Assignment 6

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Download all latex codes from

https://github.com/https://github.com/sujal100/Probability_and_Random_variable/blob/main/exercise_6/exercise_6_main_tex.tex

1 Problem [CSIR NET(JUNE-2017) MATHS-STATISTICS (Q-50)]

$X_1, X_2, \cdots$ are independent identically distributed random variables having common density f. Assume $f(x) = f(-x)$ for all $x \in \mathbb{R}$. Which of the following statements is correct?

a) $\frac{1}{n}(X_1 + \cdots + X_n) \to 0$ in probability
b) $\frac{1}{n}(X_1 + \cdots + X_n) \to 0$ almost surely
c) $\Pr\left(\frac{1}{\sqrt{n}}(X_1 + \cdots + X_n) < 0\right) \to \frac{1}{2}$
d) $\sum_{i=1}^{n} X_i$ has the same distribution as $\sum_{i=1}^{n}(-1)^i X_i$

2 Solution

We using

a)(1) *Convergence in probability :* Let $X_1, X_2, \ldots$ be an infinite sequence of random variables, and let Y be another random variable. Then the sequence $\{X_n\}$ converges in probability to Y, if

$$\forall \epsilon > 0, \lim_{n\to\infty} \Pr(|X_n - Y| \geq \epsilon) = 0, \tag{2.0.1}$$

and we write

$$X_n \xrightarrow{P} Y. \tag{2.0.2}$$

(2) *Convergence in almost surely :* Let $X_1, X_2, \ldots$ be an infinite sequence of random variables. We shall say that the sequence $\{X_i\}$ converges with probability 1 (or converges almost surely (a.s.)) to a random variable Y, if

$$\Pr\left(\lim_{n\to\infty} X_n = Y\right) = 1 \tag{2.0.3}$$

and we write

$$X_n \xrightarrow{a.s.} Y \tag{2.0.4}$$

(3) *Strong law of large number(SLLN) :* Let $X_1, X_2, \ldots$ be an infinite sequence of random variables, If $\mathbf{E}[|X_1|] < \infty$. Then, as $n \to \infty$, we have

$$\frac{S_n}{n} \xrightarrow{a.s.} \mathbf{E}[X_1] \implies \frac{S_n}{n} \xrightarrow{P} \mathbf{E}[X_1], \tag{2.0.5}$$

$$\text{where, } S_n = X_1 + \cdots + X_n \tag{2.0.6}$$

using SLLN, (B) are incorrect option.

b) *Relation between in probability and almost surely :* Let $Z, Z_1, Z_2, \ldots$ be random variables. Suppose $Z_n \to Z$ with probability 1. Then ,we say

$$Z_n \xrightarrow{a.s.} Z \implies Z_n \xrightarrow{P} Z. \tag{2.0.7}$$

(2.0.5), also in probability also hold this equation. Hence (A) is incorrect option.

c) *Central Limit Theorem :* Let $X_1, X_2, \ldots$ be i.i.d. with finite mean μ and finite variance σ^2. Let $Z \sim N(0, 1)$. Set $S_n = X_1 + \cdots + X_n$, and

$$Z_n = \frac{S_n - n\mu}{\sqrt{n\sigma^2}} \tag{2.0.8}$$

Then as $n \to \infty$, the sequence $\{Z_n\}$ converges in distribution to the Z, i.e., $Z_n \xrightarrow{D} Z$.
Consider,

$$Y = \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} \tag{2.0.9}$$

So,

$$E(Y) = E\left(\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}\right) = 0 \tag{2.0.10}$$

$$V(Y) = V\left(\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}\right) = \frac{1}{n}2n = 2 \tag{2.0.11}$$

$$Y \sim N[0, 2] \tag{2.0.12}$$

we know that,

$$f(x) = f(-x) \implies \text{Symmetry about Zero,} \tag{2.0.13}$$

So,

$$\Pr(Y < 0) = \frac{1}{2} \tag{2.0.14}$$

$$\Pr\left(\frac{1}{\sqrt{n}}(X_1 + \cdots + X_n) < 0\right) = \frac{1}{2} \tag{2.0.15}$$

Hence, (C) is incorrect option.

d) *Characteristic function :* For a scalar random variable X the characteristic function is defined as the expected value of e^{itx}, where i is the imaginary unit, and $t \in \mathbf{R}$ is the argument of the characteristic function:

$$\begin{cases} \varphi_X : \mathbb{R} \to \mathbb{C} \\ \varphi_X(t) = \mathrm{E}\left[e^{itX}\right] = \int_{\mathbb{R}} e^{itx} dF_X(x) \\ = \int_{\mathbb{R}} e^{itx} f_X(x) dx = \int_0^1 e^{itQ_X(p)} dp \end{cases} \tag{2.0.16}$$

Here F_X is the cumulative distribution function of X,Consider, $\phi_x(t)$ is characteristic function of $X_i, i = 1, \ldots, n$.

$$f(x) = f(-x) \implies \phi_x(t) = \phi_{-x}(t) \tag{2.0.17}$$

Therefore,

$$\phi_{\sum_{i=1}^n X_i}(t) = \phi_{X_1 + \ldots + X_n}(t) = \phi_{X_1}(t) \cdots \phi_{X_n}(t) \tag{2.0.18}$$

$$= [\phi_x(t)]^n \tag{2.0.19}$$

similarly,

$$\phi_{\sum_{i=1}^n (-1)^i X_i}(t) = \phi_{-X_1} + \phi_{X_2} + \ldots + \phi_{(-1)^n X_n}(t) \tag{2.0.20}$$

$$= \phi_{-X_1}(t) \cdot \phi_{X_2}(t) \cdots \phi_{(-1)^n X_n}(t) \tag{2.0.21}$$

$$= [\phi_x(t)]^n \tag{2.0.22}$$

$$\phi_{\sum_{i=1}^n X_i}(t) = \phi_{\sum_{i=1}^n (-1)^i X_i}(t) \tag{2.0.23}$$

$\therefore \sum_{i=1}^n X_i$ has same distribution as $\sum_{i=1}^n (-1)^i X_i$. Hence, only (D) is correct option.