

Dynamic Programming Solution to Distributed Storage Operation and Design

Junjie Qin and Ram Rajagopal

Abstract—Energy storage provides an important way to average temporal variability of intermittent energy generation. Grid level distributed storage enables additional spatial averaging effect by sending stored energy through the network. However, the problem of optimal storage operation in the network, coupling storage and network constraints with randomness of renewable generation, is challenging. An efficient solution method to this problem based on dynamic programming is presented in this paper. This approach also leads to analytical expressions for the expected system cost and the optimal control parametrized by dual variables, which can be computed by solving a simple deterministic convex program. These analytical expressions simplify tasks such as network storage system design. The approach is illustrated with numerical examples.

Index Terms—Distributed energy storage, dynamic programming.

I. INTRODUCTION

ENERGY storage is an important tool to mitigate temporal intermittency of renewable generation. It is much more environmentally friendly compared to conventional approaches such as fast ramping generation and has the potential to significantly bring down the system cost by reducing the curtailment [1]–[3]. A key characteristic of renewable resources such as wind and solar is that they are distributed: the capacities and locations of renewable generation is largely constrained by the availability of the corresponding natural resources, and cannot be concentrated in a single facility like thermal generation. Distributed energy storage systems are then advantageous to centralized ones, especially in transmission networks where congestion is an issue.

The problem of energy storage operation over a network with renewable generations is challenging. Conventional approaches solve the deterministic controls using the forecast with fixed reserve margins. For example, the ‘3- σ ’ rule [4] assumes the net demand (load minus the wind) equals the forecast plus 3σ , where σ is the standard deviation of the forecast error. Such operation rules can be overly conservative when renewable penetration is deep leading to large σ values [5], [6]. Stochastic programming based operation rules have been proposed as an alternative approach to worst case operation rules [7] (*cf.* related dispatch literature [8]–[10]). For storage operation problems, these methods typically require a multiple

stage formulation, which suffers from curse of dimensionality and can be difficult to implement in real world large scale applications.

Dynamic programming (DP) provides a general framework for solving multiple-stage optimization and control problems [11] and historically has been applied to problems such as hydrothermal coordination within a deterministic formulation [12]. Recent applications of (stochastic) DP for renewable integration demonstrate DP’s utility and flexibility in incorporating a wide variety of stochastic models for the renewable generation processes. It is also observed that DP is extremely useful in proving analytical and structural results for certain power system engineering problems leading to simple rules for the control in scenarios with deep renewable penetration [2], [13]–[15]. However, most of these studies focus on aggregated models without explicit consideration of the complexity arising from the electric network. Here we illustrate an approach to enable DP to handle network constraints.

This paper addresses the distributed energy storage operation problem with dynamic programming. We focus on an important case where the bulk conventional generation has been decided and distributed storages are operated to average the intermittent generation over the network. We refer this case as intra-hour averaging because the corresponding practical scenario is that the storage is used to smooth the renewable generation over a relatively short period when the conventional generations are contracted to provide certain amount of electric energy.

The paper contributes to the existing literature in two ways. First it develops a procedure to incorporate network constraints in the DP framework, yet maintains the important characteristic of DP-based approaches that the optimal cost-to-go function and control can often be obtained explicitly. Here the cost-to-go functions have a closed form parametrized by dual variables, which in turn can be solved from a deterministic convex program. Indeed, the method is general in the sense that it handles constraints that are linear in the state and control variables. Thus additional constraints such as one that involves state variables over different stages (*e.g.* ramping constraints) can be treated within our framework. Second, it illustrates that the analytical form for the system cost from DP analysis can be used to give simple solutions to other planning and design problems. The example we consider is the network storage design problem: in a electric network with no storage or few amount of storage, where to install storage devices and what size of the storage device should be installed at a particular location. It turns out the problem is reduced to a convex problem which can be solved efficiently.

J. Qin is with the Institute for Computational and Mathematical Engineering and the Stanford Sustainable Systems Lab, Stanford University, CA, 94305 (e-mail: jqin@stanford.edu).

R. Rajagopal is with the Department of Civil and Environmental Engineering and the Stanford Sustainable Systems Lab, Stanford University, CA, 94305 (e-mail: ramr@stanford.edu). R. Rajagopal is supported by the Powell Foundation Fellowship.

