ENPM 667

PROBLEM SET - 4

NAME - SUHAS NAGARAJ
UID - 119505373

PROBLEM - 1

Given :- $\dot{x}(t) = \frac{1}{12}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} x(t) + e^{t/2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$

To show :- If the system is controllable or not

→ The given system is of the form

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t)$$

$$\therefore A(t) = \begin{bmatrix} \frac{5}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{5}{12} \end{bmatrix} \quad \text{and} \quad B(t) = \begin{bmatrix} e^{t/2} \\ e^{t/2} \end{bmatrix}$$

The grammian of ~~matrix is~~ controllability, its represented by $W_c(t_0, t_f)$ determines the controllability of the system. If the matrix is fully ranked (or invertible) then the system is controllable.

The grammian of controllability for a LTV system is given by :-

$$W_c(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t_0, \tau)\, B(\tau)\, B^T(\tau)\, \Phi^T(t_0, \tau)\, d\tau.$$

Here, as A matrix is constant (not time dependent) the state transition matrix $\Phi(t_0, \tau)$ is given by

$$\Phi(t_0, \tau) = \begin{bmatrix} e^{A_{11}(t_0-\tau)} & e^{A_{12}(t_0-\tau)} \\ e^{A_{21}(t_0-\tau)} & e^{A_{22}(t_0-\tau)} \end{bmatrix}$$

$$\therefore \Phi(t_0, \tau) = \begin{bmatrix} e^{\frac{5}{12}(t_0-\tau)} & e^{\frac{1}{12}(t_0-\tau)} \\ e^{\frac{1}{12}(t_0-\tau)} & e^{\frac{5}{12}(t_0-\tau)} \end{bmatrix}$$

$$\therefore \Phi(t_0, \tau) \cdot B(\tau) = \begin{bmatrix} e^{\frac{5}{12}(t_0-\tau)} & e^{\frac{1}{12}(t_0-\tau)} \\ e^{\frac{1}{12}(t_0-\tau)} & e^{\frac{5}{12}(t_0-\tau)} \end{bmatrix} \begin{bmatrix} e^{\frac{\tau}{2}} \\ e^{\frac{\tau}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} e^{\frac{5}{12}t_0 + \frac{\tau}{12}} + e^{\frac{t_0}{12} - \frac{5}{12}\tau} \\ e^{\frac{t_0}{12} - \frac{5}{12}\tau} + e^{\frac{5}{12}t_0 + \frac{\tau}{12}} \end{bmatrix}$$

$$\Phi(t_0, \tau) \cdot B(\tau) = \left(e^{\frac{5}{12}t_0 + \frac{\tau}{12}} + e^{\frac{t_0}{12} - \frac{5}{12}\tau}\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Taking the transpose of the above matrix

$$\left[\Phi(t_0, \tau) \cdot B(\tau)\right]^T = B^T(\tau) \cdot \left(\Phi(t_0, \tau)\right)^T$$

$$= \left(e^{\frac{5}{12}t_0 + \frac{\tau}{12}} + e^{\frac{t_0}{12} - \frac{5}{12}\tau}\right) \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Substituting the values

$$W_c(t_0, t_f) = \int_{t_0}^{t_f} \left(e^{\frac{5}{12}t_0 + \frac{\tau}{12}} + e^{\frac{t_0}{12} - \frac{5}{12}\tau}\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \left(e^{\frac{5}{12}t_0 + \frac{\tau}{12}} + e^{\frac{t_0}{12} - \frac{5}{12}\tau}\right) \begin{bmatrix} 1 & 1 \end{bmatrix} d\tau$$

$$W_c(t_0, t_f) = \int_{t_0}^{t_f} \left(e^{\frac{5}{12}t_0 + \frac{\tau}{12}} + e^{\frac{t_0}{12} - \frac{5}{12}\tau}\right)^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Here we can observe that, even after integration, the 2 columns of the grammian matrix will be linearly dependent [as column 1 = column 2] This shows that the grammian matrix $W_c(t_0, t_f)$ is not invertible and hence the system is uncontrollable

★ ALTERNATE PROOF FOR PROBLEM - 1 USING Matrix $C = [B \,\vdots\, AB \,\vdots\, \cdots \; A^{n-1}B]$

We know that $A = \begin{bmatrix} \frac{5}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{5}{12} \end{bmatrix}$ and $B(t) = \begin{bmatrix} e^{\frac{t}{2}} \\ e^{\frac{t}{2}} \end{bmatrix}$

