# CS61B Lecture #33

**Today's Readings:** Graph Structures: *DSIJ*, Chapter 12

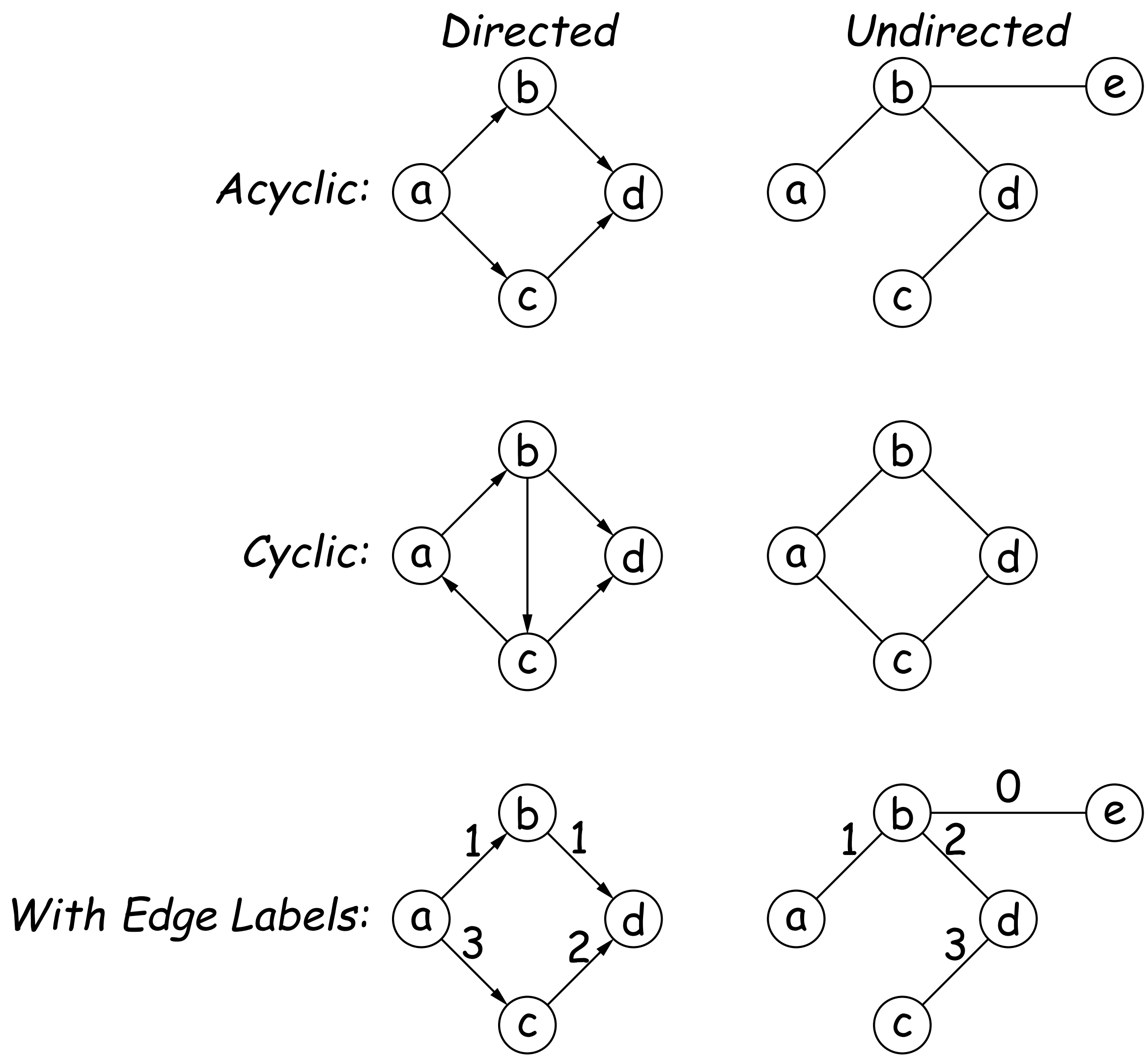
# Why Graphs?

