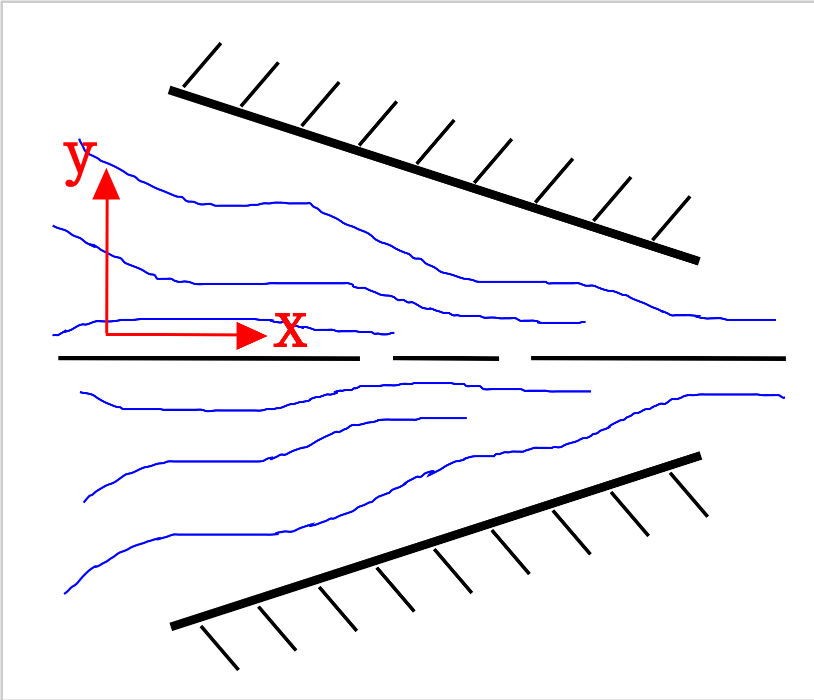
Material Derivative

Introduction



No continuum mechanics course can claim to be complete without a discussion of material derivatives. The material derivative computes the time rate of change of any quantity such as temperature or velocity (which gives acceleration) for a portion of a material moving with a velocity, $\mathbf{v}$. If the material is a fluid, then the movement is simply the flow field.

The sketch to the right shows a fluid flowing through a converging nozzle. Clearly any particle of fluid speeds up as it flows along the decreasing cross-section path. But if you were to focus on a single (x, y) point in space, you would not notice this acceleration because the fluid velocity at that point appears constant. The problem with this interpretation is that you are not following the same particle to see it speeding up over time The material derivative effectively corrects for this confusing effect to give a true rate of change of a quantity.

There are in fact many other names for the material derivative. They include total derivative, convective derivative, substantial derivative, substantive derivative, and still others.

Calculation of the Material Derivative

There are many symbolic representations of the material derivative. The most popular is $\frac{D()}{Dt}$, although $\frac{d()}{dt}$ may also be used. The quantity being differentiated goes inside the parentheses. Both representations are shorthand for

$$\frac{D()}{Dt} = \frac{\partial()}{\partial t} + \mathbf{v} \cdot \frac{\partial()}{\partial \mathbf{x}}$$

Note that the last derivative is with respect to $\mathbf{x}$, not $\mathbf{X}$. This makes the equation ideal for Eulerian based applications such as fluid mechanics. It is arrived at by applying the chain rule to Eulerian quantities that are dependent on time and position. For example, the material derivative of temperature, T, is

$$\frac{d}{dt}T(t,x,y,z) = \frac{\partial T}{\partial t} + \left(\frac{\partial T}{\partial x}\right)\left(\frac{\partial x}{\partial t}\right) + \left(\frac{\partial T}{\partial y}\right)\left(\frac{\partial y}{\partial t}\right) + \left(\frac{\partial T}{\partial z}\right)\left(\frac{\partial z}{\partial t}\right)$$

$$= \frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z}$$

$$= \frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T$$

$$= \frac{DT}{Dt}$$

This is written in tensor notation as

$$\frac{DT}{Dt} = T_{,t} + v_i T_{,i}$$

A second example: the material derivative of velocity gives acceleration.

$$\mathbf{a} = \frac{d}{dt}\mathbf{v}(t,x,y,z) = \frac{\partial \mathbf{v}}{\partial t} + v_x\frac{\partial \mathbf{v}}{\partial x} + v_y\frac{\partial \mathbf{v}}{\partial y} + v_z\frac{\partial \mathbf{v}}{\partial z}$$

$$= \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}$$

$$= \frac{D\mathbf{v}}{Dt}$$

This is written in tensor notation as

$$a_i = \frac{D\mathbf{v}}{Dt} = v_{i,t} + v_k v_{i,k}$$

Note that i is the free index here. So the above equation for the material derivative of velocity is actually three equations, one for each component.

$$a_x = \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}$$

$$a_y = \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}$$

$$a_z = \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}$$

Material Derivative Particulars

Note how important it is to write $(v_k v_{i,k})$ not $(v_i v_{i,k})$. This dictates which components are automatically summed over the three dimensions.

Also, it helps to apply a rigorous mathematical interpretation to each partial derivative in order to minimize any confusion. For example, $\frac{\partial \mathbf{v}}{\partial t}$ implies that x, y, and z are held constant. So this term captures the transient changes in the flow field at each fixed point in space. If all the fluid is always flowing past a point at the same velocity, then this term is zero. On the other hand, the $\mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{x}}$ term implies that the calculations are done at a 'snapshot' in time.

Material Derivative Example

This example involves a ball being thrown straight up into the air. The usual description of its position, y, at any time, t is

$$y = Y + v_o t - \frac{1}{2} g\, t^2$$

and its velocity is

$$v = \frac{dy}{dt} = v_o - g\, t$$

and its acceleration is

$$a = \frac{dv}{dt} = -g$$

So not surprisingly (at all), its acceleration is constant and equal to the acceleration of gravity. This was easy because the position at any instant is an explicit function of time and its initial position, Y.

But what if instead of position as a function of time, the velocity field was given as a function of position? This is typical of fluid flows. The velocity field could then be written as

$$v_y = \pm\sqrt{v_o^2 - 2g(y - y_o)}$$

where the positive root applies to the ball rising, and the negative one represents it falling. It is not readily apparent at all in this case that the velocity equation, as a function of position, actually represents a constant acceleration equal to gravity. But it does. To see this, apply the material derivative.

First, note that this problem is 1-D, so things simplify to

$$a = \frac{Dv_y}{Dt} = \frac{\partial v_y}{\partial t} + v_y \frac{\partial v_y}{\partial y}$$

$\frac{\partial v_y}{\partial t}$ is zero because the velocity, as described here, is not transient. It does not change with time at any given location. v_y is given above. And $\frac{\partial v_y}{\partial y}$ is

$$\frac{\partial v_y}{\partial y} = \frac{\mp g}{\sqrt{v_o^2 - 2g(y - y_o)}}$$

And multiplying $v_y \frac{\partial v_y}{\partial y}$ out gives

$$\begin{aligned} v_y \frac{\partial v_y}{\partial y} &= \pm\sqrt{v_o^2 - 2g(y - y_o)} \left(\frac{\mp g}{\sqrt{v_o^2 - 2g(y - y_o)}} \right) \\ &= -g \end{aligned}$$

So this does indeed give the exact same constant acceleration of gravity as before.