Traction Vectors

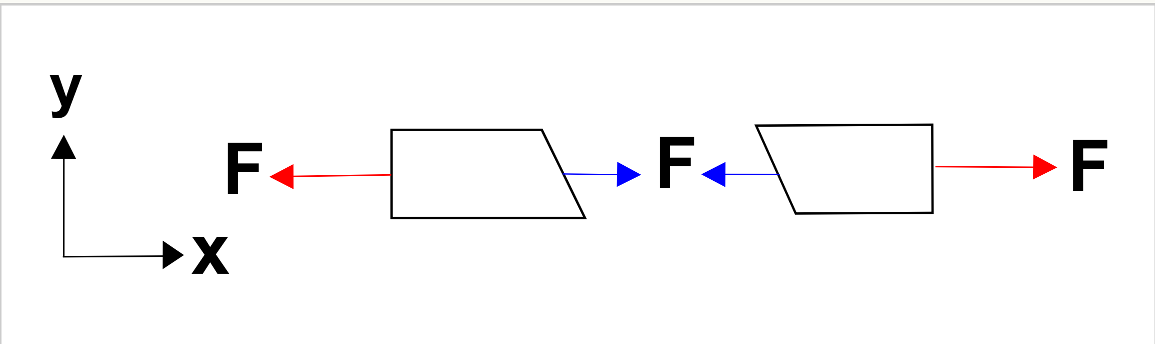
Introduction

The traction vector, $\mathbf{T}$, is simply the internal force vector on a cross-section divided by that cross-section's area.

$$\mathbf{T} = \frac{\mathbf{F}_{\text{internal}}}{\text{Area}}$$

So $\mathbf{T}$ has units of stress, like MPa, but it is absolutely a vector, not a stress tensor. So all the usual rules for vectors apply to it. For example, dot products, cross products, and coordinate transforms can be applied.

Calculating a Traction Vector



The object above has a 400 mm^2 cross sectional area and is being pulled in tension by a 4,000 N force in the x-direction. So

$$\mathbf{F}_{\text{internal}} = 4,000\,\mathbf{i}\,\text{N}$$

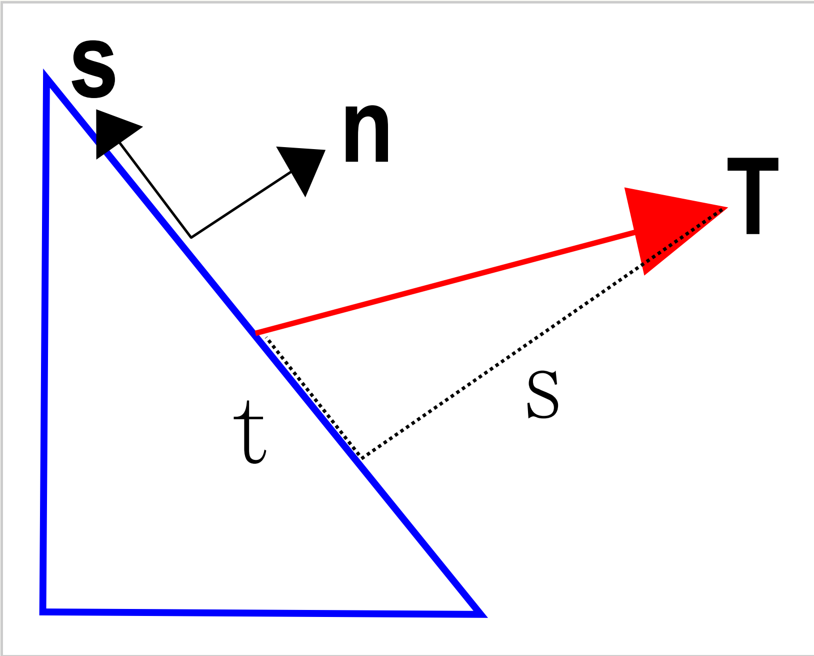
It is cut (virtually) at an angle of 30° , so the applicable area in this case is

$$A = \frac{400\ \text{mm}^2}{\cos(30°)} = 462\ \text{mm}^2$$

And the traction vector is

$$\mathbf{T} = \frac{1}{462\ \mathrm{mm}^2}\ 4{,}000\,\mathbf{i}\,\mathrm{N} = 8.66\,\mathbf{i}\,\mathrm{MPa}$$

