

Problem 12.529

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Question

Question:

Let \mathbf{M} be a 3×3 matrix and suppose that 1, 2 and 3 are the eigenvalues of \mathbf{M} . If

$$\mathbf{M}^{-1} = \frac{\mathbf{M}^2 - \mathbf{M} + \mathbf{I}_3}{\alpha}$$

for some scalar $\alpha \neq 0$, then α is equal to ____.

Solution

SOLUTION Solution:

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Let λ be an eigenvalue of \mathbf{M} , and \mathbf{v} the corresponding eigenvector:

$$\mathbf{M}\mathbf{v} = \lambda\mathbf{v}$$

Applying both sides to the given identity:

$$\mathbf{M}^{-1}\mathbf{v} = \frac{\mathbf{M}^2 - \mathbf{M} + \mathbf{I}_3}{\alpha}\mathbf{v}$$

Step-by-Step Calculation

Since polynomial expressions in \mathbf{M} act on eigenvectors by substituting λ for \mathbf{M} , we have:

$$\mathbf{M}^2 \mathbf{v} = \lambda^2 \mathbf{v}, \quad \mathbf{I}_3 \mathbf{v} = \mathbf{v}$$

Thus,

$$\mathbf{M}^{-1} \mathbf{v} = \frac{\lambda^2 - \lambda + 1}{\alpha} \mathbf{v}$$

But

$$\mathbf{M}^{-1} \mathbf{v} = \frac{1}{\lambda} \mathbf{v}$$

Therefore,

$$\frac{1}{\lambda} = \frac{\lambda^2 - \lambda + 1}{\alpha} \implies \alpha = \lambda(\lambda^2 - \lambda + 1)$$

Frame Title

For the three eigenvalues, compute α :

For $\lambda = 1$:

$$\alpha = 1 \times (1^2 - 1 + 1) = 1$$

For $\lambda = 2$:

$$\alpha = 2 \times (2^2 - 2 + 1) = 2 \times (4 - 2 + 1) = 6$$

For $\lambda = 3$:

$$\alpha = 3 \times (9 - 3 + 1) = 3 \times 7 = 21$$

Thus, there is no single α that satisfies all three cases, so the expression does not hold for all eigenvalues simultaneously.