Astrostatistics: Monday 18 Feb 2019

https://github.com/CambridgeAstroStat/PartIII-Astrostatistics-2019

- Example Sheet online, Ex Class Tue Feb 19, 1pm MR5
- Today: continue Bayesian computation / Monte Carlo Methods / MCMC
- MacKay: Ch 29-30; Bishop: Ch 11; Gelman
- Givens & Hoeting "Computational Statistics" (Free download through Cambridge Library iDiscover)
- Hogg & DFM, 2017 "Data analysis recipes: Using Markov Chain Monte Carlo." https://arxiv.org/abs/1710.06068

Markov Chain Monte Carlo (MCMC)

Mapping the Posterior $P(\theta|D)$

- Markov Chain Monte Carlo (MCMC)
- Just did: 1D Metropolis algorithm
- Now:
  - Drawing Multivariate Gaussian random variables
  - N-D Metropolis Algorithm
  - Rules of thumb for proposal scale
  - assessing convergence (G-R Ratio)
  - Metropolis-Hastings algorithm
  - Gibbs sampling

d-dim Metropolis Algorithm: Posterior $P(\theta \mid D)$, Symmetric Proposal/Jump dist'n $J(\theta^* \mid \theta) = J(\theta \mid \theta^*)$

1. Choose a random starting point θ_0

2. At step i = 1…N, propose a new parameter value $\theta^* \sim N(\theta_{i-1}, \mathbf{\Sigma_p})$. The proposal distr'n is $J(\theta^* \mid \theta_{i-1}) = N(\theta^* \mid \theta_{i-1}, \mathbf{\Sigma_p})$

3. Evaluate ratio of posteriors at proposed vs current values. $r = P(\theta^* \mid \mathbf{y}) / P(\theta_{i-1} \mid \mathbf{y})$.

4. Accept θ^* with probability min(r,1): $\theta_i = \theta^*$. If not accept, stay at same value $\theta_i = \theta_{i-1}$ for the next step & include in chain.

5. Repeat steps 2-4 until reach some measure of convergence (G-R) and gather enough independent samples to compute your inference (reduce Monte Carlo error)

Multi-parameter Bayesian inference: Gaussian example: Gelman BDA Sec 3.2 - 3.3

Sampling distribution: $y_i \sim N(\mu, \sigma^2)$ $\qquad i = 1 \ldots n$

Likelihood Function: $P(\boldsymbol{y}|\mu,\sigma^2) = \prod_{i=1}^{n} N(y_i|\mu,\sigma^2)$

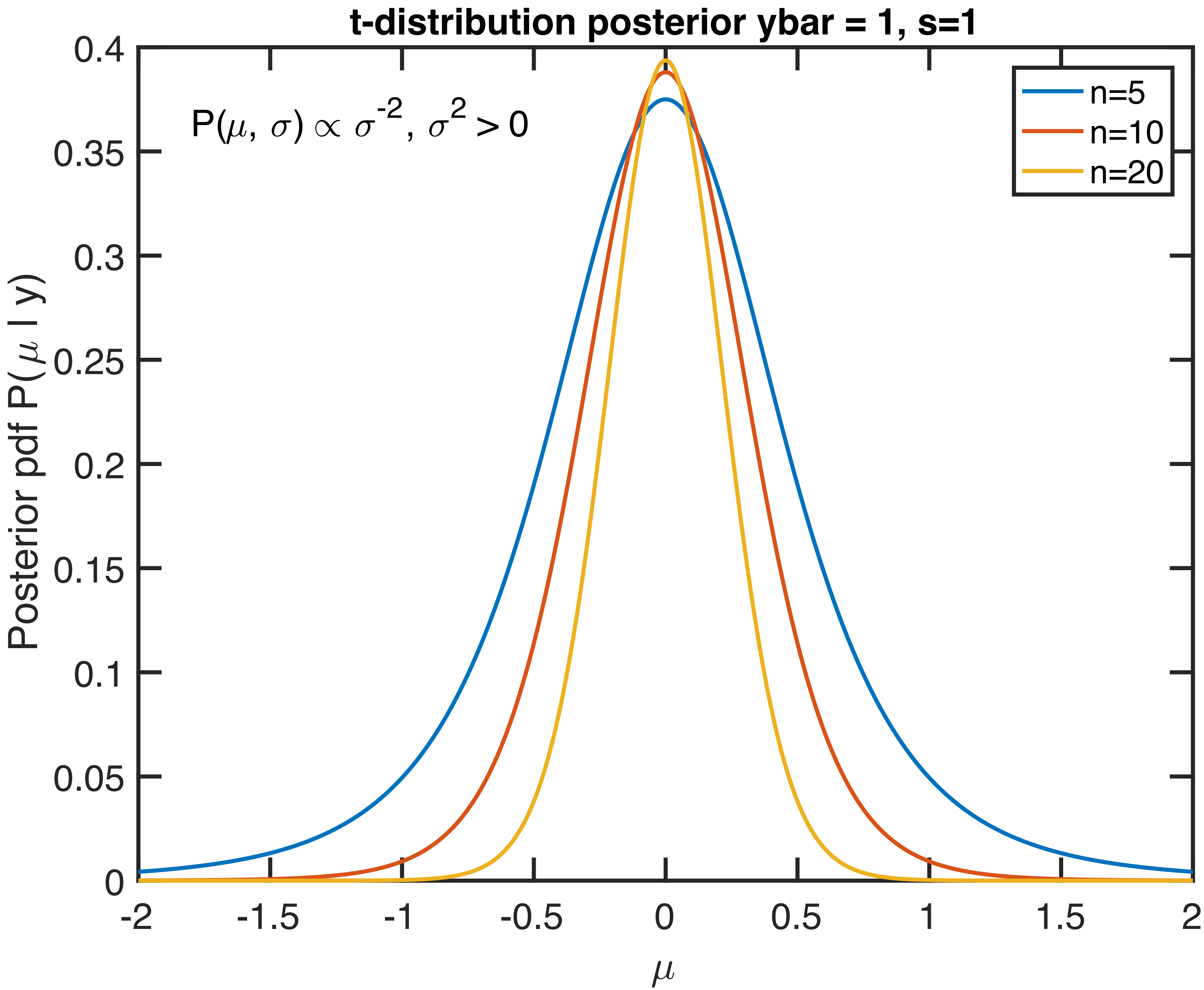
Prior: $P(\mu) \propto 1 \qquad P(\sigma^2) \propto \sigma^{-2}, \sigma^2 > 0$

