Astrostatistics: 13 Mar 2019

https://github.com/CambridgeAstroStat/PartIII-Astrostatistics-2019

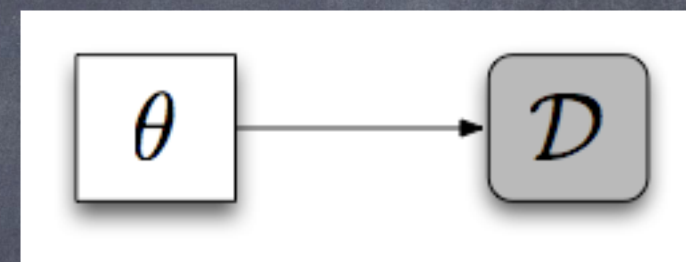
- Make-Up Lecture, Thu 14 Mar, 11am-12: MR13
- Example Class, Fri 15 Mar 1pm: MR 12
- Today:
- Hierarchical Bayes & Shrinkage Estimators

Aside on Product of Gaussian densities

Review:
Hierarchical Bayes and Probabilistic Graphical Models:
a visual way to understand complex statistical models

Simple Bayes:

$$\mathcal{D}|\,\theta \sim \mathrm{Model}(\theta)$$

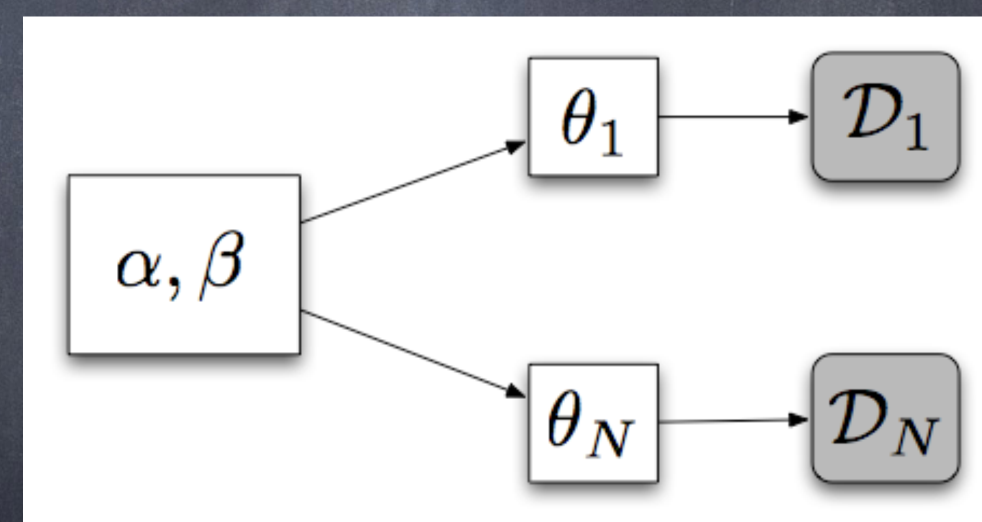


$$P(\theta|\,\mathcal{D}) \propto P(\mathcal{D}|\,\theta)P(\theta)$$

Hierarchical Bayes:

$$\mathcal{D}_i|\,\theta_i \sim \mathrm{Model}(\theta_i)$$

$$\theta_i|\,\alpha,\beta \sim \mathrm{PopModel}(\alpha,\beta)$$



$$P(\{\theta_i\},\alpha,\beta|\{\mathcal{D}_i\}) \propto \left[\prod_{i=1}^{N} P(\mathcal{D}_i|\,\theta_i)P(\theta_i|\alpha,\beta)\right]P(\alpha,\beta)$$

Build up complexity by layering conditional probabilities

Review: Hierarchical Bayesian "Normal-Normal" Model

Level 1: Population Distribution of Latent Variables (Absolute Mags)

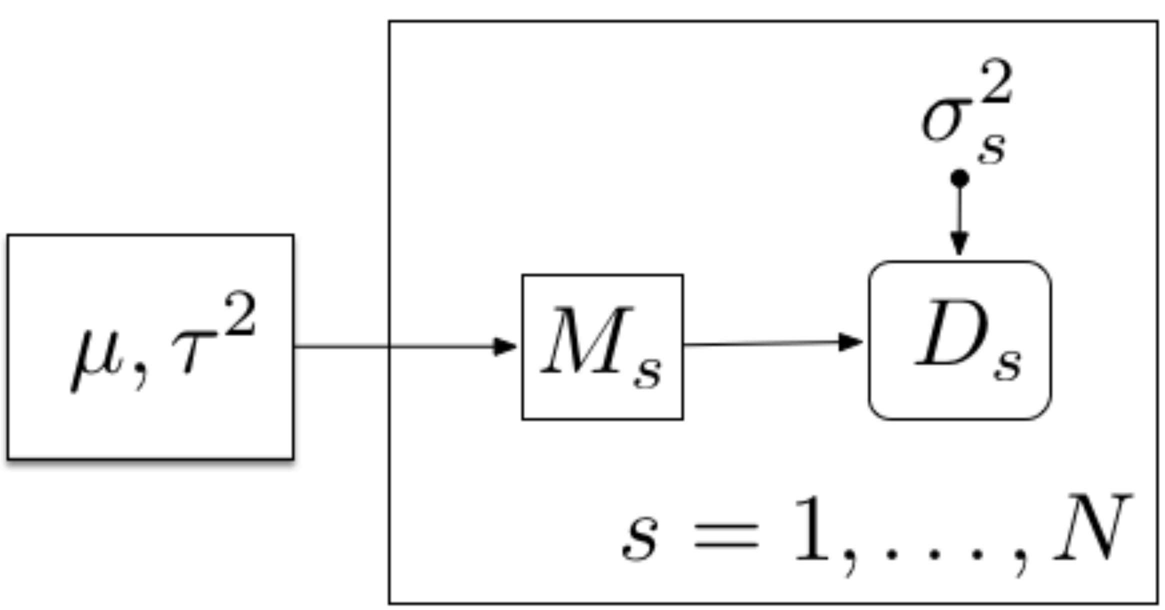
$$M_s \sim N(M_0, \tau^2) \quad s = 1 \dots N$$

Level 2 : Measurement Error Process

$$D_s | M_s \sim N(M_s, \sigma_s^2) \quad s = 1 \dots N$$

Joint Probability Density of Data, Latent Variables, **H**yperparameters

$$P(\{D_s\}, \{M_s\}, \boldsymbol{H} = M_0, \tau^2)$$



Joint factors into Conditional and Marginal densities based on Model Assumptions

Hierarchical vs Regular Bayes

- Could regard as just a general Bayesian inference problem in a very high dimensional parameter space, e.g.

$$\boldsymbol{\theta} = \{M_1, \ldots, M_N; M_0, \tau^2\} = \{\boldsymbol{M}; M_0, \tau^2\}$$

$$P(\boldsymbol{\theta}|\boldsymbol{D}) \propto P(\boldsymbol{D}|\boldsymbol{\theta})P(\boldsymbol{\theta})$$

$$P(\boldsymbol{\theta}|\boldsymbol{D}) \propto P(\boldsymbol{D}|\boldsymbol{M})P(\boldsymbol{M}|M_0, \tau^2)P(M_0, \tau^2)$$

$$P(\boldsymbol{\theta}|\boldsymbol{D}) \propto \left[\prod_{s=1}^{N} P(D_s|M_s)P(M_s|M_0, \tau^2)\right] P(M_0, \tau^2)$$

- However, special hierarchical structure is useful for modelling, estimation, and computation

- For large N, wouldn't want to do an N+2 dimensional Metropolis MCMC!

