Problem 12.217

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Question

Question : Given the surface

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3,$$

find the directional derivative at $P(1, 2, 3)$ in the direction of the vector $\mathbf{P}$, where O is the origin.

Solution

SOLUTION

we represent the partial derivative as :

$$\frac{\partial f}{\partial x} = f_x(x, y)$$

the directional derivative of $f(x, y)$ in the direction of the unit vector $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ is

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b,$$

and in three dimensions,

$$D_{\mathbf{u}} F(x, y, z) = \nabla \mathbf{F}(\mathbf{x}, \mathbf{y}, \mathbf{z})^\top \mathbf{u},$$

where $\nabla \mathbf{F}$ is the gradient of F.

Step-by-Step Calculation

Let

$$F(x, y, z) = x^2 + \frac{y^2}{4} + \frac{z^2}{9}.$$

Compute the gradient:

$$\nabla \mathbf{F} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)^\top = \begin{pmatrix} 2x & \frac{y}{2} & \frac{2z}{9} \end{pmatrix}^\top$$

At $P(1, 2, 3)$:

$$\nabla \mathbf{F}(\mathbf{1}, \mathbf{2}, \mathbf{3}) = \begin{pmatrix} 2 & 1 & \frac{2}{3} \end{pmatrix}^\top$$

Step-by-Step Calculation

Step 3: Unit Direction Vector

The position vector from the origin to the point $P(1, 2, 3)$ is

$$\mathbf{P} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

To get the unit vector,

$$||\mathbf{P}||^2 = \mathbf{P}^\top \mathbf{P} = 1^2 + 2^2 + 3^2 = 14$$

$$\hat{\mathbf{P}} = \frac{\mathbf{P}}{||\mathbf{P}||}$$

So,

$$\hat{\mathbf{P}} = \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right)^\top$$

Step-by-Step Calculation

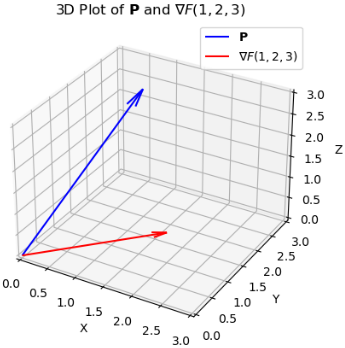
By the theorem :

$$D_{\mathbf{u}}F = \nabla \mathbf{F}^\top \hat{\mathbf{P}}$$

$$= \frac{2}{\sqrt{14}} + \frac{2}{\sqrt{14}} + \frac{2}{\sqrt{14}}$$

$$= \frac{6}{\sqrt{14}}$$

$$\boxed{\frac{6}{\sqrt{14}}}$$


3D Plot of **P** and $\nabla F(1,2,3)$
P
$\nabla F(1,2,3)$
X
Y
Z
0.0
0.5
1.0
1.5
2.0
2.5
3.0