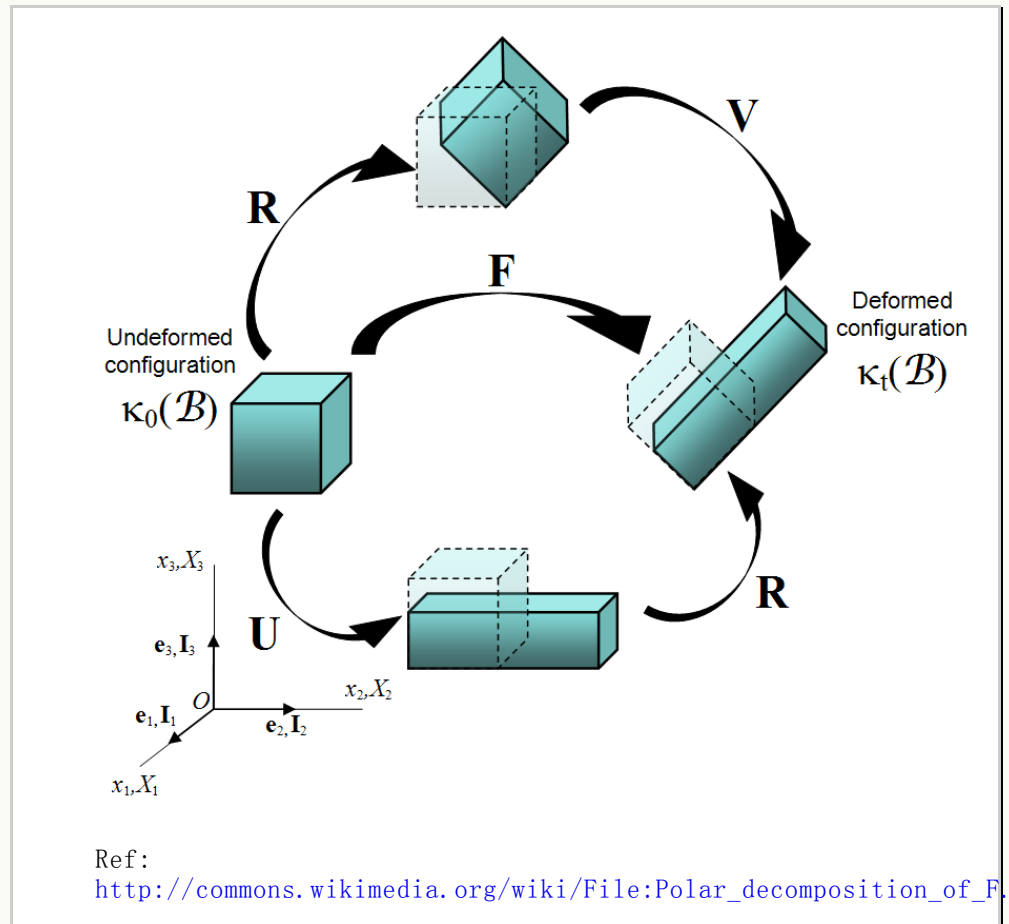


Introduction

The polar decomposition concept was introduced on the previous [deformation gradient](#) page. In it, we saw through example that \mathbf{F} can be written as either $\mathbf{R} \cdot \mathbf{U}$ or $\mathbf{V} \cdot \mathbf{R}$. In each case, \mathbf{R} is the rotation matrix, and \mathbf{U} and \mathbf{V} are symmetric matrices describing the deformations.

The purpose of this page is to show how to compute a polar decomposition in the general 3-D case. That process will introduce another new concept: the square root of a matrix. But first...