<u>Posterior</u>

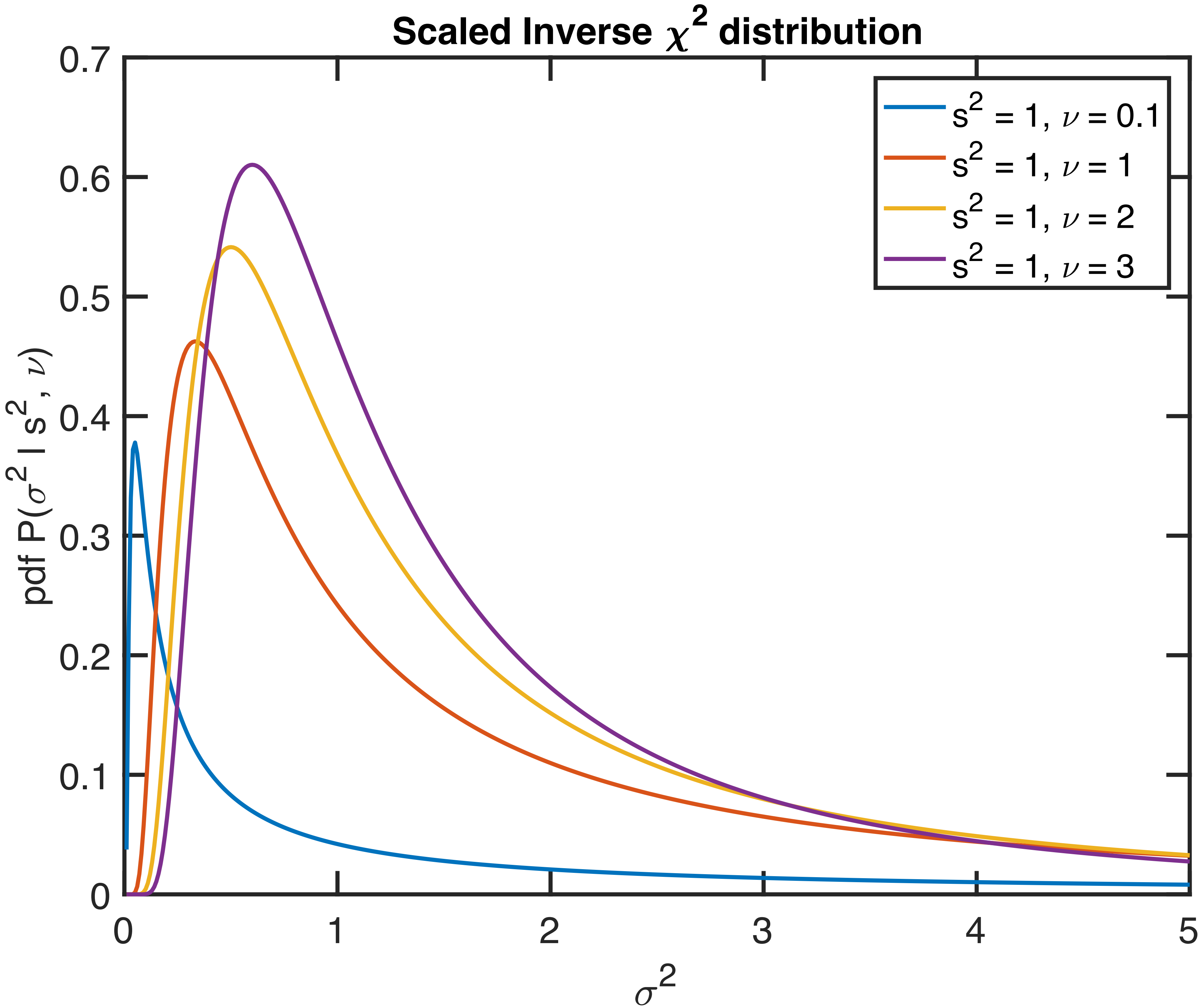
$$P(\mu,\sigma^2|\boldsymbol{y}) \propto (\sigma^2)^{-(n+2)/2} \exp\left(-\tfrac{1}{2\sigma^2}[(n-1)s^2 + n(\bar{y}-\mu)^2]\right)$$

Sufficient Statistics: $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i \qquad s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$

