# 10

# Three-Dimensional Linear Elastostatics

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## §10.1. Introduction

We move now from the easy ride of Poisson problems and Bernoulli-Euler beams to the tougher road of elasticity in three dimensions. This Chapter summarizes the governing equations of linear elastostatics. Various notational systems are covered in sufficient detail to help readers with the literature of the subject, which is enormous and spans over two centuries. The governing equations are displayed in a Strong Form Tonti diagram for visualization convenience.

The classical single-field variational principle of Total Potential Energy is derived in this Chapter as prelude to mixed and hybrid variational principles, which are presented in following Chapters.[1]

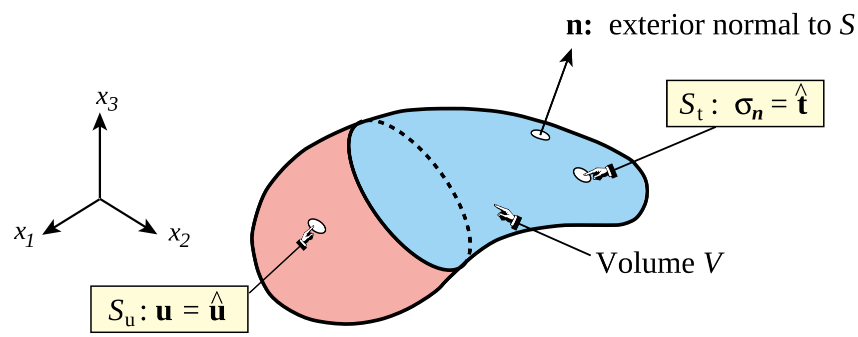


FIGURE 10.1. A linear-elastic body of volume $V$ in static equilibrium. The body surface $S : S_t \cup S_u$ is split into $S_t$, on which surface tractions are prescribed, and $S_u$, on which surface displacements are prescribed.

## §10.2. The Governing Equations

Consider a linearly elastic body of volume $V$, which is bounded by surface $S$, as shown in Figure 10.1. The body is referred to a three dimensional, rectangular, right-handed Cartesian coordinate system $x_i \equiv \{x_1, x_2, x_3\}$. The body is in *static* equilibrium under the action of body forces $b_i$ in $V$, prescribed surface tractions $\hat{t}_i$ on $S_t$ and prescribed displacements $\hat{u}_i$ on $S_u$, where $S_t \cup S_u \equiv S$ are two complementary portions of the boundary $S$. This separation of boundary conditions and source data is displayed in more detail in Figure 10.2.

The three unknown internal fields are *displacements* $u_i$, *strains* $e_{ij} = e_{ji}$ and *stresses* $\sigma_{ij} = \sigma_{ji}$. All of them are defined in $V$. In the absence of internal interfaces the three fields may be assumed to be continuous and piecewise differentiable; see, e.g., the expository chapter by Gurtin [310]. At internal interfaces (for example a change in material) certain strain and stress components may be discontinuous, but such “jump conditions” are ignored in the present treatment.

The three known or data fields are the body forces $b_i$, prescribed surface tractions $\hat{t}_i$ and prescribed displacements $\hat{u}_i$. These are given in $V$, on $S_t$, and on $S_u$, respectively.

The equations that link the various volume fields are called the *field equations* of elasticity. Those linking volume fields (evaluated at the surface) to prescribed surface fields are called *boundary conditions*. The ensemble of field equations and boundary conditions collectively represents the *governing equations* of elastostatics.

[1] The material in this and next two chapters is mostly taken from the Adavanced Variational Methods in Mechanics 1991 course, complemented with additional material on problem-solving.

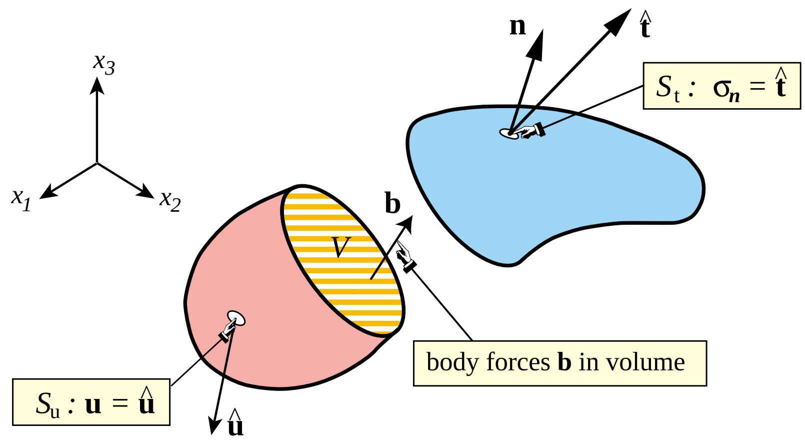


FIGURE 10.2. Showing in more detail the separation of the surface $S$ of the elastic body into two complementary regions: $S_t$ and $S_u$.

**Remark 10.1**. The field equations are generally partial differential equations (PDEs), although for elasticity the constitutive equations become algebraic. The classical boundary conditions are algebraic relations.

**Remark 10.2**. The separation of $S$ into traction-specified $S_t$ and displacement-specified $S_u$ may be more complex than the simple surface partition of the Poisson problem. This is because $\hat{t}_i$ and $\hat{u}_i$ are now *vectors* with several components. These may be specified at the same surface point in various combinations. This happens in many practical problems. For example, one may consider a portion of $S$ where a pressure (normal) force is applied whereas the tangential displacement components are zero. Or a bridge roller support: the displacement normal to the rollers is precluded (a displacement condition) but the tangential displacements are free (a traction condition). This mixture of force and displacement conditions over the same surface element would complicate the notation considerably. In this exposition we shall use the "union of" notation $S \equiv S_t \cup S_u$ for notational simplicity but the presence of such complications should be kept in mind.

### §10.2.1. Direct Tensor Notation

In the foregoing description we used the so-called *component notation* or *indicial notation* for fields. More precisely, the notation appropriate to Rectangular Cartesian Coordinates (RCC). For example, the single symbol $u_i$ is equivalent to writing the three components $u_1, u_2, u_3$ of the displacement field $\mathbf{u}$. We now review the so-called *direct tensor notation* or *compact tensor notation*.

Scalars, which are zero-dimensional tensors, are represented by non-boldface Roman or Greek symbols. Example: $\rho$ for mass density and $g$ for the acceleration of gravity.

Vectors, which are one-dimensional tensors, are represented by **boldface** symbols. These will be usually lowercase letters unless common usage dictates the use of uppercase symbols.[2] For example:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \qquad \mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}, \tag{10.1}$$

identify the vectors of displacements, body forces and surface tractions, respectively.

[2] This happens in electromagnetics: tradition since Maxwell has kept field vectors such as **E** (electric field) and **B** (magnetic field) in uppercase.

Two-dimensional tensors are represented by **underlined boldface** lowercase symbols. These will usually be lowercase Roman or Greek letters. For example

$$\underline{\mathbf{e}} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ & e_{22} & e_{23} \\ \text{symm} & & e_{33} \end{bmatrix} \equiv e_{ij}, \qquad \underline{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ \text{symm} & & \sigma_{33} \end{bmatrix} \equiv \sigma_{ij}, \tag{10.2}$$

denote the strain and stress tensors, respectively. The *transpose* of a second order tensor, denoted by $(.)^T$ is obtained by switching the two indices. A tensor is *symmetric* if it equates its transpose. Both the stress and strain tensors are symmetric: $\underline{\boldsymbol{\sigma}} = \underline{\boldsymbol{\sigma}}^T$ or $\sigma_{ij} = \sigma_{ji}$. Likewise $\underline{\mathbf{e}} = \underline{\mathbf{e}}^T$ or $e_{ij} = e_{ji}$.

