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ENPM 667
PROBLEM SET 3

PROBLEM 1

The spectral norm of a matrix A is defined as

$$\|A\| = \max_{\|X\|=1} \|Ax\|$$

where A is any $m \times n$ matrix and x is a vector with unit norm (unit vector)

To prove $\rightarrow$ $$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} \quad \text{— } (1)$$

Here $x \neq 0$ signifies that x is a non zero vector
This non zero vector "x" can be split as

$$x = a y$$

where a is a scalar value and y is a unit vector

Substituting this in equation (1)

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|y\|=1} \frac{\|Aay\|}{\|ay\|}$$

But $\|Aay\| = |a| . \|Ay\|$ and $\|ay\| = |a| . \|y\|$

$$\therefore \|A\| = \max_{\|y\|=1} \frac{\cancel{|a|}\,\|Ay\|}{\cancel{|a|}\,\|y\|}$$

$$\therefore \|A\| = \max_{\|y\|=1} \frac{\|Ay\|}{\|y\|}$$

But as y is a unit vector, $\|y\| = 1$

$$\therefore \|A\| = \max_{\|y\|=1} \|Ay\|$$

$$\|A\| = \|A\|$$

Hence Proved

$\Longrightarrow$ For any $n \times 1$ vector x, prove that

$$\|Ax\| \leq \|A\|\,\|x\|$$

we have proven that $\displaystyle\max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \|A\|$

The maximum value of $\dfrac{\|Ax\|}{\|\cancel{A}x\|}$ is always greater than or equal to any particular value of $\dfrac{\|Ax\|}{\|x\|}$

Hence $$\max_{x \neq 0} \frac{\|Ax\|}{\|x\|} \geq \frac{\|Ax\|}{\|x\|}$$

But $$\max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \|A\|$$

$$\therefore \|A\| \geq \frac{\|Ax\|}{\|x\|}$$

$$\therefore \|A\| \cdot \|x\| \geq \|Ax\|$$

Hence proved

$\Rightarrow$ Prove that $\|AB\| \leq \|A\| \, \|B\|$

where A and B are conformable matrices

wkt
$$\|AB\| = \max_{x \neq 0} \frac{\|ABx\|}{\|x\|}$$

$$\|AB\| = \max_{x = 0} \frac{\|A(Bx)\|}{\|x\|}$$

But $\|A(Bx)\| \leq \|A\| \, \|Bx\|$

$$\therefore \|AB\| \leq \max_{x \neq 0} \frac{\|A\| \, \|Bx\|}{\|x\|}$$

But $\|Bx\| \leq \|B\| \, \|x\|$

$$\|AB\| \leq \max_{x \neq 0} \frac{\|A\| \, \|B\| \, \cancel{\|x\|}}{\cancel{\|x\|}}$$

$$\therefore \|AB\| \leq \|A\| \cdot \|B\|$$

As the term do not depend on x anymore

PROBLEM - 2

Show that for $n \times n$ matrix A, $\sigma(A) \leq \|A\|$

where $\sigma(A) = \max \{ |\lambda| : \lambda$ is an eigen value of $A \}$

Proof: Let $\lambda_1, \lambda_2, \lambda_3 \dots \lambda_n$ be the eigen values of A. Then the spectral radius of A, $\sigma(A)$, is the maximum of the eigen values of A (maximum in terms of magnitude)

Each eigen value has eigen vectors associated with it, such that

$$A x_i = \lambda_i x_i \quad \text{—} \ 1$$

where λ_i is the eigen value and x_i is the corresponding eigen vector.

Here, the maximum absolute value of λ_i is $\sigma(A)$

Consider LHS of the equation 1. Considering $\lambda = \sigma(A)$ and taking its norm,

