MATH 344-1 Notes

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Chapter 1

Metric Spaces

1.1 Learning Objectives

- What is a metric and metric space?
- What is an open ball?
- How do open balls lead to the definition of open sets?
- What are some examples of metrics?
- What are some examples of open sets?

1.2 Definitions

Definition: Metrics and Metric Spaces

A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$ that satisfies the following conditions:

- **Non-negative**: $d(x,y) \geq 0$ for all $x,y \in X$
- **Symmetric**: $d(x,y) = d(y,x)$ for all $x,y \in X$
- **Non-degenerate**: $d(x,y) = 0$ if and only if $x = y$
- **Triangle Inequality**: $d(x,y) + d(y,z) \geq d(x,z)$ for all $x,y,z \in X$

A set X equipped with a particular metric is called a **metric space**.

Definition: Open Balls

Let X be equipped with a metric d. Given a point $p \in X$ and a positive, real radius r, we define the **open ball** centered at p with radius r to be the set of all points whose distance from p is less than r. I.e.

$$B_r(p) = \{x \in X \mid d(p,x) < r\}$$

Definition: Open Sets

Let X be a metric space. A subset U of X is said to be **open in** X if and only if, for each $x \in U$, there is a positive, real radius r so that the open ball centered at x of radius r is entirely contained in U – i.e., so $B_r(x) \subseteq U$.

1.3 Examples

Example: Euclidean Distance

The **Euclidean Distance** is defined as follows: If $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ are points in $\mathbb{R}^n$ then the **Euclidean distance** between X and Y is

$$d(X, Y) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$$

$\mathbb{R}^n$ with the Euclidean distance as its metric is called the n-**dimensional Euclidean Space**.

Example: Discrete Metric

A more extreme metric is the **discrete metric** and is defined as follows:

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Example: Open Sets in a Discrete Metric Space

If X is a discrete metric space (that is, equipped with the discrete metric), then every subset of X is open in X.

To show why, let $U \subseteq X$ be given. Let $x \in U$ also be given. Notice that in the discrete metric, balls are given as shown:

$$B_r(x) = \begin{cases} \{x\} & 0 < r \leq 1 \\ U & r > 1 \end{cases}$$

Thus, if we choose r such that $0 < r \leq 1$, then we see that $B_r(x) = \{x\} \subseteq U$, which tells us that U is open in X.

Example: Trivial Open Sets

For a metric space X, the empty set $\emptyset$ and the entire set X are always open sets in X.

Example: Open Balls are Indeed Open

Let X be a metric space. We have that any open ball in X is open.

To show why, let's start by having center $p \in X$ and radius $r > 0$ be given. We want to show that for any $x \in B_r(p)$, we are able to find some other radius $\epsilon > 0$ such that $B_\epsilon(x) \subset B_r(p)$. Let $d \equiv d(p,x)$. Our claim is that $B_{r-d}(x) \subset B_r(p)$.

To prove the claim, we start by letting $y \in B_{r-d}(x)$. We eventually want to show that $d(y,p) < r$, which will guarantee that $y \in B_r(p)$. By the triangle inequality, we see that

$$d(y,p) \leq \underbrace{d(y,x)}_{<r-d} + \underbrace{d(x,p)}_{d} < r$$

Which proves what we wanted to show.

Thus, we have that for any $y \in B_{r-d}(x)$, we get $y \in B_r(p)$. Therefore, $B_{r-d}(x) \subset B_r(p)$, which tells us that $B_r(p)$ is open.

1.4 Discussion

Remark: Why we started with metric spaces

Topology is rather abstract, which makes metric spaces a better starting point; the ideas from metric spaces (e.g., openness) can be generalized to topological spaces, which makes grasping the concepts a little easier to do.

Chapter 2

Topological Spaces

2.1 Learning Objectives

- What is a topology?
- When given a topological space, what does it mean for a set to be open?
- What are some examples of topologies?
  - What are the trivial and discrete topologies?
- What does it mean for a topology to be finer or coarser than another topology?
- What does it mean for two topologies to be comparable?

2.2 Definitions

Definition: Topology

A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\emptyset$ and X are in $\mathcal{T}$.
2. The union of the elements of *any* subcollection of $\mathcal{T}$ is in $\mathcal{T}$; i.e., $\mathcal{T}$ is closed under arbitrary unions.
   a) More precisely, given a collection of sets $\{U_i\}_{i\in I}$, where I is some index set and $U_i \in \mathcal{T}$, we have that $\bigcup_{i\in I} U_i \in \mathcal{T}$.
3. The intersection of the elements of any *finite* subcollection of $\mathcal{T}$ is in $\mathcal{T}$; i.e., $\mathcal{T}$ is closed under finite intersections.
   a) More precisely, given any finite number of sets $U_1, \ldots, U_n \in \mathcal{T}$, we have that $U_1 \cap \ldots \cap U_n \in \mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a **topological space**.

Definition: Open Set

If X is a topological space with topology $\mathcal{T}$, we say that a subset U of X is an **open set** of X if U belongs to the collection $\mathcal{T}$.

Definition: Finer and Coarser Topologies

Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supset \mathcal{T}$, we say that $\mathcal{T}'$ is **finer** than $\mathcal{T}$; if $\mathcal{T}'$ *properly* contains $\mathcal{T}$, then we say that $\mathcal{T}'$ is **strictly finer** than $\mathcal{T}$.

We also say that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$, or **strictly coarser**, in these two respective situations.

Finally, we say that $\mathcal{T}$ is **comparable** with $\mathcal{T}'$ if either $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T} \supset \mathcal{T}'$.

2.3 Theorems

Theorem: deMorgan's Laws

deMorgan's law essentially states that complements transform unions into intersections and vice-versa. Putting it in a more mathematical statement, we see:

$$X \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (X \setminus A_i) \quad \text{and} \quad X \setminus \bigcup_{i \in I} A_i = \bigcap_{i \in I} (X \setminus A_i)$$

Proof We find that both statements are merely statements of logic.

For the first statement, let's say that $x \in X \setminus \bigcap_{i \in I} A_i$. Notice that this just means that $x \notin \bigcap_{i \in I} A_i$, which further means that there exists at least one $i \in I$ such that $x \notin A_i$. I.e., we see that there exists at least one $i \in I$ such that $x \in X \setminus A_i$, which means that $x \in \bigcup_{i \in I} (X \setminus A_i)$. Therefore we have that

$$X \setminus \bigcap_{i \in I} A_i \subseteq \bigcup_{i \in I} (X \setminus A_i)$$

Now let's say that $x \in \bigcup_{i \in I} (X \setminus A_i)$. This means that there exists at least one $i \in I$ such that $x \in X \setminus A_i$, or equivalently, such that $x \notin A_i$. As a result, it must be the case, then, that $x \notin \bigcap_{i \in I} A_i$, which is equivalent to $x \in X \setminus \bigcap_{i \in I} A_i$ Therefore we have that

$$\bigcup_{i \in I} (X \setminus A_i) \subseteq X \setminus \bigcap_{i \in I} A_i$$

Combining both inclusions results in the equality that we were after.

For the second statement, let's say that $x \in X \setminus \bigcup_{i \in I} A_i$. Again, this just means that $x \notin \bigcup_{i \in I} A_i$, which further means that that for all $i \in I$, we must have $x \notin A_i$. I.e., we see that for all $i \in I$, we must have $x \in X \setminus A_i$, which means that $x \in \bigcap_{i \in I} (X \setminus A_i)$. Therefore we have that

$$X \setminus \bigcup_{i \in I} A_i \subseteq \bigcap_{i \in I} (X \setminus A_i)$$

Now let's say that $x \in \bigcap_{i \in I}(X \setminus A_i)$. This means that for all $i \in I$, we have $x \in X \setminus A_i$, or equivalently, such that $x \notin A_i$. As a result, it must be the case, then, that $x \notin \bigcup_{i \in I} A_i$, which is equivalent to $x \in X \setminus \bigcup_{i \in I} A_i$ Therefore we have that

$$\bigcap_{i \in I}(X \setminus A_i) \subseteq X \setminus \bigcup_{i \in I} A_i$$

Combining both inclusions results in the equality that we were after.

2.4 Examples

Example: Metric Topology

A metric space has an induced topology that we call the **metric topology**. Notice that the condition for openness in this topology is determined by the metric; e.g., $\mathbb{R}^2$ with the euclidean metric has the condition of openness via open balls.

Example: The Standard Topology on $\mathbb{R}$

Let $X = \mathbb{R}$, and let $\mathcal{T}$ to be all unions of open intervals. We have that $\mathcal{T}$ is a topology.

To show why, it is clear that $\emptyset$ and $\mathbb{R}$ are in $\mathcal{T}$. Furthermore, it is clear that $\mathcal{T}$ is closed under unions since it follows by construction. All that is left to do is to show that $\mathcal{T}$ is closed under finite intersections.

Take $I_1, \ldots, I_n \in \mathcal{T}$. Note that $I_1 \cap I_2$ is still an open interval (where we are taking $\emptyset$ to be open), thus it follows by induction that $I_1 \cap \ldots \cap I_n$ is still an open interval as well. Since we have $\mathcal{T}$ to be all unions of open intervals, it follows that $I_1 \cap \ldots \cap I_n \in \mathcal{T}$.

We give this topology on $\mathbb{R}$ a special name – it is called the **standard topology** on $\mathbb{R}$.

Example: Discrete Topology

Let X be any set. The collection of *all* subsets of X is a topology on X – called the **discrete topology**.

To show why, we have right away that $\emptyset$ and X are in $\mathcal{T}$ since both $\emptyset, X \subset X$. Furthermore, the second condition is met right away as well since the union of any subcollection of $\mathcal{T}$ is still a subset of X. All that is left to do is to show that $\mathcal{T}$ is closed under finite intersections.

Take $U_1, \ldots, U_n \in \mathcal{T}$. Notice that $U_1 \cap \ldots \cap U_n$ is a subset of each of the sets $U_1, \ldots, U_n$ (this can be proved rigorously via induction), which are themselves subsets of X. Hence it follows that $U_1 \cap \ldots \cap U_n$ is a subset of X, which means that $U_1 \cap \ldots \cap U_n \in \mathcal{T}$.

Example: Trivial Topology

Let X be any set. The collection consisting of X and $\emptyset$ only is a topology on X – called the **indiscrete topology**, or the **trivial topology**. This is clear to see as $\mathcal{T}$ consists of X and $\emptyset$ only, which immediately results in the three conditions to be met.

Example: Finite Complement Topology

Let X be a set, and let $\mathcal{T}_f$ be the collection of all subsets U of X such that $X \setminus U$ is either finite or is all of X. It follows that $\mathcal{T}_f$ is a topology on X, called the **finite complement topology**.

To show why, it is clear to see that both X and $\emptyset$ are in $\mathcal{T}_f$; $X \setminus X = \emptyset$ is finite and $X \setminus \emptyset = X$ is all of X.

Let us now take $\{U_i\}_{i \in I}$ to be an indexed family of nonempty elements of $\mathcal{T}_f$. Our goal is to show that $\bigcup_{i \in I} U_i \in \mathcal{T}_f$, which just amounts to showing that $X \setminus \bigcup_{i \in I} U_i$ is finite. Notice that DeMorgan's law tells us

$$X \setminus \bigcup_{i \in I} U_i = \bigcap_{i} (X \setminus U_i)$$

Because we have each $U_i \in \mathcal{T}$, it follows that $X \setminus U_i$ is finite by construction. Thus, we have that $\bigcap_{i \in I}(X \setminus U_i)$ is finite as arbitrary intersections of finite sets are finite, which takes care of the second condition.

Let us now take $U_1, \ldots, U_n$ to be a finite collection of nonempty elements of $\mathcal{T}_f$. We want to show that $U_1 \cap \ldots \cap U_n \in \mathcal{T}_f$. Again, this just amounts to showing that $X \setminus U_1 \cap \ldots \cap U_n$ is finite. In this case, DeMorgan's law tells us

$$X \setminus U_1 \cap \ldots \cap U_n = \bigcup_{i=1}^{n} (X \setminus U_i)$$

Since we know from above that $X \setminus U_i$ is finite, we have that $\bigcup_{i=1}^{n}(X \setminus U_i)$ is finite as a finite union of finite sets is still finite. This finishes the proof.

Example: Countable Complement Topology

Let X be a set, and let $\mathcal{T}_c$ be the collection of all subsets U of X such that $X \setminus U$ is either countable or is all of X. It follows that $\mathcal{T}_c$ is a topology on X.

To show why, it is clear to see that both X and $\emptyset$ are in $\mathcal{T}_c$; $X \setminus X = \emptyset$ is finite (which is countable) and $X \setminus \emptyset = X$ is all of X.

Let us now take $\{U_i\}_{i \in I}$ to be an indexed family of nonempty elements of $\mathcal{T}_c$. Our goal is to show that $\bigcup_{i \in I} U_i \in \mathcal{T}_c$, which just amounts to showing that $X \setminus \bigcup_{i \in I} U_i$ is countable. Notice that DeMorgan's law tells us

$$X \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} (X \setminus U_i)$$

Because we have each $U_i \in \mathcal{T}$, it follows that $X \setminus U_i$ is countable by construction. Thus, we have that $\bigcap_{i\in I}(X \setminus U_i)$ is countable as arbitrary intersections of countable sets are still countable; such a statement is true by noting that $\bigcap_{i\in I}(X \setminus U_i)$ is a subset of all $\{U_i\}_{i\in I}$, and a subset of a countable set is still countable.

Let us now take $U_1, \ldots, U_n$ to be a finite collection of nonempty elements of $\mathcal{T}_c$. We want to show that $U_1 \cap \ldots \cap U_n \in \mathcal{T}_c$. Again, this just amounts to showing that $X \setminus U_1 \cap \ldots \cap U_n$ is countable. In this case, DeMorgan's law tells us

$$X \setminus U_1 \cap \ldots \cap U_n = \bigcup_{i=1}^{n}(X \setminus U_i)$$

Since we know from above that $X \setminus U_i$ is countable, we have that $\bigcup_{i=1}^{n}(X \setminus U_i)$ is countable as a finite (in fact, we can have a countable amount) union of countable sets is still countable. This finishes the proof.

Example: Topology of a 3-element Set

Let X be a three-element set, $X = \{a, b, c\}$. There are many possible topologies on X – a few of which are shown in Figure 2.1. Notice that the diagram in the upper left-hand corner indicates the trivial topology, and the diagram in the lowe right-hand corner indicates the discrete topology.

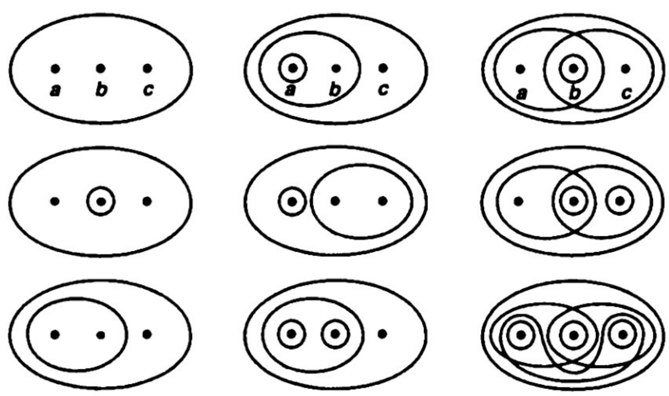

Figure 2.1: Some Possible Topologies on X

The two diagrams shown in Figure 2.2 are examples of non-topologies in X.

Figure 2.2: Some Possible Non-Topologies on X

The left diagram is not a topology because it is not closed under arbitrary unions, and the right diagram is not a topology because it is not closed under finite intersections.

Example: Comparing Topologies on $\mathbb{R}$

Let $X = \mathbb{R}$. Compare the following topologies on $\mathbb{R}$:

$$\mathcal{T}_{\text{discrete}}, \mathcal{T}_{\text{trivial}}, \mathcal{T}_{\text{standard}}, \mathcal{T}_{\text{cofinite}}$$

Here the standard topology on $\mathbb{R}$ means the same thing as the Euclidean topology.

We see that

$$\mathcal{T}_{\text{trivial}} \subset \mathcal{T}_{\text{cofinite}} \subset \mathcal{T}_{\text{standard}} \subset \mathcal{T}_{\text{discrete}}$$

One can immediately expect $\mathcal{T}_{\text{trivial}}$ and $\mathcal{T}_{\text{discrete}}$ to be the coarsest and finest topologies, respectively. This leaves us with finding where $\mathcal{T}_{\text{cofinite}}$ and $\mathcal{T}_{\text{standard}}$ should be placed inclusion-wise.

We can immediately see that for any open interval (a,b) in $\mathcal{T}_{\text{standard}}$, we see that its complement is not finite – thus, we have that $\mathcal{T}_{\text{cofinite}}$ cannot contain $\mathcal{T}_{\text{standard}}$. We now want to see whether $\mathcal{T}_{\text{standard}}$ contains $\mathcal{T}_{\text{cofinite}}$. This amounts to showing that every open set in $\mathcal{T}_{\text{cofinite}}$ is also open in $\mathcal{T}_{\text{standard}}$.

Let U be any open set in $\mathcal{T}_{\text{cofinite}}$. By definition, this means that $\mathbb{R} \setminus U$ is either finite or the $\emptyset$. Assuming that it is finite (the $\emptyset$ is trivially open in any topology), we have that

$$\mathbb{R} \setminus U = \{x_1, \ldots, x_n\} \quad \text{where} \quad n \in \mathbb{Z}_+$$

Furthermore, let us assume that the finite point set is ordered from smallest to largest – that is, $x_1 \leq \ldots \leq x_n$. With this, we see that U must be of the following form:

$$U = (-\infty, x_1) \cup (x_1, x_2) \cup \ldots \cup (x_{n-1}, x_n) \cup (x_n, \infty)$$

which is just a union of open intervals, which is also open itself. Thus, U is open in $\mathcal{T}_{\text{standard}}$ as well.

As a result, we are left with $\mathcal{T}_{\text{cofinite}} \subset \mathcal{T}_{\text{standard}}$.

2.5 Discussion

Remark: Open sets in a Topology

Notice that the notion of open sets differs from what we originally know them to be. When working with topologies, we first need to figure out the subsets that will make up the topology (there could be many possibilities). Once such subsets are found or determined, we have that they are automatically open by definition.

Remark: On the Third Condition of a Topology

We can restate the third condition instead to be that $\mathcal{T}$ is closed under *pairwise intersections* – that is, if $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$.

Using induction, we can very easily show that this re-stated condition is equivalent to the original third equation.

Remark: Finite Complement Topology

This topology is also referred to as the **cofinite topology**; Cofinite refers to the subsets of a set X whose complements in X are finite.

Thus, we have the cofinite topology to include all the cofinite subsets of X and the empty set – slightly different from the definition given in the original example.

Remark: Finer and Coarser Topologies

Finally, this section ended with the concept of finer or coarser topologies. In its essence, we can think of a topological space as being something like a bucket of gravel – the pebbles and all unions of collections of pebbles being the open sets. If we now smash the pebbles into smaller ones, the collection of open sets has been enlarged, and the topology, like the gravel, is said to have been made finer by the operation. Such an operation for a topology would correspond to partitioning subsets into smaller subsets (while keeping the original subsets).

Furthermore, the introduction of finer or coarser topologies brought forth the notion of topologies being comparable. It should be noted that topologies in general need not be comparable.

Remark: Topologies Induced from Metrics

We can show that for any finite sets, we can only induce a topology from a metric if the metric is discrete – that is, the only topology induced from metrics for any finite set is the discrete topology.

Chapter 3

Basis for a Topology

3.1 Learning Objectives

- What is a basis for a topology?
- What does it mean for a topology to be generated by a basis?
- What are some common examples of topologies on $\mathbb{R}$?
- What is subbasis and the topology that it generates?
- Why can we express every open set in a topological space as a union of basis elements? [Lemma 13.1]
- How can we obtain a basis for a given topology? [Lemma 13.2]
- How can we determine if topologies are finer than each other via bases? [Lemma 13.3]
- What are the three common topologies on $\mathbb{R}$, and what do their corresponding bases consist of? [Topologies on $\mathbb{R}$]

3.2 Definitions

Definition: Basis for a Topology

If X is a set, then a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets X (called **basis elements**) such that

1. For each $x \in X$, there is at least one basis element B containing x.
2. If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

Definition: Topology Generated by a Basis

Given a basis $\mathcal{B}$, we define the **topology $\mathcal{T}$ generated by $\mathcal{B}$** as follows: A subset U of X is said to be open in X (that is, $U \in \mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.

Note that each basis element is itself an element of $\mathcal{T}$.

Definition: Topologies on $\mathbb{R}$

If $\mathcal{B}$ is the collection of all open intervals in $\mathbb{R}$, i.e., of the form

$$(a,b) = \{x \mid a < x < b\}$$

Then the topology generated by $\mathcal{B}$ is called the **standard topology** on $\mathbb{R}$.

If $\mathcal{B}'$ is the collection of all half-open intervals of the form

$$[a,b) = \{x \mid a \leq x < b\}$$

where $a < b$, then the topology generated by $\mathcal{B}'$ is called the **lower limit topology** on $\mathbb{R}$. When $\mathbb{R}$ is given the lower limit topology, we shall denote it by $\mathbb{R}_\ell$.

Finally, let K denote the set of all numbers of the form $\frac{1}{n}$ for $n \in \mathbb{Z}_+$, and let $\mathcal{B}''$ be the collection of all open intervals (a,b), along with all sets of the form $(a,b) \setminus K$. The topology generated by $\mathcal{B}''$ will be called the K-**topology** on $\mathbb{R}$. When $\mathbb{R}$ is given this topology, we shall denote it by $\mathbb{R}_K$.

Definition: Subbasis Topology

A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X.

The **topology generated by the subbasis** $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

3.3 Theorems

Lemma: 13.1

Let X be a set; let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

Proof Let's say that we are given any collection of elements of $\mathcal{B}$, which we shall denote as $\{B_i\}_{i \in I}$ where I is some indexing set. By definition, we have basis elements to be themselves an element of $\mathcal{T}$, which means that $B_i \in \mathcal{T}$ for all $i \in I$. Since $\mathcal{T}$ is a topology, we have that the union of these basis elements are also in $\mathcal{T}$ – that is, we have $\bigcup_{i \in I} B_i \in \mathcal{T}$. Because we have showed that the union of an arbitrary collection of elements of $\mathcal{B}$ was contained $\mathcal{T}$, we have that the collection of all unions of elements of $\mathcal{B}$ is a subset of $\mathcal{T}$.

Conversely, let us now be given any $U \in \mathcal{T}$, where $\mathcal{B}$ is the basis for $\mathcal{T}$. Since U is open, we have that for each $x \in U$, there is a basis element $B_x \in \mathcal{B}$ such that $x \in B_x$ and $B_x \subset U$. Since

$B_x \subset \bigcup_{x\in U} B_x$, it follows that every $x \in U$ must also be contained in $\bigcup_{x\in U} B_x$ – that is,

$$U \subset \bigcup_{x\in U} B_x$$

Furthermore, since we have that $B_x \subset U$ for all $x \in U$, it follows as well that

$$\bigcup_{x\in U} B_x \subset U$$

which results in $U = \bigcup_{x\in U} B_x$. I.e., we have that U equals a union of elements of $\mathcal{B}$, which tells us that $\mathcal{T}$ is a subset of the collection of all unions of elements of $\mathcal{B}$.

Putting everything together proves our lemma.

Lemma: 13.2

Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each $x \in U$, there is an element $C \in \mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.

Proof The proof will come in two main parts: the first deals with showing that $\mathcal{C}$ is indeed a basis; the second deals with showing that given a topology $\mathcal{T}$ of X, the topology generated by $\mathcal{C}$ equals $\mathcal{T}$.

We need to show that $\mathcal{C}$ is a basis. The first condition is met easily: since X is itself an open set, the hypothesis tells us that for any $x \in X$, there is an element $C \in \mathcal{C}$ such that $x \in C \subset \mathcal{C}$. For the second condition, we let x to belong to $C_1 \cap C_2$, where $C_1, C_2 \in \mathcal{C}$. Since we have both C_1 and C_2 are open, it follows that $C_1 \cap C_2$ is open as it is a finite intersection of open sets. Therefore, we have by hypothesis again that we are able to find an element $C_3 \in \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.

Now we want to show that the topology generated by $\mathcal{C}$, say $\mathcal{T}'$, equals $\mathcal{T}$. We first start by noting that if U belongs to $\mathcal{T}$, then by definition, U must be open. If $x \in U$, then the hypothesis tells us that there is an element $C \in \mathcal{C}$ such that $x \in C \subset U$. Thus, it follows by definition of a topology being generated by a basis that $U \in \mathcal{T}'$ – i.e., we have $\mathcal{T} \subset \mathcal{T}'$.

Conversely, let's say that $W \in \mathcal{T}'$. By [Lemma 13.1], we have that W equals a union of elements of $\mathcal{C}$. Because we know that each element of $\mathcal{C}$ belongs to $\mathcal{T}$, and $\mathcal{T}$ is a topology, it follows that $W \in \mathcal{T}$ – i.e., we have $\mathcal{T}' \subset \mathcal{T}$.

Putting everything together results in $\mathcal{T} = \mathcal{T}'$.

Lemma: 13.3

Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then the following are equivalent:

1. $\mathcal{T}'$ is finer and $\mathcal{T}$

2. For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Proof

(2) $\implies$ (1) Recall that $\mathcal{T}'$ being finer than $\mathcal{T}$ means that $\mathcal{T}' \supset \mathcal{T}$. Thus, given an element $U \in \mathcal{T}$, we wish to show that $U \in \mathcal{T}'$. Let $x \in U$. Since we know that $\mathcal{B}$ generates $\mathcal{T}$, there is an element $B \in \mathcal{B}$ such that $x \in B \subset U$. The second condition of this lemma tells us that there exists an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$. Thus, we have $B' \subset B \subset U$, and by the definition of a topology being generated by a basis, we further have that $U \in \mathcal{T}'$.

(1) $\implies$ (2) Let's now say that we are given $x \in X$ and $B \in \mathcal{B}$, with $x \in B$. Now B belongs to $\mathcal{T}$ by definition and $\mathcal{T} \subset \mathcal{T}'$ by the first condition; therefore, we have $B \in \mathcal{T}'$. Since $\mathcal{T}'$ is generated by $\mathcal{B}'$, it follows that there is an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Lemma: 13.4

The topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.

Proof We shall let $\mathcal{T}$, $\mathcal{T}'$, and $\mathcal{T}''$ be the topologies of $\mathbb{R}$, $\mathbb{R}_\ell$ and $\mathbb{R}_K$, respectively. Given a basis element (a,b) for $\mathcal{T}$ and a point $x \in (a,b)$, we have that the basis element $[x,b)$ for $\mathcal{T}'$ contains x and lies in (a,b). However, given the basis element $[x,d)$ for $\mathcal{T}'$, we find that there is no open interval (a,b) such that it contains x and lies in $[x,d)$ (a will always be to the left of x). Thus, we find that $\mathcal{T}'$ is strictly finer than $\mathcal{T}$.

We now focus our attention to $\mathbb{R}_K$ and $\mathbb{R}$. Given a basis element (a,b) for $\mathcal{T}$ and a point x of (a,b), this same interval is a basis element for $\mathcal{T}''$ that contains x. However, given the basis element $B = (-1,1) \setminus K$ for $\mathcal{T}''$ and the point 0 of B, we find that there is no open interval that contains 0 and lies in B – any open interval that contains 0 will contain numbers of the form $\frac{1}{n}$ for $n \in \mathbb{Z}_+$, which would make it impossible for such an interval to lie in B. Thus, we find that $\mathcal{T}''$ is strictly finer than $\mathcal{T}$.

Finally, we want to show that the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable. To do so, we just need to find an open set in each that is not open in the other. To start, we see that $[2,3)$ is open in $\mathbb{R}_\ell$, but not in $\mathbb{R}_K$. Furthermore, we have that $\mathbb{R} \setminus K$ is open in $\mathbb{R}_K$, but not in $\mathbb{R}_\ell$ since every open set containing 0 contains numbers of the form $\frac{1}{n}$ for $n \in \mathbb{Z}_+$.

3.4 Examples

Example: Example of a Basis in $\mathbb{R}^2$

Let $X = \mathbb{R}^2$. Let $\mathcal{B}$ be the collection of all balls in the plane. Then $\mathcal{B}$ satisfies both conditions of a basis; indeed, the first condition is met as we can always put a ball around any $x \in X$, and the second condition is illustrated in Figure 3.1

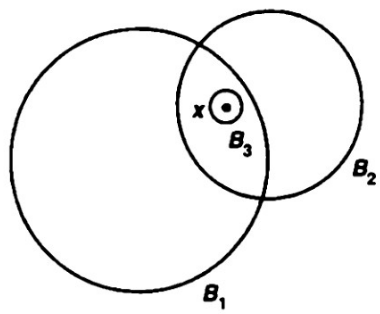

Figure 3.1: A Visual of the Second Condition

In the topology generated by $\mathcal{B}$, we also have that the definition of openness can be rewritten as follows: a subset U of the plane is open if every $x \in U$ lies within some ball contained in U – this is the definition of openness that we learned in the metric spaces section earlier on.

Example: Another Example of a Basis in $\mathbb{R}^2$

Let $X = \mathbb{R}^2$. Let $\mathcal{B}'$ be the collection of all rectangular regions (interiors of rectangles) in the plane, where the rectangles have sides parallel to the coordinate axes. Then $\mathcal{B}'$ satisfies both conditions of a basis; indeed, the first condition is met as we can always put a rectangular region around any $x \in X$, and the second condition is illustrated in Figure 3.2

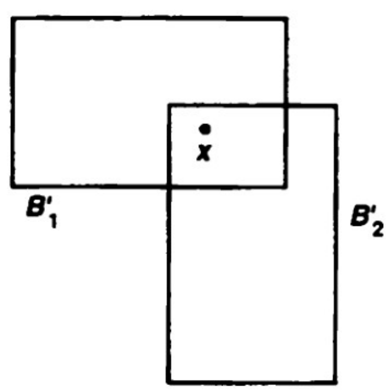

Figure 3.2: A Visual of the Second Condition

One can see that the second condition is trivial since the intersection of any two basis elements is either another rectangular region (hence, a basis element) or the empty set.

In fact, we have that the basis $\mathcal{B}$ from the previous example generates the same topology as the collection $\mathcal{B}'$ of all rectangular regions. Figure 3.3 illustrates the proof.

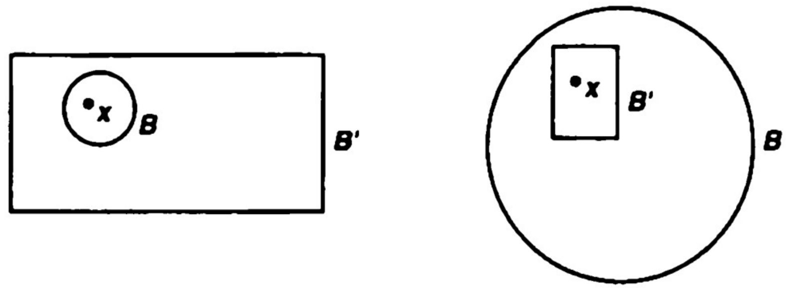

Figure 3.3: A Visual Proof

Example: Basis for a Discrete Topology

If X is any set, then the collection of all one-point subsets of X is a basis for the discrete topology on X.

The first condition of a basis is met trivially – for any $x \in X$, it is contained in the one-point subset $\{x\}$. For the second condition, we have that x will only belong to one one-point subset – i.e., the intersection of any two one-point subsets is $\emptyset$. Thus, the second condition is vacuously met (there is nothing to check!).

Example: Basis for $\mathbb{R}_\ell$

We want to show that the collection of all half-open intervals of the form

$$[a,b) = \{x \mid a \leq x < b\}$$

where $a < b$, is indeed a basis of $\mathbb{R}$.

For the first condition (fullness), we let $x \in \mathbb{R}$. It is clear to see that $x \in [x, x+1)$, but there are many more other possibilities.

For the second condition (smallness), we let $x \in [a,b) \cap [c,d)$, where $a < c$, WLOG. We shall focus on the case when $b < d$ to save some time. Notice that if the intersection is not $\emptyset$, then we get $[a,b) \cap [c,d) = [c,b)$. Thus, we have $x \in [c,b)$, which fulfills the smallness condition. If we have $\emptyset$ instead, then we see that the smallness condition is met vacuously.

Example: Topology Generated by a Basis Verification

We want to show that the collection $\mathcal{T}$ generated by the basis $\mathcal{B}$ is indeed a topology on X.

$\boxed{\emptyset, X \in \mathcal{T}}$

For this condition, we need to check that both $\emptyset$ and X are open. We start by noting that $\emptyset$ satisfies the defining conditions of openness vacuously (there is nothing to check since $\emptyset$ has no elements).

Now to show that X is open, we need to show that for each $x \in X$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset X$. This immediately follows from the fact that we have a basis $\mathcal{B}$ – by definition, each basis element is a subset of X, and for each $x \in X$, there is at least one basis element B containing x.

$\boxed{\mathcal{T} \text{ is closed under unions}}$

For this condition, let us take an indexed family $\{U_i\}_{i\in I}$, of elements of $\mathcal{T}$. We want to show that $\mathcal{T}$ is closed under unions – that is, we want to show that

$$U = \bigcup_{i\in I} U_i$$

belongs to $\mathcal{T}$. To do so, we need to check that U is open.

Let's say that we are given some $x \in U$. It follows that there is at least one index i such that $x \in U_i$. Since we know that $U_i \in \mathcal{T}$, it follows that U_i is open, which further means (by the definition of openness via a basis) that there exists some basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U_i \subset U$. Hence, it follows that U is open.

$\boxed{\mathcal{T} \text{ is closed under finite intersections}}$

For this condition, we want to show that $\mathcal{T}$ is closed under finite intersections. If we have n elements $U_1, \ldots, U_n$ of $\mathcal{T}$, then all we need to do is to show that $U_1 \cap \ldots \cap U_n \in \mathcal{T}$, which requires us to check that $U_1 \cap \ldots \cap U_n$ is open. We shall do so via induction.

For $n = 1$, we are done since $U_1 \in \mathcal{T}$ by construction. Let us check for $n = 2$. Our goal is to show that $U_1 \cap U_2 \in \mathcal{T}$. Given $x \in U_1 \cap U_2$, we can choose basis elements B_1, B_2 such that $x \in B_1$, $x \in B_2$ and $B_1 \subset U_1$, $B_2 \subset U_2$ since we know both U_1 and U_2 to be open. Now because we are given a basis, we have its second condition enables us to choose a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$. A visual of this condition is shown in Figure 3.4

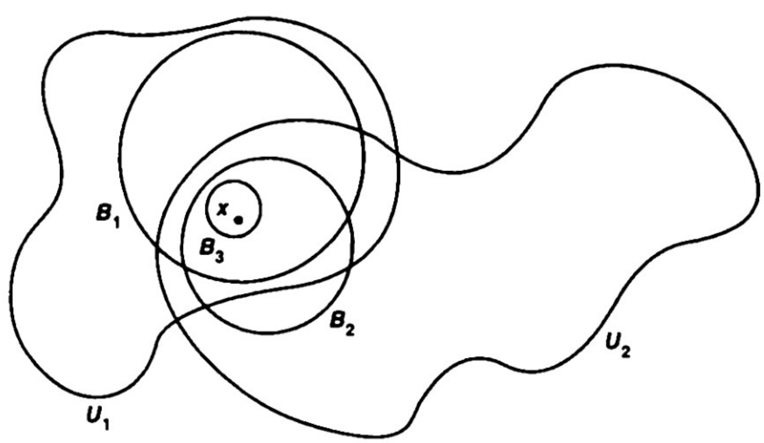

Figure 3.4: A Visual on the Second Condition

Thus, we have that $x \in B_3$ and $B_3 \subset U_1 \cap U_2$, which tells us that $U_1 \cap U_2$ is open – that is, we have $U_1 \cap U_2 \in \mathcal{T}$.

Finally, we suppose that it is true for $n-1$ and prove it for n. Notice that

$$U_1 \cap \ldots \cap U_n = (U_1 \cap \ldots \cap U_{n-1}) \cap U_n$$

By our induction hypothesis, we have that $U_1 \cap \ldots \cap U_{n-1} \in \mathcal{T}$. Further notice that this reduces us down to the $n = 2$ case, which we have already solved for. Thus, we have the intersection of $U_1 \cap \ldots \cap U_{n-1}$ and U_n also belongs to $\mathcal{T}$.

Thus, we have checked that the collection of open sets generated by a basis $\mathcal{B}$ is a topology.

Example: Subbasis Topology Verification

It suffices to show that the collection $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ is a basis. By doing so, [Lemma 1] tells us that the collection $\mathcal{T}$ of all unions of elements of $\mathcal{B}$ is a topology.

We start with the first condition for a basis. For any given $x \in X$, we have that it belongs to some element of $\mathcal{S}$, and hence, to an element of $\mathcal{B}$.

For the second condition, we let

$$B_1 = S_1 \cap \ldots \cap S_m \quad \text{and} \quad B_2 = S_1' \cap \ldots \cap S_n'$$

be two elements of $\mathcal{B}$. We now let

$$B_3 = B_1 \cap B_2 = (S_1 \cap \ldots \cap S_m) \cap (S_1' \cap \ldots \cap S_n')$$

which we also see to be a finite intersection of elements of $\mathcal{S}$, so we have that $B_3 \in \mathcal{B}$. Since $B_3 = B_1 \cap B_2$, it is also the case that $B_3 \subset B_1 \cap B_2$, which satisfies the second condition.

3.5 Discussion

Remark: On Basis

Basis can be referred to as a **basis of subsets**, and their elements can also be referred to as a **basic open set**.

Furthermore, the first and second condition for a basis is sometimes referred to as **fullness** and **smallness**, respectively.

Remark: Why do we care about working with a basis?

For each of the examples that we discussed in Section 12, we were able to specify the topology by describing the entire collection $\mathcal{T}$ of open sets. However, this is usually too difficult – in most cases, one specifies instead a smaller collection of subsets X (i.e., a basis) and defines the topology in terms of being generated by such a basis.

Remark: On Open Sets

When we are working with open sets, we **always** need to ask which with respect to which basis our sets are open.

Remark: Openness in a basis and its implications

When working with a topology, we have by definition its elements to be open. However, when we are working with a basis, we now have a condition for openness. Thus, it makes sense for a topology to be generated by a basis, since the condition for openness inherited by a basis is what will generate which sets within X will be open or not – i.e., the choice of basis will determine which sets are open because of the condition for openness that we associate with such a basis, which will further determine what the topology on X will be.

Remark: Standard Topology on $\mathbb{R}$

Whenever we consider $\mathbb{R}$, we shall suppose the topology on $\mathbb{R}$ to be the standard one unless we specifically state otherwise.

Remark: On [Lemma 13.1]

This lemma states that every open set U in X can be expressed as a unions of basis elements. However, the expression for U is not unique.

Remark: On [Lemma 13.2]

This lemma is particularly important as it allows for us to obtain a basis for a given topology.

The main point is that a basis should include open sets that are *small* enough to capture arbitrary nearness to any given point. Thus, every open neighborhood of x contains a basic open neighborhood of x.

Remark: On [Lemma 13.3]

The main importance of this lemma is that it allows for us to determine whether one topology is finer than another.

The direction of the inclusion may be difficult to remember. It may be easier to remember if we recall the analogy between a topological space and a truckload full of gravel. Think of the pebbles as the basis elements of the topology; after the pebbles are smashed to dust, the dust particles are the basis elements of the new topology. The new topology is finer than the old one, and each dust particle was contained inside a pebble, as the criterion states.

Chapter 4

Closed Sets and Limit Points

4.1 Learning Objectives

- What is an open neighborhood?
- What are open sets? Closed sets?
- What are the interior, closure, and boundary of a set?
- What is a limit point? An accumulation point?
  - How are limit points related to the closure?
- How is convergence defined in a topological space?
- What does it mean for a space to be Hausdorff? T_1?
  - How are these two spaces related?
  - What are some examples of both spaces?

4.2 Definitions

Definition: Open Neighborhood

An **open neighborhood of a point** x in a topological space X is any open subset of X that contains x.

Definition: Closed Sets

A subset A of a topological space X is said to be **closed** if the set $X \setminus A$ is open.

Definition: Clopen Sets

Sets that are both closed and open are said to be **clopen**.

Definition: Interior of a Set

Let's say that we are given a subset A of a topological space X. The **interior** of A is defined as the union of all open sets contained in A. The interior of A is denoted by Int A or by A°.

We can also define Int A to be the largest open subset of X that is contained in A.

By design, we see that if $U \in X$ is open and $U \subset A$, then $U \subset \text{Int } A$.

Definition: Closure of a Set

Let's say that we are given a subset A of a topological space X. The **closure** of A is defined as the intersection of all closed sets containing in A. The closure of A is denoted by Cl A or by $\overline{A}$.

