Numerical solution of generalized Black–Scholes model

Group 16

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Introduction: The Black Scholes Equation

- We assume one risk-free asset and one risky asset constitutes a arbitrage-free frictionless-market.
- The stock price S of the unit risky asset follows the following stochastic differential equation at time τ:

$$dS = (\mu - D)Sd\tau + \sigma SdW \tag{1}$$

- Using Itô's lemma and eliminating randomness, we derive Black–Scholes equation. The Black–Scholes Model for evaluating European call option price $C(S, \tau)$ is given as

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial^2 S} + (r - D)\frac{\partial C}{\partial S} - rC = -\frac{\partial C}{\partial \tau} \qquad S > 0,\ \tau \in (0, T) \tag{2}$$

with the final condition

$$C(S, T) = max(S - K, 0) \qquad S \in [0, \infty]$$

Introduction: The Black Scholes Equation

- The Black–Scholes equation, in which σ, r and D are constants, can be easily reduced to standard heat equation.

$$C(S, \tau) = Sexp(-D(T - \tau))N(d_1) - Kexp(-r(T - \tau))N(d_2) \quad (3)$$

where,

$$d_1 = \frac{lnS - lnK + (r - D + \frac{1}{2}\sigma^2)(T - \tau)}{\sigma\sqrt{T - \tau}}$$
$$d_2 = d_1 - \sigma\sqrt{T - \tau}$$

and $N(y)$ is the cumulative standard normal distribution.

Introduction: The Black Scholes Equation

- The above mentioned transformation is not possible when the parameters σ, r and D are not constants.
- This paper presents a numerical scheme that efficiently tackles the above mentioned case.
- A simultaneous discretization in space and time using High-Order Difference approximation with Identity Expansion (HODIE) scheme in space direction and two-step backward differentiation for temporal discretization is implemented.

The Black Scholes PDE

- The generalized Black–Scholes model for evaluating European call option price $C(S,\tau)$ is

$$\frac{1}{2}\sigma^2(S,\tau)S^2\frac{\partial^2 C}{\partial^2 S} + (r(S,\tau) - D(S,\tau))\frac{\partial C}{\partial S} - r(S,\tau)C = -\frac{\partial C}{\partial \tau} \qquad S > 0, \tag{4}$$

with the final condition

$$C(S,T) = max(S-K,0) \qquad S \in [0,\infty]$$

and the boundary conditions

$$C(0,\tau) = 0$$
$$C(S,\tau) \to S \;\; as \;\; S \to \infty$$

where S is the asset price, τ is the time variable, $\sigma(S,\tau)$ is the market volatility, $r(S,\tau)$ is the interest rate and $D(S,\tau)$ is the dividend yield of the asset.

The Scheme

- The scheme starts with applying the transformation $\tau = \mathrm{T}$ - t thus converting the final value conditions to initial value boundary conditons followed by making the resulting non-differentiable initial conditions smooth.
- The ϵ-neighborhood at the points of non-differtiablity are approximated using polynomials.
- The scheme then truncates the asset price domain from $[0, \infty)$ to $[0, S_{max}]$ inorder to apply a numerical scheme.

The Scheme

- The resulting PDE is given by, Lu(S,t)

$$\equiv \frac{\partial u}{\partial t} + \frac{1}{2}\hat{\sigma}^2(S,t)S^2\frac{\partial^2 u}{\partial^2 S} + (\hat{r}(S,t) - \hat{D}(S,t))\frac{\partial u}{\partial S} - \hat{r}(S,t)u = f(S,t) \tag{5}$$

$$where \;\; S \in (0, S_{\mathrm{max}}), \quad t \in (0, T)$$

with the initial condition

$$u(S,0) = \phi(S), \qquad S \in [0, S_{max}]$$

and the boundary conditions

$$u(0,t) = 0, \qquad t \in [0, T] \;\; and$$

$$u(S_{max}, t) = S_{max} e^{(-\int_0^t \hat{D}(S_{max},q)dq) - Kexp(-\int_0^t \hat{r}(S_{max},q)dq)}, t \in [0, T]$$

The Scheme

- We partition the space Ω into the discretization Ω_h^k with M intervals along the space direction, with grid spacing h, and with N intervals along the time direction, each with grid spacing k.
- The fully discritized scheme on this mesh Ω_h^k is given by

$$\beta_{m,1}^n(\delta_t U_m^n) + \beta_{m,2}^n(\delta_t U_{m+1}^n) + [\alpha_{m,-}^n U_{m-1}^n + \alpha_{m,c}^n U_m^n + \alpha_{m,+}^n U_{m+1}^n] \\ = \beta_{m,1}^n f_m^n + \beta_{m,2}^n f_{m+1}^n$$

  for $m = 1, 2, \ldots, M$ and $n = 1, 2, \ldots, N$.
- From second time level onwards, the time direction is discretized using two-step backward differentiation formula and backward Euler's formula is used for the solution at the first time level.

The Scheme

- Space discretization is done using the classical HODIE scheme with three stencil points and two auxiliary points.
- The HODIE coefficients $\alpha^n_{m,-}, \alpha^n_{m,c}$ and $\alpha^n_{m,+}$ are the three coefficients of approximate solution U at the three stencil points at n^{th} time level.
- The HODIE coefficients $\beta^n_{m,1}$ and $\beta^n_{m,2}$ are the two coefficients for identity expansion at the two auxiliary points at n^{th} time level.

Numerical Experiments

- The paper discusses two European options following Black Scholes model and uses the given numerical scheme to approximate their solutions.
- These two options are special cases of the Black Scholes model, for which the closed form solution is available.
- Hence, the maximum absolute error $\hat{E}^{M,N}_{max}$ and the Root Mean Square error $\hat{E}^{M,N}_{rms}$ and the corresponding order of convergence $\hat{p}^{M,N}_{max}$ and $\hat{p}^{M,N}_{rms}$ can be calucated from the analytical solution, and the ones obtained from the numerical scheme.