We know that $\|A\| = \max\limits_{x \neq 0} \dfrac{\|Ax\|}{\|x\|}$

But $\max\limits_{x \neq 0} \dfrac{\|Ax\|}{\|x\|} \geq \dfrac{\|Ax\|}{\|x\|}$

$$\|A\| \geq \frac{\|Ax\|}{\|x\|}$$

$$\|Ax\| \leq \|A\| \, \|x\| \quad - (2)$$

Consider RHS of the equation (1)

$$\|\gamma x\| = \|\sigma(A) x\| = |\sigma(A)| \, \|x\| \quad - (3)$$

Combining equation (2) and (3)

$$\cancel{|\sigma(A)|} \; |\gamma| \, \|x\| = \sigma(A) \, \|x\| = \|Ax\| \leq \|x\| \, \|A\|$$

$$\sigma(A) \, \cancel{\|x\|} \leq \cancel{\|x\|} \, \|A\|$$

$$\sigma(A) \leq \|A\|$$

Hence proved

PROBLEM - 3

Show that

$$\frac{d}{dt} A^{-1}(t) = -A^{-1}(t) \, \dot{A}(t) \, A^{-1}(t)$$

where $A(t)$ is a continously differentiable $n \times n$ matrix function that is invertible at each 't'

Solution :- Let $X = A(t)$ and $Y = A^{-1}(t)$

Such that $XY = A(t)\,A^{-1}(t) = I$

Now differentiating on both sides

$$\frac{d}{dt}(XY) = \frac{d}{dt}(I)$$

$$X\frac{dY}{dt} + \frac{dX}{dt}\cdot Y = 0$$

Here $\dfrac{dX}{dt} = \dot{A}(t)$

$X = A(t)$

$Y = A^{-1}(t)$

$$\therefore X\cdot\frac{dY}{dt} = -\frac{dX}{dt}\cdot Y$$

$$A(t)\,\frac{d}{dt}\left(A^{-1}(t)\right) = -\dot{A}(t)A^{-1}(t)$$

Multiplying by $A^{-1}(t)$ on both sides

$$A^{-1}(t)\,A(t)\,\frac{d}{dt}\left(A^{-1}(t)\right) = -A^{-1}(t)\,\dot{A}(t)A^{-1}(t)$$

$$\therefore \frac{d\left(A^{-1}(t)\right)}{dt} = -A^{-1}(t)\cdot\dot{A}(t)\cdot A^{-1}(t)$$

Hence Proved

PROBLEM - 4

Solve $\dot{x} = ax(t) + b(t)\,u(t)$ using laplace transforms with initial condition $x(0)$.

$$\dot{x} = ax(t) + b(t)\,u(t)$$

$$\dot{x} - ax(t) = b(t)\,u(t)$$

Taking laplace transform on both sides

$$L(\dot{x}) - L(ax(t)) = L(b(t).u(t))$$

$$sX(s) - X(0) - aX(s) = L(b(t).u(t))$$

Let us represent the RHS of the above equation as $T(s)$ where $T(s) = L(b(t).u(t))$ and

$$L^{-1}(T(s)) = b(t).u(t).$$

$$\therefore\ sX(s) - X(0) - aX(s) = T(s)$$

$$X(s)[s-a] - X(0) = T(s)$$

$$X(s)[s-a] = X(0) + T(s)$$

$$X(s) = \frac{1}{s-a}X(0) + \frac{1}{s-a}T(s)$$

Taking inverse laplace on both sides,

$$L^{-1}(X(s)) = L^{-1}\left(\frac{X(0)}{s-a}\right) + L^{-1}\left(\frac{T(s)}{s-a}\right)$$

$$x(t) = X(0)\, L^{-1}\left(\frac{1}{s-a}\right) + L^{-1}\left(\frac{T(s)}{s-a}\right)$$

$$x(t) = X(0).\, e^{at} + L^{-1}\left(\frac{T(s)}{s-a}\right)$$

$$x(t) = X(0).\, e^{at} + b(t).u(t) \times L^{-1}\left(\frac{1}{s-a}\right)$$

$$x(t) = X(0).\, e^{at} + b(t).u(t).\, e^{at}$$

PROBLEM - 5

Define state variables such that

$$y^{(n)}(t) + a_{n-1} t^{-1} y^{(n-1)}(t) + \cdots + a_1 t^{-n+1} y^{(1)}(t) + a_0 t^{-n} y(t) = 0$$

where $y^{(n)}(t) = \frac{d^n y(t)}{dt^n}$, can be written as

LSE $\dot{x}(t) = t^{-1} A\, x(t)$ where A is constant $n \times n$ matrix

Solution :- Let the state variables be

$x_1, x_2 \ldots x_n$ such that

$$x_1 = t^{-n} y(t)$$

$$x_2 = t^{-n+1} y^{(1)}(t)$$

$$x_3 = t^{-n+2} y^{(2)}(t)$$

$$\vdots$$

$$x_{n-1} = t^{-n+(n-1-1)} y^{n-2}(t) = t^{-2} y^{n-2}(t)$$

$$x_n = t^{-1} y^{n-1}(t)$$

Taking derivative of state variables

$$\rightarrow \quad x_1' = \frac{d}{dt}\left(t^{-n} y(t)\right) = y^{(1)}(t)\, t^{-n} + y(t)\,(-n)\, t^{-n-1}$$

