Uniform distribution for 2-Dimensional R.V.

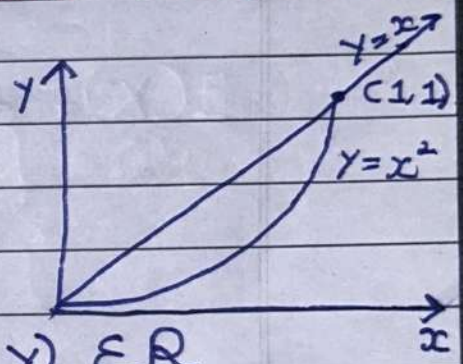
The 2 dim r.v. (x, y) is uniformly distributed over a region R with PDF given by

$$f(x,y) = \begin{cases} \dfrac{1}{\text{Area}(R)} & ; (x,y) \in R \\ 0 & ; \text{otherwise} \end{cases}$$

Problems

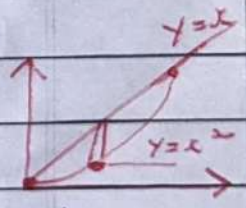
Q: Suppose the 2 dim r (x, y) is uniformly distributed over the region between $y = x$ and $y = x^2$. Find the marginal pdfs $g(x)$ and $h(y)$

Sol^n $y = x$, $y = x^2$



$$\text{Area} = \int_0^1 \int_{x^2}^{x} dy\, dx$$

$$f(x,y) = \begin{cases} \dfrac{1}{\text{Area}(R)} & ; (x,y) \in R \\ 0 & \text{otherwise} \end{cases}$$

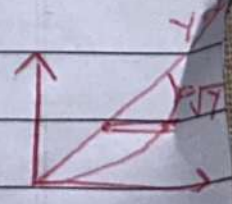


y limit

$$g(x) = \int_{x^2}^{x} 6\,dy = \begin{cases} 6x - 6x^2 & ; 0 \leq x \leq 1 \\ 0 & ; \text{otherwise} \end{cases}$$

x limit

$$h(y) = \int_{y}^{\sqrt{y}} 6\,dx = \begin{cases} 6\sqrt{y} - 6y & ; 0 \leq y \leq 1 \\ 0 & ; \text{otherwise} \end{cases}$$



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$$\text{Area} \int_0^1 \int_{x^2}^{x} dy\, dx$$

$$= \int_0^1 y \Big|_{x^2}^{x} dx$$

$$= \int_0^1 x - x^2 \, dx$$

$$= \frac{x^2}{2} - \frac{x^3}{3} \Big|_0^1$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right) - (0)$$

$$= \frac{1}{6}$$

To check valid pdf

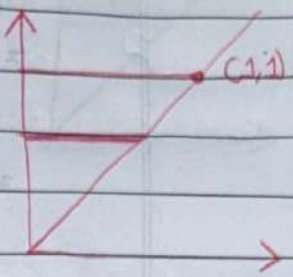
$$f(x,y) = \begin{cases} \frac{1}{\text{Area}} = 6 & ; (x,y) \in R \\ 0 & ; \text{Otherwise} \end{cases}$$

$$\iint f(x,y)\, dy\, dx = 1$$

$\therefore$ Valid pdf

Q2- Suppose the 2-dim r.v (X, Y) is uniformly dim r.v (X, Y) is uniformly distributed over the region $R = \{(x, y) \mid 0 < x < y < 1\}$

Sol^n Area = double integration not required



Area of Δ

$$= \int_0^1 \int_0^x dy\, dx$$

$$= \int_0^1 y \Big|_0^x dx$$

$$= \int_0^1 x\, dx$$

$$= \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}$$

& Area $= \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$ direct

$$f(x, y) = \begin{cases} 2 & ; (x, y) \in R \\ 0 & ; \text{Otherwise} \end{cases}$$

$$g(x) = \int_x^1 2\, dy = 2\,[1 - x] = \cancel{2 - 2x} \; \cancel{2 - 2x \; 0 < x}$$

$$= 2 - 2x$$

$$g(x) = \begin{cases} 2 - 2x & ; 0 \leq x < 1 \\ 0 & ; \text{otherwise} \end{cases}$$

$$h(y) = \int_0^y 2\, dx$$

$$= 2y$$

$$\begin{cases} 2y & ; 0 \leq y \leq 1 \\ 0 & ; \text{Otherwise} \end{cases}$$

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$$E(X) = \int x \cdot g(x)\,dx = \int_0^1 x \cdot 2\,(1-x)\,dx$$

$$= \int_0^1 2x - 2x^2 \, dx$$

$$= x^2 - \frac{2x^3}{3}\Big|_0^1$$

$$= \left(1 - \frac{2}{3}\right) - (0)$$

$$= \frac{1}{3}$$

$$E(Y) = \int y\,h(y)\,dy = \int_0^1 y \cdot (2y)\,dy$$

$$= \int_0^1 2y^2\,dy = \frac{2y^3}{3}\Big|_0^1 = \frac{2}{3}$$

$$E(X^2) = \int_0^1 x^2 g(x)\,dx = \int_0^1 2x^2 - 2x^3\,dx$$

$$= \frac{2x^3}{3}\Big|_0^1 - \frac{2x^4}{4}\Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

$$E(Y^2) = \int_0^1 y^2 \cdot 2y\,dy = \int 2y^3\,dx = \frac{2y^4}{4}\Big|_0^1 = \frac{1}{2}$$

$$V(X) = E(X^2) - (E(X))^2$$

$$= \frac{1}{6} - \left(\frac{1}{3}\right)^2$$

$$= \frac{1}{18}$$

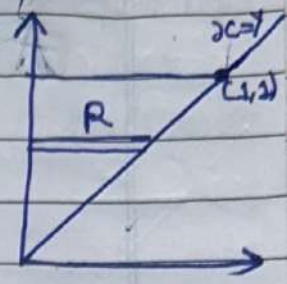
$$V(Y) = E(Y^2) - (E(Y))^2$$

$$= \frac{1}{2} - \left(\frac{2}{3}\right)^2$$

$$= \frac{1}{18}$$

x is from 0 to Y

$$E(x,Y) = \iint xY \cdot f(x,Y)\, dx\, dY$$



$$= \int_0^1 \int_0^Y 2xY\, dx\, dY$$

$$= \int_0^1 2Y \frac{x^2}{2}\Big|_0^Y dY$$

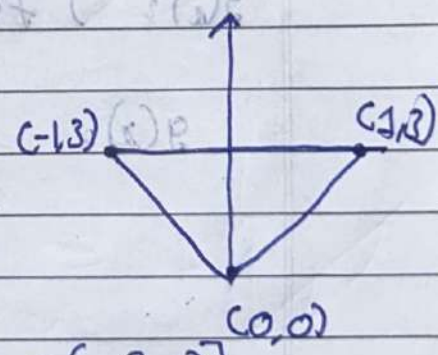
$$= \int_0^1 Y(Y^2 - 0)\, dY$$

$$= \int_0^1 Y^3\, dY = \frac{Y^4}{4}\Big|_0^1 = \frac{1}{4}$$

$$\rho_{xY} = \frac{E(xY) - E(x)E(Y)}{\sqrt{V(x)\,V(Y)}} = \frac{1/4 - 2/3 \times 1/3}{\sqrt{1/18 \times 1/18}} = \frac{1}{2}$$

$$= \left(\frac{1}{4} - \frac{2}{9}\right) \times 18$$

Q3 Suppose that a two dim r.v (x, Y) is uniformly distributed over the Δ as shown in the figure. Find $g(x)$, $h(Y)$, ρ_{xY}



Soln: Area = 3

$A \equiv (-1, 3)$

$B \equiv (1, 3)$

$C \equiv (0, 0)$

$$\text{Area} = \frac{1}{2}\left[-1(3-0) + 1(0-3) + 0(3-3)\right]$$

$$= \frac{1}{2}\left[-3 + (-3)\right]$$

$$= \frac{-6}{2} = |-3| = 3 \text{ sq units}$$

$m = \frac{y_2 - y_1}{x_2 - x_1}$ $\quad y - y_1 = m(x - x_1)$

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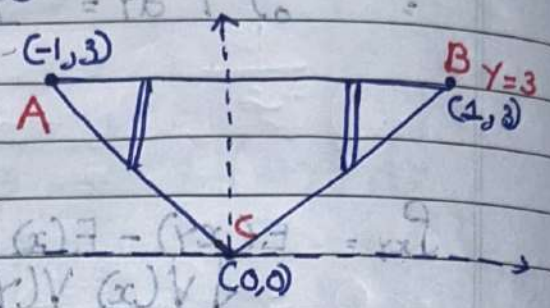
$$f(x,y) = \begin{cases} \frac{1}{3} & ; (x,y) \in R \\ 0 & ; \text{otherwise} \end{cases}$$

$$g(x) = \int f(x,y)\,dy = \int \frac{1}{3}\,dy$$



$y = mx + c$

BC

$y = 3x$

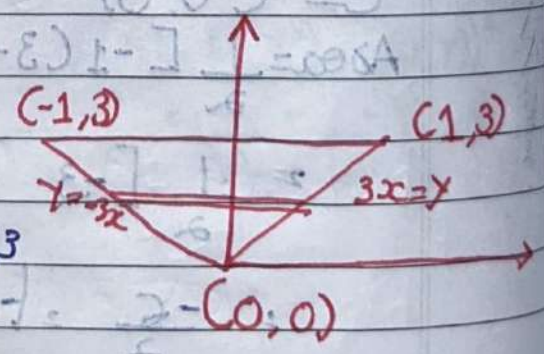
$$g(x) = \int f(x,y)\,dy = \int_{3x}^{x} 1/3\,dy = \frac{1}{3}(3 - 3x) = 1 - x \; ; 0 \leq x$$

$$g(x) = \int f(x,y)\,dy = \int_{x}^{3} \frac{1}{3}\,dy = \frac{(3+3x)}{3} = 1 + x$$

$$g(x) = \begin{cases} 1 - x & ; 0 \leq x \\ 1 + x & ; 0 \geq x \end{cases}$$

$$h(y) = \int f(x,y)\,dx$$

$$= \int_{-y/3}^{y/3} \frac{1}{3}\,dx = \frac{2y}{9} \; ; 0 \leq y \leq 3$$



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$$g(x) = \begin{cases} 1-x : 0 \le x \\ 1+x : 0 \ge x \\ 0 : \text{otherwise} \end{cases} \qquad h(y) = \begin{cases} \frac{2y}{9} : 0 \le y < 3 \\ 0 : \text{otherwise} \end{cases}$$

$$E(x) = \int_0^3 \int_{-y/3}^{y/3} x \, dx \, dy$$

$$= \int_0^3 \frac{x^2}{2} \Big|_{-y/3}^{y/3} dy$$

$$= \frac{1}{2} \int_0^3 \left(\frac{y}{3}\right)^2 - \left(\frac{-y}{3}\right)^2 dy$$

$$= \frac{1}{2} \int 0 \, dy$$

$$= 0$$

$$E(Y) = \int Y . h(Y) \, dY = \int_0^3 Y . \frac{2Y}{9} \, dY$$

$$= \frac{2}{3} \frac{Y^3}{9} \Big|_0^3$$

$$= \cancel{\frac{2 \cdot 3^3}{3^2}} = \cancel{6} = 2$$

$$E(X^2) = \int_0^3 \int_{-y/3}^{y/3} x^2 \, dx \, dy$$

$$= \int_0^3 \frac{x^3}{3} \Big|_{-y/3}^{y/3} dy$$

$$= \int_0^3 \frac{1}{3} \left[\left(\frac{y}{3}\right)^3 - \left(\frac{-y}{3}\right)^3 \right] dy$$

$$= \frac{1}{3} \int_0^3 \left(\frac{2y^3}{3} \, dy \right)$$

$$= \frac{2}{9} \frac{y^4}{4} \Big|_0^3 = \frac{1}{18} (81 - 0) = \frac{9}{2} = 4.5$$

$V(X) = \frac{1}{6}$ $V(Y) = \frac{1}{2}$

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$$E(Y^2) = \int_0^3 Y^2 \cdot \frac{2Y}{9}\, dy$$

$$= \frac{2}{3^2} \left. \frac{Y^4}{4} \right|_0^3$$

$$= \frac{1}{6 \times 3} (3^4 - 0)$$

$$= \frac{3^3}{2 \times 3} = \frac{27}{2} = \frac{9}{2}$$

$$\rho_{xy} = \frac{E(XY) - E(X)\,E(Y)}{\sqrt{V(X)\,V(Y)}} = 0$$

But it is not independent

$$f(x, y) \neq g(x)\, g(y)$$

Distributions

Discrete Distributions

1 Binomial Distribution:

Toss a coin 5 times getting 2 heads,

Eg, if you toss a coin 5 time

p(Two heads)

$P(H) = p \quad P(T) = q$

$$^5C_2 \cdot p^2 q^{5-2}$$

Only for sucess and failure,

i) Any trial results in sucess and failure

ii) There are finite number of trials and each trial is independent

iii) The prob of success is same in each trial

(Sum trials are called Bernoullian trials)

Then $B(n, p, x)$ is the prob of exactly 'x' succes in n Bernoullian trials and 'p' being the problem of success in each trial is given by

$$B(n, p, x) = {}^nC_x p^x q^{n-x} \quad ; x = 0, 1, 2, 3 \ldots n$$

x	Probability
0	q^n
1	npq^{n-1}
2	$nC_2 p^2 q^{n-2}$
3	${}^nC_3 p^3 q^{n-3}$
4	${}^nC_4 p^4 q^{n-4}$
⋮	
n-2	
n-1	nqp^{n-1}
n	p^n

$$\Sigma = q^n + npq^{n-1} + \ldots + nqp^{n-1} + p^n = 1$$

$$= (p+q)^n = 1$$

Q1 Examples of Binomial distribution:

$B(n, p, x) = {}^nC_x\, p^x\, q^{n-x}$

Q1 Six coins are tossed. Find the prob of getting

i) exactly 3 heads

ii) Atleast 3 heads

iii) atleast one head

$n=6,\ p=q=1/2$

i) $P(\text{exactly 3 heads}) = B(6, 1/2, 3) = {}^6C_3\,(1/2)^3\,(1/2)^{6-3}$

ii) $P(\text{atleast 3 heads}) = B(6, 1/2, 3) + B(6, 1/2, 4), B(6, 1/2, 5), B(6, 1/2, 6)$

$= {}^6C_3(1/2)^3(1/2)^3 + {}^6C_4(1/2)^4(1/2)^2 + {}^6C_5(1/2)^5 + {}^6C_6(1/2)$

$= \frac{21}{32}$

iii) $P(\text{at least one head}) = 1 - B(6, 1/2, 0) = 1 - \frac{1}{64} = \frac{63}{64}$

Q2 The prob that a pen manufactured by a factory be defective is 1/10. If 12 such pens are manufactured What is the prob that

i) exactly 2 are defective

ii) atleast 2 are defective

iii) None are defective

Sol^n $n = 12$, $p = 1/10$, $q = 9/10$

i) exactly 2 are defective

$= \cancel{B(1/10)^2\,(9/10)^{10}} = B(12, 1/10, 2)$

$= {}^{12}C_2\,(1/10)^2\,(9/10)^{10}$

$= \cancel{0.12552}\ = 0.230127770$

ii) $P(x \geq 2) = 1 - P(X \leq 1) = 1 - (P(X=0) + P(X=1))$

$= 1 - [B(12, 1/10, 0) + B(12, 1/10, 1)]$

$= 0.341$

iii) $P(X=0) = B(12, 1/10, 0) = {}^{12}C_0 \, (1/10)^0 \, (9/10)^{12}$

$= 0.2824$

Mean and Variance:

$\mu = E(X) = \sum x P(X) = \sum_{x=0}^{n} x \cdot {}^nC_x p^x q^{n-x}$

X	P(X)
0	$p^0 q^n$
1	npq^{n-1}
⋮	
n	

$\mu = npq^{n-1} + 2\,{}^nC_2\, p^2 q^{n-2} + 3\,{}^nC_3\, p^3 q^{n-3} + \dots + (n-1)\,n\,p^{n-1} q + nq\,p^n$

$\mu = np(q^{n-1} + (n-1)pq^{n-2} + (n-1)C_2\, p^2 q^{n-3} + \dots q(n-1)p^{n-2} + p^{n-1})$

$(p+q)^n = p^n + {}^nC_1 pq^{n-1} + {}^nC_2 p^2 q^{n-2} + \dots + q^n$

$(p+q)^{n-1} = p^{n-1} + {}^{n-1}C_1 pq^{n-2} + \dots + q^{n-1}$

$\mu = np(p+q)^{n-1} = np$

$E(X^2) = \sum x^2\, {}^nC_x p^x q^{n-x} = \sum x(x-1)\, {}^nC_x p^x q^{n-x} + \sum x\, {}^nC_x p^x q^{n-x}$

$E(X^2) = \sum x(x-1)\, {}^nC_x p^x q^{n-x} + \sum x\, {}^nC_x p^x q^{n-x}$

$= 2\,{}^nC_2 p^2 q^{n-3} + 3 \times 2 \times {}^nC_3 p^3 q^{n-3} + \dots + n(n-1)p^n + np$

$= 2\,\frac{n(n-1)}{2} p^2 q^{n-2} + \frac{3 \times 2 \times n(n-1)(n-2)}{3 \times 2} p^3 q^{n-3} + \dots + n(n-1)p^n + np$

$= n(n-1)p^2 \{ q^{n-2} + (n-2)pq^{n-3} + (n-2)C_2\, p^2 q^{n-4} + \dots p^{n-2} \} + np$

$= n(n-1)p^2 (p+q)^{n-2} = n(n-1)p^2 + np$

$$V(X) = E(X^2) - (E(X))^2$$
$$= n(n-1)p^2 + np - (np)^2$$
$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$
$$= np - np^2$$
$$= np(1-p) = npq$$

Problems

Q1 The mean and varience of a binomial distribution are 4 & 4/3. Find $P(X \geq 1)$

Given $np = 4$ and $npq = 4/3$

$npq = 4/3$

$np = 4$

$$\frac{\cancel{np}q}{\cancel{np}} = \frac{4}{3} \times \frac{1}{4}$$

$$q = \frac{1}{3}$$

$$p = \frac{2}{3}$$

$$n \times \frac{2}{3} = 4$$

$n = 6$

$\therefore n = 6 \quad p = 2/3 \quad q = 1/3$

$\therefore P(X \geq 1) = 1 - P(X < 1) = 1 - P(X = 0) = 1 - B(6, 2/3, 0)$
$= 0.9982$

Q2 Out of 800 families with 5 children each, how many families would you expect to have
a) 3 boys
b) 5 girls
c) either 2 boys

a $P(X=3) = B(5, 1/2, 3) = {}^5C_3 (1/2)^3 (1/2)^2 = \frac{5}{16}$

$= 800 \times \frac{5}{16} = 250$ families will have 3 boys

b $P(5 \text{ girl}) = P(x=0) = B(5, 1/2, 0) = \cancel{800 \times}$
$= {}^5C_0 (1/2)^0 (1/2)^5$
$= 1/32$

$= 800 \times \frac{1}{32} = 25$ families with 5 girls

c) P(either 2 boys or 3 boys)
$P(X=2) + P(x=3) = B(5, 1/2, 2) + B(5, 1/2, 3)$
$= 5/8$

Q3 In playing with an opponent of equal ability which is more probable
a) winning 3 games out of 4 or 5 out of 8.
b) winning atleast 3 games out of 4 or atleast 5 out of 8. $P = q = 1/2$

a winning 3 out of 4 — winning 5 out of 8
$n=4$ $x=3$ — $n=8$, $x=5$
$B(4, 1/2, 3) = 1/4$ — $B(8, 1/2, 5) = \frac{7}{32} = 0.218$

b) P(atleast 3 out of 4) — P(atleast 5 out of 8)
$n=4$ — $n=8$
$P(X \geq 3) = P(X=3) + P(X=4)$ — $P(X \geq 5) = P(X=5) + P(X=6) + P(X=7) + P(X=8)$
$= B(4, 1/2, 3) + B(4, 1/2, 4)$ — $= 0.363$
$= 0.3125$

Winning at least 5 out of 8 is more profitable

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Q4 An airline knows that 5% of the people making reservations on an certain flight will not turn up. So their policy is to sell 52 tickets for a flight that can only hold 50 passengers.

What is the prob that there will be a seat available for every passenger who turns up

$n = 52$ $x =$ NO. of passenders will not turn up

$\therefore p = 5\% = 0.05$

$q = 0.95$

p (there will be a seat available for every one who turns up)

$\therefore p(X \geq 2) = 1 - [P(X=0) + p(X=1)]$

$= 1 - [B(52, 0.05, 0) + B(52, 0.05, 1)]$

$= 0.7405$

Q5 In a bombing event there is a 50% chance that a bomb will strike the target. Two direct hits are needed to destroy the target ~~hits~~ completely.

