CMPE 185 Autonomous Mobile Robots

Mobile Robot Kinematics

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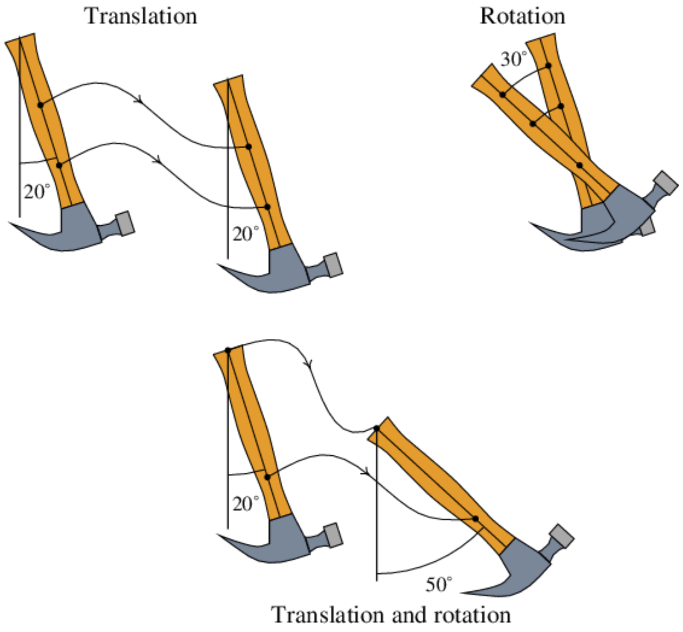
Kinematics

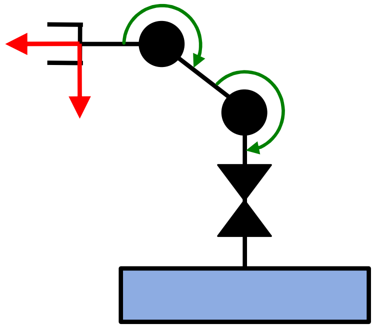
- Definition of kinematics
  - "Description of the motion of points, bodies or systems of bodies"
  - …without consideration of the causes of motion (=> dynamics)
  - Required for kinematic simulation and control



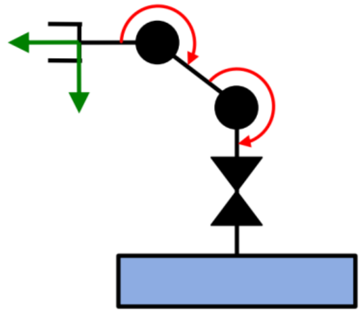
- Types of motion of single bodies
  - Translation
  - Rotation
  - Combined motion

Kinematics

- **Forward kinematics**
  - Given a set of actuator positions, determine the corresponding pose



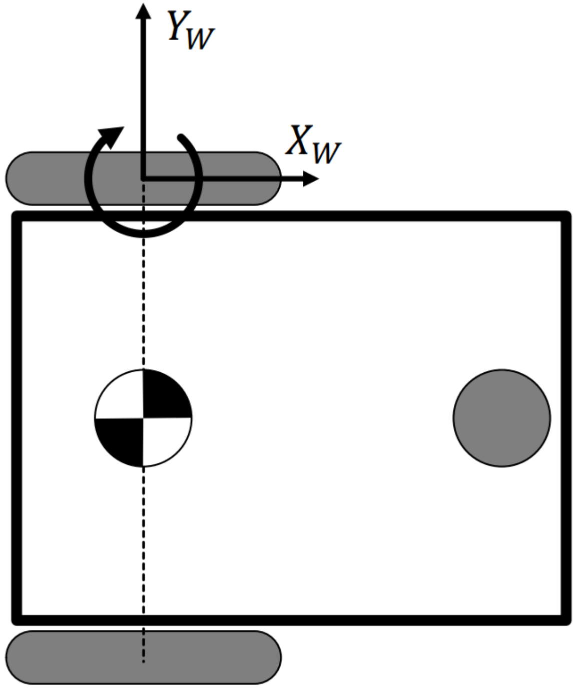
- **Inverse kinematics**
  - Given a desired pose, determine the corresponding actuator positions



