Astrostatistics: Wed 20 Feb 2019

https://github.com/CambridgeAstroStat/PartIII-Astrostatistics-2019

- Example Sheet online, Ex Class Tue Feb 19, 1pm MR5
- Today: continue Bayesian computation / Monte Carlo Methods / MCMC
- MacKay: Ch 29-30; Bishop: Ch 11; Gelman
- Givens & Hoeting “Computational Statistics” (Free download through Cambridge Library iDiscover)
- Hogg & DFM, 2017 “Data analysis recipes: Using Markov Chain Monte Carlo.” https://arxiv.org/abs/1710.06068

Markov Chain Monte Carlo (MCMC)

Mapping the Posterior P(θ |D)

- Markov Chain Monte Carlo (MCMC)
- Now:
  - Drawing Multivariate Gaussian random variables
  - N-D Metropolis Algorithm
  - Rules of thumb for proposal scale
  - assessing convergence (G-R Ratio)
  - Metropolis-Hastings algorithm
  - Gibbs sampling

d-dim Metropolis Algorithm: Posterior $P(\theta \mid D)$, Symmetric Proposal/Jump dist'n $J(\theta^* \mid \theta) = J(\theta \mid \theta^*)$

1. Choose a random starting point θ_0

2. At step i = 1…N, propose a new parameter value $\theta^* \sim N(\theta_{i-1}, \mathbf{\Sigma_p})$. The proposal distr'n is $J(\theta^* \mid \theta_{i-1}) = N(\theta^* \mid \theta_{i-1}, \mathbf{\Sigma_p})$

3. Evaluate ratio of posteriors at proposed vs current values. $r = P(\theta^* \mid \mathbf{y}) / P(\theta_{i-1} \mid \mathbf{y})$.

4. Accept θ^* with probability min(r,1): $\theta_i = \theta^*$. If not accept, stay at same value $\theta_i = \theta_{i-1}$ for the next step & include in chain.

5. Repeat steps 2-4 until reach some measure of convergence (G-R) and gather enough independent samples to compute your inference (reduce Monte Carlo error)

Tuning d-dim Metropolis

- $\theta^* \sim N(\theta_i, \mathbf{\Sigma_p})$: if proposal scale $\mathbf{\Sigma_p}$ is too large, will get too many rejections and not go anywhere. If proposal scale too small, you will accept very many small moves: inefficient random walk

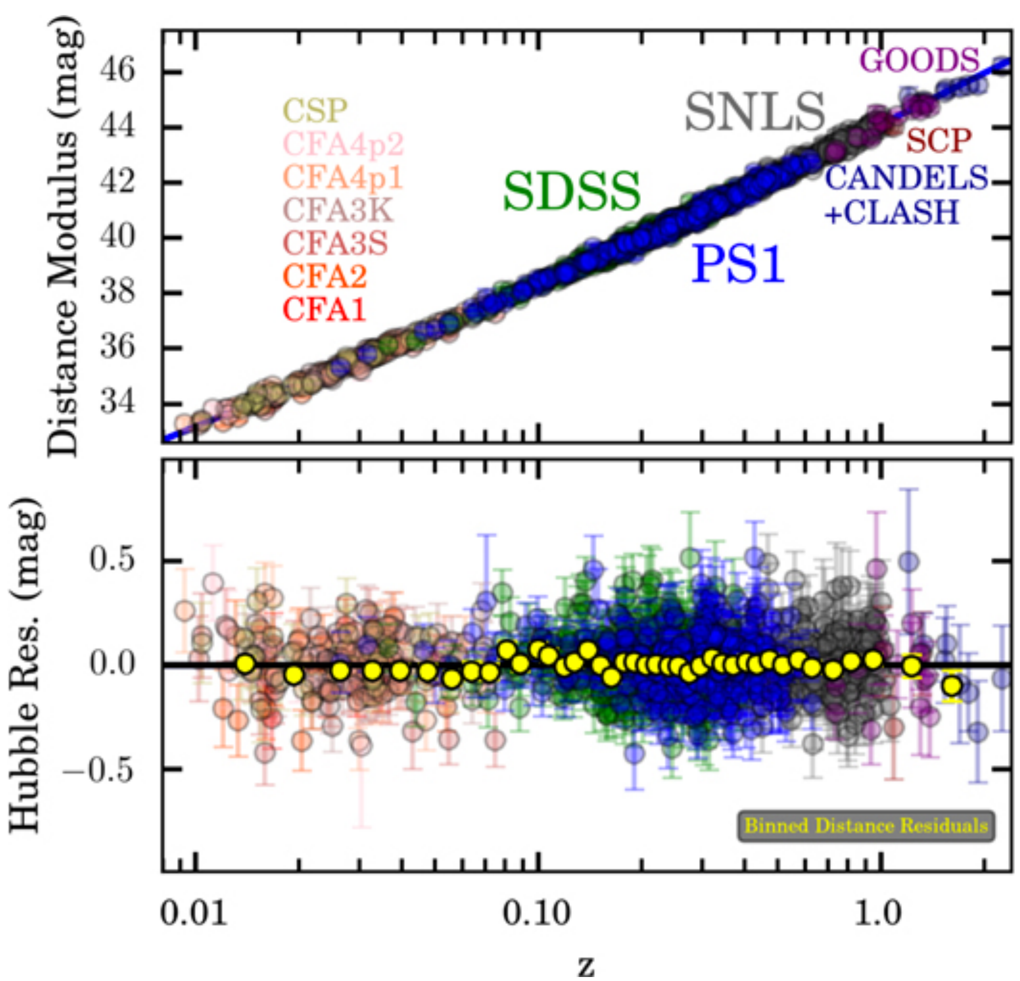
- Laplace Approximation: $$P(\boldsymbol{\theta}|\boldsymbol{D}) \approx N(\boldsymbol{\theta}|\,\hat{\boldsymbol{\theta}}, \boldsymbol{\Sigma})$$

$$\hat{\boldsymbol{\theta}} = \text{ posterior mode} \qquad (\boldsymbol{\Sigma}^{-1})_{ij} = \frac{\partial^2 \log P(\boldsymbol{\theta}|\boldsymbol{D})}{\partial\theta_i\, \partial\theta_j}\Big|_{\hat{\boldsymbol{\theta}}}$$

- Choose $\mathbf{\Sigma_p} = c^2\,\mathbf{\Sigma}$: $$c \approx 2.4/\sqrt{d}$$

- Scale Proposal to aim for an acceptance ratio of 44% in 1D, 23% in d > 5

Astrostatistics Case Study:
Supernova Cosmology



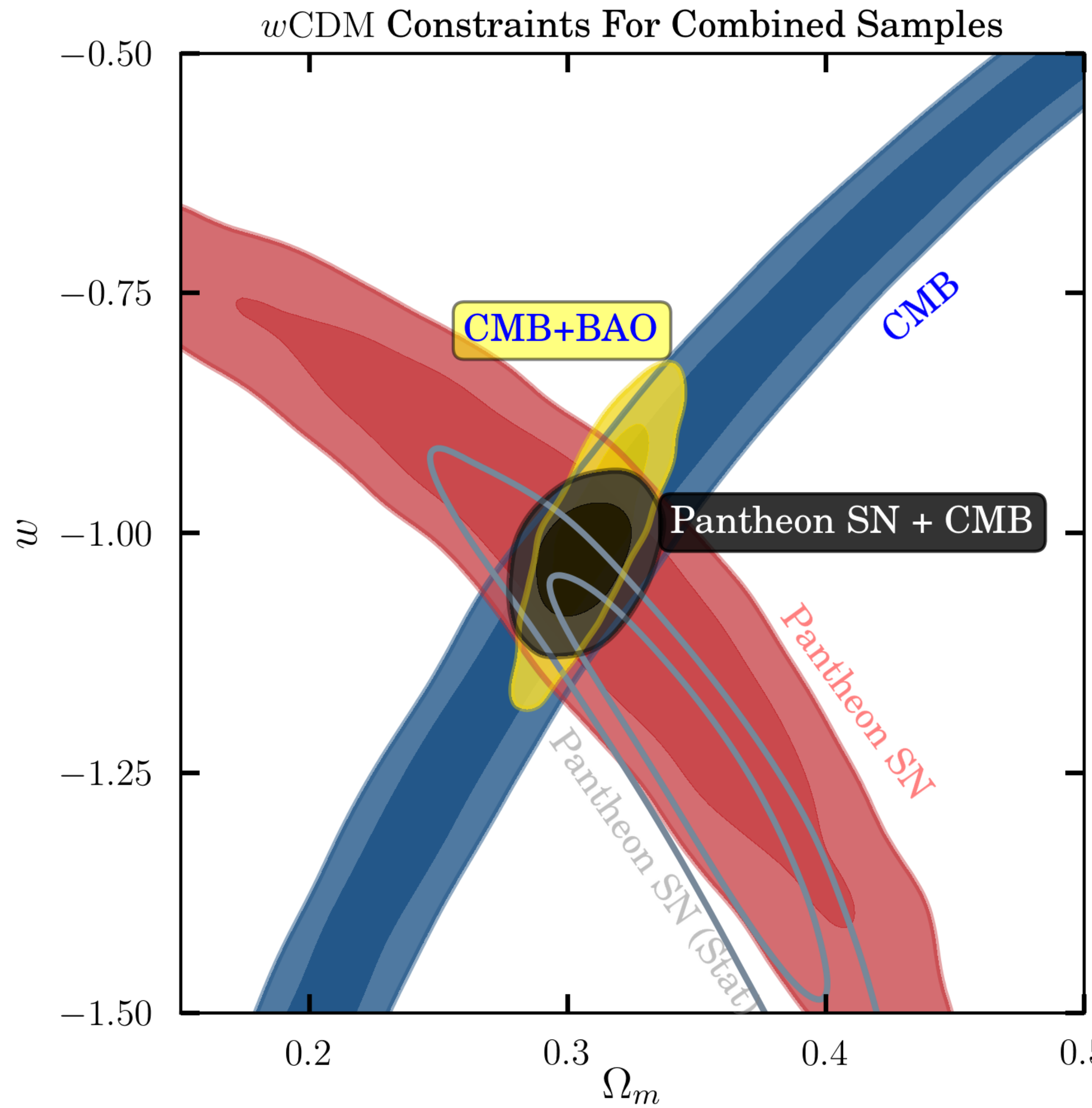
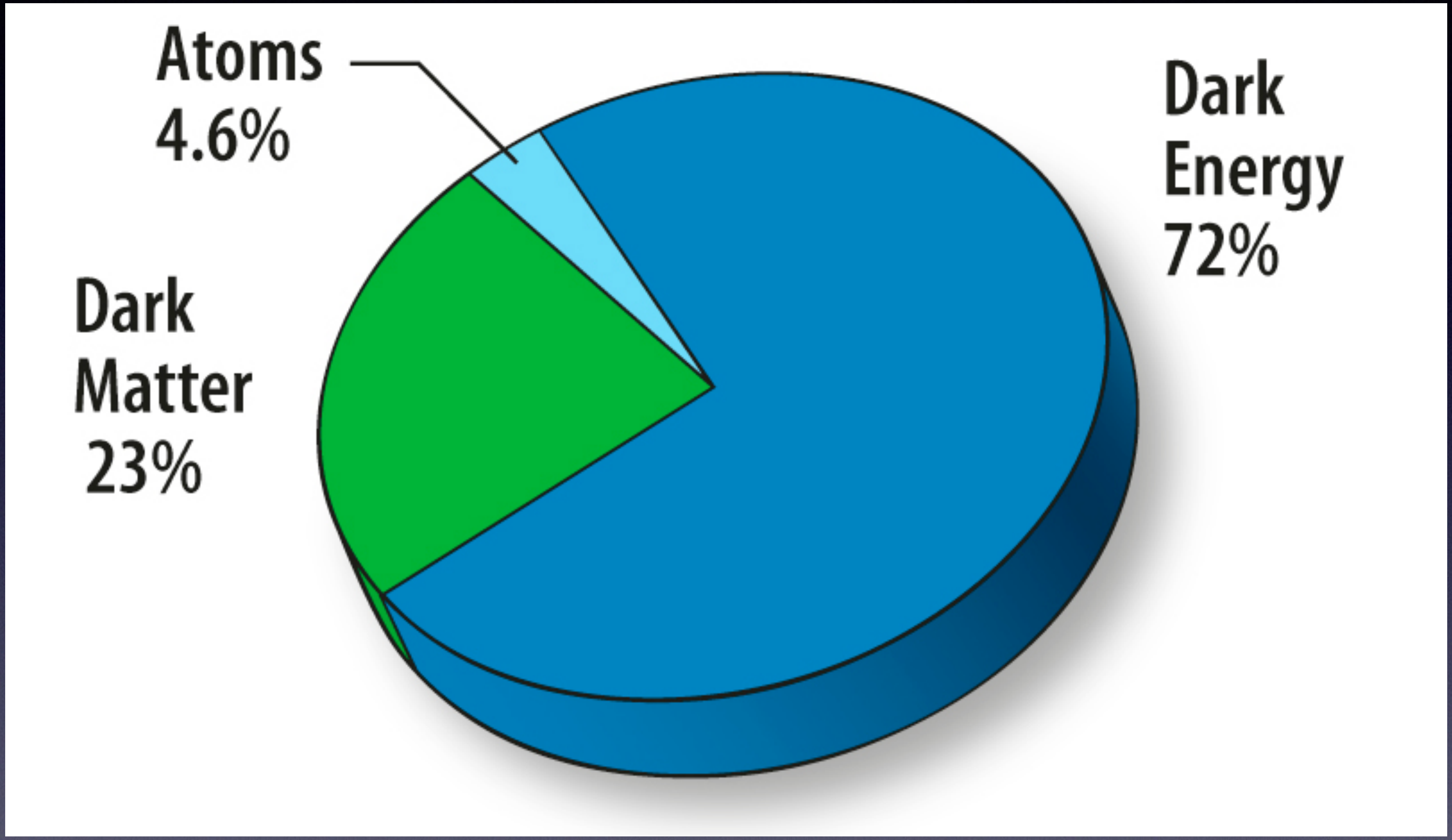


