

Problem 12.217

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Question

Question : Given the surface

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3,$$

find the directional derivative at $P(1, 2, 3)$ in the direction of the vector \mathbf{P} , where O is the origin.

Solution

SOLUTION

we represent the partial derivative as :

$$\frac{\partial f}{\partial x} = f_x(x, y)$$

the directional derivative of $f(x, y)$ in the direction of the unit vector $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ is

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b, \quad (1.1)$$

and in three dimensions,

$$D_{\mathbf{u}}F(x, y, z) = \nabla \mathbf{F}(\mathbf{x}, \mathbf{y}, \mathbf{z})^\top \mathbf{u}, \quad (1.2)$$

where $\nabla \mathbf{F}$ is the gradient of F .

Step-by-Step Calculation

Let

$$F(x, y, z) = x^2 + \frac{y^2}{4} + \frac{z^2}{9}. \quad (1.3)$$

Compute the gradient:

$$\nabla \mathbf{F} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)^{\top} = \left(2x \quad \frac{y}{2} \quad \frac{2z}{9} \right)^{\top} \quad (1.4)$$

At $P(1, 2, 3)$:

$$\nabla \mathbf{F}(\mathbf{1}, \mathbf{2}, \mathbf{3}) = \left(2 \quad 1 \quad \frac{2}{3} \right)^{\top} \quad (1.5)$$

Step-by-Step Calculation

Step 3: Unit Direction Vector

The position vector from the origin to the point $P(1, 2, 3)$ is

$$\mathbf{P} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad (1.6)$$

To get the unit vector,

$$\|\mathbf{P}\|^2 = \mathbf{P}^\top \mathbf{P} = 1^2 + 2^2 + 3^2 = 14 \quad (1.7)$$

$$\hat{\mathbf{P}} = \frac{\mathbf{P}}{\|\mathbf{P}\|} \quad (1.8)$$

So,

$$\hat{\mathbf{P}} = \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right)^\top$$

Step-by-Step Calculation

By the theorem :

$$D_{\mathbf{u}}F = \nabla \mathbf{F}^{\top} \hat{\mathbf{P}} \quad (1.9)$$

$$= \frac{2}{\sqrt{14}} + \frac{2}{\sqrt{14}} + \frac{2}{\sqrt{14}} \quad (1.10)$$

$$= \frac{6}{\sqrt{14}} \quad (1.11)$$

$$\boxed{\frac{6}{\sqrt{14}}}$$

3D Plot of \mathbf{P} and $\nabla F(1, 2, 3)$

