ACM/IDS 104 - Problem Set 2 - MATLAB Problems

Before writing your MATLAB code, it is always good practice to get rid of any leftover variables and figures from previous scripts.

```
clc; clear; close all;
```

Problem 5 (10 points) Fundamental Matrix Subspaces

Your task for this problem is to write a function that takes a matrix `A` as its argument, and outputs four matrices: `K, I, cK` and `cI` where:

- Columns of `K` form a basis of the kernel of `A`. If $\ker A = \{0\}$, then `K` must be a zero vector of the appropriate dimension.
- Columns of `I` form a basis of the image of `A`. If $\mathrm{im} A = \{0\}$, then `I` must be a zero vector of the appropriate dimension.
- Columns of `cK` form a basis of the cokernel of `A`. If $\mathrm{coker} A = \{0\}$, then `cK` must be a zero vector of the appropriate dimension.
- Columns of `cI` form a basis of the coimage of `A`. If $\mathrm{coim} A = \{0\}$, then `cI` must be a zero vector of the appropriate dimension.

Move to the bottom of this livescript to write the function.

Now, let us test our function:

```
A = magic(6); % feel free to define A as you like
[K, I, cK, cI] = subspacer(A); % this is how you call a MATLAB function
disp(K);
```

```
   -0.4714
   -0.4714
    0.2357
    0.4714
    0.4714
   -0.2357
```

```
disp(I);
```

```
   -0.4082    0.5574    0.0456   -0.4182    0.3092
   -0.4082   -0.2312    0.6301   -0.2571   -0.5627
   -0.4082    0.4362    0.2696    0.5391    0.1725
   -0.4082   -0.3954   -0.2422   -0.4590    0.3971
   -0.4082    0.1496   -0.6849    0.0969   -0.5766
   -0.4082   -0.5166   -0.0182    0.4983    0.2604
```

```
disp(cK);
```

```
    0.5000
   -0.0000
   -0.5000
   -0.5000
   -0.0000
    0.5000
```

```
disp(cI);
```

```
   -0.4082    0.6234   -0.3116    0.2495   -0.2511
   -0.4082   -0.6282    0.3425    0.1753   -0.2617
   -0.4082   -0.4014   -0.7732   -0.0621    0.1225
   -0.4082    0.1498    0.2262   -0.4510   -0.5780
   -0.4082    0.1163    0.2996    0.6340    0.3255
   -0.4082    0.1401    0.2166   -0.5457    0.6430
```

START HERE by writing the function:

```
function [K, I, cK, cI] = subspacer(A)
%{
This is the MATLAB function syntax.
-> [K, I, cK, cI] are the outputs of the function.
-> "subspacer" is the name of the function. (you can change that if
                            you wish but make sure you change
                            every function call as well!)
-> A is the argument of the function.
%}
[m, n] = size(A);
r = rank (A);
%{
We start by finding out the dimensions and rank of A.
Let us consider the matrix K. There exist 2 cases:
1) The kernel is trivial i.e. kerA = {0}
2) The kernel is not trivial -> Hint: use null()
Complete the following if/else statement.
%}
if r == n % this condition is done for you
    K = zeros(n, 1);
else
    K = null(A);
end
%{
Now, let us consider the matrix cK.
As above, there exist 2 cases. Remember, you can use ' to
transpose a matrix.
Write a similar if/else statement to produce cK.
%}
if rank(A') == m
    cK = zeros(m, 1);
else
    cK = null(A');
end
%{
For the image I and coimage cI, there exists only 1 condition
we must test, and that is if rankA = 0. With this in mind,
complete the following if/else statement.
-> Hint: orth() is useful here.
%}
```

```
if r == 0
    I = zeros(m, 1);
    cI = zeros(n, 1);
else
    I = orth(A);
    cI = orth(A');
end
end
```

P-Set 2

1. a) $\boxed{\text{W is not a subspace}}$ of vector space V.

W is not closed under scalar addition. We prove this with a counterexample.

$A_1 = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$ $A_2 = \begin{pmatrix} 5 & 4 \\ 10 & 8 \end{pmatrix}$

$A_1 + A_2 = \begin{pmatrix} 8 & 10 \\ 11 & 10 \end{pmatrix}$ $\det|A_1+A_2| = -30 \neq 0$

b) $\boxed{\text{W is a subspace.}}$

1. The 0 vector $\in W$. ✓
2. Closed under scalar addition
   - $tr(A_1+A_2) = tr A_1 + tr A_2 = 0$. We know this b/c matrix addition is the sum of the respective (i,j) entries ✓
3. Closed under scalar mult.
   - $c \in \mathbb{R}$
   - $tr(cA) = c\,tr(A) = 0$ ✓

c) $\boxed{\text{W is not a subspace.}}$

W is not closed under scalar addition. We prove this with a counterexample.

