

Principal Strains & Invariants

Search

home > deformation & strain > principal strain



Introduction

This page covers standard coordinate transformations, principal strains, and strain invariants. Everything here applies regardless of the type of strain tensor, so both ϵ and \mathbf{E} will be used here.

Coordinate transformations of 2nd rank tensors were discussed on this [coordinate transform page](#). The transform applies to any strain tensor, or stress tensor for that matter. It is written as

$$\mathbf{E}' = \mathbf{Q} \cdot \mathbf{E} \cdot \mathbf{Q}^T$$

Everything below follows from two facts: First, the tensors are symmetric. Second, the above coordinate transformation is used.

2-D Principal Strains

In 2-D, the transformation equations are

$$\epsilon'_{xx} = \epsilon_{xx} \cos^2 \theta + \epsilon_{yy} \sin^2 \theta + 2 \left(\frac{\gamma_{xy}}{2} \right) \sin \theta \cos \theta$$

$$\epsilon'_{yy} = \epsilon_{xx} \sin^2 \theta + \epsilon_{yy} \cos^2 \theta - 2 \left(\frac{\gamma_{xy}}{2} \right) \sin \theta \cos \theta$$

$$\frac{\gamma'_{xy}}{2} = (\epsilon_{yy} - \epsilon_{xx}) \sin \theta \cos \theta + \left(\frac{\gamma_{xy}}{2} \right) (\cos^2 \theta - \sin^2 \theta)$$

The equations are written in terms of $\gamma/2$ to emphasize that half of all the shear values are used in the transformation equations.

[This page](#) performs full 3-D tensor transforms, but can still be used for 2-D problems.. Enter values in the upper left 2x2 positions and rotate in the 1-2 plane to perform transforms in 2-D. The screenshot below shows a case of pure shear rotated 45° to obtain the principal strains. Note also how the \mathbf{Q} matrix transforms.

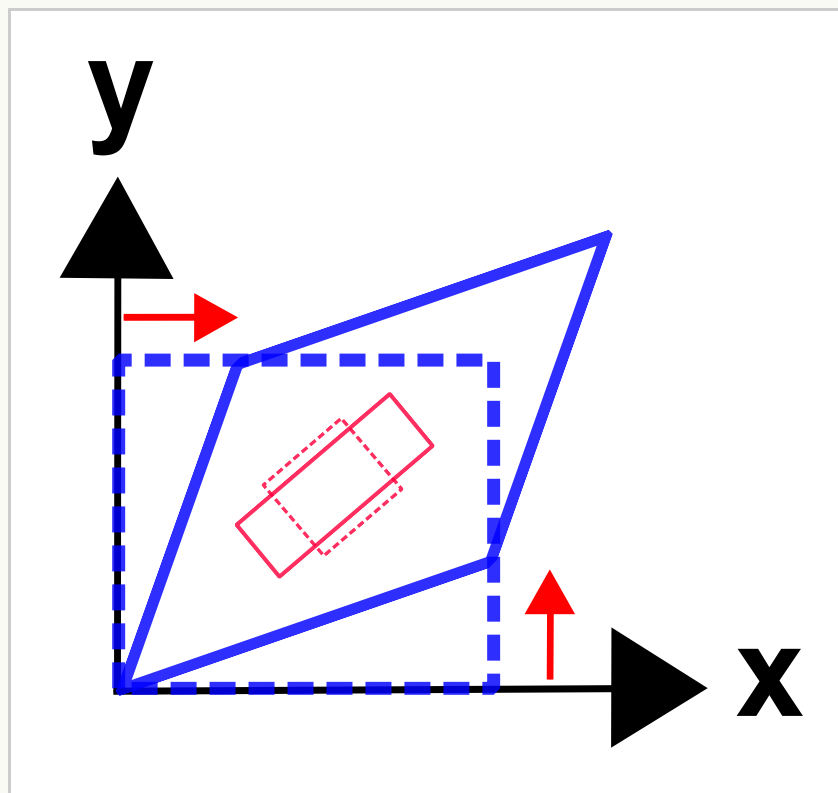
Tensor Transforms

Tensor			Transform Matrix		
1	0	0	0.7071068	0.7071068	0
0	-1	0	-0.7071068	0.7071068	0
0	0	0	0	0	1

Rotate ☒ 1 - 2
 following ☐ 2 - 3
 plane... ☐ 3 - 1

degrees

The figure below shows the deformed shapes corresponding to the pure shear case in the tensor transform webpage example. The blue square aligned with the axes clearly undergoes shear. But the red square inscribed in the larger blue square only sees simple tension and compression. These are the principal values of the pure shear deformation in the global coordinate system.