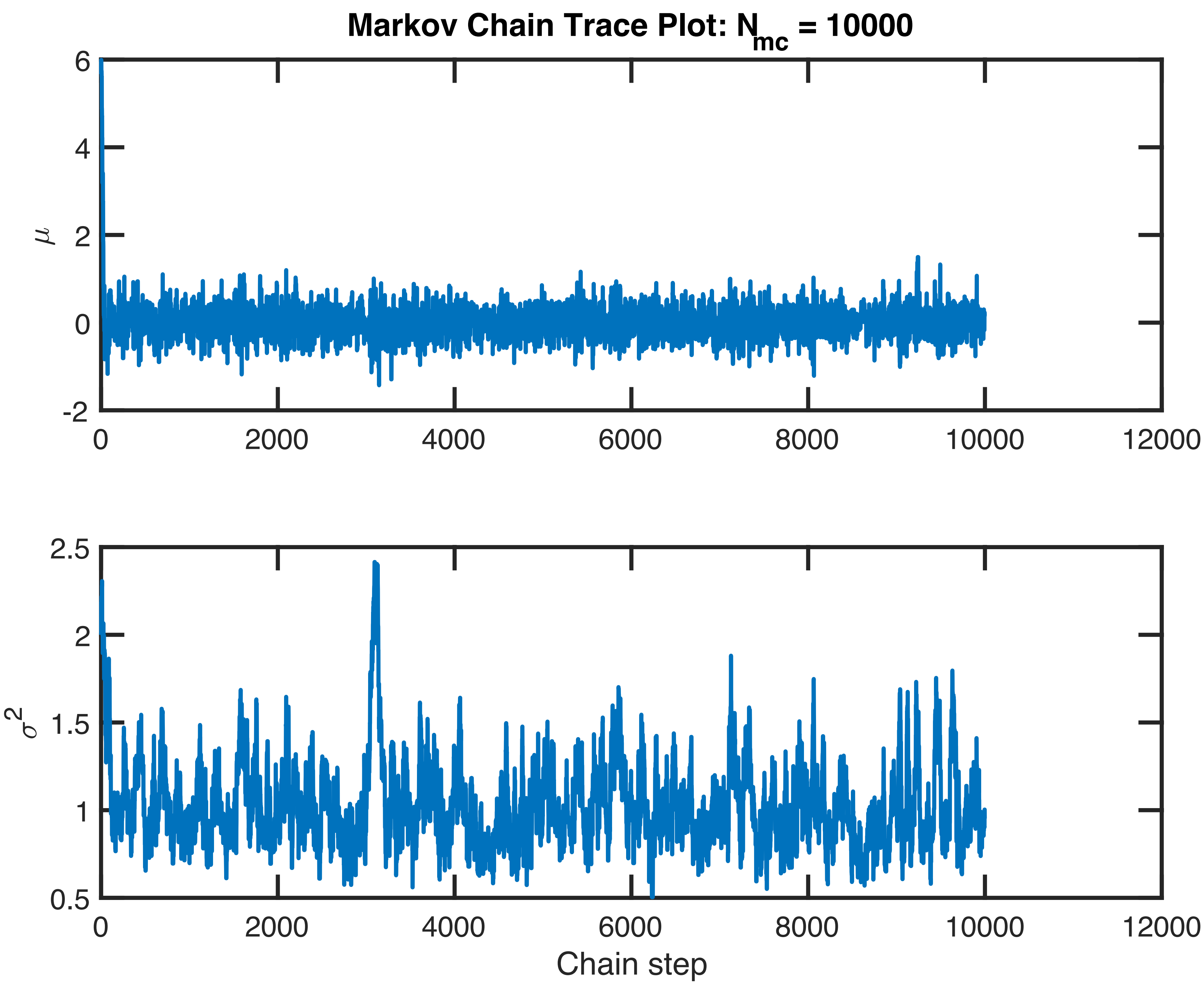
Before: Posterior Distribution of a Gaussian Mean
Analytic Result



Posterior Distribution $P(\sigma^2 \mid y)$ follows a Scaled Inverse χ^2 distribution (~ inverse gamma)



Metropolis 2D Code Example (metropolis2.m)



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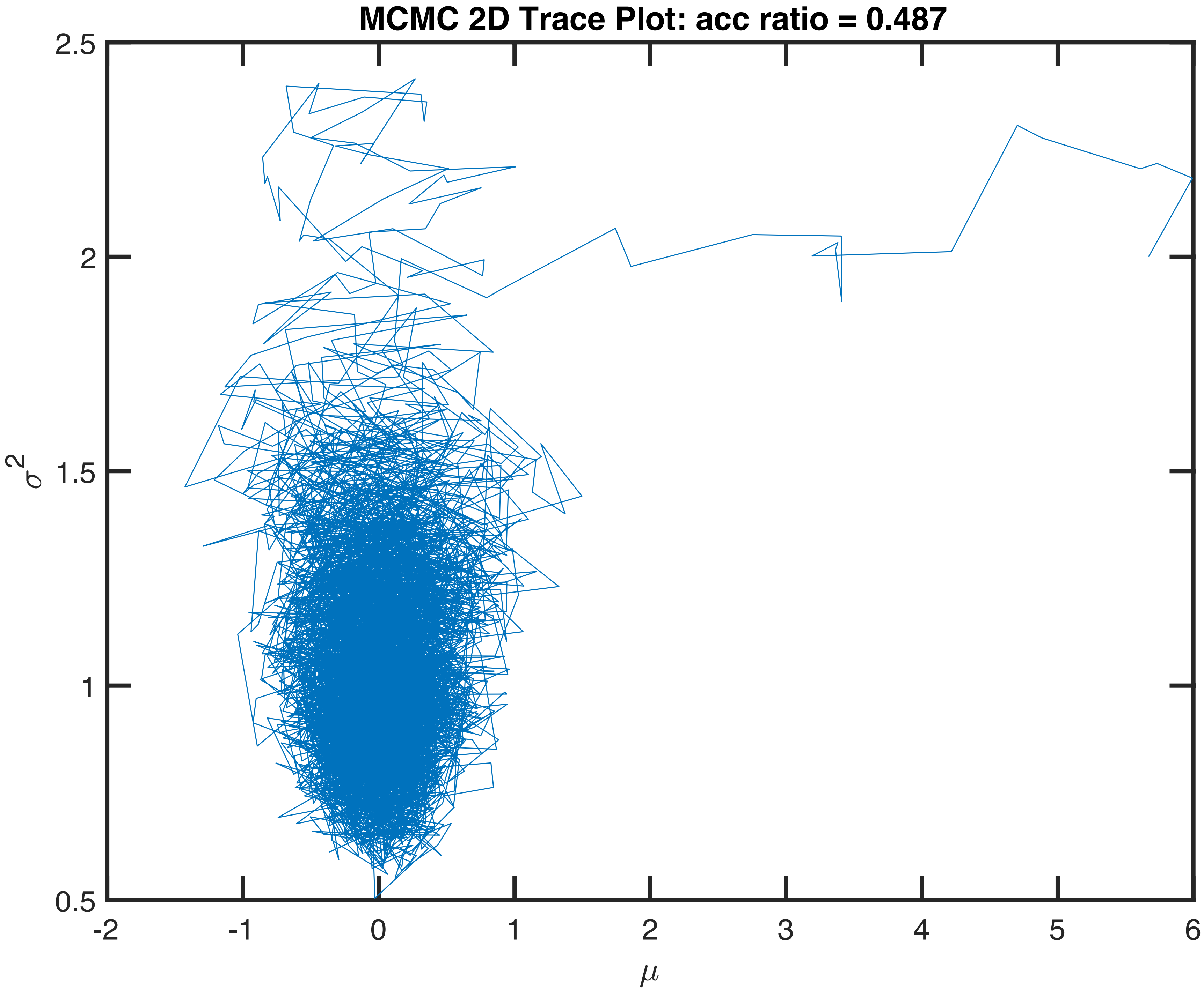

