Midterm

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1 Problem 1

Assume that the total size of the cake is 1. Denote the physical size of the pieces by x_1, x_2, x_3. Since $x_1 + x_2 + x_3 = 1$and all $x_i \geq 0$, the solution space is just a triangle. Assign the three people as A,B,C, then we can a traingle where each elementary triangle is an ABC triangle, the A, B, C present the "ownership" of the vertice. A similar triangulation of finer mesh can also be labelled in this way.

2 Problem 2

Initiate LH Algorithm:

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}, B = \begin{pmatrix} 1 & 5 \\ 4 & 2 \end{pmatrix},$$

P	x1	x2	t3	t4	=
3	1	4	1	0	1
4	5	2	0	1	1

Q	r1	r2	y3	y4	=
1	1	0	3	2	1
2	0	1	1	3	1

$$L(x) = \{1, 2\}, L(y) = \{3, 4\}$$

Second step:

P	x1	x2	t3	t4	=
3	1	4	1	0	1
4	9	0	-1	2	1

Q	r1	r2	y3	y4	=
1	1	0	3	2	1
2	0	1	1	3	1

$$L(x) = \{1, 3\}, L(y) = \{3, 4\}$$

Third step:

$$L(x) = \{1, 3\}, L(y) = \{1, 4\}$$

P	x1	x2	t3	t4	=
3	1	4	1	0	1
4	9	0	-1	2	1

Q	r1	r2	y3	y4	=
1	1	0	3	2	1
2	-1	3	0	7	2

P	x1	x2	t3	t4	=
3	0	18	5	-1	4
4	9	0	-1	2	1

Q	r1	r2	y3	y4	=
1	1	0	3	2	1
2	-1	3	0	7	2

Fourth step:

$$L(x) = \{3, 4\}, L(y) = \{1, 4\}$$

Fifth step:

P	x1	x2	t3	t4	=
3	0	18	5	-1	4
4	9	0	-1	2	1

Q	r1	r2	y3	y4	=
1	3	-2	7	0	1
2	-1	3	0	7	2

$$L(x) = \{3, 4\}, L(y) = \{1, 2\}$$

So we have a mixed strategy equilibrium.

3 Problem 3

(a)

- Budget constraint :

  $px^i \leq pw^i$

- Individual optimality :

  $x^{i*} \in argmax\{u_i(x^i) : px^i \leq pw^{iT}, x^i \geq 0\}$, where $u^1 = [3, 0, 0, 0, 0]^T$ $u^2 = [0, 4, 2, 0, 0]^T$ $u^3 = [0, 2, 1, 0, 0]^T$

- Market clearance:

  $\sum_{i \in M} x^i \leq \sum_{i \in M} w^{iT}$

To get the maximum utility, the agent will put his money on the good where he get most utility with every unit of money since his budget constraints to the equation $px^i \leq pw^i$. So if $x_j^{i*} > 0$, that means $\frac{u_j^i}{p_j}$ is the maximum.

(b)

- Normalization:
  - Everything is owned by someone: Condition fulfilled.
  - Everything is liked by someone: Elimite the 5_{th} good. Then we have 4 goods at the market $N = \{1,2,3,4\}$. The initial endowment of the agents $w^1 = \big(0,2,0,1\big), w^2 = \big(0,2,1,0\big), w^3 = \big(1,0,0,3\big)$,
  - Normalization: For the 2_{nd} and 4_{th} good, divide by total amount. The initial endowment of the agents $w^1 = \big(0,1/2,0,1/4\big), w^2 = \big(0,1/2,1,0\big), w^3 = \big(1,0,0,3/4\big)$,

- Atomization:
  - Every agent owns one item : replace agent 1, 2, 3 by $1_1, 1_2, 2_1, 2_2, 3_1, 3_2$,, then we have 6 markets agents $M = \{1_1, 1_2, 2_1, 2_2, 3_1, 3_2\}$, where $\begin{cases} u^{1_1} = \big[3,0,0,0\big]^T \\ u^{1_2} = \big[3,0,0,0\big]^T \\ u^{2_1} = \big[0,4,2,0\big]^T \\ u^{2_2} = \big[0,4,2,0\big]^T \\ u^{3_1} = \big[0,2,0,1\big]^T \\ u^{3_2} = \big[0,2,0,1\big]^T \end{cases}$