Matrix $C = [B \,\vdots\, AB \,\vdots\, \cdots \; A^{n-1}B]$

Here As $n = 2$, only B and AB column vectors exist

$$B = \begin{bmatrix} e^{\frac{t}{2}} \\ e^{\frac{t}{2}} \end{bmatrix} \qquad AB = \begin{bmatrix} \frac{5}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{5}{12} \end{bmatrix} \begin{bmatrix} e^{\frac{t}{2}} \\ e^{\frac{t}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{12} e^{\frac{t}{2}} + \frac{1}{12} e^{\frac{t}{2}} \\ \frac{1}{12} e^{\frac{t}{2}} + \frac{5}{12} e^{\frac{t}{2}} \end{bmatrix}$$

$$AB = \begin{bmatrix} \frac{e^{\frac{t}{2}}}{2} \\ \frac{e^{\frac{t}{2}}}{2} \end{bmatrix}$$

Substituting

$$C = \left[B \;\vdots\; AB \right]$$

$$C = \begin{bmatrix} e^{\frac{t}{2}} & \frac{e^{\frac{t}{2}}}{2} \\ e^{\frac{t}{2}} & \frac{e^{\frac{t}{2}}}{2} \end{bmatrix}$$

The system is said to be controllable if this matrix C is fully ranked. In this case, column 2 of matrix C is $\frac{1}{2}$ of column 1 of the matrix. Hence, the columns are linearly dependent. Therefore, the system is not controllable

PROBLEM - 2

Given :-
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \gamma_1(t) & \gamma_2(t) \\ -\gamma_2(t) & \gamma_1(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

To solve :- Compute $\Phi(t_0, t_f)$ and solution to the state space equation.

→ The given state space equation is of the form

$$\vec{\dot{x}}(t) = A(t)\,\vec{x}(t)$$

Hence $A(t) = \begin{bmatrix} \gamma_1(t) & \gamma_2(t) \\ -\gamma_2(t) & \gamma_1(t) \end{bmatrix}$

The solution to this state space equation is given by

$$\vec{x}(t) = \Phi(t_0, t_f)\,\vec{X}_0$$

Where $\vec{X}_0$ is the initial state of the system

This can be solved by eigen value - Eigenvector method where the state transition matrix is given by

$$\Phi(t_0, t_f) = V \Lambda V^{-1} \quad \text{— (1)}$$

where V is a matrix of eigen vectors of A as its columns and Λ is a diagonal matrix given by $\Lambda = \begin{bmatrix} e^{\int_{t_0}^{t_f} \cancel{A}\, T_1\, dt} & 0 \\ 0 & e^{\int_{t_0}^{t_f} T_2\, dt} \end{bmatrix}$

where T_1 and T_2 are eigen values of A

To find the eigen values and eigen vectors of A, consider $|A - \lambda I| = 0$

$$\left| \begin{bmatrix} \gamma_1(t) & \gamma_2(t) \\ -\gamma_2(t) & \gamma_1(t) \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} \gamma_1(t) - \lambda & \gamma_2(t) \\ -\gamma_2(t) & \gamma_1(t) - \lambda \end{vmatrix} = 0$$

$$(\gamma_1(t) - \lambda)^2 + (\gamma_2(t))^2 = 0$$

$$\gamma_1^2(t) - 2\gamma_1(t)\lambda + \lambda^2 + \gamma_2^2(t) = 0.$$

$$\therefore \lambda^2 - 2\lambda(\gamma_1(t)) + \left[\gamma_1^2(t) + \gamma_2^2(t)\right] = 0.$$

Taking roots of this equation

$$a = 1 \qquad b = -2\gamma_1(t) \qquad c = \gamma_1^2(t) + \gamma_2^2(t)$$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2\gamma_1(t) \pm \sqrt{\cancel{4\gamma_1^2(t)} - \cancel{4\gamma_1^2(t)} - 4\gamma_2^2(t)}}{2}$$

$$= \frac{2\gamma_1(t) \pm 2i\gamma_2(t)}{2}$$

$$\lambda_{1,2} = \gamma_1(t) \pm i\gamma_2(t)$$

$$\therefore \left.\begin{aligned} \lambda_1 &= \gamma_1(t) + i\gamma_2(t) \\ \lambda_2 &= \gamma_1(t) - i\gamma_2(t) \end{aligned}\right\} \quad - (2)$$

