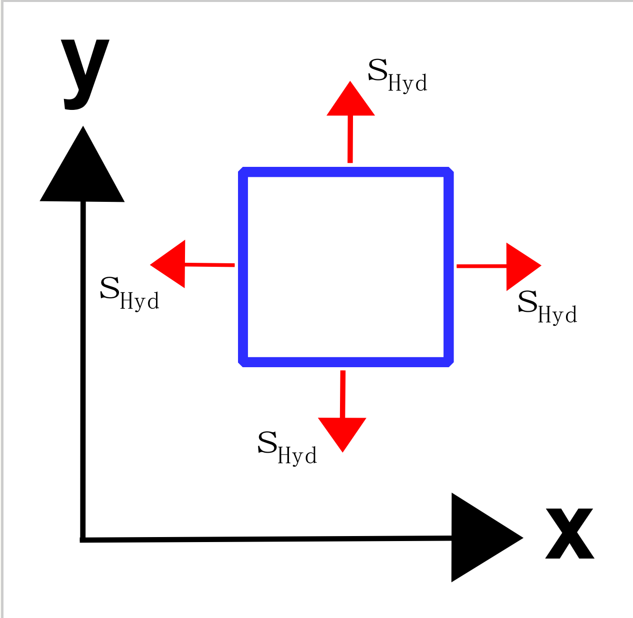
Hydrostatic & Deviatoric Stresses

Introduction

This page introduces hydrostatic and deviatoric stresses. The two are subsets of any given stress tensor, which, when added together, give the original stress tensor back. The hydrostatic stress is related to volume change, while the deviatoric stress is related to shape change.



Hydrostatic Stress

Hydrostatic stress is simply the average of the three normal stress components of any stress tensor.

$$\sigma_{\text{Hyd}} = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}$$

There are many alternative ways to write this.

$$\sigma_{\text{Hyd}} = \frac{1}{3}\text{tr}(\boldsymbol{\sigma}) = \frac{1}{3}I_1 = \frac{1}{3}\sigma_{kk}$$

It is a scalar quantity, although it is regularly used in tensor form as

$$\boldsymbol{\sigma}_{\text{Hyd}} = \begin{bmatrix} \sigma_{\text{Hyd}} & 0 & 0 \\ 0 & \sigma_{\text{Hyd}} & 0 \\ 0 & 0 & \sigma_{\text{Hyd}} \end{bmatrix}$$

Hydrostatic Stress Example

Given the following stress tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} 50 & 30 & 20 \\ 30 & -20 & -10 \\ 20 & -10 & 10 \end{bmatrix}$$

The hydrostatic stress is

$$\sigma_{\text{Hyd}} = \frac{50 + (-20) + 10}{3} = 13.3$$

which can be written as

$$\boldsymbol{\sigma}_{\text{Hyd}} = \begin{bmatrix} 13.3 & 0 & 0 \\ 0 & 13.3 & 0 \\ 0 & 0 & 13.3 \end{bmatrix}$$

And that's all there is to it.

Hydrostatic Stresses and Coordinate Transformations

This could not be simpler. Hydrostatic stresses do not change under coordinate transformations. This is easily accepted in light of the fact that σ_{Hyd} is a function of I_1. Also

$$\boldsymbol{\sigma}_{\text{Hyd}} = \begin{bmatrix} \sigma_{\text{Hyd}} & 0 & 0 \\ 0 & \sigma_{\text{Hyd}} & 0 \\ 0 & 0 & \sigma_{\text{Hyd}} \end{bmatrix}$$

contains equal amounts of stress in all three directions. And since the tensor does not change under any transformation, this means that no shear stresses ever arise, so every direction is a principal direction with σ_{Hyd} stress.

Hydrostatic Stress and Pressure

Pressure is simply the negative of hydrostatic stress. The negative aspect is often confusing. It is why we talk about atmospheric pressure as 30 inches of Hg, a positive number, even though atmospheric pressure is in fact a negative stress because it is compressive. So using P for pressure...

$$P = -\sigma_{\text{Hyd}} = -\frac{(\sigma_{11} + \sigma_{22} + \sigma_{33})}{3}$$

The stress tensor containing pressure, P, is

$$\boldsymbol{\sigma}_{\text{Hyd}} = \begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix}$$

Of course, it is rare to talk about pressure unless the hydrostatic stress is compressive, which corresponds to a positive pressure.

Also, unless one is working with boundary layer flows over aircraft, automobiles, etc, then the stress state in the air is one of hydrostatic stress alone, without any shear stresses. And the hydrostatic stress is compressive, which is a positive pressure.

Hydrostatic Stress in Incompressible Materials

This is almost a nontopic because hydrostatic stresses usually have no impact on incompressible materials at all, so there is very little to discuss. This also means that one cannot determine the hydrostatic stress based on strain or a deformation gradient. And this is why finite element programs often fudge by making the incompressible material ever-so-slightly compressible.

Deviatoric Stress

Deviatoric stress is what's left after subtracting out the hydrostatic stress. The deviatoric stress will be represented by $\boldsymbol{\sigma}'$. For example

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \boldsymbol{\sigma}_{\text{Hyd}}$$

In tensor notation, it is written as

$$\sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}$$

And in terms of pressure, it is written as

$$\sigma'_{ij} = \sigma_{ij} + P\,\delta_{ij}$$

Deviatoric Stress Example

Given the following stress tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} 50 & 30 & 20 \\ 30 & -20 & -10 \\ 20 & -10 & 10 \end{bmatrix}$$

The hydrostatic stress is

$$\sigma_{\text{Hyd}} = \frac{50 + (-20) + 10}{3} = 13.3$$

which can be written as

$$\boldsymbol{\sigma}_{\text{Hyd}} = \begin{bmatrix} 13.3 & 0 & 0 \\ 0 & 13.3 & 0 \\ 0 & 0 & 13.3 \end{bmatrix}$$

Subtracting the hydrostatic stress tensor from the total stress gives

$$\boldsymbol{\sigma}' = \begin{bmatrix} 50 & 30 & 20 \\ 30 & -20 & -10 \\ 20 & -10 & 10 \end{bmatrix} - \begin{bmatrix} 13.3 & 0 & 0 \\ 0 & 13.3 & 0 \\ 0 & 0 & 13.3 \end{bmatrix} = \begin{bmatrix} 36.7 & 30.0 & 20.0 \\ 30.0 & -33.3 & -10.0 \\ 20.0 & -10.0 & -3.3 \end{bmatrix}$$

Note that the result is traceless. Its first invariant equals zero. Or put another way, the hydrostatic stress of a deviatoric stress tensor is zero.

An interesting aspect of a traceless tensor is that it can be formed entirely from shear components. For example, a coordinate system transformation can be found to express the deviatoric stress tensor in the above example as shear stress exclusively. In the screenshot here, the above deviatoric stress tensor was input into the webpage, and then the coordinate system was rotated until the following stress tensor was obtained.

Tensor Transforms

Tensor			Transform Matrix		
-0.000005(	49.369495	-13.56070	0.9004007	-0.325245	-0.288960
49.369495	0.00001	-3.477229	0.4350547	0.6779863	0.5925083
-13.56070	-3.477229	0.0000086	0.0032023	-0.659208	0.751955

Rotate following plane...

- ◉ 1 - 2
- ○ 2 - 3
- ○ 3 - 1

90 degrees

Rotate

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Principal Stresses

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