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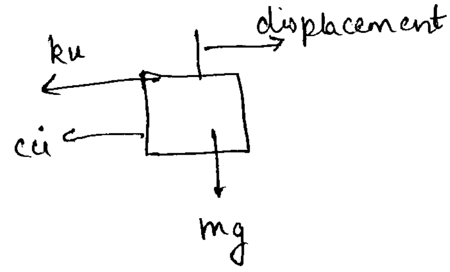
Q1 a) FBD.



By Newtons 2nd Law,

$$-ku = c\dot{u} - P(t) + m\ddot{u}$$

$$P(t) = m\ddot{u} + c\dot{u} + ku$$

as $P(t) = 0$, we have $m\ddot{u} + c\dot{u} + ku = 0$

b) $\ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = 0$ (dividing by m on both sides)

sol^n :- $\dot{u} = \lambda e^{\lambda t}$ $\quad \ddot{u} = \lambda^2 e^{\lambda t}$

substituting the value here, we get,

$$\lambda^2 e^{\lambda t} + \frac{c\lambda}{m} e^{\lambda t} + \frac{k}{m} e^{\lambda t} = 0$$

$$e^{\lambda t}\left(\lambda^2 + \frac{c\lambda}{m} + \frac{k}{m}\right) = 0$$

as $e^{\lambda t}$ cannot be equal to 0,

$$\lambda^2 + \frac{c\lambda}{m} + \frac{k}{m} = 0$$

on solving, we get,

$$\lambda = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - \frac{4k}{m}}}{2}$$

$$\Rightarrow \lambda = \frac{-\frac{c}{m} \pm \sqrt{\frac{c^2 - 4mk}{m^2}}}{2}$$

$$\lambda = \frac{-c \pm \sqrt{c^2-4mk}}{2m}$$

for a solⁿ to exist $m \neq 0$,

3 cases,

Case 1 : $c^2 - 4mk > 0$

eqⁿ has real ~~&~~ roots and overdamped system,

$$\lambda_1 = \frac{-c + \sqrt{c^2-4mk}}{2m}$$

$$\lambda_2 = \frac{\cancel{c+}\ -c - \sqrt{c^2-4mk}}{2m}$$

solⁿ ⇒ $u = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}$ where a_1 & a_2 are constants

Case 2 $c^2 - 4mk = 0$

eqⁿ has single root and system is critically damped

$$\lambda = \frac{-c}{2m}$$

solⁿ :- $u = (a_1 + a_2 t) e^{\lambda t}$ where a_1 & a_2 are constants.

Case 3 $c^2 - 4mk < 0$

eqⁿ has imaginary roots and is underdamped.

$$\lambda_1 = \frac{-c + i\sqrt{4mk - c^2}}{2m}$$

$$\lambda_2 = \frac{-c - i\sqrt{4mk - c^2}}{2m}$$

~~soln :~~ $\cancel{\Rightarrow e^{-c/2m} (a \cos \lambda_1 t + b \sin \lambda_2 t)}$

soln :-

$$\Rightarrow \quad u = e^{-c/2m} \left(a \cos \frac{\sqrt{4mk - c^2}}{2m} t + b \sin \frac{\sqrt{4mk - c^2}}{2m} t \right)$$

d) let $\dot{u} = z$

$\therefore \quad \ddot{u} = \dot{z}$

equating this in the initial eqn we get

$\cancel{m\ddot{z} + c\dot{z} = 0}$ and

$\cancel{\dot{u} = z}$

$\cancel{m\dot{z}}$ $\quad m\dot{z} + cz + ku = 0$

and $\quad \dot{u} = z$.

Q3)

ODE $\quad \dfrac{d^2u}{dx^2} + 10u = 2 \qquad 0 < x < 1$

BC $\quad u(0) = 10 \qquad \dfrac{du}{dx}(1) = 0$.

let $u = Ae^{\alpha x}$

$\dfrac{du}{dx} = A\alpha e^{\alpha x} \qquad \dfrac{d^2u}{dx^2} = A\alpha^2 e^{\alpha x}$

for Homogenous solⁿ,

$\dfrac{d^2u}{dx^2} + 10u = 0. \qquad A\alpha^2 e^{\alpha x} + 10(Ae^{\alpha x}) = 0$

$Ae^{\alpha x}(\alpha^2 + 10) = 0 \Rightarrow \alpha = \pm\sqrt{10}\,i$ is the form of $p+iq$ and $p-iq$ with $p=0$ $q=\sqrt{10}$