Wheeled Kinematics

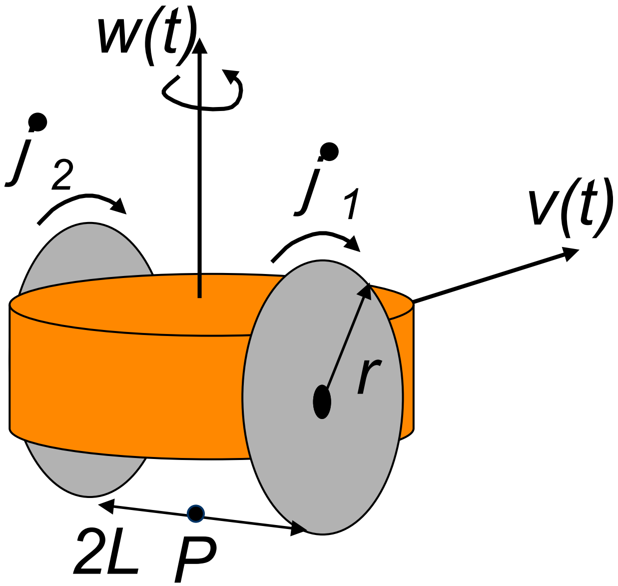
- Not all degrees of freedom of a wheel can be actuated or have encoders
- Wheels can impose **differential constraints** that complicate the computation of kinematics

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \dot{\varphi} r \\ 0 \end{bmatrix} \begin{array}{l} \leftarrow \text{rolling constraint} \\ \leftarrow \text{no-sliding constraint} \end{array}$$



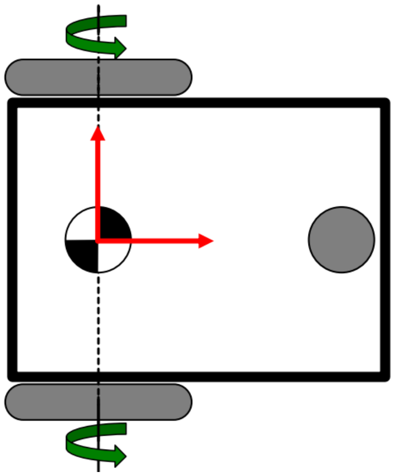
Wheeled Kinematics

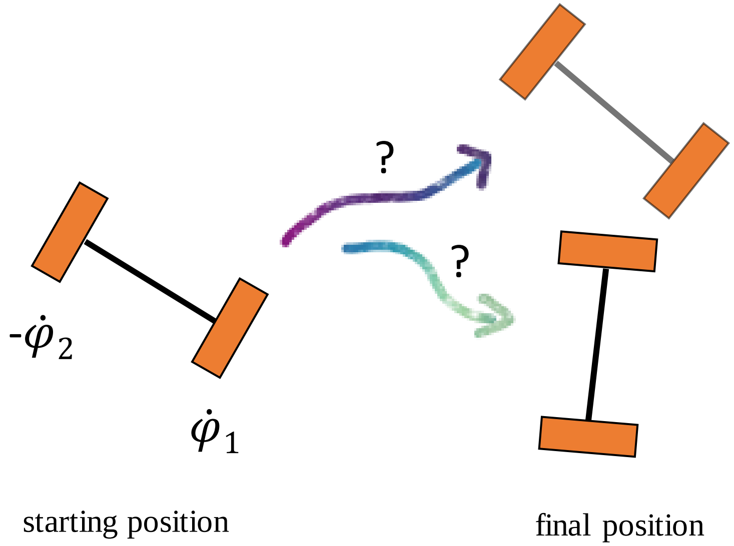
- P: center of the robot
- r: radius of the wheel
- $2L$: length of the axles
- v: linear velocity of the robot
- w: angular (rotational) velocity of the robot
- $\dot{\varphi}_1$: rotational speed of the right wheel
- $\dot{\varphi}_2$: rotational speed of the left wheel



Differential Kinematics

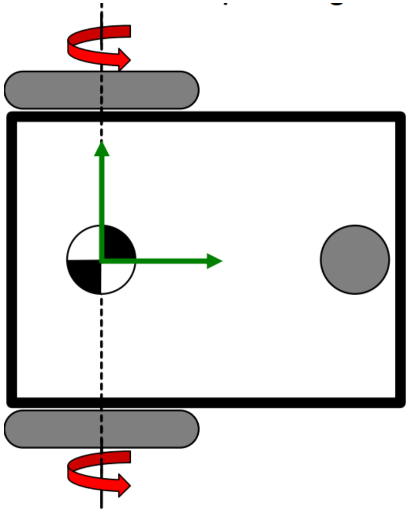
- Differential forward kinematics
  - Given the wheels' speed inputs - $\dot{\varphi}_1$ and $\dot{\varphi}_2$, determine the robot's velocity $\dot{\xi}_I$ in the global frame

