MA473 Term Paper Report

Numerical solution of generalized Black–Scholes model

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1 Introduction

We assume one risk-free asset and one risky asset constitutes a market which is frictionless and without any arbitrage opportunity. The stock price S of the unit risky asset follows the following stochastic differential equation at time τ:

$$dS = (\mu - D)Sd\tau + \sigma SdW \tag{1}$$

Here μ is the drift rate, D is the dividend yield, σ is the market volatility and dW is the increment of a standard Wiener process. Now by using Itô's lemma and by eliminating the randomness in a complete market, we derive Black–Scholes equation. The Black–Scholes model for evaluating European call option price $C(S, \tau)$ is given as

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial^2 S} + (r - D)\frac{\partial C}{\partial S} - rC = -\frac{\partial C}{\partial \tau} \qquad S > 0,\ \tau \in (0, T) \tag{2}$$

with the final condition

$$C(S, T) = max(S - K, 0) \qquad S \in [0, \infty]$$

where τ is the time variable, r is the risk free interest rate, T is the expiry time of the contract and K is the strike price. The Black–Scholes equation, in which σ, r and D are constants, can be easily reduced to standard heat equation.

$$C(S, \tau) = Sexp(-D(T - \tau))N(d_1) - Kexp(-r(T - \tau))N(d_2) \tag{3}$$

where

$$d_1 = \frac{lnS - lnK + (r - D + \frac{1}{2}\sigma^2)(T - \tau)}{\sigma\sqrt{T - \tau}}$$
$$d_2 = d_1 - \sigma\sqrt{T - \tau}$$

and the cumulative standard normal distribution function $N(y)$ is defined as

$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} exp(-\frac{1}{2}x^2)dx$$

But when these parameters σ, r and D are dependent on stock price S and time variable τ, such simplification of the problem is not possible.
In this paper a numerical scheme which efficiently tackles the degeneracy concern of

the Black–Scholes equation without performing logarithmic transform of the equation, without truncating the domain so as to exclude the point of degeneracy and without using nonuniform mesh near the point of degeneracy is presented. We implement simultaneous discretization in space and time using High-Order Difference approximation with Identity Expansion (HODIE) scheme in space direction and two-step backward differentiation for temporal discretization.

2 The Black–Scholes partial differential equation

The generalized Black–Scholes model for evaluating European call option price $C(S,\tau)$ is

$$\frac{1}{2}\sigma^2(S,\tau)S^2\frac{\partial^2 C}{\partial^2 S}+(r(S,\tau)-D(S,\tau))\frac{\partial C}{\partial S}-r(S,\tau)C=-\frac{\partial C}{\partial \tau} \qquad S>0,\ \tau\in(0,T) \tag{4}$$

with the final condition

$$C(S,T)=max(S-K,0) \qquad S\in[0,\infty]$$

and the boundary conditions

$$\begin{aligned} &C(0,\tau)=0\\ &C(S,\tau)\to S\ \ as\ \ S\to\infty \end{aligned}$$

where S is the asset price, τ is the time variable, $\sigma(S,\tau)$ is the market volatility, $r(S,\tau)$ is the interest rate and $D(S,\tau)$ is the dividend yield of the asset. σ, r and D are assumed to be continuous and bounded functions of space and time variables within the contract period.

The initial conditions of the above problems are not smooth, we need to ensure the convergence of the solution obtained from finite difference scheme. Hence we replace a small ϵ-neighborhood of the point of singularity by a ninth degree polynomial so that the payoff is a fourth order smooth function.
For European option, define a function $\psi(x)$ as

$$\psi(x)=\begin{cases} x & for\ \ x\geq\epsilon\\ c_0+c_1x+c_2x^2+...+c_9x^9 & for\ -\epsilon<x<\epsilon\\ 0 & for\ \ x\leq-\epsilon \end{cases}$$

where $\epsilon>0$ is a small constant and c_i, $i=0,1,...,9$ are the constant coefficients. By using the following conditions

$$\begin{aligned} \psi(-\epsilon)=\psi^{'}(-\epsilon)=\psi^{''}(-\epsilon)=\psi^{'''}(-\epsilon)=\psi^{(4)}(-\epsilon)=0\\ \psi(\epsilon)=\epsilon,\ \ \psi^{'}(\epsilon)=1,\ \ \psi^{''}(\epsilon)=\psi^{'''}(\epsilon)=\psi^{(4)}(\epsilon)=0 \end{aligned}$$

we obtain the values of the coefficients c_i, $i = 0, 1, ..., 9$.