- For expressing non-hierarchically related items
- Examples:
  - Networks: pipelines, roads, assignment problems
  - Representing processes: flow charts, Markov models
  - Representing partial orderings: PERT charts, makefiles

# Some Terminology

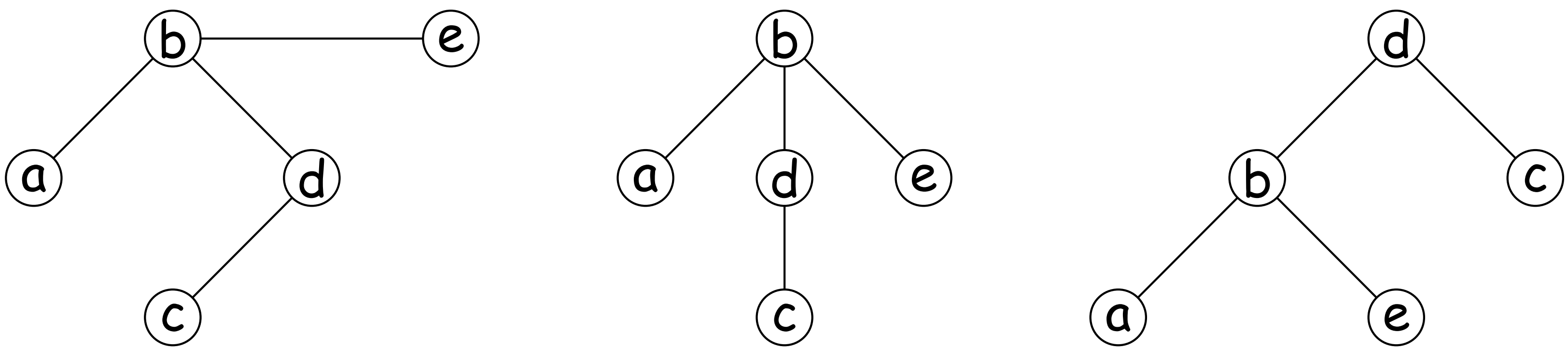
- A *graph* consists of
  - A set of *nodes* (aka *vertices*)
  - A set of *edges:* pairs of nodes.
  - Nodes with an edge between are *adjacent*.
  - Depending on problem, nodes or edges may have *labels* (or *weights*)
- Typically call node set $V = \{v_0, \ldots\}$, and edge set $E$.
- If the edges have an order (first, second), they are *directed edges,* and we have a *directed graph (digraph),* otherwise an *undirected graph*.
- Edges are *incident* to their nodes.
- Directed edges *exit* one node and *enter* the next.
- A *cycle* is a path without repeated edges leading from a node back to itself (following arrows if directed).
- A graph is *cyclic* if it has a cycle, else *acyclic*. Abbreviation: Directed Acyclic Graph—*DAG*.

# Some Pictures



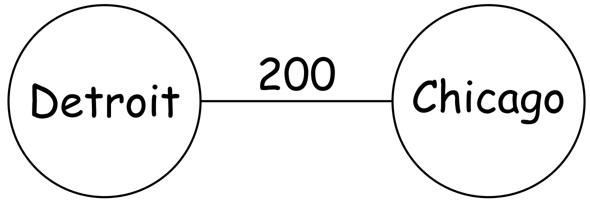
# Trees are Graphs

- A graph is *connected* if there is a (possibly directed) path between every pair of nodes.
- That is, if one node of the pair is *reachable* from the other.
- A DAG is a (rooted) tree iff connected, and every node but the root has exactly one parent.
- A connected, acyclic, undirected graph is also called a *free tree.* Free: we're free to pick the root; e.g.,

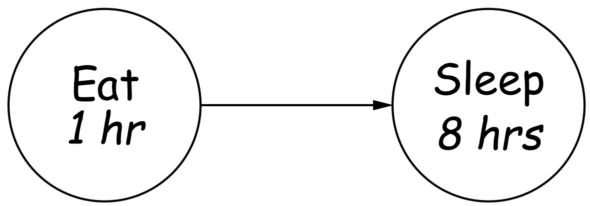


# Examples of Use

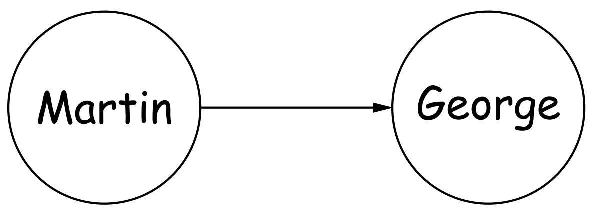
- Edge = Connecting road, with length.




