# CS61B Lecture #16: Complexity



## What Are the Questions?

- Cost is a principal concern throughout engineering:

  "An engineer is someone who can do for a dime what any fool can do for a dollar."

- Cost can mean
  - *Operational cost* (for programs, time to run, space requirements).
  - *Development costs*: How much engineering time? When delivered?
  - *Maintenance costs*: Upgrades, bug fixes.
  - *Costs of failure*: How robust? How safe?
- Is this program *fast enough*? Depends on:
  - *For what purpose;*
  - *For what input data.*
- How much *space* (memory, disk space)?
  - Again depends on what input data.
- How will it *scale*, as input gets big?



## Enlightening Example

**Problem:** Scan a text corpus (say $10^8$ bytes or so), and find and print the 20 most frequently used words, together with counts of how often they occur.

- Solution 1 (Knuth): Heavy-Duty data structures
  - Hash Trie implementation, randomized placement, pointers galore, several pages long.
- Solution 2 (Doug McIlroy): UNIX shell script:

```
tr -c -s '[:alpha:]' '[\n*]' < FILE | \
sort | \
uniq -c | \
sort -n -r -k 1,1 | \
sed 20q
```

- Which is better?
  - #1 is much faster,
  - but #2 took 5 minutes to write and processes 100MB in $\approx 50$ sec.
  - I pick #2.
- In very many cases, almost anything will do: Keep It Simple.



## Cost Measures (Time)

- *Wall-clock or execution* time
  - You can do this at home:

    ```
    time java FindPrimes 1000
    ```
  - Advantages: easy to measure, meaning is obvious.
  - Appropriate where time is critical (real-time systems, e.g.).
  - Disadvantages: applies only to specific data set, compiler, machine, etc.
- *Dynamic statement counts* of # of times statements are executed:
  - Advantages: more general (not sensitive to speed of machine).
  - Disadvantages: doesn't tell you actual time, still applies only to specific data sets.
- *Symbolic execution times:*
  - That is, *formulas* for execution times as functions of input size.
  - Advantages: applies to all inputs, makes scaling clear.
  - Disadvantage: practical formula must be approximate, may tell very little about actual time.



## Asymptotic Cost

- Symbolic execution time lets us see *shape* of the cost function.
- Since we are approximating anyway, pointless to be precise about certain things:
  - *Behavior on small inputs*:
    - Can always pre-calculate some results.
    - Times for small inputs not usually important.
    - Often more interested in *asymptotic behavior* as input size becomes very large.
  - *Constant factors* (as in "off by factor of 2"):
    - Just changing machines causes constant-factor change.
- How to abstract away from (i.e., ignore) these things?

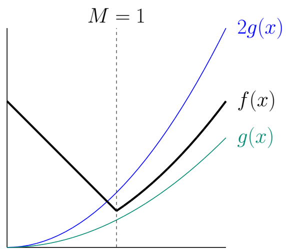


## Handy Tool: Order Notation

- Idea: Don't try to produce specific functions that specify size, but rather *families of functions with similarly behaved magnitudes*.
- Then say something like "$f$ is bounded by $g$ if it is in $g$'s family."
- For any function $g(x)$, the functions $2g(x)$, $0.5g(x)$, or for any $K > 0$, $K \cdot g(x)$, all have the same "shape". So put all of them into $g$'s family.
- Any function $h(x)$ such that $h(x) = K \cdot g(x)$ for $x > M$ (for some constant $M$) has $g$'s shape "except for small values." So put all of these in $g$'s family.
- For upper limits, throw in all functions whose absolute value is everywhere $\leq$ some member of $g$'s family. Call this set $O(g)$ or $O(g(n))$.
- Or, for lower limits, throw in all functions whose absolute value is everywhere $\geq$ some member of $g$'s family. Call this set $\Omega(g)$.
- Finally, define $\Theta(g) = O(g) \cap \Omega(g)$—the set of functions *bracketed in magnitude by* two members of $g$'s family.

## Big Oh

- Goal: Specify bounding from above.




- Here, $f(x) \leq 2g(x)$ as long as $x > 1$,
- So $f(x)$ is in $g$'s "bounded-above family," written

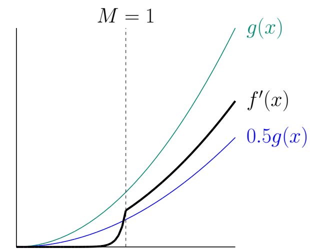
$$f(x) \in O(g(x)),$$

- . . . even though (in this case) $f(x) > g(x)$ everywhere.



## Big Omega

- Goal: Specify bounding from below:




- Here, $f'(x) \geq \frac{1}{2}g(x)$ as long as $x > 1$,
- So $f'(x)$ is in $g$'s "bounded-below family," written

$$f'(x) \in \Omega(g(x)),$$

- . . . even though $f(x) < g(x)$ everywhere.