Now, to find the eigen vectors, consider the equation

$Ax = \lambda x$, For λ_1 :-

$$\begin{bmatrix} \gamma_1(t) & \gamma_2(t) \\ -\gamma_2(t) & \gamma_1(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (\gamma_1(t) + i\gamma_2(t))\, x_1 \\ (\gamma_1(t) + i\gamma_2(t))\, x_2 \end{bmatrix}$$

$$\cancel{\gamma_1(t)x_1} + \gamma_2(t)x_2 = \cancel{\gamma_1(t)x_1} + i\gamma_2(t)x_1$$

$$\cancel{\gamma_2(t)}x_2 = i\cancel{\gamma_2(t)}x_1$$

$$x_2 = ix_1$$

If $x_1 = 1$, $x_2 = i$

$$\therefore V_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} \longrightarrow \text{for } \lambda_1 \quad - (3)$$

Now, for λ_2 :-

$$\begin{bmatrix} \gamma_1(t) & \gamma_2(t) \\ -\gamma_2(t) & \gamma_1(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (\gamma_1(t) - i\gamma_2(t))\, x_1 \\ (\gamma_1(t) - i\gamma_2(t))\, x_2 \end{bmatrix}$$

$$\cancel{\gamma_1(t)}x_1 + \gamma_2(t)x_2 = \cancel{\gamma_1(t)}x_1 - i\gamma_2(t)x_1$$

$$\cancel{\gamma_2(t)}x_2 = -i\cancel{\gamma_2(t)}x_1$$

$x_2 = -i x_1$

If $x_1 = 1$, $x_2 = -i$

$\therefore V_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ — (4) $\longrightarrow$ for τ_2

Now, the matrix V is given by

$V = [V_1 \mid V_2]$

$V = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$ — (5)

$V^{-1} = \dfrac{\text{Adj } V}{|V|} = \dfrac{1}{-i-i} \begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix} = \dfrac{1}{-2i} \begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}$

$V^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2i} \end{bmatrix}$ — (6)

Substituting (2) (5) and (6) in equation (1)

$\Phi(t_f, t_0) = V \Lambda V^{-1}$

$= \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} e^{\int_{t_0}^{t_f} \tau_1 dt} & 0 \\ 0 & e^{\int_{t_0}^{t_f} \tau_2 dt} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2i} \end{bmatrix}$

Here let $e^{\int_{t_0}^{t_f} \tau_1 dt} = \Lambda_1$ and

$e^{\int_{t_0}^{t_f} \tau_2 dt} = \Lambda_2$

$$\therefore \quad \Phi(t_0,t_f) \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2i} \end{bmatrix}$$

$$= \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_1 i & -\Lambda_2 i \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2i} \end{bmatrix}$$

$$\Phi(t_0,t_f) = \begin{bmatrix} \frac{\Lambda_1}{2} + \frac{\Lambda_2}{2} & \frac{\Lambda_1}{2i} - \frac{\Lambda_2}{2i} \\ \frac{\Lambda_1 i}{2} - \frac{\Lambda_2 i}{2} & \frac{\Lambda_1}{2} + \frac{\Lambda_2}{2} \end{bmatrix}$$

$\Rightarrow$ Here $\frac{1}{i} = -i$

$$\therefore \phi(t_0,t_f) = \begin{bmatrix} \frac{\Lambda_1}{2} + \frac{\Lambda_2}{2} & -\left(\frac{\Lambda_1 i}{2} - \frac{\Lambda_2 i}{2}\right) \\ \frac{\Lambda_1 i}{2} - \frac{\Lambda_2 i}{2} & \frac{\Lambda_1}{2} + \frac{\Lambda_2}{2} \end{bmatrix}$$

where $\Lambda_1 = e^{\int_{t_0}^{t_f} \gamma_1 \, dt}$ and $\Lambda_2 = e^{\int_{t_0}^{t_f} \gamma_2 \, dt}$

where $\gamma_1 = \delta_1(t) + i\delta_2(t)$ and $\gamma_2 = \delta_1(t) - i\delta_2(t)$

Hence, the solution for the system is given by

$$\vec{x}(t) = \Phi(t_0,t_f) \cdot \vec{X}(t_0)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{\Lambda_1}{2} + \frac{\Lambda_2}{2} & -\left(\frac{\Lambda_1 i}{2} - \frac{\Lambda_2 i}{2}\right) \\ \frac{\Lambda_1 i}{2} - \frac{\Lambda_2 i}{2} & \frac{\Lambda_1}{2} + \frac{\Lambda_2}{2} \end{bmatrix} \begin{bmatrix} X_1(t_0) \\ X_2(t_0) \end{bmatrix}$$