- Edge = Must be completed before; Node label = time to complete.

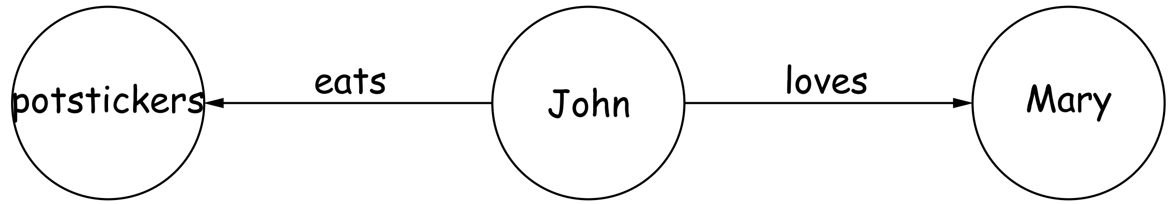



- Edge = Begat

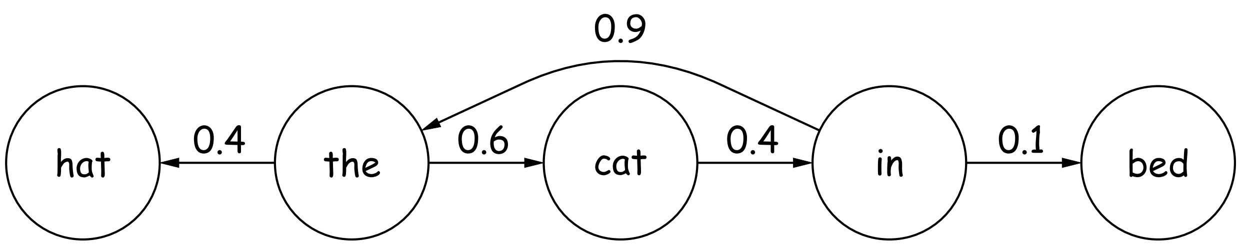


# More Examples

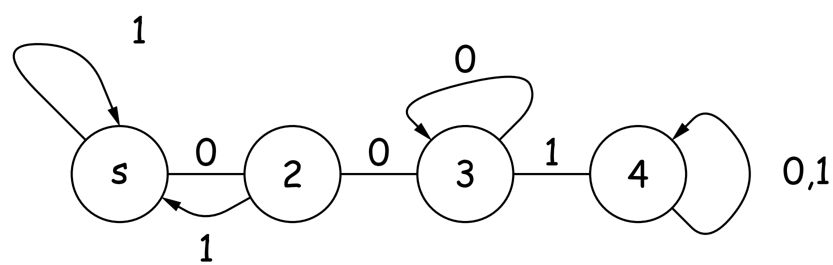
- Edge = some relationship




- Edge = next state might be (with probability)

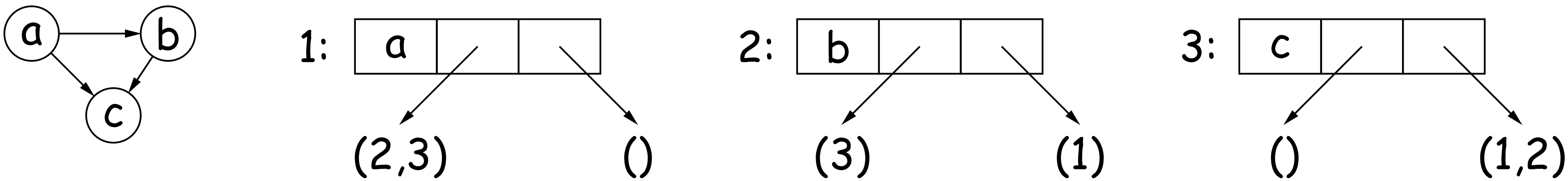



- Edge = next state in state machine, label is triggering input. (Start at s. Being in state 4 means "there is a substring '001' somewhere in the input".)



