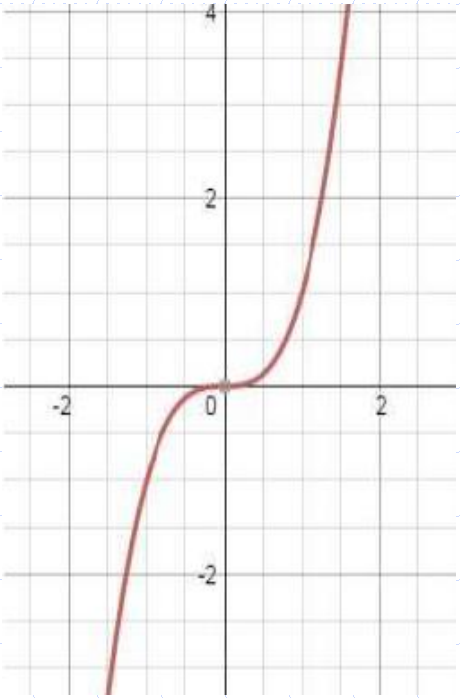
Lesson 1: Math Review

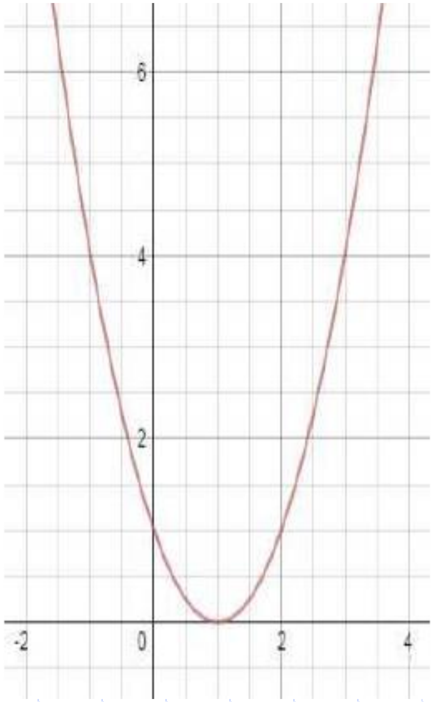
Increasing/Nondecreasing functions

- A function is increasing if its graph climbs steadily upward. More precisely:
  - Defintion. A function f on the real line is increasing if, whenever $x_1 < x_2$, $f(x_1) < f(x_2)$.
  - Defintion. A function f on the real line is nondecresing if, whenever $x_1 \le x_2$, $f(x_1) \le f(x_2)$.
- Example: $f(x) = x^3$ is an increasing function.
- Question: Is $f(x) = x^2$ increasing?



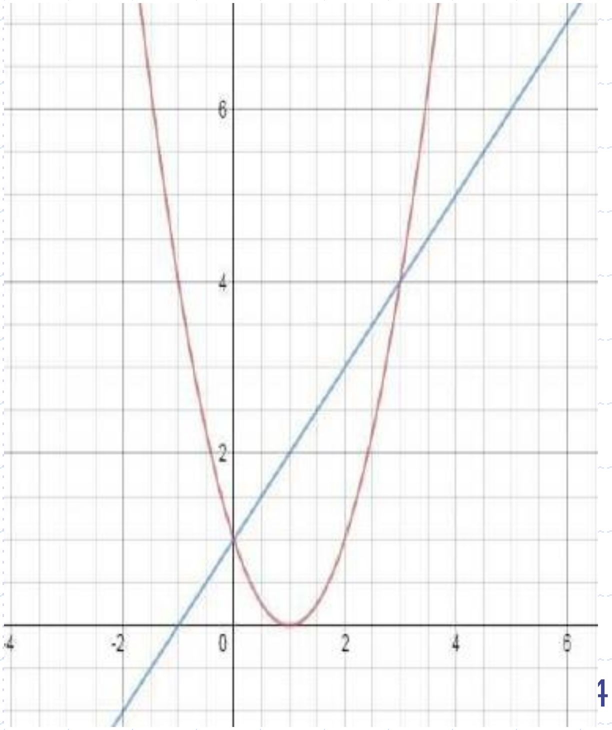
Eventually Nondecreasing Functions

- A function is eventually nondecreasing if for all values beyond a certain point on the x-axis, the graph steadily climbs. More precisely,
  - Definition. A function f is eventually nondecreasing if for some real number x_0, f is increasing on $[x_0,\infty)$. In other words, for some x_0 we have that whenever $x_0 \leq x_1 \leq x_2$, then $f(x_0) \leq f(x_1) \leq f(x_2)$.
- Example: $f(x) = (x - 1)^2$.



