Problem 12.113

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Question

Question : The area of the region bounded by the parabola $y = x^2 + 1$ and the straight line $x + y = 3$ is:

a) $59/6$

b) $9/2$

c) $10/3$

d) $7/6$

Solution

SOLUTION Solution:
Step 1: Write equations in matrix (quadratic) form

$$y = x^2 + 1 \tag{1.1}$$

In matrix form: $$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}^T \mathbf{x} + 1 = 0 \tag{1.2}$$

Step-by-Step Calculation

Step 2: Parametric representation of the line:

$x + y = 3 \implies y = 3 - x$

Let the line in parametric vector form be:

$$\mathbf{X} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.3}$$

Step-by-Step Calculation

Step 3: Substitute into the matrix equation and solve for λ

Substitute $x = \lambda$, $y = 3 - \lambda$ into the matrix equation.

$$\left(\begin{pmatrix} \lambda \\ 3 - \lambda \end{pmatrix}\right)^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ 3 - \lambda \end{pmatrix} + 2 \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}^T \begin{pmatrix} \lambda \\ 3 - \lambda \end{pmatrix} + 1 = 0 \tag{1.4}$$

Frame Title

Calculate each component:

$$\begin{pmatrix} \lambda \\ 3-\lambda \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ 3-\lambda \end{pmatrix} = \lambda^2 \tag{1.5}$$

$$2 \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}^T \begin{pmatrix} \lambda \\ 3-\lambda \end{pmatrix} = 2 \cdot \left(0 \cdot \lambda + \left(-\frac{1}{2} \right) (3-\lambda) \right) = -(3-\lambda) = -3 + \lambda \tag{1.6}$$

So, the equation becomes:

$$\lambda^2 - 3 + \lambda + 1 = 0 \implies \lambda^2 + \lambda - 2 = 0 \tag{1.7}$$

Solving this quadratic equation:

$$(\lambda + 2)(\lambda - 1) = 0 \implies \lambda = -2, 1 \tag{1.8}$$

So, the intersection points are:

$$\mathbf{X} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}, \mathbf{X} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.9}$$

Step-by-Step Calculation

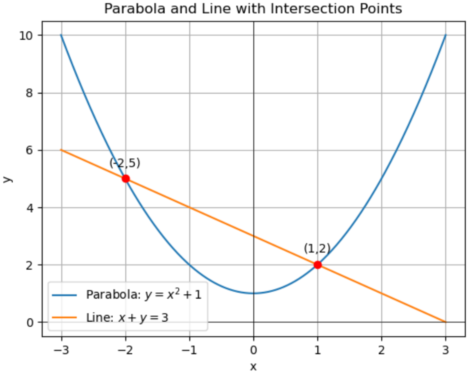
Step 4: Area Calculation

The area between the curves can be written as:

$$A = \int_{-2}^{1} \left[(3 - x) - (x^2 + 1)\right]\, dx = \int_{-2}^{1} (2 - x - x^2)\, dx \tag{1.10}$$

Integrate:

$$\begin{aligned}
A &= \int_{-2}^{1} (2 - x - x^2)\, dx && (1.11)\\
&= \left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-2}^{1} && (1.12)\\
&= \left(2(1) - \frac{1}{2}(1)^2 - \frac{1}{3}(1)^3\right) - \left(2(-2) - \frac{1}{2}(-2)^2 - \frac{1}{3}(-2)^3\right) && (1.13)\\
&= (2 - 0.5 - 0.333) - (-4 - 2 + 2.666) && (1.14)\\
&= 1.167 - (-3.334) && (1.15)\\
&= 4.5 = \frac{9}{2} && (1.16)
\end{aligned}$$


Parabola and Line with Intersection Points
10
8
6
4
2
0
y
(-2,5)
(1,2)
Parabola: $y = x^2 + 1$
Line: $x + y = 3$
−3
−2
−1
0
1
2
3
x