$$= t^{-1}\left(y^{(1)}(t)\, t^{-n+1} - n\, y(t)\, t^{-n}\right)$$

$$x_1' = t^{-1}\left(x_2 - n x_1\right)$$

$$\rightarrow \quad x_2' = \frac{d}{dt}\left(t^{-n+1} y^{(1)}(t)\right) = y^{(2)}(t)\, t^{-n+1} + y^{(1)}(t)\,(-n+1)\, t^{-n+1-1}$$

$$= t^{-1}\left(y^{(2)}(t)\, t^{-n+2} + y^{(1)}(t)\,(-n+1)\, t^{-n+1}\right)$$

$$x_2' = t^{-1}\left(x_3 + (-n+1)\, x_2\right)$$

$$\rightarrow \quad x_3' = \frac{d}{dt}\left(t^{-n+2} y^{(2)}(t)\right) = y^{(3)}(t)\, t^{-n+2} + y^{(2)}(t)\,(-n+2)\, t^{-n+2-1}$$

$$= t^{-1}\left(y^{(3)}(t)\, t^{-n+3} + y^{(2)}(t)\,(-n+2)\, t^{-n+2}\right)$$

$$x_3' = t^{-1}\left[x_4 + (-n+2)\, x_3\right]$$

$$\rightarrow \quad x_{n-1}' = y^{(n-1)}(t)\, t^{-2} + y^{(n-2)}(-2)\, t^{-2}\, t^{-1}$$

$$= t^{-1}\left(y^{(n-1)}(t)\, t^{-1} + y^{(n-2)}(-2)\, t^{-2}\right)$$

$$x_{n-1}' = t^{-1}\left(x_n + (-2)\, x_{n-1}\right)$$

$$\rightarrow \quad x_n' = y^{n-1+1}(t)\, t^{-1} + y^{n-1}(-1)\, t^{-1} \cdot t^{-1}$$

$$x_n' = t^{-1}\left(y^n(t) + (-1)\, y^{n-1} t^{-1}\right) = t^{-1}\left(y^n(t) - x_{n-1}\right)$$

$$\vec{\dot{X}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \cancel{t}^{-1}(x_2 + (-n)x_1) \\ t^{-1}(x_3 + (-n+1)x_2) \\ t^{-1}(x_4 + (-n+2)x_3) \\ \vdots \\ t^{-1}(x_n + (-2)x_{n-1}) \\ t^{-1}(y^n(t)) - x_n) \end{bmatrix} = t^{-1} \begin{bmatrix} x_2 + (-n)x_1 \\ x_3 + (-n+1)x_2 \\ x_4 + (-n+2)x_3 \\ \vdots \\ x_n + (-2)x_{n-1} \\ y^n(t) - x_{n-1} \end{bmatrix}$$

Matrix A

$$\vec{\dot{X}} = t^{-1} \begin{bmatrix} -n & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & (-n+1) & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & (-n+2) & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & -2 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

As $y^n(t) = -a_{n-1}x_n - a_{n-2}x_{n-1} \cdots - a_0 x_1$

$\therefore \underline{\vec{\dot{x}}(t) = t^{-1} A \vec{x}(t)}$

where $\vec{\dot{x}}(t) = [\dot{x}_1 \ \dot{x}_2 \ \dot{x}_3 \ \cdots \ \dot{x}_{n-1} \ \dot{x}_n]^T$

$\vec{x}(t) = [x_1 \ x_2 \ x_3 \ \cdots \ x_{n-1} \ x_n]^T$

PROBLEM 6

Prove that $\frac{\partial}{\partial \tau} \Phi(t,\tau) = -\Phi(t,\tau) A(\tau)$

Here $\Phi(t,\tau)$ is the transition matrix.

