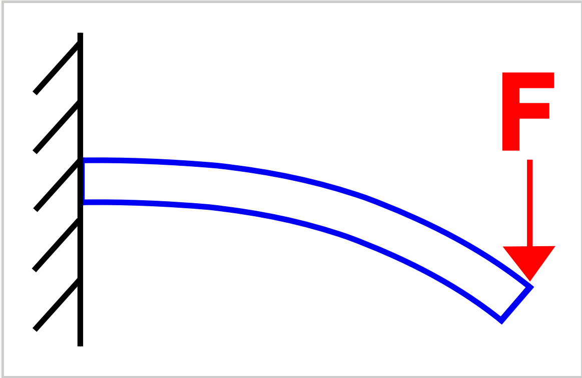
Beam Bending

Search Continuum Mechanics Website

Search

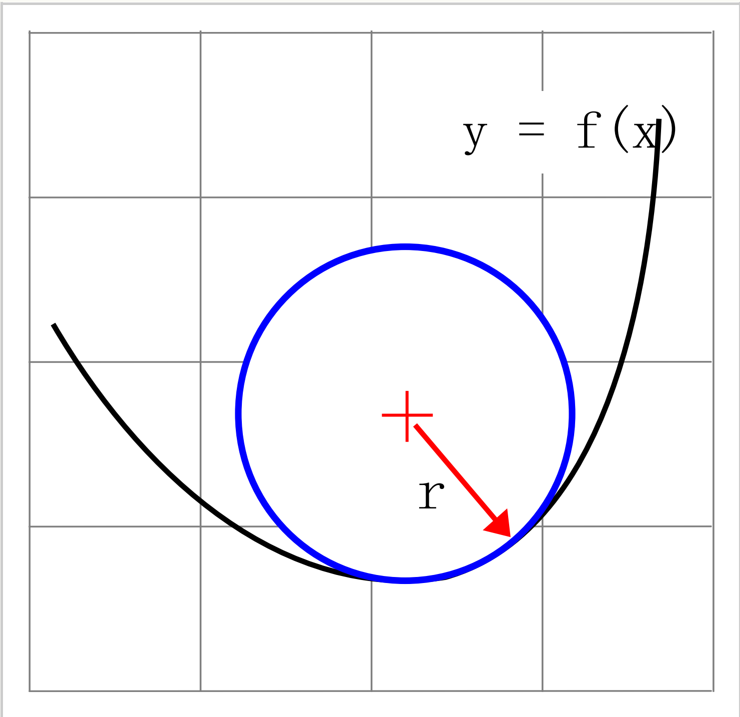
Introduction



This page reviews classical beam bending theory, which is an important consideration in nearly all structural designs and analyses. Though less apparent, it is also relevant to column buckling as well. And that is in fact the second motive behind this page, to lay the foundation for the upcoming discussion of column buckling theory.

The intent here is not to cover all aspects of beam bending. In particular, topics such as determining the neutral axis, the parallel axis theorem, and computing beam deflections are not covered.

Radius of Curvature



The radius of curvature is fundamental to beam bending, so it will be reviewed here. It is usually represented by the Greek letter ρ, and can be thought of as the radius of a circle having the same curvature as a portion of the graph, a curve in the road, or most any other path. When the path is straight, ρ is infinite, and when the path has a sharp curve in it, ρ is small.

We will eventually need an analytical expression for radius of curvature, so it will be developed here. Start with any function, $y = y(x)$, as shown in the figure. Recall that arc length, s, is related to ρ through $\rho\theta = s$, where θ is the angle of the arc. In differential form, this is $\rho\, d\theta = ds$. Now divide both sides by dx.

$$\rho \frac{d\theta}{dx} = \frac{ds}{dx}$$

We need expressions for $\frac{d\theta}{dx}$ and $\frac{ds}{dx}$. Starting with $\frac{d\theta}{dx}$, recall that

$$\tan(\theta) = \frac{dy}{dx}$$

so

$$\theta = \mathbf{Tan}^{-1}(y')$$

Differentiate with respect to x to get $\frac{d\theta}{dx}$.