Figure 20. Confidence contours at 68% and 95% for the Ω_m and w cosmological parameters for the wCDM model. Constraints from CMB (blue), SN - with systematic uncertainties (red), SN - with only statistical uncertainties (gray-line), and SN+CMB (purple) are shown.

Cosmological Energy Content



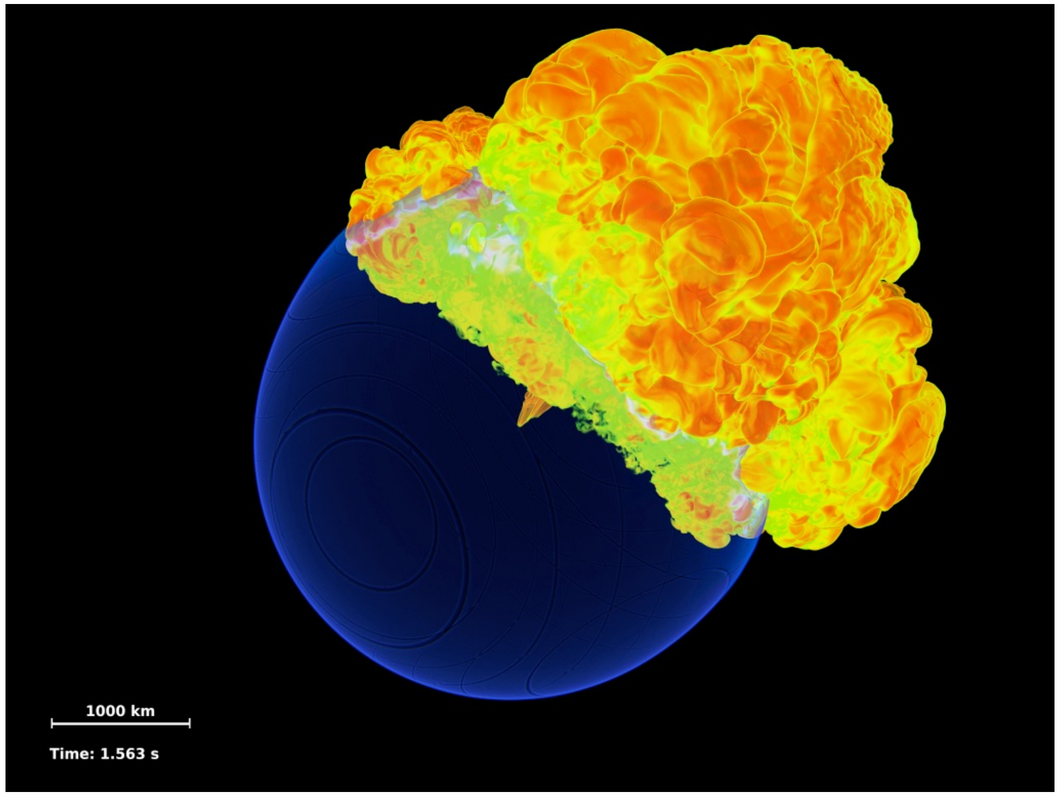
What is Dark Energy?

Dark Energy Equation of state $P = w\rho$

Is $w + 1 = 0$? (Cosmological Constant: $w = -1$)

Type Ia Supernovae (SN Ia) are Almost Standard Candles

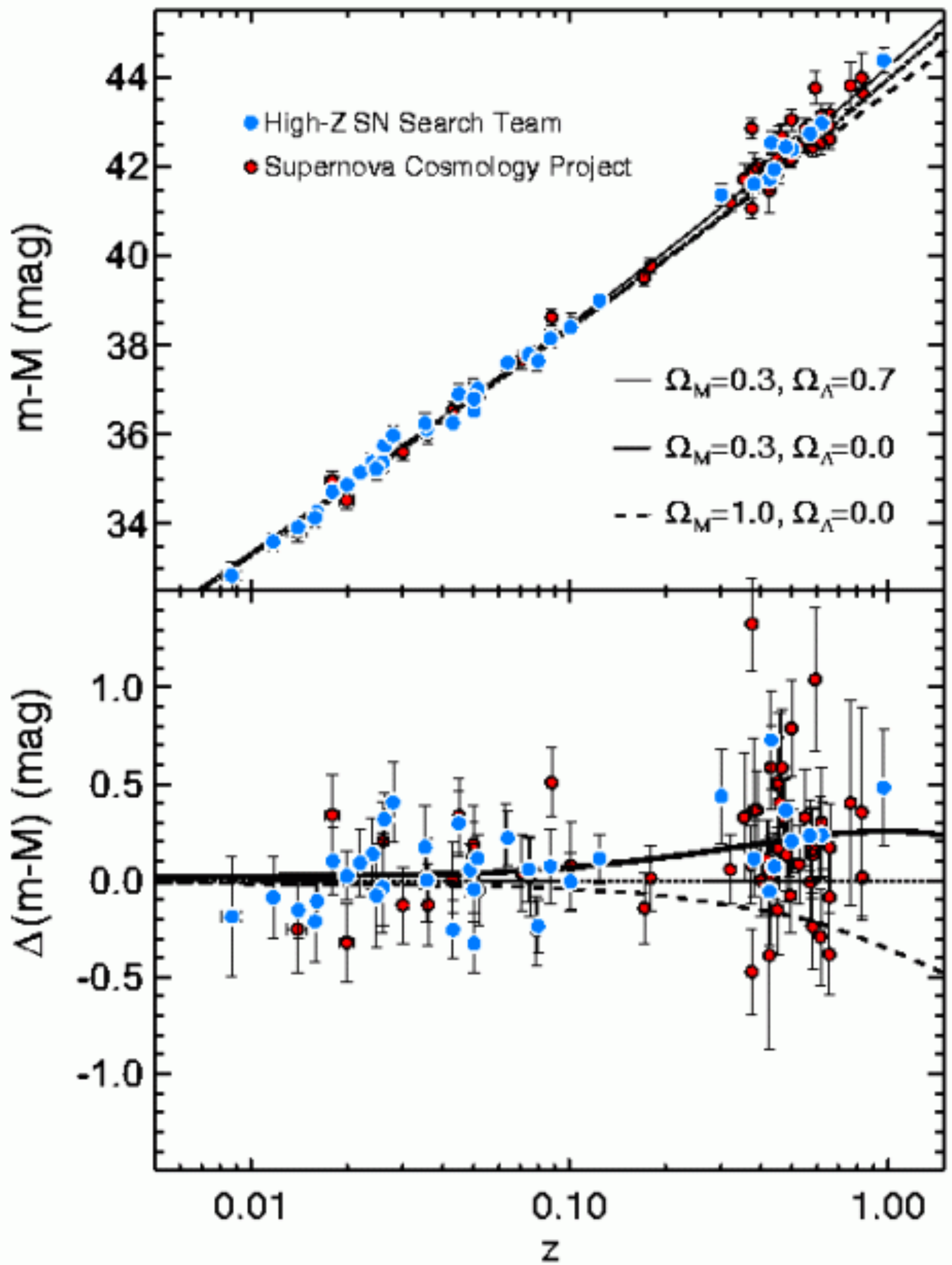
- Progenitor: C/O White Dwarf Star accreting mass leads to instability
- Thermonuclear Explosion: Deflagration/Detonation
- Nickel to Cobalt to Iron Decay + radiative transfer powers the light curve
- General Idea, but Theoretical Astrophysics simulations cannot quantitatively reproduce realistic observations (use empirical models)