Estimitate the minimum NO. of bombs to be dropped to give 99% or better chance of completely destroying the target

$p = q = \frac{1}{2}$ $x =$ number of bomb to be dropped

$p(X \geq 2) \geq 0.99$

$1 - p(X=0) - p(X=1) \geq 0.99$

$p(X=0) + p(X=1) \leq 0.01$

$${}^nC_0 (1/2)^0 (1/2)^n + {}^nC_1 (1/2)(1/2)^{n-1} \leq 0.01$$

$$\left(\frac{1}{2}\right)^n + n\left(\frac{1}{2}\right)^n \leq 0.01$$

$n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$

× × × × × × × × × × ✓

Sometimes,

$p = 0.00001$ $\quad n = 10000$

$x = 9985$

p is small, n is large

2 Poisson Distribution

Binomial distⁿ tends to poisson distⁿ under following conditions

i) $n \to \infty$

ii) $p \to 0$

iii) $np = \alpha$, $\alpha > 0$

$$P(x=k) = {}^nC_k \cdot p^k q^{n-k}$$

$$= \frac{n(n-1)(n-2)\ldots(n-(k-1))(\alpha/n)^k (1-\alpha/n)^{n-k}}{k!}$$

$$= \frac{\alpha^k}{k!} \frac{(1-1/n)(1-2/n)\ldots(1-\frac{(k-1)}{n})(1-\frac{\alpha}{n})^n}{(1-\alpha/n)^k}$$

As $\lim_{n \to \infty}$

$$= \frac{\alpha^k}{k!} \lim_{n\to\infty} (1-1/n)(1-2/n)\ldots(1-\frac{k-1}{n})$$

$$= \frac{\alpha^k}{k!} \lim_{n\to\infty} \frac{(1-1/n)(1-2/n)\ldots(1-\frac{k-1}{n})(1-\frac{\alpha}{n})^n}{\lim_{n\to\infty}(1-\frac{\alpha}{n})^k}$$

$$= \frac{\alpha^K}{K!} \lim_{n\to\infty} (1-\alpha/n)^n$$

$$\boxed{= \frac{\alpha^K}{K!} e^{-\alpha}}$$

$$Y = (1-\alpha/n)^n$$

$$\log Y = \lim_{n\to\infty} \log(1-\alpha/n)^n$$

$$\log Y = \lim_{n\to\infty} n \log(1-\alpha/n)$$

$$= \lim_{n\to\infty} \frac{\log(1-\alpha/n)}{\frac{1}{n}}$$

$$= \frac{\frac{1}{1-\frac{\alpha}{n}} \cdot \frac{\alpha}{n^2}}{\frac{-1}{n^2}}$$

$$= -\alpha$$

Poisson Distribution

The PMF is given by

$$P(X=x) = \frac{e^{-\alpha}\alpha^x}{x!} \quad x = 0, 1, 2, 3, 4, \ldots$$

Where α is a parameter and $\alpha = np$

Mean and Variance

$$\mu = E(x) = \sum x.P(x) = \sum_{x=0}^{\infty} x.\frac{e^{-\alpha}\alpha^{x}}{x!}$$

$$= 0 + e^{-\alpha}\left\{\frac{\alpha}{1} + \frac{2\alpha^2}{2!} + \frac{3\alpha^3}{3!} + \dots\right\}$$

$$= \alpha e^{-\alpha}\left\{1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \dots\right\}$$

$$= \alpha e^{-\alpha} . e^{\alpha}$$

$$= \alpha$$

Note $e^{x} = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$$E(x^2) = \sum x^2 . p(x) = \sum_{x=0}^{\infty} x^2 . \frac{e^{-\alpha}\alpha^{x}}{x!} = \sum \frac{x(x-1)e^{-\alpha}\alpha^{x}}{x!} + \sum \frac{x e^{-\alpha}\alpha^{x}}{x!}$$

$$= e^{-\alpha}\left\{\frac{2\alpha^2}{2!} + \frac{3.2\alpha^3}{3!} + \frac{4\times3\alpha^4}{4!} + \dots\right\} + \alpha$$

$$= \alpha^2 e^{-\alpha}\left\{1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \dots\right\} + \alpha$$

$$= \alpha^2 . e^{-\alpha} . e^{\alpha} + \alpha = \alpha^2 + \alpha$$

$$V(X) = E(X^2) - (E(X))^2 = \alpha^2 + \alpha - \alpha^2 = \alpha$$

$$E(X) = V(X) = \alpha$$

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Problems

Q1 If the number of accidents occuring on a highway eachday is a Poisson variate with mean equal to 3,
What is the prob that no accident occur.

$\alpha = 3$

$P(\text{no accidents occur}) = P(X=0) = \frac{e^{-3} \cdot 3^0}{0!} = 0.049787$

Q2 Suppose 2% of the items produced by a machine are defective. find the prob that there are 3 defective item in a sample of 100 items

$n = 100 \quad p = 0.02 \quad \alpha = np = 2$

$P(X=3) = \frac{e^{-2} 2^3}{3!} = 0.180447044$

Using Poisson distⁿ

Binomial:

$P(X=3) = {}^{100}C_3 \cdot (0.02)^3 (0.98)^{100-3} = 0.182275941$

Q3 In a Poisson distⁿ if $P(X=2) = 2/3 \; P(X=1)$.
Then find $P(X=0)$, $P(X=3)$ and $P(X \geq 3)$

Solⁿ $P(X=2) = \frac{2}{3} P(X=1) = \frac{e^{-\alpha} \alpha^2}{2!} = \frac{2}{3} \frac{e^{-\alpha} \alpha}{1!}$

$\therefore \alpha = 4/3$

$P(X=0) = \frac{e^{-4/3} (4/3)^0}{0!}$ $\qquad$ $P(X=3) = \frac{e^{-4/3} (4/3)^3}{3!}$

$= 0.2636$ $\qquad$ $= 0.1041$

$\therefore P(X \geq 3) = 1 - P(X<3) = 1 - P(X=0) - P(X=1) - P(X=2)$

$= 0.4765$

Whenever p is small, n is large Poison's distribution

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Q4 Suppose a container contains 1000 particles. The prob that a particle escapes from the container is 0.0004. What is the prob that more than 5 such escapes occur?

$$P(X>5) = 1 - P(X \leq 5)$$
$$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)]$$

$n = 10000$
$p = 0.0004$
$\alpha = np = 4$

$$= 1 - \left[\frac{4^0 \cdot e^{-4}}{0!} + \frac{4 \cdot e^{-4}}{1!} + \frac{4^2 \cdot e^{-4}}{2!} + \frac{4^3 \cdot e^{-4}}{3!} + \frac{4^4 \cdot e^{-4}}{4!} + \frac{4^5 \cdot e^{-4}}{5!}\right]$$

$$= 1 - e^{-4}\left[1 + 4 + \frac{16}{2} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!}\right]$$

$= 1 - 0.785130387$
$= 0.2149$

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Q5 In a factory producing machines there is a small chance of $\frac{1}{500}$ for any machines to be defective

The machines are supplied in a packet of 10. Calculate approximate number of packets containing

i) No defective

ii) 1 defective

iii) 2 defective in a lot of 20000 packet

$p = \frac{1}{500}$

$n = 10$

$\alpha = np = 0.02$

i) $p(X=0) = \frac{e^{-0.02} \times (0.02)^0}{0!} = 0.9802$

to get probability of 20,000 packets

$= 0.9802 \times 20,000 = 19603.97347$ packets containg zero defective

ii) 1 defective

$p(X=1) = \frac{e^{-0.02} \times (0.02)^1}{1!} = 0.0196$

$= 0.0196 \times 20000 = 392$ packet contains 1 defective

iii) 2 defective

$p(X=2) = \frac{e^{-0.02} \times (0.02)^2}{2!} = 0.000196$

$0.000196 \times 20,000 = 3.9 \sim (4)$ packet contain 2 deffective

$n \geq 100$, Poison

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Q6 Suppose that the suicide rate in a certain state is 4 suicides per one million inhabitants per month. Find prob that in a certain town of population 5,00,000 there will be atmost 4 suicides in a month.
What is the prob. that during one year, there are at least 2 months in which more than 2 suicides occurs?

$n = 5,00,000$, $p = \frac{4}{10\,00000}$

$\alpha = np = 20 \times 10^{-1} = 2$

X: number of sucides per month

$p(X \leq 4) = e^{-2}\left(1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!}\right)$

$= 0.9473$

Y: No. of months in which more than 4 suicides occur

~~$n = 10000$ $p = 0.0004$, $\alpha = np = 4$~~

~~$p(X > 5) = 1 - P(X \leq 5)$~~

~~$= 1 - P(X=0) - P(X=1) - P(X=2) - P(X=3) - P(X=4) - P(X=5)$~~

~~$= 1 - e^{-4}\left\{1 + 4 + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!}\right\}$~~

~~$= 0.2149$~~

$n = 12$, $p = 1 - 0.943 = 0.05287$ $np =$

$Y \sim B(12, 0.00527, Y)$

$p(Y \geq 2) = 1 - P(Y \leq 1)$

$= 1 - \{P(Y=0) + P(Y=1)\}$

$= 1 - \{{}^{12}C_0 (0.527)^0 (0.9473)^{12} - {}^{12}C_1 (0.0527)(0.9473)^{11}\}$

$= 0.1292$

Continuous distributions

1. Normal distribution:

A continous r.v. X is said to follow normal distribution if its pdf is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} \quad ; -\infty < x < \infty$$

$$\sigma > 0$$

μ & σ are parameter

To check $f(x)$ is a valid pdf

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \quad ; \text{ put } \frac{x-\mu}{\sigma} = z$$

$$dx = \sigma\, dz$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, \sigma\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{-z^2/2}\, dz$$

even function

$$= \frac{1}{\sqrt{2\pi}} \cdot 2 \times \int_{0}^{\infty} e^{-z^2/2}\, dz$$

Let $\frac{z^2}{2} = t$

$$\frac{2z\,dz}{2} = dt$$

$$z\,dz = dt$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-t} \frac{dt}{\sqrt{2t}} = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t} \cdot t^{-1/2}\, dt = \frac{1}{\sqrt{\pi}} \cdot \Gamma(1/2) = 1$$

Note: $\int_{0}^{\infty} e^{-x} x^{n-1}\, dx = \Gamma n$, $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(n+1) = n\Gamma(n)$

Replace, x with $-x$ interval $(-c, c)$

if $f(-x) = f(x) \Rightarrow$ even $\quad 2\int_0^c f(x)\,dx$

if $f(-x) = -f(x) =$ odd

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Mean & Variance

Mean :

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

put $\frac{x-\mu}{\sigma} = Z$

$dx = \sigma\,dz$

$$= \frac{1}{\sigma\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} (\sigma Z + \mu) \cdot e^{-Z^2/2}\, \sigma\,dz$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \underbrace{\int_{-\infty}^{\infty} \sigma Z e^{-Z^2/2}\,dz}_{\text{odd}} \to 0 + \int_{-\infty}^{\infty} \underbrace{\mu \cdot e^{-Z^2/2}\,dz}_{\text{even}} \right.$$

~~interval (0, 2c)~~

$$= \frac{1}{\sqrt{2\pi}} \mu \cdot 2\int_0^{\infty} e^{-t} \frac{dt}{\sqrt{2t}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2\mu}{\sqrt{2}} \int_0^{\infty} e^{-t}\, t^{-1/2}\,dt$$

$$= \mu \cdot \frac{1}{\sqrt{\pi}} \cdot \Gamma 1/2 = \mu$$

To check even/odd $(0, 2c) \to$ even

$f(2c-x) = f(x) \to$ even $\quad \to (0, c)$

$-f(x) \to$ odd $\quad x \to 0$

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Mean

$$E(X) = \int_{-\infty}^{\infty} \frac{x}{\sigma\sqrt{2\pi}} \cdot e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx = \mu$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x) = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$= \underline{\text{Put}} \quad \frac{x-\mu}{\sigma} = Z \qquad x = \sigma Z + \mu$

$dx = \sigma\, dz$

$$= \int_{-\infty}^{\infty} (\sigma Z + \mu)^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \int_{-\infty}^{\infty} \frac{(\sigma Z+\mu)^2}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}Z^2} \sigma\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma Z + \mu)^2 \cdot e^{-Z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[(\sigma Z)^2 + \mu^2 + 2\sigma Z\right] \cdot e^{-Z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\infty} \sigma^2 Z^2 e^{-Z^2/2}\, dz + \mu^2 \int_{-\infty}^{\infty} e^{-Z^2/2}\, dz + \cancel{2\sigma\mu \int_{-\infty}^{\infty} Z e^{-Z^2/2}\, dz} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ 2\int_0^{\infty} \sigma^2 Z^2 e^{-Z^2/2}\, dz + \mu^2 2 \cdot \int_0^{\infty} e^{-Z^2/2}\, dz \right\}$$

put $\frac{Z^2}{2} = t$

$Z\,dz = dt$

$$= \sqrt{\frac{2}{\pi}}\, \sigma^2 \int_0^{\infty} 2t \cdot e^{-t} \frac{dt}{\sqrt{2t}} + \sqrt{\frac{2}{\pi}} \cdot \mu^2 \int_0^{\infty} \frac{e^{-t}}{\sqrt{2t}}\, dt$$

$$= \sqrt{2/\pi}\, \sigma^2 \sqrt{2} \int_0^{\infty} e^{-t} t^{1/2}\, dt + \sqrt{2/\pi}\, \frac{\mu^2}{\sqrt{2}} \int_0^{\infty} e^{-t} t^{-1/2}\, dt$$

$$= \sqrt{\frac{2}{\pi}} \cdot \sigma^2 \sqrt{2} \int_0^\infty e^{-t} t^{1/2} dt + \sqrt{\frac{2}{\pi}} \cdot \frac{\mu^2}{\sqrt{2}} \int_0^\infty e^{-t} t^{-1/2} dt$$

$$= \sqrt{\frac{2}{\pi}} \sqrt{2} \sigma^2 \cdot \Gamma(3/2) \sqrt{2} + \frac{\mu^2}{\sqrt{\pi}} \Gamma(1/2)$$

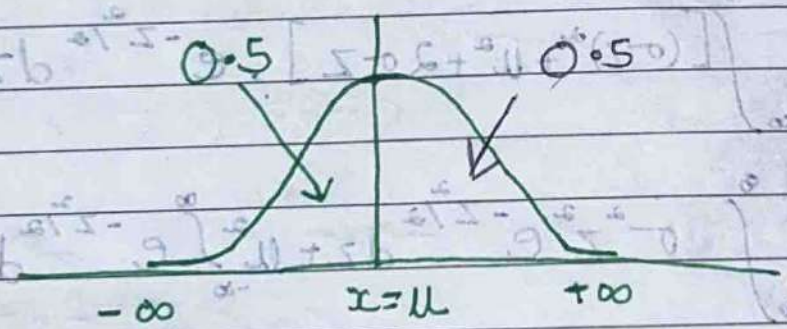
$$= \frac{2}{\sqrt{\pi}} \sigma^2 \Gamma(3/2) + \frac{\mu^2}{\sqrt{\pi}} \Gamma(1/2)$$

$$= \frac{2}{\sqrt{\pi}} \sigma^2 \times \frac{1}{2} \times \sqrt{\pi} + \frac{\mu^2}{\sqrt{\pi}} \cdot \sqrt{\pi} = \sigma^2 + \mu^2$$

$$V(X) = E(X^2) - (E(X))^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$$

Notation: $X \sim N(\mu, \sigma^2)$

The graph of $f(x)$ is symmetric about the line $x = \mu$, i.e $(P(X < \mu)) = P(X > \mu) = 0.5$



Standard Normal Variable

A r.v Z, which has $\mu = 0$ and $\sigma^2 = 1$ is the standard normal variable.

its pdf is given by

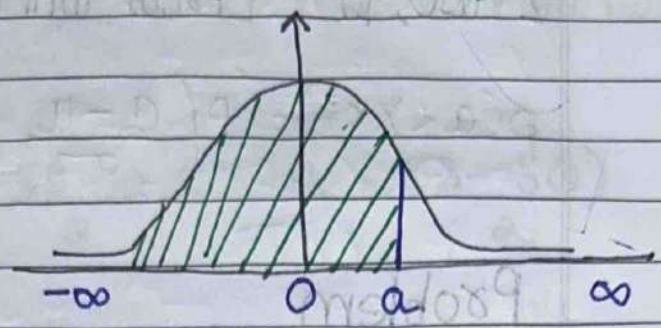
$$f(Z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-Z^2/2} \quad ; \quad -\infty < Z < \infty$$

we write $Z \sim N(0, 1)$

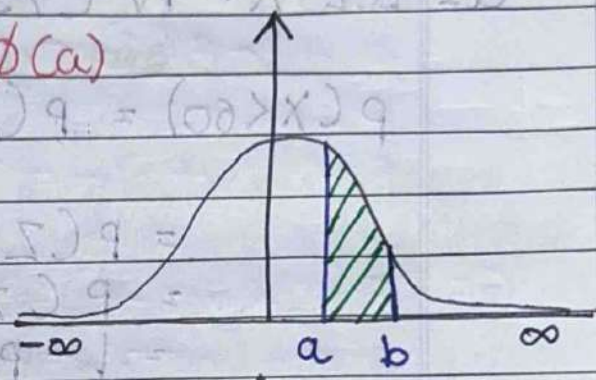
cdf of Z is given by $\phi(Z) = P(Z \leq z)$

$\phi(a) = P(Z \leq a)$

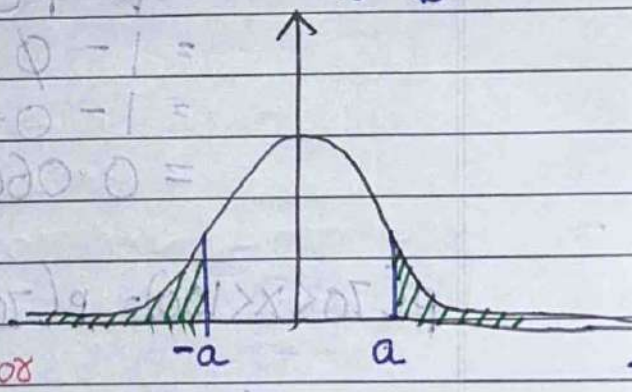
Let us assume the graph



$P(a < Z < b) = \phi(b) - \phi(a)$



$$
\begin{aligned}
P(-a) &= P(Z \leq -a) \\
&= P(Z \geq a) \\
&= 1 - P(Z < a) \\
&= 1 - \phi(a)
\end{aligned}
$$



Since the table values (prob) are given only for standard normal distribution

Standard distribution $\longrightarrow$ Standard Normal distribution

$X \sim N(\mu, \sigma^2)$ can be converted to standard normal distribution by substituting $Z = \frac{X - \mu}{\sigma}$

proof in random function

if ~~exe~~ fowa some distribution, then Y is another random variable, X^2, $\frac{X-\mu}{2}$, then what will be its distribution

So basically if you have pdf(x) then find pdf related to x

$X \sim N(\mu, \sigma^2)$ can be converted to standard normal distribution by substituting $Z = \frac{X-\mu}{\sigma}$ then $Z \sim N(0,1)$ [Proof will do latex]

$$p(a<X<b) = P\left(\frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right) = P(Z<Z<Z)$$

Problem

Q1 Let $X \sim N(75, 100)$, Find $P(X<60)$, $P(70<X<100)$

$$p(X<60) = P\left(\frac{X-\mu}{\sigma} < \frac{60-\mu}{\sigma}\right) = P\left(Z < \frac{60-75}{10}\right)$$

$$= P(Z < -1.5)$$
$$= P(Z > 1.5$$
$$= 1 - P(Z < 1.5)$$
$$= 1 - \phi(1.5)$$
$$= 1 - 0.93319$$
$$= 0.0668$$

$$p(70<X<100) = P\left(\frac{70-75}{10} < \frac{X-\mu}{\sigma} < \frac{100-75}{10}\right) = P(-0.5<Z<2.5)$$

$$= P(-0.5 < Z < 2.5)$$
$$= \phi(2.5) - \phi(-0.5)$$
$$= \phi(2.5) - (1 - \phi(0.5))$$
$$= \phi(2.5) + \phi(0.5) - 1$$
$$= (0.9938 + 0.6915) - 1$$
$$= 0.6853$$

Q2 Suppose the temperature is normally distributed with $\mu = 50°C$ and $\sigma^2 = 4°C$. What is prob that temp will lie between 48°C to 53°C

$X \sim N(50, 4)$

$P(48 < X < 53) = P\left(\frac{48-50}{2} < \frac{X-\mu}{\sigma} < \frac{53-50}{2}\right)$

$= P(-1 < Z < 1.5)$

$= \phi(1.5) - \phi(-1) = \phi(1.5) - [1 - \phi(1)]$

$= \phi(1.5) + \phi(1) - 1 = 0.7745$

Q3 Suppose the life length of 2 electronic device say D_1 and D_2 have distributions $X_1 \sim N(40, 36)$ and $X_2 \sim (45, 9)$ respectively.