Product with Transpose

Recall from [this page on matrices](#) that the product of a matrix with its transpose is always a symmetric matrix. Plugging the polar decomposition into this gives a rather surprising result.

$$\mathbf{F}^T \cdot \mathbf{F} = (\mathbf{R} \cdot \mathbf{U})^T \cdot (\mathbf{R} \cdot \mathbf{U}) = \mathbf{U}^T \cdot \mathbf{R}^T \cdot \mathbf{R} \cdot \mathbf{U}$$

But as with \mathbf{Q} , here \mathbf{R}^T is the inverse of \mathbf{R} (This is discussed on the [next page](#)). So

$$\mathbf{R}^T \cdot \mathbf{R} = \mathbf{R}^{-1} \cdot \mathbf{R} = \mathbf{I}$$

And $\mathbf{F}^T \cdot \mathbf{F}$ reduces to

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$$(\mathbf{R} \cdot \mathbf{U}) = \mathbf{U}^T \cdot \mathbf{R}^T \cdot \mathbf{R} \cdot \mathbf{U} = \mathbf{U}^T \cdot \mathbf{U}$$



The result is called the "Right Cauchy-Green Deformation Tensor," and sometimes

represented by \mathbf{C} , but I don't like this because it hides the true physics behind the letter \mathbf{C} , which is $\mathbf{U}^T \cdot \mathbf{U}$. So I won't be using it.

Alternatively, one can do $\mathbf{F} \cdot \mathbf{F}^T$ using $\mathbf{F} = \mathbf{V} \cdot \mathbf{R}$ to obtain

$$\mathbf{F} \cdot \mathbf{F}^T = (\mathbf{V} \cdot \mathbf{R}) \cdot (\mathbf{V} \cdot \mathbf{R})^T = \mathbf{V} \cdot \mathbf{R} \cdot \mathbf{R}^T \cdot \mathbf{V}^T = \mathbf{V} \cdot \mathbf{V}^T$$



And this result is called the "Left Cauchy-Green Deformation Tensor," and sometimes represented by \mathbf{B} , but I don't like this because it also hides the true physics behind the letter \mathbf{B} , which is $\mathbf{V} \cdot \mathbf{V}^T$. So I won't be using it either.

The surprising result here is that the rotation matrix, \mathbf{R} , has been eliminated from the problem in each case. But a new challenge has been introduced, that of finding \mathbf{U} from $(\mathbf{U}^T \cdot \mathbf{U})$ and \mathbf{V} from $(\mathbf{V} \cdot \mathbf{V}^T)$. This is the source of the need to compute square roots of matrices. Recall that \mathbf{U} and \mathbf{V} are both symmetric, so $\mathbf{U} = \mathbf{U}^T$ and $\mathbf{U}^T \cdot \mathbf{U} = \mathbf{U} \cdot \mathbf{U}$. Likewise $\mathbf{V} = \mathbf{V}^T$ and $\mathbf{V} \cdot \mathbf{V}^T = \mathbf{V} \cdot \mathbf{V}$. Finally, $\mathbf{U} \cdot \mathbf{U}$ is sometimes written as \mathbf{U}^2 and $\mathbf{V} \cdot \mathbf{V}$ is sometimes written as \mathbf{V}^2 .

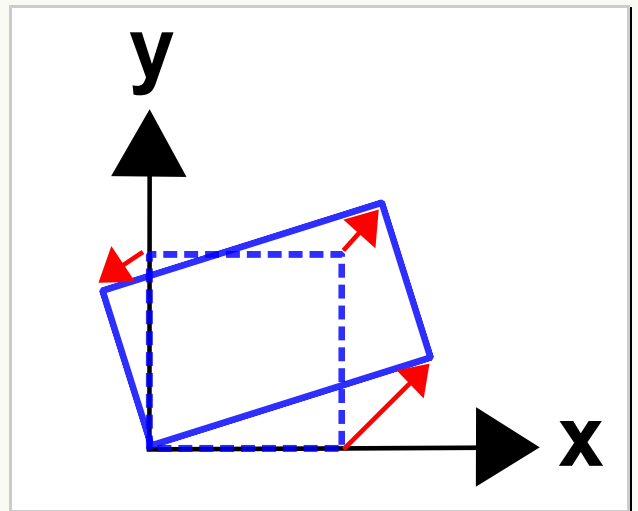
Determination of \mathbf{R} & \mathbf{U}

