Pumping Lemma

Discuss on — (1) Regular Languages
(2) Context free grammars.

I Pumping Lemma for Regular Languages

- why ? (i) $L = \{ a^n b^n \mid n \geq 0 \}$ not regular.

- Applications :- 1, Used to show Language is regular or not.

2, Procedure to show not regular

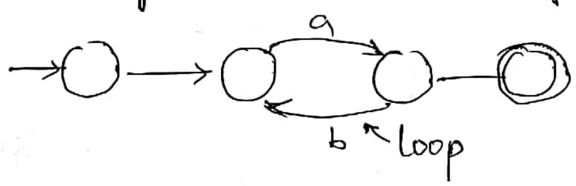
(i) if L is a regular, It satisfies Lemma

(ii) if L is not satisfies Lemma $\Rightarrow$ It is a not Regular.

- Regular Languages :-

$L \rightarrow$
(i) if Language is finite $\Rightarrow$ R.E $\Rightarrow$ R.L.
(ii) if infinite : $L = \{ ab, abab, ababab, \ldots \}$.

We can solve by Finite Automata if Loop can generate FA.
no Loop cannot " FA.



Lemma : Let L be a regular Language, Then there exist a constant n (which depends on L) such that for every string $w \in L$ such that $|w| \geq n$, we can break 'w' into three strings $w = xyz$ such that :

$\therefore$ n = no. of states of FA = Pumping Length.

(i) $y \neq \epsilon$ or ≥ 1

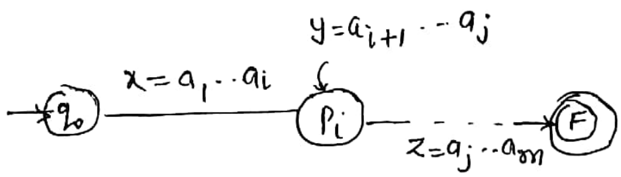
(ii) $|xy| \leq n$

(iii) For all $k \geq 0$, the string $xy^kz \in L$

$\therefore x = a_1 a_2 \ldots a_i$

$y = a_{i+1}, a_{i+2} \ldots a_j$

$z = a_{j+1}, a_{j+2} \ldots a_m$



Problems:

(1) Prove that $L = \{0^n 1^n / n \geq 1\}$

(i) It is not a finite Language $\Rightarrow$ not regular Language.

(ii) As per pumping Lemma.

(i) Assume L = is Regular, n = no. of states

(ii) Let $w = 0^n 1^n$ thus $|w| = 2n \geq n$

(iii) Let $w = xyz$ where $|xy| \leq n$

(iv) Break string w into three strings such that

$|xy| \leq n$, (i) $|y| \geq 1$ — (ii)

$\therefore x = 0^a, \; y = 0^b, \; z = 0^c 1^n$ where $a+b \leq n$

$b \geq = 1$

$c \geq 0$

$\therefore a+b+c = n$.

But condition (iii) $k = 0$:

$xy^0z = xz = 0^a 0^c 1^n \quad \therefore a+c \neq n$.

Hence $w \notin L \Rightarrow$ It is not a regular.

Method 2 $\quad L = \{0^n 1^n / n \geq 1\}$

$L = \{01, 0011, 000111 \ldots\}$.

$n = 2$

Let $w = 0011$

$x \; y \; z$.

Case 1. $|xy| \leq n \quad \therefore \cancel{1+1 = 2 \leq 2} \quad 2 \leq 2$ ✓

Case 2 $|y| \geq 1 \quad \cancel{1} \geq 1$ ✓

case 3 $\neq xy^kz. \quad \therefore k \geq 0$.

$\therefore$ Let $k = 2 \quad \therefore xy^2z = 0(0)(1)^2 1$

$= 001011 \notin L$ Hence not regular

(2) Prove that $L = \{a^{n^2} / n \geq 1\}$ is not regular

$L = \{a^1, a^4, a^9, a^{16}, a^{25} \ldots\}$

$n = 2$.

