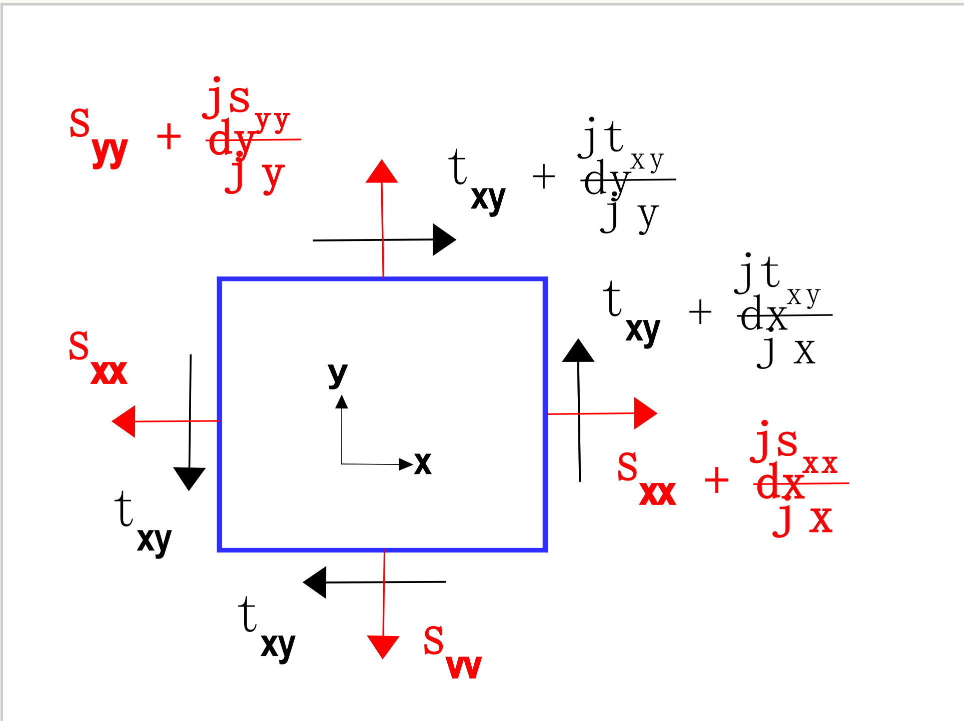
Equilibrium

Introduction

This page is all about $\sum \mathbf{F} = m\,\mathbf{a}$, except we will express the forces as stresses acting on differential sized areas. The first example will be 2-D, to minimize the complexity. Then the equations will be developed in 3-D. Finally, several examples will be presented.

2-D Equilibrium

The 2-D differential object is shown in the sketch at the right. The idea is to sum all the forces on it and set them equal to $m\,\mathbf{a}$. This can be done one component at a time, so start with the x-direction. The forces consist of



- σ_{xx} acting on face dy in the $-x$ direction
- τ_{xy} acting on face dx in the $-x$ direction
- $\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx$ acting on face dy in the $+x$ direction
- $\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} dy$ acting on face dx in the $+x$ direction
- Plus "body forces". These include any forces due to gravity, magnetism, etc, and are summarized simply as $\rho f_x dx dy$ where f_x is force per unit mass

The mass is density times volume: $\rho\, dx dy$.

Acceleration is simply a_x, although it is perfectly permissible to use the material derivative: $\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x}$.

Summing all this up gives

$$-\sigma_{xx}dy - \tau_{xy}dx + \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}dx\right)dy + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y}dy\right)dx + \rho f_x dxdy = \rho\, dxdy\, a_x$$