Hierarchical Bayesian "Normal-Normal" Model

Level 1: Population Distribution of Latent Variables (Absolute Mags)

$$M_s \sim N(M_0, \tau^2)$$

Population Dist'n / Prior

Level 2 : Measurement Error Process

$$D_s | M_s \sim N(M_s, \sigma_s^2)$$

Measurement Likelihood

Joint Probability Density of ALL THE THINGS:
Data, Latent Variables, Hyperparameters

$$P(\{D_s\}, \{M_s\}, \boldsymbol{H}) = \left[\prod_{s=1}^{N} P(D_s | M_s) P(M_s | M_0, \tau^2) \right] \times P(\boldsymbol{H})$$

(Derive on board)

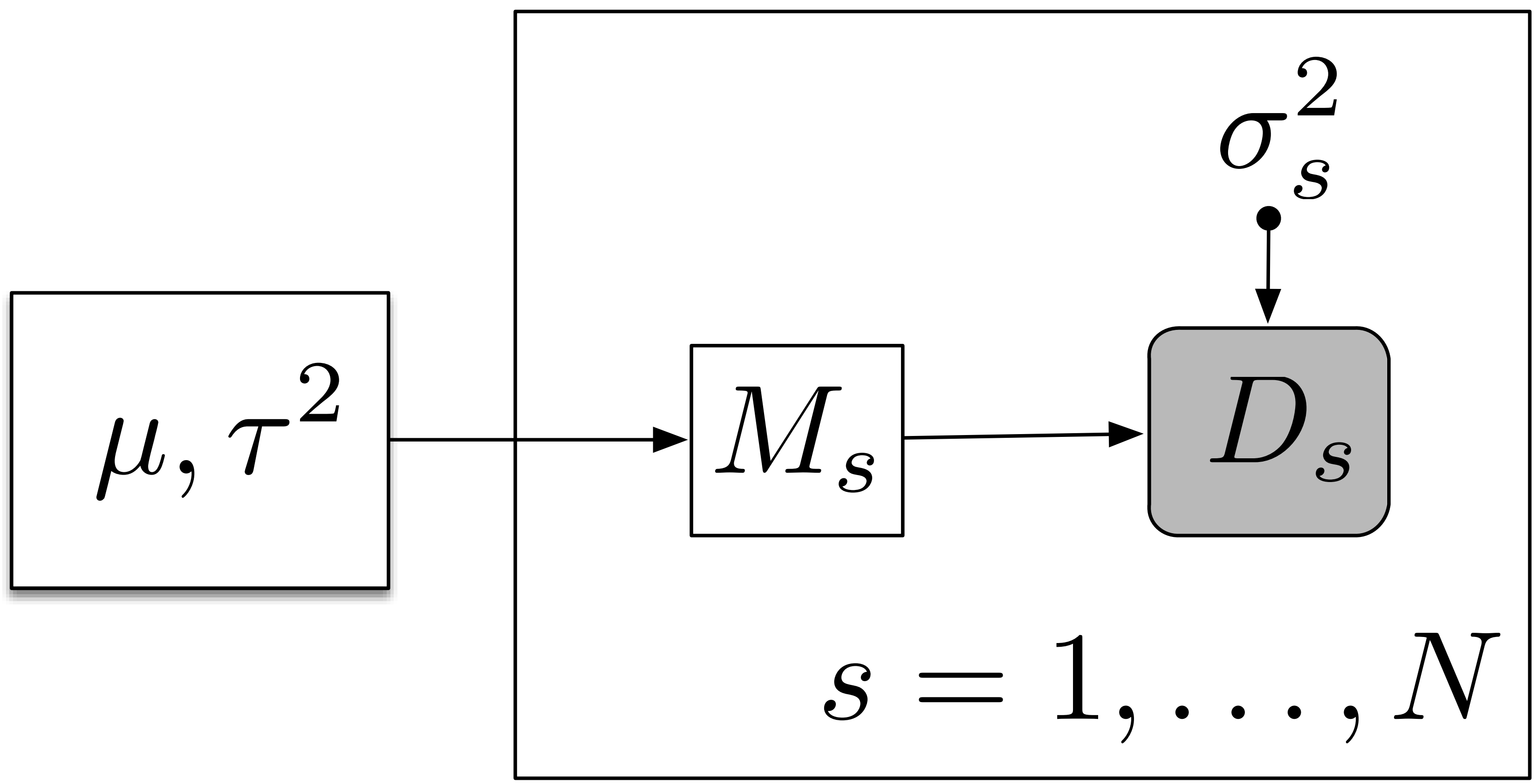
Measurement Likelihood

Population Dist'n / Prior

Hyperprior

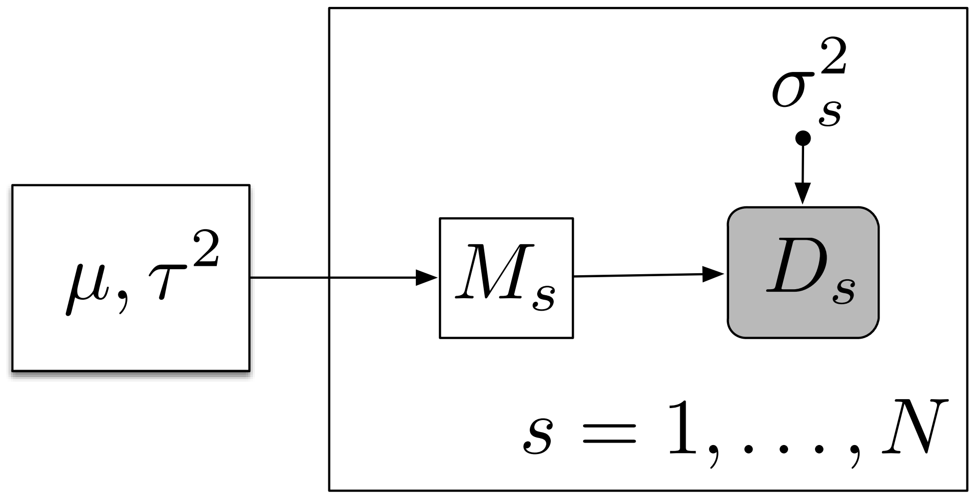
Sampling the Hierarchical Bayesian posterior

$$P(\{M_s\}, \boldsymbol{H} | \{D_s\}) \propto \Big[\prod_{s=1}^{N} P(D_s|M_s) P(M_s|M_0, \tau^2) \Big] \times P(\boldsymbol{H})$$

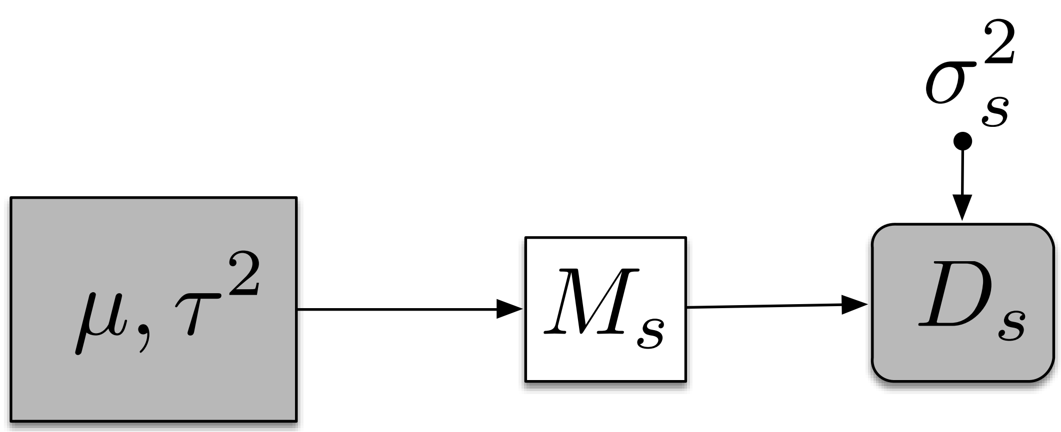


Utilises Conditional Independence structure of PGM to derive conditional posterior densities

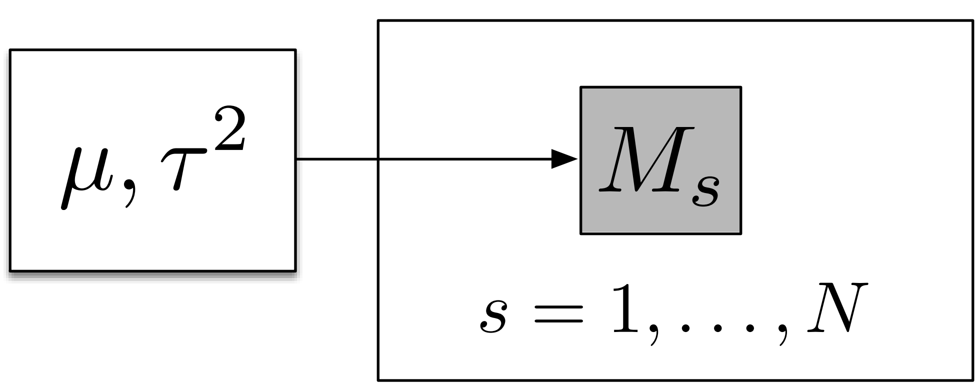
Gibbs sampling & Hierarchical Bayes



1. For s = 1 … N: Sample Latent Variables Conditional on Data and **Hyperparameters**



2. Sample Hyperparameters from Conditional on Data and Latent Variables:



Gibbs sampling & Hierarchical Bayes

Utilises Conditional Independence structure of PGM to derive conditional posterior densities

$$P(\{M_s\}, \boldsymbol{H} | \{D_s\}) \propto \Big[\prod_{s=1}^{N} P(D_s | M_s) P(M_s | M_0, \tau^2) \Big] \times P(\boldsymbol{H})$$

