Introduction to Probability

Random experiment:

An experiment which results in different possible outcomes when repeated under identical conditions

Eg: Tossing a coin, Rolling a dice

Sample Space: The set of all possible outcomes of a random experiment [S]

$S = \{H,H, HT, TH, TT\}$

The elements of S are called sample points.

A Sample space can be finite or infinite

Finite

Sample space = $\{H, T\}$

infinite sample spac
getting 2 conse cutive H

Event is a subset of a sample space.

Eg : $S = \{1, 2, 3, 4, 5, 6\}$

$A = \{2, 4, 6\}$; even number

If S has n elements the S has 2^n Subsets

Mutually exclusievely

Two events A & B are mutually exclusively if they ar disjoint ; Canot occur simultaneously

Example: In tossing of a single coin, head and
→ tail are mutually exclsively

$S = \{1, 2, 3, 4, 5, 6\}$

→ $A = \{2, 4, 6\}$ $B = \{1, 3, 5\}$; Odd even

A & B are mutually exclusive.

if they are disjoint, i.e $A \cap B = O$

→ Equally likely outcome: If all the outcomes of a random experiment have equal chance of Occurence then it is said to be equally likely

fair & not biased

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$E = S = \{1, 2, 3, 4, 5, 6\}$

Then it has 2^n subsets

Mutually exclusively

A & B are mutually exclusive if $A \cap B = \emptyset$

They camot occur simultanously

Example: In tossing of a single coin, head and tail are mutually exclusively

Equally likely outocomes

All the outcomes of a random experiment have equal chance of outcome

eg Tossing a coin $\{T, H\}$ 50 : 50

Exhaustive cases: The total number of possible outcomes of a random experiment is called exhaustive case 2^n

In tossing of a coin 3 times $= 2^3 = 8$

In $\boxed{\text{rolling a dice 2 times}} = 6^2 = 6^2$

The NO. of outcomes favourable to A.

Favourable cases: An outcome is said to be favourable to an event A, if it belongs to 'A'. The NO. of outcome favourable to A is called favourable outcome

$S = \{HH, TT, HT, TH\}$

A: atleast one head,

Then HH, HT, TH are favourable case

Probability [Clasical Approach]

If an event A can occur in m different ways out of a total 'n' ways,
all are equally likely & mutually exclusievly

$$P(A) = \frac{m}{n}$$

Q1 Example: A pair of dice are rolled. What is the probability that
i> Sum is greater than 8
ii> Sum is neither 7 or 11

FC A: Sum is greater than 8
~~EO~~: {36, 45, 46, 54, 55, 56, 63, 64, 66}

$P(A) = \frac{10}{6^2} = \frac{10}{36}$

B: Sum is neither 7 nor 11
$P(B) = 1 - [P(Sum = 7) + P(Sum = 11)]$
$= 28/36$

Q2 From 6 positive & 8 negative numbers
4 numbers are chosen at random,
P(product is possitive)
i> All 4 possitive number
ii> All 4 negative numbers
iii> 2 possitive & 2 negative
Total cases: ${}^{14}C_4$

FC: ${}^{6}C_4 + {}^{8}C_4 + ({}^{6}C_2 \cdot {}^{8}C_2)$

Req probability $= \frac{{}^{6}C_4 + {}^{8}C_4 + [{}^{6}C_2 \cdot {}^{8}C_2]}{{}^{14}C_4} = 0.5044$

~~If~~ Nothis is given in the question go for without replacement

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3 A box contain 6 red, 4 white, 5 black → balls

4 blocks are selected at random without replacement

P(Among the balls drawn Aleast one ball of each colour)

FC: 1^{st} ball red
2^{nd} ball white
3^{rd} black
4^{th} any

$2R, 1W, 1B = {}^6C_2 \cdot {}^4C_1 \cdot {}^5C_1$

$2W\ 1R, 1B = {}^4C_2 \cdot {}^6C_1 \cdot {}^5C_1$

$2B, 1R, 1W = {}^5C_2 \cdot {}^6C_1 \cdot {}^4C_1$

$E.C = {}^{15}C_4 = 1365$

$P = FC/EC = 0.527472527$

Axiomatic Definition:

Axiomatic Defination

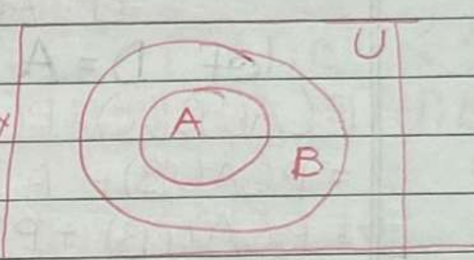
Let S be the sample space & A be an event.

We associate a real number P(A) to the event 'A' called as the probability of A satisfying the following conduction

i> $P(A) \geq 0$
$P(S) = 1$
$P(A \cup B) = P(A) + P(B)$ $P(A \cap B) = 0$ → Kolmogorov's axiom

Result: If $A \subseteq B$, then $P(A) \leq P(B)$.

$B = A \cup (B \cap \bar{A})$
As A & $B \cap \bar{A}$ are mutually exclusively



$P(B) = P(A) + P(B \cap \bar{A})$
any probability ≥ 0
Hence the proff

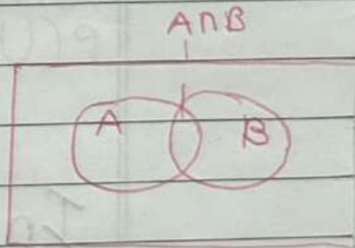
Lecture 02

Addion Law of Probability

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof

We known: $A \cup B$
$A \cup B = A \cup (\bar{A} \cap B)$



Since mutually exclusivel
$P(A \cup B) = P(A) + P(\bar{A} \cap B)$ (i)

$P(B) = (A \cap B) \cup (\bar{A} \cap B)$ (ii)
as $A \cap B$ & $(\bar{A} \cap B)$ are mutually exclusive
$P(B) = P(A \cap B) + P(\bar{A} \cap B)$ (ii)
$P(B) - P(A \cap B) = P(\bar{A} \cap B)$ (iii)
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ [From (i) & (iii)]

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Q1 One number is chosen at random from 1 to 50
Find P(chosen number is divisible by 6 or 8)

Let:
A: number mod 6 = 0
B: number mod 8 = 0

[24, 48]

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 8/50 + 6/50 - 2/50 = 6/25$

Theorem 3 events

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C)$
$+ P(A \cap B \cap C)$

Let $D = A \cup B$

$P(A \cup B \cup C) = P(D \cup C) = P(D) + P(C) - P(D \cap C)$
$= P(A \cup B) + P(C) - P((A \cup B) \cap C)$
$= P(A \cup B) + P(C) - P((A \cap C) \cup (B \cap C))$
$= P(A) + P(B) - P(A \cap B) + P(C) - [P(A \cap C) + P(B \cap C)$
$- P((A \cap C) \cap (B \cap C))]$
$= P(A) + P(B) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Note in General
$P(\cup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + P(A_i \cap A_j \cap A_k)$
$+ \ldots + (-1)^{n-1} (P \cap A_i)$

In general

$P(\cup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k)$
$+ \ldots + (-1)^{n-1} P(\cap A_i)$

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Q3 Given : $P(A) = 3/4$ & $P(B) = 3/8$

To Prove

i) $3/4 \leq P(A \cup B)$

ii) $1/8 \leq P(A \cap B) \leq 3/8$

iii) $3/8 \leq P(A \cap \bar{B}) \leq 5/8$

Solution

i) $A \subseteq A \cup B$

$P(A) \leq P(A \cup B)$

$\frac{3}{4} \leq P(A \cup B)$

ii) $A \cap B \subseteq B$ — Upper bound

$P(A \cap B) \leq P(B)$

$P(A \cap B) \leq 3/8$

$P(A \cup B) \leq 1$

$P(A) + P(B) - P(A \cap B) \leq 1$

$\frac{3}{4} + \frac{3}{8} - 1 \leq P(A \cap B)$

$\frac{1}{8} \leq P(A \cap B)$

i) & ii)

$\frac{1}{8} \leq P(A \cap B) \leq 3/8$

iii) $P(A \cap \bar{B})$

$A \cap \bar{B} \subseteq \bar{B}$

$P(A \cap \bar{B}) \leq P(\bar{B}) = 1 - 3/8 = 5/8$

$P(A \cap \bar{B}) \leq 5/8$

$A = (A \cap \bar{B}) \cup (A \cap B)$

$P(A) = P(A \cap \bar{B}) + P(A \cap B)$

$P(A \cap \bar{B}) = P(A) - P(A \cap B)$

$P(A \cap \bar{B}) = 3/4 - 3/8$ [max]

$P(A \cap \bar{B}) \geq 3/8$

$\frac{3}{8} \leq P(A \cap \bar{B}) \leq 5/8$

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Q4 A bag contains 40 tickets numbered 1,2,3,4... 40
Out of which 4 are drawn at random & are arranged
in ascending order ($T_1 < T_2 < T_3 < T_4$). What is the prob o
T_3 being 25

select 4 ticket & then arrange

25, 1, 16, 30 $1 < 16 < 25 < 30$

$\text{Total} = {}^{40}C_4$

2 tickets from 1 to 24
1 ticket from 26 to 40
select 25

$$\text{Req probability} = \frac{{}^{24}C_2 \times 1 \times {}^{15}C_1}{{}^{40}C_4} = 0.0453003$$

Q5 A problem is given to 3 students A, B and C
whose chances of solving are 1/2, 3/4 & 1/4 re
What is the prob that the problem will be solv
if all of them try independently

Req probability = (1 - P(none of them solve

$= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$

$= \cancel{1/2}\ 1 - [1/2 \times 1/4 \times 3/4]$

$= \frac{29}{32}$

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Q6 Find the probabitiy that birthday of 12 person will fall in 12 different callelor

Req probability $= \frac{12!}{12^{12}}$

Q7 Suppose that 4 digits 1, 2, 3 & 4 are written in random order. What is the prob that at least one digit occupy its proper place

$= 1 - P$(None of the digits occur proper place)

$= 1 - \frac{9}{4!}$

$= 1 - 9/4! = 5/8$

2143
3142
2413

3142
3412
3421

4123
4312
4321

Q A 5 digits number is formed using the digits 0, 1, 2, 3, 4 without repetition

P (the No. formed is divisible by 4

$4 \times 4 \times 3 \times 2 \times 1 = 4 \times 4! = 96$

Last 2 digit is divisible by 4

_ _ _ 04 — 3!

_ _ _ 40 — 3!

_ _ _ 24 → The NO is divisible — 2×2!

_ _ _ 12 — 2×2!

_ _ _ 20 — 3!

_ _ _ 32 — 2×2!

$= \frac{3! + 3! + 2\times2! + 2\times2! + 3! + 2\times2!}{96} = \frac{30}{96}$

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Q9 A person 'A' speaks truth in 75% case & person 'B' is 80%. What is the % of the case where they are contradicting

$$P(A \cap \bar{B}) + P(\bar{A} \cap B) = 3/4 \times 1/5 + 1/4 \times 4/5$$
$$= 7/20 = 1/5$$
$$= 0.35$$

Q10 Each of the 2 persons toss 3 coins. What is the probability that they get same no. of heads

P(Getting 0 head = 1/8
1 head = 3/8
2 head = 3/8
3 heads = 1/8

$P(0 \text{ head or } 1 \text{ head or } 3 \text{ head}) = \frac{1}{8} \times \frac{1}{8} + \frac{3}{8} \times \frac{3}{8} + \frac{3}{8} \times \frac{3}{8} + \frac{1}{8} \times \frac{1}{8} = \frac{5}{16}$

Q11 A box contains tags marked 1, 2, 3, ... n. Two tags are chosen at random. Find the prob that the tags will be consecutive integers if they are chosen

i) Without replacement

ii) With replacement

Total : $n(n-1)$

exhaustive case $\left\{ \begin{matrix} (1,2) & (2\ 3) & (3\ 4) & & (n-1\ \ n) \\ (2,1) & (3\ 2) & \ldots & & (n\ \ n-1) \end{matrix} \right\} \rightarrow 2(n-1)$

$$P = \frac{2(n-1)}{n(n-1)} = \frac{2}{n}$$

ii Total = n^2

$$P(\ ) = \frac{2(n-1)}{n^2}$$

Or Suppose that n digits are written down in random order. What is the prob. that none of the digits occupy its proper position.

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Lecture 03

Q12 Suppose that the each of n men in a party throws his hats in the middle of the room. The hats are mixed up and then each one randomly selects a hat. What is the prob that none of the men select his own hat?

Supposed there are n letters, placed at random in n envelop. Show that the prob of each letter will be placed in wrong envelop is $\sum_{k=2}^{n} (-1)^k \frac{1}{k!}$

hat $A_i \to i^{th}$ person gets his hat $P(A_1) = \frac{1 \times (n-1)!}{n!}$

$P(A_i) = \frac{(n-1)!}{n!} = \frac{1}{n}$

$P(A_i \cap A_j) = \frac{1 \times 1 \times (n-2)!}{n!} = \frac{1}{n(n-1)}$

2 persons getting their own hat

$P(A_1 \cap A_2 \cap \dots \cap A_n) = \frac{1}{n!}$

Req prob = 1 − P(atleast one person gets his own hat)

$= 1 - P(A_1 \cup A_2 \cup \dots \cup A_n)$

$= 1 - \{\sum P(A_i) - \sum P(A_i \cap A_j) + \dots + (-1)^{n-1} P(\cap A_i)\}$

$= 1 - \left\{ {}^nC_1 \frac{1}{n} - {}^nC_2 \frac{1}{n(n-1)} + {}^nC_3 \frac{1}{n(n-1)(n-2)} \dots + (-1)^{n-1} \frac{1}{n!} \right\}$

$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \dots + (-1)^n \frac{1}{n!}$

$= \sum_{k=2}^{n} (-1)^k \frac{1}{k!}$

Q13 A $(2n)$ digit number which starts with 2 and whose digits are all prime numbers are considered at random. Find prob that sum of any 2 adj digits is also a prime

Sol^n: Total cases

$n = 1 = 2\ \underset{2,3,5,7}{\_\_} = 4^1$ $\quad$ $n = 3\ \_\ \_\ \_\ \_\ \_\ \ 4^5$

$n = 2 = 2\ \_\ \_\ \_\ \ 4^3$

Total : 4^{2n-1}

$n = 1$ $\quad 2\ \underline{3}\ (3, 5)$ $\quad$ $n = 3$ $\quad 2\ \underset{3,5}{\_}\ \underset{2}{\_}\ \underset{3,5}{\_}\ \underset{2}{\_}\ \underset{3,5}{\_}\ \ 2^3$

$n = 2$ $\quad 2\ \underset{3,5}{\_}\ \underset{2}{\_}\ \underset{3,5}{\_}\ \ 2^2$

Thus for n, $= 2^n$

Req probability $= \dfrac{2^n}{4^{2n-1}}$

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Q14 Person 'A' can hit a target 3 time in 5 shots
$B \to 2$ times in 5 shots, $C \to 3$ times in 4 shots.
they fire at a target, the prob that 2 shots hit.

