Assignment 6

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Download all latex codes from

https://github.com/https://github.com/sujal100/Probability_and_Random_variable/blob/main/exercise_6/exercise_6_main_tex.tex

1 Problem [CSIR NET(JUNE-2017) MATHS-STATISTICS (Q-50)]

$X_1, X_2, \cdots$ are independent identically distributed random variables having common density f. Assume $f(x) = f(-x)$ for all $x \in \mathbb{R}$. Which of the following statements is correct?

a) $\frac{1}{n}(X_1 + \cdots + X_n) \to 0$ in probability
b) $\frac{1}{n}(X_1 + \cdots + X_n) \to 0$ almost surely
c) $P\left(\frac{1}{\sqrt{n}}(X_1 + \cdots + X_n) < 0\right) \to \frac{1}{2}$
d) $\sum_{i=1}^{n} X_i$ has the same distribution as $\sum_{i=1}^{n}(-1)^i X_i$

2 Solution

In this question $f(x) = f(-x) \implies$ Symmetry about Zero, $x \in \mathbb{R}$.
We using

1) Convergence in probability,
2) Convergence in almost surely,
3) relationship is between convergence in probability and convergence in probability almost surely,
4) Strong law of large number (SLLN)

1) Let $X_1, X_2, \ldots$ be an infinite sequence of random variables, and let Y be another random variable. Then the sequence $\{X_n\}$ converges in probability to Y, if for all $\epsilon > 0, \lim_{n\to\infty} P(|X_n - Y| \geq \epsilon) = 0$, and we write $X_n \xrightarrow{P} Y$.
2) Let $X_1, X_2, \ldots$ be an infinite sequence of random variables. We shall say that the sequence $\{X_i\}$ converges with probability 1 (or converges almost surely (a.s.)) to a random variable Y, if $P(\lim_{n\to\infty} X_n = Y) = 1$ and we write $X_n \xrightarrow{a.s.} Y$
3) Let $Z, Z_1, Z_2, \ldots$ be random variables. Suppose $Z_n \to Z$ with probability 1. Then $Z_n \to Z$ in probability. That is, if a sequence of random variables converges almost surely, then it converges in probability to the same limit and we say $Z_n \xrightarrow{a.s.} Z \implies Z_n \xrightarrow{P} Z$.
4) Let $X_1, X_2, \ldots$ be an infinite sequence of random variables, If $\mathbf{E}[|X_1|] < \infty$. Then, as $n \to \infty$, we have $\frac{S_n}{n} \xrightarrow{a.s.} \mathbf{E}[X_1] \implies \frac{S_n}{n} \xrightarrow{P} \mathbf{E}[X_1]$, where $S_n = X_1 + \cdots + X_n$.

a) using SLLN, (B) are incorrect option.
b) also in probability also hold this equation. Hence (A) are incorrect option.
c) $X_1, X_2, \cdots, X_n$ are independent identically distribute $\sim N[0, 2]$
Consider,
$$Y = \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} \tag{2.0.1}$$
So,
$$E(Y) = E\left(\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}\right) = 0 \tag{2.0.2}$$
$$V(Y) = V\left(\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}\right) = \frac{1}{n}2n = 2 \tag{2.0.3}$$
$$Y \sim N[0, 2] \tag{2.0.4}$$
we know that $f(x)$ is symmetric about zero. So,
$$P(Y < 0) = \frac{1}{2} \tag{2.0.5}$$
$$P\left(\frac{1}{\sqrt{n}}(X_1 + \cdots + X_n) < 0\right) = \frac{1}{2} \tag{2.0.6}$$
Hence, (C) is incorrect option.
d) Consider $\phi_x(t)$ is characteristic function of $X_i, i = 1, \ldots, n$.
$$f(x) = f(-x) \implies \phi_x(t) = \phi_{-x}(t) \tag{2.0.7}$$

Therefore,

$$\phi_{\sum_{i=1}^{n} X_i}(t) = \phi_{X_1+\ldots+X_n}(t) = \phi_{X_1}(t)\cdots\phi_{X_n}(t) \tag{2.0.8}$$

$$= [\phi_x(t)]^n \tag{2.0.9}$$

similarly,

$$\phi_{\sum_{i=1}^{n}(-1)^i X_i}(t) = \phi_{-X_1} + \phi_{X_2} + \ldots + \phi_{(-1)^n X_n}(t) \tag{2.0.10}$$

$$= \phi_{-X_1}(t)\cdot\phi_{X_2}(t)\cdots\phi_{(-1)^n X_n}(t) \tag{2.0.11}$$

$$= [\phi_x(t)]^n \tag{2.0.12}$$

$$\phi_{\sum_{i=1}^{n} X_i}(t) = \phi_{\sum_{i=1}^{n}(-1)^i X_i}(t) \tag{2.0.13}$$

Therefore, $\sum_{i=1}^{n} X_i$ has same distribution as $\sum_{i=1}^{n}(-1)^i X_i$.
Hence, only (D) is correct option.