

CS61B Lecture #32

Today:

- Pseudo-random Numbers (Chapter 11)
- What use are random sequences?
- What *are* "random sequences"?
- Pseudo-random sequences.
- How to get one.
- Relevant Java library classes and methods.
- Random permutations.

Why Random Sequences?

- Choose statistical samples
- Simulations
- Random algorithms
- Cryptography:
 - Choosing random keys
 - Generating streams of random bits (e.g., stream ciphers encrypt messages by xor'ing reproducible streams of pseudo-random bits with the bits of the message.)
- And, of course, games

What Is a "Random Sequence"?

- How about: "a sequence where all numbers occur with equal frequency"?
 - Like 1, 2, 3, 4, ...?
- Well then, how about: "an unpredictable sequence where all numbers occur with equal frequency?"
 - Like 0, 0, 0, 1, 1, 2, 2, 2, 2, 2, 3, 4, 4, 0, 1, 1, 1, ...?
- Besides, what is wrong with 0, 0, 0, 0, ... anyway? Can't that occur by random selection?

Pseudo-Random Sequences

- Even if definable, a “truly” random sequence is difficult for a computer (or human) to produce.
- For most purposes, need only a sequence that satisfies certain statistical properties, even if deterministic.
- Sometimes (e.g., cryptography) need sequence that is *hard* or *impractical* to predict.
- *Pseudo-random sequence*: deterministic sequence that passes some given set of statistical tests.
- For example, look at lengths of *runs*: increasing or decreasing contiguous subsequences.
- Unfortunately, statistical criteria to be used are quite involved. For details, see Knuth.

Generating Pseudo-Random Sequences

- Not as easy as you might think.
- Seemingly complex jumbling methods can give rise to bad sequences.
- *Linear congruential method* is a simple method used by Java:

$$X_0 = \text{arbitrary seed}$$

$$X_i = (aX_{i-1} + c) \bmod m, \quad i > 0$$

- Usually, m is large power of 2.
- For best results, want $a \equiv 5 \bmod 8$, and a, c, m with no common factors.
- This gives generator with a *period of m* (length of sequence before repetition), and reasonable *potency* (measures certain dependencies among adjacent X_i .)
- Also want bits of a to “have no obvious pattern” and pass certain other tests (see Knuth).
- Java uses $a = 25214903917$, $c = 11$, $m = 2^{48}$, to compute 48-bit pseudo-random numbers. It's good enough for many purposes, but not *cryptographically secure*.

What Can Go Wrong (I)?

- Short periods, many impossible values: E.g., a , c , m even.
- Obvious patterns. E.g., just using lower 3 bits of X_i in Java's 48-bit generator, to get integers in range 0 to 7. By properties of modular arithmetic,

$$\begin{aligned} X_i \bmod 8 &= (25214903917X_{i-1} + 11 \bmod 2^{48}) \bmod 8 \\ &= (5(X_{i-1} \bmod 8) + 3) \bmod 8 \end{aligned}$$

so we have a period of 8 on this generator; sequences like

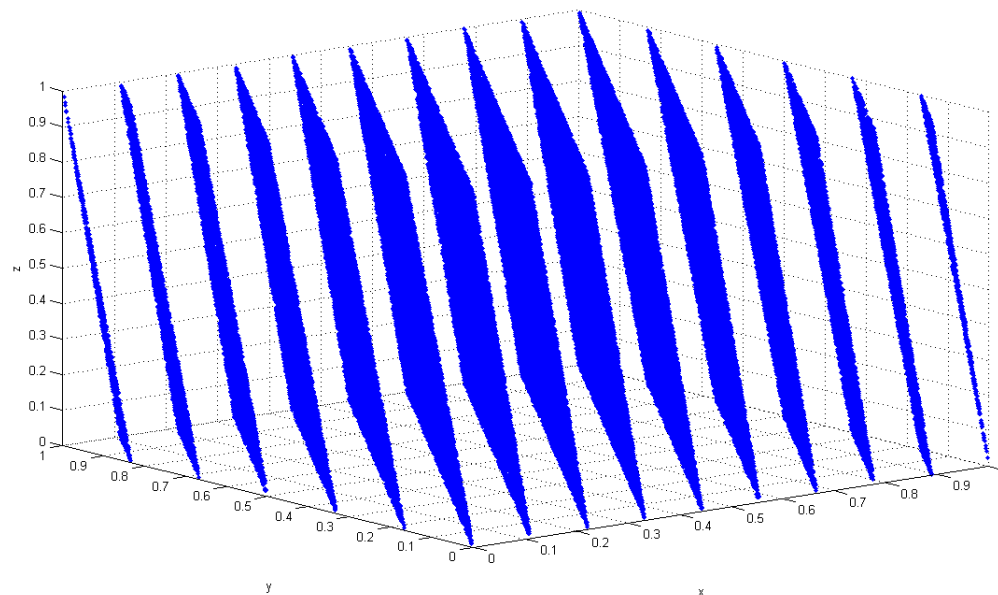
$$0, 1, 3, 7, 1, 2, 7, 1, 4, \dots$$

are impossible. This is why Java doesn't give you the raw 48 bits.

