1)-(-2)(-2)\Big) + 1\Big((3)(1)-(-3)(-2)\Big)$$

$$= -1(3-4)+1(3-6)$$

$$= 1+(-3)$$

$$\underline{C_{13} = -2}$$

$$C_{14} = \begin{vmatrix} 0 & 1 & -2 \\ 3 & -3 & 4 \\ -2 & 1 & -2 \end{vmatrix} \times (-1)^{1+4}$$

$$= -1\left[0-1\Big((3)(-2)-(4)(-2)\Big)+(-2)\Big((3)(1)-(-3)(-2)\Big)\right]$$

$$= -1\left[-1(-6+8)-2(3-6)\right]$$

$$= -1\left[-2+6\right]$$

$$\underline{C_{14} = -4}$$

$\therefore$ Determinant of the matrix $= M_{11}C_{11} + M_{12}C_{12} + M_{13}C_{13} + M_{14}C_{14}$

$$= 1(0) + 0 + 2(-2) + 3(-4)$$

$$= -4 - 12$$

$$\boxed{Det = -16}$$

b) $\begin{vmatrix} gc & ge & a+ge & gb+ge \\ 0 & b & b & b \\ c & e & e & b+e \\ a & b & b+f & b+d \end{vmatrix}$

$\Rightarrow$ Simplifying the above matrix using row and column operations, before finding the determinant

* Column 3 = Column 3 - Column 2
* Column 4 = Column 4 - Column 2

$\}$ No effect on the determinant

$$\begin{vmatrix} gc & ge & a & gb \\ 0 & b & 0 & 0 \\ c & e & 0 & b \\ a & b & f & d \end{vmatrix}$$

* Taking out 'b' from Row 2 $\rightarrow$ Determinant multiplied by 'b'

* Interchange Row 1 and Row 2 $\rightarrow$ Determinant multiplied by (-1)

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ gc & ge & a & gb \\ c & e & 0 & b \\ a & b & f & d \end{vmatrix} \times (-1) \times (b)$$

Expanding :-

$$= \left[0 - 1 \times \begin{vmatrix} gc & a & gb \\ c & 0 & b \\ a & f & d \end{vmatrix} + 0 - 0 \right] \times (-b)$$

$$= -1 \left[gc(0 - fb) - a(cd - ab) + gb(cf - 0) \right] \times (-b)$$

$$= b \left[gc(-fb) - a(cd - ab) + gb(cf) \right]$$

$$= b \left[\cancel{-gcfb} - acd + a^2 b + \cancel{gbcf} \right]$$

$$\boxed{= -abcd + a^2 b^2 \Rightarrow \text{Determinant of the matrix}}$$

Problem 2

Using the properties of determinants, solve with a minimum of calculations, the following equations for x:

$$\begin{vmatrix} x & a & a & 1 \\ a & x & b & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix} = 0$$

⇒ As the 'x' terms are in the diagonal, during expansion of the matrix to find the determinant, the 'x' terms will get multiplied. Hence, the final equation will be a cubic equation in terms of 'x'. Therefore x will have <u>3 roots</u>.

⇒ To find the values of x, we can use properties of determinants, which will minimize the steps.

→ Case 1 → $x = a$, the matrix becomes :-

$$\begin{vmatrix} a & a & a & 1 \\ a & a & b & 1 \\ a & b & a & 1 \\ a & b & c & 1 \end{vmatrix} = 0$$

Taking out a from Column 1 (as a is common)

$$a\begin{vmatrix} 1 & a & a & 1 \\ 1 & a & b & 1 \\ 1 & b & a & 1 \\ 1 & b & c & 1 \end{vmatrix} = 0$$

Here ~~a~~ Column 1 = Column 4 :

Therefore Determinant = 0 ⟹ LHS = RHS

Hence 'a' is one of the roots of 'x'.

[This is because of the property " If two or more rows (or columns) of a matrix are equal, then its determinant is zero]

Case 2 → $x = b$, the matrix becomes :-

$$\begin{vmatrix} b & a & a & 1 \\ a & b & b & 1 \\ a & b & b & 1 \\ a & b & c & 1 \end{vmatrix} = 0$$

Here, Row 2 = Row 3, Hence determinant is zero.