Recall the example from the previous page where the object is transformed from a square to the position shown in the figure. The equations to do this are

$$\begin{aligned} x &= 1.300X - 0.375Y \\ y &= 0.750X + 0.650Y \end{aligned}$$

and the deformation gradient is

$$\mathbf{F} = \begin{bmatrix} 1.300 & -0.375 \\ 0.750 & 0.650 \end{bmatrix}$$



So $\mathbf{F}^T \cdot \mathbf{F}$ is

$$\mathbf{F}^T \cdot \mathbf{F} = \begin{bmatrix} 1.300 & 0.750 \\ -0.375 & 0.650 \end{bmatrix} \begin{bmatrix} 1.300 & -0.375 \\ 0.750 & 0.650 \end{bmatrix} = \begin{bmatrix} 2.250 & 0.000 \\ 0.000 & 0.563 \end{bmatrix} = \mathbf{U}^T \cdot \mathbf{U}$$

This is the simplest of all scenarios - a diagonal matrix. In this case, the \mathbf{U} matrix is just the square-roots of the diagonal values.

$$\mathbf{U} = \begin{bmatrix} \sqrt{2.25} & 0.0 \\ 0.0 & \sqrt{0.563} \end{bmatrix} = \begin{bmatrix} 1.5 & 0.0 \\ 0.0 & 0.75 \end{bmatrix}$$

which is exactly the \mathbf{U} matrix result from the previous page. Once \mathbf{U} is determined, the

Final step is to multiply \mathbf{F} through by the inverse of \mathbf{U} to get \mathbf{R} .

$$\mathbf{F} \cdot \mathbf{U}^{-1} = \mathbf{R} \cdot \mathbf{U} \cdot \mathbf{U}^{-1} = \mathbf{R}$$

And the inverse of \mathbf{U} is very simple in this case because it is diagonal.

$$\mathbf{U}^{-1} = \begin{bmatrix} 0.667 & 0.0 \\ 0.0 & 1.333 \end{bmatrix}$$

So \mathbf{R} is

$$\mathbf{R} = \mathbf{F} \cdot \mathbf{U}^{-1} = \begin{bmatrix} 1.300 & -0.375 \\ 0.750 & 0.650 \end{bmatrix} \begin{bmatrix} 0.667 & 0.0 \\ 0.0 & 1.333 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.500 \\ 0.500 & 0.866 \end{bmatrix}$$

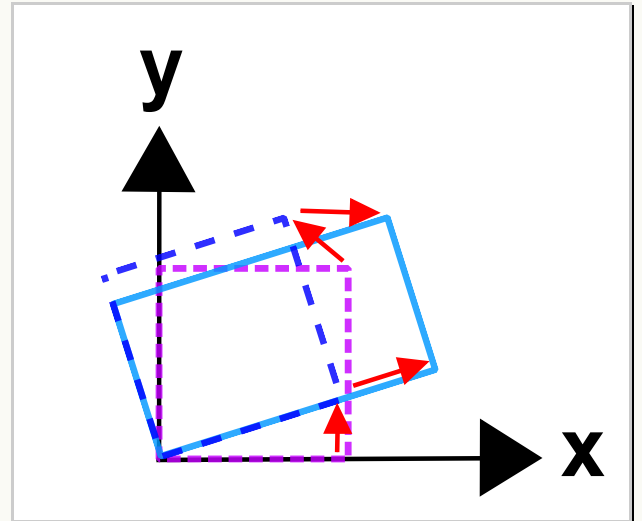
This concludes this example of determining \mathbf{R} and \mathbf{U} from a deformation matrix, \mathbf{F} . However, it was deceptively simple in this case because \mathbf{U} was a diagonal matrix, i.e., no shear. The next example will show the more typical, and much more complex case of performing a polar decomposition when shear is present in the problem.