Numerical Experiments

- The various errors discussed above are given by

$$\hat{E}_{max}^{M,N} = \max_{0 \le m \le M} |u^{m,n}(S_m, t_N) - U^{m,n}(S_m, t_N)|$$

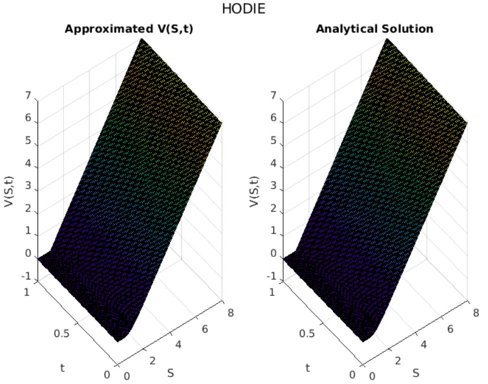
$$\hat{E}_{rms}^{M,N} = \sqrt{\frac{\sum_{m=0}^{M} [(u^{m,n}(S_m, t_N) - U^{m,n}(S_m, t_N))^2]}{M+1}}$$

$$\hat{p}_{max}^{M,N} = \log_2\left(\frac{\hat{E}_{max}^{M,N}}{\hat{E}_{max}^{2M,2N}}\right)$$

$$\hat{p}_{rms}^{M,N} = \log_2\left(\frac{\hat{E}_{rms}^{M,N}}{\hat{E}_{rms}^{2M,2N}}\right)$$

Example

- **Example 1**: Consider the Black-Scholes equation for European Call option price with $\hat{\sigma}(S,t) = 0.4$, $\hat{r}(S,t) = 0.04$, $\hat{D}(S,t) = 0.02$, $T = 1$, and $K = 1$. Take $S_{max} = 8$, and $\epsilon = 10^{-6}$.
- The analytical and the numerical solution of the above example is,



Example

- The value at $t = 0$ obtained form the numerical solution is,

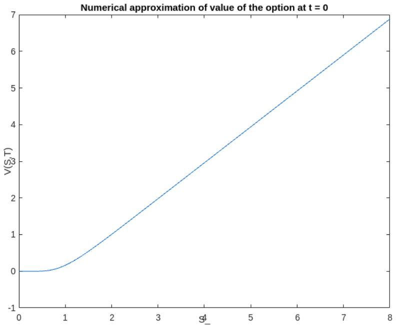


Figure 1: Numerical plot at time t=0

Example

- The plot of the absolute and RMS errors vs time is given by,

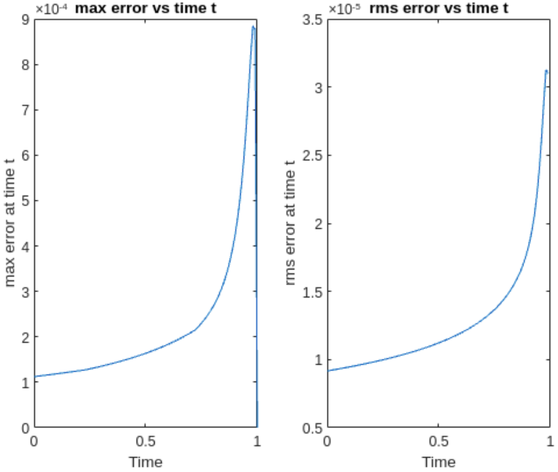


Figure 2: Absolute and RMS errors vs Time

Example

- The plot of the maximum absolute error vs the mesh size is,

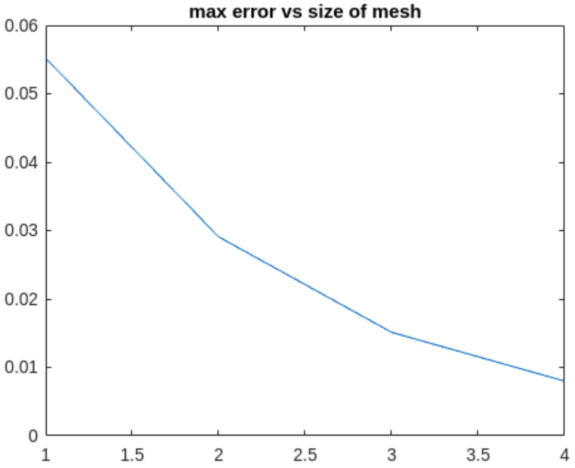


Figure 3: Max Absolute error vs mesh size

Example

- The plot of the RMS error vs the mesh size is,

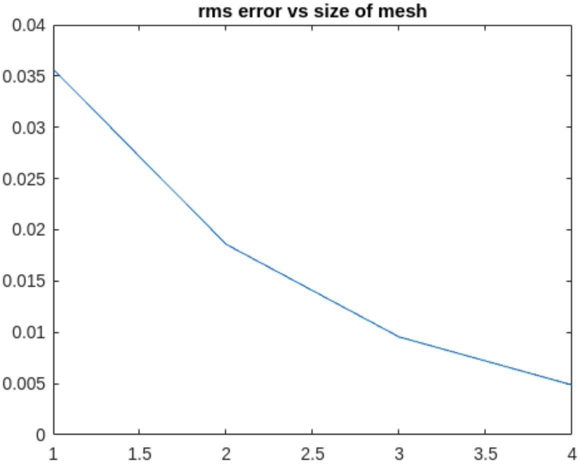


Figure 4: RMS error vs mesh size

Thank You.