Assignment 3

Sujal - AI20BTECH11020

Download all python codes from

https://github.com/sujal100/Probability_and_Random_variable/tree/main/exercise_3/codes

and latex codes from

https://github.com/https://github.com/sujal100/Probability_and_Random_variable/blob/main/exercise_3/exercise_3_main_tex.tex

1 Problem [GATE(2015)MA-11]

In an experiment, a fair die is rolled until two sixes are obtained in succession. The probability that the experiment will end in the fifth trial is equal to
(A) $\frac{125}{6^5}$ (B) $\frac{150}{6^5}$ (C) $\frac{175}{6^5}$ (D) $\frac{200}{6^5}$

2 Solution

Let Consider, Bernoulli random variables say X. Here, $P(X = n)$ refer to the probability that experiment ends in exactly n^{th} rolls. Thus, problem is asking for p_5. We note that

$$p_1 = 0, p_2 = \frac{1}{6^2} \tag{2.0.1}$$

For $n > 2$, we remark that the first roll is either a 6 or it isn't. If it is, then the second roll can't be a 6. That leads to the recursion

$$p_n = \frac{1}{6} \times \frac{5}{6} \times p_{n-2} + \frac{5}{6} \times p_{n-1} \tag{2.0.2}$$

Put $n = 0$, we get

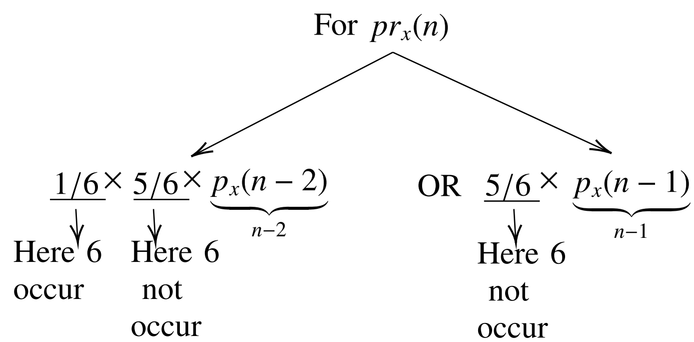
$$p_2 = \frac{5}{6^2} p_0 + \frac{5}{6} p_1 \tag{2.0.3}$$

$$\frac{1}{6^2} = \frac{5}{6^2} p_0 \tag{2.0.4}$$

$$p_0 = \frac{1}{5} \tag{2.0.5}$$

Now, put n=n+2 We have,

$$p_{n+2} = \frac{5}{6^2} p_n + \frac{5}{6} p_{n+1}, p_0 = \frac{1}{5}, p_1 = 0 \tag{2.0.6}$$



Taking, Z-transform both side,

$$Z[p_{n+2}] = \frac{5}{6^2} Z[p_n] + \frac{5}{6} Z[p_{n+1}] \tag{2.0.7}$$

$$z^2 P(z) - z^2 p_0 - z p_1 = \frac{5}{6^2} P(z) + \frac{5}{6}[zP(z) - zp_0] \tag{2.0.8}$$

Inserting initial conditions,

$$z^2 P(z) - \frac{z^2}{5} = \frac{5}{6^2} P(z) + \frac{5z}{6} P(z) - \frac{z}{6} \tag{2.0.9}$$

$$P(z) = \frac{\left(\frac{z^2}{5} - \frac{z}{6^2}\right)}{\left(z^2 - \frac{5z}{6} - \frac{5}{6^2}\right)} \tag{2.0.10}$$

$$\frac{5P(z)}{z} = \frac{36z - 5}{36z^2 - 5z - 5} \tag{2.0.11}$$

$$\frac{5\sqrt{745}P(z)}{36} = \frac{(5+\sqrt{745})z}{(72z+\sqrt{745}-5)} \tag{2.0.12}$$
$$-\frac{(\sqrt{745}-5)z}{(-72z+\sqrt{745}+5)} \tag{2.0.13}$$

We know that $Z^{-1}\left(\frac{z}{z-a}\right) = a^n$
So, taking Z-inverse transform both side,we get

$$\frac{5\sqrt{745}}{36}Z^{-1}\left(P(z)\right) = \frac{(5+\sqrt{745})}{72}Z^{-1}\left(\frac{z}{z-\frac{(5-\sqrt{745})}{72}}\right) \tag{2.0.14}$$
$$-\frac{(5-\sqrt{745})}{72}Z^{-1}\left(\frac{z}{z-\frac{(\sqrt{745}+5)}{72}}\right) \tag{2.0.15}$$

$$\frac{5\sqrt{745}}{36}p_n = \frac{(5+\sqrt{745})}{72}\left(\frac{(5-\sqrt{745})}{72}\right)^n \tag{2.0.16}$$
$$-\frac{(5-\sqrt{745})}{72}\left(\frac{(\sqrt{745}+5)}{72}\right)^n \tag{2.0.17}$$

$$p_n = \left(\frac{1}{2\sqrt{745}}+\frac{1}{10}\right)\left(\frac{(5-\sqrt{745})}{72}\right)^n \tag{2.0.18}$$
$$-\left(\frac{1}{2\sqrt{745}}-\frac{1}{10}\right)\left(\frac{(\sqrt{745}+5)}{72}\right)^n \tag{2.0.19}$$

so,

$$p_5 = \left(\frac{1}{2\sqrt{745}}+\frac{1}{10}\right)\left(\frac{(5-\sqrt{745})}{72}\right)^5 \tag{2.0.20}$$
$$-\left(\frac{1}{2\sqrt{745}}-\frac{1}{10}\right)\left(\frac{(\sqrt{745}+5)}{72}\right)^5 \tag{2.0.21}$$

$$p_5 = 0.001146095297972871 \tag{2.0.22}$$

Hence (C) is correct option.