$z = aaaa \quad |z| \geq 2$

Let $z = aaaa$

$x \; y \; z$

$\therefore |xy| \leq 2 \quad \therefore \cancel{1+1 \leq 2}$ ✓ $\quad 1+1 \leq 2$ ✓

$|y| \geq 1 \quad \cancel{1} \geq 1$ ✓

$z = xy^kz \in L$ for $k \geq 0$

$\therefore k = 2 \quad aaaaaa = a^6 \notin L$ not RL

Problem 3: show that $L = \{a^p / p$ is a prime number$\}$ is not regular.

$L = \{a, aa, aaa, aaaaa \ldots\}$ Prime numbers 1, 2, 3, 5, 7, 11 --

Assume pumping constant $n = 2$

Let $w = aaa$ $|w| \geq n$ $\therefore 3 \geq 2$ ✓

split $w = \underset{x}{a}\,\underset{y^k}{a}\,\underset{z}{a}$ $\therefore |xy| \leq n$ $\therefore 2 \leq 2$ ✓

$|y| \geq 1$ $1 \geq 1$ ✓

$w = xy^kz \in L$ for all $k \geq 0$

$\therefore$ $w = aaa \in L$ $\therefore k = 1$

$w = a(a)^2a$ $k = 2$

$aaaa \notin L$, not a RL

Problem 4: show that $L = \{ww^R / w \in (a+b)^*\}$ is not regular

$L = \{abba, aabbaa, baab \ldots\}$

Assume pumping constant $n = 3$.

Let $w = abba$ $|w| \geq n$ $4 \geq 3$ ✓

split $w = \underset{x}{a}\,\underset{y^k}{b}\,\underset{z}{ba}$ $\therefore |xy| \leq n$ $\therefore 2 \leq 3$

$|y| \geq 1$ $1 \geq 1$

$w = abba$ $k = 1$

$w = abbba \notin L$ $k = 2$, Non in R.L.

Problem 5: show that $L = \{ww / w$ in $(a+b)^*\}$ is not regular

$L = \{abab, baba, \ldots\}$.

$z = \underset{x}{b}\,\underset{y^k}{ab}\,\underset{z}{a}$

$z = b(ab)^2a$ $k = 2$

$bababa \notin L$, Not R.L

Problem 6. $L = \{a^n b^{2n} / n > 0\}$ is not regular.

$L = \{abb, aabbbb, aaabbbbbb \ldots\}$

$z = \underset{x}{a}\,\underset{y}{b}\,\underset{z}{b}$ $\therefore z = xy^kz = a(b)^2b$ $\therefore k = 2$

$= abbb \notin L$

Problem 7. show that $L = \{ 0^{i^2} / i \geq 1 \}$ is not regular

$L = \{ 0, 0^4, 0^9, 0^{16}, 0^{25} \ldots \}$

$\therefore z = \underset{x}{0}\underset{y}{00}\underset{z}{0}$ $\quad \therefore z = xy^kz = 0(00)^2 0 \quad \therefore k = 2$

$= 000000 \notin L$, not R.L

Problem 8: show that following is not regular

$L = \{ 0^i 1^j / \gcd(i,j) = 1 \}$

$\gcd(i,j) = 1$	
i	j
1	1
1	2
1	3
2	3
3	4

$L = \{ 01, 011, 0111, 00111 \ldots \}$

$\therefore z = 00111$ $\quad \therefore$

$z = \underset{x}{0}\underset{y^k}{0}\underset{z}{111}$ $\quad \therefore k = 2 = 0(0)^2 111$

$= 000111 \notin L$, not RL ;

Problem 9 $\quad L = \{ a^n b a^n / n \geq 0 \}$

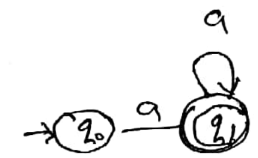
$L = \{ b, aba, aabaa \ldots \}$

$z = \underset{x}{a}\underset{y}{b}\underset{z}{a}$ $\quad \therefore k = 2. \quad a(b)^2 a = abba \notin L$, Not RL.