Cleaning up terms that cancel, and dividing through by $dxdy$ gives

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho f_x = \rho\, a_x$$

And summing forces in the y-direction leads to

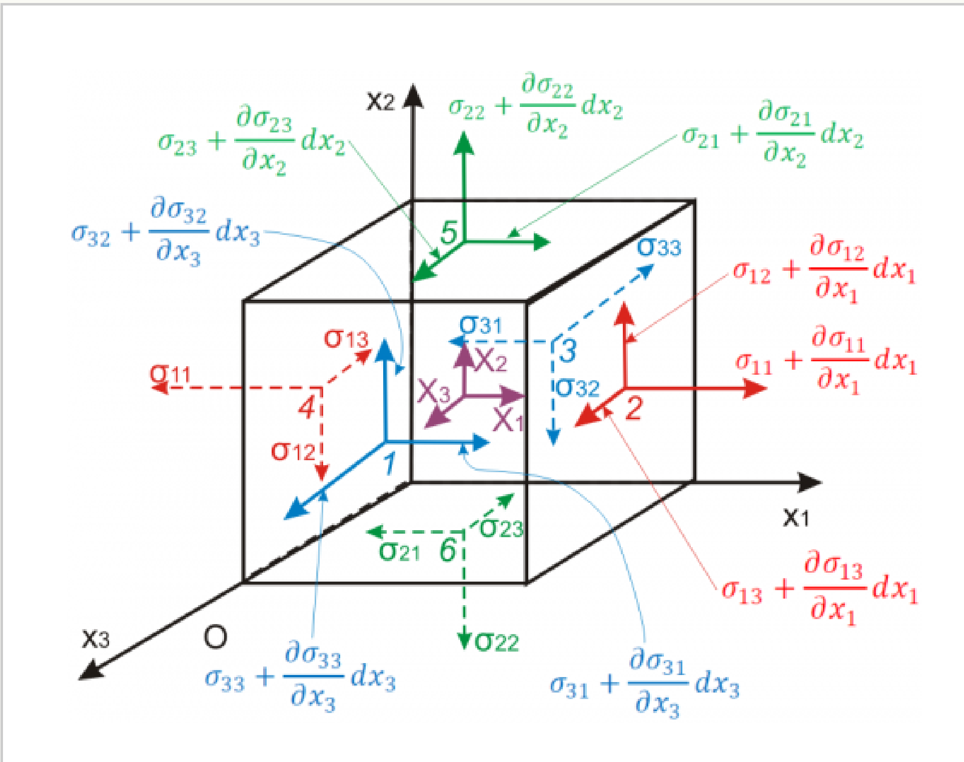
$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \rho f_y = \rho\, a_y$$

It is interesting how the equations tie together changes in all the different stress components, making them interdependent on each other.

An object is said to be in equilibrium when the right hand sides (RHS) of the equations are zero.

3-D Equilibrium

The process in 3-D is the same in principle, only there are more components involved. Performing the same exercise of summing forces in the x-direction and setting them equal to the x-direction acceleration goes as follows. (This time, an x_1, x_2, x_3 coordinate system is used.)



The forces consist of

- σ_{11} acting on face dx_2dx_3 in the $-x$ direction
- σ_{21} acting on face dx_1dx_3 in the $-x$ direction
- σ_{31} acting on face dx_1dx_2 in the $-x$ direction

- $\sigma_{11} + \frac{\partial \sigma 11}{\partial x 1} dx_1$ acting on face $dx_2 dx_3$ in the $+x$ direction

- $\sigma_{21} + \frac{\partial \sigma 21}{\partial x 2} dx_2$ acting on face $dx_1 dx_3$ in the $+x$ direction

- $\sigma_{31} + \frac{\partial \sigma 31}{\partial x 3} dx_3$ acting on face $dx_1 dx_2$ in the $+x$ direction

- The body force is $\rho f_x dx_1 dx_2 dx_3$ where f_x is force per unit mass

The mass is density times volume: $\rho\, dx_1 dx_2 dx_3$.

Acceleration is simply a_x, although it is perfectly permissible to use the material derivative: $\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x}$.

Summing all this up gives

$$\frac{\partial \sigma_{11}}{\partial x_1} dx_1 dx_2 dx_3 + \frac{\partial \sigma_{21}}{\partial x_2} dx_1 dx_2 dx_3 + \frac{\partial \sigma_{31}}{\partial x_3} dx_1 dx_2 dx_3 + \rho f_x = \rho\, dx_1 dx_2 dx_3 a_x$$

Dividing through by the differential volumes gives

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + \rho f_x = \rho\, a_x$$

And since the stress tensor is symmetric...

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + \rho f_x = \rho\, a_x$$

The complete set of equations is

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + \rho f_x = \rho\, a_x$$

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + \rho f_y = \rho\, a_y$$

$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho f_z = \rho\, a_z$$

All this is written in matrix and tensor notation as

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f} = \rho \mathbf{a} \qquad\qquad \sigma_{ij,j} + \rho f_i = \rho\, a_i$$

Or one could write the acceleration as the material derivative.