## Big Theta

- In the two previous slides, we not only have $f(x) \in O(g(x))$ and $f'(x) \in \Omega(g(x))$,. . .
- . . . but also $f(x) \in \Omega(g(x))$ and $f'(x) \in O(g(x))$.
- We can summarize this all by saying $f(x) \in \Theta(g(x))$ and $f'(x) \in \Theta(g(x))$.



## Aside: Various Mathematical Pedantry

- Technically, if I am going to talk about $O(\cdot)$, $\Omega(\cdot)$ and $\Theta(\cdot)$ as sets of functions, I really should write, for example,

$$f \in O(g) \quad \text{instead of} \quad f(x) \in O(g(x))$$

- In effect, $f(x) \in O(g(x))$ is short for $\lambda\, x.\ f(x) \in O(\lambda\, x.\ g(x))$.
- The standard notation outside this course, in fact, is $f(x) = O(g(x))$, but personally, I think that's a serious abuse of notation.



## How We Use Order Notation

- Elsewhere in mathematics, you'll see $O(\ldots)$, etc., used generally to specify bounds on functions.
- For example,

$$\pi(N) = \Theta(\frac{N}{\ln N})$$

which I would prefer to write

$$\pi(N) \in \Theta(\frac{N}{\ln N})$$

(Here, $\pi(N)$ is the number of primes less than or equal to $N$.)

- Also, you'll see things like

$$f(x) = x^3 + x^2 + O(x) \qquad \text{(or } f(x) \in x^3 + x^2 + O(x)\text{)},$$

meaning that $f(x) = x^3 + x^2 + g(x)$ where $g(x) \in O(x)$.

- For our purposes, the functions we will be bounding will be *cost functions:* functions that measure the amount of execution time or the amount of space required by a program or algorithm.



## Why It Matters

- Computer scientists often talk as if constant factors didn't matter at all, only the difference of $\Theta(N)$ vs. $\Theta(N^2)$.
- In reality they do matter, but at some point, constants always get swamped.

| $n$ | $16 \lg n$ | $\sqrt{n}$ | $n$ | $n \lg n$ | $n^2$ | $n^3$ | $2^n$ |
|---|---|---|---|---|---|---|---|
| 2 | 16 | 1.4 | 2 | 2 | 4 | 8 | 4 |
| 4 | 32 | 2 | 4 | 8 | 16 | 64 | 16 |
| 8 | 48 | 2.8 | 8 | 24 | 64 | 512 | 256 |
| 16 | 64 | 4 | 16 | 64 | 256 | 4,096 | 65,636 |
| 32 | 80 | 5.7 | 32 | 160 | 1024 | 32,768 | $4.2 \times 10^9$ |
| 64 | 96 | 8 | 64 | 384 | 4,096 | 262,144 | $1.8 \times 10^{19}$ |
| 128 | 112 | 11 | 128 | 896 | 16,384 | $2.1 \times 10^9$ | $3.4 \times 10^{38}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 1,024 | 160 | 32 | 1,024 | 10,240 | $1.0 \times 10^6$ | $1.1 \times 10^9$ | $1.8 \times 10^{308}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $2^{20}$ | 320 | 1024 | $1.0 \times 10^6$ | $2.1 \times 10^7$ | $1.1 \times 10^{12}$ | $1.2 \times 10^{18}$ | $6.7 \times 10^{315,652}$ |

- For example: replace column $n^2$ with $10^6 \cdot n^2$ and it still becomes dominated by $2^n$.

## Some Intuition on Meaning of Growth

- How big a problem can you solve in a given time?
- In the following table, left column shows time in microseconds to solve a given problem as a function of problem size $N$.
- Entries show the *size of problem* that can be solved in a second, hour, month (31 days), and century, for various relationships between time required and problem size.
- $N$ = problem size.

| Time ($\mu$sec) for problem size $N$ | Max $N$ Possible in 1 second | 1 hour | 1 month | 1 century |
|---|---|---|---|---|
| $\lg N$ | $10^{300000}$ | $10^{1000000000}$ | $10^{8\cdot 10^{11}}$ | $10^{10^{14}}$ |
| $N$ | $10^6$ | $3.6 \cdot 10^9$ | $2.7 \cdot 10^{12}$ | $3.2 \cdot 10^{15}$ |
| $N \lg N$ | $63000$ | $1.3 \cdot 10^8$ | $7.4 \cdot 10^{10}$ | $6.9 \cdot 10^{13}$ |
| $N^2$ | $1000$ | $60000$ | $1.6 \cdot 10^6$ | $5.6 \cdot 10^7$ |
| $N^3$ | $100$ | $1500$ | $14000$ | $150000$ |
| $2^N$ | $20$ | $32$ | $41$ | $51$ |



## Using the Notation

- Can use this order notation for any kind of real-valued function.
- We will use them to describe cost functions. Example:

```
/** Find position of X in list L, or -1 if not found. */
int find(List L, Object X) {
   int c;
   for (c = 0; L != null; L = L.next, c += 1)
      if (X.equals(L.head)) return c;
   return -1;
}
```

- Choose representative operation: number of `.equals` tests.
- If $N$ is length of $L$, then loop does *at most* $N$ tests: *worst-case time* is $N$ tests.
- In fact, total # of instructions executed is roughly proportional to $N$ in the worst case, so can also say worst-case time is $O(N)$, regardless of units used to measure.
- Use $N > M$ provision (in defn. of $O(\cdot)$) to ignore empty list.