    The initial endowment of the agents $\begin{cases} w^{1_1} = \big(0,2,0,0\big) \\ w^{1_2} = \big(0,0,0,1\big) \\ w^{2_1} = \big(0,2,0,0\big) \\ w^{2_2} = \big(0,0,1,0\big) \\ w^{3_1} = \big(1,0,0,0\big) \\ w^{3_2} = \big(0,0,0,3\big) \end{cases}$
  - Every item is owned by one agent: Rename the same type of items own by different agents and equalize the utilities by an agent on them, then we have 6 goods at the market $N = \{1, 2_1, 2_2, 3_1, 3_2, 4\}$. $\begin{cases} w^{1_1} = \big(0,2,0,0,0,0\big) \\ w^{1_2} = \big(0,0,0,0,1,0\big) \\ w^{2_1} = \big(0,0,2,0,0,0\big) \\ w^{2_2} = \big(0,0,0,1,0,0\big) \\ w^{3_1} = \big(1,0,0,0,0,0\big) \\ w^{3_2} = \big(0,0,0,0,0,3\big) \end{cases}$ here $\begin{cases} u^{1_1} = \big[3,0,0,0,0,0\big]^T \\ u^{1_2} = \big[3,0,0,0,0,0\big]^T \\ u^{2_1} = \big[0,4,4,2,0,0\big]^T \\ u^{2_2} = \big[0,4,4,2,0,0\big]^T \\ u^{3_1} = \big[0,2,2,0,1,1\big]^T \\ u^{3_2} = \big[0,2,2,0,1,1\big]^T \end{cases}$

- Normalization revision: Divide by its size, we get $\begin{cases} w^{1_1} = \Big(0,1,0,0,0,0\Big) \\ w^{1_2} = \Big(0,0,0,0,1,0\Big) \\ w^{2_1} = \Big(0,0,1,0,0,0\Big) \\ w^{2_2} = \Big(0,0,0,1,0,0\Big) \\ w^{3_1} = \Big(1,0,0,0,0,0\Big) \\ w^{3_2} = \Big(0,0,0,0,0,1\Big) \end{cases}$

- Since everything is liked by someone (The goods that no one likes are eliminated), and an edge from i to j means agent i likes item j, there is non-zero indegree.

(c)
In "Linear Utility Market", the goods are initialely distributed to the agents, but they don't have endowment of money.
In "Fisher Market", there is a seller who owns all the goods, but the agents have endowment of money to buy the goods.
These differences lead to future differences in market clearance and budget constraints.

4 Problem 1

- Bidder 1 get q1 and bidder 2 get q2,
  $p_1 = v_2q_1 + v_3q_2 - v_2q_2 = 26$,
  $p_2 = v_1q_1 + v_3q_2 - v_1q_1 = 12$

- We have $\begin{cases} u_1(q_1) \geq 0 \\ u_1(q_1) \geq u_1(q_2) \\ u_2(q_2) \geq 0 \\ u_2(q_2) \geq u_2(q_1) \\ u_3(q_1) \leq 0 \\ u_3(q_2) \leq 0 \end{cases} \Rightarrow \begin{cases} v_1q_1 - p_1 \geq 0 \\ v_1q_1 - p_1 \geq v_1q_2 - p_2 \\ v_2q_2 - p_2 \geq 0 \\ v_2q_2 - p_2 \geq v_2q_1 - p_1 \\ v_3q_1 - p_1 \leq 0 \\ v_3q_2 - p_2 \leq 0 \end{cases} \Rightarrow \begin{cases} 14 \leq p_1 \leq 36 \\ 6 \leq p_2 \leq 14 \\ 14 \leq p_1 - p_2 \leq 18 \end{cases}$

- Truthful bidding under the GSP protocol is optimal for the buyers since no bidder would change its bid to improve its utility. In this case, if all start by bidding their true vale, this is already a Nash equilibrium.