In the reminder of the paper, Section II introduces the problem formulation. Section III derives the DP solution to the distributed storage operation problem, followed by an application to network design problem in Section IV. The method is illustrated by numerical examples in Section V. Section VI concludes the paper.

II. PROBLEM FORMULATION

The power grid is modeled as a graph $G(\mathcal{V}, \mathcal{E})$, whose vertices represent nodes or buses, and edges represent transmission lines. Define $n = |\mathcal{V}|$ and $m = |\mathcal{E}|$. The intra-hour time interval is discretized into T stages, indexed by $t \in \{0, 1, \dots, T\}$. Given an economic dispatch decision, for each vertex $v \in \mathcal{V}$, let $d_i(t) \in \mathbb{R}$ be defined as $d_i(t) = l_i(t) - g_i(t) - w_i(t)$ representing the net demand at node i in time period t , where $g_i(t)$ is dispatched power injection from a conventional generator, $w_i(t)$ is power injection from a wind generator, and $l_i(t)$ is power consumed by a load. If a particular component is not presented at bus i , the corresponding quantity is set to 0. In case $d_i(t) < 0$, it represents surplus. Each line $e \in \mathcal{E}$ is associated with a capacity C_e . The power flow from node i to node j in time period t is denoted as $f_{ij}(t)$. In vector notation, the flow is $\mathbf{f}(t) = \{f_e(t)\}_{e \in \mathcal{E}}$, and the line capacity constraint is $|\mathbf{f}(t)| \leq \mathbf{C}$, for any t .

Notation. For each nodal variable, $z_i(t)$, let $\mathbf{z}(t)$ be the vector of $z_i(t)$ for each node, and let $\mathbf{z} = [\mathbf{z}(0)^\top \dots \mathbf{z}(T-1)^\top]^\top$.

A. DC Power Flow

We utilize a DC power flow approximation to the actual AC flow [12], [16]. Let the network incidence matrix $\nabla \in \mathbb{R}^{n \times m}$ be defined as

$$\nabla_{k,e} = \begin{cases} 1 & \text{if } e = (k, j), j \in \mathcal{V}, \\ -1 & \text{if } e = (j, k), j \in \mathcal{V}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the nodal imbalance at time period t is

$$\mathbf{s}(t) = \mathbf{d}(t) + \mathbf{u}(t) - \nabla \mathbf{f}(t).$$

For bus i , $s_i(t)$ may be covered by a fast-ramping generation if positive, and represents energy to be extracted and disposed if negative. Under the DC power flow assumptions, for purely inductive transmission lines, the nodal phase angle $\boldsymbol{\theta}(t)$ is related with flow linearly:

$$\mathbf{f}(t) = \mathbf{B}_f \boldsymbol{\theta}(t),$$

where $\mathbf{B}_f = \text{diag}(b_e) \nabla^\top$, and $-b_e < 0$ is the susceptance of line e . Fix the phase angle at the slack bus to be 0, let $\tilde{\boldsymbol{\theta}}(t) \in \mathbb{R}^{n-1}$ be the remaining phase angles, $\tilde{\nabla} \in \mathbb{R}^{(n-1) \times m}$ be the network incidence matrix with the row corresponding to the slack bus deleted. Denote $\tilde{\mathbf{B}}_f = \text{diag}(b_e) \tilde{\nabla}^\top$, $\Delta = \nabla \mathbf{B}_f$. Then

$$\mathbf{s}(t) = \mathbf{d}(t) + \mathbf{u}(t) - \Delta \tilde{\boldsymbol{\theta}}(t).$$

B. Energy Storage

To simplify the presentation, we use an idealized storage model. The capacity of storage at bus i denoted B_i , for each $i \in \mathcal{V}$. If there is no storage at bus i , we require $B_i = 0$. Thus the storage level at bus i satisfies $x_i(t) \in [0, B_i]$ for all t . The dynamics of the idealized storage is modeled as

$$\mathbf{x}(t+1) = \mathbf{x}(t) + \mathbf{u}(t), \quad t = 0, 1, \dots, T-1,$$

where $u_i(t)$ is the control (*i.e.*, charging or discharging operation) of storage at bus i during time period t . Due to energy conservation and storage capacity, $u_i(t) \in [-x_i(t), B_i - x_i(t)]$.