We can also define $\overline{A}$ to be the smallest closed subset of X that contains A.

By design, we see that if $C \in X$ is closed and $C \supset A$, then $\overline{A} \subset C$.

Definition: Boundary of a Set

Let's say that we are given a subset A of a topological space X. The **boundary** of A, denoted as ∂A is the complement of Int A in $\overline{A}$ – that is,

$$\partial A \equiv \overline{A} \setminus \text{Int } A$$

An alternative definition that is commonly given is the following:

$$\partial A \equiv \overline{A} \cap (\overline{X \setminus A})$$

Definition: Limit Point

If A is a subset of the topological space X and if x is a point of X, we say that x is a **limit point** of A if every neighborhood of x intersects A in some point *other than x itself* – that is, if U is an open neighborhood of a limit point x of A, then $U \cap A$ must contain at least one point besides x.

Said differently, x is a limit point of A if it belongs to the closure of $A \setminus \{x\}$. Notice that the point x may lie in A or not; for this definition, it does not matter.

Definition: Accumulation Point

Let X be a topological space and $(x_0, x_1, x_2, \ldots)$ be a sequence in X. We say that a given $x \in X$ is an **accumulation point** of the sequence if, for each open neighborhood U of x, there exist infinitely many $n \in \mathbb{N}$ such that $x_n \in U$.

Definition: Image of a Sequence

The **image** of a given sequence in a topological space X is the subset of X consisting of all elements that appear in that sequence.

Definition: Convergence

In an arbitrary topological space, we say that a sequence of points $x_1, x_2, \ldots$ (commonly denoted as (x_n)) of the space X **converges** to the point x (called the **limit** of the sequence) of X provided that, corresponding to each neighborhood U of x, there is a positive integer N such that $x_n \in U$ for $n \geq N$.

If X is a metric space, then we say that a sequence of points $x_1, x_2, \ldots$ of the space X **converges** to the point x of X (written as $x_n \to x$) if and only if: for all $\epsilon > 0$, there exists an index N such that $d(x_n, x) < \epsilon$ for all $n \geq N$. I.e., we have $x_n \to x$ when: for each $\epsilon > 0$, the sequence $x_1, x_{2,\ldots}$ is *eventually* contained in the open ball of radius ϵ centered at 0.

By eventually, we mean that there is some index N so that the statement is true for all x_n with $n \geq N$.

Definition: Hausdorff Space

A topological space X is called a **Hausdorff space** if for each pair x_1, x_2 of distinct points of X, there exists neighborhoods U_1, U_2 of x_1, x_2, respectively, that are disjoint – i.e., $U_1 \cap U_2 = \emptyset$

Definition: T_1

The T_1 axiom is the condition that finite point sets be closed.

4.3 Theorems

Theorem: 17.1

Let X be a topological space. Then the following conditions hold:

1. $\emptyset$ and X are closed.
2. Arbitrary intersections of closed sets are closed.
3. Finite union of closed sets are closed.

Proof

1 $\emptyset$ and X are closed because they are the complements of the open sets X and $\emptyset$, respectively.

2 Given a collection of closed sets $\{A_i\}_{i\in I}$, where I is some index set, we want to show that $\bigcap_{i\in I} A_i$ is closed. Equivalently, we want to show that its complement $X \setminus \bigcap_{i\in I} A_i$ is open. We see

by DeMorgan's law that

$$X \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (X \setminus A_i)$$

Since the sets $X \setminus A_i$ are open by definition, the right side of this equation represents an arbitrary union of open sets, which we know to be open. Therefore, we have that arbitrary intersections of closed sets are closed.

$\boxed{3}$ Similarly, if we have a finite collection of sets $A_1, \dots, A_n$, then we want to show that $\bigcup_{i=1}^{n} A_i$ is closed. Equivalently, we want to show that its complement $X \setminus \bigcup_{i=1}^{n} A_i$ is open. We see by DeMorgan's law again that

$$X \setminus \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X \setminus A_i)$$

Since the sets $X \setminus A_i$ are open by definition, the right side of this equation represents a finite intersection of open sets, which we know to be open. Therefore, we have that finite unions of closed sets are closed.

Theorem: Alternative Characterizations

Let A be a subset of a topological space X.

(a) A point $x \in X$ is in the interior of A if and only if there is an open neighborhood of x that is entirely contained in A.

(b) A point $x \in X$ is in the closure of A if and only if every open neighborhood of x has a nonempty intersection with A.

(c) A point $x \in X$ is in the boundary of A if and only if every open neighborhood of x has a nonempty intersection with A and with $X \setminus A$.

Proof

Theorem: 17.2

Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

Proof

$\boxed{\Longrightarrow}$ Let's assume that A is closed in Y. By definition, it follows that $Y \setminus A$ is open in Y, which by definition again means that $Y \setminus A$ must equal the intersection of an open set U of X with Y. Thus, we have that $Y \setminus A = Y \cap U$. From here, we see the following:

$$Y \setminus (Y \setminus A) = Y \setminus (Y \cap U) \iff A = Y \cap (X \setminus U)$$

Furthermore, it follows that $X \setminus U$ is closed in X as $X \setminus (X \setminus U) = U$ is open in X. Thus, we see that A equals the intersection of a closed set of X with Y.

$\boxed{\Longleftarrow}$ Let's now assume that $A = Y \cap C$, where C is closed in X. It follows by definition that $X \setminus C$ is open in X, meaning that $Y \cap (X \setminus C)$ is open in Y, which follows by definition of the subspace topology. This is where we note that

$$Y \setminus A = Y \setminus (Y \cap C) \iff Y \setminus A = Y \cap (X \setminus C)$$

Furthermore, if follows that $X \setminus C$ is open in X since $X \setminus (X \setminus C) = C$ is closed. Thus, we have that $Y \setminus A$ is open in Y by definition of the subspace topology, which implies that A is closed in Y by definition as well.

Theorem: 17.3

Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.

$\boxed{\text{Proof}}$ If A is closed in Y, then we see by [Theorem 17.2] that there exists some closed set C of X such that $A = Y \cap C$. Since we know that Y is closed in X, we have that $Y \cap C$ is also closed in X due to arbitrary intersections of closed sets of X being closed in X. Thus, we have that A is closed in X.

Theorem: 17.4

Let Y be a subspace of X, let A be a subset of Y, let $\overline{A}$ denote the closure of A in X. Then the closure of A in Y equals $\overline{A} \cap Y$.

$\boxed{\text{Proof}}$ Let B denote the closure of A in Y. We know that the set $\overline{A}$ is closed in X, which means that $Y \cap \overline{A}$ is closed in Y by [Theorem 17.2]. Because $A \subset Y$ and $A \subset \overline{A}$, we have that $A \subset Y \cap \overline{A}$. We now note that since B is the closure of A in Y, it follows by definition that B equals the intersection of *all* closed subsets of Y containing A. Thus, it must be the case that $B \subset (Y \cap \overline{A})$.

On the other hand, we know that B is closed in Y. Thus, by [Theorem 17.2], we have that $B = Y \cap C$ for some closed set C of X. Since B contains A, it must be the case that C contains A as well. As a result, we have that C is a closed set of X containing A, which means that it is in the collection of closed sets containing A. By definition, $\overline{A}$ is the intersection of such a collection, which means that $\overline{A} \subset C$. Therefore, we have that $(Y \cap \overline{A}) \subset (Y \cap C) = B$.

Putting everything together gives us that $B = Y \cap \overline{A}$.

Theorem: 17.5

Let A be a subset of the topological space X.

(a) Then $x \in \overline{A}$ if and only if every open set U containing x intersects A.

(b) Supposing the topology of X is given be a basis, then $x \in \overline{A}$ if and only if every basis element B containing x intersects A.

Proof

$\boxed{\text{a}}$ We shall prove this statement via contrapositive – that is, we are proving the following: $x \notin \overline{A}$ if and only if there exists an open set U containing x that does not intersect with A.

($\implies$) If $x \notin \overline{A}$, then we have that $U = X \setminus \overline{A}$ is an open set containing x that does not intersect A, as desired.

($\impliedby$) If there exists an open set U containing x which does not intersect A, then we see that $X \setminus U$ is a closed set that contains A. By definition of the closure $\overline{A}$, we have that $\overline{A} \subset (X \setminus U)$. Since $x \notin X \setminus U$, it follows that $x \notin \overline{A}$.

$\boxed{\text{b}}$ This statement follows readily from (a).

($\implies$) If every open set containing x intersects A , then so does every basis element B containing x, because B is an open set.

($\impliedby$) If every basis element containing x intersects A, then so does every open set U containing x, because U contains a basis element that contains x.

Theorem: 17.6

Let A be a subset of the topological space X, and let A' be the set of all limit points of A. Then

$$\overline{A} = A \cup A'$$

Proof

$\boxed{A \cup A' \subset \overline{A}}$ Let's suppose that $x \in A \cup A'$. If $x \in A$, then we see that $x \in \overline{A}$ since $A \subset \overline{A}$ by definition. Now if $x \in A'$, then we see that every neighborhood of x intersects A (in a point different from x). Therefore, by [Theorem 17.5], we have that $x \in \overline{A}$. Hence, $A' \subset A$. Putting everything together results in $A \cup A' \subset \overline{A}$.

$\boxed{\overline{A} \subset (A \cup A')}$ We now suppose $x \in \overline{A}$. Our goal is to show that $x \in A \cup A'$. If x happens to lie in A, it is trivial that $x \in A \cup A'$. Let's now suppose that $x \in \overline{A}$ and $x \notin A$. Since $x \in \overline{A}$, we know by by [Theorem 17.5] that every neighborhood U of x intersects A; because $x \notin A$, we have the set U must intersect A at a point different from x. Thus, $x \in A'$ by definition, and we have $x \in A \cup A'$.

Putting everything together results in $\overline{A} = A \cup A'$.

Corollary: 17.7

A subset of a topological space is closed if and only if it contains all its limit points.

Proof

($\implies$) The set A is closed if and only if $A = \overline{A}$, meaning that A contains all its limits points by [Theorem 17.6].

($\impliedby$) If A contains all its limit points, then we see that $A' \subset A$. Thus, we have that $A \cup A' = A$, which equivalently means that $\overline{A} = A$ by [Theorem 17.6].

Theorem: 17.8

Every finite point set in a Hausdorff space X is closed.

Proof Notice that every finite point set can be viewed as the union of one-point sets – e.g., $\{a,b,c\} = \{a\} \cup \{b\} \cup \{c\}$. Thus, it suffices to show that every one-point set $\{x_0\}$ is closed since every finite point set will then be a finite union of closed sets, which is still closed.

If x is a point of X different from x_0, then x and x_0 have disjoint neighborhoods U and V, respectively. Since U does not intersect $\{x_0\}$, the point x cannot belong to the closure of the set $\{x_0\}$. As a result, the closure of the set $\{x_0\}$ is $\{x_0\}$ itself, so that it is closed.

Theorem: 17.9

Let X be a space satisfying the T_1 axiom; let A be a subset of X. Then the point x is limit point of A if and only if every neighborhood of x contains infinitely many points of A.

Proof

$\boxed{\implies}$ For this direction, we shall prove by contradiction. Suppose that x is a limit point of A. Suppose as well that some neighborhood U of x intersects A in only finitely many points. Then we have that U also intersects $A \setminus \{x\}$ in finitely many points, which we shall denote as $x_1, \ldots, x_m$. It follows by the T_1 axiom that $\{x_1, \ldots, x_m\}$ is closed as it is a finite point set. Thus, the set $X \setminus \{x_1, \ldots, x_m\}$ is an open set of X. From this, we have that

$$U \cap (X \setminus \{x_1, \ldots, x_m\})$$

is a neighborhood of x that does not intersect with the set $A \setminus \{x\}$; we have essentially constructed a set that removes all the intersection points of $A \setminus \{x\}$ and U, while maintaining such a set to be open. This contradicts the assumption that x is a limit point of A since we have found a neighborhood of x that does not intersect A in some other point other than x itself.

$\boxed{\impliedby}$ Now, if every neighborhood of x intersects A in infinitely many points, it certainly must intersect A in some point other than x itself. Hence, x must be a limit point of A.

Theorem: 17.10

If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.

Proof Let's suppose that (x_n) is a sequence of points of X that converges to x. If we now take any other point $y \neq x$, then X being Hausdorff allows for us to find two disjoint neighborhoods U and V of x and y, respectively. Since we know that U contains x_n for all but finitely many values of n – that is, there is a positive integer N such that $x_n \in U$ for all $n \geq N$ – we have that it is impossible for the set V to also contain x_n for all but finitely many values of n since it is disjoint to U. Thus, (x_n) cannot converge to y. Since y was arbitrary, we have that (x_n) converges to at most one point of X.

Theorem: 17.11

Every simply ordered set is a Hausdorff space in the order topology. The product of two Hausdorff spaces is a Hausdorff space. A subspace of a Hausdorff space is a Hausdorff space.

Proof

Subspace of a Hausdorff Space is Hausdorff Let X be a Hausdorff space and $Y \subset X$ be a subspace of X. Let $x, y \in Y$ be two distinct points. Our goal is to show that there exists open neighborhoods of x and y, respectively, in Y such that these neighborhoods are disjoint.

Since x, y are also distinct elements in X, we see that X being Hausdorff tells us that there exists open neighborhoods U and V of x and y, respectively, in X such that $U \cap V = \emptyset$. With this, $x \in U$ and $x \in Y$ tells us that $x \in Y \cap U$; similarly, we have that $y \in Y \cap V$.

Notice that both $Y \cap U$ and $Y \cap V$ are open in Y since U and V are open in X. Furthermore, we have that

$$(Y \cap U) \cap (Y \cap V) = \underbrace{(Y \cap Y)}_{Y} \cap \underbrace{(U \cap V)}_{\emptyset} = \emptyset$$

Putting everything together, we see that $Y \cap U$ and $Y \cap V$ are open neighborhoods of x and y, respectively, in Y that are also disjoint. Hence Y is Hausdorff.

Product of Two Hausdorff Spaces is Hausdorff
Let X and Y be two Hausdorff spaces. Our goal is to show that $X \times Y$ is Hausdorff. We start by letting $\mathbf{x}_1, \mathbf{x}_2 \in X \times Y$ be any two distinct points – that is, $\mathbf{x}_1 = (x_1, y_1)$ and $\mathbf{x}_2 = (x_2, y_2)$ with $x_1 \neq x_2$ and $y_1 \neq y_2$.

Because X is Hausdorff, we see that there exists open neighborhoods U_1, U_2 of x_1, x_2 in X such that $U_1 \cap U_2 = \emptyset$; similarly, because Y is Hausdorff, we see that there exists open neighborhoods V_1, V_2 of y_1, y_2 in Y such that $V_1 \cap V_2 = \emptyset$. From this, we get that $U_1 \times V_1$ and $U_2 \times V_2$ are two open neighborhoods of $\mathbf{x}_1$ and $\mathbf{x}_2$ that are disjoint from each other:

$$(U_1 \times V_1) \cap (U_2 \times V_2) = \underbrace{(U_1 \cap U_2)}_{\emptyset} \times \underbrace{(V_1 \cap V_2)}_{\emptyset} = \emptyset$$

Thus, it follows that $X \times Y$ is Hausdorff.

4.4 Examples

Example: Some Examples of Closed Sets

The subset $[a, b]$ of $\mathbb{R}$ is closed because its complement

$$\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, +\infty)$$

is open. Similarly, we see that $[a, +\infty)$ is closed because its complement $(-\infty, a)$ is open. Notice that $[a, b)$ of $\mathbb{R}$ is neither open nor closed.

In the place $\mathbb{R}^2$, the set

$$\{x \times y \mid x \geq 0 \text{ and } y \geq 0\}$$

is closed because its complement is the union of the two sets

$$(-\infty, 0) \times \mathbb{R} \quad \text{and} \quad \mathbb{R} \times (-\infty, 0)$$

each of which is a product of open sets of $\mathbb{R}$ and is therefore open in $\mathbb{R}^2$.

In the finite complement topology on a set X, the closed sets consists of X itself and all finite subsets of X.

In the discrete topology on the set X, every set is open. Thus it follows that every set is closed as well.

Example: Trivial Clopen Sets

Given any topological space X, we know from [Theorem 17.1] that $\emptyset$ and X are clopen. We call these the **trivial clopen subsets**.

Example: On the Interior and Closure of a Set

Let's say that we are given a subset A of a topological space X. It is clear to see that $\text{Int}\, A$ is an open set (an arbitrary union of open sets is still open). It is also clear to see that $\overline{A}$ is a closed set (an arbitrary intersection of closed sets is still closed).

Furthermore, we see by definition that

$$\text{Int}\, A \subset A \subset \overline{A}$$

If A is open, then we see that $A = \text{Int}\, A$. Indeed, A being open means that it is in the union of all open sets contained in A (note that A contains itself). Thus, the union of all open sets contained in A must equal A – the union of any subsets of A cannot be larger than A itself!

Similarly, if A is closed, then we see that $A = \overline{A}$. Indeed, A being closed means that it is in the intersection of all closed sets containing A. Thus, the intersection of all closed sets containing A must equal A – the intersection of a collection of sets cannot be larger than each of the individual sets!

Example: The Boundary of a Set is Closed in X

Let A be a subset of a topological space X. We want to show that ∂A is closed in X. To do so, we note that

$$\partial A = \overline{A} \setminus \operatorname{Int} A = \overline{A} \cap (X \setminus \operatorname{Int} A)$$

Further notice that $X \setminus \operatorname{Int} A$ is closed since $X \setminus (X \setminus \operatorname{Int} A) = \operatorname{Int} A$ is open. Since arbitrary intersections of closed sets are still closed, we see that ∂A is closed as well.

Example: Some Examples of Closures of Sets

Let $X = \mathbb{R}$. If $A = (0,1]$, then $\overline{A} = [0,1]$. This follows from seeing that every neighborhood of 0 intersects A, while every point outside $[0,1]$ has a neighborhood disjoint from A. Similar arguments apply to the following subsets of X.

If $B = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$, then we have that $\overline{B} = \{0\} \cup B$.

If $C = \{0\} \cup (1,2)$, then $\overline{C} = \{0\} \cup [1,2]$.

If $\mathbb{Q}$ is the set of rational numbers, then $\overline{Q} = \mathbb{R}$.

If $\mathbb{Z}_+$ is the set of positive integers, then $\overline{Z}_+ = \mathbb{Z}_+$.

If $\mathbb{R}_+$ is the set of positive reals, then closure of $\mathbb{R}_+$ is the set $\mathbb{R}_+ \cup \{0\}$.

Consider the subspace $Y = (0,1]$ of the real line $\mathbb{R}$. The set $A = (0, \frac{1}{2})$ is a subset of Y. Its closure in $\mathbb{R}$ is the set $[0, \frac{1}{2}]$, and its closure in Y is the set $[0, \frac{1}{2}] \cap Y = (0, \frac{1}{2}]$.

Example: Examples of Limit Points

Consider the real line $\mathbb{R}$. If $A = (0,1]$, then the point 0 is a limit point of A and so is the point $\frac{1}{2}$ – in fact, every point of the interval $[0,1]$ is a limit point of A, but no other points of $\mathbb{R}$ is a limit point of A.

If $B = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$, then 0 is the only limit point of B. Every other point x of $\mathbb{R}$ has a neighborhood that either does not intersect B at all (in particular, for $x \notin B$ and $x \neq 0$), or it intersects B only in the points x itself (in particular, for $x \in B$ and $x \neq 0$).

If $C = \{0\} \cup (1,2)$, then the limit points of C are the points of the interval $[1,2]$.

If $\mathbb{Q}$ is the set of rational numbers, every point of $\mathbb{R}$ is a limit point of $\mathbb{Q}$ – this follows from $\mathbb{Q}$ being dense in $\mathbb{R}$.

If $\mathbb{Z}_+$ is the set of positive integers, no point of $\mathbb{R}$ is a limit point of $\mathbb{Z}_+$.

If $\mathbb{R}_+$ is the set of positive reals, then every point of $\{0\} \cup \mathbb{R}_+$ is a limit point of $\mathbb{R}_+$.

Example: T_1-Spaces

We have that $\mathbb{R}$ with the cofinite topology is a T_1- space. This is because in the cofinite topology, sets whose complements are finite are defined to be open – meaning that set that are finite are closed in the cofinite topology. Hence, $\mathbb{R}$ with the cofinite topology is a T_1-space.

$\mathbb{R}$ with the trivial topology (where the only open subsets are $\emptyset$ and $\mathbb{R}$ itself) is not a T_1- space. This is because we have that $\emptyset$ and $\mathbb{R}$ are the only closed sets in the trivial topology. Thus, singleton sets (and thus, finite point sets) are not closed.

$\mathbb{R}$ with the lower limit topology is T_1. Because the lower limit topology is finer than the standard topology, we have that every subset of $\mathbb{R}$ that is closed in the standard topology is also closed in the lower limit topology. Since singleton sets are closed in the standard topology, they are also closed in the lower limit topology as well. Hence T_1.

Example: Hausdorff Spaces

Given a metric space X with the metric topology, we have that X is Hausdorff. To show why, we need to show that for all distinct points $x, y \in X$, there exists an open neighborhood U and V of x and y ,respectively, so that $U \cap V \emptyset$. Let us define $d \equiv d(x,y)$. It can be shown easily that $B_{\frac{d}{2}}(x) \cap B_{\frac{d}{2}}(y) = \emptyset$, which tells us that X is Hausdorff.

If we now consider $\mathbb{R}$ with the cofinite topology, then we have that it is not Hausdorff. The claim is that every pair of cofinite sets intersects nonemptily. To show why, let F and G be finite sets, which means that $\mathbb{R} \setminus F$ and $\mathbb{R} \setminus G$ are cofinite. DeMorgan's law tells us that

$$(\mathbb{R} \setminus F) \cap (\mathbb{R} \setminus G) = \mathbb{R} \setminus (F \cup G)$$

Notice that $F \cup G$ is still finite, meaning that $\mathbb{R} \setminus (F \cup G) \neq \emptyset$. Thus, $\mathbb{R}$ with the cofinite topology is not Hausdorff.

Example: Converging Sequences

In Hausdorff spaces, we have shown in [Theorem 17.10] that all sequences converge to at most one point in that space. However, for arbitrary topological spaces, it is entirely possible that a single sequence converges to multiple points.

Take $X = \mathbb{R}$ with the trivial topology and the sequence $(1, 1, 1, \ldots)$. We have that the sequences converges to 1, but also to 2 – in fact, the sequences converges to any point in $\mathbb{R}$! This is because the only open neighborhood that contains each element in the sequence (that is, 1) is $\mathbb{R}$. Thus, it follows by the definition of [convergence] that our sequence converges to any point in $\mathbb{R}$.

4.5 Discussion

Remark: Topologies and Closed Sets

As we saw with open sets, a set closed in Y can also be closed in X – however, it is not always the case. There is, though, a special situation in which every set closed in Y is also closed in X – this is [Theorem 17.3].

Remark: Some Convenient Terminology

We shall say that a set A **intersects** a set B if the intersection $A \cap B$ is not empty.

Remark: On the Alternative Characterization

If the topology on X is generated by a basis, then we have that the [alternative characterizations] all hold if the term *open neighborhood* is replaced in each occurrence by the term *basic open neighborhood*, which is an open neighborhood that is also a basic open set.

In fact, this replacement holds true all the time – that is, whenever we are working with open sets, we can equivalently work with basic open sets.

Remark: On [Theorem 17.8]

Notice that [Theorem 17.8] essentially tells us that if X is a Hausdorff topological space, then X is also a T_1-space.

Remark: On the Definition of T_1

The definition of T_1 can instead be stated to as follows: a T_1-space is a topological space in which all singleton subsets are closed.

Indeed, we see that any finite point sets can be represented as the union of singleton sets. Thus, if the singleton sets are closed, then so are finite point sets.

Chapter 5

The Subspace Topology

5.1 Learning Objectives

- What is the subspace topology?
- What are the open and closed sets in the subspace topology?
- How do closures and interiors behave in subspaces?
- What are some examples of subspace topologies?

5.2 Definitions

Definition: Subspace Topology

Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, call the **subspace topology**.

With this topology, Y is called a **subspace** of X; its open sets consist of all intersection of open sets of X with Y.

Definition: Open Sets in Subspace

If Y is a subspace of X, we say that a set U is **open in** Y (or open *relative* to Y) if it belongs to the topology of Y; this implies in particular that it is a subset of Y. We say that U is **open in** X if it belongs to the topology of X.

Definition: Closed Sets in a Subspace

If Y is a subspace of X, we say that a set A **closed in** Y if A is a subset of Y and if A is closed in the subspace topology of Y – that is, if $Y \setminus A$ is open in Y.

Definition: Convex Sets

Given an *ordered* set X, let us say that a subset Y of X is **convex** in X if for each pair of points $a < b$ of Y, the entire interval a, b of points of X lies in Y.

5.3 Theorems

Lemma: 16.1

If $\mathcal{B}$ is a basis for the topology of X then the collection

$$\mathcal{B}_Y = \{Y \cap B \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Proof We start by noting that $\mathcal{B}_Y$ is a collection of open sets of Y, where the topology on Y is the subspace topology. From here, we shall utilize [Lemma 13.2]; our goal is to show that for each open set $Y \cap U$ of Y (where U is an open set of X) and each $y \in Y \cap U$, there is an element $Y \cap B \in \mathcal{B}_Y$ such that $y \in Y \cap B \subset Y \cap U$.

Let's say that we are given any U open in X, $Y \cap U$ open in Y, and $y \in Y \cap U$. By the definition of openness in X (under the basis $\mathcal{B}$), we have that there is a basis element $B \in \mathcal{B}$ such that $y \in B \subset U$.

Notice that $B \subset U$ implies that $Y \cap B \subset Y \cap U$. Furthermore, knowing that $y \in Y \cap U$ and $y \in B \subset U$ implies as well that $y \in Y \cap B \subset Y \cap U$. Thus, by [Lemma 13.2], we have that $\mathcal{B}_Y$ is a basis for the subspace topology on Y.

Lemma: 16.2

Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Proof Since U is open in Y, we have that there exists some V open in X so that $U = Y \cap V$. Since Y and V are both open in X, and X is a topological space, we have that $Y \cap V$ is open in X as well (finite intersection of open sets is open). Thus, we have that U is open in X.

Theorem: 16.3

If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Proof Let us start by examining the general basis element for the subspace topology on $A \times B$. Such a basis element if of the following form: $(A \times B) \cap (U \times V)$, where $U \times V$ is the general basis element for $X \times Y$, where U is open in X and V is open in Y. We can see that

$$(A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V)$$

Now since $A \cap U$ and $B \cap V$ are the general open sets for the subspace topologies on A and B, respectively, we have the set $(A \cap U) \times (B \cap V)$ to be the general basis element for the product topology on $A \times B$.

From this, we can conclude that the bases for the subspace topology on $A \times B$ and for the product topology on $A \times B$ are the same. Hence, the topologies are the same as well.

Theorem: 16.4

Let X be an ordered set in the order topology; let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.

Proof

Theorem: Closures and Interiors in Subspaces

Let X be a topological space and let $A \subset U \subset X$. The following hold:

(a) If U is open in X, then the interior of A in U equals the interior of A in X.

(b) If U is closed in X, then the closure of A in U equals the closure of A in X

Proof

a Let us start by noting that Int A is defined to be the union of all open subsets (either in U or X) contained in A. We shall start by supposing that we have some open subset V in U that is contained in A. We want to show that V is also an open subset in X contained in A. It is immediate to see that the containment condition is trivially met, so we just need to show that V is open in X. Note that being open in U (which is a subspace of X) means that there exists some open set W of X such that $V = U \cap W$. Since both U and W are open sets of X, and finite intersections of open sets are open, we have that $U \cap W$ (hence V) is open in X. Thus, we have that

$$\text{Int } A \text{ in } U \subset \text{Int } A \text{ in } X$$

Let us now suppose that we have some open subset Z in X that is contained in A. We want to show that Z is also an open subset in U contained in A. Again, the containment condition is trivially met by construction, so we just need to show that Z is open in U. Because Z is contained in A, which is a subset of U (i.e., $Z \subset A \subset U$), we have that $U \cap Z = Z$. Therefore, we see that Z itself is open in U. Thus, we have that

$$\text{Int } A \text{ in } U \supset \text{Int } A \text{ in } X$$

Putting everything together gives us

$$\text{Int } A \text{ in } U = \text{Int } A \text{ in } X$$

b Let us now let U to be closed in X. Furthermore, let us define the following:

$$\text{Cl}_U A = \text{ closure of } A \text{ in } U \quad \text{and} \quad \text{Cl}_X A = \text{ closure of } A \text{ in } X$$

We first want to show that $\text{Cl}_U\ A \subset \text{Cl}_X\ A$. To do so, notice that $\text{Cl}_X\ A$ and U are both closed in X and contain A. Thus, it follows that $\text{Cl}_X\ A \cap U$ is closed in U under the subspace topology and contains A (c.f. [Theorem 17.2]). By definition, we have that $\text{Cl}_U\ A$ is the smallest set closed in U containing A, meaning that $\text{Cl}_U\ A \subset \text{Cl}_X\ A \cap U$.

From here, we note that $\text{Cl}_X\ A \cap U \subset \text{Cl}_X\ A$ since the intersection must be contained in each of the set that are being intersected. Putting everything together results in

$$\text{Cl}_U\ A \subset \text{Cl}_X\ A \cap U \subset \text{Cl}_X\ A$$

We now want to show that $\text{Cl}_X\ A \subset \text{Cl}_U\ A$. We start by noting that $\text{Cl}_U\ A$ is closed in U, and that U is closed in X. Thus, by the transitivity of closedness, we see that $\text{Cl}_U\ A$ is closed in X. From here, we note that $A \subset \text{Cl}_U\ A$, which tells us that $\text{Cl}_U\ A$ is a closed set in X that also contains A. Since $\text{Cl}_X\ A$ is defined to be the smallest closed set in X that also contains A, it follows that $\text{Cl}_X\ A \subset \text{Cl}_U\ A$. Putting everything together gives us

$$\text{Cl}_U\ A = \text{Cl}_X\ A$$

5.4 Examples

Example: Subspace Topology

Let us show that the [subspace topology] $\mathcal{T}_Y$ is indeed a topology. For the first condition, we see that $\emptyset, Y \in \mathcal{T}_Y$ because

$$\emptyset = Y \cap \emptyset \quad \text{and} \quad Y = Y \cap X$$

where $\emptyset$ and X are elements of $\mathcal{T}$.

We now want to show that it is closed under finite intersections. Let's say that we are given a finite amount of open sets $U_1, \ldots, U_n \in \mathcal{T}$. Our goal is to show that $(Y \cap U_1) \cap \ldots \cap (Y \cap U_n) \in \mathcal{T}_Y$. To do so, we start by noting that because $\mathcal{T}$ is a topology, it follows that $U_1 \cap \ldots \cap U_n \in \mathcal{T}$. Thus, we have $Y \cap (U_1 \cap \ldots \cap U_n) \in \mathcal{T}_Y$. However, notice that

$$Y \cap (U_1 \cap \ldots \cap U_n) = (Y \cap U_1) \cap \ldots \cap (Y \cap U_n)$$

which means that $(Y \cap U_1) \cap \ldots \cap (Y \cap U_n) \in \mathcal{T}_Y$.

Finally, we want to show that it is closed under arbitrary unions. Let's say that we are given a collection of open sets $\{U_i\}_{i \in I}$, where I is some indexing set. Our goal is to show that $\bigcup_{i \in I}(Y \cap U_i) \in \mathcal{T}_Y$. To do so, we start by noting that because $\mathcal{T}$ is a topology, it follows that $\bigcup_{i \in I} U_i \in \mathcal{T}$. Thus, we have $Y \cap (\bigcup_{i \in I} U_i) \in \mathcal{T}_Y$. However, notice that

$$Y \cap \left(\bigcup_{i \in I} U_i \right) = \bigcup_{i \in I} (Y \cap U_i)$$

which means that $\bigcup_{i \in I}(Y \cap U_i) \in \mathcal{T}_Y$.

Example: Open and Closed Sets in Different Subspace Topologies

Decide whether the set $A = [0,1)$ is open, closed, or neither in the specified subset B of $X = \mathbb{R}$. Here $\mathbb{R}$ is equipped with the standard topology and B is equipped with the corresponding subspace topology.

$\boxed{B = \mathbb{R}}$ In this case, we see that the corresponding subspace topology of B is exactly the standard topology on $\mathbb{R}$. In this topology, we have that A is neither open nor closed.

$\boxed{B = [0,\infty)}$ In this case, we see that $B \cap (-1,1) = A$. Since $(-1,1)$ is open in $\mathbb{R}$ under the standard topology, it follows that $B \cap (-1,1) = A$ is open in B equipped with the subspace topology by definition. Notice that A is not closed in B since there is no closed set in $\mathbb{R}$ whose intersection with B equals A.

$\boxed{B = [0,1) \cup [2,3)}$ In this topology, we see that A is both open and closed in B. We have that $[2,3) = B \cap [2,3]$, which tells us that $[2,3)$ is closed in B. This further means that $A = B \setminus [2,3)$ must be open. To show that A is closed, we have that $A = B \cap [0,1]$, which results in A being closed for the same reason as with $[2,3)$.

Example: Two Subspace Topologies on a Set

Let X be a topological space and let $A \subset B \subset X$. Equip B with the subspace topology as a subset of X. The set A can also be equipped with the subspace topology as a subset of X. However, A can alternatively be equipped with the subspace topology as a subset of B. We can show that these two topologies on A are equal.

Let us start by denoting $\mathcal{T}_{A_B}$ to be the subspace topology on A in B. Let $\mathcal{T}_{A_X}$ denote the subspace topology on A in X. In particular, we see that

$$\mathcal{T}_{A_B} = \{A \cap U \mid U \text{ open in } B\}$$
$$\mathcal{T}_{A_X} = \{A \cap V \mid V \text{ open in } X\}$$

Since B is a subspace of X, we see that U being open in B means that there exists some open set W in X such that $U = B \cap W$. Adding on the fact that $A \subset B$, we see that

$$A \cap U = A \cap (B \cap W) = (A \cap B) \cap W = A \cap W$$

Thus, we get that

$$\mathcal{T}_{A_B} = \{A \cap W \mid W \text{ open in } X\}$$
$$\mathcal{T}_{A_X} = \{A \cap V \mid V \text{ open in } X\}$$

which we can see is exactly the same collection. Thus, both topologies on A are equal.

Example: Subspace Topology of $[0,1]$

Consider the subset $Y = [0,1]$ of the real line $\mathbb{R}$, in the subspace topology. Again, the subspace topology has as basis all sets of the form $Y \cap (a,b)$, where (a,b) is a basic open interval in $\mathbb{R}$. Such a set is of one of the following types:

$$Y \cap (a,b) = \begin{cases} (a,b) & \text{if } a \text{ and } b \text{ are in } Y \\ [0,b) & \text{if only } b \text{ is in } Y \\ (a,1] & \text{if only } a \text{ is in } Y \\ Y \text{ or } \emptyset & \text{if neither } a \text{ nor } b \text{ is in } Y \end{cases}$$

By definition, each of these sets is open in Y. But the sets of the second and third type are not open in the larger space $\mathbb{R}$.

Example: Subspace Topology of $\mathbb{Z}$

We claim that the subspace topology on $\mathbb{Z}$ (as a subspace of $\mathbb{R}$) is the discrete topology.

To start, we note that the open sets in $\mathbb{Z}$ under the subspace topology are all of the following form: $\mathbb{Z} \cap U$, where U is an open set in $\mathbb{R}$. One can see that this collection of open sets results in all possible subsets of $\mathbb{Z}$ by noting that we can put an open interval centered at each $x \in \mathbb{Z}$ with a radius of $\frac{1}{2}$. Since unions of open sets are still open, we find that any possible subset of $\mathbb{Z}$ is obtained by taking the union of the open intervals that we constructed which contains the points in the subset of $\mathbb{Z}$; the resulting intersection of $\mathbb{Z}$ with such a union results in the subset we wanted. Thus, we have that all the subsets of $\mathbb{Z}$ are therefore open under the subspace topology, which tells us that the subspace topology is the same as the discrete topology.

Example: Subspace Topology of $\mathbb{Q}$

We claim that the subspace topology on $\mathbb{Q}$ (as a subspace of $\mathbb{R}$) is *not* the discrete topology.

Let us suppose that $\mathbb{Q}$ under the subspace topology is the discrete topology. Then this means that we have some open set U in $\mathbb{R}$ such that $\mathbb{Q} \cap U = \{q\}$ for some $q \in \mathbb{Q}$ – indeed, if $\mathbb{Q}$ is the discrete topology under the subspace topology, then any subset of $\mathbb{Q}$ is open and can be written as the intersection of an open set in $\mathbb{R}$ and $\mathbb{Q}$. Since we know that U is an open set in $\mathbb{R}$ that contains q, we find that there exists some $t \in \mathbb{R}$ such that $(q-t, q+t) \subset U$. However, since we also know that $\mathbb{Q}$ is dense in $\mathbb{R}$, we are always able to find another rational number in between $q-t$ and q as well as q and $q+t$. Thus, we end up getting that $\mathbb{Q} \cap U \neq \{q\}$ as there are always more rational numbers that we can find in between any open interval that contains q.

Hence, we reach a contradiction and have that $\mathbb{Q}$ under the subspace topology is *not* the discrete topology.

5.5 Discussion

Remark: Closed Set in the Subspace Topology

Notice that [Theorem 17.2] gives us an alternative definition of a closed set in the subspace topology: the closed subsets of Y are exactly of the form $Y \cap C$, where C is closed in X.

Remark: Closure Notation

When dealing with a topological space X and a subspace Y, we must be careful in taking closures of sets. If A is a subset of Y, the closure of A in Y and the closure of A in X will in general be different. In such a situation, we reserve the notation $\overline{A}$ to stand for the closure of A in X.

However, we have that [Theorem 17.4] tells us that the closure of A in Y can be expressed in terms of $\overline{A}$.

Remark: Topologies and Open Sets

Note a set open in Y can also be open in X – however, it is not always the case. There is, though, a special situation in which every set open in Y is also open in X – this is [Lemma 16.2].

Chapter 6

The Product Topology

6.1 Learning Objectives

- What are finite products of sets?
- What are the properties for intersections, unions, and complements for a finite product of sets?
- What is the product topology?
- What are projections?

6.2 Definitions

Definition: Finite Product of Sets

Given a finite list of sets $X_1, \ldots, X_r$, we define their **product** to consist of all r-tuples of the form $(x_1, x_2, \ldots, x_r)$ where $x_i \in X_i$ for all i. Notationally:

$$\prod_{i=1}^{r} X_i = X_1 \times \ldots \times X_r \equiv \{(x_1, \ldots, x_r) \mid x_1 \in X_1, \ldots, x_r \in X_r\}$$

Definition: Rectangular Subset

A **rectangular subset** is a special kind of subset of a product that has the form:

$$A_1 \times \ldots \times A_r \subset X_1 \times \ldots \times X_r \quad \text{where} \quad A_1 \subset X_1, \ldots, A_r \subset X_r$$

Definition: Product Topology

Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is an open subset X and V is an open subset of Y. I.e., we have that a basic open set in the product topology is none other than a product of open sets.

We can extend this definition to more than two topological spaces. Let $X_1, \ldots, X_r$ be a finite list of topological spaces. The collection of all sets of the form:

$$U_1 \times \ldots \times U_r$$

where we have U_i to be open in X_i for all i, forms a basis for a topology on $X_1 \times \ldots \times X_r$ that we call the product topology.

Definition: Projections

Let $\text{proj}_1 : X \times Y \to X$ be defined by the equation

$$\text{proj}_1(x, y) = x$$

and let $\text{proj}_2 : X \times Y \to Y$ be defined by the equation

$$\text{proj}_2(x, y) = y$$

The maps proj_1 and proj_2 are called the **projections** of $X \times Y$ onto its first and second factors, respectively.

We can also extend this to more than two topological spaces. Let $X_1, \ldots, X_r$ be a finite list of topological spaces. Then the projection of $X_1 \times \ldots \times X_r$ onto its ith factor is the map $\text{proj}_i : X_1 \times \ldots \times X_r \to X_i$ defined by the equation

$$\text{proj}_i(x_1, \ldots, x_r) = x_i$$

Definition: Sequences in Products

If $X_1, \ldots, X_r$ is a product space, then a sequence in this space might be written as: $(X^0, X^1, \ldots)$, where

$$X^0 = (x_1^0, \ldots, x_r^0) \quad X^1 = (x_1^1, \ldots, x_r^1) \quad \ldots$$

are each elements in $X_1, \ldots, X_r$.