The solution to a homogenous eq^n of this type is

$A_1 e^{0x}\cos(\sqrt{10}\,x) + A_2 e^{0x}\sin(\sqrt{10}\,x)$

thus $U_h(x) = A_1\cos\sqrt{10}\,x + A_2\sin\sqrt{10}\,x$

for particular solution,

$U_p(x) = c_1$ since 2 is a constant

we get $\dfrac{d^2}{dx^2}c_1 + 10c_1 = 2$

$\Rightarrow 10c_1 = 2 \Rightarrow c_1 = \dfrac{1}{5} \Rightarrow U_p(x) = \dfrac{1}{5}$

$U_{comp}(x) = U_h(x) + U_p(x)$

$= \boxed{A_1\cos\sqrt{10}\,x + A_2\sin\sqrt{10}\,x + \dfrac{1}{5}}$

Applying Boundary conditions,

given that $U(0) = 10$ & $\left.\frac{du}{dx}\right|_1 = 0$

$$U(0) = 10 = A_1 + 0 + \frac{1}{5} \Rightarrow A_1 = 10 - \frac{1}{5} \Rightarrow \frac{49}{5}$$

$$\frac{du}{dx} = -A_1\sqrt{10}\sin\sqrt{10}x + \sqrt{10}A_2\cos\sqrt{10}x + 0.$$

at value of $x = 1$, we get

$$-A_1\sqrt{10}\sin\sqrt{10} + \sqrt{10}A_2\cos\sqrt{10} = 0.$$

$$\Rightarrow A_2\cos\sqrt{10} = A_1\sin\sqrt{10}.$$

$$\Rightarrow A_2 = A_1\tan\sqrt{10}$$

$$A_2 = \frac{49}{5}\tan\sqrt{10}.$$

Thus we obtain,

$$\boxed{\frac{49}{5}\cos\sqrt{10}x + \frac{49}{5}\tan\sqrt{10}\sin\sqrt{10}x + \frac{1}{5}}$$

Q4) a) given $\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$

If $\sin\theta \approx \theta$ then,

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \quad = f(\theta)$$

for proving the equation to be linear we need to prove
$$f(a\theta_1 + b\theta_2) = af(\theta_1) + bf(\theta_2)$$

$$f(\theta_1) = \frac{d^2\theta_1}{dt^2} + \frac{g}{l}\theta_1 \Rightarrow af(\theta_1) = a\left(\frac{d^2\theta_1}{dt^2} + \frac{g}{l}\theta_1\right)$$

RHS

$$f(\theta_2) = \frac{d^2\theta_2}{dt^2} + \frac{g}{l}\theta_2 \qquad bf(\theta_2) = b\left(\frac{d^2\theta_2}{dt^2} + \frac{g}{l}\theta_2\right)$$

$$f(a\theta_1 + b\theta_2) = \frac{d^2}{dt^2}(a\theta_1 + b\theta_2) + \frac{g}{l}(a\theta_1 + b\theta_2)$$

$$\Rightarrow \frac{a\,d^2\theta_1}{dt^2} + \frac{b\,d^2\theta_2}{dt^2} + a\frac{g}{l}\theta_1 + b\frac{g}{l}\theta_2$$

$$\Rightarrow a\left(\frac{d^2\theta_1}{dt^2} + \frac{g}{l}\theta_1\right) + b\left(\frac{d^2\theta_2}{dt^2} + \frac{g}{l}\theta_2\right) \quad LHS$$

thus, LHS = RHS.

This implies that the function is linear.

c) solution of the differential eq^n for small angle approximation.

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

Auxiliary eq^n :→ $m^2 + \frac{g}{l} = 0$

$$\Rightarrow m^2 = -\frac{g}{l} \Rightarrow m = \pm\sqrt{\frac{g}{l}}\,i$$

solution of this equation is of the form of
$A_1\sin\omega t + B_1\cos\omega t$, thus we get,

$$\theta(t) = c_1 \cos\left(\sqrt{\frac{g}{l}}\, t\right) + c_2 \sin\left(\sqrt{\frac{g}{l}}\, t\right)$$

here c_1 & c_2 are constants.

$g = 9.8\ m/s^2$. $\quad l = 0.5\ m \quad \theta(0) = 3^\circ. \quad \dot{\theta}(0) = 0.$

$\theta(0) = 3^\circ$.

