Assignment 4

Sujal - AI20BTECH11020

Download all latex codes from

https://github.com/https://github.com/sujal100/EE3900/blob/main/Assignment4/Assignment4.tex

Download all python codes from

https://github.com/https://github.com/sujal100/EE3900/blob/main/Assignment4/codes/code.py

1 Problem

(Linear forms Q-2.22) Find the shortest distance between the lines

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}, \tag{1.0.1}$$

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \tag{1.0.2}$$

2 Solution

In the given problem

$$\mathbf{A}_1 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 7 \\ -6 \\ 1 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}. \tag{2.0.1}$$

The lines will intersect if

$$\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 7 \\ -6 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{2.0.2}$$

$$\implies \lambda_1 \begin{pmatrix} 7 \\ -6 \\ 1 \end{pmatrix} - \lambda_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \tag{2.0.3}$$

$$\implies \begin{pmatrix} 7 & 1 \\ -6 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \tag{2.0.4}$$

Row reducing the augmented matrix,

$$\begin{pmatrix} 7 & 1 & 4 \\ -6 & -2 & 6 \\ 1 & 1 & 8 \end{pmatrix} \xleftrightarrow{R_3 \leftrightarrow R_1} \begin{pmatrix} 1 & 1 & 8 \\ -6 & -2 & 6 \\ 7 & 1 & 4 \end{pmatrix} \tag{2.0.5}$$

$$\xleftrightarrow[R_3=-7R_1+R_3]{R_2=6R_1+R_2} \begin{pmatrix} 1 & 1 & 8 \\ 0 & 4 & 54 \\ 0 & -6 & -52 \end{pmatrix} \xleftrightarrow{R_2=\frac{R_2}{4}} \begin{pmatrix} 1 & 1 & 8 \\ 0 & 1 & \frac{27}{2} \\ 0 & -6 & -52 \end{pmatrix} \tag{2.0.6}$$

$$\xleftrightarrow{R_3=6R_2+R_3} \begin{pmatrix} 1 & 1 & 8 \\ 0 & 1 & \frac{27}{2} \\ 0 & 0 & 29 \end{pmatrix} \tag{2.0.7}$$

The above matrix has $rank = 3$. Hence, the lines do not intersect. Note that the lines are not parallel but they lie on parallel planes. Such lines are known as *skew* lines as can be seen in Fig 0.

$\therefore$ The distance between given two lines is (using equation (2.0.4))

$$\left\lVert \begin{pmatrix} 7 & 1 \\ -6 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} - \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \right\rVert \tag{2.0.8}$$

We know that, the minimizer of $\|\mathbf{Ax} - \mathbf{B}\|$ is given by the solution to the normal equations $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{B}$. Since

$$\begin{pmatrix} 7 & 1 \\ -6 & -2 \\ 1 & 1 \end{pmatrix}^T \begin{pmatrix} 7 & 1 \\ -6 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 86 & 20 \\ 20 & 6 \end{pmatrix} \tag{2.0.9}$$

$$\begin{pmatrix} 7 & 1 \\ -6 & -2 \\ 1 & 1 \end{pmatrix}^T \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.10}$$

the normal equations give us the following system of equations

$$\begin{pmatrix} 86 & 20 \\ 20 & 6 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.11}$$

whose solution is $\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$. The minimum distance between this two lines is, thus,

$$\left\|\begin{pmatrix}7 & 1\\-6 & -2\\1 & 1\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}4\\6\\8\end{pmatrix}\right\| = \left\|\begin{pmatrix}4\\6\\8\end{pmatrix}\right\| \tag{2.0.12}$$

$$= \sqrt{116} \tag{2.0.13}$$

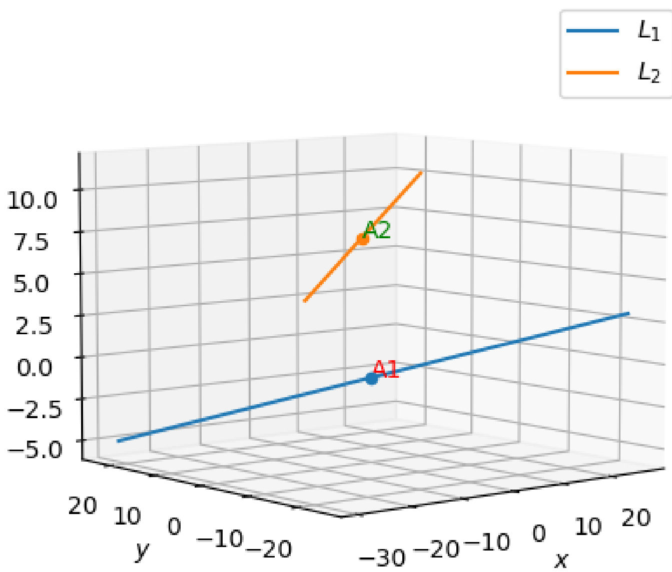

Fig. 0: plot of lines