$$c_0 = \frac{35}{256}\epsilon, \quad c_1 = \frac{1}{2}, c_2 = \frac{35}{64\epsilon}, \quad c_4 = -\frac{35}{128\epsilon^3}$$
$$c_6 = \frac{7}{64\epsilon^5}, \quad c_8 = \frac{5}{256\epsilon^7}, \quad c_3 = c_5 = c_7 = c_9 = 0$$

The function $\phi(S) = \psi(S - K)$, smooths the payoff.

Finally, by changing the final condition to $\hat{C}(S,T) = \phi(S)$ and applying the transformation $\tau = T - t$, where t is new time variable. The above Final boundary value problem is changed into Initial boundary value problem as given below:

$$Lu(S,t) \equiv \frac{\partial u}{\partial t} + \frac{1}{2}\hat{\sigma}^2(S,t)S^2\frac{\partial^2 u}{\partial^2 S} + (\hat{r}(S,t) - \hat{D}(S,t))\frac{\partial u}{\partial S} - \hat{r}(S,t)u = f(S,t) \quad (5)$$

$$where \;\; S \in (0, S_{\max}), \quad t \in (0,T)$$

with the initial condition

$$u(S,0) = \phi(S), \qquad S \in [0, S_{max}]$$

and the boundary conditions

$$u(0,t) = \hat{g_1}(t), \qquad t \in [0,T] \;\; and$$
$$u(S_{max},t) = \hat{g_2}(t), \qquad t \in [0,T]$$

where

$$\hat{g_1}(t) = 0 \;\; and$$
$$\hat{g_2}(t) = S_{max}exp(-\int_0^t \hat{D}(S_{max},q)dq) - Kexp(-\int_0^t \hat{r}(S_{max},q)dq) \quad t \in [0,T]$$

3 Discretization

We partition the space Ω into the discretization Ω_h^k with M intervals along the space direction, with grid spacing h, and with N intervals along the time direction, each with grid spacing k. A point in the grid is donated by (S_m, t_n).

The fully discritized scheme on this mesh Ω_h^k is given by,

$$\beta_{m,1}^n(\delta_t U_m^n) + \beta_{m,2}^n(\delta_t U_{m+1}^n) + [\alpha_{m,-}^n U_{m-1}^n + \alpha_{m,c}^n U_m^n + \alpha_{m,+}^n U_{m+1}^n] = \beta_{m,1}^n f_m^n + \beta_{m,2}^n f_{m+1}^n \quad (6)$$

for $m = 1, 2, \ldots, M$ and $n = 1, 2, \ldots, N$.
Where,

$$\begin{aligned}
\delta_t U_m^n &= (U_m^n - U_m^{n-1})/k, \quad n = 1 \\
\delta_t U_m^n &= (\frac{3}{2}U_m^n - 2U_m^{n-1} + \frac{1}{2}U_m^n)/k, \quad n = 2, 3, ..., N \\
U_m^0 &= \phi_m, \quad m = 0, 1, ...M. \\
U_{\cup}^n &= \hat{g_1^n}, \quad n = 0, 1, 2, ..., N \\
U_M^n &= \hat{g_1^n}, \quad n = 0, 1, 2, ..., N
\end{aligned}$$

Here, second time level onwards, the time direction is discretized using two-step backward differentiation formula and backward Euler's formula is used for the solution at the first time level.
Space discretization is done using the classical HODIE High-Order Difference approximation with Identity Expansion scheme with three stencil points and two auxiliary points. Which is discussed in the paper.

