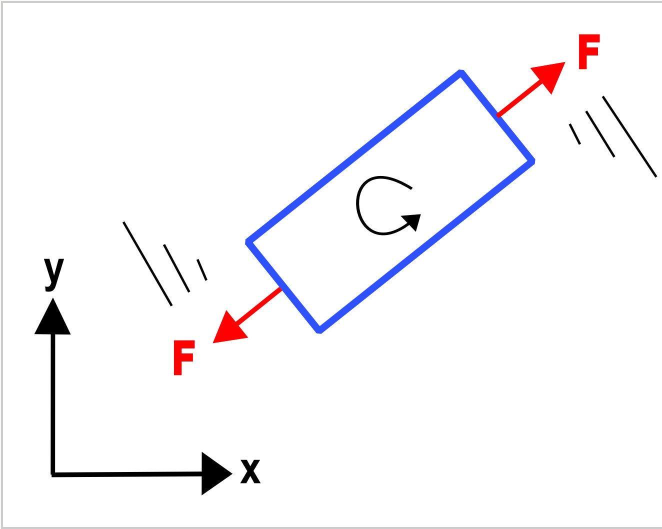
Corotational Derivatives

Introduction



We talked at the bottom of the energetic conjugates page about how certain pairs of stress and strain tensors can be used together in Hooke's Law, but that $\dot{\boldsymbol{\sigma}}$ and $\mathbf{D}$ should not be used together in the rate form.

This is because the Cauchy stress and rate of deformation tensor behave incompatibly in the presence of rigid body rotations like that shown at the right. It is probably easiest to understand the situation where the object has been stretched to a fixed amount that is then held constant while it continues to rotate.

As it rotates, the stress changes from xx-dominated to yy-dominated, so it is clearly changing with time, and therefore $\dot{\boldsymbol{\sigma}} \neq 0$. However, the rate of deformation tensor is zero because no deformations are taking place, only rotations. So there is a clear mismatch occurring between the two tensors, which would otherwise be ideally suited to each other.

Stress and Strain

Start with the most general form of linearized material behavior. This is

$$\boldsymbol{\sigma}^{\text{PK2}} = \mathbf{C} : \mathbf{E}$$

The stiffness tensor, $\mathbf{C}$, is 3x3x3x3 and can represent any material behavior, isotropic or orthotropic, as long as it is linear. The 2nd Piola-Kirchhoff stress and Green strain tensors are paired together because of their compatibility, i.e., both are defined in the reference configuration. (It is also convenient that the two are energetic conjugates, although this is not critical if the strains are small.)

The next step is to substitute the transformation from Cauchy stress to 2nd Piola-Kirchhoff stress. This gives

$$J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-T} = \mathbf{C} : \mathbf{E}$$

and then solve for the Cauchy stress

$$\boldsymbol{\sigma} = \frac{1}{J} \mathbf{F} \cdot \left(\mathbf{C} : \mathbf{E}\right) \cdot \mathbf{F}^T$$

Now take the time derivative

$$\dot{\boldsymbol{\sigma}} \;=\; -\left(\frac{\dot{J}}{J^2}\right)\mathbf{F}\cdot\,(\mathbf{C}:\mathbf{E})\cdot\mathbf{F}^T + \frac{1}{J}\,\dot{\mathbf{F}}\cdot\,(\mathbf{C}:\mathbf{E})\cdot\mathbf{F}^T +$$

$$\frac{1}{J}\,\mathbf{F}\cdot\,\left(\mathbf{C}:\dot{\mathbf{E}}\right)\cdot\mathbf{F}^T + \frac{1}{J}\,\mathbf{F}\cdot\,(\mathbf{C}:\mathbf{E})\cdot\dot{\mathbf{F}}^T$$

And substitute in several identities

$$\mathrm{tr}(\mathbf{D}) = \frac{\dot{J}}{J} \qquad \dot{\mathbf{F}} = \mathbf{L}\cdot\mathbf{F} \qquad \dot{\mathbf{F}}^T = \mathbf{F}^T\cdot\mathbf{L}^T \qquad \dot{\mathbf{E}} = \mathbf{F}^T\cdot\mathbf{D}\cdot\mathbf{F}$$

to get

$$\dot{\boldsymbol{\sigma}} \;=\; -\,\mathrm{tr}(\mathbf{D})\frac{1}{J}\mathbf{F}\cdot\,(\mathbf{C}:\mathbf{E})\cdot\mathbf{F}^T + \frac{1}{J}\,\mathbf{L}\cdot\mathbf{F}\cdot\,(\mathbf{C}:\mathbf{E})\cdot\mathbf{F}^T +$$

$$\frac{1}{J}\,\mathbf{F}\cdot\,\left(\mathbf{C}:(\mathbf{F}^T\cdot\mathbf{D}\cdot\mathbf{F})\right)\cdot\mathbf{F}^T + \frac{1}{J}\,\mathbf{F}\cdot\,(\mathbf{C}:\mathbf{E})\cdot\mathbf{F}^T\cdot\mathbf{L}^T$$