$P(A) = 3/5$ $\quad$ $P(B) = 2/5$ $\quad$ $P(C) = 3/4$

P(2 shots) = P(A & B hit, C doesn't
C & B hits, A doesn't
A & C hits, B doesn't)

$= P(A)P(B)P(\bar{C}) + P(A)P(\bar{B})P(C) + P(\bar{A})P(B)P(C)$

$= \frac{3}{5} \times \frac{2}{5} \times \frac{1}{4} + \frac{3}{5} \times \frac{3}{5} \times \frac{3}{4} + \frac{2}{5} \times \frac{2}{5} \times \frac{3}{4} = \frac{9}{20}$

Q15 The coefficient abc of a quadratic eqⁿ
$ax^2 + bx + c$ are obtained by throwing a die
Find P(real)

$b^2 - 4ac \geq 0$

Fix $b = 1$

$1 \geq 4ac$ $\quad$ No favorable cases. $\quad$ 0

No possible case for a, c,

$b = 2$ $\quad$ for (a,c) are {(1,1)} $\quad$ 2

$b = 3$ $\quad$ {(1,1), (1,2) (2,1)} $\quad$ 3

$9 \geq 4ac$

b	1	2	3	4	5	6	
FC to (a,c)	0	1	3	8	14	17	= 43

$b=4$ {(1,1) (1,2) (2,1) (3,1) (1,4) (4,4) (2,2)} $\quad$ 8

b = 4 : C

Req probability = $\frac{43}{6^3}$

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Q.16 Three number are chosen at random without replacement from {1, 2, 3, ..., 10}. Find prob that minimum of the chosen number is 3 $\boxed{\text{or}}$ the maximum of the chosen number is 7

Union law

Sol^n: Total case

A Minimum is 3 P(A) =

Maximum is 7 P(B)

Total cases: ${}^{10}C_3$

$P(A) = \frac{{}^7C_2}{{}^{10}C_3}$

Minimum is 3

3 _ _ 4, 5, 6, 7, 8, 9, 10 ${}^7C_2$

Max (B) = ${}^6C_2$

_ _ 7

1, 2, 3, 4, 5, 6

$P(B) = {}^6C_2 / {}^{10}C_3$

Intersection//

$A \cap B$: 3 number ${}^3C_1$

3 _ 7

4, 5, 6

$A \cup B = P(A) + P(B) - P(A \cap B) = \frac{{}^7C_2 + {}^6C_2 - {}^3C_1}{{}^{10}C_3} = \frac{11}{40}$

We will prove this by Mathematical induction;

Q17 Prove that $P(\bigcap_{i=1}^{n} A_i) \geq \cancel{H/40} \sum_{i=1}^{n} P[A_i] - (n-1)]$

$P(A_1 \cap A_2) = P(_1) + P(A_2) - P(A_1 \cup A_2)$

$\geq P(A_1) + P(A_2) - 1$ [max value] (i)

$\therefore P(A(A_1 \cup P(A_2)) \leq 1$

substitute 2

$P(\bigcap_{i=1}^{2} A_i) \geq \sum_{i=1}^{2} P(A_i) - (2-1)$, Result holds for $n=2$

Assume that result is true for $n=k$ i.e

$P(\bigcap_{i=1}^{k} A_i) \geq \sum_{i=1}^{k} P(A_i) - (k-1)$ (*)

consider $P(\bigcap_{i=1}^{k+1} A_i) = P(\bigcap_{i=1}^{k} A_i \cap A_{k+1})$ Using (i)

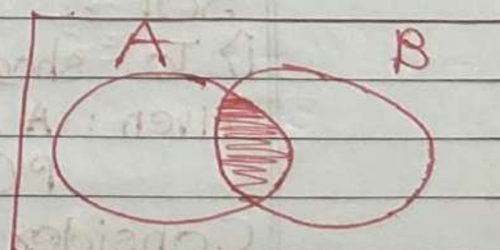
$\geq = P(\bigcap_{i=1}^{k} A_i) + P(A_{k+1}) - 1$

$\geq = \sum_{i=1}^{k+1} P(A_i) - (k+1-1)$

The conditional Probability

The conditional probability of event 'B' given that an event 'A' already happened, is denoted by B(B/A) and is given by

$P(B/A) = \frac{P(A \cap B)}{P(A)}$



$P(A/B) = \frac{P(A \cap B)}{P(B)}$

If A & B are independent events then $P(A \cap B) = P(A) . P(B)$, i.e $P(A/B) = P(A)$

Problems related to conditional probability

If A & B are events with $P(A) = 1/3$, $P(B) = 1/4$, $P(A \cup B) = 1/2$, then

$$P(A \cap B) = \frac{1}{12}$$

i) $P(A|B) = \frac{1/12}{1/4} = 1/3$

ii) $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/12}{1/3} = 1/4$

iii) $P(A \cap \bar{B}) = P(A) - P(A \cap B) = 1/4$

iv) $P(A/\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})} = \frac{1/4}{3/4} = \frac{1}{3}$

Lecture 04

Q2 If A & B are independent, then prove that
i) A and $\bar{B}$
ii) $\bar{A}$ and B
iii) $\bar{A}$ and $\bar{B}$ are also independent

Sol^n:
i) To show $P(A \cap \bar{B}) = P(A) \cdot P(\bar{B})$
Given: A & B are independent
$P(A \cap B) = P(A)\, P(B)$
Consider
$A = (A \cap \bar{B}) \cup (A \cap B) = P(A \cap \bar{B}) + P(A \cap B)$
=

$P(A \cap \bar{B}) = P(A) - P(A \cap B)$
$= P(A) - P(A).P(B)$
$= P(A) [1 - P(B)]$
$= P(A) P(\bar{B})$

Thus A & $\bar{B}$ are independent

ii) $B = (B \cap \bar{A}) \cup (A \cap B)$
$P(B) = P(\bar{A} \cap B) + P(A \cap B)$
$P(\bar{A} \cap B) = P(B) - P(A \cap B)$
$= P(B) - P(A) P(B)$
$= P(B) (P(\bar{A}))$

iii) $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$
$= 1 - \{P(A) + P(B) - P(A \cap B)\}$
$= 1 - \{P(A) + P(B) - P(A) P(B)\}$
$= [1 - P(A)] - P(B) [1 - P(A)]$
$= (1 - P(A)) (1 - P(B))$
$= P(\bar{A}) P(\bar{B})$

Q2 Person A & B throw alternatively a pair of dice Person A wins if he gets Sum = 6 before B gets sum = 7 and B wins if he gets sum = 7 before A gets sum = 6. If A begins, find his chances of winning.

Total = 6^2

FC: $\{(2,4), (3,3), (4,2) (1,5), (5,1)\}$

$P(A) = 5/6^2$ $\quad$ $P(\bar{A}) = 31/6$

Sum = 7

FC $= \{(1\,6) (6\,1) (2\,5) (5\,2) (3\,4) (4\,3)\}$

$P(B) = 6/6^2 =$ $\quad$ $P(\bar{B}) = 30/6$

A begins
P(A win) = P(A getting sum = 6 in the first toss) or
P(A getting sum ≠ 6 in first toss & B getting sum ≠ 7 and A getting sum = 6 or...)

$$= P(A) + P(\bar{A}).P(\bar{B}).P(A) + P(\bar{A})P(\bar{B})P(\bar{A})P(\bar{B})P(A)...$$

$$= \frac{5}{36} + \frac{31}{36}\cdot\frac{30}{36}\cdot\frac{5}{36} + \frac{31}{36}\cdot\frac{30}{36}\cdot\frac{31}{36}\cdot\frac{30}{36}\cdot\frac{5}{36}$$

$$= \frac{5}{36}\left(1 + \left(\frac{31\cdot30}{36}\right) + \left(\frac{31}{36}\cdot\frac{36}{36}\right)^2 + \ldots\right)$$

$$= \frac{5}{36}\left(\frac{1}{1-\frac{30}{36}\times\frac{31}{36}}\right) = \frac{30}{\cancel{36}} = \frac{30}{61}$$

If B begins,

$$= \frac{6}{36} + \frac{30}{36}\times\frac{31}{36}\times\frac{6}{36} + \frac{30}{36}\times\frac{31}{36}\times\frac{30}{36}\times\frac{31}{36}\times\frac{6}{36} + \ldots$$

$$= \frac{6}{36}\left[1 + \frac{30\times31}{36^2} + \frac{(30\times31)^2}{36^4} + \ldots\right]$$

$$= \frac{6}{36}\left[\frac{1}{1-\frac{30\times31}{36^2}}\right] = \frac{36}{61}$$

Prove that $P(\bigcap_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} P[A_i] - (n-1)$

Let us assume for $n=2$

LHS

$P(A_1 \cap A_2)$

RHS, $\sum_{i=1}^{n} P(A_i) - (n-1) = \sum_{i=1}^{2} P(A_i) - (2-1)$

$= P(A_1) + P(A_2) - 1$

$\therefore P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1$

Thus true for $n=2$

Now consider $n=k$

$$P(\bigcap_{i=1}^{k} A_i) \geq \sum_{i=1}^{k} P(A_i) - (k-1) \quad *$$

Now lets check for $n=k+1$

$$P(\bigcap_{i=1}^{k+1} A_i) \geq \sum_{i=1}^{k+1} P(A_i) - (k+1-1)$$

$$\therefore P(\bigcap_{i=1}^{k+1} A_i) = P(\bigcap_{i=1}^{k} A_i \cap A_{k+1}) \quad \text{using 1}$$

$$\geq P(\bigcap_{i=1}^{k} A_i) + P(A_{k+1}) - 1 \quad \text{substitution } *$$

$$\geq \sum_{i=1}^{k} P(A_i) - (k-1) + P(A_{k+1}) - 1$$

$$\geq \sum_{i=1}^{k+1} P(A_i) - (k+1-1)$$

pap

Date:

Q4 There are 2 bags. One contains 3
& 4 White balls.
The other bag contains 4 black and
white balls.
A die is rolled.
If 1 or 3 turnsup, Ball is chosen fro
1st bag and if any other number turns
then the ball is chosen from 2nd Bag.

Find probability of chosing black ba

$P(\text{Bag One}) = 1 \text{ or } 3 = 2/6$

∴ P(selecting a black ball)

= P(Getting 1 or 3 on the dice & Selecting a black b
from Ist bag)

or

getting 2 4, 6 on the dice & select
a ball from the 2nd bag

$= \frac{2}{6} \times \frac{3}{7} + \frac{4}{6} \times \frac{4}{7} = \frac{11}{21}$

Q5 Three dice are rolled simultaneously. Let A be the event that sum of the digits shown is 6 and B be the event that all 3 digits are different. Check whether A & B are independant

$P(A) = \frac{10}{6^3}$

F.C to sum = 6 are {(1,1,4), (1,4,1), (4,1,1), (2,2,2), (1,2,3), (2,1,3), (2,3,1), (3,1,2), (3,2,1), (1,3,2)}

F.C to

1st die, 2nd die, 3rd die

$P(B) = \frac{(6 \times 5 \cdot 4)}{6^3}$

$A \cap B$ = Sum = 6 & all 3 are different

{(1,2,3), (2,1,3), (2,3,1), (3,1,2), (3,2,1), (1,3,2)}

$P(A \cap B) = \frac{6}{6^3} = \frac{1}{6^2} = \frac{1}{36}$

If they are independent

$P(A) = \frac{10}{6^3}$ $P(B) = \frac{20}{36}$

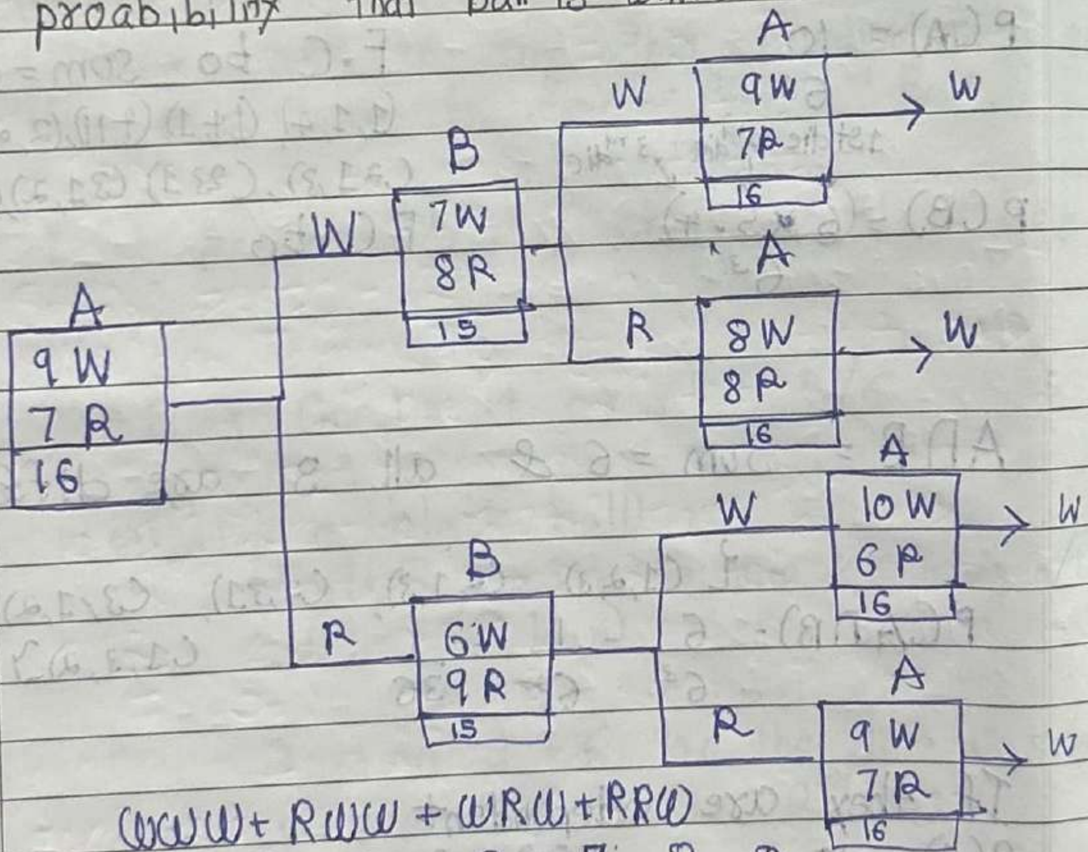
If $P(A \cap B) = P(A) \cdot P(B)$ # independant

But in this cas

$= \frac{10}{6^3} \times \frac{20}{36} = \frac{200}{6^5}$

$\therefore P(A)\, P(B) \neq P(A \cap B)$

Q6 Urn A contains 9 white and 7 red balls. Urn B contains 8 red & 6 white balls. A ball is randomly drawn from urn A and placed in urn B & then a ball is selected from urn A. What is the proabibility that ball is white



WWW + RWW + WRW + RRW

Req prob = $\frac{9}{16} \times \frac{7}{15} \times \frac{9}{16} + \frac{7}{16} \times \frac{8}{15} \times \frac{8}{16} +$

$\frac{7}{16} \times \frac{6}{15} \times \frac{10}{16} + \frac{7}{16} \times \frac{6}{15} \times \frac{9}{16}$

$= \frac{71}{128}$

Q7 Suppose $P(A) = 0.4$

$P(A \cup B) = 0.7$ & $P(B) = P.$

i For what choices of p are A & B mutually exclusive?

ii) For what choices of 'p' are A & B independent

Solution:

i) $P(A \cup B) = p(A) + p(B)$ // mutually exclusive

$0.7 = p(A) + p(B)$

$0.7 = 0.4 + p(B)$

$\therefore p(B) = 0.3$

independent $\rightarrow p(A \cap B)$

ii) $p(A \cup B) = p(A) + p(B) - p(A)\,p(B)$

$0.7 = 0.4 + x - 0.4 \times x$

$0.3 = 1.4\,x$

$x = 0.3$

if it is independent

$p(A \cup B) = p(A) + p(B) - p(A \cap B)$

$p(A \cup B) = p(A) + p(B) - p(A) \cdot p(B)$

$= 0.4 + p(B) - p(B) \times 0.4$

$0.7 = 0.4 + 0.6\,p(B)$

$0.3 = 0.6\,p(B)$

$p(B) = 0.5$

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Q8 Two defective tubes got mixed up with 2 good ones. The tubes are tested one by one until the defectives are found.

