# Midterm

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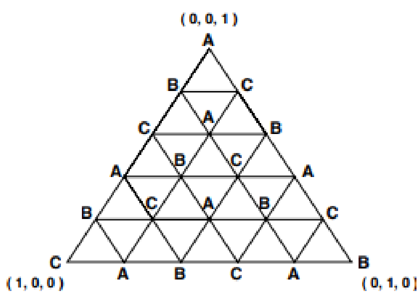
## 1 Problem 1

Assume that the total size of the cake is 1. Denote the physical size of the pieces by $x_1, x_2, x_3$. Since $x_1 + x_2 + x_3 = 1$and all $x_i \geq 0$, the solution space S is just a triangle. Assign the three people as A,B,C, then we can a traingle where each elementary triangle is an ABC triangle, the A, B, C present the "ownership" of the vertice. A similar triangulation of finer mesh can also be labelled in this way.




We obtain a new auxiliary labelling of the triangulation by 1's, 2's, and 3's by doing the following: since each point in the triangle corresponds to a set of cuts of cake, go to each vertex, and ask the owner of that vertex, "Which piece would you choose if the cake were cut with this cut-set?" Label that vertex by the number of the piece that is desired.

Since no one would ever choose an empty piece, each side of $S$ is missing one label corresponding to the piece that is empty. Hence the Sperner labelling condition is satisfied.

By Sperner's lemma, there must be a (1, 2, 3)-elementary simplex in the triangulation. This means that we have found 3 very similar cut-sets in which different people choose different pieces of cake. Carry out this procedure for a sequence of finer and finer triangulations, each time yielding smaller and smaller (1, 2, 3)-triangles. By compactness of the triangle and decreasing size of the triangles, there must be a convergent subsequence of triangles converging to a single point. Such a point corresponds to a cut-set in which the players are satisfied with different pieces.

## 2 Problem 2

Initiate LH Algorithm:

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}, B = \begin{pmatrix} 1 & 5 \\ 4 & 2 \end{pmatrix},$$

| P | x1 | x2 | t3 | t4 | = |
|---|---|---|---|---|---|
| 3 | 1 | 4 | 1 | 0 | 1 |
| 4 | 5 | 2 | 0 | 1 | 1 |

| Q | r1 | r2 | y3 | y4 | = |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 3 | 2 | 1 |
| 2 | 0 | 1 | 1 | 3 | 1 |

$$L(x) = \{1, 2\}, L(y) = \{3, 4\}$$

Second step:

| P | x1 | x2 | t3 | t4 | = |
|---|---|---|---|---|---|
| 3 | 1 | 4 | 1 | 0 | 1 |
| 4 | 9 | 0 | -1 | 2 | 1 |

| Q | r1 | r2 | y3 | y4 | = |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 3 | 2 | 1 |
| 2 | 0 | 1 | 1 | 3 | 1 |

$$L(x) = \{1, 3\}, L(y) = \{3, 4\}$$

Third step:

| P | x1 | x2 | t3 | t4 | = |
|---|---|---|---|---|---|
| 3 | 1 | 4 | 1 | 0 | 1 |
| 4 | 9 | 0 | -1 | 2 | 1 |

| Q | r1 | r2 | y3 | y4 | = |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 3 | 2 | 1 |
| 2 | -1 | 3 | 0 | 7 | 2 |

$$L(x) = \{1, 3\}, L(y) = \{1, 4\}$$

Fourth step:

| P | x1 | x2 | t3 | t4 | = |
|---|---|---|---|---|---|
| 3 | 0 | 18 | 5 | -1 | 4 |
| 4 | 9 | 0 | -1 | 2 | 1 |

| Q | r1 | r2 | y3 | y4 | = |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 3 | 2 | 1 |
| 2 | -1 | 3 | 0 | 7 | 2 |

$$L(x) = \{3, 4\}, L(y) = \{1, 4\}$$

Fifth step:

| P | x1 | x2 | t3 | t4 | = |
|---|---|---|---|---|---|
| 3 | 0 | 18 | 5 | -1 | 4 |
| 4 | 9 | 0 | -1 | 2 | 1 |

| Q | r1 | r2 | y3 | y4 | = |
|---|---|---|---|---|---|
| 1 | 3 | -2 | 7 | 0 | 1 |
| 2 | -1 | 3 | 0 | 7 | 2 |

$$L(x) = \{3, 4\}, L(y) = \{1, 2\}$$

So we have a mixed strategy equilibrium.

# 3 Problem 3

### (a)

- Budget constraint :
  $px^i \leq pw^i$
- Individual optimality :
  $x^{i*} \in argmax\{u_i(x^i) : px^i \leq pw^{iT}, x^i \geq 0\}$, where $u^1 = \left[3, 0, 0, 0, 0\right]^T$ $u^2 = \left[0, 4, 2, 0, 0\right]^T$ $u^3 = \left[0, 2, 1, 0, 0\right]^T$
- Market clearance:
  $\sum_{i \in M} x^i \leq \sum_{i \in M} w^{iT}$

To get the maximum utility, the agent will put his money on the good where he get most utility with every unit of money since his budget constraints to the equation $px^i \leq pw^i$. So if $x_j^{i*} > 0$, that means $\frac{u_j^i}{p_j}$ is the maximum.