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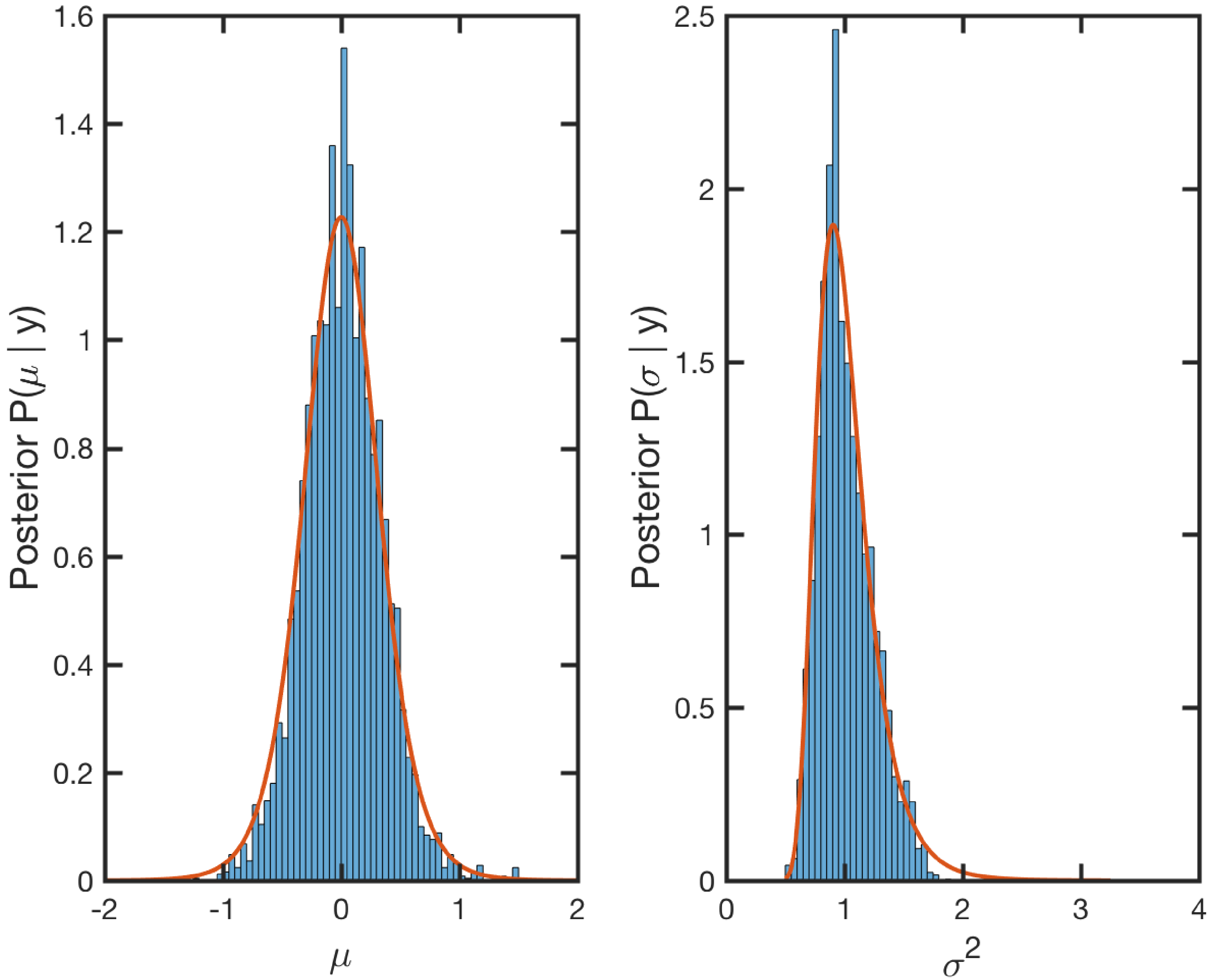

