

**Numerical simulation of thermo-elasticity, inelasticity and rupture in
membrane theory**

by

Michael James Taylor

B.A.(The Johns Hopkins University) 2003
M.S. (University of California, Berkeley) 2005

A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Engineering - Mechanical Engineering

in the

GRADUATE DIVISION
of the
UNIVERSITY OF CALIFORNIA, BERKELEY

Committee in charge:
Professor David J. Steigmann, Chair
Professor Tarek I. Zohdi
Professor Francisco Armero

Fall 2008

UMI Number: 3386160

All rights reserved

INFORMATION TO ALL USERS

The quality of this reproduction is dependent upon the quality of the copy submitted.

In the unlikely event that the author did not send a complete manuscript and there are missing pages, these will be noted. Also, if material had to be removed, a note will indicate the deletion.

UMI[®]

Dissertation Publishing

UMI 3386160

Copyright 2009 by ProQuest LLC.

All rights reserved. This edition of the work is protected against unauthorized copying under Title 17, United States Code.

ProQuest[®]

ProQuest LLC
789 East Eisenhower Parkway
P.O. Box 1346
Ann Arbor, MI 48106-1346

The dissertation of Michael James Taylor is approved:

Chair

Date

Date

Date

University of California, Berkeley

Fall 2008

**Numerical simulation of thermo-elasticity, inelasticity and rupture in
membrane theory**

Copyright 2008

by

Michael James Taylor

PREVIEW

Abstract

Numerical simulation of thermo-elasticity, inelasticity and rupture in membrane theory

by

Michael James Taylor

Doctor of Philosophy in Engineering - Mechanical Engineering

University of California, Berkeley

Professor David J. Steigmann, Chair

Two distinct two-dimensional theories for the modeling of thin elastic bodies are developed. These are demonstrated through numerical simulation of various types of membrane deformation.

The work includes a continuum thermomechanics-based theory for wrinkled thin films. The theory takes into account single-layer sheets as well as composite membranes made of multiple lamina. The resulting model is applied to the study of entropic elastic elastomers as well as Mylar/aluminum composite films. The latter has direct application in the area of solar sails. Several equilibrium deformations are illustrated numerically by applying the theory of dynamic relaxation to a finite difference discretization based on Green's theorem.

In addition, a shell theory based on the peridynamic theory of Silling is developed.

Peridynamics is a reformulation of classical continuum theory particularly suited to the modeling of damage and fracture. This theory is extended to include viscoelasticity and viscoplasticity. Several dynamic simulations are presented using a mesh-free explicit code.

Professor David J. Steigmann
Dissertation Committee Chair

To my parents, James and Linda.

Thank you.

Contents

List of Figures	v
List of Tables	ix
I Introduction	1
II Membrane Thermoelasticity	4
1 Introduction	5
2 Finite Thermoelasticity	7
3 Thermomechanical Weak Forms	12
3.1 Balance of Linear Momentum	13
3.2 Balance of Energy	14
4 Membrane Theory	15
4.1 Local Kinematic Constraints	24
5 Constitutive Theory	34
5.1 Entropic Elastic Materials	34
5.2 Hencky Materials	39
5.3 Wrinkling	40
5.3.1 Entropic Elastic Materials	43
5.3.2 Hencky Materials	44
5.4 Bi-layer Membranes	45
5.4.1 Two Hencky Layers	48
6 Numerical Solution Scheme	51
6.1 Introduction	51
6.2 Nondimensionalization	52

6.3	Spatial Discretization	53
6.4	Time Discretization	62
6.5	Complete Algorithm	63
7	Numerical Simulations	64
7.1	Material Properties	65
7.2	Additional Notes	65
7.3	Effect of Membrane Composition on Natural Width	66
7.4	Shear and Stretch of a Rectangular Sheet	66
7.4.1	Vulcanized Rubber	68
7.4.2	Mylar/Aluminum Composite	71
7.5	Square Sheet Subjected to Uniform Pressure	73
7.5.1	Vulcanized Rubber	75
7.5.2	Mylar/Aluminum Composite	77
7.6	Axial Stretch of a Square Sheet with a Traction/Heat Flux Free Hole	79
7.6.1	Vulcanized Rubber	79
7.6.2	Mylar/Aluminum Composite	81
7.7	Square Sheet with Central Hole Displaced Transversely and Twisted	85
7.7.1	Vulcanized Rubber	88
7.7.2	Mylar/Aluminum Composite	90
7.8	Simple Shear of a Square Sheet with Fixed Circular Hub Subjected to Uniform Pressure	93
7.8.1	Vulcanized Rubber	93
7.8.2	Mylar/Aluminum Composite	96
III	Peridynamic Analysis of Fracture, Viscoplasticity, and Viscoelasticity in Thin Bodies	100
8	Introduction	101
9	Peridynamic Theory	103
9.1	Three Dimensional Theory	103
9.2	Two Dimensional Theory for Thin Bodies	106
10	Constitutive Modeling	113
10.1	Damage	113
10.2	Viscoelasticity	114
10.3	Viscoplasticity	115
11	Numerical Solution Scheme	118
11.1	Membrane Model	118
11.2	Plate Model	120
11.3	Choosing Bond Stiffness and Horizon Size	121

