Assignment 6

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Download all latex codes from

https://github.com/https://github.com/sujal100/Probability_and_Random_variable/blob/main/exercise_6/exercise_6_main_tex.tex

1 Problem [CSIR NET(JUNE-2017) MATHS-STATISTICS (Q-50)]

$X_1, X_2, \cdots$ are independent identically distributed random variables having common density f. Assume $f(x) = f(-x)$ for all $x \in \mathbb{R}$. Which of the following statements is correct?

a) $\frac{1}{n}(X_1 + \cdots + X_n) \to 0$ in probability
b) $\frac{1}{n}(X_1 + \cdots + X_n) \to 0$ almost surely
c) $P\left(\frac{1}{\sqrt{n}}(X_1 + \cdots + X_n) < 0\right) \to \frac{1}{2}$
d) $\sum_{i=1}^{n} X_i$ has the same distribution as $\sum_{i=1}^{n}(-1)^i X_i$

2 Solution

In this question $f(x) = f(-x) \implies$ Symmetry about Zero, $x \in \mathbb{R}$.
We using convergence in almost surely and Strong law of large number (SLLN)
Suppose X_n, X are random variables on the same probability space. Then,

1) If $X_n \xrightarrow{\text{a.s.}} X$, then $X_n \xrightarrow{P} X$.
2) If $X_n \xrightarrow{P} X$ so that $\sum_n \mathbf{P}(|X_n - X| > \delta) < \infty$ for every $\delta > 0$, then $X_n \xrightarrow{a.B} X$
3) (SLLN) Let X_n be i.i.d with $\mathbf{E}[|X_1|] < \infty$. Then, as $n \to \infty$, we have $\frac{S_n}{n} \xrightarrow{\text{a.s.}} \mathbf{E}[X_1] \implies \frac{S_n}{n} \xrightarrow{\text{P}} \mathbf{E}[X_1]$, where $S_n = X_1 + \cdots + X_n$.

a) using SLLN, (B) are incorrect option.
b) also in probability also hold this equation. Hence (A) are incorrect option.
c) $X_1, X_2, \cdots, X_n$ are independent identically distribute $\sim N[0, 2]$
Consider,

$$Y = \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} \tag{2.0.1}$$

So,

$$E(Y) = E\left(\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}\right) = 0 \tag{2.0.2}$$

$$V(Y) = V\left(\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}\right) = \frac{1}{n}2n = 2 \tag{2.0.3}$$

$$Y \sim N[0, 2] \tag{2.0.4}$$

we know that $f(x)$ is symmetric about zero. So,

$$P(Y < 0) = \frac{1}{2} \tag{2.0.5}$$

$$P\left(\frac{1}{\sqrt{n}}(X_1 + \cdots + X_n) < 0\right) = \frac{1}{2} \tag{2.0.6}$$

Hence, (C) is incorrect option.

d) Consider $\phi_x(t)$ is characteristic function of $X_i, i = 1, \ldots, n$.

$$f(x) = f(-x) \implies \phi_x(t) = \phi_{-x}(t) \tag{2.0.7}$$

Therefore,

$$\phi_{\sum_{i=1}^{n} X_i}(t) = \phi_{X_1+\ldots+X_n}(t) = \phi_{X_1}(t) \cdots \phi_{X_n}(t) \tag{2.0.8}$$

$$= [\phi_x(t)]^n \tag{2.0.9}$$

similarly,

$$\phi_{\sum_{i=1}^{n}(-1)^i X_i}(t) = \phi_{-X_1} + \phi_{X_2} + \ldots + \phi_{(-1)^n X_n}(t) \tag{2.0.10}$$

$$= \phi_{-X_1}(t) \cdot \phi_{X_2}(t) \cdots \phi_{(-1)^n X_n}(t) \tag{2.0.11}$$

$$= [\phi_x(t)]^n \tag{2.0.12}$$

$$\phi_{\sum_{i=1}^{n} X_i}(t) = \phi_{\sum_{i=1}^{n}(-1)^i X_i}(t) \tag{2.0.13}$$

Therefore, $\sum_{i=1}^{n} X_i$ has same distribution as $\sum_{i=1}^{n}(-1)^i X_i$.
Hence, only (D) is correct option.