PROBLEM - 3

Given :-

The moment of inertia of the space shuttle robot arm $= J_1$

The moment of inertia of the space shuttle $= J_2$

Equations of motion :-

$$J_1 \ddot{q}_1 = \tau \quad -(1)$$

$$J_2 \ddot{q}_2 = \tau \quad -(2)$$

Here τ is the input

Selection of state :-

$$\vec{x} = \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix}$$

[As equations (1) and (2) are 2^{nd} order differential equation wrt q, we select $q^{(1)}$ and $q^{(0)}$ as the state variables]

State space equation :-

General form :-

$$\dot{\vec{x}} = A\vec{x} + B\vec{U}$$

ie

$$\begin{bmatrix} \dot{q}_1 \\ \ddot{q}_1 \\ \dot{q}_2 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \tau/J_1 \\ \dot{q}_2 \\ \tau/J_2 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ 0 \\ \dot{q}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \tau/J_1 \\ 0 \\ \tau/J_2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \dot{q}_1 \\ \ddot{q}_1 \\ \dot{q}_2 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J_1} \\ 0 \\ \frac{1}{J_2} \end{bmatrix} \begin{bmatrix} \tau \end{bmatrix}$$

This is the system in state space form

* Checking the controllability of the system :-

A LTI system is said to be controllable if

$$\text{rank} \begin{bmatrix} B & \vdots & AB & \vdots & A^2B & \vdots & \cdots & A^{n-1}B \end{bmatrix} = n$$

where n is number of state variables.

$\therefore$ Here, $n = 4$

$$B = \begin{bmatrix} 0 \\ \frac{1}{J_1} \\ 0 \\ \frac{1}{J_2} \end{bmatrix} \qquad AB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{J_1} \\ 0 \\ \frac{1}{J_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{J_1} \\ 0 \\ \frac{1}{J_2} \\ 0 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{J_1} \\ 0 \\ \frac{1}{J_2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{J_1} \\ 0 \\ \frac{1}{J_2} \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$A^3 B = A \cdot A^2 B$

Here as $A^2 = \vec{0}$, $A^3 B = \vec{0}$

Plugging the values in the controllability matrix

$$C = [B \vdots AB \vdots A^2B \vdots A^3B]$$

$$C = \begin{bmatrix} 0 & \frac{1}{J_1} & 0 & 0 \\ \frac{1}{J_1} & 0 & 0 & 0 \\ 0 & \frac{1}{J_2} & 0 & 0 \\ \frac{1}{J_2} & 0 & 0 & 0 \end{bmatrix}_{\substack{n \times n \\ 4 \times 4}}$$

Here, colum 3 = column 4. So all the columns of matrix C are not linearly independent
Hence the rank of C is less than 4
As rank $[C] < n$ ie rank $[C] < 4$, the system is uncontrollable

The system given above is not controllable, but it can stile be stabilizable. There can be gain matrix k such that the eigen values of $A + B_k K$ have negative real part, making the system stable

As the system is not controllable, it is possible to seperate the controllable part of the system by appropriate similarity transformation.

$\Rightarrow$ In our case rank $\left[B \,\vdots\, AB \right] = 2$ is the controllable subspace with dimension 2

The similarity transformation matrix is defined as

$$S_{n=4} = \left[B \,\vdots\, AB \,\vdots\, S_{n-2} \right] = \left[B \,\vdots\, AB \,\vdots\, S_2 \right]$$

$$S_{n=4} = \left[V_1 \,\vdots\, V_2 \,\vdots\, S_{n-2} \right] = \left[V_1 \,\vdots\, V_2 \,\vdots\, S_2 \right]$$

where S_{n-2} or S_2 is a matrix contains 2 linearly independent vectors chosen so that S is non singular.

Hence, there exist a non singular matrix S such that

$$\hat{A} = S^{-1} A S = \left[\begin{array}{c|c} A_1 & A_{12} \\ \hline 0 & A_2 \end{array} \right]$$

A_1: $n_R \times n_R \dashrightarrow 2\times 2$

A_2: $(n-n_R)\times(n-n_R) \dashrightarrow 2\times 2$

$$\hat{B} = S^{-1} B = \left[\begin{array}{c} B_1 \\ \hline 0 \end{array} \right]$$

B_1: $n_R \times m$

Here the pair A_1, B_1 is controllable and the pair $\hat{A}, \hat{B}$ is the standard form of Uncontrollable system

PROBLEM - 4

Given :- 2nd order system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} u$$

Find $u = k_1 x_1 + k_2 x_2$ such that the closed loop system has poles at $s = -2, 2$