12 Numerical Simulations	123
12.1 Tearing of a Rubber Sheet	123
12.2 Axial Stretch of Viscoelastic Rubber Sheet with Edge Crack	124
12.3 Penetration of Viscoelastic Rubber Sheet	127
12.4 Viscoplasticity Examples	130
12.5 Axial Stretch of Viscoplastic Metallic Foil	136
12.6 Penetration of Viscoplastic Metallic Foil	137
12.7 Axial Stretch of Viscoplastic Metallic Foil with Edge Crack	146
12.8 Axial Stretch of a Nonlinear Elastic Plate	151
 IV Summary and Discussion	 154
Bibliography	157

List of Figures

4.1	Membrane in Reference Configuration	17
4.2	Membrane in Deformed Configuration	17
6.1	Finite-difference mesh based on Green's theorem	54
7.1	Natural Width for Different Membrane Compositions	67
7.2	Max Principal Stretch in Shear/Stretch of Vulcanized Rubber at Room Temperature	68
7.3	Wrinkling in Shear/Stretch of Vulcanized Rubber at Room Temperature . .	69
7.4	Max Principal Stretch in Shear/Stretch of Vulcanized Rubber with External Heat Supply	70
7.5	Wrinkling in Shear/Stretch of Vulcanized Rubber with External Heat Supply	70
7.6	Temperature in Shear/Stretch of Vulcanized Rubber with External Heat Supply	71
7.7	Maximum Principal Stretch in Shear/Stretch of Composite Membrane at Room Temperature	72
7.8	Wrinkling in Shear/Stretch of Composite Membrane at Room Temperature	72
7.9	Maximum Principal Stretch in Shear/Stretch of Composite Membrane with External Heat Supply	73
7.10	Wrinkling in Shear/Stretch of Composite Membrane with External Heat Supply	74
7.11	Temperature in Shear/Stretch of Composite Membrane with External Heat Supply	74
7.12	Maximum Principal Stretch in Pressurized Vulcanized Rubber at Room Temperature	75
7.13	Maximum Principal Stretch in Pressurized Vulcanized Rubber with External Heat Supply	76
7.14	Temperature in Pressurized Vulcanized Rubber with External Heat Supply	76
7.15	Maximum Principal Stretch in Pressurized Composite at Room Temperature	77
7.16	Maximum Principal Stretch in Pressurized Composite with External Heat Supply	78
7.17	Temperature in Pressurized Composite with External Heat Supply	78
7.18	Maximum Principal Stretch in Vulcanized Rubber under Axial Stretch at Room Temperature	80