Two product operations may be defined between second-order tensors. The *scalar product* or *inner product* is a scalar, which in terms of components is defined as[3]

$$\underline{\boldsymbol{\sigma}} : \underline{\mathbf{e}} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij} e_{ij} = \sigma_{ij} e_{ij}. \tag{10.3}$$

With $\underline{\boldsymbol{\sigma}}$ and $\underline{\mathbf{e}}$ as stress and strain tensors, respectively, $\underline{\boldsymbol{\sigma}} : \underline{\mathbf{e}}$ is twice the strain energy density $\mathcal{U}$.

The *tensor product* or *open product* of two second order tensors is a second-order tensor defined by the composition rule:

$$\text{if} \quad \underline{\mathbf{p}} = \underline{\boldsymbol{\sigma}} \cdot \underline{\mathbf{e}}, \quad \text{then} \quad p_{ij} = \sum_{k=1}^{3} \sigma_{ik} e_{kj} = \sigma_{ik} e_{kj}. \tag{10.4}$$

This is exactly the same rule as the matrix product. For matrices the dot is omitted. Some authors also omit the dot for tensors.

Four-dimensional tensors are represented by **underlined boldface uppercase** symbols. In elasticity the tensor of elastic moduli provides the most important example:

$$\underline{\mathbf{E}} \equiv E_{ijk\ell}, \tag{10.5}$$

The components of $\underline{\mathbf{E}}$ form a $3 \times 3 \times 3 \times 3$ hypercube with $3^4 = 81$ components, so the whole thing cannot be displayed so compactly as (10.2).

Operators that map vectors to vectors are usually represented by boldface uppercase symbols. An ubiquitous operator is nabla: $\nabla$, which should be boldface except that the symbol is not available in bold in the math font used here. Applied to a scalar function, say $\phi$, it produces its gradient:

$$\nabla\phi = \mathbf{grad}\,\phi \equiv \phi_{,i} = \frac{\partial\phi}{\partial x_i} = \begin{bmatrix} \dfrac{\partial\phi}{\partial x_1} \\ \dfrac{\partial\phi}{\partial x_2} \\ \dfrac{\partial\phi}{\partial x_3} \end{bmatrix}. \tag{10.6}$$

[3] Some textbooks use the notation $\underline{\boldsymbol{\sigma}}..\underline{\mathbf{e}}$ for the scalar $\sigma_{ij} e_{ji}$, but this is unnecessary as it is easily expressed with : by simply bytransposing the second tensor.

Applying nabla to a vector via the dot product yields the divergence of the vector:

$$\nabla \cdot \mathbf{u} = \mathbf{div}\,\mathbf{u} \equiv u_{i,i} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}. \tag{10.7}$$

Applying nabla to a second order tensor yields the divergence of a tensor, which is a vector. For example:

$$\nabla \cdot \underline{\boldsymbol{\sigma}} \equiv \mathbf{div}\ \boldsymbol{\sigma} = \sigma_{ij,j} = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} = \begin{bmatrix} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \dfrac{\partial \sigma_{23}}{\partial x_3} \\ \dfrac{\partial \sigma_{31}}{\partial x_1} + \dfrac{\partial \sigma_{32}}{\partial x_2} + \dfrac{\partial \sigma_{33}}{\partial x_3} \end{bmatrix} \tag{10.8}$$

Applying $\nabla$ to a vector via the cross product yields the curl or spin operator. This operator is not needed in classical elasticity but it appears in applications that deal with rotational fields such as fluid dynamics with vorticity, or corotational structural mechanics.

### §10.2.2. Matrix Notation

Matrix notation is a modification of direct tensor notation in which everything is placed in matrix form, with some trickery used if need be. The main advantages of matrix notation are historical compatibility with finite element formulations, and ready computer implementation in symbolic or numeric form.[4]

The representation of scalars, which may be viewed as $1 \times 1$ matrices, does not change. Neither does the representation of vectors because vectors are column (or row) matrices.

Two-dimensional *symmetric* tensors are converted to one-dimensional arrays that list only the independent components (six in three dimensions, three in two dimensions). The operation is called *tensor-to-vector casting*, and the result is called a *cast vector*. Component order in a cast vector is a matter of convention, but usually the tensor diagonal components are listed first followed by the off-diagonal ones. A factor of 2 may be applied to the latter, as the strain vector example below shows. The tensor is then represented as if were an actual vector, that is by non-underlined **boldface** lowercase Roman or Greek letters.

For the strain and stress tensors the casting process produces the 6-vectors

$$\underline{\mathbf{e}} \equiv \mathbf{e} = \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{31} \\ 2e_{12} \end{bmatrix}, \qquad \underline{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix}, \tag{10.9}$$

Note that off-diagonal (shearing) components of the strain vector are scaled by 2, but that no such scaling applies to the off-diagonal (shear) stress components. The idea behind the scaling is to

[4] Particularly in high level languages such as *Matlab*, *Mathematica* or *Maple*, which directly support matrix operators.

*maintain inner product equivalence* so that for example, the strain energy density is simply

$$\mathcal{U} = \tfrac{1}{2}\underline{\boldsymbol{\sigma}} : \underline{\mathbf{e}} = \tfrac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}e_{ij} = \tfrac{1}{2}\sigma_{ij}e_{ij} = \tfrac{1}{2}\boldsymbol{\sigma}^T\mathbf{e} \\ = \tfrac{1}{2}\big(\sigma_{11}e_{11} + \sigma_{22}e_{22} + \sigma_{33}e_{33} + 2\sigma_{31}e_{31} + 2\sigma_{23}e_{23} + 2\sigma_{12}e_{12}\big). \tag{10.10}$$

Four-dimensional tensors are cast to square matrices and denoted by matrix symbols, that is, **non-underlined boldface uppercase** Roman or Greek letters. Indices are appropriately collapsed to reflect symmetries and maintain product equivalence. Rather than stating boring rules, the example of the elastic moduli tensor is given to illustrate the mapping technique.

The stress-strain relation for linear elasticity in component notation is $\sigma_{ij} = E_{ijk\ell}\,e_{k\ell}$, and in compact tensor form $\underline{\boldsymbol{\sigma}} = \underline{\mathbf{E}} \cdot \underline{\mathbf{e}}$. We would of course like to have $\boldsymbol{\sigma} = \mathbf{E}\,\mathbf{e}$ in matrix notation. This can be done by defining the $6 \times 6$ elastic modulus matrix

$$\mathbf{E} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ & & E_{33} & E_{34} & E_{35} & E_{36} \\ & & & E_{44} & E_{45} & E_{46} \\ & & & & E_{55} & E_{56} \\ \text{symm} & & & & & E_{66} \end{bmatrix} \tag{10.11}$$

The components $E_{pq}$ of $\mathbf{E}$ are related to the components $E_{ijk\ell}$ of $\underline{\mathbf{E}}$ through an appropriate mapping that preserves the product relation. For example: $\sigma_{11} = E_{1111}e_{11} + E_{1122}e_{22} + E_{1133}e_{33} + E_{1112}e_{12} + E_{1121}e_{21} + E_{1113}e_{13} + E_{1131}e_{31} + E_{1123}e_{23} + E_{1132}e_{32}$ maps to $\sigma_{11} = E_{11}e_{11} + E_{12}e_{22} + E_{13}e_{33} + E_{14}\,2e_{23} + E_{15}\,2e_{31} + E_{16}\,2e_{12}$, whence $E_{11} = E_{1111}$, $E_{14} = E_{1123} + E_{1132}$, etc.

