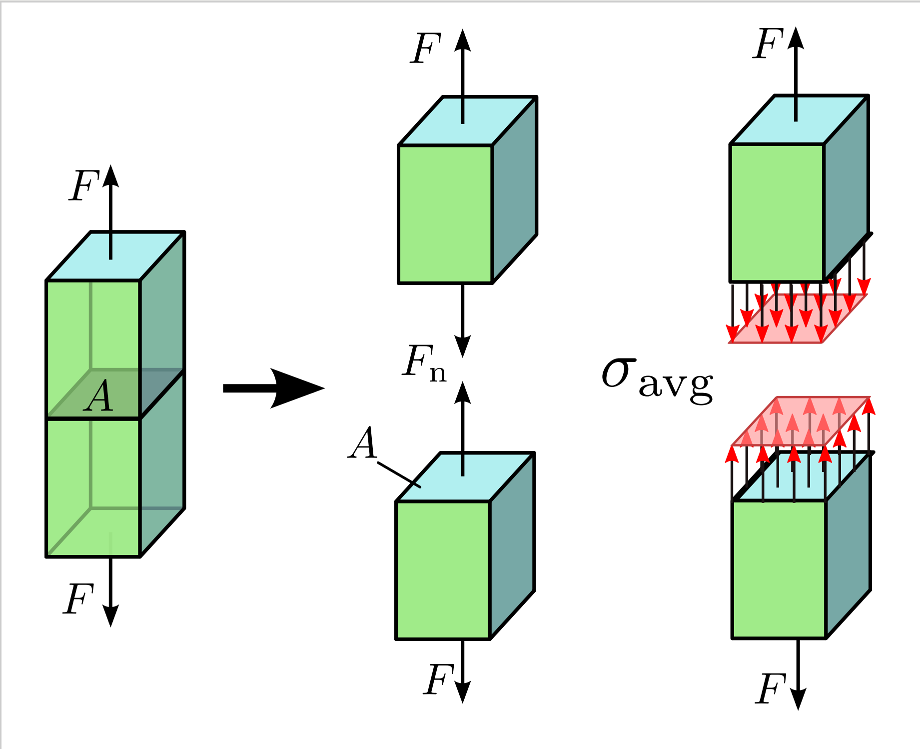
Stress

Introduction

This page reviews the familiar stress tensor. Stress is always simply ***Force/Area***, but some complexity does arrise because the relative orientation of the force vector to the surface normal dictates the type of stress. When the force vector is normal to the surface, as shown to the right, the stress is called normal stress and represented by σ.



Ref: http://en.wikipedia.org/wiki/File:Axial_stress.svg

When the force vector is parallel to the surface, the stress is called shear stress and represented by τ. When the force vector is somewhere in between, then its normal and parallel components are used as follows.

$$\sigma = \frac{F_{\text{normal}}}{A} \qquad \text{and} \qquad \tau = \frac{F_{\text{parallel}}}{A}$$

Of course, things can get complicated in nonlinear problems with large deformations (and rotations) because the final deformed area may be different from the initial area, among other things. We will ignore all these complexities for now and assume that before-and-after differences are negligible. So there will be no need to specify whether the force and area are for undeformed or deformed conditions.

Component Definitions

Many components are needed to capture a complete stress state. Consider the object in the 2-D example here that is being pulled in simple tension, though not in a direction parallel to any global axis. Standard practice is to (virtually) cut it perpendicular to the global axes as shown. This first cut results in an area with unit normal

