LECTURE-3 & 4.

SEQUENCES, SERIES

FUNCTIONS, LIMITS,

CONTINUITY.

Lecture 2

Sequence: An infinite set of complex numbers indexed by $\mathbb{N}$.

Eg: $\{1/n\}$, $\{i^n\}$

Subsequence: A subset of a sequence indexed by a strictly increasing sequence of natural numbers is called a subsequence of the given sequence.

Eg: $\{1/2n\}$, $\{i^{4k}\}$

$$\begin{matrix} \{1/2n\} & \{i^{4k}\} \\ \cap\text{ı} & \cap\text{ı} \\ \{1/n\} & \{i^n\} \end{matrix}$$

A sequence is said to converge to a limit "l" if given $\varepsilon > 0$ $\exists\, N > 0 \ni$

$$|z_n - l| < \varepsilon \quad \forall\, n \geq N.$$

[Denoted as $\lim_{n \to \infty} z_n = l$

Eg: $\{1/n\}$ converges to 0.

REMARK: $\{z_n\}$ converges to $z \Leftrightarrow \{\mathrm{Re}(z_n)\}$ converges to $\mathrm{Re}\, z$ and $\{\mathrm{Im}\, z_n\}$ converges to $\mathrm{Im}\, z$.

(Indeed, $z_n \to z \Rightarrow$ given $\varepsilon > 0$ $\exists N > 0 \ni$
$|z_n - z| < \varepsilon \;\forall n \geq N$

$$|\operatorname{Re}(z_n - z)| \leq |z_n - z| < \varepsilon$$

$$|\operatorname{Im}(z_n - z)| \leq |z_n - z| < \varepsilon$$

so $\operatorname{Re} z_n \to \operatorname{Re} z$ & $\operatorname{Im} z_n \to \operatorname{Im} z$

Conversely, if $\operatorname{Re} z_n \to \operatorname{Re} z$ & $\operatorname{Im} z_n \to \operatorname{Im} z$ then,

given $\varepsilon > 0$, choose $N \gg 0 \ni$

$$\left.\begin{array}{r} |\operatorname{Re} z_n - \operatorname{Re} z| < \varepsilon/2 \\ |\operatorname{Im} z_n - \operatorname{Im} z| < \varepsilon/2 \\ \forall n \geq N \end{array}\right\} \Rightarrow |z_n - z| = \sqrt{\varepsilon^2/4 + \varepsilon^2/4} = \varepsilon/\sqrt{2} < \varepsilon \;\forall n \geq N.$$

as required.)

<u>Cauchy sequence</u>: A sequence $\{z_n\}$ is said to be <u>Cauchy</u> if given $\varepsilon > 0$ $\exists N > 0$

$$\ni |z_n - z_m| < \varepsilon \quad \forall n, m \geq N$$

REMARK: (1) $\{z_n\}$ is Cauchy $\Leftrightarrow$ $\{\operatorname{Re} z_n\}$, $\{\operatorname{Im} z_n\}$ are Cauchy sequences (of real numbers)
(Pf same as earlier)

2. Every Cauchy seq in $\mathbb{C}$ converges.

(Pf: $\{z_n\}$ is Cauchy $\Rightarrow$ $\{\mathrm{Re}\, z_n\}$ $\{\mathrm{Im}\, z_n\}$ are Cauchy sequences (of real nos)

$\Rightarrow$ $\mathrm{Re}\, z_n \to a$, $\mathrm{Im}\, z_n \to b$

Hence $z_n = \mathrm{Re}\, z_n + i\,\mathrm{Im}\, z_n \longrightarrow a+ib$.)

3. If the sequence $\{z_n\}$ converges to l then $\{|z_n|\}$ converges to $|l|$.

(Pf: Given $\varepsilon > 0$ $\exists$ $N > 0$ $\ni$ $|z_n - l| < \varepsilon$ $\forall$ $n \geq N$

Since $\big||z_n| - |l|\big| \leq |z_n - l|$

we have $|z_n| \to |l|$)

Converse is not true! (Unless $l = 0$).