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f} = \rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \right) \qquad \sigma_{ij,j} + \rho f_i = \rho \left(v_{i,t} + v_k v_{i,k} \right)$$

Equilibrium in Cylindrical Coordinates

The equilibrium equations in cylindrical coordinates contain several additional terms, such as $\frac{\sigma_{\theta\theta}}{r}$ and $\frac{\sigma_{\theta z}}{r}$, that further complicate matters.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma_{rr}\right) + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} - \frac{\sigma_{\theta\theta}}{r} + \rho f_r = \rho a_r$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma_{r\theta}\right) + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{\sigma_{r\theta}}{r} + \rho f_\theta = \rho a_\theta$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma_{rz}\right) + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \rho f_z = \rho a_z$$

Centripetal Acceleration

It is possible to get a quick, rough estimate of the circumferential stress level in a tire undergoing axisymmetric centripetal forces during a high speed limit test. The radial acceleration equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma_{rr}\right) + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} - \frac{\sigma_{\theta\theta}}{r} + \rho f_r = \rho a_r$$

The radial acceleration is

$$a_r = -\frac{V^2}{r}$$

The other terms are expected to be negligible, except $\sigma_{\theta\theta}/r$. Setting these two equal to each other gives

$$-\frac{\sigma_{\theta\theta}}{r} = -\rho\frac{V^2}{r}$$

This simplifies to

$$\sigma_{\theta\theta} = \rho V^2$$

So for a tire spinning at 200 kph (= 55.55 m/s), with rubber density equal to 1,150 kg/m^3, the circumferential stress should be around

$$\sigma_{\theta\theta} \;=\; (1150\,\mathrm{kg/m^3})(55.55\,\mathrm{m/s})^2 \;=\; 3,500,000\;\mathrm{Pa} \;=\; 3.5\;\mathrm{MPa}$$

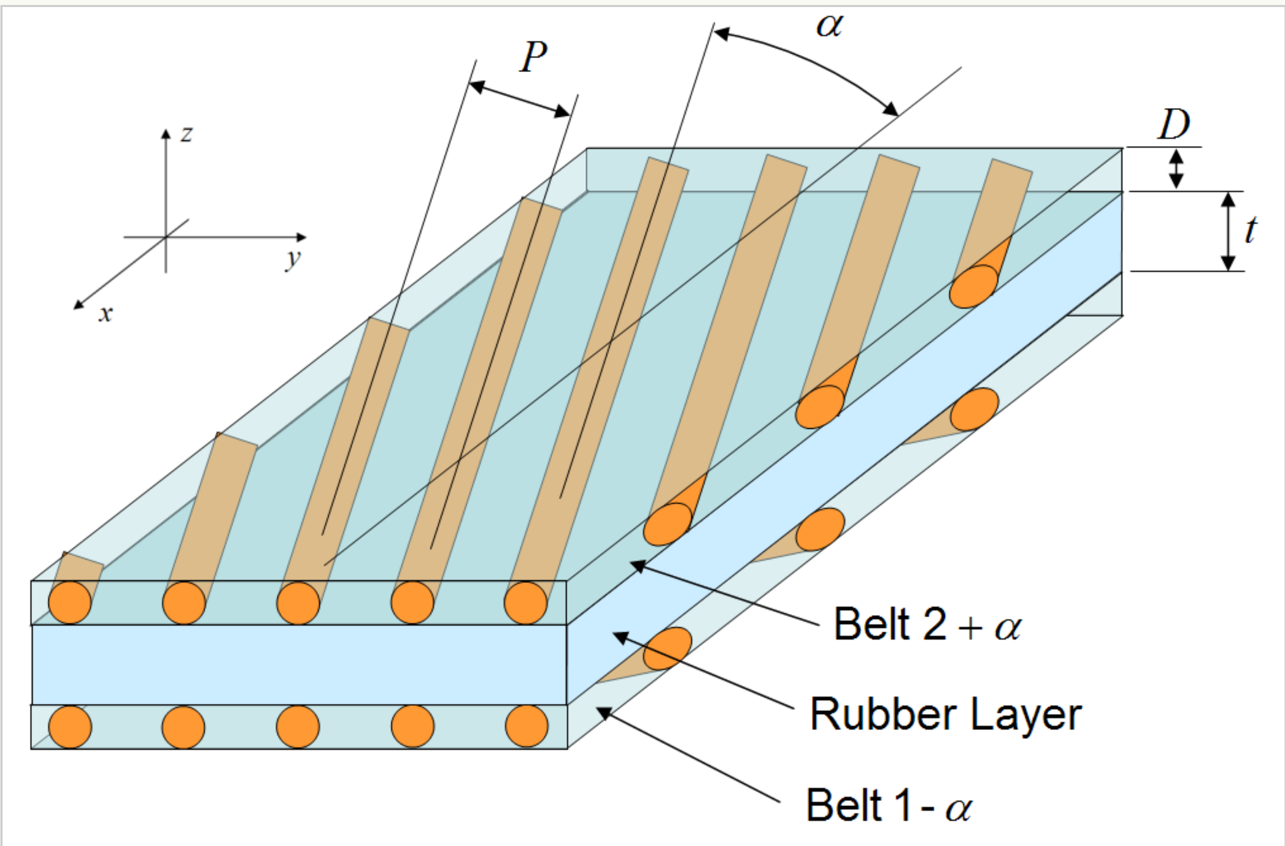
For the steel in the NSTs, the density is 7,800 kg/m^3, and the circumferential stress should be around

$$\sigma_{\theta\theta} \;=\; (7800\,\mathrm{kg/m^3})(55.55\,\mathrm{m/s})^2 \;=\; 24,000,000\;\mathrm{Pa} \;=\; 24\;\mathrm{MPa}$$

