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ENPM 667

PROBLEM SET - 1

Problem - 1

$$dF(x,y) = \left(\frac{1}{x^2+2} + \frac{\alpha}{y}\right) dx + \left(xy^\beta + 1\right) dy$$

→ This is a first degree first order equation and it is of the form

$$dF(x,y) = A(x,y)\,dx + B(x,y)\,dy$$

we know that $\frac{\partial A(x,y)}{\partial y} = \frac{\partial B(x,y)}{\partial x}$ if equation is exact.

$$\therefore \frac{\partial}{\partial y}\left(\frac{1}{x^2+2} + \frac{\alpha}{y}\right) = \frac{\partial}{\partial x}\left(xy^\beta + 1\right)$$

$$0 + \alpha\left(\frac{-1}{y^2}\right) = \frac{\partial}{\partial x}\left(xy^\beta + 1\right)$$

$$-\frac{\alpha}{y^2} = y^\beta$$

$$-\alpha y^{-2} = y^\beta$$

Equating the powers and co efficient on both sides,

$\alpha = -1$ and $\beta = -2$

$$\therefore dF(x,y) = \left(\frac{1}{x^2+2} - \frac{1}{y}\right) dx + \left(\frac{x}{y^2} + 1\right) dy$$

To solve this equation, consider:

$$dF(x,y) = \int A(x,y)\,dx + G(y) = C_1$$

$$\int\left(\frac{1}{x^2+2} - \frac{1}{y}\right)dx + G(y) = C_1$$

$$\int\frac{1}{x^2+2}\,dx - \int\frac{1}{y}\,dx + G(y) = C_1$$

$$\int\frac{1}{x^2+2}\,dx - \frac{x}{y} + G(y) = C_1$$

To solve $\int\frac{1}{x^2+2}\,dx$, consider $\frac{x}{\sqrt{2}} = a$

$$x = \sqrt{2}a \implies dx = \sqrt{2}\,da$$

$$\int\frac{1}{x^2+2}\,dx = \int\frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)} = \frac{1}{2}\int\frac{1}{\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)} = \frac{1}{2}\int\frac{1}{a^2+1}dx$$

$$= \frac{\sqrt{2}}{2}\int\frac{1}{a^2+1}\,da = \frac{1}{\sqrt{2}}\int\frac{1}{a^2+1}\,da$$

This is of the form $\int\frac{1}{x^2+1}\,dx = \tan^{-1}(x) + C$

$$\therefore \int\frac{1}{x^2+2}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}(a) = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C_2$$

$$\therefore dF(x,y) = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{x}{y} + G(y) + C_2 = C_1$$

Differentiate the above equation with respect to y and equate it to $B(x,y)$

$$0 + \frac{d}{dy}\left(-\frac{x}{y}\right) + \frac{d}{dy} G(y) + 0 = \frac{x}{y^2} + 1$$

$$\frac{x}{y^2} + \frac{d}{dy} G(y) = \frac{x^2}{y} + 1$$

$$\frac{d}{dy} G(y) = 1 \longrightarrow \text{Integrate}$$

$$G(y) = y + C_3 \longrightarrow \text{Substitute back in the equation}$$

$$\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{x}{y} + y + C_3 = C_1$$

$$\boxed{\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{x}{y} + y = C} \quad [\text{where } C = C_1 - C_3]$$

Problem - 2

$$R\frac{dq}{dt} + \frac{q}{c} = V(t) \quad \text{where } V(t) = V_0 \sin \omega t$$

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{V_0 \sin \omega t}{R} \quad \text{— (1)}$$

$$\frac{dq}{dt} = \frac{V_0 \sin \omega t}{R} - \frac{q}{RC}$$

$$dq = \left(\frac{V_0 \sin \omega t}{R} - \frac{q}{RC}\right) dt$$

$$\left(\frac{V_0 \sin \omega t}{R} - \frac{q}{RC}\right) dt - dq = 0$$

This is of the form $A(x,y)\,dx + B(x,y)\,dy = 0$

$$\therefore \frac{\partial}{\partial q}\left(\frac{V_0}{R}\sin\omega t - \frac{q}{RC}\right) = -\frac{1}{RC}$$

$$\frac{\partial}{\partial t}(-1) = 0$$

Hence As $\frac{\partial A(x,y)}{\partial y} \neq \frac{\partial B(x,y)}{\partial x}$, the equation is not exact.

In order to make the equation exact, a ~~multiplication~~ Integrating factor should be multiplied on both sides.