7.19 Wrinkling in Vulcanized Rubber under Axial Stretch at Room Temperature	80
7.20 Wrinkling Close-Up in Vulcanized Rubber under Axial Stretch at Room Temperature	81
7.21 Maximum Principal Stretch in Vulcanized Rubber under Axial Stretch with Heat Supply	82
7.22 Wrinkling in Vulcanized Rubber under Axial Stretch with Heat Supply . . .	82
7.23 Wrinkling Close-Up in Vulcanized Rubber under Axial Stretch with Heat Supply	83
7.24 Temperature in Vulcanized Rubber under Axial Stretch with Heat Supply .	83
7.25 Maximum Principal Stretch in Composite under Axial Stretch at Room Temperature	84
7.26 Wrinkling in Composite under Axial Stretch at Room Temperature	84
7.27 Wrinkling Close-Up in Composite under Axial Stretch at Room Temperature	85
7.28 Maximum Principal Stretch in Composite under Axial Stretch with Heat Supply	86
7.29 Wrinkling in Composite under Axial Stretch with Heat Supply	86
7.30 Wrinkling Close-Up in Composite under Axial Stretch with Heat Supply . .	87
7.31 Temperature in Composite under Axial Stretch with Heat Supply	87
7.32 Maximum Principal Stretch in Vulcanized Rubber with Displaced Central Hub at Room Temperature	88
7.33 Wrinkling in Vulcanized Rubber with Displaced Central Hub	89
7.34 Maximum Principal Stretch in Vulcanized Rubber with Displaced Central Hub with External Heat Supply	89
7.35 Temperature in Vulcanized Rubber with Displaced Central Hub with External Heat Supply	90
7.36 Maximum Principal Stretch in Composite with Displaced Central Hub at Room Temperature	91
7.37 Wrinkling in Composite with Displaced Central Hub	91
7.38 Maximum Principal Stretch in Composite with Displaced Central Hub with External Heat Supply	92
7.39 Temperature in Composite with Displaced Central Hub with External Heat Supply	92
7.40 Maximum Principal Stretch in Vulcanized Rubber in Pressurized Simple Shear at Room Temperature	94
7.41 Wrinkling in Vulcanized Rubber in Pressurized Simple Shear at Room Temperature	94
7.42 Maximum Principal Stretch in Vulcanized Rubber in Pressurized Simple Shear with External Heat Supply	95
7.43 Wrinkling in Vulcanized Rubber in Pressurized Simple Shear with External Heat Supply	95
7.44 Temperature in Vulcanized Rubber in Pressurized Simple Shear with External Heat Supply	96
7.45 Maximum Principal Stretch in Composite in Pressurized Simple Shear at Room Temperature	97

7.46	Wrinkling in Composite in Pressurized Simple Shear at Room Temperature	97
7.47	Maximum Principal Stretch in Composite in Pressurized Simple Shear with External Heat Supply	98
7.48	Wrinkling in Composite in Pressurized Simple Shear with External Heat Supply	98
7.49	Temperature in Composite in Pressurized Simple Shear with External Heat Supply	99
9.1	Reference Configuration in the Peridynamic Theory	104
9.2	Peridynamic Bond in the Current Configuration	105
10.1	Illustration of Elastic-Plastic Peridynamic Bond in Sublayer Method	116
10.2	Example Force vs. Extension Plot for Elastic-Plastic Peridynamic Bond in Sublayer Method	116
12.1	Tearing of Rubber Sheet, $t = 0.005s$	124
12.2	Tearing of Rubber Sheet, $t = 0.012s$	125
12.3	Tearing of Rubber Sheet, $t = 0.019s$	125
12.4	Tearing of Rubber Sheet, $t = 0.025s$	126
12.5	Tearing of Rubber Sheet, $t = 0.039s$	126
12.6	Axial Stretch of Viscoelastic Rubber Sheet with Edge Crack, $t = 0s$	127
12.7	Axial Stretch of Viscoelastic Rubber Sheet with Edge Crack, $t = 0.025s$	128
12.8	Axial Stretch of Viscoelastic Rubber Sheet with Edge Crack, $t = 0.052s$	128
12.9	Axial Stretch of Viscoelastic Rubber Sheet with Edge Crack, $t = 0.063s$	129
12.10	Axial Stretch of Viscoelastic Rubber Sheet with Edge Crack, $t = 0.066s$	129
12.11	Axial Stretch of Viscoelastic Rubber Sheet with Edge Crack, $t = 0.067s$	130
12.12	Penetration of Viscoelastic Rubber Sheet, $t = 0s$	131
12.13	Penetration of Viscoelastic Rubber Sheet, $t = 0.012s$	131
12.14	Penetration of Viscoelastic Rubber Sheet, $t = 0.025s$	132
12.15	Penetration of Viscoelastic Rubber Sheet, $t = 0.039s$	132
12.16	Penetration of Viscoelastic Rubber Sheet, $t = 0.052s$	133
12.17	Penetration of Viscoelastic Rubber Sheet, $t = 0.056s$	133
12.18	Penetration of Viscoelastic Rubber Sheet, $t = 0.066s$	134
12.19	Penetration of Viscoelastic Rubber Sheet, $t = 0.079s$	134
12.20	Penetration of Viscoelastic Rubber Sheet, $t = 0.093s$	135
12.21	Penetration of Viscoelastic Rubber Sheet, $t = 0.106s$	135
12.22	Bond Force vs. Extension in Metal Sheet Subjected to Central Impact	136
12.23	Axial Stretch of Dynamic Plastic Metal Sheet. Color Shows % Bonds Yielded. $t = 0s$.	137
12.24	Axial Stretch of Dynamic Plastic Metal Sheet. Color Shows % Bonds Yielded. $t = 0.009s$.	138
12.25	Axial Stretch of Dynamic Plastic Metal Sheet. Color Shows % Bonds Yielded. $t = 0.011s$.	138
12.26	Axial Stretch of Dynamic Plastic Metal Sheet. Color Shows % Bonds Yielded. $t = 0.016s$.	139