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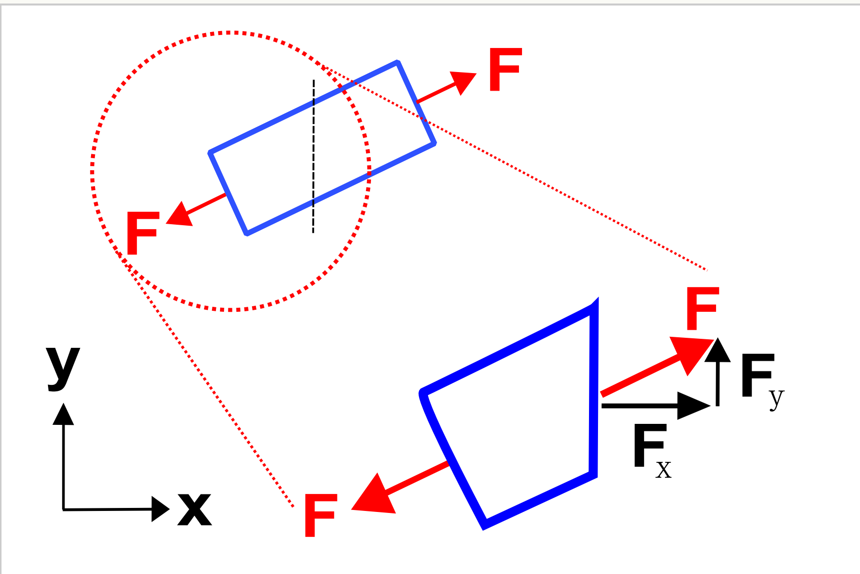
force on this area contains both normal and parallel components. The stresses are defined as



$$\sigma_{xx} = \frac{F_x}{A_x} \qquad \text{and} \qquad \tau_{xy} = \frac{F_y}{A_x}$$

Note how the two subscripts on the stress variables match those on the force and area components with one subscript coming from each.

Alternately, one could (virtually) cut the object horizontally to produce a surface with an outward normal in the y-direction. This leads to

$$\sigma_{yy} = \frac{F_y}{A_y} \qquad \text{and} \qquad \tau_{yx} = \frac{F_x}{A_y}$$

If a numerical example were worked out, one would notice an amazing result. It is that $\tau_{xy} = \tau_{yx}$. This will always be true in order to maintain rotational equilibrium. This is discussed in more detail next.

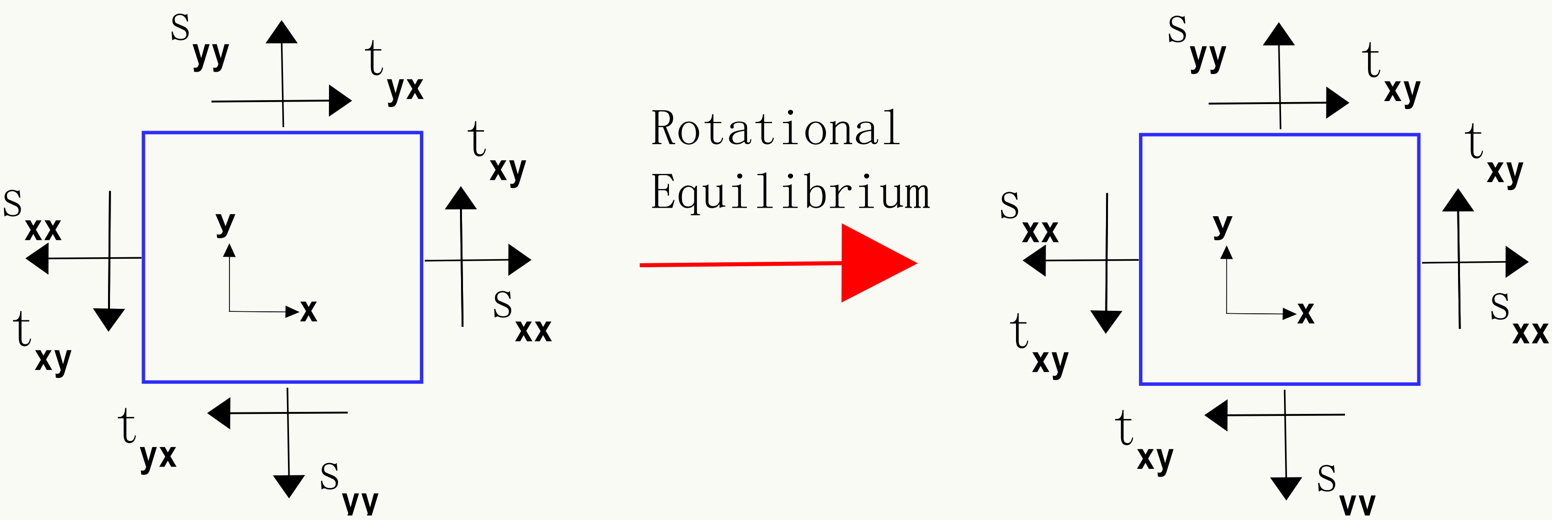
Equilibrium

The complete (2D) stress state at a point is shown below. The key difference between the left and right figures is the shear stresses. But they will be discussed later.

