Problem 2.10.24

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September 28, 2025

Question

Question: Let $\mathbf{x}$,$\mathbf{y}$ and $\mathbf{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\frac{\pi}{3}$. If $\mathbf{a}$ is a non-zero vector perpendicular to $\mathbf{x}$ and $\mathbf{y} \times \mathbf{z}$ and $\mathbf{b}$ is a non-zero vector perpendicular to $\mathbf{y}$ and $\mathbf{z} \times \mathbf{x}$, then

(A) $\mathbf{b} = (\mathbf{b} \cdot \mathbf{z})(\mathbf{z} - \mathbf{x})$

(B) $\mathbf{a} = (\mathbf{a}.\mathbf{y})(\mathbf{y} - \mathbf{z})$

(C)$\mathbf{a}.\mathbf{b} =$ -$(\mathbf{a}.\mathbf{y})(\mathbf{b}.\mathbf{z})$

(D)$\mathbf{a} =$ -$(\mathbf{a}.\mathbf{y})(\mathbf{z} - \mathbf{y})$

Solution

Solution: as mentioned in the question :

$$||\mathbf{x}|| = \sqrt{2}, ||\mathbf{y}|| = \sqrt{2}, ||\mathbf{z}|| = \sqrt{2}$$

$$\mathbf{x}^\top \mathbf{y} = 1, \mathbf{x}^\top \mathbf{z} = 1, \mathbf{y}^\top \mathbf{z} = 1 \tag{1.1}$$

Solution

If $\mathbf{a}$ is a non-zero vector perpendicular to $\mathbf{x}$ and $\mathbf{y} \times \mathbf{z}$ this implies :

$$\mathbf{a}^\top \mathbf{x} = 0, \mathbf{a}^\top (\mathbf{y} X \mathbf{z}) = 0$$

this implies that $\mathbf{x}$ and $\mathbf{y}X\mathbf{z}$ lie in the same plane and their vector product is parallel to $\mathbf{a}$
so the vector product of

$$\mathbf{a} X (\mathbf{x} X (\mathbf{y} X \mathbf{z})) = 0$$

vector product

defining : triple vector product

$$(\mathbf{x} X (\mathbf{y} X \mathbf{z})) = (\mathbf{x}^\top \mathbf{z})\mathbf{y} - (\mathbf{x}^\top \mathbf{y})\mathbf{z}$$

Solution

by equation (1.1) : $\mathbf{x}^\top \mathbf{z} = 1, \mathbf{x}^\top \mathbf{y} = 1$

$$\mathbf{a} X(\mathbf{x} X(\mathbf{y} X \mathbf{z})) = (\mathbf{x}^\top \mathbf{z})(\mathbf{y} X \mathbf{a}) - (\mathbf{x}^\top \mathbf{y})(\mathbf{z} X \mathbf{a}) = \mathbf{a} X(\mathbf{y} - \mathbf{z}) = 0$$

as $\mathbf{a}$ is parallel to $\mathbf{y} - \mathbf{z}$ so we can say that :

$$\mathbf{a} = k(\mathbf{y} - \mathbf{z})$$

k is a constant

Solution

If $\mathbf{b}$ is a non-zero vector perpendicular to $\mathbf{y}$ and $\mathbf{z} \times \mathbf{x}$ this implies :

$$\mathbf{b}^\top \mathbf{y} = 0, \mathbf{b}^\top (\mathbf{z} X \mathbf{x}) = 0$$

this implies that $\mathbf{x}$ and $\mathbf{y} X \mathbf{z}$ lie in the same plane and their vector product is parallel to $\mathbf{a}$
so the vector product of

$$\mathbf{b} X (\mathbf{y} X (\mathbf{z} X \mathbf{x})) = 0$$

Solution

by equation (1.1) : $\mathbf{x}^\top \mathbf{y} = 1, \mathbf{z}^\top \mathbf{y} = 1$

