

Assignment 7

Sujal - AI20BTECH11020

Download all latex codes from

https://github.com/sujal100/Probability_and_Random_variable/blob/main/exercise_6/exercise_6_main_tex.tex

1 PROBLEM [CSIR NET(JUNE-2017) MATHS-STATISTICS (Q-104)]

Let $\{X_n, n \geq 1\}$ be i.i.d. uniform $(-1, 2)$ random variables. Which of the following statements are true?

- a) $\frac{1}{n} \sum_{i=1}^n X_i \rightarrow 0$ almost surely
- b) $\left\{ \frac{1}{2n} \sum_{i=1}^n X_{2i} - \frac{1}{2n} \sum_{i=1}^n X_{2i-1} \right\} \rightarrow 0$ almost surely
- c) $\sup \{X_1, X_2, \dots\} = 2$ almost surely
- d) $\inf \{X_1, X_2, \dots\} = -1$ almost surely

2 SOLUTION

We use convergence in almost surely and Strong law of large number (SLLN)

- 1) Let X_1, X_2, \dots be an infinite sequence of random variables. We shall say that the sequence $\{X_i\}$ converges with probability 1 (or converges almost surely (a.s.)) to a random variable Y , if

$$\Pr\left(\lim_{n \rightarrow \infty} X_n = Y\right) = 1 \quad (2.0.1)$$

$$\text{and we write, } X_n \xrightarrow{a.s.} Y \quad (2.0.2)$$

- 2) Let X_n be i.i.d with $\mathbf{E}[|X_1|] < \infty$. Then, as $n \rightarrow \infty$, we have

$$\frac{S_n}{n} \xrightarrow{a.s.} \mathbf{E}[X_1] \implies \frac{S_n}{n} \xrightarrow{P} \mathbf{E}[X_1] \quad (2.0.3)$$

$$\text{, where } S_n = X_1 + \dots + X_n \quad (2.0.4)$$

also,

$$X_i \xrightarrow{a.s.} X \implies g(X_i) \xrightarrow{a.s.} g(X) \quad (2.0.5)$$

a)

$$\frac{1}{n} (X_1 + \dots + X_n) \rightarrow E(X) \in (-1, 2) \quad (2.0.6)$$

$$\text{as } n \rightarrow \infty, \quad (2.0.7)$$

according to strong law of large numbers (SLLN).

So, option (A) is incorrect.

- b) using this 2.0.5, we solve as

$$\left\{ \frac{1}{2n} \sum_{i=1}^n X_{2i} - \frac{1}{2n} \sum_{i=1}^n X_{2i-1} \right\} \xrightarrow{a.s.} \left\{ \frac{nX}{2n} - \frac{nX}{2n} \right\} \quad (2.0.8)$$

$$= 0 \quad (2.0.9)$$

option (B) is correct.

- c) A lower bound of a subset S of a partially ordered set (P, \leq) is an element a of P such that

$$a \leq x \quad \forall x \in S \quad (2.0.10)$$

A lower bound a of S is called an infimum (or greatest lower bound, or meet) of S if for all lower bounds y of S in P , $y \leq a$ (a is larger than or equal to any other lower bound). Using $X_i \xrightarrow{a.s.} X$ this, we conclude that

$$\sup \{X_1, X_2, \dots\} = 2 \text{ almost surely} \quad (2.0.11)$$

- d) Similarly, an upper bound of a subset S of a partially ordered set (P, \leq) is an element b of P such that

$$b \geq x \quad \forall x \in S \quad (2.0.12)$$

An upper bound b of S is called a supremum (or least upper bound, or join) of S if for all upper bounds z of S in P , $z \geq b$ (b is less than any other upper bound). Again using $X_i \xrightarrow{a.s.} X$ this, we conclude that

$$\inf \{X_1, X_2, \dots\} = -1 \text{ almost surely} \quad (2.0.13)$$

Hence (B), (C) and (D) are correct options.