From the properties of the transition matrix,

$$\Phi^{-1}(t,\tau) = \Phi(\tau,t) \quad — 1$$

and $\Phi(t,\tau) . \Phi^{-1}(t,\tau) = I \quad — 2$

Differentiate equation 2 with respect to τ partially,

$$\frac{\partial}{\partial \tau}\left[\Phi(t,\tau) . \Phi^{-1}(t,\tau)\right] = \frac{\partial}{\partial \tau}(I)$$

From 1 $\Phi^{-1}(t,\tau) = \Phi(\tau,t)$

$$\therefore \frac{\partial}{\partial \tau}\left[\Phi(t,\tau) . \Phi(\tau,t)\right] = \frac{\partial}{\partial \tau}(I)$$

$$= \frac{\partial}{\partial \tau}\Phi(t,\tau) . \Phi(\tau,t) + \Phi(t,\tau)\frac{\partial}{\partial \tau}\Phi(\tau,t)$$

$$= 0$$

$$\therefore \frac{\partial}{\partial \tau}\Phi(t,\tau) . \Phi(\tau,t) = -\Phi(t,\tau) . \frac{\partial}{\partial \tau}\Phi(\tau,t)$$

$$\therefore \cancel{\frac{\partial}{\partial \tau} \Phi(t,\tau) \cdot \Phi(\tau,t) = -\frac{\partial}{\partial \tau} \Phi(\tau,t) \cdot \Phi(t,\tau)}$$

Consider the term $\frac{\partial}{\partial \tau} \Phi(\tau, t)$

From the defination of transition matrix

$$\frac{d}{dt} \Phi(t, t_0) = A(t)\, \Phi(t, t_0)$$

Hence $\frac{\partial}{\partial \tau} \Phi(\tau, t) = A(\tau)\, \Phi(\tau, t)$

Substituting

$$\frac{\partial}{\partial \tau} \Phi(t,\tau) \cdot \cancel{\Phi(\tau,t)} = -\Phi(t,\tau) \cdot \left[A(\tau)\, \cancel{\Phi(\tau,t)}\right]$$

$$\therefore \frac{\partial}{\partial \tau} \Phi(t,\tau) = -\Phi(t,\tau)\, A(\tau)$$

Hence proved.

PROBLEM - 7

Compute state transition matrix for

$$A(t) = \begin{bmatrix} 1 & 0 \\ 1 & \eta(t) \end{bmatrix}$$

where η is a bounded and continous function of t

Solution :- Given :- Matrix $A(t) = \begin{bmatrix} 1 & 0 \\ 1 & \eta(t) \end{bmatrix}$

When the input $u(t)$ is zero, the state space equation is given by :-

$$\vec{\dot{X}} = A(t)\vec{X} \quad \text{—} \ (1)$$

Let $\vec{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{\dot{X}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$

The solution to the above equation is given by

$$\vec{X} = \Phi(t, t_0)\,\vec{X_0} \quad \text{—} \ (2)$$

where $\vec{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{X_0} = \begin{bmatrix} X_{10} \\ X_{20} \end{bmatrix}$

To Find $\rightarrow \Phi(t, t_0)$

Consider equation (1)

$$\vec{\dot{X}} = A(t)\,\vec{X}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & \eta(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 + \eta(t)\, x_2 \end{bmatrix}$$

$$\therefore \dot{x}_1 = x_1 \quad \text{--- (3)}$$

$$\dot{x}_2 = x_1 + \eta(t)\, x_2 \quad \text{--- (4)}$$

Consider equation (3)

$$\dot{x}_1 = \frac{dx_1}{dt} = x_1$$

$$\therefore \frac{1}{x_1}\, dx_1 = dt$$

Integrating on both sides by taking the limits x_{10} to x_1 on LHS and t_0 to t on RHS as x_1 at time t_0 is x_{10}

$$\therefore \int_{x_{10}}^{x_1} \frac{1}{x_1}\, dx_1 = \int_{t_0}^{t} dt$$

$$\ln(x_1) - \ln(x_{10}) = t - t_0$$

$$\ln\left(\frac{x_1}{x_{10}}\right) = t - t_0$$

$$\frac{x_1}{x_{10}} = e^{t-t_0}$$

$$x_1 = x_{10}\, e^{t-t_0} \quad \text{--- (5)}$$

Consider equation 4

$$\dot{X}_2 = \frac{dX_2}{dt} = x_1 + \eta(t)\, x_2 = X_{10}\, e^{t-t_0} + \eta(t)\, x_2$$