If electronic device is to be used for

i) 45 hour period and more

ii) 48 hour period and more

Which device is prefered

Solution

$X_1 \sim N(40, 36)$ $\quad$ $X_2 \sim N(45, 9)$

$P(X_1 \geq 45)$ $\quad$ $P(X_2 \geq 45)$

$P\left(Z \geq \frac{45-40}{6}\right)$ $\quad$ $P\left(Z \geq \frac{45-45}{3}\right)$

$P(Z \geq 0.8333)$ $\quad$ $P(Z \geq 0) = 1 - P(Z \leq 0)$

$= 1 - P(Z \leq 0.8333)$ $\quad$ $= 1 - 0.5 = 0.5$

$= 1 - 0.7967$

$= 0.2033$ $\quad$ # Prefered

$P\left(Z \geq \frac{48-40}{6}\right) = P(Z \geq 1.333)$ $\quad$ $P\left(Z \geq \frac{48-45}{3}\right) = P(Z \geq 1)$

$= 1 - P(Z \leq 1.33)$ $\quad$ $= 1 - P(Z \leq 1) = 1 - 0.8413$

$= 1 - 0.9082 = 0.0918$ $\quad$ $= 0.1587$

prefered

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Q4 If $X \sim N(75, 25)$, find $P(X > 80 | X > 77)$

Sol^n

$$P(X>80|X>77) = \frac{(P(X>80)) \cap P(X>77)}{P(X>77)}$$

$$= \frac{P(X>80)}{P(X>77)} = \frac{P\left(Z > \frac{80-75}{5}\right)}{P\left(Z > \frac{77-75}{5}\right)} = \frac{P(Z>1)}{P(Z>0.4)}$$

$$= \cancel{1-\phi(1)} = \frac{1-P(Z<1)}{1-P(Z<0.4)} = \frac{1-0.8413}{1-0.6554}$$

$$= 0.4605$$

Q5 The height of 500 students are found to have normal distribution out of 500 students 258 are found to be within 2 cm of mean height of 170 cm. Find σ

$\mu = 170$

$P(170-2 < X < 170+2) = 258/500 = 0.5160$

$P(168 < X < 172) = 0.5160$

$P\left(\frac{168-170}{\sigma} < Z < \frac{172-170}{\sigma}\right)$

$P(-2/\sigma < Z < 2/\sigma) = 0.5160$

$= \phi$

$\phi(2/\sigma) - P(-2/\sigma) = 0.516$

$\phi(2/\sigma) - [1 - \phi(2/\sigma)] = 0.5160$

$\phi(2/\sigma) + \phi(2/\sigma) - 1 = 0.5160$

$2\phi(2/\sigma) = 1.5160$

$\phi(2/\sigma) = \cancel{0.2580}\ 0.7580$

$\frac{2}{\sigma} = \phi^{-1}(0.7580) = 0.7$

$\sigma = \frac{2}{0.7} = 2.8571$

Note $f'(x)=0$ and $f''(x)<0$ at $x=\mu$

$\therefore f(x)$ has maximum value at μ and the max value is $\frac{1}{\sigma\sqrt{2\pi}}$

Problems:

Q1 The montly income of 10000 persons is found to be normally distributed wit $\mu=7500$ and $\sigma=500$

i) What % of the people have income exceeding 6680?

ii) What is the lowest income among richest 100?

i) $X \sim N(7500, 500^2)$

$P(X>6680) = P\left(Z>\frac{6680-7500}{500}\right) = P(Z>-1.64)$

$\cancel{= 1-P(Z<1.64) = 1-0.9495 = 0.0505}$

$= P(Z<1.64) = P(Z>-1.64)$

$= 0.9495$

$= 94.95\%$

ii) $P(X>c) = \frac{100}{10000} = 0.01$

$P\left(Z>\frac{c-\mu}{\sigma}\right) = 0.01$

$P\left(Z>\frac{c-7500}{500}\right) = 0.01$

$P\left(Z<\frac{c-7500}{500}\right) = 1-0.01 = 0.99$

$\frac{c-7500}{500} = \phi^{-1}(0.99)$

$\frac{c-7500}{500} = 2.33$

$c = 8665$

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Q2 In an exam the pass % is 45 % and distinction percentage is 9%. The minimum marks and distinction marks are 40 and 74 respectively. Assuming that marks are normally distributed, find μ and σ^2

Sol^n

$X \sim N(\mu, \sigma^2)$

$P(X > 40) = 0.45$ $\qquad$ $P(X > 74) = 0.09$

$P(X < 40) = 0.55$ $\qquad$ $P(X < 74) = 0.91$

$P\left(Z < \frac{40-\mu}{\sigma}\right) = 0.55$ $\qquad$ $P\left(Z < \frac{74-\mu}{\sigma}\right) = 0.91$

$P\left(Z < \frac{40-\mu}{\sigma}\right) = 0.55$ $\qquad$ $\frac{74-\mu}{\sigma} = \phi^{-1}(0.91)$

$\frac{40-\mu}{\sigma} = \phi^{-1}(0.55)$ $\qquad$ $\frac{74-\mu}{\sigma} = 1.34$

$\frac{40-\mu}{\sigma} = 0.13$ $\qquad$ $74 - \mu = 1.34\sigma$

$40 = 0.13\sigma + \mu$ $\qquad$ $74 = \mu + 1.34\sigma$

~~$\sigma = 24.7033$~~

~~$\mu = 36.7886$~~

$\sigma = 28.0992$ $\qquad$ $\sigma^2 = 789.5650$

$\mu = 36.3471$

If you can't see the value, #-ve

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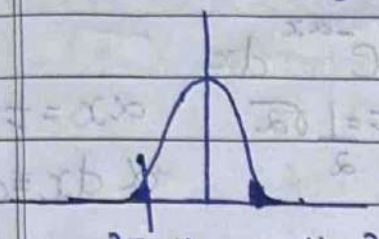
Q3 In a normally distribution 7% of the items are below 35 and 89% of the items are under 63.
Find mean and varience

$p(X<35) = 0.07$ $\qquad$ $P(X<63) = 0.89$

$p(Z < \frac{35-\mu}{\sigma}) = 0.07$ $\qquad$ $p(Z < \frac{63-\mu}{\sigma}) = 0.89$



$\frac{35-\mu}{\sigma}$ $\qquad$ $\frac{\mu-35}{\sigma}$

$p(Z > \frac{\mu-35}{\sigma}) = 0.07$ $\qquad$ $p(X<63) = 0.89$

$p(Z < \frac{\mu-35}{\sigma}) = 1-0.07$ $\qquad$ $p(Z < \frac{63-\mu}{\sigma}) = 0.89$

$p(Z < \frac{\mu-35}{\sigma}) = 0.93$ $\qquad$ $\frac{63-\mu}{\sigma} = \phi^{-1}(0.89)$

$\frac{\mu-35}{\sigma} = \phi^{-1}(0.93)$ $\qquad$ $\frac{63-\mu}{\sigma} = 1.23$

$\frac{\mu-35}{\sigma} = 1.48$

$-35 = -\mu + 1.48\sigma$ $\qquad$ $63 = \mu + 1.23\sigma$

$\mu = 50.2915$

$\sigma = 10.3321$ $\qquad$ $\sigma^2 =$

$$\int_0 e^{-x} x^{n-1} dx = \Gamma n$$

$$\Gamma n = (n-1)!$$

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Exponential Distribution:

A continuous random variable X is said to follow exponential distribution with parameter $\alpha > 0$ if the pdf is given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & ; x > 0 \\ 0 & ; \text{otherwise} \end{cases}$$

To check valid pdf

$$\int_0^\infty \alpha e^{-\alpha x} dx = 1$$

Mean and variance:

$$E(X) = \int_0^\infty x . f(x) dx = \int_0^\infty x . \alpha e^{-\alpha x} dx$$

$$E(X) = \int_0^\infty t e^{-t} \times \frac{1}{\alpha} dt = \frac{1}{\alpha} \int_0^\infty t e^{-t} dt = \frac{1}{2} \Gamma 2$$

$\alpha x = t$

$\alpha dx = dt$

$$E(X) = \frac{1}{\alpha} \times 1! = \frac{1}{\alpha}$$

$$E(x^2) = \int_0^\infty x^2 \alpha . e^{-\alpha x} . dx = \int t . \frac{t}{\alpha} e^{-t} \frac{dt}{\alpha}$$

$$= \int_0^\infty \frac{t^2 e^{-t}}{\alpha^2} dt = \frac{1}{\alpha^2} \int_0^\infty t^2 e^{-t} dt = \frac{1}{\alpha^2} (\Gamma 3)$$

$$= \frac{1}{\alpha^2} 2! = \frac{2}{\alpha^2}$$

$$V(X) = E(x^2) - (E(X))^2 = \frac{2}{\alpha^2} - \left(\frac{1}{\alpha}\right)^2 = \frac{2}{\alpha^2} - \frac{1}{\alpha^2} = \frac{1}{\alpha^2}$$

Q1 The life length of a certain device is a r.v having exponential distribution with mean = 200. Find prob that such a device will work for

i) At most 100 hours

ii) anywhere between 400 and 600 hours

$$\text{Mean}(\mu) = \frac{1}{\alpha} = 200 ; \quad \alpha = 1/200 = 0.005$$

$$f(x) = \begin{cases} 0.005 e^{-0.005 x} & ; x > 0 \\ 0 & ; \text{elsewher} \end{cases}$$

i) $P(x<100) = \int_0^{100} 0.005e^{-0.005x}\,dx = 0.3935$

Let $u = -0.005x$ $\quad u = -0.5$ to 0

$du = -0.005\,dx$

$= \int_{-0.5}^{0} 0.005\,e^{u} \cdot \frac{-1}{0.005}\,du = \int_{-0.5}^{0} -e^{u}\,du$

$= -[e^{0} - e^{-0.5}] = -(-0.3935) = 0.3935$

ii) $P(400<x<600) = \int_{400}^{600} 0.005\,e^{-0.005x}\,dx = 0.0855$

Q2 If X is expential distⁿ with mean $= 1/2$

then $P(1<X<3) =$ _____

$P(X>0.5) =$ _____

$\text{Mean}(u) = \frac{1}{\alpha} = \frac{1}{2}$; $\alpha = 2$

$$f(x) = \begin{cases} 2e^{-2x} & ; x>0 \\ 0 & ; \text{otherwise} \end{cases}$$

$P(1<x<3) = \int_1^3 2e^{-2x}\,dx = 0.1329$

$p(X>0.5) = \int_{0.5}^{\infty} 2e^{-2x}\,dx = 2\int_{0.5}^{\infty} e^{-2x}\,dx$

$u = -2x$ $\quad x = 0.5$ $\quad x = \infty$

$du = -2\,dx$ $\quad u = -1.0$ $\quad u = -\infty$

$= \int_{-\infty}^{-1} -e^{u}\,du = -[e^{-1} - e^{-\infty}] = -[0.3679]$

Q3 The time in sales/day is exponentially distributed with ₹800 as the average. If sales tax is at the rate of 6%, find prob that the increase in sales tax will exceed ₹30 per day.

$$f(x) = \begin{cases} \frac{1}{800} e^{-1/800\,x} & ; x>0 \\ 0 & ; \text{elsewhere} \end{cases}$$

$p(x>500) = \int_0^{\infty} 1/800\, e^{-1/800\,x}\,dx$

$= 0.5353$

$100 - 6$

$? - 30$

500

$\frac{100 \times 30}{6}$

$= 500$

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3. Gama Distribution:

A continuous r.v X is said to follow gama distribution with parameter $\gamma>0$, $\alpha>0$, if its pdf is given by

$$f(x) = \begin{cases} \frac{\alpha}{\Gamma\gamma} \cdot e^{-\alpha x} (\alpha x)^{\gamma-1} & ; x>0 \\ & ; \text{otherwise} \end{cases}$$

If $\gamma=1$, then Gama distribution tends to exponential distribution

Mean & Variance

$$E(X) = \int_0^\infty x \cdot \frac{\alpha}{\Gamma\gamma} (\alpha x)^{\gamma-1} \cdot e^{-\alpha x} \cdot dx$$

$\alpha x = t$

$\alpha dx = dt$

$$= \frac{1}{\Gamma\gamma} \int_0^\infty (\alpha x)^{\gamma} \cdot e^{-\alpha x} \, dx$$

$$= \frac{1}{\Gamma\gamma} \int_0^\infty t^{\gamma} \cdot e^{-t} \frac{dt}{\alpha}$$

$$= \frac{1}{\Gamma\gamma} \cdot \frac{1}{\alpha} \int_0^\infty e^{-t} t^{\gamma} \, dt = \frac{1}{\Gamma\gamma} \cdot \frac{1}{\alpha} \Gamma(\gamma+1)$$

$$= \frac{\gamma \Gamma\gamma}{\Gamma\gamma \times \alpha} = \frac{\gamma}{\alpha}$$

$$E(x^2) = \int_0^\infty x^2 \cdot \frac{\alpha}{\Gamma\gamma} (\alpha x)^{\gamma-1} e^{-\alpha x} \, dx$$

$$= \int_0^\infty \frac{1}{\Gamma\gamma} x (\alpha x)^{\gamma} e^{-\alpha x} \, dx$$

$$= \frac{1}{\Gamma\gamma} \int_0^\infty \frac{t}{\alpha} t^{\gamma} e^{-t} \frac{dt}{\alpha} = \frac{1}{\alpha^2 \Gamma\gamma} \int_0^\infty e^{-t} t^{\gamma+1}$$

$$= \frac{1}{\alpha^2 \Gamma\gamma} \Gamma(\gamma+2) = \frac{(\gamma+1)\gamma\Gamma\gamma}{\alpha^2 (\Gamma\gamma)} = \frac{(\gamma+1)(\gamma)}{\alpha^2}$$

$$V(X) = E(X^2) - (E(X))^2 = \frac{(\gamma+1)(\gamma)}{\alpha^2} - \frac{\gamma^2}{\alpha^2} = \frac{\gamma}{\alpha^2}$$

Problems

Q1 The number of accidents per day in a city is Gama distribution with mean = 6 and variance = 18. Find Probability that there will be

i) more than 8 accidents

ii) Between 5 and 8 accidents on a particular day

$\frac{\gamma}{\alpha} = 6$, $\frac{\gamma}{\alpha^2} = 18$

$\gamma = 6\alpha$ $\gamma = 18\alpha^2$

$\frac{\gamma}{\gamma} = \frac{18^3 \alpha^2}{6\alpha} = 1$

$1 = 3\alpha$

$\alpha = 1/3$

$\gamma = 6\alpha = 6 \times \frac{1}{3} = 2$

$$f(x) = \begin{cases} 1/9\, e^{-x/3} \cdot x & ; x > 0 \\ 0 & ; \text{otherwise} \end{cases}$$

i) $P(x > 8) = \int_8^{\infty} \frac{1}{9} x\, e^{-x/3}\, dx$

ii) $P(5 < x < 8) = \int_5^8 \frac{1}{9}\, x \cdot e^{-x/3} = 0.2468$

5 Chi- Square χ^2 distribution :

$$f(x) = \begin{cases} \frac{1}{2^{n/2}\,\Gamma(n/2)} e^{-x/2} . x^{n/2-1} & ; x>0 \\ 0 & ; \text{otherwise} \end{cases}$$

$n>0$ is a parameter

n is known as degree of freedom

Notation : $X \sim \chi^2(n)$

mean : $E(x) = \int_0^\infty x . \frac{1}{2^{n/2}\,\Gamma(n/2)} . e^{-x/2} . x^{n/2-1} dx$

$$= \frac{1}{2^{n/2}\,\Gamma(n/2)} \int_0^\infty e^{-t} (2t)^{n/2} \, 2dt$$

$\frac{x}{2} = t$

$dx = 2dt$

$\frac{x}{2} = t$

$x = 2t$

$1 = 2\frac{dt}{dx}$

$dx = 2dt$

$$= \frac{1}{2^{n/2}\,\Gamma(n/2)} . 2^{n/2+1} \int_0^\infty e^{-t} t^{n/2} dt = \frac{2}{\Gamma(n/2)} \Gamma(n/2+1)$$

$$= \frac{2\, n/2\, \Gamma(n/2)}{\Gamma(n/2)} = n$$

$$E(X^2) = \int_0^\infty x^2 . \frac{1}{2^{n/2}\,\Gamma(n/2)} e^{-x/2} . x^{n/2-1} dx$$

Put $\frac{x}{2} = t$

$dx = 2dt$

$$= \frac{1}{2^{n/2}\,\Gamma(n/2)} \int_0^\infty e^{-t} (2t)^{n/2+1} \, 2dt$$

$$= \frac{4}{\Gamma(n/2)} \int_0^\infty e^{-t} t^{n/2+1} dt = \frac{4}{\Gamma(n/2)} . \Gamma(n/2+2) = \frac{4 \times (n/2+1) n/2\, \Gamma(n/2)}{\Gamma(n/2)}$$

$= (n+2)n$

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$V(X) = n(n+2) - n^2 = 2n$

$\therefore$ Read table

$P(X < 7.88) = 0.995$

Find $P(X < 25)$ when $n = 15$

$P(X < 25) = 0.95$ $\quad n = 15$

$P(X < 67.5)$ when $n = 100$ $\quad X^2_{0.005} = 67.5$

$= 0.005$

$P(X < 90)$ when $n = 100$ is $\underline{0.25}$ $\quad X^2_{0.25} = 90$

$X^2_{0.005}$

Room NO. 310A

Function of Random Variables

Function of one dimensional Random Variables

Discrete: If X is discrete r.v and $Y = H(X)$ is a continuous function of X, then Y is also a discrete r.v

Suppose the possible X values are $x_1, x_2, x_3, \ldots, x_n$ with respective probabilities $P(x_1), P(x_2), P(x_3) \ldots P(x_n)$ then possible values of Y are $H(x_1), H(x_2) \ldots H(x_n)$

Example:

1)

X	-1	0	1
P(X)	1/3	1/2	1/6

Then, $Y = 3X + 1$ has pmf

1)	-2	1	4
P(Y)	1/3	1/2	1/6

2) If $Y = x^2$, then pmf of Y is

Y	1	0
P(Y)	$\frac{1}{3} + \frac{1}{6}$	1/2

Add P(-1) and P(1)

Y	1	0
P(Y)	1/2	1/2

3 Given

X	1	2 .....
P(X)	1/2	$1/2^2$

$$Y = \begin{cases} -1 ; & X \text{ is even} \\ 1 ; & X \text{ is odd} \end{cases}$$

Y	-1	1
P(Y)		

$$P(Y=1) = P(X=2 \cdot X=4 \ldots)$$

$$= \frac{1}{2^2} + \frac{1}{2^4} + \ldots$$

$$= \frac{\frac{1}{2^2}}{1-\frac{1}{2^2}} = \frac{1}{2^2}\left[1 + \frac{1}{2^2} + \frac{1}{2^4} + \ldots\right]$$

$$= \frac{1}{2^2}\left[\frac{1}{1-\frac{1}{4}}\right]$$

$$= \frac{1}{2^2}\left[\frac{1}{\frac{3}{4}}\right]$$

$$= \frac{1}{4} \times \frac{4}{3} = \frac{1}{3}$$

$$P(Y=-1) = 1 - P(Y) = \frac{2}{3}$$

Y	1	-1
P(Y)	1/3	2/3

(1+2)n

$$g(y) = \begin{cases} \frac{2}{9}(y-1) ; & 1<y<4 \\ 0 ; & \text{otherwise} \end{cases}$$

Continuous r.v

Let X be a continuous r.v with pdf $f(x)$ and $H(x)$ is a continuous function Hence $Y = H(x)$ is a continuous r.v

To obtain pdf of Y, i.e $g(y)$ we follow the steps given below.

1) Obtain cdf of Y i.e $g(y) = P(Y \leq y)$
2) Differentiate $G(y)$ w.r to y we get $g(y)$
3) Determine the range space of Y st $g(y) > 0$.

eg)

If $f(x) = \begin{cases} 2x ; & 0<x<1 \\ 0 ; & \text{otherwise} \end{cases}$, and $Y = 3X+1$, find pdf of Y

$$G(y) = P(Y \leq y)$$
$$= P(3X+1 \leq y)$$
$$= P\left(X \leq \frac{y-1}{3}\right)$$
$$= \int_0^{\frac{y-1}{3}} 2x\,dx = 2\left.\frac{x^2}{2}\right|_0^{\frac{y-1}{3}} = \left(\frac{y-1}{3}\right)^2$$

$$g(y) = G'(y) = 2\left(\frac{y-1}{3}\right) \times \frac{1}{3} = \frac{2}{9}(y-1)$$

$(0 < x < 1$

$1 < \frac{y-1}{3} < 1$ $\quad \therefore \int_1^4 \frac{3}{9}(y-1)\,dy = 1$

$1 < y < 4$

alternate method

Or

$$G(Y) = P\left(X \leq \frac{Y-1}{3}\right)$$

$\because P(a<x<b) = F(b) - F(a)$

$$= F\left(\frac{Y-1}{3}\right)$$

$$g(Y) = G'(Y) = F'\left(\frac{Y-1}{3}\right) \cdot \frac{1}{3}$$

$$= f\left(\frac{Y-1}{3}\right) \times \frac{1}{3}$$

$$\therefore f(X) = 2x$$

$$= 2 \times \left(\frac{Y-1}{3}\right) \times \frac{1}{3}$$

$$= \frac{2}{9}(Y-1)$$

Q2 Given $f(x) = \begin{cases} 2x & ; 0<x<1 \\ 0 & \text{otherwise} \end{cases}$, Find pdf of $Y = e^{x}$

$$G(Y) = P(Y \leq y) = P(e^{-x} \leq y)$$

$$= P\left(\frac{1}{y} \leq e^{x}\right)$$

$$= P(\log(1/y) \leq x)$$

$$= \int_{\log 1/y}^{1} 2x\,dx$$

$$= 2\frac{x^2}{2}\Big|_{\log(1/y)}^{1} = 1 - (\log(1/y))^2$$

$$= 1 - (\log(1/y))^2$$

$$g(Y) = G'(Y) = -2\left(\log \frac{1}{Y}\right) \cdot \frac{1}{\frac{1}{Y}} \cdot \frac{-1}{Y^2}$$

$$= \frac{2}{Y} \log(1/Y)$$

$$0 < x < 1$$

$$0 < \log \frac{1}{Y} < 1$$

$$0 < -\log Y < 1$$

$$0 > \log Y > -1$$

$$e^0 > Y > e^{-1}$$

$$\frac{1}{e} < Y < 1$$

Or

$$P(\log 1/Y \leq X) =$$

$$= F(\infty) - F(\log 1/Y)$$

$$G(Y) = 1 - F(\log 1/Y)$$

$$g(Y) = G'(Y) = -F'(\log 1/Y) \cdot Y \cdot (-1/Y^2)$$

$$g(Y) = f(\log 1/Y) \times 1/Y$$

$$= 2 \log(1/Y) \times \frac{1}{Y}$$

$$= \frac{2}{Y} \log(1/Y)$$

$$g(Y) = \begin{cases} \frac{2}{Y} \log(1/Y) & ; \ 1/e < Y < 1 \\ 0 & ; \ \text{Otherwise} \end{cases}$$

So

Q3 If $X \sim N(0, \sigma^2)$ then find pdf of $Y = |X|$

Sol^n

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-0}{\sigma}\right)^2} \quad ; -\infty < x < \infty$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2\sigma^2}}$$

$$G(Y) = P(Y \leq y) = P(|X| \leq y)$$
$$= P(-y \leq X \leq y)$$

$$G(Y) = F(Y) - F(-Y)$$

$$g(Y) = F'(Y) - F'(-Y)(-1)$$
$$= f(Y) + f(-Y)$$
$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{Y^2}{2\sigma}} + \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(-Y)^2}{2\sigma}}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-Y^2}{2\sigma^2}} \times 2 = \frac{\sqrt{2}\sqrt{2}\, e^{-\frac{Y^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$$

$$= \frac{\sqrt{2}\, e^{-\frac{Y^2}{2\sigma^2}}}{\sigma\sqrt{\pi}}$$

$-\infty < x < \infty$

$Y = |X|$, $0 < Y < \infty$

So its limit should be

Result: Let X be a continuous r.v with pdf $f(x)$.
Let $Y = X^2$. Then pdf of Y is
$$g(y) = \frac{1}{2\sqrt{y}}\left(f(\sqrt{y} + f(-\sqrt{y})\right)$$
proof

$$G(y) = P(Y \leq y) = P(X^2 \leq y)$$

$$= P(-\sqrt{y} < X < \sqrt{y})$$

$$G(y) = F(\sqrt{y}) - F(-\sqrt{y})$$

$$g(y) = G'(y) = F'(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - F'(-\sqrt{y}) \cdot \frac{-1}{2\sqrt{y}}$$

$$= \frac{f(\sqrt{y})}{2\sqrt{y}} + \frac{f(-\sqrt{y})}{2\sqrt{y}}$$

$$= \frac{1}{2\sqrt{y}}\left(f(\sqrt{y}) + f(-\sqrt{y})\right)$$

Problems

Q1 Suppose $f(x) = \begin{cases} 2xe^{-x^2} & ; 0 < x < \infty \\ 0 & ; \text{otherwise, find} \end{cases}$

Find pdf of $Y = X^2$

We have $g(y) = \frac{1}{2\sqrt{y}}\left(f(\sqrt{y}) + f(-\sqrt{y})\right)$

$$= \frac{1}{2\sqrt{y}}\left(2 \cdot \sqrt{y}\, e^{-y} + \underset{\text{-ve number}}{0}\right)$$

$$= \frac{1}{2\sqrt{y}}\left(2 \cdot \sqrt{y}\, e^{-y}\right) = e^{-y} \quad ; 0 < x < \infty$$

Verify:

$G(y) = P(Y \leq y)$

$= P(X^2 \leq y)$

$= P(-\sqrt{y} \leq X \leq \sqrt{y})$

$= \int_0^{\sqrt{y}} 2x e^{-x^2} dx$

$x^2 = t$

$2x dx = dt$

$G(y) = \int_0^y e^{-t} dt = -e^{-t}\Big|_0^y$

$= 1 - e^{-y}$

$g(y) = 0 - (-1) e^{-y} = e^{-y}$

Q2 Given $f(x) = \begin{cases} \frac{2}{9}(x+1) & ; -1 < x < 1 \\ 0 & ; \text{otherwise} \end{cases}$

Find pdf of $Y = X^2$

Sol^n:

$g(y) = \frac{1}{2\sqrt{y}} \left(f(\sqrt{y}) + f(-\sqrt{y}) \right)$

$= \frac{1}{2\sqrt{y}} \left(\frac{2}{9}(\sqrt{y}+1) - \frac{2}{9}(-\sqrt{y}+1) \right)$

$= \frac{2}{9\sqrt{y}} \quad ; 0 < y < 1$

~~Q3 Le~~

$$\begin{cases} \frac{2}{9\sqrt{y}} & ; 0 < y < 1 \\ 0 & ; \text{Otherwise} \end{cases}$$

Q3 Let $X \sim U(-1,1)$ Find pdf of $Y = X^2$
(X is uniformly distributed in the interval -1 and 1

Sol^n:

$$f(x) = \begin{cases} \frac{1}{2} & ; -1 < X < 1 \\ 0 & ; \text{otherwise} \end{cases}$$

$$g(y) = \frac{1}{2\sqrt{y}} \left(f(\sqrt{y}) + f(-\sqrt{y}) \right)$$

$$= \frac{1}{2\sqrt{y}} \left(1/2 + 1/2 \right)$$

$$= \frac{1}{2\sqrt{y}}$$

$$g(y) = \begin{cases} \frac{1}{2\sqrt{y}} & ; 0 < y < 1 \end{cases}$$

Theorem: Let X be a continuous r.v with pdf $f(x)$.
Suppose $Y = H(x)$ is a strictly monoto (increasing or decreasing) function of X: then Y has pdf

$$g(y) = f(x) \left| \frac{dy}{dx} \right|$$

where $x = H^{-1}(y)$ i.e x is expressed in terms of Y

Problem

Q1 Suppose X is uniformly distributed over $(0,1)$
Find of $Y = \frac{1}{X+1}$

$$f(x) = \begin{cases} 1 & ; 0 < x < 1 \\ 0 & ; \text{otherwise} \end{cases}$$

$Y = \frac{1}{X+1}$ is straitcly decreasing function

$$X = \frac{1}{Y} - 1 = \frac{1-Y}{Y}$$

$$X = \frac{1}{Y} - 1$$

$$\frac{dX}{dY} = \frac{-1}{Y^2}$$

$$g(y) = f(1/Y - 1) \cdot \left| \frac{dx}{dY} \right|$$

$$= \frac{1}{\frac{1}{Y} - 1}$$

$$= 1 \cdot \frac{1}{Y^2}$$

$\cancel{g(Y) = f(1)}$

$0 < x < 1$

$0 < \frac{1}{Y} - 1 < 1$

$$g(Y) = 1 \cdot \frac{1}{Y^2} ; \frac{1}{2} < Y < 1$$

Note: Let X be a continuous r.v with pdf $f(x) = \frac{1}{\pi} \frac{1}{1+x^2}$; $-\infty < x < \infty$

Then X is said to follow Cauchy's distrib

Q2 If X is uniformly distributed over $(-\pi/2, \pi/2)$ find of $Y = \tan X$.
(show that $Y = \tan X$ follow Cauchy distribution

Sol^n: $f(x) = \begin{cases} 1/\pi & ; -\pi/2 < x < \pi/2 \\ 0 & ; \text{otherwise} \end{cases}$

$Y = \tan X$ is strictly increasing function

$X = \tan^{-1}(Y)$ $\qquad \frac{dx}{dY} = \frac{1}{1+Y^2}$

$f(\tan^{-1} Y) = 1/\pi$

$g(Y) = f(\tan^{-1} Y) \left| \frac{dx}{dY} \right|$

$= \frac{1}{\pi} \cdot \frac{1}{1+Y^2}$; $-\infty < Y < \infty$

// Thus follow cauch's distribution

Q3 If $X \sim N(\mu, \sigma^2)$, then show that $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$ and

$$Y = Z^2 = \left(\frac{X-\mu}{\sigma}\right)^2 \sim \chi^2(1)$$

Sol^n:

$$f(x) = \left\{ \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \right. \quad ; -\infty < x < \infty$$

$$G(z) = P(Z \leq z) = P\left(\frac{X-\mu}{\sigma} \leq z\right)$$

$$= P(X \leq z\sigma + \mu)$$

$$= F(\sigma z + \mu) - F(-\infty)$$

$$= F(\sigma z + \mu) \qquad F(-\infty) = 0$$

$$g(z) = G'(z) = F'(\sigma z + \mu)\,\sigma = f(\sigma z + \mu)\,\sigma$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{\sigma z + \mu - \mu}{\sigma}\right)^2} \sigma = \frac{1}{2\pi} e^{-z^2/2} \sim N(0,1)$$

$Y = Z^2$ where $Z \sim N(0,1)$, $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$$g(y) = \frac{1}{2\sqrt{y}} \left(f(\sqrt{y}) + f(-\sqrt{y}) \right)$$

$$= \frac{1}{2\sqrt{y}} \times \frac{1}{\sqrt{2\pi}} \left\{ e^{-y/2} + e^{-y/2} \right\}$$

$$= \frac{1}{\sqrt{y}} \times \frac{1}{\sqrt{2\pi}} e^{-y/2}$$

$\chi^2_{(n)}$ distribution

$$f(x) = \frac{x^{n/2-1} e^{-x/2}}{2^{n/2} \Gamma(n/2)}$$

Substitute $n=1$

$$\chi^2_{(1)} = \frac{x^{-1/2} e^{-x/2}}{\sqrt{2}\sqrt{\pi}} = \frac{e^{-x/2}}{\sqrt{2\pi x}}$$

$\therefore Y = Z^2 \sim \chi^2_{(1)}$

Moment Generating Function (MGF)

Let X be a r.v. The moment generating function (mgf) of X denoted by $M_X(t)$ and is defined as

$$M_X(t) = E(e^{tX})$$

If X is discrete then $M_X(t) = \sum e^{tx} p(x)$

If X is continuous then $M_X(t) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\, dx$

$E(X^n)$, $n = 1, 2, 3, 4 \ldots$ are called moments of X about origin (Raw moments)

The first moment about origin

or

first raw moment is the mean

$E(X-\mu)^n$, $n = 1, 2, \ldots$ are called moments about the mean μ. (central Moment)

The second moment about μ i.e $E(X-\mu)^2$ is variance

We know $M_X(t) = E(e^{tx})$

$$= E\left(1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \dots\right)$$

$$= 0 + tE(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \dots \quad (i)$$

From (i) we can note that $E(X^n)$ is the coeff of $\frac{t^n}{n!}$ in the expansion of $M_X(t)$

$$M_X'(t) = E(X) + tE(X^2) + \frac{t^2}{2}E(X^3)$$

$$M_X'(0) = E(X) + 0 = E(X)$$

$$M_X''(t) = 0 + E(X^2) + tE(X^3) + \dots$$

$$M_X''(0) = E(X^2)$$

In general $M_X^{(n)} = E(X^n)$

Q1 Given $M_x(t) = \frac{3}{3-t}$, find σ.

Sol^n : $M_x'(t) = \frac{3}{(3-t)^2} = 3(3-t)^{-2}$

$E(X) = M_x'(0) = \frac{3}{(3-0)^2} = \frac{1}{3}$

$M''_{(x)}(t) = 3(-2)(3-t)^{-3}(-1)$

$= \frac{6}{(3-t)^3}$

$M_x''(0) = E(X^2) = \frac{6}{3^3} = \frac{2}{9}$

$\sigma = \sqrt{E(X^2) - (E(x))^2} = \sqrt{2/9 - 1/9} = 1/3$

Q1 Find MGF of Binomial distribution and hence find mean and variance.

Soln :

$P(X=x) = {}^nC_x p^x q^{n-x}$; $x = 0, 1, 2 .. n$

$M_x(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx} \cdot {}^nC_x p^x q^{n-x}$

$= q^n + e^t \, {}^nC_1 p q^{n-1} + e^{2t} \, {}^nC_2 p^2 q^{n-2} + \dots e^{tn} p^n$

$= q^n + (e^t p) \, {}^nC_1 q^{n-1} + (e^t p)^2 \, {}^nC_2 q^{n-2} + \dots (e^t p)^n$

$M_x(t) = (q + e^t p)^n$

$M'_x(t) = n(q + e^t p)^{n-1} e^t p$

$E(X) = M'_x(0) = np$

$M''(t) = np\left\{e^t (n-1)(q + e^t p)^{n-2} e^t p + (q + e^t p)^{n-1} e^t\right\}$

$E(X^2) = M''_x(0) = np\{(n-1)p + 1\}$

$V(X) = E(X^2) - (E(X))^2 = npq$

Q2 Find mgf of poisson distribution & hence find mean and variance.

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx}\left(\frac{e^{-\alpha}\alpha^x}{x!}\right)$$

$$= e^{-\alpha}\sum_{x=0}^{\infty}\left(\frac{(e^t\alpha)^x}{x!}\right)$$

$$= e^{-\alpha}\left\{1 + \frac{e^t\alpha}{1} + \frac{(e^t\alpha)^2}{2!} + \frac{(e^t\alpha)^3}{3!} + \dots\right\}$$

$$= \cancel{e^{-\alpha}e^{e^t}}$$

$$= \cancel{e^{-\alpha}e^{e^t\alpha}} = e^{-\alpha}e^{e^t\alpha} = e^{\alpha(e^t-1)}$$

$$M_X'(t) = e^{\alpha(e^t-1)}\,\alpha e^t$$
$$= \alpha . e^{\alpha(e^t-1)} . e^t$$

$$E(X) = M_X'(0) = \alpha$$

$$M_X''(t) = \alpha\{e^t e^{\alpha(e^t-1)}\alpha e^t + e^{\alpha(e^t-1)}e^t\}$$

$$E(X^2) = M_X''(0) = \alpha^2 + \alpha$$

$$V(X) = E(X^2) - (E(X))^2 = \alpha^2 + \alpha - \alpha^2 = \alpha$$

Q3 Let X be the outcome when a fair dice is tossed, Find mgf of X & hence find mean and variance

Sol.

X	1	2	3	4	5	6
P(X)	1/6	1/6	1/6	1/6	1/6	1/6

$$M_X(t) = E(e^{tx}) = \sum e^{tx} P(x)$$

$$= e^t \cdot \frac{1}{6} + e^{2t} \frac{1}{6} + e^{3t} \cdot \frac{1}{6} + e^{4t} \cdot \frac{1}{6} + e^{5t} \times \frac{1}{6} + e^{6t} \times \frac{1}{6}$$

$$= \frac{1}{6} \cdot \{ e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t} \}$$

$$M_X'(t) = \frac{1}{6} \{ e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t} \}$$

$$M_X'(0) = \frac{1}{6} \{ 1 + 2 + 3 + 4 + 5 + 6 \} = \frac{21}{6}$$

$$M_X''(t) = \frac{1}{6} \{ e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t} \}$$

$$E(X^2) = \frac{1}{6} \{ 1 + 4 + 9 + 16 + 25 + 36 \}$$

Q4. Find the mgf of exponential distribution. Hence find mean and variance

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & ; 0 < x < \infty \\ 0 & ; \text{otherwise} \end{cases}$$

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} \cdot \alpha e^{-\alpha x} \, dx$$

$$= \alpha \int_0^\infty e^{tx - \alpha x} \, dx = \alpha \int_0^\infty e^{x(t-\alpha)} \, dx$$

$$= \alpha \left. \frac{e^{x(t-\alpha)}}{t-\alpha} \right|_0^\infty$$

$$= \frac{\alpha}{\alpha - t} = \frac{1}{1 - t/\alpha} = (1 - t/\alpha)^{-1}$$

$$(1-x)^{-1} = 1 + x + x^2 + \dots$$

$$M'_X(t) = \cancel{1-\alpha} \; 1 + \frac{t}{\alpha} + \frac{t^2}{\alpha^2} + \dots$$

$E(X) =$ coefficient of t in eqn

$$\mu = \frac{1}{\alpha}$$

$$E(X^2) = \frac{2}{\alpha^2}$$

ribution

Q5 Find the mgf of uniform distribution in the interval (a, b). Hence find mean & variance

Sol^n: $f(x) = \begin{cases} \frac{1}{b-a} & ; a<x<b \\ 0 & ; \text{otherwise} \end{cases}$

$$M_x(t) = E(e^{tx}) = \int_a^b \frac{1}{b-a} e^{xt} dx$$

$$= \frac{1}{t(b-a)} (e^{bt} - e^{at})$$

$$M'_x(t) = \frac{1}{b-a} \cdot \frac{t(e^{bt} b - a e^{at}) - (e^{bt} - e^{at})}{t^2}$$

Not Possible

$$M_x(t) = \frac{1}{b-a} \left\{ \left(1 + bt + \frac{(bt)^2}{2!} + \frac{(bt)^3}{3!} + \dots\right) - \left(1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!}\right) \right.$$

$$= \frac{1}{b-a} \left\{ (b-a)t + \frac{(b^2-a^2)}{2!} t^2 + \frac{(b^3-a^3)}{3!} t^3 + \dots \right\}$$

$$= 1 \left\{ t + \frac{(a+b)t^2}{2!} + \frac{(b^2+ab+a^2)}{3!} t^3 + \dots \right\}$$

$E(X)$ = coefficient of t in the expansion

$$M_x(t) = \frac{b+a}{2}$$

$E(X^2)$ = coeff of $\frac{t^2}{2!}$ in $M_x(t) = \frac{b^2+ab+a^2}{3}$

Q6 Find Mgf of Gama distribution

$$f(x) = \begin{cases} \dfrac{e^{-\alpha x}\,\alpha\, x^{\gamma-1}}{\Gamma\gamma} & ; \; 0 < x < \infty \\ 0 & \end{cases}$$

$$M_X(t) = E(e^{tx}) =$$

$$= \int_0^\infty \frac{e^{tx}\, e^{-\alpha x}\, \alpha\, x^{\gamma-1}}{\Gamma\gamma}\, dx$$

$$= \frac{\alpha^\gamma}{\Gamma\gamma} \int_0^\infty e^{-x(\alpha-t)}\, x^{\gamma-1}\, dx$$

put $x(\alpha - t) = Y$

$$dx = \frac{dY}{(\alpha - t)}$$

$$= \frac{\alpha^\gamma}{\Gamma\gamma} \frac{1}{(\alpha-t)^\gamma} \int_0^\infty e^{-Y}\, Y^{\gamma-1}$$

$$= \left(\frac{\alpha}{\alpha - t}\right)^\gamma$$

Q4 Find MGF of normal distribution & hence find its mean & variance

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad ; -\infty < x < \infty$$

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2\sigma^2}(x^2+\mu^2-2x\mu-2\sigma^2tx)} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\mu}{2\sigma^2}} \int_{-\infty}^{\infty} e^{\frac{-1}{2\sigma^2}(x^2-2x(\mu+\sigma^2t))} dx$$

completing the square

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\mu}{2\sigma^2}} \int_{-\infty}^{\infty} e^{\frac{-1}{2\sigma^2}(x-(\mu+\sigma^2t))^2} \cdot e^{\frac{(\mu+\sigma^2t)^2}{2\sigma^2}}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{\mu t+\frac{\sigma^2t^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2t))^2} dx$$

Put $\frac{x-(\mu+\sigma^2t)}{\sqrt{2}\sigma} = Y$

$dx = \sqrt{2}\sigma dY$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{\mu t+\frac{\sigma^2t^2}{2}} \int_{-\infty}^{\infty} e^{-Y^2} \sqrt{2}\sigma dY$$

$$= \frac{1}{\sqrt{\pi}} e^{\mu t+\frac{\sigma^2t^2}{2}} \underbrace{2\int_0^{\infty} e^{-Y^2} dY}_{\to \frac{2\sqrt{\pi}}{2}}$$

Put $Y^2 = Z$

$2Y dY = dZ$

$$= \frac{1}{\sqrt{\pi}} e^{\mu t+\frac{\sigma^2t^2}{2}} \cdot 2 \cdot \frac{\sqrt{\pi}}{2}$$

$$M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$M_x'(t) = e^{\mu + \frac{\sigma^2 t^2}{2}} (\mu + \sigma^2 t)$$

$$M_x'(0) = \mu$$

$$M_x''(t)$$

$$E(X^2) = M_x''(0) = \mu^2 + \sigma^2 - \mu^2$$

$$V(X) = \sigma^2$$

Find mgf of Standard normal distribut

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{tx} e^{-\frac{x^2}{2}} dx$$

$$M_x(t) = e^{t^2/2}$$

$$E(X) = M_x'(0) = 0$$

Q9 Find mgf of χ^2 distribution and hence find and variance

Sol^n:

$$f(x) = \begin{cases} \frac{1}{2^{n/2}\Gamma(n/2)} e^{-x/2} \cdot x^{n/2-1} & ; x>0 \\ 0 & ; \text{otherwise} \end{cases}$$

$$M_x(t) = \int_0^{\infty} e^{tx} \cdot \frac{1}{2^{n/2}\Gamma(n/2)} \cdot e^{-x/2} \cdot x^{n/2-1} \cdot dx$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^{\infty} e^{-x(1/2-t)} \cdot x^{n/2-1} dx$$

Put $x(1/2-t) = y$

$(1/2-t)\,dx = dy$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^{\infty} e^{-y} \cdot \frac{y^{n/2-1}}{(1/2-t)^{n/2-1}} \cdot \frac{dy}{(1/2-t)}$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)(1/2-t)^{n/2}} \int_0^{\infty} e^{-y} \cdot y^{n/2-1} dy$$

$$= \frac{1}{2^{n/2}\cancel{\Gamma(n/2)}(1/2-t)^{n/2}} \cdot \cancel{\Gamma(n/2)}$$

$$= \frac{1}{2^{n/2}(1/2-t)^{n/2}} = \frac{1}{(1-2t)^{n/2}} = (1-2t)^{-n/2}$$

$M'_x(t) =$

$E(X) = M_x'(0) = n$

$M_x''(t) =$

$E(X^2) = M_x''(0) = n^2 + 2n$

$V(X) = E(X^2) - (E(X))^2 = n^2 + 2n - n^2 = 2n$

Theorem 1:

Let X be a r.v and let $Y = \alpha X + \beta$ where α, β are constants. Then MGF of Y is

$M_Y(t) = e^{t\beta} M_X(\alpha t)$

Proof

$M_Y(t) = E(e^{tY}) = E(e^{t(\alpha x + \beta)})$

$= E(e^{t\beta + (t\alpha x)}) = E(e^{t\beta} \cdot e^{t\alpha x})$

$= e^{t\beta} E(e^{t\alpha x}) = e^{t\beta} E(e^{t\alpha x})$

$M_X(t) = E(e^{tx})$

$M_X(\alpha t) = E(e^{\alpha t x})$

$= e^{t\beta} M_X(\alpha t)$

Theorem 2:
Suppose X & Y are 2 independent r.v with mgf $M_X(t)$ and $M_Y(t)$ respectively. Then $M_Z(t) = M_X(at)\, M_Y(bt)$ where $Z = aX + bY$, a and b are constants

Proof:

$$M_Z(t) = E(e^{tZ}) = E(e^{t(aX+bY)}) = E(e^{atX} \cdot e^{btY})$$

As x and y are independent

$$M_Z(t) = E(e^{atX})\, E(e^{btY}) = M_X(at) \cdot M_Y(bt)$$

Q1 Problems

Let $M_X(t) = (0.4e^t + 0.6)^8$.

Find $E(X)$, $V(X)$ and $M_Y(t)$ where $Y = 3X + 2$

You can differentiate, substitute 0

Find $E(X)$, $E(X^2)$, $V(X)$

or

We known if X is Binominal distribution then $M_X(t) = (pe^t + q)^n$

By comparing we get

$p = 0.4$ $\quad q = 0.6$ $\quad n = 8$

$E(X) = np = 8 \times 0.4 = 3.2$

$V(X) = npq = 8 \times 0.4 \times 0.6 = \cancel{2.08}\ 1.92$

$\alpha = 2, \beta = 3$

$$M_Y(t) = e^{t\beta} M_X(\alpha t) = e^{2t} M_X(3t) = e^{2t}(0.4e^{3t} + 0.6)^8$$

$$= e^{2t}(0.4e^{3t} + 0.6)^8$$

Q2 If $X \sim N(\mu, \sigma^2)$, $Y = aX + b$, find $M_Y(t)$. Hence show that $Y \sim N(a\mu + b, a^2\sigma^2)$

Soln:

$$M_Y(t) = e^{tb} M_X(at)$$

We known $M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$$M_Y(t) = e^{tb} \cdot e^{a\mu t + \frac{a^2\sigma^2 t^2}{2}} = e^{tb + a\mu b + \frac{a^2\sigma^2 t^2}{2}}$$

$$= e^{(a\mu + b)t + \frac{a^2 t^2 \sigma^2}{2}}$$

Comparing with $M_X(t)$,

Y has mean $= a\mu + b$ & variance $= a^2\sigma^2$

$$Y \sim N(a\mu + b, a^2\sigma^2)$$

Q3 Given $P(X = x) = pq^{x-1}$; $x = 1, 2, 3, 4, 5 \ldots$ Find $M_X(t)$. The r.v X is said to follow geometric distribution). Hence find mean and variance.

Sol^n:

$$M_X(t) = E(e^{tx}) = \sum_{x=1}^{\infty} e^{tx} \cdot \boxed{p q^{x-1}} \rightarrow \text{pdf}$$

$$= pq^0 p e^t + pq^1 p e^{2t} + pq^2 p e^{3t} + \ldots$$

$$= pe^t \{1 + e^t q + e^{2t} q^2 + \ldots\}$$

$$= pe^t \left(\frac{1}{1 - e^t q}\right)$$

$M_X(t)' =$

$$E(X) = M_X'(0) = \frac{1}{p}$$

$M_X''(t) =$

$$E(X^2) = M_X''(0) = \frac{p + 2q}{p^2}$$

$$V(X) = q/p^2$$

Q4 If $M_X(t) = e^{2t(1+t)}$, what is the pdf of $Y = \left(\frac{X-2}{2}\right)^2$. Also, find mgf of Y.