In 2-D, the principal strain orientation, θ_P , can be computed by setting $\gamma'_{xy} = 0$ in the above shear equation and solving for θ to get θ_P , the principal strain angle.

$$0 = (\epsilon_{yy} - \epsilon_{xx}) \sin \theta_P \cos \theta_P + \left(\frac{\gamma_{xy}}{2} \right) (\cos^2 \theta_P - \sin^2 \theta_P)$$

This gives

$$\tan(2\theta_P) = \frac{\gamma_{xy}}{\epsilon_{xx} - \epsilon_{yy}} = \frac{2\epsilon_{xy}}{\epsilon_{xx} - \epsilon_{yy}}$$

The transformation matrix, \mathbf{Q} , is

$$\mathbf{Q} = \begin{bmatrix} \cos \theta_P & \sin \theta_P \\ -\sin \theta_P & \cos \theta_P \end{bmatrix}$$

Inserting this value for θ_P back into the equations for the normal strains gives the principal values. They are written as ϵ_{max} and ϵ_{min} , or alternatively as ϵ_1 and ϵ_2 .

$$\epsilon_{max}, \epsilon_{min} = \frac{\epsilon_{xx} + \epsilon_{yy}}{2} \pm \sqrt{\left(\frac{\epsilon_{xx} - \epsilon_{yy}}{2} \right)^2 + \left(\frac{\gamma_{xy}}{2} \right)^2}$$

They could also be obtained by using $\mathbf{E}' = \mathbf{Q} \cdot \mathbf{E} \cdot \mathbf{Q}^T$ with \mathbf{Q} based on θ_P .



Principal Strain Notation

Principal strains can be written as ϵ_1 , ϵ_2 , and ϵ_3 . Only one subscript is usually used in this case to differentiate the principal strain values from the normal strain components: ϵ_{11} , ϵ_{22} , and ϵ_{33} .



2-D Principal Strain Example

Start with the strain tensor

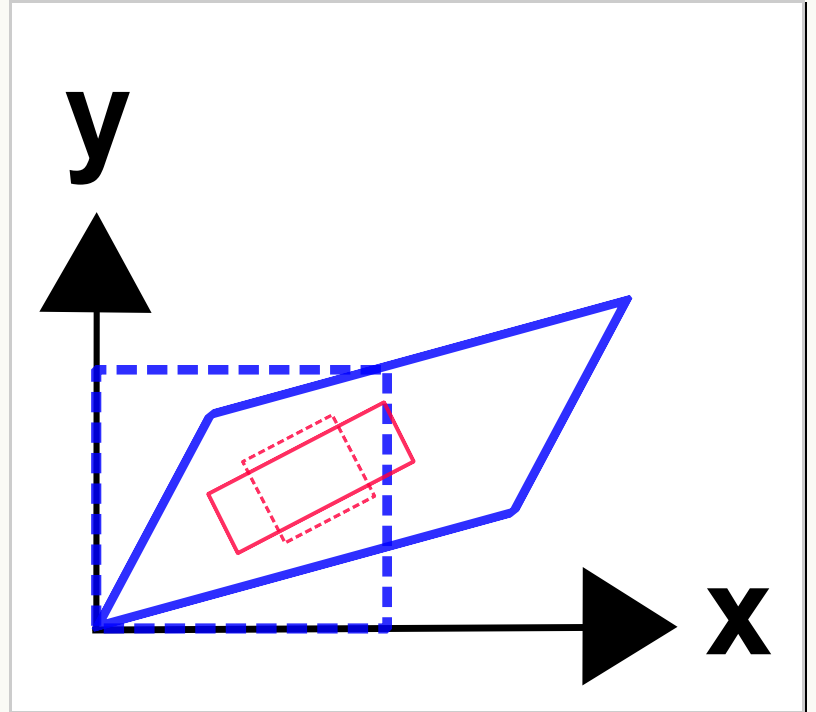
$$\boldsymbol{\epsilon} = \begin{bmatrix} 0.50 & 0.30 \\ 0.30 & -0.20 \end{bmatrix}$$