Credit: FLASH Center

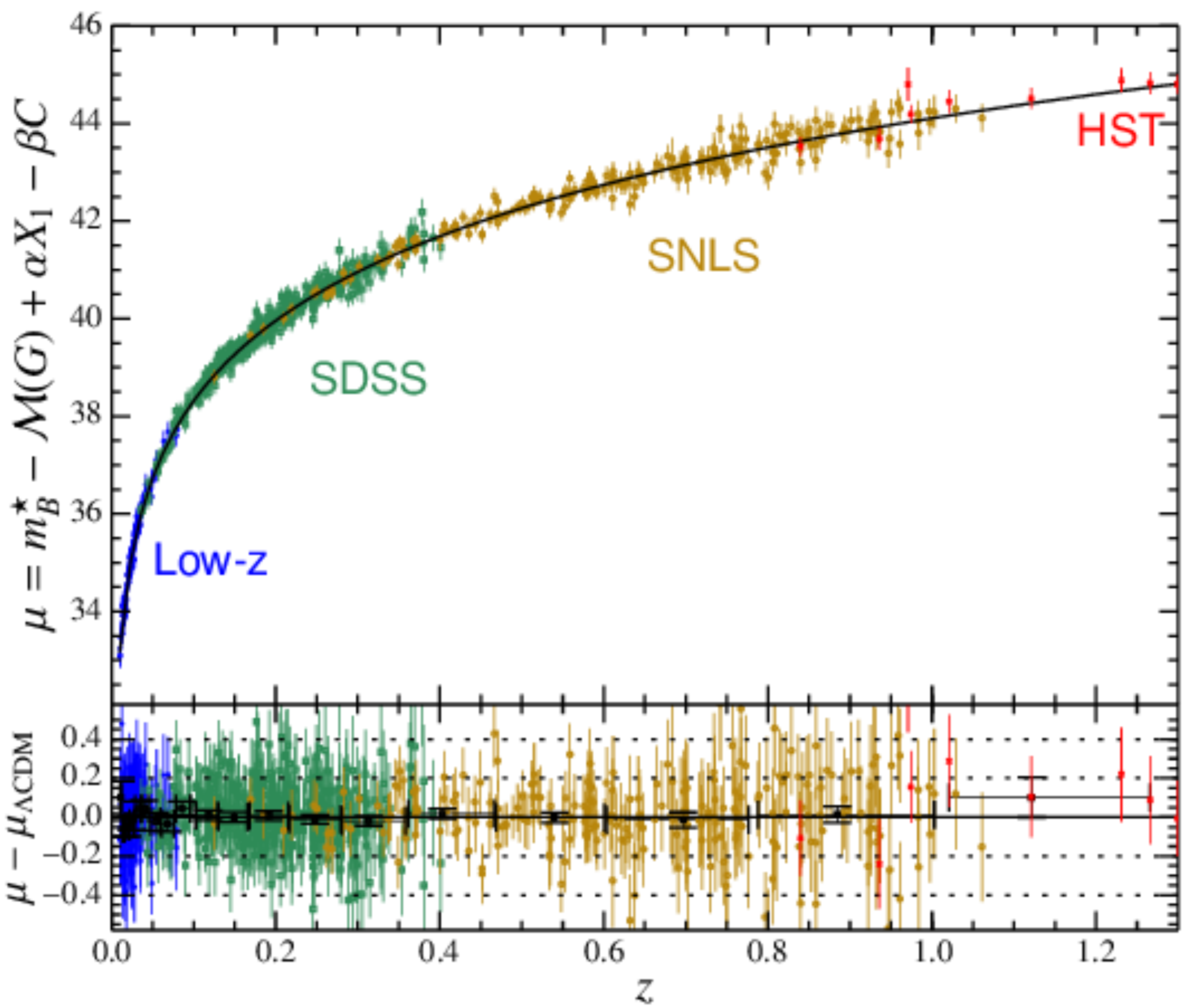
SN Ia Hubble Diagram
(Distance Moduli vs. z):

The Universe is accelerating ($\Omega_\Lambda > 0$)!



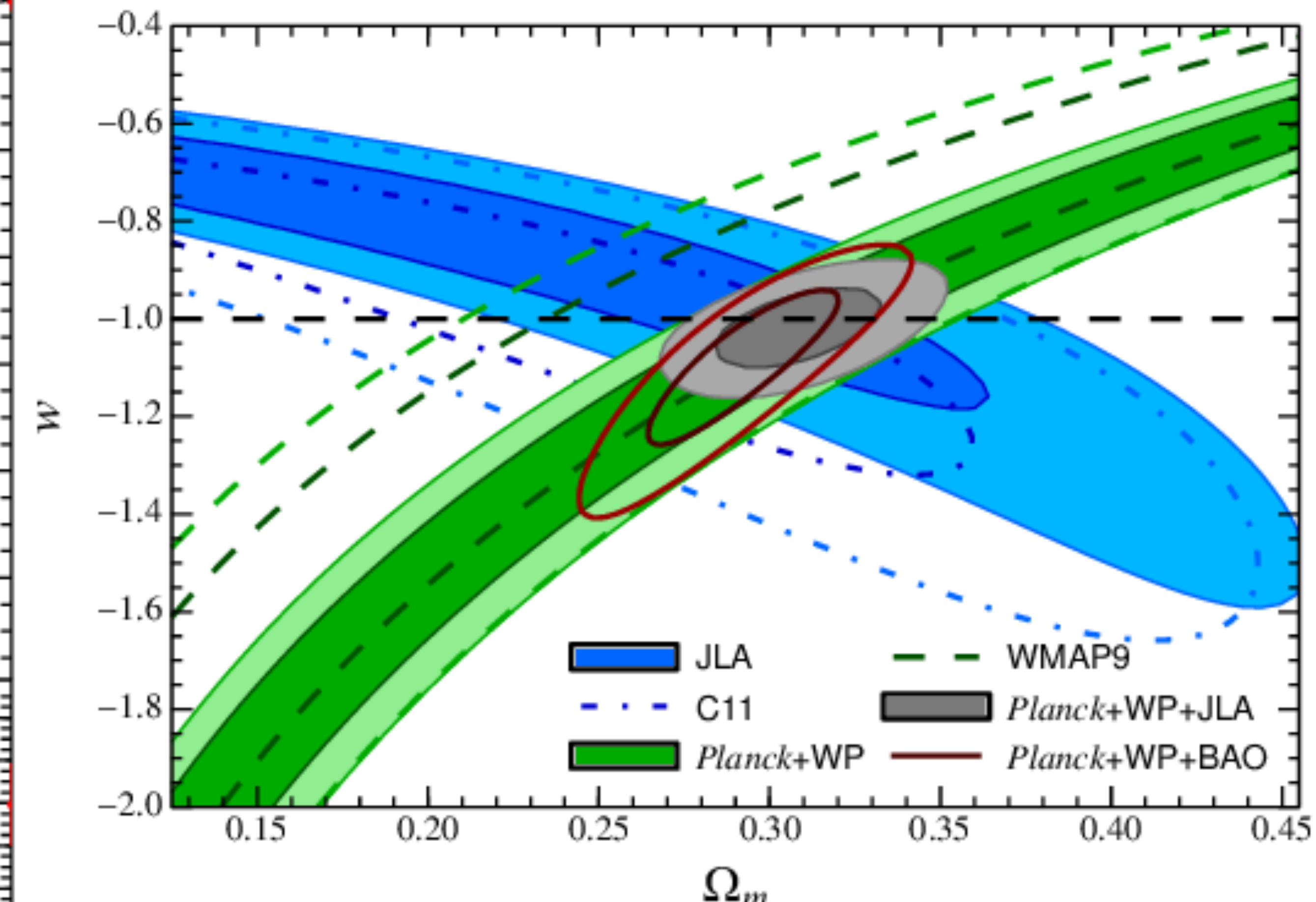
Type Ia SN Cosmology

Hubble Diagram
Modern SN Ia Surveys



Joint Lightcurve Analysis
(JLA, Betoule et al. 2014)

Cosmological Constraints



$w = -1.027 \pm 0.055$ (stat+sys)

Cosmology with standard candles

For us: Simplify:
Let's say Supernovae are *standard* candles

$$M_s \sim N(M_0, \sigma^2_{\rm int})$$ Population Distribution

$$m_s = M_s + \mu(z_s; \boldsymbol{\theta})$$ (Log) Inverse Square Law

Assume apparent magnitudes and redshifts
$\{m_s, z_s\}$ measured perfectly

$$\boldsymbol{\theta} = (H_0, \Omega_M, \Omega_L, w)$$ Cosmological Parameters

Theoretical Distances

1 Comoving Distance

The dimensionless comoving distance $\tilde{d}$ to an object with redshift z is:

$$\tilde{d}(z; \Omega_M, \Omega_\Lambda, w) = \int_0^z \frac{dz'}{\sqrt{\Omega_M(1+z')^3 + \Omega_k(1+z')^2 + \Omega_\Lambda(1+z')^{3(w+1)}}} \tag{1}$$

where Ω_M is the matter density, Ω_Λ is the dark energy density, the curvature is $\Omega_k \equiv 1 - \Omega_M - \Omega_\Lambda$, and w is the equation-of-state parameter of dark energy.

In terms of the scale factor $a \equiv (1+z)^{-1}$, the integral can be expressed as:

$$\tilde{d}(z; \Omega_M, \Omega_\Lambda, w) = \int_{(1+z)^{-1}}^1 \frac{da'/a'}{\sqrt{\Omega_M/a' + \Omega_k + \Omega_\Lambda/a'^{(1+3w)}}} \tag{2}$$

(derived from standard FLRW metric)

Theoretical Luminosity Distances (Arbitrary Curvature)

2 Luminosity Distance

The theoretical cosmological distance to an object with redshift z inferred from the standard candle method are given by the luminosity distance d_L. The dimensionless distance is:

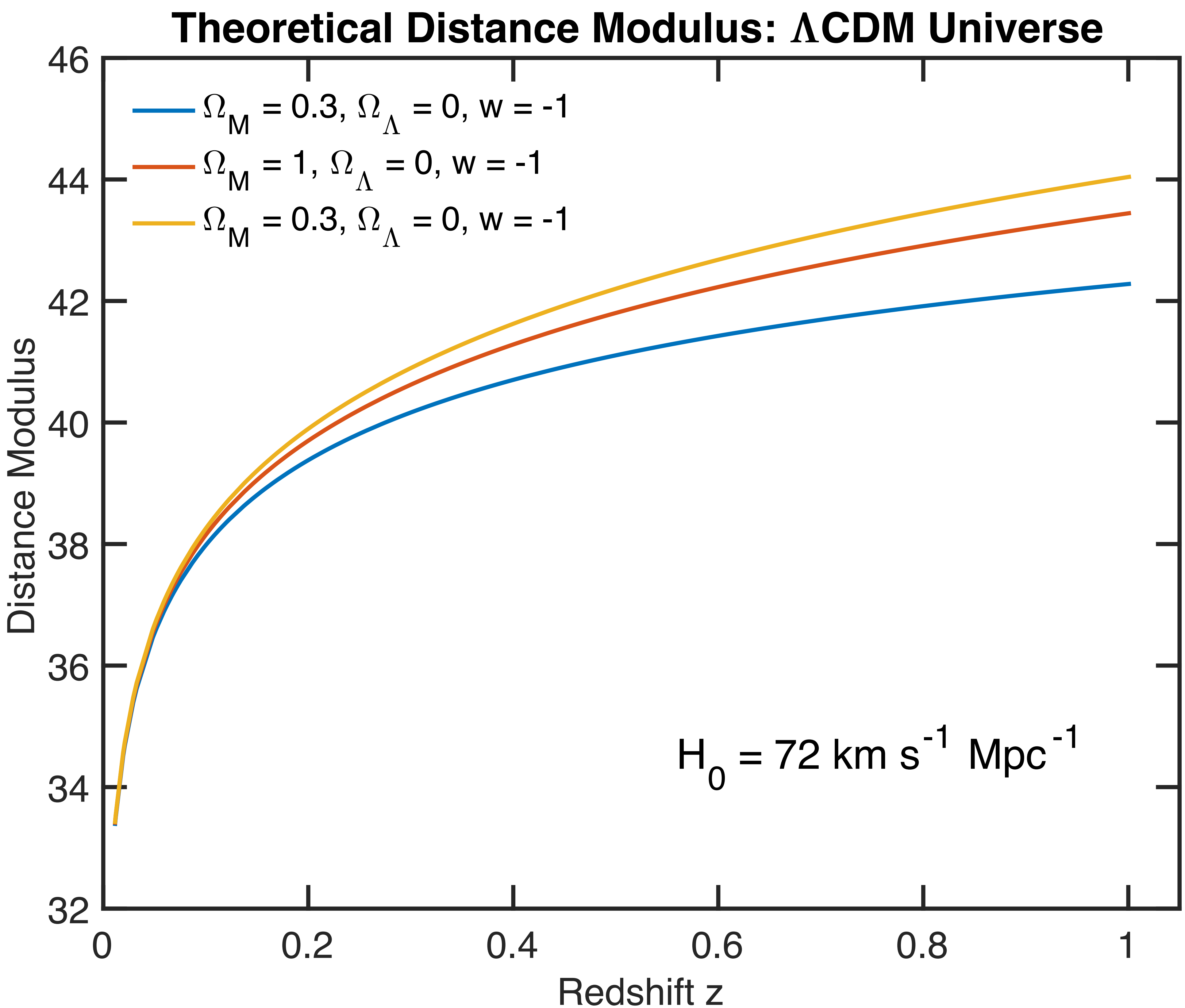
$$\tilde{d}_L(z;\Omega_M,\Omega_\Lambda,w) = (1+z)\begin{cases} |\Omega_k|^{-\frac{1}{2}}\sinh[\sqrt{|\Omega_k|}\tilde{d}(z;\Omega_M,\Omega_\Lambda,w), & \Omega_k > 0 \\ \tilde{d}(z;\Omega_M,\Omega_\Lambda,w), & \Omega_k = 0 \\ |\Omega_k|^{-\frac{1}{2}}\sin[\sqrt{|\Omega_k|}\tilde{d}(z;\Omega_M,\Omega_\Lambda,w), & \Omega_k < 0 \end{cases} \tag{3}$$