Growth Rates of Functions

- Some functions grow faster than others. Example: $f(x) = (x - 1)^2$ and $g(x) = x + 1$. Notice when $x = 3$, the quadratic function overtakes the linear function. We say f is asymptotically greater than g. (f "≥"g) These ideas will be used to evaluate and compare running times of algorithms.



Mathematical Induction

- The idea: Suppose you wish to prove that some statement $\varphi(n)$, which asserts something about each whole number n, is true for every n. For example, to prove that for all $n \geq 0$, $n < 2^n$, we would use "$n < 2^n$" as our statement $\varphi(n)$.
- We wish to show that this statement holds for every n. Suppose now that we can prove two things:
  A. that, $\varphi(0)$ is true (in our example, this would mean that we can prove $0 < 2^0$);
  B. that, for any n, if $\varphi(n)$ happens to be true, then $\varphi(n + 1)$ must also be true (in our example, this would mean that, if it happens to be true that $n < 2^n$, then it must be true that $n + 1 < 2^{n+1}$).
- Mathematical Induction says that, if you can prove both A. and B., then you have proven that, for every n, $\varphi(n)$ is indeed true.

Standard Induction

- Suppose $\varphi(n)$ is a statement depending on n. If
  - $\varphi(0)$ is true, and
  - under the assumption that $n \geq 0$ and $\varphi(n)$ is true, you can prove that $\varphi(n + 1)$ is also true,

  then $\varphi(n)$ holds true for all natural numbers n.

General Induction

- Suppose $\varphi(n)$ is a statement depending on n and suppose $k \geq 0$ is an integer. If
  - $\varphi(k)$ is true, and
  - under the assumption that $n \geq k$ and $\varphi(n)$ is true, you can prove that $\varphi(n + 1)$ is also true,

  then $\varphi(n)$ holds true for all natural numbers $n \geq k$.
- In General Induction, the step in the proof where $\varphi(k)$ is verified is called the *Basis Step*. The second step, where $\varphi(n + 1)$ is proved assuming $\varphi(n)$, is called the *Induction Step*. As we reason during this second step, we will typically need to make use of $\varphi(n)$ as an assumption; in this context, $\varphi(n)$ is called the *induction hypothesis*.

Total Induction

- Suppose $\varphi(n)$ is a statement depending on n and $k \geq 0$. If
  - $\varphi(k)$ is true, and
  - under the assumption that $n > k$ and that each of $\varphi(k)$, $\varphi(k + 1), \ldots, \varphi(n - 1)$ are true, you can prove that $\varphi(n)$ is also true,

  then $\varphi(n)$ holds true for all $n \geq k$.

Example of Mathematical Induction

- Prove φ(n): $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

For the Basis Step, notice that $\phi(1)$ is the statement

$$\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$$

which is obviously true. For the Induction Step, we assume $\phi(n)$ is true, and we prove $\phi(n+1)$. $\phi(n+1)$ is the following statement:

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

To prove $\phi(n+1)$ is true, we follow these steps:

$$\begin{aligned}
\sum_{i=1}^{n+1} i &= \left(\sum_{i=1}^{n} i\right) + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \qquad \text{(by Induction Hypothesis)} \\
&= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\
&= \frac{(n+1)(n+2)}{2}
\end{aligned}$$

Example of Mathematical Induction

Suppose $T(1) = c$, $T(n) \geq 2T(n-2)$; $S(1) = c$, $S(n) = 2S(n-2)$.

Prove $T(n) \geq S(n)$ for all $n \geq 1$

Proof. Proceed by induction on n to show $\varphi(n)$: $T(n) \geq S(n)$.

Basis step: $\varphi(1)$: $T(1) \geq S(1)$ is true.

Induction step: Assume $T(k) \geq S(k)$ whenever $1 \leq k < n$, prove $T(n) \geq S(n)$.

$$T(n) \geq 2T(n-2) \geq 2S(n-2) = S(n)$$

So $T(n) \geq S(n)$ for all $n \geq 1$.

The Division Algorithm

(1) Suppose m, n are positive integers. Dividing n by m gives a quotient and remainder.