6.3 Theorems

Theorem: 15.1

If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then the collection

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

We can extend this to more generally be the following: If the topologies on $X_1, \ldots, X_r$ are generated by the bases $\mathcal{B}_1, \ldots, \mathcal{B}_r$, then the collection of all sets of the form

$$B_1 \times \ldots \times B_r$$

where $B_i \in \mathcal{B}_i$ for all i is a basis for the topology of $X_1 \times \ldots \times X_r$

Proof For this proof, we shall apply [Lemma 13.2]. Let W be some open set of $X \times Y$, and let $x \times y$ be some point of W. By definition of the product topology, we have that W being open means that there exists a basis element $U \times V$ such that $x \times y \in W \subset U \times V$. Now because $\mathcal{B}$ and $\mathcal{C}$ are bases for X and Y, respectively, we can choose an element B of $\mathcal{B}$ such that $x \in B \subset U$, and an element C of $\mathcal{C}$ such that $y \in C \subset V$. From this, it follows that $x \times y \in B \times C \subset W$. Thus, we have that the collection $\mathcal{D}$ meets the criterion of [Lemma 13.2], so we have that $\mathcal{D}$ is a basis for $X \times Y$.

Theorem: 15.2

The collection

$$\mathcal{S} = \{\text{proj}_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\text{proj}_2^{-1}(V) \mid V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof Let us start by denoting $\mathcal{T}$ to be the product topology on $X \times Y$; let $\mathcal{T}'$ be the topology generated by $\mathcal{S}$. Our goal is to show that $\mathcal{T} = \mathcal{T}'$.

Because every element of $\mathcal{S}$ belongs to $\mathcal{T}$, it follows that any arbitrary unions of these elements, or finite intersections of these elements, or even arbitrary unions of finite intersections of these elements also belong to $\mathcal{T}$. Thus, we have that $\mathcal{T}' \subset \mathcal{T}$.

On the other hand, we have that every basis element $U \times V$ for the topology $\mathcal{T}$ is a finite intersection of elements of $\mathcal{S}$, since

$$\begin{aligned}\text{proj}_1^{-1}(U) &= \{x \times y \in X \times Y \mid \text{proj}_1(x \times y) \in U\} \\ &= \{x \times y \in X \times Y \mid x \in U\}\end{aligned}$$

$$\begin{aligned}\text{proj}_2^{-1}(V) &= \{x \times y \in X \times Y \mid \text{proj}_2(x \times y) \in V\} \\ &= \{x \times y \in X \times Y \mid y \in V\}\end{aligned}$$

From this, we can see that

$$\text{proj}_1^{-1}(U) \cap \text{proj}_2^{-1}(V) = \{x \times y \in X \times Y \mid x \in U \text{ and } y \in V\} = U \times V$$

Therefore, we see that $U \times V$ belongs to $\mathcal{T}'$, so that $\mathcal{T} \subset \mathcal{T}'$ as well.

Putting everything together results in $\mathcal{T} = \mathcal{T}'$.

Theorem: Product of Closed Subsets

Let $X_1, \ldots, X_r$ be topological spaces. If A_i is a closed subset of X_i for each i, then $A_1 \times \ldots \times A_r$ is closed in $X_1 \times \ldots \times X_r$.

In fact, we see that this holds for arbitrary products of closed sets.

Proof It suffices to show that $(X_1 \times \ldots \times X_r) \setminus (A_1 \times \ldots \times A_r)$ is open. We know that

$$(X_1 \times \ldots \times X_r) \setminus (A_1 \times \ldots \times A_r) = [(X_1 \setminus A_1) \times X_2 \times \ldots \times X_r] \cup \ldots \cup [X_1 \times \ldots \times X_{r-1} \times (X_r \setminus A_r)]$$

Since each A_i is a closed subset of X_i for each i, we get that $X_i \setminus A_i$ is open for each i as well. Thus, we have $[(X_1 \setminus A_1) \times X_2 \times \ldots \times X_r], \ldots, [X_1 \times \ldots \times X_{r-1} \times (X_r \setminus A_r)]$ to all be open in $X_1 \times \ldots \times X_r$. Because $X_1 \times \ldots \times X_r$ is a topology, we have that arbitrary unions of open sets will remain open, which tells us that $(X_1 \times \ldots \times X_r) \setminus (A_1 \times \ldots \times A_r)$ is open. Hence, $A_1 \times \ldots \times A_r$ is closed in $X_1 \times \ldots \times X_r$.

6.4 Examples

Example: Product Topology

Let us check that the basis given in the [Product Topology Definition] is indeed a basis. Note that it suffices to only prove for the product between two topological spaces since the r-amount case follows immediately from induction.

The first condition requires us to show that for each $(x, y) \in X \times Y$, there is at least one basis element containing (x, y); this condition is trivially met since $X \times Y$ is itself a basis element.

The second condition requires us to show that if (x, y) belongs to the intersection of two basis elements, say $U_1 \times V_1$ and $U_2 \times V_2$, then there is a basis element $U_3 \times V_3$ containing (x, y) such that $U_3 \times V_3 \subset (U_1 \times V_1) \cap (U_2 \times V_2)$; this condition follows almost immediately since we have that

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$$

Since finite intersections of open sets are open, we have that we can let $U_3 = U_1 \cap U_2$ and $V_3 = V_1 \cap V_2$, which satisfies the second condition.

Example: Standard vs. Product Topology on Euclidean Space

The set $\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R}$ has its standard topology induced from the Euclidean metric. This has a basis of open balls. Alternatively, we can equip this set with the product topology, where each copy of $\mathbb{R}$ has been equipped with its standard topology. This has a basis of open rectangles. We claim that these are the same topologies.

Open balls are open in the product topology since for every point in an arbitrary open ball, we can find some open rectangle contained in the ball that also contains our point. Thus, the product topology is at least as fine as the standard topology. The same can be said for open rectangles being open in the standard topology ; any point in an arbitrary open rectangle is contained within some open ball, where the open ball is contained in the open rectangle. Thus, the standard topology is at least as fine as the product topology. Putting both results together gives us that both topologies are the same.

Example: Sorgenfrey Plane

Let $\mathbb{R}_\ell$ denote $\mathbb{R}$ with the lower-limit topology. The topological space $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is referred to as the **Sorgenfrey Plane**. Let's examine open subsets of the Sorgenfrey plane.

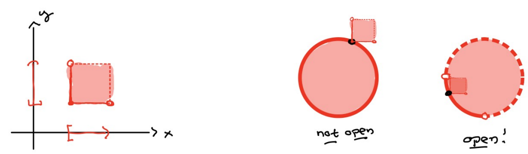

Figure 6.1: Open Subsets in the Sorgenfrey plane

Let's examine the subspace topology on the line defined by $y = x$ in $\mathbb{R}_\ell \times \mathbb{R}_\ell$. Something open in this subspace looks like an interval whose lower endpoint is included but the upper endpoint is not – like a slanted lower limit topology.

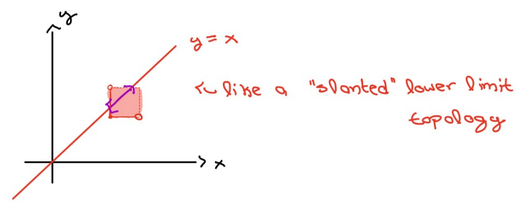

Figure 6.2: Subspace topology of $y = x$

Let's examine the line defined by $y = -x$ in $\mathbb{R}_\ell \times \mathbb{R}_\ell$. In this case, we see that any point on the line is open – that is, we get a discrete topology.

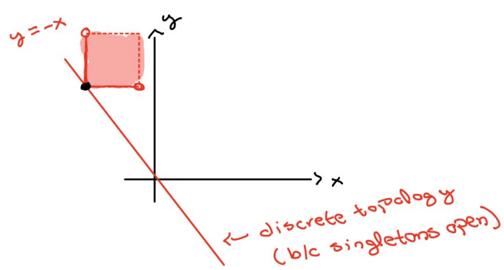

Figure 6.3: Subspace topology of $y = -x$

In fact, we see that if the line has a negative slope, then the line inherits the discrete topology; when it is positive, we see that the line inherits the lower limit topology.

6.5 Discussion

Remark: Intersections in Products

Intersection of rectangular sets behave nicely in products. Let $A_1 \times \ldots \times A_r$ and $B_1 \times \ldots \times B_r$ be rectangular subsets of a product of sets. Then

$$(A_1 \times \ldots \times A_r) \cap (B_1 \times \ldots \times B_r) = (A_1 \cap B_1) \times \ldots \times (A_r \cap B_r)$$

In particular, we have that an intersection of rectangular subsets is still rectangular. A similar equality holds for an *arbitrary* intersection of rectangular subsets instead of just pairwise intersection.

Remark: Unions in Products

Unlike with intersections, we *cannot* just substitute $\cap$'s with $\cup$'s in the equality given for intersections in products. For example:

$$(\{0\} \times \{0\}) \cup (\{1\} \times \{1\}) \neq (\{0\} \cup \{1\}) \times (\{0\} \cup \{1\})$$

This is also evidence that a union of rectangular subsets need not be rectangular.

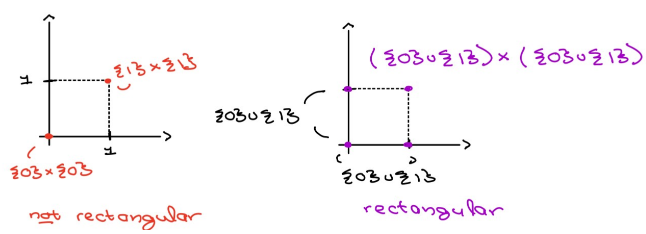

Figure 6.4: A visual of the example given

Remark: Complements in Products

Complements of rectangular subsets also behave a bit less nicely. For example:

$$(\mathbb{R} \times \mathbb{R}) \setminus (\{0\} \times \{0\}) \neq (\mathbb{R} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$$

This is also evidence that a complement of rectangular subset need not be rectangular.

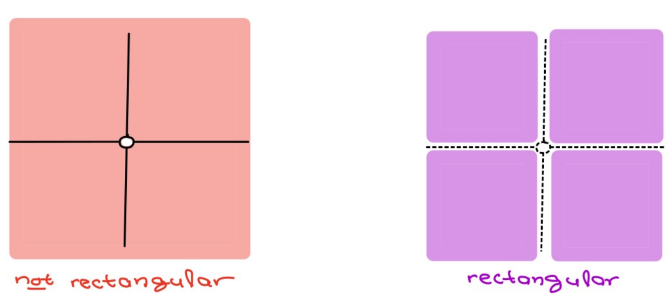

Figure 6.5: A visual of the example given

The correct equality of rectangular subsets is as follows:

$$(X_1 \times \ldots \times X_r) \setminus (A_1 \times \ldots \times A_r) = [(X_1 \setminus A_1) \times X_2 \times \ldots \times X_r] \cup \ldots \cup [X_1 \times \ldots \times X_{r-1} \times (X_r \setminus A_r)]$$

In fact, we see that this result holds for arbitrary products (since the proof only relies on DeMorgan's law, which works for arbitrary products).

Remark: On the Basis for a Product Topology

It is important to note that the basis for a product topology is not a topology on $X \times Y$. For example take Figure (6.6)

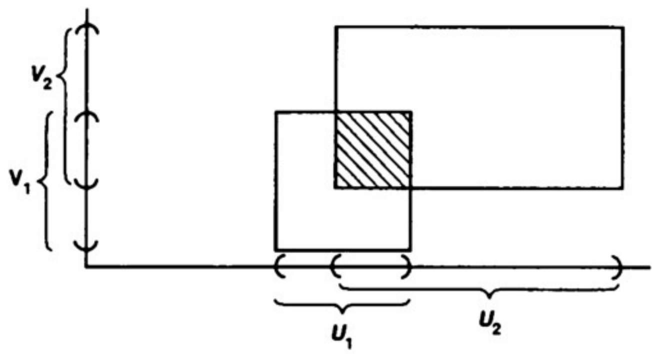

Figure 6.6: Visual of two open sets in X and Y

The union of the two rectangles in Figure (6.6) is not a product of two sets, so it cannot belong to $\mathcal{B}$; however, we have that it is still open in $X \times Y$ since for every point in the union, we can always find a smaller rectangle that is within the union that also contains our point.

However, the intersection of any two basic open sets is a basic open set.

Chapter 7

More Product Topology

7.1 Learning Objectives

- What is the box topology?
- What is the product topology for arbitrary products?
  - How do the box topology and product topology differ? Which one is preferred?
- Is the inverse image of a projection open?
- What is convergence like in arbitrary products?

7.2 Definitions

Definition: I-tuple and Coorindate

Let I be an index set. Given a set X, we define a I-**tuple** of elements of X to be a function $\mathbf{x} : I \to X$. If i is an element of I, we denote the value of $\mathbf{x}$ at i by x_i rather than $\mathbf{x}(i)$; we call it the ith **coordinate** of $\mathbf{x}$. We often denote the function $\mathbf{x}$ itself by the symbol

$$(x_i)_{i\in I}$$

which is as close as we can come to a "tuple notation" for an arbitrary index set I. We denote the set of all I-tuples of elements of X by X^I.

Definition: Cartesian Product

Let $\{X_i\}_{i\in I}$ be an indexed family of sets; let $X = \bigcup_{i\in I} X_i$. The **cartesian product** of this indexed family, denoted by

$$\prod_{i\in I} X_i$$

is defined to be the set of all I-tuples $(x_i)_{i\in I}$ of elements of X such that $x_i \in X_i$ for each $i \in I$. That is, it is the set of all functions

$$\mathbf{x} : I \to \bigcup_{i\in I} X_i$$

such that $\mathbf{x}(i) \in X_i$ for each $i \in I$.

Definition: Arbitrary Products

Let I be an index set. If $\{X_i\}_{i\in I}$ is an indexed collection of sets, then we can define their product (also known as the **cartesian product**) as follows :

$$\prod_{i\in I} X_i = X_1 \times X_2 \times X_3 \times \ldots = \{(x_1, x_2, x_3, \ldots) \mid x_1 \in X_1, x_2 \in X_2, x_3 \in X_3, \ldots\}$$

The elements of this product, say $\mathbf{x}$, are entirely determined by their coordinates x_i where $i \in I$. We notate this by writing these elements in terms of their coordinates:

$$\mathbf{x} = (x_i)_{i\in I} = (x_1, x_2, x_3, \ldots)$$

Definition: Box Topology

Let $\{X_i\}_{i\in I}$ be an indexed set of topological spaces and $X = \prod_{i\in I} X_i$. The **box topology** on X is the topology generated by the basis of sets of the form:

$$\prod_{i\in I} U_i \quad \text{where } U_i \text{ is open in } X_i \text{ for each } i \in I$$

Definition: Product Topology (Arbitrary Products)

Let $\{X_i\}_{i\in I}$ be an indexed set of topological spaces and $X = \prod_{i\in I} X_i$. The **product topology** on X is the topology generated by the basis of the following form:

$$\prod_{i\in I} U_i \quad \text{where } U_i \text{ is open in } X_i \text{ for each } i \in I$$

and

$$U_i = X_i \text{ for all but finitely many } i$$

Definition: Alternative Definition of the Product Topology

Let $\mathcal{S}_i$ denote the collection

$$\mathcal{S}_i = \{\mathrm{proj}_i^{-1}(U_i) \mid U_i \text{ is open in } X_i\}$$

and let $\mathcal{S}$ denote the union of these collections,

$$\mathcal{S} = \bigcup_{i\in I} \mathcal{S}_i$$

The topology generated by the subbasis $\mathcal{S}$ is called the **product topology**. In this topology, $\prod_{i\in I} X_i$ is called a **product space**.

Definition: Projections

Let $\{X_i\}_{i \in I}$ be an indexed set of topological spaces and $X = \prod_{i \in I} X_i$. For each fixed index $k \in I$, the **projection onto the kth component** is the function $\text{proj}_k : X \to X_k$, with the formula:

$$\text{proj}_k(\mathbf{x}) = \text{proj}_k(x_1, x_2, x_3, \ldots) = x_k$$

7.3 Theorems

Theorem: Comparison of the Box and Product Topologies

Let $\{X_i\}_{i \in I}$ be an indexed set of topological spaces and $X = \prod_{i \in I} X_i$. The box topology on X has as basis all sets of the form $\prod_{i \in I} U_i$, where U_i is open in X_i for each $i \in I$. The product topology on X has as basis all sets of the form $\prod_{i \in I} U_i$, where U_i is open in X_i for each $i \in I$ *and* $U_i = X_i$ except for finitely many values of $i \in I$.

Proof To compare the box and product topologies (using the alternative definition), we consider the basis $\mathcal{B}$ that $\mathcal{S}$ generates. Using what we found in [Theorem 15.2], we can gather that the collection $\mathcal{B}$ consist of all finite intersections of elements of $\mathcal{S}$. If we intersect elements belonging to the same one of the sets $\mathcal{S}_i$, we do not get anything new, because

$$\text{proj}_i^{-1}(U_i) \cap \text{proj}_i^{-1}(V_i) = \text{proj}_i^{-1}(U_i \cap V_i)$$

that is, the intersection of two elements of $\mathcal{S}_i$, or of finitely many such elements, is again an element of $\mathcal{S}_i$.

We get something new only when we intersect elements from *different* sets $\mathcal{S}_i$. The typical element of the basis $\mathcal{B}$ can thus be described as follows: Let $i_1, \ldots, i_n$ be a finite set of distinct indices from the index set I, and let U_{i_j} be an open set in X_{i_j} for $j = 1, \ldots, n$. Then

$$B = \text{proj}_{i_1}^{-1}(U_{i_1}) \cap \text{proj}_{i_2}^{-1}(U_{i_2}) \cap \ldots \cap \text{proj}_{i_n}^{-1}(U_{i_n})$$

is the typical element of $\mathcal{B}$.

Now a point $\mathbf{x} = (x_i)_{i \in I}$ is in B if and only if its i_1th coordinate is in U_{i_1}, its i_2th coordinate is in U_{i_2}, and so on. There is no restriction whatever on the ith coordinate of $\mathbf{x}$ it i is not one of the indices $i_1, \ldots, i_n$. As a result, we can write B as the product where U_i denotes the entire space X_i if $i \neq i_1, \ldots, i_n$.

Theorem: 19.2

Suppose the topology on each space X_i is given by a basis $\mathcal{B}_i$. The collection of all sets of the form

$$\prod_{i \in I} B_i$$

where $B_i \in \mathcal{B}_i$ for each $i \in I$, will serve as a basis for the box topology on $\prod_{i \in I} X_i$.

The collection of all sets of the same form, where $B_i \in \mathcal{B}_i$ for finitely many indices i and $B_i = X_i$ for the remaining indices, will serve as a basis for the product topology on $\prod_{i \in I} X_i$.

Proof

Theorem: 19.3

Let A_i be a subspace of X_i, for each $i \in I$. Then $\prod_{i \in I} A_i$ is a subspace of $\prod_{i \in I} X_i$ if both products are given in the box topology, or if both products are given in the product topology.

Proof

Theorem: 19.4

If each space X_i is Hausdorff, then $\prod_{i \in I} X_i$ is a Hausdorff space in both the box and product topologies.

Proof

Theorem: 19.5

Let $\{X_i\}_{i \in I}$ be an indexed set of topological spaces and $X = \prod_{i \in I} X_i$. Let $A_i \subset X_i$ for each $i \in I$. If X is given either the product or the box topology, then

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i}$$

Proof

Theorem: 19.6

Let $f : A \to \prod_{i \in I} X_i$ be given by the equation

$$f(a) = (f_i(a))_{i \in I}$$

where $f_i : A \to X_i$ for each i. Let $X = \prod_{i \in I} X_i$ have the product topology. Then the function f is continuous if and only if each function f_i is continuous.

Proof

Theorem: Inverse Image Under a Projection

Let $\{X_i\}_{i\in I}$ be an indexed set of topological spaces and $X = \prod_{i\in I} X_i$. The inverse image of an open subset of X_k under proj_k is open in the product topology (and therefore also in the at least as fine box topology).

Proof Let us start by fixing $k \in I$ and having $U_k \subset X_k$ be an open subset. The inverse image of U_k under proj_k is defined to be the set of all point in the domain X whose output lies in U_k; i.e., it is the set of all points whose kth coordinate is in U_k. We can write this as follows:

$$\text{proj}_k^{-1}(U_k) = \prod_{i\in I} U_i \quad \text{where} \quad U_i = X_i \text{ when } i \neq k$$

For $i = k$, we already have that $U_k \subset X_k$.

Notice that $\text{proj}_k^{-1}(U_k)$ is an arbitrary product of open sets where $U_i = X_i$ for all but $i = k$. Thus, $\text{proj}_k^{-1}(U_k)$ is a basic open set in the product topology (for arbitrary products), which means that it is open.

Theorem: Convergence of Sequences in Products

Let $\{X_i\}_{i\in I}$ be an indexed set of topological spaces and $X = \prod_{i\in I} X_i$. Consider a sequence

$$\mathbf{x}^0 = (x_i^0)_{i\in I} \quad \mathbf{x}^1 = (x_i^1)_{i\in I} \quad \mathbf{x}^2 = (x_i^2)_{i\in I} \quad \ldots$$

and a point $\mathbf{x} = (x_i)_{i\in I}$ to which we would like to determine whether the sequence converges.

Then $\mathbf{x}^n \to \mathbf{x}$ in the product topology if and only if $x_i^n \to x_i$ for each index $i \in I$.

Note that this theorem holds for when I is a finite indexing set.

Proof

$\Longrightarrow$ Let $\mathbf{x}^n \to \mathbf{x}$ in the product topology. Let $i \in I$ be given, and also let an open neighborhood $U_i \subset X_i$ of x_i be given. Our goal is to show that $x_i^n \to x_i$.

In order for our sequence to converge to x_i, it follows by definition that there exists some positive integer N such that $x_i^n \in U_i$ for all $n \geq N$. From here, we note that the [inverse image] of U_i under proj_i, i.e. $\text{proj}_i^{-1}(U_i)$, is open in X. Further note that $\text{proj}_i^{-1}(U_i)$ consists of all points in X such that their ith coordinate is in U_i. Since U_i was given to be an open neighborhood of x_i, we see that $x_i \in U_i$ by construction. Thus, we end up getting that $\mathbf{x} \in \text{proj}_i^{-1}(U_i)$ – that is, $\text{proj}_i^{-1}(U_i)$ is an open neighborhood of $\mathbf{x}$.

Now because we were given $\mathbf{x}^n \to \mathbf{x}$, it follows that there exists some positive integer N such that $\mathbf{x}^n \in \text{proj}_i^{-1}(U_i)$ for all $n \geq N$. Notice that this implies that $x_i^n \in U_i$ for all $n \geq N$, which means that $x_i^n \to x_i$.

$\Longleftarrow$ Now let's assume that $x_i^n \to x_i$ for all $i \in I$. Our goal is to show that $\mathbf{x}^n \to \mathbf{x}$.

Let us start by letting $U = \prod_{i\in I} U_i$ be a basic open neighborhood of $\mathbf{x}$ – always note that we can interchange "open" and "basic". Because U_i is an open neighborhood of x_i and because $x_i^n \to x_i$, we have that for all $i \in I$, there exists N_i so $x_i^n \in U_i$ for all $n \geq N_i$. This is a little worrisome since we eventually want to take the maximum of all these N_i so that we can use it for U; if there are infinitely many N_i, then we might not have a maximum!

Thankfully, we have by definition of the product topology that there only exists a finite amount of situations where $U_i \neq X_i$. In the case that $U_i = X_i$, we can take $N_i = 0$ since any sequence x_i^n is contained in X_i. Thus, we can pick $N_i = 0$ for all but finitely many i (those i where $U_i \neq X_i$). We can then set $N = \max\{\text{remaining } N_i\}$, which results in us having that $\mathbf{x}^n \in U$ for all $n \geq N$. Hence, we get that $\mathbf{x}^n \to \mathbf{x}$.

7.4 Examples

Example: Box Topology

We want to show that the basis given in [box topology] definition is indeed a basis.

Let $\{X_i\}_{i\in I}$ be an indexed set of topological spaces and $X = \prod_{i\in I} X_i$. We see that for each $x \in X$, there is indeed at least one basis element containing x – X itself is a basis element by definition since X_i is open trivially for all $i \in I$. Thus, the first condition for a basis is met.

As for the second condition, we want to show that if x belongs to the intersection of two basis elements, say $B_1 = \prod_{i\in I} U_i$ and $B_2 = \prod_{i\in I} V_1$, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$. In fact, we see that this condition follows immediately as the intersection of B_1 and B_2 is itself another basis element:

$$B_3 = \left(\prod_{i\in I} U_i\right) \cap \left(\prod_{i\in I} V_i\right) = \prod_{i\in I} (U_i \cap V_i)$$

Where $U_i \cap V_i$ are both open in X_i due to then being a finite intersection of open sets.

Example: Sequences in the Box Topology

Let's consider the sequence $\mathbf{x}^n$ of points in $\mathbb{R}^\omega = \prod_{i=0}^{\infty} \mathbb{R}$ shown below:

$$\mathbf{x}^1 = (1,1,1,\ldots) \quad \mathbf{x}^2 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots\right) \quad \mathbf{x}^3 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\right) \quad \ldots$$

Seems natural that we have the convergence $\mathbf{x}^n \to (0,0,0,\ldots)$ because, for each fixed index, we have a sequence that converges to 0 ... Right?

Unfortunately, **not** in the box topology. We can find a basic open neighborhood of $(0,0,0,\ldots)$ that contain **none** of the $\mathbf{x}^n$ above:

$$\prod_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)$$

Example: Closure of $\mathbb{R}^\infty$

We define $\mathbb{R}^\infty \subset \mathbb{R}^\omega$ to consist of all elements whose coordinates are eventually zero. Precisely,

$$\mathbb{R}^\infty = \{\mathbf{x} \in \mathbb{R}^\omega \mid \text{there exists index } K \text{ so that } x_k = 0 \text{ for all } k \geq K\}$$

The closure of $\mathbb{R}^\infty$ in the product topology on $\mathbb{R}^\omega$, denoted as $\mathrm{Cl}\,\mathbb{R}^\infty$, is all of $\mathbb{R}^\omega$.

To show why, we start by letting $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ be a given element of $\mathbb{R}^\omega$. Our goal is to show that $\mathbf{x} \in \mathrm{Cl}\,\mathbb{R}^\infty$. We know that [Theorem 17.5] tells us that we must equivalently show that every basic open set U containing $\mathbf{x}$ intersects $\mathbb{R}^\infty$.

Let $U = \prod_{i \in I} U_i$ be any basic open neighborhood of $\mathbf{x}$ – again, we can always interchange "open" with "basic". We know by definition of the product topology that there can only be finitely many $i \in I$ such that $U_i \neq \mathbb{R}$; let $i_1, \ldots, i_n$ be all such i and define $N = \max\{i_1, \ldots, i_n\}$. From this, it follows that for all $i > N$, we have $U_i = \mathbb{R}$.

We now define a sequence $\mathbf{x}'$ of elements in $\mathbb{R}^\infty$ as follows:

$$x_i' \equiv \begin{cases} x_i & i \leq N \\ 0 & i > N \end{cases}$$

We can see that $\mathbf{x}' \in U \cap \mathbb{R}^\infty$, which means that $U \cap \mathbb{R}^\infty$ is non-empty. Since our choice of U was arbitrary, we have that every basic open set in $\mathbb{R}^\omega$ containing $\mathbf{x}$ intersects $\mathbb{R}^\infty$. Thus, $\mathbf{x} \in \mathrm{Cl}\,\mathbb{R}^\infty$.

Furthermore, since our choice of $\mathbf{x}$ was arbitrary, we have that every element of $\mathbb{R}^\omega$ is in the closure of $\mathbb{R}^\infty$. Therefore, we see that $\mathbb{R}^\omega = \mathrm{Cl}\,\mathbb{R}^\infty$, as desired.

Alternatively, for each element $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ of $\mathbb{R}^\omega$, we can identify a sequence $\mathbf{x}^n$ of elements in $\mathbb{R}^\infty$ so that $\mathbf{x}^n \to \mathbf{x}$ in the product topology of $\mathbb{R}^\omega$:

$$\begin{aligned} \mathbf{x}^1 &= (x_1, 0, \ldots) \\ \mathbf{x}^2 &= (x_1, x_2, 0, \ldots) \\ \mathbf{x}^3 &= (x_1, x_2, x_3, 0, \ldots) \\ &\vdots \\ \mathbf{x} &= (x_1, x_2, x_3, \ldots) \end{aligned}$$

We can see that this sequence converges in the product topology of $\mathbb{R}^\omega$ since we have that each $x_i^n \to x_i$ (recall [convergence] of sequences in products).

To show why $\mathbb{R}^\omega$ is equal to $\mathrm{Cl}\,(\mathbb{R}^\infty)$, we note that every open neighborhood of $\mathbf{x}$ contains points in this sequence (by construction, essentially), which are also points in $\mathbb{R}^\infty$. So $\mathbf{x}$ is in $\mathrm{Cl}\,(\mathbb{R}^\infty)$, and since this choice of $\mathbf{x}$ was arbitrary, we have that $\mathbb{R}^\omega$ is equal to $\mathrm{Cl}\,(\mathbb{R}^\infty)$.

7.5 Discussion

Remark: Inverse Images and Products

Let $\{X_i\}_{i\in I}$ be an indexed set of topological spaces and $X = \prod_{i\in I} X_i$. For any collection of subsets $\{U_i\}_{i\in I}$ in X, we have

$$\begin{aligned}\bigcap_{i\in I} \operatorname{proj}_i^{-1}(U_i) &= \{\mathbf{x} \in X \mid \operatorname{proj}_i(\mathbf{x}) \in U_i \text{ for all } i \in I\} \\ &= \{\mathbf{x} \in X \mid x_i \in U_i \text{ for all } i \in I\} = \prod_{i\in I} U_i\end{aligned}$$

Remark: On [Theorem 19.1]

For finite products, we see that both the box and product topologies are exactly the same. Also, we have that the [Box Topology] is by definition **at least as fine** as the product topology.

Remark: Convention

Whenever we consider the product $X = \prod_{i\in I} X_i$, we shall assume it is given the product topology unless we specifically state otherwise.

Remark: Topologies on Arbitrary Products

On an arbitrary product, the natural candidate for a basis (i.e., the basis for the box topology) yields a topology that is sadly not suitable when discussing convergence of sequences.

We instead have an alternative topology (i.e., the product topology for arbitrary products) that is designed so convergence works as expected; it can be naturally derived to avoid "infinitely shrinking" products of open sets.

Chapter 8

Continuous Functions

8.1 Learning Objectives

- What does it mean for a function to be continuous at a point?
  - How does this compare to the usual continuity definition?
  - What is the equivalent definition? [Equivalent Definition]
- What is a homeomorphism?
- What is a homeomorphism invariant?
- What are all the different ways that we can construct a continuous function? [Theorem 18.2]
- How does continuity work with products? [Theorem 18.4]
- What is the Pasting Lemma? [Theorem 18.3]

8.2 Definitions

Definition: Image of a Function

If A is a subset of the domain X, then its image (denoted as $f(A)$) is

$$f(A) = \{f(x) \in Y \mid x \in A\}$$

In words, $f(A)$ is the set of all outputs of points from the specified subset A.

Definition: Inverse Image of a Function

If $f : X \to Y$ is a function and B is a subset of the codomain Y, then its **inverse image** is

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}$$

In words, $f^{-1}(B)$ is the set of all points in the domain X whose outputs lie in the specified subset B of the codomain.

Definition: Injection, Surjection, Bijection

Let $f : X \to Y$ be a function between two sets X and Y. We say that f is an **injection** if each output corresponds to *at most* one input. Symbolically,

$$\forall x, y \in X, f(a) = f(b) \implies a = b \iff \forall x, y \in X, a \neq b \implies f(a) \neq f(b)$$

We say that f is a **surjection** if each element in the codomain is an output of the function. Symbolically,

$$\forall y \in Y, \exists x \in X \text{ such that } f(x) = y$$

We say that f is a **bijection** if it is both an injection and surjection.

Definition: Inverse Function

An inverse to a function $f : X \to Y$ is a function $f^{-1} : Y \to X$ so that f and f^{-1} undo one another in the sense that

$$f \circ f^{-1} = i_Y \quad \text{and} \quad f^{-1} \circ f = i_X$$

Where $i_Y : Y \to Y$ and $i_X : X \to X$ are the identity functions on Y and X, respectively. On the level of elements, the two equalities above are the same as saying that

$$f(f^{-1}(y)) = y \quad \forall y \in Y \quad \text{and} \quad f^{-1}(f(x)) = x \quad \forall x \in X$$

Definition: Continuity at a Point

Let $f : X \to Y$ be a function between topological spaces. Then f is **continuous at a given** $x \in X$ if and only if, for each open neighborhood V of $f(x)$, there exists an open neighborhood U of x so that $f(U) \subset V$.

If the topologies of X and Y happen to be generated by bases, then we obtain an equivalent definition by replacing "open neighborhood" with "basic open neighborhood" in each occurrence above.

Definition: Homeomorphisms

Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both the function f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called a **homeomorphism**.

If there is a homeomorphism between topological spaces X and Y, then we say that X and Y are **homeomorphic**.

Definition: Homeomorphism Invariant

A **homeomorphism invariant** is a property of topological spaces that remains unchanged among homeomorphic spaces.

Definition: Topological Property

Any property of X that is entirely expressed in terms of the topology of X (that is, in terms of the open sets of X) yields, via the bijective correspondence $f : X \to Y$, the corresponding property for the space Y. Such a property of X is called the **topological property** of X.

Definition: Topological Imbedding

Suppose that $f : X \to Y$ is an injective continuous map, where X and Y are topological spaces. Let Z be the image set $f(X)$, considered as a subspace of Y; then the function $f' : X \to Z$ obtained by restricting the range of f is bijective. If f' happens to be a homeomorphism of X with Z, we say that the map $f : X \to Y$ is a **topological imbedding**, or simply an **imbedding**, of X in Y.

8.3 Theorems

Theorem: Invertible Functions

Let $f : X \to Y$ be a function between two sets. We say that f is a bijection if and only if it is invertible, meaning that it has an inverse.

Proof We shall prove this by first showing that f is injective if and only if f admits a left inverse $g : Y \to X$ where $g \circ f = i_X$.

Recall that f being injective tells us that we have for all $x, y \in X$

$$f(x) = f(y) \implies x = y$$

If g is the left inverse of f, then we have that

$$g(f(x)) = x \quad \forall x \in X$$

If we assume that f is injective, then we want to show that g is indeed the left inverse of f. To do so, it just amounts to showing that g is indeed a well-defined function. We see that

$$f(x_1) = f(x_2) \implies x_1 = x_2 \iff g(f(x_1)) = g(f(x_2))$$

meaning that g is indeed a left inverse of f, given that f is injective.

Now, if g is the left inverse of f, then we want to show that f is injective. Since g is well defined, we see that

$$f(x_1) = f(x_2) \implies g(f(x_1)) = g(f(x_2)) \iff x_1 = x_2$$

which tells us that f is indeed injective.

We shall now prove that f is surjective if and only if f admits a right inverse $g : Y \to X$ where $f \circ g = i_Y$.

Recall that f being surjective tells us that we have for all $y \in Y$, there exists some $x \in X$ such that $f(x) = y$. Let us define $g : Y \to X$ to be the function which maps each $y \in Y$ to such $x \in X$; if there is more than one x, then the function g maps y to one of them chosen in an arbitrary way – this excludes the possibility that g maps y to two distinct values, in which it wouldn't be a function. It follows that

$$f(g(y)) = f(x) = y \quad \forall y \in Y$$

which tells us that g is indeed the right inverse of f.

Now, if g is the right inverse of f, then we want to show that f is surjective. Let $y \in Y$ be given. We want to show that there exists some $x \in X$ such that $y = f(x)$. Since g is well defined, we see that $g(y) = x$ for some $x \in X$. Furthermore, since g is the right inverse of f, we have that

$$y = f(g(y)) = f(x)$$

which tells us that f is indeed surjective.

Putting everything together, we see that f is bijective if and only if f has both a left and right inverse – that is, f is bijective if and only if f has an inverse.

Theorem: Equivalent Definition of Continuity

Let $f : X \to Y$ be a function between topological spaces. Then the following definitions are equivalent. When either definition is satisfied, we simply say that f is **continuous**.

(a) at each point: f is continuous at each point x in its domain

(b) inverse images of open sets: $f^{-1}(V)$ is open in X whenever V is open in Y.

Proof

(a) $\implies$ (b)

Let f be continuous at each $x \in X$. Let $V \subset Y$ be open. Our goal is to show that $f^{-1}(V)$ is open in X. This means that for each $x \in f^{-1}(V)$, we can to find some open neighborhood U of x so that $U \subset f^{-1}(V)$.

Let $x \in f^{-1}(V)$ be given. Because we know that V is an open neighborhood of $f(x)$ and f is continuous at x, there exists an open neighborhood U so that $f(U) \subset V$. By definition, we have that for all $x \in U$, we get $f(x) \in V$. Notice that

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\}$$

which tells us that all of U is included in $f^{-1}(V)$ – that is, we have that $U \subset f^{-1}(V)$. This completes this direction.

(b) $\implies$ (a)

Now, let $f^{-1}(V)$ be open whenever V is open. Let $x \in X$ be given. Our goal is to show that f is continuous at x.

Let V be an open neighborhood of $f(x)$. Thus, it follows that $f^{-1}(V)$ is open by assumption, and it also contains x – that is, $f^{-1}(V)$ is an open neighborhood of x in X. If we pick $U = f^{-1}(V)$, then we see that

$$f(U) = \{f(x) \in Y \mid x \in U\} = \{f(x) \in Y \mid x \in f^{-1}(V)\}$$

which tells us that $f(x) \in V$ for all $x \in U$. Thus, we have that $f(U) \subset V$.

Theorem: 18.1

Let X and Y be topological spaces; let $f : X \to Y$. Then the following are equivalent:

1. f is continuous.
2. For every subset A of X, we have that $f(\overline{A}) \subset \overline{f(A)}$.
3. For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.

Proof We shall show that (1) $\implies$ (2) $\implies$ (3) $\implies$ (1).

(1) $\implies$ (2)

Let $x \in \mathrm{Cl}\ A$ be given. Then $f(x)$ represents some arbitrary element of $f(\mathrm{Cl}\ A)$. Our goal is to show that $f(x) \in \mathrm{Cl}\ f(A)$, which is equivalent to showing that every open neighborhood that contains $f(x)$ intersects $f(A)$ non-trivially.

Let V be any open neighborhood of $f(x)$. Since we know that f is continuous, V being open in Y implies that $f^{-1}(V)$ is also open in X. Furthermore, we see that V containing $f(x)$ implies as well that $f^{-1}(V)$ contains x – thus, $f^{-1}(V)$ is an open neighborhood of x. Since we assumed $x \in \mathrm{Cl}\ A$, it follows that $f^{-1}(V) \cap A \neq \emptyset$. Notice that

$$f^{-1}(V) \cap A = \{x \in X \mid x \in A \text{ and } x \in f^{-1}(V)\}$$

which tells us that for any $x \in f^{-1}(V) \cap A$, it must be the case that $f(x) \in V \cap f(A)$. Now since $f^{-1}(V) \cap A \neq \emptyset$, we have that $V \cap f(A) \neq \emptyset$. Hence, we can conclude that $f(x) \in \mathrm{Cl}\ f(A)$, which results in $f(\mathrm{Cl}\ A) \subset \mathrm{Cl}\ f(A)$.

(2) $\implies$ (3)

Let B be a closed set of Y and let $A = f^{-1}(B)$. We want to show that A is closed in X. In this case, it suffices to show that $A = \overline{A}$. By elementary set theory, we have that

$$f(A) = f(f^{-1}(B)) \subset B$$

Therefore, we see that if $x \in \overline{A}$, then

$$f(x) \in f(\overline{A}) \subset \overline{f(A)} \subset \overline{B} = B$$

Where the first inclusion results from our assumption, and the second inclusion results from the fact that $A \subset B$ implies $\overline{A} \subset \overline{B}$. From here, we note that $f(x) \in B$ is equivalent to saying that $x \in f^{-1}(B) = A$. Thus, we are left with $\overline{A} \subset A$. Since $A \subset \overline{A}$ by definition, it follows that $A = \overline{A}$, which tells us that $f^{-1}(B)$ is closed.

(3) $\implies$ (1)

Let B be a closed set of Y. We are given that $f^{-1}(B)$ is closed in X and want to show that f is continuous – that is, for every open set V of Y, we have $f^{-1}(V)$ to be open in X.

It follows that $Y \setminus V$ is closed in Y, which results in $f^{-1}(Y \setminus V)$ by our assumption. Furthermore, we have that

$$f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$$

which results in $f^{-1}(V)$ to be open in X. Thus, we have that f is continuous.

BONUS: (1) $\implies$ (3)

Let f be continuous and $B \subset Y$ be a closed set. It follows that $Y \setminus B$ is open. Since f is continuous, we have that $f^{-1}(Y \setminus B)$ is open as well. Notice that

$$X \setminus f^{-1}(B) = f^{-1}(Y \setminus X)$$

which results in $f^{-1}(B)$ is closed in X.