$$\frac{d\theta}{dx} = \frac{y''}{1 + (y')^2}$$

To obtain $\frac{ds}{dx}$, start with

$$ds^2 = dx^2 + dy^2$$

Divide both sides by dx^2

$$\frac{ds^2}{dx^2} = 1 + \frac{dy}{dx}^2$$

and take the square root of both sides

$$\frac{ds}{dx} = \sqrt{1 + (y')^2}$$

Combining all this together gives

$$\rho \frac{y''}{1 + (y')^2} = \sqrt{1 + (y')^2}$$

And rearrange a little to obtain the popular expression.

$$\frac{1}{\rho} = \frac{y''}{\left[1 + (y')^2\right]^{3/2}}$$

Interestingly, any expression involving the radius of curvature seems to always have it appear in the denominator. And this is no exception, even when it is a defining equation.

Also interesting is the fact that many mechanics applications involve bending, but on a small scale. The beam bending discussed here is no exception. In such cases, the best approach is to define the x-axis along the beam such so that the y deflections, and more importantly the deformed slope, y', will both be small. If $y' << 1$, then y' can be neglected in the above equation. This produces a much simpler expression.

$$\frac{1}{\rho} \approx y''$$

3-D Radius of Curvature

A very handy, and general 3-D expression for ρ is

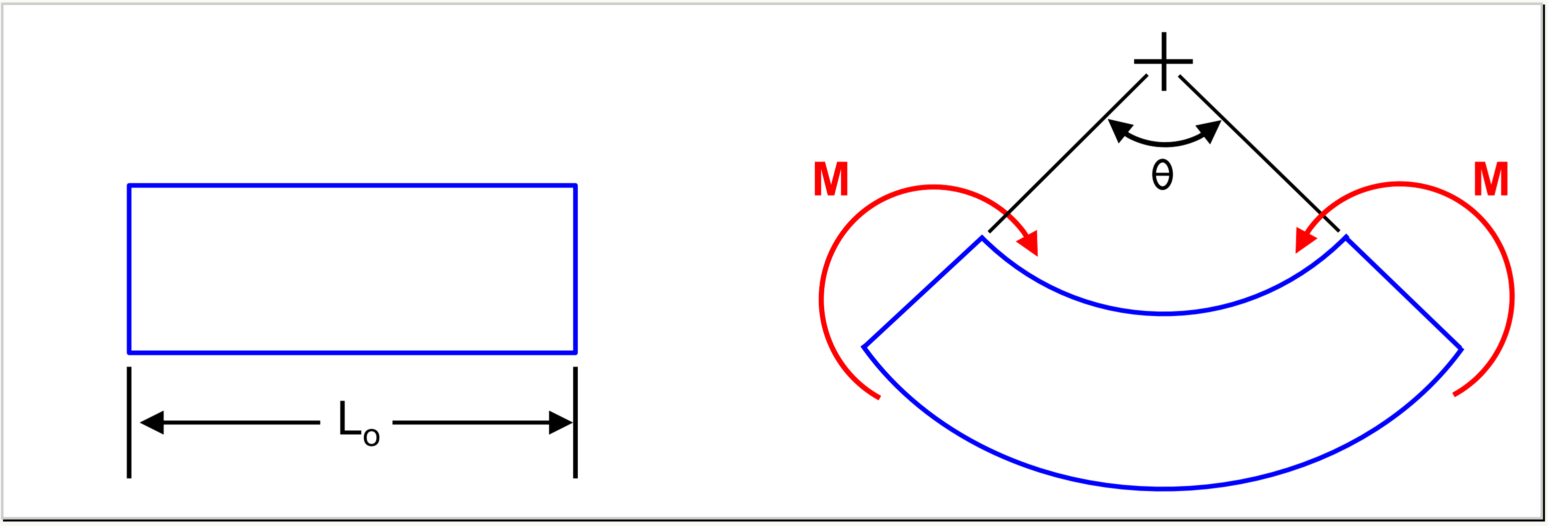
$$\frac{1}{\rho} = \frac{|\mathbf{v}' \times \mathbf{v}''|}{|\mathbf{v}'|^3}$$