As LHS = RHS, $x = b$ satisfies the equation.

Hence, 'b' is one of the roots of 'x'

[Property :- If two or more rows (or columns) of a matrix are equal, then its determinant is zero.'

* This property is an extension of the zero property of the determinant which states " If one or more rows | columns of a matrix are all zeros, then the determinant is zero". We can perform row | column operations to get the all zero row | column in this case]

Case - 3 → $x = c$, the matrix becomes

$$\begin{vmatrix} c & a & a & 1 \\ a & c & b & 1 \\ a & b & c & 1 \\ a & b & c & 1 \end{vmatrix} = 0$$

Here, as Row 3 = Row 4, the determinant of the matrix is zero. As LHS = RHS, ~~the~~

$x = c$ satisfies the equation.

Hence, 'c' is one of the roots of 'x'

[Property :- If two or more columns (or rows) of a matrix are equal, then its determinant is zero]

⇒ Hence $x = a$, $x = b$ and $x = c$ are the solutions for x.

Problem 3

$$A = \begin{bmatrix} 1 & \gamma_1 & 0 \\ \gamma_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ where γ_1 and γ_2 are non zero complex numbers

a) Eigen Values and Eigen Vectors of A,

we know that

$$Ax = \lambda x \quad — (1)$$

$$(Ax - \lambda x) = \vec{0}$$

$$x(A - I\lambda) = \vec{0}$$

Here $x \neq \vec{0}$, Hence $|A - \lambda I| = 0$

$$\text{Det}\left(\begin{bmatrix} 1 & \gamma_1 & 0 \\ \gamma_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda \right) = 0$$

$$\begin{vmatrix} 1-\lambda & \gamma_1 & 0 \\ \gamma_2 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

Expanding with respect to Row 3

$$(1-\lambda)\left[(1-\lambda)(1-\lambda) - (\gamma_1)(\gamma_2)\right] = 0$$

$$(1-\lambda)\left[(1-\lambda)^2 - \gamma_1\gamma_2\right] = 0$$

Here $1-\lambda = 0$ and $((1-\lambda)^2 - \gamma_1\gamma_2) = 0$

$1 - \lambda = 0$

$\underline{\lambda = 1}$

$(1-\lambda)^2 - \gamma_1\gamma_2 = 0$

$(1-\lambda)^2 = \gamma_1\gamma_2$

$1 - \lambda = \pm\sqrt{\gamma_1\gamma_2}$

$\therefore \underline{\lambda = 1 \mp \sqrt{\gamma_1\gamma_2}}$

$\therefore$ The Eigen values are $\tau_1 = 1$, $\tau_2 = 1 + \sqrt{\gamma_1 \gamma_2}$

and $\tau_3 = 1 - \sqrt{\gamma_1 \gamma_2}$

$\Rightarrow$ Finding Eigen Vectors,

$\rightarrow$ Consider $\tau_1 = 1$ $\rightarrow$ substituting in equation 1

$$AX = \tau_1 X$$

$$\begin{bmatrix} 1 & \gamma_1 & 0 \\ \gamma_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Multiplying the Matrices and equating,

$x_1 + \gamma_1 x_2 = x_1$	$\gamma_2 x_1 + x_2 = x_2$	$x_3 = x_3$
$\therefore \gamma_1 x_2 = 0$	$\gamma_2 x_1 = 0$	$\therefore x_3$ can take any value
$x_2 = 0$	$x_1 = 0$	Let $x_3 = 1$

$\therefore$ The eigen vector is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = e_1$$

$\rightarrow$ Consider $\tau_2 = 1 + \sqrt{\gamma_1 \gamma_2}$

$$AX = \tau_2 X$$

$$\begin{bmatrix} 1 & \gamma_1 & 0 \\ \gamma_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 + \sqrt{\gamma_1 \gamma_2} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Multiplying the matrices and equating

* $x_1 + \gamma_1 x_2 = \left(1 + \sqrt{\gamma_1 \gamma_2}\right) x_1$

$\cancel{x_1} + \gamma_1 x_2 = \cancel{x_1} + x_1 \sqrt{\gamma_1 \gamma_2}$

$\gamma_1 x_2 = x_1 \sqrt{\gamma_1 \gamma_2}$

$\div$ by $\sqrt{\gamma_1}$ on both sides

$\sqrt{\gamma_1}\, x_2 = \sqrt{\gamma_2}\, x_1$.