SERIES:

A sum of the form $\sum_{n=0}^{\infty} z_n$ is called a series.

Corresponding to a series, we associate its sequence of partial sums:

Let $\sum_{n=0}^{\infty} z_n$ be a series. Then the k-th partial sum $S_k := \sum_{n=0}^{k} z_n$.

Eg: $\sum_{n=0}^{\infty} z^n$, $S_k = 1 + z + \cdots + z^{k-1} = \dfrac{1-z^k}{1-z}$ $(z \neq 1)$

The series $\sum_{n=0}^{\infty} z_n$ is said to converge to l if the sequence of partial sums $\{S_k\}$ converges.

Eg: $\sum_{n=0}^{\infty} z^n$, $S_k = \frac{1-z^k}{1-z}$ (for $z \neq 1$)

if $|z|<1$ then $|z|^k \to 0$; hence so does the sequence z^k (by Remark (3) above).
Thus, $\lim_{k\to\infty} \frac{1-z^k}{1-z} = \frac{1}{1-z}$

Remark: 1 If $\sum_{n=0}^{\infty} z_n$ converges then $\sum_{n=k}^{\infty} z_n$ is a null sequence; in particular, $\{z_n\}$ is a null sequence
(Indeed, $\sum_{n=0}^{\infty} z_n = l \Leftrightarrow$ given $\varepsilon > 0$ $\exists$ $N>0$ $\ni$

$$|l - S_k| < \varepsilon \quad \forall k > N$$

Fixing k, $S_m - S_k = m^{th}$ partial sum of $\sum_{n=k+1}^{\infty} z_n$. $\forall m>k$.
$\left|\sum_{n=k+1}^{\infty} z_n\right| = \left|\lim_{m\to\infty} S_m - S_k\right| = |l - S_k| < \varepsilon \quad \forall k > N$.)

In above Eg: $\sum_{n=0}^{\infty} z^n$ does not converge for $|z| \geq 1$ since $\{z^n\}$ is not a null sequence.

Remark 2: "The series $\sum_{n=0}^{\infty} z_n$ cgs $\Leftrightarrow$ $\{S_k\}$ is Cauchy; ie, $\sum_{n=0}^{\infty} z_n$ cgs iff given $\varepsilon>0$ $\exists$ $N>0$ $\ni \left|\sum_{k=n+1}^{m} z_n\right| < \varepsilon$. $\forall m>n>N$
(Easy)

Definition: <u>Absolutely convergent series.</u>

A series $\sum_{n=0}^{\infty} z_n$ is said to be absolutely convergent if $\sum_{n=0}^{\infty} |z_n|$ is convergent.

Eg: $\sum_{n=0}^{\infty} \frac{i^n}{n^2}$.

<u>Remark</u>(3): A series which is absolutely convergent is also convergent (ie $\boxed{\text{absolutely cgt} \Rightarrow \text{cgt}}$)

(Warning: For <u>sequences</u>: cgt $\not\Rightarrow$ abs cgt).

Indeed, if $\sum_{n=0}^{\infty} |z_n|$ is cgt, then by earlier Remark (2) $\sum_{n=m+1}^{k} |z_n| < \varepsilon \quad \forall\, k > m > N$.

$$\therefore \left| \sum_{n=m+1}^{k} z_n \right| \le \sum_{n=m+1}^{k} |z_n| < \varepsilon \quad \forall\, k > m > N.$$

Hence, $\sum_{n=0}^{\infty} z_n$ is cgt.

Remark (4): If $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=0}^{\infty} w_n$ are cgt

then so is $\sum_{n=0}^{\infty} (z_n \pm w_n)$ & $\sum_{n=0}^{\infty} c z_n$.