i) What is the probability that the last defective tube is obtained in the second test

ii) " third test"

iii) " fourth test"

i) Second test

i) We should find the dective in first go & the the 2^{nd} one in 2^{nd} test

p(1^{st} defective tube in the 1^{st} test
2^{nd} defective tube in 2^{nd} test)

$$= \frac{2}{4} \times \frac{1}{3} = \frac{2}{12} = \frac{1}{6}$$

ii) third test

One deffective & One non deffective selected in first two test
& in 3^{rd} test we will find the 3^{rd} defective

= 1D, 1G, 1D or 1G, 1D, 1, D, ~~1G~~

$$= \left(\frac{2}{4} \times \frac{2}{3} + \frac{2}{4} \times \frac{2}{3}\right) \times \frac{1}{2} = \frac{1}{3}$$

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iii) fourth test

(DGGD + GGDD + GDGD)

$$= \left(\frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} + \frac{2}{4} \times \frac{1}{3} \times \frac{2}{2} \times 1 + \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2}\right) = \frac{1}{2}$$

29 Consider families of n number of childeren.
Let E be an event that the family has both boys and girls
Let B be an event that there is atmost one girl in the family.
Find the values of n for which E & B are Independent assuming p(Boy child) = 1/2

Sol^n

$p(E) =$

How many, boys or girl not given

$$p(E) = 1 - p(\text{all boys or all girls})$$

$$= 1 - \left[\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \cdots + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \cdots\right]$$

$$= 1 - \frac{2}{2^n}$$

$${}^nC_1 \times \frac{1}{2} \times \frac{1}{2^{n-1}}$$

$p(B) =$ at most one girl

$= p$ (0 girl & n boys or 1 girl & (n-1) boys

all n boys   1st girl   B G ...

$$= \frac{1}{2^n} + \frac{1}{2} \times \left(\frac{1}{2}\right)^{n-1} + \cdots \frac{1}{2} \times \frac{1}{2} \times \left(\frac{1}{2}\right)^{n-2} + \cdots \left(\frac{1}{2}\right)^{n-1} \frac{1}{2}$$

$$= \frac{1}{2^n} + \frac{1}{2} \times \left(\frac{1}{2}\right)^{n-1} + \frac{1}{2} \times \frac{1}{2} \times \left(\frac{1}{2}\right)^{n-2} + \cdots \left(\frac{1}{2}\right)^{n-1} \times \frac{1}{2}$$

$$= \frac{1}{2^n} + \frac{n}{2^n} = \frac{n+1}{2^n}$$

$p(E \cap B) = p$ (One girl & $(n-1)$ boys)

$$= \frac{1}{2}\left(\times \frac{1}{2^{n-1}} + \dots\right) n \; \frac{1}{2^{n-1}} \times \frac{1}{2} = \frac{n_{C_1}}{2^n}$$

Now for what value, E & B are independent

$p(E \cap B) = p(E) \times p(B)$

$$\frac{n}{2^n} = \left(1 - \frac{2}{2^n}\right)\left(\frac{n+1}{2^n}\right)$$

$$\frac{\cancel{n} - \cancel{n} - 1}{2^n} = 1 - \frac{2}{2^n}$$

$$\frac{-1}{\cancel{2^n}} = \frac{2^n - 2}{\cancel{2^n}}$$

$$1 = 2^n$$

$$\log 1 = n \log 2$$

$$n =$$

$$\frac{n}{2^n} = \left(1 - \frac{2}{2^n}\right)\left(\frac{n+1}{2^n}\right) =$$

$$2^n n = (2^n - 2)(n+1)$$

$$\cancel{2^n n} = \cancel{2^n n} - 2n + 2^n - 2$$

$$\cancel{2n + 2^n + 2 = 0}$$

$$0 = 2^n - 2n - 2$$

$$\log(2) = n \log(2) - 2 \log(n)$$

$$(1-n) \log 2 =$$

$$\frac{n}{2^n} = \frac{(2^n-2)}{2^n} \cdot \frac{(n+1)}{2^n}$$

$$2^n n = 2^n(n+1) - 2(n+1)$$

$$\cancel{2^n n} = \cancel{2^n n} + 2^n - 2n - 2$$

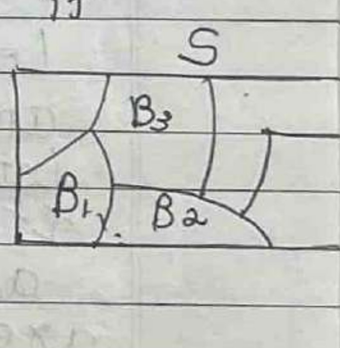
$$0 = 2^n - 2n - 2$$

$$2 = 2^n - 2n \qquad 1 = 2^{n-1} - n \qquad n=1$$

$$n = 3 \qquad 2^{n-1} = n+1 \qquad n=2$$

$$n=3$$

Partition: The event $B_1, B_2, B_3, \dots B_K$ represent a partition of the sample space S if



i) $B_i \cap B_K = \emptyset$ for all $i = j$

ii) $\bigcup_{i=1}^{K} B_i = S$

iii) $P(B_i) \geq 0$ for all i

$S = \{HH, HT, TH, TT\}$

Eg: Getting exactly 2 heads : B_1

Getting exactly 2 tails : B_2

Getting exactly One head or One tail: B_3

Total Probability Theorem

Let A be an event w.r to S and $B_1, B_2, B_3, \dots B_K$ be a partition of S Then

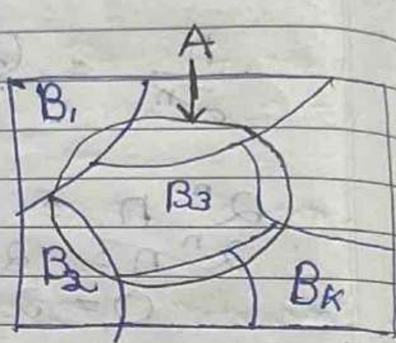
$$P(A) = P(A/B_1)P(B_1) + P(A/B_2)P(B_2) + \dots + P(A/B_K)P(B_K)$$

proof

We know $A = A \cap S$

$A = A \cap (B_1 \cup B_2 \cup B_3 \cup B_4 \ldots \cup B_K)$

$A = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3)$
$\ldots \cup (A \cap B_K)$



These are mutually exclusive

$p(A) = p(A|B_1)\,p(B_1) + p(A|B_2)\,p(B_2) + \ldots + p(A|B_K)\,p(B_K)$

Bayes's Theorem:

Let S be the sample space and A be an event.
Let $B_1, B_2, B_3, B_4, \ldots B_K$ form a portion of S.
If the conditional probabilities $p(A|B_i)$ and $P(B_i)$ $i = 1, 2, 3, 4, 5 \ldots k$ are all known then

$$p(B_i/A) = \frac{P(A|B_i)\,P(B)}{\sum_{i=1}^{K} P(A|B_i)\,P(B_i)}$$

proof:

$$p(B_i|A) = \frac{P(A \cap B_i)}{P(A)} = \frac{P(A|B_i)\,P(B_i)}{P(A)}$$

$$= \frac{P(A|B_i)\,P(B_i)}{\sum_{i=1}^{K} P(A|B_i)\,P(B_i)}$$

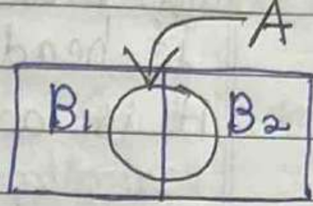
By total prob theorem.

Problems

Q1 Find the prob that a randomly chosen year has 53 Sundays.
If it has 53 sundays, what is the prob that the chosen year is a leap year?

Let

B_1: leap year
B_2: Non leap year
A: has 53 sundays



$p(B_1) = 1/4$ $\quad p(B_2) = 3/4$ $\quad p(A/B_1) = 2/7$

$p(A|B_2) = 1/7$

$p(A) = p(A|B_1) \cdot P(B_1) + P(A/B_2) \cdot P(B_2)$

$= \frac{2}{7} \times \frac{1}{4} + \frac{1}{7} \times \frac{3}{4}$

2 Sunday
{SM, MT, TW, WT, TF, FS SS}
check only 2 extra days in LY

$p(B_1/A) = \frac{p(A/B_1)\ P(B_1)}{P(A)} = \frac{2/7 \times 1/4}{2/7 \times 1/4 + 1/7 \times 3/4} = \frac{2}{28}$ 5

$p(A) = P(A|B_1) \cdot p(B_1) + p(A|B_2)\ p(B_2) = 2/7 \times 1/4 + 1/7 \times 3/4$

$p(B_1|A) = \frac{P(A/B_1)\ P(B_1)}{P(A)} = \frac{2/7 \times 1/4}{2/7 \times 1/4 + 1/7 \times 3/4} = \frac{2}{5}$

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Date: / /

Q2 A bag contain 3 coins. One of which heads and other 2 are normal not biased

A coin is chosen at random from the bag and tossed 4 times continously.
If head appears all 4 times, What prob that it is a 2 headed coin?

B_1: Selecting a 2 headed coin
B_2: normal
A: Getting 4 time head

$p(B_1) = 1/3$
$p(A/B_1) = 1$

$p(B_2) = 2/3$
$p(A/B_2) = 1/2 \times 1/2 \times 1/2 \times 1/2$
$= 1/2^4$

$$p(B_1/A) = \frac{p(A/B_1)\,p(B_1)}{p(A/B_1)\,p(B) + p(A/B_2)\,p(B_2)}$$

$$= \frac{1 \times 1/3}{1 \times 1/3 + 1/2^4 \times 2/3} = \frac{8}{9}$$

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Date: / /

Q3 Box 1 contain 4 black & 5 green balls and Box 2 contain 5 black & 4 green

3 balls are randomly drawn from box 1 without replacement & transfered into box 2. Then a ball is drawn from box 2 and is found to be green.
What is prob that 2 green & 1 black are transfered from box 1 initially

A: Selected ball from box 2 is Green
B_1: Selecting 2 green & 1 black
B_2: remaing posibilite of 3 balls
B_2: 3 green
B_3: 3 black
B_4: Selecting 2 black 1 green

$P(B_1) = \frac{{}^5C_2 \cdot {}^4C_1}{{}^9C_3}$ $\quad P(B_2) = {}^5C_3 / {}^9C_3$

$P(B_3) = \frac{{}^4C_3}{{}^9C_3}$ $\quad P(B_4) = \frac{{}^4C_2 \times {}^5C_1}{{}^9C_3}$

Now 6G & 6B are present

$P(A/B_1) = \frac{{}^6C_1}{{}^{12}C_1}$

5G 7B

$\boxed{P(A/B_4) = 5/12}$

7green, 5black

$P(A/B_2) = \frac{{}^7C_1}{12}$

4G 8 black

$P(A/B_3) = 4/12$

$P(B_1/A) = \frac{P(A/B_1)\, P(B)}{P(A/B_1) P(B_1) + P(A/B_2) P(B_2) + P(A/B_3) P(B_3) + P(A/B_4) P(B_4)}$

$= 60/71$

Date: / /

Q4 It is suspected that a patient has one of the 3 diseases A_1, A_2, A_3. Suppose that population suffering from these illness are in the ratio $2:1:1$

The patient is given a test which turns out to be +ve in 25% of the cases of A_1

50% of cases of A_2
90% of case of A_3

Given that out of 3 tests taken by the patient 2 are +ve, then find prob for each of disease.

Sol^n Let A_i : patient has disease A_i, $i = 1, 2, 3$
B : Out of 3 tests conducted 2 results in +ve

$P(A_1) = 2/4 \qquad P(A_2) = 1/4 \qquad P(A_3) = 1/4$

$P(B/A_1) = P(\text{Out of } 3 \text{ test } 2 \text{ are } +ve)$
$= P(I^{st} +ve, 2^{nd} +ve, 3^{rd} -ve,$
or
$I^{st} +ve, 2^{nd} -ve, 3^{rd} +ve$
or
$I^{st} -ve, 2^{nd} +ve, 3^{rd} +ve)$
$= 3 \times 0.25 \times 0.25 \times 0.75$

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$p(B|A_2) = 3 \times 0.5 \times 0.5 \times 0.5 =$

$p(B/A_3) = 3 \times 0.9 \times 0.9 \times 0.1$

$p(A_1|B) = \dfrac{p(B/A_1)\, p(A_1)}{\sum_{i=1}^{3} p(B|A_i)\, p(A_i)} = \dfrac{375}{1199}$

$p(A_2/B) =$ $= 0.4170$

$= 0.2705$

Q Doctor travel

train : 3/10 — train & late 1/4

Bus : 1/5 — Bus 1/3

car : 1/2 — car 1/2

What is Probability he come by train

$p(T) = 3/10$

$p(B) = 1/5$

$p(C) = 1/2$

L : Arrive late

$p(L/T) = 1/4$

$p(L/B) = 1/3$

$p(L/C) = 1/2$

$$p(T/L) = \frac{p(L/T) \cdot p(T)}{p(L/T) \cdot p(T) + p(L|B) \cdot p(B) + p(L/C) \cdot p(C)}$$

Why are we doing this?
→ mean mode median

$\cancel{P(A) \times P}$ $\frac{3}{6} \times 0.3$

$= \frac{3}{6} \times 0.3 + \frac{1}{6} \times 0.1 + \frac{1}{6} \times 0.1$

$= 0.692307$

$= 0.692$

$= 0.7$

Random Variable

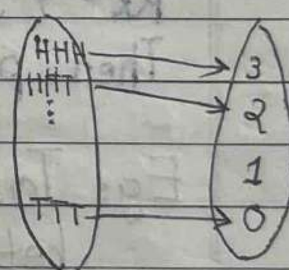
A random variable X is a function which assigns a real number to every sample point $f \in S$

$f : S \to r$

The set of such values are the range space of random variable X, denoted by R_x

Example: A coin is tossed 3 times

S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}



Let X: No of heads

Then $R_x = \{0, 1, 2, 3\}$

X = 0, {TTT}

X = 1, {THT, HTT, TTH}

X = 2, {HHT, HTH, THH}

X = 3 {HHH}

A real No. to every sample point

Example: Roll a dice 2 times

X: Sum of the numbers appeared on the die.

$R_x = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$\{X = 2\} = \{(1,1)\}$

$\{X = 3\} = \{(1,2), (2,1)\}$

$\{X = 4\} = \{ \quad \}$

$\{X = 12\} = \{(6,6)\}$

Types of Random variable:

1) Discrete random Variable

Let X be a r.v. if the number of possible values of X is finite or countably infinite, then X is a discrete random variable.