Sol^n: Given $M_X(t) = e^{2t(1+t)} = e^{(2t)+(2t^2)}$

We know the above mgf is mgf of Normal dist^n $\left(M_X(t) = e^{\mu t + \sigma^2 t^2/2}\right)$

By comparing, we get $X \sim N(2, 4)$

$$Z = \frac{X-2}{2} \sim N(0,1)$$

$$Y = Z^2 = \left(\frac{X-2}{2}\right)^2 \sim \chi^2_{(1)}$$

pdf of χ^2 is equal to $\frac{1}{\sqrt{2\pi}} e^{-x/2} x^{-1/2}$

when $n=1$

pdf of Y is $\frac{1}{\sqrt{2\pi}} e^{-x/2} x^{-1/2} : 0 < x < \infty$

MGF of χ^2 is $(1-2t)^{-n/2}$

$\therefore M_Y(t) = (1-2t)^{-1/2}$

Q5) If X_1, X_2, X_3 are independent r.v. and $Z = X_1 + X_2 + X_3$ and mgf of X_1, X_2 & X_3 are $(1-2t)^{-3}$, $(1-2t)^{-2}$, and $(1-2t)^{-1/2}$. Find mean and variance of $\frac{Z+1}{2}$

OR

Sol^n: $M_Z(t) = M_{X_1}(t)\, M_{X_2}(t)\, M_{X_3}(t)$

$= (1-2t)^{-3} (1-2t)^{-2} (1-2t)^{-1/2}$

$= (1-2t)^{-11/2}$

Comparing $M_Z(t)$ with mgf of χ^2 we get $(1-2t)^{n/2}$

$n = 11$

$E(Z) = n = 11$

$V(Z) = 2n = 22$

$M_Z'(t) =$

$E(Z) = M_Z'(0) = 11$

$M_Z''(t) =$

$E(Z^2) = M_Z''(0) =$

$V(Z) = 22$

$E\left(\frac{Z+1}{2}\right) = \frac{1}{2} E(Z) + E(1/2) = \frac{1}{2} \cdot 11 + \frac{1}{2} = \frac{12}{2} = 6$

$V(ax+b) = a^2 V(x)$

$V\left(\frac{Z+1}{2}\right) = \frac{1}{4} V(Z) = \frac{1}{4} \times 22 = 11/2$

Q6 Given $M_{X_1}(t) = e^{2t(1+t)}$, $M_{X_2}(t) = e^{3t(1+t)}$, $M_{X_3}(t) = e^{4t(1+t)}$ and X_1, X_2 and X_3 are indep. r.v. Find pdf of $Z = 4X_1 + X_2 + 2X_3$. Also find $E(Z/2)$ and $V(Z/2)$

Sol^n:

$M_Z(t) = M_{X_1}(4t) \cdot M_{X_2}(t) + M_{X_3}(2t)$

Or

$M_Z'(t) = = e^{2(4t)(1+4t)} \cdot e^{3t(1+t)} \cdot e^{4(2t)(1+2t)}$

$E(Z) = M_Z'(0) = 19$

$= e^{19t + 15t^2}$

mgf of normal distribution, $Z \sim N(19, 102)$

$M''(t)$

$E(Z^2) = M''(0)$

$V(Z) = 102$

$E(Z) = 19$

$V(Z) = 102$

$E(Z/2) = 19/2$

$V(Z/2) = \frac{102}{4}$

Reproductive Property

Let $X_1, X_2, X_3 \ldots X_n$ be n independent r.v having identical dist^n. If $Z = X_1 + X_2 + X_3 \ldots X_n$ has same dist^n, then the dist^n is said to have reproductive property

Theorem 1:
Let $X_1, X_2, X_3 \ldots X_n$ are n independent r.v's having gama distribution with parameter (α, γ) $(\alpha, \gamma_2) \ldots (\alpha, \gamma_n)$. Then $Y = \sum X_i = X_1 + X_2 + X_3 + \ldots + X_n$ also has gama distribution with parameter $(\alpha, X_i =$ $X_1 + X_2 + X_3 + X_4 + X_5 + \ldots + X_n$ also has gama distribution with parameter $(\alpha, \sum \gamma_i)$
(Gama distribution has resproductive property

$$M_Y(t) = M_{X_1}(t) \ldots M_{X_n}(t)$$

$$= \left(\frac{\alpha}{\alpha - t}\right)^{\gamma_1} \ldots \left(\frac{\gamma}{\gamma - t}\right)^{\gamma_n}$$

$$= \left(\frac{\alpha}{\alpha - t}\right)^{\sum(\gamma_i)} = \left(\frac{\alpha}{\alpha - t}\right)^{\gamma_1 + \gamma_2 + \gamma_3 + \gamma + + \ldots \gamma_n}$$

$$Y \sim G(\alpha, \sum \gamma_i)$$

Theorem 2: Let X_i $(i = 1, 2, 3, 4, 5, 6, 7, 8, 9 \ldots n)$ be independent r.v. with $X_i \sim N(\mu_i, \sigma_i^2)$
Then $Y = \sum a_i X_i = a_1 X_1 + a_2 X_2 + a_3 X_3 + \ldots + a_n X_n$ is also Normal with mean $\sum a_i \mu_i$ and variance $\sum a_i^2 \sigma_i^2$.
Hence Normal distribution has reproductive property.

Proof: $M_Y(t) = M_{X_1}(a_1 t) M_{X_2}(a_2 t) \ldots M_n(a_n t)$

$$= e^{a_1 \mu_1 t + \frac{\sigma_1^2 a_1^2 t^2}{2}} \cdot e^{a_2 \mu_2 t + \frac{\sigma_2^2 a_2^2 t^2}{2}} \ldots e^{a_n \mu_n t + \frac{\sigma_n^2 a_n^2 t^2}{2}}$$

$$= e^{a_1 \mu_1 t + \frac{\sigma_1^2 a_1^2 t^2}{2} + a_2 \mu_2 t + \frac{\sigma_2^2 a_2^2 t^2}{2} + a_3 \mu_3 t + \frac{\sigma_3^2 a_3^2 t^2}{2} + \ldots}$$

$$= e^{(a_1 \mu_1 + a_2 \mu_2 + \ldots + a_n \mu_n) t + \frac{t^2}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \ldots + \sigma_n^2)}$$

$$\therefore Y \sim N\left(\sum_{i=1} a_i \mu_i, \sum \sigma_i^2 a_i^2\right)$$

Theorem 3: Poison distribution has re reproductive property.
i.e $X_i \sim P(\alpha_i)$
$\Rightarrow Y = \sum X_i \sim P(\sum \alpha_i)$

$$M_Y(t) = e^{\alpha_1 (e^t - 1)} \cdot e^{\alpha_2 (e^t - 1)} \ldots e^{\alpha_n (e^t - 1)}$$

$$= e^{(e^t - 1)(\sum \alpha_i)}$$, Y has Poisson's distⁿ with parameter $\sum \alpha_i$

$X_1 \sim N(\mu_1, \sigma_1^2)$
$X_2 \sim N(\mu_2, \sigma_2^2)$

$Y = a_1 X_1 + a_2 X_2 \sim N(a_1 \mu_1 + a_2 \mu_2, a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + a_3^2 \sigma_3^2)$

Theorem 4:

χ^2 dist^n has reproductive reproductive property

i.e $X_i \sim \chi^2(n_i) \Rightarrow Y = \sum X_i \sim \chi^2(\sum n_i)$

Proof:

$$M_X(t) = (1-2t)^{-n_1/2} (1-2t)^{-n_2/2} \dots (1-2t)^{-n_n/2}$$

$$= (1-2t)^{-[n_1/2 + n_2/2 + n_3/2 + n_4/2 + \dots]}$$

$$= (1-2t)^{-\sum n_i/2}$$

Y has χ^2 dist^n with parameter $\sum n_i$

Problems:

Q1) Let $X \sim N(\underset{\mu_1}{0}, 3^2)$ and $Y \sim (\underset{\mu_2}{0}, 3^2)$. Then Find $P(5 < 3X + 4Y < 10)$

Sol^n:

Let $U = \overset{aX}{3X} + \overset{bY}{4Y}$

$U \sim N(0, 3^2 \times 3^2 + 4^2 \times 3^2)$

$U \sim N(0, 225)$

$P(5 < U < 10)$

$P\left(\frac{5-0}{15} < Z < \frac{10-0}{15}\right)$

$P(1/3 < Z < 2/3)$

$P(0.33 < Z < 0.67)$

$= \phi(0.67) - \phi(0.33)$

$= 0.7486 - 0.6293$

$= 0.1193$

Q2 The marks obtained by the students in mathematics, physics and chemistry in an exam are normally distributed with mean 52, 50 and 48 and standard deviations 10, 8 and 6 respectively. Find prob that a ~~total~~ ~~of~~ student selected at random has secured a total of

(i) 180 or above
ii) 135 or above

Let: X Marks in Mathematics $X \sim N(52, 10^2)$
Y: Physics $Y \sim N(50, 8^2)$
Z: chemistry $Z \sim N(48, 6^2)$

$T = X + Y + Z$

$T \sim N(52 + 50 + 48, 10^2 + 8^2 + 6^2)$
$T \sim N(150, 200)$

i) $P(T > 180) = P\left(Z > \frac{180 - 150}{\sqrt{200}}\right)$

$= 1 - P(Z < 2.12)$
$= 1 - \phi(2.12)$
$= 0.017$

ii) $P(T < 135) = P\left(Z < \frac{135 - 150}{\sqrt{200}}\right)$

$= P(Z < -1.06)$
$= \phi(-1.06)$
$= 1 - \phi(1.06)$
$= 0.14457$

Q3 Find mgf of $X \sim U(-a, a)$ and hence find $E(X^{2n})$.

Sol^n:

$$f(x) = \begin{cases} \frac{1}{2a} & ; -a < x < a \\ 0 & ; \text{otherwise} \end{cases}$$

$$M_X(t) = E(e^{tx}) = \int_{-a}^{a} e^{tx} \cdot \frac{1}{2a} \, dx$$

$$= \frac{1}{2at} (e^{at} - e^{-at})$$

$$= \frac{1}{2at} \left(\left\{ 1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \dots \right\} - \left\{ 1 - at + \frac{a^2t^2}{2!} - \frac{a^3t^3}{3!} + \dots \right\} \right)$$

$$= \frac{1}{2at} \left(2 \left(at + \frac{(at)^3}{3!} + \frac{(at)^5}{5!} + \dots \right) \right)$$

$$= \frac{1}{at} \left(1 + \frac{(at)^2}{3!} + \frac{(at)^4}{5!} + \frac{(at)^6}{7!} + \dots \frac{(at)^{2n}}{(2n+1)!} \right)$$

$$= 1 + \frac{(at)^2}{3!} + \frac{(at)^4}{5!} + \dots + \frac{a^{2n} \cdot t^{2n}}{(2n+1)!} \frac{(2n)!}{(2n)!}$$

$$E(X^{2n}) = \text{coeff of } \frac{t^{2n}}{(2n)!} \text{ in } M_X(t) = \frac{a^{2n}}{2n+1}$$

Q4 Let $X \sim N(\mu, \sigma^2)$. Then show that
$E((X-\mu)^{2n}) = 1.3.5.\ldots(2n-1)\sigma^{2n}$

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Let $Y = X - \mu$

We know $E(X^n)$ is coeff of $\frac{t^n}{n!}$ in $M_X(t)$.

Then $E[(X-\mu)^{2n}]$ is coeff of $\frac{t^{2n}}{(2n)!}$ in $M_Y(t)$

$\cancel{= 1 + \sigma^2}$

$$M_Y(t) = e^{-\mu t} . M_X(t)$$

$$\boxed{\begin{array}{l} Y = \alpha X + \beta \\ M_Y(t) = e^{t\beta} M_X(\alpha t) \end{array}}$$

$$= e^{-\mu t} . e^{\mu t + \frac{\sigma^2 t^2}{2}} = e^{\frac{\sigma^2 t^2}{2}}$$

$$= 1 + \frac{\sigma^2 t^2}{2} + \left(\frac{\sigma^2 t^2}{2}\right)^2 \frac{1}{2!} + \ldots + \left(\frac{\sigma^2 t^2}{2}\right)^n \times \frac{1}{n!} \frac{(2n)!}{(2n)!} + \ldots$$

$$E((X-\mu)^{2n}) = \frac{\sigma^{2n}(2n)!}{2^n n!} = \frac{\sigma^{2n} 1.3.5.\ldots(2n-1)\, 2.4.6.\ldots 2n}{2^n n!}$$

$$= \frac{\sigma^{2n} 1.3.5.7.\ldots(2n-1) . 2^n [1.2.3.4.5.6.\ldots n]}{2^n n!}$$

$$= \frac{\sigma^{2n} 1.3.5.7.\ldots(2n-1) . 2^n . n!}{2^n . n!}$$

$$= \sigma^{2n} 1.3.5.7.\ldots(2n-1)$$

Q5 Suppose X has pdf $f(x) = \frac{1}{2} e^{-|x|}$; $-\infty < x < \infty$, find mgf of X. and hence find mean and variance.

Sol^n:

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{2} e^{-|x|} \, dx$$

$$= \frac{1}{2} \int_{-\infty}^{0} e^{tx} e^{x} \, dx + \frac{1}{2} \int_{0}^{\infty} e^{tx} e^{-x} \, dx$$ 75

$$= \frac{1}{2} \left(\frac{1}{1+t} + \frac{1}{1-t} \right) = \frac{1}{2} \frac{[1-t+1+t]}{(1-t^2)}$$ 90

2 methods

$$= \frac{2}{2(1-t^2)} = \frac{1}{1-t^2}$$

$$= (1-t^2)^{-1} = 1 + t^2 + t^4 + \dots$$

$M_X' t = \frac{1}{1-t^2}$	$M_X(t) = (1-t^2)^{-1}$ $= 1 + t^2 + t^4 + \dots$
$M_X'(t) =$ $E(X) = M_X'(0) =$	 $E(X) =$ coef of $t = 0$
$M_X''(t) =$ $E(X^2) = M_X''(0) =$ $V(X) =$	$E(X^2) =$ coef of $\frac{t^2}{2!} = 2! \left(\frac{t^2}{2!}\right)$ $= 2$ $V(X) = E(X^2) - (E(X))^2$ $= 2 - 0 = 2$

Q6 Suppose the r.v has pdf $f(x) = \lambda e^{-\lambda(x-a)}$, $x \geq a$, find mgf. Hence find mean and variance.

Sol^n

$$M_x(t) = \int_a^\infty e^{tx} \lambda e^{-\lambda(x-a)} dx = \frac{\lambda e^{at}}{\lambda - t}$$

$$M_x'(0) = E(X) = a + \frac{1}{\lambda}$$

$$V(X) = \frac{1}{\lambda^2}$$

Q7 Let X be a r.v having values 0, 1, 2, 3, 4.....
$p(x) = ab^x$, $a, b > 0$,
$a + b = 1$. Find mgf
If $E(X) = m_1$ and $E(X^2) = m_2$, show that
$m_2 = m_1(2m_1 + t_1)$

Sol^n.

$$M_x(t) = \sum_{x=0}^{\infty} e^{tx} ab^x = \frac{a}{1 - be^t}$$

$$E(X) = m_1 = M_x'(0) = \frac{b}{a}$$

$$E(X^2) = m_2 = M_x''(0) = \frac{b(1+b)}{a^2}$$

Only for exponential & geometric distribution
A r.v X is said to have memoryless property if $P(X > s+t \mid X > s) = P(X > t)$

if $(P(X > s+t \mid X > s) = P(X > t))$

Example,

1 Show that Exponential distn has memoryless property

$P(X > s+t \mid X > s) = P(X > t)$

Example

1) Show that exponential distribution has memoryless property

Sol^n:

$$P(X > s+t \mid X > s) = \frac{P(X > s+t \cap X > s)}{P(X > s)}$$

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & ; x > 0 \\ 0 & ; \text{otherwise} \end{cases} \qquad = \frac{P(X > s+t)}{P(X > s)}$$

$$= \frac{\int_{s+t}^{\infty} \alpha e^{-\alpha x}\, dx}{\int_{s}^{\infty} \alpha e^{-\alpha x}\, dx} = \frac{e^{-(s+t)\alpha}}{e^{-\alpha s}} = e^{-\alpha t}$$

$$P(x > t) = \int_{t}^{\infty} \alpha e^{-\alpha x}\, dx = e^{-\alpha t}$$

$$= P(X > s+t \mid x > s)$$

$$= P(x > t)$$

2 Show that dist^n has memoryless propert

Sol^n:

$$P(x) = pq^{x-1} \quad ; \quad x = 1, 2, \ldots\ldots$$

$$P(x > s+t \mid x > s) = \frac{P(x > s+t)}{P(x > s)}$$

$$= \frac{\sum_{x=s+t+1}^{\infty} pq^{x-1}}{\sum_{x=s+1}^{\infty} pq^{x-1}}$$

$$= \frac{pq^{s+t+0} + pq^{s+t+1} + \ldots\ldots}{pq^{s} + pq^{s+1} + \ldots\ldots}$$

$$= \frac{pq^{s+t}(1 + q + q^2 + \ldots\ldots)}{pq^{s}(1 + q + \ldots\ldots)}$$

$$= q^t$$

$$P(x > t) = \sum_{x=t+1}^{\infty} pq^{x-1} = pq^t + pq^{t+1} + \ldots = pq^t(1 + q + q^2 + \ldots)$$

$$= pq^t(1-q)^{-1} = pq^t p^{-1} = q^t$$

Function of 2D random variables

Suppose (X, Y) is a 2dim continuous r.v. with pdf $f(x, y)$ defined over a region R.
Let $Z = H_1(X, Y)$ and $W = H_2(X, Y)$
Suppose H_1 and H_2 satisfy the following conditions

$Z = X^2 + Y^2$ $\quad$ $Z = X - Y$

$\sqrt{X^2+Y^2}$

i) The equation $Z = H_1(x, y)$ and $W = H_2(x, y)$ may be uniquely solved for x and y interms of Z & W.
i.e
$$X = G(Z, W) \text{ and } Y = G_2(Z, W)$$

We have 2DV H_1 & H_2 which depend on X&Y

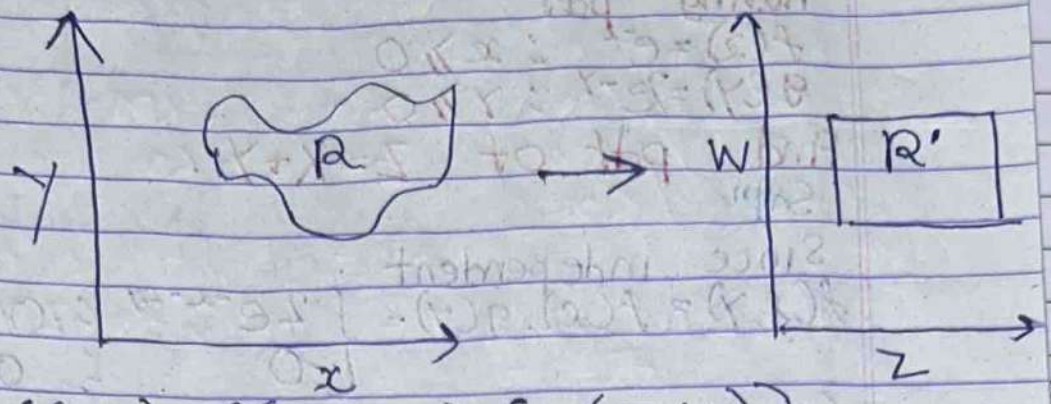
Suppose H_1 and H_2 satisfy the following conditions.

i) The equations
a) $Z = H_1(X, Y)$
b) $W = H_2(X, Y)$
may be uniquely solved for X and Y interms of Z and W
ie
a') $X = G_1(Z, W)$
b') $Y = G_2(Z, W)$

ii) The partial derivatives
$$\frac{\partial y}{\partial w}, \frac{\partial y}{\partial z}, \frac{\partial x}{\partial w}, \frac{\partial x}{\partial z}$$
exits and are continuous
Then, Joint pdf of Z and W
ie,
$g(Z, W)$ is given by

$$g(Z, W) = \begin{cases} f(x, y)\,|J| & : (Z, W) \in R \\ 0 & : \text{otherwise} \end{cases}$$

$$g(z,w) = \begin{cases} f(x,y)|J| & : (z,w) \in R \\ 0 & : \text{otherwise} \end{cases}$$



$f(x,y) = f(G_1(z,w), G_2(z,w))$

and Jacobian

$$J = \frac{\partial(x,y)}{\partial(z,w)} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}$$

Procedure to obtain pdf of $Z = H_1(X,Y)$

1) Introduce another r.v. $W = H_2(X,Y)$
2) Obtain pdf of Z, W i.e $g(z,w)$
3) Obtain marginal pdf of Z ie, $g(z)$ (Integrate $g(z,w)$ wrto w)

Problems

Q1 Two independent r.v X and Y having pdf

$f(x) = e^{-x} \; ; x \geq 0$

$g(y) = 2e^{-y} \; ; y \geq 0$

Find pdf of $Z = X + Y$

Sol^n

Since independent

$$f(x,y) = f(x).g(y) = \begin{cases} 2e^{-x-2y} & ; 0 \leq x, y \leq \infty \\ 0 & ; \text{otherwise} \end{cases}$$

Given

$Z = X + Y$, Let $W = Y$

Introduce a new random variable $W = Y$, $W = X - Y$

$X = Z - W$, $Y = W$

$$J = \begin{vmatrix} \frac{\partial X}{\partial Z} & \frac{\partial X}{\partial W} \\ \frac{\partial Y}{\partial Z} & \frac{\partial Y}{\partial W} \end{vmatrix} = 1$$

Joint pdf of Z and W

i.e, $g(z,w) = \begin{cases} 2e^{-(z+w)} & ; \\ 0 & ; \text{otherwise} \end{cases}$

$0 \leq y < \infty$ $\quad$ $0 \leq x < \infty$

$0 \leq w \leq \infty$ $\quad$ $0 \leq z - w \leq \infty$

$w \leq z \leq \infty$

$0 \leq y \leq \infty$ $\quad$ $0 \leq x \leq \infty$ $\quad$ $0 \leq x \leq \infty$

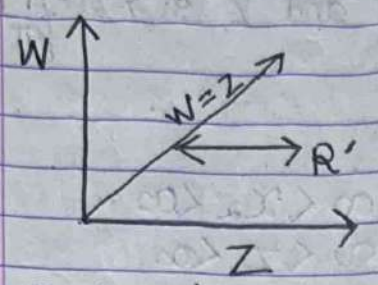
$0 \leq w \leq \infty$ $\quad$ $0 \leq z - w \leq \infty$ $\quad$ $0 \leq z - w \leq \infty$

$w \leq z \leq \infty$ $\quad$ $-z \leq -w \leq \infty$

$z \geq w \geq -\infty$

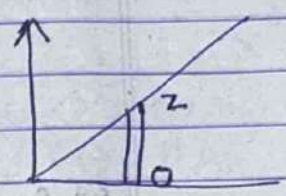
Find common $\quad$ $-\infty \leq w \leq z$

& $0 \leq w \leq \infty$



Joint pdf of Z and W i.e $g(z,w)=$

$$\begin{cases} 2e^{-(z+w)} & ; 0 \leq w \leq z \leq \infty \\ 0 & ; \text{otherwise} \end{cases}$$



Marginal pdf of (z)

Integrate wrt. w.

$$g(z) = \int_0^z 2e^{-(z+w)}\, dw = 2e^{-z} \frac{e^{-w}}{-1}\Big|_0^z = 2e^{-z} - 2e^{-2z}$$

Q2 If X_1 and X_2 are 2 independent r.Vs having standard normal distribution then find pdf of $Y = \frac{X_1}{X_2}$ | show that $Y = X_1/X_2$ has Cauchy's distribution

Sol^n

$X_1 \sim N(0,1)$ $\qquad$ $X_2 \sim N(0,1)$

$f(x_1) = \frac{1}{\sqrt{2\pi}} e^{-x_1^2/2}$; $-\infty < x < \infty$ $\qquad$ $f(x_2) = \frac{1}{\sqrt{2\pi}} e^{-x_2^2/2}$; $-\infty < x_2 < \infty$

$$f(x_1, x_2) = f(x_1) f(x_2) = \frac{1}{2\pi} \cdot e^{-1/2(x_1^2 + x_2^2)} \quad ; -\infty \leq x_1, x_2 \leq \infty$$

Given $Y = X_1/X_2$ $\qquad$ Let $Z = X_2$

$X_1 = YZ$ $\qquad$ $X_2 = Z$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial Y} & \frac{\partial X_1}{\partial Z} \\ \frac{\partial x_2}{\partial Y} & \frac{\partial X_2}{\partial Z} \end{vmatrix} = Z$$

The joint pdf of Y and Z, $g(Y,Z) = \frac{1}{2\pi}|z| e^{-\frac{z^2}{2}(y^2+1)}$

limit ?