Further, $\sum_{n=0}^{\infty} z_n \pm \sum_{n=0}^{\infty} w_n = \sum_{n=0}^{\infty} (z_n \pm w_n)$

$$\sum_{n=0}^{\infty} c z_n = c \sum_{n=0}^{\infty} z_n.$$

Comparison test: Let $\sum_{n=0}^{\infty} |a_n|$ be a convergent series (of real numbers).

If $|z_n| \leq |a_n| \ \forall \ n > N_1$ then

$$\sum_{n=0}^{\infty} z_n \text{ also converges.}$$

Pf: $\sum_{n=0}^{\infty} |a_n|$ converges, so given $\varepsilon > 0 \ \exists \ N_2 > 0 \ \ni$

$$\sum_{k=n+1}^{m} |a_n| < \varepsilon \ \forall \ m \geq n > N_2$$

Consider $|S_n - S_m|$ where S_k is the k^{th} partial sum.

$$|S_n - S_m| = \left|\sum_{k=m+1}^{n} z_k\right| \leq \underbrace{\sum_{k=m+1}^{n} |z_k| \leq \sum_{k=m+1}^{n} |a_n|}_{} < \varepsilon$$

$\forall \ n,m \geq \max\{N_1, N_2\}$

$\Rightarrow \{S_n\}$ is Cauchy hence convergent. ☒

Infact, $\sum_{n=0}^{\infty} z_n$ is absolutely convergent.

Ratio test: Let $\sum_{n=0}^{\infty} z_n$ be a series such that $\lim_{n \to \infty} \frac{|z_{n+1}|}{|z_n|}$ exists and is equal to 'l'. Then

(i) $\sum_{n=0}^{\infty} z_n$ converges if $l < 1$

(ii) $\sum_{n=0}^{\infty} z_n$ diverges if $l > 1$

(iii) when $l=1$, the series may or may not converge.

Pf: (i) Let $l<1$. Then given $\varepsilon>0$ $\exists$ $N>0$

$$\ni \left| \frac{|z_{n+1}|}{|z_n|} - l \right| < \varepsilon \quad \forall\ n \geq N$$

In particular, $\dfrac{|z_{n+1}|}{|z_n|} < l+\varepsilon \quad \forall\ n \geq N$

Since $l<1$, we can choose $\varepsilon>0$ small enough such that $l+\varepsilon<1$

$$\text{ie} \quad \frac{|z_{n+1}|}{|z_n|} < \underbrace{l+\varepsilon}_{\rho} < 1 \quad \forall\ n \geq N$$

$$\therefore \quad |z_{n+1}| < \rho |z_n| \quad \forall\ n \geq N$$

Thus, $|z_n| < \rho|z_{n-1}| < \rho^2|z_{n-2}| < \cdots < \rho^{n-N}|z_N|$

$\forall\ n \geq N$

If $\rho<1$ then $\sum \rho^n$ is cgt; so by comparison test $\sum_{n=0}^{\infty} z_n$ cgs.

(ii) $l>1$ then $\dfrac{|z_{n+1}|}{|z_n|} > l-\varepsilon \quad \forall\ n \geq N$

Choose $\varepsilon > 0$, small enough $\ni$ $l > l - \varepsilon > 1$

So, $\dfrac{|z_{n+1}|}{|z_n|} > l - \varepsilon > 1 \quad \forall\, n \geq N$

$\therefore |z_{n+1}| > |z_n| \quad \forall\, n \geq N$

In particular $|z_n|$ is not a null sequence.
Hence $\sum_{n=0}^{\infty} z_n$ does not converge. ▨

(iii) same as in real case.

<u>Root test:</u> Let $\sum_{n=0}^{\infty} z_n$ be a series such that

$\limsup \sqrt[n]{|z_n|} = l$. Then

(i) if $l < 1$, $\sum_{n=0}^{\infty} z_n$ converges

(ii) if $l > 1$ $\sum_{n=0}^{\infty} z_n$ diverges

(iii) if $l = 1$ the series may or may not converge.