For a given value of the Hubble constant, the dimensionful luminosity distance is

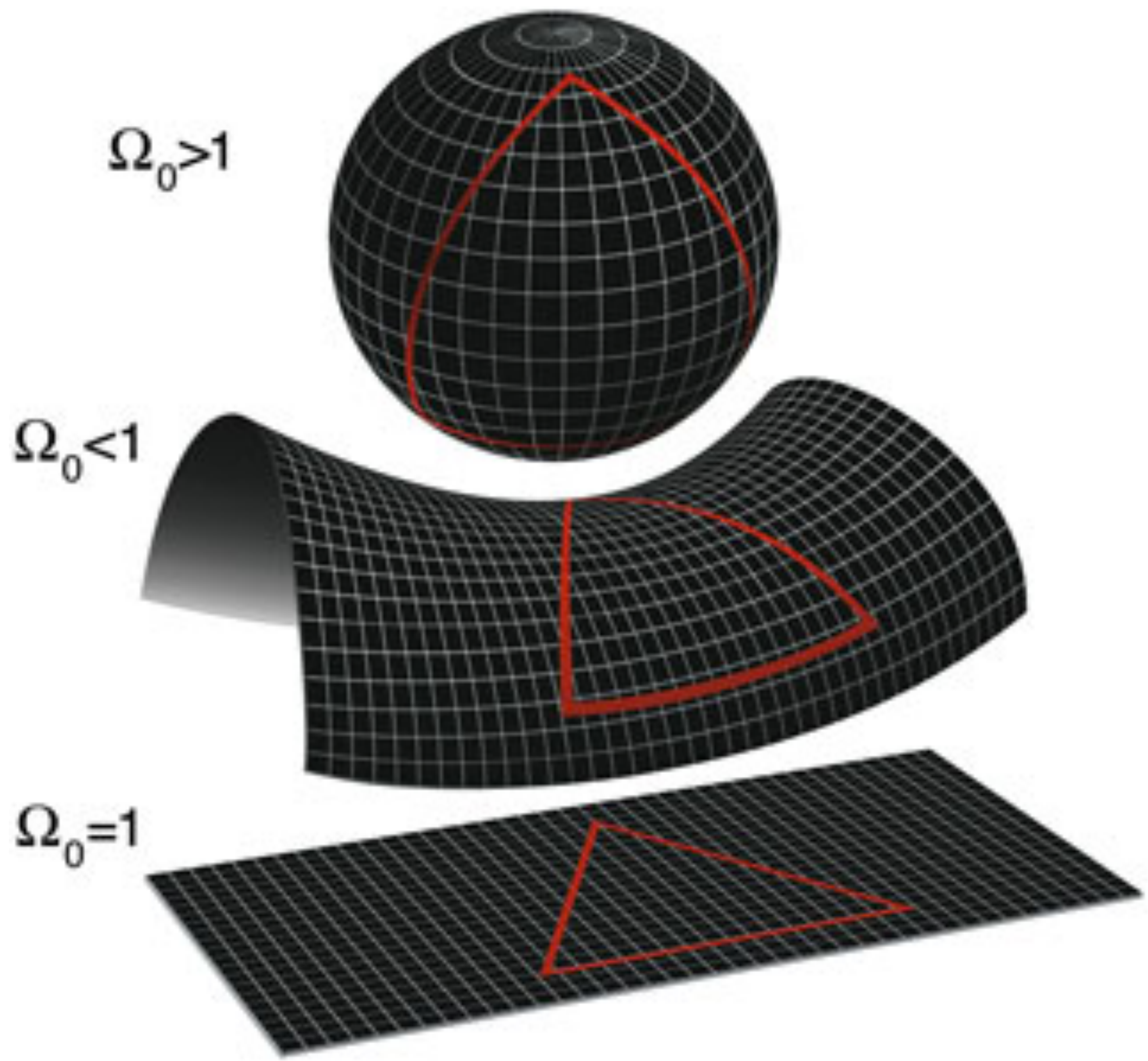
$$d_L(z;\Omega_M,\Omega_\Lambda,w,H_0) = \frac{c}{H_0}\tilde{d}_L(z;\Omega_M,\Omega_\Lambda,w) \tag{4}$$

(derived from standard FLRW metric)

Theoretical Distance Modulus
(Arbitrary Curvature)



