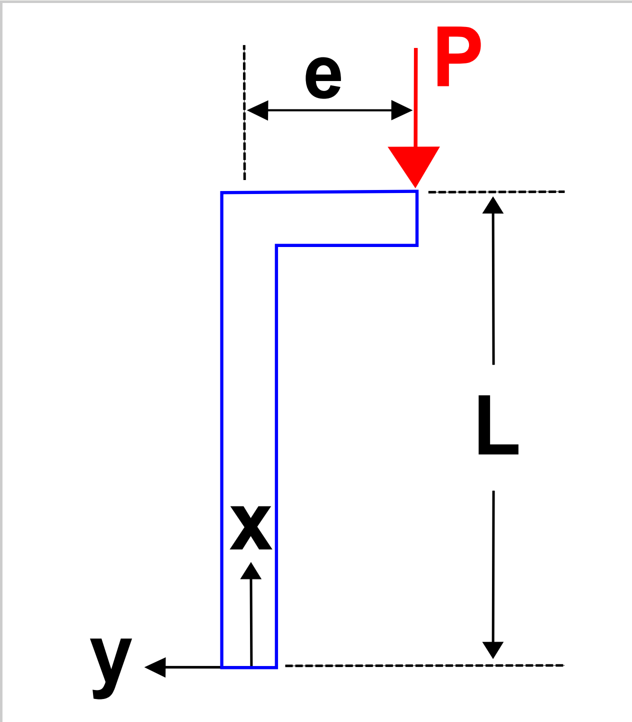
Buckling of Eccentrically Loaded Columns

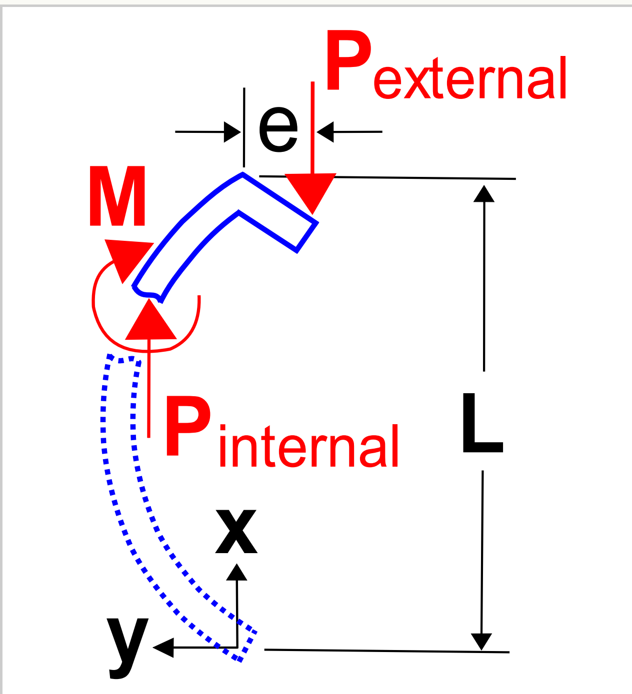
Introduction



Like classical column buckling theory, the buckling of columns under eccentric (offset) loads is also a topic of unique complexity. It is unique in that the analysis leads to nonlinear dependences of beam deflections and stresses on the applied load. As shown in the figure, a load, P, is eccentric when its line of action is offset a distance, e, from the column. This produces a bending moment equal to $P*e$ that exists in addition to the usual deflection-related moment in the classical theory.

Curiously, the mechanics of eccentric column buckling can be considered more straight-forward than those of the classical theory. For example, unlike classical column buckling, failure here can indeed be related to the material's yield strength. In fact, it will be shown that under eccentric loading conditions, failure will always occur due to stresses exceeding the material's yield strength before the load reaches its so-called critical value, P_{cr}.

Analytical Solution - Both Ends Pinned



Analysis of the buckling of eccentrically loaded columns begins just as that of classical Euler Buckling Theory, with the beam bending equation.

$$E\,I\,y'' = M$$

However, this time the bending moment is slightly more complex. It is now $M = -P\,(y+e)$, where P is the compressive load, y is the column deflection, and e is the offset distance of the load relative to the

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eccentricity.

Inserting this expression for M in the above differential equation produces

$$E\,I\,y'' + P\,y = -P\,e$$

which has the following general solution

$$y = A\sin\left(\sqrt{\frac{P}{E\,I}}\;x\right) + B\cos\left(\sqrt{\frac{P}{E\,I}}\;x\right) - e$$

where A and B are constants determined from the boundary conditions.