Finding ~~Multiplication~~ Integrating factor :-

As the initial equation is of the form

$$\frac{dy}{dx} + P(x)\cdot y = Q(x)$$

the integrating factor is a function of x alone and is given by,

$$\mu(x) = \exp\left\{\int f(x)\,dx\right\}, \text{ where } f(x) = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right)$$

$$\therefore \mu(t) = \exp\left\{\int f(t)\,dt\right\}$$

$$f(t) = -\frac{1}{1}\left[-\frac{1}{RC} - 0\right]$$

$$\therefore f(t) = \frac{1}{RC}$$

$$\therefore \mu(t) = \exp\left\{\int \frac{1}{RC}\,dt\right\} = \exp\left\{\frac{t}{RC}\right\} = \underline{e^{\frac{t}{RC}}}$$

[The integrating factor is also got by $\exp\left\{\int P\,dt\right\}$ where $P = \frac{1}{RC}$]

Multiplying integrating factor on both sides of equation 1

$$e^{\frac{t}{RC}} \frac{dq}{dt} + \frac{q}{RC} e^{\frac{t}{RC}} = \frac{V_0}{R} \sin \omega t \times e^{\frac{t}{RC}}.$$

The LHS is of the form $u\,dv + v\,du = d(uv)$

$$\therefore e^{\frac{t}{RC}} \frac{dq}{dt} + \frac{q}{RC} e^{\frac{t}{RC}} = \frac{d}{dt}\left(e^{\frac{t}{RC}} \times q\right).$$

$$\therefore \frac{d}{dt}\left(e^{\frac{t}{RC}} \times q\right) = \frac{V_0}{R} e^{\frac{t}{RC}} \sin \omega t$$

Integrating on both sides and applying limits $0 \to t$

$$\left(e^{\frac{t}{RC}} \times q\right) = \frac{V_0}{R} \int_0^t e^{\frac{t}{RC}} \sin \omega t$$

To solve $\int_0^t e^{\frac{t}{RC}} \sin \omega t$, I have used integration by parts method, ie $\int_a^b u\,dv = uv\Big]_a^b - \int_a^b v\,du$

Let $a = \int_0^t e^{\frac{t}{RC}} \sin \omega t$

Here $u = e^{\frac{t}{RC}}$ $\quad dv = \sin \omega t$

$du = \frac{e^{\frac{t}{RC}}}{RC}$ $\quad v = \left(-\frac{\cos \omega t}{\omega}\right).$

$$\therefore a = \left[e^{\frac{t}{RC}} \times \left(-\frac{\cos \omega t}{\omega}\right)\right]_0^t - \int_0^t \left(-\frac{\cos \omega t}{\omega}\right)\left(\frac{e^{\frac{t}{RC}}}{RC}\right)$$

$$= \left[e^{\frac{t}{RC}}\left(-\frac{\cos \omega t}{\omega}\right)\right]_0^t - \frac{1}{\omega RC} \int_0^t (-\cos \omega t)\, e^{\frac{t}{RC}}$$

Here let $b = \int_0^t e^{\frac{t}{RC}} (-\cos \omega t)$

$u = e^{\frac{t}{RC}}$ $\quad dv = -\cos \omega t$

$du = \frac{e^{\frac{t}{RC}}}{RC}$ $\quad v = -\frac{\sin \omega t}{\omega}$

$$\therefore b = \left[e^{\frac{t}{RC}} \left(-\frac{\sin wt}{w} \right) \right]_0^t - \int_0^t \left(-\frac{\sin wt}{w} \right) \frac{e^{\frac{t}{RC}}}{RC}$$

$$= \left[e^{\frac{t}{RC}} \left(-\frac{\sin wt}{w} \right) \right]_0^t + \frac{1}{wRC} \int_0^t e^{\frac{t}{RC}} \cdot \sin wt$$

But $\int_0^t e^{\frac{t}{RC}} \sin wt = a$

$$\therefore b = \left[e^{\frac{t}{RC}} \left(-\frac{\sin wt}{w} \right) \right]_0^t + \frac{1}{wRC} \times a$$

Substituting back in equation.

$$a = \left[e^{\frac{t}{RC}} \left(-\frac{\cos wt}{w} \right) \right]_0^t - \frac{1}{wRC} \left[\left[\frac{e^{\frac{t}{RC}} (-\sin wt)}{w} \right]_0^t + \frac{a}{wRC} \right]$$

$$a = \left[e^{\frac{t}{RC}} \left(-\frac{\cos wt}{w} \right) \right]_0^t + \left[\frac{1}{w^2 RC} e^{\frac{t}{RC}} \sin wt \right]_0^t - \frac{a}{w^2 R^2 C^2}$$

$$a\left(1 + \frac{1}{w^2 R^2 C^2} \right) = e^{\frac{t}{RC}} \left(\frac{-\cos wt}{w} \right) + \frac{1}{w} + \frac{1}{w^2 RC} e^{\frac{t}{RC}} \sin wt$$

$$a\left(\frac{w^2 R^2 C^2 + 1}{w^2 R^2 C^2} \right) = \frac{1}{w} + \frac{e^{\frac{t}{RC}}}{w} \left(\frac{\sin wt}{wRC} - \cos wt \right).$$