# Representation

- Often useful to number the nodes, and use the numbers in edges.
- *Edge list representation:* each node contains some kind of list (e.g., linked list or array) of its successors (and possibly predecessors).




- *Edge sets:* Collection of all edges. For graph above:

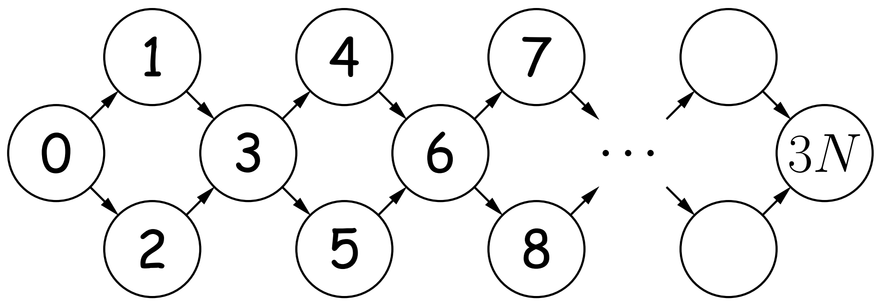
$$\{(1,2),(1,3),(2,3)\}$$

- *Adjacency matrix:* Represent connection with matrix entry:

$$\begin{array}{cc} & \begin{array}{ccc} \mathit{1} & \mathit{2} & \mathit{3} \end{array} \\ \begin{array}{c} \mathit{1} \\ \mathit{2} \\ \mathit{3} \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

# Traversing a Graph

- Many algorithms on graphs depend on traversing all or some nodes.
- Can't quite use recursion because of cycles.
- Even in acyclic graphs, can get combinatorial explosions:




Treat 0 as the root and do recursive traversal down the two edges out of each node: $\Theta(2^N)$ operations!

- So typically try to visit each node constant # of times (e.g., once).

# Recursive Depth-First Traversal of a Graph

- Can fix looping and combinatorial problems using the "bread-crumb" method used in earlier lectures for a maze.
- That is, *mark* nodes as we traverse them and don't traverse previously marked nodes.
- Makes sense to talk about *preorder* and *postorder*, as for trees.

```
void preorderTraverse(Graph G, Node v)
{
  if (v is unmarked) {
    mark(v);
    visit v;
    for (Edge(v, w) ∈ G)
      traverse(G, w);
  }
}
```

```
void postorderTraverse(Graph G, Node v)
{
  if (v is unmarked) {
    mark(v);
    for (Edge(v, w) ∈ G)
      traverse(G, w);
    visit v;
  }
}
```

# Recursive Depth-First Traversal of a Graph (II)

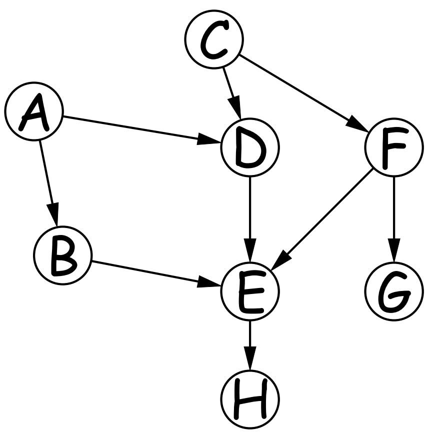
- We are often interested in traversing *all* nodes of a graph, not just those reachable from one node.
- So we can repeat the procedure as long as there are unmarked nodes.

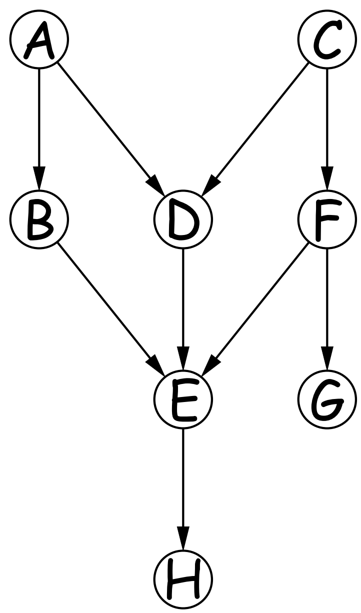
```
void preorderTraverse(Graph G) {
   for (v ∈ nodes of G) {
      preorderTraverse(G, v);
   }
}

void postorderTraverse(Graph G) {
   for (v ∈ nodes of G) {
      postorderTraverse(G, v);
   }
}
```

# Topological Sorting

**Problem:** Given a DAG, find a linear order of nodes consistent with the edges.

- That is, order the nodes $v_0,\ v_1,\ \ldots$ such that $v_k$ is never reachable from $v_{k'}$ if $k' > k$.
- Gmake does this. Also PERT charts.