$-\infty < x_1 < \infty$ $\quad$ $-\infty < x_2 < \infty$

$-\infty < Y_Z < \infty$ $\quad$ $-\infty < Z < \infty$

$-\infty < Y < \infty$

$$g(Y,Z) = \begin{cases} \frac{1}{2\pi}|z| e^{-\frac{z^2}{2}(y^2+1)} & ; -\infty < Z, Y < \infty \\ 0 & ; \text{otherwise} \end{cases}$$

Pdf of Y is $g(Y) = \int_{-\infty}^{\infty} \frac{1}{2\pi}|z| e^{-\frac{z^2}{2}(y^2+1)} dz$

$= \frac{2}{2\pi} \int_0^{\infty} z e^{-\frac{z^2}{2}(y^2+1)}$ $\quad$ $\frac{1}{2} z^2(y^2+1) = t$

$= \frac{1}{\pi} \int_0^{\infty} \frac{e^{-t}}{1+y^2} dt = \frac{1}{\pi(1+y^2)}$ $\quad$ $z(1+y^2)dz = dt$

$; -\infty < Y < \infty$

$g(Y) = \frac{1}{\pi(1+y^2)}$ has Cauchy's distribution

$\theta = \tan^{-1} Y/X$ $\quad$ $|J|$ in polar co-ordinate $= |R|$

Q5 Given $f(x,y) = \begin{cases} 4xye^{-(x^2+y^2)} & ; x \geq 0, y \geq 0 \\ 0 & ; \text{Otherwise} \end{cases}$

Find pdf of $R = \sqrt{X^2+Y^2}$

When X^2+Y^2

Polar co-ordinates

Sol^n: Let θ be another r.v.

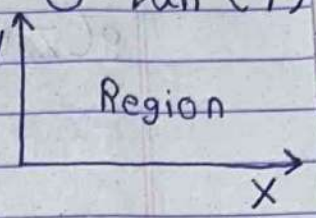
$X = R\cos\theta$

$Y = R\sin\theta$

$|J| = |R|$

Joint pdf of R and θ is

$\theta = \tan^{-1}(Y/X)$



$$g(r,\theta) = \begin{cases} 4r^2\cos\theta\sin\theta\, e^{-r^2}|r| & ; 0 < r < \infty \\ & ; 0 < \theta < \pi/2 \\ 0 & ; \text{Otherwise} \end{cases}$$

pdf of R is $\pi/2$

$$g(r) = \int_0^{\pi/2} 4r^2\cos\theta\sin\theta\, e^{-r^2}|r|\, d\theta$$

$$= 4r^3e^{-r^2}\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta$$

$$= 4r^3e^{-r^2}\cdot\frac{1}{2}$$

$$= 2r^3e^{-r^2}$$

$$g(r) = \begin{cases} 2r^3e^{-r^2} & ; 0 < r < \infty \\ 0 & ; \text{Otherwise} \end{cases}$$

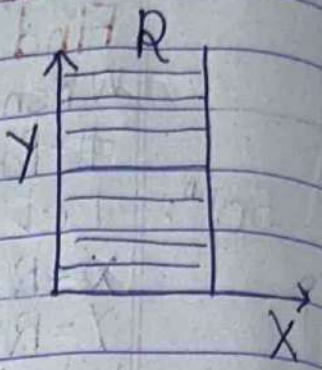
OR:

Let $V = Y$

$X = \sqrt{R^2 - V^2}$, $Y = V$

Q6 Given $f(x,y) = \begin{cases} \frac{x}{2} e^{-y} & ; 0<x<2, \; y>0 \\ 0 & ; \text{otherwise} \end{cases}$

Find pdf of $Z = X+Y$



Sol^n ~~Fin~~

Let $W = Y$

$X = Z - W$

$|J| = 1$

$$g(z,w) = \begin{cases} \frac{z-w}{2} e^{-w} & ; \\ 0 & ; \text{otherwise} \end{cases}$$

$0 < x < 2$ $\qquad$ $0 < y < \infty$

$0 < z - w < 2$ $\qquad$ $0 < w < \infty$

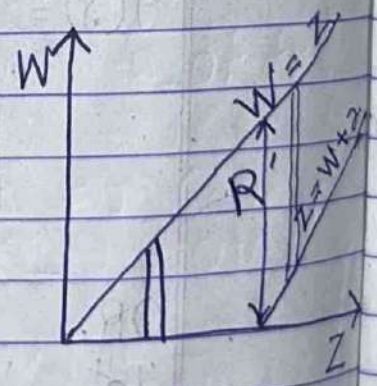
$w < z < 2 + w$ ✓

$-z < -w < 2 - z$

$-z < -w < 2 - z$

$z > w > z - 2$

$$g(z,w) = \begin{cases} \frac{z-w}{2} e^{-w} & ; 0 < w < z < w+2 \\ 0 & ; \text{otherwise} \end{cases}$$



pdf of Z, $0<Z<2$

$$g(Z) = \int_0^Z \left(\frac{Z-W}{2}\right) e^{-W}\, dW$$

$$= \frac{1}{2}\left[\int_0^Z Z e^{-W}\, dW - \int_0^Z W e^{-W}\, dW\right]$$

$$= \frac{1}{2}\left\{ Z \frac{e^{-W}}{-1}\Big|_0^Z - \left[\frac{We^{-W} - e^{-W}}{-1}\right]_0^Z \right\}$$

$$= \frac{Z + e^{-Z} + 1}{2}$$

$2<Z<\infty$ $\quad g(Z) = \sum_{Z-2}^{3} ($

$$g(Z) = \int_{Z-2}^{Z} \left(\frac{Z-W}{2}\right) e^{-W}\, dW$$

$$= \frac{1}{2}\left\{ Z \int_{Z-2}^{Z} e^{-W}\, dW - \int_{Z-2}^{Z} W e^{-W}\, dW \right\}$$

$$= \frac{1}{2}\left\{ Z \frac{e^{-W}}{-1}\Big|_{Z-2}^{Z} - \left[\frac{We^{-W} - e^{-W}}{-1}\right]_{Z-2}^{Z} \right\}$$

$$= \frac{e^{-(Z-2)} + e^{-Z}}{2}$$

Student's t distribution: [William Sealy Gosset developed]

Let $U \sim N(0,1)$, $V \sim \chi^2(\gamma)$

Then $f(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$; $-\infty < u < \infty$

and $f(v) = \frac{1 \times e^{-v/2} v^{\gamma/2 - 1}}{\Gamma\gamma/2 \cdot 2^{\gamma/2}}$; $0 < v < \infty$

$$= \frac{e^{-v/2} v^{\gamma/2-1}}{2^{\gamma/2} \Gamma\gamma/2} \quad ; 0 < v < \infty$$

Joint pdf of U and V is

$$f(u,v) = f(u) \cdot f(v) = \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-\left(\frac{u^2+v}{2}\right)}}{2^{\gamma/2}} \cdot \frac{v^{\gamma/2-1}}{\Gamma\gamma/2} \quad ; -\infty < u < \infty \; ; 0 < v < \infty$$

define a r.v

$T = \frac{U}{\sqrt{v/\gamma}}$ and $Z = V$

$\Rightarrow U = T\sqrt{\frac{V}{\gamma}}$ or $U = T\sqrt{\frac{Z}{\gamma}}$ and $V = Z$

Then pdf of T is

$$\boxed{f(t) = \frac{\Gamma\gamma+1}{\sqrt{\pi}\,\Gamma\gamma/2\,\gamma^{1/2}\left(1+t^2/\gamma\right)^{\frac{\gamma+1}{2}}} \quad ; -\infty < t < \infty}$$

Notation: $T \sim t(\gamma)$, γ is degree of freedom

NOTE: T distribution tends to Cauchy's distribution with $\gamma = 1$

Q1 Problems

Let T has t distribution with 14 degrees of freedom. Then find $P(|T| > 2.228)$

Sol^n: $P(|T| > 2.228) = 1 - P(|T| < 2.228)$

$= 1 - P(-2.228 < T < 2.228)$

$= 1 - [\phi(2.228) - \phi(-2.228)]$

$= 1 - [\phi(2.228) - (1 - \phi(2.228))]$

$= 1 - \phi(2.228) + 1 - \phi(2.228)$

$= 2 - (\phi(2.228)) \times 2$

$= 2 - 2 \times 0.975$

$= 0.05$

Q2 Let T has t distribution with degree of freedom 14.

Determine b st $P(-b < T < b) = 0.9$

$P(-b < T < b) = 0.9$

$\phi(b) - \phi(-b) = 0.9$

$\phi(b) - (1 - \phi(b)) = 0.9$

$2\phi(b) - 1 = 0.9$

$2\phi(b) = 1.9$

$\phi(b) = 0.95$

$b = \phi^{-1}(0.95)$

$b = 1.76$

F distribution [Ronald Fisher and George W Snedecor]

let $u \sim \chi^2(\gamma_1)$ and $V \sim \chi^2(\gamma_2)$ are independent r.v.s

Joint pdf of U, V is

$$f(u,v) = \frac{e^{-\left(\frac{u+v}{2}\right)} u^{\frac{\gamma_1}{2}-1} v^{\frac{\gamma_2}{2}-1}}{2^{\frac{\gamma_1+\gamma_2}{2}} \cdot \Gamma\gamma_1/2 \, \Gamma\gamma_2/2}$$

Define

$$F = \frac{u/\gamma_1}{v/\gamma_2} = \frac{\gamma_2}{\gamma_1}\frac{u}{v}$$

Let $V = Z$. Then $u = \frac{Z\gamma_1 F}{\gamma_2}$ and $Z = V$

pdf of F is

$$g(F) = \frac{\sqrt{\frac{\gamma_1+\gamma_2}{2}} \left(\frac{\gamma_1}{\gamma_2}\right)^{\frac{\gamma_1}{2}} F^{\frac{\gamma_1}{2}-1}}{\sqrt{\gamma_1/2}\,\sqrt{\gamma_2/2}\left(\frac{F\gamma_1}{\gamma_2}+1\right)^{\frac{\gamma_1+\gamma_2}{2}}} \quad ; 0 < F < \infty$$

Notation: $F \sim F(\gamma_1, \gamma_2)$

Problems

Q1 Let X_1 and X_2 a independent r.v.s having same dist^n with pdf $f(x) = e^{-x}$; $x \geq 0$. The show that $Z = \frac{X_1}{X_2}$ has F distribution (i.e. $F(2,2)$)

Sol^n

$$f(x_1, x_2) = f(x_1)f(x_2) = \begin{cases} e^{-(x_1+x_2)} & ; 0 \leq x_1, x_2 < \infty \\ 0 & ; \text{otherwise} \end{cases}$$

$Z = \frac{X_1}{X_2}$, Let $V = X_2$. Then $X_1 = ZV$

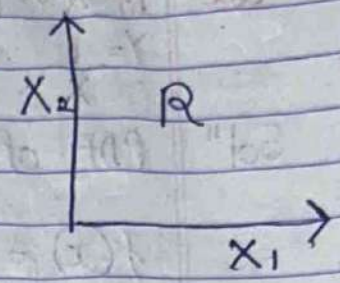
$0 \leq x_1 < \infty$ $\quad$ $0 \leq x < \infty$

$0 \leq zv < \infty$ $\quad$ $0 \leq v \leq \infty$

$0 \leq z < \infty$

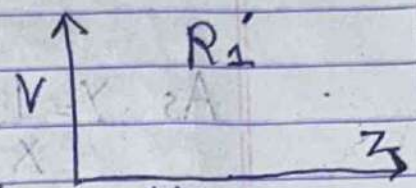
$Z = \frac{X_1}{X_2}$, Let $V = X_2$.

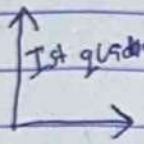
Then $X_1 = ZV$



$$|J| = \begin{vmatrix} \frac{\partial X_1}{\partial Z} & \frac{\partial X_1}{\partial V} \\ \frac{\partial X_2}{\partial Z} & \frac{\partial X_2}{\partial V} \end{vmatrix}$$

$$= |V|$$





$$g(Z, V) = \begin{cases} e^{-V(Z+1)} |V| & ; 0 \leq V, Z < \infty \\ 0 & ; \text{otherwise} \end{cases}$$

$$g(Z) = \int_0^\infty e^{-V(Z+1)} V \cdot dV$$

$$= \left[\frac{V \cdot e^{-V(Z+1)}}{-(Z+1)} - \frac{e^{-V(Z+1)}}{(Z+1)^2} \right]_0^\infty$$

$$= \frac{1}{(Z+1)^2}$$

$$F(\gamma_1, \gamma_2) = \frac{\sqrt{\frac{\gamma_1+\gamma_2}{2}} \left(\gamma_1/\gamma_2\right)^{\frac{\gamma_1}{2}} F^{\frac{\gamma_1}{2}-1}}{\sqrt{\gamma_1/2}\ \sqrt{\gamma_2/2} \left(\frac{F\gamma_1}{\gamma_2}+1\right)^{\frac{\gamma_1+\gamma_2}{2}}}$$

$$F(2,2) = \frac{1}{(F+1)^2}$$

$$Z \sim F(2,2)$$

Q2 If $X \sim F(\gamma_1, \gamma_2)$ then show that $Y = \frac{1}{X} \sim F(\gamma_2, \gamma_1)$

Sol^n PDF of X is

$$f(x) = \frac{\Gamma\left(\frac{\gamma_1+\gamma_2}{2}\right)\left(\frac{\gamma_1}{\gamma_2}\right)^{\gamma_1/2} x^{\frac{\gamma_1}{2}-1}}{\Gamma\left(\frac{\gamma_1}{2}\right)\Gamma\left(\frac{\gamma_2}{2}\right)\left(\frac{x\gamma_1}{\gamma_2}+1\right)^{\frac{\gamma_1+\gamma_2}{2}}} \quad ; 0 < x < \infty$$

As $Y = \frac{1}{X}$ is ↓ing function

$$g(y) = f(1/y) \cdot \left|\frac{dx}{dy}\right|$$

$$= \frac{\Gamma\left(\frac{\gamma_1+\gamma_2}{2}\right)\left(\frac{\gamma_1}{\gamma_2}\right)^{\gamma_1/2}\left(\frac{1}{y}\right)^{\frac{\gamma_1}{2}-1}}{\Gamma(\gamma_1/2)\,\Gamma(\gamma_2/2)\left(\frac{\gamma_1}{y\gamma_2}+1\right)^{\frac{\gamma_1+\gamma_2}{2}}} \cdot \frac{1}{y^2} \qquad x = 1/y, \quad \frac{dx}{dy} = \left|\frac{-1}{y^2}\right|$$

$$= \frac{\Gamma\left(\frac{\gamma_1+\gamma_2}{2}\right)\left(\frac{\gamma_1}{\gamma_2}\right)^{\gamma_1/\gamma_2} y^{\frac{-\gamma_1}{\gamma_2}+1-2} \cdot (y\gamma_2)^{\frac{\gamma_1+\gamma_2}{2}}}{\Gamma(\gamma_1/2)\,\Gamma(\gamma_2/2)\,(\gamma_1+y\gamma_2)^{\frac{\gamma_1+\gamma_2}{2}}}$$

$$= \frac{\Gamma\left(\frac{\gamma_1+\gamma_2}{2}\right)\left(\frac{\gamma_1}{\gamma_2}\right)^{\frac{\gamma_1}{2}} y^{\frac{-\gamma_1}{2}-1+\frac{\gamma_1}{2}+\frac{\gamma_2}{2}} (\gamma_2)^{\frac{\gamma_1}{2}}}{\Gamma(\gamma_1/2)\,\Gamma(\gamma_2/2)\,(\gamma_1+y\gamma_2)^{\frac{\gamma_1+\gamma_2}{2}}}$$

$$= \cancel{\Gamma\left(\frac{\gamma_1+\gamma_2}{2}\right) \gamma_2^{\gamma_1/2} \gamma_2}$$

$$= \frac{\Gamma\left(\frac{\gamma_1+\gamma_2}{2}\right) \gamma_1^{\gamma_1/2} \gamma_2^{\gamma_2/2} Y^{\gamma_2/2-1}}{\Gamma\left(\frac{\gamma_1}{2}\right)\Gamma\left(\frac{\gamma_2}{2}\right)(\gamma_1+Y\gamma_2)^{\frac{\gamma_1+\gamma_2}{2}} \left(\frac{\gamma_1}{\gamma_1}\right)^{\frac{\gamma_1+\gamma_2}{2}}}$$

$$= \frac{\Gamma\left(\frac{\gamma_1+\gamma_2}{2}\right) \gamma_1^{\gamma_1/2} \gamma_2^{\gamma_2/2} Y^{\frac{\gamma_2}{2}-1} \cdot \gamma_1^{-\frac{\gamma_1}{2}} \gamma_1^{-\frac{\gamma_2}{2}}}{\Gamma\left(\frac{\gamma_1}{2}\right)\Gamma\left(\frac{\gamma_2}{2}\right)\left(1+\frac{Y\gamma_2}{\gamma_1}\right)^{\frac{\gamma_1+\gamma_2}{2}}}$$

$$= \frac{\Gamma\left(\frac{\gamma_1+\gamma_2}{2}\right) \left(\frac{\gamma_2}{\gamma_1}\right)^{\gamma_2/2} (Y)^{\frac{\gamma_2}{2}-1}}{\Gamma\left(\frac{\gamma_1}{2}\right)\Gamma\left(\frac{\gamma_2}{2}\right)\left(1+\frac{Y\gamma_2}{\gamma_1}\right)^{\frac{\gamma_1+\gamma_2}{2}}}$$

$$\therefore Y = \frac{1}{X} \sim F(\gamma_2, \gamma_1)$$

Q1 Obtain $F(10,15)$ for $p=0.95$

Ans $P(F \leq a) = 0.95$

$a = 2.54$

Q2 Obtain $F(15,9)$ for $p=0.99$

Ans Ans: 4.96

Q3 Find $F(8,7)$ for $p=0.05$

$P(F \leq a) = 0.05$

$P(1/a \leq 1/f) = 0.05$

$P(1/f \leq 1/a) = 1 - 0.05 = 0.95$

We know

$\frac{1}{f} \sim F(7,8)$ if $F \sim F(8,7)$

$\frac{1}{a} = 3.50$, $a = 1/3.50$

Q4 If F has F distribution with $\gamma_1 = 5$ and $\gamma_2 = 10$, Find a, b such that $P(F \leq a) = 0.05$ and $P(F \leq b) = 0.95$

$F \sim F(5, 10)$
$P(F \leq b) = 0.95$
$b = 3.33$

$P(F \leq a) = 0.05$
$P(1/a \leq 1/F) = 0.05$
$P(1/F \leq 1/a) = 1 - 0.05$

We known $\frac{1}{F} \sim F(10, 5)$

$\frac{1}{a} = 4.74$

$a = 1/4.74$

Sampling

→ Sample: A finite subset of a population
→ Sample size: No of individuals in a sample
→ Sampling: Process of obtaining a sample
sampling with replacement
default ★ sampling without replacement

Random Sampling:
In which the sample units are selected at random.

Defination:
Let X be a random variable with certain probability distribution having mean μ and variance σ^2.

Let $X_1, X_2, X_3 \ldots X_n$ be n independent r.v each having the same distribution as that of X
(independent & identical distribution)
(iid)

Then, $(X_1, X_2, X_3 \ldots X_n)$ is called a random sample of size n from X.
Here
μ = population mean
σ^2 = population variance

Statistic:

A real valued function of one or more random variables which does not depend on any unknown parameter is called satistics

Example
$Y = \Sigma x_i$ is a satistiss
$Z = \frac{X - \mu}{\sigma}$ is a satistics if μ and σ are Known.