12.27	Axial Stretch of Dynamic Plastic Metal Sheet. Color Shows % Bonds Yielded. $t = 0.023s$	139
12.28	Axial Stretch of Dynamic Plastic Metal Sheet. Color Shows % Bonds Yielded. $t = 0.03s$	140
12.29	Axial Stretch of Dynamic Plastic Metal Sheet. Color Shows % Bonds Yielded. $t = 0.037s$	140
12.30	Axial Stretch of Dynamic Plastic Metal Sheet. Color Shows % Bonds Yielded. $t = 0.047s$	141
12.31	Penetration of Viscoplastic Metallic Sheet, $t = 0s$	141
12.32	Penetration of Viscoplastic Metallic Sheet, $t = 0.021s$	142
12.33	Penetration of Viscoplastic Metallic Sheet, $t = 0.0069s$	142
12.34	Penetration of Viscoplastic Metallic Sheet, $t = 0.01s$	143
12.35	Penetration of Viscoplastic Metallic Sheet, $t = 0.015s$	143
12.36	Penetration of Viscoplastic Metallic Sheet, $t = 0.021s$	144
12.37	Penetration of Viscoplastic Metallic Sheet, $t = 0.028s$	144
12.38	Penetration of Viscoplastic Metallic Sheet, $t = 0.035s$	145
12.39	Penetration of Viscoplastic Metallic Sheet, $t = 0.041s$	145
12.40	Penetration of Viscoplastic Metallic Sheet, $t = 0.047s$	146
12.41	Axial Stretch of Viscoplastic Metallic foil with Edge Crack, $t = 0s$	147
12.42	Axial Stretch of Viscoplastic Metallic foil with Edge Crack, $t = 0.0025s$	147
12.43	Axial Stretch of Viscoplastic Metallic foil with Edge Crack, $t = 0.0041s$	148
12.44	Axial Stretch of Viscoplastic Metallic foil with Edge Crack, $t = 0.0054s$	148
12.45	Axial Stretch of Viscoplastic Metallic foil with Edge Crack, $t = 0.0077s$	149
12.46	Axial Stretch of Viscoplastic Metallic foil with Edge Crack, $t = 0.011s$	149
12.47	Axial Stretch of Viscoplastic Metallic foil with Edge Crack, $t = 0.013s$	150
12.48	Axial Stretch of Viscoplastic Metallic foil with Edge Crack, $t = 0.016s$	150
12.49	Axial Stretch of a Nonlinear Elastic Plate, $t = 0s$	151
12.50	Axial Stretch of a Nonlinear Elastic Plate, $t = 0.011s$	152
12.51	Axial Stretch of a Nonlinear Elastic Plate, $t = 0.027s$	152
12.52	Axial Stretch of a Nonlinear Elastic Plate, $t = 0.054s$	153

List of Tables

7.1	Material Properties Used in Single-Layer Examples	65
7.2	Material Properties Used in Bi-Layer Examples	65

Acknowledgments

I owe a debt of gratitude to my advisor and mentor, Professor David Steigmann. I consider working with someone with his kindness and mastery of mechanics to be a great privilege. What I have learned from him extends well beyond what is contained within the following pages.

Several other professors also deserve acknowledgment. Chief among those is Tarek Zohdi. He played a big role in many of the key moments of my graduate career ranging from my decision to come to Cal all the way to being a member of this dissertation committee. Thanks to him and Professors G.I. Barenblatt, George Johnson and David Bogey for their confidence in me to continue towards this goal. Many thanks to Professor Francisco Armero for being on this committee and for offering helpful suggestions.

As a graduate student, you often receive a boost (or several) from the people in the trenches with you. In that spirit, I'd like to thank David Powell. Over the years, we've had probably hundreds of conversations ranging from finite elements to soap bubbles. Each one helped me understand the subject, whatever it was, just a little bit more.

I had the pleasure of working with Scott McCormick, Pete Graham and Michael Neuffer for a summer in the Hesse Hall shop and in two semesters teaching the shock absorber lab. These gentlemen do a terrific job supporting, maintaining, and improving student labs. From day one they made me feel a part of their team. I am grateful for them letting a theoretical fellow such as myself in on their fun.

On a more personal level, I'd like to offer big thanks to my parents. They always were behind me one hundred percent and encouraged me to work hard and aim high. That

I am here writing this is due, in large part, to their hard work as well. For that, I am grateful.