$$dX_2 = \left[x_1 + \eta(t)\, x_2\right] dt = \left[X_{10}\, e^{t-t_0} + \eta(t)\, x_2\right] dt$$

$$\left[x_1 + \eta(t)\, x_2\right] dt - dX_2 = 0 \quad \text{— 6}$$

This is of the form $A(x,y)\,dx + B(x,y)\,dy = 0$

$$\therefore\ A(t, X_2) = X_{10}\, e^{t-t_0} + \eta(t)\, x_2$$

$$B(t, X_2) = -1$$

Checking if the equation is exact :-

$$\frac{\partial A(t, X_2)}{\partial X_2} = \frac{\partial}{\partial X_2}\left[X_{10}\, e^{t-t_0} + \eta(t)\, X_2\right] = \eta(t)$$

$$\frac{\partial B(t, X_2)}{\partial t} = \frac{\partial}{\partial t}(-1) = 0$$

As $\dfrac{\partial A}{\partial X_2} \neq \dfrac{\partial B}{\partial t}$, the equation is not exact

In order to make it exact, we have to multiply an integrating factor.

Finding integrating factor :-

Integrating factor $\mu(t) = e^{\int_{t_0}^{t} F(t)}$ where

$$F(t) = \frac{1}{B}\left(\frac{\partial A}{\partial X_2} - \frac{\partial B}{\partial t}\right)$$

$$= \frac{1}{-1}\left(\eta(t) - 0\right)$$

$$F(t) = -\eta(t)$$

$\therefore$ Integrating factor $\mu(t) = e^{\int_{t_0}^{t} -\eta(t)}$

Multiplying the integrating factor to equation (6)

$$\left(X_{10} e^{t-t_0} + \eta(t) X_2\right) e^{\int_{t_0}^{t} -\eta(t)} dt - e^{\int_{t_0}^{t} -\eta(t)} dX_2 = 0$$

$$\overbrace{\left[X_{10} e^{t-t_0} . e^{-\int_{t_0}^{t} \eta(t)} dt + e^{\int_{t_0}^{t} -\eta(t)} . \eta(t) X_2 \, dt\right]}^{A(t, X_2)} - \underbrace{e^{\int_{t_0}^{t} -\eta(t)} dX_2}_{B(t, X_2)} = 0$$

The solution to this equation is found out by following steps

Consider $U(t, X_2) = \int_{t_0}^{t} A(t, X_2)\, dt + G(X_2)$

$$U(t, X_2) = \int_{t_0}^{t} X_{10}\, e^{t-t_0} \cdot e^{\int_{t_0}^{t} -\eta(t)} dt + \int_{t_0}^{t} e^{\int_{t_0}^{t} -\eta(t)} \cdot \eta(t) X_2\, dt$$

$$+ G(X_2) \quad — (7)$$

Consider $\int_{t_0}^{t} X_{10}\, e^{t-t_0} \cdot e^{\int_{t_0}^{t} -\eta(t)} dt$

$$\Rightarrow \int_{t_0}^{t} X_{10}\, e^{t-t_0-\int_{t_0}^{t}\eta(t)} dt$$

Here let $t - t_0 - \int_{t_0}^{t} \eta(t) = a$

then $\dfrac{da}{dt} = \dfrac{d}{dt}\left(t - t_0 - \int_{t_0}^{t} \eta(t)\right)$

$$\frac{da}{dt} = 1 - 0 - \eta(t)$$

$$\therefore dt = \frac{da}{1-\eta(t)}$$

Also the limits of integration are 't_0' to 't' w.r.t 't'

Finding limits w.r.t 'a' :-

at $t = t$, $\underline{a = a}$

$t = t_0$, $a = t_0 - t_0 - \int_{t_0}^{t_0} \eta(t) = \underline{0}$

$$\therefore \int_{t_0}^{t} X_{10}\, e^{t-t_0-\int_{t_0}^{t}\eta(t)} dt = \int_{0}^{a} \frac{X_{10}}{1-\eta(t)} \cdot e^{a}\, da$$

$$= \frac{X_{10}}{1-\eta(t)}\left(e^{a} - e^{0}\right)$$

$$= \frac{x_{10}}{1-\eta(t)} \left[e^{t - t_0 - \int_{t_0}^{t} \eta(t)} - 1 \right] \quad — (8)$$