Theorem: Equivalent Definition of Homeomorphism

Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. Then f is a homeomorphism if and only if we have $f(U)$ is open if and only if U is open. That is, another way to defined a homeomorphism is to say that it is a bijective correspondence $f : X \to Y$ such that $f(U)$ is open if and only if U is open.

Proof

Theorem: 18.2 (Rules for Constructing Continuous Functions)

Let X, Y, and Z be topological spaces.

(a) (Constant Function) If $f : X \to Y$ maps all of X into the single point b of Y – that is, $f(x) = b$ for all $x \in X$, then f is continuous.

(b) (Inclusion) If A is a subspace of X (equipped with the subspace topology), the inclusion function $i_A : A \to X$, given by $i_A(x) = x$ for all $x \in A$, is continuous.

(c) (Composites) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$, defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$, is continuous.

(d) (Restricting the Domain) If $f : X \to Y$ is continuous, and if A is a subspace of X, then the restricted function $f|_A : A \to Y$ is continuous.

(e) (Restricting or Expanding the Range) Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the function $h : X \to Z$ obtained by expanding the range of f is continuous.

(f) (Local Formulation of Continuity) The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_i such that $f|_{U_i}$ is continuous for each $i \in I$, where I is some index set.

Proof

a
Let $V \subset Y$ be open. We want to show that $f^{-1}(V)$ is open in X. Notice that if V contains b, then we end up getting that $f^{-1}(V) = X$; however, if V does not contain b, then $f^{-1}(V) = \emptyset$. In either case, we have that $f^{-1}(V)$ is open since both X and $\emptyset$ are trivially open.

b
Let $U \subset X$ be open. We want to show that $i_A^{-1}(U)$ is open in A in the subspace topology. Notice that $i_A^{-1}(U) = U \cap A$, and since A is equipped with the subspace topology, we see that $U \cap A$ is open in A by definition since U is open in X. Thus, $i_A^{-1}(U)$ is open in A in the subspace topology.

c
Let $W \subset Z$ be open. Our goal is to show that $(g \circ f)^{-1}(W)$ is open in X. Notice that $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$. Since we are given that W is open in Z, we have that $g^{-1}(W)$ is open in Y because g is continuous. Furthermore, since f is continuous, we see that $f^{-1}(g^{-1}(W))$ must be open as well. Therefore, we see that $(g \circ f)$ is continuous.

d
Notice that $f|A = f \circ i_A$, and since both are continuous, their composition is continuous as well.

e
Restricting range: Let $V \cap Z$ be given where V is open in Y; notice that $V \cap Z$ is an arbitrary open set in Z under the subspace topology. We want to show that g is continuous, which means that we need to show that $g^{-1}(V \cap Z)$ is open in X. Notice that Z is not necessarily open in Y, which could pose problems for showing that $g^{-1}(V \cap Z)$ is open in X.

Let us look at $g^{-1}(V \cap Z)$. This is equivalent to saying that $g^{-1}(V) \cap g^{-1}(Z)$. Since Z is defined to be a subspace of Y containing the image set $f(X)$, it follows that $g^{-1}(Z) = X$. Thus, we have that $g^{-1}(V) \cap g^{-1}(Z) = g^{-1}(V) \cap X = g^{-1}(V)$. From here, we see the following:

$$g^{-1}(V) = \{x \in X \mid g(x) \in V\}$$
$$f^{-1}(V) = \{x \in X \mid f(x) \in V\}$$

Since g is obtained by restricting the range of f, we see that $g(x) = f(x)$ for all $x \in X$, which results in $g^{-1}(V) = f^{-1}(V)$. Since f is continuous, it follows that $f^{-1}(V)$ is open, which further results in $g^{-1}(V)$ to be open as well. Thus, we have that $g^{-1}(V \cap Z)$ is open in X. Hence g is continuous.

Expanding range: Now to show that $h : X \to Z$ is continuous if Z has Y as a subspace, we note that h is the composite of the map $f : X \to Y$ and $i_Y : Y \to Z$. We have already shown that the composition of continuous function are continuous, which results in h to be continuous.

f
By hypothesis, we can write X as a union of open sets U_i, such that $f|_{U_i}$ is continuous for each $i \in I$. We now let V be an open set in Y. We want to show that $f^{-1}(V)$ is open. Notice that $f|_{U_i}$

is constructed by restricting the function f to U_i, meaning that

$$(f|_{U_i})^{-1}(V) = \{x \in U_i \mid f(x) \in V\} = f^{-1}(V) \cap U_i$$

since all the expression above represent the set of those points x lying in U_i for which $f(x) \in V$. Because we are given $f|_{U_i}$ to be continuous, we have that $f^{-1}(V) \cap U_i$ is open in U_i, hence open in X. Let us now take a look $f^{-1}(V)$ more closely knowing that X can be written as a union of open set U_i:

$$\begin{aligned} f^{-1}(V) &= \{x \in X \mid f(x) \in V\} \\ &= \{x \in \bigcup_{i \in I} U_i \mid f(x) \in V\} \\ &= \{x \in X \mid x \in U_i \text{ and } f(x) \in V \text{ for at least one } i \in I\} \\ &= \bigcup_{i \in I} (f^{-1}(V) \cap U_i) \end{aligned}$$

Since each $f^{-1}(V) \cap U_i$ is open in X for all $i \in I$, and arbitrary unions of open sets are still open, we have that $f^{-1}(V)$ is open. Thus, f is continuous.

Theorem: 18.3 (The Pasting Lemma)

Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$, defined by setting $h(x) = f(x)$ if $x \in A$, and $h(x) = g(x)$ if $x \in B$.

Proof Let f and g be continuous. Our goal is to show that h is continuous. Let $D \subset Y$ be closed. We want to show that $h^{-1}(D)$ is closed as well, which will tell us h is continuous. Notice that

$$h^{-1}(D) = [h^{-1}(D) \cap A] \cup [h^{-1}(D) \cap B] = f^{-1}(D) \cup g^{-1}(D)$$

We have that $f^{-1}(D)$ and $g^{-1}(D)$ are closed in A and B, respectively. Since A and B are closed in X, it follows that $f^{-1}(D)$ and $g^{-1}(D)$ are both closed in X as well by transitivity of closedness. Hence, it follows that $h^{-1}(D)$ is closed as well since we have a finite union of closed sets, which finishes our proof.

This argument works for finite unions, but does not work for arbitrary unions since we cannot guarantee that arbitrary unions of closed sets are closed. Furthermore, we have that this theorem can be rephrased in terms of open subsets.

Theorem: Maps into Products

Let $Y = \prod_{i \in I} Y_i$ be a product of topological spaces. Let $f : X \to Y$ be a function given by the equation:

$$\mathbf{f}(t) = (f_1(t), f_2(t), f_3(t), \ldots)$$

which we can further write in terms of its **component functions** f_i:

$$\mathbf{f} = (f_i)_{i \in I} \quad \text{where} \quad f_i = \mathrm{proj}_i \circ \mathbf{f}$$

We say that $\mathbf{f}$ is continuous if and only if its component functions are all continuous, assuming that the codomain has been equipped with the product topology.

Proof

$\implies$

Assume that $\mathbf{f}$ is continuous. We want to show that each of their component functions f_i are continuous for all $i \in I$. Indeed, we see that they are continuous since such functions are defined as $f_i = \text{proj}_i \circ \mathbf{f}$, which is the composition of two continuous functions.

$\impliedby$

Assume $f_i : X \to Y$ is continuous for all $i \in I$. Let $U = \prod_{i \in I} U_i$ be a basic open set in the product topology. Our goal is to show that $\mathbf{f}$ is continuous, which we can do by showing that $\mathbf{f}^{-1}(U)$ is open.

We start by noting that

$$f_i^{-1} = (\text{proj}_i \circ \mathbf{f})^{-1} = \mathbf{f}^{-1} \circ \text{proj}_i^{-1}$$

Thus, it follows that

$$\mathbf{f}^{-1}(U) = \mathbf{f}^{-1}\left(\prod_{i \in I} U_i\right) = \mathbf{f}^{-1}\left(\bigcap_{i \in I} \text{proj}_i^{-1}(U_i)\right) = \bigcap_{i \in I} (\mathbf{f}^{-1} \circ \text{proj}_i^{-1})(U_i) = \bigcap_{i \in I} f_i^{-1}(U_i)$$

Since we are working in a product topology, it follows that

$$\bigcap_{i \in I} f_i^{-1}(U_i) = \bigcap_{\substack{i \in I \\ U_i \neq Y}} f_i^{-1}(U_i)$$

Since each f_i is continuous for all $i \in I$, we have that $f_i^{-1}(U_i)$ is open for all $i \in I$. Furthermore, since we only have finitely many U_i such that $U_i \neq Y_i$, it follows that $\mathbf{f}^{-1}(U)$ is open since a finite intersection of open sets is still open. Thus, we have that $\mathbf{f}$ is continuous.

Theorem: Continuity and Limits

Let $f : X \to Y$ be a continuous function between topological spaces. If $x_n \to x$ is a convergent sequence in X, then $f(x_n) \to f(x)$ in Y.

Proof Let $x_n \to x$. Our goal is to show that $f(x_n) \to f(x)$. Let V be an open neighborhood of $f(x)$. We want to show that $f(x_n) \in V$ for all $n \geq N$. Notice that $f^{-1}(V)$ is an open neighborhood of x because f is continuous. Because $x_n \to x$, we have that there exists some positive integer N such that $x_n \in f^{-1}(V)$ for all $n \geq N$. Notice that $x_n \in f^{-1}(V) \iff f(x_n) \in V$.

Theorem: Hausdorffness and Continuity

Let X be a topological space and let A be a subset of X so that $\mathrm{Cl}\, A = X$. Let $f : X \to Y$ and $g : X \to Y$ be a pair of continuous functions into a topological space Y that **agree** on A, meaning that $f(x) = g(x)$ for all $x \in A$.

If Y is Hausdorff, then f and g must agree on all of X.

Proof Towards a contradiction (TAC), assume there exists some $p \in X$ so that $f(p) \neq g(p)$. Because Y is Hausdorff, there exists disjoint open neighborhoods, say U containing $f(p)$ and V containing $g(p)$. Notice that both $f^{-1}(U)$ and $g^{-1}(V)$ are open neighborhoods of p.

If we take their intersection, then we still end up with an open neighborhood of p – that is, $f^{-1}(U) \cap g^{-1}(V)$ is still an open neighborhood of p. Since $p \in \mathrm{Cl}\, A$, this implies that $A \cap (f^{-1}(U) \cap g^{-1}(V)) \neq \emptyset$. Let us pick any $x \in A \cap (f^{-1}(U) \cap g^{-1}(V))$. This means that $f(x) \in U$ and $g(x) \in V$. Because $x \in A$, we have by assumption that $f(x) = g(x)$, which results in $U \cap V \neq \emptyset$. However, we have that $U \cap V = \emptyset$, so we reach a contradiction.

Therefore, f and g must agree on all of X.

8.4 Examples

Example: Unit Circle – Cutting and Glueing

Let us consider the following functions between the unit circle S and the half open interval (insert image). Assume both are equipped with their Euclidean topologies.

Cutting

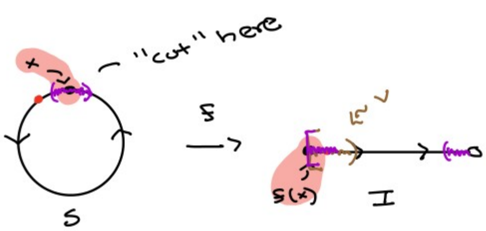

Figure 8.1: Open Neighborhoods of x after Cutting

Notice that an open neighborhood of x in S contains points that are left and right of x, but an open neighborhood of x in the half-open interval consists of a smaller half-open interval. In Figure (8.1), the open neighborhood of x in S is defined as U, and the open neighborhood of x in the half open interval is defined as V. If our function f is the action of cutting the unit circle at x, then we can see in Figure (8.1) that the image of U under this function is not contained in V – there are values at the right-end of our half-open interval that are not contained in V.

Cutting

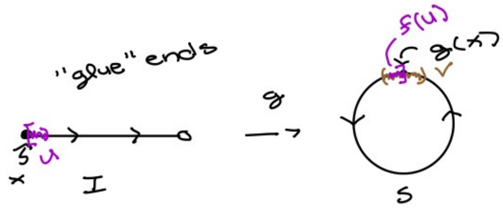

Figure 8.2: Open Neighborhoods of x after Glueing

In Figure (8.2), the open neighborhood of x in the half-open interval is defined as U, and the open neighborhood of x in S is defined as V. In this scenario, our function g is now defined to be the action of glueing the half-open interval into a unit circle. We can see in Figure (8.2) that the image of U under this function is contained in V this time around. Thus, g is continuous.

In general, we have that "cutting" is something that is not continuous. However, "glueing" is continuous. This example also shows that for a continuous function, its inverse need not be continuous as well.

Example: Projections are Continuous Functions

Let $X = \prod_{i \in I} X_i$ be a product of topological spaces. We fix an index $k \in I$ and consider: $\text{proj}_k : X \to X_k$. These functions are always continuous, whether we equip X with the product topology or the box topology.

This can be easily proven by noting that the inverse image of an open subset of X_k under proj_k is open in the product topology (and therefore also in the at least as fine box topology). I.e., we have that whenever $V \subset X_k$ is open in X_k, $\text{proj}_k^{-1}(V)$ is open in X. Thus, proj_k is continuous.

Example: Continuity of a Function (Codomain given by a basis)

Let $f : X \to Y$ be a function between topological spaces where the topology on Y is generated by a basis. Assume $f^{-1}(B)$ is open whenever B is a basic open subset of Y. We want to show that f is continuous.

Let $V \subset Y$ be open. Since Y is generated by a basis, we have that $V = \bigcup_{i \in I} B_i$ where B_i are some basic open subsets. Using the properties of inverse images, we see that

$$f^{-1}(V) = f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i)$$

Since each $f^{-1}(B_i)$ is open by assumption, we see that $f^{-1}(V)$ is a union of open subsets, which is still open. Thus, $f^{-1}(V)$ is open in X.

Example: Continuity in the Discrete and Trivial Topology

Let $f : X \to Y$ be a function between topological spaces. We want to show that

(a) f is automatically continuous if X has the discrete topology.

(b) f is automatically continuous if Y has the trivial topology.

$\boxed{\text{a}}$

Let's say that we are given some V that is open in Y. It follows that $f^{-1}(V)$ must be some subset of X, but since we have that X has the discrete topology, all subsets of X are open. Hence, we see that $f^{-1}(V)$ is open as well, meaning that f is continuous.

$\boxed{\text{b}}$

Let's now say that Y has the trivial topology – this means that the only open sets are $\emptyset$ and Y itself. It is clear to see that $f^{-1}(\emptyset) = \emptyset$, which is always an open set in X. We also see that $f^{-1}(Y) = X$, which is also an open set in X. Hence, we see that f is continuous as well.

Example: Continuity From the Right

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

This is evidently discontinuous at $x = 0$ when $\mathbb{R}$ is given the standard topology. However it is continuous at $x = 0$ as a function $f : \mathbb{R}_\ell \to \mathbb{R}$. In other words, it is continuous when the domain is equipped with the lower- limit topology. Thus, we see that continuity from the right is the same as continuity when the domain is equipped with the lower-limit topology.

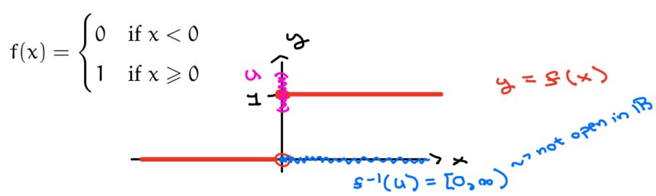

Figure 8.3: Continuity from the Right

Example: A Non-example of a Homeomorphism

Glueing and Cutting are bijective, but we do not end up with a homeomorphism The issue is that glueing is continuous, but cutting is not.

8.5 Discussion

Remark: Some Properties of Inverse Image

Let $f : X \to Y$ be a function. Let $\{B_i\}_{i\in I}$ be an indexed set of subsets of the codomain Y. Then

$$f^{-1}\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f^{-1}(B_i) \quad \text{and} \quad f^{-1}\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} f^{-1}(B_i)$$

It is as if the inverse image commutes with unions and intersections.

Remark: Some Properties of Images

Let $f : X \to Y$ be a function. Let $\{A_i\}_{i\in I}$ be an indexed set of subsets of the domain X. Then

$$f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i)$$

Notice that the same cannot be said of intersections in this case – we can say

$$f\left(\bigcap_{i\in I} A_i\right) \subset \bigcap_{i\in I} f(A_i)$$

Take for example $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ as its formula. Let $A = (-\infty, 0]$ and $B = [0, \infty)$. Then we see that

$$f(A \cap B) = \{0\} \quad \text{but} \quad f(A) \cap f(B) = [0, \infty)$$

Remark: On Continuity

Continuity of a function depends not only upon the function f itself, but also on the topologies specified for its domain and range.

Remark: Differences Between Definitions of Continuity

Notice that Munkres uses the definition of continuity that involves inverse images. However, we introduced the definition of continuity that involves images; the main reason being that this definition is more intuitive to understand. All the other definitions of continuity (say, in analysis) were framed using images, so it seems natural that the topological definitions follows in those same footsteps.

In analysis, a function $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous at a particular point $p \in \mathbb{R}$ if and only if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |f(x) - f(p)| < \epsilon \text{ whenever } |x - p| < \delta$$

We can phrase this instead in terms of open balls. In $\mathbb{R}$, an open ball of radius r centered at

p is really jus the open interval $B_r(p) = (p-r, p+r)$. We can further rephrase this in terms of inequalities: a point x is in $B_r(p)$ if and only if $|x-p| < r$.

Using this change of phrase, we can give an alternative definition for continuity. We say that $f : \mathbb{R} \to \mathbb{R}$ is continuous at p if and only if:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } f(x) \in B_\epsilon(f(p)) \text{ whenever } x \in B_\delta(p)$$

This definition is more in line with the topological definition we had for continuity.

Although the definition of continuity involving images is more intuitive, it is not as useful as the definition of continuity involving inverse images. This is because inverse images have much nicer properties than images, so it makes it easier to work with when solving problems.

Remark: Why we need the inverse function to be continuous

The condition that f^{-1} is continuous says that for each open set U of X, the inverse image of U under the map $f^{-1} : Y \to X$ is open in Y. However, notice that the inverse image of U under the map f^{-1} is the same as the image of U under the map f.

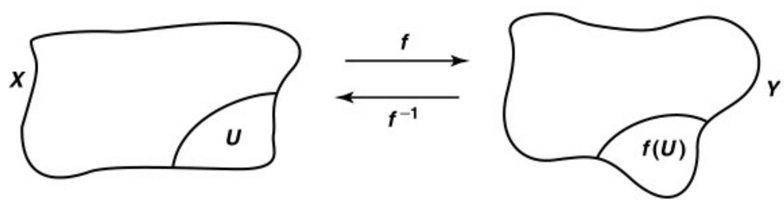

Figure 8.4: A visual for homeomorphisms

Remark: On Equivalent Definitions of Homeomorphisms

These equivalent definitions shows that a homeomorphism $f : X \to Y$ gives us a bijective correspondence not only between X and Y, but also between the collection of open sets of X and of Y.

Remark: Analogy of Homeomorphism to Isomorphism

In algebra, we have studied the notion of an isomorphism between algebraic objects such as groups or rings. An isomorphism is a bijective correspondence that preserves the algebraic structure involved.

The analogous concept in topology is that of homeomorphism; it is a bijective correspondence that preserves the topological structure involved.

Remark: More Examples

More examples can be found on §18 of Munkres.

Chapter 9

The Quotient Topology

9.1 Learning Objectives

- What is an equivalence relation?
  - What are some examples of equivalence relations?
- What is an equivalence class?
  - What are some examples of equivalence classes?
- How is a quotient topology defined?
- How is a quotient space defined?
  - How is the quotient space and quotient topology related?
- What is a quotient map?

9.2 Definitions

Definition: Equivalence Relation

Given a set X, an **equivalence relation** on X is a way of declaring when pairs x and y are *equivalent* to one another, denoted by $x \sim y$, so that the following conditions are satisfied.

- **Reflexivity**: $x \sim x$ for all x.
- **Symmetry**: If $x \sim y$, then $y \sim x$.
- **Transitivity**: If $x \sim y$ and $y \sim z$, then $x \sim z$.

Definition: Equivalence Classes

Given an equivalence relation on a set, we can partition that set into what we call **equivalence classes**, which are maximal subsets consisting of elements that are all equivalent to one another. If x is an element, then we use $[x]$ to denote the equivalence class containing x.

Definition: Saturated Sets

We say that a subset C of X is **saturated** (with respect to the surjective map $p : X \to Y$) if C contains every set $p^{-1}(\{y\})$ that it intersects. Thus, C is saturated if it equals the complete inverse image of a subset of Y.

Definition: Quotient Map

Let X and Y be topological spaces. Let $p : X \to Y$ be a *surjective* map. The map p is said to be a **quotient map** provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

In other words, p is a quotient map provided that it is *continuous* and, for each $U \subset Y$, if $p^{-1}(U)$ is open in X, then U must be open in Y.

Furthermore, we can equivalently say p is a quotient map if Y is the quotient space corresponding to p.

To say that p is a quotient map is equivalent to saying that p is continuous and p maps *saturated* open sets of X to open sets of Y (or saturated closed sets of X to closed sets of Y).

Definition: Quotient Topology

Let X be a toplogical space and Y a *set*. Let $p : X \to Y$ be a surjective map. The **quotient topology** induced by p on the codomain Y is the *finest* topology on Y so that p is continuous, in the sense that the quotient topology contains every other topology on Y that makes p continuous.

In other words, there exists exactly one topology $\mathcal{T}$ on Y relative to which p is a quotient map; it is called the **quotient topology** induced by p

Definition: Quotient Space

Let X be a toplogical space and Y a *set*. Let $p : X \to Y$ be a surjective map. When the codomain Y is equipped with the quotient topology, then it is called the **quotient space** corresponding to p.

9.3 Theorems

Theorem: 22.1

Let $p : X \to Y$ be a quotient map; let A be a subspace of X that is saturated with respect to p; let $q : A \to p(A)$ be the map obtained by restricting p.

1. If A is either open or closed in X, then q is a quotient map.
2. If p is either an open map or a closed map, then q is a quotient map.

Proof

Theorem: 22.2

Let $p : X \to Y$ be a quotient map. Let Z be a space and let $g : X \to Z$ be a map that is constant on each set $p^{-1}(\{y\})$, for $y \in Y$. Then g induces a map $f : Y \to Z$ such that $f \circ p = g$. The induced map f is continuous if and only if g is continuous; f is a quotient map if and only if g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, dashed, "f"'] & Z \end{tikzcd}$$

Proof

Corollary: 22.3

Let $g : X \to Z$ be a surjective continuous map. Let X^* be the following collection of subsets of X:

$$X^* = \{g^{-1}(\{z\}) \mid z \in Z\}$$

Give X^* the quotient topology.

(a) The map g induces a bijective continuous map $f : X^* \to Z$, which is a homeomorphism if and only if g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ X^* \arrow[r, "f"'] & Z \end{tikzcd}$$

(b) If Z is Hausdorff, then so is X^*

Proof

Theorem: Equivalent Quotient Spaces

Let $p : X \to Y$ be a quotient map. Then Y is homeomorphic to the quotient space X^* determined by the equivalence relation: $x \sim y$ if and only if $p(x) = p(y)$.

Proof Consider this diagram of maps:

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "p"] & \\ X^* \arrow[r, "\overline{p}"'] & Y \end{tikzcd}$$

We define $\pi(x) = [x]$ for all $x \in X$. Let us call $\overline{p} : X^* \to Y$ to be defined by $\overline{p}([x]) = p(x)$. We claim that $\overline{p}$ is a homeomorphism between X and Y, which means that we need to check three conditions: the first is that $\overline{p}$ is well-defined, the second is that $\overline{p}$ is a bijection, and the third is that $\overline{p}$ and $\overline{p}^{-1}$ are continuous.

For the first and second conditions can be seen via the following equivalences:

$$[x] = [y] \iff x \sim y \iff p(x) = p(y)$$

where the second equivalence follows from our assumption. Going from left-to-right gives us that $\overline{p}$ is well-defined, Going from right-to-left gives us that $\overline{p}$ is injective. Furthermore, $\overline{p}$ is surjective since p is a surjection. Thus, $\overline{p}$ is a bijection, which tells us that it has an inverse $\overline{p}^{-1}$.

We now need to verify the third condition – that is, we need to show that both $\overline{p}$ and $\overline{p}^{-1}$ are continuous. We start by noting that X^* is equipped with the quotient topology induced by π and Y is equipped with the quotient topology induced by p. Further notice that $p = \overline{p} \circ \pi$.

We first go about showing that $\overline{p}$ is continuous. Let $V \subset Y$ be open. Our goal is to show that $\overline{p}^{-1}(V)$ is open in X^* in the quotient topology induced by π. This means that we need to show that $\pi^{-1}(\overline{p}^{-1}(V))$ is open in X. Notice that

$$\pi^{-1}(\overline{p}^{-1}(V)) = (\overline{p} \circ \pi)^{-1}(V) = p^{-1}(V)$$

Since we know that p is continuous, we find that $p^{-1}(V)$ is open in X, which means that $\pi^{-1}(\overline{p}^{-1}(V))$ is open in X. Thus, we have that $\overline{p}$ is continuous.

We now need to show that $\overline{p}^{-1}$, which exists since $\overline{p}$ is a bijection, is continuous. Let $W \subset X^*$ be open. We want to show that $(\overline{p}^{-1})^{-1}(W)$ is open in Y under the quotient topology induced by p. Now because the inverse of a function is unique, we see that $(\overline{p}^{-1})^{-1}(W) = \overline{p}(W)$. Thus, it suffices to check that $\overline{p}(W)$ is open in Y under the quotient topology induced by p.

To do so, we shall utilize the fact that p is a quotient map; because p is a quotient map, we have that $\overline{p}(W)$ is open in Y if and only if $p^{-1}(\overline{p}(W))$ is open in X. Notice that

$$p = \overline{p} \circ \pi \iff \pi = \overline{p}^{-1} \circ p \implies p^{-1}(\overline{p}(W)) = (\overline{p}^{-1} \circ p)^{-1}(W) = \pi^{-1}(W)$$

Since π itself is a quotient map (namely, continuous), we have that W being open in X^* implies that $\pi^{-1}(W)$ is open in X. Thus, we have that $p^{-1}(\overline{p}(W))$ is open in X, which results in $\overline{p}(W)$ to be open in Y. With this, we see that $\overline{p}^{-1}$ is continuous.

With all three conditions proved, we have that $\overline{p}$ is a homeomorphism between X and Y

9.4 Examples

Example: Equivalence Relations and Classes

Take $X = \mathbb{R}$. We can deem $x \sim y$ if and only if x and y differ by an integer. E.g.

$$1 \sim 3 \quad \text{and} \quad \pi \sim \pi + 5 \quad \text{and} \quad 1 \nsim 1.5$$

This is clearly an equivalence relation. Any x differs with itself by 0, meaning that $x \sim x$. If $x \sim y$, it is clear that $y \sim x$ since differing by an integer is commutative. Finally, if $x \sim y$ and $y \sim z$, then we get $x \sim z$ since integers are closed via addition/subtraction.

As for the equivalence classes for this given equivalence relation, we see that

$$[1] = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \quad \text{and} \quad [\pi] = \{\ldots, \pi - 2, \pi - 1, \pi, \pi + 1, \pi + 2\}$$

Another equivalence relation that we'll frequently look at is one that is associated to a given function $f : X \to Y$ between sets. It is an equivalence relation on the domain X, where we deem $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$ – that is, inputs with the same output are deemed equivalent.

One can see that such a relation, is indeed an equivalence relation. We get that $x \sim x$ immediately due to our function being well-defined. If $x \sim y$, it is clear that $y \sim x$ since equality is itself symmetric. Finally, if $x \sim y$ and $y \sim z$, then we get $x \sim z$ since equality is again transitive.

Example: Quotient Topology

Let $I = [0, 1]$ be the unit interval with the subspace topology. Consider the equivalence relation on I that deems $0 \sim 1$ and nothing else extra Let I^* denote the set of all equivalence classes. We can think of I^* as a unit circle S^1.

Let us figure out the equivalence classes

$$[0] = [1] = \{0, 1\} \quad \text{and} \quad [x] = \{x\} \text{ where } x \in R \text{ and } x \neq 0, 1$$

We can relate I and I^* using a special function, denoted p, which sends an element in I to its corresponding equivalence class – that is, $p(x) = [x]$. We can see that 0 and 1 get sent to the same equivalence class, and the rest of the values get sent to their own unique equivalence class.

In this case, we see that the function p looks like the "glueing" function. Now, if we declare $V \subset I^*$ to be open, then we would want $p^{-1}(V)$ to be open as well – that is, we want p to be continuous. We also want to declare the other implication – that is, if $p^{-1}(V)$ is open, then we would want V to be open in I^* as well. With both of these declaration, we have defined the quotient topology on I^*.

Example: Two Special Quotient Maps

Two special kinds of quotient maps are the *open maps* and the *closed maps*. Recall that a map $f : X \to Y$ is said to be an **open map** if for each open set U of X, the set $f(U)$ is open in Y. If is said to be a **closed map** if for each closed set A of X, the set $f(A)$ is closed in Y.

It follows immediately from the definition that if $p : X \to Y$ is a surjective continuous map that is either open or closed, then p is a quotient map.

We shall prove the case when p is an open map. Our goal is to show that $p^{-1}(V)$ open implies V is open for some $V \subset Y$. Recall that

$$p^{-1}(V) = \{x \in X \mid p(x) \in V\}$$

Now because p is surjective, we find that $p^{-1}(V)$ will contain all the possible values of x such that every element in V is hit. Thus, it follows that $p(p^{-1}(V)) = V$. Since p is an open map, and $p^{-1}(V)$ is open, it follows that V must be open as well. Thus, we have that p is a quotient map.

We now consider the case when p is a closed map. Our goal is to show that $p^{-1}(V)$ open implies V is open for some $V \subset Y$. Since $p^{-1}(V)$ is open, it follows that $X \setminus p^{-1}(V)$ is closed. Notice that

$$X \setminus p^{-1}(V) = p^{-1}(Y \setminus V)$$

which tells us that $p^{-1}(Y \setminus V)$ must be closed as well. Since p is surjective, we see that $p(p^{-1}(Y \setminus V)) = Y \setminus V$. Furthermore, since p is a closed map, we see that $p^{-1}(Y \setminus V)$ being closed implies that $Y \setminus V$ must be closed as well. Hence, it follows that V must then be open. Thus, we have that p is a quotient map.

Example: Verifying the Quotient Topology

Recall that the [quotient topology], say $\mathcal{T}$, is defined by letting it consists of those subsets U of A such that $p^{-1}(U)$ is open in X. It is easy to check that $\mathcal{T}$ is a topology. The sets $\emptyset$ and A are open since we have that $p^{-1}(\emptyset) = \emptyset$ and $p^{1}(A) = X$ – both of which are open in X. We have that arbitrary unions of open sets in A still remain open since

$$p^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} p^{-1}(U_i)$$

Lastly, we have that finite intersections of open sets in A also remain open since

$$p^{-1}\left(\bigcap_{i=1}^{n} U_i\right) = \bigcap_{i=1}^{n} p^{-1}(U_i)$$

Example: Open maps and Quotient Maps

We examine the function from $\mathbb{R}$ into the unit circle S^1 (thought of as the subspace of $\mathbb{R}^2$ defined by $x^2 + y^2 = 1$) given below:

$$p : \mathbb{R} \to S^1 \quad \text{where} \quad p(t) = (\cos(2\pi t), \sin(2\pi t))$$

One can see clearly that p is an open map. On top of that, we can also see that p is both surjective and continuous. Thus, we have that p is a quotient map by the [third example].

Example: Equivalent Quotient Spaces

Let $\mathbb{R}^*$ be the quotient space determined by the following equivalence relation on $\mathbb{R}$:

$$x \sim y \iff x - y \in \mathbb{Z}$$

We want to show that R^* is homeomorphic to the unit circle S^1.

To do so, we shall consider the function

$$p : \mathbb{R} \to S^1 \quad \text{where} \quad p(t) = (\cos(2\pi t), \sin(2\pi t))$$

which we previously showed was a quotient map. If we can show that the given equivalence relation on $\mathbb{R}$ is equivalent to the equivalence relation on $\mathbb{R}$ using p (that is, $x \sim y \iff p(x) = p(y)$), then we have that by the theorem about equivalent quotient spaces that S^1 is homeomorphic to the quotient space of $\mathbb{R}$.

To show that these two equivalence relations are equivalent, it suffices to show that two elements being similar via the equivalence relation using p must also be similar via the equivalence relation given to us, and vice-versa. Indeed, we see this to be the case since both of the components of p have a period of 1. Thus, we have that $p(x) = p(y)$ when x and y are some multiple of the period away from each other, which is some integer away from each other – that is, it must be the case that $x - y \in \mathbb{Z}$.

Hence, we end up getting that S^1 to be homeomorphic to $\mathbb{R}^*$.

Example: 1 (Munkres)

Let X be the subspace $[0,1] \cup [2,3]$ of $\mathbb{R}$, and let Y be the subspace $[0,2]$ of $\mathbb{R}$. The map $p : X \to Y$ defined by

$$p(x) = \begin{cases} x & x \in [0,1] \\ x - 1 & x \in [2,3] \end{cases}$$

is readily seen to be surjective, continuous, and closed. Thus, it follows that p is a quotient.

However, p is not an open map since the image of the open set $[0,1]$ of X is not open in Y (there is no open interval in $\mathbb{R}$ that intersects with Y to get $[0,1]$).

Now notice that if A is the subspace $[0,1) \cup [2,3]$, then the map $q : A \to Y$ obtained by restricting p is continuous and surjective, but *not* a quotient map; the set $[2,3]$ is open in A and is saturated with respect to q, but its image is not open in Y.

Example: 2 (Munkres)

Let $\text{proj}_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the projection onto the first coordinate; then proj_1 is clearly surjective and continuous. Furthermore, we see that proj_1 is an open map; for if $U \times V$ is a nonempty basis element for $\mathbb{R} \times \mathbb{R}$, then $\text{proj}_1(U \times V) = U$ is open in $\mathbb{R}$. Thus, it follows that proj_1 carries open sets of $\mathbb{R} \times \mathbb{R}$ to open sets of $\mathbb{R}$.

However, proj_1 is not a closed map. The subset

$$C = \{x \times y \mid xy = 1\}$$

of $\mathbb{R} \times \mathbb{R}$ is closed, but $\text{proj}_1(C) = \mathbb{R} \setminus \{0\}$, which is not closed in $\mathbb{R}$.

Notice that if A is the subspace of $\mathbb{R} \times \mathbb{R}$ that is the union of C and the origin $\{0\}$, then the map $q : A \to \mathbb{R}$ obtained by restricting proj_1 is continuous and surjective, but it is still not a quotient map; for the one-point set $\{0\}$ is open in A and is saturated with respect to q, but its image is not open in $\mathbb{R}$.

Example: 3 (Munkres)

Let p be the map of the real line $\mathbb{R}$ onto the three-point set $A = \{a, b, c\}$ defined by

$$p(x) = \begin{cases} a & x > 0 \\ b & x < 0 \\ c & x = 0 \end{cases}$$

We can see that the quotient topology on A induced by p is the one indicated in Figure (9.1)

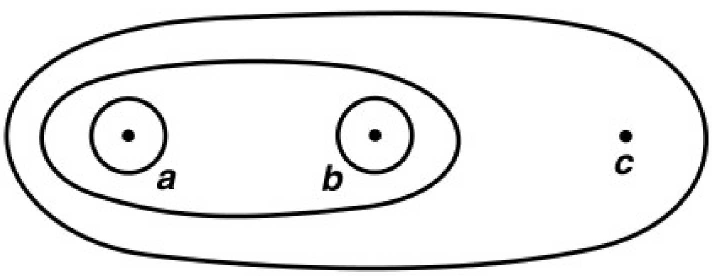

Figure 9.1: The quotient topology on A induced by p

Example: 4 (Munkres)

Let X be the closed unit ball

$$\{x \times y \mid x^2 + y^2 \leq 1\}$$

in $\mathbb{R}^2$, and let X^* be the partition of X consisting of all the one-point sets $\{x \times y\}$ for which $x^2 + y^2 < 1$, along with the set $S^1 = \{x \times y \mid x^2 + y^2 = 1\}$.

Typical saturated open sets in X are pictured by the shaded regions in Figure (9.2). One can show that X^* is homeomorphic with the subspace of $\mathbb{R}^3$ called the **unit 2-sphere**, defined by

$$S^2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$$

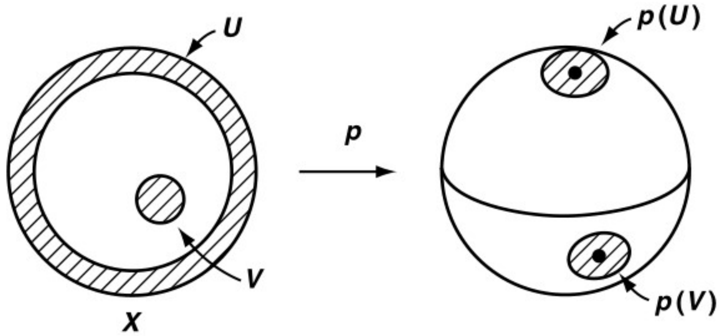

Figure 9.2: Typical saturated open sets in X

9.5 Discussion

Remark: On the Quotient Map

The condition for a quotient map is *stronger* than continuity; for continuity, we had $p^{-1}(U)$ to be open in X whenever U is open in Y, which is only one direction of implication. However, a quotient map requires an equivalence – that is, both directions of implication.

We also note that an equivalent condition for a quotient map is as follows: The map p is said to be a **quotient map** provided a subset A of Y is closed in Y if and only if $p^{-1}(A)$ is closed in X. The equivalence of these two conditions follows from the fact that

$$f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$$

Remark: On the Quotient Space X^*

The topology of X^* can be described in another way: A subset U of X^* is a collection of equivalence classes, and the set $p^{-1}(U)$ is just the union of equivalence classes belonging to U. Thus, the typical open set of X^* is a collection of equivalence classes whose union is an open set of X. One can see that this does indeed form a topology on X^*.

With this description, we can see that the surjective map $\pi : X \to X^*$, where $\pi(x) = [x]$, is a quotient map, and thus induces a quotient topology on X^*.

As a result, when X is a topological space with an equivalence relation and X^* is the corresponding set of equivalence classes, it is implied that the quotient topology on X^* is induced by the function $X \to X^*$ that sends a point to its equivalence class; that is, X^* is induced by the function $\pi : X \to X^*$, where $\pi(x) = [x]$.

One can think of X^* as having been obtained by "identifying" each pair of equivalent points. For this reason, the quotient space X^* is often called an **identification space**, or a **decomposition space**, of the space X.

Remark: On Equivalent Quotient Spaces

Let $p : X \to Y$ be a quotient map. The theorem about equivalent quotient spaces is stating that we can always produce an equivalence relation on X using the quotient map p – this equivalence relation being $x \sim y \iff p(x) = p(y)$, which states that two elements in X are similar if they are both sent to the same point via p. I.e., if p acts two elements in the same way (that is, sending them to the same point), then these two elements are similar to each other.

Notice that this equivalence relation partitions X up into equivalence classes, and we have the set of these equivalence classes X^* to be a quotient space itself; recall that the the quotient topology on X^* is induced by the function $X \to X^*$ that sends a point to its equivalence class – that is, X^* is induced by the function $\pi : X \to X^*$, where $\pi(x) = [x]$.

Not only that, but we showed that such an equivalence relation results in the quotient space Y induced by p to be homeomorphic to X^* – i.e., Y and X^* are equivalent spaces! Thus, we can always think of any quotient space as being a set of equivalent classes instead, which is what Munkres does for his definition of quotient spaces.

Munkres offers the following definition for a quotient space: Let X be a topological space, and let X^* be a partition of X into disjoint subsets whose union is X. Let $p : X \to X^*$ be the surjective map that carries each point of X to the element of X^* containing it. In the quotient topology induced by p, the space X^* is called a **quotient space** of X.

From this definition, he then states that given X^*, there is an equivalence relation on X of which the elements of X^* are the equivalence classes.