Finally, operators that can be put in vector form are usually represented by a vector symbol **boldface lowercase** whereas operators that can be put in matrix form are usually represented as matrices. Here is an example:

$$\mathbf{e} = \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{12} \\ 2e_{23} \\ 2e_{31} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} \\ \dfrac{\partial u_2}{\partial x_2} \\ \dfrac{\partial u_3}{\partial x_3} \\ \dfrac{\partial u_3}{\partial x_2} + \dfrac{\partial u_2}{\partial x_3} \\ \dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1} \\ \dfrac{\partial u_2}{\partial x_1} + \dfrac{\partial u_1}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \dfrac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial x_3} \\ 0 & \dfrac{\partial}{\partial x_3} & \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} & 0 & \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{D}\,\mathbf{u}. \tag{10.12}$$

Operator $\mathbf{D}$ is called the *symmetric gradient* in the continuum mechanics literature.[5] In the matrix notation defined above it is written as a $6 \times 3$ matrix. In direct tensor notation $\underline{\mathbf{D}} = \tfrac{1}{2}(\nabla + \nabla^T)$ is the tensor that maps $\mathbf{u}$ to $\underline{\mathbf{e}}$, and we write $\underline{\mathbf{e}} = \underline{\mathbf{D}} \cdot \mathbf{u}$. For the indicial form see below.

[5] Some books use variants of $\nabla$, such as $\bar{\nabla}$, $\hat{\nabla}$, $\nabla_S$ or $\nabla^S$, for this operator.

## §10.3. The Field Equations

Three field equations connect the internal (volume) fields.

### §10.3.1. Strain-Displacement Equations

The strain-displacement equations, also called the kinematic equations (KE) or deformation equations, yield the strain field given the displacement field. For linear elasticity the infinitesimal strain tensor $e_{ij}$ is given by

$$\boxed{e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{in } V.} \tag{10.13}$$

Here a comma denotes differentiation with respect to the space variable whose index follows.

In compact tensor notation, with $\underline{\mathbf{D}}$ as the symmetric gradient operator,

$$\boxed{\underline{\mathbf{e}} = \tfrac{1}{2}(\nabla + \nabla^T) \cdot \mathbf{u} = \underline{\mathbf{D}} \cdot \mathbf{u} \quad \text{in } V.} \tag{10.14}$$

The matrix form is $\mathbf{e} = \mathbf{D}\,\mathbf{u}$. The full form of this is given in (10.12).

The inverse problem: given a strain field, find the displacements, is not generally soluble unless *strain compatibility conditions* are verified. These are complicated second-order PDE given in any book on elasticity. This strain-to-diaplacement problem will not be considered here.

### §10.3.2. Constitutive Equations

The constitutive equations connect the stress and strain fields in $V$. These equations are intended to model the behavior of materials as continuum media. Generally they are partial differential equations (PDEs) or even integrodifferential equations in space and time. In linear elasticity, however, a considerable simplification occurs because the relation becomes *algebraic*, *linear*, and *homogeneous*. For this case the stress-strain relations may be written in component notation as

$$\boxed{\sigma_{ij} = E_{ijk\ell}\, e_{k\ell} \quad \text{in } V.} \tag{10.15}$$

The $E_{ijk\ell}$ are called *elastic moduli*. They are the components of a fourth order tensor $\mathbf{E}$ called the *elasticity tensor*. The elastic moduli satisfy generally the following symmetries

$$E_{ijk\ell} = E_{jik\ell} = E_{ij\ell k}, \tag{10.16}$$

which reduce their number from $3^4 = 729$ to $6^2 = 36$. Furthermore, if the body admits a strain energy (that is, the material is not only elastic but hyperelastic) the elastic moduli enjoys additional symmetries:

$$E_{ijk\ell} = E_{k\ell ij}, \tag{10.17}$$

which reduce their number to 21. Further symmetries occur if the material is orthotropic or isotropic. In the latter case the elastic moduli may be expressed as function of only two independent material constants, for example Young's modulus $E$ and Poisson ratio $\nu$.

In compact tensor notation:

$$\boxed{\underline{\boldsymbol{\sigma}} = \underline{\mathbf{E}} \cdot \underline{\mathbf{e}}.} \quad equationdef AVMM : Ch10 : eqn : StressStrainRelationsInCompactTensorNotation$$

In matrix form:

$$\boxed{\boldsymbol{\sigma} = \mathbf{E}\,\mathbf{e},} \tag{10.18}$$

in which $\mathbf{E}$ is the $6 \times 6$ matrix listed in (10.11).

If the elasticity tensor is invertible, the relation that connects strains to stresses is written

$$e_{ij} = C_{ijk\ell}\,\sigma_{k\ell} \quad \text{in } V. \tag{10.19}$$

The $C_{ijk\ell}$ are called *elastic compliances*. They are also the components of a fourth order tensor called the *compliance tensor*, which satisfies the same symmetries as $\mathbf{E}$. In compact tensor notation

$$\underline{\mathbf{e}} = \underline{\mathbf{C}} \cdot \underline{\boldsymbol{\sigma}} = \underline{\mathbf{E}}^{-1} \cdot \underline{\boldsymbol{\sigma}}, \tag{10.20}$$

and in matrix form

$$\mathbf{e} = \mathbf{C}\ \boldsymbol{\sigma} = \mathbf{E}^{-1}\boldsymbol{\sigma}. \tag{10.21}$$

### §10.3.3. Internal Equilibrium Equations

The internal equilibrium equations of elastostatics are

$$\boxed{\sigma_{ij,j} + b_i = \frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0 \quad \text{in } V.} \tag{10.22}$$

These follow from the linear momentum balance equations derived in books on continuum mechanics. The compact tensor notation is

$$\boxed{\nabla \cdot \underline{\boldsymbol{\sigma}} + \mathbf{b} = \mathbf{0} \quad \text{in } V.} \tag{10.23}$$

The matrix form is

$$\boxed{\mathbf{D}^T \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \quad \text{in } V.} \tag{10.24}$$

Here $\mathbf{D}^T$ is the transpose of the symmetric gradient operator given in (10.12).

## §10.4. The Boundary Conditions

The classical boundary conditions of elastostatics connect displacements and fluxes to given data to satisfy boundary compatibility and equilibrium condistions, respectively.

### §10.4.1. Surface Compatibility Equations

The surface compatibility equations, more commonly called *displacement boundary conditions*, are

$$\boxed{u_i = \hat{u}_i \quad \text{on} \;\; S_u.} \tag{10.25}$$

or in direct notation (both tensor and matrix)

$$\boxed{\mathbf{u} = \hat{\mathbf{u}} \quad \text{on} \;\; S_u.} \tag{10.26}$$

The physical meaning is that the displacement components at points of $S_u$ must match the prescribed values.