<u>Pf:</u> $\limsup \sqrt[n]{|z_n|} = l$

(by defn) $\Rightarrow$ every convergent subsequence of $\{\sqrt[n]{|z_n|}\}$
has limit $\leq l$.

(i) if $l < 1$ then there are almost finitely many terms of the sequence which are greater than l (<1)

$$\therefore \sqrt[n]{|z_n|} \leq l < r < 1 \quad \forall\, n \geq N$$

$$\therefore |z_n| < r^n \quad \forall\, n \geq N$$

$\Rightarrow$ by comparison $\sum_{n=0}^{\infty} |z_n|$ is cgt

$\Rightarrow \sum_{n=0}^{\infty} z_n$ is cgt.

(ii) $l > 1$, there exists a subsequence of $\{\sqrt[n]{|z_n|}\}$ such that it converges to $l > 1$

$\Rightarrow$ there are infinitely many terms of the sequence $\{\sqrt[n]{|z_n|}\}$ which are > 1.

$\Rightarrow \{z_n\}$ is not a null sequence.

(iii) $l = 1$ (as in real case; look for examples).

Appendix: lim sup of a real sequence.

Let $\{a_n\}$ be a sequence of real numbers.

Then $\limsup \{a_n\} = \sup\{$ limits of convergent subsequences $\}$

If limits and supremum can take value in $\mathbb{R} \cup \{\infty\}$, the above always exists.

Example 1: $\left\{ \sin \frac{n\pi}{2} \right\}$

$$= \{0, 1, 0, -1, 0, 1, 0, -1, \ldots\}$$

convergent subsequences are eventually $\begin{cases} \{0, 0, \ldots\} \\ \{1, 1, \ldots\} \\ \{-1, -1, \ldots\} \end{cases}$

$\therefore$ $\{$limits of cgt subsequences$\} = \{0, -1, 1\}$

$$\limsup \left\{ \sin \frac{n\pi}{2} \right\} = 1$$

Example 2: $\left\{ 1, \frac{1}{2}, 3 + \frac{1}{3}, \frac{1}{4}, 5 + \frac{1}{5}, \ldots \right\}$

odd term $a_{2n+1} = (2n+1) + \frac{1}{2n+1}$

even term $a_{2n} = \frac{1}{2n}$

$\{a_2, a_4, \ldots\}$ converges to 0

$\{a_1, a_3, \ldots\}$ diverges to ∞

$\limsup a_n = \infty$

Properties:

Let $\limsup\{a_n\} = l$, $\limsup\{b_n\} = l'$

(1) $\limsup\{a_n + b_n\} \leq l + l'$

(2) if $l, l' > 0$ then

$$\limsup_{n\to\infty} a_n b_n \leq l l'$$

(2) Let $l = \lim\limits_{n\to\infty} b_n > 0$. Then

$$\limsup_{n\to\infty} a_n b_n = l\left(\limsup_{n\to\infty} a_n\right).$$

Pf:- $\quad = \sup\{\lim\limits_{k\to\infty} a_{n_k} b_{n_k} / \{a_{n_k} b_{n_k}\} \text{ is cgt}\}$

$$\Updownarrow$$

$$\lim_{k\to\infty} a_{n_k} b_{n_k} = l \lim_{k\to\infty} a_{n_k} \Leftarrow \{a_{n_k}\} \text{ is cgt} \quad \left(\because \left\{\frac{a_{n_k} b_{n_k}}{b_{n_k}}\right\} \text{ is cgt after dropping '0's in } \{b_{n_k}\} \text{ if required}\right)$$

$\therefore \limsup\limits_{n\to\infty} a_n b_n = l(\limsup a_n)$.