The boundary conditions are $y=0$ at $x=0$ and $x=L$, corresponding to both ends being pinned.

The first boundary condition, $y=0$ at $x=0$, leads to the conclusion that $B=e$. And this gives

$$y = A\sin\left(\sqrt{\frac{P}{E\,I}}\;x\right) + e\left[\cos\left(\sqrt{\frac{P}{E\,I}}\;x\right) - 1\right]$$

The second boundary condition, ($y=0$ at $x=L$), gives

$$A = e\,\tan\left(\sqrt{\frac{P}{E\,I}}\;\frac{L}{2}\right)$$

And the complete solution to the differential equation is

$$y = e\left[\tan\left(\sqrt{\frac{P}{E\,I}}\;\frac{L}{2}\right)\sin\left(\sqrt{\frac{P}{E\,I}}\;x\right) + \cos\left(\sqrt{\frac{P}{E\,I}}\;x\right) - 1\right]$$

Interesting Observations

It is quite interesting to note that unlike the classically loaded column of the previous page, this eccentrically loaded column has a straight-forward, well-behaved, solution for its lateral displacements. It is equally interesting to note that the dependence of these lateral displacements on the applied load is quite nonlinear. This is highly unusual for elastic analyses.

Equally interesting is the fact that the dependence on eccentricity, e, is linear even while the dependence on load is not. In fact, the deflection of every portion of the column is directly proportional to the eccentricity. Note that the equation reduces to the scenario of classical column buckling as e goes to zero because it predicts $y=0$ everywhere even in the presence of the load.

Returning to the case of nonzero eccentricity. Yet another fascinating property of the analytical solution above is its dependence on the tangent function. Recall that the tangent function goes to infinity as its argument approaches $\pi/2$. Therefore, the solution is telling us that the column deflections become infinite whenever

$$\sqrt{\frac{P}{EI}}\frac{L}{2} = \frac{\pi}{2}$$

Solving this equation for P gives the following result, which is remarkable because it is exactly the buckling solution for classical noneccentrically loaded columns.

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

Furthermore, it is independent of the eccentricity, e. This is completely counter-intuitive. All engineering intuition believes (strongly!) that the buckling load should decrease as the eccentricity increases. We will resolve this next... and engineering intuition will be satisfied.

Stresses

The resolution to the above dilemma, namely that the critical buckling load in an eccentrically loaded column is independent of the load's eccentricity, is found in the stresses generated by the beam's deformations. To see this, begin by recognizing that the stress in the column is governed by

$$\sigma = \frac{P}{A} + \frac{My}{I}$$

where P/A is the stress due to the compressive load, and My/I is the stress due to bending. Note that y here is not the column deflection. It represents the distance from the column's neutral axis. Note also that sign conventions are being completely ignored here because everything is being treated as a positive value, when in fact, the stress is clearly compressive. Even the applied load is being taken as a positive value. This is OK. Simply interpret the stress as an absolute value.

The question is, "What is the maximum compressive stress in the column?" This is found by inserting the conditions that maximize the effects of bending. The maximum bending stress occurs at a point in the cross-section farthest from the neutral axis, i.e., $y = c$. Second, it occurs where the bending moment is a maximum along the column's length. In other words

$$\sigma_{max} = \frac{P}{A} + \frac{M_{max}\, c}{I}$$

Recall from above that $M = P\,(y + e)$ where y now represents the beam/column deflection (and once again, negative signs are being ignored). So the maximum bending moment, M_{max} corresponds to the maximum beam deflection.

$$M_{max} = P\,(y_{max} + e)$$

So y_{max} is needed. This occurs at $x = L/2$ on the beam. Evaluating the column deflection at $L/2$ gives

$$y_{max} = e\left[\sec\left(\sqrt{\frac{P}{EI}}\,\frac{L}{2}\right) - 1\right]$$

Recall that $\sec(x) = 1/\cos(x)$.

Using this result and substituting back through the equations for M_{max} and σ_{max} leads to

$$\sigma_{max} = P\left[\frac{1}{A} + \frac{e\,c}{I}\sec\left(\sqrt{\frac{P}{EI}}\,\frac{L}{2}\right)\right]$$

Like the tangent function, the secant function also goes to infinity as its argument approaches $\pi/2$. The result is that the predicted stress also approaches infinity. This leads to a new column failure criterion, namely, the column will fail when the stress equals the material's yield stress: $\sigma_{max} = \sigma_{yield}$.