C. Forecast Model

The net demand $\mathbf{d}(t)$ is random. In practice, components of $d_i(t)$, *i.e.*, $w_i(t)$, and $l_i(t)$ can be predicted with different accuracies. The dispatched conventional generation $g_i(t)$ is typically treated as deterministic, and here assumed to be given. Forecast errors of both wind and loads at the bus level are distributed as a (truncated) Gaussian random variables [17], therefore for each time period t , the net load is decomposed as

$$\mathbf{d}(t) = \hat{\mathbf{d}}(t) + \mathbf{e}(t), \quad (1)$$

where $\hat{\mathbf{d}}(t)$ is the forest and $\mathbf{e}(t) \sim N(\mathbf{0}, \Sigma_e(t))$. The covariance matrix satisfies $\Sigma_e(t) = \sigma_e(t)^2 \Sigma'_e$, where Σ'_e is a known error correlation matrix that does not depend on t .

D. Costs

The objective of storage operation is to provide a cost-effective way to ensure energy balance for each bus and each time period. Thus for time period t , it is natural to consider cost function of the form

$$h^{\text{PL}}(\mathbf{u}(t), \tilde{\boldsymbol{\theta}}(t), \mathbf{d}(t)) = \boldsymbol{\alpha}^\top (\mathbf{s}(t))^+ + \boldsymbol{\beta}^\top (\mathbf{s}(t))^- , \quad (2)$$

where the i th component of vector $\boldsymbol{\alpha}$ measures the cost or economic loss due to a unit of energy shortfall at bus i , or equivalently the cost of using a fast-ramping generation to cover such shortfall, and the i th component of vector $\boldsymbol{\beta}$ measures the cost of disposing a unit of excessive energy. Here $(x)^+ = \max(x, 0)$ and $(x)^- = \max(-x, 0)$. However, it is evident that piecewise linear cost h^{PL} does not lead to a simple control policy (*cf.* [18] for the analysis of the single bus case and [19] for the discussion of multiple bus case). Instead, we consider a surrogate quadratic cost function of the form

$$h_0^{\text{Q}}(\mathbf{u}(t), \tilde{\boldsymbol{\theta}}(t), \mathbf{d}(t)) = \mathbf{s}(t)^\top \mathbf{Q}_1 \mathbf{s}(t) + \boldsymbol{\ell}^\top \mathbf{s}(t),$$

where \mathbf{Q}_1 is a given symmetric positive definite matrix, and $\boldsymbol{\ell}$ is a given vector. The quadratic cost (3) can lead to a better cost model for the fast-ramping generation in the case of shortfall. Further, the quadratic cost, upon taking expectation, can be viewed as a risk measure when the matrix \mathbf{Q}_1 and vector $\boldsymbol{\ell}$ are chosen appropriately [20].

In addition, for storage devices currently available the number of charge/discharge cycles within the lifespan is typically very limited. Although advanced batteries can deliver more than 5000 charge/discharge cycles [21], an operation scheme

that operates the storage too frequently is clearly undesired with depreciation considered for intra-hour averaging purpose focused in this paper. Thus we impose an additional penalty term representing the life cycle cost of storage operation,

$$h_1^Q(\mathbf{u}(t), \tilde{\boldsymbol{\theta}}(t), \mathbf{d}(t)) = \mathbf{s}(t)^\top \mathbf{Q}_1 \mathbf{s}(t) + \mathbf{u}(t)^\top \mathbf{Q}_2 \mathbf{u}(t) + \ell^\top \mathbf{s}(t), \quad (3)$$

where \mathbf{Q}_2 is symmetric positive definite.