### (b)

- Normalization:
  - Everything is owned by someone: Condition fulfilled.
  - Everything is liked by someone: Elimite the $5_{th}$ good. Then we have 4 goods at the market $N = \{1, 2, 3, 4\}$. The initial endowment of the agents $w^1 = \left(0, 2, 0, 1\right), w^2 = \left(0, 2, 1, 0\right), w^3 = \left(1, 0, 0, 3\right)$,
  - Normalization: For the $2_{nd}$ and $4_{th}$ good, divide by total amount. The initial endowment of the agents $w^1 = \left(0, 1/2, 0, 1/4\right), w^2 = \left(0, 1/2, 1, 0\right), w^3 = \left(1, 0, 0, 3/4\right)$,
- Atomization:
  - Every agent owns one item : replace agent 1, 2, 3 by $1_1, 1_2, 2_1, 2_2, 3_1, 3_2$,, then we have 6 markets agents $M = \{1_1, 1_2, 2_1, 2_2, 3_1, 3_2\}$, where $\begin{cases} u^{1_1} = \left[3, 0, 0, 0\right]^T \\ u^{1_2} = \left[3, 0, 0, 0\right]^T \\ u^{2_1} = \left[0, 4, 2, 0\right]^T \\ u^{2_2} = \left[0, 4, 2, 0\right]^T \\ u^{3_1} = \left[0, 2, 0, 1\right]^T \\ u^{3_2} = \left[0, 2, 0, 1\right]^T \end{cases}$

The initial endowment of the agents $\begin{cases} w^{1_1} = \left(0,2,0,0\right) \\ w^{1_2} = \left(0,0,0,1\right) \\ w^{2_1} = \left(0,2,0,0\right) \\ w^{2_2} = \left(0,0,1,0\right) \\ w^{3_1} = \left(1,0,0,0\right) \\ w^{3_2} = \left(0,0,0,3\right) \end{cases}$

- Every item is owned by one agent: Rename the same type of items own by different agents and equalize the utilities by an agent on them, then we have 6 goods at the market

  $N = \{1, 2_1, 2_2, 3_1, 3_2, 4\}$. $\begin{cases} w^{1_1} = \left(0,2,0,0,0,0\right) \\ w^{1_2} = \left(0,0,0,0,1,0\right) \\ w^{2_1} = \left(0,0,2,0,0,0\right) \\ w^{2_2} = \left(0,0,0,1,0,0\right) \\ w^{3_1} = \left(1,0,0,0,0,0\right) \\ w^{3_2} = \left(0,0,0,0,0,3\right) \end{cases}$ here $\begin{cases} u^{1_1} = \left[3,0,0,0,0,0\right]^T \\ u^{1_2} = \left[3,0,0,0,0,0\right]^T \\ u^{2_1} = \left[0,4,4,2,0,0\right]^T \\ u^{2_2} = \left[0,4,4,2,0,0\right]^T \\ u^{3_1} = \left[0,2,2,0,1,1\right]^T \\ u^{3_2} = \left[0,2,2,0,1,1\right]^T \end{cases}$

- Normalization revision: Divide by its size, we get $\begin{cases} w^{1_1} = \left(0,1,0,0,0,0\right) \\ w^{1_2} = \left(0,0,0,0,1,0\right) \\ w^{2_1} = \left(0,0,1,0,0,0\right) \\ w^{2_2} = \left(0,0,0,1,0,0\right) \\ w^{3_1} = \left(1,0,0,0,0,0\right) \\ w^{3_2} = \left(0,0,0,0,0,1\right) \end{cases}$

- Since everything is liked by someone (The goods that no one likes are eliminated), and an edge from i to j means agent i likes item j, there is non-zero indegree.

**(c)**
In "Linear Utility Market", the goods are initialely distributed to the agents, but they don't have endowment of money.
In "Fisher Market", there is a seller who owns all the goods, but the agents have endowment of money to buy the goods.
These differences lead to future differences in market clearance and budget constraints.

# 4 Problem 4

- Bidder 1 get q1 and bidder 2 get q2,
  $p_1 = v_2 q_1 + v_3 q_2 - v_2 q_2 = 26$,
  $p_2 = v_1 q_1 + v_3 q_2 - v_1 q_1 = 12$

- We have $\begin{cases} u_1(q_1) \geq 0 \\ u_1(q_1) \geq u_1(q_2) \\ u_2(q_2) \geq 0 \\ u_2(q_2) \geq u_2(q_1) \\ u_3(q_1) \leq 0 \\ u_3(q_2) \leq 0 \end{cases} \Rightarrow \begin{cases} v_1 q_1 - p_1 \geq 0 \\ v_1 q_1 - p_1 \geq v_1 q_2 - p_2 \\ v_2 q_2 - p_2 \geq 0 \\ v_2 q_2 - p_2 \geq v_2 q_1 - p_1 \\ v_3 q_1 - p_1 \leq 0 \\ v_3 q_2 - p_2 \leq 0 \end{cases} \Rightarrow \begin{cases} 14 \leq p_1 \leq 36 \\ 6 \leq p_2 \leq 14 \\ 14 \leq p_1 - p_2 \leq 18 \end{cases}$
- Truthful bidding under the GSP protocol is optimal for the buyers since no bidder would change its bid to improve its utility. In this case, if all start by bidding their true vale, this is already a Nash equilibrium.