It is worth noting that the equation can also be written as

$$\sigma_{max} = P\left[\frac{1}{A} + \frac{e\,c}{I}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right]$$

where $P_{cr} = \pi^2\,EI\,/\,L^2$. This reveals that P_{cr} in eccentrically loaded columns is only an academic landmark. The column will actually fail when $\sigma_{max} = \sigma_{yield}$, and this will always occur at a load less than P_{cr}.

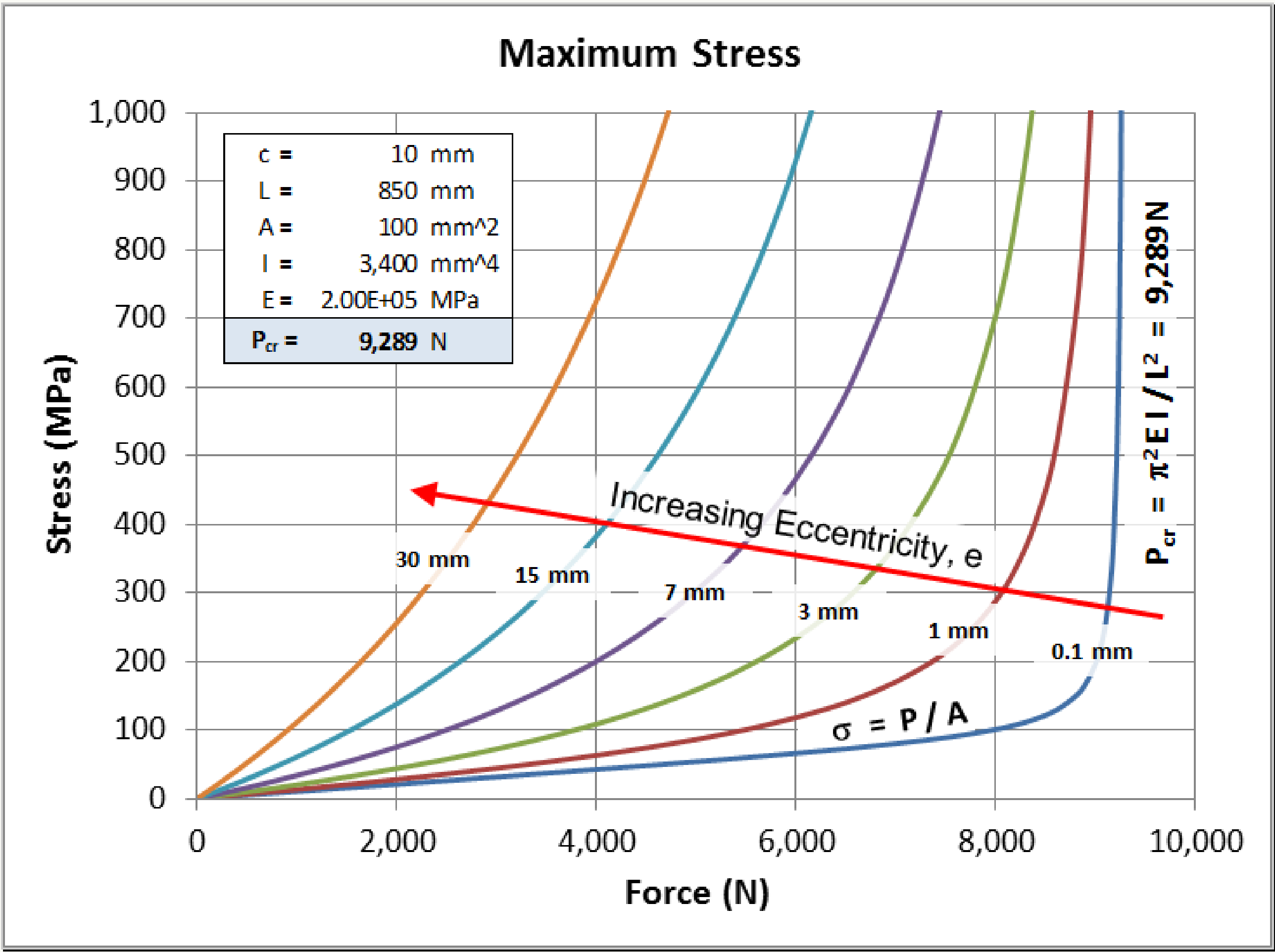
Eccentric Buckling Example

The graph here shows the predicted stress in an 850 mm long steel bar (E = 200,000 MPa) with cross-sectional area of 100 mm^2. The stress is plotted against load for a series of different eccentricities.