Normal and Shear Stresses



Normal and shear stresses are simply the components of the traction vector that are normal and parallel to the area's surface as shown in the figure. Using $\mathbf{n}$ for the unit normal vector to the surface, and $\mathbf{s}$ for the unit vector parallel to it, means that

$$\sigma = \mathbf{T}\cdot\mathbf{n} \qquad \text{and} \qquad \tau = \mathbf{T}\cdot\mathbf{s}$$

It's very important to recognize that σ and τ here are each scalar values, not full tensors. This is the natural result of the dot product operations involving $\mathbf{T}$, $\mathbf{n}$, and $\mathbf{s}$. (Dot products produce scalar results.)

The normal and shear stress values here are scalars rather than tensors because they are only two individual components of the full stress tensor.

Also, note that in 3-D, there are in fact an infinite number of $\mathbf{s}$ vectors parallel to the surface, each having a different component in-and-out of the page, so to speak. This is why it is common to specify one parallel to the page and a second perpendicular to it.

Normal and Shear Stress from a Traction Vector

Recall that the traction vector from the above example was

$$\mathbf{T} = \frac{1}{462\ \mathrm{mm}^2}\ 4{,}000\,\mathbf{i}\,\mathrm{N} = 8.66\,\mathbf{i}\,\mathrm{MPa}$$

The unit normal to the surface is

$$\mathbf{n} = (\cos 30^\circ,\ \sin 30^\circ,\ 0)$$

So the normal stress on the surface is

$$\sigma = \mathbf{T} \cdot \mathbf{n} = (8.66,\ 0,\ 0) \cdot (\cos 30^\circ,\ \sin 30^\circ,\ 0) = 7.5\,\mathrm{MPa}$$

The vector parallel to the surface is

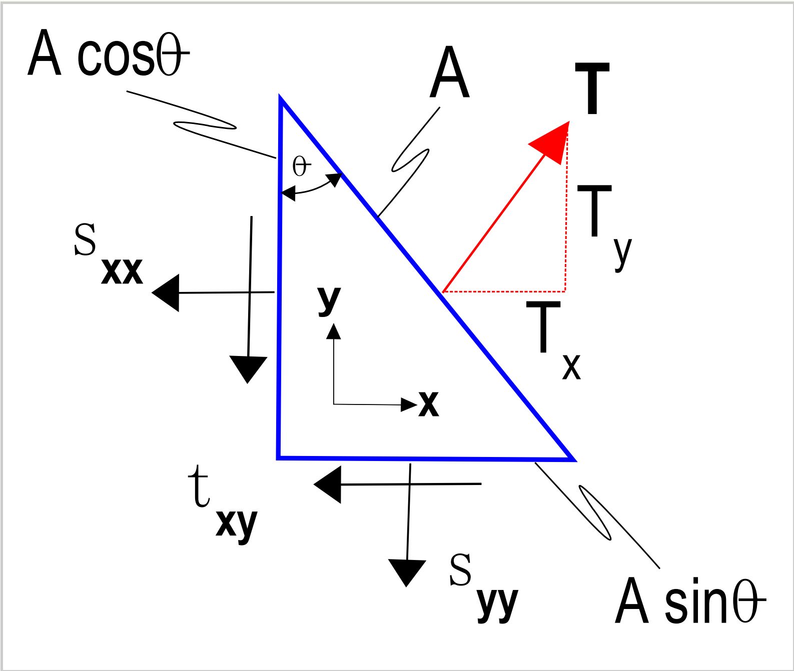
$$\mathbf{s} = (-\sin 30^\circ,\ \cos 30^\circ,\ 0)$$

The shear stress on the surface is

$$\tau = \mathbf{T} \cdot \mathbf{s} = (8.66,\ 0,\ 0) \cdot (-\sin 30^\circ,\ \cos 30^\circ,\ 0) = -4.33\,\mathrm{MPa}$$

Stress Tensors and Traction Vectors

The relationship between the traction vector and stress state at a point results directly from setting the sum of forces on an object equal to zero, i.e., imposing equilibrium.