$\Rightarrow$ Consider $\int_{t_0}^{t} e^{\int_{t_0}^{t} -\eta(t)} \cdot \eta(t) \cdot X_2 \, dt$

Here consider $\int_{t_0}^{t} -\eta(t) = b$

$$\frac{db}{dt} = \frac{d}{dt}\left(\int_{t_0}^{t} -\eta(t) \right)$$

$$\frac{db}{dt} = -\eta(t)$$

$$\therefore -db = \eta(t)\, dt$$

Finding limits of integration,

at $t = t$, $b = \underline{b}$

at $t = t_0$, $b = \int_{t_0}^{t_0} -\eta(t) = \underline{0}$

$$\therefore \int_{t_0}^{t} e^{\int_{t_0}^{t} -\eta(t)} \cdot \eta(t) \cdot X_2 \, dt = -\int_{0}^{b} e^{b} \cdot X_2 \, db$$

$$= -X_2 \left[e^{b} - e^{0} \right]$$

$$= -X_2 \left[e^{\int_{t_0}^{t} -\eta(t)} - 1 \right] \quad — (9)$$

Substituting equation (8) and (9) in equation (7)

$$U(t, X_2) = \frac{x_{10}}{1-\eta(t)}\left[e^{t-t_0-\int_{t_0}^{t}\eta(t)} - 1\right] - X_2\left[e^{\int_{t_0}^{t}-\eta(t)} - 1\right]$$

$$+ G(X_2)$$

Differentiating the above equation with respect to X_2 and equating it to $B(t, X_2)$:-

$$\frac{d}{dX_2} U(t, X_2) = B(t, X_2)$$

$$0 - \left[e^{\int_{t_0}^{t}-\eta(t)} - 1\right] = -e^{\int_{t_0}^{t}-\eta(t)} - \frac{d}{dX_2} G(X_2)$$

$$\cancel{-e^{\int_{t_0}^{t}-\eta(t)}} + \frac{d}{dX_2} G(X_2) + 1 = \cancel{-e^{\int_{t_0}^{t}-\eta(t)}}$$

$$\therefore \frac{d}{dX_2} G(X_2) = -1$$

Integrating

$$G(X_2) = -X_2 + C$$

Substituting back in the equation,

$$\frac{x_{10}}{1-\eta(t)}\left[e^{t-t_0-\int_{t_0}^{t}\eta(t)} - 1\right] - X_2\left[e^{\int_{t_0}^{t}-\eta(t)} - 1\right] - X_2 + C = 0$$

$$\frac{x_{10}}{1-\eta(t)}\left[e^{t-t_0-\int_{t_0}^{t}\eta(t)} - 1\right] - X_2 e^{\int_{t_0}^{t}-\eta(t)} + C = 0$$

At $t = t_0$, $X_2 = X_{20}$

$$\frac{x_{10}}{1-\eta(t)}\left[\cancel{e^0 - 1}^{\,0}\right] - X_{20}e^0 + C = 0$$

$$\therefore \quad C = X_{20}$$

Substituting back in the equation

$$\frac{x_{10}}{1-\eta(t)}\left[e^{t-t_0-\int_{t_0}^{t}\eta(t)} - 1\right] - X_2 e^{\int_{t_0}^{t}-\eta(t)} + X_{20} = 0$$

$$\frac{x_{10}}{1-\eta(t)}\left[e^{t-t_0-\int_{t_0}^{t}\eta(t)} - 1\right] + X_{20} = X_2 e^{\int_{t_0}^{t}-\eta(t)}$$

$$\therefore X_2 = \frac{x_{10}}{e^{\int_{t_0}^{t}-\eta(t)}\cdot(1-\eta(t))}\left[e^{t-t_0-\int_{t_0}^{t}\eta(t)} - 1\right] + \frac{X_{20}}{e^{\int_{t_0}^{t}-\eta(t)}}$$

$$X_2 = e^{\int_{t_0}^{t}\eta(t)}\cdot X_{20} + X_{10}\left[\frac{e^{t-t_0}}{1-\eta(t)} - \frac{e^{\int_{t_0}^{t}\eta(t)}}{1-\eta(t)}\right]$$