Comparing the given equation with

$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$

we get $A = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Number of state variables $= n = 2$

Controllability of the system :-

Let $C = \left[B \,\vdots\, AB \,\vdots\, A^2B \,\vdots\, \ldots A^{n-1}B \right]$ Here C is the controllability matrix

if rank $[C] = n$, then system is controllable

Here as $n = 2$, $C = \left[B \,\vdots\, AB \right]$

$$B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1+6 \\ 1+4 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

1. $C = \begin{bmatrix} 1 & 7 \\ -2 & 5 \end{bmatrix}$

The matrix is said to be fully ranked if it has linearly independent rows. The above matrix C is fully ranked as the 2 columns are linearly independent ie there is no straight relationship between the columns.

Therefore the matrix C is fully ranked making the given system Controllable and we can place the poles wherever we want

* Now, to determine $U = k_1 x_1 + k_2 x_2$ so that the closed loop system poles are at $s = -2, 2$, consider the closed loop system of the form

$$\vec{x}(t) = (A + B_k K)\,\vec{x}(t) + B_D \vec{U}_D(t)$$
$$\vec{y}(t) = (C + DK)\,\vec{x}(t)$$

~~The stability of the system is~~

Now the $(A + BK)$ represent the A matrix of closed loop system.

Consider $A_{closed} = A + BK$

$$A_{closed} = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cancel{\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ -2k_1 & -2k_2 \end{bmatrix}$$

$$A_{closed} = \begin{bmatrix} 1+k_1 & -3+k_2 \\ 1-2k_1 & -2k_2-2 \end{bmatrix}$$

To find the poles, consider

$$\det(\lambda I - A_{closed}) = 0 \quad \text{[Given as hint]}$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1+k_1 & -3+k_2 \\ 1-2k_1 & -2k_2-2 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} \lambda-1-k_1 & 3-k_2 \\ -1+2k_1 & \lambda+2k_2+2 \end{vmatrix} = 0$$

$$(\lambda-1-k_1)(\lambda+2k_2+2) - (3-k_2)(-1+2k_1) = 0$$

$$\lambda^2 + 2\lambda k_2 + 2\lambda - \lambda - 2k_2 - 2 - \lambda k_1 - 2k_1k_2 \cdots$$

$$\cdots - 2k_1 - (-3 + 6k_1 + k_2 - 2k_1k_2) = 0$$

$$\therefore r^2 + r(2k_2 + 2 - 1 - k_1) + (-2k_2 - 2 - \cancel{2k_1k_2} - 2k_1 + 3$$
$$- 6k_1 - k_2 + \cancel{2k_1k_2}) = 0$$

$$r^2 + r(2k_2 - k_1 + 1) + (-3k_2 - 8k_1 + 1) = 0$$

$\Rightarrow$ Substituting the values of r ie $r_1 = -2$ & $r_2 = 2$

~~we get~~ when $r = -2$,

$$(-2)^2 - 2(2k_2 - k_1 + 1) + (-3k_2 - 8k_1 + 1) = 0$$

$$4 - 4k_2 + 2k_1 - 2 - 3k_2 - 8k_1 + 1 = 0$$

$$3 - 7k_2 - 6k_1 = 0.$$

$$\therefore 6k_1 + 7k_2 = 3 \quad -(1)$$

when $r = 2$,

$$2^2 + 2(2k_2 - k_1 + 1) + (-3k_2 - 8k_1 + 1) = 0.$$

$$4 + 4k_2 - 2k_1 + 2 - 3k_2 - 8k_1 + 1 = 0.$$

$$k_2 - 10k_1 + 7 = 0$$

$$10k_1 - k_2 = 7 \quad -(2)$$

Solving equations 1 and 2, we get,

$$k_1 = \frac{13}{19} \quad \text{and} \quad k_2 = -\frac{3}{19}$$

∴ The state feedback control is given by

$$U = k_1 x_1 + k_2 x_2$$

$$U = \frac{13}{19} x_1 - \frac{3}{19} x_2$$

PROBLEM - 5

given :- $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U$

comparing the above equation with

$$\vec{\dot{x}} = A\vec{x} + B\vec{U}$$

we get $A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Number of state variables = 2

checking for controllability :-

$\text{rank}\left[B \,\vdots\, AB \right] = ?$

$$AB = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\text{rank} \left[B \,\vdots\, AB \right] = \text{rank} \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} = 1$$

Here, the 2 columns are linearly dependent, hence the rank $[B \,\vdots\, AB] < n$