The principal orientation is

$$\tan(2\theta_P) = \frac{2 * 0.3}{0.50 - (-0.20)}$$

$$\theta_P = 20.3^\circ$$

The principal strains are



$$\epsilon_{max}, \epsilon_{min} = \frac{0.50 - 0.20}{2} \pm \sqrt{\left(\frac{0.50 + 0.20}{2}\right)^2 + (0.30)^2}$$

$$\epsilon_{max}, \epsilon_{min} = 0.611, -0.311$$

The only limitation to using these equations for principal values is that it is not known which one applies to 20.3° and which applies to 110.3° . That's why an attractive alternative is

$$\begin{bmatrix} \epsilon'_{11} & \epsilon'_{12} \\ \epsilon'_{12} & \epsilon'_{22} \end{bmatrix} = \begin{bmatrix} \cos(20.3^\circ) & \sin(20.3^\circ) \\ -\sin(20.3^\circ) & \cos(20.3^\circ) \end{bmatrix} \begin{bmatrix} 0.50 & 0.30 \\ 0.30 & -0.20 \end{bmatrix} \begin{bmatrix} \cos(20.3^\circ) & -\sin(20.3^\circ) \\ \sin(20.3^\circ) & \cos(20.3^\circ) \end{bmatrix}$$

$$= \begin{bmatrix} 0.611 & 0.000 \\ 0.000 & -0.311 \end{bmatrix}$$

This confirms that the 0.611 principal strain value in the ϵ_{11} slot is indeed 20.3° from the X-axis. The ϵ_{22} value is 90° from the first.

3-D Principal Strains

Coordinate transforms in 3-D are

$$\begin{bmatrix} E'_{11} & E'_{12} & E'_{13} \\ E'_{12} & E'_{22} & E'_{23} \\ E'_{13} & E'_{23} & E'_{33} \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} q_{11} & q_{21} & q_{31} \\ q_{12} & q_{22} & q_{32} \\ q_{13} & q_{23} & q_{33} \end{bmatrix}$$

The second **Q** matrix is once again the transpose of the first.

[This page](#) performs tensor transforms.

Tensor Transforms

Tensor

0.517131	2.297996	2.42067
2.297996	5.149457	5.073937
2.42067	5.073937	5.333407

Transform Matrix

0.923037	0.321394	-0.21147
-0.21147	0.883022	0.418989
0.321394	-0.34202	0.883022

Rotate

☒ 1 - 2

☐ 2 - 3

☐ 3 - 1

following plane...

20

degrees

Rotate

Back

Reset

Help

And [this page](#) calculates principal values (eigenvalues) and principal directions (eigenvectors).

Eigenvalues & Eigenvectors

Input

4.420	1.35	3.509
1.35	3.832	0
3.509	0	3.207

Calculate

Eigenvectors

0.7569918	0.2669156	0.5964222
-0.0544287	0.9353492	-0.3495137
-0.6511531	0.2321164	0.7225794

Eigenvalues

7.6606977

3.753443

0.0448594

Back

Reset

Help

It's important to remember that the inputs to both pages must be symmetric. In fact, both pages enforce this.

The eigenvalues above can be written in matrix form as

$$\mathbf{E} = \begin{bmatrix} 0.243 & 0 & 0 \\ 0 & 1.256 & 0 \\ 0 & 0 & 4.338 \end{bmatrix}$$



Maximum Shear

The maximum amount of shear at any point is easy to calculate from the principal strains. It is simply

$$\gamma_{max} = \epsilon_{max} - \epsilon_{min}$$

This applies in both 2-D and 3-D. The maximum shear always occurs in a coordinate system orientation that is rotated 45° from the principal coordinate system. For the principal strain tensor above

$$\mathbf{E} = \begin{bmatrix} 0.243 & 0 & 0 \\ 0 & 1.256 & 0 \\ 0 & 0 & 4.338 \end{bmatrix}$$

The max and min principal strains are in the E_{33} and E_{11} slots, respectively. So the max shear orientation is obtained by rotating the principal coordinate system by 45° in the (1 – 3) plane.