Remark: Quotient Maps

The composite of two quotient maps is a quotient map; this fact follows from the following equation:

$$p^{-1}(q^{-1}(U)) = (q \circ p)^{-1}(U)$$

On the other hand the product of two quotient maps need not be a quotient map. This also is true for subspaces: if $p : X \to Y$ is a quotient map and A is a subspace of X, then the map $q : A \to p(A)$ obtained by restricting p need not be a quotient map. However, there are special conditions that makes q a quotient map – c.g. [Theorem 22.1]

Finally, if X is Hausdorff, then there is no reason that the quotient space X^* needs to be Hausdorff. However, there is a simple condition for X^* to satisfy the T_1 axiom; one simply requires that each element of the partition X^* be a closed subset of X.

Remark: Proofs and Examples

The proofs for the theorems/corollary as well as more examples can be found in §22 of Munkres.

Chapter 10

Connected Spaces

10.1 Learning Objectives

- What are the equivalent definitions of connected sets?
- How does connectedness apply for subspaces? [Lemma 23.1]
- How do connected subspaces behave when its larger space has a separation? [Lemma 23.2]
- How do unions of connected sets behave? [Theorem 23.3]
- Why can we say that the closure a connected set is connected? [Theorem 23.4]
- Is the image of a connected space always connected? [Lemma 23.5]
- Is a finite product of connected spaces connected? How about arbitrary products? [Theorem 23.6]

10.2 Definitions

Definition: Separation and Connected

Let X be a topological space. A **separation** of X is a pair U, V of disjoint non-empty open subsets of X whose union is X. The space X is said to be **connected** if there *does not* exists a separation of X.

10.3 Theorems

Theorem: Alternative Definition of Connectedness

A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.

Said slightly differently, we say that a space X is connected if and only if it contains no non-trivial clopen subsets.

Proof

$\boxed{\implies}$

Let's say that X is connected. Towards a contradiction, we assume that there exists a nonempty proper subset A of X such that it is both open and closed in X. We can see that $U = A$ and $V = X \setminus A$ constitute a separation of X, for they are open, disjoint, nonempty, and their union is in X. By definition, we have that X is not connected, which is a contradiction. Hence, the only subsets of X that are both open and closed in X are the empty set and X itself.

$\boxed{\impliedby}$

Let's now assume that the only subsets of X that are both open and closed in X are the empty set and X itself. Towards a contradiction, we assume that X is not connected. It follows by definition that there exists a separation of X. If U and V form this separation of X, then we see that U must be nonempty and different from X (if it were X, then we would have V to be the $\emptyset$), and we have that U is both open and closed in X since both U and $X \setminus U = V$ are open by definition. This gives us a contradiction, which tells us that X is connected.

Lemma: 23.1

If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y.

Proof We essentially want to show that the following are equivalent:

1. a pair of disjoint nonempty sets A and B of X whose union is Y and neither of which contains a limit point of the other
2. a pair of disjoint nonempty open sets A and B in Y whose union (in Y) is Y; that is, A and B are both open and closed in Y.

We start by showing that (1) $\implies$ (2). It suffices to show that both A and B are closed in Y; this is because if A is closed in Y, then $Y \setminus A = B$ is open in Y, and the same can be said to show that A is open in Y using the fact that B is also closed in Y.

WLOG, we shall show that A is closed in Y. We start by noting that $\overline{A}$ (the closure of A in X) is closed in X. Thus, we have by definition, that $Y \cap \overline{A}$ is closed in Y. If we define A' to be the set of limit points of A in X, then we see that $\overline{A} = A \cup A'$. Thus, we see that

$$Y \cap \overline{A} = (A \cup B) \cap (A \cup A') = A \cup (B \cap A') = A \cup \emptyset = A$$

The third equality follows from the fact that A and B do not contain limit points of the other. Hence, we get that A is closed in Y. Now, the same argument can be applied to show that B is closed in Y.

We now prove (2) $\implies$ (1). Since A is closed in Y, it follows that A equals the Cl A in Y. From [Theorem 17.4], we know that the Cl A in Y is equivalent to $\overline{A} \cap Y$, where $\overline{A}$ is the closure of A in X. Thus, we see that

$$A = \overline{A} \cap Y \iff A = (A \cup A') \cap (A \cup B) \iff A = A \cup (A' \cap B)$$

Since A is closed, we have that it must contain all of its limit points, which means that $A' \subset A$. Thus, we have that

$$A' \cap B \subset \underbrace{A \cap B}_{\emptyset} \implies A' \cap B = \emptyset$$

I.e., we have that B does not contain any limit points of A. Now the same argument can be made to show that A does not contain any limit points of B.

Lemma: 23.2

If the sets C and D form a separation of X, and if Y is a connected subspace of X, then Y lies entirely within either C or D.

Proof Let the sets C and D form a separation of X. By definition, we have that both C and D are open in X. Thus, it follows that the sets $C \cap Y$ and $D \cap Y$ are open in Y under the subspace topology. Since C and D are disjoint, we have that $C \cap Y$ and $D \cap Y$ must be disjoint as well. Furthermore, we can see that their union is Y:

$$(C \cap Y) \cup (D \cap Y) = (C \cup D) \cap Y = X \cap Y = Y$$

Now if $C \cap Y$ and $D \cap Y$ were both nonempty, then they would constitute a separation of Y, which would result in Y to not be connected. However, this would contradict our assumption that Y is connected, meaning one of the sets $C \cap Y$ or $D \cap Y$ must be empty. Hence, Y must entirely lie in C or in D.

Theorem: 23.3

The union of a collection of connected subspaces of X that have a point in common is connected.

Proof Let $\{A_i\}_{i \in I}$ be an indexed collection of *connected* subspaces of a given topological space X that have at least one point p in common. We want to show that their union

$$A \equiv \bigcup_{i \in I} A_i$$

is also a connected subspace.

Towards a contradiction, we shall suppose that Y is not connected. This means that there exists a separation of Y; let C and D make up this separation. Because C and D are disjoint, it follows that p must be contained either in C or in D. WLOG, let us suppose that $p \in C$. Now since A_i is connected for all $i \in I$, it follows by [Lemma 23.2] that A_i must lie entirely in C or in D; because we assumed that $p \in C$, it follows that $A_i \subset C$ for all $i \in I$. Thus, we have that $A \subset C$, which results in $A \cap D = \emptyset$, meaning that D is empty. Hence, we reach a contradiction since D is non-empty.

Let $\{A_i\}_{i \in I}$ be an indexed collection of *connected* subspaces of a given topological space X that have at least one point p in common. We want to show that their union

$$A \equiv \bigcup_{i \in I} A_i$$

is also a connected subspace.

Let $V \subset A$ be nonempty (could be trivial!) and clopen in A. We want to show that $V = A$, which means that there are no non-trivial clopen subsets of A, thus proving that A is connected.

Since V is contained in the union A, we see that $V \cap A_j \neq \emptyset$ for at least one j. Furthermore, we see that $V \cap A_j$ is clopen in A_j under the subspace topology of A since V is clopen in A. Thus, we have that $V \cap A_j$ is a non-empty clopen subset of A_j.

Now because A_j is connected, it must be the case that $V \cap A_j = A_j$; otherwise, we would have that $V \cap A_j$ is a non-trivial clopen subset of A_j – contradicting the connectedness of A_j.

Thus, it follows that $p \in V$, which results in $V \cap A_i \neq \emptyset$ for all $i \in I$, as p is a point that is common to all A_i. Now since each A_i are connected, the same argument can be used to show that $V \cap A_i = A_i$ for all $i \in I$.

Therefore, we get that

$$V \cap A = V \cap \bigcup_{i \in I} A_i = \bigcup_{i \in I} (V \cap A_i) = \bigcup_{i \in I} A_i = A$$

which results in $A \subset V$. Putting everything together results in $V = A$.

Theorem: 23.4

Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$, then B is also connected.

Said differently: if B is formed by adjoining to the connected subspace A some or all of its limit points, then B is connected.

Proof Let A be connected and let $A \subset B \subset \overline{A}$. Towards a contradiction, suppose that B is not connected. This means that there exists a separation on B; let's say that C and D form this separation. By [Lemma 23.2], it follows that the set A must lit entirely in C or D; WLOG, suppose that $A \subset C$. Then we know that $\overline{A} \subset \overline{C}$; since A is both closed and open (namely, closed), it follows that $\overline{C} = C$, which tells us that B must be contained in C since $B \subset \overline{A}$.

However, since C and D are disjoint from each other, we also have that $\overline{C} \cap D = C \cap D = \emptyset$. This implies that $B \cap D = \emptyset$ since $B \subset C$. As a result, we see that $B = C \cup D = C$ – i.e., $D = \emptyset$. Thus, we reach a contradiction since C and D form a separation on B, meaning that both D cannot be empty.

Theorem: 23.5

The image of a connected space under a continuous map is connected.

Proof: Let $f : X \to Y$ be a continuous map, and let X be connected. Our goal is to show that the image space $Z = f(X)$ is connected as well.

We start by noting that because f is continuous, the map obtained by restricting its range to Z must also be continuous (c.f. [Theorem 18.2]). If we define this map to be $g : X \to Z$, then we get that g is a surjective, continuous map.

Towards a contradiction, let's assume that Z is not connected. This means that there exists a separation of Z; let us say that A and B form this separation. By definition, we see that $Z = A \cup B$, both A and B are closed and open, A and B are disjoint, and both A and B are nonempty. Thus, we get that

$$A \cap B = \emptyset \implies \emptyset = g^{-1}(\emptyset) = g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$$

which tells us that $g^{-1}(A)$ and $g^{-1}(B)$ are disjoint sets. We also get that

$$A \cup B = Z \implies X = g^{-1}(Z) = g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$$

which tells us that the union of $g^{-1}(A)$ and $g^{-1}(B)$ is all of X. Furthermore, we see that $g^{-1}(A)$ and $g^{-1}(B)$ are both open since g is continuous, and they are both nonempty since g is surjective with both A and B being nonempty themselves.

Hence, we see that $g^{-1}(A)$ and $g^{-1}(B)$ form a separation of X, which contradicts the assumption that X is connected. Thus, we have that the image space Z is connected.

Alternative Proof: Let X be connected. Suppose, towards a contradiction, that there is a non-trivial clopen subset A of $f(X)$ with the subspace topology. Notice that A might not be clopen in Y. However, we see that this does not make an difference since we can work with the codomain restricted function $g : X \to f(X)$, which is also continuous.

Thus, we have that $g^{-1}(A)$ is open, but since A is non-trivial, we have that $A \neq f(X)$ and $A \neq \emptyset$, which implies that $f^{-1}(A) \neq X$ and $f^{-1}(A) \neq \emptyset$, respectively. Hence $f^{-1}(A)$ is a non-trivial clopen set in X, which contradicts the connectedness of X.

Theorem: 23.6

A finite cartesian product of connected spaces is connected.

Although we won't prove it here, this result generalizes to show that an arbitrary product of connected spaces is connected.

Proof: It suffices to prove that the product of two connected spaces is connected as the finite product case results immediately from induction.

Let X and Y be two connected spaces. Choose a "base point" $a \times b$ in the product $X \times Y$. Notice that the "horizontal slice" $X \times b$ is homeomorphic to X, and because X is connected, we have that $X \times b$ is connected as well. Similarly, we have that the "vertical slice" $x \times Y$ is connected as well, being homeomorphic to Y.

As a result, we see that each "T-shaped" space

$$T_x = (X \times b) \cup (x \times Y)$$

is connected, being the union of two connected spaces that have the point $x \times b$ in common.

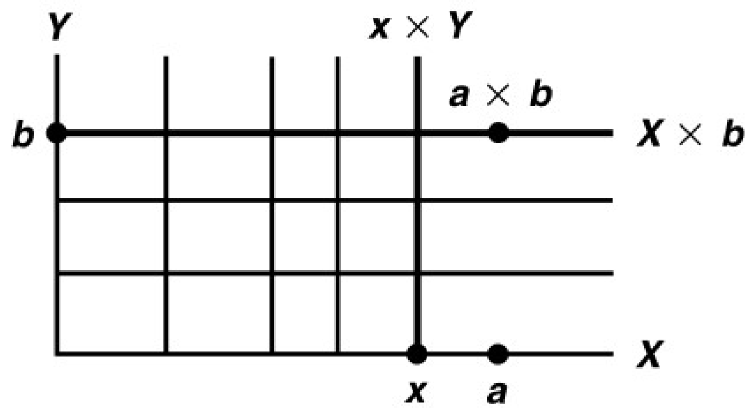

Figure 10.1: A visual of the slices and their intersections

Now form the union $\bigcup_{x \in X} T_x$ of all of these T-shaped spaces. We have that this union is connected because it is the union of a collection of connected spaces that have the point $a \times b$ in common. Furthermore, we can see that this union equals $X \times Y$, which tells us that $X \times Y$ is connected.

Theorem: A Variant of Theorem 23.4

Let X be a topological space and let $A \subset X$ be a subspace so $\overline{A} = X$. If A is connected, then X is connected.

Proof Let A be connected. Let us suppose that X is not connected. This means that there exists non-trivial clopen subsets of X; let's suppose that B is a non-trivial clopen subset of X. As a result, we have that $X \setminus B$ is also a non-trivial clopen subset of B. Since A is connected, we see that it must be entirely contained in either B or $X \setminus B$; WLOG, let's suppose that A is entirely contained in B. Since B is clopen (namely, closed) it must include all of its limit points, meaning that it must include the limit points of A; in other words, since B is closed and contains A, we have that $\overline{A} \subset B$. However, we have that $\overline{A} = X$, which is impossible since B is a non-trivial clopen subset of X. Thus, we reach a contradiction and get that X must be connected.

Another approach to the proof is to look at $B \cap A$. We see that $B \cap A$ is clopen in A under the subspace topology since B is clopen in X. We need to show that $B \cap A$ is non-trivial in A to reach a contradiction. We start by showing that $B \cap A \neq \emptyset$; since B itself is non-trivial, we can pick a point $p \in B$. B being clopen (namely, open) tells us that B is an open neighborhood of p. However, we have that $p \in \text{Cl } A = X$, which tells us that B must intersect A non-trivially – that is, we have $B \cap A \neq \emptyset$.

We now show that $B \cap A \neq A$. To do so, let's pick any point $q \in X \setminus B$. Since B is clopen (namely, closed), we see that $X \setminus B$ must then be open. Thus $X \setminus B$ is an open neighborhood of q. However, we also have that $q \in \text{Cl } A$, which tells us that $(X \setminus A) \cap A \neq \emptyset$ – that is, there is something in A that is not in B. Thus, we have that $B \cap A \neq A$.

Putting everything together gives us that $B \cap A$ is a non-trivial clopen set of A, which contradicts A being connected.

10.4 Examples

Example: Trivial Topology

Let X be a topological space under the trivial topology. We have that X is connected. In fact, we have that X is path-connected.

Example: Discrete Topology

Let X be a topological space under the discrete topology (with X having at least two elements). We have that X is disconnected (i.e., not connected) since every subset of X is open; thus, it follows that every set is closed since its complement is equal to some open set in X.

Example: Non-Connected Spaces

Let $X = [0,1] \cup [2,3]$ be a subset of $\mathbb{R}$ with the subspace topology. Under this topology, we can see that both $[0,1]$ and $[2,3]$ are clopen in X. This tells us that X is not connected.

Example: Closed intervals in $\mathbb{R}$ are Connected

Every closed interval in $\mathbb{R}$ is connected. We shall not prove this very rigorously, but will provide the main reasoning behind the proof.

Take $X = [0,3]$. We want to show that a non-empty clopen subset A must be $[0,3]$. Let's say that A contains 1. Since A is clopen (hence, open), we see that there must be some open interval, say I, that contains 1 and is contained in A. Now, since A is clopen (hence, closed), we know that I must include its limits points (since it must equal its closure), which means that its endpoints are included in I.

From here, we can apply a similar argument to the endpoints of I; if we focus our attention to the right endpoint, then we see that since A is clopen (hence, open), there must be some open interval, say I', that contains the endpoint and is contained in A. Now, since A is clopen (hence, closed), we know that I' must include its limits points (since it must equal its closure), which means that its endpoints are included in I' as well.

Continuing this process for both endpoints results in us eventually getting that $A = [0,3]$ – thus, showing that every closed interval in $\mathbb{R}$ is connected.

Example: Connectedness in the Cofinite Topology

Let X be a set equipped with the cofinite topology. We claim that X is connected if X is infinite or if X has ≤ 1 element. Furthermore, we claim that X is **disconnected** (i.e., not connected) otherwise.

Starting with the first claim, let us suppose that X is infinite. We want to show that X has no non-trivial clopen subsets. Towards a contradiction, let's assume that there exists a non-trivial clopen subset A of X. Since A is open in X under the cofinite topology, it follows that $X \setminus A$ is finite (it cannot be all of X since A is non-trivial). Also, since we know that A is also closed in X under the cofinite topology, it follows that $X \setminus A$ must be open in X. However, $X \setminus A$ being open in X means that $X \setminus (X \setminus A) = A$ must be finite (it cannot be all of X since A is non-trivial). This results in a contradiction since X is assumed to be infinite, meaning that both $X \setminus A$ and A cannot be finite. Thus, X has no non-trivial clopen subsets, meaning that X is connected.

Let us now consider the case when X has ≤ 1 element. If X has no elements (that is, $X = \emptyset$), then we are done since $\emptyset$ is trivially connected. Now if X has 1 element, then we see that the only clopen sets of X under the cofinite topology are the trivial clopen sets. Thus, we have that X is connected as well.

We now want to show that X is disconnected otherwise. Suppose that X is a finite set that has > 1 elements. Because X is finite, we end up getting that the discrete topology and cofinite topology are equivalent to each other. Since we know that the discrete topology is disconnected, we have that X is disconnected as well.

Example: Connectedness of Boundary and Interior

Suppose that A is a connected subspace of X. Does it follows that $\operatorname{Int} A$ and ∂A are connected as well? Does the converse hold?

Let $X = \mathbb{R}$ under the standard topology; let $A = [-1,0] \cup [0,1]$. We can see that A is connected as it is the union of two connected sets that share a point in common. However, we see that $\operatorname{Int} A = (-1,0) \cup (0,1)$, which is not connected. Now, let $A = (0,1)$. Be can see that $\partial A = \{0,1\}$, which is also not connected since we see that both $\{0\}$ and $\{1\}$ are nontrivial clopen subsets of A. Hence, we end up seeing that $\operatorname{Int} A$ and ∂A need not be connected if A is connected.

As for the converse, let us assume that $\operatorname{Int} A$ and ∂A are connected. Towards a contradiction, let us assume that A is connected. It follows that $\operatorname{Cl} A$ must be connected as well. If we let $A = \mathbb{Q}$, then we see that $\operatorname{Int} \mathbb{Q} = \emptyset$; indeed, there is no way that we can have an open neighborhood of any rational number such that this neighborhood is fully contained in $\mathbb{Q}$ – we can always another real number within this neighborhood . Thus, it follows that

$$\partial \mathbb{Q} = \operatorname{Cl} \mathbb{Q} \setminus \operatorname{Int} \mathbb{Q} = \mathbb{R} \setminus \emptyset = \mathbb{R}$$

Notice that both $\emptyset$ and $\mathbb{R}$ are connected, which tells us that Int $\mathbb{Q}$ and $\partial\mathbb{Q}$ are both connected. However, we know that $\mathbb{Q}$ is not connected since $\mathbb{Q} \cap (-\infty, a)$ and $\mathbb{Q} \cap (a, \infty)$ form a separation of $\mathbb{Q}$. Notice that $\mathrm{Cl}\, A = \mathrm{Int}\, A \cup \partial A$.

10.5 Discussion

Remark: On Connectedness

We can see that connectedness is a topological property since it is formulated entirely in terms of the collection of open sets of X. Said differently, we have that if X is connected, then so is any space homeomorphic to X.

Remark: More Examples

More examples can be found in §23 of Munkres.

Chapter 11

Connected Subspaces of the Real Line

11.1 Learning Objectives

- What is a path?
- What does it mean for a space to be path-connected?
- Is a path-connected space always connected? What about the converse? [Theorem 24.5]
- Is the image of a path-connected space always path-connected? [Lemma 24.6]
- If a space X is either connected or path-connected, then can be say that a homeomorphic space Y is either connected, or path-connected as well? [Theorem 24.7]

11.2 Definitions

Definition: Linear Continuum

A simply ordered set L having more than one element is called a **linear continuum** if the following hold:

1. L has the least upper bound property
2. It $x < y$, there exists z such that $x < z < y$.

Definition: Path and Path-Connected

Given points x and y of the space X, a **path** in X from x to y is a continuous map $\gamma : [a,b] \to X$ of some closed interval $[a,b] \subset \mathbb{R}$ with the subspace topology into X, such that $\gamma(a) = x$ and $\gamma(b) = y$.

A space X is said to be **path connected** if every pair of points of X can be joined by a path in X.

11.3 Theorems

Theorem: 24.1

If L is a linear continuum in the order topology, then L is connected, and so are the intervals and rays in L.

Proof

Corollary: 24.2

The real line $\mathbb{R}$ is connected and so are intervals and rays in $\mathbb{R}$.

Proof

Theorem: 24.3 (Intermediate Value Theorem)

Let $f : X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and it r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$

Note that the intermediate value theorem of calculus is the special case of this theorem that occurs when we take X to be a closed interval in $\mathbb{R}$ and Y to be $\mathbb{R}$.

Proof

Theorem: Path Connected Implies Connected

A path-connected space X is connected.

Proof Towards a contradiction, let's assume that X is not connected. This means that there exists a separation of X. Let A and B form this separation. Now, let $f : [a, b] \to X$ be any path in X. We know from [Theorem 23.5] that the image of a connected space under a continuous map is connected – that is, we have that $f([a, b])$ is connected. By [Lemma 23.2], we get that $f([a, b])$ must be entirely contained in either A or B. Thus, we have that there is no path in X joining a point of A to a point of B, which is a contradiction to our assumption that X is path-connected.

Another way to prove this theorem is as follows: Suppose that there is a non-trivial clopen subset A of X. We shall pick $p \in A$ and $q \in X \setminus A$. Since we know that X is path-connected, we see that there exists a path $\gamma : [a, b] \to X$ such that $\gamma(a) = p$ and $\gamma(b) = q$. We can see that $\gamma^{-1}(A)$ is clopen in $[a, b]$: since γ is continuous and A is clopen (hence, open), we see that $\gamma^{-1}(A)$ is open as well; also, since γ is continuous and A is clopen (hence, closed), we see that $\gamma^{-1}(A)$ is closed as well by [Theorem 18.2]. Furthermore, we see that $\gamma^{-1}(A)$ is non-trivial since A is non-trivial.

Thus, we have that $\gamma^{-1}(A)$ is a non-trivial clopen subset of $[a,b]$. However this contradicts the connectedness of $[a,b]$ Thus, X must be connected.

Theorem: Image of Path-Connected Space is Path-Connected

Let $f : X \to Y$ be continuous. Equip $f(X) \subset Y$ with the subspace topology. If X is path-connected, then $f(X)$ is path-connected.

Proof Suppose that we have a pair of points $x, y \in f(X)$. We want to show that there exists a continuous path connecting these two points. Since, $x, y \in f(X)$, it follows that there exists some $x', y' \in X$ such that $x = f(x')$ and $y = f(y')$. Because X is path-connected, we have that there is a continuous map $\gamma : [a,b] \to X$ such that $\gamma(a) = x'$ and $\gamma(b) = y'$.

It is here that we notice that $f \circ \gamma$ is continuous (since the composition of two continuous functions is still continuous) such that

$$(f \circ \gamma)(a) = x \quad \text{and} \quad (f \circ \gamma)(b) = y$$

which tells us that $f \circ \gamma$ is a continuous path on $f(X)$ connecting the two points x and y.

Corollary:

Connectedness and path-connectedness are [hoemomorphism invariants].

Proof Let X be homeomorphic to Y and X be connected. We want to show that Y must be connected as well. Since X and Y are homeomorphic, there exists a homeomorphism $f : X \to Y$. Since f is bijective (namely, surjective), we see that $f(X) = Y$. Since X is connected and f is continuous, we see that $f(X) = Y$ must be connected as well.

The argument for path-connectedness can be made by replacing the word "connected" above with "path-connected" instead.

11.4 Examples

Example: $\mathbb{R}^n$ is path-connected

Let $\mathbb{R}^n$ be equipped with the standard topology. We want to show that it is path-connected. To do so, we start by letting $P, Q \in \mathbb{R}^n$ be given. The path $\gamma : [0,1] \to \mathbb{R}^n$ given by

$$\gamma(t) = (1-t)P + tQ$$

We can see that $\gamma(0) = P$ and $\gamma(1) = Q$, and also see that γ is continuous since addition and scalar multiplication of points in $\mathbb{R}^n$ is continuous.

Example: Topologist's Sine Curve

Consider the **topologist's sine curve** $\mathcal{S}$ sketched below. It is the union:

$$\mathcal{S} = I \cup J$$

Where

$$I = \{(0, y) \mid -1 \leq y \leq 1\} \quad \text{and} \quad J = \left\{(x, y) \mid y = \sin\left(\frac{1}{x}\right) \text{ and } x > 0\right\}$$

We can see that J is connected since it is the image of the connected set $(0, 1]$ under a continuous map. Thus, we see that the closure of J in $\mathbb{R}^2$ is also connected, and we can see that $\text{Cl}\, J = \mathcal{S}$ – i.e., we have that $\mathcal{S}$ is connected.

However, we see that $\mathcal{S}$ is not path-connected. The idea is that I and J represent two path-connected sets, but there does not exist a continuous path from any point in J to any point in I. For J, there is only one path that any path will have to follow – this being on the sine curve itself. Every path should have a finite length, but if we try to traverse a path on J that goes to I, then every path must have infinite length. Thus, we see that there are no paths that connect points in J to points in I. Thus, $\mathcal{S}$ is not path-connected.

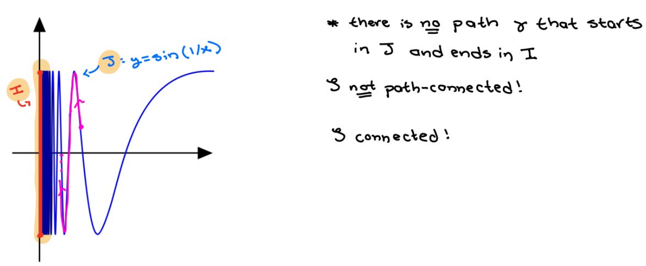

Figure 11.1: A visual of the Topologist's Sine Curve

11.5 Discussion

Remark: Path-Connectedness and Connectedness

Connectedness and path-connectedness are **not** generally the same notion.

Remark: More Examples

The proofs for the first three theorems can be found in §24 of Munkres. Furthermore, more examples can be found in §24 as well.

Chapter 12

Components and Local Connectedness

12.1 Learning Objectives

- What are connected components?
- What are path-components?
- How do connected components divide up a space? [Theorem 25.1]
  - Can we say the same thing for path-components? [Theorem 25.2]
- What is the equivalent definition of locally connected spaces? [Theorem 25.3]
- What is the equivalent definition of locally path-connected spaces? [Theorem 25.4]
- Do all path components lie in a component? When do path components coincide with components? [Theorem 25.5]

12.2 Definitions

Definition: Components

Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence classes are called the **components** (or the "connected components") of X.

Equivalent definition: let X be a topological space and p be a point in X. The **connected component** $C(p)$ of X containing p is the union of all connected subspaces of X that contain p.

Definition: Path Components

We define another equivalence relation on the space X by defining $x \sim y$ if there is a path in X from x to y. The equivalence classes are called **path components** of X.

Equivalent definition: let X be a topological space and p be a point in X. The **path components** of X containing p is the union of all path-connected subspaces of X that contain p.

Definition: Locally (Path) Connected

A space X is said to be **locally connected at** $x \in X$ if for every open neighborhood U of x, there is a connected open neighborhood V of x contained in U. If X is locally connected at each of its points, X is said to be **locally connected**.

Similarly, a space X is said to be **locally path connected at** $x \in X$ if for every open neighborhood U of x, there is a path-connected open neighborhood V of x contained in U. If X is locally path-connected at each of its points, X is said to be **locally path connected**.

12.3 Theorems

Theorem: 25.1

The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.

Proof Since we know that the components of X are equivalence classes, it follows that these components are disjoint and their union is X by construction. Furthermore, we see that each connected subspace A of X intersects with only one of these components: if A intersects the components C_1 and C_2 of X, say in points x_1 and x_2, respectively, then it follows that $x_1 \sim x_2$ by definition as they are both contained in the connected subspace A; by the transitivity property of our equivalence relation, we see that all the points in C_1 are equivalent to C_2, which means that $C_1 = C_2$.

The proof for why components are connected is as follows: for each point q of $C(p)$, we know that $p \sim q$. Thus, there is a connected subspace A_q containing both p and q. By the result that we just proved, it follows that $A_q \subset C(p)$. Thus, it follows that

$$\bigcup_{q \in C(p)} A_q \subset C(p)$$

However, we know that for any point $x \in C(p)$, it must be the case that there exists at least one $q \in C(p)$ such that $x \in A_q$; thus, we have that

$$C(p) \subset \bigcup_{q \in C(p)} A_q$$

Hence, we see that

$$C(p) = \bigcup_{q \in C(p)} A_q$$

and since the subspaces A_q are connected and have the point p in common, we get that their union (i.e. $C(p)$) is connected.

Theorem: 25.2

The path components of X are path-connected disjoint subspaces of X whose union is X, such that each nonempty path-connected subspace of X intersects only one of them.

Proof The proof for this theorem is almost the exact same as the proof given in [Theorem 25.1] – just replace "component" with "path-component" and "connected" with "path-connected".

Theorem: 25.3

A space X is locally connected if and only if for every open set U of X, each component of U is open in X.

Proof Let X be locally connected and $U \subset X$ be open. Let $p \in U$ be given. Our goal is to show that $C(p)$ is open in X, where $C(p)$ is the connected component of U containing p. This amounts to showing that for all $q \in C(p)$, we have some open neighborhood of q that is contained in $C(p)$.

Let $q \in C(p)$ be given. Because X is locally connected, we see that there exists a connected open neighborhood V of q such that $V \subset U$. By [Theorem 25.1], we see that V being connected and containing q means that V is entirely contained in $C(p)$; that is, V is in the union of all connected subspaces of U that contain p. Thus, $V \subset C(p)$. Notice that another argument we can make is by looking at $C(p) \cup V$; this is a union of two connected spaces – both of which contains q. Thus, by [Theorem 23.3], we see that $C(p) \cup V$ is also a connected subspace of U containing p. By maximality, we see that $V \subset C(p)$. Either way, we see that $C(p)$ is open.

Conversely, let's suppose that components of open sets in X are open. Given a point x of X and a neighborhood U of x, we let C to be the component of U containing x. Since every component is connected, we have that C is a connected open neighborhood of x that is contained in U. Hence, X is locally connected.

Theorem: 25.4

A space X is locally path-connected if and only if for every open set U of X, each path component of U is open in X.

Proof Let X be locally path-connected and $U \subset X$ be open. Let $p \in U$ be given. For the sake of this proof, we shall denote $C(p)$ to be the path-component of U containing p. Our goal is to show that $C(p)$ is open in X. This amounts to showing that for all $q \in C(p)$, we have some open neighborhood of q that is contained in $C(p)$.

Let $q \in C(p)$ be given. Because X is locally path-connected, we see that there exists a path-connected open neighborhood V of q such that $V \subset U$. By [Theorem 25.2], we see that V being path-connected and containing q means that V is entirely contained in $C(p)$; that is, V is in the union of all connected subspaces of U that contain p. Thus, $V \subset C(p)$.

Conversely, let's suppose that path-components of open sets in X are open. Given a point x of X and a neighborhood U of x, we let C to be the path-component of U containing x. Since every path-component is path-connected, we have that C is a connected open neighborhood of x that is contained in U. Hence, X is locally connected.

Theorem: 25.5

If X is a topological space, each path component of X lies in a component of X. If X is locally path-connected, then the components and the path-components of X are the same.

Proof Let C be a component of X; let x be a point of C; let P be the path component of X containing x. Since all path components are path-connected, and path connected sets are connected, it follows that P must be connected; thus, it follows that P is in the union of connected components of X that contain x – that is, $P \subset C$. Our goal is to show that if X is locally path-connected, then $P = C$.

Towards a contradiction, let's suppose that $P \subsetneq C$ (i.e., P is a proper subset of C). This means that there exists path components of X that are different from P and intersect C; let Q denote the union of all such path components. Using a similar argument that we did with P, we see that each of the path components that make up Q all are contained in C. Thus, we see that

$$C = P \cup Q$$

Now, because X is locally path-connected, each path component of X is open in X. Thus, we see that both P (which is a path-component) and Q (which is a union of path-components) are open in X. As a result, we see that P and Q constitute a separation of C, which contradicts the fact that C is connected.

Theorem: Connected Components

(a) Connected components are connected.

(b) Every connected component of X is closed in X.

(c) Any pair of connected components must either be equal **or** disjoint.

Proof

a

This is easily proved using [Theorem 23.3]; since a connected component $C(p)$ of X containing p is the union of all connected subspaces of X that contain p, we have that all of the connected subspaces in this union to contain the common point p. Thus, by [Theorem 23.3], we have that their union must also be connected.

b

In general, we know that any set is contained in its closure; that is,

$$C(p) \subset \overline{C(p)}$$

Now, because $C(p)$ is connected, we have that its closure $\overline{C(p)}$ must be connected as well by [Theorem 23.4].

Notice that $\overline{C(p)}$ is connected and contains the point p; thus, it follows by definition that

$$\overline{C(p)} \subset C(p)$$

Hence, we get that $C(p) = \overline{C(p)}$, which results in $C(p)$ to be closed.

c

It suffices to prove that if two connected components share a point in common, then the two connected components must be equal (else, they have no other choice but to be disjoint from each other).

Let's say that we have two connected components $C(p)$ and $C(q)$ that share a common point, say r. It follows by [Theorem 23.3] that their union must be connected as well. Since $C(q) \cup C(p)$ contains both p and q, it follows by the definition of a connected component that

$$C(p) \cup C(q) \subset C(p) \quad \text{and} \quad C(p) \cup C(q) \subset C(q)$$

This results in the following

$$C(p) \cup C(q) \subset C(p) \cap C(q) \implies C(p) = C(q)$$

Hence, we find that two connected components that share a common point must be equal.

Theorem:

The number of connected components is a homeomorphism invariant.

Proof Let X be a topological space. We have shown in [Theorem 25.1] that the set of all components of X is equivalent to the set of all equivalence classes under the following equivalence relation: $x \sim y$ if there is a connected subspace of X containing both x and y. Thus, we have that the number of connected components is equivalent to the number of equivalence classes of X.

Now, let X be homeomorphic to Y. This means that there exists a homeomorphism $f : X \to Y$. Furthermore, we see that there exists a reduced homeomorphism $g : X^* \to Y^*$, where X^* and Y^* correspond to the set of equivalence classes of X and Y under the equivalence relation mentioned earlier; we indeed get a homeomorphism since restricting the domain/codomain of a continuous function still remains continuous.

Now since g is a homeomorphism, it follows that it is bijective. Thus, we get that X^* and Y^* must have the same cardinality: injective implies that $|X^*| \leq |Y^*|$, and surjective implies that $|X^*| \geq |Y^*|$. I.e., we see that the number of equivalence classes (equivalently, the number of connected components) for both spaces must be the same.

12.4 Examples

Example: Verifying the Equivalence Relation for Components

Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y.

We can see that reflexivity ($x \sim x$) and symmetry (if $x \sim y$, then $y \sim x$) are both obvious; reflexivity follows by knowing that singleton sets are connected, and symmetry follows from our definition being symmetric.

Now to verify the transitivity condition, let us start by assuming that $x \sim y$ and $y \sim z$; this means that x and y are contained in a connected subspace A of X, and y and z are contained in a connected subspace B of X. Since we know that A and B have a point in common (this point being y), it follows from [Theorem 23.3] that $A \cup B$ is a subspace of X that is also connected. Now because $A \cup B$ contains x, y, and z, we see that $x \sim z$, which verifies this condition.

Example: Verifying the Equivalence Relation for Path Components

Given X, define an equivalence relation on X by setting $x \sim y$ if there is a path in X from x to y.

To show that this is an equivalence relation, we first not that if there exists a path $f : [a,b] \to X$ from x to y whose domain is the interval $[a,b]$, then there is also a path g from x to y having the closed interval $[c,d]$ as its domain; this follows from the fact that any two closed intervals in $\mathbb{R}$ are homeomorphic. I.e., we are free to define a path using any closed interval as our domain.

Now, we have that reflexivity ($x \sim x$) holds for all $x \in X$ due to the existence of a constant path $f : [a,b] \to X$ defined by the equation $f(t) = x$ for all t.

Symmetry (if $x \sim y$, then $y \sim x$) follows from the fact that if $f : [0,1] \to X$ is a path from x to y, then the "reverse path" $g : [0,1] \to X$ defined by $g(t) = f(1-t)$ is a path from y to x.

Finally, transitivity is proved as follows: let $f : [0,1] \to X$ be a path from x to y, and let $g : [1,2] \to X$ be a path from y to z. We can "paste f and g together" to get a path $h : [0,2] \to X$ from x to z; the path h will be continuous by the "pasting lemma", that is by [Theorem 18.3].

Example: T vs. t

Show that a lower-case t is not homeomorphic to an upper-case T. Think of each as the union of two closed line segments (no thickness) in $\mathbb{R}^2$ with the subspace topology.

Towards a contradiction, assume that there exists a homeomorphism $f : t \to T$.

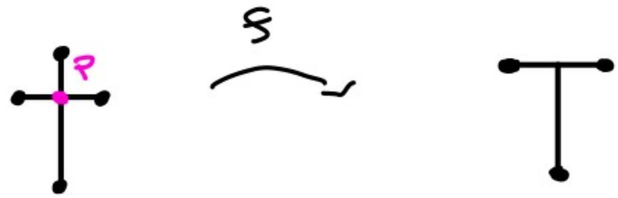

Figure 12.1: The two subspaces

There is a restricted homeomorphism that we can consider: $g : t \setminus \{p\} \to T \setminus \{f(p)\}$. Notice that g is still a homeomorphism since domain/codomain restrictions of continuous functions are still continuous.

We see that t has four connected components with $\{p\}$ removed. However, for T, we see that we can have at most three connected components. Thus, we know that t and T cannot be homeomorphic since they should have the same number of connected components otherwise.

Example: Connected Components are Not Always Open

Consider the topologist's sine curve $\mathcal{S}$, or really, consider the subspace $\hat{\mathcal{S}}$ obtained by cropping $\mathcal{S}$ as shown in Figure (12.2).

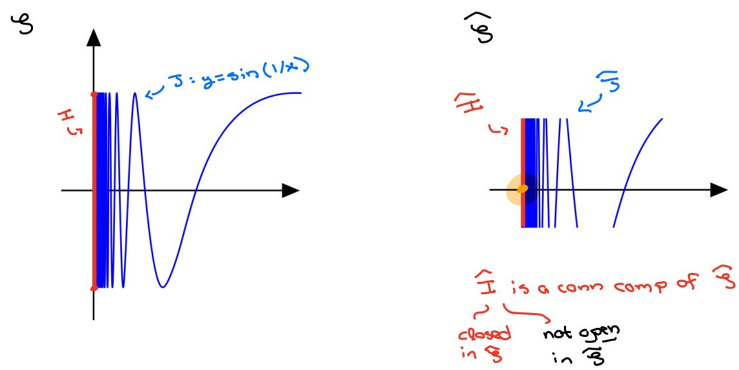

Figure 12.2: The two subspaces in question

We can see that this cropping results in $\hat{\mathcal{S}}$ to be composed up of connected components; in particular, we see that $\hat{I}$ is a connected component of $\hat{\mathcal{S}}$. We know that $\hat{I}$ is closed in $\hat{\mathcal{S}}$, but, $\hat{I}$ is not open in $\hat{\mathcal{S}}$ since any neighborhood of any point in $\hat{I}$ is not entirely contained in $\hat{I}$.

Example: Locally Connected

Topologist's sine curve is not locally connected; using the same argument as in the [previous example]. However, we see that $\mathbb{R}$ is locally connected.

Figure 12.3: The two subspaces in question

Example: Connected Components of $\mathbb{R}_\ell$

Let $\mathbb{R}_\ell$ be $\mathbb{R}$ with the lower-limit topology. Find its connected components.

Let $p \in \mathbb{R}_\ell$ be given. Let $C(p)$ be the connected component containing p. Recall that the connected component is the union of all connected subspaces of $\mathbb{R}_\ell$ containing p itself. We claim that $C(p) = \{p\}$.

For any $a < p$, we see that $[a, p) \cap C(p)$ is itself clopen in $C(p)$ under the subspace topology since we know that $[a, p)$ is clopen in $\mathbb{R}_\ell$. However, $[a, p) \cap C(p)$ cannot equal to $C(p)$ since $[a, p)$ does not contain p. Because $C(p)$ is connected, we see that the intersection must be $\emptyset$, so we see that $a \notin C(p)$. I.e., if we have that $a < p$, then a is not in $C(p)$.

