Assignment 6

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1 Problem [CSIR NET(JUNE-2017) MATHS-STATISTICS (Q-50)]

$X_1, X_2, \cdots$ are independent identically distributed random variables having common density f. Assume $f(x) = f(-x)$ for all $x \in \mathbb{R}$. Which of the following statements is correct?

a) $\dfrac{1}{n}(X_1 + \cdots + X_n) \to 0$ in probability

b) $\dfrac{1}{n}(X_1 + \cdots + X_n) \to 0$ almost surely

c) $\Pr\left(\dfrac{1}{\sqrt{n}}(X_1 + \cdots + X_n) < 0\right) \to \dfrac{1}{2}$

d) $\sum_{i=1}^{n} X_i$ has the same distribution as $\sum_{i=1}^{n}(-1)^i X_i$

2 Solution

We using

a)(1) Let $X_1, X_2, \ldots$ be an infinite sequence of random variables, and let Y be another random variable. Then the sequence $\{X_n\}$ converges in probability to Y, if

$$\forall \epsilon > 0, \lim_{n \to \infty} \Pr\left(|X_n - Y| \geq \epsilon\right) = 0, \quad (2.0.1)$$

and we write

$$X_n \xrightarrow{P} Y. \quad (2.0.2)$$

(2) Let $X_1, X_2, \ldots$ be an infinite sequence of random variables. We shall say that the sequence $\{X_i\}$ converges with probability 1 (or converges almost surely (a.s.)) to a random variable Y, if

$$\Pr\left(\lim_{n \to \infty} X_n = Y\right) = 1 \quad (2.0.3)$$

and we write

$$X_n \xrightarrow{a.s.} Y \quad (2.0.4)$$

(3) Let $X_1, X_2, \ldots$ be an infinite sequence of random variables, If $\mathbf{E}\left[|X_1|\right] < \infty$. Then, as $n \to \infty$, we have

$$\frac{S_n}{n} \xrightarrow{a.s.} \mathbf{E}\left[X_1\right] \implies \frac{S_n}{n} \xrightarrow{P} \mathbf{E}\left[X_1\right], \quad (2.0.5)$$

$$\text{where, } S_n = X_1 + \cdots + X_n \quad (2.0.6)$$

using SLLN, (B) are incorrect option.

b) Let $Z, Z_1, Z_2, \ldots$ be random variables. Suppose $Z_n \to Z$ with probability 1. Then $Z_n \to Z$ in probability. That is, if a sequence of random variables converges almost surely, then it converges in probability to the same limit and we say

$$Z_n \xrightarrow{a.s.} Z \implies Z_n \xrightarrow{P} Z. \quad (2.0.7)$$

also in probability also hold this equation. Hence (A) are incorrect option.

c) $X_1, X_2, \cdots, X_n$ are independent identically distribute $\sim N[0, 2]$
Consider,

$$Y = \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} \quad (2.0.8)$$

So,

$$E(Y) = E\left(\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}\right) = 0 \quad (2.0.9)$$

$$V(Y) = V\left(\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}\right) = \frac{1}{n}2n = 2 \quad (2.0.10)$$

$$Y \sim N[0, 2] \quad (2.0.11)$$

we know that,

$$f(x) = f(-x) \implies \text{Symmetry about Zero}, \quad (2.0.12)$$

So,

$$\Pr\left(Y < 0\right) = \frac{1}{2} \quad (2.0.13)$$

$$\Pr\left(\frac{1}{\sqrt{n}}(X_1 + \cdots + X_n) < 0\right) = \frac{1}{2} \quad (2.0.14)$$

Hence, (C) is incorrect option.

d) For a scalar random variable X the characteristic function is defined as the expected value of e^{itX}, where i is the imaginary unit, and $t \in \mathbf{R}$ is the argument of the characteristic function:

$$\begin{cases} \varphi_X : \mathbb{R} \to \mathbb{C} \\ \varphi_X(t) = \mathrm{E}\left[e^{itX}\right] = \int_{\mathbb{R}} e^{itx} dF_X(x) \\ = \int_{\mathbb{R}} e^{itx} f_X(x) dx = \int_0^1 e^{itQ_X(p)} dp \end{cases} \quad (2.0.15)$$

Here F_X is the cumulative distribution function of X, Consider, $\phi_x(t)$ is characteristic function of $X_i, i = 1, \ldots, n$.

$$f(x) = f(-x) \implies \phi_x(t) = \phi_{-x}(t) \quad (2.0.16)$$

Therefore,

$$\phi_{\sum_{i=1}^n X_i}(t) = \phi_{X_1 + \ldots + X_n}(t) = \phi_{X_1}(t) \cdots \phi_{X_n}(t) \quad (2.0.17)$$

$$= [\phi_x(t)]^n \quad (2.0.18)$$

similarly,

$$\phi_{\sum_{i=1}^n (-1)^i X_i}(t) = \phi_{-X_1} + \phi_{X_2} + \ldots + \phi_{(-1)^n X_n}(t) \quad (2.0.19)$$

$$= \phi_{-X_1}(t) \cdot \phi_{X_2}(t) \cdots \phi_{(-1)^n X_n}(t) \quad (2.0.20)$$

$$= [\phi_x(t)]^n \quad (2.0.21)$$

$$\phi_{\sum_{i=1}^n X_i}(t) = \phi_{\sum_{i=1}^n (-1)^i X_i}(t) \quad (2.0.22)$$

$\therefore \sum_{i=1}^n X_i$ has same distribution as $\sum_{i=1}^n (-1)^i X_i$. Hence, only (D) is correct option.