$\theta(0) = 3^\circ = c_1 + 0 \Rightarrow c_1 = 3.$

$$\dot{\theta}(t) = -c_1 \sqrt{\frac{g}{l}} \sin\left(\sqrt{\frac{g}{l}}\, t\right) + c_2 \sqrt{\frac{g}{l}} \cos\left(\sqrt{\frac{g}{l}}\, t\right)$$

$\dot{\theta}(0) = 0$

$\Rightarrow \quad 0 = c_2 \sqrt{\frac{g}{l}} \cos(0) \Rightarrow c_2 \sqrt{\frac{g}{l}} = 0$

$\Rightarrow \quad C_2 = 0.$

plugging the value in equation we get,

$\theta(t) = 3 \cos \sqrt{\frac{g}{l}}\, t \Rightarrow g = 9.8$
$l = 0.5\ m.$

$\theta(t) = 3 \cos \sqrt{19.6}\, t$.

$\Rightarrow \boxed{\theta(t) = 3 \cos 1.4\, t}$

this is the sol^n of the pendulum under small angle approximation.

Q6)

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \quad \text{steady state}$$

$$0 \le x \le a$$

$$0 \le y \le b.$$

a) Boundary condⁿ

$u(x,0) = 0 \qquad u(x,b) = 5$

$u(0,y) = 0 \qquad u(a,y) = 0.$

$u(x,y) = X(x)\,Y(y) \rightarrow$ Laplace eqⁿ.

substituting $u_{xx} = x''y$ & $u_{yy} = xy''$.

$x''y + xy'' = 0$

$x''y = -xy''$

$\frac{x''}{x} = -\frac{y''}{y}.$

since independent variables are used to separate the 2 sides we introduce a constant of separation. λ

$\frac{x''}{x} = -\frac{y''}{y} = \lambda$

$\cancel{\frac{x''}{y''}}$ $\qquad x'' = \lambda x \qquad x'' - \lambda x = 0 \quad — (1)$

$\qquad y'' = -\lambda y \qquad y'' + \lambda y = 0 \quad — (2).$

b) The boundary condⁿ

$u(x,0) = 0 \Rightarrow x(0)\,y(0) = 0. \quad x(0) = 0$ or $y(0) = 0$

$u(x,b) = 5 \Rightarrow x(x)\,y(b) = 5$

$u(0,y) = 0 \rightarrow x(0)\,Y(y) = 0 \Rightarrow x(0) = 0$ or $y(y) = 0.$

$u(a,y) = 0 \rightarrow x(a)\,Y(y) = 0 \rightarrow X(a) = 0$ or $Y(y) = 0.$

for non trivial solⁿ we need to ignore constant 0 function

New boundary condⁿ →

$X(0) = 0 \quad Y(0) = 0 \quad X(a) = 0$

eqⁿ (1) ⇒ $X'' - \lambda X = 0$ (BC $X(0) = 0$ $X(a) = 0$)

eqⁿ (2) ⇒ $y'' - \lambda y = 0$ (BC − $Y(0) = 0$)

4^{th} BC, ⇒ $u(x, b) = 5$.

now for eigen value problem

The eqⁿ of X give rise to the 2 point boundary value problem,

$X'' - \lambda X = 0 \quad x(0) = 0 \quad x(a) = 0$.

eigen value, $\lambda = \sigma^2 = \frac{n^2\pi^2}{a^2}$ $n = 1, 2, 3 \ldots$

$y'' + \frac{n^2\pi^2}{a^2} y = 0$ $[\, y(0) = 0 \,]$ —— (3)

eigen function

$X_n = \sin \frac{n\pi x}{a}$ $n = 1, 2, 3 \ldots$

c) from eqⁿ (3), characteristic eqⁿ.