This big equation can be shortened by recognizing that many of the terms contain $\frac{1}{J}\,\mathbf{F}\cdot\,(\mathbf{C}:\mathbf{E})\cdot\mathbf{F}^T$, which is just the Cauchy stress, $\boldsymbol{\sigma}$.

$$\dot{\boldsymbol{\sigma}} = -\mathrm{tr}(\mathbf{D})\,\boldsymbol{\sigma} + \mathbf{L}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot\mathbf{L}^T + \frac{1}{J}\,\mathbf{F}\cdot\,\left(\mathbf{C}:(\mathbf{F}^T\cdot\mathbf{D}\cdot\mathbf{F})\right)\cdot\mathbf{F}^T$$

And a little more clean-up.

$$\dot{\boldsymbol{\sigma}} - \mathbf{L}\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot\mathbf{L}^T = -\mathrm{tr}(\mathbf{D})\,\boldsymbol{\sigma} + \frac{1}{J}\,(\mathbf{F}\cdot\mathbf{F}\cdot\mathbf{C}\cdot\mathbf{F}^T\cdot\mathbf{F}^T):\mathbf{D}$$

The term involving $\mathrm{tr}(\mathbf{D})$ is usually neglected because the trace is negligibly small in most cases. In fact, it is identically zero for incompressible materials.

The term involving $\mathbf{C}$ represents a rigid body rotation of the stiffness tensor. The constituants include $\frac{1}{J}\,(\mathbf{F}\cdot\mathbf{F}\cdot\mathbf{C}\cdot\mathbf{F}^T\cdot\mathbf{F}^T)$ and are replaced by $\mathbf{C}'$.

The left hand side (LHS) is called the Lie Derivative and represented by multiply symbols. Two of these are $\overset{\nabla}{\boldsymbol{\sigma}}$ and $\overset{o}{\boldsymbol{\sigma}}$. Of these, $\overset{\nabla}{\boldsymbol{\sigma}}$ is preferred because $\overset{o}{\boldsymbol{\sigma}}$ is usually used for something else. That something else is the so-called Jaumann derivative, which is very closely related and will be discussed shortly.

So the entire equation is written simply as

$$\overset{\nabla}{\boldsymbol{\sigma}} = \mathbf{C}':\mathbf{D}$$

If the object is rotating, but not deforming, then $\mathbf{D} = 0$, and this leaves

$$\overset{\nabla}{\boldsymbol{\sigma}} \;=\; \dot{\boldsymbol{\sigma}} - \mathbf{L}\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot\mathbf{L}^T \;=\; 0 \qquad \text{when } \mathbf{D} = 0$$

Now we see what it takes to compensate for the fact that $\dot{\boldsymbol{\sigma}} \neq 0$. It is the two terms involving the velocity gradient, $\mathbf{L}$. Furthermore, since $\mathbf{D} = 0$, then $\mathbf{L}$ reduces to $\mathbf{W}$ because $\mathbf{L} = \mathbf{D} + \mathbf{W}$. This leaves

$$\overset{\nabla}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \mathbf{W} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \mathbf{W}^T = 0 \qquad \text{when } \mathbf{D} = 0$$

But $\mathbf{W}^T = -\mathbf{W}$ because $\mathbf{W}$ is antisymmetric. And the resulting formula has another specific name. It is the Jaumann derivative, $\overset{o}{\boldsymbol{\sigma}}$.

$$\overset{o}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \mathbf{W} \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \mathbf{W} = 0 \qquad \text{when } \mathbf{D} = 0$$

Although the Lie derivative appears to be most general, the Jaumann derivative seems to be the most popular. For example, it is common to see

$$\overset{o}{\boldsymbol{\sigma}} = \mathbf{C}' : \mathbf{D}$$

The apostrophe is often dropped, although it is understood from the context that $\mathbf{C}$ must in fact be $\mathbf{C}'$.