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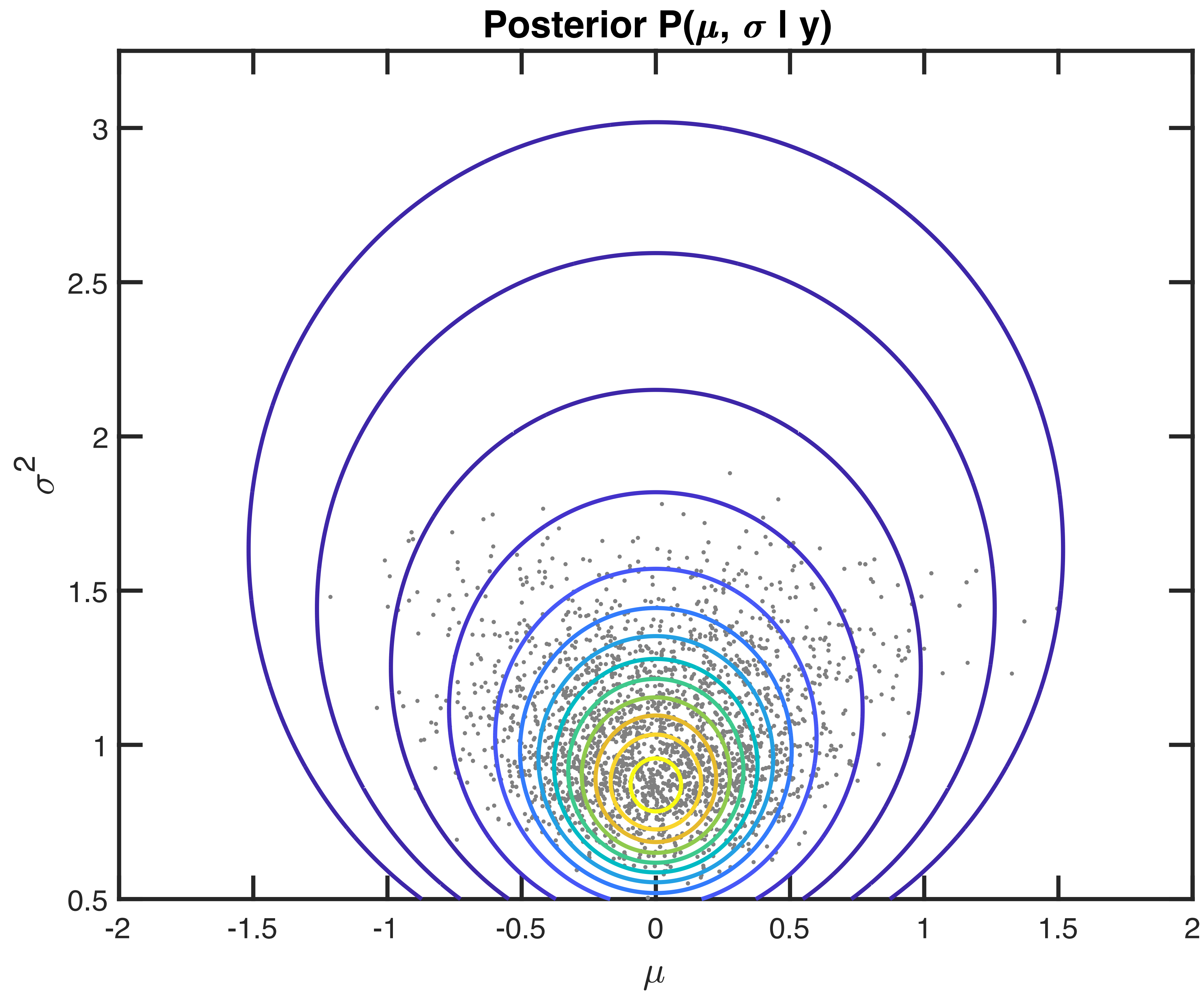


Tuning d-dim Metropolis

- $\theta^* \sim N(\theta_i, \mathbf{\Sigma_p})$: if proposal scale $\mathbf{\Sigma_p}$ is too large, will get too many rejections and not go anywhere. If proposal scale too small, you will accept very many small moves: inefficient random walk

- Laplace Approximation: $$P(\boldsymbol{\theta}|\boldsymbol{D}) \approx N(\boldsymbol{\theta}|\,\hat{\boldsymbol{\theta}}, \boldsymbol{\Sigma})$$

