

Assignment 6

Sujal - AI20BTECH11020

Download all latex codes from

https://github.com/sujal100/Probability_and_Random_variable/blob/main/exercise_6/exercise_6_main_tex.tex

1 PROBLEM [CSIR NET(JUNE-2017) MATHS-STATISTICS (Q-50)]

X_1, X_2, \dots are independent identically distributed random variables having common density f . Assume $f(x) = f(-x)$ for all $x \in \mathbb{R}$. Which of the following statements is correct?

- $\frac{1}{n} (X_1 + \dots + X_n) \rightarrow 0$ in probability
- $\frac{1}{n} (X_1 + \dots + X_n) \rightarrow 0$ almost surely
- $P\left(\frac{1}{\sqrt{n}} (X_1 + \dots + X_n) < 0\right) \rightarrow \frac{1}{2}$
- $\sum_{i=1}^n X_i$ has the same distribution as $\sum_{i=1}^n (-1)^i X_i$

2 SOLUTION

In this question $f(x) = f(-x) \implies$ Symmetry about Zero, $x \in \mathbb{R}$.

We using

- Convergence in probability,
 - Convergence in almost surely,
 - relationship is between convergence in probability and convergence in almost surely,
 - Strong law of large number (SLLN)
- 1) Let X_1, X_2, \dots be an infinite sequence of random variables, and let Y be another random variable. Then the sequence $\{X_n\}$ converges in probability to Y , if

$$\forall \epsilon > 0, \lim_{n \rightarrow \infty} P(|X_n - Y| \geq \epsilon) = 0, \quad (2.0.1)$$

and we write

$$X_n \xrightarrow{P} Y. \quad (2.0.2)$$

- 2) Let X_1, X_2, \dots be an infinite sequence of random variables. We shall say that the sequence

$\{X_i\}$ converges with probability 1 (or converges almost surely (a.s.)) to a random variable Y , if

$$P\left(\lim_{n \rightarrow \infty} X_n = Y\right) = 1 \quad (2.0.3)$$

and we write

$$X_n \xrightarrow{a.s.} Y \quad (2.0.4)$$

- 3) Let Z, Z_1, Z_2, \dots be random variables. Suppose $Z_n \rightarrow Z$ with probability 1. Then $Z_n \rightarrow Z$ in probability. That is, if a sequence of random variables converges almost surely, then it converges in probability to the same limit and we say

$$Z_n \xrightarrow{a.s.} Z \implies Z_n \xrightarrow{P} Z. \quad (2.0.5)$$

- 4) Let X_1, X_2, \dots be an infinite sequence of random variables, If $\mathbf{E}[|X_1|] < \infty$. Then, as $n \rightarrow \infty$, we have

$$\frac{S_n}{n} \xrightarrow{a.s.} \mathbf{E}[X_1] \implies \frac{S_n}{n} \xrightarrow{P} \mathbf{E}[X_1], \quad (2.0.6)$$

$$\text{where, } S_n = X_1 + \dots + X_n \quad (2.0.7)$$

- using SLLN, (B) are incorrect option.
- also in probability also hold this equation. Hence (A) are incorrect option.
- X_1, X_2, \dots, X_n are independent identically distributed $\sim N[0, 2]$

Consider,

$$Y = \frac{X_1 + X_2 + \dots + X_n}{\sqrt{n}} \quad (2.0.8)$$

So,

$$E(Y) = E\left(\frac{X_1 + X_2 + \dots + X_n}{\sqrt{n}}\right) = 0 \quad (2.0.9)$$

$$V(Y) = V\left(\frac{X_1 + X_2 + \dots + X_n}{\sqrt{n}}\right) = \frac{1}{n} 2n = 2 \quad (2.0.10)$$

$$Y \sim N[0, 2] \quad (2.0.11)$$

we know that $f(x)$ is symmetric about zero. So,

$$P(Y < 0) = \frac{1}{2} \quad (2.0.12)$$

$$P\left(\frac{1}{\sqrt{n}}(X_1 + \dots + X_n) < 0\right) = \frac{1}{2} \quad (2.0.13)$$

Hence, (C) is incorrect option.

d) Consider, $\phi_x(t)$ is characteristic function of $X_i, i = 1, \dots, n$.

$$f(x) = f(-x) \implies \phi_x(t) = \phi_{-x}(t) \quad (2.0.14)$$

Therefore,

$$\phi_{\sum_{i=1}^n X_i}(t) = \phi_{X_1 + \dots + X_n}(t) = \phi_{X_1}(t) \cdots \phi_{X_n}(t) \quad (2.0.15)$$

$$= [\phi_x(t)]^n \quad (2.0.16)$$

similarly,

$$\phi_{\sum_{i=1}^n (-1)^i X_i}(t) = \phi_{-X_1} + \phi_{X_2} + \dots + \phi_{(-1)^n X_n}(t) \quad (2.0.17)$$

$$= \phi_{-X_1}(t) \cdot \phi_{X_2}(t) \cdots \phi_{(-1)^n X_n}(t) \quad (2.0.18)$$

$$= [\phi_x(t)]^n \quad (2.0.19)$$

$$\phi_{\sum_{i=1}^n X_i}(t) = \phi_{\sum_{i=1}^n (-1)^i X_i}(t) \quad (2.0.20)$$

$\therefore \sum_{i=1}^n X_i$ has same distribution as $\sum_{i=1}^n (-1)^i X_i$.

Hence, only (D) is correct option.