Artificial Intelligence 1
Assignment7: Propositional Logic
– Given Dec 8, Due Dec 18 –

Problem 7.1 (PL Concepts) 30 pt

Which of the following statements are true? In each case, give an informal argument why it is true or a counter-example.

1. Every satisfiable formula is valid.
2. Every valid formula is satisfiable.
3. If A is satisfiable, then $\neg A$ is unsatisfiable.
4. If $A \models B$, then $A \wedge C \models B \wedge C$.
5. Every admissible inference rule is derivable.
6. If $\vdash$ is sound for $\models$ and $\{A, B\} \vdash C$, then C is satisfiable if A and B are.

Solution:

1. Not true. Counter-example: p is satisfiable (put $\varphi(p) = \mathtt{T}$) but not valid (falsified by $\varphi(p) = \mathtt{F}$).

2. True. Assume F is valid. Then F is satisfied by all assignments. We know (This is a subtle step that can easily be overlooked.) that there is at least one assignment a. (Even if there are no propositional variables, we would still have the empty assignment.) So a must satisfy F and therefore F is satisfiable.

3. Not true. Counter-example: p is satisfiable (put $\varphi(p) = \mathtt{T}$), but $\neg p$ is also satisfiable (put $\varphi(p) = \mathtt{F}$).

4. True. Assume $A \vDash B$ (H) and an assignment φ such that $I_\varphi(A \wedge C) = \mathtt{T}$ (A).
   We need to show that also $I_\varphi(A \wedge C) = \mathtt{T}$ (G).
   By definition, (A) yields $I_\varphi(A) = \mathtt{T}$ (A1) and $I_\varphi(C) = \mathtt{T}$ (A2).
   By definition of (H), we obtain from (A1) that $I_\varphi(B) = \mathtt{T}$ (B).
   Then we obtain (G) from its definition and (B) and (A2).

5. Not true. Counter-example: The empty derivation relation has no inference rules and thus no derivable formulas. Then any rule with non-empty set of assumptions is admissible. But no rule is derivable.

6. Not true. The assumptions do show that $A, B \vDash C$. So if we have an assignment that satisfies both A and B, then that assignment also satisfies C and thus C is satisfiable. But we only know that A and B are satisfiable by some assignments, not necessarily the same one. A counter-example, is $A = p$, $B = \neg p$, C any unsatisfiable formula. Then $A, B \vDash C$ holds (because there are no assignments that satisfy both A and B), and A and B but not C are satisfiable.

Problem 7.2 (Equivalence of CSP and SAT) 30 pt

We consider

- CSPs (V, D, C) with finite domains as before
- SAT problems (V, A) where V is a set of propositional variables and A is a propositional formula over V.

We will show that these problem classes are equivalent by reducing their instances to each other.

1. Given a SAT instance $P = (V, A)$, define a CSP instance $P' = (V', D', C')$ and two bijections
   - f mapping satisfying assignments of P to solutions of P'
   - f' the inverse of f

We already know that binary CSPs are equivalent to higher-order CSPs. Therefore, it is sufficient to give a higher-order CSP.

2. Given a CSP instance (V, D, C), define a SAT instance (V', A') and bijections as above

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Solution:

1. We define P' by $V' = V$, $D_v = \{\mathtt{T}, \mathtt{F}\}$ for every $v \in V$, and $C = \{A\}$, i.e., C contains the single higher-order constraint that holds if an assignment to V' (seen as an propositional assignment to V) satisfies A.
f and f' are the identity.

2. We define P' as follows. V' contains variables p'_{va} for every $v \in V$ and $a \in D_v$. The intuition behind p'_{va} is that v has value a.
A' is the conjunction of the following formulas:

   - for all $v \in V$ with $D_v = \{a_1, \ldots, a_n\}$, the formula $p'_{va_1} \vee \ldots \vee p'_{va_n}$ (i.e., v must have at least one value)
   - for all $v \in V$, and $a, b \in D_v$ with $a \neq b$, the formula $p'_{va} \Rightarrow \neg p'_{vb}$ (i.e., v can have at most one value)
   - for all C_{vw} and $(a, b) \notin C_{vw}$, the formula $\neg(p'_{va} \wedge p'_{wb})$ (i.e., every constraint must be satisfied)

   The bijection f maps a solution α of P to a A'-satisfying propositional assignment φ for V' as follows: for all v, a, we put $\varphi(p'_{va}) = \mathtt{T}$ if $\alpha(v) = a$ and $\varphi(p'_{va}) = \mathtt{F}$ otherwise.
   The inverse bijection f' maps an A'-satisfying assignment φ to a solution α of P as follows: for all v we put $\alpha(v) = a$ where a is the unique value for which $\varphi(p'_{va}) = \mathtt{T}$.

Problem 7.3 (Calculi Comparison) 60 pt

Prove (or disprove) the validity of the following formulae in i) Natural Deduction ii) Tableau and iii) Resolution.

1. $(P \wedge Q) \Rightarrow (P \vee Q)$ (to be done in the tutorial, not part of grading)
2. $((A \vee B) \wedge (A \Rightarrow C) \wedge (B \Rightarrow C)) \Rightarrow C$
3. $((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$

Solution: ND

1.

(1)	1	$(P \wedge Q)$	Assumption
(2)	1	P	$\wedge E_\ell$ (on 1)
(3)	1	$(P \vee Q)$	$\vee I_\ell$ (on 2)
(4)		$(P \wedge Q) \Rightarrow (P \vee Q)$	$\Rightarrow I$ (on 1 and 3)

2.