So, the system is uncontrollable

Closed loop system :-

$$\dot{\vec{X}} = (A + BK)\,\vec{X}$$

Charecteristic equation to find the poles,

$$\det \left[\lambda I - A_{closed} \right] = 0$$

where $A_{closed} = A + BK$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}$$

$$A_{closed} = \begin{bmatrix} -1 & 0 \\ k_1 & k_2 + 2 \end{bmatrix}$$

Substituting,

$$\left| \begin{bmatrix} \tau & 0 \\ 0 & \tau \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ k_1 & k_2+2 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} \tau + 1 & 0 \\ -k_1 & \tau - k_2 - 2 \end{vmatrix} = 0$$

$$(\tau+1)(\tau - k_2 - 2) = 0$$

$$\cancel{\tau^2 - k_2\tau - 2\tau + \tau - k_2 - 2 = 0}$$

$$\cancel{\tau^2 - \tau(k_2+1) - (k_2+2) = 0}$$

$\therefore \quad \tau + 1 = 0$

$\tau = -1 \quad — \quad (1)$

Also $\quad \tau - k_2 - 2 = 0$

$\tau = k_2 + 2 \quad — \quad (2)$

Therefore one of the poles of the system is always at -1 whereas the other pole can be placed anywhere by controlling the value of k_2

* Can be closed loop poles can be placed at -2 ?

→ One of the pole of the closed loop system is at -1 and this cannot be changed. But the other pole can be placed at -2 by selecting k_2 as

$k_2 = -4$

Therefore the system poles can be placed at $-1, -2$ but it cannot be placed at $-2, -2$

* Can this system be stabilized ?

Yes, this system can be stabilized as one pole is already at left half plane as it has a fixed value of -1 (-ve real value) and the other pole can be placed at left half plane by selecting appropriate values of k_2

If $k_2 < -2$ then the system is stable as the poles will have -ve real parts.

Problem 6

Given :- $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

Comparing the above equation with

$\dot{\vec{X}} = A\vec{X} + B\vec{U}$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Number of state variables, $n = 2$

Checking for controllability :-

rank $[B \,\vdots\, AB] = ?$

$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$AB = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

rank $[B \,\vdots\, AB]$ = rank $\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} = 1$

The 2 columns of the above matrix are lineary dependent, hence the matrix is <u>not</u> fully ranked

$\therefore$ The system is not controllable

* Closed loop system :-

$\dot{\vec{X}} = (A + BK)\vec{X}$

Equation to find poles of the closed loop system :-

$$\det(\gamma I - A_{closed}) = 0$$

where $A_{closed} = A + BK$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ k_1 & k_2+2 \end{bmatrix}$$

$$\therefore \det(\gamma I - A_{closed}) = \det\left[\begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ k_1 & k_2+2 \end{bmatrix}\right] = 0$$

$$= \begin{vmatrix} \gamma-1 & 0 \\ -k_1 & \gamma-k_2-2 \end{vmatrix} = 0$$

$$\therefore (\gamma-1)(\gamma-k_2-2) = 0$$

$$\therefore \gamma = 1 \text{ and } \gamma = k_2 + 2$$

Here, one of the poles of the system is fixed at 1 and it cannot be changed by varying the values of k_1 or k_2

Hence the system is un stable and is also unstabilizable as it has one pole in right half plane and it cannot be changed.

Can one pole be placed at -2?

Yes one of the pole can be placed at -2 by selecting the value of k_2 as -4. But the other pole is fixed at +1, hence the poles can be (1, -2) but cannot be (-2, -2)

PROBLEM 7

Given :- $y(t)$ is output of an LTI system with input $u(t)$. $x(t_0) = 0$.

To prove :- $\dot{y}(t)$ is the output of the LTI system when input is $\dot{u}(t)$

⇒ ~~genera~~ The state space representation of an LTI system is given by :-

$$\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t)$$

$$y(t) = C\vec{x}(t) + D\vec{u}(t)$$

Taking Laplace transform of the state space equations,

$$L\left(\dot{\vec{x}}(t)\right) = L\left(A\vec{x}(t)\right) + L\left(B\vec{u}(t)\right)$$

$$\left[s\vec{x}(s) - \cancel{\vec{x}(0)}^{\,0}\right] = A\vec{x}(s) + B\vec{u}(s)$$

$$\therefore\; s\vec{x}(s) = A\vec{x}(s) + B\vec{u}(s)$$

$$s\vec{x}(s) - A\vec{x}(s) = B\vec{u}(s).$$

$$\cancel{\vec{x}(s)\left[sI-A\right] = B\vec{u}(s)}$$

$$\cancel{\vec{x}(s) = \left[sI-A\right]^{-1}B\vec{u}(s)}$$

$$\cancel{s-A}$$

$$\left[sI-A\right]\vec{x}(s) = B\vec{u}(s)$$

Multiplying $[sI-A]^{-1}$ on left ~~side~~ on both sides.