$$\sigma_{xx}\ A\ \cos\theta + \tau_{xy}\ A\ \sin\theta = T_x\ A$$

$$\tau_{xy}\ A\ \cos\theta + \sigma_{yy}\ A\ \sin\theta = T_y\ A$$

The area, A, cancels out of both sides leaving

$$\sigma_{xx}\ \cos\theta + \tau_{xy}\ \sin\theta = T_x$$

$$\tau_{xy}\ \cos\theta + \sigma_{yy}\ \sin\theta = T_y$$

but $\cos\theta$ and $\sin\theta$ are the components of the unit normal to the surface, $\mathbf{n} = (\cos\theta, \sin\theta)$, that $\mathbf{T}$ is acting on.

Replacing the $\cos\theta$ and $\sin\theta$ with n_x and n_y gives

$$\sigma_{xx}\, n_x + \tau_{xy}\, n_y = T_x$$

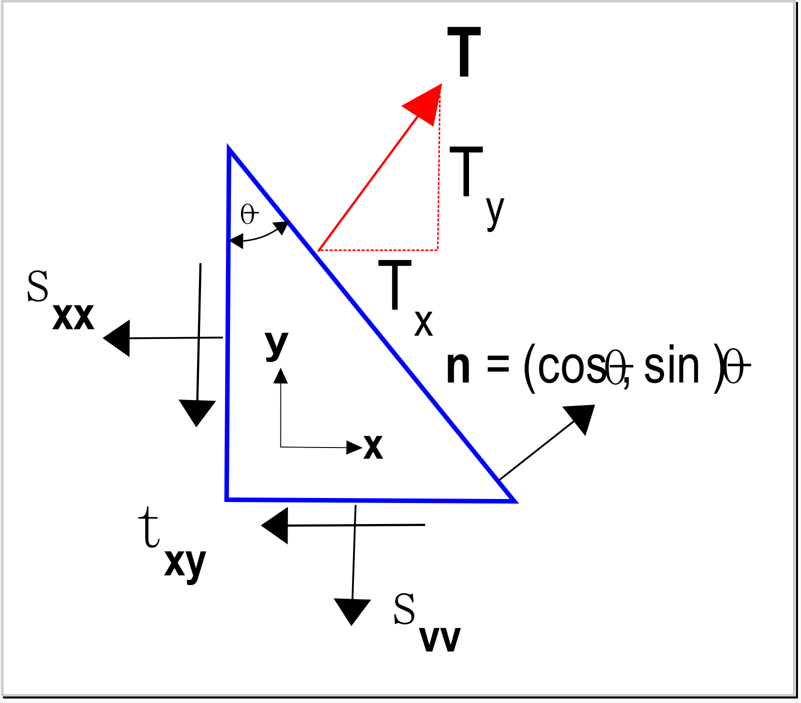
$$\tau_{xy}\, n_x + \sigma_{yy}\, n_y = T_y$$

Both equations can be summarized as

$$\mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n}$$

or in tensor notation as

$$T_i = \sigma_{ij}\, n_j$$



The above equations are very useful, compact, matrix and tensor notation representations of the equilibrium equations. The full equations, in 3-D, are

$$\sigma_{xx}\, n_x + \tau_{xy}\, n_y + \tau_{xz}\, n_z = T_x$$

$$\tau_{yx}\, n_x + \sigma_{yy}\, n_y + \tau_{yz}\, n_z = T_y$$

$$\tau_{zx}\, n_x + \tau_{zy}\, n_y + \sigma_{zz}\, n_z = T_z$$

The tensor notation term, $\sigma_{ij}\, n_j$, leads to nine separate stress components. For example, both σ_{xz} and σ_{zx} are present above, and both are always equal. This is in fact common in all equations involving stress and strain.