(2) Example: Divide 17 by 3: Quotient is 5 and remainder is 2. Using integer division and Java mod notation, we can write:

1. quotient $= 17/3$
2. remainder $= 17 \,\%\, 3$

Using mathematical notation:

1. quotient $= \lfloor 17/3 \rfloor$
2. remainder $= 17 \mod 3$

We can write:

$$17 = \text{quotient} \cdot 3 + \text{remainder} = \lfloor 17/3 \rfloor \cdot 3 + 17 \mod 3.$$

(3) In general, for any positive integers m, n, there are unique q, r so that

$$n = mq + r \qquad \text{and} \qquad 0 \leq r < m.$$

In other words

$$n = m \cdot \lfloor \frac{n}{m} \rfloor + n \mod m.$$

Calculus

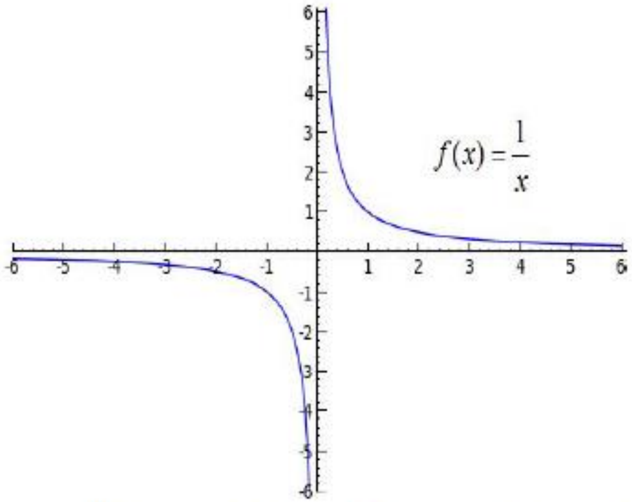
For this Algorithms course, it is not necessary to have an in-depth understanding of calculus, but it *is* important to know a few of the simple concepts and formulas, which we review here. The two concepts to be familiar with are:

(1) Limits at infinity. Example: $\lim_{n\to\infty}(n+1)/n^2 = 0$.

(2) Derivative formulas. Example: $\frac{d}{dx}x^2 - x + 1 = 2x - 1$.

Limits at Infinity

Consider the following graph of $f(x) = \frac{1}{x}$:



As x gets bigger and bigger, $f(x)$ gets closer and closer to 0. We write

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

Since we will be working with the set **N** of natural numbers, instead of the set **R** of real numbers, we will express this limit as

$$\lim_{n \to \infty} \frac{1}{n} = 0.$$

Limits at Infinity

- What is the limit of $f(n) = \frac{n}{n-1}$?

We can compute this limit algebraically by factoring from numerator and denominator the reciprocal of the highest power of n that occurs in the expression:

$$\begin{aligned}\lim_{n\to\infty}\frac{n}{n-1} &= \lim_{n\to\infty}\left(\frac{n}{n-1}\cdot\frac{1/n}{1/n}\right)\\ &= \lim_{n\to\infty}\frac{1}{1-\frac{1}{n}}\\ &= 1.\end{aligned}$$

Derivatives

The derivative of a function $f(x)$, which is written in any of these ways: $f'(x)$, $\frac{d}{dx}f(x)$, $\frac{dy}{dx}$, represents the *slope of the line tangent to the graph of f at the point* (x, y).

For example:

Derivatives

- Please see CalculusReference.pdf

There are a number of convenient formulas for computing derivatives of familiar functions:

(1) $\frac{d}{dx}\, a = 0$ for any real number a.

(2) $\frac{d}{dx}\, x^r = rx^{r-1}$, for any real number $r \neq 0$.

(3) $\frac{d}{dx}\, 2^x = 2^x \ln 2$

(4) $\frac{d}{dx}\, \log x = \frac{1}{x} \cdot \log e$

(5) For any functions $f(x), g(x)$ (whose derivatives exist) and real numbers a, b:

(a) (Linearity Rule) $\frac{d}{dx}(af(x) + bg(x)) = a\frac{d}{dx}\, f(x) + b\frac{d}{dx}\, g(x)$

(b) (Product Rule) $\frac{d}{dx}(f(x) \cdot g(x)) = f(x) \cdot \frac{d}{dx}\, g(x) + g(x) \cdot \frac{d}{dx}\, f(x)$

(c) (Reciprocal Rule) $\frac{d}{dx}(\frac{1}{f(x)}) = \frac{-f'(x)}{[f(x)]^2}$

Eventually Increasing Functions and Derivatives

Fact. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function (and assume its derivative exists everywhere). For any interval (a, b) (where a could possibly be $-\infty$, b could be ∞):

(1) if $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b)

(2) if $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b).