$$a = \frac{R^2 w^2 C^2}{1 + R^2 w^2 C^2} \left[\frac{1}{w} + \frac{e^{\frac{t}{RC}}}{w} \left(\frac{\sin wt}{wRC} - \cos wt \right) \right.$$

$$= \frac{R^2 w C^2}{(1 + R^2 w^2 C^2)} \left[1 + e^{\frac{t}{RC}} \left(\frac{\sin wt}{wRC} - \cos wt \right) \right]$$

$$a = \frac{R^2 w C^2}{(1 + R^2 w^2 C^2)\, wR\cancel{C}} \left[wRC + e^{\frac{t}{RC}} \left(\sin wt - wRC \cos wt \right) \right.$$

But

$$e^{\frac{t}{RC}} \times q = \frac{V_0}{R} \times a$$

$$\therefore e^{\frac{t}{RC}} \times q = \frac{V_0}{\cancel{R}} \times \left[\frac{R C}{(1+R^2w^2c^2)} \left[wRC + e^{\frac{t}{RC}} (\sin wt - wRC \cdot \cos wt) \right] \right]$$

$$e^{\frac{t}{RC}} \times q = \frac{V_0 C}{1+R^2w^2c^2} \left[wRC + e^{\frac{t}{RC}} (\sin wt - wRC \cdot \cos wt) \right]$$

$$q = \frac{V_0 C}{1+R^2w^2c^2} \left[\frac{wRC}{e^{\frac{t}{RC}}} + \frac{\cancel{e^{\frac{t}{RC}}}}{\cancel{e^{\frac{t}{RC}}}} (\sin wt - wRC \cdot \cos wt) \right]$$

$$\boxed{\therefore q = \frac{V_0 C}{1+R^2w^2c^2} \left\{ \frac{wRC}{e^{\frac{t}{RC}}} + \sin wt - wRC \cdot (\cos wt) \right]}$$

Problem 3

$$\frac{dy}{dx} = \frac{-(2x^2+y^2+x)}{xy}$$

$$xy\, dy = -(2x^2+y^2+x)\, dx$$

$$(2x^2+y^2+x)\, dx + xy\, dy = 0 \quad \text{— (1)}$$

This is of the form $A(x,y)\, dx + B(x,y)\, dy = 0$

For an first-degree first-order ODE to be exact

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$$

$$\therefore \frac{\partial A}{\partial y} = \frac{\partial}{\partial y}(2x^2+y^2+x) = 2y$$

$$\frac{\partial B}{\partial x} = \frac{\partial}{\partial x}(xy) = y$$

Hence, as $\frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x}$, the ODE is not exact.

In order to make it exact, we have to multiply the equation by an integrating factor, that is given by

$$u(x) = \exp\left\{\int f(x)\,dx\right\}$$

where $f(x) = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right)$.

$$= \frac{1}{xy}(2y-y)$$

$$f(x) = \frac{y}{xy} = \frac{1}{x}$$

$$\therefore u(x) = \exp\left\{\int f(x)\,dx\right\} = \exp\left\{\int \frac{1}{x}\,dx\right\}$$

$$= \exp\{\ln x\}$$

$$u(x) = x$$

Multiplying $u(x)$ in equation ①

$$(2x^2+y^2+x)\,x\,.\,dx + (xy \times x)\,dy = 0.$$

$$\therefore (2x^3+xy^2+x^2)\,dx + (x^2y)\,dy = 0$$

Now $A_2(x,y) = 2x^3+xy^2+x^2$ and $B_2(x,y) = x^2y$

To solve, consider

$$\int A_2(x,y)\,dx + G(y) = C_1$$

$$\int (2x^3 + xy^2 + x^2)\,dx + G(y) = C_1$$

$$\left(\frac{2x^4}{\cancel{4}\,2} + \frac{x^2y^2}{2} + \frac{x^3}{3}\right) + G(y) = C_1$$

Differentiate the above equation with respect to y and equate it to $B_2(x,y)$

$$0 + \int \frac{x^2y^2}{2}\,dy + \int \frac{x^3}{3}\,dy + \frac{d}{dy}G(y) = x^2y$$

$$\frac{x^2}{\cancel{2}}(\cancel{2}y) + 0 + \frac{d}{dy}G(y) = x^2y$$

$$\therefore \frac{d}{dy}G(y) = 0.$$

$$G(y) = 0 + C_2.$$

Substituting back in the equation,

$$\frac{x^4}{2} + \frac{x^2y^2}{2} + \frac{x^3}{3} + 0 + C_2 = C_1$$

$$\boxed{\frac{x^4}{2} + \frac{x^2y^2}{2} + \frac{x^3}{3} = C}$$ where $C = C_1 - C_2$

Problem - 4

a) $\frac{d^2f}{dt^2} + \frac{2df}{dt} + 5f = 0$ with $f(0) = 1$ and $f'(0) = 0$

→ The equation is of the form $(D^2 + 2D + 5)f = 0$.