The max shear value itself is

$$\begin{aligned} \gamma_{max} &= \epsilon_{max} - \epsilon_{min} \\ &= 4.338 - 0.243 \\ &= 4.095 \end{aligned}$$

The manual way of computing principal strains is to solve a cubic equation for the three principal values. The equation results from setting the following determinant equal to zero. The λ values, once computed, will equal the principal values of the strain tensor.

$$\begin{vmatrix} E_{11} - \lambda & E_{12} & E_{13} \\ E_{12} & E_{22} - \lambda & E_{23} \\ E_{13} & E_{23} & E_{33} - \lambda \end{vmatrix} = 0$$

This can be expanded out to give

$$\begin{aligned}
& (E_{11} - \lambda)[(E_{22} - \lambda)(E_{33} - \lambda) - E_{23}^2] - \\
& E_{12}[E_{12}(E_{33} - \lambda) - E_{23}E_{13}] + \\
& E_{13}[E_{12}E_{23} - (E_{22} - \lambda)E_{13}] = 0
\end{aligned}$$

Invariants

...and expanded out even further to give

$$\begin{aligned}
\lambda^3 - (E_{11} + E_{22} + E_{33})\lambda^2 + (E_{11}E_{22} + E_{22}E_{33} + E_{33}E_{11} - E_{12}^2 - E_{13}^2 - E_{23}^2)\lambda - \\
(E_{11}E_{22}E_{33} - E_{11}E_{23}^2 - E_{22}E_{13}^2 - E_{33}E_{12}^2 + 2E_{12}E_{13}E_{23}) = 0
\end{aligned}$$



"Now consider this..."

No matter what coordinate transformation you apply to the strain tensor, its principal strains had better be the same three values. And the only way for this to happen in the above equation is for the equation itself to always be the same, no matter the transformation. This means that the combinations of strain components, which serve as coefficients of the λ 's, must be invariant under coordinate transformations. Their values must not change. And that's why they are called invariants.

So the equation can be written as

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

where

$$I_1 = E_{11} + E_{22} + E_{33}$$

$$I_2 = E_{11}E_{22} + E_{22}E_{33} + E_{33}E_{11} - E_{12}^2 - E_{13}^2 - E_{23}^2$$

$$I_3 = E_{11}E_{22}E_{33} - E_{11}E_{23}^2 - E_{22}E_{13}^2 - E_{33}E_{12}^2 + 2E_{12}E_{13}E_{23}$$

Although probably not initially apparent, the invariants are actually familiar quantities.

$$I_1 = \text{tr}(\mathbf{E})$$

$$I_2 = \begin{vmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{vmatrix} + \begin{vmatrix} E_{11} & E_{13} \\ E_{13} & E_{33} \end{vmatrix} + \begin{vmatrix} E_{22} & E_{23} \\ E_{23} & E_{33} \end{vmatrix}$$

$$I_3 = \det(\mathbf{E})$$

They can be written in tensor notation as

$$I_1 = E_{kk}$$

$$I_2 = \frac{1}{2} \left[(E_{kk})^2 - E_{ij}E_{ij} \right]$$

$$I_3 = \epsilon_{ijk}E_{i1}E_{j2}E_{k3}$$

I_3 is in tensor notation, but no one should actually calculate a determinant based on tensor notation rules because it is very inefficient.



Physical Interpretation of Invariants

The physical interpretation of the invariants depends on what tensor the invariants are computed from. For any stress or strain tensor, I_1 is directly related to the hydrostatic component of the that tensor. This is universal.

I_2 tends to be related more to the deviatoric aspects of stress and strain, although not exclusively.

Finally, I_3 does not seem to have any physical significance as the determinant of a stress or strain tensor. But it does when applied to the deformation gradient. In that case, $I_3 = V_F/V_o$, the ratio of deformed to initial volume, which is 1 for incompressible materials like rubber.