To find the values of x_1 and x_2, assume the value of one and derive the value for the other

$\therefore$ Let $x_1 = \sqrt{\gamma_1}$,

then $\cancel{\sqrt{\gamma_1}}\, x_2 = \sqrt{\gamma_2} \times \cancel{\sqrt{\gamma_1}}$

$x_2 = \sqrt{\gamma_2}$

* $x_3 = \left(1 + \sqrt{\gamma_1 \gamma_2}\right) x_3$

$\cancel{x_3} = \cancel{x_3} + x_3 \sqrt{\gamma_1 \gamma_2}$

$\therefore x_3 \cdot \sqrt{\gamma_1 \gamma_2} = 0$

$\therefore x_3 = 0$, As $\sqrt{\gamma_1 \gamma_2} \neq 0$

$\therefore$ The eigen vector is,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \sqrt{\gamma_1} \\ \sqrt{\gamma_2} \\ 0 \end{bmatrix} = e_2$$

→ Consider $\gamma_3 = 1 - \sqrt{\gamma_1 \gamma_2}$

$Ax = \gamma_3 x$

$$\begin{bmatrix} 1 & \gamma_1 & 0 \\ \gamma_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 - \sqrt{\gamma_1 \gamma_2} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Expanding and equating

$$x_1 + \gamma_1 x_2 = \left(1 - \sqrt{\gamma_1 \gamma_2}\right) x_1$$

$$\cancel{x_1} + \gamma_1 x_2 = \cancel{x_1} - x_1 \sqrt{\gamma_1 \gamma_2}$$

$\div$ both side by $\sqrt{\gamma_1}$

$$\sqrt{\gamma_1}\, x_2 = -x_1 \sqrt{\gamma_2}$$

$\therefore$ Let $x_1 = \sqrt{\gamma_1}$

then $\cancel{\sqrt{\gamma_1}}\, x_2 = -\cancel{\sqrt{\gamma_1}} \sqrt{\gamma_2}$

$$x_2 = -\sqrt{\gamma_2}$$

* $x_3 = \left(1 - \sqrt{\gamma_1 \gamma_2}\right) x_3$

$$x_3 = x_3 - \sqrt{\gamma_1 \gamma_2}\, x_3$$

$$\therefore x_3 \sqrt{\gamma_1 \gamma_2} = 0$$

$\therefore x_3 = 0$ As $\sqrt{\gamma_1 \gamma_2} \neq 0$

$\therefore$ The eigen vector is :-

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \sqrt{\gamma_1} \\ -\sqrt{\gamma_2} \\ 0 \end{bmatrix} = e_3$$

b) Conditions for the Eigen values to be Real :-

$\Rightarrow$ The eigen values are $\tau_1 = 1$, $\tau_{2,3} = 1 \pm \sqrt{\gamma_1 \gamma_2}$

$\rightarrow$ $\tau_1 = 1$, here eigen value is already requal to a real number 1

$\rightarrow$ $\tau_{2,3} = 1 \pm \sqrt{\gamma_1 \gamma_2}$, ~~here the γ_1 and γ_2~~

here, the term $\sqrt{\gamma_1 \gamma_2}$ should be real in order for the eigen values to be real.