What Can Go Wrong (II)?

Bad potency leads to bad correlations.

- The infamous IBM generator RANDU: $c = 0$, $a = 65539$, $m = 2^{31}$.
- When RANDU is used to make 3D points: $(X_i/S, X_{i+1}/S, X_{i+2}/S)$, where S scales to a unit cube, ...
- ...points will be arranged in parallel planes with voids between. So "random points" won't ever get near many points in the cube:



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Additive Generators

- Additive generator:

$$X_n = \begin{cases} \text{arbitrary value}, & n < 55 \\ (X_{n-24} + X_{n-55}) \bmod 2^e, & n \geq 55 \end{cases}$$

- Other choices than 24 and 55 possible.
- This one has period of $2^f(2^{55} - 1)$, for some $f < e$.
- Simple implementation with circular buffer:

```
i = (i+1) % 55;  
X[i] += X[(i+31) % 55]; // Why +31 (55-24) instead of -24?  
return X[i]; /* modulo  $2^{32}$  */
```

- where $X[0 \dots 54]$ is initialized to some “random” initial seed values.

Cryptographic Pseudo-Random Number Generators

- The simple form of linear congruential generators means that one can predict future values after seeing relatively few outputs.
- Not good if you want *unpredictable* output (think on-line games involving money or randomly generated keys for encrypting your web traffic.)
- A *cryptographic pseudo-random number generator (CPRNG)* has the properties that
 - Given k bits of a sequence, no polynomial-time algorithm can guess the next bit with better than 50% accuracy.
 - Given the current state of the generator, it is also infeasible to reconstruct the bits it generated in getting to that state.

Cryptographic Pseudo-Random Number Generator Example

- Start with a good *block cipher*—an encryption algorithm that encrypts blocks of N bits (not just one byte at a time as for Enigma). AES is an example.
- As a seed, provide a key, K , and an initialization value I .
- The j^{th} pseudo-random number is now $E(K, I + j)$, where $E(x, y)$ is the encryption of message y using key x .

Adjusting Range and Distribution

- Given raw sequence of numbers, X_i , from above methods in range (e.g.) 0 to 2^{48} , how to get uniform random integers in range 0 to $n - 1$?
- If $n = 2^k$, is easy: use top k bits of next X_i (bottom k bits not as "random")
- For other n , be careful of slight biases at the ends. For example, if we compute $X_i / (2^{48}/n)$ using all integer division, and if $(2^{48}/n)$ gets rounded down, then you can get n as a result (which you don't want).
- If you try to fix that by computing $(2^{48}/(n - 1))$ instead, the probability of getting $n - 1$ will be wrong.

Adjusting Range (II)

- To fix the bias problems when n does not evenly divide 2^{48} , Java throws out values after the largest multiple of n that is less than 2^{48} :

```
/** Random integer in the range 0 .. n-1, n>0. */
int nextInt(int n) {
    long X = next random long ( $0 \leq X < 2^{48}$ );
    if (n is  $2^k$  for some  $k$ )
        return top  $k$  bits of X;

    int MAX = largest multiple of  $n$  that is  $< 2^{48}$ ;
    while ( $X_i \geq \text{MAX}$ )
        X = next random long ( $0 \leq X < 2^{48}$ );
    return  $X_i / (\text{MAX}/n)$ ;
}
```

Arbitrary Bounds

- How to get arbitrary range of integers (L to U)?
- To get random float, x in range $0 \leq x < d$, compute

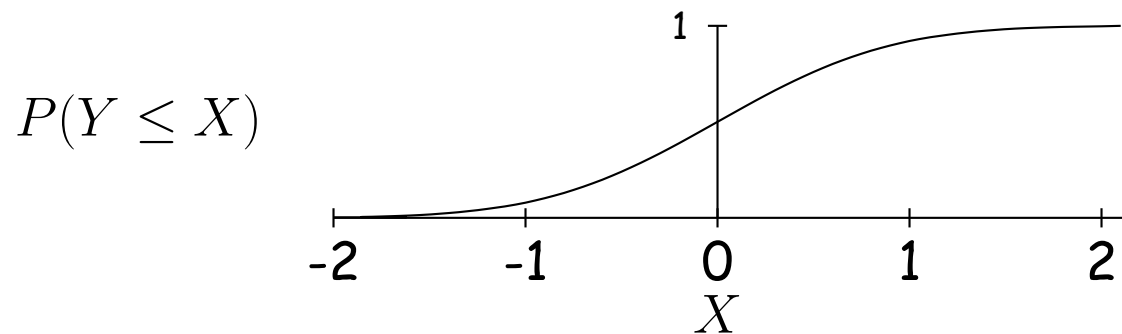
```
return d*nextInt(1<<24) / (1<<24);
```