Probability mass function (p.m.f)

Let X be a discrete r.v with $R_x = \{x_1, x_2, \ldots\}$

Then prob mass function of X is a function $p(x_i) = p(X = x_i)$ satisfying the following condition

i) $p(x_i) \geq 0$

ii) $\sum p(x_i) = 1$

Eg: Tossing a coin 3 Times

let X be a discrete r.V. with $R_x = \{x_1, x_2, x_3, \ldots\}$

Then prob mass function of X is a functio

Eg: Tossing a coin 3 times

Let X: No. of heads

$R_x = \{0, 1, 2, 3\}$

Example : Roll a dice

Continous Random Variable

A r.v is said to be continuous if it can take all possible values between certain limits

Eg $f(x) = x^2 ; 0 < x < 1$
eg. life length of an electrical device

Let X be a continuous r.v. Then pdf of X is a function $f(x)$ satisfying the following condition
X can be -ve
But
i) $f(x) \geq 0$
ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$
iii) $P(a<x<b) = \int_a^b f(x)\,dx$ for any $-\infty < a < b < \infty$

Cumulative distribution function

Let X be a r.v (Discrete or Continuous)
The cdf of X is denoted by $F(x)$ is equal to
$F(x) = P(X \leq x)$

case i) Let X be discrete r.v
If X takes values $x_1, x_2, x_3 \dots x_n$ with respective probabilities $p(x_1), p(x_2) \dots p(x_n)$, then

$$F(x) = \begin{cases} 0 & ; -\infty < x < x_1 \\ p(x_1) & ; x_1 \leq x < x_2 \\ p(x_2) & ; x_2 \leq x < x_3 \\ \vdots & \vdots \\ p(x_1) + p(x_2) + \dots p(x_n) & ; x \geq x_n \end{cases}$$

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Case ii : X is continuous

$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$

properties

1) $F(-\infty) = 0$

2) $F(\infty) = 1$

3) $P(a<x<b) = \int_a^b f(x)dx = \int_{-\infty}^{b} f(x)dx - \int_{-\infty}^{a} f(x)dx$

$= F(b) - F(a)$

4) $f(x) = \frac{d}{dx} F(x)$

Only in continuous

5) $P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b) = P(a < x < b)$

(If X is continuous r.v)

Q1 From a lot of 10 machines containing 3 defectives a sample of 4 is drawn at random without replacement. Let X denote number of defectives. Find p.m.f, $F(x)$, $P(0<x<2)$. Draw the graph of pmf $p(x)$ & cdf (x)

X = No. of defective

X	P(x)
0	1/6
1	1/2
2	3/10
3	1/10
	1

$P(X=0) = P$(out of 10 machine selects 4, in that all 4 are non defective $= {}^7C_4 / {}^{10}C_4$

$P(X=1) = ({}^7C_3 \cdot {}^3C_1) / {}^{10}C_4$

$P(X=2) = ({}^7C_2 \cdot {}^3C_2) / {}^{10}C_4$

$P(X=3) = ({}^7C_1 \cdot {}^3C_3) / {}^{10}C_4$

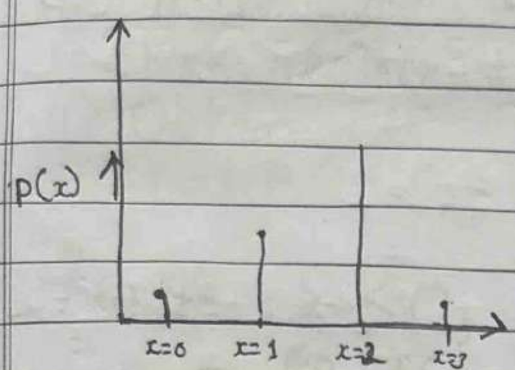
$$F(x) = \begin{cases} 0 & x < 0 \\ 1/6 & 0 \leq x < 1 \\ 2/3 & 1 \leq x < 2 \\ 29/30 & 2 \leq x < 3 \\ 1/1 & x \geq 3 \end{cases}$$

$P(0 < x < 2) = \cancel{1/2}$

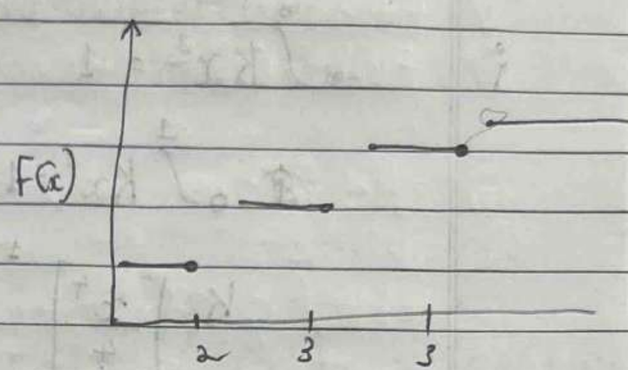
$= P(X = 1)$

$= 1/2$

PMF



CDF F(x)



Q2 Given

x	0	1	2	3	4	5	6
$p(x)$	k	3k	5k	7k	9k	11k	13k

Find k such that the above is a valid pmf.

Find $P(3 < x \leq 6)$; $P(x \geq 5)$; $P(x < 4)$

$\sum P(x) = 1$

$k + 3k + 5k + 7k + 9k + 11k + 13k = 1$

$k = \frac{1}{49}$

$P(3 < x \leq 6) = P(X = 4, 5, 6) = (9 + 11 + 13)k = 39/49$

$P(x \geq 5) = P(x = 5, 6) = (11 + 13)k = 24/49$

$P(x < 4) = P(X = 0, 1, 2, 3) = 16/49$

Q3 Given $f(x) = \begin{cases} Kx^3 & ; \ 0 < x < 1 \\ 0 & ; \ \text{otherwise} \end{cases}$

i) Find K such that $f(x)$ is pdf

ii) find $p(1/4 < x < 3/4)$

iii) find $p(X > 0.8)$

iv) Find CDF

i $\int_{-\infty}^{\infty} Kx^3 = 1$

$\int_{0}^{1} Kx^3 = 1$

$K \left[\frac{x^4}{4}\right]_0^1 = 1$

$K \times \frac{1}{4} = 1$

$K = 4$

ii $p(1/4 < x < 3/4) = \int_{1/4}^{3/4} 4x^3 \, dx = 5/16$

iii) $p(x > 0.8) = \int_{0.8}^{1} 4x^3 \, dx = \frac{369}{625}$

iv $F(x)$

$x < 0, \quad F(x) = 0$

$0 < x < 1 \quad F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{x} f(x)dx$

$= \int_{0}^{x} 4x^3 dx = x^4$

$x \geq 1 \quad F(x) = 1$

$\therefore F(x) = \begin{cases} 0 & ; \ x < 0 \\ x^4 & ; \ 0 \leq x < 1 \\ 1 & ; \ x \geq 1 \end{cases}$

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Now you can also calculate using CDF

$p(1/4 < x < 3/4) = F(3/4) - F(1/4)$

$= (3/4)^4 - (1/4)^4$

Q4 Diameter of an electric cable is assumed to be a continuous r.v. with pdf $f(x) = \begin{cases} 6x(1-x) & ; 0 \le x \le 1 \\ 0 & ; \text{otherwise} \end{cases}$

i) Check whether $f(x)$ is a valid pdf

ii) Obtain CDF

iii) Find $P(X \le 1/2 \mid 1/3 \le X \le 2/3)$

i $\cancel{\int_{-\infty}^{\infty} 6x(1-}$ $x < 0$, $F(x) = 0$

$0 < x < 1$, $F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{0}^{x} 6x(1-x)\,dx$

$= \cancel{\int_{-\infty}^{\infty} f(x)}$

$= \int_{0}^{x} (6x - 6x^2)\,dx$

$= 3x^2 - 2x^3$

$x \ge 1$, $F(x)$, $F(x) = 1$

iii) $P(X \le 1/2 \mid 1/3 \le X \le 2/3)$

$$= \frac{P(X \le 1/2 \cap \frac{1}{3} \le X \le 2/3)}{P(1/3 \le X \le 2/3)}$$

$$= \frac{P(1/3 \le x \le 1/2)}{P(1/3 \le x \le 2/3)}$$

$$= \frac{\int_{1/3}^{1/2} 6x(1-x)\,dx}{\int_{1/3}^{2/3} 6x(1-x)\,dx} = 1/2$$

or

$$\frac{F(1/2) - F(1/3)}{F(2/3) - F(1/3)} = \frac{[3(1/2)^2 - 2(1/2)^3] - [3(1/2)^2 - 2(1/3)^3]}{[3(2/3)^2 - 2(2/3)^3] - [3(1/3)^2 - 2(1/3)^3]}$$

$= 1/2$

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Q5 $f(x) = \begin{cases} ax ; & 0 < x < 1 \\ a & 1 < x < 2 \\ -ax + 3a & 2 < x < 3 \\ 0 & \text{otherwise} \end{cases}$

i) Find a such that $f(x)$ is a valid pdf

ii) Find CDF

i) We know $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (-ax+3a)\,dx = 1$

$\cancel{a\left[\frac{x^2}{2}\right]_0^1 + a\left[ax\right]_1^2 + -a\left[\frac{x^2}{2}\right]_2^3 + 3a\left[x\right]_2^3 = 1}$

$\cancel{a \times \frac{1}{2} + a - a\left[\frac{5}{4}\right] + 3a = 1}$

$\int_0^1 ax + \int_1^2 a\,dx + \int_2^3 (-ax+3a)\,dx = 1$

$a \times \frac{1}{2} + a + \frac{1}{2}a = 1$

$2a = 1$

$a = 1/2$

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$x < 0$, $F(x) = 0$

$x < x < 1$ $\quad F(x) = \int_0^x x/2 \, dx = x^2/4$

$1 < x < 2$ $\quad F(x) = \int_{-\infty}^x f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^x f(x)\,dx$

$= \int_0^1 x/2 \, dx + \int_1^x 1/2 \, dx$

$= (1/2\, x - 1/4)$

$2 < x < 3$ $\quad F(x) = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^x -x/2 + 3/2 \, dx$

$= \int_0^1 x/2\,dx + \int_1^2 1/2\,dx + \int_2^x -\frac{x}{2} + \frac{3}{2}\,dx$

$= \frac{3}{4} + -\frac{1}{2}\left[\frac{x^2}{2}\right]_2^x + \frac{3}{2}\left[x\right]_2^x$

$= \frac{3}{4} - \frac{1}{4}\left[(x^2 - 2^2)\right] + \frac{3}{2}(x-2)$

$= \frac{3}{4} - \frac{1}{4}(x^2 - 4) + \frac{3}{2}(x-2)$

$= \frac{3 - x^2 + 4 + 6(x-2)}{4}$

$= \frac{3 - x^2 + 4 + 6x - 12}{4} = \frac{-x^2 + 6x - 5}{4}$

$$F(x) = \begin{cases} 0 & : \quad x < 0 \\ \frac{x^2}{4} & : \quad 0 < x < 1 \\ 1/2\, x - 1/4 & \quad 1 < x < 2 \\ -\frac{x^2}{4} + \frac{3x}{2} - \frac{5}{4} & \quad 2 < x < 3 \\ 1 & \quad x \geq 3 \end{cases}$$

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Q6 A r.v assume 4 Value with probabil

$\frac{1+3k}{4}, \frac{1-k}{4}, \frac{1+2}{4}, \frac{1-4k}{4},$

For what value of k is prob distn

This does'nt work

$\sum P(x) = \frac{1+3k+1-k+1+2k+1-4k}{4} = 4/4 = 1$

$P(x) \geq 0$

$\frac{1+3k}{4} \geq 0$ & $\frac{1-k}{4} \geq 0, \frac{1+2k}{4} \geq 0, \frac{1-4k}{4} \geq 0$

$K \geq \frac{-1}{3}$ $\quad \frac{K}{9} \leq 1$ $\quad K \geq -1/2$ $\quad K \leq 1/4$

$\therefore \frac{-1}{3} \leq K \leq 1/4$

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Q7 A r.V X has pmf $p(x=k) = \frac{c}{2^k}$, $k=0,1,2$

i) Find c ii) Find cdf

iii) p (X is even) (iv) $p(x \geq 5)$

$$\sum_{k=0}^{\infty} \frac{c}{2^k} = 1$$

$$, c = 1$$

ii)

$x < 0$ $F(x) = 0$

$0 \leq x < 1$ $F(x) = P(x=0) = 1/2$

$1 \leq x < 2$ $F(x) = P(0) + P(x=1) = 1/2 + 1/2^2$

$k \leq x < k+1$ $F(x) = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{k+1}} = \frac{2^{k+1}-1}{2^{k+1}}$

$$F(x) = \begin{cases} 0 & 0 \\ \frac{2^{k+1}-1}{2^{k+1}}, & \cancel{x<0} \; k \leq x < k+1 \end{cases}$$

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Lecture 7:

A random Variable

Probability density function $f(x)$

i) $f(x) \geq 0$

ii) $\int_{-\infty}^{\infty} f(x) = 1$

$P(a < x < b) = \int_a^b f(x)$

Cumulative distn function

CDF $F(x)$

$F(x) = P(X \leq x)$

$F(x) = \sum_{X \leq x} P(x)$ $F(x) = \int_{-\infty}^{x} f(x)\,dx$

Q1 If pmf of a r.v. is given by $P(X=r) = Kr^3$, $r = 1, 2, 3, 4$.

Find $P(1/2 < x < 5/2)$

r	$P(X=r)$
1	K
2	$8K$
3	$27K$
4	$64K$

$\sum P(K=r) = 1$

$K + 8K + 27K + 64K = 1$

$100K = 1$

$K = 1/100$

$P(1/2 < x < 5/2) = P(X=1) + P(X=2)$

$= 0.01 + 0.08$

$= 0.09$

Q2 If $f(x) = \begin{cases} \frac{2x}{a} & ; 0 \leq x \leq 1 \\ 0 & ; \text{Otherwise} \end{cases}$ find a s.t $f(x)$ is a valid pdf

Sol^n:

$$\int_0^1 \frac{2x}{a}\,dx = 1$$

$$\frac{2}{a}\left[\frac{x^2}{2}\right]_0^1 = 1$$

$$\frac{1}{a}[1-0] = 1$$

$$\frac{1}{a} = 1$$

$$a = 1$$

Q3 $F(x) = \begin{cases} 1-(1+x)e^{-x} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$, Find pdf

$$f(x) = \frac{d}{dx}F(x) = \frac{d}{dx}(1-(x+1)e^{-x})$$

$$= -\frac{d}{dx}(xe^{-x} + e^{-x})$$

$$= -\{e^{-x} + -xe^{-x} + -e^{-x}\}$$

$$= xe^{-x}$$

$$f(x) = \begin{cases} xe^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$${}_{a}\int^{b} \frac{1}{b-a}\,dx = 1$$

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Uniform Distribution

A continuous random variable X is said to be uniformly distributed over an interval (a,b) if its pdf is given by $f(x) = \begin{cases} \frac{1}{b-a} & ; a<x<b \\ 0 & \text{otherwise} \end{cases}$

Q1 A r.v X is uniformly distributed over $-1 \leq x \leq 1$
Find $P(|X - 1/2| > 1/4)$

$$f(x) = \begin{cases} \frac{1}{b-a} & ; a<x<b \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{1-(-1)} & -1<x<1 \\ 0 & \text{Otherwise} \end{cases}$$

$$= \begin{cases} 1/2 & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

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$P(|X-1/2| > 1/4) = 1 - P(|X-1/2| < 1/4)$

$= 1 - P(-1/4 < X - 1/2 < 14)$

$= 1 - P(1/4 < X < 3/4)$

$= 1 - \int_{1/4}^{3/4} 1/2 \, dx$

$= 3/4$

Q2 If r.v. K is uniformly distributed over (0, 5). What is prob that the roots of eq^n $4x^2 + 4xK + K + 2 = 0$ are real

$$\text{pdf} = \begin{cases} \frac{1}{5} & : 0 \leq K \leq 5 \\ 0 & : \text{otherwise} \end{cases}$$

$b^2 - 4ac \geq 0$

$(4K)^2 - 4(4)(K+2) \geq 0$

$16K^2 - 16(K+2) \geq 0$

$16K^2 - 16K - 32 \geq 0$

$K^2 - K - 2 \geq 0$

$K^2 - 2K + K - 2$

$K(K-2) + 1(K-2) \geq 0$

$(K+1)(K-2) \geq 0$

$\therefore K \geq -1$ & $K \geq 2$ $\quad \therefore K \geq 2$

$K \leq -1$ & $K \leq 2$ $\quad \therefore K \leq -1$

$P(\text{roots are real}) = P(K \geq 2) + P(K \leq -1)$

$= \int_2^5 1/5 \, dx = 3/5$

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Q3 Suppose X is uniformly distributed over $(-a, a)$ where $a>0$. Whenever possible determine 'a' so that following conditions are satisfied

i> $P(X>1) = 1/3$ ii> $P(X<1) = 1/2$
iii> $P(X<1/2) = 0.7$ iv> $P(|X|<1) = P(|X|>1)$