The graph reveals several important properties of eccentric column buckling. The story begins with the lowest (blue) curve for the near-zero eccentricity of 0.1 mm. This curve approaches the theoretical limit where the stress is simply

P/A until it suddenly goes to infinity at $P = P_{cr}$. Of course, any object will yield, and soon fail, when its stress goes to infinity, and this is consistent with the noneccentric buckling theory in which the column fails at $P = P_{cr}$.

Second, the load at which the stress grows rapidly, decreases with increasing offset, e, of the load. Therefore any such eccentrically loaded column will fail before P reaches P_{cr} because the stress in it will, at some point, exceed its yield strength. This is the property of eccentrically loaded columns that satisfies intuition even though P_{cr} is independent of e.



Dimensionless Parameters

The equation for stress can be cast in a nondimensional form, permitting all solutions to be plotted as a series of dimensionless parameters in a single graph. This is accomplished by first returning to the following equation.

$$\sigma_{max} = P\left[\frac{1}{A} + \frac{e\,c}{I}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right]$$

Next, factor area, A, out of the bracketed term

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{e\,c\,A}{I}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right]$$

and move it to the left hand side.

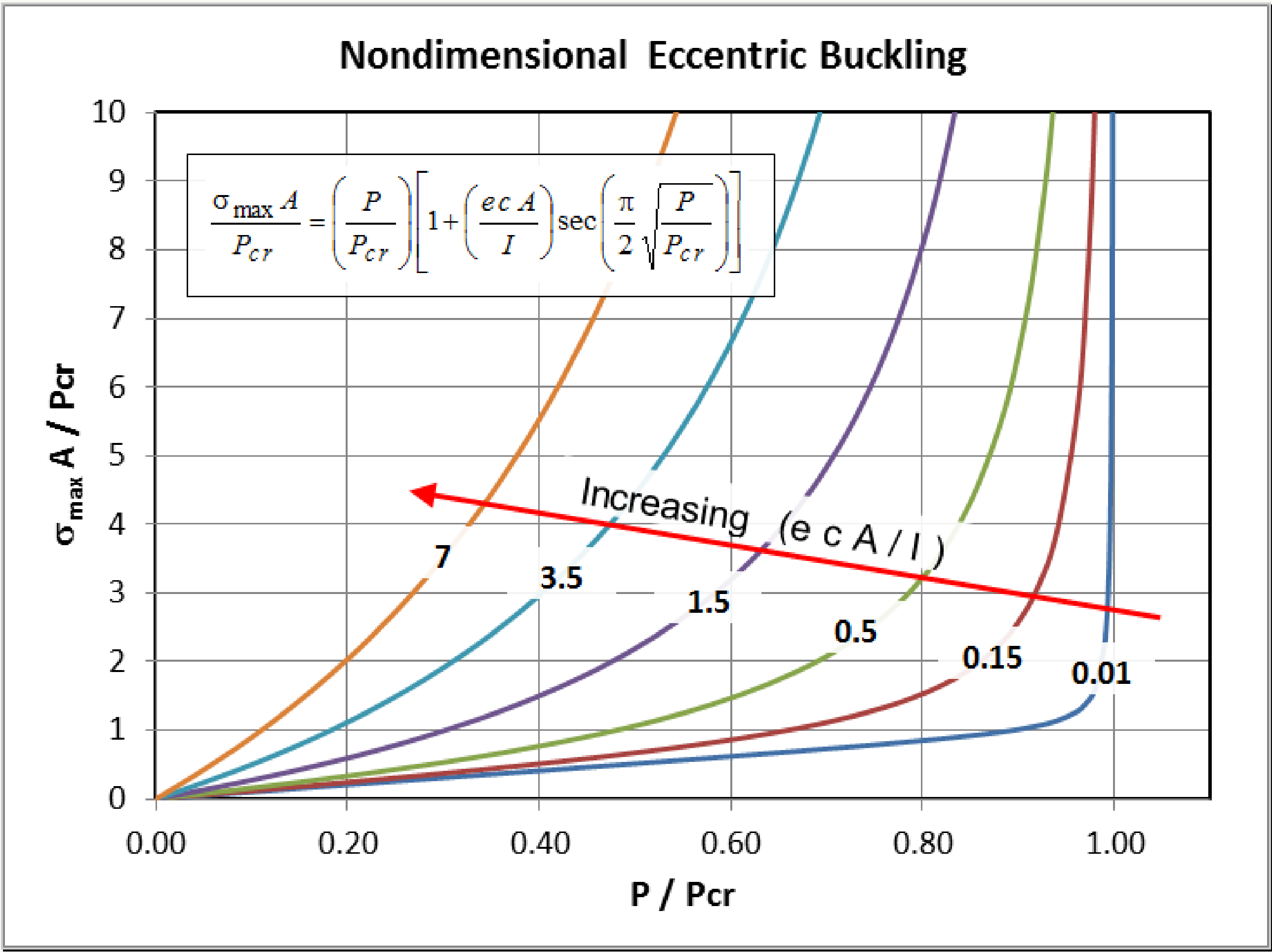
$$\sigma_{max}\, A = P \left[1 + \frac{e\, c\, A}{I} \sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right]$$