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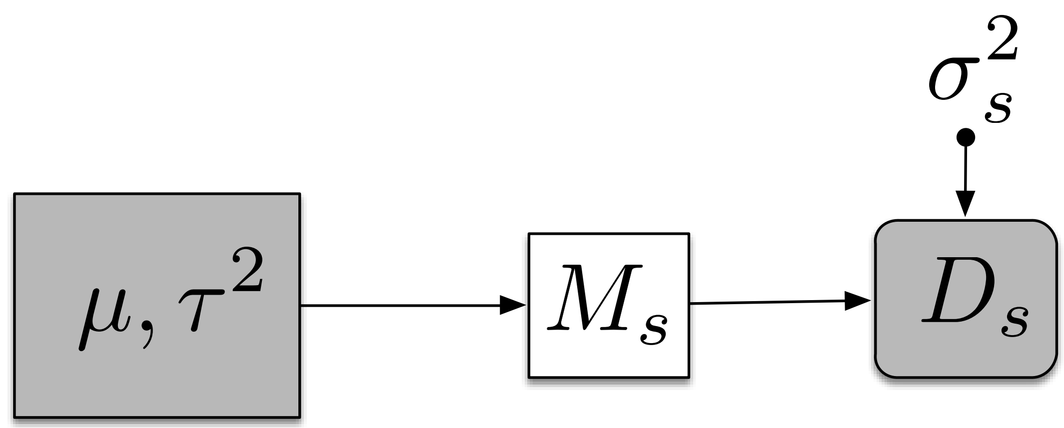

$$P(M_s | \, \mu, \tau^2, D_S) \propto P(D_s | \, M_s) \times P(M_s | M_0, \tau^2)$$

$$\propto N(D_s | \, M_s, \sigma^2) \times N(M_s | M_0, \tau^2)$$

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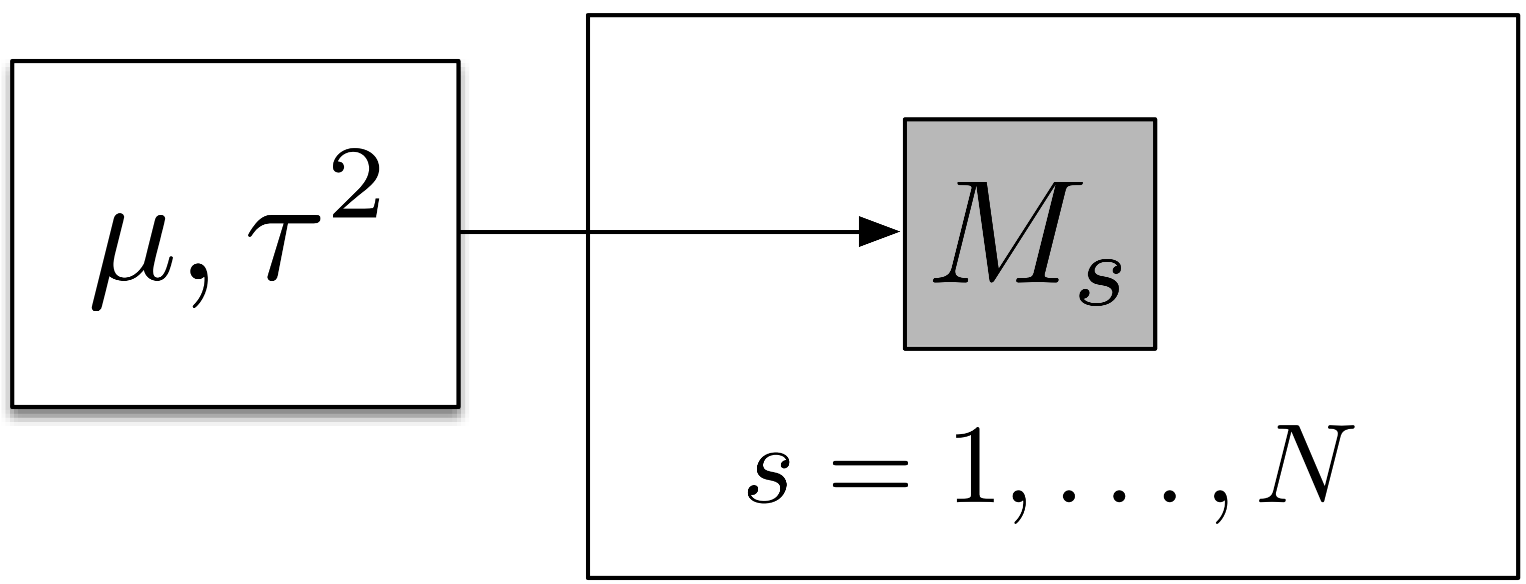

Hyperprior: $P(\mu, \tau^2) \propto (\tau^2)^{-1/2}$

$$P(M_0, \tau^2 | \{M_s\}; \{D_s\}) = P(M_0, \tau^2 | \{M_s\}) = P(M_0 | \tau^2, \{M_s\}) P(\tau^2 | \{M_s\})$$

(See Example Sheet 1, Problem 4) (Gaussian) (Inv-χ^2)

Gibbs sampling & Hierarchical Bayes

Utilises Conditional Independence structure of PGM to derive conditional posterior densities

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(Gaussian) (Inv-χ^2)

Hyperprior: $P(\mu, \tau^2) \propto (\tau^2)^{-1/2}$

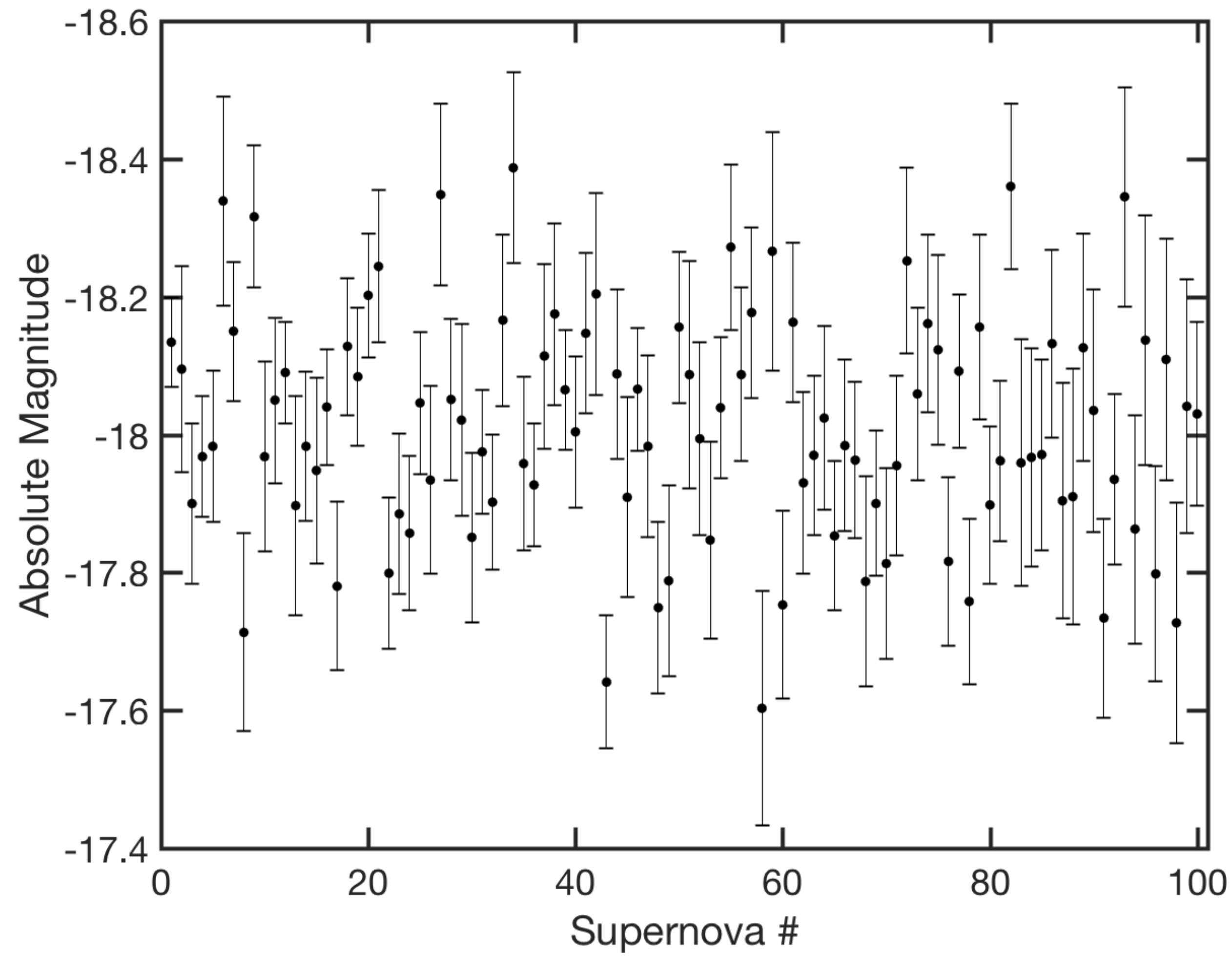
Conditional Posteriors: $\tau^2 | \{M_s\} \sim \text{Inv-}\chi^2 \left(N - 2, \frac{(N-1)}{(N-2)} s^2 \right)$