4 Numerical experiments

In the paper, the author has taken the example of two European options following Black Scholes model and used the above given numerical scheme to find out the solution to the two options. These two options are form a special case of the Black Scholes model, for which the closed form solution is available, and hence, we can calculate the maximum absolute error $\hat{E}_{max}^{M,N}$ and the Root Mean Square error $\hat{E}_{rms}^{M,N}$ and the corresponding order of convergence $\hat{p}_{max}^{M,N}$ and $\hat{p}_{rms}^{M,N}$ from the original values, and the ones obtained from the numerical scheme. The expressions are given as,

$$\begin{aligned}
\hat{E}_{max}^{M,N} &= \max_{0 \le m \le M} |u^{m,n}(S_m, t_N) - U^{m,n}(S_m, t_N)| \\
\hat{E}_{rms}^{M,N} &= \sqrt{\frac{\sum_{m=0}^{M}[(u^{m,n}(S_m, t_N) - U^{m,n}(S_m, t_N))^2]}{M+1}} \\
\hat{p}_{max}^{M,N} &= \log_2(\frac{\hat{E}_{max}^{M,N}}{\hat{E}_{max}^{2M,2N}}) \\
\hat{p}_{rms}^{M,N} &= \log_2(\frac{\hat{E}_{rms}^{M,N}}{\hat{E}_{rms}^{2M,2N}})
\end{aligned}$$

Example 1: Consider the Black-Scholes equation for European Call option price with $\hat{\sigma}(S, t) = 0.4$, $\hat{r}(S, t) = 0.04$, $\hat{D}(S, t) = 0.02$, $T = 1$, and $K = 1$. Take $S_{max} = 8$, and $\epsilon = 10^{-6}$.