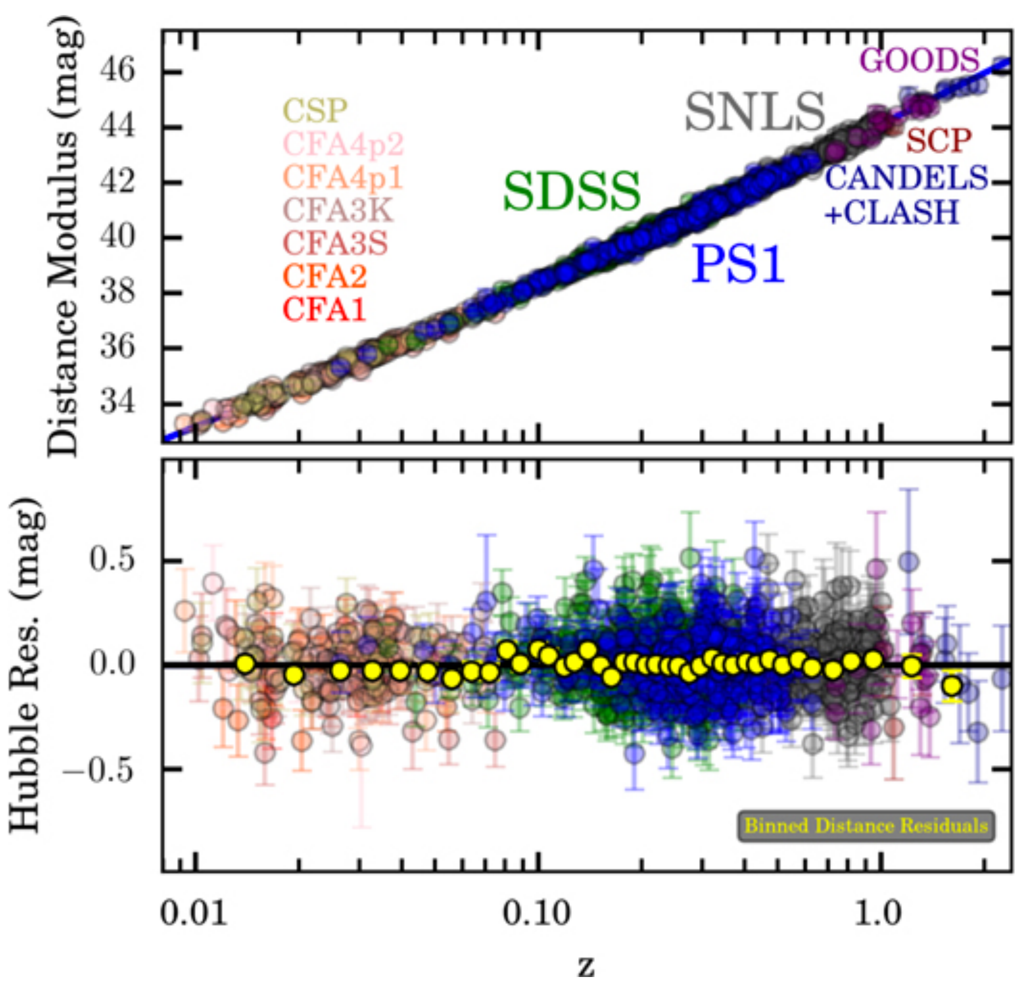
  $$\hat{\boldsymbol{\theta}} = \text{ posterior mode} \qquad (\boldsymbol{\Sigma}^{-1})_{ij} = \frac{\partial^2 \log P(\boldsymbol{\theta}|\boldsymbol{D})}{\partial\theta_i\,\partial\theta_j}\bigg|_{\hat{\boldsymbol{\theta}}}$$

- Choose $\mathbf{\Sigma_p} = c^2\,\mathbf{\Sigma}$: $$c \approx 2.4/\sqrt{d}$$

- Aim for an acceptance ratio of 44% in 1D, 23% in d > 5

Astrostatistics Case Study:

Supernova Cosmology



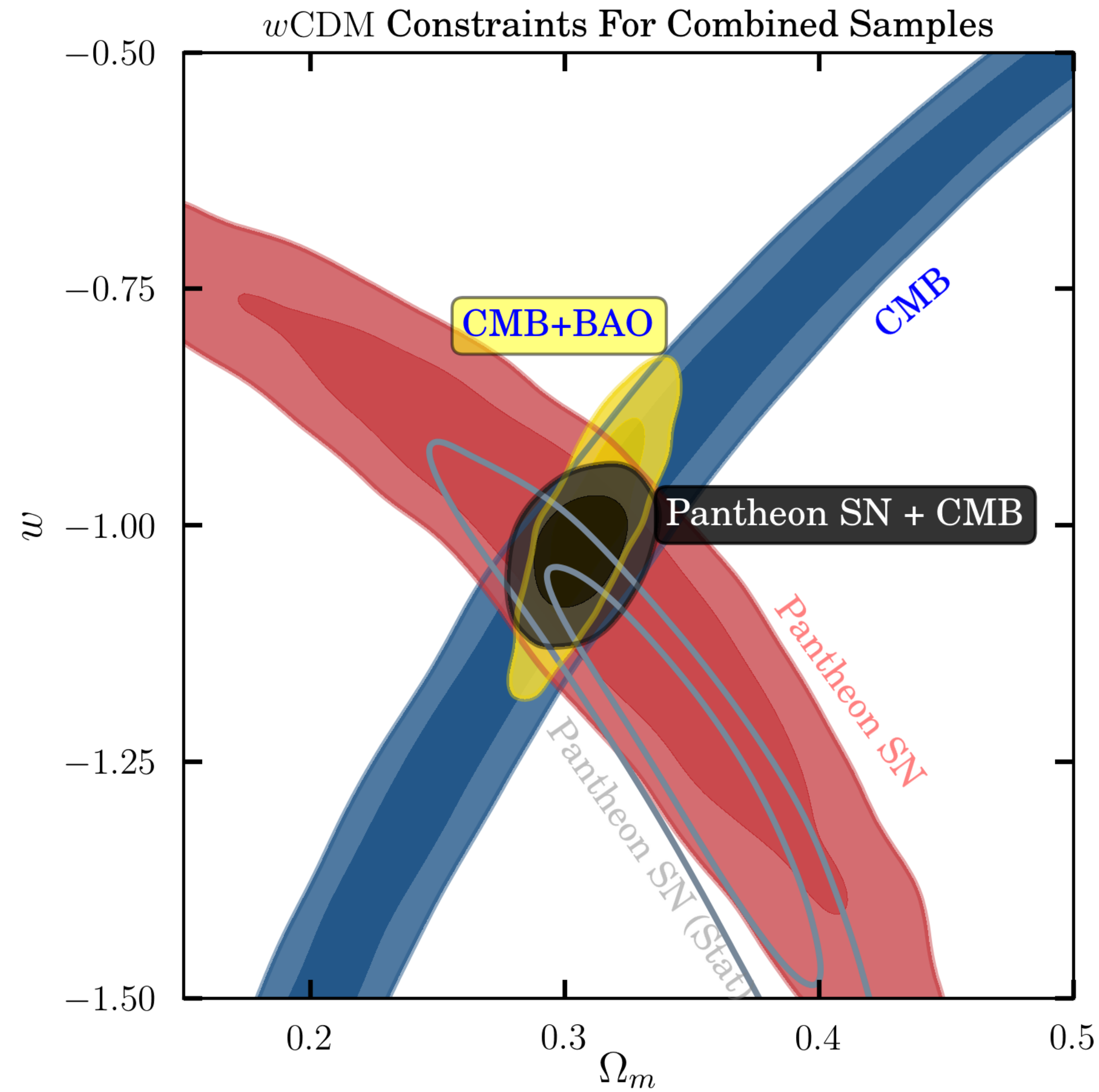
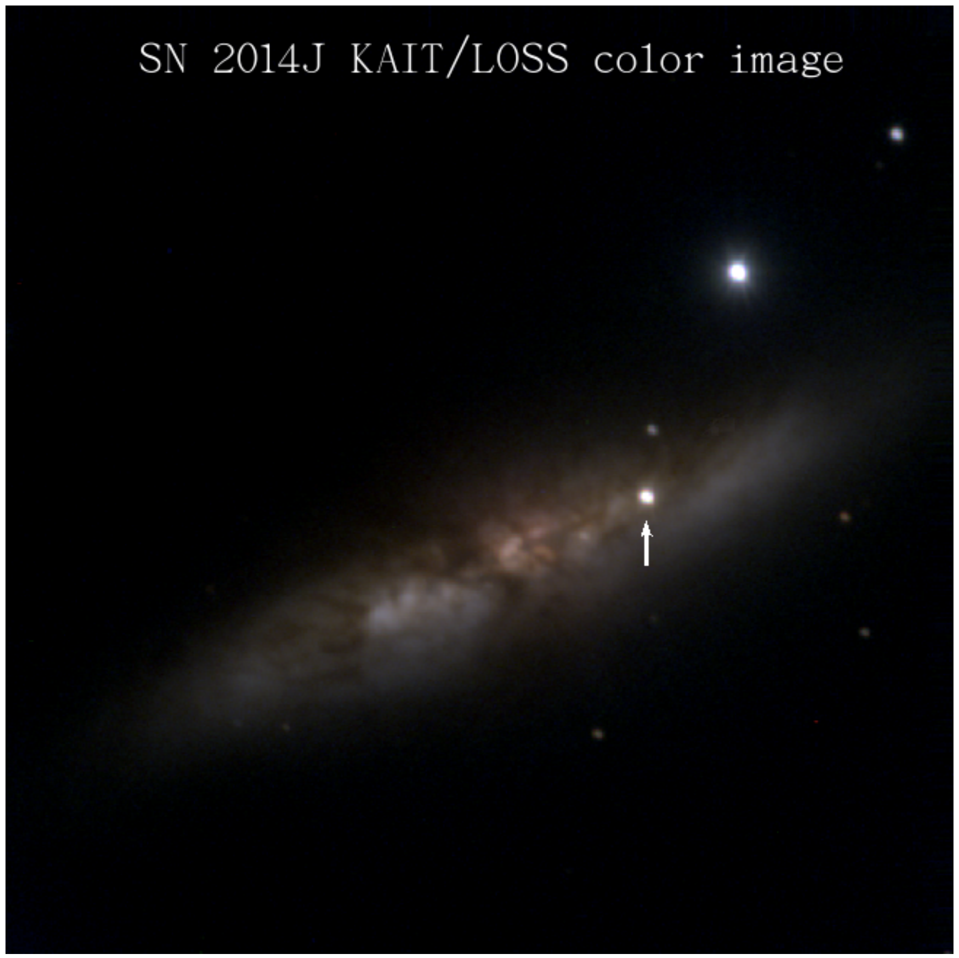
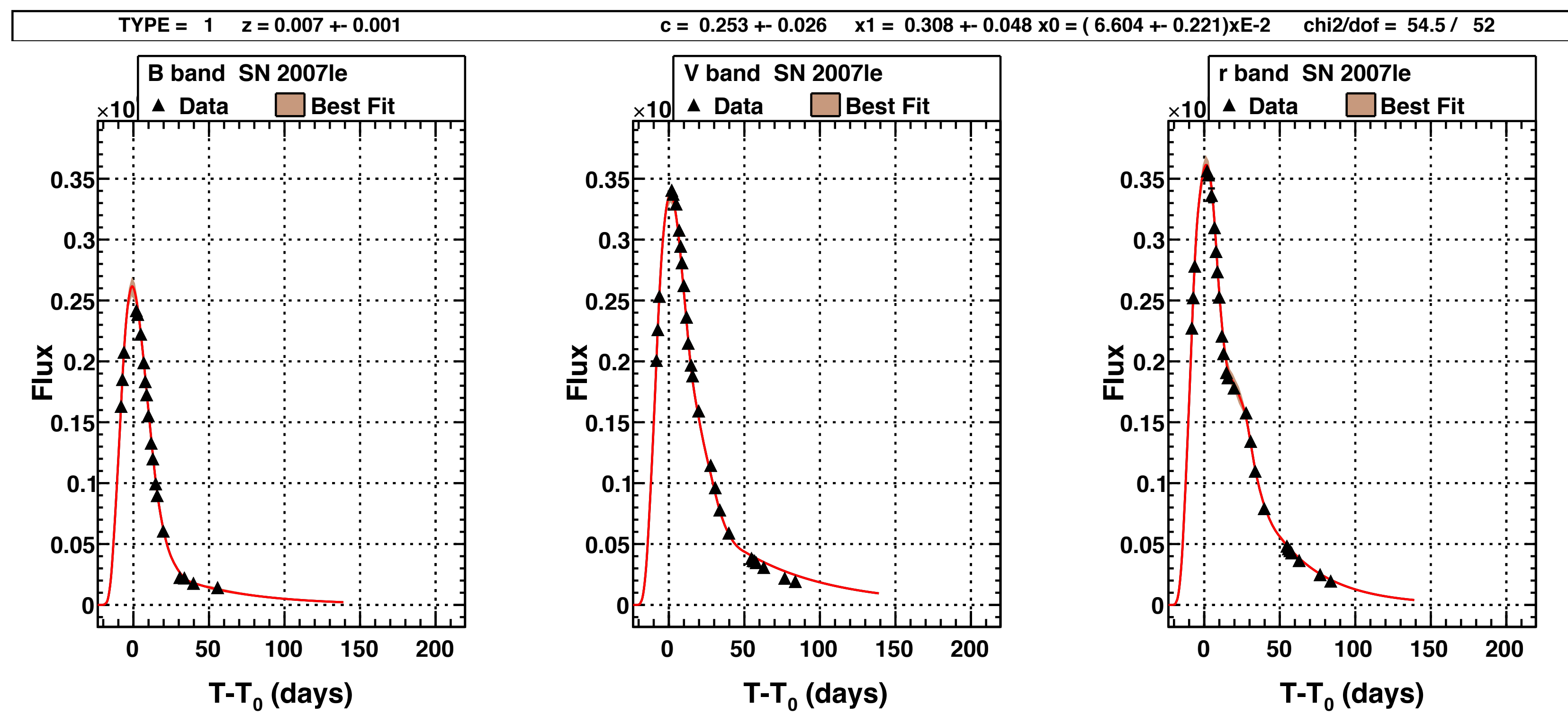