→ The above equation can also be expressed in form of an auxillary equation $ar^2 + br + c = 0$ where

$a = 1$, $b = 2$ and $c = 5$.

$\therefore \quad r^2 + 2r + 5 = 0.$

to find r, $\because \quad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$r = \frac{-2 \pm \sqrt{2^2 - (4 \times 1 \times 5)}}{2 \times 1}$$

$$r = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$$r = -1 \pm 2i$$

$\therefore r_1 = -1 + 2i$ and $r_2 = -1 - 2i$

This solution is of the form $r = \alpha \pm \mu i$

where $\mu = 2$ and $\alpha = -1$

∴ The complementary function is given by

$$f_c = C_1 e^{\sigma t} \cos \omega t + C_2 e^{\sigma t} \sin \omega t$$

$$f_c = C_1 e^{-t} \cos 2t + C_2 e^{-t} \sin 2t \quad — ①$$

Now, to find the values of C_1 and C_2, use initial condition $f(0) = 1$ and $f'(0) = 0$

$$f_c(0) = 1 = C_1 e^0 \cos 0 + C_2 e^0 \sin 0$$

$$1 = C_1$$

∴ Substituting $C_1 = 1$ in equation ①

$$f_c = e^{-t} \cos 2t + C_2 e^{-t} \sin 2t$$

differentiate the above equation

$$f'_c = e^{-t}(-\sin 2t)(2) + (-e^{-t}) \cos 2t + C_2 [e^{-t}(\cos 2t).2 + (-e^{-t}) \sin 2t]$$

Now, applying $f'(0) = 0$

$$f'(0) = 0 = e^0(-\sin 0)(2) + (-e^0) \cos 0 + C_2 [e^0 (\cos 0).2 + (-e^0) \sin 0]$$

$$0 = -1 + C_2 [2]$$

$$\therefore 2C_2 = 1$$

$$C_2 = \frac{1}{2}$$

$\therefore f_c = e^{-t}\cos 2t + \frac{1}{2}\cdot e^{-t}\cdot \sin 2t$

As the RHS of initial equation is 0, the complementary solution is the general solution of the equation.

$$\therefore \boxed{f(t) = e^{-t}\cos 2t + \frac{1}{2}\cdot e^{-t}\cdot \sin 2t}$$

b) $\frac{d^2 f}{dt^2} + 2\frac{df}{dt} + 5f = e^{-t}\cos(3t)$

The above equation is of the form :-

$(D^2 + 2D + 5) f = e^{-t}\cos(3t)$

As the RHS of the equation $\neq 0$, there exists a particular integral such that general solution

$f(t) = CF + PI$

→ To find complementary function, consider auxillary equation $r^2 + 2r + 5 = 0$, which is of the form

$ar^2 + br + c = 0.$

→ To find r ⟶ $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$r = \frac{-2 \pm \sqrt{2^2 - (4\times1\times5)}}{2\times1}$

$= \frac{-2 \pm \sqrt{4 - 20}}{2}$

$r = \frac{-2 \pm \sqrt{-16}}{2}$

$\therefore r = \frac{-2 \pm 4i}{2}$

$r = -1 \pm 2i$

This is of the form $r = \gamma \pm \mu i$ where $\mu = 2$ and $\gamma = -1$

$\therefore$ The complementary function is given by

$f_c = C_1 e^{\gamma t} \cos \mu t + C_2 e^{\gamma t} \sin \mu t$

$f_c = C_1 e^{-t} \cos 2t + C_2 e^{-t} \sin 2t$ — (1)

→ To find particular integral (f_P) :-

As the RHS of the initial equation is a combination of exponential and cosine term, the trial function is of the form

$f_p = e^{at}(A \sin bt + B \cos bt)$

where $a = -1$ and $b = 3$

$\therefore f_P = e^{-t}(A \sin 3t + B \cos 3t)$ — (2)

$f'_p = e^{-t}(A \cos 3t\,(3) + B(-\sin 3t)(3)) + (-e^{-t})(A \sin 3t + B \cos 3t)$

$f'_P = e^{-t}(3A \cos 3t - 3B \sin 3t - A \sin 3t - B \cos 3t)$.