Sol^n:

$$f(x) = \begin{cases} \frac{1}{2a} & ; -a < x < a \\ 0 & \text{Otherwise} \end{cases}$$

i $P(X>1) = 1/3$

$$\int_1^a \frac{1}{2a}\,dx = \frac{1}{3}$$

$$\frac{1}{2a}[x]_1^a = \frac{1}{3}$$

$$\frac{1}{2a}[a-1] = \frac{1}{3}$$

$$3a-3 = 2a$$

$$a = 3$$

ii> $P(X<1) = 1/2$

$$\int_{-a}^1 \frac{1}{2a}\,dx = \frac{1}{2}$$

$$\frac{1}{\cancel{2}a}[x]_{-a}^1 = \frac{1}{\cancel{2}}$$

$$\frac{1}{a}[1+a] = 1$$

$$1+\cancel{a} = \cancel{a}$$

∴ Ans is not possible

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Q8 iii) $P(X < 1/2) = 0.7$

$$\int_{-a}^{1/2} \frac{1}{2a}\, dx = 0.7$$

$$\frac{1}{2a}[1/2 + a] = 0.7$$

$$\frac{1}{2a}\frac{[1+2a]}{2} = 0.7$$

$$1 + 2a = 2.8a$$

$$1 = 0.8a$$

$$a = \frac{1}{0.8} = \frac{10}{8} = \frac{5}{4}$$

iv) $P(|X|<1) = P(|X|>1)$

$P(|X|<1) = 1 - P(|X|<1)$

$2P(|X|<1) = 1$

$P(|X|<1) = 1/2$

$P(-1<X<1) = \frac{1}{2}$

$$\int_{-1}^{1} \frac{1}{2a}\, dx = \frac{1}{2}$$

$$\frac{1}{a}[1+1] = 1$$

$$\boxed{2 = a}$$

Expected Valve

Let X be discrete r.v with values $X_1, X_2 \ldots X_n$ & respective probabilites $p(X_i)$, $i = 1, 2, \ldots$

The expected value of X denoted by $E(X)$ or μ is defined as

$$E(X) = \sum_{i=1}^{\infty} x_i P(x_i)$$

If X is a continuous r.v with pdf $f(x)$ then

$$E(X) = \int_{-\infty}^{\infty} x . f(x)\, dx$$

Propertie

1) If $x = c$, a constant then $E(X) = C$

$$E(X) = P(x_1) \times C + P(x_2).C + P(x_3).C + \ldots P(x_n) C$$

$$= C[P(x_1) + P(x_2) + P(x_3) + \ldots P(x_n)]$$

$$= C$$

2) $E(cX) = cE(X)$

3) $E(aX+b) = aE(X) + b$

Variance: Variance of a random variable X is denoted by $V(X)$ or σ^2, and is defined as

$$V(X) = E(X - E(X))^2$$
$$= E(X^2 + (E(X))^2 - 2X \cdot E(X))$$
$$= E(X^2) + (E(X))^2 - 2E(X)E(X)$$
$$= E(X^2) - (E(X))^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx \qquad E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

Note: $V(X) \geq 0$

$\sqrt{V(X)}$ is known as standard deviation of X

Properties

1> $V(C) = E(C - E(C))^2 = E(C - C)^2 = E(0)^2$
$= 0$

2> $V(aX+b) = E((aX+b) - E(aX+b))^2$
$= E(aX + \cancel{b} - aE(X) - \cancel{b})$
$= E(a(X - E(X)))^2$
$= a^2 E(X - E(X))^2$
$= a^2 V(X)$

Problem

Q1 A Student takes a multiple choice test containing 2 questions. The first one has 3 choices and the second one has 5 choice. The student chooses answer at random for each of the 2 questions.
Let X denote the NO. of right answer find $E(X)$ & $V(X)$

X	0	1	2
$p(x)$			

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$p(X=0) = P(0 \text{ right answer}) = \frac{{}^{2}C_{1}}{{}^{3}C_{1}} = \frac{2}{3} \times \frac{4}{5}$

$= \frac{8}{15}$

$P(X=1) = P(1 \text{ right answer}) = \frac{1}{3} \times \frac{4}{5} + \frac{2}{3} \times \frac{1}{5}$

$= \frac{6}{15} = \frac{2}{5} = 0.4$

$E(X) = \sum x P(x) = 0 \times \frac{8}{15} + 1 \times \frac{6}{15} \times 2 \times \frac{1}{15} = \frac{8}{15}$

$E(X^2) = \sum x^2 p(x) = 0 \times \frac{8}{15} + 1 \times \frac{6}{15} + 2^2 \times \frac{1}{15}$

$= \frac{2}{3}$

$V(X) = E(X^2) - (E(X))^2 = \frac{2}{3} - \left(\frac{8}{15}\right)^2 = \frac{86}{225}$

Q2 Suppose X is a r.v. with $E(X) = 10$, $V(X) = 25$. For what +ve values of a and b, $Y = aX - b$ has expectation zero and variance one?

Sol^n ~~$Y = aX - b$~~

~~$E(Y) = E(aX - b) - E(b)$~~

~~$E(Y) = E(aX - b) - b$~~

$Y = aX - b$

$E(Y) = E(aX - b) = aE(X) - b = 10a - b$

1) $0 = 10a - b$

~~$V(Y) = V(aX - b) = a^2 E(aX)$~~

$V(Y) = V(aX - b) = a^2 V(X) = a^2 . 25$

$1 = a^2 \times 25$

2) $a = 1/5$

From ① & ② $b = 2$

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Q3 A person enters the game by paying ₹1
Three dice he has to roll.
If 6 appears Once he gets ₹1
twice ₹2
thrice ₹8
otherwise nothing
Is the game fair? [i.e expected gain is zero
If not how much should the person receive in order to make the game fair when he gets 6 thrice.

Soln:

X: Gain from the profit

X	-1	0	1	7
p(X)	$\frac{125}{216}$	75/216	15/216	1/216

$p(-1) = P(\text{getting other than 6 all 3 times})$

$= \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{125}{216}$

$p(0) = \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times 3 = \frac{25 \times 3}{216}$ [One time 6]

$= \frac{75}{216}$

$P(1) = p(2 \text{ times } 6) = \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times 3 = \frac{15}{216}$

$p(7) = p(3 \text{ times } 6) = 1/216$

$$E(X) = \frac{-1}{} \times \frac{125}{216} + 0 \times \frac{75}{216} + 1 \times \frac{15}{216} + 7 \times \frac{1}{216}$$

$$= -\frac{103}{216}$$

$$0 = -1 \times \frac{125}{216} + 0 + \frac{15}{216} + x \times \frac{1}{216}$$

$$0 = -\frac{55}{108} + \frac{x}{216}$$

$$110 = x$$

Thus we should get $(110+1) =$ ₹111 for the game to be fair

Q4 Evaluate $E(X)$ & $V(X)$

X	8	12	16	20	24
$p(X)$	1/8	1/6	3/8	1/4	1/12

Sol^n

$$E(X) = \sum P(x_i)\, x_i = 8 \times 1/8 + 12 \times 1/6 + 16 \times 3/8 + 20 \times 1/4 + 24 \times 1/12$$

$$= 16$$

$$E(X^2) = \sum x^2 p(x) = 276$$

$$V(X) = E(X^2) - (E(X))^2 = 276 - 16^2 = 20$$

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Q5 Suppose that an device has a life leng[th] X (in 1000 hours) which is considered as a continuous r.v with $f(x) = e^{-x}, x>$
Suppose the cost of manufacturing one such item is 2 dollars and the manufactu[rer] sells the item for 5 dollars.
But guarantees a total refund if $X \leq 0$[.9]
What is manufacturer's expected profit or gain?

Sol^n

Let P be the profit $P = \begin{cases} 3 & : X > 0.9 \\ -2 & : X \leq 0.9 \end{cases}$

$E(p) = 3\,P(X > 0.9) - 2\,P(X \leq 0.9)$

$= \cancel{3 \int_{0.9}^{1000}}$

$= 3\left[1 - P(x < 0.9)\right] - 2P(X \leq 0.9)$

$= 3 - 3P(x \leq 0.9) - 2P(X \leq 0.9)$

$= 3 - 5\,P(x \leq 0.9)$

$= 3 - 5\int_0^{0.9} e^{-x}\,dx$

$= 0.0328$

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Q7 A coin is tossed till a first head appears.
Let X denotes the NO. of tosses.
Let $P(H) = p$ & $p(T) = q$
Find $E(X)$ & $V(X)$

Sol^n:

X	1	2	3	....
$p(x)$	p	pq	qqp	

X = NO. of tossed required to get heads

$$E(x) = \sum x_i P(x_i) = p + 2pq + 3q^2p + \dots\dots$$

$$= p(1 + 2q + 3q^2 + \dots\dots)$$

$$(1-x)^{-n} = 1 + nC_1 + {}^{n+1}C_2 x^2 + \dots\dots$$

$$(1-x)^{-1} = 1 + x + x^2 + \dots\dots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2$$

$$= p(1 + 2q + 3q^2 + 4q^3 + \dots\dots)$$

$$= p(1-q)^{-2}$$

$$= p(p)^{-2}$$

$$= 1/p$$

$E(X^2) =$

$= \sum x_i^2 . p(X_i) =$

X	1	2	3	.....	k
$p(X)$	p	qp	q^2p		$q^{k-1}p$

$= \sum X^2 p(x) = \sum x^2 . q^{x-1} p$

$= \sum x(x-1) q^{x-1} p + \sum x q^{x-1} p$

$\rightarrow \frac{1}{p}$

$= p \sum x(x-1) q^{x-1} + 1/p$

$= p \{2q + 6q^2 + 12q^3 + \dots\} + 1/p$

$= 2pq \{1 + 3q + 6q^2 + \dots\}$

$= 2pq \{(1-q)^{-3}\} + \frac{1}{p}$

$= 2pq \times (p)^{-3} + \frac{1}{p} = 2 \times \cancel{p} \times q \times \frac{1}{\cancel{p^3} p^2} + \frac{1}{p}$

$= \frac{2q}{p^2} + \frac{1}{p} = \frac{p+2q}{p^2}$

$V(X) = \frac{2q}{p^2} + \frac{1}{p} - \left(\frac{1}{p}\right)^2 = \frac{2q + 1 - 1}{p^2}$

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$$V(x) = E(X^2) - (E(X))^2$$

$$= \frac{2q+p}{p^2} - \frac{1}{p^2}$$

$$= \frac{2q-(1-p)}{p^2} = \frac{2q-q}{p^2} = \frac{q}{p^2}$$

Q8 Given $f(X) = \begin{cases} x e^{-x^2/2} & ; x > 0 \\ 0 & ; \text{otherwise} \end{cases}$

Find $E(X)$ & $V(X)$:

Sol^n:

$$E(X) = \int_{-\infty}^{\infty} x . f(x) dx$$

$$= \int_{-\infty}^{0} x . x e^{-x^2/2} dx$$

$$= \int_{-\infty}^{0} x^2 e^{-x^2/2} dx$$

$$\frac{x^2}{2} = t \qquad x^2 = 2t$$

$$x = \sqrt{2t}$$

$$\cancel{2} x^2 = 2t$$

$$\cancel{2} x = \cancel{2} \frac{dt}{dx}$$

$$x dx = dt$$

$$\int 2 \times \frac{x}{2} . e^{-x^2/2} x . x dx$$

$$\int 2 \times \frac{x}{2} . e^{-t} dt = \int \sqrt{2t} e^{-t} dt$$

Gama $\int e^{-x} x^{n-1} dx = \Gamma n$

$$\int x^2 . e^{-x^2/2} dx$$

$$\frac{x^2}{2} = t \qquad \frac{x^2}{2} = t$$

$$x^2 = 2t \qquad x^2 = 2t$$

$$2x = 2\frac{dt}{dx} \qquad x = \sqrt{2t}$$

$$x\,dx = dt$$

$$\int \sqrt{2t} . e^{-t} . dt$$

$$\int_0^\infty \sqrt{2} . t^{1/2} . e^{-t} dt$$

Note

$$\int_0^\infty e^{-x} x^{n-1} dx = \Gamma n,$$

$$\Gamma n+1 = n \Gamma n$$

$$\Gamma 1/2 = \sqrt{\pi}$$

$$\sqrt{2} \times \int_0^\infty t^{1/2} e^{-t} dt = \sqrt{2}\ \Gamma 3/2$$

$$= \sqrt{2} \times \frac{1}{2} \times \sqrt{\pi} = \sqrt{\frac{\pi}{2}}$$

$$E(X^2) = \int_0^\infty x^2 \cdot x e^{-x^2/2} \, dx$$

$$= \int_0^\infty x^3 e^{-x^2/2} \, dx$$

$$\frac{x^2}{2} = t$$

$$dt = x \, dx$$

$$= \int 2 \frac{x^2}{2} \cdot e^{-x^2/2} \cdot x \, dx$$

$$= \int 2t \cdot e^{-t} \cdot dt$$

$$= 2 \int_0^\infty t \cdot e^{-t} \, dt = 2 \int_0^\infty e^{-t} t^1 \, dt$$

$$= 2\Gamma 2 = 2 \times 1\Gamma 1 = 2$$

$$V(X) = E(X^2) - (E(X))^2 = 2 - \pi$$

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Chebyshev's Inequality

Let X be a r.v. Then for any +ve real No. ε and a constant c,

$$p(|X-c| \geq \varepsilon) \leq \frac{1}{\varepsilon^2} E(X-c)^2$$

$$p(|X-c| < \varepsilon) \geq 1 - \frac{1}{\varepsilon^2} E(X-c)^2$$

proof : We have $|X-C| \geq \varepsilon$

$$\frac{(X-c)^2}{\varepsilon^2} \geq 1$$

$$p(|X-c| \geq \varepsilon) \leq 1 = \int_{-\infty}^{\infty} f(x)\,dx$$

$$\leq \int_{-\infty}^{\infty} \frac{(x-c)^2}{\varepsilon^2} f(x)\,dx$$

$$\cancel{\leq \int_{-\infty}^{\infty} f(x)\,dx} = \leq \frac{1}{\varepsilon^2} \int_{-\infty}^{\infty} (x-c)^2 f(x)\,dx$$

$$\leq \frac{1}{\varepsilon^2} E[(X-C)^2]$$

$$\therefore p(|X-c| \geq \varepsilon) \leq \frac{1}{\varepsilon^2} E[(X-C)^2]$$

$$\therefore P(|X-c| < \varepsilon) \geq 1 - \frac{1}{\varepsilon^2} E[(X-C)^2]$$

Note:

Substituting $c = \mu$ and $\varepsilon = K$ in chebyshev's inequality we get

$$p(|X-\mu| \geq K) \leq \frac{1}{K^2} E[(X-\mu)]^2 = \frac{\sigma^2}{K^2}$$

$$P(|X-\mu| < K) \geq 1 - \frac{\sigma^2}{K^2}$$

$$p(-K+\mu < X < K+\mu) \geq 1 - \frac{\sigma^2}{K^2}$$

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Q1 Problem

Two dice are rolled. Let X denote sum of the NO. showing up. Verify Chebyshev's in equality for $P(|X-7| \geq 3)$

Sol^n:

X	2	3	4	5	6	7	8	9	10	11	12
P(x)	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

$E(X) = \sum x P(x) = 7$

$E(X^2) = \sum x^2 P(X)$

$V(X) = E(X^2) - [E(X)]^2 = 35/6$

$$P(|X-7| \geq 3) \leq \frac{\sigma^2}{K^2}$$

$$\therefore P(|X-7| \geq 3) \leq \frac{\sigma^2}{K^2} \leq \frac{35/6}{9} \leq 0.648$$