### §10.4.2. Surface Equilibrium Equations

The surface equilibrium equations, more commonly called *stress boundary conditions*, and also *traction boundary conditions*, are

$$\boxed{\sigma_{ij}\, n_j = \hat{t}_i \quad \text{on} \;\; S_t,} \tag{10.27}$$

in which $n_j$ are the components of the external unit normal $\mathbf{n}$ at points of $S_t$ where tractions are specified; see Figure 10.2. Note that

$$\sigma_{ni} = \sigma_{ij} n_j = t_i, \tag{10.28}$$

are the components of the *internal traction vector* $\mathbf{t} \equiv \boldsymbol{\sigma}_n$. The physical interpretation of the stress boundary condition is that the internal traction vector must equal the prescribed traction vector. Or: the net flux $t_i - \hat{t}_i$ on $S_t$ vanishes, component by component (a balnce or conservation law). In compact tensor form

$$\boxed{\mathbf{t} = \boldsymbol{\sigma}_n = \underline{\boldsymbol{\sigma}} \cdot \mathbf{n} = \hat{\mathbf{t}}.} \tag{10.29}$$

Stating (10.27) in a matrix form that uses the stress cast vector $\boldsymbol{\sigma}$ defined in (10.9) requires some care. It would be incorrect to write either $\hat{\mathbf{t}} = \boldsymbol{\sigma}^T \mathbf{n}$ or $\hat{\mathbf{t}} = \mathbf{n}^T \boldsymbol{\sigma}$ because $\boldsymbol{\sigma}$ is $6 \times 1$ and $\mathbf{n}$ is $3 \times 1$. Not only are these vectors non-conforming but even augmenting $\mathbf{n}$ with 3 zeros those products are scalars. The correct matrix form is a bit contrived:

$$\hat{\mathbf{t}} = \begin{bmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \end{bmatrix} = \begin{bmatrix} n_1 & 0 & 0 & 0 & n_3 & n_2 \\ 0 & n_2 & 0 & n_3 & 0 & n_1 \\ 0 & 0 & n_3 & n_2 & n_1 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = \mathbf{P}_n\, \boldsymbol{\sigma}, \tag{10.30}$$

in which $\mathbf{P}_n$ is the $3 \times 6$ normal-projection matrix shown above.

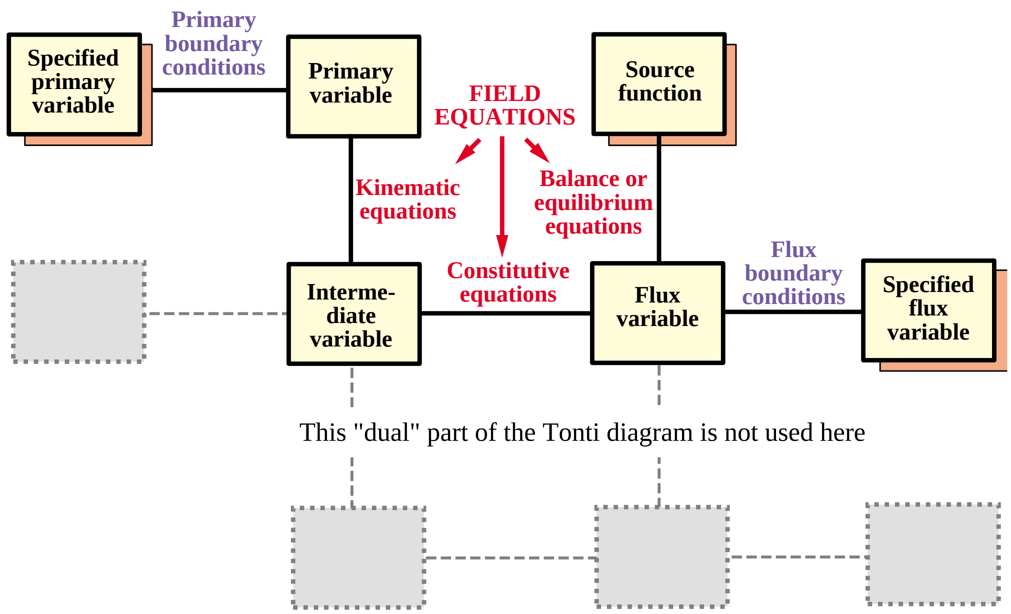


FIGURE 10.3. The general configuration of the Tonti diagram. Upper portion reproduced from Chapter 2. The diagram portion shown in dashed lines, which represents the so-called dual or potential equations, is not used in this book.

## §10.5. Tonti Diagrams

The Tonti diagram was introduced in Chapter 6 to represent the field equations of a mathematical model in graphical form. The general configuration of the expanded form of that diagram ("expanded" means that it shows boundary conditions) is repeated in Figure 10.3 for convenience. This diagram lists generic names for the "box occupants" and the connecting links.

Boxes and box-connectors drawn in solid lines are said to constitute the *primal* formulation of the governing equations. Dashed-lines boxes and connectors that appear in the bottom of the figure pertain to the so-called *dual* formulation in terms of potentials, which will not be used in this book.[6]

Figure 10.4 shows the primal formulation of the linear elasticity problem represented as a Tonti diagram. For this particular problem the displacements are the primary (or primal) variables, the strains the intermediate variables, and the stresses the flux variables. The source variables are the body forces. The prescribed configuration variables are prescribed displacements on $S_u$ and the prescribed flux variables are the surface tractions on $S_t$.

Tables 10.1 and 10.2 lists the generic names for the components of the Tonti diagram, as well as those specific for the elasticity problem. Table 10.3 summarizes the governing equations of linear elastostatics written down in three notational schemes.

[6] In the dual formulation the intermediate and flux variable exchange roles, so that boundary conditions of flux type are linked to the intermediate variable of the primal formulation. In this way it is possible, for instance, to specify strain boundary conditions in elasticity: just go for the dual formulation.

**Table 10.1 Abbreviations for Tonti Diagram Box Contents**

| *Acronym* | *Meaning* | *Alternate names in literature* |
|---|---|---|
| PV | Primary variable | Primal variable, configuration variable, "across" variable |
| IV | Intermediate variable | First intermediate variable, auxiliary variable |
| FV | Flux variable | Second intermediate variable, "through" variable |
| SV | Source variable | Internal force variable, production variable |
| PPV | Prescribed primary variable | |
| PFV | Prescribed flux variable | |

**Table 10.2 Abbreviations for Tonti Diagram Box Connectors**

| *Acronym* | *Generic name* | *Name(s) given in the elasticity problem* |
|---|---|---|
| KE | Kinematic equations | Strain-displacement equations |
| CE | Constitutive equations | Stress-strain equations, material equations |
| BE | Balance equations | Internal equilibrium equations |
| PBC | Primary boundary conditions | Displacement BCs |
| FBC | Flux boundary conditions | Stress BCs, traction BCs |

**Table 10.3 Summary of Elastostatic Governing Equations**

| *Acr* | *Valid* | *Compact or direct tensor form* | *Matrix form* | *Component (indicial) form* |
|---|---|---|---|---|
| KE | in $V$ | $\underline{\mathbf{e}} = \frac{1}{2}(\nabla + \nabla^T) \cdot \mathbf{u} = \underline{\mathbf{D}} \cdot \mathbf{u}$ | $\mathbf{e} = \mathbf{D}\,\mathbf{u}$ | $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ |
| CE | in $V$ | $\underline{\boldsymbol{\sigma}} = \underline{\mathbf{E}} \cdot \underline{\mathbf{e}}$ | $\boldsymbol{\sigma} = \mathbf{E}\,\mathbf{e}$ | $\sigma_{ij} = E_{ijk\ell}\, e_{k\ell}$ |
| BE | in $V$ | $\nabla \cdot \underline{\boldsymbol{\sigma}} + \mathbf{b} = \mathbf{0}$ | $\mathbf{D}^T \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$ | $\sigma_{ij,j} + b_i = 0$ |
| PBC | on $S_u$ | $\mathbf{u} = \hat{\mathbf{u}}$ | $\mathbf{u} = \hat{\mathbf{u}}$ | $u_i = \hat{u}_i$ |
| FBC | on $S_t$ | $\underline{\boldsymbol{\sigma}} \cdot \mathbf{n} = \boldsymbol{\sigma}_n = \mathbf{t} = \hat{\mathbf{t}}$ | $\mathbf{P}_n\,\boldsymbol{\sigma} = \boldsymbol{\sigma}_n = \mathbf{t} = \hat{\mathbf{t}}$ | $\sigma_{ij} n_j = \sigma_{ni} = t_i = \hat{t}_i$ |