Defination:

Let $(X_1, X_2, X_3, X_4, X_5 \ldots X_n)$ be a random sample of size n from X. Then, the statistic $\bar{X} = \frac{\Sigma X_i}{n}$ is called sample mean and
$S^2 = \frac{\Sigma (X_i - \bar{X})^2}{n}$ is called the sample Vari

Distribution of $\bar{X}$ and $\frac{ns^2}{\sigma^2}$

$Y = aX + b$

$M_Y(t) = e^{bt} M_X(at)$

Theorem: Given $E(X) = \mu$ and $V(x) = \sigma^2$ and $(X_1, X_2, X_3 \ldots X_n)$ be a random sample of size n from X.
Let $\bar{X}$ be the sample mean. Then
$E(\bar{X}) = \mu$ and $V(\bar{X}) = \frac{\sigma^2}{n}$

Proof:

$$E(\bar{X}) = E\left(\frac{\sum X_i}{n}\right) = \frac{1}{n} E(X_i)$$

$$= \frac{1}{n}[E(X_1) + E(X_2) + E(X_3) + \ldots + E(X_n)]$$

$$= \frac{1}{n}[\mu + \mu + \mu + \ldots + \mu = \frac{n\mu}{n} = \mu$$

$$V(\bar{X}) = V\left(\frac{\sum X_i}{n}\right) = \frac{1}{n^2}[V(X_1) + V(X_2) + \ldots + V(X_n)]$$

$$= \frac{1}{n^2}[\sigma^2 + \sigma^2 + \ldots \sigma^2] = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

Distribution of $\bar{X}$ and ns^2/σ^2:

Let $(X_1, X_2, X_3 \ldots X_n)$ be a random sample of size n from X where $X \sim N(\mu, \sigma^2)$ and $Y = \bar{X} = \frac{\sum X_i}{n}$.

Then,

$M_Y(t) = M_{X_1}(t/n) \cdot M_{X_2}(t/n) \ldots M_{X_n}(t/n)$

$Y = aX + b$
$M_Y(t) = e^{bt} M_X(at)$
$X \sim N(\mu, \sigma^2)$
$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{n}}$

$$M_Y(t) = e^{\mu t/n + \frac{\sigma^2 t^2}{2n^2}} \ldots e^{\frac{\mu t}{n} + \frac{\sigma^2 t^2}{2n^2}}$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2n}}$$

$$Y = \bar{X} \sim N(\mu, \sigma^2/n)$$

$$Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

$\frac{ns^2}{\sigma^2} \sim \chi^2_{(n-1)}$ where $s^2 = \frac{\Sigma(X_i - \bar{X})^2}{n}$

$X \sim N(\mu, \sigma^2)$; $\frac{X-\mu}{\sigma} \sim N(0,1)$, $\left(\frac{X-\mu}{\sigma}\right)^2 \sim \chi^2_{(1)}$

Problems

Q1 Let $\bar{X}$ be the mean of a random sample of size 25 from a distribution which is $N(75, 100)$. Find $P(71 < \bar{X} < 79)$.

Sol^n $X \sim N(75, 100)$ $\quad n = 25$

$\bar{X} \sim N\left(75, \frac{100}{25}\right)$

$\bar{X} \sim N(75, 4)$ $\quad Z = \frac{\bar{X} - 75}{2}$

$P(71 < \bar{X} < 79) = P\left(\frac{71-75}{2} < Z < \frac{79-75}{2}\right)$

$P(-2 < Z < 2) = \phi(2) - \phi(-2)$

$= \phi(2) - (1 - \phi(2))$

$= 2\phi(2) - 1$

$= 0.9544$

Q2 Let S^2 be the variance of a random sample of size 6 and distribution is $N(\mu, 12)$.
Evaluate $P(2.3 < S^2 < 22.2)$

$S^2 = \frac{\Sigma(X_i - \bar{X})^2}{n}$

Solⁿ:

$Y = \frac{nS^2}{\sigma^2} \sim \chi^2_{(n-1)}$

$Y = \frac{6S^2}{12} = \frac{S^2}{2} \sim \chi^2_{(n-1)}$

$P(2.3 < S^2 < 22.2) = P\left(\frac{2.3}{2} < \frac{S^2}{2} < \frac{22.2}{2}\right)$

$= P(1.15 < Y < 11.1)$

$= \chi^2_5(11.1) - \chi^2_5(1.15)$

$= 0.95 - 0.05$

$= 0.90$

Q3 Let $\bar{X}$ and s^2 be the mean and variance of a random sample of size 25 from the distribution $N(3, 100)$. Evalute $P\{0 < \bar{X} < 6, 55.2 < s^2 < 145.6\}$

Solⁿ

$X \sim N(3, 100)$ $\qquad n = 25$

$\bar{X} = N\left(3, \frac{100}{25}\right)$

$Z = \frac{\bar{X} - 3}{2} \sim N(0, 1)$

$P(0 < \bar{X} < 6) = P\left(\frac{0-3}{2} < Z < \frac{6-3}{2}\right)$

$= P(-1.5 < Z < 1.5) = 2\phi(1.5) - 1$

$P(55.2 < S^2 < 145.6)$ $\quad Y = \frac{ns^2}{\sigma^2}$ $\quad \frac{25S^2}{100} = \frac{S^2}{4}$

$= \frac{S^2}{4} \sim \chi^2_{(24)}$

$P\left(\frac{55.2}{4} < \frac{S^2}{4} < \frac{145.6}{4}\right) = \chi^2_{24}(\ ) - \chi^2_{24}(\ )$

$=$

Q5 Let $\bar{X}$ be mean of the random sample of size 5. from a normal distribution with $\mu = 0$ and $\sigma^2 = 125$ Find c such that $P(\bar{X} < c) = 0.9$

$X \sim N(0, 125)$

$\bar{X} \sim N\left(0, \frac{125}{5}\right)$ $\quad \frac{X-0}{\sqrt{\frac{125}{5}}}$ $\quad \frac{X-0}{\sqrt{25}}$

$\bar{X} \sim N(0, 25)$

$P(\bar{X} < c) = 0.9$

$P\left(\frac{\bar{X}-0}{5} < \frac{c-0}{5}\right) = 0.9$

$P(Z < c/5) = 0.9$

$\frac{c}{5} = \phi^{-1}(0.90)$

$\frac{c}{5} = 1.29$

$c = 6.45$

Q6 Let $\bar{X}$ be the mean of random sample of size n from a normal distribution with mean μ and $\sigma^2 = 100$. Find n such that $P(\mu - 5 < \bar{X} < \mu + 5) = 0.954$

$X \sim N(\mu, 100)$

$\bar{X} \sim N(\mu, \frac{100}{n}) \Rightarrow \frac{X - \mu}{\sqrt{\frac{100}{n}}} \sim N(0, 1)$

$P(\mu + 5 < X < \mu + 5) = 0.954$

$P(-\frac{\sqrt{n}}{2} < Z < \frac{\sqrt{n}}{2}) = 0.954$

$\cancel{\phi(\sqrt{n}/2) - \phi(-\sqrt{n}/2) = 0.954}$

$\cancel{2\phi(\sqrt{n}/2}$

$\phi(\frac{\sqrt{n}}{2}) - \phi(-\frac{\sqrt{n}}{2}) = 0.954$

$\phi(\frac{\sqrt{n}}{2}) - (1 - \phi(\frac{\sqrt{n}}{2})) = 0.954$

$2\phi(\frac{\sqrt{n}}{2}) - 1 = 0.954$

$2\phi(\frac{\sqrt{n}}{2}) = 1.954$

$\phi(\frac{\sqrt{n}}{2}) = 0.9770$

$\frac{\sqrt{n}}{2} = \phi^{-1}(0.977)$

$n = 16$

Even if $X_1, X_2, X_3, \ldots X_n$ follows some other distribution, $\bar{X}$ follows Normal distributis $n \to \infty$

Central Limit Theorem:

Let $X_1, X_2, \ldots X_n$ denote a random sample of size n from X having a distribution which has has mean μ and variance σ^2. Then $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ or $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$ or $Y = \frac{\Sigma X_i - n\mu}{\sigma\sqrt{n}}$

or

$$Y = \frac{\Sigma X_i - n\mu}{\sigma\sqrt{n}} \sim N(0,1)$$

or

Then $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ or $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

limiting distribution $N(0,1)$

proof:

$$M_Y(t) = E(e^{tY})$$

$$= E\left(e^{t\left(\frac{\Sigma X_i - n\mu}{\sigma\sqrt{n}}\right)}\right)$$

$$= E\left(e^{\frac{t}{\sigma\sqrt{n}}[(X_1-\mu)+(X_2-\mu)+\ldots+(X_n-\mu)]}\right)$$

$$= E\left(e^{\frac{t}{\sigma\sqrt{n}}(X_1-\mu)}\right) E\left(e^{\frac{t}{\sigma\sqrt{n}}(X_2-\mu)}\right) \ldots E\left(e^{\frac{t}{\sigma\sqrt{n}}(X_n-\mu)}\right)$$

each X_i has same distribution as that of X

$$= \left[E\left(e^{\frac{t}{\sigma\sqrt{n}}(X-\mu)}\right)\right]^n$$ as each X_i has same dist^n as that of X

$$= \left[E\left(1 + \frac{t}{\sigma\sqrt{n}}(X-\mu) + \frac{t^2}{2!\,\sigma^2 n}(X-\mu)^2 + \ldots\right)\right]^n$$

$$= \left[E(1) + \frac{t}{\sigma\sqrt{n}} E(X-\mu) + \frac{t^2}{2!\,\sigma^2 n} E(X-\mu)^2 + \ldots\right]^n$$

$$= \left[1 + \frac{t}{\sigma\sqrt{n}}(\mu-\mu) + \frac{t^2}{2!\,\sigma^2 n}\sigma^2 + \frac{t^3}{3!}\frac{E(X-\mu)^3}{n\sqrt{n}\,\sigma^3} + \ldots\right]^n$$

apply $n \to \infty$

$$= \left[1 + \frac{t^2}{2n} + O(n^{-3/2})\right]^n = e^{t^2/2} \quad \text{as } n \to \infty$$

Y has limiting dist^n $N(0,1)$. $\boxed{\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a}$

Problem

Q1 Let $\bar{X}$ denote the mean of the ~~random~~ sample of size 100 from $\chi^2_{(150)}$. Find $P(49 < \bar{X} < 51)$

$X \sim \chi^2_{(150)}$ $\chi^2_{(50)}$, Variance $= \cancel{300}\ 100$

mean $= 150$

$\bar{X} \sim N\left(150, \frac{100}{100}\right) \to \bar{X} \sim N(150, 1)$

$Z = \frac{\bar{X} - 150}{1} \sim N(0,1)$

$P(49 < \bar{X} < 51) = P\left(\frac{49-150}{\sqrt{3}} < Z < \frac{51-150}{\sqrt{3}}\right)$

$X \sim \chi^2_{(50)}$

mean $= 50$ Variance $= 100$

$\bar{X} \sim N\left(50, \frac{100}{100}\right)$

$\bar{X} \sim N(50, 1)$

$Z = \frac{\bar{X} - 50}{1}$

$P(49 < \bar{X} < 51) = P\left(\frac{49-50}{1} < Z < \frac{51-1}{1}\right)$

$= P(-1 < Z < 1)$

$= \phi(1) - \phi(-1) = 2\phi(1) - 1$

$= 0.68268$

Q2 Let $\bar{X}$ denote the mean of random sample of size 128 from gama distⁿ with $\gamma = 2$ and $\alpha = 1/4$.
Find $P(7 < \bar{X} < 9)$

Solⁿ $E(X) = \frac{\gamma}{\alpha} = 8$ $\quad V(X) = \frac{\gamma}{\alpha^2} = 32$

$X \sim G(8, 32)$

$\bar{X} \sim N\left(8, \frac{32}{128}\right)$ $\quad Z = \frac{X-8}{\sqrt{\frac{32}{128}}} = \frac{X-8}{\sqrt{\frac{1}{4}}}$

$= \frac{X-8}{0.5}$

$P(7 < \bar{X} < 9) = P\left(\frac{7-8}{0.5} < Z < \frac{9-8}{0.5}\right)$

$= P(-2 < Z < 2) = \phi(2) - \phi(-2) = 2\phi(2) - 1$
$= 0.954$

Q3 Suppose X_J, $J = 1, 2, 3, 4, \ldots 50$ having poisson distribution with $\lambda = 0.03$.
Let $S = X_1 + X_2 + X_3 + \ldots + X_{50}$. Find $(S \geq 3)$

Solⁿ $X \sim P(0.03, 0.03)$ $\quad$ mean = variance = 0.03

We known $Y = \frac{\sum X_i - n\mu}{\sigma\sqrt{n}} \sim N(0, 1)$

$Y = \frac{S - 50(0.03)}{\sqrt{0.03}\sqrt{50}} = \frac{S - 1.5}{1.224}$

$P(S > 3) = P\left(\frac{S-1.5}{1.224} \geq \frac{3-1.5}{1.224}\right)$

$= 1 - \phi(1.224) = \cancel{1.1123}$ 0.1123

Q4 A computer in adding numbers rounds each number to its nearest integer. Suppose that total all rounding errors are independent & uniformly distributed over $(-0.5, 0.5)$

i) If 1500 numbers are added, What is the prob that the magnitude of the total error exceeds 15?

ii) How many numbers may be added together so that magnitude of total error is less than 10 with prob 0.9

Sol^n $f(x) = \begin{cases} 1 ; & -0.5 < x < 0.5 \\ 0 ; & \text{otherwise} \end{cases}$

$$E(X) = \int_{-0.5}^{0.5} 1 \cdot x \, dx$$

$$= \left. \frac{x^2}{2} \right|_{-0.5}^{0.5}$$

$$= \frac{(0.5)^2 - (-0.5)^2}{2}$$

$$= 0$$

$$\mu = 0$$

$$E(X^2) = \int_{-0.5}^{0.5} 1 \cdot x^2 \, dx$$

$$= \left. \frac{x^3}{3} \right|_{-0.5}^{0.5}$$

$$= \frac{(0.5)^3 - (-0.5)^3}{3}$$

$$= \frac{(0.5)^3 \times 2}{3}$$

$$= 1/12$$

$$V(X) = 1/12 - 0 = 1/12$$

i) Total error

$$S = \sum_{i=1}^{150} X_i$$

$$Y = \frac{\sum X_i - n\mu}{\sigma\sqrt{n}} \sim N(0, 1)$$

$$= \frac{S - 1500 \times 0}{\frac{1}{\sqrt{12}} \times \sqrt{1500}} = \frac{S}{11.18} \sim N(0, 1)$$

Req probability $= P(|S| > 15)$

$= 1 - P(|S| < 15)$

$= 1 - P(-15 < S < 15)$

$= 1 - P\left(\frac{-15-0}{11.18} < Y < \frac{15-0}{11.18}\right)$

$= 1 - P(-1.3417 < Y < 1.3417)$

$= \cancel{1-P}$

$1 - [\phi(1.34) - \phi(-1.34)]$

$= 1 - [0.9099 - 0.0901]$

$= 0.18024$

ii) $P(|S| < 10) = 0.9$ $\qquad Y = \frac{\sum X_i - n\mu}{\sigma\sqrt{n}}$

$P(-10 < S < 10) = 0.9$ $\qquad \sim N(0,1)$

$P\left(\frac{-10}{\sqrt{\frac{n}{12}}} < Y < \frac{10}{\sqrt{\frac{n}{12}}}\right) = 0.9$ $\qquad Y = \frac{S-0}{\frac{1}{\sqrt{12}}\sqrt{n}}$

$\phi\left(\frac{10}{\sqrt{\frac{n}{12}}}\right) - \phi\left(\frac{-10}{\sqrt{\frac{n}{12}}}\right) = 0.9$

$2\phi\left(\frac{\sqrt{12}}{\sqrt{n}} \cdot 10\right) - 1 = 0.9$

$\phi\left(\frac{10\sqrt{12}}{\sqrt{n}}\right) = \frac{1.9}{2} = 0.95$

$\frac{10\sqrt{12}}{\sqrt{n}} = \phi^{-1}(0.95) = 1.65$

$n = 440.77$

Q5 A distⁿ with unknown μ and $\sigma^2 = 1.5$ is considered. How large a sample should be taken from the distⁿ in order that prob will be 0.95 that the sample mean will be reaching 0.5 of the population mean

Solⁿ

$$P(\mu - 0.5 < \bar{X} < \mu + 0.5) = 0.95$$

$$X \sim N(\mu, 1.5)$$

$$\bar{X} \sim N\left(\mu, \frac{1.5}{n}\right)$$

$$Z = \frac{\bar{X} - \mu}{\sqrt{\frac{1.5}{n}}} \sim N(0,1)$$

$$P(\mu - 0.5 < \bar{X} < \mu + 0.5) = 0.95$$

$$P\left(\frac{\mu - 0.5 - \mu}{\sqrt{\frac{1.5}{n}}} < Z < \frac{\mu + 0.5 - \mu}{\sqrt{\frac{1.5}{n}}}\right) = 0.95$$

$$P\left(\frac{-0.5\sqrt{n}}{\sqrt{1.5}} < Z < \frac{0.5\sqrt{n}}{\sqrt{1.5}}\right) = 0.95$$

$$2\phi\left(\frac{0.5\sqrt{n}}{\sqrt{1.5}}\right) = 1 + 0.95 = 1.95$$

$$\frac{0.5\sqrt{n}}{\sqrt{1.5}} = \phi^{-1}(0.975)$$

$$n = 23.04$$

Q6 Let $X_1, X_2, X_3 \ldots X_{25}$.
$Y_1, Y_2, Y_3 \ldots Y_{25}$.
be 2 independent random samples from the distribution $N(3, 16)$ and $N(4, 9)$ respectively. Evaluate $P\left(\frac{\bar{X}}{\bar{Y}} > 1\right)$

Sol^n

$P\left(\frac{\bar{X}}{\bar{Y}} > 1\right) = P(\bar{X} > \bar{Y})$

$= P(\bar{X} - \bar{Y} > 0)$

$X \sim N(3, 16)$

$\bar{X} \sim N\left(3, \frac{16}{25}\right)$

$Y \sim N(4, 9)$

$\bar{Y} \sim N\left(4, \frac{9}{25}\right)$

$\bar{X} - \bar{Y} \sim N\left(3-4, \frac{16}{25} + \frac{9}{25}\right)$

$\bar{X} - \bar{Y} \sim N(-1, 1)$

$Z = \frac{\bar{X} - \bar{Y} + 1}{1} \sim N(0, 1)$

$P(Z > \frac{0+1}{1})$

$= P(Z > 1)$

$= 1 - P(Z < 1)$

$= 1 - \phi(1)$

$= 0.158$

Interval Estimation:

Let X be a r.v with some distribution depending on an unknown parameter θ.

An estimate of θ can be given by 2 magnitudes within which θ can lie is called an interval estimate of parameter θ

The process of obtaining an interval estimate of θ is called interval estimation.

An interval with one or both of its endpoints as random variable is called random interval

Eg: $G(X_1, X_2, X_3 \ldots X_n) < \theta < H(X_1, X_2 \ldots X_n)$

Confidence coefficient and confidence interval:

Let θ be an unknown parameter to be determined by a random sample $(X_1, X_2, X_3, X_4 \ldots X_n)$ of size n.

The confidence interval for θ is a random interval containg θ with high probability say $(1-\alpha)$

Here $(1-\alpha)$ is called as confidence coefficient

If $P(a(x_1, x_2, \ldots x_n) < \theta < H(x_1, x_2, x_3 \ldots x_n)) = 1-\alpha$

then

$(a(x_1, x_2, x_3 \ldots x_n), H(x_1, x_2, x_3 \ldots x_n))$

is called an $(1-\alpha)$ 100% confidence interval for θ

Note: Let $X_1, X_2, X_3, X_4 \ldots X_n$ be a random sample from $N(\mu, \sigma^2)$.

i) $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ for μ where σ^2 is known.

ii) $T = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n-1}}} \sim T_{(n-1)}$ for μ when σ^2 is unknown

iii) $Y = \frac{\sum (x_i - \mu)^2}{\sigma^2} \sim \chi^2_{(n)}$ for σ^2 when μ is known

iv) $Y = \frac{nS^2}{\sigma^2} \sim \chi^2_{(n-1)}$ for σ^2 when μ is unknown

Confidence interval for mean μ when σ^2 is known.

Let $X_1, X_2, X_3 \ldots X_n$ be a random sample of size n from $N(\mu, \sigma^2)$. Then

$$\bar{X} = \frac{\sum X_i}{n} \sim \left(\mu, \frac{\sigma^2}{n}\right)$$

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

Let confidence coefficient be $1-\alpha$

find 'a' such that $P(-a < Z < a) = 1-\alpha$

$$P(-a < Z < a) = 1-\alpha$$

$$P\left(-a < \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} < a\right) = 1-\alpha$$

$$P\left(\frac{-a\sigma}{\sqrt{n}} < \bar{X}-\mu < \frac{a\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$P\left(-\bar{X}-\frac{a\sigma}{\sqrt{n}} < -\mu < \frac{a\sigma}{\sqrt{n}} - \bar{X}\right) = 1-\alpha$$

$$P\left(\bar{X}+\frac{a\sigma}{\sqrt{n}} > \mu > -\frac{a\sigma}{\sqrt{n}} + \bar{X}\right) = 1-\alpha$$

$$P\left(\bar{X}-\frac{a\sigma}{\sqrt{n}} < \mu < \bar{X}+\frac{a\sigma}{\sqrt{n}}\right) = 1-\alpha$$

confidence interval for μ, when an observed sample mean $\bar{X}$, variance σ^2 and sample size n is given

Problems

Q1 Let observed value of $\bar{X}$, mean of random sample of size 20 from the distribution $N(\mu, 80)$ be 81.2. Find a 95% confidence interval for μ

Soln $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

find 'a' St $P(-a < Z < a) = 0.95$

$$2\phi(a) - 1 = 0.95$$

$$\phi(a) = \frac{1.95}{2} = 0.975$$

$$a = \phi^{-1}(0.975)$$

$$a = 1.96$$

$$P\left(-a < \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} < a\right) = 0.95$$

$$P\left(-1.96 < \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} < a\right) = 0.95$$

$$P\left(-1.96\frac{(\sqrt{80})}{\sqrt{20}} < \bar{X}-\mu < 1.96\frac{\sqrt{80}}{\sqrt{20}}\right) = 0.95$$

$$P\left(-81.2 - \frac{1.96\sqrt{80}}{\sqrt{20}} < -\mu < -81.2 + \frac{1.96\sqrt{80}}{\sqrt{20}}\right) = 0.95$$

So the interval is $(77.82, 85.12)$

or $\left(\bar{X} - \frac{a\sigma}{\sqrt{n}}, \bar{X} + \frac{a\sigma}{\sqrt{n}}\right)$

$\bar{X} = 81.2$
$\sigma = \sqrt{80}$
$n = 20$

Q2 A sample of size 25 yields $\bar{X} = 78.3$ Obtain a 99.6 % confidence interval for μ.