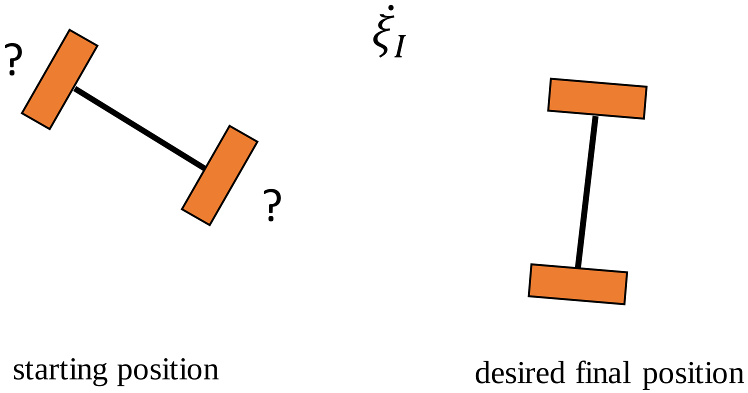




Differential Kinematics

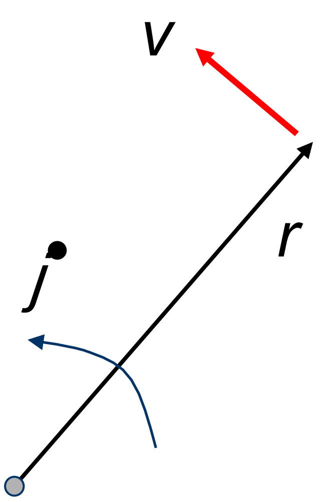
- Differential inverse kinematics
  - Given the desired velocities of the robot in the global frame, determine the corresponding wheels’ speed input





Differential Forward Kinematics

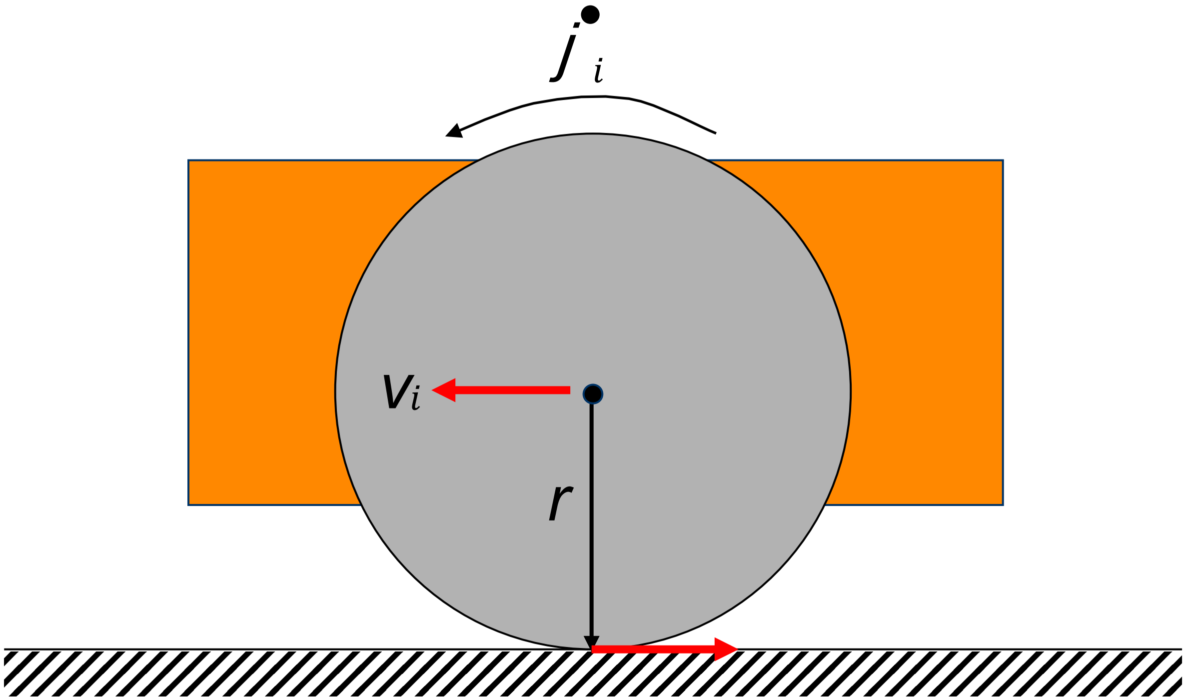
- Before we continue, we need to understand the relation between angular velocity and linear velocity