$f''_P = e^{-t}[3A(-\sin 3t)(3) - 3B(\cos 3t)(3) - A(\cos 3t)(3) - B(-\sin 3t)(3)] + (-e^{-t})[3A \cos 3t - 3B \sin 3t - A \sin 3t - B \cos 3t]$.

$$f_p'' = e^{-t}\Big(3B\sin 3t - 3A\cos 3t - 9A\sin 3t - 9B\cos 3t -$$
$$3A\cos 3t + 3B\sin 3t + A\sin 3t + B\cos 3t\Big)$$

$$f_p'' = e^{-t}\cos 3t\,(-3A - 9B - 3A + B) + e^{-t}\sin 3t\,(3B + A + 3B$$
$$-9A)$$

$$f_p'' = e^{-t}\cos 3t\,[-6A - 8B] + e^{-t}\sin 3t\,[6B - 8A]$$

Substituting f_p'', f_p' and f_p value in the initial

equation $f'' + 2f' + 5f = e^{-t}\cos 3t$

$$\Rightarrow \Big[e^{-t}\cos 3t\,[-6A - 8B] + e^{-t}\sin 3t\,[6B - 8A]\Big] + 2\Big[e^{-t}\cos 3t\,(3A - B)$$
$$+ e^{-t}\sin 3t\,(-3B - A)\Big] + 5\Big[e^{-t}B\cos 3t + e^{-t}A\sin 3t\Big]$$

$$= e^{-t}\cos 3t$$

$$\Rightarrow e^{-t}\cos 3t\,\big[-6A - 8B + 6A - 2B + 5B\big] + e^{-t}\sin 3t\,\big[6B - 8A$$
$$-6B - 2A + 5A\big] = e^{-t}\cos 3t + 0 \times e^{-t}\sin 3t$$

$\Rightarrow$ Equating the coefficients on LHS and RHS

* $-6A - 8B + 6A - 2B + 5B = 1$

$\therefore -5B = 1$

$B = -\dfrac{1}{5}$

* $6B - 8A - 6B - 2A + 5A = 0$

$-5A = 0 \Rightarrow A = 0$

Substituting values of A and B in equation 2

$$f_p = e^{-t}(A \sin 3t + B\cos 3t)$$

$$f_p = e^{-t}\left(0 - \frac{1}{5} \times \cos 3t\right)$$

$$f_p = \frac{-e^{-t}}{5} \cos 3t \quad - \text{ 3}$$

$\Rightarrow$ General solution, $f = f_c + f_p$ [ie $f =$ 1 + 3]

$$\therefore f = C_1 e^{-t} \cos 2t + C_2 e^{-t} \sin 2t - \frac{e^{-t}}{5} \cos 3t \quad - \text{ 4}$$

To find the values of C_1 and C_2, apply the initial condition $f(0) = 0$ and $f'(0) = 0$.

$$\rightarrow f(0) = 0 = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 - \frac{e^0}{5} \cos 0$$

$$0 = C_1 - \frac{1}{5}$$

$$\therefore C_1 = \frac{1}{5}$$

$$\rightarrow f'(t) = C_1\left[e^{-t}(-\sin 2t)(2) - e^{-t}\cos 2t\right] + C_2\left[e^{-t}(2\cos 2t) - e^{-t}\sin 2t\right] - \left[\frac{e^{-t}(-\sin 3t)(3)}{5} - \frac{e^{-t}\cos 3t}{5}\right]$$

$$f'(0) = 0 = C_1\left[e^0(\sin 0) \times 2 - e^0 \cos 0\right] + C_2\left[e^0(2 \times \cos 0) - e^0 \sin 0\right] - \left[\frac{e^0(-\sin 0)(3)}{5} - \frac{e^0 \cos 0}{5}\right]$$

$$f'(0) = 0 = -C_1 + 2C_2 + \frac{1}{5}$$

$$= -\frac{1}{5} + 2C_2 + \frac{1}{5}$$

$\therefore \; 2C_2 = 0$

$C_2 = 0$

Substituting the values of C_1 and C_2 in equation (4)

$$f = \frac{1}{5}\left(e^{-t}\cos 2t\right) + 0 - \frac{e^{-t}}{5}\cos 3t$$

$$\boxed{f = \frac{e^{-t}}{5}\cos 2t - \frac{e^{-t}}{5}\cos 3t}$$

Problem 5

$y''(t) + y(t) = \sin(2t)$ Using Laplace transforms

Initial conditions $\rightarrow$ $y(0) = 2$, $y'(0) = 1$

$\rightarrow$ Take laplace transform of the equation on both sides :-

$$L\{y''(t) + y(t)\} = L\{\sin 2t\}$$

$$\Rightarrow \left[s^2\bar{y}(s) - s\,y(0) - y'(0)\right] + \bar{y}(s) = \frac{2}{s^2+4}$$