In order for a square root term to be real, the value inside the square root should be positive. Hence $(\gamma_1 \times \gamma_2)$ should be a positive real number.

c) Two vectors are said to be orthogonal if the dot product of one with the transpose of the other is equal to zero,

$\rightarrow$ In this problem, the dot product of e_1 and e_3 and the dot product of e_1 and e_2 are zero

Hence they are Orthogonal

$\rightarrow$ ~~To prove that e_2 and e_3 are orthogonal~~

→ Conditions for e_2 and e_3 to be orthogonal :

* If γ_1 and γ_2 are real numbers

$$e_2 \times e_3^T = 0$$

$$\begin{bmatrix} \sqrt{\gamma_1} \\ \sqrt{\gamma_2} \\ 0 \end{bmatrix} \begin{bmatrix} \sqrt{\gamma_1} & -\sqrt{\gamma_2} & 0 \end{bmatrix} = 0$$

$$\gamma_1 - \gamma_2 = 0$$

$\underline{\gamma_1 = \gamma_2}$ → If γ_1 and γ_2 are real numbers

* If γ_1 and γ_2 are complex numbers,

$e_2 \times e_3^{CT} = 0$ where e_3^{CT} is the conjugate transpose of e_3

$$\begin{bmatrix} \sqrt{\gamma_1} \\ \sqrt{\gamma_2} \\ 0 \end{bmatrix} \begin{bmatrix} \sqrt{\gamma_1^c} & -\sqrt{\gamma_2^c} & 0 \end{bmatrix} = 0$$

where γ_1^c is conjugate of γ_1 and γ_2^c is conjugate of γ_2

$$\sqrt{\gamma_1}\sqrt{\gamma_1^c} - \sqrt{\gamma_2}\sqrt{\gamma_2^c} = 0$$

$$\sqrt{\gamma_1\gamma_1^c} - \sqrt{\gamma_2\gamma_2^c} = 0$$

$$|\gamma_1| - |\gamma_2| = 0$$

$\underline{|\gamma_1| = |\gamma_2|}$ → If we assume γ_1 and γ_2 as complex numbers.

Problem 4

$$A = \begin{bmatrix} 2 & -3 & 1 & 3 \\ 1 & 4 & -3 & -3 \\ 5 & 3 & -1 & -1 \\ 3 & -6 & -3 & 1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}$$

For LU decomposition of A, consider A in the form,

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & L_{32} & 1 & 0 \\ L_{41} & L_{42} & L_{43} & 1 \end{bmatrix} \times \begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{bmatrix} \quad \text{—} (1)$$

$$LU = \begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ L_{21}U_{11} & L_{21}U_{12}+U_{22} & L_{21}U_{13}+U_{23} & L_{21}U_{14}+U_{24} \\ L_{31}U_{11} & L_{31}U_{12}+L_{32}U_{22} & L_{31}U_{13}+L_{32}U_{23}+U_{33} & L_{31}U_{14}+L_{32}U_{24}+U_{34} \\ L_{41}U_{11} & L_{41}U_{12}+L_{42}U_{22} & L_{41}U_{13}+L_{42}U_{23}+L_{43}U_{33} & L_{41}U_{14}+L_{42}U_{24}+L_{43}U_{34}+U_{44} \end{bmatrix}$$

