Problem 12.113

Sujal Rajani

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Question

Question : The area of the region bounded by the parabola $y = x^2 + 1$ and the straight line $x + y = 3$ is:

a) $\frac{59}{6}$

b) $\frac{9}{2}$

c) $\frac{10}{3}$

d) $\frac{7}{6}$

Solution

SOLUTION Solution:
Step 1: Write equations in matrix (quadratic) form

$$y = x^2 + 1$$

In matrix form: $\quad \mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}^T \mathbf{x} + 1 = 0$

Step-by-Step Calculation

Step 2: Parametric representation of the line:

$x + y = 3 \implies y = 3 - x$

Let the line in parametric vector form be:

$$\mathbf{X} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Step-by-Step Calculation

Step 3: Substitute into the matrix equation and solve for λ

Substitute $x = \lambda$, $y = 3 - \lambda$ into the matrix equation.

$$\left(\begin{pmatrix} \lambda \\ 3-\lambda \end{pmatrix}\right)^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ 3-\lambda \end{pmatrix} + 2 \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}^T \begin{pmatrix} \lambda \\ 3-\lambda \end{pmatrix} + 1 = 0$$

Frame Title

Calculate each component:

$$\begin{pmatrix} \lambda \\ 3-\lambda \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ 3-\lambda \end{pmatrix} = \lambda^2$$

$$2\begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}^T \begin{pmatrix} \lambda \\ 3-\lambda \end{pmatrix} = 2 \cdot \left(0 \cdot \lambda + \left(-\frac{1}{2}\right)(3-\lambda)\right) = -(3-\lambda) = -3+\lambda$$

So, the equation becomes:

$$\lambda^2 - 3 + \lambda + 1 = 0 \implies \lambda^2 + \lambda - 2 = 0$$

Solving this quadratic equation:

$$(\lambda+2)(\lambda-1) = 0 \implies \lambda = -2, 1$$

So, the intersection points are:

$$\mathbf{X} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}, \mathbf{X} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Step-by-Step Calculation

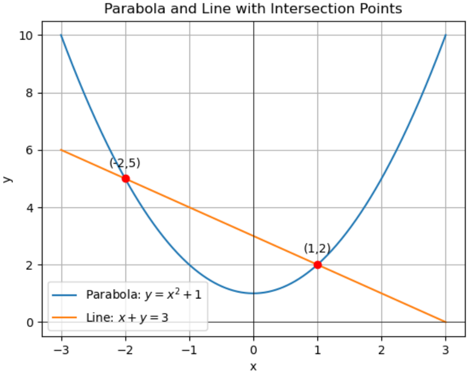
Step 4: Area Calculation

The area between the curves can be written as:

$$A = \int_{-2}^{1} \left[(3-x)-(x^2+1)\right]\,dx = \int_{-2}^{1}(2-x-x^2)\,dx$$

Integrate:

$$\begin{aligned}
A &= \int_{-2}^{1}(2-x-x^2)\,dx \\
&= \left[2x-\frac{1}{2}x^2-\frac{1}{3}x^3\right]_{-2}^{1} \\
&= \left(2(1)-\frac{1}{2}(1)^2-\frac{1}{3}(1)^3\right)-\left(2(-2)-\frac{1}{2}(-2)^2-\frac{1}{3}(-2)^3\right) \\
&= (2-0.5-0.333)-(-4-2+2.666) \\
&= 1.167-(-3.334) \\
&= 4.5 = \frac{9}{2}
\end{aligned}$$


Parabola and Line with Intersection Points
10
8
6
4
2
0
y
(-2,5)
(1,2)
Parabola: $y = x^2 + 1$
Line: $x + y = 3$
−3
−2
−1
0
1
2
3
x