$\llcorner$ 10

Substituting equations 5 and 10 back in equation 2

$$\vec{X} = \Phi(t, t_0)\, \vec{X_0}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_{10} \cdot e^{t-t_0} \\ x_{20} \cdot e^{\int_{t_0}^{t} \eta(t)} + x_{10}\left[\dfrac{e^{t-t_0}}{1-\eta(t)} - \dfrac{e^{\int_{t_0}^{t} \eta(t)}}{1-\eta(t)} \right] \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{t-t_0} & 0 \\ \dfrac{e^{t-t_0}}{1-\eta(t)} - \dfrac{e^{\int_{t_0}^{t} \eta(t)}}{1-\eta(t)} & e^{\int_{t_0}^{t} \eta(t)} \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

Comparing,

$$\Phi(t, t_0) = \begin{bmatrix} e^{t-t_0} & 0 \\ \dfrac{e^{t-t_0}}{1-\eta(t)} - \dfrac{e^{\int_{t_0}^{t} \eta(t)}}{1-\eta(t)} & e^{\int_{t_0}^{t} \eta(t)} \end{bmatrix}$$

PROBLEM - 8

Compute matrix exponential e^{At} for

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & -2 \end{bmatrix}$$

Solution - Solved using eigen value method

Finding eigen values :-

we know that

$$AX = \lambda X$$

$$AX - \lambda X = 0$$

$$X\left[A - \lambda I\right] = 0$$

As $X \neq 0$

$$|A - \lambda I| = 0$$

$$\cancel{\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & 2 \end{vmatrix} -}$$

$$\begin{vmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ 6 & 5 & -2-\lambda \end{vmatrix} = 0$$

$$(0-\lambda)\left[-\lambda(-2-\lambda) - 5\right] - 1\left[0-6\right]$$

$$= -\lambda\left[2\lambda + \lambda^2 - 5\right] + 6$$

$$= -\lambda^3 - 2\lambda^2 + 5\lambda + 6 = 0$$

$$= \lambda^3 + 2\lambda^2 - 5\lambda - 6 = 0$$

Solving for λ

$$(\lambda+3)(\lambda+1)(\lambda-2)=0$$

$\therefore \lambda_1 = -3$

$\lambda_2 = -1$

$\lambda_3 = 2$

Finding the currosponding eigen vector for eigen values λ_1, λ_2 and λ_3 :-

$$AX_1 = \lambda_1 X_1$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_2 = -3x_1$

$x_3 = -3x_2$

Let $x_1 = 1$

$x_2 = -3$

$x_3 = -3(-3) = 9$

$$\therefore X_1 = \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix}$$

$$AX_2 = \lambda_2 X_2$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\therefore$ $x_2 = -x_1$

$x_3 = -x_2$

Let $x_1 = 1$

$x_2 = -1$

$x_3 = 1$

$$\therefore X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$AX_3 = \lambda_3 X_3$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_2 = 2x_1$

$x_3 = 2x_2$

Let $x_1 = 1$

$x_2 = 2$

$x_3 = 4$

$$\therefore X_3 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

The matrix P formed by taking the eigen vectors ~~as~~ as its columns is given by

$$P = \left[X_1 \;\vdots\; X_2 \;\vdots\; X_3 \right]$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -3 & -1 & 2 \\ 9 & 1 & 4 \end{bmatrix}$$

Finding P^{-1}

$$P^{-1}_{ij} = \frac{C_{ji}}{|P|}(-1)^{i+j}$$ were $|P|$ is determinent of P and C_{ji} is the determinent excluding row j and colum i

$$|P| = 1\begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix} - 1\begin{vmatrix} -3 & 2 \\ 9 & 4 \end{vmatrix} + 1\begin{vmatrix} -3 & -1 \\ 9 & 1 \end{vmatrix}$$

$$= (-4-2) - 1(-12-18) + 1(-3+9)$$

$$= -6 + 30 + 6$$

$$|P| = 30$$

Now $$P^{-1}_{11} = \frac{C_{11}}{|P|} = \frac{(-1\times 4)-(1\times 2)}{30} = \frac{-4-2}{30} = \frac{-6}{30}$$

$$P^{-1}_{12} = -\frac{C_{21}}{|P|} = \frac{-[(1\times 4)-(1\times 1)]}{30} = \frac{-(4-1)}{30} = \frac{-3}{30}$$

$$P^{-1}_{13} = \frac{C_{31}}{|P|} = \frac{(1\times2)-(-1\times1)}{30} = \frac{2+1}{30} = \frac{3}{30}$$