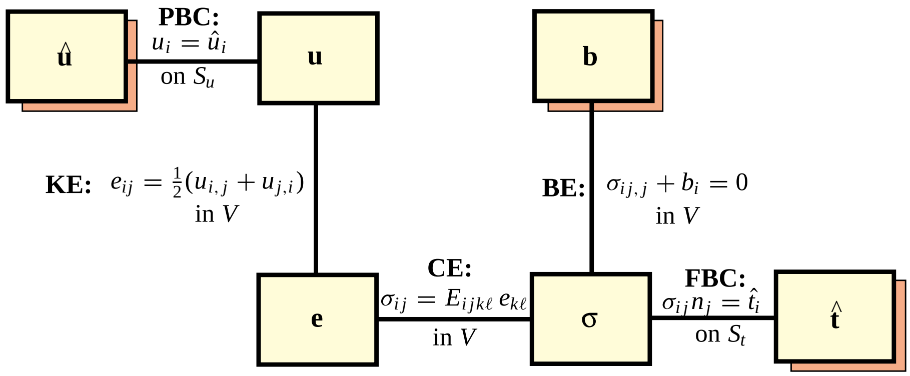


FIGURE 10.4. Strong Form Tonti diagram for linear elastostatics. Governing equations are written near links in indicial form.

## §10.6. Other Notational Conventions

To facilitate comparison with older textbooks and papers, the governing equations are restated below in two more alternative forms: in "grad/div" notation, and in full form.

### §10.6.1. Grad-Div Direct Tensor Notation

This is a variation of the "nabla" direct tensor notation. Symbols **grad** and **div** are used instead of $\nabla$ and $\nabla\cdot$ for gradient and divergence, respectively, whereas **symm grad** means the symmetric gradient operator $\mathbf{D} = \frac{1}{2}(\nabla + \nabla^T)$. The notation is slightly mode readable but takes more room.

$$\boxed{\begin{array}{lcl} \text{KE:} & \underline{\mathbf{e}} = \textbf{symm grad}\,\mathbf{u} & \text{in } V, \\ \text{CE:} & \underline{\boldsymbol{\sigma}} = \underline{\mathbf{E}}\ \underline{\mathbf{e}} & \text{in } V, \\ \text{BE:} & \textbf{div}\,\underline{\boldsymbol{\sigma}} + \mathbf{b} = \mathbf{0} & \text{in } V, \\ \text{PBC:} & \mathbf{u} = \hat{\mathbf{u}} & \text{on } S_u, \\ \text{FBC:} & \underline{\boldsymbol{\sigma}} \cdot \mathbf{n} = \boldsymbol{\sigma}_n = \mathbf{t} = \hat{\mathbf{t}}, & \text{on } S_t. \end{array}} \tag{10.31}$$

### §10.6.2. Full Notation

In the full-form notation everything is spelled out. No ambiguities of interpretation can arise; consequently this works well as a notation of last resort, and also as a "comparison template" against one can check out the meaning of more compact notations. It is also useful for programming in low-order languages.

The full form has, however, two major problems. First, it can become quite voluminous when higher order tensors are involved. Notice that most of the equations below are truncated because there is no space to state them fully. Second, compactness encourages visualization of essentials: long-windedness can obscure the forest with too many trees. Anyway, here they are:

$$\boxed{\begin{array}{lcr} \text{KE:} & e_{11} = \dfrac{\partial u_1}{\partial x_1}, \quad e_{12} = e_{21} = \tfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right), \quad \ldots & \text{in } V, \\ \text{CE:} & \sigma_{11} = E_{1111} e_{11} + E_{1112} e_{12} + \ \ldots \text{ (7 more terms)}, \quad \sigma_{12} = \quad \ldots & \text{in } V, \\ \text{BE:} & \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} + b_1 = 0, \ \ldots & \text{in } V, \\ \text{PBC:} & u_1 = \hat{u}_1, \quad u_2 = \hat{u}_2, \quad u_3 = \hat{u}_3 & \text{on } S_u, \\ \text{FBC:} & \sigma_{11}\, n_1 + \sigma_{12}\, n_2 + \sigma_{13}\, n_3 = \hat{t}_1, \quad \ldots & \text{on } S_t. \end{array}} \tag{10.32}$$

## §10.7. Solving Elastostatic Problems

By *solving an elastostatic problem* we mean meant to find the *displacement*, *strain* and *stress* fields that satisfy *all governing equations*. That is, the field equations and the boundary conditions.

Under mild assumptions of primary interest to mathematicians, the elastostatic problem has one and only one solution. There are, however, practical exceptions where the solution is not unique. Two instances:

1. *"Free Floating" Structures*. The displacement field is not unique but strains and stresses are. Example are aircraft structures in flight and space structures in orbit.

2. *Incompressible Materials*. The mean (hydrostatic) stress field is not determined from the displacements and strains. Determination of the hydrostatic stress field depends on the stress boundary conditions, which may be insufficient in some cases.

An analytical solution of the elastostatic problem is only possible for very simple cases. Most practical problems require a numerical solution. Most numerical methods involve a *discretization* process through which an approximate solution with a finite number of degrees of freedom is constructed.

### §10.7.1. Discretization Methods

Discretization methods of highest importance in computational mechanics can be grouped into three classes: finite difference, finite element, and boundary methods.

*Finite Difference Method* (FDM). The governing differential equations are replaced by difference expressions based on the field values at nodes of a finite difference grid. Although FDM remains important in fluid mechanics and in dynamic problems for the time dimension, it has been largely superseded by the finite element method in a structural mechanics in general and elastostatics in particular.

*Finite Element Method* (FEM). This is the most important "volume integral" method. One or more of the governing equations are recast to hold in some *average* sense over subdomains of simple geometry. This recasting is often done in terms of *variational forms* if variational principles can be readily constructed, as is the case in elastostatics. The procedure for constructing the simplest class of these principles is outlined in the next section.

*Boundary Methods*. Under certain conditions the field equations between volume fields can be eliminated in favor of *boundary unknowns*. This dimensionality reduction process leads to integro-differential equations taken over the boundary $S$. Discretization of these equations through finite element or collocation techniques leads to the so-called *boundary element methods* (BEM).

Further discussion on the use of these methods for simulating structural systems can be found in Chapter 1 of the Advanced Finite Element methods (AFEM) book [245]. Some more specialized approaches, e.g., meshfree and spectral discretizations, are desccribed there.

| *Step** | *Operation* | *Description* |
|---|---|---|
| **1** | **Select master field(s)** | One or more of the unknown internal fields (displaceents, strains and stresses in the elasticity problem) are chosen as masters. |
| **2** | **Select weak connections** | Selected strong links are weakened. Slave fields may have to be split if necessary. If so, duplicated slave fields are connected by weak links. |
| **3** | **Construct Weak Form (WF)** | A Weak Form is establsihed by choosing weights for the weak links and integrating over their domains (volume or surface). |
| **4** | **Replace weights by alleged variations and homogenize** | WF weight functions are replaced by master field variations propagated through the strong links as appropriate. Master field variations are homogenized using integration by parts (the divergence theorem in 2D or 3D) as necessary. |
| **5** | **Find Variational Form (VF)** | The VF is obtained as the functional whose first variation reproduces the varied-WF of Step 4. This process may involve trial and error, e.g., playing with alleged variations and their signs. If no VF can be found, one may have to be content with the WF of Step 3 as basis to develop numerical approximations. |
| * Steps 4-5 are typically the most difficult and time-consuming ones. There is no guarantee that Step5 will be successful. | | |

FIGURE 10.5. Steps to construct a Variational Form (VF) starting from the field-decomposed Strong Form (SF).