Acctual Ans:

$$P(|X-7| \geq 3) = 1 - P(|X-7| < 3)$$
$$= 1 - P(4 < X < 10)$$
$$= 1 - P(X = 5, 6, 7, 8, 9)$$
$$= 1 - \left\{\frac{24}{36}\right\} = \frac{1}{3} = 0.33$$

$$\therefore \frac{1}{3} < 0.648$$

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Q2 Given $f(x) = \begin{cases} 2e^{-2x} & : x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Verify chebyshev's inequality for obtaining $p(|X-\mu| \geq 1)$

$$E(X) = \int_0^\infty x \cdot 2e^{-2x}\,dx = 2\int_0^\infty x e^{-2x}\,dx = 1/2$$

$$E(X^2) = \int_0^\infty x^2 \cdot 2e^{-2x}\,dx = 2\int_0^\infty x^2 e^{-2x}\,dx$$

$$V(X) = E(X^2) - (E(X))^2 = 1/4$$

$$p(|X - 1/2| \geq 1) \leq \frac{\sigma^2}{k^2} \leq \frac{1/4}{1} = \frac{1}{4}$$

Actual answer

$$P(|X-1/2| \geq 1) = 1 - P(|X - 1/2| < 1)$$
$$= 1 - P(-1/2 < x < 3/2)$$
$$= 1 - \int_0^{3/2} 2e^{-2x}\,dx$$

$$= 1 - 0.95021$$
$$= 0.49787068$$

$\therefore 0.497 < 1/4$

$$\int_0^\infty x e^{-2x}\,dx = x\int_0^\infty e^{-2x} - \int_0^\infty (1) e^{-2x}\,dx$$

put $-2x = u$ $\qquad = \frac{-x(e^{-2x})}{2} + \frac{e^{-2x}}{2}$

$-2 = \frac{du}{dx}$

$dx = \frac{-1}{2}du$

$$= x\int_0 \frac{-1}{2}e^u\,du - \int_0^\infty \frac{-1}{2}e^u\,du = \frac{-x}{2}e^u + \frac{e^u}{2}$$

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Q3 Given $\mu = 10$, $\sigma^2 = 4$.
Find a lower bond for $P(5 \leq X \leq 15)$

$P(|X-10| \geq K) \leq \frac{4}{K^2}$

$P(|X-10| < K) \geq 1 - \frac{4}{K^2}$

$P(10-K < X < 10+K) \geq 1 - \frac{4}{K^2}$

when $K = 5$

$P(5 < X < 15) \geq 1 - \frac{4}{25} \geq \frac{21}{25}$

$P(5 < X < 15) = \frac{21}{25}$

Q3 Given $\mu = 3$, $\sigma^2 = 3/4$. Find a lower bond for $P(2, X < 4)$

$P(|X-3| \geq K) \leq \frac{3/4}{K^2}$

$P(|X-3| < K) \geq 1 - \frac{3}{4K^2}$

$P(3-K < X < 3+K) \geq 1 - \frac{3}{4K^2}$

$K = 1$

$P(2 < X < 4) \geq 1 - \frac{3}{4} \geq \frac{1}{4}$

Q5 Verify Chebyshev's inequality for $K = 2$ &

$f(x) = \begin{cases} 1/4 & : 2 < x < 6 \\ 0 & : \text{Otherwise} \end{cases}$

$E(x) = \int_2^6 1/4 \, x \, dx = 4$

$E(X^2) = \int_2^6 x^2 \, 1/4 \, dx = \frac{52}{3}$

$V(X) = \frac{52}{3} - 16 = 4/3$

$P(|X-4| \geq 2) \leq \frac{4/3}{K^2} \leq \frac{1}{3}$

Actual answer

$P(2 < x < 6)$

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(Q) Two dimensional random variable

Joint prob mass function:

Let (x,y) be a 2 dim r.v. Then $P_{ij} = (X = x_i, Y = y_j)$ is said to be joint prob mass function if $(X = x_i, Y = y_j)$ is said to be joint prob

i) $P_{ij} = P(X = X_i, Y = Y_j)$

ii) $\sum_{ij} P(x_{ij}) = 1$

X\Y	Y_1	Y_2	Y_3	Y_J	$P(x_i)$
x_1					$P(x_1)$
x_2					$P(x_2)$
x_3					
⋮					
x_i			P_{ij}		
$q(y_j)$	$q(y_1)$	$q(y_2)$			1

2D

Marginal Random variable → 1D

Marginal prob dist:

Let X takes the values $x_1, x_2, x_3, x_4 \ldots x_n$ & Y takes the value $Y_1, Y_2, Y_3 \ldots Y_m$

$P(x_i) = P(X = X_i) = P(X = x_i\ Y = Y_1, \text{or } X = X_i, Y = Y_2 \text{ or} \ldots)$

$= \sum_j P(x_i\ Y_j)$

x	$P(x_i)$
x_1	$P(x_1)$
x_2	$P(x_2)$
x_3	⋮
⋮	⋮
x_n	$P(x_n)$

Y	$q(Y_j)$
Y_1	$q(Y_1)$
Y_2	$q(Y_2)$
Y_3	⋮
⋮	⋮
Y_m	$q(Y_m)$

Marginal prob dist^n of Y is $q(Y_j) = \sum_i P(x_i\ Y_j)$

Margin Join Prob density function

Let (x, y) be a continuous r.v assuming all possible values in some region R

The joint pdf $f(x,y)$ is function satisfying

i) $f(x,y) \geq 0$

ii) $\iint_R f(x,y)\,dx\,dy = 1$

iii) $P(a \leq X \leq b,\ c \leq Y \leq d)$

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Marginal PDF

Marginal pdf of X given by

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

Marginal pdf of Y is given by

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Conditional prob of X for a given $Y = y_j$ is given by $P(X_i \mid Y_j)$

$$\frac{P(x_i\, y_j)}{q(y_j)}$$

$$P(Y_j \mid X_i) = \frac{P(x_i\, y_j)}{p(x_i)}$$

Continuous

$$g(x|y) = \frac{f(x,y)}{h(y)} \qquad h(y|x) = \frac{f(x,y)}{g(x)}$$

Prob(X), for a given Y — Prob Y for a given X

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Note: let (x,y) be a 2 dimension R.V. Then X & Y are independent

Discrete ✡

If any one value is not independent

___ X ___ = ___

Continuous =

$f(x,y) = g(x).h(y) \rightarrow$ continuous

$P(x_i, y_j) = P(x_i)\, q(y_j) \rightarrow$ discrete

Problems 1

Q1 A coin is tossed 3 times. Let X denote 0 or 1 according as a tail or a head appears in the first toss.
Let Y denotes the number of tails occur.
Determine the joint pmf, marginal pmf of x & Y
$P(x \leq 0)$, $P(x \leq 0, Y \leq 2)$, $P(X+Y \leq 2)$ $P(X=1 | Y=2)$

X | Y	0	1	2	3	P(x)
0	0	1/8	2/8	1/8	4/8
1	1/8	2/8	1/8	0	4/8
q(y)	1/8	3/8	3/8	1/8	1

$P_{00} = P(X=0, Y=0) = P($Tail appears in first toss && 0 tails appear)

$P(01) = P(X=0, Y=1) = P($Tail appear in first toss & total 1 tail)

Marginal Pdf

X	P(X)
0	4/8
1	4/8

Y	q(y)
0	1/8
1	3/8
2	3/8
3	1/8
	1

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$p(x \leq 0) = P(X=0) = 1/2 = (0 + 1/8 + 2/8 + 1/8) = 4/8 = 1/2$

$p(X \leq 0, Y \leq 2) = 0 + 1/8 + 2/8 = 3/8$

$P(X+Y \leq 2) = P(X=0, Y=0,1,2$
$X=1, Y=0,1$
$X=2, Y=0)$

$= 0 + 1/8 + 2/8 + 1/8 + 2/8$
$= 6/8 = 3/4$

$P(X=1 | Y=2) = \dfrac{P(X=1, Y=2)}{P(Y=2)} = \dfrac{(1/8)}{(3/8)} = \dfrac{1}{3}$

$P(1,1) = \dfrac{3}{8} \times \dfrac{4}{8} = \dfrac{12}{64} = \dfrac{6}{32} = \dfrac{3}{16} \neq \dfrac{2}{8}$

Q2 Given $f(x,y) = \begin{cases} x^2 + \dfrac{xy}{3} & ; 0 \leq x \leq 1, \; 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$

check whether $f(x,y)$ is a valid pdf. Find marginal pdf.
Find $P(X > 1/2)$, $P(Y < X)$, $P(Y < 1/2 \mid X < 1/2)$, $g(x/y)$, $h(y|x)$, $P(X+Y > 1)$. Check whether X and Y are independent

valid pdf

~~P(X>1/2)~~ $\neq \int_0^2 \int_0^1 \left(x^2 + \dfrac{xy}{3}\right) dx\, dy = \int_0^2 \dfrac{x^3}{3} + \dfrac{x^2 y}{2} \Big|_0^1 dy$

$= \left\{ \left[\dfrac{1}{3} + \dfrac{y}{6}\right] - [0] \right\} dy = \int_0^2 \dfrac{1}{3} + \dfrac{y}{6} \, dy$

$= \left[\dfrac{1}{3} y + \dfrac{y^2}{12}\right]_0^2 = 1$

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Marginal pdf of X is $g(x) = \int_0^2 x^2 + \frac{xy}{3}\, dy$

$$= \left[x^2 y + \frac{xy^2}{6}\right]_0^2$$

$$= \left[\left(x(2) + x\frac{4}{6}\right) - (0)\right]$$

$$= 2x + \frac{2x}{3}$$

$$g(x) = \begin{cases} 2x^2 + \frac{2x}{3} & : 0 \le x \le 1 \\ 0 & : \text{otherwise} \end{cases}$$

$$h(y) = \int_0^1 x^2 + \frac{xy}{3}\, dx = \begin{cases} \frac{1}{3} + \frac{y}{6} & : 0 \le y \le 2 \\ 0 & : \text{otherwise} \end{cases}$$

$$P(X > 1/2) = P(X \ge 1/2,\ 0 \le y \le 2)$$

$$= \int_0^2 \left(\int_{1/2}^1 (x^2 + xy)\, dx\right) dy$$

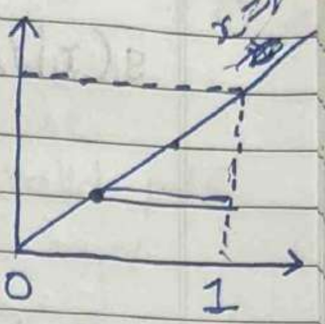
$$= \int_0^2 \left(\left[\frac{x^3}{3} + \frac{yx^2}{2}\right]_{1/2}^1\right) dy \quad = 9/6$$

$$\cancel{= \int_0^2 \left\{\left(\frac{1}{8} + \frac{y}{8}\right) dy\right\} = \left(\frac{1}{8}y + \frac{y^2}{16}\right]_0^2}$$

$$= \frac{1}{2}$$

1> x is from y to 1
2> y is from 0 to 2

$$P(x<y) = \int_0^1\int_y^1 x^2 + \frac{xy}{3}\,dx\,dy$$



$$= \int_0^1\int_y^1 x^2 + \frac{xy}{3}\,dx\,dy$$

$$= \int_0^1 \left[\left(\frac{x^3}{3} + \frac{x^2y}{6}\right)\right]_y^1 dy$$

$$= \int_0^1 \left[\left\{\frac{1}{3} + \frac{y}{6}\right\} - \left\{\frac{y^3}{3} + \frac{y^3}{6}\right\}\right] dy$$

$$= \int_0^1 \left(\frac{1}{3} + \frac{y}{6} - \frac{y^3}{2}\right) dy$$

$$= \int_0^1 \frac{1}{3} + \frac{y}{6} - \frac{y^3}{2}\,dy = \frac{7}{24}$$

$$p(y<1/2 \mid x<1/2) = \frac{P(y<1/2, x<1/2)}{p(x<1/2)}$$

$$= \frac{P(0<y<1/2, 0<x<1/2)}{p(x<1/2)}$$

$$= \frac{P(0<x<1/2, 0<x<1/2)}{1-\frac{1}{6}}$$

$$= \left[\left(\int_0^{1/2}\int_0^{1/2} x^2 + \frac{xy}{3}\,dx\,dy\right)\right] \div \frac{5}{6}$$

$$= \left[\int_0^{1/2}\left[\frac{x^3}{3} + \frac{x^2y}{6}\right]_0^{1/2} dy\right] \div \frac{5}{6}$$

$$= \int_0^{1/2}\left(\frac{1}{24} + \frac{1}{24}y\right) - 0\,dy = \frac{5}{192} \times \frac{1}{6} = \frac{5}{32}$$

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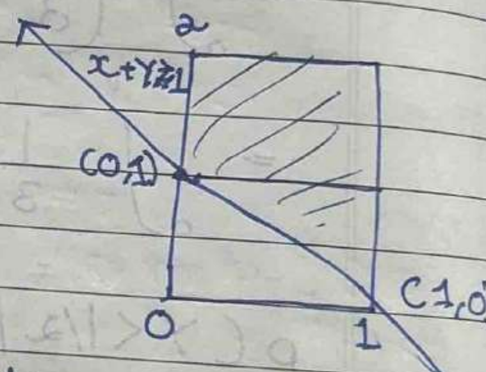
$$g(x|y) = \frac{f(x,y)}{h(y)} = \frac{x^2 + \frac{xy}{3}}{\frac{1}{3} + \frac{y}{6}} = \frac{6x^2 + 2xy}{2+y} \quad ; \; 0 \leq x \leq 1, \; 0 \leq y \leq 2$$

$$h(y/x) = \frac{f(x,y)}{g(x)} = \frac{x^2 + \frac{xy}{3}}{2x^2 + \frac{2}{3}x} = \frac{3x^2 + xy}{6x^2 + 2x} \quad ; \; 0 \leq x \leq 1, \; 0 \leq y \leq 2$$

$P(X+Y \geq 1)$



$= 1 - P(X+Y \leq 1)$

$$= 1 - \int_0^1 \int_0^{1-y} x^2 + \frac{xy}{3} \, dx \, dy$$

$$= 1 - \int_0^1 \frac{x^3}{3} + \frac{x^2 y}{6} \, dy = 1 - \int_0^1 \left\{ \frac{x^3}{3} + \frac{x^2 y}{6} \right\}_0^{1-y}$$

$$= 1 - \int_0^1 \left\{ \frac{(1-y)^3}{3} + \frac{(1-y)^2 y}{6} \right\} dy = \frac{65}{72}$$

$f(x,y) \neq g(x)h(y)$ Not independent

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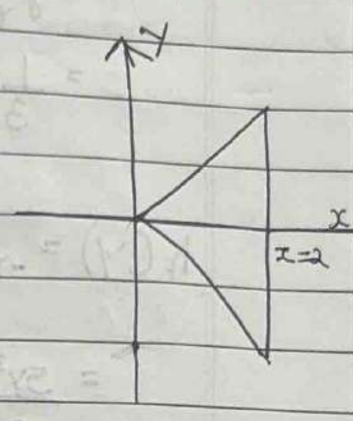
Q3 Given $f(x,y) = \begin{cases} kx(x-y) ; & 0<x<2 \quad -x<y<x \\ 0 & ; \text{Otherwise} \end{cases}$

Find k such that $f(x,y)$ is valid pdf.