Comparing Matrix A and Matrix LU,

I

* $\boxed{U_{11} = 2}$
* $L_{21} U_{11} = 1$

$\therefore \boxed{L_{21} = \frac{1}{2}}$

* $\boxed{U_{12} = -3}$
* $L_{21} U_{12} + U_{22} = 4$

$\therefore \boxed{U_{22} = 11/2}$

* $\boxed{U_{13} = 1}$
* $\boxed{U_{14} = 3}$
* $L_{21} U_{13} + U_{23} = -3$

$\therefore \boxed{U_{23} = -\frac{7}{2}}$

* $L_{21} U_{14} + U_{24} = -3$

$\boxed{U_{24} = -\frac{9}{2}}$

II

* $L_{31} U_{11} = 5$

$\boxed{L_{31} = \frac{5}{2}}$

* $L_{31} U_{12} + L_{32} U_{22} = 3$

$\boxed{L_{32} = \frac{21}{11}}$

* $L_{31} U_{13} + L_{32} U_{23} + U_{33} = -1$

$\boxed{U_{33} = \frac{35}{11}}$

* $L_{31} U_{14} + L_{32} U_{24} + U_{34} = -1$

$\boxed{U_{34} = \frac{1}{11}}$

* $L_{41} U_{11} = 3$

* $\boxed{\therefore L_{41} = \frac{3}{2}}$

* $L_{41} U_{12} + L_{42} U_{22}$

$\boxed{L_{42} = \frac{-3}{11}}$

III

* $L_{41} U_{13} + L_{42} U_{23} + L_{43} U_{33} = -3$

$L_{43} = \frac{-60}{11} \times \frac{11}{35}$

$\boxed{L_{43} = \frac{-12}{7}}$

* $L_{41} U_{14} + L_{42} U_{24} + L_{43} U_{34} + U_{44} = 1$

$\boxed{U_{44} = \frac{-32}{7}}$

∴ Substituting the values back in equation 1

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 5/2 & 21/11 & 1 & 0 \\ 3/2 & -3/11 & -12/7 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & 11/2 & -7/2 & -9/2 \\ 0 & 0 & 35/11 & 1/11 \\ 0 & 0 & 0 & -32/7 \end{bmatrix}$$

LU Decomposition of A

i) $b = (-4 \ \ 1 \ \ 8 \ \ -5)^T$

$AX = b$

$LUX = b$

$Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 5/2 & 21/11 & 1 & 0 \\ 3/2 & -3/11 & -12/7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 8 \\ -5 \end{bmatrix}$$

Expanding,

* $\boxed{y_1 = -4}$
* $\frac{1}{2} y_1 + y_2 = 1$

  $\boxed{y_2 = 3}$
* $\frac{5}{2} y_1 + \frac{21}{11} y_2 + y_3 = 8$

  $\boxed{y_3 = \frac{135}{11}}$
* $\frac{3}{2} y_1 - \frac{3}{11} y_2 - \frac{12}{7} y_3 + y_4 = -5$

  $\boxed{y_4 = \frac{160}{7}}$

$$\Rightarrow \quad y = \begin{bmatrix} -4 \\ 3 \\ 135/11 \\ 160/7 \end{bmatrix}$$

But $\quad Ux = y$

$$\therefore \begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & 11/2 & -7/2 & -9/2 \\ 0 & 0 & 35/11 & 1/11 \\ 0 & 0 & 0 & -32/7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 135/11 \\ 160/7 \end{bmatrix}$$

Expanding,

* $-32/7\ x_4 = \dfrac{160}{7}$

$\therefore \boxed{x_4 = -5}$

* $\dfrac{35}{11} x_3 + \dfrac{1}{11} x_4 = \dfrac{135}{11}$

$\boxed{x_3 = 4}$

* $\dfrac{11}{2} x_2 - \dfrac{7}{2} x_3 - \dfrac{9}{2} x_4 = 3$

$\boxed{x_2 = -1}$

* $2x_1 - 3x_2 + x_3 + 3x_4 = -4$

$\boxed{x_1 = 2}$

$$\therefore X = \begin{bmatrix} 2 \\ -1 \\ 4 \\ -5 \end{bmatrix}$$

ii) $b = (-10 \quad 0 \quad -3 \quad -24)^T$

$AX = b$

$LUX = b$

$LY = b$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 5/2 & 21/11 & 1 & 0 \\ 3/2 & -3/11 & -12/7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ -3 \\ -24 \end{bmatrix}$$

Expanding,

* $\boxed{y_1 = -10}$
* $\frac{1}{2} y_1 + y_2 = 0$

  $\therefore \boxed{y_2 = 5}$
* $\frac{5}{2} y_1 + \frac{21}{11} y_2 + y_3 = -3$

  $\boxed{y_3 = \frac{137}{11}}$
* $\frac{3}{2} y_1 - \frac{3}{11} y_2 - \frac{12}{7} y_3 + y_4 = -24$