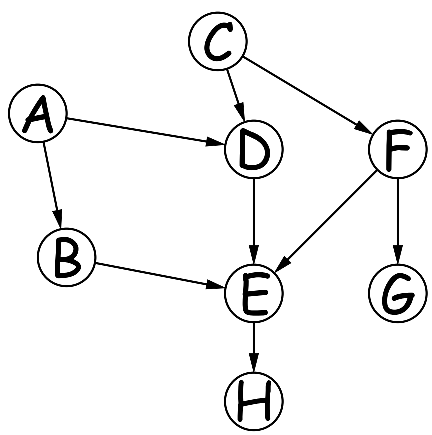


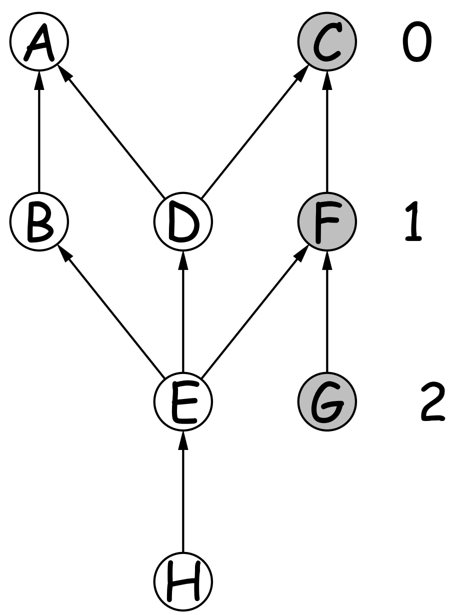




| | | |
|---|---|---|
| A | C | C |
| C | A | F |
| | | G |
| B | F | A |
| D | D | B |
| F | B | |
| E | G | D |
| G | E | E |
| H | H | H |

# Sorting and Depth First Search

- **Observation**: Suppose we *reverse the links* on our graph.
- If we do a recursive DFS on the reverse graph, starting from node H, for example, we will find all nodes that must come *before* H.
- When the search reaches a node in the reversed graph and there are no successors, we know that it is safe to put that node first.
- In general, a *postorder* traversal of the reversed graph visits nodes only after all predecessors have been visited.




Numbers show post-order traversal order starting from G: everything that must come before G.

# General Graph Traversal Algorithm

```
COLLECTION_OF_VERTICES fringe;

fringe = INITIAL_COLLECTION;
while (!fringe.isEmpty()) {
  Vertex v = fringe.REMOVE_HIGHEST_PRIORITY_ITEM();

  if (!MARKED(v)) {
    MARK(v);
    VISIT(v);
    For each edge(v,w) {
      if (NEEDS_PROCESSING(w))
        Add w to fringe;
    }
  }
}
```