Invariant Example

This example will start with a random strain tensor and demonstrate that the invariants are invariant under coordinate transformations. Start with the strain tensor

$$\mathbf{E} = \begin{bmatrix} 0.50 & 0.30 & 0.20 \\ 0.30 & -0.20 & -0.10 \\ 0.20 & -0.10 & 0.10 \end{bmatrix}$$

Its invariants are

$$I_1 = 0.50 + (-0.20) + 0.10$$

$$= 0.40$$

$$I_2 = (0.50)(-0.20) + (-0.20)(0.10) + (0.10)(0.50) - (0.30)^2 - (0.20)^2 - (-0.10)^2$$

$$= -0.21$$

$$I_3 = \det(\mathbf{E})$$

$$= -0.028$$

Now rotate the coordinate system by some random amount shown in the screenshot.

Tensor Transforms

Tensor			Transform Matrix		
0.308041	-0.105634	0.273464	0.863536	0.451927	-0.223755
-0.105634	-0.345531	0.0349123	-0.218394	0.735085	0.641835
0.273464	0.0349123	0.437490	0.454543	-0.505381	0.733471

Rotate ☐ 1 - 2 degrees

following ☐ 2 - 3

plane... ☒ 3 - 1

The transformed strain tensor is now

$$\mathbf{E}' = \begin{bmatrix} 0.308 & -0.106 & 0.273 \\ -0.106 & -0.346 & 0.035 \\ 0.273 & 0.035 & 0.437 \end{bmatrix}$$

And the "new" invariants are

$$I_1 = 0.308 + (-0.346) + 0.437$$

$$= 0.40$$

$$I_2 = (0.308)(-0.346) + (-0.346)(0.437) + (0.437)(0.308) - (-0.106)^2 - (0.273)^2 - (0.035)^2$$

$$= -0.21$$

$$I_3 = \det(\mathbf{E})$$

$$= -0.028$$

So they are indeed invariant.

Just for grins... Let's compute the principal strains and then recompute the invariants to demonstrate again that they are invariant.

The principal strains are

Eigenvalues & Eigenvectors

Input			Eigenvectors		
.5	.3	.2	0.9161354	0.2888504	0.2779593
.3	-.2	-.1	-0.375470	0.8611946	0.3425871
.2	-.1	.1	-0.140420	-0.418221	0.8974256
<input type="button" value="Calculate"/>					
Eigenvalues					
			0.6552685	-0.370576	0.1153083

Eigenvalues & Eigenvectors

Input			Eigenvectors		
.308	-.106	.273	0.7551055	-0.220196	-0.617518
-.106	-.346	.035	0.1956547	0.9746741	-0.108303
.273	.035	.437	0.625727	-0.039039	0.7790646
<input type="button" value="Calculate"/>					
Eigenvalues					
			0.1156535	-0.371167	0.6545135

$$\mathbf{E}' = \begin{bmatrix} 0.655 & 0 & 0 \\ 0 & -0.371 & 0 \\ 0 & 0 & 0.115 \end{bmatrix}$$

Curiously, the two different starting points lead to the principal values being listed in different orders. This is typical and has no special meaning. The final, physical answer is the same. The **Q** will accommodate the different listing orders.

Finally, calculating the invariants one last time using the principal values gives

$$I_1 = 0.655 + (-0.371) + 0.115$$

$$= 0.40$$

$$I_2 = (0.655)(-0.371) + (-0.371)(0.115) + (0.115)(0.655)$$

$$= -0.21$$

$$I_3 = \det(\mathbf{E}) = (0.655)(-0.371)(0.115)$$

$$= -0.028$$

And so it works again. Note how the invariant calculations are rather simple when all the off-diagonal terms are zero. The determinant is especially easy.