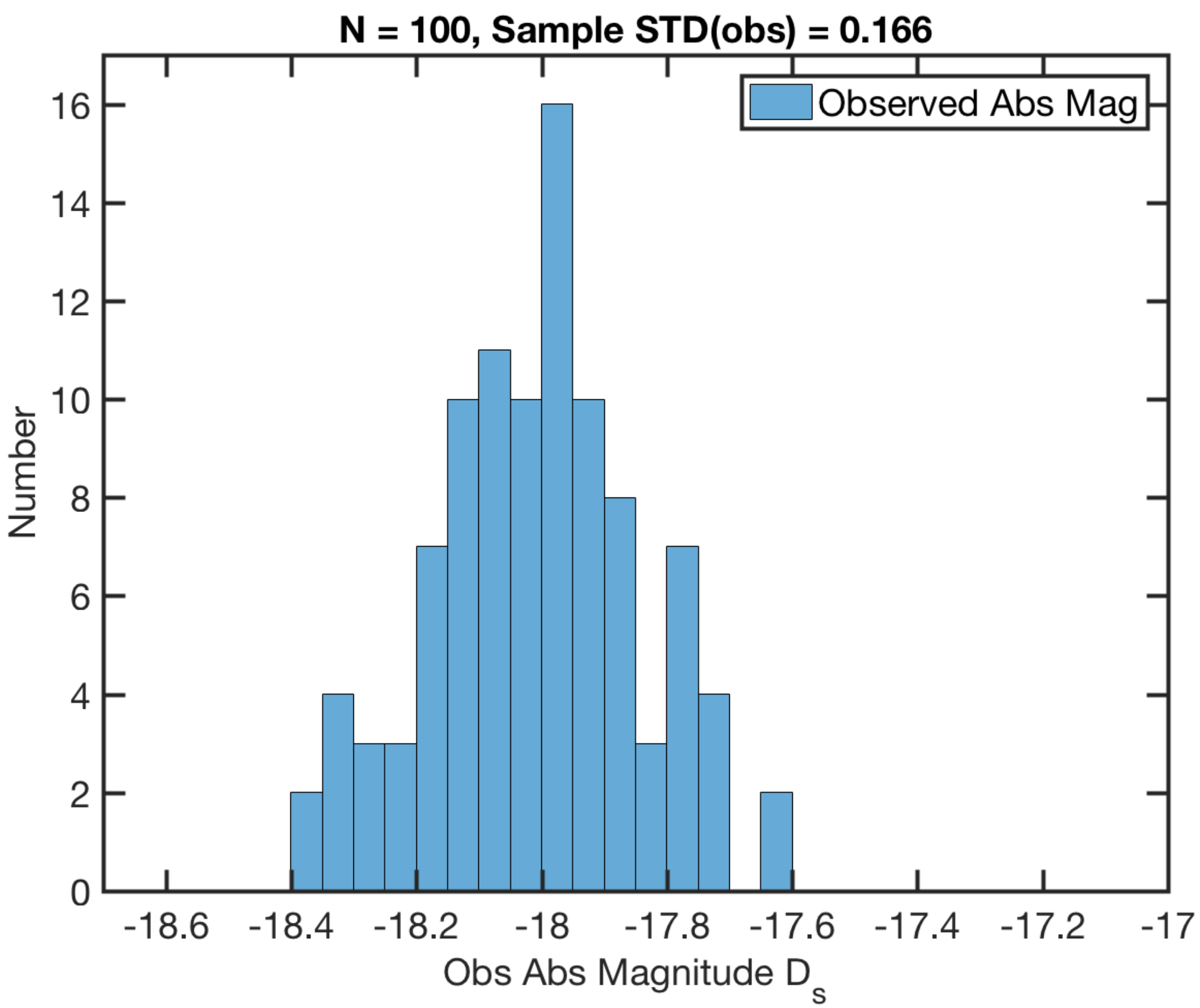
$$M_0 | \tau^2; \{M_s\} \sim N(\bar{M}, \tau^2 / N)$$

$$\bar{M} \equiv \frac{1}{N} \sum_{s=1}^{N} M_s \qquad s^2 \equiv \frac{1}{N-1} \sum_{s=1}^{N} (M_s - \bar{M})^2$$