  $\boxed{y_4 = \frac{96}{7}}$

$$\therefore y = \begin{bmatrix} -10 \\ 5 \\ 137/11 \\ 96/7 \end{bmatrix}$$

But $UX = y$

$$\begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & 11/2 & -7/2 & -9/2 \\ 0 & 0 & 35/11 & 1/11 \\ 0 & 0 & 0 & -32/7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 5 \\ 137/11 \\ 96/7 \end{bmatrix}$$

Expanding,

* $-\frac{32}{7} x_4 = \frac{96}{7}$

$\boxed{x_4 = -3}$

* $\frac{35}{11} x_3 + \frac{x_4}{11} = \frac{137}{11}$

$\boxed{x_3 = 4}$

* $\frac{11}{2} x_2 - \frac{7}{2} x_3 - \frac{9}{2} x_4 = 5$

$\boxed{x_2 = 1}$

* $2x_1 - 3x_2 + x_3 + 3x_4 = -10$

$\boxed{x_1 = -1}$

$$\therefore X = \begin{bmatrix} -1 \\ 1 \\ 4 \\ -3 \end{bmatrix}$$

$\Rightarrow$ Det $(A) = 160$

$|A| = |L||U|$

$|A| = 1 \times |U|$

$\therefore |A| = |U|$

$|U| = \cancel{2} \times (\cancel{11}/\cancel{2}) \times (\overset{5}{\cancel{35}}/\cancel{11}) \times (-32/\cancel{7})$

$|U| = -160$

$\therefore |A| = -160$

$\Rightarrow$ Confirming $|A| = -160$ by direct calculation

$$A = \begin{bmatrix} 2 & -3 & 1 & 3 \\ 1 & 4 & -3 & -3 \\ 5 & 3 & -1 & -1 \\ 3 & -6 & -3 & 1 \end{bmatrix}$$

$$|A| = A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13} + A_{14}C_{14}$$

$$C_{11} = \begin{vmatrix} 4 & -3 & -3 \\ 3 & -1 & -1 \\ -6 & -3 & 1 \end{vmatrix} \times (-1)^{1+1}$$

$$= 4(-1-3) - (-3)(3-6) - 3(-9-6)$$

$$C_{11} = -16 - 9 + 45 = \underline{20}$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & -3 & -3 \\ 5 & -1 & -1 \\ 3 & -3 & 1 \end{vmatrix}$$

$$= -1\left[1(-1-3) + 3(5+3) - 3(-15+3)\right]$$

$$= -1[-4 + 24 + 36]$$

$$C_{12} = \underline{-56}$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 4 & -3 \\ 5 & 3 & -1 \\ 3 & -6 & 1 \end{vmatrix}$$

$$= 1(+3-6) - 4(5+3) - 3(-30-9)$$

$$C_{13} = \underline{82}$$

$$C_{14} = (-1)^{1+4} \begin{vmatrix} 1 & 4 & -3 \\ 5 & 3 & -1 \\ 3 & -6 & -3 \end{vmatrix}$$

$$= -1 \left[1(-9-6) - 4(-15+3) - 3(-30-9) \right]$$

$$= -1 \left[-15 + 48 + 117 \right]$$

$$\underline{C_{14} = -150}$$

$\Rightarrow$ Substituting,

$$|A| = 2(20) - 3(-56) + 1(82) + 3(-150)$$

$$= 40 + 168 + 82 - 450$$

$$|A| = -160$$

Hence $|A| = -160$ is confirmed by direct calculation.

<u>Problem 5</u>

$$\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

i) The above matrix is symmetric and hence normal, <u>so it is diagonalizable</u>.

ii) Let the matrix be A, Such that

$$A = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

wkt $AX = \lambda X$

$$AX - \lambda X = 0$$

$$X[A - I\lambda] = 0$$

$$\therefore |A - I\lambda| = 0 \quad \text{As } X \neq 0$$

$$\therefore \begin{vmatrix} -1-\lambda & 2 & 2 \\ 2 & -1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{vmatrix} = 0$$