Now if $a > p$, then we can consider the interval $[a, a+1)$. We see that $[a, a+1) \cap C(p)$ is clopen in $C(p)$ under the subspace topology, but again not equal to $C(p)$ since $[a, a+1)$ does not contain p. Thus, we have that the intersection must be $\emptyset$. As a results we also find that $a \notin C(p)$. I.e., if we have that $a > p$, then a is not in $C(p)$.

From this, we see that $C(p)$ must be $\{p\}$, which tells us that $\mathbb{R}_\ell$ is very disconnected.

12.5 Discussion

Remark: More Examples

More examples can be found on §25 of Munkres.

Chapter 13

Compact Spaces

13.1 Learning Objectives

- What is an open covering and how does it relate to the definition of compactness?
- What is a tubular neighborhood?
- What is the finite intersection property?
- How does compactness behave with subsets? [Lemma 26.1]
- Are all subspaces of a compact set compact? If not, then what condition can we put on these subspaces so that they always are? [Theorem 26.2]
- Are all compact subspaces closed? If not, then what condition can we put on the overall space so that they always are? [Theorem 26.3]
- Why can we always find disjoint neighborhoods of compact subspaces in a Hausdorff space? Why do we need these subspaces to be compact? [Theorem 26.4]
- Is the image of a compact space always compact? [Theorem 26.5]
- If the domain of a function is compact and the codomain is Hausdorff, then is it required to show that the function's inverse is continuous when showing whether or not the function is a homeomorphism. [Theorem 26.6]
- Is the product of finitely many compact spaces compact? How about for arbitrary products? [Theorem 26.7]
- What is the Tube Lemma? [Lemma 26.8]

13.2 Definitions

Definition: Open Covering

A collection $\mathcal{A}$ of subsets of a space X is said to **cover** X, or be a **covering** of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an **open covering** of X if its elements are open subsets of X.

For subspaces, we have the following: if Y is a subspace of X, a collection $\mathcal{A}$ of subsets of X is said to **cover** Y if the union of its elements *contains* Y.

Definition: Compact Spaces

A space X is said to be compact if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Definition: Tubular Neighborhoods

Let $X \times Y$ be a product of two topological spaces. Then a **tubular neighborhood** of a **slice** $X \times \{y\}$ is a set of the form $X \times V$, where V is an open neighborhood of y.

Definition: Finite Intersection Property

A collection $\mathcal{C}$ of subsets of X is said to have the **finite intersection property** if for every finite subcollection

$$\{C_1, \ldots, C_n\}$$

of $\mathcal{C}$, the intersection $C_1 \cap \ldots \cap C_n$ is nonempty.

13.3 Theorems

Lemma: 26.1

Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.

Proof Suppose that Y is compact and $\mathcal{A} = \{A_i\}_{i \in I}$ is an open covering of Y by sets open in X. We want to show that there exists a finite subcollection in $\mathcal{A}$ that covers Y.

We start off by using the fact that $\mathcal{A}$ is an open covering of Y by sets open in X; by definition of what a covering is for a subspace, we have that

$$Y \subset \bigcup_{i \in I} A_i$$

Which is equivalent to saying that

$$Y = Y \cap \left(\bigcup_{i \in I} A_i \right) = \bigcup_{i \in I} (A_i \cap Y)$$

Thus, we see that the collection $\mathcal{A}' = \{A_i \cap Y \mid i \in I\}$ is an open covering of Y by sets open in Y.

Since Y is compact, we see that it has a finite subcollection in $\mathcal{A}'$, say

$$\{A_{i_1} \cap Y, \ldots, A_{i_n} \cap Y\}$$

that covers Y; that is,

$$\bigcup_{j=1}^{n}(A_{i_j} \cap Y) = Y$$

Notice that

$$Y = \bigcup_{j=1}^{n}(A_{i_j} \cap Y) = Y \cap \left(\bigcup_{j=1}^{n} A_{i_j}\right)$$

which tells us that

$$Y \subset \bigcup_{j=1}^{n} A_{i_j}$$

Thus, we end up getting that $\{A_{i_1}, \ldots, A_{i_n}\}$ is a finite subcollection of $\mathcal{A}$ that covers Y.

Conversely, suppose that every covering of Y by sets open in X contains a finite subcollection covering Y; we wish to prove that Y is compact.

Let $\mathcal{A}' = \{A'_i\}_{i \in I}$ be any open covering of Y by sets open in Y. Our goal is to show that there exists a finite subcollection of $\mathcal{A}'$ that covers Y. For each i, we can choose a set A_i open in X such that

$$A'_i = A_i \cap Y$$

by definition of A'_i being open in Y under the subspace topology. We can see that the collection $\mathcal{A} = \{A_i\}_{i \in I}$ is a covering of Y by sets open in X:

$$Y = \bigcup_{i \in I} A'_i = \bigcup_{i \in I}(A_i \cap Y) = Y \cap \bigcup_{i \in I} A_i \implies Y \subset \bigcup_{i \in I} A_i$$

By our hypothesis, we have that there exists some finite subcollection of $\mathcal{A}$, say $\{A_{i_1}, \ldots, A_{i_n}\}$, such that it covers Y. Thus, we have by definition of a covering for a subspace that

$$Y \subset \bigcup_{j=1}^{n} A_{i_j}$$

Notice that

$$Y = Y \cap \left(\bigcup_{j=1}^{n} A_{i_j}\right) = \bigcup_{j=1}^{n}(Y \cap A_{i_j}) = \bigcup_{j=1}^{n} A'_{i_j}$$

From this, we get that $\{A'_{i_1}, \ldots, A'_{i_n}\}$ is a subcollection of $\mathcal{A}'$ that covers Y.

Theorem: Variation of [Lemma 26.1]

Let X be a topological space and Y a subspace of X. Let A be a subset such that $A \subset Y \subset X$. A is compact in X if and only if A is compact in Y.

Proof Let's say that A is compact in X. By [Lemma 26.1], we have that every covering of A by sets open in X contains a finite subcollection covering A. We want to show that A is compact in Y, meaning that for every covering of A by sets open in Y, there exists a finite subcollection covering A.

Let $\mathcal{U}$ be an open covering of A by sets open in Y. By definition of what a covering is for a subspace, we see that

$$A \subset \bigcup_{U \in \mathcal{U}} U$$

By definition of the subspace topology, we see that for each $U \in \mathcal{U}$, there exists some open set V of X such that $U = V \cap Y$. Let us denote the collection of all such open sets V of X as $\mathcal{V}$. We can see that

$$A \subset \bigcup_{U \in \mathcal{U}} U \iff A \subset \bigcup_{V \in \mathcal{V}} (V \cap Y) \iff A \subset \left(\bigcup_{V \in \mathcal{V}} V \right) \cap Y \implies A \subset \bigcup_{V \in \mathcal{V}} V$$

which tells us that $\mathcal{V}$ is an open covering of A by sets open in X.

Since A is compact in X, we see that there exists some finite subcollection $\{V_1, \ldots, V_n\}$ of $\mathcal{V}$ such that it covers A. Hence, it follows that

$$A \subset \bigcup_{i=1}^{n} V_i \text{ and } A \subset Y \implies A \subset \left(\bigcup_{i=1}^{n} V_i \right) \cap Y \iff A \subset \bigcup_{i=1}^{n} (V_i \cap Y) \iff A \subset \bigcup_{i=1}^{n} U_i$$

which tells us that $\{U_1, \ldots, U_n\}$ is a finite subcollection of $\mathcal{U}$ such that it openly covers A in Y. Since our choice of open cover $\mathcal{U}$ in Y was arbitrary, we have that A must be compact in Y.

Let's now say that A is compact in Y. We want to show that A is compact in X, meaning that for every covering of A by sets open in X, there exists a finite subcollection covering A.

Let $\mathcal{U}$ be an open covering of A by sets open in X. By definition of what a covering is for a subspace, we see that

$$A \subset \bigcup_{U \in \mathcal{U}} U$$

Since $A \subset Y$ as well, we see that

$$A \subset \left(\bigcup_{U \in \mathcal{U}} U \right) \cap Y \iff A \subset \bigcup_{U \in \mathcal{U}} (U \cap Y)$$

Under the subspace topology, we have that $U \cap Y$ is open in Y for each $U \in \mathcal{U}$. Thus, we have that $\{U \cap Y \mid U \in \mathcal{U}\}$ is an open covering of A by sets open in Y.

Since A is compact in Y, we see that there exists a finite subcollection $\{U_1 \cap Y, \ldots, U_n \cap Y\}$ such that it covers A. Hence it follows that

$$A \subset \bigcup_{i=1}^{n} (U_i \cap Y) \iff A \subset \left(\bigcup_{i=1}^{n} U_i \right) \cap Y \implies A \subset \bigcup_{i=1}^{n} U_i$$

which tells us that $\{U_1, \ldots, U_n\}$ is a finite subcollection of $\mathcal{U}$ such that it covers A. Since our choice of open cover $\mathcal{U}$ in X was arbitrary, we have that A must be compact in X.

Theorem: 26.2

Every closed subspace of a compact space is compact.

Proof Let X be compact and $A \subset X$ be closed. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of A. We want to show that $\mathcal{U}$ admits a finite subcover of A.

To do so, we note that $\mathcal{U} \cup \{X \setminus A\}$ is an open cover of X. Since X is compact, it follows that such an open cover admits a finite subcover of X. If this subcover contains the set $X \setminus A$, then we can discard $X \setminus A$; otherwise, we can leave the subcover alone. The resulting subcover is a finite subcover of $\mathcal{U}$ that covers A.

Theorem: 26.3

Every compact subspace of a Hausdorff space is closed.

Proof Let Y be a compact subspace of the Hausdorff space X. We shall prove that $X \setminus Y$ is open, so that Y is closed.

Let x_0 be a point of $X \setminus Y$. We show that there is a neighborhood of x_0 that is disjoint from Y. For each point $y \in Y$, we are always able to choose disjoint neighborhoods U_y and V_y of the points x_0 and y, respectively (using the Hausdorff condition). We can see by construction that the collection $\{V_y \mid y \in Y\}$ is a covering of Y by the sets open in X; since Y is compact, we see that there are finitely many of them $V_{y_1}, \dots, V_{y_n}$ that cover Y.

Let us now consider the following two sets:

$$V = V_{y_1} \cup \dots \cup V_{y_n} \quad \text{and} \quad U = U_{y_1} \cap \dots \cap U_{y_n}$$

We can see clearly that both U and V are open; namely, we see that compactness of Y is what allowed for us to obtain a *finite* intersection of the corresponding neighborhoods U_{y_j} of x_0. We can see as well that V contains Y, and it is disjoint from the open set U; for if z is a point of V, then $z \in V_{y_i}$ for some i. Hence $z \notin U_{y_i}$, and so $z \notin U$.

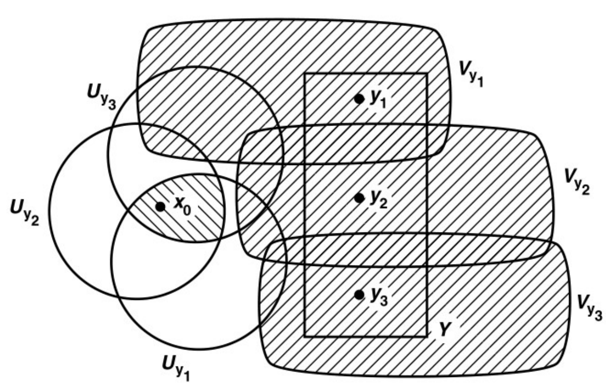

Figure 13.1: A visual on the disjointness of U

As a result, we see that U is a neighborhood of x_0 disjoint from Y, as desired. Thus, we see that Y is closed in X.

Lemma: 26.4

If Y is a compact subspace of the Hausdorff space X and x_0 is not in Y, then there exist disjoint open sets U and V of X containing x_0 and Y, respectively.

Proof This result is proved while proving [Theorem 26.3].

Theorem: Extending Lemma 26.4

Let X be Hausdorff. Let K and L be disjoint compact subsets of X. Then there exists disjoint open neighborhoods U of K and V of L.

Proof For all $x \in K$, we have that there exists disjoint open neighborhoods U_x of x and $V_x \subset L$ (this follows from [Lemma 26.4]). $\{U_x\}_{x\in K}$ is an open cover of K. Since K is compact, we see that it admits a finite subcover: $\{U_{x_1}, \dots, U_{x_n}\}$.

Define

$$U = \bigcup_{i=1}^{n} U_{x_i} \quad \text{and} \quad V = \bigcap_{i=1}^{n} V_{x_i}$$

We can see that U is an open neighborhood of K and V is an open neighborhood of L since any V_x contains L. We have that $U \cap V = \emptyset$, which results in K and L to be disjoint.

Theorem: 26.5

The image of a compact space under a continuous map is compact.

Proof Let $f : X \to Y$ be a continuous function and X is compact. We want to show that $f(X)$ is compact.

Let's say that we are given some open cover of $f(X)$, say $\mathcal{A} = \{A_i\}_{i\in I}$. Our goal is to show that $\mathcal{A}$ admits a finite subcover of $f(X)$.

Then we see that $\{f^{-1}(A_i)\}_{i\in I}$ are an open cover of X. Now since X is compact, we see that there exists a finite subcover of X, say $\{f^{-1}(A_{i_1}), \dots, f^{-1}(A_{i_n})\}$. From here, we see that $\{A_{i_1}, \dots, A_{i_n}\}$ is an open cover of $f(X)$.

$$f(X) \subset \bigcup_{i\in I} A_i \iff X \subset f^{-1}\left(\bigcup_{i\in I} A_i\right) \iff X \subset \bigcup_{i\in I} f^{-1}(A_i)$$

Theorem: 26.6

Let $f : X \to Y$ be a bijective continuous function. If the domain X is compact and the codomain Y is Hausdorff, then f is a homeomorphism.

Proof All we need to do is to show that f^{-1} is continuous so that f is a homeomorphism. To do so, it suffices to show that f is a closed map – that is, images of closed sets are closed. Doing so will results in f^{-1} to be continuous.

If A is closed in X, then A must be compact by [Theorem 26.2]. Therefore, we have by [Theorem 26.5] that $f(A)$ is compact in Y. Now since Y is Hausdorff, it follows that $f(A)$ is closed in Y by [Theorem 26.3], which results in f to be a closed map.

We will now go about showing that $f^{-1} : Y \to X$, which is defined since f is bijective, is continuous. We shall do so by showing that $(f^{-1})^{-1}(A)$ is closed in Y whenever A is closed in X. Now since f is bijective, we get

$$(f^{-1})^{-1}(A) = f(A)$$

is closed in Y because f is a closed map

Theorem: 26.7

The product of finitely many compact spaces is compact.

Proof It suffices to show that the result holds for a product of two compact sets; the case for the product of finitely many compact spaces follows immediately from induction.

Let X and Y be compact spaces. Let $\mathcal{A}$ be an open covering of $X \times Y$. Our goal is to show that there exists a finite subcollection of $\mathcal{A}$ such that it openly covers $X \times Y$.

Given $y \in Y$, it follows that the space $X \times \{y\}$ is homeomorphic to the space X (using the same map as in [Lemma 26.8]). Since X was compact, it follows that the slice $X \times \{y\}$ is compact, which allows for us to find a finite subcollection $A_1, \ldots, A_n$ of $\mathcal{A}$ such that it openly covers $X \times \{y\}$. As a result, we see that

$$N = A_1 \cup \ldots \cup A_n$$

is an open set containing $X \times \{y\}$.

By [Lemma 26.8], it follows that N contains a tube $X \times V$ about $X \times \{y\}$, where V is open in Y. It must then be the case that $X \times V$ is covered by finitely many elements $A_1, \ldots, A_n$ of $\mathcal{A}$. As a result, we have that for each $b \in Y$, we can choose a neighborhood V_b of b such that the tube $X \times V_b$ can be covered by finitely many elements of $\mathcal{A}$. The collection of all the neighborhoods V_b is an open covering of Y; therefore by compactness of Y, there exists a finite subcollection

$$\{V_1, \ldots, V_k\}$$

that openly covers Y. The union of the tubes

$$X \times V_1, \ldots, X \times V_k$$

is all of $X \times Y$, and since each may be covered by finitely many elements of $\mathcal{A}$, it follows that $X \times Y$ can also be covered by finitely many elements of $\mathcal{A}$.

Thus, we see that $X \times Y$ is compact.

Lemma: Homeomorphic Spaces to Slices

Let X and Y be topological spaces. WLOG, consider the slice $X \times \{y\}$. We have that $X \times \{y\}$ is homeomorphic to X.

Proof Let us consider the following map:

$$f : X \to X \times \{y\} \quad \text{where} \quad f(x) = x \times y \quad \forall x \in X$$

We can see that this map is clearly bijective. Hence, we know that there exists an inverse of f given as follows:

$$f^{-1} : X \times \{y\} \to X \quad \text{where} \quad f^{-1}(x \times y) = x \quad \forall x \in X$$

We can see that f^{-1} is clearly continuous since it is the projection onto X, which we already know is a continuous map. To show that f is continuous, let us consider any basic open set $U \times \{y\}$ in $X \times \{y\}$, where U is an open set of X. It follows that $f^{-1}(U \times \{y\}) = U$, which we know is open. Thus, we have that f is continuous.

Putting everything together results in us getting that f is indeed a homeomorphism, which further results in $X \times \{y\}$ being homeomorphic to X.

Lemma: 26.8 (The Tube Lemma)

Consider the product space $X \times Y$, where X is compact. If N is an open set of $X \times Y$ containing the slice $X \times \{y\}$ of $X \times Y$, then N contains some tube $X \times V$ about $X \times \{y\}$, where V is a neighborhood of y in Y.

In other words, there is a neighborhood V of y in Y such that N contains the entire set $X \times V$.

Proof Let X be compact and $y \in Y$ be given. Let N be an open neighborhood of $X \times \{y\}$ be given. Our goal is to show that there is a neighborhood V of y in Y such that N contains the entire set $X \times V$.

Let us first start by covering $X \times \{y\}$ by basis elements $U \times W$ (for the topology of $X \times Y$) lying in N. By the [lemma] above, we see that $X \times \{y\}$ is homeomorphic to X. Thus, we have that X being compact implies that $X \times \{y\}$ must be compact as well. Therefore, we can cover $X \times \{y\}$ by finitely many such basis elements:

$$U_1 \times W_1, \ldots, U_n \times W_n$$

This is where we note that we assume that each basis element $U_i \times W_i$ actually intersects $X \times \{y\}$, since otherwise that basis element would be superfluous; we could discard it from the finite collection and still have a covering of $X \times \{y\}$.

We now define

$$V = W_1 \cap \ldots \cap W_n$$

The set V is open, and it contains y because each set $U_i \times W_i$ intersects $X \times \{y\}$. We claim that the sets $U_i \times W_i$, which were chosen to cover the slice $X \times \{y\}$, actually cover the tube $X \times V$ –

that is, we want to show that

$$X \times V \subset \bigcup_{i=1}^{n} (U_i \times W_i)$$

To verify this claim, we want to show that for any point $a \times b \in X \times V$, we have that $a \times b \in U_i \times W_i$ for some i. Let $a \times b$ be a point of $X \times V$. Consider the point $a \times y$ of the slice $X \times \{y\}$ having the same x-coordinate as this point. Since $\{U_i \times W_i\}_{i \in I}$ is a finite open cover of $X \times \{y\}$, we have that $a \times y$ belongs to $U_i \times W_i$ for some i, so that $a \in U_i$. However, since $b \in V$, we see that $b \in W_j$ for *every* j. Thus, it follows that $a \times b \in U_i \times W_i$, which results in $\{U_i \times W_i\}_{i \in I}$ to be an open cover of $X \times V$.

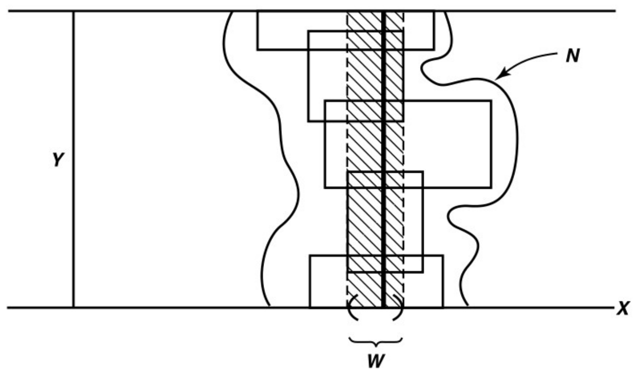

Figure 13.2: A visual of the Tube Lemma

Theorem: 26.9

Let X be a topological space. Then X is compact if and only if every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ of all the elements of $\mathcal{C}$ is nonempty.

Proof Given a collection $\mathcal{A}$ of subsets of X, let

$$\mathcal{C} = \{X \setminus A \mid A \in \mathcal{A}\}$$

be the collection of their complements. Then the following statements hold:

1. $\mathcal{A}$ is a collection of open sets if and only if $\mathcal{C}$ is a collection of closed sets.
2. The collection $\mathcal{A}$ covers X if and only if the intersection $\bigcap_{C \in \mathcal{C}} C$ of all the elements of $\mathcal{C}$ is empty.
3. The finite subcollection $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$ covers X if and only if the intersection of the corresponding elements $C_i = X \setminus A_i$ of $\mathcal{C}$ is empty.

The first statement follows immediately by definition of open and closed sets. The second and third follow from DeMorgan's law:

$$X \setminus \left(\bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} (X \setminus A_i)$$

where some collection of $\{A_i\}_{i \in I}$ covers X if $X \setminus (\bigcup_{i \in I} A_i) = \emptyset$. From here, the proof will proceed in two steps: take the contrapositive of the theorem, and then the complement of the sets.

The state that X is compact is equivalent to saying: "Given any collection $\mathcal{A}$ of open subsets of X, if $\mathcal{A}$ covers X, then some finite subcollection of $\mathcal{A}$ covers X." This statement is equivalent to the contrapositive, which is the following: "Given any collection $\mathcal{A}$ of open subsets of X, if no finite subcollection of $\mathcal{A}$ covers X, then $\mathcal{A}$ does not cover X."

Letting $\mathcal{C}$ be, as earlier, the collection $\{X \setminus A \mid A \in \mathcal{A}\}$ and applying $(1)-(3)$, we see that this statement is in turn equivalent to the following: "Given any collection $\mathcal{C}$ of closed sets of X, if every finite intersection of elements of $\mathcal{C}$ is nonempty, then the intersection of all the elements of $\mathcal{C}$ is nonempty." This is just the condition of our theorem.

Theorem: Heine-Borel

A subset of $\mathbb{R}^n$ with the standard topology is compact in $\mathbb{R}^n$ if and only if it is both closed and **bounded** (which means that there is a finite upper limit to how far apart points in that subspace can be).

Proof

13.4 Examples

Example: Not all Subsets of a Compact Space are Compact

We shall consider the open covers of the circle S and of the circle minus its north pole.

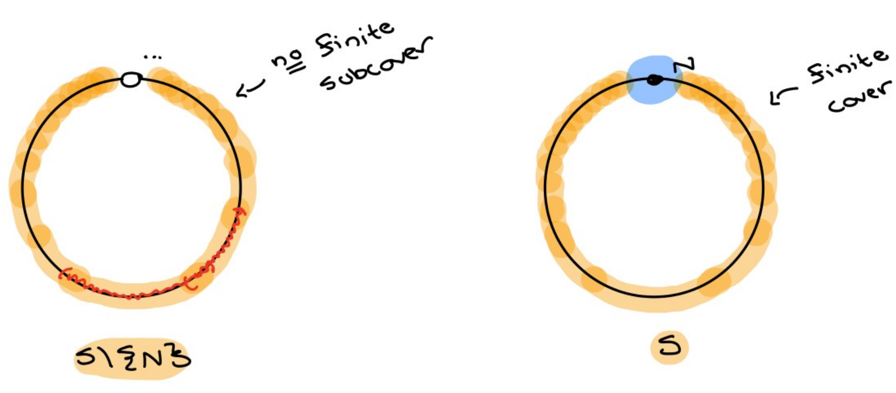

Figure 13.3: The two subspaces in question

For the circle minus its north pole, we can construct a cover by adding smaller and smaller open sets of the circle such that it eventually approaches the north pole, but never reaches it.

For the circle with the north pole, we see that we can use the same cover as above, but with another open set that contains the north pole. In fact, we see that the open set that contains the north pole contains infinitely many other open sets that are in the cover, Thus, we see that such sets are redundant, meaning that we can remove them from our cover. What results is that we get a finite cover on the circle.

However, for the circle without the north pole, we see that there cannot exist a finite subcover – the infinite open sets that lead up to the north pole were vital towards our cover for this subspace.

This example tells us that not all subsets of a compact set are compact; the condition that sets be closed was key being able to avoid a situation where there were infinite open sets that led up to a limit point (in our case, the north pole).

Example: Compact Sets in the Cofinite Topology

Let X be a set equipped with the cofinite topology. Prove that every subset of X is compact in X.

Let A be any subset of X. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be any open cover of A. Our goal is to show that there exists a finite subcollection of $\mathcal{U}$ that covers A.

Let us focus on U_i for some fixed index $i \in I$. Since X is equipped with the cofinite topology, we have that U_i being open implies that $X \setminus U_i$ is finite. I.e., we see that there are only finitely many points in X that are *not* contained in U_i. In particular, this means that there are finitely many points in A that are not contained in U_i; in the case that all points of A are in U_i, then we have that A is compact since U_i itself is a finite subcover of A. Let's say that $x_1, \ldots, x_n$ are the points in A that are *not* contained in U_i. Because $\mathcal{U}$ is an open cover of A, each of the points $x_1, \ldots, x_n$ are contained in some open set $\{U_{i_1}, \ldots, U_{i_n}\}$ in the cover.

Thus, we see that

$$\{U_i, U_{i_1}, \ldots, U_{i_n}\}$$

is a finite subcollection of $\mathcal{U}$ that covers A. Hence, A is compact in X.

Example: Tubular Neighborhoods

We give an open neighborhood of $\mathbb{R} \times \{0\}$

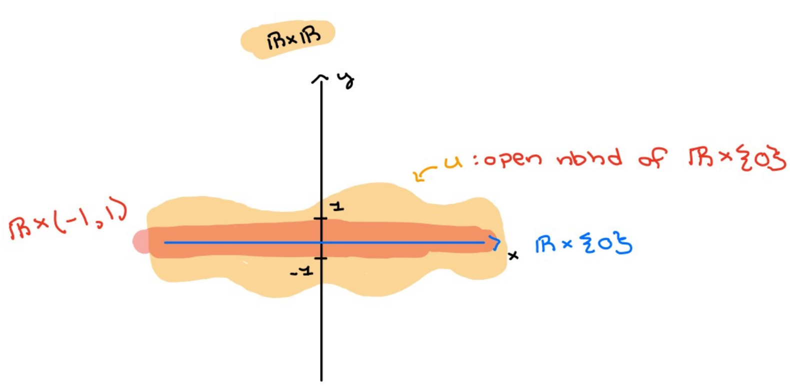

Figure 13.4: An example of a tubular neighborhood

However, not all neighborhoods of $\mathbb{R} \times \{0\}$ contain a tubular neighborhood:

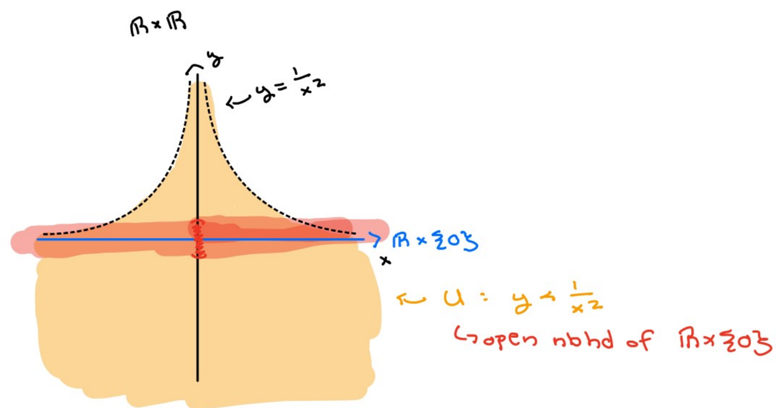

Figure 13.5: An example where no tubular neighborhoods exist

13.5 Discussion

Remark: [Lemma 26.1]

This lemma states A being compact in X means that A is also compact in the subspace topology.

Remark: Cofinite Topology

This topology is not Hausdorff, and we see that there are sets that are not closed, but compact.

Remark: [Lemma 26.4]

This lemma tells us that Hausdorffness automatically gives us a stronger condition: we can separate points from compact sets. In fact, this lemma can be extended (as we did with [Theorem 26.4]) to say that disjoint subsets of a compact set can be separated.

Remark: [Theorem 26.7]

We note that both spaces being compact was important: X being compact allowed for us to find an open tubular neighborhood around each slice $X \times \{y\}$ for all $y \in Y$ via [Lemma 26.8]. Notice that each of these tubular neighborhoods can be covered by a finite subcollection of $\mathcal{A}$. Why does compactness of Y matter then? Notice that we have an infinite amount of tubular neighborhoods, and an infinite union of finite sets may not be finite. Thus, Y being compact allows for us to see that there are only a finite amount of tubular neighborhoods needed to cover Y (really, $X \times Y$); these tubular neighborhoods are each covered by a finite subcollection of $\mathcal{A}$, and a finite union of finite sets will remain finite. Hence, we are left with a finite subcollection of $\mathcal{A}$ that also covers $X \times Y$.

We have shown in this theorem that the product of finitely many compact spaces is compact. It can be shown as well that arbitrary products of compact spaces are compact as well; however, the proof is much more difficult and has its own name – *Tychonoff's Theorem*.

Remark: [Theorem 26.9]

A special case of this theorem occurs when we have a **nested sequence**

$$C_1 \supset C_2 \supset \ldots \supset C_n \supset C_{n+1} \supset \ldots$$

of closed sets in a compact space X. If each of the sets C_n is nonempty, then the collection $\mathcal{C} = \{C_n\}_{n\in\mathbb{Z}_+}$ automatically has the finite intersection property. Then the intersection

$$\bigcap_{n\in\mathbb{Z}_+} C_n$$

is nonempty.

Remark: Compactness in a Space vs. Period

If A is a subset of a topological space X, then there are two ways we have talked about A being compact.

First We can talk about A being *compact in* X. This has to do with open covers involving subsets open in X.

Second We can talk about A being *compact as a topological space* in its own right equipped with the subspace topology. This has to do with open covers involving subsets open in A.

Importantly, these two notions are the *same* (as shown in [Lemma 26.1])! So, when we say that "A is compact in X", we might as well say that "A is compact with its subspace topology". The point is that the containing X does not really matter beyond the subspace topology that it gives.

However, the property of being closed *does* depend on the containing space. For example, let's compare: the interval $(0,1)$ is closed in $(0,1)$, but it is *not* closed in $\mathbb{R}$. Closedness in $(0,1)$ does not guarantee closedness in $\mathbb{R}$. The interval $[\frac{1}{3}, \frac{2}{3}]$ is compact in $(0,1)$, so it *must* also be compact in $\mathbb{R}$. Why? Because compactness is an *inherent* property of $[\frac{1}{3}, \frac{2}{3}]$. This is what makes compactness so special.

Remark: More Examples

More examples can be found in §26 of Munkres.

Chapter 14

Local Compactness

14.1 Learning Objectives

- What does it mean for a space to be locally compact?
  - What are the equivalent definitions of local compactness and how do they relate to each other? [Theorem 29.2]
- What is a one-point compactification of a topological space X?
  - When can we say that such a one-point compactification exists? [Corollary 29.4]
  - Are these one-point compactifications Hausdorff? [Theorem 29.1]
- Are closed or open subspaces of a locally compact space also locally compact? How about for when the larger space is locally compact Hausdorff?

14.2 Definitions

Definition: Locally Compact

Munkres provides the following definition: A topological space X is said to be **locally compact at** x if there is some compact subspace C of X that contains a neighborhood of x. If X is locally compact at each of its points, X is said simply to be **locally compact**.

In lecture, though, we are given the following: A topological space X is **locally compact at a given point** $p \in X$ if and only if every open neighborhood of p contains a *compact neighborhood* of p, which is a compact set that contains an open neighborhood of p. A topological space X is **locally compact** if and only if it is locally compact at each of its points.

Definition: One-point Compactification

If Y is a compact Hausdorff space and X is a proper subspace of Y whose closure equals Y, then Y is said to be a **compactification** of X.

A **one-point compactification** of a non-compact topological space X is any *compact* topological space Y with a distinguished point $\star$ so that $Y \setminus \{\star\} \cong X$ (Here $\cong$ means homeomorphic). I.e., if $Y \setminus X$ equals a single point, then Y is called the **one-point compactification** of X.

14.3 Theorems

Theorem: 29.1

Let X be a space. Then X is locally compact Hausdorff if and only if there exists a space Y satisfying the following conditions:

1. X is a subspace of Y
2. The set $Y \setminus X$ consists of a single point
3. Y is a compact Hausdorff space.

If Y and Y' are two spaces satisfying these conditions, then there is a homeomorphism of Y with Y' that equals the identity map on X.

Proof The full details can be seen in pages 183 – 184.

Theorem: 29.2

Let X be a Hausdorff space. Then X is locally compact if and only if given $x \in X$, and given a neighborhood U of x, there is a neighborhood V of x such that $\text{Cl } V$ is compact and $\text{Cl } V \subset U$.

Proof

$\implies$

Let X be locally compact. Let $x \in X$ and a neighborhood U of x in X be given. We want to show that there is a neighborhood V of x such that $\text{Cl } V$ is compact and $\text{Cl } V \subset U$.

Since we have that X is locally compact Hausdorff, it follows from [Theorem 29.1] that there exists a Hausdorff one-point compactification Y of X. We shall define C to be the set $Y \setminus U$; it follows immediately that C is closed in Y. By [Theorem 26.2], it follows that C is compact in Y as well. We know from [Lemma 26.4] that there exists disjoint open sets V and W such that they contain x and C, respectively.

With this, we see that the closure $\text{Cl } V$ of V in Y is compact (as it is closed in Y). Furthermore, we see that $\text{Cl } V$ is disjoint from W. To show why, let's say that $w \in W$ is given. Because W is an open neighborhood of w that does not intersect V, we see that w cannot be a limit point of V. Hence, we find that no point $w \in W$ can be a limit point of V, which tells us that $W \cap \text{Cl } V = \emptyset$. From this, it immediately follows that $\text{Cl } V$ and C are disjoint from each other as $C \subset W$. Thus, we have that for each $x \in \text{Cl } V$, it follows that $x \notin C$, which implies that $x \in U$. Thus, $\text{Cl } V \subset U$ as intended.

$\impliedby$

Conversely, let's say that $x \in X$ and an open neighborhood U of x is given. Furthermore, we are also given that there exists a neighborhood V of x such that $\text{Cl } V$ is compact and $\text{Cl } V \subset U$. If we define $C = \text{Cl } V$, then we see that it is a compact set containing a neighborhood of x. Since this is true for all $x \in X$, we see that X is locally compact.

Corollary: 29.3

Let X be locally compact Hausdorff; let A be a subspace of X. If A is closed in X or open in X, then A is locally compact.

Proof Suppose that A is closed in X. We want to show that for any $x \in A$, we can find some compact subspace of A that contains a neighborhood of x in A. Since X is locally compact, we see that there exists some compact subspace C of X that contains some neighborhood U of x in X. Under the subspace topology, we can see that $C \cap A$ is closed in C since we are given that A is closed in X. Hence, it follows that $C \cap A$ must also be compact as C is compact. Furthermore, we can see that $C \cap A$ contains the open neighborhood $U \cap A$ of x in A. Thus, we have that $C \cap A$ is a compact subspace of A that contains a neighborhood of x in A, which tells us that A is locally compact. Notice that we have not used the Hausdorffness condition here.

Suppose now that A is open in X. Given $x \in A$, we apply [Theorem 29.2] to choose a neighborhood V of x in X such that $\mathrm{Cl}\ V$ is compact and $\mathrm{Cl}\ V \subset A$. With this, we see that $C = \mathrm{Cl}\ V$ is a compact subspace of A containing the neighborhood V of x in A. Thus, A is locally compact.

Corollary: 29.4

A space X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact Hausdorff.

Proof This follows from [Theorem 29.1] and [Corollary 29.3]

Theorem: Hausdorff One-Point Compactification

Let X be a non-compact topological space having *Hausdorff* one-point compactifications Y_1 and Y_2 with distinguished points $\star_1 \in Y_1$ and $\star_2 \in Y_2$.

Then Y_1 is homeomorphic to Y_2.

Proof We start by noting that because Y_1 and Y_2 are one-point compactifications of X with distinguished points $\star_1$ and $\star_2$, respectively, it follows that

$$Y_1 \setminus \{\star_1\} \cong X \quad \text{and} \quad Y_2 \setminus \{\star_2\} \cong X$$

Since $\cong$ is an equivalence relation, we have that it must be transitive. Thus, we see that

$$Y_1 \setminus \{\star_1\} \cong Y_2 \setminus \{\star_2\}$$

which tells us that there exists a homeomorphism $f : Y_1 \setminus \{\star_1\} \to Y_2 \setminus \{\star_2\}$ between them.

Let us construct another function $g : Y_1 \to Y_2$, where

$$g(\star_1) = \star_2 \quad \text{and} \quad g(x) = f(x) \quad \forall x \in Y_1 \text{ s.t. } x \neq \star_1$$

Our claim is that g is a homeomorphism. Indeed, it is clear to see that g is bijective since we have constructed it to be so (note that f is bijective). Our goal is to show that both g and g^{-1} are continuous. However, it suffices to only show that g is continuous since its domain is compact and codomain is Hausdorff [Theorem 26.6].

Let $C \subset Y_2$ be closed. Our goal is to show that $g^{-1}(C)$ is closed. Because Y_2 is compact, C being closed in Y_2 means that it must be compact in Y_2 as well. We shall now look at two cases:

Case 1: $\star_2 \notin C$

In this case, we see that $C \subset Y_2 \setminus \{\star_2\}$. By the [variation of Lemma 26.1], we have that C is compact in $Y_2 \setminus \{\star_2\}$ as well. Furthermore, we see that $g^{-1}(C) = f^{-1}(C)$. Because $f^{-1} : Y_2 \setminus \{\star_2\} \to Y_1 \setminus \{\star_1\}$ is continuous and C is compact in $Y_2 \setminus \{\star_2\}$, we find that $f^{-1}(C)$ is compact in $Y_1 \setminus \{\star_1\}$. Thus, we get that $f^{-1}(C)$ is compact in the larger space Y_1 by the [variation of Lemma 26.1]. Now since Y_1 is Hausdorff, it follows that $f^{-1}(C)$ is closed. I.e., we see that $g^{-1}(C)$ is closed.

Case 2: $\star_2 \in C$

In this case, we see that $Y_2 \setminus C$ is contained in $Y_2 \setminus \{\star_2\}$. Furthermore, we have that it is open in Y_2 since C is closed in Y_2, meaning that it is also open in $Y_2 \setminus \{\star_2\}$ under the subspace topology. Because $f : Y_1 \setminus \{\star_1\} \to Y_2 \setminus \{\star_2\}$ is continuous, we have that $f^{-1}(Y_2 \setminus C)$ is open in $Y_1 \setminus \{\star_1\}$. Since Y_1 is Hausdorff (hence T_1), we see that all singleton sets are closed. Thus, $Y_1 \setminus \{\star_1\}$ is open, and by the transitivity of openness, we see that $f^{-1}(Y_2 \setminus C)$ is open in Y_1. Now because $Y_2 \setminus C$ does not contain $\star_2$, we have that $g^{-1}(Y_2 \setminus C)$ is equal to $f^{-1}(Y_2 \setminus C)$. Notice that

$$\begin{aligned} f^{-1}(Y_2 \setminus C) &= g^{-1}(Y_2 \setminus C) \\ &= \{x \in Y_1 \setminus \{\star_1\} \mid g(x) \in Y_2 \setminus C\} \\ &= \{x \in Y_1 \setminus \{\star_1\} \mid g(x) \notin C\} \\ &= \{x \in Y_1 \setminus \{\star_1\} \mid x \notin g^{-1}(C)\} \\ &= Y_1 \setminus g^{-1}(C) \end{aligned}$$

Notice that C includes $\star_2$, meaning that $g^{-1}(C)$ contains $\star_1$. Now, because $f^{-1}(Y_2 \setminus C)$ is open in Y_1, it follows that $g^{-1}(C)$ must be closed in Y_1.

In either case, we see that g is continuous, which results in g to be a homeomorphism between Y_1 and Y_2.

Theorem: Variation of [Corollary 29.3]

Let X be locally compact and U be an open subspace of X. Then U is locally compact.

If X is also Hausdorff, then any closed subspaces of X are locally compact.

