Proof of the AVL Tree Height Theorem

Notice first that for all k,

(*) $$2F_k > F_{k+1}$$

This follows because $2F_k = F_k + F_k > F_k + F_{k-1} = F_{k+1}$.

Lemma. If an AVL tree has height *h* and has *n* nodes, then $n \geq F_{h+1}$, where *F* is the Fibonacci sequence

We prove the Lemma by induction on height. When h = 0, n = 1 and the result is obvious. Assume the result holds for heights less than h.

Let T be AVL with height h and n nodes. Let T_L and T_R be the left and right subtrees of T; let n_L be the number of nodes in T_L and n_R the number of nodes in T_R; let h_L be the height of T_L and h_R the height of T_R. Assume $h_L \geq h_R$. Then $h = h_L + 1$

Case 1: $h_L = h_R$

$$\begin{aligned} n &= n_L + n_R + 1 \\ &\geq F_{hL+1} + F_{hR+1} + 1 \\ &\geq 2F_{hL+1} + 1 \\ &\geq F_{hL+2} + 1 \\ &= F_{h+1} + 1 \\ &\geq F_{h+1} \end{aligned}$$

Case 2: $h_L = h_R + 1$

$$\begin{aligned} n &= n_L + n_R + 1 \\ &\geq F_{hL+1} + F_{hR+1} + 1 \\ &= F_h + F_{h-1} + 1 \\ &= F_{h+1} + 1 \\ &\geq F_{h+1} \end{aligned}$$