$f_1(x) = 1$
$f_2(x) = 0.5 + 1.5x$

$(f_1+f_2)(x) = 1.5x + 1.5$
$(f_1+f_2)(0) \cdot (f_1+f_2)(1) = 1.5(3) = 4.5 \neq 1$

d) $\boxed{\text{W is a subspace.}}$

1. If $f(t)=0$, then $f(\frac{1}{2})=0=\int_0^1 f(t)dt$ ✓

2. Closed under scalar addition.

$f_1(t), f_2(t) \in W$, $f_1(t)+f_2(t)=g(t)$

$$g\left(\tfrac{1}{2}\right)=\int_0^1 g(t)dt=\int_0^1 f_1(t)dt+\int_0^1 f_2(t)dt=f_1\left(\tfrac{1}{2}\right)+f_2\left(\tfrac{1}{2}\right) \checkmark$$

3. Closed under scalar mult.

$c \in \mathbb{R}$, $g(t)=cf(t)$

$$g\left(\tfrac{1}{2}\right)=\int_0^1 g(t)dt=\int_0^1 cf(t)dt=c\int_0^1 f(t)dt=cf\left(\tfrac{1}{2}\right) \checkmark$$

e) $\boxed{\text{W is a subspace.}}$

1. If $v(x,y)=\begin{pmatrix}0\\0\end{pmatrix}$, $\nabla \cdot v=0$ ✓

2. Closed under scalar addition

$$v_1(x,y)=\begin{bmatrix}v_{1,1}(x,y)\\v_{1,2}(x,y)\end{bmatrix} \quad v_2(x,y)=\begin{bmatrix}v_{2,1}(x,y)\\v_{2,2}(x,y)\end{bmatrix}$$

$$w(x,y)=v_1(x,y)+v_2(x,y)$$

$$\nabla \cdot w=\frac{\partial(v_{1,1}+v_{2,1})}{\partial x}+\frac{\partial(v_{1,2}+v_{2,2})}{\partial y}$$

$$\nabla \cdot w=\frac{\partial v_{1,1}}{\partial x}+\frac{\partial v_{2,1}}{\partial x}+\frac{\partial v_{1,2}}{\partial y}+\frac{\partial v_{2,2}}{\partial y}$$

$$\nabla \cdot w=\nabla \cdot v_1+\nabla \cdot v_2=0+0=0 \checkmark$$

3. Closed under scalar mult.

$w(x,y)=cv(x,y)$, $c \in \mathbb{R}$

$$\nabla \cdot w=\frac{\partial(cv_1)}{\partial x}+\frac{\partial(cv_2)}{\partial y}=c\frac{\partial v_1}{\partial x}+c\frac{\partial v_2}{\partial x}=0 \checkmark$$

2. a) To determine whether p_1, p_2, and p_3 are linearly independent, we can put the coefficients of the quadratics in a matrix and check if rank A = # polynomials = 3

$$c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x) = 0$$

$$c_1(x^2-3) + c_2(2-x) + c_3(x^2+2x+1) = 0$$

$$(c_1+c_3)x^2 + (-c_2+2c_3)x + (-3c_1+2c_2+c_3) = 0$$

To achieve lin. ind., all coeff. must be 0.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ -3 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The system has rank 3 so only triv. sol'n exists $\Rightarrow$ $\boxed{\text{they are lin. independent}}$

b) To span $P^{(2)}$ we need 3 lin ind. polynomials. We have shown this is true so it does span $P^{(2)}$

c) Since p_1, p_2, and p_3 are lin. independent and span $P^{(2)}$, it is a basis of $P^{(2)}$.