As $L\{\sin 2t\} = \frac{2}{s^2+2^2}$

$$\Rightarrow (s^2+1)\,\bar{y}(s) - s\,y(0) - y'(0) = \frac{2}{s^2+4}$$

Substituting values of $y(0)$ and $y'(0)$

$$\Rightarrow (s^2+1)\,\bar{y}(s) - 2s - 1 = \frac{2}{s^2+4}$$

$$\Rightarrow (s^2+1)\ \bar{y}(s) = \frac{2}{s^2+4} + 2s + 1$$

$$= \frac{2 + 2s(s^2+4) + (s^2+4)}{s^2+4}$$

$$(s^2+1)\ \bar{y}(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+4)}$$

$$\bar{y}(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)}$$

The RHS of the above equation can be re-written as

$$\frac{2s^3+s^2+8s+6}{(s^2+1)(s^2+4)} = \frac{As+B}{(s^2+4)} + \frac{Cs+D}{(s^2+1)} \quad \text{—— (i)}$$

$$\frac{2s^3+s^2+8s+6}{(s^2+4)(s^2+1)} = \frac{(As+B)(s^2+1) + (Cs+D)(s^2+4)}{(s^2+4)(s^2+1)}$$

Equating the numerators,

$$2s^3+s^2+8s+6 = (As+B)(s^2+1) + (Cs+D)(s^2+4)$$

$$2s^3+s^2+8s+6 = As^3 + As + Bs^2 + B + Cs^3 + 4Cs + Ds^2 + 4D$$

Equating the co efficients

$s^3 \Rightarrow A + C = 2$

$s^2 \Rightarrow B + D = 1$

$s \Rightarrow A + 4C = 8$

$c \Rightarrow B + 4D = 6.$

To, get the values of A, B, C and D:-

$$\begin{array}{r} A+4C=8 \\ (-)\ A+C=2 \\ \hline 0+3C=6. \end{array}$$

$C = 6/3$

$\underline{C = 2}$

$A+C=2$

$A+2=2$

$\underline{A=0}$

$$\begin{array}{r} B+4D=6 \\ (-)\ B+D=1 \\ \hline 3D=5 \end{array}$$

$\underline{D = \frac{5}{3}}$

$B+D=1$

$B = 1-D$

$= 1 - \frac{5}{3}$

$\underline{B = -\frac{2}{3}}$

Substituting the values back in equation ①

$$\frac{0.s+\left(-\frac{2}{3}\right)}{(s^2+4)} + \frac{2s+5/3}{(s^2+1)} = \bar{y}(s)$$

$$-\frac{1}{3}\left(\frac{2}{s^2+4}\right) + \frac{2s}{s^2+1} + \frac{5}{3}\times\frac{1}{s^2+1} = \bar{y}(s)$$

Taking inverse-laplace on both sides,

$$-\frac{1}{3}(\sin 2t) + 2\cos t + \frac{5}{3}\sin t = y(t)$$

As $\frac{a}{s^2+a^2} = L\{\sin at\}$ and $\frac{s}{s^2+a^2} = L\{\cos at\}$

$$\therefore \boxed{y(t) = -\frac{\sin 2t}{3} + 2\cos t + \frac{5}{3}\sin t}$$

Problem 6

$$\frac{d^3y}{dx^3} + \frac{3d^2y}{dx^2} + \frac{3\,dy}{dx} + y = 30e^{-x}$$

→ The equation is of the form

$$(D^3 + 3D^2 + 3D + 1)y = 30e^{-x}$$

→ As the RHS of the equation $\neq 0$, the general solution is given by

$$y = y_c + y_p$$

⇒ To find complementary function y_c :-

The auxillary equation is

$$r^3 + 3r^2 + 3r + 1 = 0.$$

$$= (r+1)^3 = 0$$

$$\therefore r = -1, -1, -1$$

As all the roots of the auxillary equation are equal, the complementary function is of the form

$$y_c(x) = C_1 e^{rx} + C_2 x e^{rx} + C_3 x^2 e^{rx}$$

$$y_c(x) = C_1 e^{-x} + C_2 x e^{-x} + C_3 x^2 e^{-x}$$

$$y_c(x) = e^{-x}(C_1 + C_2 x + C_3 x^2)$$

To find PI $y_p(x)$, consider the equation

$$(D^3+3D^2+3D+1)y = 30e^{-x}$$

where D is the differential operator

$$y_p = \frac{1}{D^3+3D^2+3D+1} \times 30e^{-x}$$

$$y_p = \frac{30e^{-x}}{(D+1)^3}$$

The above equation is of the form $\frac{e^{ax}\, v(x)}{f(D)}$

where $v(x) = 30$, $a = -1$ and $f(D) = (D+1)^3$

But $\frac{e^{ax}\, v(x)}{f(D)}$ can be rewritten as $e^{ax} \frac{v(x)}{f(D+a)}$

$$y_p = \frac{30e^{-x}}{(D-1+1)^3} = \frac{30e^{-x}}{D^3}$$

Here $\frac{1}{D^3}$ can be replaced by triple integral I^3 ($\int\int\int$)