## Be Careful

- It's also true that the worst-case time is $O(N^2)$, since $N \in O(N^2)$ also: Big-Oh bounds are loose.
- The worst-case time is $\Omega(N)$, since $N \in \Omega(N)$, but that does *not* mean that the loop *always* takes time $N$, or even $K \cdot N$ for some $K$.
- Instead, we are just saying something about the *function* that maps $N$ into the *largest possible* time required to process any array of length $N$.
- To say as much as possible about our worst-case time, we should try to give a $\Theta$ bound: in this case, we can: $\Theta(N)$.
- But again, that still tells us nothing about *best-case* time, which happens when we find `X` at the beginning of the loop. Best-case time is $\Theta(1)$.



## Effect of Nested Loops

- Nested loops often lead to polynomial bounds:

```
for (int i = 0; i < A.length; i += 1)
   for (int j = 0; j < A.length; j += 1)
      if (i != j && A[i] == A[j])
         return true;
return false;
```

- Clearly, time is $O(N^2)$, where $N$ = `A.length`. *Worst-case time* is $\Theta(N^2)$.
- Loop is inefficient though:

```
for (int i = 0; i < A.length; i += 1)
   for (int j = i+1; j < A.length; j += 1)
      if (A[i] == A[j]) return true;
return false;
```

- Now worst-case time is proportional to

$$N - 1 + N - 2 + \ldots + 1 = N(N-1)/2 \in \Theta(N^2)$$

(so asymptotic time unchanged by the constant factor).



## Recursion and Recurrences: Fast Growth

- Silly example of recursion. In the worst case, both recursive calls happen:

```
/** True iff X is a substring of S */
boolean occurs(String S, String X) {
  if (S.equals(X)) return true;
  if (S.length() <= X.length()) return false;
  return
    occurs(S.substring(1), X) ||
    occurs(S.substring(0, S.length()-1), X);
}
```

- Define $C(N)$ to be the worst-case cost of `occurs(S,X)` for `S` of length $N$, `X` of fixed size $N_0$, measured in # of calls to `occurs`. Then

$$C(N) = \begin{cases} 1, & \text{if } N \le N_0, \\ 2C(N-1)+1 & \text{if } N > N_0 \end{cases}$$

- So $C(N)$ grows exponentially:

$$C(N) = 2C(N-1)+1 = 2(2C(N-2)+1)+1 = \ldots = \underbrace{2(\cdots 2}_{N-N_0} \cdot 1+1)+\ldots+1$$
$$= 2^{N-N_0} + 2^{N-N_0-1} + 2^{N-N_0-2} + \ldots + 1 = 2^{N-N_0+1} - 1 \in \Theta(2^N)$$



## Binary Search: Slow Growth

```
/** True X iff is an element of S[L .. U]. Assumes
 *  S in ascending order, 0 <= L <= U-1 < S.length. */
boolean isIn(String X, String[] S, int L, int U) {
  if (L > U) return false;
  int M = (L+U)/2;
  int direct = X.compareTo(S[M]);
  if (direct < 0) return isIn(X, S, L, M-1);
  else if (direct > 0) return isIn(X, S, M+1, U);
  else return true;
}
```

- Here, worst-case time, $C(D)$, (as measured by # of calls to `.compareTo`), depends on size $D = U - L + 1$.
- We eliminate `S[M]` from consideration each time and look at half the rest. Assume $D = 2^k - 1$ for simplicity, so:

$$\begin{aligned} C(D) &= \begin{cases} 0, & \text{if } D \le 0, \\ 1 + C((D-1)/2), & \text{if } D > 0. \end{cases} \\ &= \underbrace{1+1+\ldots+1}_{k} + 0 \\ &= k = \lg(D+1) \in \Theta(\lg D) \end{aligned}$$

## Another Typical Pattern: Merge Sort

```
List sort(List L) {
  if (L.length() < 2) return L;
  Split L into L0 and L1 of about equal size;
  L0 = sort(L0);  L1 = sort(L1);
  return Merge of L0 and L1
}
```

Merge ("combine into a single ordered list") takes time proportional to size of its result.

- Assuming that size of L is $N = 2^k$, worst-case cost function, $C(N)$, counting just merge time (which is proportional to # items merged):

$$\begin{aligned} C(N) &= \begin{cases} 0, & \text{if } N < 2; \\ 2C(N/2) + N, & \text{if } N \geq 2. \end{cases} \\ &= 2(2C(N/4) + N/2) + N \\ &= 4C(N/4) + N + N \\ &= 8C(N/8) + N + N + N \\ &= N \cdot 0 + \underbrace{N + N + \ldots + N}_{k = \lg N} \\ &= N \lg N \end{aligned}$$

- In general, can say it's $\Theta(N \lg N)$ for arbitrary $N$ (not just $2^k$).