$$r\dot{j} = v$$

Differential Forward Kinematics

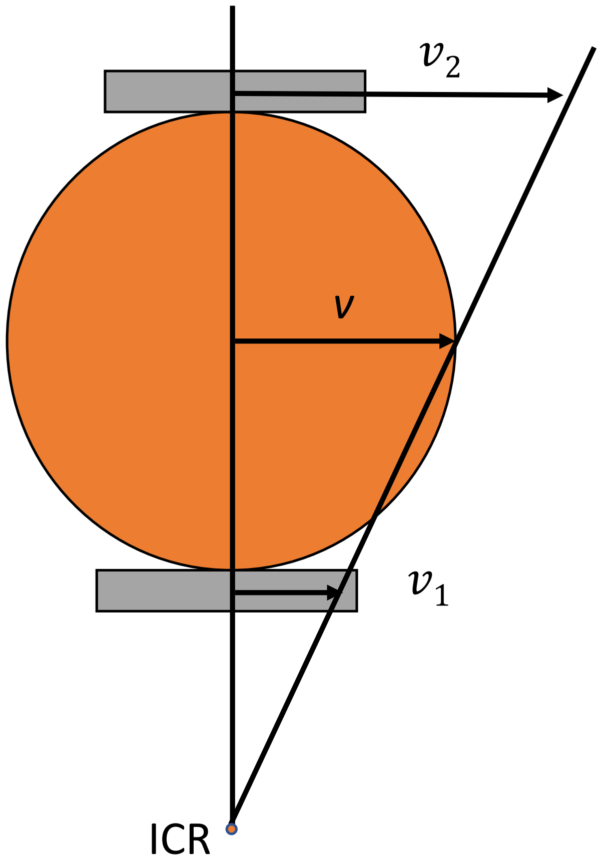
- Apply this to a wheel on the robot
- Kinematic constraint: $r\dot{\varphi}_i = v_i$