The fact that the tire is actually a nonhomogeneous composite probably makes the actual values significantly different from these estimates.

NST Shear Equilibrium

This example relates interply shear, γ_{xz}, which is present between the NSTs and peaks at the belt edge, to intraply shear stress, τ_{xy}, in the plane of the NSTs.



The main governing equilibrium equation for this situation is

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho f_x = \rho\, a_x$$

Assume that several terms in the equation are negligible, leaving only

$$\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0$$

The interply rubber layer develops shear, called γ_{xz}. Therefore the shear stress is

$$\tau_{xz} = G\gamma_{xz}$$

Now focus on the NST2. The shear stress, τ_{xz}, in the shear layer is the shear stress on the bottom surface of the NST2. But the shear stress on the top of the NST2 is about zero. So the the change in shear stress through the thickness of the NST2 is

$$\frac{\partial \tau_{xz}}{\partial z} = \frac{\tau_{\text{top}} - \tau_{\text{bottom}}}{D} = \frac{0 - G\gamma_{xz}}{D} = -\left(\frac{G}{D}\right)\gamma_{xz}$$

Substituting this into the equilibrium equation gives

$$\frac{\partial \tau_{xy}}{\partial y} - \left(\frac{G}{D}\right)\gamma_{xz} = 0$$

So the intraply shear in the NST can be related to the interply shear strain as

$$\tau_{xy} = \int \left(\frac{G}{D}\right)\gamma_{xz}\, dy$$

Granted, this equation my not be very useful by itself. But it is essential to the general analytical solution for the stresses and strains in the belts.

Equilibrium and the Speed of Stress Waves

It turns out that the equilibrium equation is very useful to the estimation of the speed of stress waves in materials. The process starts by pulling in a few seemingly unrelated topics. For starters, recall that the wave equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

where u represents displacement, and c is the speed of the stress waves in the material - effectively the speed of sound in the material. (And this is the focus of this discussion.)

Now bring up the following equilibrium equation

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho f_x = \rho\, a_x$$

and neglect the shear and body force terms, leaving only

$$\frac{\partial \sigma_{xx}}{\partial x} = \rho\, a_x$$

And now substitute several relationships. Begin by noting that $a_x = \frac{\partial^2 u}{\partial t^2}$ just like in the wave equation.

Next, note that σ_{xx} in the equilibrium equation is related to ϵ_{xx} by

$$\sigma_{xx} = E\epsilon_{xx}$$

for the case of uniaxial tension. But then, ϵ_{xx} is related to the displacements through

$$\epsilon_{xx} = \frac{\partial u}{\partial x}$$

Again, this is for the simple case of uniaxial tension. So stress can be related to displacements by

$$\sigma_{xx} \;=\; E\epsilon_{xx} \;=\; E\frac{\partial u}{\partial x}$$

And

$$\frac{\partial \sigma_{xx}}{\partial x} = E\frac{\partial^2 u}{\partial x^2}$$

Substituting all this into the equilibrium equation gives

$$E\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2}$$

Or

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{(E/\rho)} \frac{\partial^2 u}{\partial t^2}$$

Now the big finish.... Comparing this to the wave equation shows that

$$c^2 = \frac{E}{\rho} \qquad \text{or} \qquad c = \sqrt{\frac{E}{\rho}}$$

And that is the relationship for the speed of a uniaxial stress wave through a material, its speed of sound!

Speed of Sound in Materials

For steel, $E = 200(10)^9$ Pa and $\rho = 7,800\ \mathrm{kg/m^3}$. So this gives

$$c \;=\; \sqrt{\frac{E}{\rho}} \;=\; \sqrt{\frac{200(10)^9\ \mathrm{Pa}}{7,800\ \mathrm{kg/m^3}}} \;=\; 5\ \mathrm{km/s} \;=\; 5\ \mathrm{m/ms}$$

For aluminum, $E = 70(10)^9$ Pa and $\rho = 2,800\ \mathrm{kg/m^3}$. So this gives

$$c \;=\; \sqrt{\frac{E}{\rho}} \;=\; \sqrt{\frac{70(10)^9\ \mathrm{Pa}}{2,800\ \mathrm{kg/m^3}}} \;=\; 5\ \mathrm{km/s} \;=\; 5\ \mathrm{m/ms}$$

By coincidence, the speed of sound through both steel and aluminum is the same.