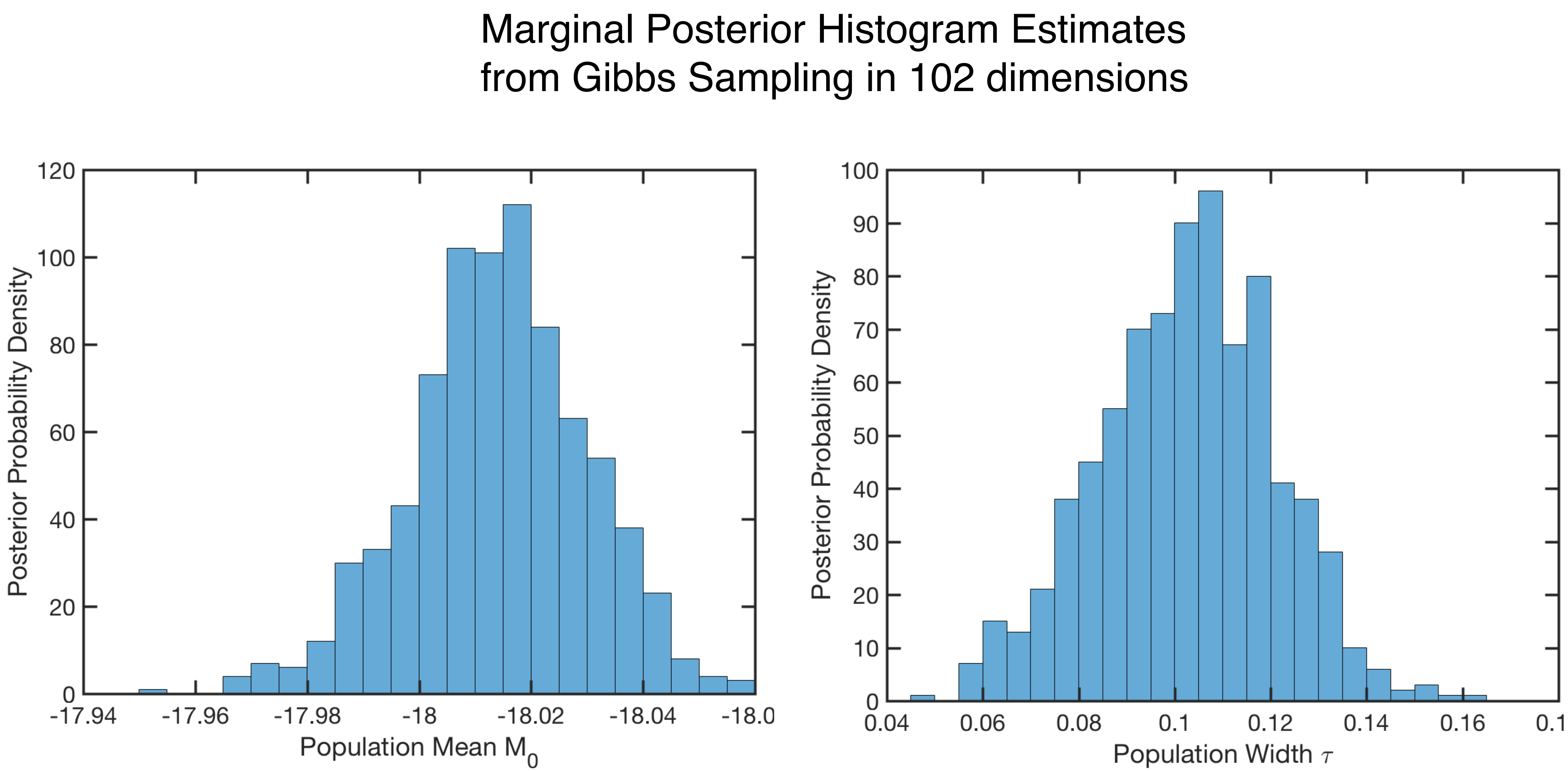
Code/Data Demo

100 Supernovae with
app mag and
distance measurements



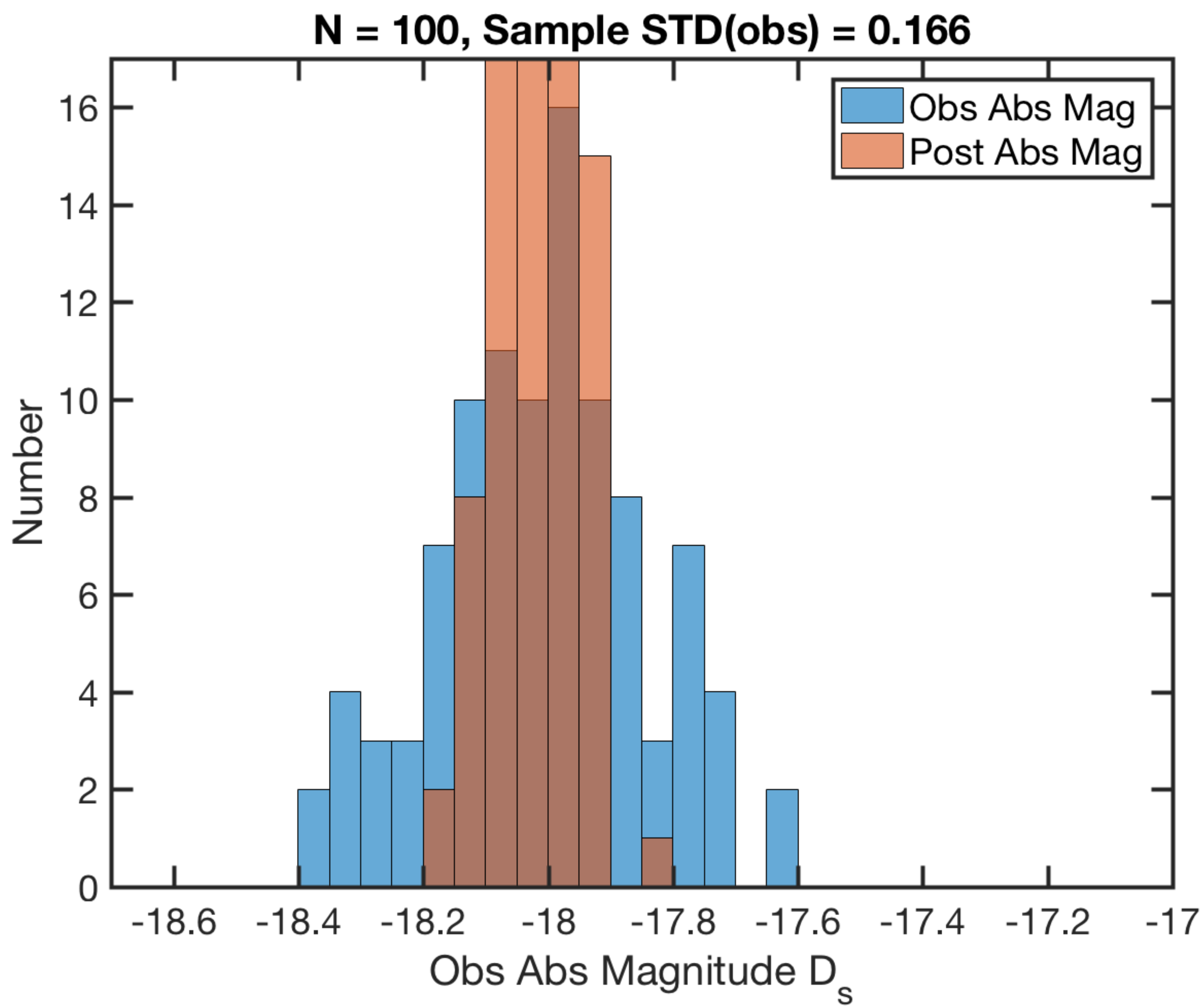


Hyperparameter Inference



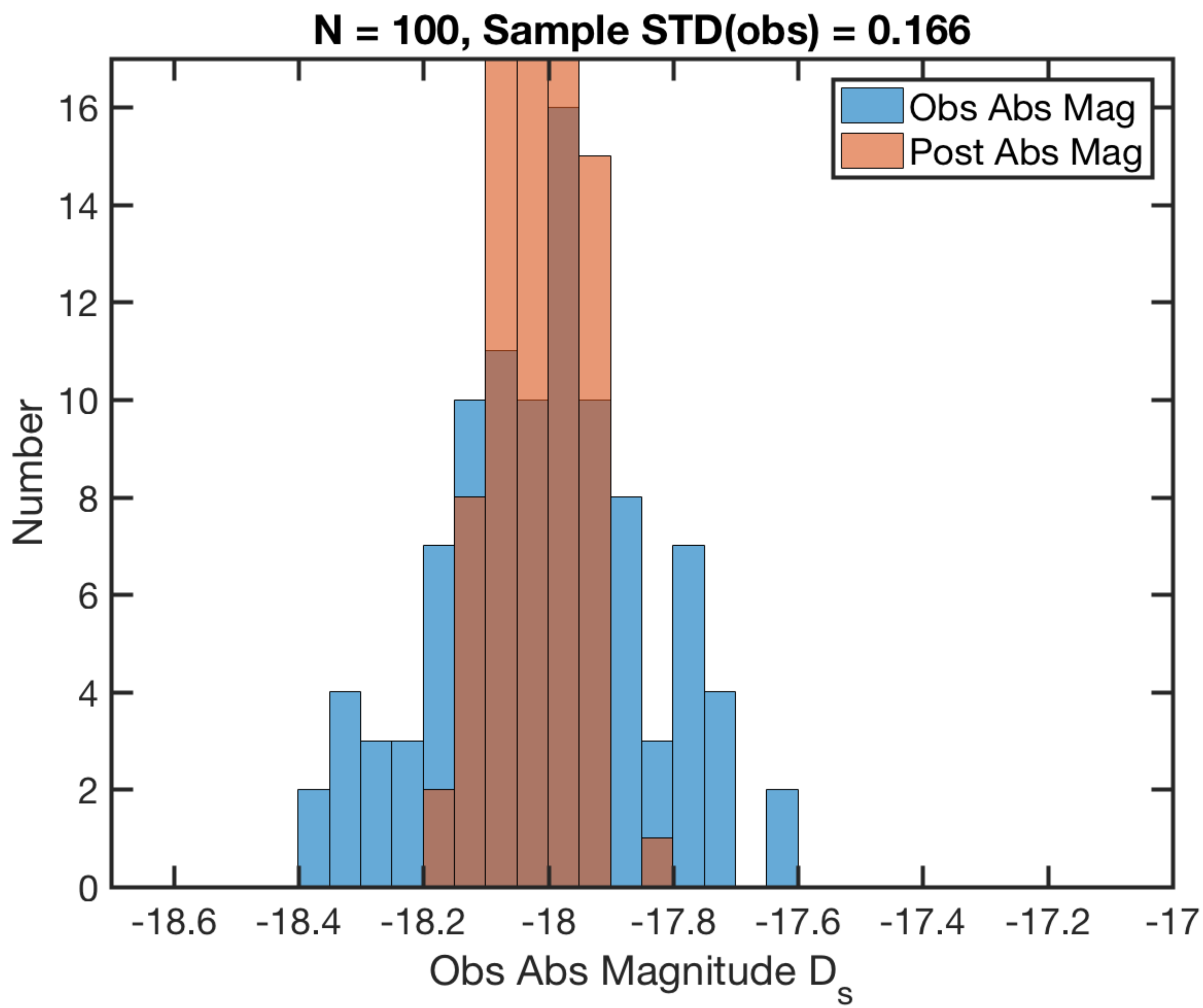
Latent Variable Inference

Histogram of $\mathbb{E}[M_s|\boldsymbol{D}]$ Estimates from MCMC



Latent Variable(s) Inference

Histogram of $\mathbb{E}[M_s|\boldsymbol{D}]$ posterior estimates from MCMC



Latent Variable(s) Inference

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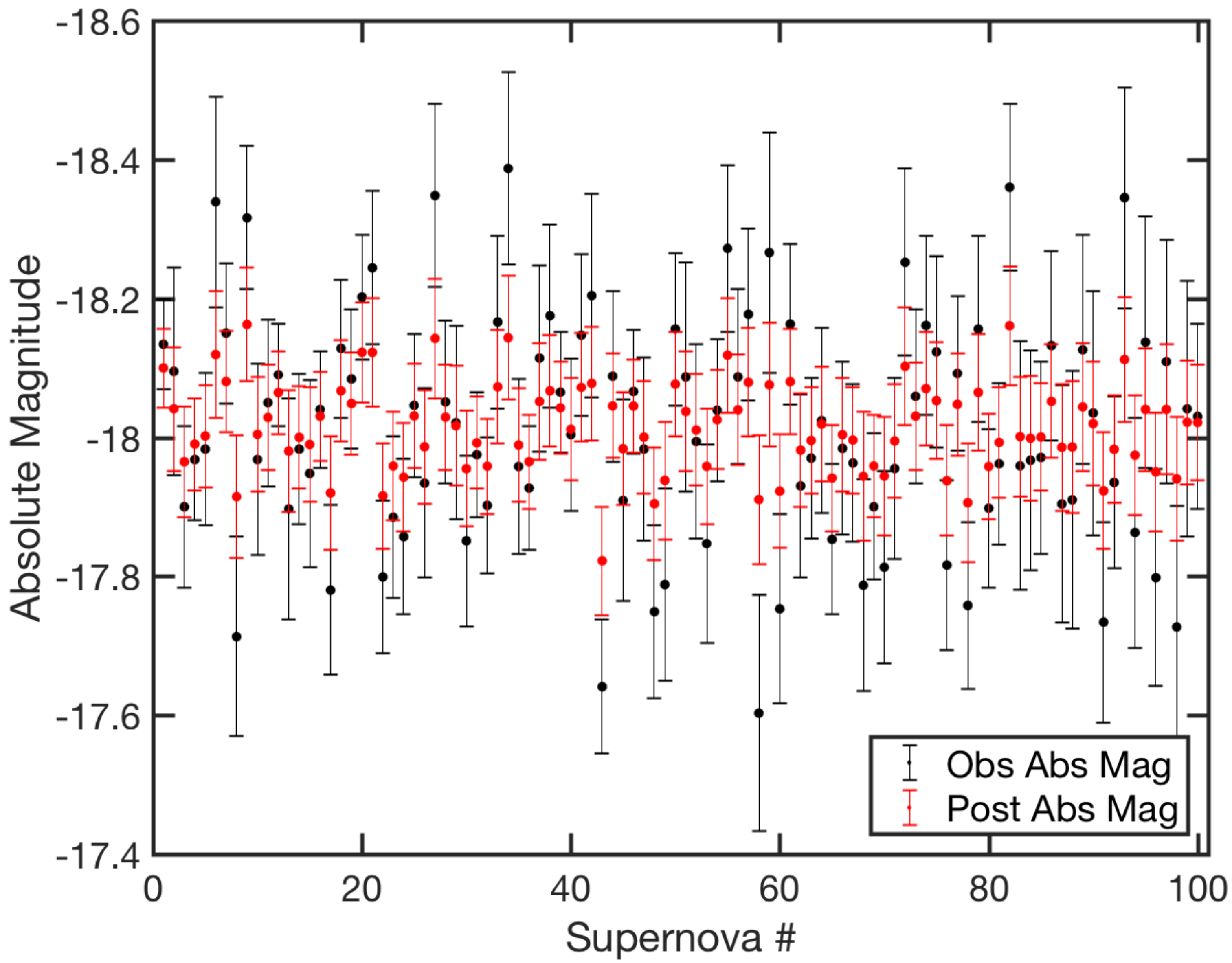
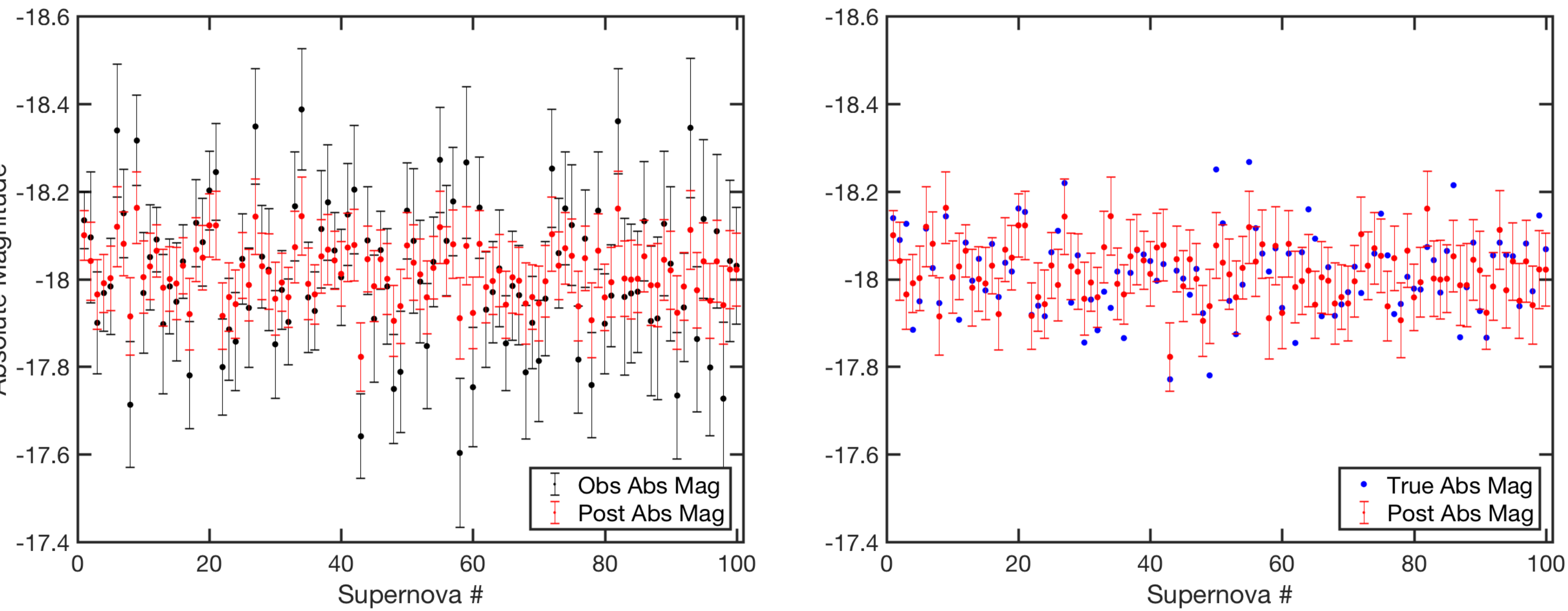


What's going on here?

Latent Variable(s) Inference

Comparing the Data, Posterior Latent Mags vs. true Mags



What's going on here?

HB models: Partial Pooling, Shrinkage, and "Borrowing of Strength"

- Common Sense Procedure:
  - Analyze each individual object's data D_i separately and get each individual MLE_i estimate (with error)
  - "Plug-in" all $\{MLE_i\}$ to estimate population hyperparameters

- Problem: Each individual θ_i estimate may be unbiased but collectively give a biased estimate of population (e.g. variance).
- Solution: Use HB to model and infer individuals & population simultaneously and get better estimates of both

Aside on Shrinkage (derivation on board)

HB models: Partial Pooling, Shrinkage, and "Borrowing of Strength"

Loredo & Hendry (2010)