(1)	1	$(A \vee B) \wedge ((A \Rightarrow C) \wedge (B \Rightarrow C))$	Assumption
(2)	1	$(A \vee B)$	$\wedge E_\ell$ (on 1)
(3)	1	$(A \Rightarrow C) \wedge (B \Rightarrow C)$	$\wedge E_r$ (on 1)
(4)	1	$(A \Rightarrow C)$	$\wedge E_\ell$ (on 3)
(5)	1	$(B \Rightarrow C)$	$\wedge E_r$ (on 3)
(6)	1,6	A	Assumption
(7)	1,6	C	$\Rightarrow E$ (on 4 and 6)
(8)	1,8	B	Assumption
(9)	1,8	C	$\Rightarrow E$ (on 5 and 8)
(10)	1	C	$\vee E$ (on 2, 7 and 9)
(11)		$((A \vee B) \wedge (A \Rightarrow C) \wedge (B \Rightarrow C)) \Rightarrow C$	$\Rightarrow I$ (on 1 and 10)

3.

(1)		$(P \vee \neg P)$	TND
(2)	2	P	Assumption
(3)	2,3	$(P \Rightarrow Q) \Rightarrow P)$	Assumption
(4)	2	$((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$	$\Rightarrow I$ (on 3 and 2)
(5)	5	$\neg P$	Assumption
(6)	5,6	$(P \Rightarrow Q) \Rightarrow P)$	Assumption
(7)	5,6,7	P	Assumption
(8)	5,6,7	F	FI (on 5 and 7)
(9)	5,6,7	Q	FE (on 8)
(10)	5,6	$P \Rightarrow Q$	$\Rightarrow I$ (on 7 and 9)
(11)	5,6	P	$\Rightarrow E$ (on 6 and 10)
(12)	5	$((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$	$\Rightarrow$ I (on 6 and 11)
(13)		$((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$	$\vee E$ (on 1, 4 and 12)

Solution: Tableau

1.

(1)	$(P \wedge Q) \Rightarrow (P \vee Q)^F$	
(2)	$(P \wedge Q)^T$	(from 1)
(3)	$(P \vee Q)^F$	(from 1)
(4)	P^T	(from 2)
(5)	Q^T	(from 2)
(6)	P^F	(from 3)
	close on P	

2.

(1)	$((A \vee B) \wedge (A \Rightarrow C) \wedge (B \Rightarrow C)) \Rightarrow C^F$						
(2)	$(A \vee B) \wedge (A \Rightarrow C) \wedge (B \Rightarrow C)^T$						(from 1)
(3)	C^F						(from 1)
(4)	$(A \vee B)^T$						(from 2)
(5)	$(A \Rightarrow C)^T$						(from 2)
(6)	$(B \Rightarrow C)^T$						(from 2)
(7)	A^T			B^T			(split on 6)
(8)	A^F close on A	C^T close on C	(split on 5)	B^F close on B	C^T close on C	(split on 4)	

3.

(1)	$((P \Rightarrow Q) \Rightarrow P) \Rightarrow P^F$		
(2)	$(P \Rightarrow Q) \Rightarrow P)^T$		(from 1)
(3)	P^F		(from 1)
(4)	$P \Rightarrow Q^F$	P^T	(split on 2)
(5)	P^T (from 4)	close on P	
(6)	Q^F (from 4)		
	close on P		

Solution:

Resolution

1. $(P \wedge Q) \Rightarrow (P \vee Q)$: We negate and build a CNF:

$$\begin{aligned} &(P \wedge Q) \wedge \neg(P \vee Q) \\ \equiv &P \wedge Q \wedge \neg P \wedge \neg Q \end{aligned}$$

yielding clauses $\{P^T\}, \{Q^T\}, \{P^F\}, \{Q^F\}$
Resolving the two red clauses yields $\{\}$.

2. $((A \vee B) \wedge (A \Rightarrow C) \wedge (B \Rightarrow C)) \Rightarrow C$: We negate and build a CNF:

$$\begin{aligned} &((A \vee B) \wedge (A \Rightarrow C) \wedge (B \Rightarrow C)) \wedge \neg C \\ \equiv &(A \vee B) \wedge (\neg A \vee C) \wedge (\neg B \vee C) \wedge \neg C \end{aligned}$$

yielding clauses $\{A^T, B^T\}, \{A^F, C^T\}, \{B^F, C^T\}, \{C^F\}$.
Resolving yields:

$$\begin{aligned} \{A^F, C^T\} + \{C^F\} &\Longrightarrow \{A^F\} \\ \{B^F, C^T\} + \{C^F\} &\Longrightarrow \{B^F\} \\ \{A^T, B^T\} + \{A^F\} &\Longrightarrow \{B^T\} \\ \{B^T\} + \{B^F\} &\Longrightarrow \{\} \end{aligned}$$

3. $((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$: We negate and build a CNF:

$$\begin{aligned} &((P \Rightarrow Q) \Rightarrow P) \wedge \neg P \\ \equiv &(\neg(P \Rightarrow Q) \vee P) \wedge \neg P \\ \equiv &((P \wedge \neg Q) \vee P) \wedge \neg P \\ \equiv &((P \vee P) \wedge (\neg Q \vee P) \wedge \neg P \end{aligned}$$

yielding clauses $\{P^T\}, \{Q^F, P^T\}, \{P^F\}$.
Resolving the two red clauses yields $\{\}$.