Curvature of the Universe

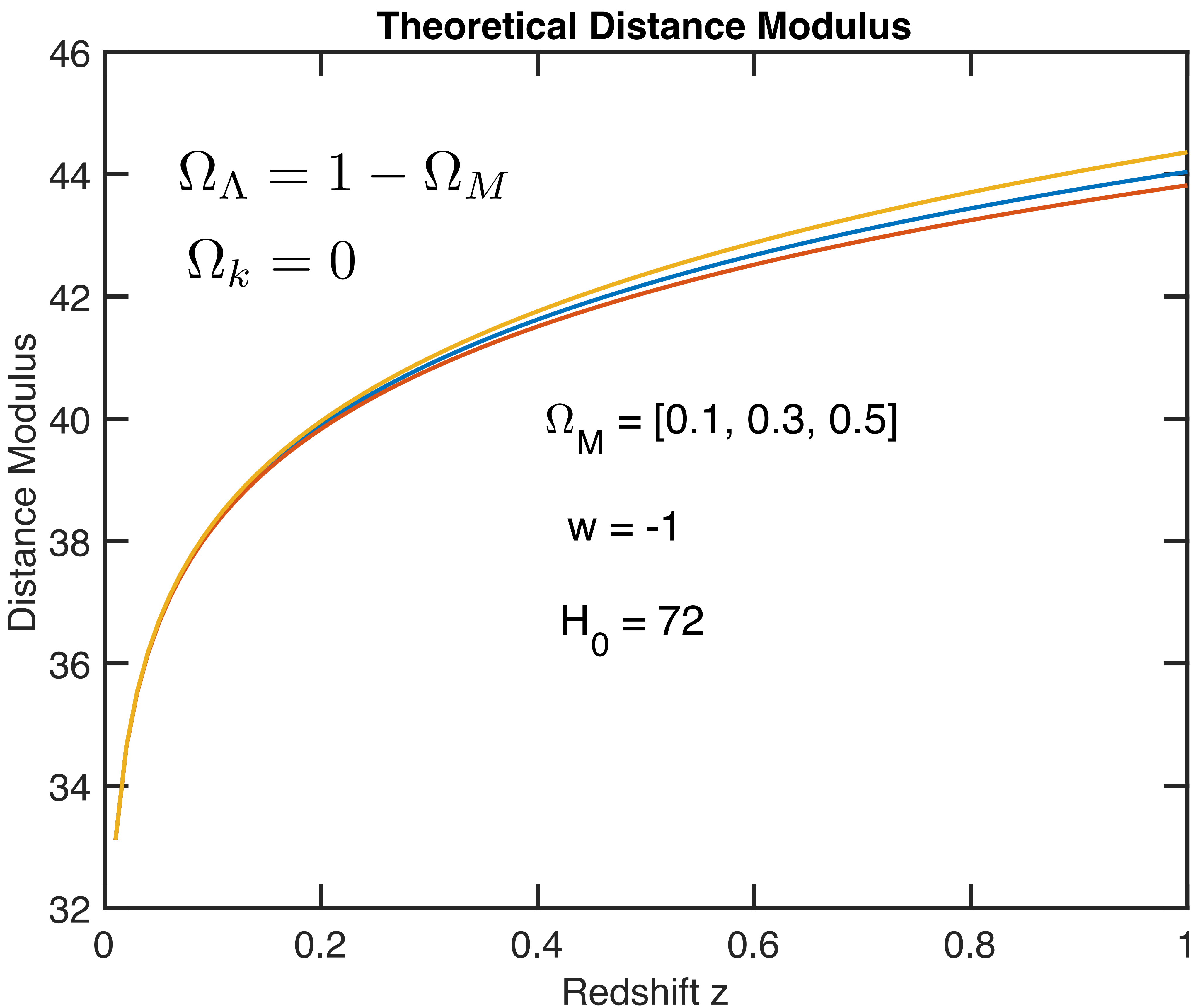


$$\Omega_k > 0$$

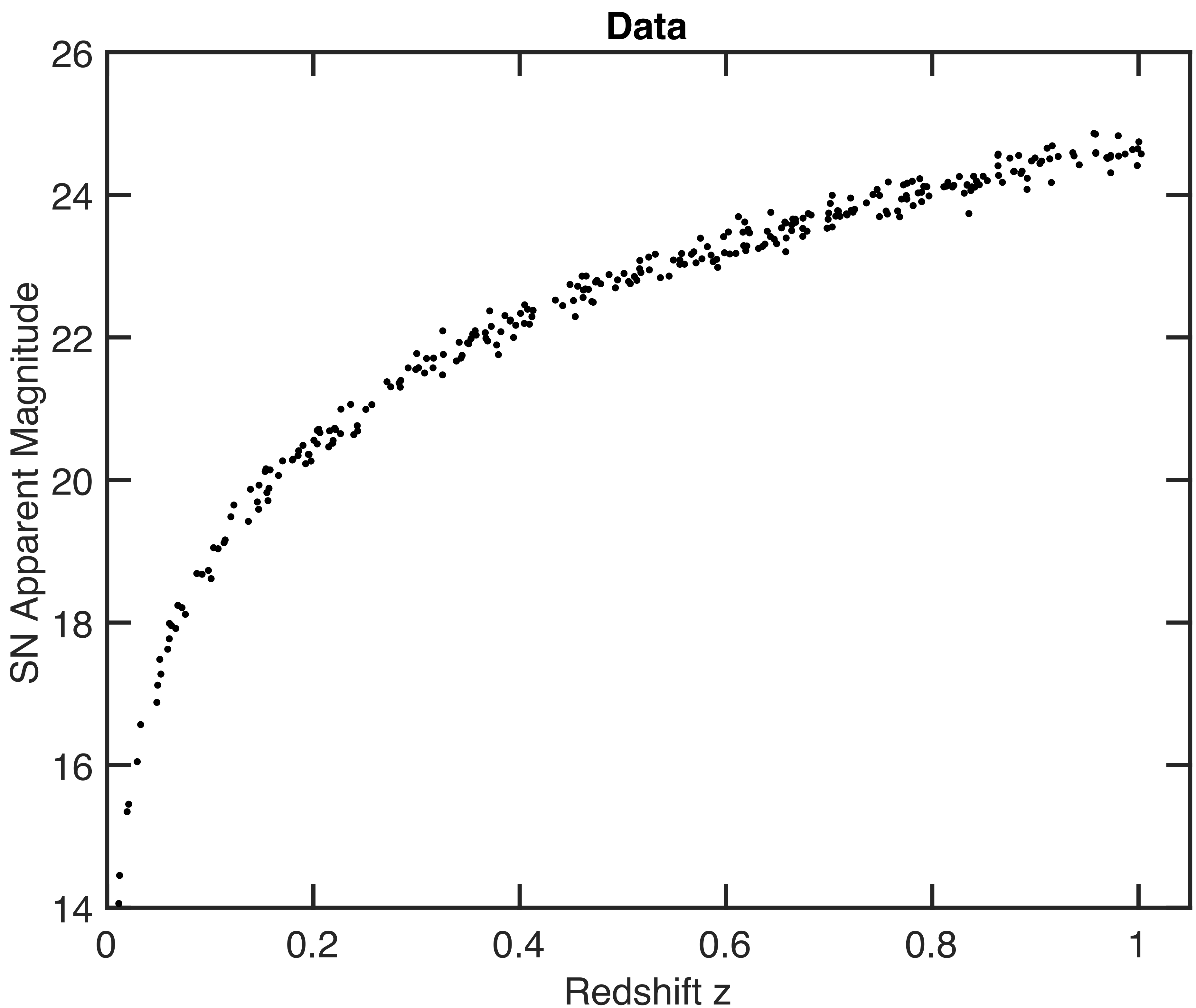
$$\Omega_k < 0$$

$$\Omega_k = 0$$

Theoretical Distance Modulus



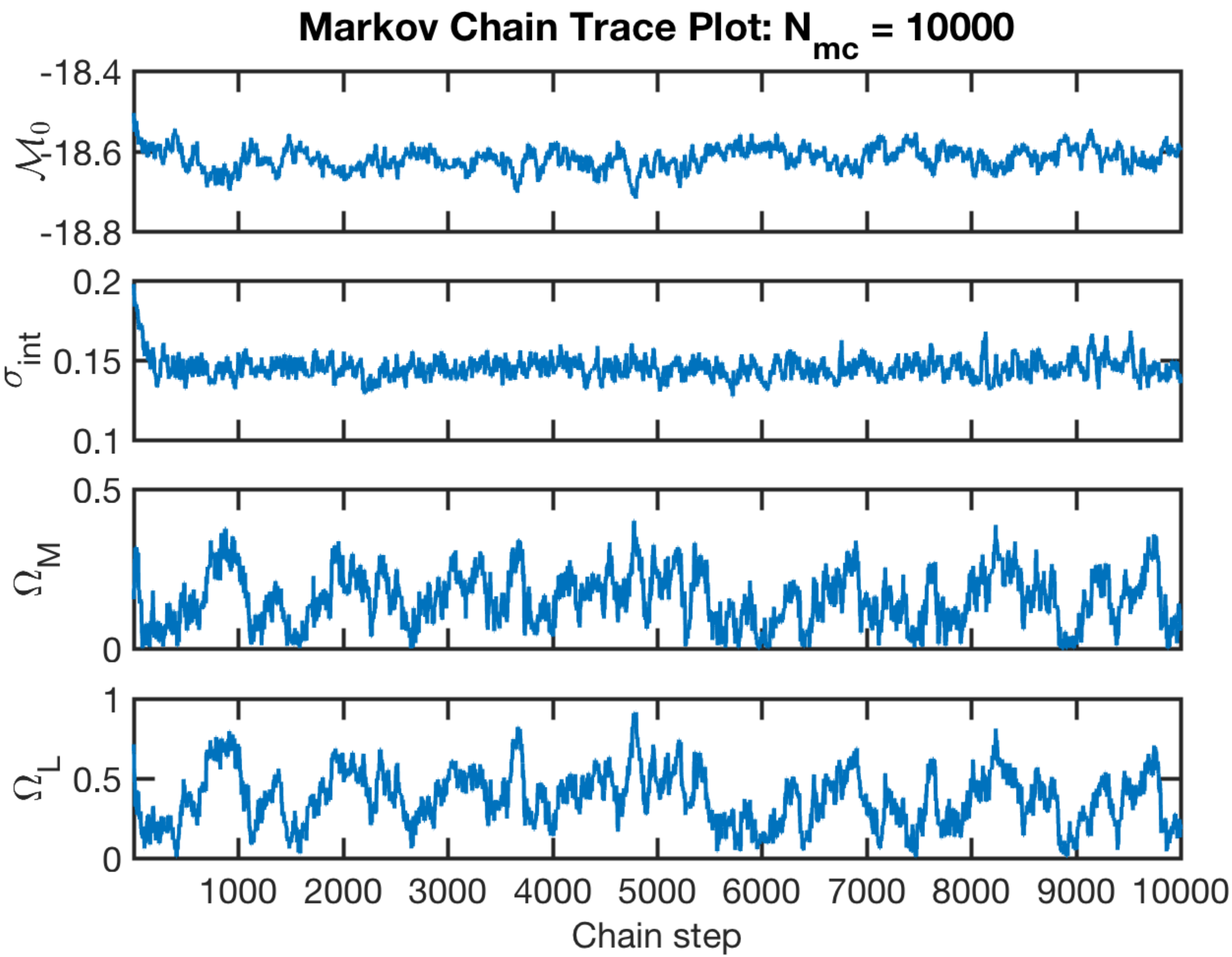
Idealised supernova dataset



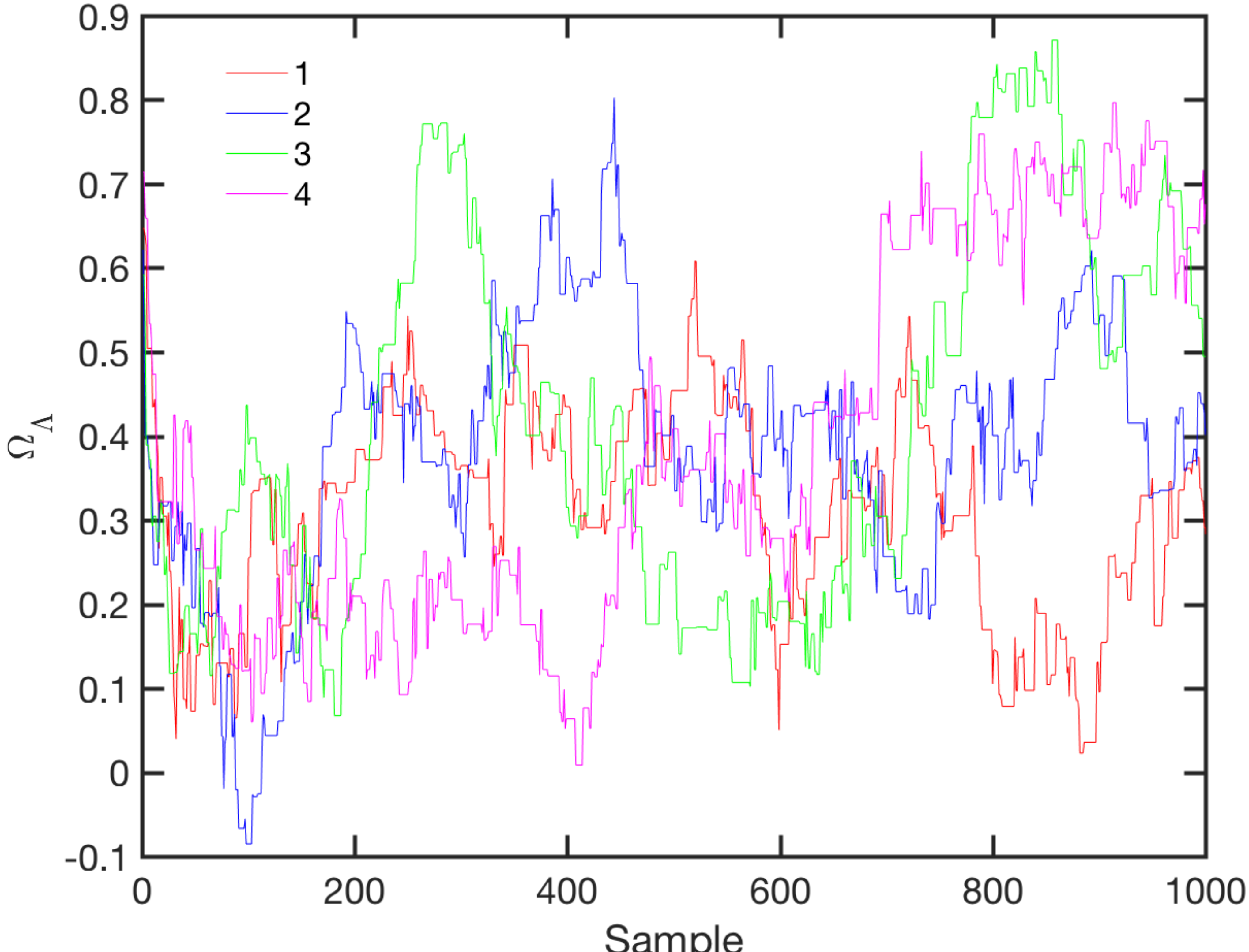
First assume $w = -1$

Write down model & posterior

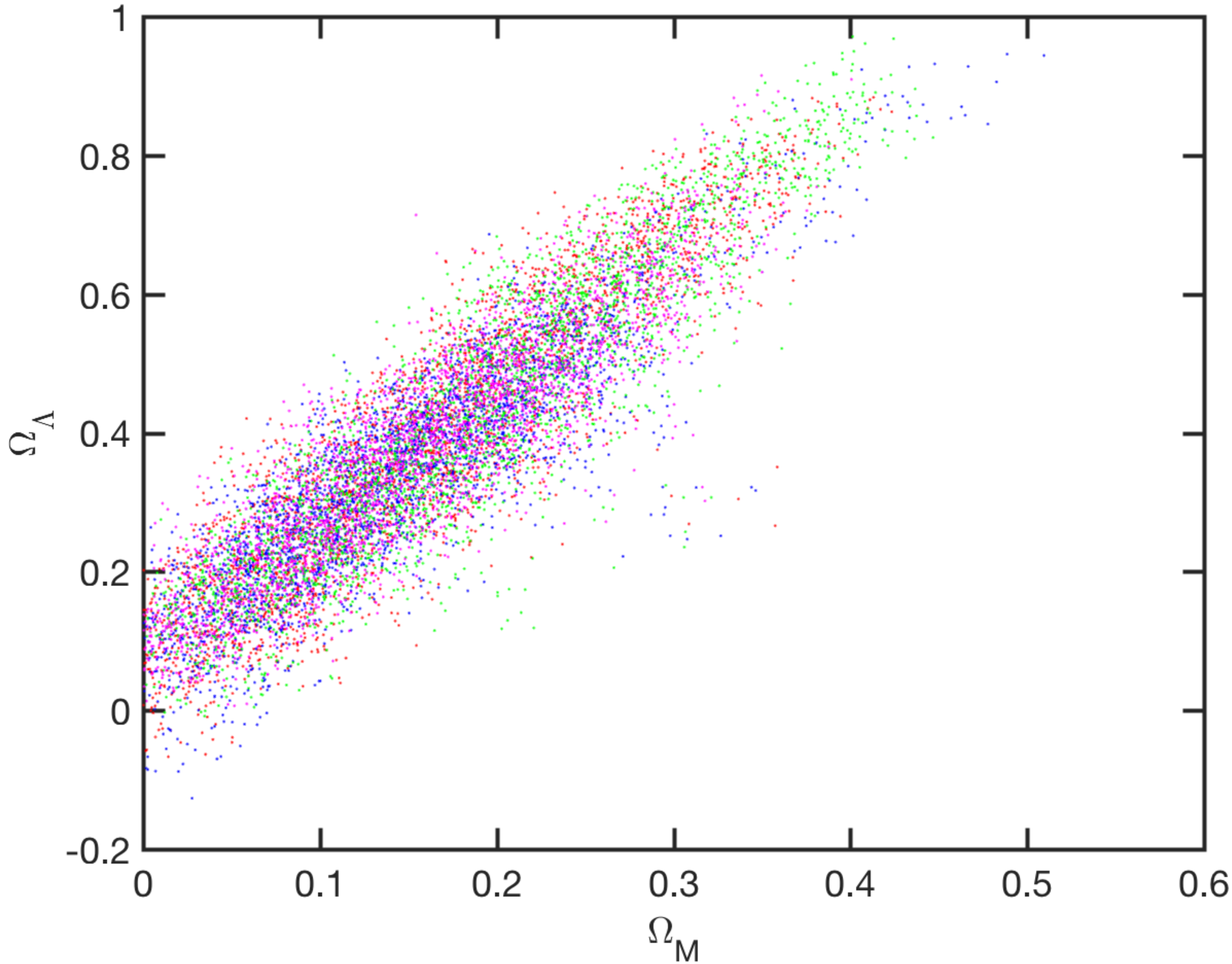
4D Metropolis: one chain



Multiple Independent Chains



Multiple Independent Chains



Assessing Convergence with multiple chains: Gelman-Rubin (G-R) ratio

Monitoring convergence of each scalar estimand

Suppose we have simulated m parallel sequences, each of length n (after discarding the first half of the simulations). For each scalar estimand ψ, we label the simulation draws as ψ_{ij} ($i = 1, \ldots, n; j = 1, \ldots, m$), and we compute B and W, the between- and within-sequence variances:

$$B = \frac{n}{m-1}\sum_{j=1}^{m}(\overline{\psi}_{.j} - \overline{\psi}_{..})^2, \text{ where } \overline{\psi}_{.j} = \frac{1}{n}\sum_{i=1}^{n}\psi_{ij}, \quad \overline{\psi}_{..} = \frac{1}{m}\sum_{j=1}^{m}\overline{\psi}_{.j}$$