E. Distributed Storage Operation Problem

Summarizing elements in previous subsections, the Distributed Storage Operation (**DSO**) problem is

$$\text{minimize} \quad \mathbb{E} \left[\sum_{t=0}^{T-1} h_1^Q(\mathbf{u}(t), \tilde{\boldsymbol{\theta}}(t), \mathbf{d}(t)) \right] \quad (4a)$$

$$\text{subject to} \quad \mathbf{x}(t+1) = \mathbf{x}(t) + \mathbf{u}(t), \quad (4b)$$

$$-\mathbf{x}(t) \leq \mathbf{u}(t) \leq \mathbf{B} - \mathbf{x}(t), \quad (4c)$$

$$-\mathbf{C} \leq \tilde{\mathbf{B}}_f \tilde{\boldsymbol{\theta}}(t) \leq \mathbf{C}. \quad (4d)$$

F. Related Literature

We contrast our model with that of other studies. Our setup is closely related to [19]. By relaxing inequality constraints and considering an infinite horizon problem, the focus of [19] is the limiting behavior of the system. Thus their approach does not retain the practicality of this work. The **DSO** problem is also related to but differs from [22], [23], which address deterministic OPF with storage. Here we consider the complexity due to randomness, and concentrate on enabling DP-based approach to handle network constraints.

III. DP APPROACH TO DSO

Equipped with a quadratic cost function and linear dynamics, the problem is closely related to linear-quadratic/linear-quadratic-gaussian controller problems, which are well studied in term of the problem structure and solution approaches. However, standard LQ/LQG controller results do not apply to the case with hard constraints on the state/control. Utilizing a special version of the KKT condition [24] restricted to the affine subspace in which (4b) holds, we reformulate the problem into a standard LQ problem by minimizing the Lagrangian with respect to the primal (control) variables. This yields an explicit formula for the Lagrange dual function. The resulting dual problem turns out to be a convex QP which can be solved efficiently.

We start by restating the problem in a more compact matrix notation:

$$\text{minimize} \quad \mathbb{E} \left[\sum_{t=0}^{T-1} \frac{1}{2} \mathbf{v}(t)^\top \mathbf{Q} \mathbf{v}(t) + \ell_{v0}(t)^\top \mathbf{v}(t) + a_0(t) \right] \quad (5a)$$

$$\text{subject to} \quad \mathbf{x}(t+1) = \mathbf{x}(t) + \mathbf{F} \mathbf{v}(t), \quad (5b)$$

$$\mathbf{H}_x \mathbf{x}(t) + \mathbf{H}_v \mathbf{v}(t) \leq \mathbf{c}, \quad (5c)$$

where

$$\mathbf{v}(t) = \begin{bmatrix} \mathbf{u}(t) \\ \tilde{\boldsymbol{\theta}}(t) \end{bmatrix}, \quad \mathbf{Q} = 2 \begin{bmatrix} \mathbf{Q}_1 + \mathbf{Q}_2 & -\mathbf{Q}_1 \boldsymbol{\Delta} \\ -\boldsymbol{\Delta}^\top \mathbf{Q}_1 & \boldsymbol{\Delta}^\top \mathbf{Q}_1 \boldsymbol{\Delta} \end{bmatrix},$$

$$\ell_{v0}(t) = \begin{bmatrix} \mathbf{I} \\ -\boldsymbol{\Delta}^\top \end{bmatrix} (\ell + 2 \mathbf{Q}_1 \mathbf{d}(t)),$$

$$a_0(t) = \mathbf{d}(t)^\top \mathbf{Q}_1 \mathbf{d}(t) + \ell^\top \mathbf{d}(t),$$

$$\mathbf{H}_x = \begin{bmatrix} -\mathbf{I} \\ \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{H}_v = \begin{bmatrix} -\mathbf{I} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\tilde{\mathbf{B}}_f \\ \mathbf{0} & \tilde{\mathbf{B}}_f \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} \mathbf{0} \\ \mathbf{B} \\ \mathbf{C} \\ \mathbf{C} \end{bmatrix},$$

and $\mathbf{F} = [\mathbf{I} \quad \mathbf{0}]$. Note it is easy to verify \mathbf{Q} is symmetric positive definite. Let $\boldsymbol{\lambda}(t)$ be the lagrange multiplier vector corresponding to the inequality constraints in (4). Then the Lagrangian is