Traction Vector from Stress Tensor

Given the stress tensor (in MPa)

$$\boldsymbol{\sigma} = \begin{bmatrix} 50 & 10 & 30 \\ 10 & 95 & 20 \\ 30 & 20 & 15 \end{bmatrix}$$

Calculate the traction vector on a surface with unit normal $\mathbf{n} = (0.400,\ 0.600,\ 0.693)$.

$$\begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix} = \begin{bmatrix} 50 & 10 & 30 \\ 10 & 95 & 20 \\ 30 & 20 & 15 \end{bmatrix} \begin{Bmatrix} 0.400 \\ 0.600 \\ 0.693 \end{Bmatrix} = \begin{Bmatrix} 46.79 \\ 74.86 \\ 34.40 \end{Bmatrix}$$

So $\mathbf{T} = 46.79\,\mathbf{i} + 74.86\,\mathbf{j} + 34.40\,\mathbf{k}\,\mathrm{MPa}$.

If the area is 100 mm^2, then the force on it would be $\mathbf{F} = 4,679\,\mathbf{i} + 7,486\,\mathbf{j} + 3,440\,\mathbf{k}\,\mathrm{N}$.

Stress Transforms

This section introduces an aspect of coordinate transformations of stress tensors that is a subset of the general case, which comes later. It does so by combining different equations involving the traction vector.

Recall that the normal and shear stresses on a surface are related to the traction vector by

$$\sigma = \mathbf{T} \cdot \mathbf{n} \qquad \text{and} \qquad \tau = \mathbf{T} \cdot \mathbf{s}$$

Recall that the normal and shear stresses here are just scalar quantities on the surface, not a full stress tensor.

But we also saw that the traction vector is related to the full stress tensor by

$$\mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n}$$

Substituting this equation for $\mathbf{T}$ into the above ones gives

$$\sigma = \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} \qquad \text{and} \qquad \tau = \mathbf{s} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}$$

In tensor notation, the equations are

$$\sigma = \sigma_{ij}\, n_i\, n_j \qquad \text{and} \qquad \tau = \sigma_{ij}\, s_i\, n_j$$

These represent very useful relationships between the stress tensor in the global coordinate system and the normal and shear stress components at <u>any</u> other orientation.

Stress Transform Example

Recall the above stress tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} 50 & 10 & 30 \\ 10 & 95 & 20 \\ 30 & 20 & 15 \end{bmatrix}$$

We had calculated the traction vector on a surface with unit normal $\mathbf{n} = (0.400,\, 0.600,\, 0.693)$. This time, calculate the normal and shear stresses on this surface.

The normal stress on the surface is

$$\begin{aligned} \sigma &= \{0.400 \;\; 0.600 \;\; 0.693\} \begin{bmatrix} 50 & 10 & 30 \\ 10 & 95 & 20 \\ 30 & 20 & 15 \end{bmatrix} \begin{Bmatrix} 0.400 \\ 0.600 \\ 0.693 \end{Bmatrix} \\ &= 87.47 \text{ MPa} \end{aligned}$$

In order to compute a shear stress, we first need a specific one of an infinite number of unit vectors parallel to the surface. Let's choose $\mathbf{s} = (-0.832,\, 0.555,\, 0.000)$. A dot product will verify that this vector is perpendicular to $\mathbf{n}$.

$$\tau = \{-0.832\ \ 0.555\ \ 0.000\} \begin{bmatrix} 50 & 10 & 30 \\ 10 & 95 & 20 \\ 30 & 20 & 15 \end{bmatrix} \begin{Bmatrix} 0.400 \\ 0.600 \\ 0.693 \end{Bmatrix}$$

$$= 2.62 \text{ MPa}$$

So there is very little shear on this face <u>in the given s direction</u>. But this doesn't mean that there is no shear on the face at all. To see this, choose a second direction parallel to the surface and perpendicular to the first s. Obtain this by crossing the unit normal vector with the first tangential vector.

$$\mathbf{n} \times \mathbf{s} = (0.400\,\mathbf{i} + 0.600\,\mathbf{j} + 0.693\,\mathbf{k}) \times (-0.832\,\mathbf{i} + 0.555\,\mathbf{j} + 0.000\,\mathbf{k})$$

$$= -0.385\,\mathbf{i} - 0.576\,\mathbf{j} + 0.721\,\mathbf{k}$$

So the shear in the direction perpendicular to the first is

$$\tau = \{\text{-}0.385\ \ \text{-}0.576\ \ 0.721\} \begin{bmatrix} 50 & 10 & 30 \\ 10 & 95 & 20 \\ 30 & 20 & 15 \end{bmatrix} \begin{Bmatrix} 0.400 \\ 0.600 \\ 0.693 \end{Bmatrix}$$

$$= -36.33 \text{ MPa}$$

So there is a good bit of shear stress in this perpendicular direction. And the negative value indicates that it is in the direction opposite of the s direction.