The analytical and the numerical solution of the above example is,

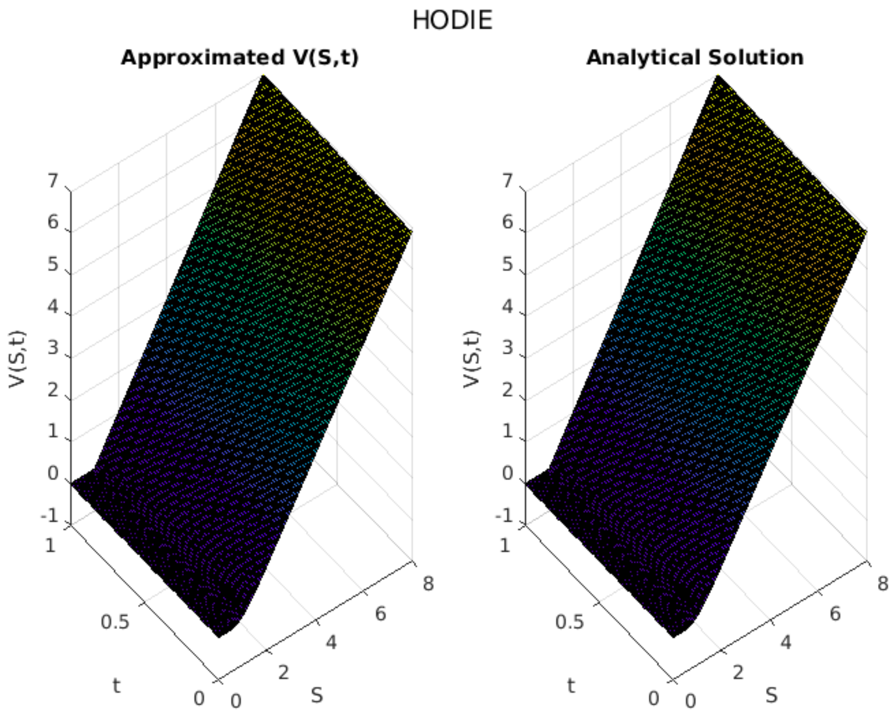

Figure 1: The analytical and the numerical solutions

The value at $t = 0$ obtained form the numerical solution is,

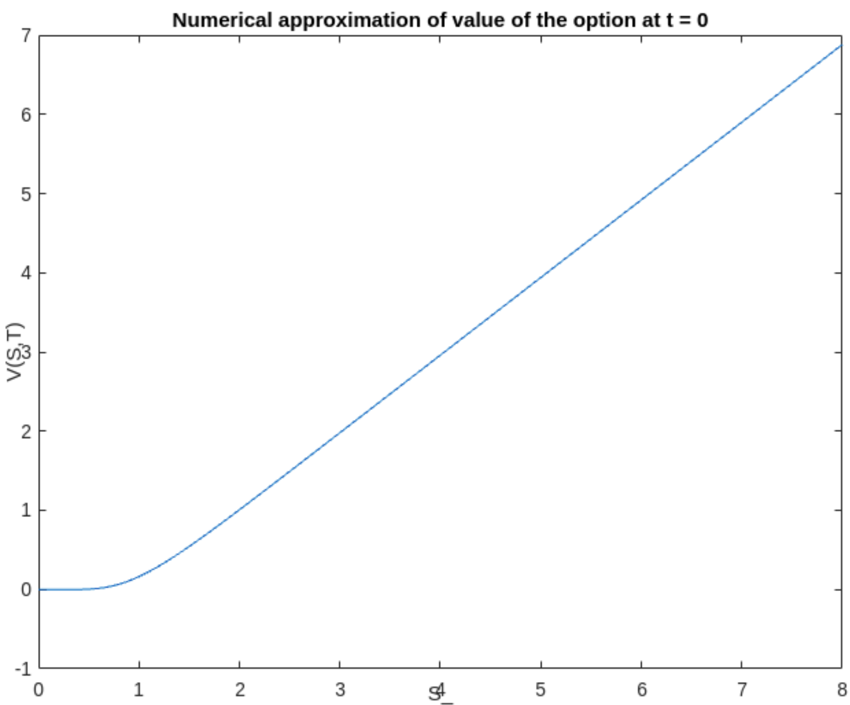

Figure 2: Numerical plot at time t=0

The plot of the absolute and RMS errors vs time is given by,

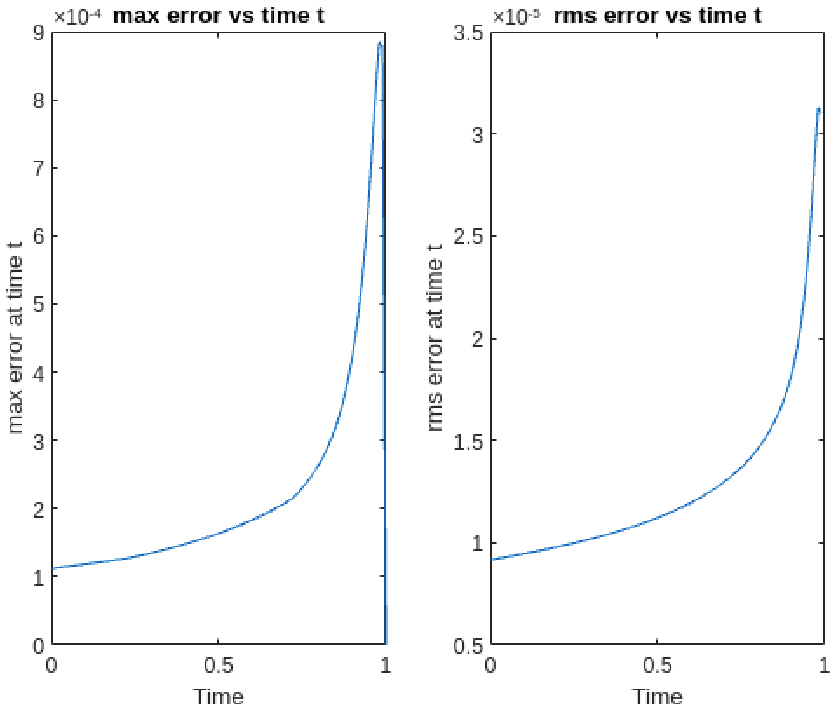