To find coordinates, augment original matrix with $\vec{b} = (0,0,1)^T$ and solve for c_1, c_2, and c_3.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ -3 & 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/8 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & 1/8 \end{array}\right]$$

So, $q(x) = -\frac{1}{8} p_1(x) + \frac{1}{4} p_2(x) + \frac{1}{8} p_3(x)$

3, a) 1. Show zero vector is in subspace
- The zero vector is analagous to a generalized Fib. sequence with $x_1 = x_2 = 0$

2. Closed under scalar addition
- $g = f_1 + f_2$
- We will show for n^{th} term of g
  - $g_n = x_n + y_n = g_{n-1} + g_{n-2}$

$$g_n = x_n + y_n = (x_{n-1} + x_{n-2}) + (y_{n-1} + y_{n-2})$$
$$= (x_{n-1} + y_{n-1}) + (x_{n-2} + y_{n-2})$$
$$g_n = g_{n-1} + g_{n-2} \checkmark$$

3, Closed under scalar mult,
- $g = \alpha f$, $\alpha \in \mathbb{R}$

$$g_n = \alpha x_n = \alpha(x_{n-1} + x_{n-2})$$
$$= \alpha x_{n-1} + \alpha x_{n-2}$$
$$g_n = g_{n-1} + g_{n-2} \checkmark$$

b) The dimension of generalized Fibonacci is $\boxed{2}$ since every other term of sequence $n \geq 3$ can be written as a lin combination of first 2 entries

Thus the basis is $\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$ (n rows in each)

c) $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \vdots & 0 & 0 \\ & \vdots & \vdots \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$ coordinates : $(1,1)$

4. $A = \begin{bmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & \vdots & & \vdots \\ n^2-n+1 & n^2-n+2 & \cdots & n^2 \end{bmatrix}$

This matrix is the same matrix from Pset 1 which we proved had rank 2. The matrix row reduces to the following form:

$$A = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & -n & -2n & \cdots & -n(n-1) \\ \vdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

• The ker A is when $Ax = 0$. We can solve this system of equations accordingly:

$R_2/-n \sim$

$$A \left[\begin{array}{ccccc|c} 1 & 2 & 3 & \cdots & n & 0 \\ 0 & 1 & 2 & \cdots & n-1 & 0 \\ \vdots & 0 & 0 & \cdots & 0 & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right]$$

$x_1 = -3x_3 - 4x_4 + \ldots - nx_n$

$x_2 = -2x_3 - 3x_4 + \ldots - (n-1)x_n$

Our basis for kernel will have $(n-2)$ vectors with n entries each.

The solution to $Ax=0$ is as such: $\left\{ x_3 \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ -3 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \ldots + x_n \begin{pmatrix} -n \\ -(n-1) \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$

The solution to $A\vec{x}=\vec{0}$ is also the basis of our kernel.

$$\ker A = \left\{ \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ -3 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} -n \\ -(n-1) \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$$

For general size n: $\ker A = \left\{ V : \begin{array}{l} V_1 = -x \\ V_2 = -(x-1) \\ V_x = 1 \end{array} \right\}$

set of vectors v w/ these props. →

- The image of A is the two columns with pivot positions in RREF form (column space).

$$\mathrm{Im}\, A = \left\{ \begin{pmatrix} 1 \\ n+1 \\ \vdots \\ n^2-n+1 \end{pmatrix}, \begin{pmatrix} 2 \\ n+2 \\ \vdots \\ n^2-n+2 \end{pmatrix} \right\}$$

- cokernel of A

$$A^T = \begin{bmatrix} 1 & n+1 & \cdots & 1+n(n-1) \\ 2 & n+2 & \cdots & 2+n(n-1) \\ \vdots & \vdots & & \vdots \\ n & 2n & \cdots & n+n(n-1) \end{bmatrix}$$

Row reduce $\Rightarrow$

$R_2 \sim R_2 - 2R_1$

$$\begin{bmatrix} 1 & n+1 & \cdots & 1+n(n-1) \\ 0 & -n & \cdots & -n(n-1) \\ & & 0 & \end{bmatrix}$$

We know the rest of rows are 0 because we have already shown that only the first 2 rows are lin. independent.

$R_2 \sim R_2 / -n$

$$\begin{bmatrix} 1 & n+1 & \cdots & 1+n(n-1) \\ 0 & 1 & \cdots & n-1 \\ & & 0 & \end{bmatrix}$$

coker $A^T = A^T x = 0$

Our basis for coker A will have $(n-2)$ vectors w/ n entries each.

$$\boxed{\text{coker } A : \text{ For gen size } n : \ker A^T = \left\{ V : \begin{array}{l} v_1 = -1 - n(x-1) \\ v_2 = -(x-1) \\ v_x = 1 \end{array} \right\}}$$

set of vectors v w/ these prop.

$$\boxed{\text{coIm } A = \text{First two row of } A = \left\{ \begin{array}{c} (1, 2, \ldots, n) \\ (n+1, n+2, \ldots 2n) \end{array} \right\}}$$