Transformation Tip

This transformation "trick" could be used to compute the normal and shear stresses on all six faces of a cube at any random orientation, and in the process, perform a complete coordinate transformation of a stress tensor. But it's actually easier to do $\boldsymbol{\sigma}' = \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T$ just as is the case for strain tensors.

But the opposite is also true. The stress tensor can be replaced with the strain tensor to obtain

$$\epsilon_{\text{normal}} = \mathbf{n} \cdot \boldsymbol{\epsilon} \cdot \mathbf{n} \qquad \text{and} \qquad \gamma/2 = \mathbf{s} \cdot \boldsymbol{\epsilon} \cdot \mathbf{n}$$

Or in tensor notation as

$$\epsilon_{\text{normal}} = \epsilon_{ij}\, n_i\, n_j \qquad \text{and} \qquad \gamma/2 = \epsilon_{ij}\, s_i\, n_j$$

This works because since both stress and strain are tensors, then any math operation that applies to one also applies to the other.

For a quick math review, note that $s_i\, n_j$ in the above equations can be interpreted as a diadic product of the two vectors, $\mathbf{s} \otimes \mathbf{n}$. And then this result is "double dotted" with the stress or strain tensor to obtain the final scalar shear value. So the calculation could be written as

$$\tau = \mathbf{s} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} = \boldsymbol{\sigma} : (\mathbf{s} \otimes \mathbf{n})$$

(The same could also be done to compute the normal stress as well.)

This diadic product for shear arises so often in metal plasticity that it is represented by the single letter $\mathbf{p}$, and named the Schmidt tensor after the engineer who studied metal plasticity in the early 1900's.

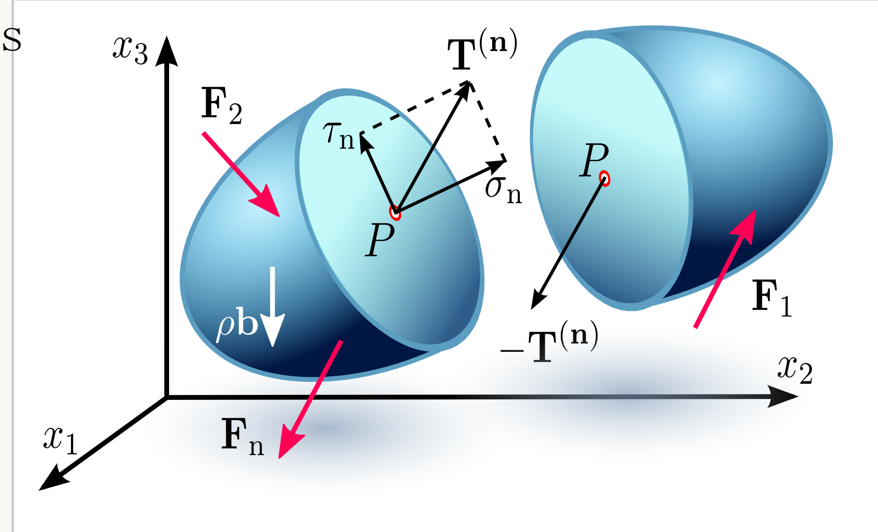
$$\mathbf{p} = \mathbf{s} \otimes \mathbf{n} = \begin{bmatrix} s_1 n_1 & s_1 n_2 & s_1 n_3 \\ s_2 n_1 & s_2 n_2 & s_2 n_3 \\ s_3 n_1 & s_3 n_2 & s_3 n_3 \end{bmatrix}$$

Forces on Cross Sections

Before closing, recall one more time that the force on a cross-section is

$$\mathbf{F} = \int \mathbf{T}\, dA = \int \boldsymbol{\sigma} \cdot \mathbf{n}\, dA$$

Both forms turn up often in literature.



Ref:
http://commons.wikimedia.org/wiki/File:stress_vector.svg