- Consider $f(x) = (x - 1)^2$.
  $f'(x) = 2x - 2 = 2(x - 1)$
  $f'(x)$ is positive when $x > 1$ and is negative when $x < 1$. By the fact, we may conclude:
  - $f(x)$ is increasing on $(1,\infty)$
  - $f(x)$ is decreasing on $(-\infty, 1)$.

Detecting Growth Rates Using Limits

We can tell linear functions $f(n) = an + b$ always grow more slowly than the quadratic $g(n) = n^2$ because the quotient $f(n)/g(n)$ tends to 0 as n becomes large:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{an+b}{n^2} = \lim_{n\to\infty} \frac{\frac{a}{n} + \frac{b}{n^2}}{1} = 0.$$

Example: Show that 5n + 3 grows more slowly than n^2

Solution: $$\lim_{n\to\infty} \frac{5n+3}{n^2} = \lim_{n\to\infty} \frac{\frac{5}{n} + \frac{3}{n^2}}{1} = 0$$

(continued)

On the other hand, all quadratic functions always grow at the same rate. We can see this using limits: If $f(n) = an^2 + bn + c$ and $g(n) = dn^2 + en + r$, where $a \neq 0$ and $d \neq 0$, then the quotient $f(n)/g(n)$ tends to a nonzero number:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{an^2 + bn + c}{dn^2 + en + r} = \lim_{n\to\infty} \frac{a + \frac{b}{n} + \frac{c}{n^2}}{d + \frac{e}{n} + \frac{r}{n^2}} = \frac{a}{d} \neq 0.$$

Example: Show that $3n^2 + 7$ grows at the same rate as $5n^2 - n$.

Solution: $$\lim_{n\to\infty} \frac{3n^2 + 7}{5n^2 - n} = \lim_{n\to\infty} \frac{3 + \frac{7}{n^2}}{5 - \frac{1}{n}} = \frac{3}{5} \neq 0$$

Theta, Little-oh, Little-omega

Suppose f(n) and g(n) are functions. Then

- f(n) is o(g(n)) ("f(n) is little-oh of g(n)") if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$
  [f(n) grows much more slowly than g(n)]

- f(n) is ω(g(n)) ("f(n) is little-omega of g(n)") if g(n) is o(f(n))
  [f(n) grows much faster than g(n)]

- f(n) is Θ(g(n)) ("f(n) is theta of g(n)") if for some nonzero number ***r***
  $\lim_{n\to\infty} \frac{f(n)}{g(n)} = r$ [f(n) grows at the same rate as g(n)]

The classes of functions represented by o, ω, and Θ are called complexity classes.

Note: It is theoretically possible that limits of this kind may not exist. This situation almost never arises in the context of determining running times of algorithms, and so we do not attempt to handle this special case in this course.

Examples

- $5n + 3$ is $o(n^2)$
- $3n^2 + 7$ is $\Theta(n^2)$
- n^2 is $\omega(n)$.

Standard Complexity Classes

- The most common complexity classes used in analysis of algorithms are, in increasing order of growth rate:

  $$\Theta(1), \Theta(\log n), \Theta(n^{1/k}), \Theta(n), \Theta(n\log n), \Theta(n^k)\ (k > 1),$$

  $$\Theta(2^n), \Theta(n!), \Theta(n^n)$$

  Functions that belong to classes in the first row are known as *polynomial time bounded.*
- Verification of the relationships between these classes sometimes requires the use of *L'Hopital's Rule*

L'Hopital's Rule. Suppose f and g have derivates (at least when x is large) and their limits as $x \to \infty$ are either both 0 or both infinite. Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

as long as these limits exist.

Example using L'Hopital

Problem. Show that $\log n$ is $o(\sqrt{n})$.

Solution. We show that the limit of the quotient is 0.

$$\lim_{n\to\infty} \frac{\log n}{n^{1/2}} = \lim_{n\to\infty} \frac{\frac{1}{n} \cdot \log e}{(1/2)n^{-(1/2)}} = \lim_{n\to\infty} \frac{2\log e}{n^{1/2}} = 0.$$