Competitive Exams:

Problem set:

1) $L = \{ a^n / n \geq 1 \}$ Regular pattern



2) $L = \{ a^{2n} / n \geq 1 \}$ "

3) $L = \{ a^n b^m / n, m \geq 1 \}$ Regular

4) $L = \{ a^n b^n / n \leq 10^{10} \}$ — Finite ⇒ Regular

5) $L = \{ a^i b^{i^2} / i, j \geq 1 \}$ — Not Regular — no pattern

6,

Conclusions: 1) Finite ?

2) Infinite ? → yes → genarate FA

3) Consisting "pattern" or not ?

if yes ⟹ R.L.

II Pumping Lemma for CFL (Context Free Language)

— Pumping Lemma used to prove that certain Languages are not Context Free Languages.

Let L(G) is CFL, then the following conditions must be satisfied.

i) every $z \in L(G)$ with $|z| \geq n$ / n is natural number and $z = uvwxy$ for some string.

ii) $|vx| \geq 1$

iii) $|vwx| \leq n$

iv) $uv^kwx^ky \in L$ for all $k \geq 0$.

Problems: 1) Prove that $L = \{a^i b^i c^i / i \geq 1\}$ is not CFL.

Ans: i) Assume L = CFL and n is a natural number.

$\therefore L = \{abc, aabbcc, aaabbbccc, aaaabbbbcccc \ldots\}$

Let n=10.

ii) Let $z = aaaabbbbcccc$ $\therefore |z| \geq n$

split z such that $z = uvwxy$.

case i) v and x contain same symbol or not same symbol

$z = \underbrace{aa}_{u}\underbrace{aa}_{v}\underbrace{bbbb}_{w}\underbrace{c}_{x}\underbrace{ccc}_{y}$

$\therefore |vx| \geq 1$ $\therefore 2+1 \geq 1$ ✓

$|vwx| \leq n$ $\therefore 2+4+1 = 7 \leq 10$ ✓

$uv^kwx^ky \in L$ for all $k \geq 0$

$\therefore k=1$ aa aa bbbb cc cc $\in L$

$k=2$ aa aaaa bbbb ccccc $\notin L$ $\Rightarrow$

Not in CFL.

Problem 2 show that $L = \{a^{n^2} / n \geq 1\}$ is not CFL

$\therefore L = \{a^1\ a^4\ \underline{a^9}\ a^{16}\ a^{25} \ldots \}$.

Let $L = \{ \ldots\ aaaaaaaaa\ \ldots \}$

$\therefore n = 3$

$\therefore$ Let $z = aaa\ aaa\ aaa$ $\quad \therefore |z| \geq 3$.

split into $z = uvwxy$

$$\underbrace{aa}_{u}\ \underbrace{a}_{v}\ \underbrace{a}_{w}\ \underbrace{a}_{x}\ \underbrace{a\ aaa}_{y}$$

$\therefore |vx| = 2 \geq 1$ ✓

$|vwx| = 1+1+1 \leq 3$ ✓

$z = uv^kwx^ky \in L$ for $k \geq 0$.

$\therefore k = 2$ $\quad aaa\,(a)^2 a\,(a)^2 aaa$

$aaa\ aaa\ aa\ aaa = a^{11} \notin L$ Not CFL

Problem 3

show that following Language is not CFL

$L = \{a^{n!} / n \geq 0\}$.

$L = \{a, aa, aaaaaa, a^{24} \ldots \}$.

Let $n = 5$.

