12.113

AI25BTECH11035 - SUJAL RAJANI

Problem:
The area of the region bounded by the parabola $y = x^2 + 1$ and the straight line $x + y = 3$ is:

a) $\frac{59}{6}$
b) $\frac{9}{2}$
c) $\frac{10}{3}$
d) $\frac{7}{6}$

Solution:

Step 1: Write equations in matrix (quadratic) form

$$y = x^2 + 1 \implies x^2 - y + 1 = 0$$

$$\text{In matrix form:} \quad \mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}^T \mathbf{x} + 1 = 0$$

Step 2: Parametric representation of the line:
$x + y = 3 \implies y = 3 - x$
Let the line in parametric vector form be:

$$\mathbf{X} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So,

$$x = \lambda, \quad y = 3 - \lambda$$

Step 3: Substitute into the matrix equation and solve for λ
Substitute $x = \lambda$, $y = 3 - \lambda$ into the matrix equation.

$$\left(\begin{pmatrix} \lambda \\ 3 - \lambda \end{pmatrix} \right)^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ 3 - \lambda \end{pmatrix} + 2 \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}^T \begin{pmatrix} \lambda \\ 3 - \lambda \end{pmatrix} + 1 = 0$$

Calculate each component:

$$\begin{pmatrix} \lambda \\ 3 - \lambda \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ 3 - \lambda \end{pmatrix} = \lambda^2$$

$$2 \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}^T \begin{pmatrix} \lambda \\ 3 - \lambda \end{pmatrix} = 2 \cdot \left(0 \cdot \lambda + \left(-\frac{1}{2} \right) (3 - \lambda) \right) = -(3 - \lambda) = -3 + \lambda$$

So, the equation becomes:

$$\lambda^2 - 3 + \lambda + 1 = 0 \implies \lambda^2 + \lambda - 2 = 0$$

Solving this quadratic equation:

$$(\lambda + 2)(\lambda - 1) = 0 \implies \lambda = -2, 1$$

So, the intersection points are:

$$\mathbf{X} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}, \mathbf{X} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Step 4: Area Calculation

The area between the curves can be written as:

$$A = \int_{-2}^{1} \left[(3 - x) - (x^2 + 1)\right] dx = \int_{-2}^{1} (2 - x - x^2)\, dx$$

Integrate:

$$\begin{aligned}
A &= \int_{-2}^{1} (2 - x - x^2)\, dx \\
&= \left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-2}^{1} \\
&= \left(2(1) - \frac{1}{2}(1)^2 - \frac{1}{3}(1)^3\right) - \left(2(-2) - \frac{1}{2}(-2)^2 - \frac{1}{3}(-2)^3\right) \\
&= (2 - 0.5 - 0.333) - (-4 - 2 + 2.666) \\
&= 1.167 - (-3.334) \\
&= 4.5 = \frac{9}{2}
\end{aligned}$$

Final Answer:

$$\boxed{\frac{9}{2}}$$