Figure 3: Absolute and RMS errors vs Time

The plot of the maximum absolute error vs the mesh size is,

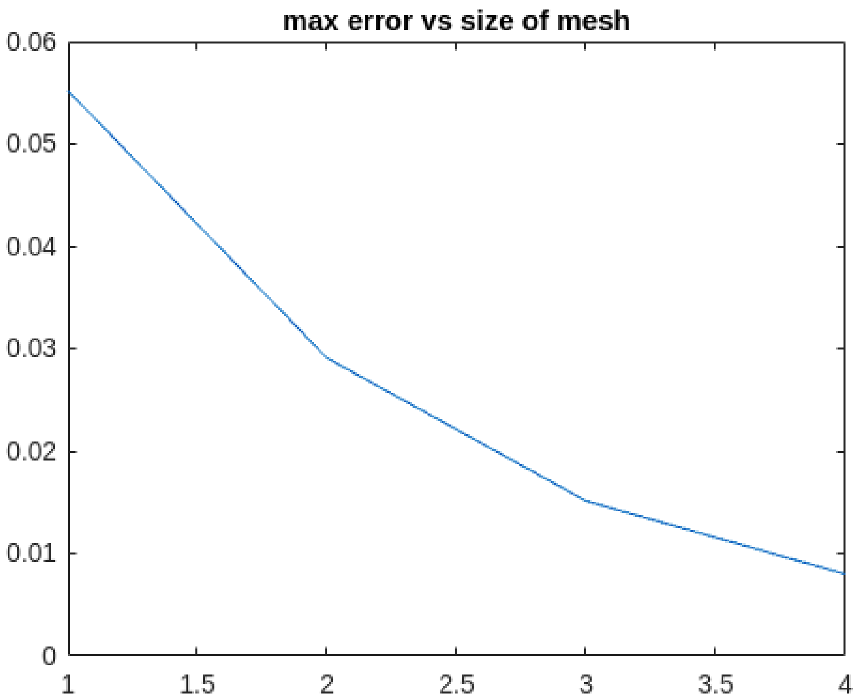

Figure 4: Max Absolute error vs mesh size

The plot of the RMS error vs the mesh size is,

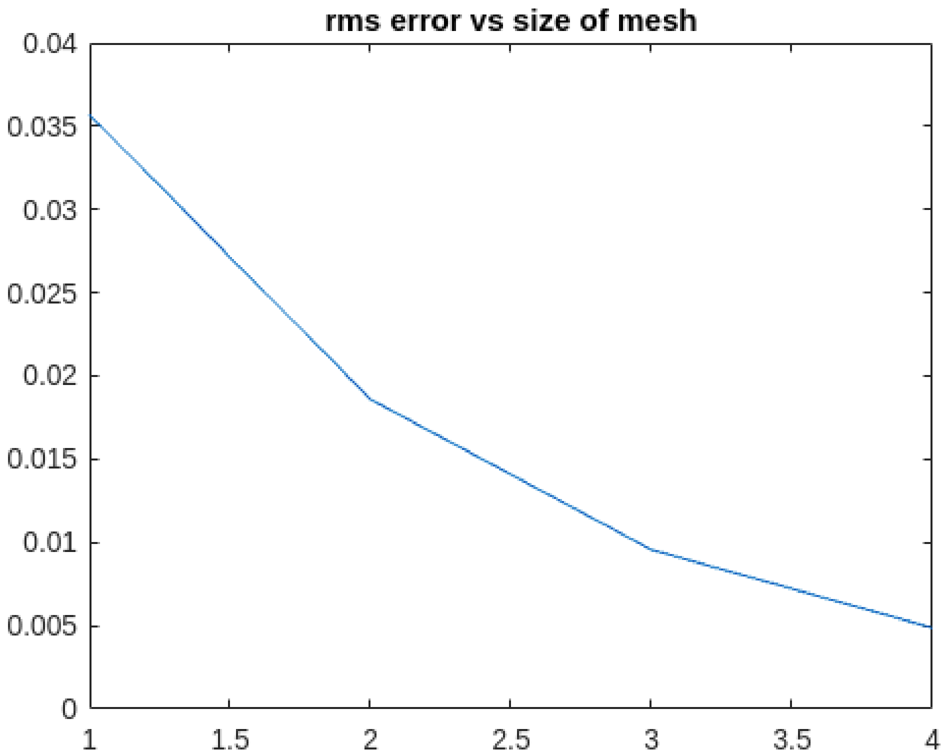

Figure 5: RMS error vs mesh size

A table with calculated errors and their convergence ratios for this example is given below.