(3) $\limsup c\{a_n\} = cl$. $\quad (c \geq 0)$

Pf:-

$$\limsup c\{a_n\} = \sup\{\lim_{k\to\infty} c a_{n_k} / \{c a_{n_k}\} \text{ is cgt}\}$$

$$\Updownarrow$$

$$\lim_{k\to\infty} c a_{n_k} = c \lim_{k\to\infty} a_{n_k} \Leftarrow \{a_{n_k}\} \text{ is cgt}$$

$$\therefore \limsup\{c a_{n_k}\} = \sup\{c \lim_{k\to\infty} a_{n_k} / \{a_{n_k}\} \text{ is cgt}\}$$

$$= c \sup\{\lim_{k\to\infty} a_{n_k} / \{a_{n_k}\} \text{ is cgt}\}$$

$[\because c > 0)$

$$= c \limsup\{a_{n_k}\}.$$

Let $k \in \mathbb{N}$

(B) $\limsup \{a_n^{1/k}\} = \left(\limsup\{a_n\}\right)^{1/k}$

Pf: Recall that the map $\mathbb{R}^{>0} \longrightarrow \mathbb{R}$, $\alpha \longmapsto \alpha^{1/k}$ is continuous.

and for $x, y \in \mathbb{R}^{>0}$, $x < y \Leftrightarrow x^k < y^k$.
$(\because x^k - y^k = (x-y)(x^{k-1} + x^{k-2}y + \cdots + y^{k-1}))$

$$\limsup\{a_n^{1/k}\} = \sup\left\{\lim_{j\to\infty} a_{n_j}^{1/k} \,/\, \{a_{n_j}^{1/k}\} \text{ is cgt}\right\}$$

$\Updownarrow$ $(\because \alpha \mapsto \alpha^k$ is cont$)$

$$\lim_{j\to\infty} a_{n_j}^{1/k} = \left(\lim_{j\to\infty} a_{n_j}\right)^{1/k} \Leftarrow \{(a_{n_j}^{1/k})^k\} \text{ is cgt}$$

$(\because \alpha \mapsto \alpha^{1/k}$ is cont$)$

Further $x < y \Leftrightarrow x^k < y^k$ (for $x, y \in \mathbb{R}^{>0}$)

$$\Rightarrow \sup\left\{\left(\lim_{j\to\infty} a_{n_j}\right)^{1/k} \,/\, \{a_{n_j}\} \text{ is cgt}\right\}$$
$$= \left(\sup\left\{\lim_{j\to\infty} a_{n_j} \,/\, \{a_{n_j}\} \text{ is cgt}\right\}\right)^{1/k}.$$

∎

§ Functions, limits and continuity.

Our beacon light once again is $\mathbb{R}$.

Function : $f : \mathbb{C} \to \mathbb{C}$

$$x+iy \mapsto u+iv.$$

Recall, $\mathrm{Re} : \mathbb{C} \to \mathbb{R}$; $\mathrm{Im} : \mathbb{C} \to \mathbb{R}$

$$x+iy \mapsto x \qquad\qquad x+iy \mapsto y$$

Composing f with these fns we get,

$$\mathrm{Re}(f) := \mathrm{Re} \circ f \quad ; \quad \mathrm{Im}(f) := \mathrm{Im} \circ f$$

We may think of $f : \mathbb{C} \to \mathbb{C}$ as a fn from $\mathbb{R}^2 \to \mathbb{R}^2$, in fact as two functions from $\mathbb{R}^2 \to \mathbb{R}$,

namely,

$$u : \mathbb{R}^2 \to \mathbb{R}$$
$$(x, y) \mapsto \mathrm{Re}(f)(x+iy)$$

$$v : \mathbb{R}^2 \to \mathbb{R}$$
$$(x, y) \mapsto \mathrm{Im}(f)(x+iy)$$

Thus, $f(x+iy) = u(x,y) + i v(x,y)$

(WARNING: the domain of f is $\mathbb{C}$, while of u, v is $\mathbb{R}^2$. The above equality is not equality of functions!! It is under the identification of $\mathbb{C}$ with $\mathbb{R}^2$.)