## §10.8. Constructing Variational Forms

Finite element methods for the elasticity problem are based on *Variational Forms*, or VFs, of the foregoing Strong Form (SF) equations. Although the SF is unique, there are many VFs.[7] As explained in previous Chapters, the search for a VF begins by selecting one or more master fields, and weakening one or more links. This process produces a set of equations called the Weak Form, or WF, which may be viewed as an midway stop between the SF and the VF.

The end result of the process is the construction of a *functional* $\Pi$ that contains integrals of the known and unknown fields. Associated with the functional is a *variational principle:* setting the first variation $\delta\Pi$ to zero recovers the strong form of the weakened field equations as Euler-Lagrange equations, and the strong form of the weakened BCs as natural boundary conditions.

Here is a summary of the VF construction steps: (1) select master(s), (2) select weak links, (3) construct the Weak Form (WF), (3) make the WF the variation of an alleged functional, (5) find the functional to establish the Variational Form (VF). Those steps are summarized in Figure 10.5, and generically described below keeping the elasticity equations in mind.

The steps are illustrated with the construction of the single-field primal elasticity functional, called the Total Potential Energy or TPE.

### §10.8.1. Step 1: Select Master Field(s)

One or more of the unknown internal fields

$$u_i, \quad e_{ij}, \quad \sigma_{ij}, \tag{10.33}$$

are chosen as masters. A master (also called *primary*, *varied*, or *parent*) field is one that is subjected to the $\delta$ process of Variational Calculus. Fields that are not masters, *i.e.* not subject to variation,

[7] There is in fact an *infinite* number of functionals, parametrizable by a *finite* number of parameters, as shown in the expository survey [222]. Most books and papers give the impression that there is only a finite number. What is true is that only a finite number of *canonical* functionals exist; they are listed in the next Chapter for the elasticity problem.

are called *slave*, *secondary*, or *derived*. The *owner* (also called *parent* or *source*) of a slave field is the master from which it comes from.

If only one master field is chosen, the resulting variational principle (obtained after going through Steps 2–5) is called *single-field*, and *multifield* otherwise.

A known or data field (for example: body forces or surface tractions in elastostatics) cannot be a master field because it is not subject to variation, and is not a secondary field because it does not derive from others. Hence we see that *fields can only be of three types*: master, slave, or data.

### §**10.8.2. Step 2: Select Weak Connections**

Given a master field, consider the equations that link it to other known and unknown fields. These are called the *connections* of that field. Classify these connections into two types:

*Strong connection*. The connecting relation is enforced *point by point* in its original form. For example if the connection is a PDE or an algebraic equation we use it as such. Also called *a priori* enforcement. When applied to a boundary condition, a strong connection is also referred to as an *essential constraint* or *essential B.C.*

*Weak connection*. The connection relationship is enforced only in an *average* or *mean* sense through the use of a weight or test function, or of a distributed Lagrange multiplier. Also called *a-posteriori* enforcement. When applied to a boundary condition, a weak connection is also referred to as a *natural constraint* or *natural B.C.*

A general rule to keep in mind is that *a slave field must be reachable from its master owner through strong connections*.

If there is more than one master field (*i.e.* we are constructing a multifield principle), the foregoing definitions *must be applied to each master field in turn*. In other words, we must consider the connections that "emanate" from each of the master fields. The end result is that the same field may appear more than once. For example in elasticity the strain field $\mathbf{e}$ may appear up to three times: (1) as a master field, (2) as a slave field derived from displacements, and (3) as a slave field derived from stresses. If this causes a slave field to have multiple masters, that field must be *duplicated* in such a way that each duplicate has one and only one owner. The duplicates are then connected with a weak link. Such a complication cannot occur with single-field principles.

**Remark 10.3**. There is usually limited freedom as regards the choice of strong vs. weak connections. The key test comes when one tries to form the total variation in Steps 4-5. If this happens to be the exact variation of a functional, the choice is admissible. Else is back to the drawing board.

### §**10.8.3. Step 3: Construct A Weak Form**

Establish a Weak Form (WF) by multiplying the weak link residuals by weighting functions and integrating over the approrpriate domains: volume for field equations, surface for boundary conditions.

Once all choices of Steps 1–3 have been made, the remaining manipulations are technical in nature, and essentially consist of applying the tools and techniques of vector, tensor and variational calculus: Lagrange multipliers, integration by parts, homogenization of variations, surface integral splitting, and so on. Since the number of operational combinations is huge, the techniques are best illustrated through specific examples.

The end result of these gyrations should be a *variational statement*

$$\boxed{\delta\Pi = 0,} \tag{10.34}$$

where the symbol $\delta$ here embodies variations with respect to *all master fields*. Steps 4–5 are designed to get there.

### §10.8.4. Step 4: Replace Weights by Alleged Variations and Homogenize

Once all choices of Steps 1–3 have been made, the remaining manipulations are technical in nature, and essentially consist of applying the tools and techniques of vector, tensor and variational calculus: Lagrange multipliers, integration by parts (called the divergence theorem in 2D or 3D) homogenization of variations, surface integral splitting, and so on. Since the number of operational combinations is huge, the techniques are best illustrated through specific examples.

The end result of all these gyrations should be a *variational statement*

$$\boxed{\delta\Pi = 0,} \tag{10.35}$$

where the symbol $\delta$ here embodies variations with respect to *all master fields*, and $\Pi$ (if it exist) is the target functional.

### §10.8.5. Step 5: Find Variational Form (VF)

With luck, the variational statement constructed in Step 4 will be recognized as the *exact variation* of a functional $\Pi$, such as the one in §10.9.4 below. If one arrives at this happy end, the variational statement becomes a true variational principle and the movie fades into the sunset before the credits roll in. The expression $\delta\Pi = 0$ now states a *variational principle*. If this is a new principle for an important problem, your name may forever live attached to that $\Pi$.

We now illustrate the foregoing steps with the detailed derivation of the most important single-field VF in elastostatics: the principle of Total Potential Energy or TPE.

## §10.9. Derivation of Total Potential Energy Principle

### §10.9.1. A Long Journey Starts with the First Step

The departure point for deriving the classical TPE principle is the WF diagrammed in Figure 10.6. Such modifications are briefly explained in the figure label and in the text below. The displacement field $u_i$ is the only master. The strain and stress fields are slaves. The slave-provenance notation introduced in Chapter 3 is used: the owner of a slave field is marked by a superscript. For example, $\mathbf{e}^u = \mathbf{D}\,\mathbf{u}$ means "$\mathbf{e}^u$ is owned by $\mathbf{u}$" through the strong KE link.