Determination of \mathbf{V} & \mathbf{R}

Return to the same example as before. The coordinate mapping equations are

$$\begin{aligned} x &= 1.300X - 0.375Y \\ y &= 0.750X + 0.650Y \end{aligned}$$

This time, performing the $\mathbf{V} \cdot \mathbf{R}$ decomposition begins with $\mathbf{F} \cdot \mathbf{F}^T$.



$$\mathbf{F} \cdot \mathbf{F}^T = \begin{bmatrix} 1.300 & -0.375 \\ 0.750 & 0.650 \end{bmatrix} \begin{bmatrix} 1.300 & 0.750 \\ -0.375 & 0.650 \end{bmatrix} = \begin{bmatrix} 1.831 & 0.731 \\ 0.731 & 0.985 \end{bmatrix} = \mathbf{V} \cdot \mathbf{V}^T$$

This is the more common situation in which the resulting matrix is not diagonal. This greatly complicates the process of finding its square root. As was the case with \mathbf{U} , recall that \mathbf{V} is also symmetric, so $\mathbf{V} = \mathbf{V}^T$ and $\mathbf{V} \cdot \mathbf{V}^T = \mathbf{V} \cdot \mathbf{V}$.

Finding Square Roots of (Symmetric) Matrices

The following process of finding square roots of matrices only applies to symmetric ones. Otherwise, the square root will contain imaginary numbers that are of little use in the non-imaginary world of continuum mechanics.

The steps are as follows:

1. Transform the symmetric matrix to its principal orientation
2. Take the square roots of the diagonal components

2. Take the square roots of the diagonal components
3. Rotate back to the original orientation

Step 1 is relatively easy to accomplish for the 2-D example here. (A 3-D example will be presented later.) The rotation angle is

$$\tan 2\theta = \left(\frac{2a_{12}}{a_{11} - a_{22}} \right) = \left(\frac{2 * 0.731}{1.831 - 0.985} \right) \longrightarrow \theta = 30^\circ$$

Applying the 30° rotation to $\mathbf{V} \cdot \mathbf{V}^T$ is done as follows

$$\begin{bmatrix} \cos 30^\circ & \sin 30^\circ \\ -\sin 30^\circ & \cos 30^\circ \end{bmatrix} \begin{bmatrix} 1.831 & 0.731 \\ 0.731 & 0.985 \end{bmatrix} \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} = \begin{bmatrix} 2.250 & 0.0 \\ 0.0 & 0.563 \end{bmatrix}$$

which also should not be a surprise that the principal values of $\mathbf{V} \cdot \mathbf{V}^T$ are the same as those of $\mathbf{U}^T \cdot \mathbf{U}$. So the transformed \mathbf{V}' matrix is

$$\mathbf{V}' = \begin{bmatrix} \sqrt{2.25} & 0.0 \\ 0.0 & \sqrt{0.563} \end{bmatrix} = \begin{bmatrix} 1.5 & 0.0 \\ 0.0 & 0.75 \end{bmatrix}$$

The final step is to transform this result back by -30° to the original orientation.

$$\mathbf{V} = \begin{bmatrix} \cos(-30^\circ) & \sin(-30^\circ) \\ -\sin(-30^\circ) & \cos(-30^\circ) \end{bmatrix} \begin{bmatrix} 1.5 & 0.0 \\ 0.0 & 0.75 \end{bmatrix} \begin{bmatrix} \cos(-30^\circ) & -\sin(-30^\circ) \\ \sin(-30^\circ) & \cos(-30^\circ) \end{bmatrix} = \begin{bmatrix} 1.313 & 0.325 \\ 0.325 & 0.938 \end{bmatrix}$$

And we arrive back at the original \mathbf{V} matrix applied on the previous page. (Amazing how that works out!)

As a check of the square-root process...

$$\mathbf{V} \cdot \mathbf{V} = \begin{bmatrix} 1.313 & 0.325 \\ 0.325 & 0.938 \end{bmatrix} \begin{bmatrix} 1.313 & 0.325 \\ 0.325 & 0.938 \end{bmatrix} = \begin{bmatrix} 1.831 & 0.731 \\ 0.731 & 0.985 \end{bmatrix}$$

So this is correct.



