12.529

AI25BTECH11035 - SUJAL RAJANI

Question:
Let $\mathbf{M}$ be a 3×3 matrix and suppose that $1, 2$ and 3 are the eigenvalues of $\mathbf{M}$. If

$$\mathbf{M}^{-1} = \frac{\mathbf{M}^2 - \mathbf{M} + \mathbf{I}_3}{\alpha}$$

for some scalar $\alpha \neq 0$, then α is equal to ____.

Solution:
Let λ be an eigenvalue of $\mathbf{M}$, and $\mathbf{v}$ the corresponding eigenvector:

$$\mathbf{M}\mathbf{v} = \lambda\mathbf{v}$$

Applying both sides to the given identity:

$$\mathbf{M}^{-1}\mathbf{v} = \frac{\mathbf{M}^2 - \mathbf{M} + \mathbf{I}_3}{\alpha}\mathbf{v}$$

Since polynomial expressions in $\mathbf{M}$ act on eigenvectors by substituting λ for $\mathbf{M}$, we have:

$$\mathbf{M}^2\mathbf{v} = \lambda^2\mathbf{v}, \qquad \mathbf{I}_3\mathbf{v} = \mathbf{v}$$

Thus,

$$\mathbf{M}^{-1}\mathbf{v} = \frac{\lambda^2 - \lambda + 1}{\alpha}\mathbf{v}$$

But

$$\mathbf{M}^{-1}\mathbf{v} = \frac{1}{\lambda}\mathbf{v}$$

Therefore,

$$\frac{1}{\lambda} = \frac{\lambda^2 - \lambda + 1}{\alpha} \implies \alpha = \lambda(\lambda^2 - \lambda + 1)$$

For the three eigenvalues, compute α:
For $\lambda = 1$:

$$\alpha = 1 \times (1^2 - 1 + 1) = 1$$

For $\lambda = 2$:

$$\alpha = 2 \times (2^2 - 2 + 1) = 2 \times (4 - 2 + 1) = 6$$

For $\lambda = 3$:

$$\alpha = 3 \times (9 - 3 + 1) = 3 \times 7 = 21$$

Thus, there is no single α that satisfies all three cases, so the expression does not hold for all eigenvalues simultaneously.