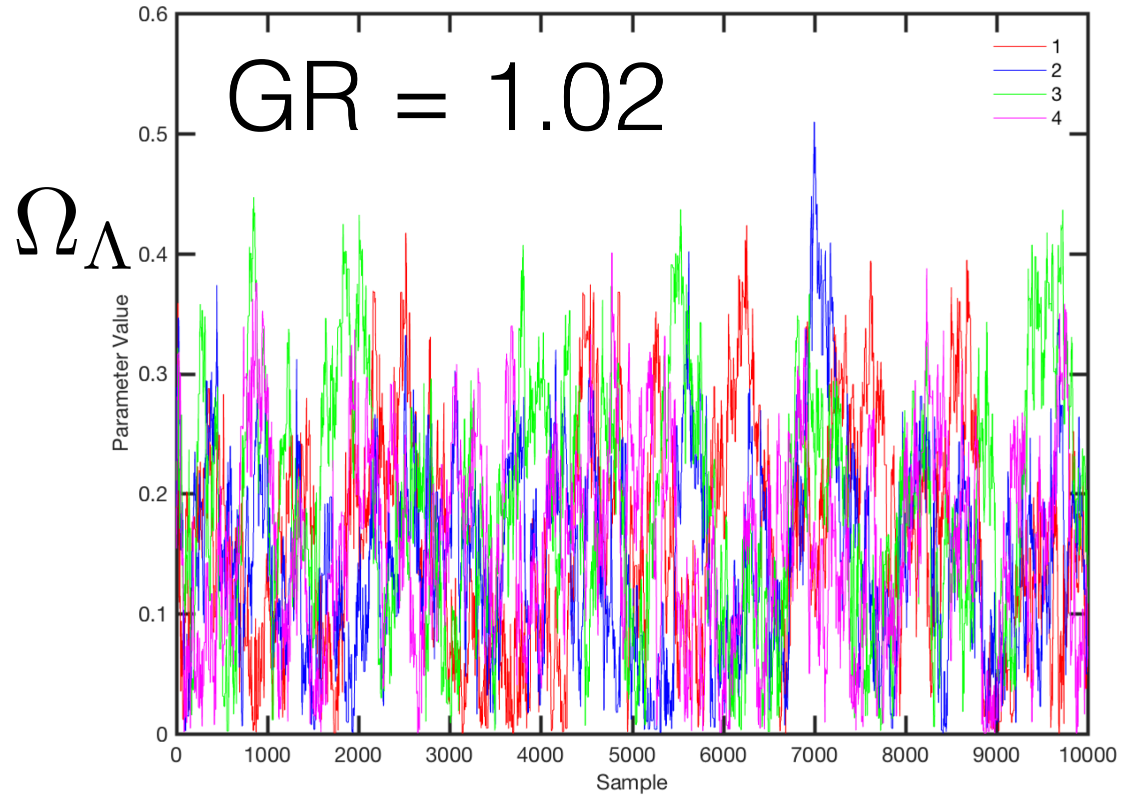
$$W = \frac{1}{m}\sum_{j=1}^{m}s_j^2, \text{ where } s_j^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\psi_{ij} - \overline{\psi}_{.j})^2.$$

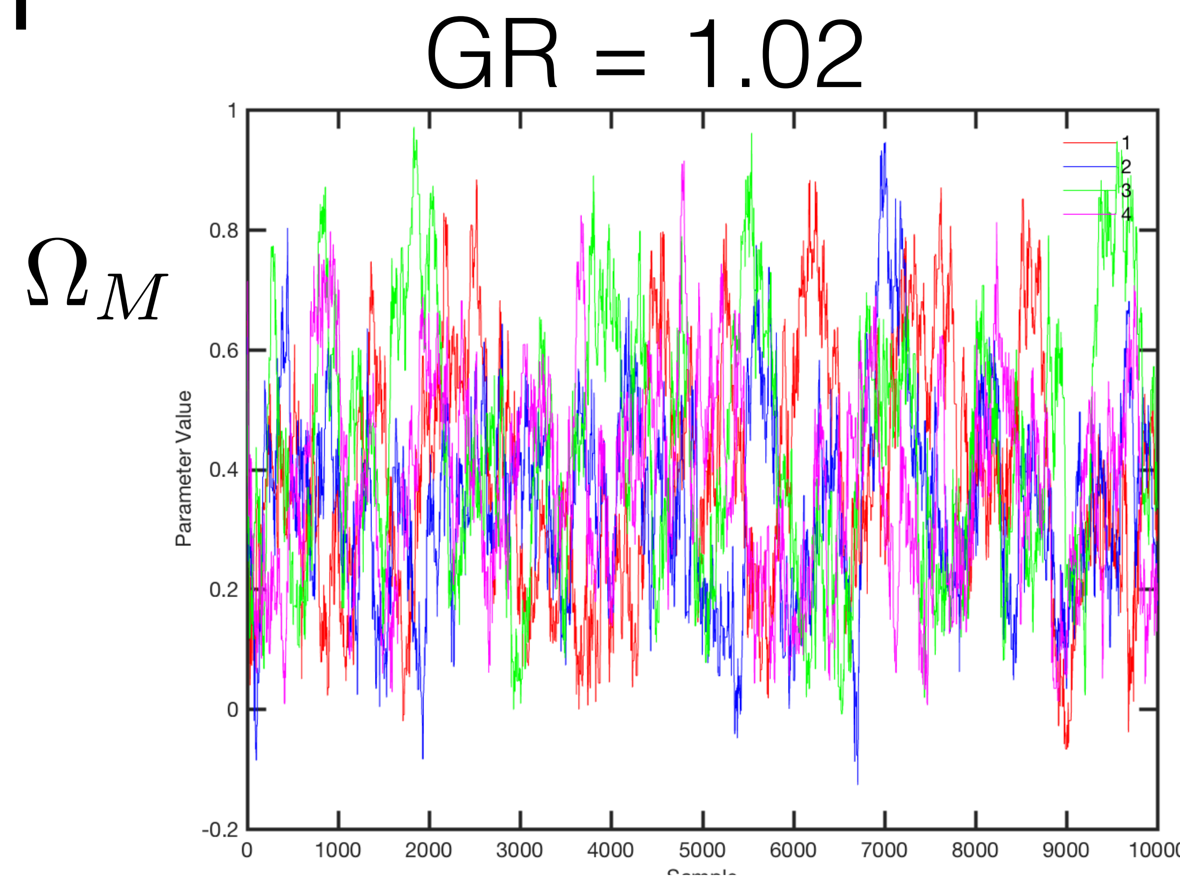
We can estimate $\text{var}(\psi|y)$, the marginal posterior variance of the estimand, by a weighted average of W and B, namely

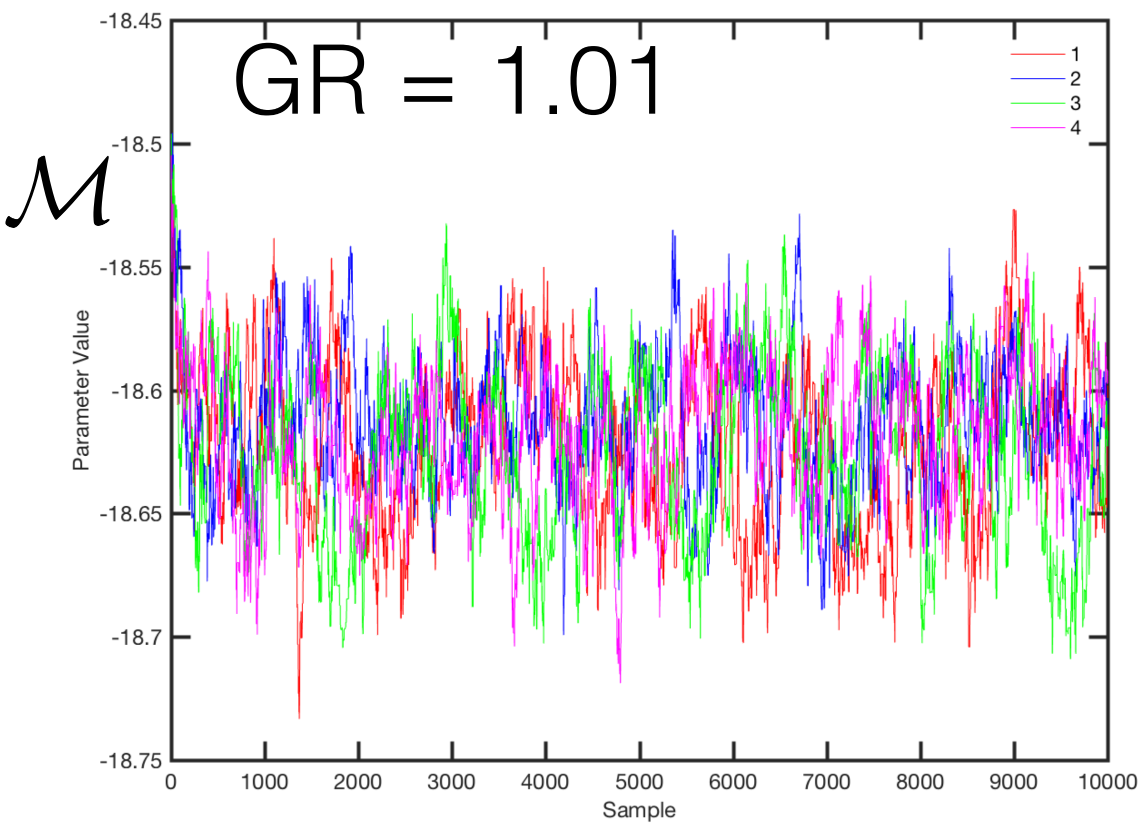
$$\widehat{\text{var}}^+(\psi|y) = \frac{n-1}{n}W + \frac{1}{n}B. \tag{11.3}$$