$$\mathcal{L}(\mathbf{x}, \mathbf{v}, \boldsymbol{\lambda}) = \mathbb{E} \left[\sum_{t=0}^{T-1} \frac{1}{2} \mathbf{v}(t)^\top \mathbf{Q} \mathbf{v}(t) + \ell_{v0}(t)^\top \mathbf{v}(t) + a_0(t) + \boldsymbol{\lambda}(t)^\top (\mathbf{H}_x \mathbf{x}(t) + \mathbf{H}_v \mathbf{v}(t) - \mathbf{c}) \right],$$

which simplifies to

$$\mathcal{L}(\mathbf{x}, \mathbf{v}, \boldsymbol{\lambda}) = \mathbb{E} \left[\sum_{t=0}^{T-1} \frac{1}{2} \mathbf{v}(t)^\top \mathbf{Q} \mathbf{v}(t) + \ell_v(\boldsymbol{\lambda}; t)^\top \mathbf{v}(t) + \ell_x(\boldsymbol{\lambda}; t)^\top \mathbf{x}(t) + a_1(\boldsymbol{\lambda}; t) \right],$$

where $\ell_x(\boldsymbol{\lambda}; t) = \mathbf{H}_x^\top \boldsymbol{\lambda}(t)$, $\ell_v(\boldsymbol{\lambda}; t) = \ell_{v0}(t) + \mathbf{H}_v^\top \boldsymbol{\lambda}(t)$, and $a_1(\boldsymbol{\lambda}; t) = a_0(t) - \mathbf{c}^\top \boldsymbol{\lambda}(t)$.

The problem of minimizing the Lagrangian with respect to \mathbf{x} and \mathbf{v} , i.e., for fixed $\boldsymbol{\lambda}$, finding (\mathbf{x}, \mathbf{v}) pair that

$$\text{minimize} \quad \mathcal{L}(\mathbf{x}, \mathbf{v}, \boldsymbol{\lambda}) \quad (6a)$$

$$\text{subject to} \quad \mathbf{x}(t+1) = \mathbf{x}(t) + \mathbf{F} \mathbf{v}(t), \quad (6b)$$

is solved with DP. Let the cost-to-go functions be

$$\begin{aligned} J(\boldsymbol{\lambda}, \mathbf{x}(t); t) &= \inf_{\mathbf{v}(t)} \mathbb{E} \left[\frac{1}{2} \mathbf{v}(t)^\top \mathbf{Q} \mathbf{v}(t) + \ell_v(\boldsymbol{\lambda}; t)^\top \mathbf{v}(t) + \ell_x(\boldsymbol{\lambda}; t)^\top \mathbf{x}(t) \right. \\ &\quad \left. + a_1(\boldsymbol{\lambda}; t) + J(\mathbf{x}(t) + \mathbf{F} \mathbf{v}(t); t+1) \right] \end{aligned}$$

for $t = 0, \dots, T-1$, with $J(\boldsymbol{\lambda}, \mathbf{x}(T); T) = 0$.

Theorem III.1. (i). For each $\boldsymbol{\lambda}$ and $t = 0, \dots, T$, the cost-to-go function is

$$J(\boldsymbol{\lambda}, \mathbf{x}(t); t) = \mathbf{p}(\boldsymbol{\lambda}; t)^\top \mathbf{x}(t) + r(\boldsymbol{\lambda}; t),$$

where

$$\mathbf{p}(\boldsymbol{\lambda}; t) = \mathbf{p}(\boldsymbol{\lambda}; t+1) + \ell_x(\boldsymbol{\lambda}; t), \quad (7)$$

$$\begin{aligned} r(\boldsymbol{\lambda}; t) &= r(\boldsymbol{\lambda}; t+1) + \hat{a}_1(\boldsymbol{\lambda}; t) \\ &\quad - \frac{1}{2} (\hat{\ell}_v(\boldsymbol{\lambda}; t) + \mathbf{F}^\top \mathbf{p}(t+1))^\top \mathbf{Q}^{-1} (\hat{\ell}_v(\boldsymbol{\lambda}; t) + \mathbf{F}^\top \mathbf{p}(t+1)), \end{aligned} \quad (8)$$

with $\mathbf{p}(\boldsymbol{\lambda}; T) = \mathbf{0}$, $r(\boldsymbol{\lambda}; T) = 0$, $\hat{a}_1(\boldsymbol{\lambda}; t) = \mathbb{E}[a_1(\boldsymbol{\lambda}; t)]$, and $\hat{\ell}_v(\boldsymbol{\lambda}; t) = \mathbb{E}[\ell_v(\boldsymbol{\lambda}; t)]$.