$$y_p = 30e^{-x} I^3$$

$$= 30e^{-x} \int\int\int$$

$$= 30e^{-x} \int\int x$$

$$= 30e^{-x} \int \frac{x^2}{2}$$

$$y_p = 30e^{-x} \times \frac{x^3}{2\times 3}$$

$\therefore PI = y_p = 5x^3 e^{-x}$

General solution

$$\boxed{y = y_c + y_p = e^{-x}(c_1 + c_2 x + c_3 x^2) + 5x^3 e^{-x}}$$

Problem 7

a) $y'' - y = x^n$

The general solution for the above equation is :-

$y = y_c + y_p$

$\Rightarrow$ To find complementary function y_c, consider auxillary equation,

$r^2 - 1 = 0$

$(r+1)(r-1) = 0$

$r = 1$ and $r = -1$

The complementary equation is of the form

$y_c = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

$= c_1 e^{x} + c_2 e^{-x}$

→ Now to find Particular Integral PI, assume a solution of the form

$$y_p = k_1(x)e^x + k_2(x)e^{-x} \quad \text{--- (1)}$$

→ Additional condition imposed :-

$$k_1'(x)e^x + k_2'(x)e^{-x} = 0$$

→ Differentiate equation (1)

$$y_p' = k_1'(x)e^x + e^x k_1(x) + k_2'(x)e^{-x} - e^{-x}k_2(x)$$

$$y_p' = e^x k_1(x) - e^{-x}k_2(x) + \underline{e^x k_1'(x) + e^{-x}k_2'(x)}$$

$$y_p' = e^x k_1(x) - e^{-x}k_2(x) + 0$$

→ Differentiate the above equation,

$$y_p'' = k_1(x)e^x + e^x k_1'(x) - k_2'(x)e^{-x} + k_2(x)e^{-x}$$

$$y_p'' - y_p = \cancel{k_1(x)e^x} + e^x k_1'(x) - k_2'(x)e^{-x} + \cancel{k_2(x)e^{-x}} - \cancel{k_1(x)e^x} - \cancel{k_2(x)e^{-x}}$$

$$\therefore y_p'' - y_p = k_1'(x)e^x - k_2'(x)e^{-x} = x^n \quad \text{--- (2)}$$

and → $k_1'(x)e^x + k_2'(x)e^{-x} = 0. \quad \text{--- (3)}$

Adding (3) + (2) → $2k_1'(x)e^x = x^n$

$$k_1'(x) = \frac{x^n}{2e^x}$$

$$k_1'(x) = \frac{x^n e^{-x}}{2}$$

Integrating on both sides,

$$k_1(x) = \int \frac{x^n e^{-x}}{2}$$

$$k_1(x) = \frac{1}{2}\left[-e^{-x}\, n! \sum_{m=0}^{n} \frac{x^m}{m!}\right]$$

As $\int \frac{x^n e^{-x}}{\cancel{2}} dx = -e^{-x}\, n! \sum_{m=0}^{n} \frac{x^m}{m!}$ → Formula

$$\therefore k_1(x) = \frac{-e^{-x}}{2}\, n! \sum_{m=0}^{n} \frac{x^m}{m!} \quad — (4)$$

⟹ Subtracting ((3) - (2))

$$2k_2'(x)e^{-x} = -x^n$$

$$k_2'(x) = -\frac{x^n e^x}{2}$$

Integrating the above equation

$$k_2(x) = -\int \frac{x^n e^x}{2}$$

But $\int x^n e^x dx = \left[\sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!} x^k\right] e^x$ → Formula

$$\therefore k_2(x) = -\frac{e^x}{2}\left[\sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!} x^k\right]$$

$$k_2(x) = \frac{-e^x}{2}\, n! \sum_{k=0}^{n} (-1)^{n-k} \frac{x^k}{k!} \quad — (5)$$

Substituting (4) and (5)

$$y_P = \left[\int \frac{x^n e^{-x}}{2}\right] e^{x} + \left[\int \frac{-x^n e^{x}}{2}\right] e^{-x}$$

$$= \left[-\frac{e^{-x}}{2}\, n! \sum_{m=0}^{n} \frac{x^m}{m!}\right] e^{x} + e^{-x}\left[-\frac{e^{x}}{2}\, n! \sum_{k=0}^{n} (-1)^{n-k} \frac{x^k}{k!}\right]$$

