AI 1 2022/23
Assignment9: First-Order Logic
– Given Jan. 12, Due Jan. 22 –

Problem 9.1 (Induction) 20 pt

Use structural induction on terms and formulas to define a function C that maps every term/formula to the number of occurrences of free variables. For example, $C(\forall x.P(x,x,y,y,z)) = 3$ because the argument has 2 free occurrences of y and 1 of z.

Hint: Use an auxiliary function $C'(V,A)$ that takes the set V of bound variables and a term/formula A. Define C' by structural induction on A. Then define $C(A) = C'(\emptyset, A)$.

Solution: C' is defined as follows for terms

- variables X: $C'(V,X) = 0$ if $X \in V$ and $C'(V,X) = 1$ if $X \notin V$
- applications of n-ary function symbol f: $C'(V, f(t_1, ..., t_n)) = \Sigma_i C'(V, t_i)$

and for formulas

- applications of n-ary predicate symbol p: $C'(V, p(t_1, ..., t_n)) = \Sigma_i C'(V, t_i)$
- nullary connectives: $C'(V,T) = C'(V,F) = 0$
- unary connectives: $C'(V, \neg A) = C'(V,A)$
- binary connectives: $C'(V, A_1 \wedge A_2) = C'(V, A_1 \vee A_2) = C'(V, A_1 \rightarrow A_2) = C'(V,A_1) + C'(V,A_2)$
- quantifiers: $C'(V, \forall x.A) = C'(V, \exists x.A) = C'(V \cup \{x\}, A)$

This definition exhibits the typical pattern of structural induction:

- An additional argument (V) is used to track the bound variables.
- When recursing into a quantifier that argument is updated by adding the bound variable x. (In general, additional information about could be added, e.g., whether it is bound by $\forall$ or $\exists$.)
- At the leafs of the syntax tree (the base cases of the induction, here the variables), the additional argument is used.
- The main function is defined by initializing the additional argument (here with $\emptyset$).

Problem 9.2 (First-Order Semantics) 30 pt

Let $= \in \Sigma_2^p$, $P \in \Sigma_1^p$ and $+ \in \Sigma_2^f$. We use the semantics of first-order logic without equality.

Prove or refute the following formulas semantically. That means you must show that $I_\varphi(A) = T$ for all models I and assignments φ (without using a proof calculus) or to give some I, φ such that $I_\varphi(A) = F$.

1. $P(X)$
2. $\forall X.\forall Y. = (+(X,Y), +(Y,X))$
3. $\exists X.(P(X) \Rightarrow \forall Y.P(Y))$
4. $P(Y) \Rightarrow \exists X.P(X)$

Solution: Let φ be any value function.

1. Not valid. One out of many counter-examples is given by domain $\mathbb{N}$, $I(P) = \{0\}$, and $\varphi(X) = 1$.
2. Not valid. A counter-model is $\mathcal{I}_\varphi(=) = \emptyset$ with an arbitrary domain.
3. Valid:

$$\begin{aligned}
&\mathcal{I}_\varphi(\exists X.(P(X) \Rightarrow \forall Y.P(Y))) = \top\\
\Leftrightarrow&\text{There is some } a \in \mathcal{D}_\mathcal{I} \text{ s.t. } \mathcal{I}_\varphi((P(a) \Rightarrow \forall Y.P(Y))) = \top\\
\Leftrightarrow&\text{There is some } a \in \mathcal{D}_\mathcal{I} \text{ s.t. } \mathcal{I}_\varphi(\neg(P(a) \wedge \neg\forall Y.P(Y))) = \top\\
\Leftrightarrow&\text{There is some } a \in \mathcal{D}_\mathcal{I} \text{ s.t. } \mathcal{I}_\varphi(P(a) \wedge \neg\forall Y.P(Y)) = \bot\\
\Leftrightarrow&\text{There is some } a \in \mathcal{D}_\mathcal{I} \text{ s.t. } \mathcal{I}_\varphi(P(a)) = \bot \text{ or } \mathcal{I}_\varphi(\neg\forall Y.P(Y)) = \bot\\
\Leftrightarrow&\text{There is some } a \in \mathcal{D}_\mathcal{I} \text{ s.t. } \mathcal{I}_\varphi(P(a)) = \bot \text{ or } \mathcal{I}_\varphi(\forall Y.P(Y)) = \top\\
\Leftrightarrow&\text{There is some } a \in \mathcal{D}_\mathcal{I} \text{ s.t. } \mathcal{I}_\varphi(P(a)) = \bot \text{ or for all } b \in \mathcal{D}_\mathcal{I} : \ \mathcal{I}_\varphi(P(b)) = \top
\end{aligned}$$

   Now the last statement holds because if the left side does not hold, then the right side must hold.

Problem 9.3 (Natural Deduction) 25 pt

Let $R \in \Sigma_2^p, P \in \Sigma_1^p, c \in \Sigma_0^f$.

Prove the following formula in Natural Deduction:

$$((\forall X.\forall Y.R(Y,X) \Rightarrow P(Y)) \ \wedge \ (\exists Y.R(c,Y))) \Rightarrow P(c)$$

Solution:

1(Assumption)[1]	$(\forall X.\forall Y.R(Y,X) \Rightarrow P(Y)) \;\wedge\; (\exists Y.R(c,Y))$
2 $\wedge$ -Elimination on 1	$\forall X.\forall Y.R(Y,X) \Rightarrow P(Y)$
3 $\wedge$ -Elimination on 1	$\exists Y.R(c,Y)$
4$\forall$-Elimination on 2	$\forall Y.R(Y,X) \Rightarrow P(Y)$
5$\forall$-Elimination on 4	$R(c,X) \Rightarrow P(c)$
6$\forall$-Introduction on 5	$\forall X.R(c,X) \Rightarrow P(c)$
7(Assumption)[2]	$R(c,d)$
8$\forall$-Elimination on 6	$R(c,d) \Rightarrow P(c)$
9 $\Rightarrow$ -Elimination on 8, 7	$P(c)$
10$\exists$-Elimination[2] on 3, 9	$P(c)$
11 $\Rightarrow$ -Introduction[1] on 10	$((\forall X.\forall Y.R(Y,X) \Rightarrow P(Y)) \;\wedge\; (\exists Y.R(c,Y))) \Rightarrow P(c)$