N	10x2^2	10x2^3	10x2^4	10x2^5
M	2^6	2^7	2^8	2^9
E_{max}	5.52e-2	2.91e-2	1.51e-2	0.8e-2
p_{max}		0.9246	0.9435	0.9206
E_{rms}	3.57e-2	1.86e-2	9.6e-3	4.9e-3
p_{rms}		0.9406	0.9603	0.9727

Example 2: Consider the Black-Scholes equation for European Call option price with $\hat{\sigma}(S,t) = 0.4$, $\hat{r}(S,t) = 0.02$, $\hat{D}(S,t) = 0.04$, $T = 1$, and $K = 1$. Take $S_{max} = 8$, and $\epsilon = 10^{-6}$.

The analytical and the numerical solution of the above example is,

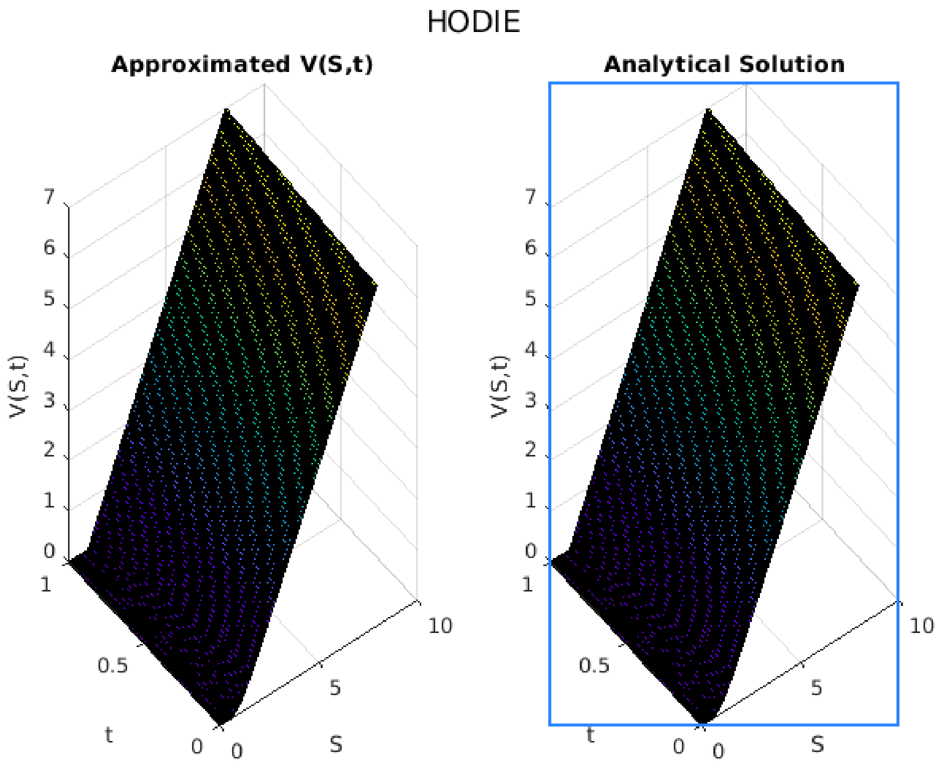

Figure 6: The analytical and the numerical solutions

The value at $t = 0$ obtained form the numerical solution is,

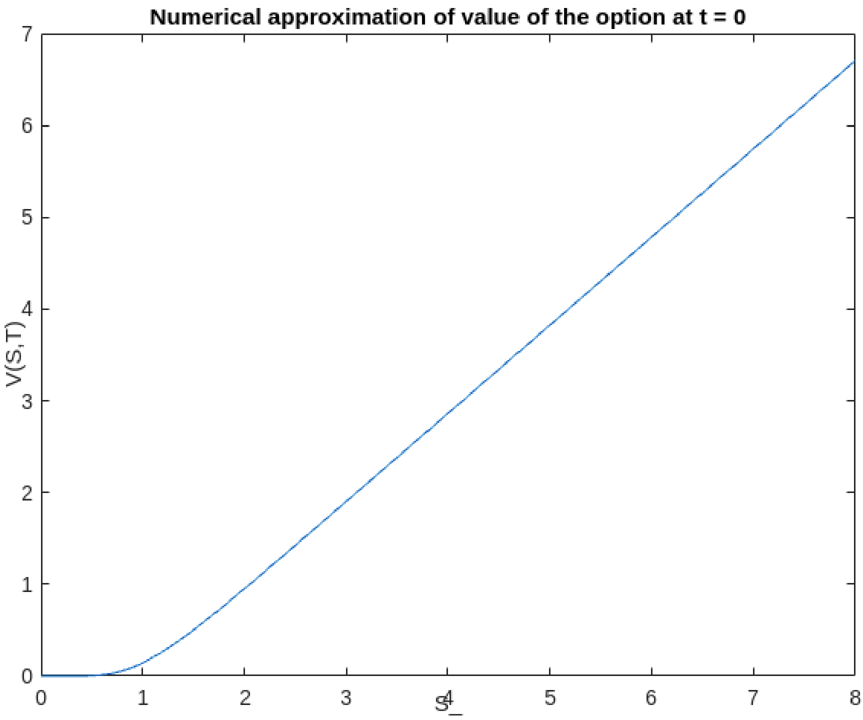

Figure 7: Numerical plot at time t=0

The plot of the absolute and RMS errors vs time is given by,

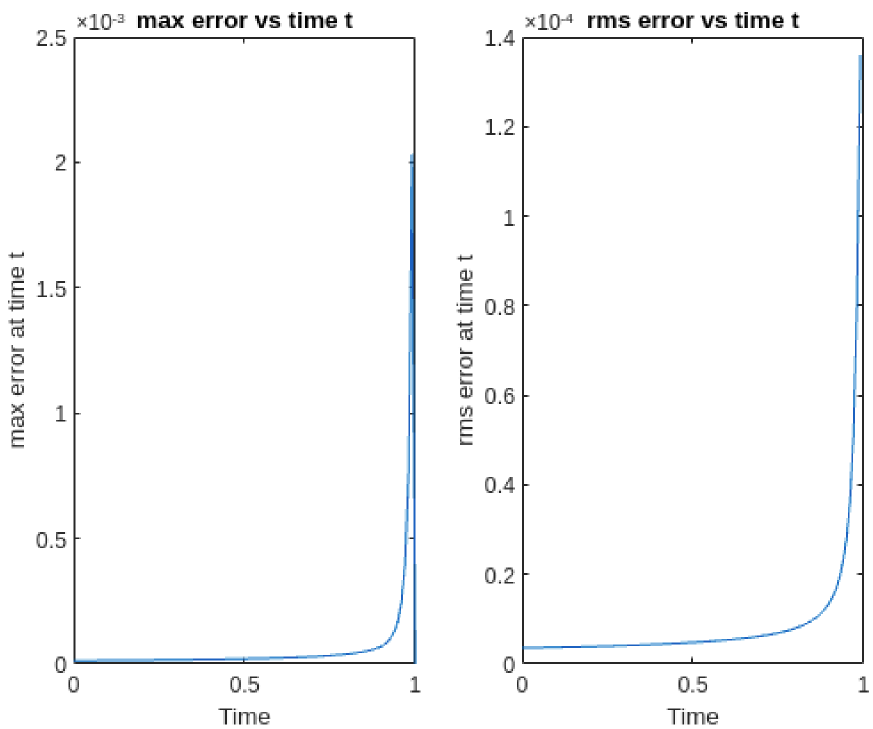

Figure 8: Absolute and RMS errors vs Time

The plot of the maximum absolute error vs the mesh size is,