$$\mathbf{bX}(\mathbf{y}X(\mathbf{z}X\mathbf{x})) = (\mathbf{y}^\top \mathbf{x})(\mathbf{z}X\mathbf{b}) - (\mathbf{y}^\top \mathbf{z})(\mathbf{x}X\mathbf{b}) = \mathbf{b}X(\mathbf{z} - \mathbf{x}) = 0$$

as $\mathbf{b}$ is parallel to $\mathbf{z} - \mathbf{x}$ so we can say that :

$$\mathbf{b} = m(\mathbf{z} - \mathbf{x})$$

m is a constant

Solution

now we check options one by one :
option (a)
$\mathbf{b} = (\mathbf{b}.\mathbf{z})(\mathbf{z} - \mathbf{x})$
we are putting b on both sides=

$$\begin{aligned} m(\mathbf{z} - \mathbf{x}) &= (m(\mathbf{z} - \mathbf{x})^\top \mathbf{z})(\mathbf{z} - \mathbf{x}) \\ m(\mathbf{z} - \mathbf{x}) &= m(\mathbf{z}.\mathbf{z} - \mathbf{x}^\top \mathbf{z})(\mathbf{z} - \mathbf{x}) \\ m(\mathbf{z} - \mathbf{x}) &= m(\mathbf{z} - \mathbf{x}) \end{aligned}$$

option A is correct .

Solution

option (b)
$\mathbf{a} = (\mathbf{a}^\top \mathbf{y})(\mathbf{y} - \mathbf{z})$
we are putting a on both sides=

$$k(\mathbf{y} - \mathbf{z}) = (k(\mathbf{y} - \mathbf{z})^\top \mathbf{y})(\mathbf{y} - \mathbf{z})$$
$$k(\mathbf{y} - \mathbf{z}) = (k(\mathbf{y}^\top \mathbf{y} - \mathbf{y}^\top \mathbf{z})(\mathbf{y} - \mathbf{z})$$

$$k(\mathbf{y} - \mathbf{z}) = k(\mathbf{y} - \mathbf{z})$$

option B is correct .

Solution

option (c)
$\mathbf{a}^\top\mathbf{b}$ = -$(\mathbf{a}^\top\mathbf{y})(\mathbf{b}^\top\mathbf{z})$
we are putting a and b on both sides=

$$\begin{aligned} k(\mathbf{y}-\mathbf{z})m(\mathbf{z}-\mathbf{x}) = -(m(\mathbf{z}-\mathbf{x}).\mathbf{z})(k(\mathbf{y}-\mathbf{z})^\top\mathbf{y}) \\ -km = -km \end{aligned}$$

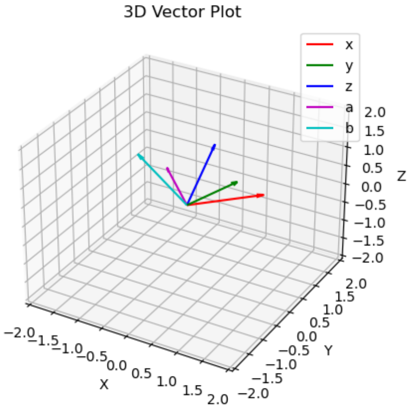
option C is correct .
option D is correct because option B is correct.

Solution

for plotting we are assuming the position vector :

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{z} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

fragile



C Code

```c
#include <stdio.h>
#include <math.h>

// Vector operations
typedef struct {
    double x, y, z;
} Vector;

Vector add(Vector a, Vector b) {
    return (Vector){a.x + b.x, a.y + b.y, a.z + b.z};
}
```

C Code

```c
Vector sub(Vector a, Vector b) {
    return (Vector){a.x - b.x, a.y - b.y, a.z - b.z};
}

Vector scalarMul(double c, Vector a) {
    return (Vector){c * a.x, c * a.y, c * a.z};
}

double dot(Vector a, Vector b) {
    return a.x*b.x + a.y*b.y + a.z*b.z;
}
```