Proof Let U be an open subspace of X. Let $p \in U$ be given. Let V be an open neighborhood of p in U. We want to show that there exists a compact neighborhood of p that is contained in V. Because U is open in X, we find that V is also open in X by the transitivity of openness. Because X is locally compact, we see that there exists a compact neighborhood K of p in X that is contained in V.

Let X be also Hausdorff; let U now be a closed subspace of X. We have by [Theorem 29.2] that the definition of local compactness given by Munkres is equivalent to the definition given in lecture. Thus, we are able to utilize [cor:29.3] to get that U is also locally compact.

Lemma: Variation of [Theorem 29.2]

Let X be a compact Hausdorff topological space. Let A be a closed subset of X and U be an open neighborhood of A. Prove that there exists an open neighborhood V of A such that $\text{Cl } V \subset U$.

Proof Since both A and $X \setminus U$ are closed and X is compact, it follows that they are both compact. Furthermore, it is clear to see that A and $X \setminus U$ are disjoint since U contains A. Using the [Extension of Lemma 26.4], we find that there exists disjoint open sets V and W such that they contain A and $X \setminus U$, respectively.

Our goal is to show that $\text{Cl } V \subset U$. Let $x \in \text{Cl } V$ be given; we want to show that $x \in U$. Towards a contradiction, let's suppose that $x \in X \setminus U$. It follows that W is then some open neighborhood of x in X. Because $x \in \text{Cl } V$, we see that $W \cap V \neq \emptyset$. However, we reach a contradiction since W and V are disjoint. Hence, it must be the case that $x \in U$. Since this was true for any arbitrary $x \in \text{Cl } V$, it follows that $\text{Cl } V \subset U$.

Theorem: Another Variation of [Corollary 29.3]

Let X be a compact Hausdorff space. Then every open (and closed) subspace of X is locally compact and Hausdorff.

Proof We have shown previously that Hausdorffness translates to subspaces in general [Theorem 17.11]. We have also shown that local compactness translates to open (and closed) subspaces [thm:29.4]. Thus, it suffices to show that X is locally compact (X is already Hausdorff)

We start off by letting $p \in X$ and an open neighborhood U of p be given. By [Variation of Lemma 29.3], we see that there exists an open neighborhood V of p so that $\text{Cl } V \subset U$. Now because $\text{Cl } V$ is closed in X, it follows that $\text{Cl } V$ is compact since X is compact. Thus, we have that $\text{Cl } V$ is a compact neighborhood of p in U. Hence, we have that X is locally compact.

Theorem: Existence of One-Point Compactifications

If a space X is locally compact and Hausdorff, then there exists a one-point compactification of X.

Proof This is a direct result of [Theorem 29.1]

14.4 Examples

Example: Compactification

Sadly, many reasonable spaces are not compact. However, most reasonable spaces can be compactified! We consider the open interval $(0,1)$ and show intuitively how to add a *single* point in $(0,1)$ so that it becomes compact!

We can turn $(0,1)$ into a compact subspace of $\mathbb{R}$ by adding on the endpoints 0 and 1. However, it is possible to turn $(0,1)$ into a compact subspace using only one point; notice that $(0,1)$ is homeomorphic to a circle minus its north pole. If we were to add on the north pole, then we get a full circle, which we know is compact. In this case, the north pole was our distinguished point $\star$.

Example: Locally Compact

$\mathbb{R}^n$ is locally compact.

Let $K = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. We show that there is no compact neighborhood of 0 in $\mathbb{R} \setminus K$ with the standard topology. Let $A \subset \mathbb{R} \setminus K$ contain an open neighborhood of 0 in $\mathbb{R} \setminus K$. Then, we see that A contains one of the maximal open intervals, but not its endpoints. A is not closed in $\mathbb{R}$. The Heine-Borel theorem tells us that A is not compact in $\mathbb{R}$. Hence, we see that A is not compact in $R \setminus A$ since compactness is an inherent property.

Example: Open Disk in $\mathbb{R}^2$

Is a Hausdorff one-point compactification of the open disk $x^2+y^2 < 1$ with the standard topology guaranteed to exist? If so, can you surmise what it is?

Notice that the open disk is not compact since it is not closed in $\mathbb{R}^2$ (Heine-Borel). We start by noting that $\mathbb{R}^2$ under the standard topology is locally compact. Since the open disk is an open subspace of $\mathbb{R}^2$, we see that the open disk must also be locally compact. Furthermore, we have that the open disk is Hausdorff since $\mathbb{R}^2$ is Hausdorff. Since the open disk is locally compact and Hausdorff, we have that it must have a Hausdorff one-point compactification.

To surmise what it is, we start with a previous example. For any open interval in $\mathbb{R}$, we found that its one-point compactification resulted to be a circle; recall that any open interval in $\mathbb{R}$ is homeomorphic to a circle minus its north pole (i.e., $S^1 \setminus \{N\}$). In a similar fashion we see that any open disk in $\mathbb{R}^2$ is homeomorphic to a sphere in $\mathbb{R}^3$ minus its north pole (i.e. $S^2 \setminus \{N\}$). Thus, the one-point compactification of any open disk in R^2 is a sphere in $\mathbb{R}^3$ (that is, S^2), where the distinguished point is the north pole.

In fact, the one-point compactification of $\mathbb{R}$ is S^1 since we have that $\mathbb{R} \cong (0,1)$.

Example: Compactness implies Local Compactness

We first remark that this implication holds for the definition of local compactness given by Munkres – not in lecture! To make this implication work with the definition given in lecture, we need our space to be Hausdorff as well.

Let's say that X is a compact topological space. Our goal is to show that X is locally compact. To do so, we just need to show that for any given point $x \in X$, there is some compact subspace C of X that contains a neighborhood of x. Let U be any open neighborhood of x in X. We can see that $\text{Cl } U$ is closed in X, which results in $\text{Cl } U$ to be compact in X as well. If we define $C = \text{Cl } U$, then we see that the condition of local compactness is met. Hence X is locally compact.

Example: Discrete Topology is locally compact Hausdorff

Let X be under the discrete topology. Our goal is to show that X is locally compact Hausdorff. To show that X is Hausdorff, we start by letting two distinct points $x, y \in X$ be given. Because we are under the discrete topology, we have that $\{x\}$ and $\{y\}$ open in X, and we can see that they are disjoint as well. Thus, we get that X is Hausdorff.

Note that X is only compact under the discrete topology if and only if it is finite. Let's say that X is compact under the discrete topology. If X were not finite, then the open cover consisting of singleton point sets would have not finite subcover. Now, if X were given to be finite and under the discrete topology, we see that any open cover of X is finite, which tells us that X is compact.

With this, we can show that X is locally compact. To do so, we want to show that for any $x \in X$, there is some compact subspace C of X that contains a neighborhood of x. Let U be any finite set that contains x. We see that U must be compact in X by what we found earlier. Furthermore, we see that $\{x\} \subset U$ as U contains x, which tells us that U is a compact subspace of X that contains a neighborhood of x. Thus, X is locally compact.

14.5 Discussion

Remark: Local Properties

Usually, one says that a space X satisfies a given property "locally" if every $x \in X$ has "arbitrarily small" neighborhoods having the given property.

Remark: Local Compactness vs Compactness

In this section, we introduce another commonly used formulation of compactness. We note that it is weaker in general than compactness, though it coincides with compactness for metrizable spaces (as shown in [Theorem 28.2]).

Remark: Hausdorff One-Point Compactifications

Hausdorff one-point compactifications are unique up to homeomorphism, provided that they exist in the first place.

Remark: On [Theorem 29.2]

This theorem gives us equivalent definition of what it means for a space to be locally compact – in fact, this is the definition that is given in lecture, where Cl V is our compact neighborhood in question in the case when X is Hausdorff.

Remark: Difference between Definitions

One key thing to keep in mind is that the definition given in lecture is only equivalent to the definition that Munkres provides when we are given X to be Hausdorff. However, the definition given in lecture does not require the space to be Hausdorff.

For all the proofs in the variations (after [Corollary 29.4]), we used the definition of local compactness given in lecture – not the definition given in Munkres. However, when the space is Hausdorff, it does not matter which definition that we choose to use.

Remark: On [Corollary 29.3]

We saw that any *closed* subspace of a locally compact space is locally compact as well. However, in order to say the same for *open* subspaces, we need that the larger space to be Hausdorff as well in order to say that such open subspaces are locally compact.

Remark: More Examples and Proofs

The proofs as well as more examples can be found in §29 of Munkres.

Chapter 15

The Countability Axioms

15.1 Learning Objectives

- What does it mean for a space to be first-countable? Second-countable?
  - What is the difference between the two?
- What does it mean for a subset to be dense?
- What does it mean for a space to be Lindelöf?
  - What is the difference between Lindelöf and second-countable?
- What does it mean for a space to be separable?
- Does a space being second-countable imply that it is Lindelöf? Separable? [Theorem 30.3]
- What condition on the space X do we need in order for the following to be equivalent?
  1. X is second-countable
  2. X is separable
  3. X is Lindelöf

  Why is the condition necessary? [Extension of Theorem 30.3].

15.2 Definitions

Definition: First-Countable

A space X is said to have a **countable basis at** x if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$. A space that has a countable basis at each of its points is said to satisfy the **first countability axiom**, or to be **first-countable**.

Definition: Second-Countable

If a space X has a countable basis for its topology, then X is said to satisfy the **second countability axiom**, or to be **second-countable**.

I.e., A topological space is second-countable if its topology can be generated by a countable basis.

Definition: Dense Set

A subset A of a space X is said to be **dense** in X if $\overline{A} = X$.

Definition: Lindelöf space

A space for which every open covering contains a countable subcovering is called a **Lindelöf space**.

Definition: Separable

A space having a countable dense subset is often said to be **separable**.

15.3 Theorems

Theorem: 30.1

Let X be a topological space.

(a) Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \overline{A}$; the converse hold if X is first countable.

(b) Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is first-countable.

Proof The proof for this proof can be seen in §21 – it is just a generalization under the hypothesis of metrizability.

Theorem: 30.2

A subspace of a first-countable space is first-countable, and a countable product of first-countable spaces is first-countable.

A subspace of a second-countable space is second-countable, and a countable product of second-countable spaces is second-countable.

Proof Let us first consider the second countability axiom. If $\mathcal{B}$ is a countable basis for X, then we have that

$$\{B \cap A \mid B \in \mathcal{B}\}$$

is a countable basis for the subspace of A of X. Thus, we have that A is second-countable. Now, if $\mathcal{B}_i$ is a countable basis for the spaces X_i, then the collection of all products $\prod U_i$, where $U_i \in \mathcal{B}_i$

for finitely many values of i and $U_i = X_i$ for all other values of i, is a countable basis for $\prod X_i$.

Let us now consider the first countability axiom. Let A be a given subset of X. Let $x \in A$ be given as well. Since X is first-countable and $x \in A \subset X$, it follows that there is a countable basis $\mathcal{B}_x$ at x in the topology of X. As a result, we can see that

$$\{B \cap A \mid B \in \mathcal{B}_x\}$$

is a countable basis for x in the subspace topology of A. Since this is true for any $x \in A$, we have that A is first-countable. Now, let $\prod X_i$ be a countable product of first-countable spaces. Let $\mathbf{x} \in \prod X_i$ be any given point. Since we know that each X_i is first-countable, we have that for each coordinate x_i of $\mathbf{x}$, there is some countable basis $\mathcal{B}_{x_i}$ at x in the topology of X_i. The collection of all products $\prod U_i$, where $U_i \in \mathcal{B}_{x_i}$ for finitely many values of i and $U_i = X_i$ for all other values of i, is a countable basis for $\mathbf{x}$. Thus, we have that $\prod X_i$ is first-countable.

Theorem: 30.3

Suppose that X has a countable basis – that is, suppose that X is second countable. Then:

(a) Every open covering of X contains a countable subcollection covering X. I.e., X is Lindelöf.

(b) There exists a countable subset of X that is dense in X. I.e., X is separable.

Proof Let X be second-countable. Let $\mathcal{B} = \{B_n\}$ be a countable basis for X.

a

Let $\mathcal{U}$ be an open cover of X. We want to show that there exists a countable subcollection of $\mathcal{U}$ that still covers X. If $\mathcal{B}$ is *subordinate to the open cover* $\mathcal{U}$, meaning that every $B \in \mathcal{B}$ is contained in some $U_B \in \mathcal{U}$, then:

$$\{U_B \mid B \in \mathcal{B}\}$$

is a countable subcover of X.

Given some open cover $\mathcal{U}$ of a topological space X, and a basis $\mathcal{B}$ for the topology on X, then there exists a basis $\mathcal{B}_{\mathcal{U}} \subset \mathcal{B}$ and is subordinate to $\mathcal{U}$.

To show why, let $B \in \mathcal{B}$ be given such that it is not contained in any of the open sets in the collection $\mathcal{U}$. Let $x \in B$ be given. We know that because $\mathcal{U}$ is an open cover of X, we have that x must belong to at least one open set in this cover, say U_x. Now, since $\mathcal{B}$ is a basis for X and U_x is open in X, it follows by [Lemma 13.2] that there exists some element $B' \in \mathcal{B}$ such that $x \in B' \subset U_x$. Notice that $x \in B$ and $x \in B'$; $\mathcal{B}$ being a basis tells us that there exists a basis element B_x containing x such that $B_x \subset B \cap B'$. Doing this for all $x \in B$ results in us seeing that $B = \bigcup_{x \in B} B_x$; note that each B_x is contained in some element U_x of the open cover $\mathcal{U}$.

To recap, we have essentially broken down B into smaller basis elements, where each of these elements are contained in some element U_x of the open cover $\mathcal{U}$, and whose union is all of B. If we do this for all $B \in \mathcal{B}$ such that it is not contained in any of the open sets in the collection $\mathcal{U}$, then we end up with a finer basis $\mathcal{B}_{\mathcal{U}}$ that is subordinate to $\mathcal{U}$.

One other way that we could have proved the existence of a subordinate basis is if we were to remove all the basis elements that were not contained in some element of the open cover $\mathcal{U}$. Indeed, we see that the remaining basis elements are subordinate to the open cover $\mathcal{U}$, and they still form a basis for the topology on X; the reason being similar to the argument made earlier.

Generally, we can replace $\mathcal{B}$ with a potentially smaller basis that is subordinate to $\mathcal{U}$, and we can apply the same argument as earlier.

b

For each nonempty $B \in \mathcal{B}$, let us pick some $x_B \in B$. We claim that

$$A = \{x_B \mid B \in \mathcal{B}\}$$

is both countable and dense. It is clearly countable since $\mathcal{B}$ is countable. To show that A is dense, we want to show that $\text{Cl } A = X$. Let $x \in X$ and an open neighborhood U of x be given. Because we have that $\mathcal{B}$ is a basis, this implies that there exists $B \in \mathcal{B}$ such that $x \in B \subset U$. Thus, we have that $x_B \in U$, which results in $U \cap A \neq \emptyset$. Thus, we have that $x \in \text{Cl } A$, which tells us that $X \subset \text{Cl } A$. Hence, we see that $\text{Cl } A = X$.

Theorem: Extension of [Theorem 30.3]

Let X be a metric space. Then the following statements are equivalent:

1. X is second-countable
2. X is separable
3. X is Lindelöf

Proof From [Theorem 30.3], we can see that (1) $\implies$ (2) and (1) $\implies$ (3). It suffices to show that (2) $\implies$ (1) and (3) $\implies$ (1).

(2) $\implies$ (1)

Let A be a countable dense subset of X. We define $\mathcal{B} = \{B_q(a) \mid a \in A \text{ and } q \in \mathbb{Q}_{>0}\}$ to be the set of balls of rational radius centered at points in A. We claim that $\mathcal{B}$ is a countable basis for X.

We shall first show that $\mathcal{B}$ is countable. Let us define $B_{\mathbb{Q}_{>0}}(a)$ to be as follows:

$$B_{\mathbb{Q}_{>0}}(a) = \{B_q(a) \mid q \in \mathbb{Q}_{>0}\}$$

Since $\mathbb{Q}_{>0}$ is countable, we see that $B_{\mathbb{Q}_{>0}}(a)$ is countable as well. Thus, we see that

$$\mathcal{B} = \{B_q(a) \mid a \in A \text{ and } q \in \mathbb{Q}_{>0}\} = \bigcup_{a \in A} B_{\mathbb{Q}_{>0}}(a)$$

is a countable union of countable sets, which we know is countable.

From here, we let $x \in X$ and $r \in \mathbb{R}_{>0}$ be given. We want to show that there exists some $a \in A$ and $q \in \mathbb{Q}_{>0}$ so that $x \in B_q(a)$ and $B_q(a) \subset B_r(x)$. Since we know that A is dense in X, we have that $\overline{A} = X$; thus, it follows that for any $x \in X$ and any open neighborhood U of x, we get that $U \cap A \neq \emptyset$. Since $B_r(x)$ is an open neighborhood of x, we see that $B_r(x) \cap A \neq \emptyset$; in fact, we can do better and say that since $B_{\frac{r}{2}}(x)$ is an open neighborhood of x, we see that $B_{\frac{r}{2}}(x) \cap A \neq \emptyset$. Let $a \in B_{\frac{r}{2}}(x) \cap A$.

Our goal now is to find some $q \in \mathbb{Q}_{>0}$ such that $x \in B_q(a) \subset B_r(x)$. Since we know that $a \in B_{\frac{r}{2}}(x)$, we have that $d(a,x) < \frac{r}{2}$. To guarantee that $x \in B_q(a)$, we can choose some rational number $q \in \mathbb{Q}_{>0}$ such that $d(a,x) < q < \frac{r}{2}$; such a rational number q exists as $\mathbb{Q}$ is dense in $\mathbb{R}$. All that is left to do is to show that $B_q(a) \subset B_r(x)$ – this amounts to showing that for all $b \in B_q(a)$, we have that $b \in B_r(x)$. The triangle inequality tells us that

$$d(b,x) \leq d(b,a) + d(a,x)$$

Because $x \in B_q(a)$, we see that $d(a,x) < q < \frac{r}{2}$; the same could be said for b. Thus, we see that

$$d(b,x) \leq d(b,a) + d(a,x) < q + q < \frac{r}{2} + \frac{r}{2} = r$$

Since our choice of $b \in B_q(a)$ was arbitrary, we have that $B_q(a) \subset B_r(x)$.

Finally, we want to show that $\mathcal{B}$ is a basis that generates the topology on X. This amounts to showing that for any open set U of X and each $x \in U$, there is an element $B \in \mathcal{B}$ such that $x \in B \subset U$; notice that this is precisely [Lemma 13.2]. Let U be any open set of X; let $x \in U$ be given. Since U is open in X and X is a metric space, we see that there exists some positive real number r such that $x \in B_r(x) \subset U$. We have just shown that there exists some $a \in A$ and $q \in \mathbb{Q}_{>0}$ so that $x \in B_q(a)$ and $B_q(a) \subset B_r(x)$. If we let $B = B_q(a)$, then we have that $x \in B \subset B_r(x) \subset U$, which tells us that $\mathcal{B}$ is indeed a basis that generates the topology on X. Hence, we have that X has a countable basis – that is, X is second-countable.

$\boxed{(3) \implies (1)}$

Let X be a Lindelöf metric space. Since X is a metric space, we have that balls of some radius r, where r is a real number, are open subsets of X. Furthermore, since X is Lindelöf, we see that for every open cover $\mathcal{A}$, there exists a countable subcollection of $\mathcal{A}$ that covers X.

We start off by noting that

$$\mathcal{A} = \{B_{\frac{1}{n}}(x) \mid x \in X\}$$

is an open cover of X for all $n \in \mathbb{Z}_+$. Since X is Lindelöf, we have that there exists a countable subcollection of $\mathcal{A}$ that covers X, and this is true for all $n \in \mathbb{Z}_+$; let's denote $\mathcal{A}_n$ to be this countable covering of X by $\frac{1}{n}$-balls.

We shall now define the following:

$$\mathcal{B} = \bigcup_{n \in \mathbb{Z}} \mathcal{A}_n$$

We claim that $\mathcal{B}$ is a countable basis. It is clear to see that $\mathcal{B}$ is countable as it is a countable union of countable sets. All that is left to do is to show that $\mathcal{B}$ is a basis.

Let U be any open set of X. Let x be any element of U. Since U is open, we have that there exists some positive real number r such that $x \in B_r(x) \subset U$ – i.e., U being open in a metric space means that we can always put a ball of some positive real radius around each point in U such that this ball is contained in U. Now, because $\mathcal{A}_n$ is a countable subcover of X for all $n \in \mathbb{Z}_+$, we see that x must be contained in some ball of radius $\frac{1}{n}$ centered at some point y_n of X – that is, $x \in B_{\frac{1}{n}}(y_n)$ for all $n \in \mathbb{Z}_+$, where y_n is some element of X that depends on the particular countable subcover $\mathcal{A}_n$ and $x \in X$ that we are considering.

Our goal is to find some $n \in \mathbb{Z}_+$ such that $B_{\frac{1}{n}}(y_n) \subset B_r(x)$. It suffices to show that for all $z \in B_{\frac{1}{n}}(y_n)$, we have that $z \in B_r(x)$. Since we have a metric to work with, say d, the triangle inequality tells us that

$$d(z,x) \leq d(z,y_n) + d(y_n,x)$$

Because $x \in B_{\frac{1}{n}}(y_n)$, we see that $d(y_n,x) < \frac{1}{n}$; notice that the same can be said with z. Thus, we see that

$$d(z,x) \leq d(z,y_n) + d(y_n,x) < \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$$

In order for $z \in B_r(x)$, we need $d(z,x) < r$ – this is guaranteed to happen if $\frac{1}{n} < \frac{r}{2}$. Therefore, if we choose $n \in \mathbb{Z}_+$ such that $\frac{1}{n} < \frac{r}{2}$, then we end up getting that $z \in B_r(x)$ for all $z \in B_{\frac{1}{n}}(y_n)$ – that is, we get $x \in B_{\frac{1}{n}}(y_n) \subset B_r(x) \subset U$.

Putting everything together, we see that for all $x \in U$, there exists some $n \in \mathbb{Z}_+$ such that $x \in B_{\frac{1}{n}}(y_n) \subset U$, where y_n is some element of X that depends on the particular finite subcover $\mathcal{A}_n$ and $x \in X$ that we are considering. By [Lemma 13.2], we have that $\mathcal{B}$ is a countable basis for X. Thus, X is second-countable.

Theorem: Variation of [Theorem 30.2]

Let X be a topological space and Lindelöf. Any closed subspace of X is Lindelöf.

Proof Let A be a closed subspace of X. Let $\mathcal{U}$ be an open cover of A by sets open in A. We want to show that there exists a countable subcover of A by sets open in A. To start, we first note that for each element $U_i \in \mathcal{U}$, there exists some open set V_i in X such that $U_i = A \cap V_i$ by definition of the subspace topology. From this, we can see that $\mathcal{V} = \{V_i\}$ is an open cover of A by sets open in X.

Since A is closed in X, this means that $X \setminus A$ must be open in X as well. Thus, we get that

$$(X \setminus A) \cup \Big(\bigcup V_i \Big)$$

is an open cover of X by sets open in X. Since X is Lindelöf, it follows that there exists a countable subcover of X for our particular open cover. Let's say that such a countable subcover is as follows:

$$\Big\{ X \setminus A, V_{i_j} \mid j \in \mathbb{Z}_+ \Big\}$$

With this, we can see that $\{V_{i_j}\}_{j \in \mathbb{Z}_+}$ is still an open cover of A by sets open in X, which implies that $\{U_{i_j} = A \cap V_{i_j}\}_{j \in \mathbb{Z}_+}$ is a countable open cover of A by sets open in A. Thus, we have that A is Lindelöf.

15.4 Examples

Example: Second Axiom Implies First

The second countability axiom implies the first.

Let X be second-countable. The X has a countable basis $\mathcal{B}$ for its topology. For any points $x \in X$, we have that the subset of $\mathcal{B}$ consisting of those basis elements containing the point x is a countable basis at x. Thus, X is first-countable as well.

Example: Examples of Second-Countable Spaces

$\mathbb{R}$ is second-countable; a countable basis for $\mathbb{R}$ would be as follows:

$$\{(p,q) \mid p,q \in \mathbb{Q}\}$$

We have shown in a previous HW that this was indeed a basis. We also know that this collection is countable since $\mathbb{Q} \times \mathbb{Q}$ is countable, and the collection can be injectively mapped into $\mathbb{Q} \times \mathbb{Q}$.

Example: $\mathbb{R}_\ell$ is not Second-Countable

Let $\mathcal{B}$ be any basis for the lower-limit topology. For all $x \in \mathbb{R}$, we have that the interval $[x, x+1)$ is certainly open in the lower-limit topology. Thus, we have that there exists some $B_x \in \mathcal{B}$ so that $x \in B_x \subset [x, x+1)$. If we have that $x \neq y$, then $B_x \neq B_y$; this is because the smallest real number that in B_x is x, but the smallest number in B_y is y – thus, these sets cannot be equal. Notice that the function $f : \mathbb{R} \to \mathcal{B}$ given by $x \mapsto B_x$ is an injection, thus $\mathcal{B}$ must be uncountable (it must have a larger than or equal to cardinality than $\mathbb{R}$, which is uncountable).

Example: Dense and Separable

$\mathbb{R}$ and $\mathbb{R}_\ell$ are both separable; for $\mathbb{R}$, we see that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$. We claim that $\mathbb{Q}$ is also dense in $\mathbb{R}_\ell$. If it weren't dense, there is some point some $x \in \mathbb{R}$ such that $x \notin \overline{\mathbb{Q}}$. Thus, there is some open open interval that contains x that is disjoint from $\mathbb{Q}$, however, this cannot be the case (flesh this point out some more).

15.5 Discussion

Remark: Countable

Recall that a set is **countable** if it is bijective with a subset of $\mathbb{N}$. Countability is a particular instance of the **cardinality** or size of a set. If X is a set, then we use $|X|$ to denote its cardinality.

For example, the set $\mathbb{Q}$ of rational numbers is countable, but the set $\mathbb{R}$ of real numbers is not. In fact, we have that the real numbers have a strictly larger cardinality than the rational numbers.

Some fundamental properties:

- The *finite* product of countable sets is countable.
- Any countable union of countable sets is countable.

Notice a countable product of countable sets is not countable! The intuition is that products of sets grow far faster in size than unions of sets; products correspond to *multiplying* cardinalities while unions correspond to something closer to adding cardinalities.

Remark: On First-Countable

The most useful fact concerning spaces that satisfy this axiom is the fact that in such a space, convergent sequences are adequate to detect limit points of sets and to check continuity of functions.

Remark: On Second-Countable

Why is this axiom interesting? Well, for one thing, many familiar spaces do satisfy it. For another, it is a crucial hypothesis used in proving such theorems as the Urysohn metrization theorem.

Remark: On [Theorem 30.3]

The two properties listed in [Theorem 30.3] are sometimes taken as alternative countability axioms.

Separable does not imply second-countable – $\mathbb{R}_\ell$.

Remark: More Examples

More examples can be found in §30 in Munkres.

Chapter 16

The Separation Axioms

16.1 Learning Objectives

- What does it mean for a space to be regular? Normal?
  - What is the difference between the two?
- What is the equivalent definition of regular? Normal? [Lemma 31.1]
- How does regularity and normality behave with subspaces? [Theorem 31.2]

16.2 Definitions

Definition: Regular and Normal Sets

Suppose that one-point sets are closed in X – that is, suppose that X is T_1. Then X is said to be **regular** if for each pair consisting of a point x and a closed set B disjoint from x, there exists disjoint open sets containing x and B, respectively.

The space X is said to be **normal** if for each pair A, B of disjoint closed sets of X, there exists disjoint open sets containing A and B, respectively.

Definition: Closed Neighborhood

A **closed neighborhood** is a closed set that contains an open neighborhood.

16.3 Theorems

Lemma: 31.1

Let X be a topological space. Let one-point sets in X be closed – that is, let X be T_1.

(a) X is regular if and only if given a point $x \in X$ and an open neighborhood U of x, there is a closed neighborhood V of x such that $V \subset U$ – that is, there is an open neighborhood V of x such that $\overline{V} \subset U$.

(b) X is normal if and only if given a closed set A and an open set U containing A, there is a closed neighborhood V of A such that $V \subset U$ – that is, there is an open set V containing A such that $\overline{V} \subset U$.

Proof a

Suppose that X is regular. Let x and the neighborhood U of x be given. Let $B = X \setminus U$; we can see that B is a closed set of X that is disjoint from x. Since X is regular, we see that there exists disjoint open sets V and W containing x and B, respectively. Notice that $\overline{V}$ is a closed neighborhood of x as $\overline{V}$ is closed in X and contains the open neighborhood V of x. Further notice that the set $\overline{V}$ is disjoint from B; indeed if any $y \in B$ is given, then we see that W is an open neighborhood of y. Since W and V are disjoint from each other, we find that y cannot be in $\overline{V}$ by [Theorem 17.5]. Thus, it must be the case that $\overline{V} \subset U$, as desired.

Notice that $X \setminus W$ would also work as well; indeed, $X \setminus W$ is closed and contains the open neighborhood V of x. Thus, $X \setminus W$ is a closed neighborhood of x. Furthermore, we see that $X \setminus W$ is contained in U since W was an open neighborhood of $X \setminus U$.

Conversely, suppose that the point x and some closed set B not containing x be given. Our goal is to show that there exists disjoint open sets containing x and B, respectively. Let $U = X \setminus B$; this is an open neighborhood of x since B is closed and disjoint from x. By hypothesis, we have that there is an open neighborhood V of x such that $\overline{V} \subset U$; notice that $\overline{V}$ is our closed neighborhood of x. Since $\overline{V} = V \cup V'$, it follows that the open sets V and $X \setminus \overline{V}$ are disjoint open sets containing x and B, respectively. Therefore, we have that X is regular.

b

This proof uses the exact same argument; we just need to replace the point x with a closed set A of X throughout.

Theorem: 31.2

(a) A subspace of a Hausdorff space is Hausdorff; a product of Hausdorff spaces is Hausdorff.

(b) A subspace of a regular space is regular; a product of regular spaces is regular.

Proof

a This was already proved in [Theorem 17.11].

b

Let Y be a subspace of the regular space X. Since X is T_1, we find that Y must be T_1 as well since singleton point sets of Y are still closed in Y. Let x be a point of Y and let B be a closed subset of Y disjoint from x. By [Theorem 17.4], we have that $\mathrm{Cl}\, B$ in Y is equal to $\overline{B} \cap Y$, where $\overline{B}$ denotes the closure of B in X. Now since B is closed we see that $\mathrm{Cl}\, B = B$, which tells us that

$$B = \overline{B} \cap Y$$

Thus, it follows that $x \notin \overline{B}$ as B is disjoint from x. Using the regularity of X, we can choose disjoint open sets U and V of X containing x and B, respectively. Then $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y containing x and B, respectively.

Alternatively, we can prove that Y is regular using [Lemma 31.1]: Let Y be a subspace of the regular space X; let $x \in Y$ and an open neighborhood U of x in Y be given. By definition of the subspace topology, we see that U being open in Y means that there exists some open set V in X such that $U = V \cap Y$. Since X is regular, we see that there exists an open neighborhood W of x in X such that Cl $W \subset V$ (this is where argument for normality fails since a closed subset of Y may not be closed in X). Thus, we see that $W \cap Y$ is an open neighborhood of x in Y such that its closure Cl $W \cap Y$ in Y is contained in the open neighborhood $V \cap Y$ of x in Y – that is, (Cl $W \cap Y) \subset (V \cap Y)$ Thus, we are done.

Let X and Y be regular topological spaces. We want to show that $X \times Y$ is regular. Notice that both X and Y being T_1 results in $X \times Y$ to be T_1. We now let a point $(a, b) \in X \times Y$ and an open neighborhood U of (a, b) be given. By definition of open neighborhood, we see that there exists a basic open neighborhood $V \times W$ of (a, b) that is contained in U. X being regular tells us that there exists a closed neighborhood C of a that is contained in V. Y being regular tells us that there exists a closed neighborhood D of b that is contained in W. We find that $C \times D$ is a closed neighborhood of (a, b) that is contained in U.

16.4 Examples

Example: Separation Axioms

It is clear that a regular space is Hausdorff, and that a normal space is regular. Notice that we needed to include the condition that one-point sets be closed as a part of the definition of regularity and normality in order for this to be the case; a two-point space in the trivial topology satisfies the other part of the definitions of regularity and normality, even though it is not Hausdorff.

Hausdorff, regular, and normal are all called **separation axioms** for the reason that they involve "separating" certain kinds of sets from one another by disjoint open sets. We have used the word "separation" before, of course, when we studied connected space. But in that case, we were trying to find disjoint open sets whose *union was the entire space*. The present situation is quite different because the open sets need not satisfy this condition.

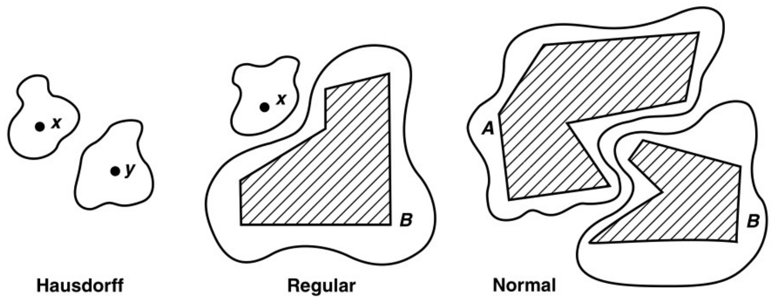

Figure 16.1: The three separation axioms illustrated

Example: Regular but not Hausdorff

Let $\mathbb{R}_K$ be $\mathbb{R}$ with the K-topology, which is the topology that is almost the standard topology, except K is deemed *closed*.

K is closed in $\mathbb{R}_K$ and $0 \notin K$. Notice that there exists open neighborhoods of 0 that does not intersect with K – e.g., $\mathbb{R} \setminus K$. However, 0 and K can't be separated by disjoint open neighborhoods. Thus, we have that $\mathbb{R}_K$ is not regular. On HW, we have already shown that $\mathbb{R}_K$ is Hausdorff.

16.5 Discussion

Remark: On [Theorem 31.2]

Notice that there is no analogous theorem for normal spaces. This is because we see that we are not guaranteed that a closed subset in Y will be closed in X – meaning that we cannot utilize the normality of X. Now if Y were given to be a closed subset of X itself, then an analogous theorem would exist as a result.

Remark:

We will use the countability and separation axioms to show that certain spaces are metric spaces, which are very strong spaces to work with.

Remark: On [Theorem 31.2]

We have that this theorem holds for arbitrary products of Hausdorff or regular spaces.

Chapter 17

Normal Spaces

17.1 Learning Objectives

- What extra conditions can be added onto a regular space to make it normal? [Theorem 32.1] and [Variation of Theorem 32.1]
- Are every metrizable spaces normal? [Theorem 32.2]
- Are compact Hausdorff spaces normal? Regular? [Theorem 32.3]
- Are locally compact Hausdorff spaces normal? Regular? [Variation of Theorem 32.3]

17.2 Theorems

Theorem: 32.1

Every regular space with a countable basis is normal.

Proof Let X be a regular space with a countable basis $\mathcal{B}$. Let A and B be disjoint closed subsets of X. We want to show that there exists disjoint open sets containing A and B. Using the regularity of X, we see that for each $x \in A$, there exists disjoint open sets U_x and W_x that contain x and B, respectively. By [Lemma 31.1], we see that there exists an open neighborhood V_x of x such that $\mathrm{Cl}\, V_x \subset U_x$. Furthermore, since X is generated by the basis $\mathcal{B}$, we see that there exists a basis element $B_x \in \mathcal{B}$ such that $x \in B_x \subset V$ – notice that B_x could be the same for multiple $x \in A$. By choosing such a basis element for each $x \in A$, we see that $\{B_x\}_{x\in A}$ is a countable covering of A whose closures do not intersect B: it is countable as $\{B_x\}_{x\in A}$ must be a subcollection of a countable set, which must still be countable; and, we see for each $x \in A$ that $B_x \subset V_x$, which implies that $\overline{B}_x \subset \overline{V}_x \subset U_x$, with each U_x being disjoint from B. Since this covering of A is countable, we can index it with the positive integers; let us denote it by $\{U_n\}_{n\in\mathbb{Z}_+}$.

Similarly, we can choose a countable collection $\{V_n\}_{n\in\mathbb{Z}_+}$ of open sets covering B, such that each set $\overline{V}_n$ is disjoint from A. The sets $U = \bigcup_{n\in\mathbb{Z}_+} U_n$ $V = \bigcup_{n\in\mathbb{Z}_+} V_n$ are open sets containing A and B, respectively, but they need not be disjoint. We perform the following simple trick to construct two open sets that are disjoint. Given n, we define

$$U_n' = U_n \setminus \left(\bigcup_{i=1}^{n} \overline{V}_i \right) \quad \text{and} \quad V_n' = V_n \setminus \left(\bigcup_{i=1}^{n} \overline{U}_i \right)$$

Note that each set U_n' is open, being the difference of an open set U_n and a closed set $\bigcup_{i=1}^n \overline{V}_i$. Similarly, each set V_n' is open. The collection $\{U_n'\}_{n\in\mathbb{Z}_+}$ covers A because each $x \in A$ belongs to U_n for some n, and we have that x belongs to *none* of the sets $\overline{V}_i$. Similarly, the collection $\{V_n'\}_{n\in\mathbb{Z}_+}$ covers B.

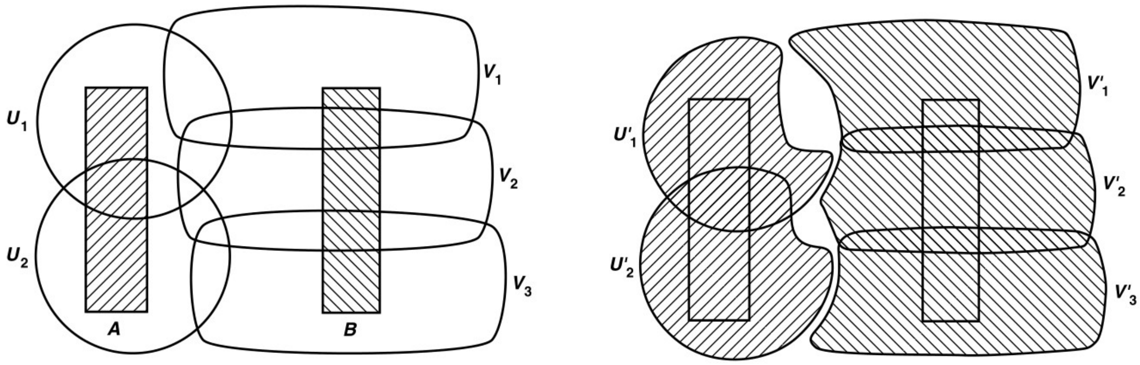

Figure 17.1: A visual on the constructed covers

Finally, the open sets

$$U' = \bigcup_{n\in\mathbb{Z}_+} U_n' \quad \text{and} \quad V' = \bigcup_{n\in\mathbb{Z}_+} V_n'$$

are disjoint. For if $x \in U' \cap V'$, then we see that $x \in U_j' \cap V_k'$ for some j and k. Suppose that $j \leq k$. It follows from the definition of U_j' that $x \in U_j$; and since $j \leq k$, it follows from the definition of V_k' that $x \notin \overline{U}_j$. A similar contradiction arises if $j \geq k$.

Theorem: Variation of [Theorem 32.1]

Let X be regular and Lindelöf. Then X is normal.

Proof Let X be regular and Lindelöf. Let A and B be disjoint and closed in X. We claim that there exists an open cover $\mathcal{U}$ of A such that, for all $U \in \mathcal{U}$, we have that $\overline{U} \cap B = \emptyset$. Likewise, we further claim that there exists an open cover $\mathcal{V}$ of B such that, for all $V \in \mathcal{V}$, we have that $\overline{V} \cap A = \emptyset$.

To show why, we have that for all $a \in A$, $X \setminus B$ is an open neighborhood of a in X. By [Lemma 31.1], we see that there exists an neighborhood U_a of a such that $\overline{U_a} \subset X \setminus B$. Thus, we see that $\overline{U_a} \cap B = \emptyset$. Let $\mathcal{U} = \{U_a \mid a \in A\}$. Likewise, we can construct the desired $\mathcal{V}$ the same way.