Symbols and Terms

It's important not to confuse $\overset{o}{\boldsymbol{\sigma}}$ with $\dot{\boldsymbol{\sigma}}$, or even $\boldsymbol{\sigma}$. The symbol $\boldsymbol{\sigma}$ is still the Cauchy stress, and $\dot{\boldsymbol{\sigma}}$ is still its time derivative. The terms with the spin tensor simply compensate for the rate of change of the Cauchy stress when rigid body rotations are present.

This topic has many different names. for example, the Jaumann derivative is also called the Jaumann stress rate, or simply the Jaumann rate. And both the Jaumann derivative and Lie derivative fall under the category of corotational derivatives, or corotational stress rates, or simply corotational rates.

And finally, there is the issue of objectivity. The rate of deformation tensor is objective because its computation is still correct in the presence of rotations. $\dot{\boldsymbol{\sigma}}$ is not considered objective because $\dot{\boldsymbol{\sigma}} \neq \mathbf{C} : \mathbf{D}$. But $\overset{o}{\boldsymbol{\sigma}}$ is objective because $\overset{o}{\boldsymbol{\sigma}} = \mathbf{C} : \mathbf{D}$. So the terms objective stress rates, or simply objective rates also turn up.

Corotational Rate Example in 2-D

Start with a 2-D rotation matrix, $\mathbf{R}$.

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

so $\dot{\mathbf{R}}$ is

$$\dot{\mathbf{R}} = \omega \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}$$

And $\mathbf{R}^T$ is

$$\mathbf{R}^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Multiplying the two together gives the spin tensor, $\mathbf{W}$.

$$\mathbf{W} = \dot{\mathbf{R}} \cdot \mathbf{R}^T \;=\; \omega \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Now assume a constant stress tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} 20 & 0 \\ 0 & 0 \end{bmatrix}$$

So as the object rotates (without deforming), the stress tensor evolves as

$$\boldsymbol{\sigma}' \;=\; \mathbf{R} \cdot \boldsymbol{\sigma} \cdot \mathbf{R}^T$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 20 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 20\;\cos^2\theta & 10\sin(2\theta) \\ 10\sin(2\theta) & 20\;\sin^2\theta \end{bmatrix}$$

And the derivative of the stress tensor is

$$\dot{\boldsymbol{\sigma}} = \omega \begin{bmatrix} -20\sin(2\theta) & 20\cos(2\theta) \\ 20\cos(2\theta) & 20\sin(2\theta) \end{bmatrix}$$

Even though the material is not deforming, the stress tensor is clearly changing with time, and $\dot{\boldsymbol{\sigma}} \neq 0$. So now compute $\mathbf{W} \cdot \boldsymbol{\sigma}$ and $\boldsymbol{\sigma} \cdot \mathbf{W}$ in order to calculate the Jaumann derivative.

$$\mathbf{W} \cdot \boldsymbol{\sigma} \;=\; \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 20\;\cos^2\theta & 10\sin(2\theta) \\ 10\sin(2\theta) & 20\;\sin^2\theta \end{bmatrix}$$

$$= \omega \begin{bmatrix} -10\sin(2\theta) & -20\;\sin^2\theta \\ 20\;\cos^2\theta & 10\sin(2\theta) \end{bmatrix}$$

And

$$\boldsymbol{\sigma} \cdot \mathbf{W} \;=\; \omega \begin{bmatrix} 20\;\cos^2\theta & 10\sin(2\theta) \\ 10\sin(2\theta) & 20\;\sin^2\theta \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \omega \begin{bmatrix} 10\sin(2\theta) & -20\;\cos^2\theta \\ 20\;\sin^2\theta & -10\sin(2\theta) \end{bmatrix}$$

And putting everything together gives

$$\dot{\boldsymbol{\sigma}} - \mathbf{W}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot\mathbf{W} = \omega\begin{bmatrix} -20\sin(2\theta) & 20\cos(2\theta) \\ 20\cos(2\theta) & 20\sin(2\theta) \end{bmatrix} - \omega\begin{bmatrix} -10\sin(2\theta) & -20\ \sin^2\theta \\ 20\ \cos^2\theta & 10\sin(2\theta) \end{bmatrix} + \omega\begin{bmatrix} 10\sin(2\theta) & -20\ \cos^2\theta \\ 20\ \sin^2\theta & -10\sin(2\theta) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So it does indeed work.

The key purpose of this example was not to show that one should always get a zero result, because one should not always. The purpose of this example was to show that if no deformation is taking place at a given instant, then the Jaumann derivative is zero, even though $\dot{\boldsymbol{\sigma}} \neq 0$.