where $\mathbf{v}$ is a vector defined parametrically as $\mathbf{v} = \mathbf{v}(x(t), y(t), z(t))$, $\mathbf{v}'$ is its first derivative, $\mathbf{v}''$ is its second derivative, and $|\ldots|$ represents the length of a vector, i.e., the square root of the sum of the squares of its components.

Bending Strains

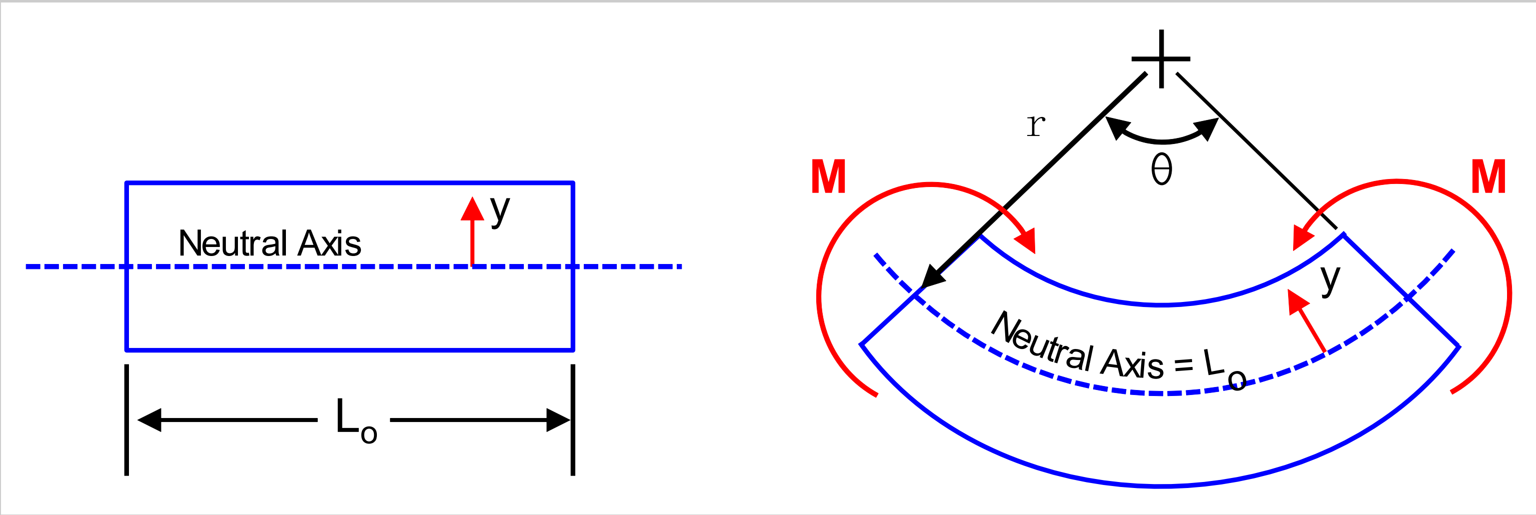
Recall that arc length, L, is related to radius of curvature, ρ, through $L = \rho\theta$, where θ is the angle.

Things become more complex when thickness is considered. In the figure below, an object of initial length, L_o, is bent as shown. Since it has finite thickness, different portions of it are stretched, or compressed, different amounts. The outer portion of a beam is stretched the most because it is farthest from that center. Mathematically, all portions are bent the same angle, θ, but ρ varies through the thickness, so the quantity $\rho\theta$ varies too, and therefore L varies as well.



The next step is to make a concious decision to avoid the confusion of having many different radii of curvature through the thickness of a bent object. This is accomplished in two steps.

First, find the one ρ that satisfies $\rho\theta = L_o$. Note that ρ is the computed result here and θ and L_o are the inputs. Note also that the length in the equation is L_o, the original, undeformed length, not the deformed one. This step establishes one unique value of ρ for a cross-section, rather than having multiple values, which could lead to much confusion.



Neutral Axis

The location in the cross-section where $\rho\theta = L_o$ is known as the Neutral Axis. It is the one place where the final deformed length is the same as the original undeformed length, so no stretching takes place... due to bending. Do not assume that the neutral axis must be in the middle of the object's cross-section. That is not necessarily the case, particularly if the object is a composite of different materials with different stiffnesses.

The "due to bending" caveat in the above paragraph is present because the object may also be simultaneously loaded in tension (or compression) that stretches it until every point in its cross-section is longer (or shorter if compressed) than the original length, L_o.

The second step is to introduce the variable y as the distance from the neutral axis to any other radius in the cross-section as shown in the figure below. As a result, the radius of curvature at any y is $(\rho - y)$ and the final length at any y is given by

$$L = (\rho - y)\theta$$

Recall that $L_o = \rho\theta$. Now the strain, ϵ, can be expressed as

$$\epsilon_x = \frac{L - L_o}{L_o} = \frac{(\rho - y)\theta - \rho\theta}{\rho\theta}$$

which simplifies to

$$\epsilon_x = -\frac{y}{\rho}$$

This is a key result for the strain in the object. It shows that the strain is zero at $y = 0$, the neutral axis, and varies linearly from it. If the object is thick, then y can take on large values, but in thin objects, it does not. This is fundamentally why thick objects have more bending stiffness than thin objects.

Also, the radius of curvature in the denomiator accounts for many effects of bending. When the object is not bent, then ρ is infinite, and the strains are naturally zero. As the object bends, ρ decreases and the equation shows that the strains will increase.

Finally, note that the strain is a normal strain and is in fact longitudinal, along the length of the beam. It is common to align the x-axis along the of beam's length, making the strain ϵ_x.

Bending Stress

Now that we have an expression for strain, developing one for stress could not be simpler. Multiply the strain by E, the elastic modulus, to obtain stress, σ_x.

$$\sigma_x = -\frac{E y}{\rho}$$

Although this step was increadibly simple, it is in fact rather profound in what has been neglected by the simplicity. Recall from the page on Hooke's Law that each normal stress component is dependent on all three normal strain components. But here, we have simply multiplied the strain by E to obtain stress. This step has a key assumption built into it... that there are no lateral loads/stresses acting on the beam. In such cases, the equations "work out" so that $\sigma_x = E\epsilon_x$ as in uniaxial tension. This occurs in most beams because they are thin relative to their length.

Bending in Plates

In plates, this step would be more complex because a plate is thin in only one direction, not two, relative to its length. The non-thin direction resists the Poisson Effect associated with tension and leads to

$$\sigma_x = \frac{E}{(1-\nu^2)}\epsilon_x$$

where ν is the materials Poisson ratio. It is clear that the stress for a given strain is higher in this case than for classical uniaxial tension.

Bending Moments and Stress

The bending moment, M_z, on the cross-section due to the stress field is computed by

$$M_z = \int_A r \times dF = -\int_A y\sigma_x dA$$

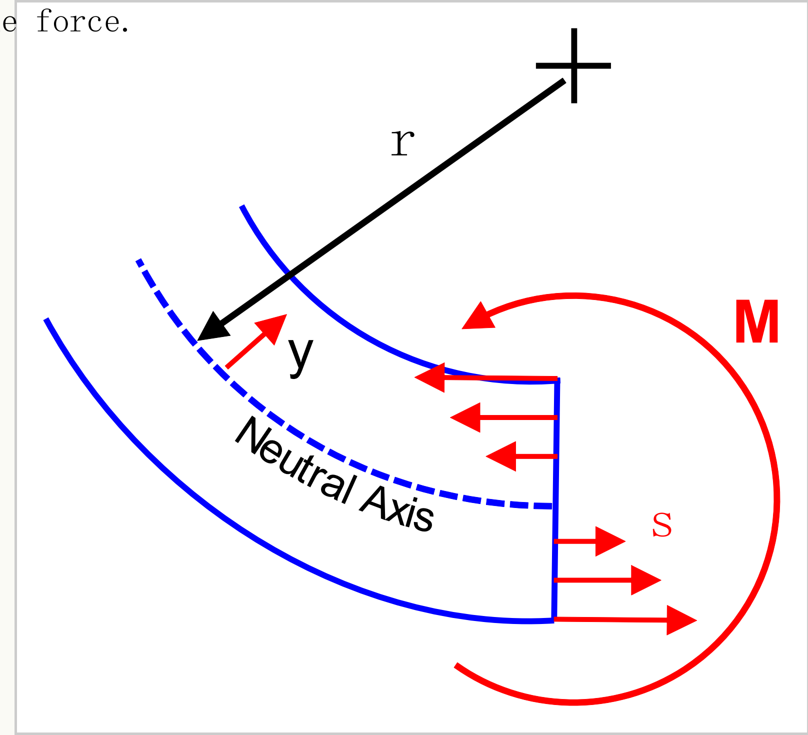
where y is the moment arm and $\sigma_x dA$ is the force.

Recall that $\sigma_x = -Ey/\rho$ and insert into the moment equation to obtain

$$M_z = -\int_A y \left[\frac{-Ey}{\rho}\right] dA = \frac{E}{\rho}\int_A y^2 dA$$

The remaining integral is very important. Note that it is entirely geometric. It is called the Moment of Inertia, I_{zz}, of the cross-section.

$$I_{zz} = \int_A y^2 dA$$



Always keep in mind that the y values are measured from the neutral axis, which corresponds to $y = 0$.

The bending moment equation can now be written as

$$M_z = \frac{E I_{zz}}{\rho}$$

which is a very important equation.

Small Bending Approximation

Recall that we discussed earlier that when the bending level is small and $y' << 1$, then ρ can be closely approximated by y''. This gives

$$M_z = E I_{zz} y'' \qquad \text{(when } y' << 1\text{)}$$

which is another very important and useful equation.

Now... go back to these two equations

$$M_z = \frac{E I_{zz}}{\rho} \qquad \text{and} \qquad \sigma_x = -\frac{Ey}{\rho}$$

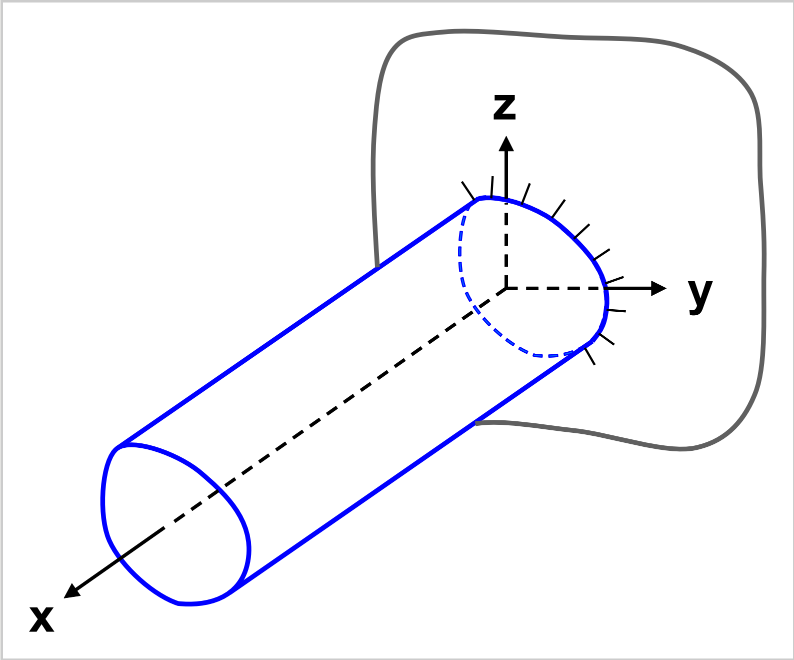
and recognize that both contain E/ρ. Solving each equation for this ratio gives