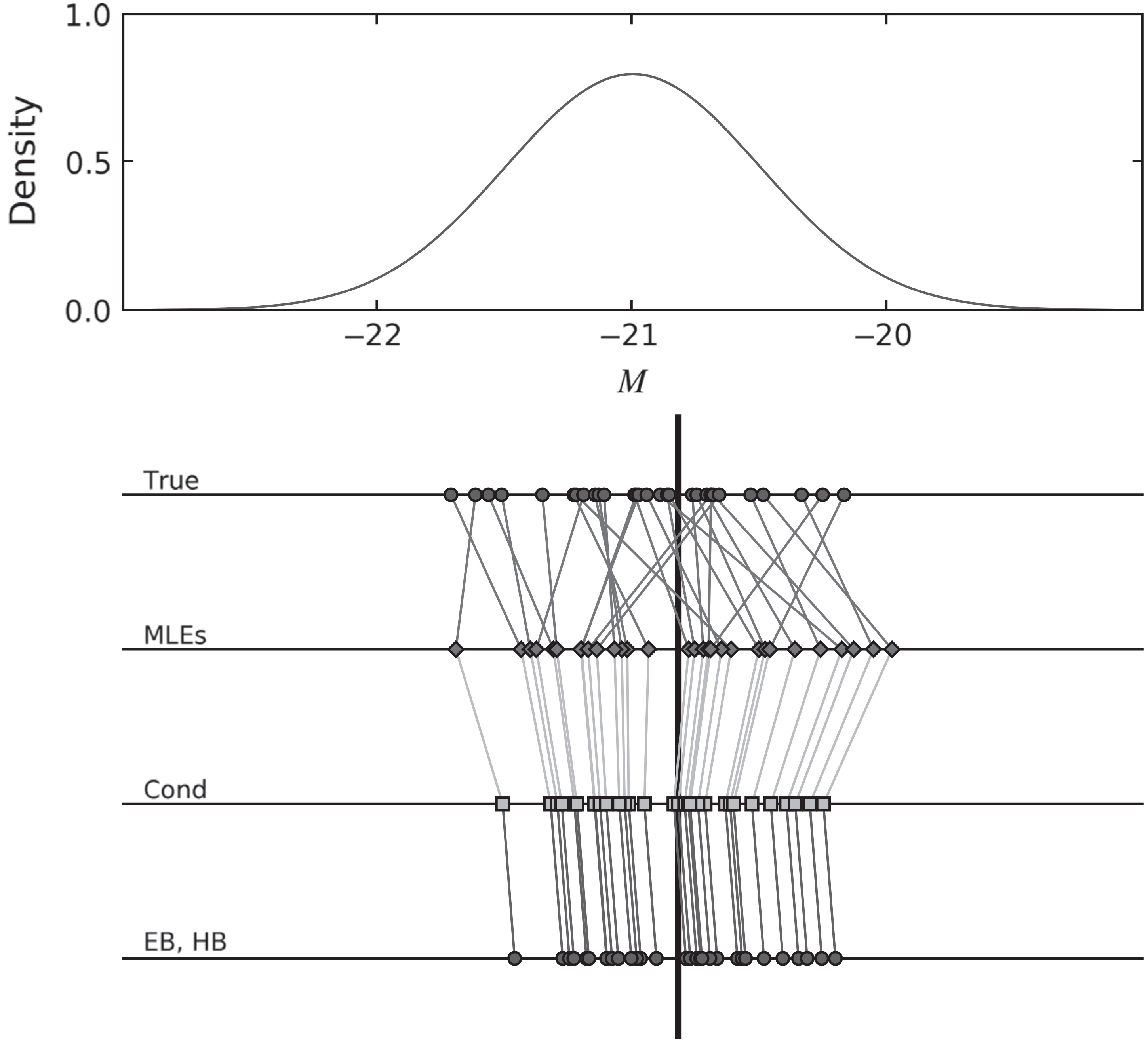
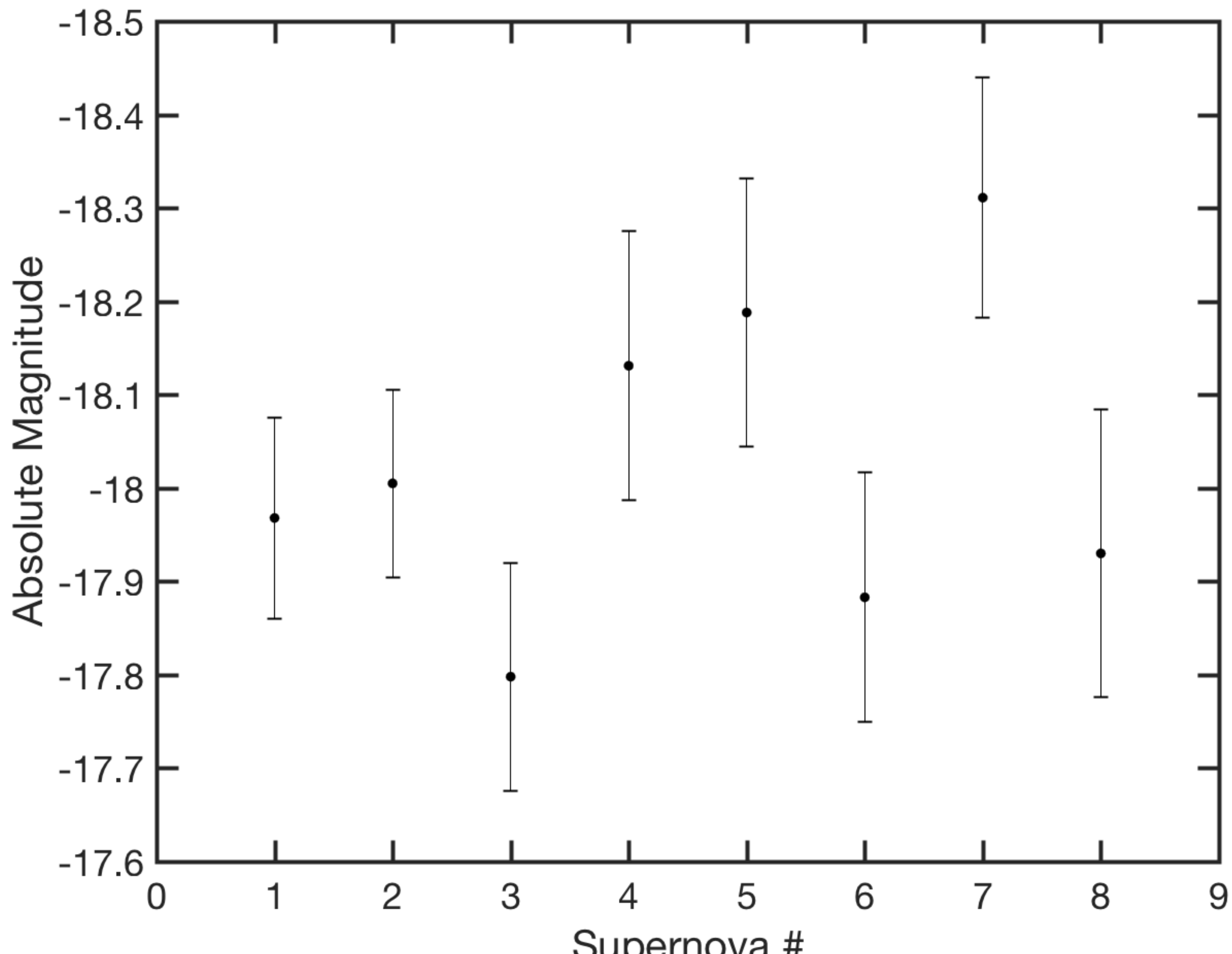


Fig. 11.2. Shrinkage in a simple normal–normal model. Top panel shows population distribution. 'True' axis shows M_i values of 30 samples. Remaining axes show estimates from measurements with $\sigma = 0.3$ normal error: MLEs, conditional (on the true mean), and empirical/hierarchical Bayes estimates.

"Bayesian Multi-level modeling of cosmological populations" in Bayesian Methods in Cosmology

How does Hierarchical Bayes implement shrinkage?

First, let's shrink the dataset



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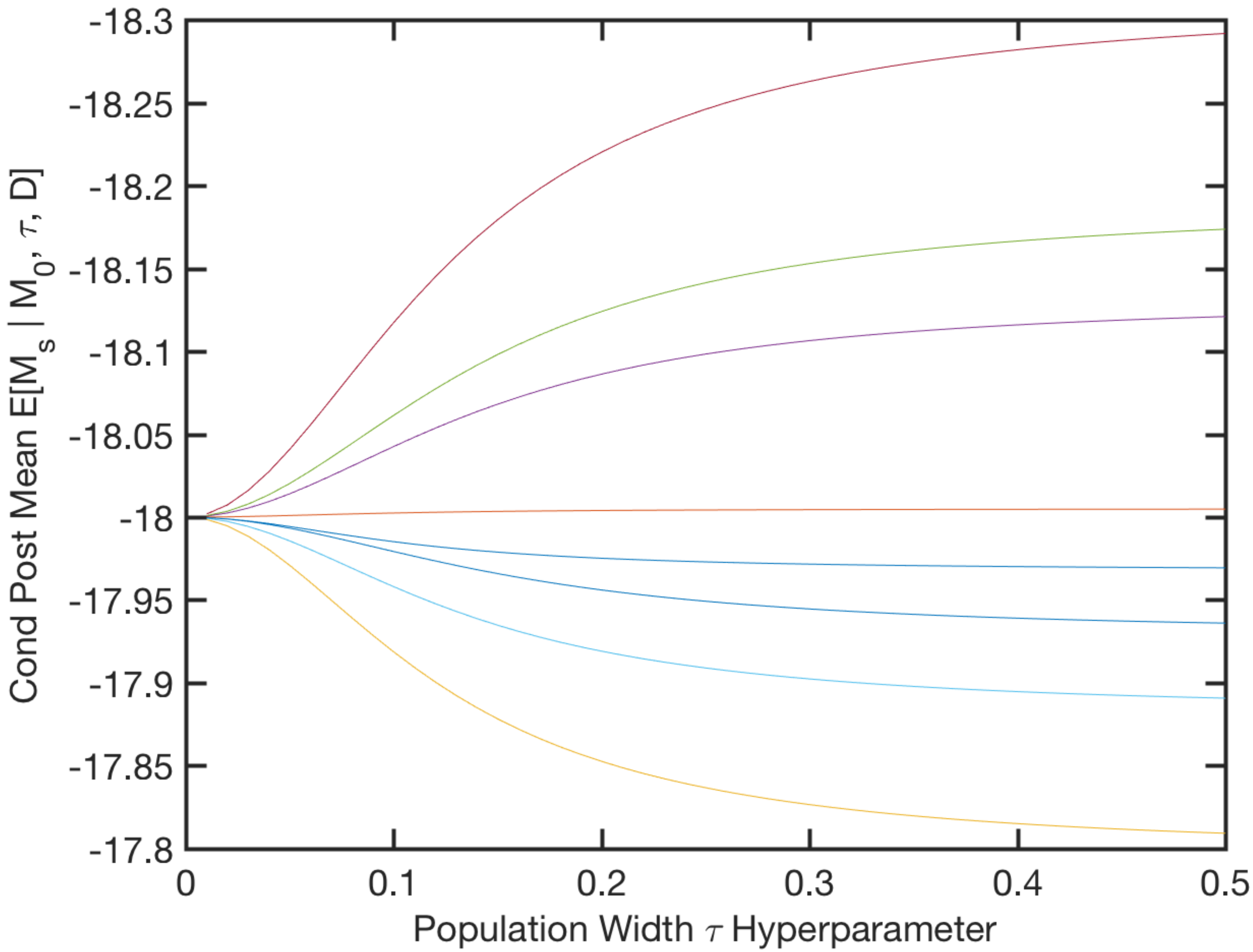