$$[sI-A]^{-1}[sI-A]\vec{x}(s) = [sI-A]^{-1}B\vec{u}(s)$$

$$\vec{x}(s) = [sI-A]^{-1}B\vec{u}(s) \quad \text{— (1)}$$

Now consider, $L(\vec{y}(t)) = L\left(C\vec{x}(t) + D\vec{u}(t)\right)$

$$\vec{Y}(s) = C\vec{X}(s) + D\vec{U}(s)$$

Substituting ①

$$\vec{Y}(s) = C\,[sI - A]^{-1} B\,\vec{U}(s) + D\,\vec{U}(s)$$

$$\vec{Y}(s) = \left[C\,[sI - A]^{-1} B + D\right]\vec{U}(s) \quad \text{— ②}$$

Now consider the input as ~~x~~ $\dot{u}$ and output as $\dot{y}$, then the state space equation becomes

$$\dot{x}(t) = A\vec{x}(t) + B\vec{u}(t)$$

$$\dot{y}(t) = C\vec{x}(t) + D\vec{\dot{u}}(t)$$

let $\dot{y}(t) = \hat{y}^{\bullet}(t)$ → estimated output

Now, take laplace transforms of the modified state space equations,

$$L(\dot{x}(t)) = L\left(A\vec{x}(t) + B\vec{\dot{u}}(t)\right)$$

$$sX(s) - \cancel{X(0)}^{\,0} = AX(s) + B\left(sU(s) - \cancel{U(0)}^{\,0}\right)$$

$$\therefore\ sX(s) = AX(s) + BsU(s)$$

$$sX(s) - AX(s) = BsU(s)$$

$$\cancel{X\bar{s}}\ \cancel{S-A}\ (SI-A)X(s) = BSU(s)$$

$$X(s) = (SI-A)^{-1}BSU(s) \quad — \ 3$$

Consider, $L(\hat{Y}(t)) = L(C\vec{X}(t) + D\vec{U}(t))$

$$\hat{Y}(s) = CX(s) + DSU(s)$$

Substituting 3

$$\hat{Y}(s) = C(SI-A)^{-1}BS\vec{U}(s) + DS\vec{U}(s)$$

$$\hat{Y}(s) = \left[C(SI-A)^{-1}B + D\right]S\vec{U}(s)$$

But from 2 we know that

$$Y(s) = \left(C(SI-A)^{-1}B+D\right)\vec{U}(s)$$

$$\therefore \hat{y}(s) = Y(s) \cdot S$$

Taking inverse laplace,

$\underline{\underline{\hat{y}(t) = \dot{y}(t)}}$ → The estimated output is equal to $\dot{y}(t)$, hence proving the statement

Hence proved

PROBLEM 8

Given :- $\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & -1 & 0 \end{bmatrix} x(t)$, $x(t_0) = x_0$

To solve :- The solution for the linear state equation

The given equation is of the form

$$\dot{\vec{x}}(t) = A\,\vec{x}(t)$$

Taking Laplace transform on both sides,

$$L(\dot{\vec{x}}(t)) = L(A\vec{x}(t))$$

$$sX(s) - x(t_0) = AX(s)$$

$$sX(s) - AX(s) = x(t_0)$$

$$(sI - A)X(s) = x(t_0)$$

Here $x(t_0) = x_0$.

$$(sI - A)X(s) = x_0$$

$$X(s) = (sI - A)^{-1} x_0 \quad \text{--- (1)}$$

Here $sI - A = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & -1 & 0 \end{bmatrix}$

$$sI - A = \begin{bmatrix} s+1 & 0 & 0 \\ 0 & s+4 & -4 \\ 0 & 1 & s \end{bmatrix}$$