The strong connections are the kinematic equations KE (in elasticity the strain-displacement equations), the constitutive equations CE, and the primary boundary conditions PBC (in elasticity the displacement boundary conditions). These are shown in Figure 10.6 as solid box-connecting lines:

$$\boxed{\text{Strong}: \quad e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \ \text{ in } V, \quad \sigma_{ij} = E_{ijk\ell}\, e_{k\ell} \ \text{ in } V, \quad u_i = \hat{u}_i \ \text{ on } S_u.} \tag{10.36}$$

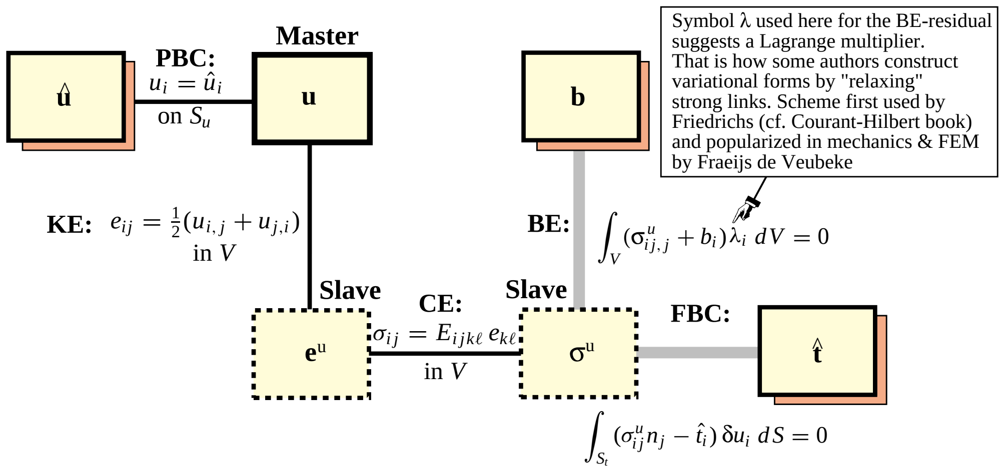


FIGURE 10.6. Tonti Diagram of the Weak Form (WF) used as departure point for deriving the TPE functional of linear elastostatics.

The *weak* connections are the balance equations BE (in elasticity the stress equilibrium equations), and the flux boundary conditions FBC (in elasticity the traction boundary conditions), These appear in Figure 10.6 as shaded lines:

$$\boxed{\text{Weak:} \qquad \sigma_{ij,j} + b_i = 0 \quad \text{in } V, \qquad \sigma_{ij} n_j = \hat{t}_i \quad \text{on } S_t.} \tag{10.37}$$

### §10.9.2. Lagrangian Glue

Now we get down to the business of variational calculus. A slight notational variation of the *residual weighting* technique of previous Chapters is used. The notation has certain interpretation advantages that will become apparent later when dealing with hybrid principles.

To treat BE as a weak connection, take the first of (10.37), replace $\sigma_{ij}$ by the slave $\sigma^u_{ij}$, multiply by a piecewise differentiable 3-vector field $\lambda_i$ and integrate over $V$:

$$\int_V (\sigma^u_{ij,j} + b_i)\, \lambda_i \, dV = 0. \tag{10.38}$$

Apply the divergence theorem to the first term in (10.38):

$$\int_V \sigma^u_{ij,j}\, \lambda_i \, dV = -\int_V \sigma^u_{ij} \lambda_{i,j} \, dV + \int_S \sigma^u_{ij}\, n_j \lambda_i \, dS. \tag{10.39}$$

For a symmetric stress tensor $\sigma^u_{ij} = \sigma^u_{ji}$ this formula may be transformed[8] to

$$\int_V \sigma^u_{ij,j} \lambda_i \, dV = -\int_V \sigma^u_{ij}\, \tfrac{1}{2}(\lambda_{i,j} + \lambda_{j,i})\, dV + \int_S \sigma^u_{ij}\, n_j \lambda_i \, dS. \tag{10.40}$$

[8] This transformation is stated in §5.5 of Sewell's book [640]. It may also be verified directly as in Exercise 10.4.

Assignation of meaning of internal energy to the second term in (10.40) suggests identifying $\lambda_i$ with the variation of the displacement field $u_i$ (a "lucky guess" that can be proved rigorously *a posteriori*):

$$\int_V \sigma^u_{ij,j}\,\delta u_i\,dV = -\int_V \sigma^u_{ij}\ \delta e^u_{ij}\,dV + \int_S \sigma^u_{ij}\,n_j\,\delta u_i\,dS, \tag{10.41}$$

in which the strain-variation symbol means

$$\delta e^u_{ij} = \tfrac{1}{2}(\delta u_{i,j} + \delta u_{j,i})\ \text{ in } V, \tag{10.42}$$

because of the strong connection $e^u_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, which if varied with respect to $u_i$ yields (10.42).

**Remark 10.4**. Although the essence of the treatment of weak connections is ultimately the same, there is far from universal agreement on terminology in the literature. The foregoing scheme is known as the *Lagrange multiplier* treatment. It closely follows Fraeijs de Veubeke (a major contributor to variational mechanics). The technique was originally introduced by Friedrichs (a disciple of Courant and Hilbert) in a mathematical context.

Other authors, primarily in fluid mechanics, favor *weight functions* (as in the treatment of the Poisson equation in previous Chapters) or *test functions*. If the WF is directly discretized, as often done in fluid mechanics, the former technique leads to weighted-residual subdomain methods (for example the Fluid Volume Method) whereas test functions lead to Galerkin and Petrov-Galerkin methods. Some authors, such as Lanczos [419], multiply directly equilibrium residuals by displacement variations, which are called then *virtual displacements*. Some music but different lyrics.

### §10.9.3. The First-Variation Pieces

Substituting (10.41) into (10.38), with $\lambda_i \to \delta u_i$, we obtain

$$\int_V \sigma^u_{ij}\,\delta e^u_{ij}\,dV - \int_V b_i\,\delta u_i\,dV - \int_S \sigma^u_{ij}\,n_j\,\delta u_i\,dS = 0. \tag{10.43}$$

The surface integral may be split as follows:

$$\int_S \sigma^u_{ij}\,n_j\,\delta u_i\,dS = \int_{S_t} \sigma^u_{ij}\,n_j\,\delta u_i\,dS + \int_{S_u} \sigma^u_{ij}\,n_j\,\cancelto{0}{\delta\hat{u}_i}\,dS = \int_{S_t} \sigma^u_{ij}\,n_j\,\delta u_i\,dS. \tag{10.44}$$

where the substitution $\delta u_i = \delta\hat{u}_i$ on $S_u$ results from the strong connection $u_i = \hat{u}_i$ on $S_u$. But $\delta\hat{u}_i = 0$ because prescribed (data) fields are not subject to variation, and the $S_u$ integral drops out. Treating the FBC weak connection with $\delta u_i$ as 3-vector weight function we obtain

$$\int_{S_t} (\sigma^u_{ij} n_j - \hat{t}_i)\,\delta u_i\,dS = 0, \qquad \text{whence} \qquad \int_{S_t} \sigma^u_{ij} n_j\,\delta u_i\,dS = \int_{S_t} \hat{t}_i\,\delta u_i\,dS. \tag{10.45}$$

### §10.9.4. A Happy Ending

Substituting (10.44) and the second of (10.45) into (10.43), we obtain the final form of the variation in the master field $u_i$, which we write (hopefully) as the variation of a functional $\Pi_{\rm TPE}$:

$$\delta\Pi_{\rm TPE} = \int_V \sigma^u_{ij}\ \delta e^u_{ij}\, dV - \int_V b_i\, \delta u_i\, dV - \int_{S_t} \hat{t}_i\ \delta u_i\, dS = 0. \tag{10.46}$$

And indeed (10.46) can be recognized[9] as the exact variation, with respect to $u_i$, of

$$\boxed{\Pi_{\rm TPE}[u_i] = \tfrac{1}{2}\int_V \sigma^u_{ij}\, e^u_{ij}\, dV - \int_V b_i\, u_i\, dV - \int_{S_t} \hat{t}_i\, u_i\, dS.} \tag{10.47}$$

This $\Pi_{\rm TPE}$ is called the *total potential energy* functional. It is often written as the difference of the strain energy and the external work functionals:

$$\boxed{\begin{array}{ll} \Pi_{\rm TPE} = U_{\rm TPE} - W_{\rm TPE}, & \text{in which} \\ U_{\rm TPE} = \tfrac{1}{2}\displaystyle\int_V \sigma^u_{ij} e^u_{ij}\, dV, & W_{\rm TPE} = \displaystyle\int_V b_i u_i\, dV + \int_{S_t} \hat{t}_i u_i\, dS. \end{array}} \tag{10.48}$$

Consequently (10.46) is a true variational principle and not just a variational statement.