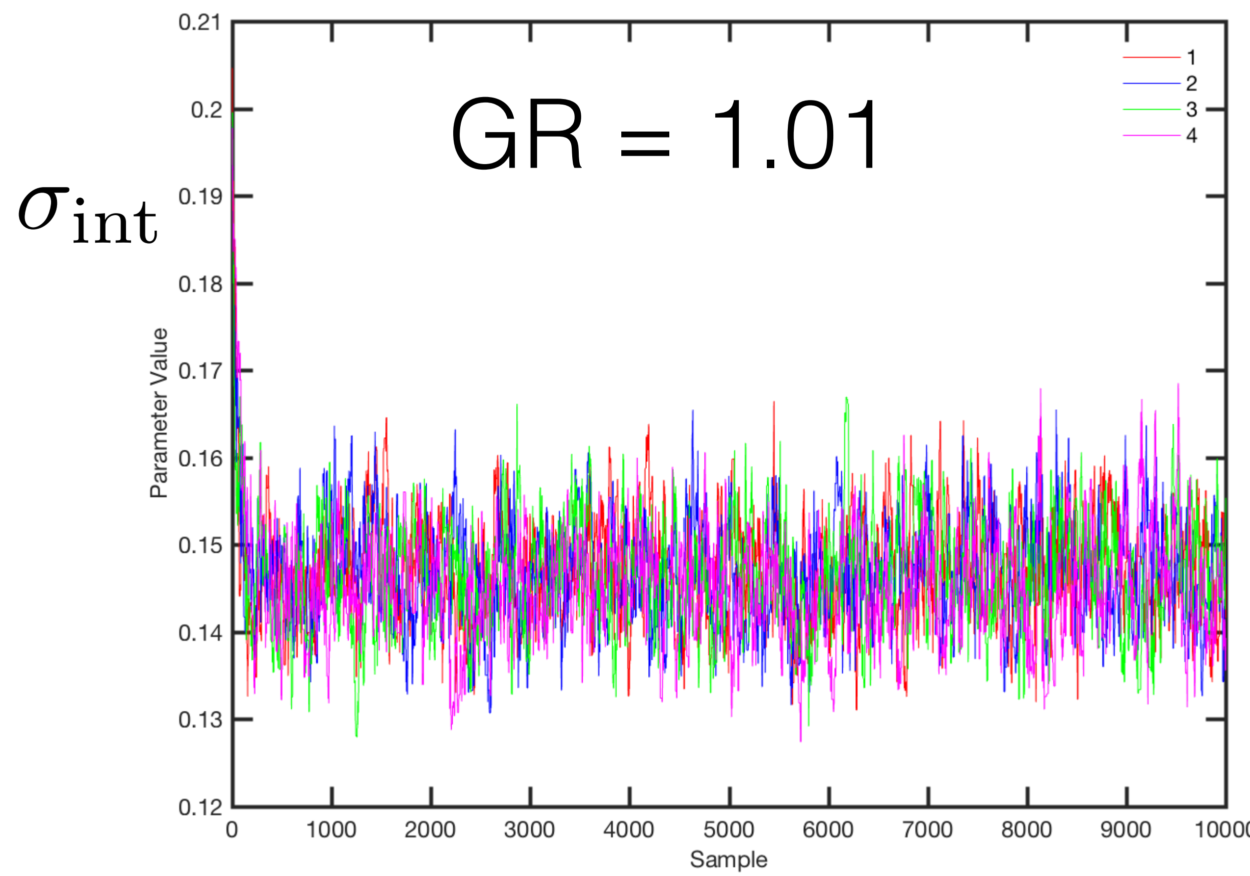
G-R ratio: $\hat{R} = \sqrt{\frac{\widehat{\text{var}}^+(\psi|y)}{W}}, \quad \approx 1$