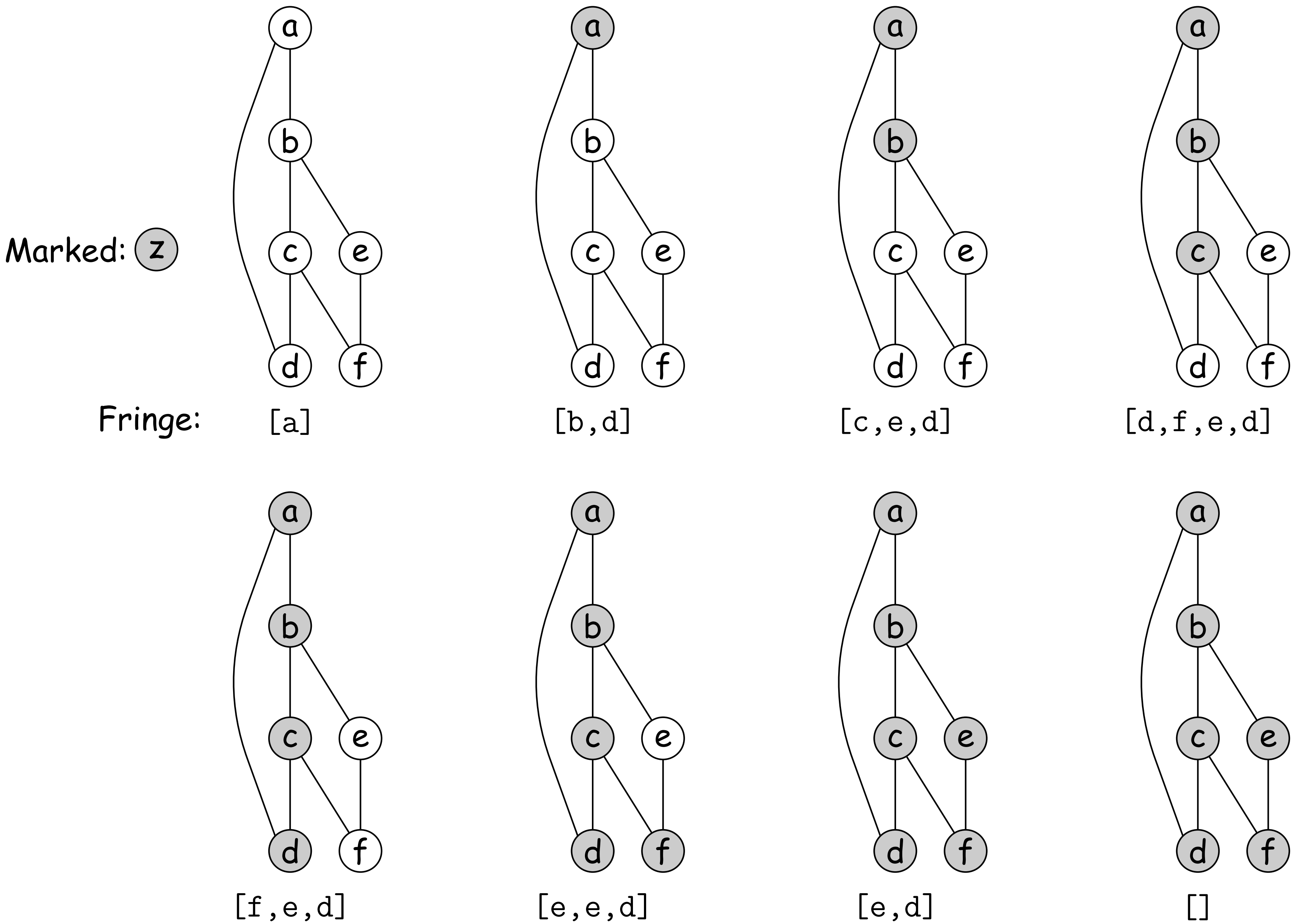
Replace *COLLECTION_OF_VERTICES*, *INITIAL_COLLECTION*, etc. with various types, expressions, or methods to different graph algorithms.