Finally, divide both sides by P_{cr}.

$$\frac{\sigma_{max}\, A}{P_{cr}} = \left(\frac{P}{P_{cr}}\right) \left[1 + \left(\frac{e\, c\, A}{I}\right) \sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right]$$

The equation is now nondimensional. It contains several dimensionless terms: $(\sigma_{max}\, A/P_{cr})$, $(e\, c\, A/I)$, and (P/P_{cr}).

All solutions of the equation can now be plotted as a series of master-curves on a single graph.



Dimensionless Parameter Calculation Example

Assume we have a 100 in long bar made of aluminum having $E = 10\text{E}6$ psi and yield strength of $\sigma_{yield} = 65,000$ psi. If the bar has $c = 0.5\,\text{in}$, $A = 1\,\text{in}^2$, and $I = 1\,\text{in}^4$, then P_{cr} is

$$P_{cr} = \frac{\pi^2 E I}{L^2} = \frac{\pi^2 * 10\text{E6 psi} * 1\,\text{in}^4}{(100\,\text{in})^2} = 9,870\,\text{lbf}$$

But! Now assume that the load is offset 3 in from the column axis. Now the column will fail at a lower load than P_{cr} because the stress in the column will reach the aluminum's yield strength. To determine how much lower, we first need to compute

$$\frac{\sigma_{yield} A}{P_{cr}} = \frac{65,000\,\text{psi} * 1\,\text{in}^2}{9,870\,\text{lbf}} = 6.59$$