Since A and B are closed and X is Lindelöf, we have that $\mathcal{U}$ and $\mathcal{V}$ both admit countable subcovers by [Variation of Theorem 30.2]. We shall denote them to be as follows:

$$\{U_i \mid i \in \mathbb{Z}_+\} \quad \text{and} \quad \{V_i \mid i \in \mathbb{Z}_+\}$$

where

$$\overline{U_i} \cap B = \emptyset \quad \text{and} \quad \overline{V_i} \cap A = \emptyset \quad \forall i \in \mathbb{Z}_+$$

We define new open covers

$$\begin{aligned} \tilde{U}_1 &= U_1 \setminus \overline{V_1} & \qquad \tilde{V}_1 &= V_1 \setminus \overline{U_1} \\ \tilde{U}_2 &= U_2 \setminus (\overline{V_1} \cup \overline{V_2}) & \tilde{V}_2 &= V_2 \setminus (\overline{U_1} \cup \overline{U_2}) \\ &\vdots \qquad\qquad\qquad \text{and} & &\vdots \\ \tilde{U}_n &= U_n \setminus \left(\bigcup_{j=1}^{n} \overline{V_j} \right) & \tilde{V}_n &= V_n \setminus \left(\bigcup_{j=1}^{n} \overline{U_j} \right) \end{aligned}$$

Notice that for each n, we have that $\bigcup_{j=1}^n \overline{U_j}$ and $\bigcup_{j=1}^n \overline{V_j}$ are both closed in X as they are a finite union of closed sets. We now set

$$U = \bigcup_{i=1}^{\infty} \tilde{U}_i \quad \text{and} \quad V = \bigcup_{j=1}^{\infty} \tilde{V}_j$$

which we see are disjoint open neighborhoods of A and B.

Theorem: 32.2

Every metrizable space is normal.

Proof Let X be a metric space with metric d. We have already seen metric spaces be Hausdorff. Let disjoint closed subsets C, D in X be given. Because we have that $X \setminus D$ is open, we have for all $c \in C$, there exists $r_c > 0$ so that

$$B_{r_c}(c) \subset X \setminus D$$

Likewise, because $X \setminus C$ is open, we have for all $d \in D$, there exists $r_d > 0$ so

$$B_{r_d}(d) \subset X \setminus C$$

Let $U = \bigcup_{c \in C} B_{\frac{r_c}{2}}(c)$ and $V = \bigcup_{d \in D} B_{\frac{r_d}{2}}(d)$. Notice that U and V are still open neighborhoods of C and D, respectively. We want to show that $U \cap V = \emptyset$.

Towards a contradiction, let us suppose that $U \cap V \neq \emptyset$. Then, there exists some $p \in U \cap V$. Furthermore, there exists some $c \in C$ and $d \in D$ such that $p \in B_{\frac{r_c}{2}}(c) \cap B_{\frac{r_d}{2}}(d)$. The triangle inequality implies that

$$d(c, d) \leq d(c, p) + d(p, d) < \frac{r_c}{2} + \frac{r_d}{2}$$

Thus us now look at two possible cases:

Case 1: $r_c < r_d$ In this case, we see that $d(c, d) < r_d$, which tells us that $c \in B_{r_d}(d)$, which contradicts the fact $B_{r_d}(d) \subset X \setminus C$.

Case 2: $r_c > r_d$ In this case, we see that $d(c, d) < r_c$, which tells us that $d \in B_{r_c}(c)$, which contradicts the fact $B_{r_c}(c) \subset X \setminus D$.

Thus, we have that any metric space is Hausdorff.

Theorem: 32.3

Every compact Hausdorff space is normal.

Proof Let X be a compact Hausdorff space. Let A and B be two disjoint closed sets in X. Our goal is to show that there exists disjoint open sets K and L containing A and B, respectively.

Since both A and B are closed in X, we have that they must be compact themselves as X is compact. Hence, by [Theorem 26.4], we see that there exists disjoint open sets K and L containing A and B, respectively.

Theorem: Variation of [Theorem 32.3]

Every locally compact Hausdorff space is regular.

Proof Let X be locally compact Hausdorff. Let disjoint point $p \in X$ and closed C of X be given. Since C is closed, we see that $X \setminus C$ is open. Furthermore, $X \setminus C$ is an open neighborhood of p. Since X is locally compact, this implies that there exists a compact neighborhood K of p in $X \setminus C$. Let $U = \text{Int } K$. Notice that U is still an open neighborhood of p. Let $V = X \setminus K$. V is an open neighborhood of C because K is compact, hence closed since X is Hausdorff. As a result, we see that $U \cap V = \emptyset$.

Theorem: 32.4

Every well-ordered set X is normal in the order topology.

Proof

17.3 Examples

Example: $\mathbb{R}_\ell$ is normal

Let two disjoint closed sets A, B of $\mathbb{R}_\ell$ be given. To show that $\mathbb{R}_\ell$ is normal, we first need to show that it is T_1; indeed, we have already shown this to be the case. Now, because $X \setminus B$ is open and contains A, we have that for all $a \in A$, there exists $x_a > a$ so $[a, x_a) \subset X \setminus B$. Let us define the following:

$$U = \bigcup_{a \in A} [a, x_a).$$

which we see is an open neighborhood of A. Similarly, because $X \setminus A$ is open and contains B, we have that for all $b \in B$, there exists $x_b > b$ so $[b, x_b) \subset X \setminus A$. Let us define the following:

$$V = \bigcup_{b \in B} [b, x_b).$$

which we see is an open neighborhood of B.

We claim that $U \cap V = \emptyset$. Towards a contradiction, let's assume that $U \cap B \neq \emptyset$. This mean that

$$[a, x_a) \cap [b, x_b) \neq \emptyset$$

WLOG, let's say that $a < b$. If the intersection is nonempty, then this means that we must have the following situation

$$a < b < x_a < x_b \implies x_a \in [b, x_b) \implies x_a \in B$$

Which is a contradiction.

Example: The Sorgenfrey Plane is not normal

Recall that the Sorgenfrey Plane was defined to be $\mathbb{R}_\ell \times \mathbb{R}_\ell$. Let us look at the line $L : y = -x$. As a subspace, we see that L has the discrete topology. We can see that L is closed in $\mathbb{R}_\ell \times \mathbb{R}_\ell$. Furthermore, every subset of L is closed in L. So, by transitivity, every subset of L is closed in $\mathbb{R}_\ell \times \mathbb{R}_\ell$. Let $A = \{\text{points in } L \text{ with rational coordinates}\}$. Let $B = \{\text{points in } L \text{ with irrational coordinates}\}$. We can see that both A and B are disjoint and closed, but not separable by disjoint open neighborhoods. The technical explanation is given in Munkres.

17.4 Discussion

Remark: More Examples and Proofs

More examples as well as the rest of the proofs can be found in §32 of Munkres.

Chapter 18

The Urysohn Lemma

18.1 Learning Objectives

- What does it mean for two subsets of a topological space to be separated by a continuous function?
- What does the Urysohn Lemma state? [Theorem 33.1]
- What are the main ideas behind the proof? [Theorem 33.1]

18.2 Definitions

Definition: Separated by a Continuous Function

If A and B are two subsets of the topological space X, and if there is a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, we say that A and B **can be separated by a continuous function**.

Definition: Completely Regular

A space X is **completely regular** if one-point sets are closed (i.e. T_1) in X and if for each point x_0 and each closed set A not containing x_0, there is a continuous function $f : X \to [0,1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.

Definition: Well-Ordered

A set A with an order relation $<$ is said to be **well-ordered** if every nonempty subset of A has a smallest element.

18.3 Theorems

Theorem: 10.1

Every nonempty finite ordered set has the order type of a section $\{1,\ldots,n\}$ of $\mathbb{Z}_+$, so it is well-ordered.

Proof We shall first show that every finite ordered set A has a smallest element. If A has one element, this is trivial. Let us now suppose that this is true for sets having $n-1$ elements; let A have n elements and let $a_0 \in A$. Then $A \setminus \{a_0\}$ has a smallest element a_1, and the smaller of $\{a_0, a_1\}$ is the smallest element of A.

We now show that there is an order-preserving bijection of A with $\{1,\ldots,n\}$ for some n. If A has one element, this fact is trivial as $f : A \to n$, where $n \in \mathbb{Z}_+$, is such a bijection. Let us now suppose that this is true for sets having $n-1$ elements; let A have n elements. Furthermore, let b be the smallest element of A. By hypothesis, there is an order-preserving bijection

$$f' : A \setminus \{b\} \to \{1,\ldots,n-1\}$$

Now, define an order-preserving bijection $f : A \to \{1,\ldots,n\}$ by setting

$$f(b) = n \quad \text{and} \quad f(x) = f'(x) \quad \text{for } x \neq b$$

Theorem: 33.1 (Urysohn lemma)

Let X be a normal space; let A and B be disjoint closed subsets of X. Let $[a,b]$ be a closed interval in the real line. Then there exists a continuous map

$$f : X \to [a,b]$$

such that $f(x) = a$ for every $x \in A$, and $f(x) = b$ for every $x \in B$.

Proof Notice that $[a,b] \cong [0,1]$; futhermore, we have that $f(x) = a$ for every $x \in A$ implies that $A \subset f^{-1}(a)$. Thus, we can rephrase the lemma as follows:

Let X be a normal space; let A and B be disjoint closed subsets of X. Let $[0,1]$ be a closed interval in the real line. Then there exists a continuous map

$$f : X \to [0,1]$$

such that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$.

The first step of the proof is to construct, using normality, a certain family U_p of open sets of X, indexed by the rational numbers. Then one uses these sets to define the continuous function f.

Step 1 Let P be the set of all rational numbers in the interval $[0,1]$ – in fact, any countable dense subset of $[0,1]$ will do providing that it contains 0 and 1. We shall define, for each $p \in P$, an open set U_p of X, in such a way that whenever $p < q$, we have

$$\overline{U_p} \subset U_q$$

Thus, the sets U_p will be simply ordered by inclusion in the same way their subscripts are ordered by the usual ordering in the real line.

Because P is countable, we can use induction to define the sets U_p (or rather, the principle of recursive definition). Arrange the elements of P in an infinite sequence in some way; for convenience, let us suppose that the numbers 0 and 1 are the first two elements of the sequence.

Now define the sets U_p, as follows: Because A is a closed set contained in the open set $X \setminus B$, the normality of X (namely, [Lemma 31.1]) allows us to choose an open set U_0 such that

$$A \subset U_0 \quad \text{and} \quad \overline{U_0} \subset X \setminus B$$

From here, we can define U_1 to be $X \setminus B$ to get that $\overline{U_0} \subset U_1$, as desired. This solves the base case for our induction.

In general, let P_n denote the set consisting of the first n rational numbers in the sequence. As our induction hypothesis, let us suppose that U_p is defined for all rational numbers p belonging to the set P_n, satisfying the condition

$$p < q \implies \overline{U_p} \subset U_q$$

Let r denote the next rational number in the sequence; we wish to define U_r.

Consider the set $P_{n+1} = P_n \cup \{r\}$. It is a finite subset of the interval $[0,1]$, and, as such, it has a simple ordering derived from the usual order relation $<$ on the real line. We know that in a finite simply ordered set, every element (other than the smallest and the largest) has an immediate predecessor and an immediate successor (see [Theorem 10.1]). The number 0 is the smallest element, and 1 is the largest element, of the simply ordered set P_{n+1}, and r is neither 0 nor 1. So, we have that r has an immediate predecessor p in P_{n+1} and an immediate successor q in P_{n+1}. The sets U_p and U_q are already defined. Thus, by the induction hypothesis, we see that $p < q$ implies that $\overline{U_p} \subset U_q$. Using the normality of X (namely, [Lemma 31.1]), we can find an open set U_r of X such that

$$\overline{U_p} \subset U_r \quad \text{and} \quad \overline{U_r} \subset U_q$$

Notice that the following condition

$$p < q \implies \overline{U_p} \subset U_q$$

holds for every pair of elements of P_{n+1}. If both elements lie in P_n, then the condition holds by the induction hypothesis. If one of them is r and the other is a point s of P_n, then either $s \le p$, in which case

$$\overline{U_s} \subset U_p \subset \overline{U_p} \subset U_r$$

or $s \ge p$, in which case

$$\overline{U_r} \subset U_q \subset \overline{U_q} \subset U_s$$

Thus, for every pair of elements of P_{n+1}, we have that the earlier condition holds.

By induction, we have that U_p is defined for all $p \in P$.

$\boxed{\text{Step 2}}$ Now that have defined U_p for all rational numbers p in the interval $[0,1]$, we can extend this definition to all rational numbers p in $\mathbb{R}$ by defining

$$\begin{aligned} U_p &= \emptyset \quad \text{if } p < 0 \\ U_p &= X \quad \text{if } p > 1 \end{aligned}$$

With this, we can still see that for any pair of rational numbers p and q,

$$p < q \implies \overline{U_p} \subset U_q$$

$\boxed{\text{Step 3}}$ Given a point x of X, let us now define $\mathbb{Q}(x)$ to be the set of those rational numbers p such that the corresponding open sets U_p contain x:

$$\mathbb{Q}(x) = \{p \in \mathbb{Q} \mid x \in U_p\}$$

Notice that this set contains no numbers less than 0, since no x is in U_p for $p < 0$ because $U_p = \emptyset$ in such a scenario. Furthermore, we see that this set contains every number greater than 1, since every x is in U_p for $p > 1$ because $U_p = X$ in such a scenario. Therefore, we see that $\mathbb{Q}(x)$ is bounded below, and its greatest lower bound is a point of the interval $[0,1]$. Knowing this, we define

$$f : X \to \mathbb{R} \quad \text{where} \quad f(x) = \inf \mathbb{Q}(x) = \inf\{p \in \mathbb{Q} \mid x \in U_p\}$$

We claim that f is the desired function.

$\boxed{\text{Step 4}}$ To verify that f is the desired function, we need to show that $f(X) = [0,1]$, $f(x) = 0$ for all $x \in A$, $f(x) = 1$ for all $x \in B$, and that f is continuous.

By the way that we constructed $\mathbb{Q}(x)$ and U_p, we see that the image of f is indeed $[0,1]$ – that is, $f(X) = [0,1]$. If $x \in A$, then we end up getting that $x \in U_p$ for every $p \geq 0$ so that $\mathbb{Q}(x) = \mathbb{Q}_{\geq 0}$; thus, we end up getting that $f(x) = \inf \mathbb{Q}(x) = 0$. Similarly, if $x \in B$, then we end up getting that $x \in U_p$ for no $p \leq 1$ – that is for $p > 1$; thus, we get that $\mathbb{Q}(x) = \mathbb{Q}_{>1}$, which results in us getting that $f(x) = \inf \mathbb{Q}(x) = 1$.

All that is left to show is that f is continuous. For this purpose, we first prove the following facts:

1. $x \in \overline{U_r} \implies f(x) \leq r$
2. $x \notin U_r \implies f(x) \geq r$

To prove (1), notice that for any $s > r$, we have that $\overline{U_r} \subset U_s$. Thus, if $x \in \overline{U_r}$, then it must be the case that $x \in U_s$ for any $s > r$. Therefore, we see that $\mathbb{Q}(x)$ contains all rational numbers greater than r; by definition, we have

$$f(x) = \inf \mathbb{Q}(x) \leq r$$

To prove (2), notice that for any $s < r$, we have that $U_s \subset \overline{U_s} \subset U_r$. Thus, if $x \notin U_r$, then it must be the case that $x \notin U_s$ for any $s < r$. Therefore, we see that $\mathbb{Q}(x)$ contains no rational numbers less than r; by definition, we have

$$f(x) = \inf \mathbb{Q}(x) \geq r$$

We can now prove that f is continuous.

Let x_0 be a given point of X; let (c,d) be an open interval in $\mathbb{R}$ containing the point $f(x_0)$. Our goal is to find a neighborhood U of x_0 such that $f(U) \subset (c,d)$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we are able to choose rational numbers p and q such that

$$x < p < f(x_0) < q < d$$

We claim that

$$U = U_q \setminus \overline{U_p}$$

is the desired open neighborhood of x_0.

It is clear to see that U is open, as $\overline{U_p}$ is closed. To show that $x_0 \in U$, we note that the contrapositive of (2) tells us that $f(x_0) < q \implies x \in U_q$; the contrapositive of (1) tells us that $f(x_0) > p \implies x \notin \overline{U_p}$. Hence, it follows that $x_0 \in U$.

We now want to show that $f(U) \subset (c,d)$. Let $x \in U$ be given. It follows that $x \in U_q \subset \overline{U_q}$, which results in $f(x) \leq q$ by (1). Furthermore, it follows that $x \notin \overline{U_p}$, which means that $x \notin U_p$, which also results in $f(x) \geq p$ by (2). Thus, we have that $f(x) \in [p,q] \subset (c,d)$. Since this is true for all $x \in U$, we see that $f(U) \subset [p,q] \subset (c,d)$, as desired.

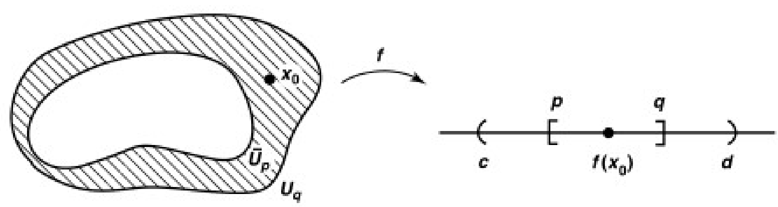

Figure 18.1: A visual on the continuity of f

Theorem: 33.2

A subspace of a completely regular space is completely regular. A product of completely regular spaces is completely regular.

Proof

18.4 Examples

Example: Completely Regular Spaces

Normal spaces are completely regular.

Example: Illustration of Step 1 in the [Urysohn Lemma] Proof

Let us suppose we started with the standard way of arranging the elements of P in an infinite sequence:

$$P = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \dots\right\}$$

After defining U_0 and U_1 as done in the proof, we would define $U_{\frac{1}{2}}$ (using the normality of X) so that

$$\overline{U_0} \subset U_{\frac{1}{2}} \quad \text{and} \quad \overline{U_{\frac{1}{2}}} \subset U_1$$

Then, we would fit $U_{\frac{1}{3}}$ in between U_0 and $U_{\frac{1}{3}}$ so that

$$\overline{U_0} \subset U_{\frac{1}{3}} \quad \text{and} \quad \overline{U_{\frac{1}{3}}} \subset U_{\frac{1}{2}}$$

Then, we would fit $U_{\frac{2}{3}}$ in between $U_{\frac{1}{2}}$ and U_1 so that

$$\overline{U_{\frac{1}{2}}} \subset U_{\frac{2}{3}} \quad \text{and} \quad \overline{U_{\frac{2}{3}}} \subset U_1$$

And so on.

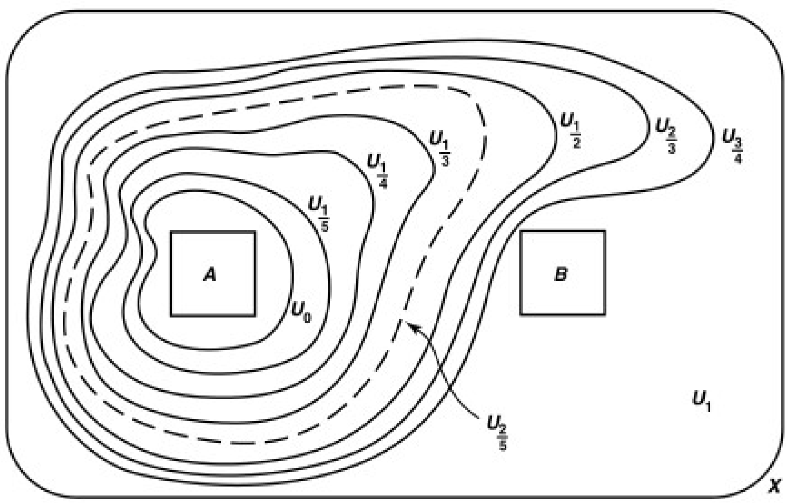

Figure 18.2: An Illustration of Step 1 in the Urysohn Lemma

18.5 Discussion

Remark: On the Urysohn Lemma

The Urysohn Lemma says that if every pair of disjoint closed sets in X can be separated by disjoint open sets, then each such pair can be separated by a continuous function.

Notice that the converse is trivial – that is, if a pair of disjoint closed sets in X can be separated by a continuous function, then each such pair can be separated by disjoint open sets; if $f : X \to (0,1)$ is the function, then $f^{-1}([0, \frac{1}{2}))$ and $f^{-1}((\frac{1}{2}, 1])$ are disjoint open sets containing A and B, respectively.

The proof of the Urysohn lemma cannot be generalized to show that in a regular space, where you can separate points from closed sets by disjoint open sets, you can also separate points from closed sets by continuous functions.

At first glance, it seems that the proof of the Urysohn lemma should go through. You take a point a and a closed set B not containing a, and you begin the proof just as before by defining $U_1 = X \setminus B$ and choosing U_0 to be an open set about a whose closure is contained in U_1 (using the regularity of X). But at the very next step of the proof, you run into difficulty. Suppose that p is the next rational number in the sequence after 0 and 1. You want to find an open set U_p such that $\overline{U_0} \subset U_p$ and $\overline{U_p} \subset U_1$ – for this, regularity is not enough.

Remark: On Completely Regular

Requiring that one be able to separate a point from a closed set by a continuous function is, in fact, a stronger condition than requiring that one can separate them by disjoint open sets. This requirement is made into a new separation axiom, which we defined to be [completely regular].

Remark: On Well-Ordered

Given a set A without an order relation, it is natural to ask whether there exists an order relation for A that makes it into a well-ordered set. If A is finite, any bijection

$$f : A \to \{1, \ldots, n\}$$

can be used to define an order relation on A; under this relation, A has the same order type as the ordered set $\{1, \ldots, n\}$. In fact, every order relation on a finite set can be obtained this way, as shown in [Theorem 10.1].

Chapter 19

The Urysohn Metrization Theorem

19.1 Learning Objectives

- What is the Urysohn Metrization Theorem? [Theorem 34.1]
- What is the Imbedding Theorem and how is it utilized in the proof for the Urysohn Metrization Theorem? [Theorem 34.2] and [Theorem 34.1]

19.2 Theorems

Theorem: 34.1 (Urysohn Metrization Theorem)

Every regular space X with a countable basis is metrizable.

Proof We shall prove that X is metrizable by imbedding X in a metrizable space Y; that is, by showing that X is homeomorphic with a subspace of Y. There are two version of this proof, and they differ in the choice of the metrizable space Y. In the first version, Y is the space $\mathbb{R}^\omega$ in the product topology, a space that we have previously proved to be metrizable (Theorem 20.5). In the second version, the space Y is also $\mathbb{R}^\omega$, but this time in the topology given by the uniform metric $\overline{\rho}$ (see §20). In each case, it turns out that our construction actually imbeds X in the subspace $[0,1]^\omega$ of $\mathbb{R}^\omega$.

We shall focus on the first version of the proof and want to prove the following:

There exists a countable collection of continuous functions $f_n : X \to [0,1]$ having the property that given any point x_0 of X and any neighborhood U of x_0, there exists an index n such that f_n is positive at x_0 and vanishes outside U.

Notice that because we are given X to be regular and second-countable, it follows by [Theorem 32.1] that X is normal.

Let $\mathcal{B} = \{B_n\}$ be a countable basis for X. For each pair n, m of indices for which $\overline{B}_n \subset B_m$, we can apply the [Urysohn Lemma] to choose a continuous function $g_{n,m} : X \to [0,1]$ such that

$$g_{n,m}(\overline{B}_n) = \{1\} \quad \text{and} \quad g_{n,m}(X \setminus B_m) = \{0\}$$

We shall now define $\mathcal{G} = \{g_{n,m}\}$.

We can see that $\mathcal{G}$ satisfies the property that we're after: given any x_0 of X and any neighborhood U of x_0, we see that there exists some basis element B_m such that $x \in B_m \subset U$. Because we know that X is regular, it follows by [Lemma 31.1] that there exists some open neighborhood V of x_0 such that $\overline{V} \subset B_m$. Because $\mathcal{B}$ is a basis for X, we see that there exists some $B_n \in \mathcal{B}$ such that $x \in B_n \subset V$. Hence, it follows that $\overline{B}_n \subset \overline{V}$, which results in $\overline{B}_n \subset B_m$.. Then, we see that n, m is a pair of indices for which the function $g_{n,m}$ is defined, which we know is positive at x_0 (in fact, it is 1) and vanishes outside U (as $X \setminus U \subset X \setminus B_m$, and we know that $g_{n,m}(X \setminus B_m) = \{0\}$). Furthermore, we can see that $\mathcal{G}$ is countable since the collection $\{g_{n,m}\}$ is indexed with a subset of $\mathbb{Z}_+ \times \mathbb{Z}_+$, which we know is countable. Therefore, we see that it can be reindexed with the positive integers, giving us the desired collection $\{f_n\}$.

By the [Imbedding Theorem], we see that the function $F : X \to \mathbb{R}^\omega$ defined by

$$F(x) = (f_n)_{n \in \mathbb{Z}_+}$$

is an imbedding of X in $\mathbb{R}^\omega$. Because $\mathbb{R}^\omega$ is metrizable, hence any of its subsets are also metrizable, we see that X must be metrizable as well – in fact, because we see that since each f_n maps X into $[0,1]$ for all $n \in \mathbb{Z}_+$, we have that F imbeds X in $[0,1]^\omega$.

Theorem: 34.2 (Imbedding Theorem)

Let X be a space in which one-point sets are closed (i.e., X is closed). Suppose that $\{f_i\}_{i \in I}$ is an indexed family of continuous functions $f_i : X \to \mathbb{R}$ satisfying the requirement that for each point x_0 of X and each neighborhood U of x_0, there is an index i such that f_i is positive at x_0 and vanishes outside U. Then the function $F : X \to \mathbb{R}^I$ defined by

$$F(x) = (f_i(x))_{i \in I}$$

is an imbedding of X in $\mathbb{R}^I$. If f_i maps X into $[0,1]$ for each i, then F imbeds X in $[0,1]^I$.

Proof Our goal is to show that F is an imbedding. Recall that an imbedding is an injection that is homeomorphic to its image.

To show that F is continuous, we first see that each of its components f_i are given to be continuous. Because $\mathbb{R}^I$ has the product topology, it follows by [Theorem 18.4] that F is continuous.

To show that F is injective, let us suppose that $x \neq y$. Since we have that X is T_1, we have that both $\{x\}$ is closed, which makes $X \setminus \{x\}$ an open neighborhood of y in X. It follows by the requirements of our functions f_i that there exists some index i such that $f_i(x) = 0$ and $f_i(y) > 0$; therefore, we get have $F(x) \neq F(y)$.

We now just need to show that F is a homeomorphism of X onto its image, which is the subspace $Z = F(X)$ of $\mathbb{R}^I$. It is clear to see that F is a continuous bijection of X with Z; thus, all that is left to show is that for each open set U in X, the set $F(U)$ is open in Z. Let z_0 be a point of $F(U)$. We want to find an open set W of Z such that

$$z_0 \in W \subset F(U)$$

Because we know that F is a bijection (namely, surjection), we see that there exists some $x_0 \in U$ such that $F(x_0) = z_0$. By the requirements of our functions f_i, we see that there exists some index N for which $f_N(x_0 > 0)$ and $f_N(X \setminus U) = \{0\}$. Take the open ray $(0, +\infty)$ in $\mathbb{R}$, and let V be the open set

$$V = \text{proj}_N^{-1}((0, +\infty))$$

of $\mathbb{R}^I$. Let $W = V \cap Z$; we can see that W is open in Z by definition of the subspace topology.

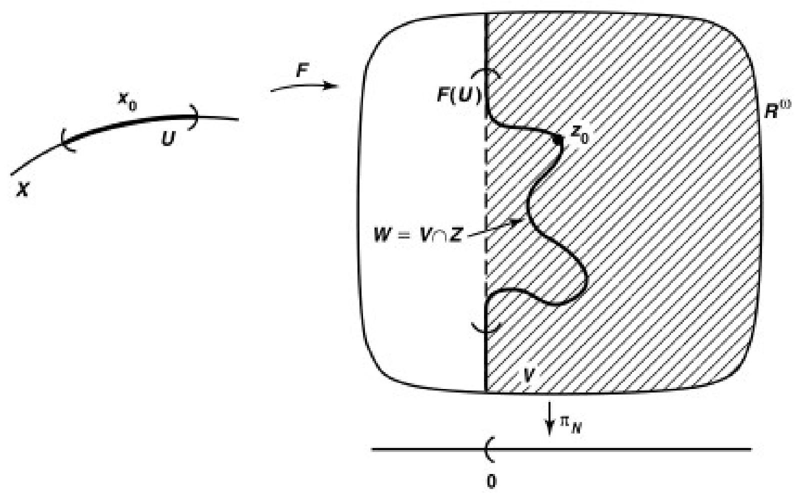

Figure 19.1: A visual on W

We claim that $z_0 \in W \subset F(U)$. To start, we are given that $z_0 \in F(U) \subset Z$. Notice that

$$\text{proj}_N(z_0) = \text{proj}_N(F(x_0)) = f_N(x_0) > 0$$

which tells us that $z_0 \in V$; thus, we see that $z \in (V \cap Z) = W$.

We now want to show that $W \subset F(U)$. Let $z \in W$ be given. Since F is a bijection (namely, a surjection), we see that there exists some $x \in X$ such that $z = F(x)$. Furthermore, we have that $z \in V$ (by definition of W), which tells us that $\text{proj}_N(z) \in (0, +\infty)$. Since $\text{proj}_N(z) = \text{proj}_N(F(x)) = f_N(x)$, it must be that case that $f_N(x) > 0$. Because f_N vanishes outside U, the point x must be in U. Thus, we see that $z = F(x)$ is in $F(U)$, as desired.

Putting everything together tells us that F is an imbedding of X in $\mathbb{R}^I$.

Theorem: 34.3

A space X is completely regular if and only if it is homeomorphic to a subspace of $[0,1]^I$ for some I.

Proof

Chapter 20

The Tietze Extension Theorem

20.1 Learning Objectives

- What is a continuous extension?
- What does the Tietze Extension Theorem say? [Theorem 35.1]
  - What are the key steps in the proof?

20.2 Definitions

Definition: Continuous Extension

Let X be a topological space and A a subspace of X. Let $f : A \to Y$ be a continuous function. A **continuous extension** of this $f : A \to Y$ to X is a continuous function $g : X \to Y$ that satisfies $g|_A = f$.

20.3 Theorems

Theorem: 35.1 (Tietze Extension Theorem)

Let X be a normal space; let A be a closed subspace of X

(a) Any continuous map of A into the closed interval $[a, b]$ of $\mathbb{R}$ may be extended to a continuous map of all of X into $[a, b]$.

(b) Any continuous map of A into $\mathbb{R}$ may be extended to a continuous map of all of X into $\mathbb{R}$.

Proof

Main Idea of the Proof

The idea of the proof is to construct a sequence of continuous functions s_n defined on the entire space X, such that the sequence s_n converges uniformly, and such that the restriction of s_n to A approximates f more and more closely as n becomes large. Then the limit function will be continuous, and its restriction to A will equal f.

Step 1

The first step is to construct a particular function g defined on all of X such that g is not too large, and such that g approximates f on the set A to a fair degree of accuracy. To be more precise, let us take the case $f : A \to [-r, r]$. We assert that there exists a continuous function $g : X \to \mathbb{R}$ such that

$$|g(x)| \leq \frac{1}{3}r \quad \forall x \in X$$
$$|g(a) - f(a)| \leq \frac{2}{3}r \quad \forall a \in A$$

The function g is constructed as follows: Divide the interval $[-r, r]$ into three equal intervals of length $\frac{2}{3}r$:

$$I_1 = [-r, -\tfrac{1}{3}r] \quad I_2 = [-\tfrac{1}{3}r, \tfrac{1}{3}r] \quad I_3 = [\tfrac{1}{3}r, r]$$

Let B and C be the subsets

$$B = f^{-1}(I_1) \quad \text{and} \quad C = f^{-1}(I_3)$$

of A. Because f is continuous, we see that

$$I_1 \cap I_3 = \emptyset \implies f^{-1}(I_1 \cap I_3) = f^{-1}(\emptyset) \iff f^{-1}(I_1) \cap f^{-1}(I_3) = \emptyset$$

which tells us that B and C are closed disjoint subset of A. Since A is given to be a closed subspace of X, we see by the transitivity of closed sets that B and C are also disjoint closed subsets of X.

By the [Urysohn Lemma], we see that there exists a continuous function

$$g : X \to [-\tfrac{1}{3}r, \tfrac{1}{3}r]$$

having the property that $g(x) = -\frac{1}{3}r$ for each $x \in B$, and $g(x) = \frac{1}{3}r$ for each $x \in C$. Then, we have that $|g(x)| \leq \frac{1}{3}r$ for all $x \in X$.

From here, we claim that for each $a \in A$

$$|g(a) - f(a)| \leq \frac{2}{3}r$$

There are three cases. If $a \in B$, then we have that

$$g(a) = -\frac{1}{3}r \quad \text{and} \quad f(a) \in I_1 \implies -r \leq f(a) \leq -\frac{1}{3}r$$

If $a \in C$, then we have that

$$g(a) = \frac{1}{3}r \quad \text{and} \quad f(a) \in I_3 \implies \frac{1}{3}r \leq f(a) \leq r$$

If $a \notin B \cup C$, then we have that both $g(a)$ and $f(a)$ are in I_2, which results in

$$-\frac{1}{3}r \leq g(a) \leq \frac{1}{3}r \quad \text{and} \quad -\frac{1}{3}r \leq f(a) \leq \frac{1}{3}r$$

In either case, we see that $|g(a) - f(a)| \leq \frac{2}{3}r$ – see Figure (20.1).

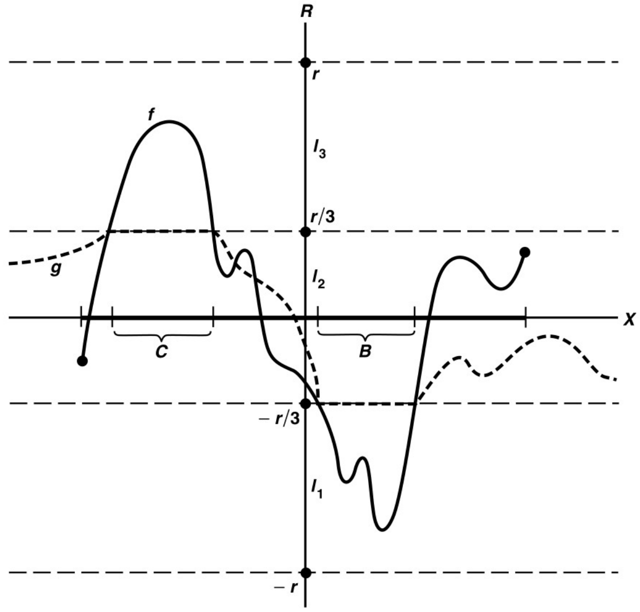

Figure 20.1

Step 2

We now prove part (a) of the Tietze theorem. WLOG, we can replace the arbitrary closed interval $[a,b]$ of $\mathbb{R}$ by the interval $[-1,1]$.

Let $f : X \to [-1,1]$ be a continuous map. Then f satisfies the hypotheses of Step 1, with $r = 1$. Therefore, we have that there exists a continuous real-valued function g_1, defined on all of X, such that

$$|g_1(x)| \leq \frac{1}{3} \quad \forall x \in X$$
$$|f(a) - g_1(a)| \leq \frac{2}{3} \quad \forall a \in A$$

Now consider the function $f - g_1$. This function maps A into the interval $[-\frac{2}{3}, \frac{2}{3}]$, so we can apply Step 1 again, letting $r = \frac{2}{3}$. We obtain a real-valued function g_2 defined on all of X such that

$$|g_2(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right) \quad \forall x \in X$$
$$|f(a) - g_1(a) - g_2(a)| \leq \left(\frac{2}{3}\right)^2 \quad \forall a \in A$$

Then we apply Step 1 to the function $f - g_1 - g_2$. And so on.

At the general step, we have the real-valued function $g_1, \ldots, g_n$ defined on all of X such that

$$|f(a) - g_1(a) - \ldots - g_n(a)| \leq \left(\frac{2}{3}\right)^n$$

for $a \in A$. Applying Step 1 to the function $f - g_1 - \ldots - g_n$, with $r = (\frac{2}{3})^n$, we obtain a real-valued function g_{n+1} defined on all of X such that

$$|g_{n+1}(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^n \quad \forall x \in X$$
$$|f(a) - g_1(a) - \ldots - g_n(a) - g_{n+1}(a)| \leq \left(\frac{2}{3}\right)^{n+1} \quad \forall a \in A$$

By induction, the functions g_n are defined for all n.

We now define

$$g(x) = \sum_{n=1}^{\infty} g_n(x)$$

for all $x \in X$. We can see with how each g_n is defined that this infinite series converges; this follows from the comparison theorem of calculus, where it converges by comparison with the geometric series

$$\frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1}$$

To show that g is continuous, we must show that the sequence

$$s_n(x) = \sum_{i=1}^{n} g_i(x)$$

converges to $g(x)$ uniformly for all $x \in X$. This fact follows at once from the "Weierstrass M-test" of analysis. Without assuming this result, one can simply note that if $k > n$, then

$$\begin{aligned}
|s_k(x) - s_n(x)| &= \left|\sum_{i=n+1}^{k} g_i(x)\right| \\
&\leq \frac{1}{3}\sum_{i=n+1}^{k}\left(\frac{2}{3}\right)^{i-1} \\
&< \frac{1}{3}\sum_{i=n+1}^{\infty}\left(\frac{2}{3}\right)^{i-1} = \left(\frac{2}{3}\right)^{n}
\end{aligned}$$

Holding n fixed and letting $k \to \infty$, we see that

$$|g(x) - s_n(x)| \leq \left(\frac{2}{3}\right)^{n}$$

for all $x \in X$. Therefore, we see that s_n converges to g uniformly.

We now want to show that $g(a) = f(a)$ for all $a \in A$ – that is, we want to show that the restriction of g to A equals f. Let

$$s_n(x) = \sum_{i=1}^{n} g_i(x)$$

be the nth partial sum of the series. Then $g(x)$ is by definition the limit of the infinite sequence $s_n(x)$ of partial sums, as we have shown earlier. Since

$$\left|f(a) - \sum_{i=1}^{n} g_i(a)\right| = |f(a) - s_n(a)| \leq \left(\frac{2}{3}\right)^{n}$$

for all $a \in A$, it follows that $s_n \to f(a)$ for all $a \in A$. Therefore, we have that $f(a) = g(a)$ for all $a \in A$.

Finally, we want to show that g maps X into the interval $[-1,1]$. This condition is in fact satisfied automatically, since we have already shown earlier that $g(x)$ converges for all $x \in X$ by the comparison theorem of calculus – that is, we have

$$|g(x)| \leq \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1} = 1$$

However, this is just a lucky accident rather than an essential part of the proof. If all we knew was that g mapped X into $\mathbb{R}$, then the map $r \circ g$, where $r : \mathbb{R} \to [-1,1]$ is the map

$$\begin{aligned} r(y) &= y && \text{if } |y| \leq 1 \\ r(y) &= \frac{y}{|y|} && \text{if } |y| \geq 1 \end{aligned}$$

would be an extension of f mapping X into $[-1,1]$.

Step 3

We now prove part (b) of the theorem, in which f maps A into $\mathbb{R}$. We can replace $\mathbb{R}$ by the open interval $(-1,1)$, since this interval is homeomorphic to $\mathbb{R}$.

Let f be a continuous map from A into $(-1,1)$. The half of the Tietze theorem already proved shows that we can extend f to a continuous map $g : X \to [-1,1]$ mapping X into the *closed* interval. How can we find a map h carrying X into the *open* interval?

Given g, let us define a subset D of X by the equation

$$D = g^{-1}(\{-1\}) \cup g^{-1}(\{1\})$$

Since g is continuous, we have that D is the finite union of closed sets of X, which is still a closed set of X. Because we have shown that $g(A) = f(A)$, which is contained in $(-1,1)$ by construction of f, we have that A must be disjoint from D.

By the [Urysohn Lemma], there is a continuous function $\phi : X \to [0,1]$ such that $\phi(D) = \{0\}$ and $\phi(A) = \{1\}$. With this, we define

$$h(x) = \phi(x)g(x)$$

Then h is continuous, being the product of two continuous functions. Also, we see that h is an extension of f, since for $a \in A$, we have

$$h(a) = \phi(a)g(a) = 1 \cdot g(a) = f(a)$$

Finally, h maps all of X into the open interval $(-1,1)$; for if $x \in D$, then $h(x) = 0 \cdot g(x) = 0$, and if $x \notin D$, then $|g(x)| < 1$, which results in us getting that $|h(x)| \leq 1 \cdot |g(x)| < 1$.

20.4 Examples

Example: Continuous Extension

Let $A = [1, \infty)$ and $X = \mathbb{R}$. Let $f : A \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$. We are able to make a continuous extension of f to all of $\mathbb{R}$ by just glueing on another continuous function to f.

Let's now consider the slightly altered: $A = (0, \infty)$ and $X = \mathbb{R}$. Let $f : A \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$. We can see in this case that there does not exist a continuous extension of f – our function diverges at 0, and there is no way that we can "glue" on another continuous function to make an extension of f continuous.

20.5 Discussion

Remark: More Examples

More examples can be seen in §35 in Munkres.