C Code

```c
Vector cross(Vector a, Vector b) {
    return (Vector){
        a.y*b.z - a.z*b.y,
        a.z*b.x - a.x*b.z,
        a.x*b.y - a.y*b.x
    };
}

void printVec(Vector v) {
    printf("(%.2f, %.2f, %.2f)", v.x, v.y, v.z);
}
```

C Code

```c
int main() {
    // Define x, y, z such that |x|=|y|=|z|=sqrt(2) and angle = 60 deg
    // Example choice:
    Vector x = {1, 1, 0}; // |x| = sqrt(2)
    Vector y = {1, 0, 1}; // |y| = sqrt(2)
    Vector z = {0, 1, 1}; // |z| = sqrt(2)
    // These satisfy xy = xz = yz = 1  cos = 1/2   = 60

    // Define a  x, y, xz  take a = cross(x, y)
    Vector a = cross(x, y);

    // Define b  y, z, zx  take b = cross(y, z);

    Vector b = cross(y, z);
```

C Code

```
printf("x = "); printVec(x); printf("\n");
  printf("y = "); printVec(y); printf("\n");
  printf("z = "); printVec(z); printf("\n");

  printf("a = "); printVec(a); printf("\n");
  printf("b = "); printVec(b); printf("\n\n");

  // Now check conditions
  Vector lhs1 = b;
  Vector rhs1 = scalarMul(dot(b, z), sub(z, x));
  printf("1) b == (b.z)(z - x) ? -> (%f,%f,%f)\n", lhs1.x-rhs1.
      x, lhs1.y-rhs1.y, lhs1.z-rhs1.z);
```

C Code

```
Vector lhs2 = a;
 Vector rhs2 = scalarMul(dot(a, y), sub(y, z));
 printf("2) a == (a.y)(y - z) ? -> (%f,%f,%f)\n", lhs2.x-rhs2.
     x, lhs2.y-rhs2.y, lhs2.z-rhs2.z);

 double lhs3 = dot(a, b);
 double rhs3 = -(dot(a, y) * dot(b, z));
 printf("3) ab == -(a.y)(b.z) ? -> LHS=%.2f, RHS=%.2f\n", lhs3
     , rhs3);
```

C Code

```
  Vector lhs4 = a;
    Vector rhs4 = scalarMul(-(dot(a, y)), sub(z, y));
    printf("4) a == -(a.y)(z - y) ? -> (%f,%f,%f)\n", lhs4.x-rhs4
        .x, lhs4.y-rhs4.y, lhs4.z-rhs4.z);

    return 0;
}
```

Python Code for Plotting

```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib as mp
mp.use("TkAgg")

# -------------------------
# Step 1: Define vectors
# -------------------------
vectors = {
    "x": np.array([1, 1, 0]),
    "y": np.array([1, 0, 1]),
    "z": np.array([0, 1, 1]),
    "a": np.array([-1, 1, 0]),
    "b": np.array([-1, 0, 1])
}
```

Python Code for Plotting

```
colors = {
    "x": "red",
    "y": "blue",
    "z": "green",
    "a": "orange",
    "b": "purple"
}

# ------------------------
# Step 2: Plot vectors
# ------------------------
fig = plt.figure(figsize=(8, 8))
ax = fig.add_subplot(111, projection="3d")
for name, vec in vectors.items():
    ax.quiver(0, 0, 0, vec[0], vec[1], vec[2],
              color=colors[name], linewidth=2, arrow_length_ratio
                  =0.1, label=name)
```

Python Code for Plotting

```python

# Step 3: Formatting
# ------------------------
ax.set_xlim([-2, 2])
ax.set_ylim([-2, 2])
ax.set_zlim([-2, 2])

ax.set_xlabel("X-axis")
ax.set_ylabel("Y-axis")
ax.set_zlabel("Z-axis")
ax.set_title("3D Vector Plot")
ax.legend()

plt.show()
```