Note that one thing that is not correct is to square each individual term of a matrix.

$$\begin{bmatrix} 1.313^2 & 0.325^2 \\ 0.325^2 & 0.938^2 \end{bmatrix} = \begin{bmatrix} 1.724 & 0.106 \\ 0.106 & 0.880 \end{bmatrix} \neq \begin{bmatrix} 1.831 & 0.731 \\ 0.731 & 0.985 \end{bmatrix}$$

Once \mathbf{V} is determined, the final step is to multiply \mathbf{F} through by the inverse of \mathbf{V} to get \mathbf{R} .

$$\mathbf{V}^{-1} \cdot \mathbf{F} = \mathbf{V}^{-1} \cdot \mathbf{V} \cdot \mathbf{R} = \mathbf{R}$$

The inverse of \mathbf{V} can be computed using [this page](#).

$$\mathbf{V}^{-1} = \begin{bmatrix} 0.833 & -0.289 \\ -0.289 & 1.166 \end{bmatrix}$$

And \mathbf{R} is

$$\mathbf{R} = \mathbf{V}^{-1} \cdot \mathbf{F} = \begin{bmatrix} 0.833 & -0.289 \\ -0.289 & 1.166 \end{bmatrix} \begin{bmatrix} 1.300 & -0.375 \\ 0.750 & 0.650 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.500 \\ 0.500 & 0.866 \end{bmatrix}$$

Which is the correct result... again.

This concludes the example of determining \mathbf{V} and \mathbf{R} from a deformation matrix, \mathbf{F} .

Review

So we have seen that given a simple coordinate mapping of

$$\begin{aligned} x &= 1.300X - 0.375Y \\ y &= 0.750X + 0.650Y \end{aligned}$$

The deformation gradient is

$$\mathbf{F} = \begin{bmatrix} 1.300 & -0.375 \\ 0.750 & 0.650 \end{bmatrix}$$

And this can be partitioned into

$$\mathbf{F} = \mathbf{R} \cdot \mathbf{U} = \begin{bmatrix} 0.866 & -0.500 \\ 0.500 & 0.866 \end{bmatrix} \begin{bmatrix} 1.50 & 0.0 \\ 0.0 & 0.75 \end{bmatrix}$$

Or alternatively

$$\mathbf{F} = \mathbf{V} \cdot \mathbf{R} = \begin{bmatrix} 1.313 & 0.325 \\ 0.325 & 0.938 \end{bmatrix} \begin{bmatrix} 0.866 & -0.500 \\ 0.500 & 0.866 \end{bmatrix}$$

In each case, the rotation is the same (because it must be). It's important to understand that the deformations are the same also. They just appear to be different because one is

imposed before the object is rotated, the other after. To me, the advantage of the $\mathbf{R} \cdot \mathbf{U}$ decomposition, is that one can easily identify the deformations because they are applied to the underformed, unrotated, un-anythinged object. So one does not need to know how much the object rotated before knowing which directions on it were deformed.

3-D Example

Here is a full-blown 3-D example. An object deforms according to

$$\begin{aligned}x &= X + 2Y \sin t + 0.5Z \\y &= -0.333X + Y - Z \sin t \\z &= X^2 \sin 2t + 1.5Z\end{aligned}$$

so the deformation gradient is

$$\mathbf{F} = \frac{\partial x_i}{\partial X_j} = \begin{bmatrix} 1 & 2 \sin t & 0.5 \\ -0.333 & 1 & -\sin t \\ 2X \sin 2t & 0 & 1.5 \end{bmatrix}$$

And the question is, "What are the conditions at $\mathbf{X} = (1, 1, 1)$ at $t = 0.25$ sec?"

Substituting the time and position coordinates in gives

$$\mathbf{F} = \frac{\partial x_i}{\partial X_j} = \begin{bmatrix} 1 & 0.495 & 0.5 \\ -0.333 & 1 & -0.247 \\ 0.959 & 0 & 1.5 \end{bmatrix}$$

Begin the process by premultiplying \mathbf{F} by its transpose. [This page on matrix multiplication](#) can be used to multiply the matrices.

$$\mathbf{F}^T \cdot \mathbf{F} = \begin{bmatrix} 2.031 & 0.162 & 2.021 \\ 0.162 & 1.245 & 0 \\ 2.021 & 0 & 2.561 \end{bmatrix} = \mathbf{U}^T \cdot \mathbf{U}$$

The next step is to find an orientation (i.e., a transformed coordinate system) that makes the result a diagonal matrix so that the square roots can be taken.

[This page on eigenvalues and eigenvectors](#) can do that.