Two-Wheeled Robot Kinematic Model

- Linear velocity of the robot is the average of the two wheel velocities

$$v = \frac{v_1 + v_2}{2} = \frac{r\dot{\varphi}_1 + r\dot{\varphi}_2}{2}$$

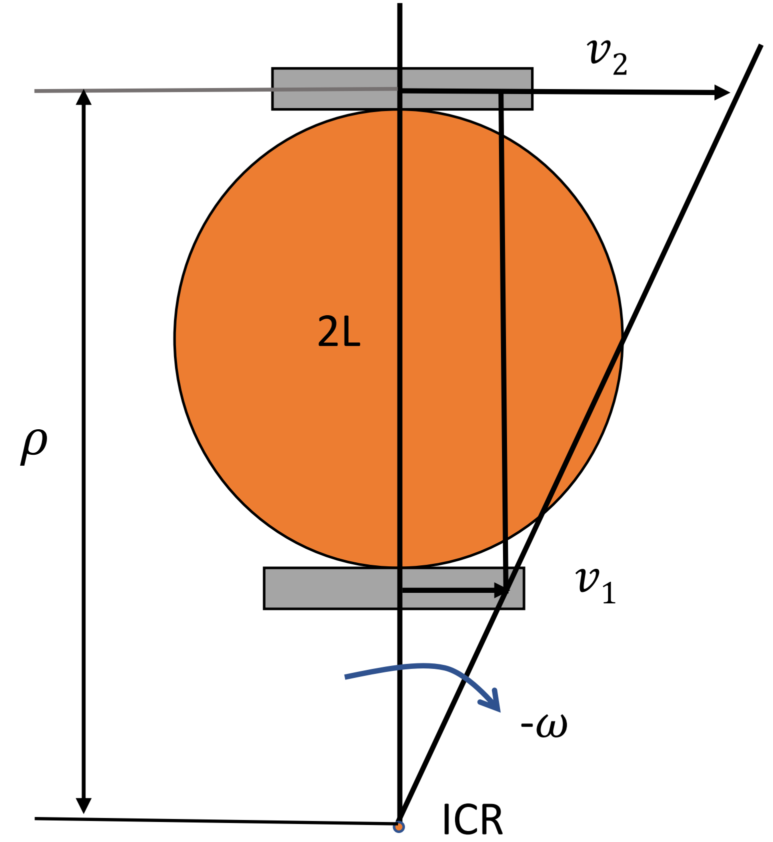


Two-Wheeled Robot Kinematic Model

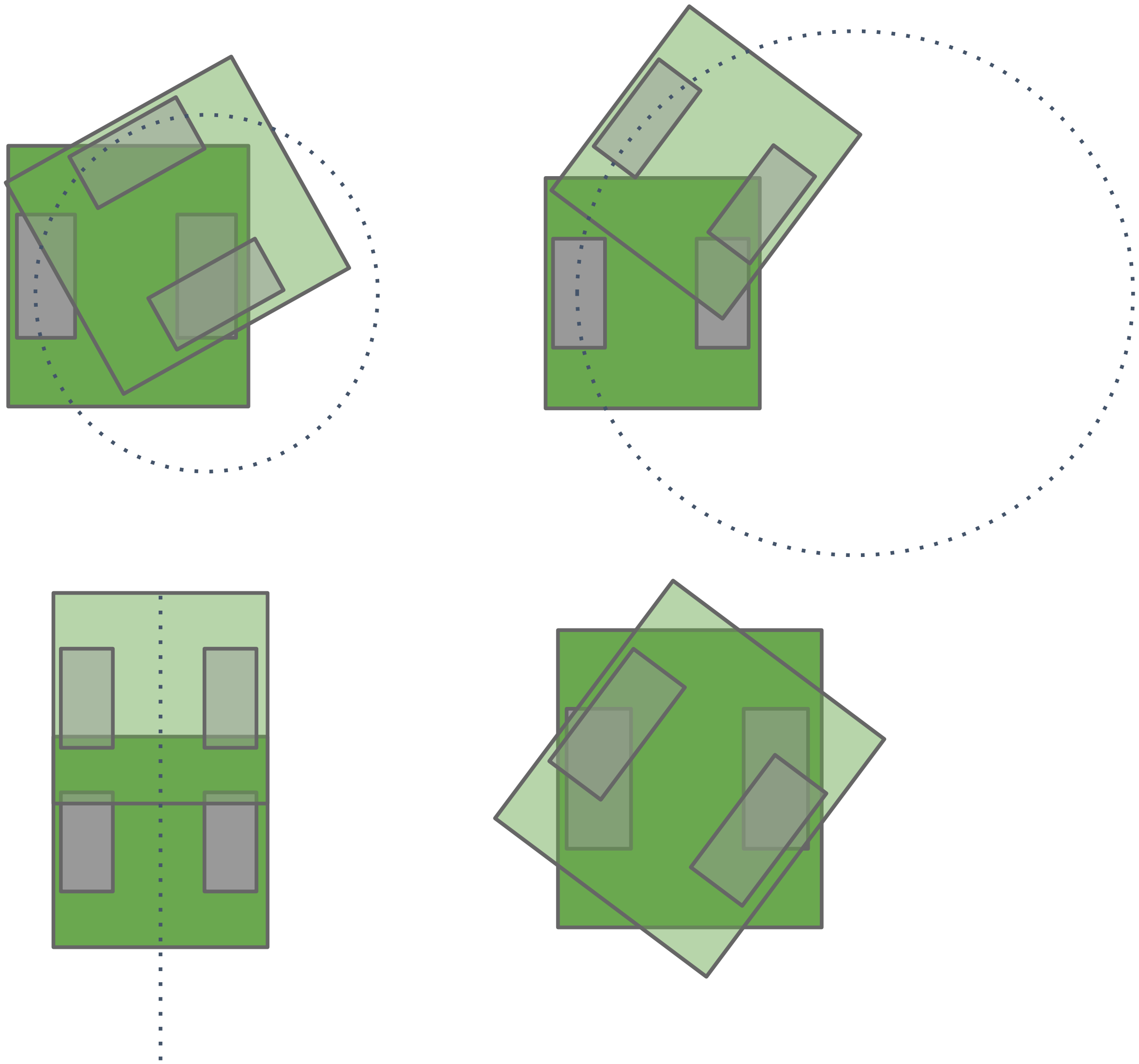
- Use the instantaneous center of rotation (ICR)
- Equivalent triangles give the angular rate of rotation

$$\omega = \frac{-v_2}{\rho} = \frac{-(v_2 - v_1)}{2L}$$

$$\omega = \frac{r\dot{\varphi}_1 - r\dot{\varphi}_2}{2L}$$



Fun Time



$$v_L = +k,\ v_R = 0$$

$$v_L = +k,\ v_R = +k$$

$$v_L = +k,\ v_R = +k'$$

$$v_L = +k,\ v_R = -k$$

Forward Kinematics

- We now know how to calculate how wheel speeds affect the robot velocities in the global coordinate frame

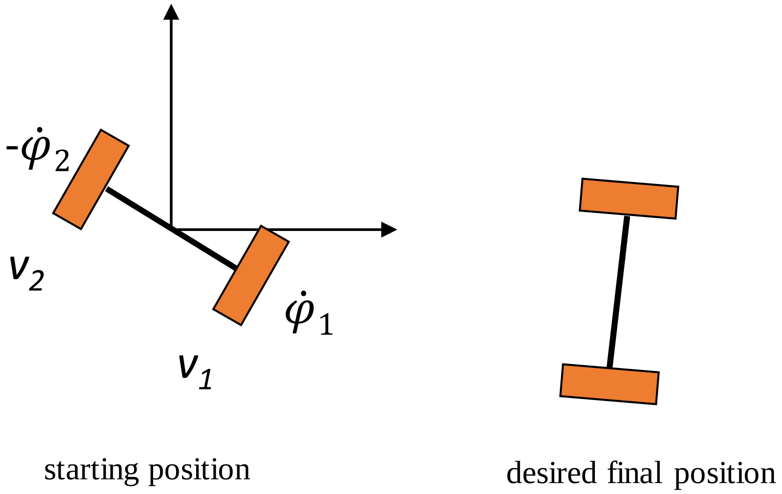
- This will be useful when we want to control the robot to track points, i.e., move to desired locations in the global coordinate frame by controlling wheel speeds