(ii). For each $\boldsymbol{\lambda}$ and $t = 0, \dots, T-1$, the optimal control policy is

$$\mathbf{v}^*(\boldsymbol{\lambda}, \mathbf{x}(t); t) = -\mathbf{Q}^{-1} (\hat{\ell}_v(\boldsymbol{\lambda}; t) + \mathbf{F}^\top \mathbf{p}(t+1)).$$

Theorem III.1 solves problem (6) parametrized by λ . Then the Lagrange dual function can be written as

$$\mathcal{L}(\mathbf{x}(\lambda), \mathbf{v}(\lambda), \lambda) = J(\mathbf{x}(0); 0) = \mathbf{p}(\lambda; 0)^\top \mathbf{x}(0) + r(\lambda; 0),$$

where $\mathbf{p}(\lambda; 0)$ and $r(\lambda; 0)$ are computed via (7) and (8), and $\mathbf{x}(0)$ is a given vector of initial storage levels. After some algebra, we arrive at the following expression

$$\mathcal{L}(\mathbf{x}(\lambda), \mathbf{v}(\lambda), \lambda) = -\frac{1}{2}\lambda^\top \mathbf{Q}_\lambda \lambda - \ell_\lambda^\top \lambda + a_2, \quad (9)$$

where \mathbf{Q}_λ is a $T \times T$ block matrix whose (t_i, t_j) -th block is

$$\mathbf{Q}_\lambda(t_i, t_j) = \begin{cases} \mathbf{H}_v \mathbf{Q}^{-1} \mathbf{H}_v^\top + t_i \mathbf{H}_x \mathbf{F} \mathbf{Q}^{-1} \mathbf{F}^\top \mathbf{H}_x^\top & \text{if } t_i = t_j, \\ \mathbf{H}_v \mathbf{Q}^{-1} \mathbf{F}^\top \mathbf{H}_x^\top + t_i \mathbf{H}_x \mathbf{F} \mathbf{Q}^{-1} \mathbf{F}^\top \mathbf{H}_x^\top & \text{if } t_j > t_i, \\ \mathbf{H}_x \mathbf{F} \mathbf{Q}^{-1} \mathbf{H}_v^\top + t_j \mathbf{H}_x \mathbf{F} \mathbf{Q}^{-1} \mathbf{F}^\top \mathbf{H}_x^\top & \text{if } t_j < t_i, \end{cases}$$

ℓ_λ is a $T \times 1$ block vector whose t -th block is

$$\ell_\lambda(t) = \mathbf{c} + \mathbf{H}_x \mathbf{x}(0) + \mathbf{H}_v \mathbf{Q}^{-1} \hat{\ell}_{v0}(t) + \sum_{\tau=0}^{t-1} \mathbf{H}_x \mathbf{F} \mathbf{Q}^{-1} \hat{\ell}_{v0}(\tau),$$

and $a_2 = \sum_{t=0}^{T-1} \hat{a}_0(t) - \frac{1}{2} \hat{\ell}_{v0}(t)^\top \mathbf{Q}^{-1} \hat{\ell}_{v0}(t)$. It can also be easily verified that $\mathcal{L}(\mathbf{x}(\lambda), \mathbf{v}(\lambda), \lambda)$ is a concave quadratic form of λ (by expanding the quadratic form into terms involving $\lambda(t)$ and checking the positivity conditions). In fact, matrix \mathbf{Q}_λ is positive semidefinite and singular. While the problem is a convex QP, up to a transformation (see Corollary IV.1) it can be solved efficiently with standard solvers.