$$P^{-1}_{21} = -\frac{C_{12}}{|P|} = -\frac{[(-3\times4)-(9\times2)]}{30} = \frac{-(-12-18)}{30} = \frac{30}{30}$$

$$P^{-1}_{22} = \frac{C_{22}}{|P|} = \frac{(1\times4)-(1\times9)}{30} = \frac{4-9}{30} = \frac{-5}{30}$$

$$P^{-1}_{23} = \frac{-C_{32}}{|P|} = -\frac{((1\times2)-(-3\times1))}{30} = -\frac{(+2+3)}{30} = \frac{-5}{30}$$

$$\cancel{P^{-1}_{33}} \quad P^{-1}_{31} = \frac{C_{13}}{|P|} = \frac{((-3)(1)-(-1\times9))}{30} = \frac{-3+9}{30} = \frac{6}{30}$$

$$P^{-1}_{32} = \frac{-C_{23}}{|P|} = \frac{-((1\times1)-(9\times1))}{30} = \frac{-(1-9)}{30} = \frac{8}{30}$$

$$P^{-1}_{33} = \frac{C_{33}}{|P|} = \frac{((1\times-1)-(-3\times1))}{30} = \frac{-1+3}{30} = \frac{2}{30}$$

Substituting

$$P^{-1} = \frac{1}{30}\begin{bmatrix} -6 & -3 & 3 \\ 30 & -5 & -5 \\ 6 & 8 & 2 \end{bmatrix}$$

we know that $e^{tA} = P\begin{pmatrix} e^{r_1 t} & 0 & 0 \\ 0 & e^{r_2 t} & 0 \\ 0 & 0 & e^{r_3 t} \end{pmatrix} P^{-1}$

$$\therefore e^{tA} = \begin{bmatrix} 1 & 1 & 1 \\ -3 & -1 & 2 \\ 9 & 1 & 4 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -6 & -3 & 3 \\ 30 & -5 & -5 \\ 6 & 8 & 2 \end{bmatrix} \times \frac{1}{30}$$

$$= \frac{1}{30} \begin{bmatrix} e^{-3t} & e^{-t} & e^{2t} \\ -3e^{-3t} & -e^{-t} & 2e^{2t} \\ 9e^{-3t} & e^{-t} & 4e^{2t} \end{bmatrix} \begin{bmatrix} -6 & -3 & 3 \\ 30 & -5 & -5 \\ 6 & 8 & 2 \end{bmatrix}$$

$$= \frac{1}{30} \begin{bmatrix} -6e^{-3t} + 30e^{-t} + 6e^{2t} & -3e^{-3t} - 5e^{-t} + 8e^{2t} & 3e^{-3t} - 5e^{-t} + 2e^{2t} \\ 18e^{-3t} - 30e^{-t} + 12e^{2t} & 9e^{-3t} + 5e^{-t} + 16e^{2t} & -9e^{-3t} + 5e^{-t} + 4e^{2t} \\ -54e^{-3t} + 30e^{-t} + 24e^{2t} & -27e^{-3t} - 5e^{-t} + 32e^{2t} & 27e^{-3t} - 5e^{-t} + 8e^{2t} \end{bmatrix}$$

$$\therefore e^{At} = \begin{bmatrix} -\frac{e^{-3t}}{5} + e^{-t} + \frac{e^{2t}}{5} & -\frac{e^{-3t}}{10} - \frac{e^{-t}}{6} + \frac{4}{15}e^{2t} & \frac{e^{-3t}}{10} - \frac{e^{-t}}{6} + \frac{e^{2t}}{15} \\ \frac{3}{5}e^{-3t} - e^{-t} + \frac{2}{5}e^{2t} & \frac{3}{10}e^{-3t} + \frac{e^{-t}}{6} + \frac{8}{15}e^{2t} & -\frac{3}{10}e^{-3t} + \frac{e^{-t}}{6} + \frac{2}{15}e^{2t} \\ \frac{9}{5}e^{-3t} + e^{-t} + \frac{4}{5}e^{2t} & -\frac{9}{10}e^{-3t} - \frac{e^{-t}}{6} + \frac{16}{15}e^{2t} & \frac{9}{10}e^{-3t} - \frac{e^{-t}}{6} + \frac{4}{15}e^{2t} \end{bmatrix}$$