First, let's look at the normal stresses, σ_{xx} and σ_{yy}. Note how the x-normal stress, σ_{xx}, is present on both the left and right sides of each square in order to maintain horizontal equilibrium. These x-normal stresses represent tension because they point out of the square. Tensile normal stresses have positive values, and compressive normal stresses have negative values.

The y-normal stresses, σ_{yy}, are also present on two surfaces, top and bottom, in order to maintain vertical equilibrium. Like σ_{xx}, σ_{yy} is also drawn to represent tension, which is positive.

The difference between the left and right pictures is that τ_{yx} in the left figure is replaced by τ_{xy} in the right figure. The left figure contains two shear stress values, τ_{xy}, which rotates the square counter-clockwise, and τ_{yx}, which rotates the square clockwise. But if the two shear values are not equal, then the square will not be in rotational equilibrium. The only way to maintain rotational equilibrium is for τ_{xy} to be equal τ_{yx}. So there is no need to have two separate variables. The right figure contains only one, τ_{xy}.



2-D Notation

Stress is in fact a tensor. Why? Because it obeys standard coordinate transformation principles of tensors. This alone appears to be enough to make it so. It can be written in any of several different forms as follows. They are all identical.

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix}$$

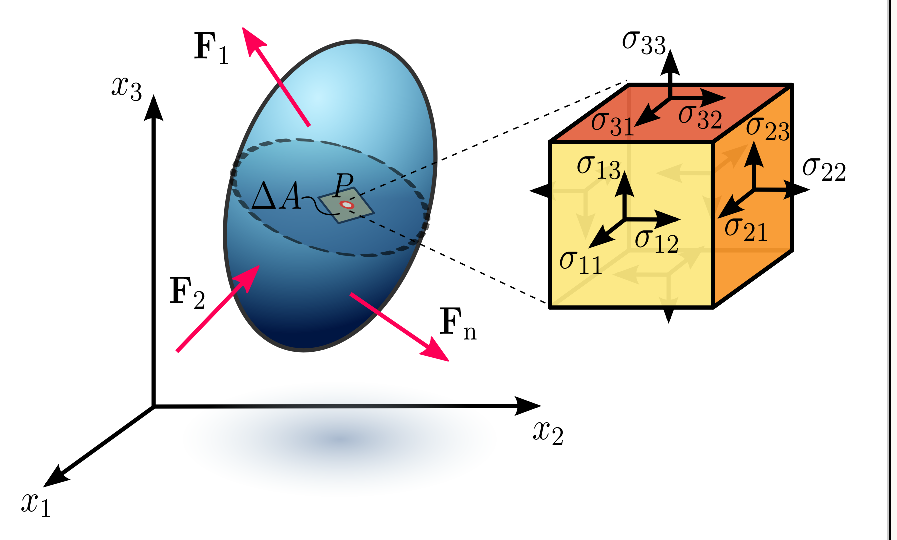
But since $\tau_{xy} = \tau_{yx}$, all the tensors can also be written as

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix}$$

Setting $\tau_{xy} = \tau_{yx}$ has the effect of making (requiring in fact) the stress tensors symmetric.

3-D Notation

All of the above conventions in 2-D also apply to the 3-D case. Notation for the 3-D case is as follows.



Ref:
http://commons.wikimedia.org/wiki/File:Stress_in_a_continuum.svg

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

But rotational equilibrium requires that $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, and $\tau_{yz} = \tau_{zy}$. This also produces symmetric tensors.

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}$$