Eigenvalues & Eigenvectors

Input			Eigenvectors		
2.0306	.162	2.0208	0.748264	-0.12095	-0.65228
.162	1.245	0	0.068240	0.992055	-0.10568
2.0208	0	2.561	0.659881	0.034566	0.750574
<input type="button" value="Calculate"/>					

Eigenvalues

0.242833

1.256143

4.337622

It reports that the principal values (eigenvalues) of $\mathbf{U}^T \cdot \mathbf{U}$ are

$$\mathbf{U}'^2 = \begin{bmatrix} 0.243 & 0 & 0 \\ 0 & 1.256 & 0 \\ 0 & 0 & 4.338 \end{bmatrix}$$

and the transformation matrix, \mathbf{Q} , used to obtain this is

$$\mathbf{Q} = \begin{bmatrix} 0.748 & -0.121 & -0.652 \\ 0.068 & 0.992 & -0.106 \\ 0.660 & 0.035 & 0.751 \end{bmatrix}$$

The next step is to take the square root of the diagonal elements of \mathbf{U}'^2 to get

$$\mathbf{U}' = \begin{bmatrix} 0.493 & 0 & 0 \\ 0 & 1.121 & 0 \\ 0 & 0 & 2.083 \end{bmatrix}$$

and transform it back to the original coordinate system. There is in fact a relatively easy way to do this. It is

$$\mathbf{U} = \mathbf{Q}^T \cdot \mathbf{U}' \cdot \mathbf{Q}$$

Multiplying this out gives

$$\mathbf{U} = \begin{bmatrix} 1.188 & 0.079 & 0.783 \\ 0.079 & 1.113 & -0.024 \\ 0.783 & -0.024 & 1.396 \end{bmatrix}$$

This shows that there is a lot of stretching in the Z direction, a little less in the X and Y directions, and a lot of shear in the $X - Z$ direction.

The inverse of \mathbf{U} is needed next. [This page can calculate inverses](#). The result is

$$\mathbf{U}^{-1} = \begin{bmatrix} 1.349 & -0.112 & -0.759 \\ -0.112 & 0.908 & 0.079 \\ -0.759 & 0.079 & 1.144 \end{bmatrix}$$

and finally, $\mathbf{R} = \mathbf{F} \cdot \mathbf{U}^{-1}$

$$\mathbf{R} = \mathbf{F} \cdot \mathbf{U}^{-1} = \begin{bmatrix} 0.914 & 0.377 & -0.148 \\ -0.374 & 0.926 & 0.049 \\ 0.156 & 0.011 & 0.988 \end{bmatrix}$$

Or one could simply use [this web page](#) to compute the $\mathbf{R} \cdot \mathbf{U}$ decomposition.

Polar Decompositions

Deformation Gradient, F

1	0.495	0.5
-0.333	1	-0.247
0.959	0	1.5

Decompose

Multiply

Rotation Matrix, R

0.914329	0.37693	-0.14807
-0.37389	0.926181	0.048932
0.155588	0.010624	0.987765

Stretch Tensor, U

1.18804	0.078701	0.782898
0.078701	1.112761	-0.02436
0.782898	-0.02436	1.395524

Up

Reset

Help

Or use [this web page](#) to compute the $\mathbf{V} \cdot \mathbf{R}$ decomposition.

Polar Decompositions

Deformation Gradient, F

1	0.495	0.5
-0.333	1	-0.247
0.959	0	1.5

Decompose

Multiply

Rotation Matrix, R

0.914329	0.37693	-0.14807
-0.37389	0.926181	0.048932
0.155588	0.010624	0.987765

Stretch Tensor, U

1.18804	0.078701	0.782898
0.078701	1.112761	-0.02436
0.782898	-0.02436	1.395524

Up

Reset

Help

Deformation Gradient, F

1	0.495	0.5
-0.333	1	-0.247
0.959	0	1.5

Decompose

Multiply

Stretch Tensor, V

1.026872	0.109033	0.654729
0.109033	1.0386	-0.28516
0.654729	-0.28516	1.630856

Rotation Matrix, R

0.914329	0.37693	-0.14807
-0.37389	0.926181	0.048932
0.155588	0.010624	0.987765

Up

Reset

Help



Deformation Gradients

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