Finding inverse of $(SI-A)$

$$(SI-A)^{-1} = \frac{\operatorname{Adj}(SI-A)}{|SI-A|}$$

$$\cancel{(SI-A)^{-1}_{11} = \frac{C_{11}}{|SI-A|}}$$

$$|SI-A| = \begin{vmatrix} s+1 & 0 & 0 \\ 0 & s+4 & -4 \\ 0 & 1 & s \end{vmatrix}$$

$$= (s+1)\left[s(s+4) + 4 \right]$$

$$= (s+1)\left[s^2 + 4s + 4 \right]$$

$$= (s+1)\cdot(s+2)^2$$

$$\operatorname{Adj}(SI-A) = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} s^2+4+4 & 0 & 0 \\ 0 & (s+1)s & 4(s+1) \\ 0 & -(s+1) & (s+1)(s+4) \end{bmatrix}$$

$$= \begin{bmatrix} (s+2)^2 & 0 & 0 \\ 0 & s(s+1) & 4(s+1) \\ 0 & -(s+1) & (s+1)(s+4) \end{bmatrix}$$

$$\therefore (sI - A)^{-1} = \frac{\operatorname{Adj}(sI-A)}{|sI-A|}$$

$$\cancel{= \frac{1}{(s+1)(s+2)^2}\begin{bmatrix} s+1 & 0 & 0 \\ 0 & s+4 & -4 \\ 0 & 1 & \end{bmatrix}}$$

$$(sI-A)^{-1} = \frac{1}{(s+1)(s+2)^2}\begin{bmatrix} (s+2)^2 & 0 & 0 \\ 0 & s(s+1) & 4(s+1) \\ 0 & -(s+1) & (s+1)(s+4) \end{bmatrix}$$

$$(sI-A)^{-1} = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{s}{(s+2)^2} & \frac{4}{(s+2)^2} \\ 0 & -\frac{1}{(s+2)^2} & \frac{s+4}{(s+2)^2} \end{bmatrix}$$

Substituting in equation 1

$$X(s) = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{s}{(s+2)^2} & \frac{4}{(s+2)^2} \\ 0 & \frac{-1}{(s+2)^2} & \frac{s+4}{(s+2)^2} \end{bmatrix} X_0$$

Taking inverse laplace transform on both sides

$$L^{-1}\{X(s)\} = L^{-1}\left\{\begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{s}{(s+2)^2} & \frac{4}{(s+2)^2} \\ 0 & -\frac{1}{(s+2)^2} & \frac{s+4}{(s+2)^2} \end{bmatrix} X_0\right\}$$

Consider $L^{-1}\{(sI-A)^{-1}\}$

$$= \begin{bmatrix} L^{-1}\left\{\frac{1}{s+1}\right\} & 0 & 0 \\ 0 & L^{-1}\left\{\frac{s}{(s+2)^2}\right\} & L^{-1}\left\{\frac{4}{(s+2)^2}\right\} \\ 0 & L^{-1}\left\{-\frac{1}{(s+2)^2}\right\} & L^{-1}\left\{\frac{s+4}{(s+2)^2}\right\} \end{bmatrix}$$

Here $L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$

$$L^{-1}\left\{\frac{s}{(s+2)^2}\right\} = L^{-1}\left\{\frac{1}{s+2} - \frac{2}{(s+2)^2}\right\}$$

$$= L^{-1}\left\{\frac{1}{s+2}\right\} - L^{-1}\left\{\frac{2}{(s+2)^2}\right\}$$

$$= e^{-2t} - 2t\,e^{-2t}$$

$$= (1-2t)\,e^{-2t}$$

$$L^{-1}\left\{\frac{4}{(s+2)^2}\right\} = e^{-2t} L^{-1}\left\{\frac{4}{s^2}\right\}$$ By considering $L^{-1}\{F(s-a)\} = e^{at} f(t)$

$$= e^{-2t} \cdot 4t$$

$$L^{-1}\left\{-\frac{1}{(s+2)^2}\right\} = e^{-2t} L^{-1}\left\{-\frac{1}{s^2}\right\}$$

$$= -e^{-2t}\,t$$

$$L^{-1}\left\{\frac{s+4}{(s+2)^2}\right\} = L^{-1}\left\{\frac{1}{s+2} + \frac{2}{(s+2)^2}\right\} = e^{-2t} + e^{-2t}\cdot 2t$$

$$= e^{-2t}(1+2t)$$

Substituting,

$$L^{-1}\left\{(sI-A)^{-1}\right\} = \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{-2t}(1-2t) & e^{-2t}.4t \\ 0 & -e^{-2t}.t & e^{-2t}(1+2t) \end{bmatrix}$$

Substituting back in equation 1

$$x(t) = \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{-2t}(1-2t) & e^{-2t}.4t \\ 0 & -e^{-2t}.t & e^{-2t}(1+2t) \end{bmatrix} x_0$$