Let $z = aaaaaa$ $\quad |z| \geq n \quad \therefore 6 \geq 5$ ✓

split into $z = uv^kwx^ky \in L$ for all $k \geq 0$

$$z = \underbrace{a}_{u}\ \underbrace{a}_{v}\ \underbrace{a}_{w}\ \underbrace{a}_{x}\ \underbrace{aa}_{y}$$

$|vx| \geq 1 \therefore 2 \geq 1$ ✓

$|vwx| \leq n \therefore 3 \leq 5$ ✓

Let $k=1$ $\quad z = aaa\,aaa \in L$ ✓

$k = 2$ $\quad z = a\,(a)^2 a\,(a)^2 aa$

$aaaaaaaa = a^8 \notin L$ Hence not in CFL

Problem 4: Show that $L = \{0^m 1^m 2^n / m \leq n \leq 2m\}$ is not CFL

$L = \{0122,\ 0011222,\ 00112222,\ \dots\}$ $m = 1, 2, 2$ $n = 2, 3, 4$

Let pumping constant $n = 4$

Let $z = 0011222$ $|z| \geq n$ $\therefore 7 \geq 4$ ✓

split $z = \underline{0}_u\underline{0}_v\underline{11}_w\underline{2}_x\underline{22}_y$ $\therefore |vx| \geq 1$ $2 \geq 1$ ✓

$|vwx| \leq n$ $4 \leq 4$ ✓

$\therefore k = 1$ $z = 0011222 \in L$ ✓

$k = 2$ $z = 0(0)^2 11 (2)^2 22$

$= 000112222 \notin L$, Hence not in CFL

Problem 5: Show that $L = \{a^i b^j / i \leq j^2\}$ is not CFL

$L = \{ab, abb, aabb, aaabb, aaaabb \dots\}$ $j = 1$ $i = 1$; $j = 2$ $i = 1, 2, 3, 4$

Let $n = 4$,

$z = aaabb$

$z = \underline{a}_u\underline{a}_v\underline{a}_w\underline{b}_x\underline{b}_y$ $|vx| \geq 1$ ✓

$|vwx| \leq n$ $3 \leq 4$ ✓

$k = 1$ $z = aaabb \in L$ ✓

$k = 2$ $z = a(a)^2 a(b)^2 b$

$= aaaabbb \notin L$, not in CFL

Problem 6 show that following language is not context-free

$L = \{a^p / p \text{ is prime number}\}$ $P = 1, 2, 3, 5, 7, \dots$

$L = \{a, a^2, a^3, a^5, a^7, a^{11}, \dots\}$

Assume pumping constant $n = 5$

Choose string $z = aaaaaaa$ $|z| \geq n$ $7 \geq 5$ ✓

split into $z = uv^k wx^k y \in L$ for $k \geq 0$

$z = \underline{aa}_u\ \underline{a}_v\ \underline{a}_w\underline{a}_x\ \underline{aa}_y$

$\therefore k = 1$ $\therefore z = uv^1 wx^1 y = aa(a)^1 a(a)^1 aa \in L$

$k = 2$ $z = uv^2 wx^2 y = aa(a)^2 a(a)^2 aa \notin L$

$k = 0$ $z = uv^0 wx^0 y = aa(a)^0 a(a)^0 aa \in L$

Not CFL

Closure properties of CFL

1) CFL are closed for CFG under Union, Concatenation, kleen closure

2, CFL closed under Intersection.

3. CFL Closed under complementation

4. CFL " " Substitution

5. CFL Closed under homomorphism

6. CFL Closed under Inverse homomorphism

7. If ~~CFL~~ L is CFL and R Regular set, then $L \cap R$ is a CFL.

Closure properties of RL (Let L_1, L_2 regular Languages) the following are regular

1) R.L is closed under

(i) union, concatenation, kleen closure
$L_1 \cup L_2$, $L_1 L_2$, L_1^*

(ii) Intersection $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$

(iii) Complementation $\overline{L_1}$

(iv) Substitution

(v) Homomorphism.

(vi) Inverse Homomorphism

(vii) Difference. $L_1 - L_2 = L_1 \cap \overline{L_2}$