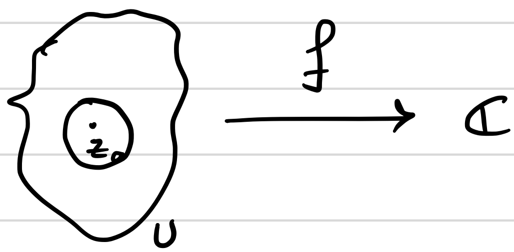
Eg: $f(z) = |z|^2$

$$f(x+iy) = x^2+y^2$$

$$u(x,y) = x^2+y^2 \; ; \; v(x,y) = 0$$

Limit of a function at a point z_0:

Let f be a function defined in a neighbourhood around z_0 (except possibly at z_0)



We say that f has a limit at z_0 if "there exists $l \in \mathbb{C} \ni$ given any $\varepsilon > 0$ $\exists\, \delta > 0$ such that $|f(z) - l| < \varepsilon$ $\forall\, 0 < |z - z_0| < \delta$".

We denote this information symbolically by writing $\lim\limits_{z \to z_0} f(z) = l$.

REMARK: $\lim\limits_{z \to z_0} f(z) = l \Leftrightarrow \lim\limits_{z \to z_0} u(x,y) = \operatorname{Re} l$ & $\lim\limits_{z \to z_0} v(x,y) = \operatorname{Im} l$ ⊠

Eg: $f(z) = |z|$

$$\lim_{z \to 0} |z| = 0$$

($\because$ given $\varepsilon > 0$, choose $\delta = \varepsilon$ then
$||z| - 0| = |z| < \varepsilon \quad \forall\ 0 \leq |z| < \varepsilon$)

Arithmetic of limits

Let f, g be two functions defined in a neighbourhood of z_0. Let $\lim_{z \to z_0} f(z) = l$ and $\lim_{z \to z_0} g(z) = l'$.

Then (i) $\lim_{z \to z_0} (f+g)(z) = l + l'$

(ii) $\lim_{z \to z_0} (f \cdot g)(z) = l l'$

(iii) if $l' \neq 0$ then $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{l}{l'}$

(iv) $\lim_{z \to z_0} c f(z) = c l$

Pf (as in the real case) $\blacksquare$

Continuity: Let f be a function defined in a neighbourhood of z_0 (including at z_0). We say that f is continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

REMARK: ① f is continuous at z_0 iff u, v are continuous at (x_0, y_0).

② f is continuous at z_0 iff whenever $z_n \to z$, the sequence $f(z_n) \to f(z)$.
(Follows, since this is true for u, v.)

③ Let f be continuous at $g(z_0)$ and g be continuous at z_0 then
$f \circ g$ is continuous at z_0.
(Proof same as in the real case).

④ By the arithmetic of limits, it follows that the if f, g are continuous at z_0 then so is $f \pm g$, fg, cf, f/g (if $g(z_0) \neq 0$).

Some examples:

① $f(z) = \bar{z}$ $\quad \lim_{z \to z_0} \bar{z} = \lim_{z \to z_0} x - iy = \lim_{(x,y) \to (x_0,y_0)} x - i \lim y$

$$= x_0 - iy_0.$$

② $f(z) = |z|$ $\quad \underbrace{\lim_{z \to z_0} |z| = |z_0|}$

$$\big||z| - |z_0|\big| \leq |z - z_0| \quad \text{(see, triangle inequality)}$$

So, given $\varepsilon > 0$, choose $\delta = \varepsilon$ then

$$\big||z| - |z_0|\big| < \varepsilon \quad \forall \ |z - z_0| < \varepsilon.$$

③ $f(z) = \dfrac{|z|}{z}$, $\quad \lim_{z \to 0} \dfrac{|z|}{z}$ does not exist, since

along the x-axis, $z = x$, $|z| = |x|$

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1 \neq \lim_{x \to 0^-} \frac{|x|}{x} = -1.$$

However, since $|z|$ & $\dfrac{1}{z}$ are continuous at $z \neq 0$ we get by the arithmetic of continuous fun that $\dfrac{|z|}{z}$ is continuous at $z \neq 0$. ∎