- Random double a bit more complicated: need two integers to get enough bits.

```
long bigRand = ((long) nextInt(1<<26) << 27) + (long) nextInt(1<<27);  
return d * bigRand / (1L << 53);
```

Generalizing: Other Distributions

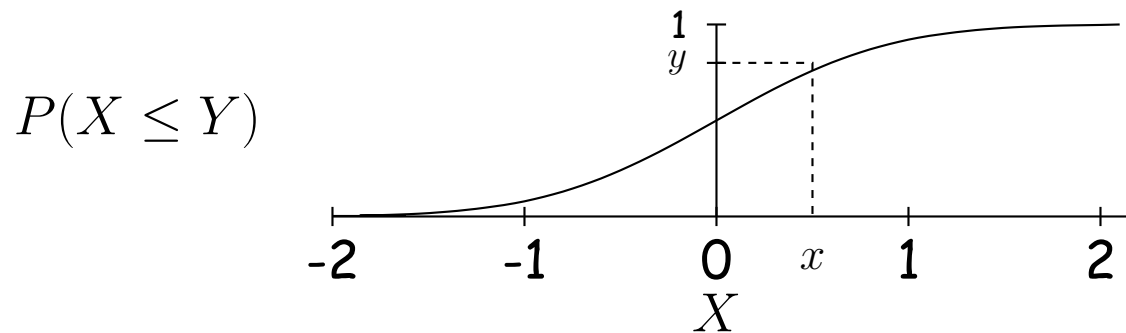
- Suppose we have some desired probability distribution function, and want to get random numbers that are distributed according to that distribution. How can we do this?
- Example: the normal distribution:



- Curve is the desired probability distribution. $P(Y \leq X)$ is the probability that random variable Y is $\leq X$.

Other Distributions

Solution: Choose y uniformly between 0 and 1, and the corresponding x will be distributed according to P .



Java Classes

- `Math.random()`: random double in $[0..1)$.
- Class `java.util.Random`: a random number generator with constructors:
 - `Random()` generator with "random" seed (based on time).
 - `Random(seed)` generator with given starting value (reproducible).
- Methods
 - `next(k)` *k*-bit random integer
 - `nextInt(n)` int in range $[0..n)$.
 - `nextLong()` random 64-bit integer.
 - `nextBoolean()`, `nextFloat()`, `nextDouble()` Next random values of other primitive types.
 - `nextGaussian()` normal distribution with mean 0 and standard deviation 1 ("bell curve").
- `Collections.shuffle(L, R)` for list *R* and Random *R* permutes *L* randomly (using *R*).

Shuffling

- A *shuffle* is a random permutation of some sequence.
- Obvious dumb technique for sorting N -element list:
 - Generate N random numbers
 - Attach each to one of the list elements
 - Sort the list using random numbers as keys.
- Can do quite a bit better:

```
void shuffle(List L, Random R) {  
    for (int i = L.size(); i > 0; i -= 1)  
        swap element i-1 of L with element R.nextInt(i) of L;  
}
```

- Example:

Swap items	0	1	2	3	4	5	Swap items	0	1	2	3	4	5
Start	A♣	2♣	3♣	A♥	2♥	3♥	$3 \iff 3$	A♣	3♥	2♥	A♥	3♣	2♣
$5 \iff 1$	A♣	3♥	3♣	A♥	2♥	2♣	$2 \iff 0$	2♥	3♥	A♣	A♥	3♣	2♣
$4 \iff 2$	A♣	3♥	2♥	A♥	3♣	2♣	$1 \iff 0$	3♥	2♥	A♣	A♥	3♣	2♣

Random Selection

- Same technique would allow us to select N items from list:

```
/** Permute L and return sublist of K>=0 randomly
 *  chosen elements of L, using R as random source. */
List select(List L, int k, Random R) {
    for (int i = L.size(); i+k > L.size(); i -= 1)
        swap element i-1 of L with element
            R.nextInt(i) of L;
    return L.sublist(L.size()-k, L.size());
}
```

- Not terribly efficient for selecting random sequence of K distinct integers from $[0..N)$, with $K \ll N$.

Alternative Selection Algorithm (Floyd)

```
/** Random sequence of K distinct integers
 * from 0..N-1, 0<=K<=N. */
IntList selectInts(int N, int K, Random R)
{
    IntList S = new IntList();

    for (int i = N-K; i < N; i += 1) {
        // All values in S are < i
        int s = R.randInt(i+1); // 0 <= s <= i < N
        if (s == S.get(j) for some j)
            // Insert value i (which can't be there
            // yet) after the s (i.e., at a random
            // place other than the front)
            S.add(j+1, i);
        else
            // Insert random value s at front
            S.add(0, s);
    }
    return S;
}
```

Example

<i>i</i>	<i>s</i>	<i>S</i>
5	4	[4]
6	2	[2, 4]
7	5	[5, 2, 4]
8	5	[5, 8, 2, 4]
9	4	[5, 8, 2, 4, 9]

selectRandomIntegers(10, 5, R)