Find $g(x)$ and $h(y)$

Sol^n:



$$= \int_0^2 \int_{-x}^{x} kx(x-y)\, dy\, dx$$

$$= k \int_0^2 \int_{-x}^{x} x^2 - xy \; dy\, dx$$

$$= k \int_0^2 \left[x^2 y - \frac{xy^2}{2} \right]_{-x}^{x} dx$$

$$= k \int_0^2 \left(x^3 - \frac{x^3}{2} \right) - \left(-x^3 - \frac{x^3}{2} \right)$$

$$= k \int_0^2 x^3 - \frac{x^3}{2} + \frac{x^3}{2} + x^3 = k \int_0^2 2x^3$$

$$= 8k$$

$$1 = 8k$$

$$k = \frac{1}{8}$$

$$g(x) = \int_{-x}^{x} \frac{1}{8} x(x-y)\, dy = \begin{cases} x^3/4 ; & 0<x<2 \\ 0 & ; \text{Otherwise} \end{cases}$$

$$h(y) = \int_0^2 \frac{1}{8} (x-y)\, dx =)$$

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$$h(Y) = \int_{Y}^{2} \frac{1}{8} x(x-Y)\,dx \qquad ; 0 \le Y \le 2$$

$$= \frac{1}{8}\int_{Y}^{2} x^2 - xY\,dx = \frac{1}{8}\left[\frac{x^3}{3} - \frac{x^2 Y}{2}\right]_Y^2 = \frac{1}{8}\left[\frac{2^3}{3} - \frac{2^2 Y}{2} - \frac{Y^3}{3} + \frac{Y^3}{2}\right]$$

$$= \frac{1}{3} - \frac{Y}{4} + \frac{Y^3}{48}$$

$$h(Y) = \int_{-Y}^{2} \frac{1}{8} x(x-Y)\,dx \qquad ; -2 \le Y \le 0$$

$$= \frac{5Y^3}{48} - \frac{Y}{4} + \frac{1}{3}$$

Expectation: Let (x, Y) be a 2 dim r.v

Discrete : $E(X) = \sum_{i}\sum_{J} x_i\, P(x_i Y_i)$

$$= \sum_{i} x_i \sum_{J} P(x_i, Y_i)$$ make 1 p & mutiply

$$= \sum_{i} P(x_i)$$ marginal pdf

$$E(Y) = \sum_{J} Y_J\, q(Y_J)$$

Continuous : $E(x) = \int_{-\infty}^{\infty}\int_{\infty}^{\infty} x\, f(x, Y)\,dx\,dy$

$$= \int_{-\infty}^{\infty} x \cdot g(x)\,dx$$

$$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Y f(x\,Y)\,dx\,dY = \int_{-\infty}^{\infty} Y \cdot h(Y)\,dY$$

$$E(xy) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x\,y)\, dx\, dy$$

$$f(x^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f(xy)\, dx\, dy$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$V(Y) = E(Y^2) - [E(Y)]^2$$

Propertio

1) $E(X+Y) = E(X) + E(Y)$

2) If X & Y are independent, then $E(XY) = E(X)\,E(Y)$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy.\, f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\; g(x).\, h(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x.\, g(x)\, dx \int_{-\infty}^{\infty} y\, h(y)\, dy$$

$$= E(X)\; E(Y)$$

3 If X & Y are independent, then $V(X+Y) = V(X) + V(Y)$

Proof:

$$V(X+Y) = E[(X+Y)^2] - E[(X+Y)]^2$$

$$= E(X^2 + Y^2 + 2XY) - [E(X) + E(Y)]^2$$

$$\cancel{= E(X^2) + E(Y^2) + 2E(X.Y)}$$

$$\cancel{- [E(X) + E(Y)]^2 - E(X) - E_2(X)\,E(}$$

$$= E(X^2) + E(Y^2) + 2E(XY) - (E(X))^2 - 2E(X)\,E(Y) - (E(Y))^2$$

$$= E(X^2) - (E(X))^2 + E(Y^2) - (E(Y))^2 + \cancel{2E(XY) - 2E(X)\,E(Y)}$$

#independent

$=$

$$= V(X) + V(Y)$$

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Problems:

Q1

XY	1	2	3
1	1/12	1/6	0
2	0	1/9	1/5
3	1/18	1/4	2/15

Find: $V(X)$, $E(Y)$, $P(X=1 | Y=2)$, $E(X+Y)$

Marginal PDF

X	P(X)
1	1/4
2	14/45
3	~~61/120~~ 79/180

Y	P(X)
1	5/36
2	19/36
3	1/3

$$E(X) = 1\times\frac{1}{4} + 2\times\frac{14}{45} + 3\times\frac{79}{180} = \frac{197}{90}$$

$$E(X^2) = 1\times\frac{1}{4} + 2^2\times\frac{14}{45} + 3^2\times\frac{79}{180} = \frac{49}{9}$$

$$V(X) = E(X^2) - (E(X))^2 = \frac{49}{9} - \left(\frac{197}{90}\right)^2 = \frac{5291}{8100}$$

$$E(Y) = 1\times\frac{5}{36} + 2\times\frac{19}{36} + 3\times\frac{1}{3} = \frac{79}{36}$$

$$E(Y^2) = 1^2\times\frac{5}{36} + 4\times 19/36 + 3^2\times 1/3 = \frac{21}{4}$$

$$V(Y) = \frac{21}{4} - \left(\frac{79}{36}\right)^2 = \frac{563}{1296}$$

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$$P(X=1|Y=2) = \frac{P(X=1, Y=2)}{P(Y=2)} = \frac{1/6}{\frac{1}{6}+\frac{1}{9}+\frac{1}{4}} = \frac{6}{19}$$

$$E(X+Y) = E(X) + E(Y) = \frac{97}{90} + \frac{79}{36} = \frac{263}{60}$$

Q2 A 2 dim random variable (X,Y) has the joint pdf given by $f(x,y) = \begin{cases} 6e^{-2x-3y} & , x, y \geq 0 \\ 0 & , \text{otherwise} \end{cases}$

Find $P(1<X<2, 2<Y<3)$, $g(x|y)$, $h(y|x)$, $E(X)$, $V(X)$

check if X & Y are independent

$$P(1<X<2, 2<Y<3) = \int_1^2 \int_2^3 6e^{-2x} \cdot e^{-3y} \, dy \, dx$$

$$= \int_1^2 6e^{-2x} \int_2^3 e^{-3y} \, dy \, dx$$

$-y = t$

$$= \int_1^2 6e^{-2x} - \int_2^3 e^{3t} \, dt$$

$-1 = \frac{dt}{dy}$

$$= \int_1^2 6e^{-2x} - \frac{1}{3} e^{-3y} \Big|_2^3$$

$-dy = dt$

$$= \int_1^2 6e^{-2x} \cdot \left(\frac{1}{3} e^{-3(3)} - \frac{1}{3} e^{-3(2)} \right) dy$$

$$= \int_1^2 6e^{-2x} \left(\frac{e^{-9}}{3} - \frac{1}{3} e^{-6} \right) dy$$

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$$P\int_1^2\int_2^3 6e^{-2x}\cdot e^{-3y}\,dy\,dx$$

$$= 6\int_1^2 e^{-2x}\int_2^3 e^{-3y}\,dy\,dx$$

$$= 6\int_1^2 e^{-2x}\left(-\frac{1}{3}e^{-3y}\right)\Big|_2^3\,dx$$

$$= -6\int_1^2 e^{-2x}\,\frac{1}{3}\,e^{-3y}\Big|_2^3\,dx$$

$$= -6\int e^{-2x}\,\frac{1}{3}\left(e^{-9}-e^{-6}\right)dx$$

$$= 2\left(e^{-6}-e^{-9}\right)\int_1^2 e^{-2x}\,dx = 2\left(e^{-6}-e^{-9}\right)\left[\frac{1}{-2}e^{-2x}\right]_1^2$$

$$= 2\left(e^{-6}-e^{-9}\right)\times-\frac{1}{2}\left[e^{-4}-e^{-2}\right)$$

$$= \left(e^{-6}-e^{-9}\right)\left(e^{-2}-e^{-4}\right)$$

$$= \left(\frac{1}{e^{6}}-\frac{1}{e^{-9}}\right)\left(\frac{1}{e^{2}}-\frac{1}{e^{4}}\right) = \cancel{\frac{1}{e^{2}}}$$

$$= 0.00027562$$

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Q. $g(x/y) = \dfrac{f(x,y)}{h(y)}$

$h(y) = \int_0^{\infty} 6e^{-2x-3y}\,dx = \int_0^{\infty} 6e^{-3y} \cdot e^{-2x}\,dx$

$= 6e^{-3y}\int e^{-2x}\,dx = 6e^{-3y}\left(\frac{-1}{2}e^{-2x}\right|_0^{\infty}$

$= -3e^{-3y}\left(\frac{1}{e^{\infty\times 2}} - \frac{1}{e^{0}}\right)$

$= -3\,e^{-3y}(-1)$

$= 3e^{-3y}$

$g(x/y) = \dfrac{f(x,y)}{h(y)} = \dfrac{6e^{-2x}e^{-3y}}{3e^{-3y}} = 2e^{-2x}$

$h(y/x) = \dfrac{f(x,y)}{g(x)}$

$g(x) = \int_0^{\infty} 6e^{-2x} \cdot e^{-3y}\,dy = 6e^{-2x}\left(-\frac{1}{3}e^{-3y}\right|_0^{\infty}$

$= 2e^{-2x}$

$h(y/x) = \dfrac{6e^{-2x} \cdot e^{-3y}}{2e^{-2x}} = 3e^{-3y}$

$f(x,y) = g(x)\,h(y)$

$= 2e^{-2x} \times 3e^{-3y} = 6e^{-2x} \cdot e^{-3y} = 6e^{-2x-3y}$

independent

$E(x) = \int_0^{\infty} 6e^{-3y}xe^{-2x}\,dx = 6e^{-3y}\int_0^{\infty} xe^{-2x}\,dx$

$u = x \quad v_1 = \int e^{-2x} = -\frac{1}{2}e^{-2x} \quad v_2 = \frac{1}{4}e^{-2x}$

$\int uv = \left(x\left(\frac{-e^{-2x}}{2}\right) - \left(\frac{e^{-2x}}{4}\right)\right|_0^{\infty}$

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$E(X) = \int_0^\infty x \cdot g(x)\,dx$ or $\int_0^\infty \int_0^\infty x \cdot f(x,y)\,dx\,dy$

$= \int_0^\infty x \cdot e^{-2x}\,dx$

$= 2 \int_0^\infty x e^{-2x}\,dx$

$x \int e^{-2x}\,dx - \int 1 \left(\int e^{-2x}\,dx \right) \cdot dx$

$x\left(\frac{-1}{2} e^{-2x}\right) - \int \frac{-1}{2} e^{-2x}\,dx$

$x\left(\frac{-1}{2} e^{-2x} - \left(\frac{-1}{2}\right)\left(\frac{-e^{-2x}}{2}\right)\right)$

$\frac{-1}{2} x e^{-2x} - \frac{e^{-2x}}{4}$

$= \frac{-1}{2}\left(x e^{-2x} + \frac{e^{-2x}}{2}\right) = \frac{-1}{2} e^{-2x}\left(x + \frac{1}{2}\right)$

$= \frac{-1}{4} e^{-2x}(2x+1) \Big|_0^\infty$

$= \left\{\frac{-1}{4} e^{-2(\infty)}(2 \times \infty + 1)\right\} + \left\{\frac{1}{4} e^{-2(0)}(2 \times 0 + 1)\right\}$

$= 0 + \frac{1}{4} = \frac{1}{4}$

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$$E(X^2) = \int_0^{\infty} x^2 e^{-2x}\, dx$$

$$= \int_0^{\infty} x^2 e^{-2x}\, dx$$

$$x^2 \int e^{-2x} dx - \int 2x \int e^{-2x} dx \quad dx$$

$$x^2 \left\{ \frac{-1}{2} e^{-2x} - \int 2x \left(\frac{-1}{2} e^{-2x} \right) dx \right.$$

$$\frac{x^2}{2} \left(-e^{-2x} \right) + \int x e^{-2x}\, dx$$

$$\frac{x^2}{2} \left(-e^{-2x} \right) \cdot \frac{-1}{2} x e^{-2x} - \frac{1}{4} e^{-2x}$$

$$= \frac{-1}{4}$$

$$V(X) = E(X^2) - (E(X))^2 = \frac{-1}{4} - \left(\frac{1}{4} \right)^2$$

$$g(x|y) = \frac{f(x,y)}{h(y)} \qquad h(y|x) = \frac{f(x,y)}{g(x)}$$

Condional Expectation

$$E(X|Y) = \int_{-\infty}^{\infty} x \cdot g(x|y)\, dx \qquad E(x|Y) = \sum x_i P(x_i|Y)$$

$$E(y|x) = \int_{-\infty}^{\infty} y \cdot h(y|x)\, dy \qquad E(y|x) = \sum y_j P(y_j|x_i)$$

Problems

Given $f(x,y) = \begin{cases} x+y ; & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 : & \text{otherwise} \end{cases}$

Find $E(X|Y)$, $E(Y|x)$, $P(X > 1/2)$, $P(Y < 1/2)$
$V(X)$, $V(Y)$

$$E(X|Y) = \int_{-\infty}^{\infty} x \cdot g(x/y)\, dx = \int_0^{\infty} x \cdot \frac{x+y}{\int_0 x+y\, dx}\, dx$$

$$= \int_0^{\infty} x \cdot \frac{x+y}{\left. yx + \frac{x^2}{2} \right|_0^1}$$

$$= \int_0^{\infty} x \quad \frac{x+y}{\left(y+\frac{1}{2}\right) - 0}$$

$$= \int_0^1 2x \cdot \frac{x+y}{2y+1}\, dx$$

$$= \int_0^1 \frac{2}{2y+1} \cdot x^2\, dx + \int_0^1 \frac{2y}{2y+1} \cdot x\, dx$$

$$= \frac{2}{2y+1} \times \frac{1}{3}(1-0) + \frac{2y}{2y+1} \times \frac{1}{2}(1-0)$$

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$$= \frac{2}{2y+1}\left(\frac{1}{3}+\frac{1}{2}\right)$$

$$= \frac{2\times 5}{6}\times\frac{1}{2y+1}$$

$$= \frac{5}{3}\times\frac{1}{2y+1}$$

$$g(x) = \int_0^1 x+y\,dy = \left(xy+\frac{y^2}{2}\right)_0^1$$

$$= \left(x+\frac{1}{2}\right) - 0$$

$$= x + 1/2$$

$$= \frac{2x+1}{2}$$

$$E(y/x) = \int y.\ h(y/x)\,dy = \int y.\ \frac{x+y}{\frac{2x+1}{2}}\,dy$$

$$= \int_0^1 2\cancel{x}.\ \frac{x+y}{2x+1}\,dy$$

$$= \frac{2}{2x+1}.\int_0^1 y\,dy + \frac{2x}{2x+1}\int_0^1 1\,dy$$

$$= \frac{2}{2x+1}\left.\frac{y^2}{2}\right|_0^1 + \frac{2x}{2x+1}\times \left.y\right|_0^1$$

$$= \frac{1}{2x+1} + \frac{2x}{2x+1}$$

$$= \frac{2x+1}{2x+1} = 1$$

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$$P(X>1/2) = \int_{1/2}^{1}\int_{0}^{1} (x+y)\,dy\,dx$$

$$= \int_{1/2}^{1} xy + \frac{y^2}{2}\Big|_0^1 \,dx$$

$$= \int_{1/2}^{1} x(1) + \frac{1}{2}$$

$$= \int_{1/2}^{1} x + \frac{1}{2}\,dx$$

$$= \frac{x^2}{2}\Big|_{1/2}^{1} + \frac{1}{2}[x]_{1/2}^{1}$$

$$= \frac{1}{2}\left(1 - \frac{1}{4}\right) + \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{3}{4} \times \frac{1}{2} + \frac{1}{4} = \frac{3}{8} + \frac{1}{4} = \frac{5}{8}$$

$$P(Y<1/2) = \int_{0}^{1}\int_{0}^{1/2} (x+y)\,dy\,dx$$

$$= \int_{0}^{1}\int_{0}^{1/2} xy + \frac{y^2}{2}\Big|_0^{1/2}$$

$$= \int_{0}^{1} x \times \frac{1}{2} + \frac{1}{8}\,dx$$

$$= \frac{x^2}{4} + \frac{x}{8}\Big|_0^1$$

$$= \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

$E(X) = \int_0^1 X\, g(x)\, dx = \int_0^1 x \cdot (x + 1/2)\, dx$

$= \int_0^1 x^2 + \frac{1}{2} x\, dx = \left. \frac{x^3}{3} + \frac{1}{2} \times \frac{x^2}{2} \right|_0^1$

$= 5/6$

$E(X^2) = \int_0^1 X^2 \cdot (x + \frac{1}{2})\, dx$

$\int_0^1 X^3 + \frac{X^2}{2}\, dX$

$\left. \frac{x^4}{4} + \frac{x^3}{6} \right|_0^1 = \frac{5}{12}$

$= \cancel{\frac{5}{12} - \left(\frac{5}{6}\right)^2}$

$\cancel{E(Y)}\ V(X) = E(X^2) - (E(X))^2 =$

$E(Y) = \int_0^1 Y\, h(Y)\, dY = \int_0^1 Y(1/2 + Y)\, dY = \cancel{3/4}\ 7/12$

$E(Y^2) = \int_0^1 Y^2 h(Y)\, dY = \int_0^1 Y^2 (1/2 + Y) \cdot dY = \cancel{1/2}\ 5/12$

$= \cancel{\frac{1}{2} - \frac{9}{16}} \quad \frac{5}{12} - \frac{49}{144} = 0.0763$

Corelation Coefficient

Let (x, y) be a 2 dim r.v. The year corelation coefficient $\rho_{x,y}$ between X & Y is defined as

$$\rho_{XY} = \frac{E[(X-E(X))(Y-E(Y))]}{\sqrt{V(x)V(Y)}} = \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)V(Y)}}$$

covarience of X, Y is defined as

$$Cov(x,y) = E(XY) - E(X)E(Y)$$

Note: If X & Y are independent then $\rho_{x,y} = 0$ since $E(XY) = E(X)E(Y)$. Then X & Y are called as uncorrelated r.v's

But the converse need not be true

☆ If $\rho_{xy} = 0$ then X and Y are not necessarily independent

Q1 Find ρ_{xy} given

Y/X	1	2	3
1	0.1	0.1	0.1
2	0.1	0.2	0.1
3	0.1	0.1	0.1

are they independent?