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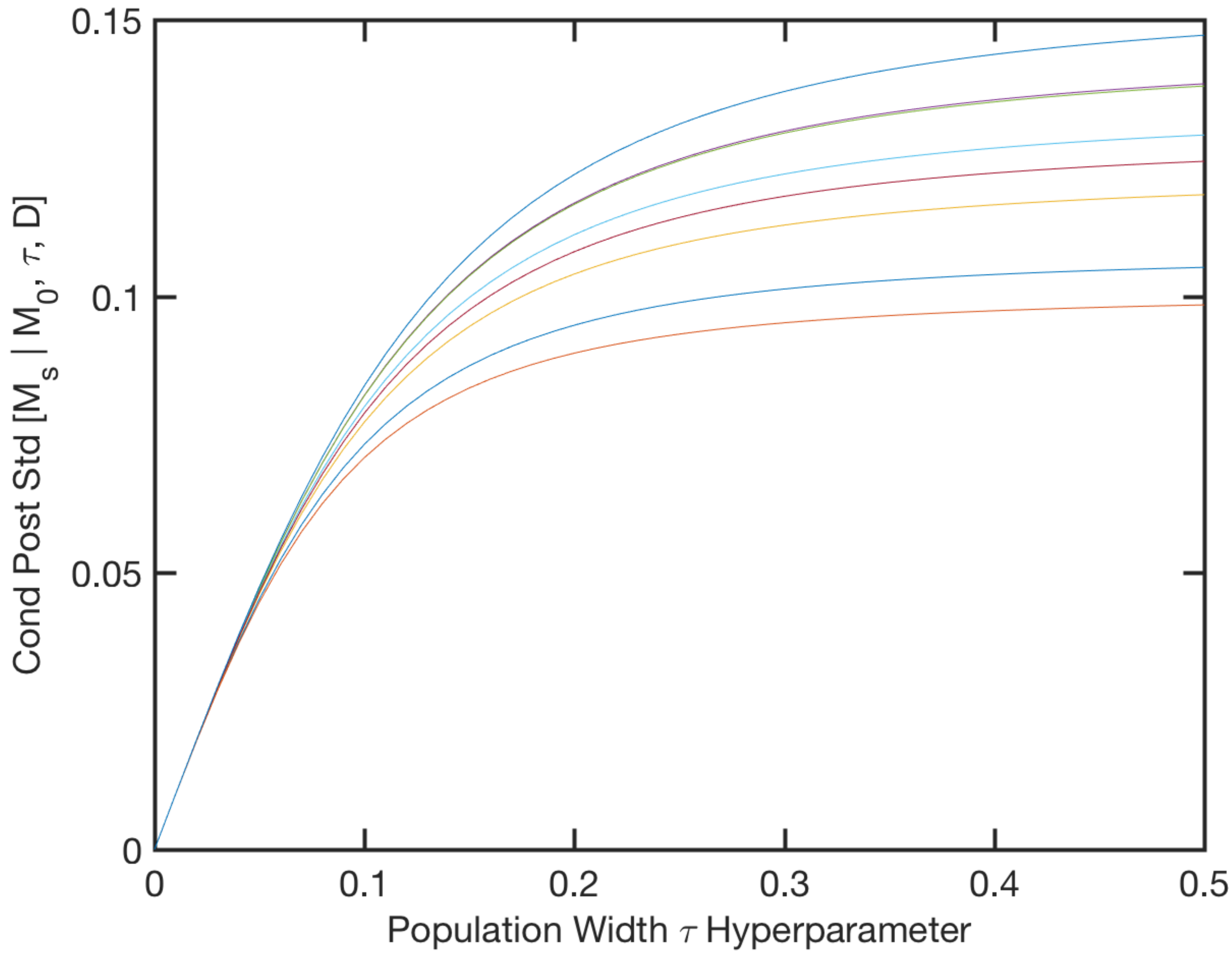

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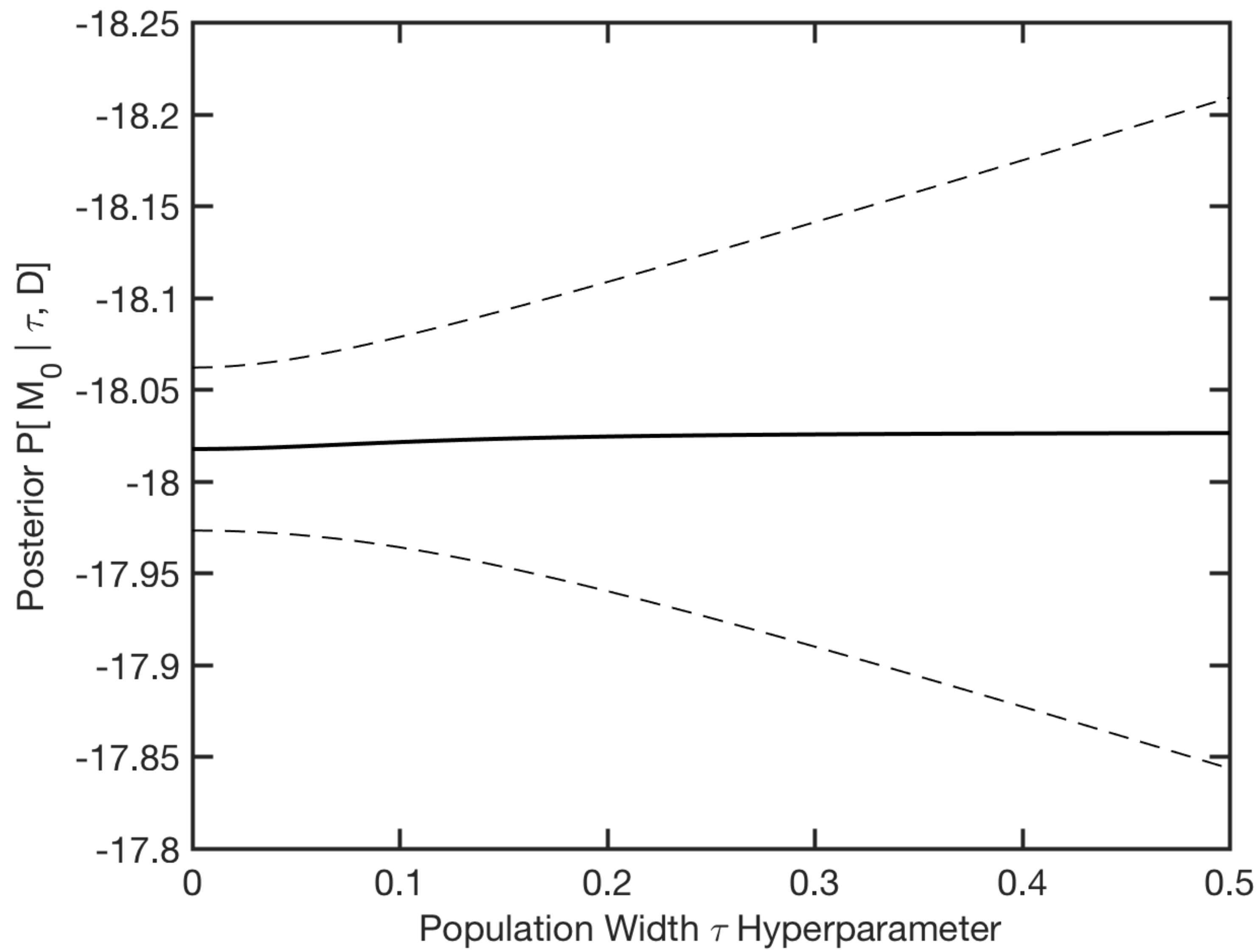

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