12.217

AI25BTECH11035 - SUJAL RAJANI

Directional Derivative at a Point on a Surface

Given the surface

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3,$$

find the directional derivative at $P(1, 2, 3)$ in the direction of the vector $\mathbf{P}$, where O is the origin.

Step 1: Theorem for Directional Derivative

we represent the partial derivative as :

$$\frac{\partial f}{\partial x} = f_x(x, y)$$

the directional derivative of $f(x, y)$ in the direction of the unit vector $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ is

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b,$$

and in three dimensions,

$$D_{\mathbf{u}} F(x, y, z) = \nabla \mathbf{F}(\mathbf{x}, \mathbf{y}, \mathbf{z})^{\top} \mathbf{u},$$

where $\nabla \mathbf{F}$ is the gradient of F.

Step 2: Find the Gradient

Let

$$F(x, y, z) = x^2 + \frac{y^2}{4} + \frac{z^2}{9}.$$

Compute the gradient:

$$\nabla \mathbf{F} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)^{\top} = \begin{pmatrix} 2x & \frac{y}{2} & \frac{2z}{9} \end{pmatrix}^{\top}$$

At $P(1, 2, 3)$:

$$\nabla \mathbf{F}(\mathbf{1}, \mathbf{2}, \mathbf{3}) = \begin{pmatrix} 2 & 1 & \frac{2}{3} \end{pmatrix}^{\top}$$

Step 3: Unit Direction Vector

The position vector from the origin to the point $P(1, 2, 3)$ is

$$\mathbf{P} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

To get the unit vector,

$$\|\mathbf{P}\|^2 = \mathbf{P}^{\top}\mathbf{P} = 1^2 + 2^2 + 3^2 = 14$$

$$\hat{P} = \frac{\mathbf{P}}{\|\mathbf{P}\|}$$

So,

$$\mathbf{P} = \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right)^{\top}$$

Step 4: Directional Derivative Formula and Calculation

By the theorem :

$$D_{\mathbf{u}}F = \nabla\mathbf{F}^{\top}\mathbf{P}$$
$$= \frac{2}{\sqrt{14}} + \frac{2}{\sqrt{14}} + \frac{2}{\sqrt{14}}$$
$$= \frac{6}{\sqrt{14}}$$

Final Answer

$$\boxed{\frac{6}{\sqrt{14}}}$$