$\Sigma X = 1$ $\quad \Sigma Y = 1$ $\quad \therefore$ Valid Pdf

$$E(X) = 1(0.3) + 2(0.4) + 3(0.3)$$
$$= 0.3 + 0.8 + 0.9$$
$$= 2$$

$$E(X^2) = 1^2(0.3) + 2^2(0.4) + 3^2(0.3)$$
$$= 4.6$$

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$E(Y) = 1(0.3) + 2(0.4) + 3(0.3)$
$= 2$

$E(Y^2) = 4.6$

$V(X) = 4.6 - 2^2 = 0.6$
$V(Y) = 0.6$

$E(X, Y) = \sum \sum xy \cdot p(x, y)$

$= \cancel{\sum \sum}$
$0.1 \times 1 + 0.1 \times 2 + 0.1 \times 3 + 2 \times 0.1 + 4 \times 0.2 + 6 \times 0.1$
$+ 3 \times 0.1 + 6 \times 0.1 + 9 \times 0.1$

$= 0.1 + 0.2 + 0.3 + 0.2 + 0.8 + 0.6 + 0.3 + 0.6 + 0.9$
$= \cancel{1 + 1 + 1.4 = 3.4 + 0.6 = 4.0}$
$= 4.0$

$\rho(x, y) = \dfrac{E(XY) - E(X)E(Y)}{\sqrt{V(X)\,V(Y)}}$

$= \dfrac{4 - \cancel{0.6 \times 0.6}\ 2 \times 2}{\sqrt{0.6 \times 0.6}} = \dfrac{4 - 4}{0.6} = 0$

But $0.1 \neq 0.3 \times 0.3$

Thus not independent

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Q2 Find $\rho_{x,y}$ if $f(x,y) = \begin{cases} 2-x-y \; ; \; 0<x<1, \; 0<y<1 \\ 0 \; : \text{otherwise} \end{cases}$

$$\rho_{XY} = \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)V(Y)}}$$

$$E(X) = \int_0^1 \int_0^1 x \cdot (2-x-y)\,dx\,dy$$

$$= \int_0^1 \int_0^1 2x - x^2 - yx \quad dx\,dy$$

$$= \int_0^1 2\frac{x^2}{2} - \frac{x^3}{3} - y\frac{x^2}{2} \Big|_0^1$$

$$= \int_0^1 x^2\Big|_0^1 - \frac{1}{3}x^3\Big|_0^1 - \frac{y}{2}\left|x^2\right|_0^1$$

$$= \int_0^1 1 - \frac{1}{3} - \frac{y}{2}$$

$$= \int_0^1 \frac{2}{3} - \frac{y}{2} = \frac{2y}{3} - \frac{y^2}{4}\Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

$$E(Y) = \int_0^1 \int_0^1 y(2-x-y)\,dy\,dx$$

$$= \int_0^1 \cancel{\frac{2y^2}{}} - xy$$

$$= \int_0^1 \frac{\cancel{2}y^2}{\cancel{2}} - \frac{xy^2}{2} - \frac{y^3}{3}\Big|_0^1$$

$$\int_0^1 1 - \frac{x}{2} - \frac{1}{3} \quad dx$$

$$= \int_0^1 \frac{2}{3} - \frac{x}{2}\,dx = \frac{5}{12}$$

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$$E(XY) = \int_0^1\int_0^1 xy(2-x-y)\,dx\,dy$$

$$= \int_0^1\int_0^1 2x^2y\,dx - x^2y\,dx - y^2x\,dx\ dy$$

$$= \int_0^1 2y\frac{x^2}{2}\Big|_0^1 - y\frac{x^3}{3}\Big|_0^1 - y^2\frac{x^2}{2}\Big|_0^1\ dy$$

$$= \int_0^1 \cancel{2}y \times \frac{1}{\cancel{2}} - \frac{y}{3} - \frac{y^2}{2}\ dy$$

$$\int_0^1 \frac{2y}{3} - \frac{y^2}{2}\ dy = \frac{1}{6}$$

$$E(X^2) = \int_0^1\int_0^1 x^2(2-x-y)\,dx\,dy$$

$$= \int_0^1\int_0^1 2x^2 - x^3 - x^2y\ dx\ dy$$

$$= \int_0^1 \frac{2x^3}{3}\Big|_0^1 - \frac{x^4}{4}\Big|_0^1 - \frac{x^3y}{3}\Big|_0^1\ dy$$

$$= \int_0^1 \frac{2}{3} - \frac{1}{4} - \frac{y}{3}\ dy$$

$$\int_0^1 \frac{5}{12} - \frac{y}{3}\,dy = \frac{5}{12}y\Big|_0^1 - \frac{y^2}{3}\Big|_0^1$$

$$= \frac{1}{4}$$

$$E(Y^2) = \int_0^1\int_0^1 y^2(2-x-y)\,dy\,dx = 1/4$$

$$V(X) = E(X^2) - (E(X))^2 = 1/4 - (5/12)^2 = 11/144$$

$$V(Y) = 11/144$$

$$\rho = \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)V(Y)}} = \frac{\frac{1}{6} - \frac{5}{12}\times\frac{5}{12}}{\frac{11}{144}}$$

$$= \frac{-1}{11}$$

$\therefore$ inversely co-related

2 defination of varience

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Q3 Prove that $V(ax \pm bY) = a^2V(x) + b^2(V(Y)) \pm 2ab\, cov(x,y)$

$V(aX \pm bY) = E(\cancel{}(ax \pm bY) - E(ax \pm bY))$

$= E((ax \pm bY) - E(ax \pm bY))^2$

$= E(ax \pm bY - \{a E(X) \pm b E(Y)\})^2$

$= E(a(X - E(X)) \pm b(Y - E(Y)))^2$

$= a^2 E(X - E(X))^2 + b^2 E(Y - E(Y))^2 \pm 2ab E((X - E(X))(Y - E(Y)))$

$= a^2 V(E) + b^2 V(Y) \pm 2ab\, cov\, Cov(X,Y)$

Q4 Two independent r.v's X_1 and X_2 have mean $(5, 12)$ and variance $(4, 9)$ respectively. Find co-varience between $3X_1 + 4X_2$ and $V = 3X_1 - X_2$

To find $cov(U,V) = E(UV) - E(U)E(V)$

Sol^n: $E((3X_1 + 4X_2)(3X_1 - X_2)) - E(3X_1 + 4X_2) \cdot E(3X_1 - X_2)$

$= E(9X_1^2 + 9X_1X_2 - 4X_2^2) - \{3E(X_1) + 4E(X_2)\}\{3E(X_1) - E(X_2)\}$

$= 9E(X_1^2) + 9E(X_1X_2) - 4E(X_2^2)$

$- \{9(E(X_1))^2 - 4(E(X_2))^2 + 12E(X_1)E(X_2) - 3E(X_1)E(X_2)$

$= 9E(X_1^2) + 9E(X_1X_2) - 4E(X_2^2) - 9(E(X_1))^2 + 4(E(X_2))^2$
$- 9E(X_1)E(X_2)$

$= [9E(X_1^2) - 9(E(X_1))^2] - [4E(X_2^2) - 4(E(X))^2]$
$+ 9E(X_1X_2) - 9E(X_1)E(X_2)$

$= 9[E(X_1^2) - (E(X_1))^2] - 4[E(X_2^2) - (E(X_2))^2] + 9\{E(X_1X_2) - E(X_1)E(X_2)\}$

$= 9 \times 4 - 4 \times 9 = 0$

$E(X_1+X_2) = E(X_1) + E(X_2)$

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Q5 If X_1, X_2, X_3 are uncorrelated r.v's with $\sigma_1 = 5, 12$ and 9. Find ρ between u & v where $U = X_1 + X_2$, $V = X_2 + X_3$

X_1, X_2, X_3 are uncorrelated

$E(X_1X_2) = E(X_1) \cdot E(X_2)$ $\quad E(X_2X_3) = E(X_2)\,E(X_3)$

$E(X_1X_3) = E(X_1)\,E(X_3)$ $\quad E($

$V(U) = V(X_1+X_2) = V(X_1) + V(X_2) = 5^2 + 12^2 = 169$

$V(V) = V(X_2) + V(X_3) = 12^2 + 9^2 = 225$

$$\rho_{UV} = \frac{E(UV) - E(U)\,E(V)}{\sqrt{V(U)\,V(V)}}$$

$$= \frac{E((X_1+X_2)(X_2+X_3)) - E(X_1+X_2)\,E(X_2+X_3)}{\sqrt{169 \times 225}}$$

$$= \frac{E(X_1X_2 + X_1X_3 + X_2^2 + X_2X_3) - \{E(X_1)E(X_3) + (E(X_2))^2 + E(X_1)E(X_2)\}}{13 \times 15}$$

$$= \frac{\cancel{E(X_1X_2)} + \cancel{E(X_1X_3)} + E(X_2^2) + \cancel{E(X_2X_3)} - \cancel{E(X_1)E(X_3)} - \cancel{E(X_2)E(X_3)} - (E(X_2))^2 - \cancel{E(X_1)E(X_2)}}{13 \times 15}$$

$= \cancel{E(X_1)E(X_2) + E(X_1)E(X_3) + E(X_2^2) + E(X_2)E(X_3)}$

$$= \frac{E(X_2^2) - (E(X_2))^2}{13 \times 15} = \frac{144}{195}$$

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Q6 If X_1, X_2, X_3 are uncorelated r.v.'s having the same standard deviations σ, find ρ_{UV} where $U = X_1 + X_2$ and $V = X_2 + X_3$

$X_1 X_2 X_3$ are un corelated

$E(X_1 X_2) = E(X_1)E(X_2)$ $\quad E(X_1 X_3) = E(X_1)E(X_3)$

$E(X_2 X_3) = E(X_2)E(X_3)$

$V(X_1) = V(X_2) = V(X_3) = \sigma^2$

$V(U) = V(X_1 + X_2) = V(X_1) + V(X_2) = 2\sigma^2$

$V(V) = V(X_2 + X_3) = 2\sigma^2$

$$\rho_{UV} = \frac{E(UV) - E(U)E(V)}{\sqrt{V(U)V(V)}}$$

$$= \frac{E((X_1+X_2)(X_2+X_3)) - E(X_1+X_2)E(X_2+X_3)}{\sqrt{4\sigma^2 \times \sigma^2}}$$

$$= \frac{E(X_1X_2 + X_1X_3 + X_2^2 + X_2X_3) - [(E(X_1)+E(X_2))(E(X_2)+E(X_3))]}{2\sigma \times \sigma}$$

$$= \frac{E(X_1)E(X_2) + E(X_1)E(X_3) + E(X_2)E(X_3) + E(X_2^2) - E(X_1)E(X_2) - E(X_1)E(X_3) - E(X_2)E(X_3) - (E(X_2))^2}{2\sigma^2}$$

$$= \frac{E(X_2^2) - (E(X_2))^2}{2\sigma^2} = \frac{V(X_2)}{2\sigma^2} = \frac{\sigma^2}{2\sigma^2}$$

$$= \frac{\cancel{\sigma^2}}{2} = \frac{1}{2}$$

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Q7 If X & Y are linearly related - i.e. $Y = aX + b$ (a and b can be +ve or -ve), Then $\rho_{XY} = \pm 1$

$$\rho_{XY} = \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)V(Y)}}$$

$$\rho[X,Y] = \frac{E(X(aX+b)) - E(X)E(aX+b)}{\sqrt{V(X)V(Y)}}$$

$$= \frac{E(aX^2+bX) - E(X)[aE(X)+b]}{\sqrt{V(X)V(aX+b)}}$$

$$= \frac{aE(X^2) + bE(X) - a[E(X)]^2 - bE(X)}{\sqrt{V(X) \cdot a^2 V(X)}}$$

$$= \frac{aE(X^2) - a(E(X))^2}{aV(X)} = \frac{a[E(X^2) - (E(X))^2]}{aV(X)}$$

$$= \frac{a V(X)}{aV(X)} = \pm 1$$

always +ve

eg. $E(X^2) \geq$ squaring, so
~~#~~ always the expected value ≥ 0

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Q8) Prove that : $-1 \leq \rho \leq 1$

We known - E(square Function) ≥ 0

consider

$$E\left[\left(\frac{X-E(X)}{\sqrt{V(X)}} \pm \frac{Y-E(Y)}{\sqrt{V(Y)}}\right)^2\right] \geq 0$$

$$E\left[\left(\frac{X-E(X)}{\sqrt{V(X)}}\right)^2 + \left(\frac{Y-E(Y)}{\sqrt{V(Y)}}\right)^2 \pm 2\left(\frac{X-E(X)}{\sqrt{V(X)}}\right)\left(\frac{Y-E(Y)}{\sqrt{V(Y)}}\right)\right] \geq 0$$

$$\frac{E(X-E(x))^2}{V(x)} + \frac{E(Y-E(Y))^2}{V(Y)} \pm \frac{2E((X-E(X))(Y-E(Y)))}{\sqrt{V(x)V(Y)}} \geq 0$$

$$\frac{V(X)}{V(X)} + \frac{V(Y)}{V(Y)} \pm 2\rho_{XY} \geq 0$$

$\cancel{2 \pm 2\rho_{XY}}$

$$2 \pm 2\rho_{XY} \geq 0$$

$$1 \pm \rho_{XY} \geq 0$$

$1+\rho \geq 0$
$\rho \geq -1$

& $1-\rho \geq 0$
$1 \geq \rho$

$$-1 \leq \rho \leq 1$$

Q9 If $U = aX + b$, $V = cY + d$

Prove that $\rho_{UV} = \pm \rho_{XY}$