Multiple Independent Chains

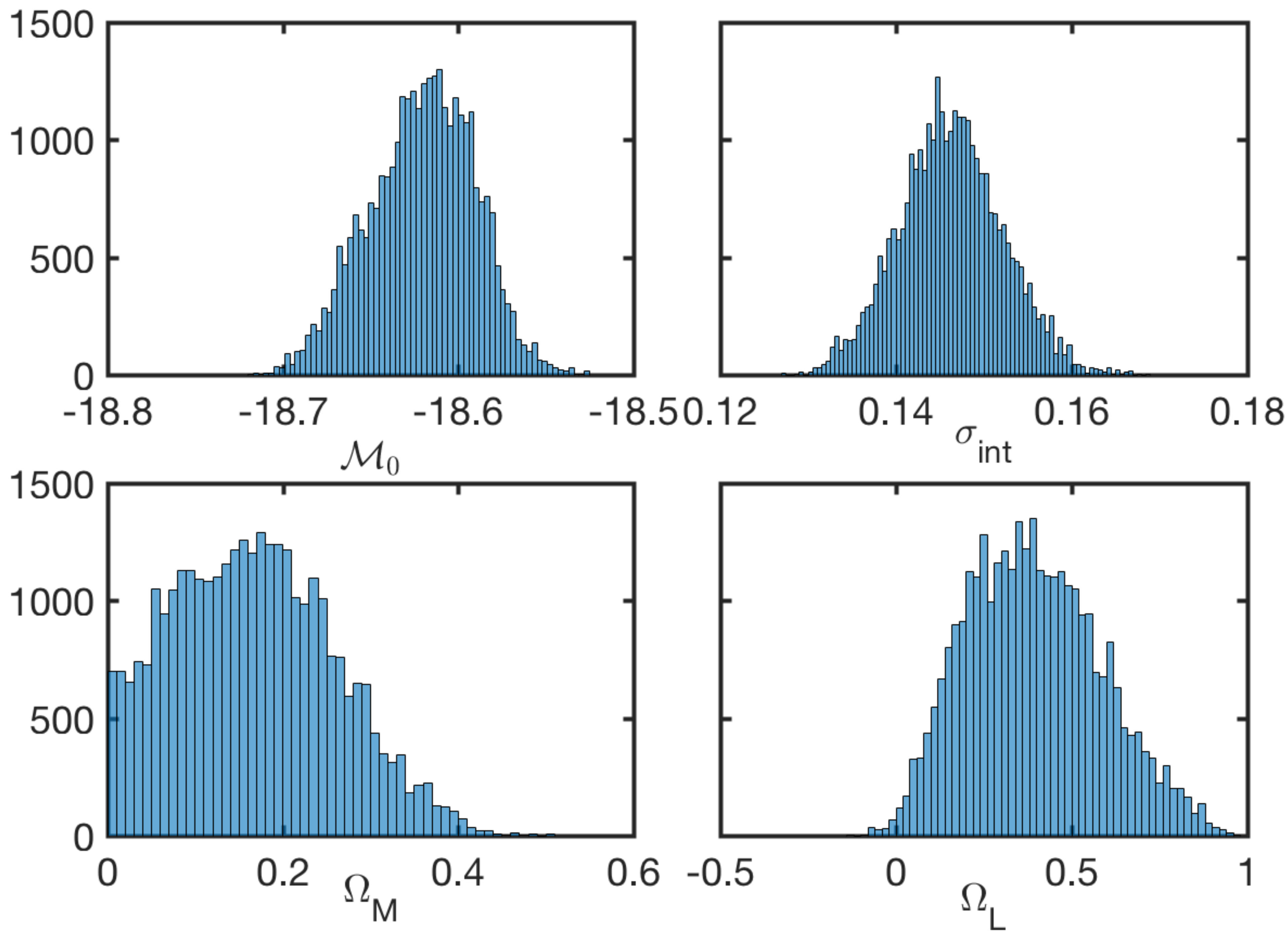




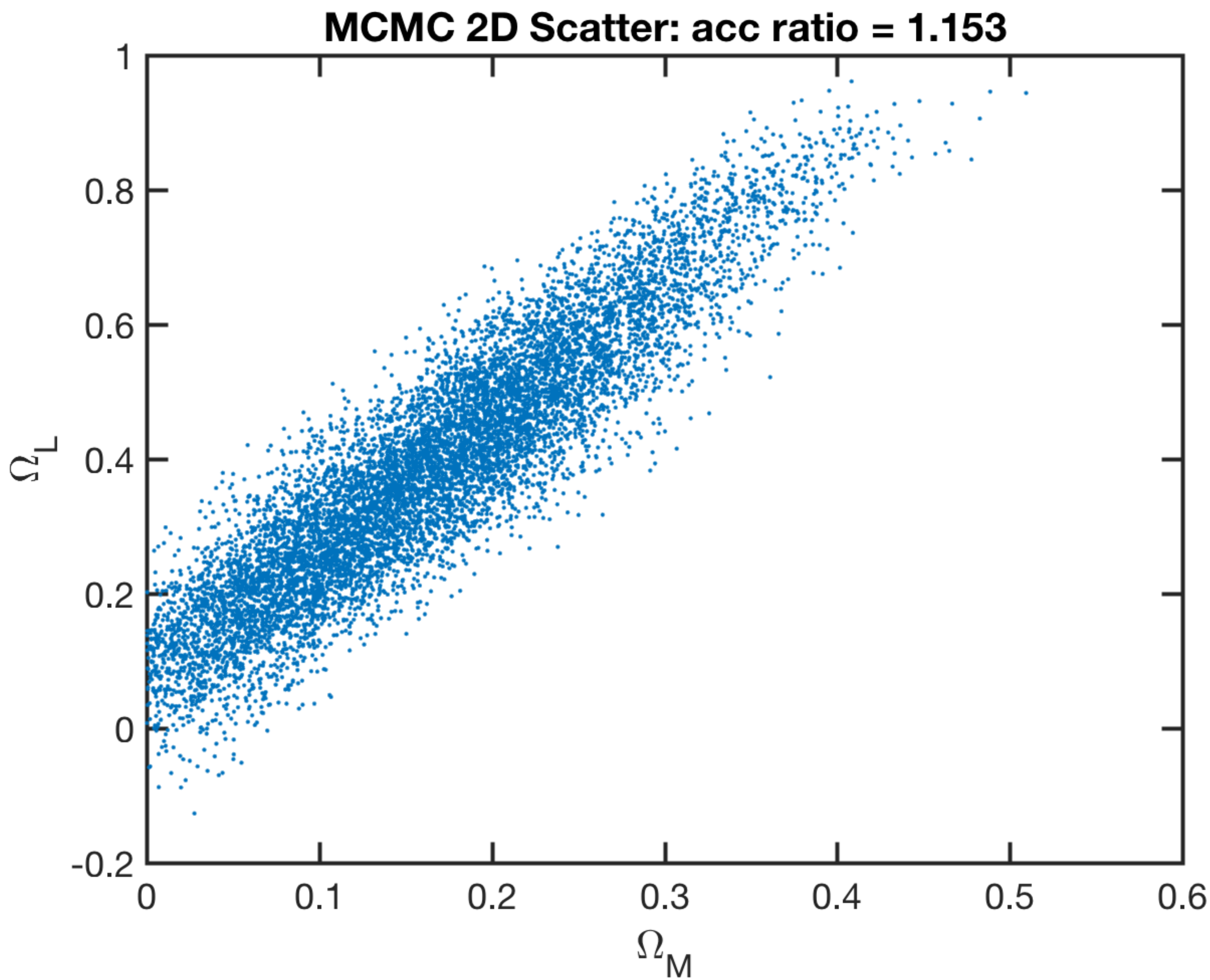




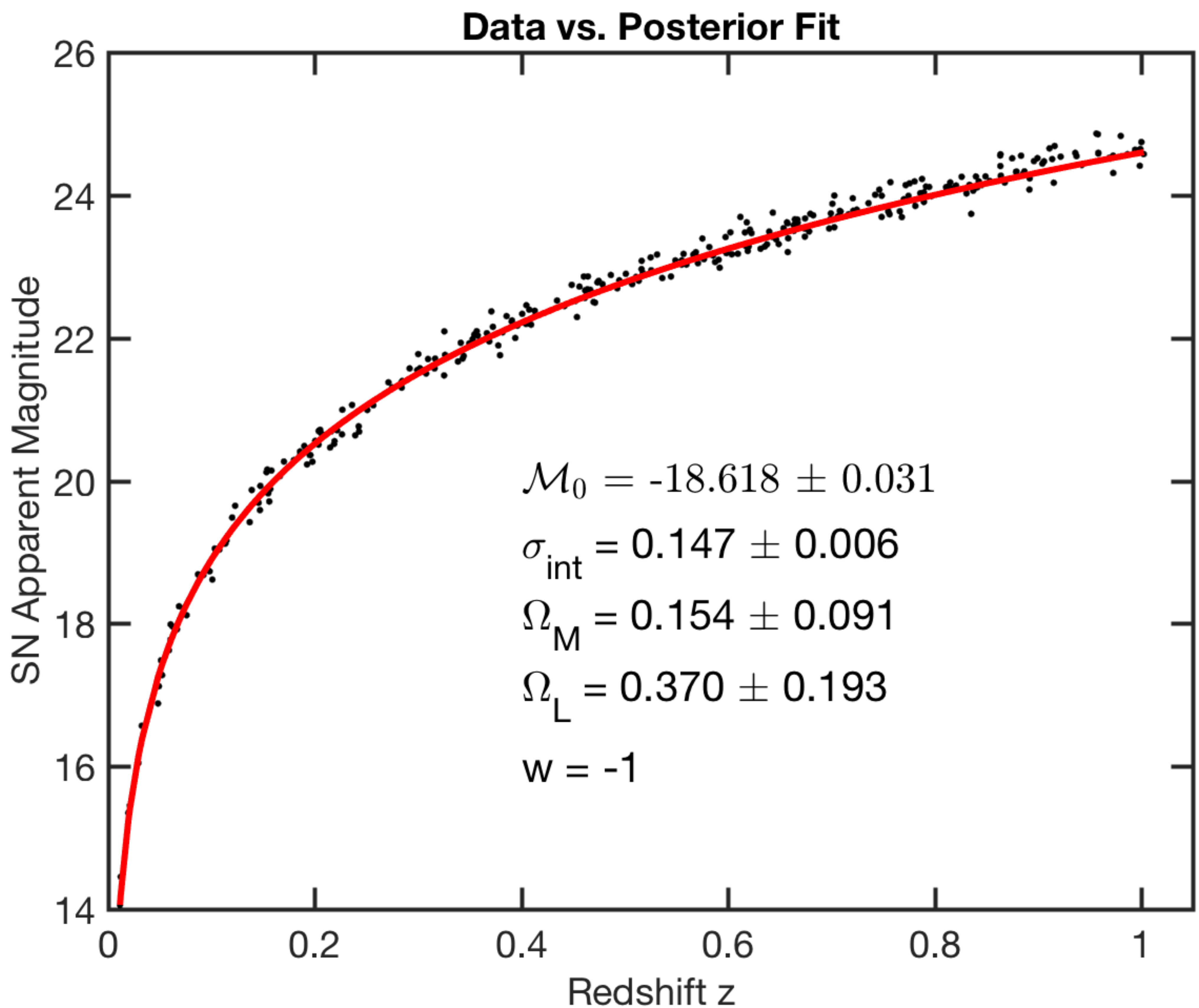
Cut burn-in (20%) and combine chains
Posterior Histograms



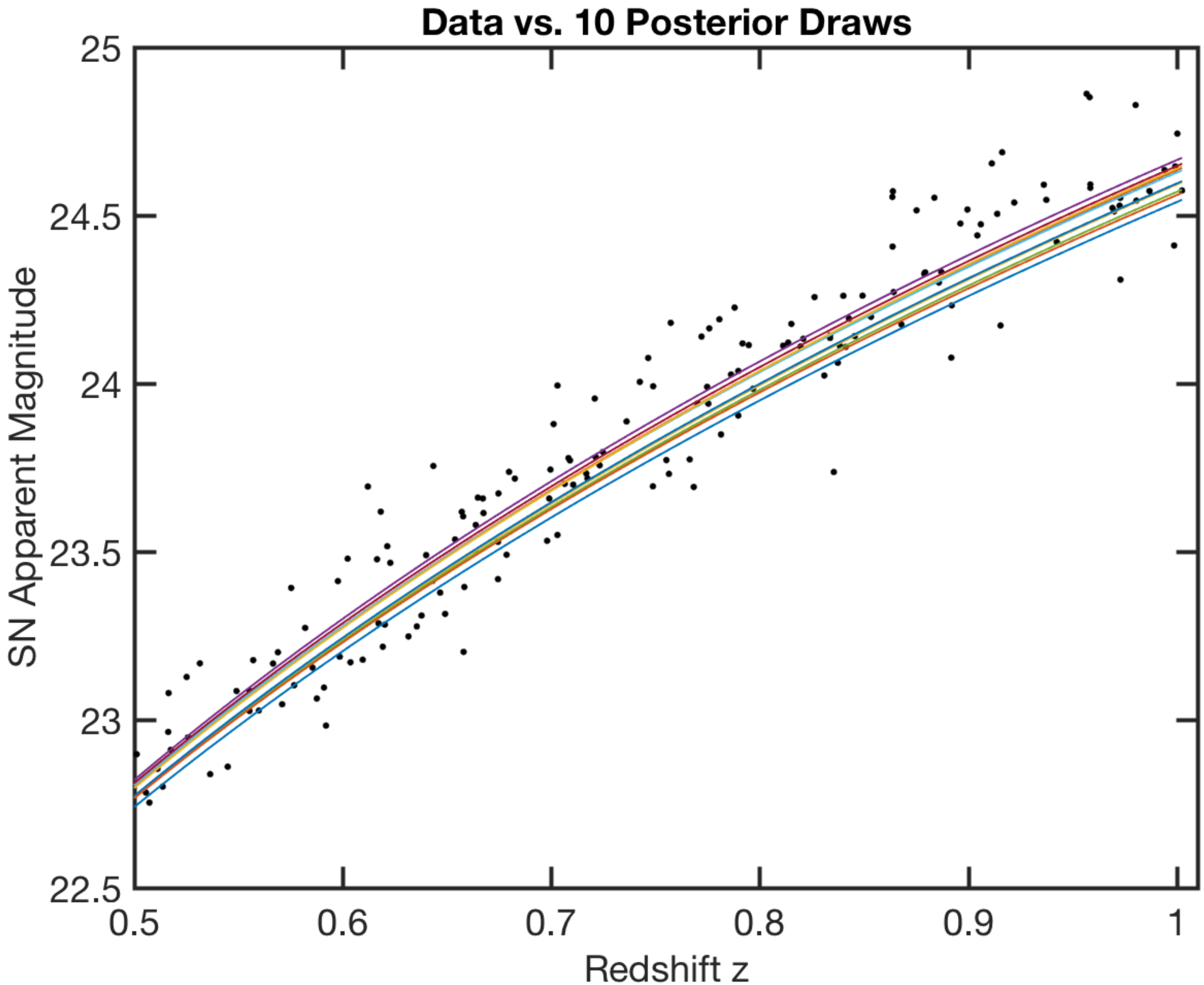
Posterior Scatter plot



Fit with posterior mean



Random Posterior Draws of parameters from chain



The deceleration parameter q_0

of the cosmological models. ”It is a striking and slightly puzzling fact that almost all current cosmological observations can be summarized by the simple statement: The jerk of the Universe equals one” [25].

Let us make Taylor expansion of the scale factor in time using the above introduced parameters

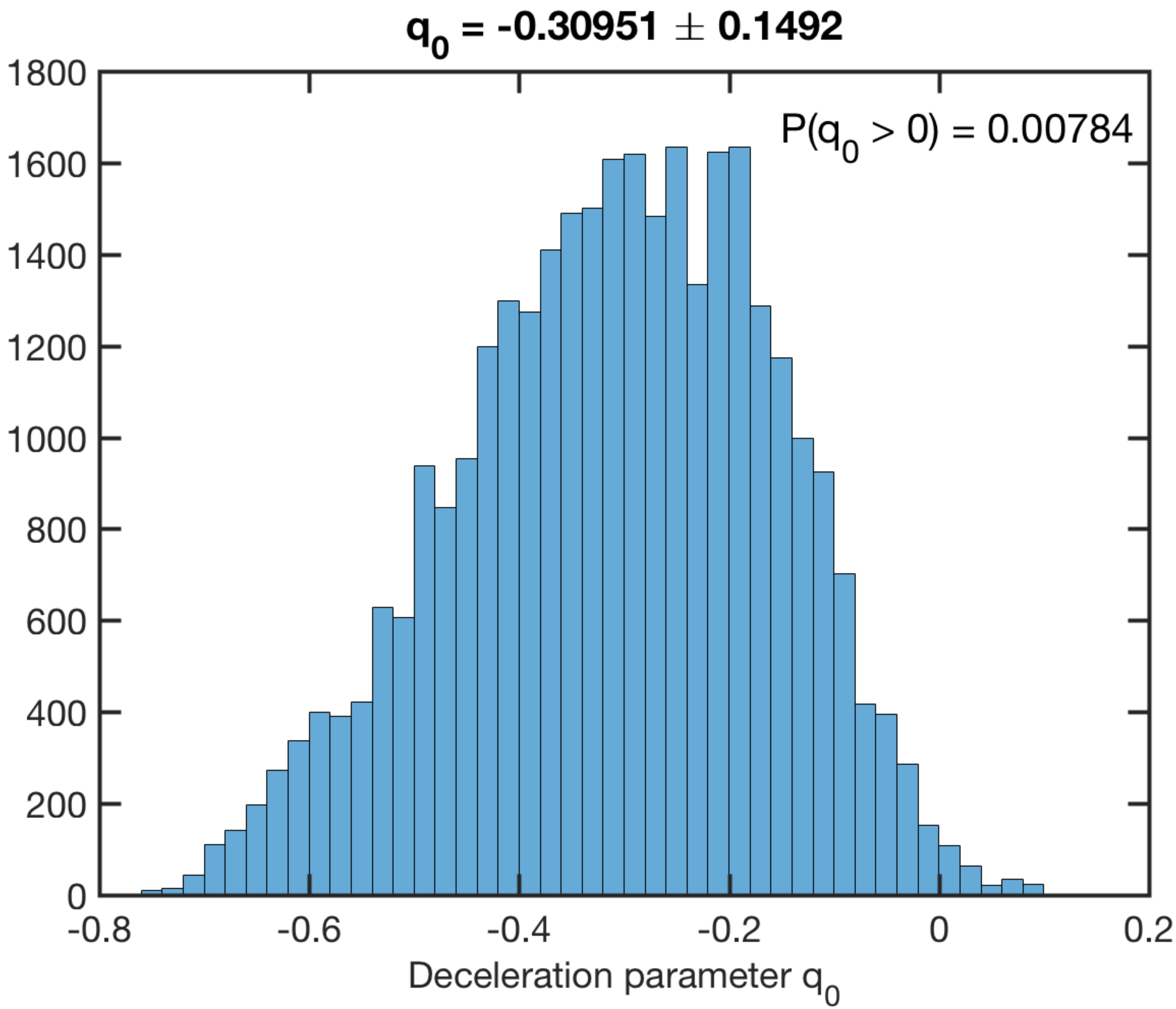
$$\begin{aligned} a(t) =& a_0 \Big[1 + H_0(t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 + \frac{1}{3!} j_0 H_0^3 (t-t_0)^3 \\ &+ \frac{1}{4!} s_0 H_0^4 (t-t_0)^4 + \frac{1}{5!} l_0 H_0^5 (t-t_0)^5 + O\left((t-t_0)^6\right) \Big]. \end{aligned} \tag{2.49}$$

a(0) = 0 (Big Bang)
a(1) = 1 (Today)

$$q_0 = \Omega_M/2 - \Omega_\Lambda$$

The posterior of derived quantities

Deceleration Parameter: $q_0 = \Omega_M/2 - \Omega_\Lambda$



$q_0 < 0$
Expansion of Universe is accelerating!