$\xi^2 + \frac{n^2\pi^2}{a^2} = 0$

has imaginary roots, $\xi = \pm \sqrt{\frac{n^2\pi^2}{a^2}}\, i$

$\xi = \pm \frac{n\pi}{a} i$

general, solⁿ for eqⁿ of y is,

$$y = c_1 e^{x} \cos\left(\frac{n\pi}{a} x\right) + c_2 e^{x} \sin\left(\frac{n\pi}{a} x\right)$$

$n = 1, 2, 3 \ldots$

$$Y_n = c_n e^{x} \cos\left(\frac{n\pi x}{a}\right) + c_n e^{x} \sin\left(\frac{n\pi x}{a}\right)$$

$$= c_n e^{x} \left[\cos\left(\frac{n\pi x}{a}\right) + \sin\left(\frac{n\pi x}{a}\right)\right]$$

$$\Rightarrow k_n \left[\cos\left(\frac{n\pi x}{a}\right) + \sin\left(\frac{n\pi x}{a}\right)\right]$$

$$u_n(x,y) = x_n(x)\, y_n(y)$$

$$\Rightarrow k_n \left[\cos\left(\frac{n\pi x}{a}\right) + \sin\left(\frac{n\pi x}{a}\right)\right]$$

$$= X \left[\cos\left(\frac{n\pi x}{a}\right) + \sin\left(\frac{n\pi x}{a}\right)\right] \quad n = 1, 2, 3.$$

The general solⁿ is

$$u(x,y) = \sum_{n=1} k_n \left[\cos\left(\frac{n\pi x}{a}\right) + \sin\left(\frac{n\pi x}{a}\right)\right] \left[\cos\left(\frac{n\pi y}{a}\right) + \sin\left(\frac{n\pi y}{a}\right)\right]$$

$n = 1, 2, 3.$

The solⁿ is specific to given boundary condⁿ to find particular solⁿ.

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$$u(x,b) = 5 = \sum_{n=1}^{\infty} k_n \left[\left(\cos\frac{n\pi x}{a} + \sin\frac{n\pi x}{a}\right)\left(\cos\frac{n\pi}{a}b + \sin\frac{n\pi b}{a}\right)\right]$$

Q19) $au' - \nu u'' = 1 \qquad 0 < x < 1$

$u(0) = 0$

$u(1) = 0$.

exact solution :-

$au' - \nu u'' = 1 \qquad$ let $u' = p \qquad u'' = p'$

on substituting we get

$-\nu p' + ap = 1$

$-\nu \frac{dp}{dx} + ap = 1$

$-\nu \frac{dp}{dx} = 1 - ap$

$\Rightarrow \int \frac{-\nu}{1-ap} dp = \int dx$

we get, $\cancel{\nu\left(\frac{\log(1-ap)}{a}\right)} = x$

$-\nu \int \frac{1}{1-ap} dp = \int dx$

$1 - ap = k \qquad -\nu \int \frac{1}{k} \times -\frac{1}{a} dk = \int dx$

$-a\,dp = dk$

$\Rightarrow dp = -\frac{1}{a} dk \qquad \frac{\nu}{a} \int \frac{1}{k} dk = \int dx$

$\Rightarrow \frac{\nu}{a} \log k = x$

log represents natural Logarithm

$\frac{\nu}{a} \log(1-ap) = x$

$\cancel{\log(1-ap)^{\nu}} = ax$

$\log(1-ap) = \frac{ax}{\nu}$

$1 - ap = e^{ax/\nu} \Rightarrow ap = 1 - e^{ax/\nu}$

$$\Rightarrow \cancel{p = \frac{1}{a}\left(1 - e^{ax/\nu}\right)}$$

$$\Rightarrow \boxed{p = \frac{1}{a} \quad 1 - e^{ax/\nu}}$$

also, $u = \int p \, dx.$

$$u = \int \frac{1 - e^{ax/\nu}}{a} \, dx$$

$$u = \int \frac{1}{a} \, dx - \frac{1}{a} \int e^{ax/\nu} \, dx.$$

$$\Rightarrow \frac{x}{a} - \frac{1}{a} \times \frac{\nu}{a} e^{ax/\nu}$$

$$\Rightarrow \frac{x}{a} - \frac{1}{a^2} \nu e^{ax/\nu} + c.$$

thus $\boxed{u = \frac{x}{a} - \frac{1}{a^2} \nu e^{ax/\nu} + c}.$

discretisation

using central diff

$$\frac{\partial u}{\partial x} = \frac{1}{2} \frac{u_{k+1}(t) - u_{k-1}(t)}{h}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{k+1}(t) - 2u_k(t) + u_{k-1}(t)}{h^2}.$$

using backward $\qquad h = \frac{1}{N+1}$

$$\frac{\partial u}{\partial x} = \frac{u_k(t) - u_{k-1}(t)}{h}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_k(t) - 2u_{k-1}(t) + u_{k-2}(t)}{h^2}.$$