# Example: Depth-First Traversal

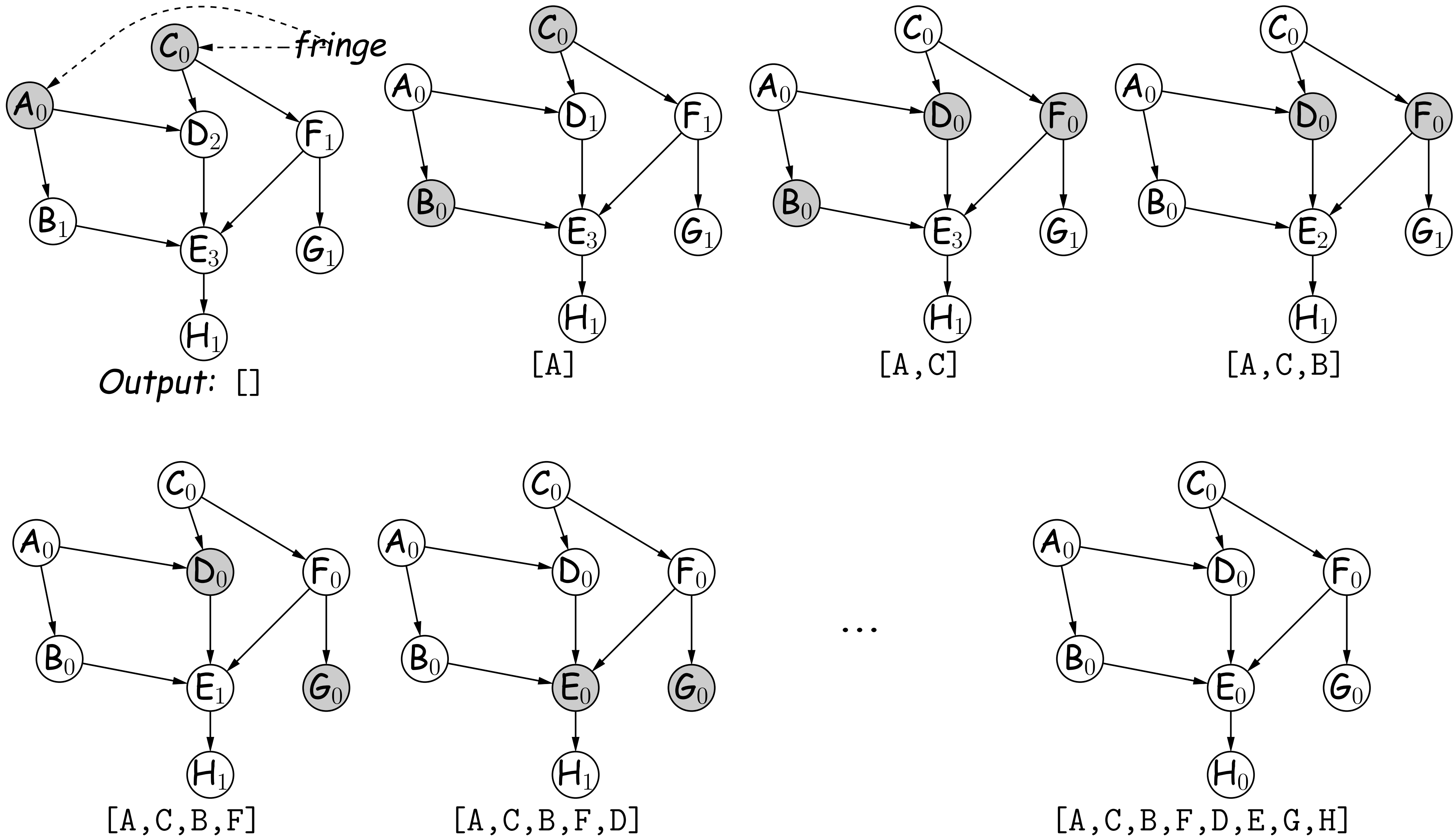
**Problem:** Visit every node reachable from $v$ once, visiting nodes further from start first.

```
Stack<Vertex> fringe;

fringe = stack containing {v};
while (!fringe.isEmpty()) {
  Vertex v = fringe.pop();

  if (!marked(v)) {
    mark(v);
    VISIT(v);
    For each edge(v,w) {
      if (!marked(w))
        fringe.push(w);
    }
  }
}
```