Theorem III.2. *Let λ^* be a solution of the dual problem*

$$\begin{aligned} & \text{maximize} \quad \mathcal{L}(\mathbf{x}(\lambda), \mathbf{v}(\lambda), \lambda) \\ & \text{subject to} \quad \lambda \geq 0. \end{aligned}$$

For $t = 0, \dots, T$, the optimal cost-to-go is

$$J(\mathbf{x}(t); t) = J(\lambda^*, \mathbf{x}(t); t).$$

For $t = 0, \dots, T-1$, the optimal control policy is

$$\nu^*(\mathbf{x}(t); t) = \nu^*(\lambda^*, \mathbf{x}(t); t).$$

Remark III.3 (Incorporation of other affine constraints). Results above do not rely on the fact inequality constraint (5c) is time invariant (i.e., \mathbf{H}_x , \mathbf{H}_v and \mathbf{c} t -independent). By incrementing \mathbf{H}_x , \mathbf{H}_v and \mathbf{c} with additional rows, Eqn. (5c) incorporates general linear constraints of state and control within each stage. Finally, the approach in fact works for general linear constraints that may involve state and control variables of all stages, e.g., $\sum_{t=0}^{T-1} \tilde{\mathbf{H}}_x(t) \mathbf{x}(t) + \tilde{\mathbf{H}}_v(t) \mathbf{v}(t) \leq \tilde{\mathbf{c}}$, where $\tilde{\mathbf{H}}_x(t)$, $\tilde{\mathbf{H}}_v(t)$ and $\tilde{\mathbf{c}}$ are arbitrary matrices and vector of appropriate dimensions. This is due to the functional form of the Lagrangian is unchanged with these types of constraints.

IV. NETWORK STORAGE DESIGN

Limited amount of distributed energy storage has been deployed up to date. To incorporate deeper renewable penetration, more storage devices need to be installed in the grid. Important questions to be addressed, however, lie in the design of distributed storage system within the current grid: which buses should we deploy a storage device, and how to decide the storage capacity? It is demonstrated that Theorem III.1 and

Theorem III.2 give an efficient approach to the network storage design problem for the purpose of intra-hour averaging.

Let \mathbf{B} be the vector of storage capacities for storage to be installed at each bus. Denote $J^*(\lambda, \mathbf{B}) = \mathcal{L}(\mathbf{x}(\lambda), \mathbf{v}(\lambda), \lambda; \mathbf{B})$, where the latter function is defined as (9), with the storage capacity specified by vector \mathbf{B} . Then $\max_{\lambda \geq 0} J^*(\lambda, \mathbf{B})$ represents the expected optimal cost given a storage design scheme \mathbf{B} . This quantity is nonincreasing in \mathbf{B} , i.e., given the network and some realization of the net demand process, larger storage devices can never do worse than smaller storage devices for the averaging purpose. To capture the tradeoff between the better averaging effect of larger storage and the lower capital cost of smaller storage, the additional cost of deploying larger storage devices must be taken into account. Let $h_B(\mathbf{B})$ be a convex nondecreasing function that models the capital cost of deploying a storage design scheme specified by \mathbf{B} . The network storage design (NSD) problem is to find \mathbf{B} that solves

$$\begin{aligned} & \text{minimize} \quad h_B(\mathbf{B}) + \kappa_T \max_{\lambda \geq 0} J^*(\lambda, \mathbf{B}) \\ & \text{subject to} \quad \mathbf{B} \geq 0, \end{aligned}$$

where scaling factor κ_T is introduced since the lifespan of a storage device can be much longer than the time period in which the system cost with storage averaging is computed.

It turns out that NSD can be solved efficiently utilizing results in the previous section. We give the equivalent convex program in the following corollary.

Corollary IV.1. *Let the unitary eigenvalue decomposition of \mathbf{Q}_λ be*

$$\mathbf{Q}_\lambda = \mathbf{U} \mathbf{\Lambda} \mathbf{U}^\top = [\mathbf{U}_1 \quad \mathbf{U}_2] \begin{bmatrix} \mathbf{D} & \\ & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{U}_1^\top \\ \mathbf{U}_2^\top \end{bmatrix},$$

where $\mathbf{D} \in \mathbb{R}^{n_q \times n_q}$ is diagonal and full rank, with $n_q = \text{rank}(\mathbf{Q}_\lambda)$, and \mathbf{U}_1 and \mathbf{U}_2 have orthogonal columns and appropriate dimensions. Let $(\mathbf{B}^*, \boldsymbol{\rho}^*)$ be the solution to