Thanks muchly to Erin Haynes for her love, support, and encouragement. Graduate school was much better with her beside me.

Finally, I'd like to acknowledge the financial support of the Department of Mechanical Engineering through the Powley Fund for Ballistics Research, grants, and guaranteed teaching positions.

PREVIEW

Part I

Introduction

This work is concerned with the modeling and numerical analysis of thin films and shells. Particularly, we are interested in deriving two-dimensional membrane and shell theories that include thermoelasticity, viscoelasticity, viscoplasticity and fracture. The models will be analyzed through numerical simulation of various equilibrium and dynamic deformations. The theories are developed and explored in the next two parts.

In Part II, we derive a theory of membrane thermoelasticity based on the familiar balance laws of continuum mechanics. The theory is developed for metallic and polymeric films and composite laminates made of combinations of both. Wrinkling and slackening are taken into account as well. A principal application of membrane thermoelasticity is to the analysis of solar sails, which consist of laminated films and reflective coatings that transfer the momentum of solar photons into thrust. Flexible polymer films with a number of industrial applications are also formed from laminae having various chemical compositions. The model is used to solve a number of equilibrium problems that demonstrate the effects of heating and mechanical loading on the stress field and consequent wrinkling pattern.

In Part III, we develop a shell model based on the peridynamic theory of S.A. Silling. The peridynamic theory was created as a reformulation of classical continuum mechanics and is especially useful in the model of damage and fracture. In peridynamics, the equation of motion is an integral equation as opposed to a differential equation. Thus, discontinuities at crack surfaces do not propose difficulties as they do in classical mechanics. This theory will be extended to encompass viscous effects (for certain rubber-like materials) and yielding (for viscoplastic materials that exhibit work-hardening). We apply a mesh-free numerical method to analyze the theory for several dynamic deformations of membranes.

In Part IV, we summarize the work and discuss the numerical results of Parts II and III. This section concludes the dissertation.

PREVIEW

Part II

Membrane Thermoelasticity

Chapter 1

Introduction

This part is concerned with the development of a theory for thermoelastic membranes and numerical simulation of illustrative equilibrium deformations. In the second chapter, we summarize some basic results in the theory of continuum finite thermoelasticity. Weak forms based on the three-dimensional equations of thermoelastic equilibrium are derived in chapter three. Chapter four develops the leading-order (membrane) theory based on these weak forms.

In chapter five, we investigate the constitutive modelling of metallic materials, polymers which exhibit thermal expansion and a special class of elastomers called entropic elastic materials. Entropic elastic materials experience the Gough-Joule effect which causes them to behave counter-intuitively in the presence of heat. A typical example of this type of material is vulcanized natural rubber. In this chapter the theory is also modified to incorporate wrinkling and the presence of multiple laminates. The particular application of a laminated sheet to the design of solar sails is discussed.

The numerical method used, dynamic relaxation, is described in chapter six. In dynamic relaxation, equilibria are considered to be long-time limits of damped dynamic problem. We discretize the problem in space using finite difference approximations based on Green's theorem and in time using an explicit central differencing scheme. Chapter seven demonstrates our numerical results, with particular focus on the effect of heat on both stretch and wrinkling.

Chapter 2

Finite Thermoelasticity

We begin by laying down the basic results for a thermoelastic material in the manner of Casey and Krishnaswamy [7]. The local equation for the balance of energy in the referential form is:

$$\rho_\kappa \dot{\epsilon} = \frac{1}{2} \mathbf{S} \cdot \dot{\mathbf{C}} + \rho_\kappa r - \text{Div} \mathbf{Q}, \quad (2.1)$$

where \mathbf{S} is the second Piola-Kirchoff Stress Tensor, \mathbf{E} is the Lagrangian strain tensor, ϵ is the internal energy per unit mass, r is the internal heat supply and \mathbf{Q} is the referential heat flux vector. Note, in addition, that the temperature in the deformed configuration is given by θ and the referential temperature gradient is given by $D\theta$.

Let the constitutive equations for a thermoelastic material be given as

$$\epsilon = \tilde{\epsilon}(\mathbf{F}, \theta) \quad \mathbf{S} = \tilde{\mathbf{S}}(\mathbf{F}, \theta) \quad \mathbf{Q} = \tilde{\mathbf{Q}}(\mathbf{F}, \theta, D\theta). \quad (2.2)$$

By enforcing invariance under superposed rigid body motion to the above constitutive laws, it can be shown that they must depend on the deformation gradient in the following way: