Matrices & Tensors

Search

home > basic math > matrices & tensors







Introduction

- If it's a physical quantity, like stress, then it's usually called a tensor. If it's not a physical quantity, then it's usually called a matrix.
- The vast majority of tensors are symmetric. One common quantity that is not symmetric, and not referred to as a tensor, is a rotation matrix.
- Tensors are in fact any physical quantity that can be represented by a scalar, vector, or matrix. Zero-order tensors, like mass, are called scalars, while 1st order tensors are called vectors. Examples of higher order tensors include stress, strain, and stiffness tensors.
- The order, or rank, of a matrix or tensor is the number of subscripts it contains. A vector is a 1st rank tensor. A 3x3 stress tensor is 2nd rank.
- Coordinate Transformations of tensors are discussed in detail here.

Identity Matrix

The identity matrix is

$$\mathbf{I} = egin{bmatrix} 1 & 0 & 0 \ 0 & 1 & 0 \ 0 & 0 & 1 \end{bmatrix}$$

Multiplying anything by the identity matrix is like multiplying by one.

Tensor Notation

The identity matrix in tensor notation is simply δ_{ij} . It is the Kronecker Delta that equals 1 when i=j and 0 otherwise.



Is It a Matrix or Not?

A note from the purests... The identity matrix is a matrix, but the Kronecker delta technically is not. δ_{ij} is a single scalar value that is either 1 or 0 depending on the values of i and j. This is also why tensor notation is not in bold, because it always refers to individual components of tensors, but never to a tensor as a whole.

Follow this link for an entertaining discussion between someone who gets it, and someone else who doesn't.



Kronecker Delta verses Dirac Delta

Don't confuse the Kronecker delta, δ_{ij} , with the Dirac delta, $\delta_{(t)}$. The Dirac delta is something totally different. It is often used in signal processing and equals 0 for all t except t=0. At t=0, it approaches ∞ such that

$$\int_{-\infty}^{\infty}f(t)\delta_{(t)}dt=f(0)$$

Transpose

The transpose of a matrix mirrors its components about the main diagonal. The transpose of matrix \mathbf{A} is written \mathbf{A}^T .



Transpose Example

$$egin{aligned} ext{If} & \mathbf{A} = egin{bmatrix} 1 & 2 & 3 \ 4 & 5 & 6 \ 7 & 8 & 9 \end{bmatrix}, & ext{then} & \mathbf{A}^T = egin{bmatrix} 1 & 4 & 7 \ 2 & 5 & 8 \ 3 & 6 & 9 \end{bmatrix} \end{aligned}$$

Tensor Notation

The transpose of A_{ij} is A_{ji} .

Determinants

The determinant of a matrix is written as $\det(\mathbf{A})$ or $|\mathbf{A}|$, and is computed as

$$|\mathbf{A}| \; = egin{array}{cccc} A_{11} & A_{12} & A_{13} \ A_{21} & A_{22} & A_{23} \ A_{31} & A_{32} & A_{33} \ \end{array}$$

$$= A_{11}(A_{22}A_{33} - A_{23}A_{32}) + A_{12}(A_{23}A_{31} - A_{21}A_{33}) + A_{13}(A_{21}A_{32} - A_{22}A_{31})$$

If the determinant of a tensor, or matrix, is zero, then it does not have an inverse.

Tensor Notation

The calculation of a determinant can be written in tensor notation in a couple different ways

$$\det(\mathbf{A}) \ = \ \epsilon_{ijk} A_{i1} A_{j2} A_{k3} \ = \ \frac{1}{6} \epsilon_{ijk} \epsilon_{rst} A_{ir} A_{js} A_{kt}$$



Determinant Example

If
$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ 5 & 7 & 4 \\ 9 & 6 & 8 \end{bmatrix}$$
, then its determinant is

$$|\mathbf{A}| = 3*(7*8-4*6) + 2*(4*9-5*8) + 1*(5*6-7*9)$$
 $= 3*32 + 2*(-4) + 1*(-33)$
 $= 96 - 8 - 33$
 $= 55$

The determinant of the product of two matrices is the same as the product of the determinants of the two matrices. In other words,

$$\det(\mathbf{A} \cdot \mathbf{B}) = \det(\mathbf{A}) * \det(\mathbf{B})$$

The determinant of a deformation gradient gives the ratio of initial to final volume of a differential element.

Inverses

The inverse of matrix \mathbf{A} is written as \mathbf{A}^{-1} and has the following very important property (see the section on matrix multiplication below)

$$\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$$

If \mathbf{B} is the inverse of \mathbf{A} , then

$$\begin{array}{lll} B_{11} &=& \left(A_{22}A_{33} - A_{23}A_{32}\right) \, / \, \det(\mathbf{A}) \\ B_{12} &=& \left(A_{13}A_{32} - A_{12}A_{33}\right) \, / \, \det(\mathbf{A}) \\ B_{13} &=& \left(A_{12}A_{23} - A_{13}A_{22}\right) \, / \, \det(\mathbf{A}) \\ B_{21} &=& \left(A_{23}A_{31} - A_{21}A_{33}\right) \, / \, \det(\mathbf{A}) \\ B_{22} &=& \left(A_{11}A_{33} - A_{13}A_{31}\right) \, / \, \det(\mathbf{A}) \\ B_{23} &=& \left(A_{13}A_{21} - A_{11}A_{23}\right) \, / \, \det(\mathbf{A}) \\ B_{31} &=& \left(A_{21}A_{32} - A_{22}A_{31}\right) \, / \, \det(\mathbf{A}) \\ B_{32} &=& \left(A_{12}A_{31} - A_{11}A_{32}\right) \, / \, \det(\mathbf{A}) \\ B_{33} &=& \left(A_{11}A_{22} - A_{12}A_{21}\right) \, / \, \det(\mathbf{A}) \end{array}$$



Inverse Example

If
$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$
 then $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0.5 & -1 \\ -6 & -1 & 5 \\ 4 & 0.5 & -3 \end{bmatrix}$ because

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & -1 \\ -6 & -1 & 5 \\ 4 & 0.5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & -1 \\ -6 & -1 & 5 \\ 4 & 0.5 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Tensor Notation

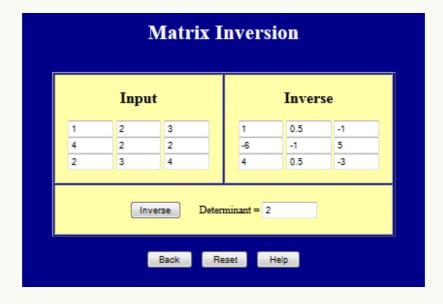
The inverse of A_{ij} is often written as A_{ij}^{-1} . Note that this is probably not rigorously correct since, as discussed earlier, neither A_{ij} nor A_{ij}^{-1} are technically matrices themselves. They are only components of a matrix. Oh well...

The inverse can be calculated using

$$A_{ij}^{-1} = rac{1}{2\det(\mathbf{A})} \epsilon_{jmn} \, \epsilon_{ipq} A_{mp} A_{nq}$$

Matrix Inverse Webpage

This page calculates the inverse of a 3x3 matrix.



Transposes of Inverses of Transposes of...

The inverse of a transpose of a matrix equals the transpose of an inverse of the matrix. Since the order doesn't matter, the double operation is abbreviated simply as A^{-T} .

$$A^{-T}=\left(A^{-1}
ight)^T=\left(A^T
ight)^{-1}$$

Transpose / Inverse Example

If
$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$
, then

$$\left(A^{-1}
ight)^T : \qquad egin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix} \qquad \overrightarrow{\operatorname{Inverse}} \qquad egin{bmatrix} 1 & 0.5 & -1 \\ -6 & -1 & 5 \\ 4 & 0.5 & -3 \end{bmatrix} \qquad \overrightarrow{\operatorname{Transpose}} \qquad egin{bmatrix} 1 & -6 & 4 \\ 0.5 & -1 & 0.5 \\ -1 & 5 & -3 \end{bmatrix}$$

So
$$\left(A^{-1}
ight)^T$$
 does indeed equal $\left(A^T
ight)^{-1}.$

Matrix Addition

Matrices and tensors are added component by component just like vectors. This is easily expressed in tensor notation.

$$C_{ij} = A_{ij} + B_{ij}$$

Matrix Multiplication (Dot Products)

The dot product of two matrices multiplies each row of the first by each column of the second. Products are often written with a dot in matrix notation as $\mathbf{A} \cdot \mathbf{B}$, but sometimes written without the dot as \mathbf{AB} . Multiplication rules are in fact best explained through tensor notation.

$$C_{ij} = A_{ik}B_{kj}$$

(Note that no dot is used in tensor notation.) The k in both factors automatically implies

$$C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + A_{i3}B_{3j}$$

which is the ith row of the first matrix multiplied by the jth column of the second matrix. If, for example, you want to compute C_{23} , then i=2 and j=3, and

$$C_{23} = A_{21}B_{13} + A_{22}B_{23} + A_{23}B_{33}$$

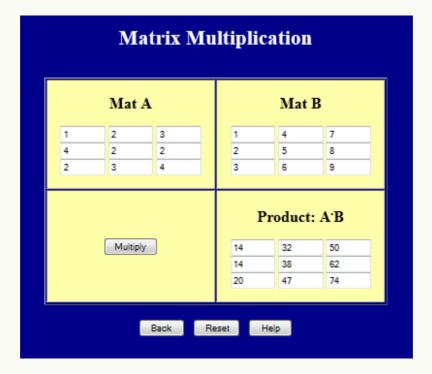


Dot Product Example

If
$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$
 and $\mathbf{B} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ then $\mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 14 & 32 & 50 \\ 14 & 38 & 62 \\ 20 & 47 & 74 \end{bmatrix}$$

This page calculates the dot product of two 3x3 matrices.





Tensor Notation and Computer Programming

Another advantage of tensor notation is that it spells out for you how to write the computer code to do it. Note how the subscripts in the FORTRAN example below exactly match the tensor notation for $C_{ij}=A_{ik}B_{kj}$. This is true for all tensor notation operations, not just this matrix dot product.

```
subroutine aa\_dot\_bb(n, a, b, c) dimension a(n, n), b(n, n), c(n, n) do i = 1, n do j = 1, n c(i, j) = 0 do k = 1, n c(i, j) = c(i, j) + a(i, k) * b(k, j) end do end do end do return end
```

Matrix Multiplication Is Not Commutative

It is very important to recognize that matrix multiplication is NOT commutative, i.e.

$$\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$$



Non-Commutativity Example

If
$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$
 and $\mathbf{B} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

then
$$\mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 14 & 32 & 50 \\ 14 & 38 & 62 \\ 20 & 47 & 74 \end{bmatrix}$$

but
$$\mathbf{B} \cdot \mathbf{A} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 31 & 31 & 39 \\ 38 & 38 & 48 \\ 45 & 45 & 57 \end{bmatrix}$$

So it is clear that $\mathbf{A} \cdot \mathbf{B}$ is not equal to $\mathbf{B} \cdot \mathbf{A}$.

Transposes and Inverses of Products

The transpose of a product equals the product of the transposes in reverse order, and the inverse of a product equals the product of the inverses in reverse order.

Note that the "in reverse order" is critical. This is used extensively in the sections on deformation gradients and Green strains.

$$(\mathbf{A} \cdot \mathbf{B})^T = \mathbf{B}^T \cdot \mathbf{A}^T$$
 and $(\mathbf{A} \cdot \mathbf{B})^{-1} = \mathbf{B}^{-1} \cdot \mathbf{A}^{-1}$

This also applies to multiple products. For example

$$(\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C})^T = \mathbf{C}^T \cdot \mathbf{B}^T \cdot \mathbf{A}^T$$



Transpose of Products Example

If
$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$
 and $\mathbf{B} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

then
$$\mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 14 & 32 & 50 \\ 14 & 38 & 62 \\ 20 & 47 & 74 \end{bmatrix}$$

and
$$(\mathbf{A} \cdot \mathbf{B})^T = \begin{bmatrix} 14 & 14 & 20 \\ 32 & 38 & 47 \\ 50 & 62 & 74 \end{bmatrix}$$

For comparison,
$$\mathbf{B}^T \cdot \mathbf{A}^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 2 & 2 & 3 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 14 & 14 & 20 \\ 32 & 38 & 47 \\ 50 & 62 & 74 \end{bmatrix}$$

Product With Own Transpose

The product of a matrix and its own transpose is always a symmetric matrix. $\mathbf{A}^T \cdot \mathbf{A}$ and $\mathbf{A} \cdot \mathbf{A}^T$ both give symmetric, although different results. This is used extensively in the sections on

deformation gradients and Green strains.



Symmetric Product Example

If
$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$

then
$$\mathbf{A} \cdot \mathbf{A}^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 2 & 2 & 3 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 14 & 14 & 20 \\ 14 & 24 & 22 \\ 20 & 22 & 29 \end{bmatrix}$$

and
$$\mathbf{A}^T \cdot \mathbf{A} = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 2 & 3 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 21 & 16 & 19 \\ 16 & 17 & 22 \\ 19 & 22 & 29 \end{bmatrix}$$

Both results are indeed symmetric, although they are unrelated to each other.

Double Dot Products

The double dot product of two matrices produces a scalar result. It is written in matrix notation as $\mathbf{A} : \mathbf{B}$. Although rarely used outside of continuum mechanics, is in fact quite common in advanced applications of linear elasticity. For example, $\frac{1}{2}\sigma : \epsilon$ gives the strain energy density in small scale linear elasticity. Once again, its calculation is best explained with tensor notation.

$$\mathbf{A}:\mathbf{B}=A_{ij}B_{ij}$$

Since the i and j subscripts appear in both factors, they are both summed to give

$$\mathbf{A}: \mathbf{B} = A_{ij}B_{ij} = A_{11}*B_{11} + A_{12}*B_{12} + A_{13}*B_{13} + A_{21}*B_{21} + A_{22}*B_{22} + A_{23}*B_{23} + A_{31}*B_{31} + A_{32}*B_{32} + A_{33}*B_{33}$$



Double Dot Product Example

If
$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$
 and $\mathbf{B} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ then

$$\mathbf{A}: \mathbf{B} = 1*1 + 2*4 + 3*7 + 4*2 + 2*5 + 2*8 + 2*3 + 3*6 + 4*9$$



Table of Contents