For rubber with $E = 1(10)^6$ Pa and $\rho = 1,150\ \mathrm{kg/m^3}$. So this gives

$$c \;=\; \sqrt{\frac{E}{\rho}} \;=\; \sqrt{\frac{1(10)^6\ \mathrm{Pa}}{1,150\ \mathrm{kg/m^3}}} \;=\; 29\ \mathrm{m/s} \;=\; 0.03\ \mathrm{m/ms}$$

Shear Wave Speeds

But we're not done! Take a look at shear waves. This time, bring up the following equilibrium equation

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \rho f_y = \rho\, a_y$$

and neglect all the terms except

$$\frac{\partial \tau_{yx}}{\partial x} = \rho\, a_y$$

And swap the subscripts on τ since it's symmetric.

Begin by substituting $a_y = \frac{\partial^2 v}{\partial t^2}$ just like in the wave equation.

And relate τ_{xy} to γ_{xy} by

$$\tau_{xy} = G\gamma_{xy}$$

And relate γ_{xy} to the displacements with the simple-shear assumption.

$$\gamma_{xy} = \frac{\partial v}{\partial x}$$

So the shear stress can be related to displacements by

$$\tau_{xy} \;=\; G\gamma_{xy} \;=\; G\frac{\partial v}{\partial x}$$

And

$$\frac{\partial \tau_{xy}}{\partial x} = G\frac{\partial^2 v}{\partial x^2}$$

Substituting all this into the equilibrium equation gives

$$G\frac{\partial^2 v}{\partial x^2} = \rho\frac{\partial^2 v}{\partial t^2}$$

Or

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{(G/\rho)}\,\frac{\partial^2 v}{\partial t^2}$$

Comparing this to the wave equation shows that

$$c^2 = \frac{G}{\rho} \qquad \text{or} \qquad c = \sqrt{\frac{G}{\rho}}$$

So for shear waves their speed depends on the shear modulus, G, not the elongation modulus, E. For incompressible materials, i.e., rubber, the shear modulus is one-third of the tension modulus, so shear waves propagate through rubber at $1/\sqrt{3}$, or 58% of the speed of uniaxial tension waves. For metals, the shear modulus is about 38% of the tension modulus. This translates to their shear wave speeds being 61% of their tension wave speeds.

Plane Wave Speeds

And finally, there are plane wave speeds. These are cases where the cross-sections of the objects are very large and hold the lateral strains constant at zero while the object undergoes tension/compression. This doesn't really apply to rubber because volume changes are involved.

To see what happens, go back to Hooke's Law for stress in terms of strain

$$\sigma_{ij} = \frac{E}{(1+\nu)} \left[\epsilon_{ij} + \frac{\nu}{(1-2\nu)} \delta_{ij} \epsilon_{kk} \right]$$

and impose the following: $\epsilon_{11} = \epsilon$, and $\epsilon_{22} = \epsilon_{33} = 0$. This gives

$$\sigma = \frac{E}{(1+\nu)} \left[\epsilon + \frac{\nu}{(1-2\nu)} \epsilon \right]$$

which simplifies to

$$\sigma = \frac{E\,(1-\nu)}{(1+\nu)(1-2\nu)} \epsilon$$

You can see that for rubber with $\nu = 0.5$, the stress required to generate any strain is infinite due to the $(1-2\nu)$ term in the denominator. This is because rubber is incompressible.

As before, substitute $\frac{\partial u}{\partial x}$ for ϵ. This gives

$$\sigma = \frac{E\,(1-\nu)}{(1+\nu)(1-2\nu)} \frac{\partial u}{\partial x}$$

And

$$\frac{\partial \sigma}{\partial x} = \frac{E\,(1-\nu)}{(1+\nu)(1-2\nu)}\,\frac{\partial^2 u}{\partial x^2}$$

Combining everything gives

$$\frac{E\,(1-\nu)}{(1+\nu)(1-2\nu)}\,\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2}$$

So the wave speed of plane waves is

$$c = \sqrt{\frac{(1-\nu)}{(1+\nu)(1-2\nu)}\left(\frac{E}{\rho}\right)}$$

So for metals with $\nu = 1/3$, the plane wave speed is 22% greater than the uniaxial tension case. But for incompressible rubber, the speed is infinite!