Inverse Kinematics

- **How to determine the speed of the wheels to obtain the desired velocities of the robot?**

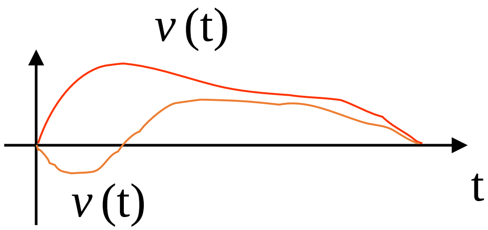
Differential Inverse Kinematics

- Given the desired velocity of the robot, determine the corresponding wheel's speed, or
- Standing in pose (x, y, θ) at time t, determine the control parameters such that the pose at time t + $\boldsymbol{\delta}t$ is (x', y', θ')



Differential Inverse Kinematics

- Finding some solutions are not hard, but finding the "best" solution is very difficult
- "Best" in the sense of
  - Quickest time
  - Most energy efficient
  - Smoothest velocity profiles
  - Etc…

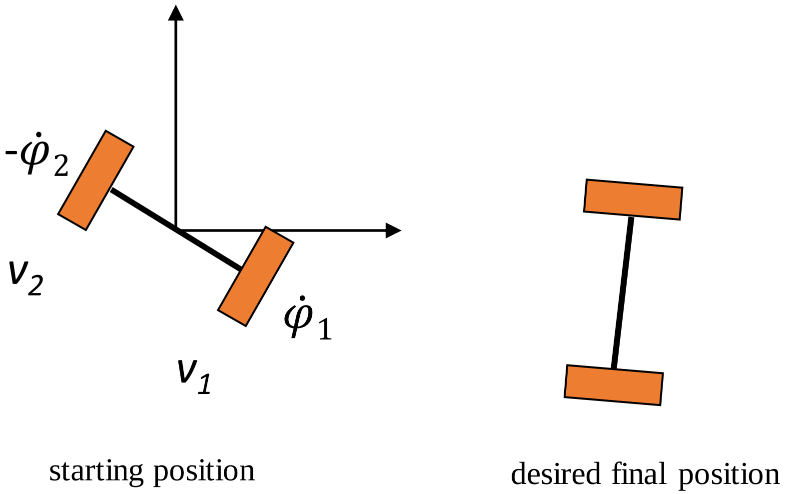


- Some solutions are not feasible for robots because of constraints – nonholonomic robots

Differential Inverse Kinematics – Decomposition

- Usually approach: decompose the problem into two operations:
  - Move in a straight line

    $v_1 = v_2$, $\omega\delta t = 0$
  - Rotate in place about center
    - $v_1 = -v_2$
    - $\theta' = \theta + \omega\delta t$



Differential Inverse Kinematics – Decomposition

- Step 1: turn so that the wheels are parallel to the line between the original and final position of the robot origin.