Sol^n $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$P(-a < Z < a) = 0.996$

$2\phi(a) = 1.996$

$\phi(a) = 0.998$

$a = \phi^{-1}(0.998) = 2.88$

$\bar{X} = 78.3$, $n = 25$, $\sigma^2 = 4$ $a = 2.88$

Interval is $\left(\bar{X} - \frac{a\sigma}{\sqrt{n}} \quad \bar{X} + \frac{a\sigma}{\sqrt{n}}\right)$

$= \left(78.3 - \frac{2.88 \times 2}{5}, 78.3 + \frac{2.88 \times 2}{5}\right)$

$\mu \in = (77.14, 79.452)$

Q3 Let $\bar{X}$ be the mean of random sample of size n from the distribution $N(\mu, 9)$. Find n such that $P(\bar{X}-1 < \mu < \bar{X}+1) = 0.9$

$P(\bar{X}-1 < \mu < \bar{X}+1) = 0.9$

$P(-\bar{X}+1 > -\mu > -\bar{X}-1) = 0.9$

$P\left(\frac{\sqrt{n}}{\sigma} > \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} > \frac{-\sqrt{n}}{\sigma}\right) = 0.9$

$= P\left(\frac{-\sqrt{n}}{3} < Z < \frac{\sqrt{n}}{3}\right) = 0.9$

$2\phi(\sqrt{n}/3) = 1.9$

$\phi \frac{\sqrt{n}}{3} = 0.95$

$n = 24.5 \approx 25$

OR

$P\left(\bar{X} - \frac{a\sigma}{\sqrt{n}} < \mu < \bar{X} + \frac{a\sigma}{\sqrt{n}}\right) = 0.9$

$\therefore P(\bar{X}-1 < \mu < \bar{X}+1) = 0.9$

$\frac{a\sigma}{\sqrt{n}} = 1$ | $a = \phi^{-1}(\ )$, $a = 1.65$

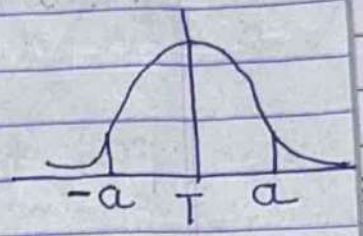
$\sqrt{n} = a\sigma$

Confidence interval for mean μ when σ^2 is unknown:

We known $Y = \frac{\bar{X}-\mu}{S/\sqrt{n-1}} \sim T_{(n-1)}$ $\quad S^2 = \frac{\Sigma(X_i-\bar{X})^2}{n}$

Suppose we want $(1-\alpha)$ 100 percent. Confidence interval for μ. We obtain 'a' such that

$$P(-a < T < a) = 1-\alpha$$



$$P\left(-a < \frac{\bar{X}-\mu}{S/\sqrt{n-1}} < a\right) = 1-\alpha$$

$$P\left(-\frac{aS}{\sqrt{n-1}} - \bar{X} < -\mu < \frac{aS}{\sqrt{n-1}} - X\right) = 1-\alpha$$

$$P\left(X - \frac{aS}{\sqrt{n-1}} < \mu < \bar{X} + \frac{aS}{\sqrt{n-1}}\right) = 1-\alpha$$

confidence interval for μ when σ^2 is unknown is $\left(\bar{X} - \frac{aS}{\sqrt{n-1}}, \bar{X} + \frac{aS}{\sqrt{n-1}}\right)$

When an observed sample mean $\bar{X}$, Sample size n and Sample variance S^2 is given.

Examples Let the random sample of size 17 from the distribution $N(\mu, \sigma^2)$ yields $\bar{X} = 4.7$ and $S^2 = 5.76$. Obtain a 90% confidence interval for mean μ.

Sol^n $Y = \frac{\bar{X}-\mu}{S/\sqrt{n-1}} \sim T_{(n-1)}$

Find 'a' st $P(-a < T < a) = 0.9$

$2\phi(a) = 1.9$

$\phi(a) = 0.95$

$a = \phi^{-1}(0.95) = 1.75$

C.I for $\mu \in \left(\bar{X} - \frac{aS}{\sqrt{n-1}} \quad \bar{X} + \frac{aS}{\sqrt{n-1}}\right)$

$\bar{X} = 4.7$, $a = 1.75$, $S^2 = 5.76$ $\quad n = 17$

$(3.65, 5.75)$

Q2 A random sample of size 15 for $N(\mu, \sigma^2)$ yields $\bar{X} = 3.2$ and $s^2 = 4.24$. Construct a 95% confidence interval for μ.

Sol^n $Y = \dfrac{\bar{X} - \mu}{s/\sqrt{n-1}} \sim T_{(n-1)}$

$$P(-a < T < a) = 0.95$$
$$2\phi(a) = 1.95$$
$$\phi(a) = 0.975$$
$$a = \phi^{-1}(0.975)$$
$$a = 2.14$$

95% C.I for $\mu = \left(\bar{X} - \dfrac{as}{\sqrt{n-1}}, \bar{X} + \dfrac{as}{\sqrt{n-1}}\right)$

$\bar{X} = 3.2$ $\quad s^2 = 4.24$ $\quad n = 15$ $\quad a = 2.14$

$(2.02, 4.38)$

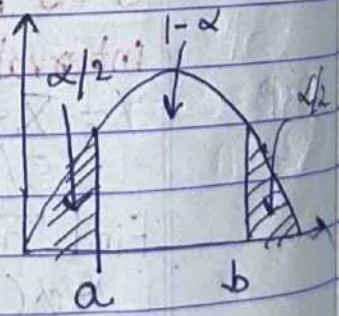
Confidence interval for σ^2 when μ is known:

$$Y = \frac{\sum (X_i - \mu)^2}{\sigma^2} \sim \chi^2_{(n)}$$

Suppose we want to find $(1-\alpha)100$ percent confidence interval for σ^2

Find a and b such that

$$P(a < Y < b) = 1 - \alpha$$



$$P(Y < a) = \frac{\alpha}{2}$$

$$P(Y < a) = \frac{\alpha}{2}$$

$$P(Y > b) = \frac{\alpha}{2} \qquad P(Y < b) = 1 - \frac{\alpha}{2}$$

$$P\left(a < \frac{\Sigma(X_i-\mu)^2}{\sigma^2} < b\right) = 1-\alpha$$

$$P\left(\frac{\Sigma(X_i-\mu)^2}{b} < \sigma^2 < \frac{\Sigma(X_i-\mu)^2}{a}\right) = 1-\alpha$$

$(1-\alpha)100$ percent C.I for σ^2 when μ is known $\left(\frac{\Sigma(X_i-\mu)^2}{b}, \frac{\Sigma(X_i-\mu)^2}{a}\right)$

Example 1:

Suppose 8.6, 7.9, 8.3, 6.4, 8.4, 9.8, 7.7, 7.8 and 7.5 are the observed values from the distribution $N(8, \sigma^2)$. Construct 90% confidence interval for σ^2

Sol^n: $Y = \frac{\Sigma(X_i-\mu)^2}{\sigma^2} \sim \chi^2(n)$ $\quad \mu = 8$

$\Sigma(X_i-\mu)^2 = 6.8$ ✓

here $1-\alpha = 0.9$

Find a and b st $P(a<Y<b) = 0.9$

$\alpha = 0.1$

or $P(Y<a) = \frac{0.1}{2}$ $\quad a = 3.33$

or $P(Y<b) = 1 - \frac{0.1}{2} = 0.95$ $\quad b = 16.9$

C.I for $\sigma^2 = \left(\frac{\Sigma(X_i-\mu)^2}{b}, \frac{\Sigma(X_i-\mu)^2}{a}\right)$

$$\sigma^2 \in \left(\frac{6.8}{16.9}, \frac{6.8}{3.33}\right)$$

$$\sigma^2 \in (0.4, 2.04)$$

Confidence interval for σ^2 when μ is unknown

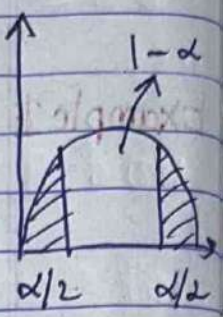
We know $Y = \frac{nS^2}{\sigma^2} \sim \chi^2_{n-1}$ $\quad S^2 = \frac{\Sigma(X_i - \bar{X})^2}{n}$

Suppose we want $(1-\alpha)$ 100 percent confidence interval for σ^2.

Find a and b such that $1-\alpha$

$$P(a < Y < b) = 1-\alpha$$

i.e find a and b satisfying



$$P(Y < a) = P(Y > b) = \frac{\alpha}{2}$$

$$P\left(a < \frac{nS^2}{\sigma} < b\right) = 1-\alpha$$

$$P\left(\frac{nS^2}{b} < \sigma^2 < \frac{nS^2}{a}\right) = 1-\alpha$$

CI for σ^2 is $\left(\frac{nS^2}{b}, \frac{nS^2}{a}\right)$

Q1 A random sample of size 15 yield $\bar{X} = 3.2$ and $S^2 = 4.24$. Construct a 90% confidence interval for σ^2.

Sol^n $Y = \frac{nS^2}{\sigma} \sim \chi^2_{n-1}$

$P(a < Y < b) = 0.9$ $\qquad 1-\alpha = 0.9$, $n = 15$

$P(Y < a) = \frac{0.1}{2} \Rightarrow a = 6.57$

$P(Y < b) = 1 - \frac{0.1}{2} \Rightarrow b = 23.7$

CI for σ^2 is $\left(\frac{nS^2}{b}, \frac{nS^2}{a}\right) = (2.68, 9.68)$

Q2 A random sample of size 9 from $N(\mu, \sigma^2)$ yield $S^2 = 7.63$. Determine 95% confidence interval for σ^2.

Sol^n $Y = \frac{nS^2}{\sigma^2} \sim \chi^2_{(n-1)}$ $\quad 1-\alpha = 0.95$, $n = 9$

$P(Y < a) = \frac{0.05}{2} \Rightarrow a = 2.18$

$P(Y < b) = 1 - \frac{0.05}{2} \Rightarrow b = 17.5$

CI $\sigma^2 = \left(\frac{nS^2}{b}, \frac{nS^2}{a}\right) = (3.92, 31.5)$

$\bar{X} \to \mu \qquad S^2 \to \sigma^2$

Point Estimation:

Let X be a random variable with some probability distribution depending on an unknown parameter θ.

Let $X_1, X_2, X_3 \ldots X_n$ be a random sample of size n from X.

If $g(X_1, X_2, X_3, \ldots X_n)$ is a function of the sample to be used for estimating the parameter θ, then 'g' is called point estimation θ

The value which 'g' assumes i.e, $g(x_1, x_2, x_3 \ldots x_n)$ is called point estimate of θ and we write

$\hat{\theta} = g(x_1, x_2, x_3 \ldots x_n)$

Let $\hat{\theta}$ be estimate for the unknown parameter θ, associated with a distribution of r.v X.

Then $\hat{\theta}$ is called an unbiased estimate of θ if $E(\hat{\theta}) = \theta$ for all θ

We say that $\hat{\theta}$ is $\boxed{\text{consistent estimates}}$ of θ,
if $\lim_{n\to\infty} P(|\hat{\theta}-\theta|>\varepsilon)=0$, $\forall \epsilon>0$
or

$$\lim_{n\to\infty} P(|\hat{\theta}-\theta|\leq\varepsilon)=1 \quad \forall \epsilon>0.$$

Theorem: let $\hat{\theta}$ be an estimate of θ based on the sample of size n if $\lim_{n\to\infty} E(\hat{\theta})=\theta$ and $\lim_{n\to\infty} V(\hat{\theta})=0$, then $\hat{\theta}$ is a consistent estimate of θ.

Q1 Show that sample mean is an unbiased and consistent estimate of population mean.

Sol^n: Let $\bar{X}=\frac{\sum X_i}{n}$

To prove $E(\bar{X})=\mu$ and $\lim_{n\to\infty} V(\bar{X})=0$

$$E(\bar{X})=E\left(\frac{1}{n}\sum X_i\right)=\frac{1}{n}E(X_1+X_2+X_3+\dots+X_n)$$

$$=\frac{\mu+\mu+\mu\dots+\mu}{n} \quad [n \text{ times}]$$

$$=\frac{\mu n}{n}=\mu,$$

X is unbiased estimate μ

$$V(\bar{X}) = V\left(\frac{\sum X_i}{n}\right) = \frac{1}{n^2} V(X_1 + X_2 + X_3 \ldots + X_n)$$

$$= \frac{1}{n^2}(\sigma^2 + \sigma^2 + \sigma^2 + \ldots + \sigma^2) \quad [n \text{ times}]$$

$$= \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

$$\lim_{n \to \infty} V(\bar{X}) = 0$$

$\bar{X}$ is consistent estimate of μ

Q2 Show that the sample variance s^2 is not an unbiased estimate of population variance.

Sol^n Let σ^2 be the population variance

Let s^2 be the sample variance

$$s^2 = \frac{\sum (X_i - \bar{X})^2}{n}$$

To prove $E(s^2) \neq \sigma^2$

$$E(s^2) = E\left(\frac{\sum (X_i - \bar{X})^2}{n}\right) = E\left(\frac{1}{n} \sum (X_i^2 + \bar{X}^2 - 2X_i\bar{X})\right)$$

$$= E\left(\frac{1}{n}\left[\sum X_i + n\bar{X}^2 - 2\bar{X}n\bar{X}\right]\right)$$

$$= E\left(\frac{1}{n}\left[\sum X_i^2 - n\bar{X}^2\right]\right)$$

$$= E\left(\frac{\sum X_i}{n} - \bar{X}^2\right) = \frac{1}{n} E(X_1^2 + X_2^2 + X_3^2 + \ldots X_n^2) - E(\bar{X}^2)$$

$X \sim N(\mu, \sigma^2)$

$\bar{X} \sim N(\mu, \sigma^2/n)$

We known $\sigma^2 = E(X^2) - [E(X)]^2$

$E(X^2) = \mu^2 + \sigma^2$

$\Rightarrow V(\bar{X}) = \sigma^2/n$

$E(\bar{X}) = \mu$

$V(\bar{X}) = E(\bar{X}^2) - (E(\bar{X}))^2$

$E(\bar{X}^2) = \frac{\sigma^2}{n} + \mu^2$

$$E(s^2) = \frac{1}{n} E(X_1^2 + X_2^2 + \ldots + X_n^2) - E(\bar{X}^2)$$

$$= \frac{1}{n}(\sigma^2 + \mu^2)^n - \left(\frac{\sigma^2}{n} + \mu^2\right)$$

$$= \sigma^2 - \frac{\sigma^2}{n} \neq \sigma^2 \Rightarrow \text{Not unbiased}$$

Q3 Show that $\bar{X}$ is a random sample of size n, and $f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & ; 0 < x < \infty \\ 0 & ; \text{otherwise} \end{cases}$

is an unbiased estimation of θ and has variance $\frac{\theta^2}{n}$ $(0 < \theta < \infty)$

(Check for consistent estimate)

Sol^n: $E(X) = \int_0^\infty x f(x)\, dx = \int_0^\infty x \cdot \frac{1}{\theta} e^{-x/\theta}\, dx$

$= \frac{1}{\theta}\left[\frac{x\, e^{-x/\theta}}{-1/\theta} - \frac{e^{-x/\theta}}{1/\theta^2}\right]_0^\infty = \theta$

$E(X^2) = \int_0^\infty x^2 \frac{e^{-x/\theta}}{\theta}\, dx =$

$= \frac{1}{\theta}\left\{\frac{x^2 \cdot e^{-x/\theta}}{-1/\theta} - \frac{2x\, e^{-x/\theta}}{1/\theta^2} + \frac{2 e^{-x/\theta}}{-1/\theta^3}\right]_0^\infty$

$= 2\theta^2$

$V(X) = 2\theta^2 - \theta^2 = \theta^2$

$X \sim D(\theta, \theta^2)$

$\bar{X} \sim N\left(\theta, \frac{\theta^2}{n}\right)$

$E(\bar{X}) = \theta \Rightarrow$ unbiased estimate

$V(\bar{X}) = \frac{\theta}{n}$

$\lim_{n \to \infty} (\bar{X}) = 0$ consistent estimate

Q4 Let $X_1, X_2, X_3, X_4 \ldots X_n$ be a random sample from a normal distribution with $\mu = 0$ and variance $\sigma^2 = \theta$, $0 < \theta < \infty$. Show that $\frac{\Sigma X_i^2}{n}$ is an unbiased & consistent estimate of θ.

sol^n

$X \sim (0, \theta)$, $\bar{x} \sim N(0, \theta/n)$

Let $Y = \frac{\Sigma X_i}{n}$

To prove: $E(Y) = \theta$ & $\lim_{n \to \infty} V(Y) = 0$

$$E(Y) = E\left(\frac{\Sigma X_i^2}{n}\right) = \frac{1}{n}(X_1^2 + X_2^2 + \ldots + X_n^2) = \frac{n\theta}{n} = \theta$$

$[V(X) = E(X^2) - (E(X))^2$

$E(X^2) = \theta$ $\Rightarrow$ It is unbiased

$$V(Y) = V\left(\frac{\Sigma X_i^2}{n}\right) = \frac{1}{n^2} V(X_1^2 + X_2^2 + \ldots + X_n^2)$$

$X \sim N(\mu, \sigma^2)$

$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

Or

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

$$K = Z^2 = \left(\frac{X - \mu}{\sigma}\right)^2 \sim \chi^2_{(1)}$$

Given $\mu = 0$, $\sigma^2 = \theta$

$K = Z^2 = \frac{X^2}{\theta} \sim \chi^2_{(1)}$

$E(K = X^2/\theta) = 1$

$\chi^2_{(n)}$

mean $= n$

var $= 2n$

$$V\left(K = \frac{X^2}{\theta}\right) = 2$$

$$V\left(\frac{X^2}{\theta}\right) = 2$$

$$V(X^2) = 2\theta^2$$

$$V(Y) = \frac{1}{n^2} V(X_1^2 + X_2^2 + \dots + X_n^2)$$

$$= \frac{1}{n^2} n(2\theta^2) = \frac{2\theta^2}{n}$$

$$\lim_{n \to \infty} V(Y) = 0$$

consistent estimate of θ

Maximum Likelihood Eliminator [MLE]

Let x be a random variable with a pdf $f(x, \theta)$
Where 'θ' is an unknown parameter.
Let $X_1, X_2, X_3 \ldots X_n$ be a random sample of size(n) from the random variable(X)
Since the random variable are independent therefore the joint pdf of n dim. random variables is :

$$= f(x_1, \theta) \cdot f(x_2, \theta) f(x_3, \theta) \ldots f(x_n, \theta)$$
$$= \prod_{i=1}^{n} f(x_i; \theta)$$

The likelihood function (L) is defined as
$$L = (X_1, X_2, X_3 \ldots X_n) = \prod_{i=1}^{n} f(x_i, \theta)$$

The maximum Likelihood Estimate (MLE) of θ say $(\hat{\theta})$ is the value of θ with maximizes the likelihood function (L) when considered as a function of (θ)

Maxima : $\frac{\partial L}{\partial \theta} = 0$

Q Let $X_1, X_2, X_3 \ldots X_n$ denote the random sample from a random variable (X) with size (n) from a distribution having pdf:

$$f(x_i, \theta) = \begin{cases} \theta^{x_i}(1-\theta)^{1-x_i} & : \theta > 0 \\ 0 & : \text{elsewhere} \end{cases}$$

Find the MLE for θ

$$f(x_i, \theta) = \begin{cases} \theta^{x_i}(1-\theta)^{1-x_i} & ; \theta > 0 \\ 0 & ; \text{elsewhere} \end{cases}$$

$$L = \prod_{i=1}^{n} f(x_i, \theta)$$

$$= \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i}$$

$$= \theta^{x_1}(1-\theta)^{1-x_1} * \theta^{x_2}(1-\theta)^{1-x_2} * \ldots * \theta^{x_n}(1-\theta)^{1-x}$$

$$= \theta^{x_1+x_2+x_3 \ldots x_n}(1-\theta)^{1+1+1+\ldots n \text{ time} - (x_1+x_2+x_3+\ldots x_n)}$$

$$= \theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i}$$

$$= \theta^{\sum_{i=1}^{n} x_i}(1-\theta)^{n-\sum_{i=1}^{n} i}$$

To maximize L:

$$\text{Log } L = \sum_{i=1}^{n} x_i \log\theta + \left(n - \sum_{i=1}^{n} x_i\right)\log(1-\theta)$$

$$\frac{\partial(\text{Log } L)}{d\theta} = \sum_{i=1}^{n} x_i \cdot \frac{1}{\theta} \left(n - \sum_{i=1}^{n} x_i\right) \frac{(-1)}{1-\theta} = 0$$

$$\frac{\Sigma x_i}{\theta} + \frac{\Sigma x_i}{1-\theta} = \frac{n}{1-\theta}$$

$$\frac{\Sigma x_i(1-\theta+\theta)}{\theta(1-\theta)} = \frac{n}{1-\theta}$$

$$\Sigma x_i = n\theta$$

$$\theta = \frac{\Sigma x_i}{n} = \bar{x}$$

random
ble (X)
tion

$\theta > 0$
lsewhere

$(1-\theta)^{1-X_1}$
$+X_2+X_3+\dots X_n)$

$g(1-\theta)$

$1) = 0$
θ

Q. Let $X_1, X_2, X_3 \dots X_n$ be random samples of random variable (x) of size (n), its pdf is :-

$$f(x,\theta) = \begin{cases} \dfrac{\theta^x e^{-\theta}}{x!} & : \theta > 0 \\ 0 & : \text{elsewhere} \end{cases}$$

Find the MLE for θ?

$$f(x,\theta) = \begin{cases} \dfrac{\theta^x e^{-\theta}}{x!} & : \theta > 0 \\ 0 & : \text{elsewhere} \end{cases}$$

$$L = \prod_{i=1}^{n} f(x_i, \theta)$$

$$= \prod_{i=1}^{n} \frac{\theta^{x_i} e^{-\theta}}{x_i!}$$

$$= \frac{\theta^{x_1} e^{-\theta}}{x_1!} * \frac{e^{x_2} e^{-\theta}}{x_2!} * \frac{e^{x_3} e^{-\theta}}{x_3!} \dots \frac{e^{x_n} e^{-\theta}}{x_n!}$$

$$= \frac{\theta^{(x_1+x_2+x_3+\dots x_n)} e^{-\theta} \cdot e^{-\theta} \dots n \text{ times}}{\prod x_i!}$$

$$= \frac{\theta^{\sum x_i} e^{-n\theta}}{\prod (x_i)!}$$

To maximise

$$\log(L) = \sum x_i \log \theta - n\theta \log e - \log(\prod x_i)$$

differentiate with respect θ

$$\frac{\partial L}{\partial \theta} = \frac{\sum x_i}{\theta} - n - 0 = 0$$

$$\frac{\sum x_i}{\theta} = n$$

$$\theta = \frac{\sum x_i}{n} = \bar{X}$$