2-D Invariants

In the 2-D world, there are only two invariants.

$$I_1 = \text{tr}(\epsilon) = \epsilon_{11} + \epsilon_{22}$$

$$I_2 = \det(\epsilon) = \epsilon_{11} * \epsilon_{22} - \epsilon_{12}^2$$

In 2-D, things are just simple enough to directly prove the invariance of invariants. We'll prove it for I_1 here. Recall the transformation equations are

$$\epsilon'_{xx} = \epsilon_{xx} \cos^2 \theta + \epsilon_{yy} \sin^2 \theta + 2 \left(\frac{\gamma_{xy}}{2} \right) \sin \theta \cos \theta$$

$$\epsilon'_{yy} = \epsilon_{xx} \sin^2 \theta + \epsilon_{yy} \cos^2 \theta - 2 \left(\frac{\gamma_{xy}}{2} \right) \sin \theta \cos \theta$$

$$\frac{\gamma'_{xy}}{2} = (\epsilon_{yy} - \epsilon_{xx}) \sin \theta \cos \theta + \left(\frac{\gamma_{xy}}{2} \right) (\cos^2 \theta - \sin^2 \theta)$$

So $\epsilon'_{xx} + \epsilon'_{yy}$ is

$$\epsilon'_{xx} + \epsilon'_{yy} = \epsilon_{xx} \cos^2 \theta + \epsilon_{yy} \sin^2 \theta + 2 \left(\frac{\gamma_{xy}}{2} \right) \sin \theta \cos \theta +$$

$$\epsilon_{xx} \sin^2 \theta + \epsilon_{yy} \cos^2 \theta - 2 \left(\frac{\gamma_{xy}}{2} \right) \sin \theta \cos \theta$$

and the shear terms cancel, while the normal strain terms combine to give

$$\begin{aligned} \epsilon'_{xx} + \epsilon'_{yy} &= \epsilon_{xx}(\cos^2 \theta + \sin^2 \theta) + \epsilon_{yy}(\sin^2 \theta + \cos^2 \theta) \\ &= \epsilon_{xx} + \epsilon_{yy} \end{aligned}$$

So it works again.



Calculating Principal Values Manually in 3-D

Here's how to compute the roots of a cubic equation... applied to principal values. Start with

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0$$

Recall that the resulting λ 's will be the principal values of the stress or strain tensor... and the invariants will need to be calculated up front.

The first step is to calculate two intermediate quantities, Q and R .

$$Q = \frac{3I_2 - I_1^2}{9} \quad R = \frac{2I_1^3 - 9I_1I_2 + 27I_3}{54}$$

Then calculate another intermediate quantity, θ .

$$\theta = \cos^{-1} \left(\frac{R}{\sqrt{-Q^3}} \right)$$

And finally, the principal values are

$$\epsilon_1 = 2 \sqrt{-Q} \cos\left(\frac{\theta}{3}\right) + \frac{1}{3}I_1$$

$$\epsilon_2 = 2 \sqrt{-Q} \cos\left(\frac{\theta + 2\pi}{3}\right) + \frac{1}{3}I_1$$

$$\epsilon_3 = 2 \sqrt{-Q} \cos\left(\frac{\theta + 4\pi}{3}\right) + \frac{1}{3}I_1$$

Note that minus signs are easily confused here because they are present in the original cubic invariant equation being solved.

Ref: <http://mathworld.wolfram.com/CubicFormula.html>

Summary

This principle of invariant quantities under coordinate transformations is in fact universal across all matrices that are symmetric and being transformed according to

$$\mathbf{A}' = \mathbf{Q} \cdot \mathbf{A} \cdot \mathbf{Q}^T$$

where \mathbf{A} is "any symmetric matrix."

And recall that the product of any matrix with its transpose is always a symmetric result, so this result would qualify. This is especially relevant to $\mathbf{F}^T \cdot \mathbf{F}$, whose invariants are used in the Mooney-Rivlin Law of rubber behavior. Mooney-Rivlin's Law and coefficients will be discussed on [this page](#). As an added teaser, we will see that the 3rd invariant of $\mathbf{F}^T \cdot \mathbf{F}$ for rubber always equals 1 because rubber is incompressible. So not only is it a constant, independent of coordinate transformations, but it is even a constant value, always equal to 1, independent of coordinate transformations and the state of deformation.



[Green Strains](#)

[Table of Contents](#)



[Hydrostatic & Deviatoric Strains](#)