$$\begin{aligned} & \text{minimize} \quad h_B(\mathbf{B}) + \frac{1}{2} \kappa_T \boldsymbol{\rho}^\top \mathbf{D}^{-1} \boldsymbol{\rho} \\ & \text{subject to} \quad \mathbf{B} \geq 0, \\ & \quad \mathbf{U}_1 \boldsymbol{\rho} \leq \tilde{\ell}_\lambda + \mathbf{R} \mathbf{B}, \end{aligned}$$

where \mathbf{R} is a $T \times 1$ block matrix whose t -th block is $\mathbf{R}(t) = [\mathbf{0} \quad \mathbf{I} \quad \mathbf{0} \quad \mathbf{0}]^\top$, and $\tilde{\ell}_\lambda = \ell_\lambda - \mathbf{R} \mathbf{B}$, which is \mathbf{B} -independent. The optimal storage design is given by \mathbf{B}^* .

V. NUMERICAL EXAMPLE

We use the network topology of IEEE 14 bus test case [25]. As an illustration, each bus is assigned with a wind process, computed using one of 14 hours sampled from 2011 BPA dataset [26], which contains data for every 5 minutes. The net demand is computed from the wind process such that the mean value of net demand over the hour is 0 for each bus. Only a forecast of the net demand, computed according to (1), is available to the operator. The forecast errors have variances that are increasing with time and are assumed to be independent across buses for simplicity. The empirical relation between forecast error and forecast horizon is calculated using data from Red Electrica Espana (cf. Figure 3 of [13]). The

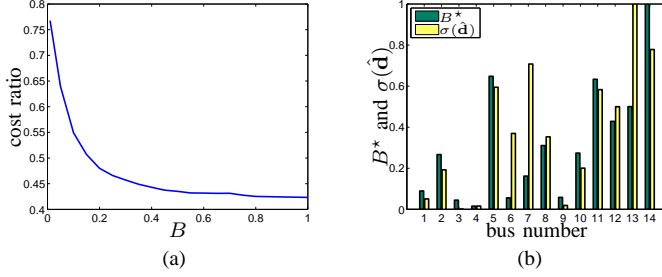


Fig. 1: (a) Ratio between costs of DP-based policy and $3\text{-}\sigma$ policy. (b) Optimal storage design (B^*) and the data standard deviation of the predicted net demand process for each bus. Storage capacities and standard deviations are normalized by their maximum values, respectively.

objective function is chosen to be uniform for each bus, i.e., $\mathbf{Q}_1 = \frac{1}{2}\alpha\mathbf{I}$, $\mathbf{Q}_2 = \frac{1}{2}\gamma_1\alpha\mathbf{I}$, and $\ell = \mathbf{0}$. The capital cost for storage is chosen to be $h_B(\mathbf{B}) = \gamma_2\alpha\sum_{i=1}^n(B_i)^+$. Since we focus on the relative costs in this section, we use $\alpha = 1$. We set $\gamma_1 = 0.02$ and $\gamma_2 = 600$ according to VOLL cost in California, and the life cycle and capital costs of energy storage reported in [27].

For the **DSO** problem, we use an uniform design of storage, i.e., each bus is assigned the same storage capacity. For a range of storage capacities, the operation cost using control computed with our approach is compared with $3\text{-}\sigma$ rule (Figure 1a). The performance of DP-based approach improves when the storage capacities are larger, relative to the $3\text{-}\sigma$ control. The control sequences computed with both approaches are also evaluated with the piecewise linear costs (2) with $\beta = \mathbf{0}$. For all values of \mathbf{B} , the DP-based control leads to cost that is at most 75% of the cost given by $3\text{-}\sigma$ control. For the **NSD** problem, we assume the storage device has a life span of 10 years. The optimal storage design is contrasted with the data standard deviation of the predicted net demand process (Figure 1b). It can be observed that while for many buses the optimal storage sizes are positively correlated with the variability of the net demand processes, such relation does not hold for certain buses due to the network.

VI. CONCLUSION

This paper formulates the distributed storage averaging problem and proposes a semi-analytical solution method based on dynamic programming. The expression for the optimal cost function obtained from the DP analysis is used to simplify the network storage design problem. In ongoing work, we generalize the method to handle other convex costs by successive quadratic approximation. A more general formulation with information update is also under consideration.

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