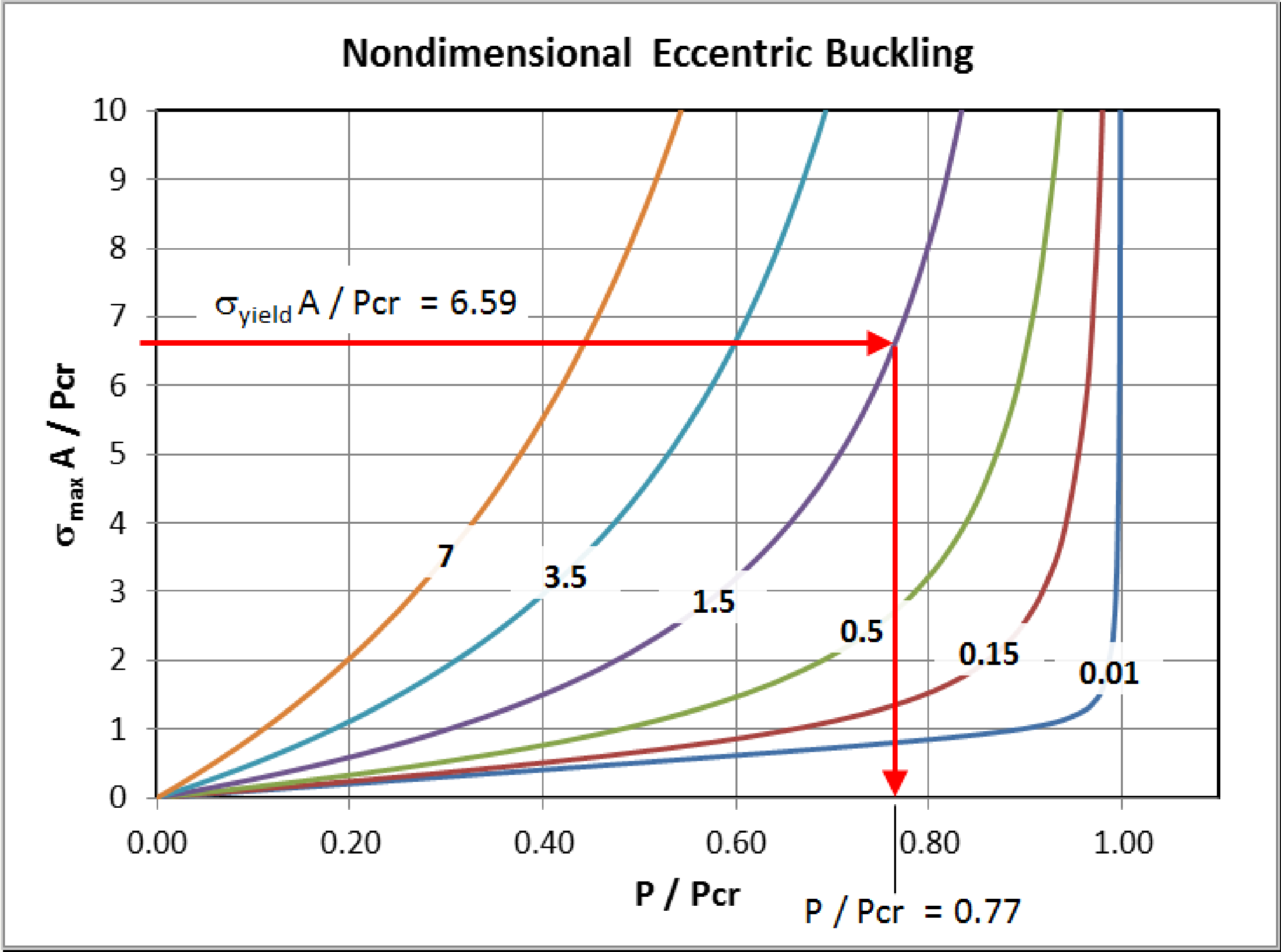
And we also need to calculate

$$\frac{e\,c\,A}{I} = \frac{3\,\text{in} * 0.5\,\text{in} * 1\,\text{in}^2}{1\,\text{in}^4} = 1.5$$

So all we need to do is start on the y-axis at a value of 6.59 and march across until we intercept the curve labeled "1.5" for ecA/I. We then turn down and find that we intercept the x-axis at $P/P_{cr} = 0.77$. The stress in the bar will therefore reach σ_{yield} at

$$P = 0.77 * 9,870\,\text{lbf} = 7,600\,\text{lbf}$$

It would be incorrect to assume that the bar can withstand 9,870 lbf compressive load if that load is offset 3 in from the bar's axis. Instead, the bar will start to yield at a load of only 7,600 lbf.



Ratio of Stresses

The dimensionless term containing stress merits additional comment. Rewriting it as

$$\frac{\sigma_{max}}{P_{cr}/A}$$

permits the denominator to be interpreted as the critical compressive stress that would cause the column to buckle (when $e = 0$). Call this σ_{cr}.

$$\frac{P_{cr}}{A} = \sigma_{cr}$$

The dimensionless stress term now becomes σ_{max}/σ_{cr}. As in the example above, it is common to equate the maximum stress in a column to the material's yield strength, σ_{yield}, in order to back-solve for P, the maximum load the column can support (not P_{cr}). This leads to the ratio

$$\frac{\sigma_{yield}}{\sigma_{cr}}$$

So the dimensionless term $\sigma_{max}A/P_{cr}$ is equivalent to, and can be interpreted as

$\sigma_{yield}/\sigma_{cr}$, the ratio of the material's yield strength to the critical stress, P_{cr}/A.

This stress ratio, $\sigma_{yield}/\sigma_{cr}$, is the focus of this discussion because of its unique interpretation. Specifically, it reflects the ratio of (i) the maximum possible load a column could carry before it yields if no buckling were to take place, to (ii) the maximum load it could actually carry when limited by buckling considerations (again, under the condition that $e = 0$). The higher this ratio, the more severe the buckling limitation is. For example, in the example above, it was equal to 6.59. If buckling could be prevented (perhaps through the use of lateral supports), then the column could support 6.59 times more load before it yields.