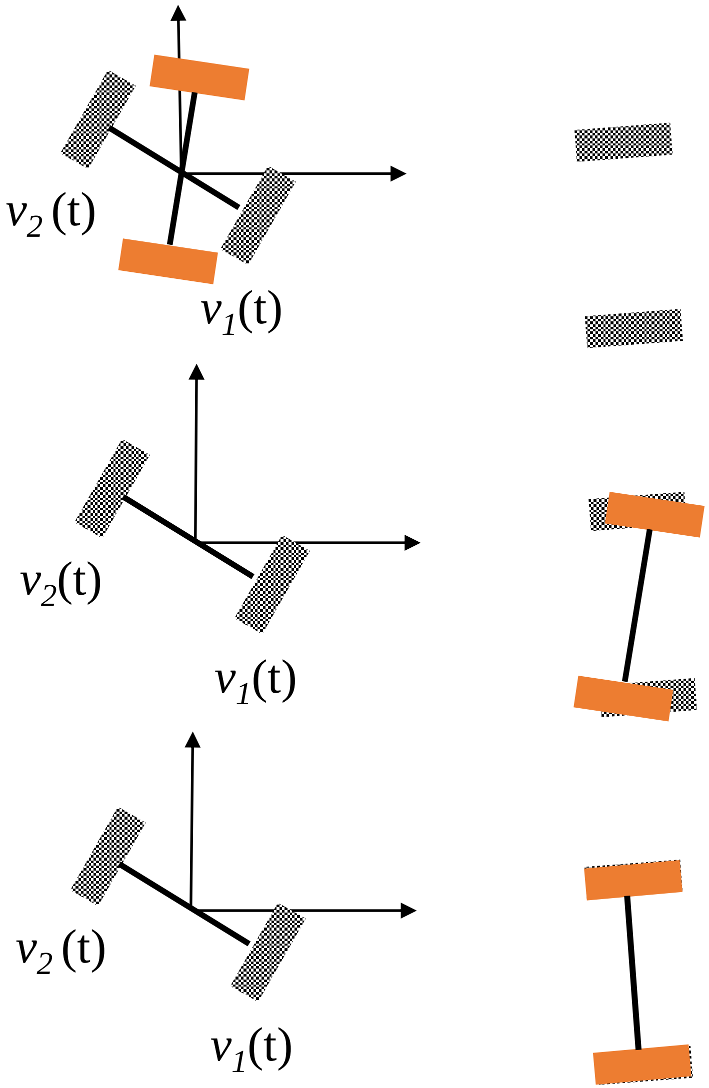
$$v_1\,(t) = -v_2(t) = v_{\max}$$

- Step 2: drive straight until the robot's origin coincides with the destination

$$v_1\,(t) = v_2\,(t) = v_{\max}$$

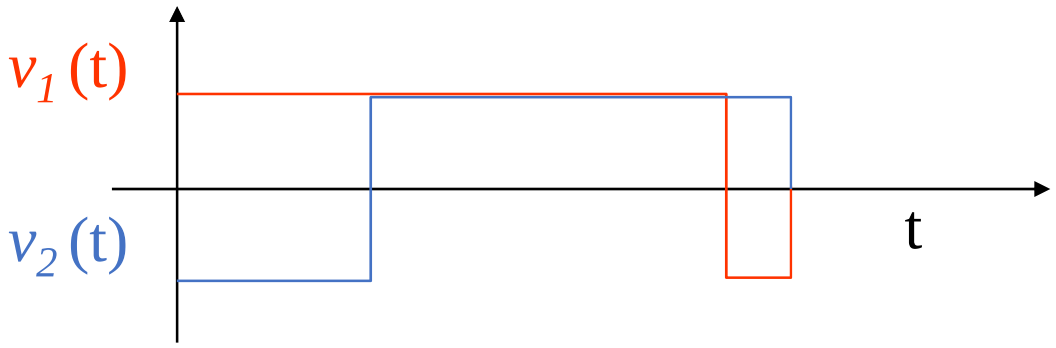
- Step 3: rotate again in order to achieve the desired final orientation

$$-v_1\,(t) = v_2\,(t) = v_{\max}$$

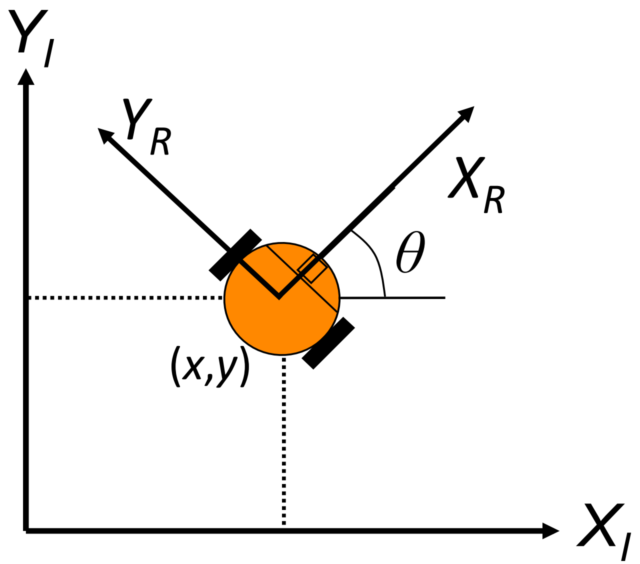


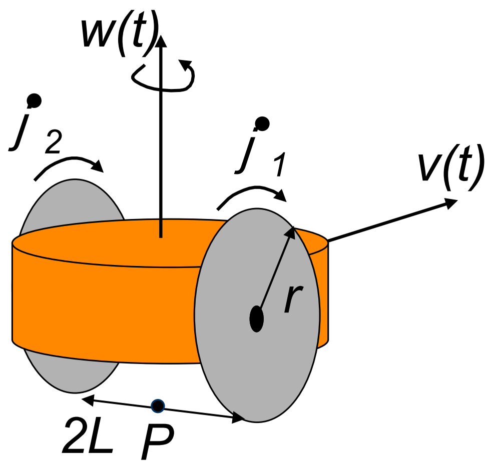
Differential Inverse Kinematics – Decomposition

- Velocity profile



- Two models





$$\begin{cases} \dot{x} = v \cos\theta \\ \dot{y} = v \sin\theta \\ \dot{\theta} = w \end{cases}$$

Design for this model!