The physical interpretation is well known: $\frac{1}{2}\sigma^u_{ij} e^u_{ij}$ is the *strain energy density* $\mathcal{U}$ in terms of displacements. Integrating this density over the volume $V$ gives the total strain energy stored in the body. In elasticity this is the only stored energy, and consequently it is also the *internal energy* $U$. Likewise, $b_i u_i$ is the *external work density* of the body forces, whereas $\hat{t}_i u_i$ is the *external work density* of the applied surface tractions. Integrating these densities over $V$ and $S_t$, respectively, and adding gives the total *external work* potential $W$.

**Remark 10.5**. What we have just gone through is called the Inverse Problem of Variational Calculus: given the governing equations (field equations and boundary conditions), find the functional(s) that have those governing equations as Euler-Lagrange equations and natural boundary conditions, respectively.

The Direct Problem of Variational Calculus is the reverse one: given a functional such as (10.47), how that the vanishing of its first variation is equivalent to the governing equations. These appear as the Euler-Lagrange equations and natural boundary conditions.

This problem is normally the first one tackled in Variational Calculus instruction in math support courses.[10] The Direct Problem is done by carrying out the foregoing steps in reverse order: form a veriation such as (10.46), integrate by parts as appropriate to homogenize variations, and use the strong connections to finally arrive at

$$\delta\Pi_{\rm TPE} = \int_V (\sigma^u_{ij,j} + b_i)\, \delta u_i\, dV + \int_{S_t} (\sigma_{ij} n_j - \hat{t}_i)\, \delta u_i\, dS. \tag{10.49}$$

Using the fundamental lemma of variational calculus[11] one then shows that $\delta\Pi_{\rm TPE} = 0$ yields the weak connections (10.37) as Euler-Lagrange equations and natural boundary conditions, respectively.

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[9] See Exercise 10.5 for the variation of the strain energy term.

[10] For example, Aerospace Math.

[11] See, e.g., §1.3 of [289].

## §10.10. The Tensor Divergence Theorem and the PVW

Recall from Chapter 7 the canonical form of the divergence theorem, which says that the vector divergence of a vector **a** over a volume is equal to the vector flux over the surface:

$$\int_V \nabla \cdot \mathbf{a}\, dV = \int_S \mathbf{a} \cdot \mathbf{n}\, dS. \tag{10.50}$$

Take $\mathbf{a} = \boldsymbol{\sigma} \cdot \mathbf{u}$, where $\boldsymbol{\sigma} = [\sigma_{ij}]$ is a symmetric stress tensor and $\mathbf{u} = [u_i]$ a displacement vector:

$$\int_V (\boldsymbol{\sigma} : \nabla\mathbf{u} + \nabla\boldsymbol{\sigma} \cdot \mathbf{u})\, dV = \int_S \boldsymbol{\sigma} \cdot \mathbf{u} \cdot \mathbf{n}\, dS. \tag{10.51}$$

Here $\nabla\mathbf{u} = [\partial u_i/\partial x_j]$ is an unsymmetric tensor called the deformation gradient. Its transpose is $\mathbf{u}^T\nabla^T = [\partial u_j/\partial x_i]$. Now $\boldsymbol{\sigma} : \nabla\mathbf{u} = (\boldsymbol{\sigma} : \nabla\mathbf{u})^T = \boldsymbol{\sigma} : \mathbf{u}^T\nabla^T = \boldsymbol{\sigma} : \frac{1}{2}(\nabla + \nabla^T) \cdot \mathbf{u} = \boldsymbol{\sigma} : \mathbf{D} \cdot \mathbf{u}$, where $\mathbf{D} = \frac{1}{2}(\nabla + \nabla^T)$. Hence

$$\int_V \boldsymbol{\sigma} : \mathbf{D} \cdot \mathbf{u}\, dV = -\int_V \nabla\boldsymbol{\sigma} \cdot \mathbf{u}\, dV + \int_S \boldsymbol{\sigma} \cdot \mathbf{u} \cdot \mathbf{n}\, dS. \tag{10.52}$$

In indicial notation this is

$$\int_V \sigma_{ij}\, \tfrac{1}{2}\Big(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\Big)\, dV = -\int_V \frac{\partial \sigma_{ij}}{\partial x_i} u_j\, dV + \int_S \boldsymbol{\sigma}_{ij} u_j n_i\, dS. \tag{10.53}$$

Recognizing that $e^u_{ij} = \frac{1}{2}(\partial u_j/\partial x_i + \partial u_i/\partial x_j)$ we finally arrive at

$$\int_V \sigma_{ij}\, e^u_{ij}\, dV = -\int_V \frac{\partial \sigma_{ij}}{\partial x_i} u_j\, dV + \int_S \boldsymbol{\sigma}_{ij} u_j n_i\, dS. \tag{10.54}$$

Taking the variation of this equation with respect to the displacements while keeping $\sigma_{ij}$ fixed yields the Principle of Virtual Work (PVW):

$$\int_V \sigma_{ij}\, \delta e^u_{ij}\, dV = -\int_V \frac{\partial \sigma_{ij}}{\partial x_i}\, \delta u_j\, dV + \int_S \boldsymbol{\sigma}_{ij}\, \delta u_j\, n_i\, dS. \tag{10.55}$$

So far $\sigma_{ij}$ and $e^u_{ij}$ are disconnected in (10.55) because no constitutive assumption has been stated in this derivation. Consequently the PVW is valid for arbitrary materials (for example, in plasticity), which underscores its generality. Setting $\sigma_{ij} = \sigma^u_{ij}$ provides the form used in §10.9.2.

## Homework Exercises for Chapter 10

## Three-Dimensional Linear Elastostatics

**EXERCISE 10.1** [A:10] Specialize the elasticity problem to a bar directed along $x_1$. Write down the field equations in indicial, tensor and matrix form.

**EXERCISE 10.2** [A:10] Justify the matrix form (10.30).

**EXERCISE 10.3** [A:20] Suppose that the displacement $\hat{\mathbf{u}}_P$ at an *internal* point $P(\mathbf{x}_P)$ is known. How can that condition be accomodated as a boundary condition on $S_u$? Hint: draw a little sphere of radius $\epsilon$ about $P$, then . . . oops I almost told the story.

**EXERCISE 10.4** [A:20] Justify passing from (10.38) to (10.39) by proving that if $\sigma_{ij}$ is symmetric, that is, $\sigma_{ij} = \sigma_{ji}$, then $\sigma_{ij}\lambda_{i,j} = \sigma_{ij}\,\frac{1}{2}(\lambda_{i,j} + \lambda_{j,i})$. Hint: one (elegant) way is to split $\lambda_{i,j} + \lambda_{j,i}$ into symmetric and antisymmetric parts; other approaches are possible.

**EXERCISE 10.5** [A:15] Prove that $\delta(\frac{1}{2}\sigma^u_{ij} e^u_{ij}) = \sigma^u_{ij}\,\delta e^u_{ij}$, where the variation $\delta$ is taken with respect to displacements $u_i$.