Expanding,

$$\Rightarrow -(1+\lambda)\left[(1+\lambda)^2 - 4\right] - 2\left[2(-1-\lambda) - 4\right]$$

$$+ 2\left[4 - 2(-1-\lambda)\right] = 0$$

$$\Rightarrow -(1+\lambda)\left[(1+\lambda)^2 - 4\right] + 12 + 4\lambda + 8 + 4 + 4\lambda = 0$$

$$\Rightarrow -(1+\lambda)^3 + 4(1+\lambda) + 24 + 8\lambda = 0$$

$$\Rightarrow -(1+\lambda)^3 + 12\lambda + 28 = 0$$

$$\Rightarrow (1+\lambda)^3 - 12\lambda - 28 = 0$$

$\Rightarrow \tau^3 + 3\tau^2 + 3\tau + 1 - 12\tau - 28 = 0$

$\tau^3 + 3\tau^2 - 9\tau - 27 = 0$

Solving, we get

$\tau = -3, -3, 3$

$\longrightarrow$ Let $\tau = -3 \rightarrow$ substituting this

$AX = \tau X$

$$\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Expanding,

$$\left.\begin{aligned} -x_1 + 2x_2 + 2x_3 &= -3x_1 \\ 2x_1 + 2x_2 - x_3 &= -3x_3 \\ 2x_1 - x_2 + 2x_3 &= -3x_2 \end{aligned}\right\} \rightarrow x_1 + x_2 + x_3 = 0$$

$\Rightarrow$ Let $\underline{x_1 = 1}$ and $\underline{x_2 = 0}$, then

$1 + 0 + x_3 = 0$

$\underline{x_3 = -1}$

$\Rightarrow$ Let $\underline{x_1 = 1}$ and $\underline{x_2 = -1}$, then

$1 - 1 + x_3 = 0$

$\underline{x_3 = 0}$

Let $\lambda = 3$

$Ax = \lambda x$

$$\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$-x_1 + 2x_2 + 2x_3 = -3x_1$$

$$-4x_1 + 2x_2 + 2x_3 = 0$$

$$-2x_1 + x_2 + x_3 = 0.$$

Here let $\underline{x_1 = 1}$, $\underline{x_2 = 1}$, then

$$-2(1) + 1 + x_3 = 0$$

$$-1 + x_3 = 0$$

$$\underline{x_3 = 1}$$

Therefore the eigen vectors for $\lambda_1 = -3$, $\lambda_2 = -3$ and $\lambda = 3$ are $e_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $e_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow$ Vector $S = [e_1 \,\vdots\, e_2 \,\vdots\, e_3]$

$$S = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

wkt $S^{-1}_{ij} = \frac{C_{ji}}{|S|}$

$$|S| = 1(0+1) - 1(-1+0) + 1(1-0)$$

$$= 1+1+1$$

$$\underline{|S| = 3}$$

$$\therefore S^{-1}_{11} = \frac{C_{11}}{3} = \frac{1}{3}$$

$$S^{-1}_{12} = \frac{C_{21}}{3} = -\frac{2}{3}$$

$$S^{-1}_{13} = \frac{C_{31}}{3} = \frac{1}{3}$$

$$S^{-1}_{21} = \frac{C_{12}}{3} = \frac{1}{3}$$

$$S^{-1}_{22} = \frac{C_{22}}{3} = \frac{1}{3}$$

$$S^{-1}_{23} = \frac{C_{32}}{3} = -\frac{2}{3}$$

$$S^{-1}_{31} = \frac{C_{13}}{3} = \frac{1}{3}$$

$$S^{-1}_{32} = \frac{C_{23}}{3} = \frac{1}{3}$$

$$S^{-1}_{33} = \frac{C_{33}}{3} = \frac{1}{3}$$

$$\therefore S^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$S^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

If a matrix is diagonalizable, then

$$A = SDS^{-1}$$

or $D = S^{-1}AS$

$$D = \frac{1}{3}\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -1-4+2 & 2+2+2 & 2-4-1 \\ -1+2-4 & 2-1-4 & 2+2+2 \\ -1+2+2 & 2-1+2 & 2+2-1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -3 & 6 & -3 \\ -3 & -3 & 6 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -3-6 & 3-3 & -3+6-3 \\ -3+3 & -6-3 & 6-3-3 \\ 3-3 & 3-3 & 3+3+3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & +3 \end{bmatrix}$$

This is the diagonal matrix which has eigen values of A on the diagonal.