# Depth-First Traversal Illustrated



# Topological Sort in Action



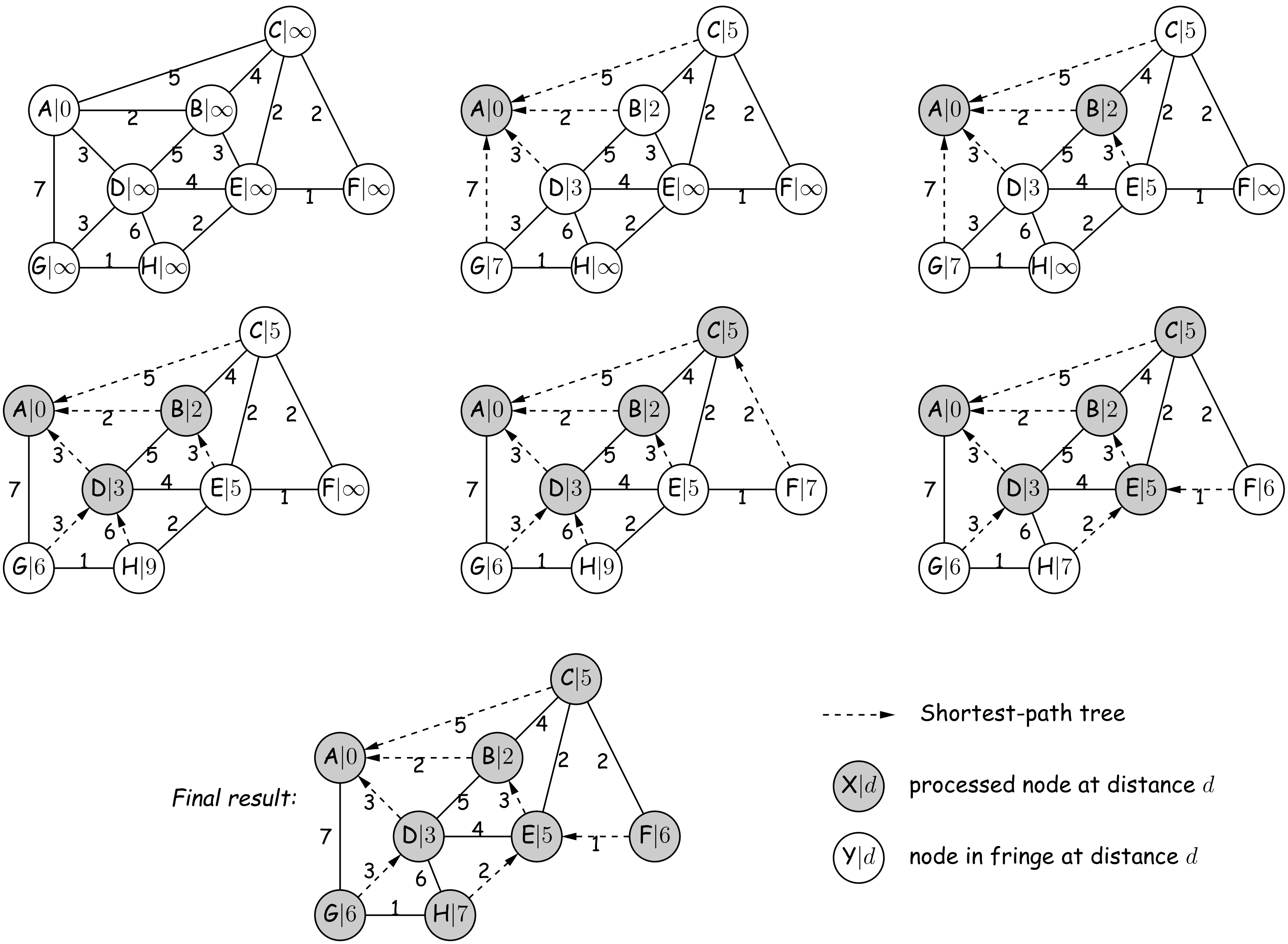
# Shortest Paths: Dijkstra's Algorithm

**Problem:** Given a graph (directed or undirected) with non-negative edge weights, compute shortest paths from given source node, $s$, to all nodes.

- "Shortest" = sum of weights along path is smallest.
- For each node, keep estimated distance from $s$, ...
- ...and of preceding node in shortest path from $s$.

```
PriorityQueue<Vertex> fringe;
For each node v { v.dist() = ∞; v.back() = null; }
s.dist() = 0;
fringe = priority queue ordered by smallest .dist();
add all vertices to fringe;
while (!fringe.isEmpty()) {
  Vertex v = fringe.removeFirst();

  For each edge(v,w) {
    if (v.dist() + weight(v,w) < w.dist())
      { w.dist() = v.dist() + weight(v,w); w.back() = v; }
  }
}
```

# Example


Final result:
Shortest-path tree
processed node at distance $d$
node in fringe at distance $d$