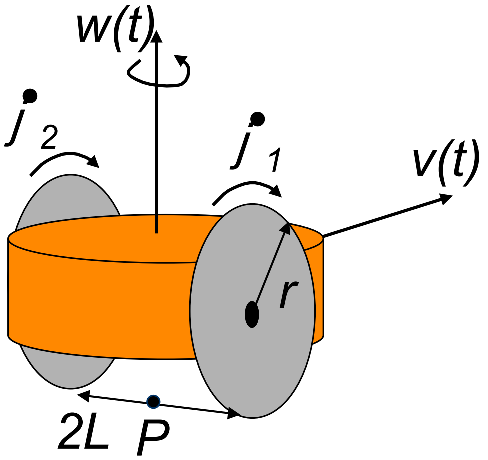
$$\begin{cases} \dot{x} = \dfrac{r}{2}(\dot{\varphi}_1 + \dot{\varphi}_2)\cos\theta \\ \dot{y} = \dfrac{r}{2}(\dot{\varphi}_1 + \dot{\varphi}_2)\sin\theta \\ \dot{\theta} = \dfrac{r}{2L}(\dot{\varphi}_2 - \dot{\varphi}_1) \end{cases}$$

Implement this model

Kinematic Model of a Simple 2D Robot

- Continuous time model:

$$
\begin{cases}
\dot{x} = \frac{r}{2}(\dot{\varphi}_1 + \dot{\varphi}_2)\cos\theta \\
\dot{y} = \frac{r}{2}(\dot{\varphi}_1 + \dot{\varphi}_2)\sin\theta \\
\dot{\theta} = \frac{r}{2L}(\dot{\varphi}_2 - \dot{\varphi}_1)
\end{cases}
$$

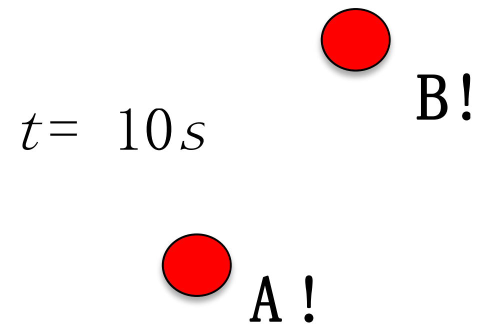


- Discrete time model

$$
\begin{cases}
x_{k+1} = x_k + \frac{r}{2}(\dot{\varphi}_{1,k} + \dot{\varphi}_{2,k})\cos\theta_k \Delta t \\
y_{k+1} = y_k + \frac{r}{2}(\dot{\varphi}_{1,k} + \dot{\varphi}_{2,k})\sin\theta_k \Delta t \\
\theta_{k+1} = \theta_k + \frac{r}{2L}(\dot{\varphi}_{2,k} - \dot{\varphi}_{1,k})\Delta t
\end{cases}
$$

From One Model to Another

- A simple task: move from A to B in 10s
- High level task!
- Control design: *v* and ω
- Commands sent to the robots: $\dot{\varphi}_1$ and $\dot{\varphi}_2$



$$v(t) = \frac{r}{2}(\dot{\varphi}_1 + \dot{\varphi}_2) \quad \Rightarrow \quad \frac{2v}{r} = \dot{\varphi}_1 + \dot{\varphi}_2$$

$$w(t) = \frac{r}{2L}(\dot{\varphi}_2 - \dot{\varphi}_1) \quad \Rightarrow \quad \frac{wL}{r} = \dot{\varphi}_2 - \dot{\varphi}_1$$

$$\dot{\varphi}_1 = \frac{2v + wL}{2r}$$

$$\dot{\varphi}_2 = \frac{2v - wL}{2r}$$

From One Model to Another

- An intuitive example
- For inputs *v* = 0, ω = *C* (a constant), find the corresponding angular wheels velocities $\dot{\varphi}_1$ and $\dot{\varphi}_2$

$$\dot{\varphi}_1 = \frac{2v + wL}{2r}$$

$$\dot{\varphi}_2 = \frac{2v - wL}{2r}$$

$$\dot{\varphi}_1 = \frac{CL}{2r}$$

$$\dot{\varphi}_2 = -\frac{CL}{2r}$$

- Thank you!