$$\frac{E}{\rho} = \frac{M_z}{I_{zz}} \qquad \text{and} \qquad \frac{E}{\rho} = -\frac{y}{\sigma_x}$$

So equate the two to obtain one of the best known equations in Mechanical Engineering.

$$\sigma_x = -\frac{M_z\, y}{I_{zz}}$$

3-D Bending



Full 3-D bending involves deflections in two directions, y and z. The strain, ϵ_x, now depends on both coordinates.

$$\epsilon_x = \frac{z}{\rho_y} - \frac{y}{\rho_z}$$

Multiply through by E to obtain stress.

$$\sigma_x = \frac{E\, z}{\rho_y} - \frac{E\, y}{\rho_z}$$

The bending moment, M_y, is calculated by integrating the stress over the cross-section with z as the moment arm.

$$M_y \;=\; \int_A z\,\sigma dA \;=\; \frac{E}{\rho_y}\int_A z^2 dA - \frac{E}{\rho_z}\int_A y\, z\, dA$$

$$= \;\frac{E}{\rho_y} I_{yy} - \frac{E}{\rho_z} I_{yz}$$

And the other moment, M_z, is calculated by integrating the stress over the cross-section with y as the moment arm.

$$M_z = -\int_A y\,\sigma dA = \frac{E}{\rho_z}\int_A y^2 dA - \frac{E}{\rho_y}\int_A y\,z\,dA$$

$$= \frac{E}{\rho_z} I_{zz} - \frac{E}{\rho_y} I_{yz}$$

The equations can be written compactly in matrix form as

$$\begin{Bmatrix} M_y \\ M_z \end{Bmatrix} = \begin{bmatrix} I_{yy} & -I_{yz} \\ -I_{yz} & I_{zz} \end{bmatrix} \begin{Bmatrix} E/\rho_y \\ E/\rho_z \end{Bmatrix}$$

This reveals that the moment of inertia is actually a tensor.

It is more useful to invert the equations to obtain the radii of curvature as functions of the moments. Doing so gives

$$\frac{E}{\rho_y} = \frac{M_y I_{zz} + M_z I_{yz}}{I_{yy} I_{zz} - (I_{yz})^2}$$

$$\frac{E}{\rho_z} = \frac{M_z I_{yy} + M_y I_{yz}}{I_{yy} I_{zz} - (I_{yz})^2}$$

And recall that

$$\sigma_x = \frac{E}{\rho_y} z - \frac{E}{\rho_z} y$$

Inserting the above equations for E/ρ_y and E/ρ_z gives

$$\sigma_x = \left[\frac{M_y I_{zz} + M_z I_{yz}}{I_{yy} I_{zz} - (I_{yz})^2}\right] z - \left[\frac{M_z I_{yy} + M_y I_{yz}}{I_{yy} I_{zz} - (I_{yz})^2}\right] y$$

and this can be written in compact matrix form as

$$\sigma_x = \frac{1}{\det(\mathbf{I})} \{-y \quad z\} \begin{bmatrix} I_{yy} & I_{yz} \\ I_{yz} & I_{zz} \end{bmatrix} \begin{Bmatrix} M_z \\ M_y \end{Bmatrix}$$

where $\det(\mathbf{I}) = I_{yy}I_{zz} - (I_{yz})^2$.

Finally, if $I_{yz} = 0$ things greatly simplify to

$$\sigma_x = \frac{M_y\, z}{I_{yy}} - \frac{M_z\, y}{I_{zz}}$$