Problem 6

$A \in R^{n \times n}$, $U \in R^{n \times k}$ and $V \in R^{n \times k}$

A, $(A + UV^T)$, $(I + V^T A^{-1} U)$ are non-singular.

Prove that :-

$$(A + UV^T)^{-1} = A^{-1} - A^{-1} U \left(I + V^T A^{-1} U \right)^{-1} V^T A^{-1}$$

Proof :-

$(A + UV^T)^{-1}$ $\Rightarrow$ Here A can be written as IA and UV^T can be written as IUV^T where I is a Identity matrix

$\therefore$ $(IA + IUV^T)^{-1}$ $\Rightarrow$ Here IUV^T can be written as $AA^{-1}UV^T$ as $I = AA^{-1}$

$$\therefore (IA + AA^{-1}UV^T)^{-1}$$

$$LHS = \left[A \left(I + A^{-1} UV^T \right) \right]^{-1}$$

$$= \left(I + A^{-1} UV^T \right)^{-1} A^{-1}$$

$$\therefore (A + UV^T)^{-1} = LHS = (I + A^{-1} UV^T)^{-1} A^{-1} \quad \text{—} \textcircled{1}$$

Consider $(I + A^{-1}UV^T)^{-1}$

By using ~~Carl~~ Carl Neumann Series expansion,

$$(I + A^{-1}UV^T)^{-1} = \sum_{k=0}^{\infty} (-1)^k (A^{-1}UV^T)^k$$

Expanding :-

$$(I + A^{-1}UV^T)^{-1} = 1 \times I - A^{-1}UV^T + (A^{-1}UV^T)^2 - \cdots\cdots$$

Substituting the above in eq^{n}. 1

$$(A + UV^T)^{-1} = \left[I - A^{-1}UV^T + (A^{-1}UV^T)^2 - \cdots\right] A^{-1}$$

$$= \left[IA^{-1} - A^{-1}UV^TA^{-1} + A^{-1}UV^TA^{-1}UV^TA^{-1} - \cdots\right]$$

$$= A^{-1} - (A^{-1}U)(V^TA^{-1}) + (A^{-1}U)(V^TA^{-1}U)(V^TA^{-1}) - \cdots]$$

$$(A + UV^T)^{-1} = A^{-1} - A^{-1}U\left[1 - V^TA^{-1}U + (V^TA^{-1}U)^2 - \cdots\right]V^TA^{-1}$$

→ Here the term $1 - V^TA^{-1}U + (V^TA^{-1}U)^2 - \cdots$ is in the form of a geometric series where

$a = I$ and $r = -V^TA^{-1}U$

→ Considering sum of m terms S_m

$$S_m = a\left(\frac{1 - r^m}{1 - r}\right) = I\left(\frac{I - (-V^TA^{-1}U)^m}{I - (-V^TA^{-1}U)}\right)$$

$$S_m = I\left(\frac{I - (-V^TA^{-1}U)^m}{1 + V^TA^{-1}U}\right)$$

$\Rightarrow$ Assuming that $|V^TA^{-1}U| < 1$, as 'm' increases the value of $(-V^TA^{-1}U)^m$ tends to zero.

Hence, Assuming this convergence

$$S_m = \frac{I}{I + V^TA^{-1}U} = (I + V^TA^{-1}U)^{-1}$$

$\Rightarrow$ Substituting the above solution

$$(A + UV^T)^{-1} = A^{-1} - A^{-1}U[S_m]V^TA^{-1}$$

$$\underline{(A + UV^T)^{-1}} = \underline{A^{-1} - A^{-1}U(I + V^TA^{-1}U)^{-1}V^TA^{-1}}$$

Hence Proved