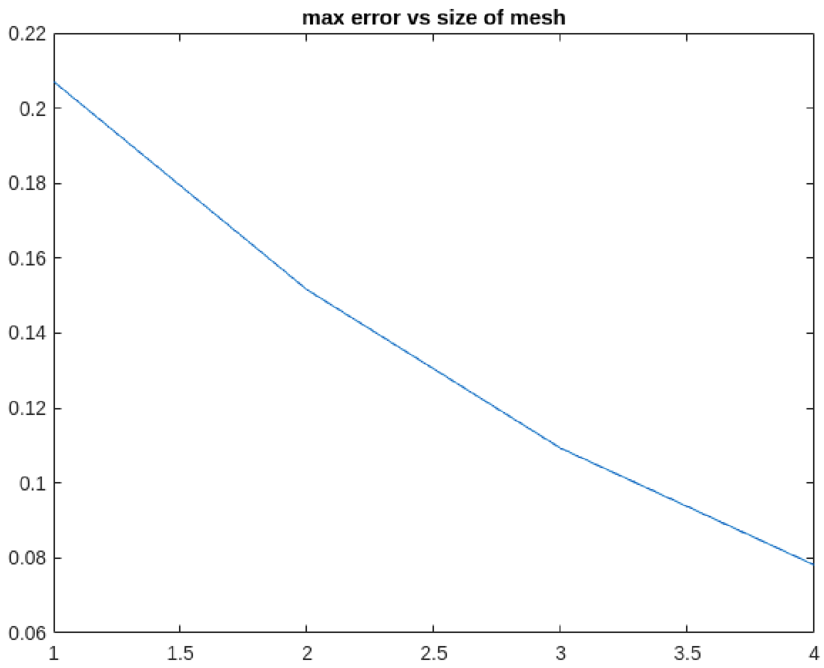

Figure 9: Max Absolute error vs mesh size

The plot of the RMS error vs the mesh size is,

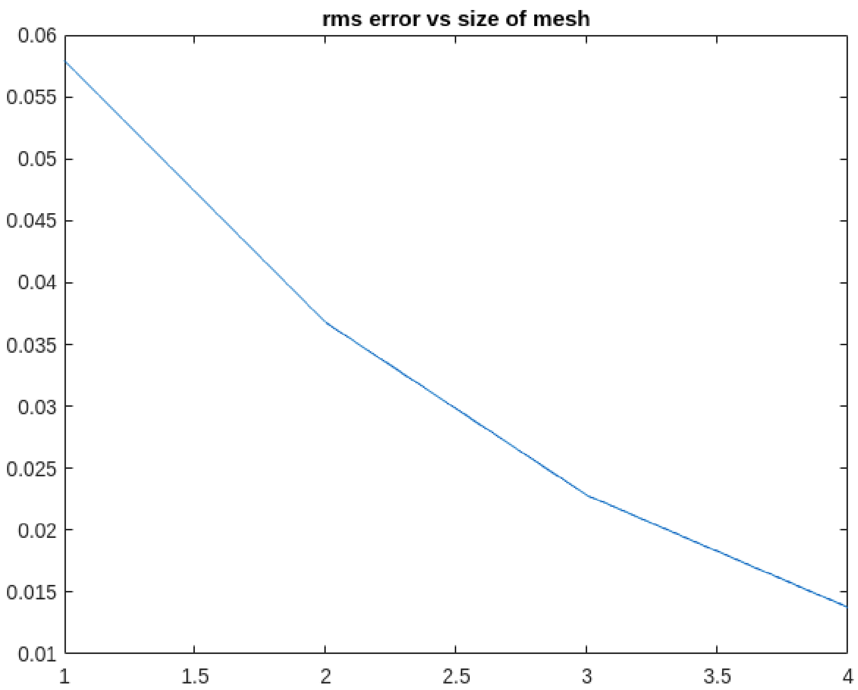

Figure 10: RMS error vs mesh size

A table with calculated errors and their convergence ratios for this example is given below.

N	10x2^2	10x2^3	10x2^4	10x2^5
M	2^6	2^7	2^8	2^9
E_{max}	8.95e-2	4.70e-2	2.44e-2	1.26e-2
p_{max}		0.9395	0.9597	0.9725
E_{rms}	5.52e-2	2.88e-2	1.48e-2	7.5e-3
p_{rms}		0.9282	0.9457	0.9586