$\therefore$ The general solution is

$$y = y_c + y_P$$

$$= c_1 e^{x} + c_2 e^{-x} + \left[-\frac{e^{-x}}{2} n! \sum_{m=0}^{n} \frac{x^m}{m!}\right] e^{x} + \left[-\frac{e^{x}}{2} n! \ldots\right.$$

$$\left.\ldots \sum_{k=0}^{n} (-1)^{n-k} \frac{x^k}{k!}\right] e^{-x}$$

$$= c_1 e^{x} + c_2 e^{-x} - \frac{n!}{2} \sum_{m=0}^{n} \frac{x^m}{m!} - \frac{n!}{2} \sum_{k=0}^{n} (-1)^{n-k} \frac{x^k}{k!}$$

$$y = c_1 e^{x} + c_2 e^{-x} - \frac{n!}{2}\left[\sum_{m=0}^{n} \frac{x^m}{m!} + \sum_{k=0}^{n} (-1)^{n-k} \frac{x^k}{k!}\right]$$

---

b) $y'' + y = \tan x \quad , \quad 0 < x < \pi/2$

$\Rightarrow$ The general solution to the above differential equation is of the form

$$y = y_c + y_P$$

$\Rightarrow$ To find the complementary solution, consider the auxillary equation :-

$$r^2 + 1 = 0$$

$r^2+1=0$

$r^2=-1$

$r=\pm\sqrt{-1}$

$r=\pm i$

This is of the form $r=\alpha\pm\beta i$ where $\alpha=0$ and $\beta=1$

$\Rightarrow$ The complementary equation is of the form

$y_c = C_1 e^{\alpha x}\cos\beta x + C_2 e^{\alpha x}\sin\beta x$

$\therefore y_c = C_1\cos x + C_2\sin x$

$\Rightarrow$ Now, to find particular integral PI; assume a solution of the form

$y_p = k_1(x)\cos x + k_2(x)\sin x \quad \text{—} \; ①$

$\Rightarrow$ Additional conditions imposed :-

$k_1'(x)\cos x + k_2'(x)\sin x = 0$

$\Rightarrow$ Differentiate the equation ①

$y_p' = -k_1(x)\sin x + k_1'(x)\cos x + k_2(x)\cos x + k_2'(x)\sin x$

But $k_1'(x)\cos x + k_2'(x)\sin x = 0$

$\therefore y_p' = -k_1(x)\sin x + k_2(x)\cos x$

$\Rightarrow$ Differentiate the above equation

$y_p'' = -k(x)\cos x - k_1'(x)\sin x - k_2(x)\sin x + k_2'(x)\cos x$

$\Rightarrow$ Substituting y''_p and y_p in the equation

$$y'' + y = \tan x$$

$$-k_1(x)\cos x - k_1'(x)\sin x - k_2(x)\sin x + k_2'(x)\cos x + \ldots$$
$$k_1(x)\cos x + k_2(x)\sin x = \tan x$$

$$\therefore\ -k_1'(x)\sin x + k_2'(x)\cos x = \tan x \quad — (2)$$

and $k_1'(x)\cos x + k_2'(x)\sin x = 0 \quad — (3)$

$\Rightarrow$ Multiply equation (2) by $\cos x$ and equation (3) by $\sin x$

$$-k_1'(x)\sin x\cos x + k_2'(x)\cos^2 x = \sin x$$
$$k_1'(x)\cos x\sin x + k_2'(x)\sin^2 x = 0$$

$\Rightarrow$ Add the above two equations

$$0 + k_2'(x)\cos^2 x + k_2'(x)\sin^2 x = \sin x$$

$$k_2'(x)\left[\cos^2 x + \sin^2 x\right) = \sin x$$

$$\therefore\ k_2'(x) = \sin x$$

$$k_2(x) = \int \sin x$$

$$k_2(x) = -\cos x$$

$\Rightarrow$ $k_1'(x)\cos x + k_2'(x)\sin x = 0$

$$k_1'(x)\cos x = -k_2'(x)\sin x$$

$$k_1'(x) = -k_2'(x)\frac{\sin x}{\cos x}$$

$$k_1'(x) = -k_2'(x)\tan x$$

But $k_2'(x) = \sin x$

$\therefore k_1'(x) = -\sin x \tan x$

$k_1(x) = \int -\sin x \tan x$

$= -\int \frac{\sin^2 x}{\cos x} = -\int \frac{1-\cos^2 x}{\cos x}$

$= -\int \frac{1}{\cos x} dx + \int \cos x$

$k_1(x) = -\int \sec x + \int \cos x$

$\therefore k_1(x) = -\ln|\sec x + \tan x| + \sin x$

Substituting the value of k_1 and k_2 in equation ①

$y_p = k_1(x) \cos x + k_2(x) \sin x$

$= [-\ln|\sec x + \tan x| + \sin x] \cos x + (-\cos x \sin x)$

$= -\ln|\sec x + \tan x| \cdot \cos x + \cancel{\cos x \sin x} - \cancel{\cos x \sin x}$

$\therefore y_p = -\ln|\sec x + \tan x| \cdot \cos x$

General solution

$y = y_c + y_p$

$\boxed{y = C_1 \cos x + C_2 \sin x - \ln|\sec x + \tan x| \cdot \cos x}$