Big-oh and Big-omega

- If f(n) *grows no faster* than g(n), we say f(n) is O(g(n)) ("big-oh")
- If f(n) *grows at least as fast* as g(n), we say f(n) is Ω(g(n)) ("big-omega")
- The Big-oh notation gives an upper bound on the growth rate of a function; The Big-omega notation gives an lower bound on the growth rate of a function.
- Limit criterion:

f(n) is O(g(n)) if

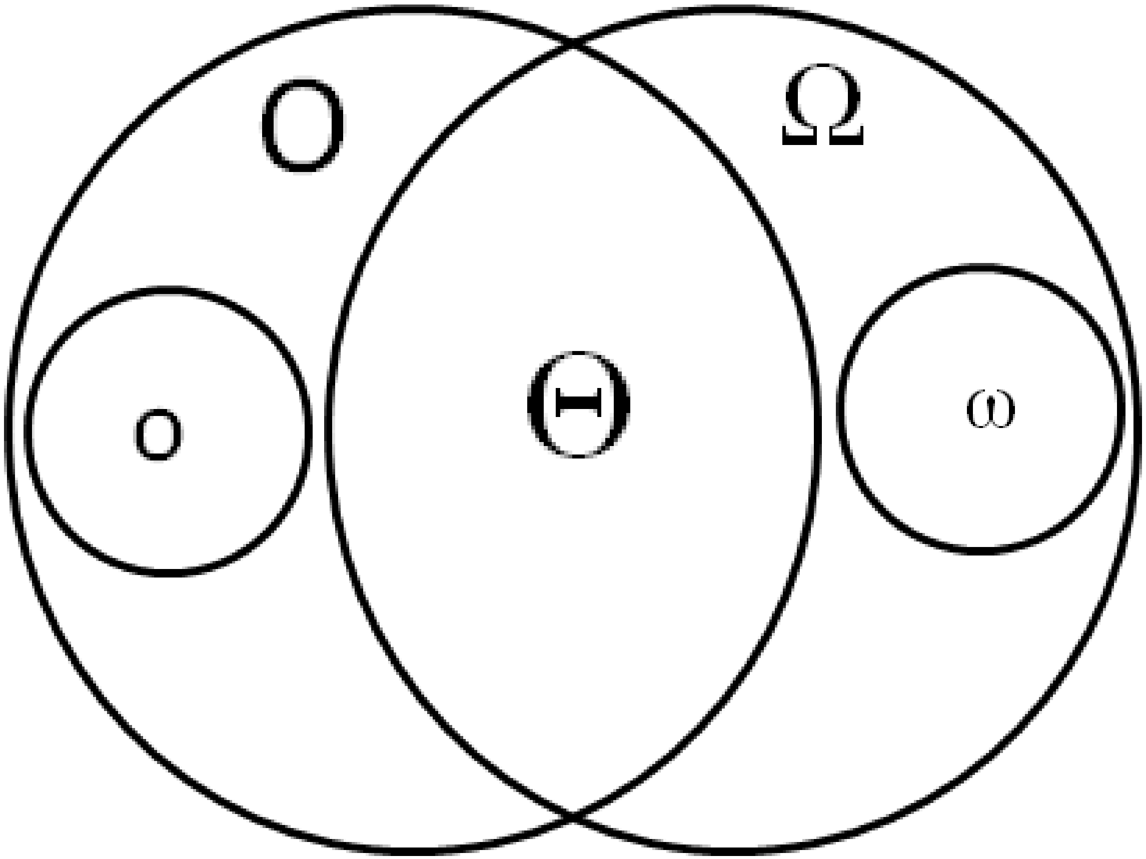
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \text{ is finite}$$

Then, f(n) is Ω(g(n)) if g(n) is O(f(n)).

Examples

- Both $2n + 1$ and $3n^2$ are $O(n^2)$
- Both $2n^2 - 1$ and $4n^3$ are $\Omega(n^2)$

Relationships Between the Complexity Classes



- Whenever f(n) is o(g(n)), f(n) is O(g(n).
- Whenever f(n) is ω(g(n)), f(n) is Ω(g(n)).
- No function is in both o and ω
- If f(n) is in both O(g(n)) and Ω(g(n)), it is in Θ(g(n)).

Summary of Criteria for Determining Complexity

$f(n)$ is $O(g(n))$	*if*	$\lim_{n\to\infty} \frac{f(n)}{g(n)}$ is finite
$f(n)$ is $\Omega(g(n))$	*if*	$\lim_{n\to\infty} \frac{g(n)}{f(n)}$ is finite
$f(n)$ is $\Theta(g(n))$	*if*	$\lim_{n\to\infty} \frac{f(n)}{g(n)}$ is nonzero
$f(n)$ is $o(g(n))$	*if*	$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$
$f(n)$ is $\omega(g(n))$	*if*	$\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$