

Assignment 6

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Download all latex codes from

https://github.com/sujal100/Probability_and_Random_variable/blob/main/exercise_6/exercise_6_main_tex.tex

1 PROBLEM [CSIR NET(JUNE-2017) MATHS-STATISTICS (Q-50)]

X_1, X_2, \dots are independent identically distributed random variables having common density f . Assume $f(x) = f(-x)$ for all $x \in \mathbb{R}$. Which of the following statements is correct?

- a) $\frac{1}{n} (X_1 + \dots + X_n) \rightarrow 0$ in probability
- b) $\frac{1}{n} (X_1 + \dots + X_n) \rightarrow 0$ almost surely
- c) $\Pr\left(\frac{1}{\sqrt{n}} (X_1 + \dots + X_n) < 0\right) \rightarrow \frac{1}{2}$
- d) $\sum_{i=1}^n X_i$ has the same distribution as $\sum_{i=1}^n (-1)^i X_i$

2 SOLUTION

We using

- a) (1) *Convergence in probability* : Let X_1, X_2, \dots be an infinite sequence of random variables, and let Y be another random variable. Then the sequence $\{X_n\}$ converges in probability to Y , if

$$\forall \epsilon > 0, \lim_{n \rightarrow \infty} \Pr(|X_n - Y| \geq \epsilon) = 0, \quad (2.0.1)$$

and we write

$$X_n \xrightarrow{P} Y. \quad (2.0.2)$$

- (2) *Convergence in almost surely* : Let X_1, X_2, \dots be an infinite sequence of random variables. We shall say that the sequence $\{X_i\}$ converges with probability 1 (or converges almost surely (a.s.)) to a random variable Y , if

$$\Pr\left(\lim_{n \rightarrow \infty} X_n = Y\right) = 1 \quad (2.0.3)$$

and we write

$$X_n \xrightarrow{a.s.} Y \quad (2.0.4)$$

- (3) *Strong law of large number (SLLN)* : Let X_1, X_2, \dots be an infinite sequence of random variables, If $\mathbf{E}[|X_1|] < \infty$. Then, as $n \rightarrow \infty$, we have

$$\frac{S_n}{n} \xrightarrow{a.s.} \mathbf{E}[X_1] \implies \frac{S_n}{n} \xrightarrow{P} \mathbf{E}[X_1], \quad (2.0.5)$$

$$\text{where, } S_n = X_1 + \dots + X_n \quad (2.0.6)$$

using SLLN, (B) are incorrect option.

- b) *Relation between in probability and almost surely* : Let Z, Z_1, Z_2, \dots be random variables. Suppose $Z_n \rightarrow Z$ with probability 1. Then $Z_n \rightarrow Z$ in probability. That is, if a sequence of random variables converges almost surely, then it converges in probability to the same limit and we say

$$Z_n \xrightarrow{a.s.} Z \implies Z_n \xrightarrow{P} Z. \quad (2.0.7)$$

also in probability also hold this equation.

Hence (A) are incorrect option.

- c) *Central Limit Theorem* : Let X_1, X_2, \dots be i.i.d. with finite mean μ and finite variance σ^2 . Let $Z \sim N(0, 1)$. Set $S_n = X_1 + \dots + X_n$, and

$$Z_n = \frac{S_n - n\mu}{\sqrt{n\sigma^2}} \quad (2.0.8)$$

Then as $n \rightarrow \infty$, the sequence $\{Z_n\}$ converges in distribution to the Z , i.e., $Z_n \xrightarrow{D} Z$.

Consider,

$$Y = \frac{X_1 + X_2 + \dots + X_n}{\sqrt{n}} \quad (2.0.9)$$

So,

$$E(Y) = E\left(\frac{X_1 + X_2 + \dots + X_n}{\sqrt{n}}\right) = 0 \quad (2.0.10)$$

$$V(Y) = V\left(\frac{X_1 + X_2 + \dots + X_n}{\sqrt{n}}\right) = \frac{1}{n}2n = 2 \quad (2.0.11)$$

$$Y \sim N[0, 2] \quad (2.0.12)$$

we know that,

$$f(x) = f(-x) \implies \text{Symmetry about Zero,} \quad (2.0.13)$$

So,

$$\Pr(Y < 0) = \frac{1}{2} \quad (2.0.14)$$

$$\Pr\left(\frac{1}{\sqrt{n}}(X_1 + \dots + X_n) < 0\right) = \frac{1}{2} \quad (2.0.15)$$

Hence, (C) is incorrect option.

- d) *Characteristic function* : For a scalar random variable X the characteristic function is defined as the expected value of e^{itx} , where i is the imaginary unit, and $t \in \mathbf{R}$ is the argument of the characteristic function:

$$\begin{cases} \varphi_X : \mathbf{R} \rightarrow \mathbf{C} \\ \varphi_X(t) = E[e^{itX}] = \int_{\mathbf{R}} e^{itx} dF_X(x) \\ = \int_{\mathbf{R}} e^{itx} f_X(x) dx = \int_0^1 e^{itQ_X(p)} dp \end{cases} \quad (2.0.16)$$

Here F_X is the cumulative distribution function of X , Consider, $\phi_x(t)$ is characteristic function of $X_i, i = 1, \dots, n$.

$$f(x) = f(-x) \implies \phi_x(t) = \phi_{-x}(t) \quad (2.0.17)$$

Therefore,

$$\phi_{\sum_{i=1}^n X_i}(t) = \phi_{X_1 + \dots + X_n}(t) = \phi_{X_1}(t) \cdots \phi_{X_n}(t) \quad (2.0.18)$$

$$= [\phi_x(t)]^n \quad (2.0.19)$$

similarly,

$$\phi_{\sum_{i=1}^n (-1)^i X_i}(t) = \phi_{-X_1} + \phi_{X_2} + \dots + \phi_{(-1)^n X_n}(t) \quad (2.0.20)$$

$$= \phi_{-X_1}(t) \cdot \phi_{X_2}(t) \cdots \phi_{(-1)^n X_n}(t) \quad (2.0.21)$$

$$= [\phi_x(t)]^n \quad (2.0.22)$$

$$\phi_{\sum_{i=1}^n X_i}(t) = \phi_{\sum_{i=1}^n (-1)^i X_i}(t) \quad (2.0.23)$$

$\therefore \sum_{i=1}^n X_i$ has same distribution as $\sum_{i=1}^n (-1)^i X_i$.
Hence, only (D) is correct option.