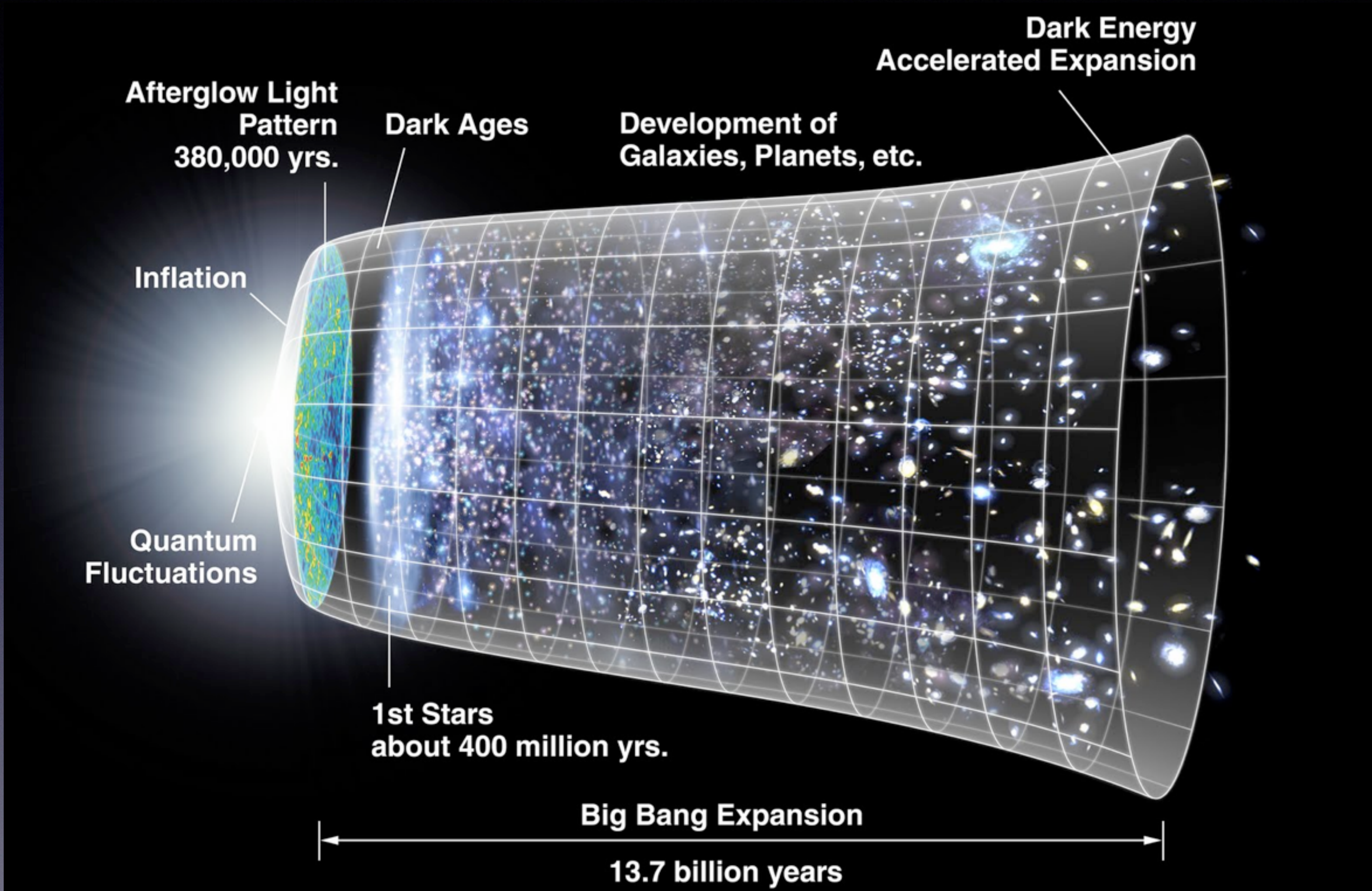
Figure 20. Confidence contours at 68% and 95% for the Ω_m and w cosmological parameters for the wCDM model. Constraints from CMB (blue), SN - with systematic uncertainties (red), SN - with only statistical uncertainties (gray-line), and SN+CMB (purple) are shown.

Supernova Time Series (Light Curves)



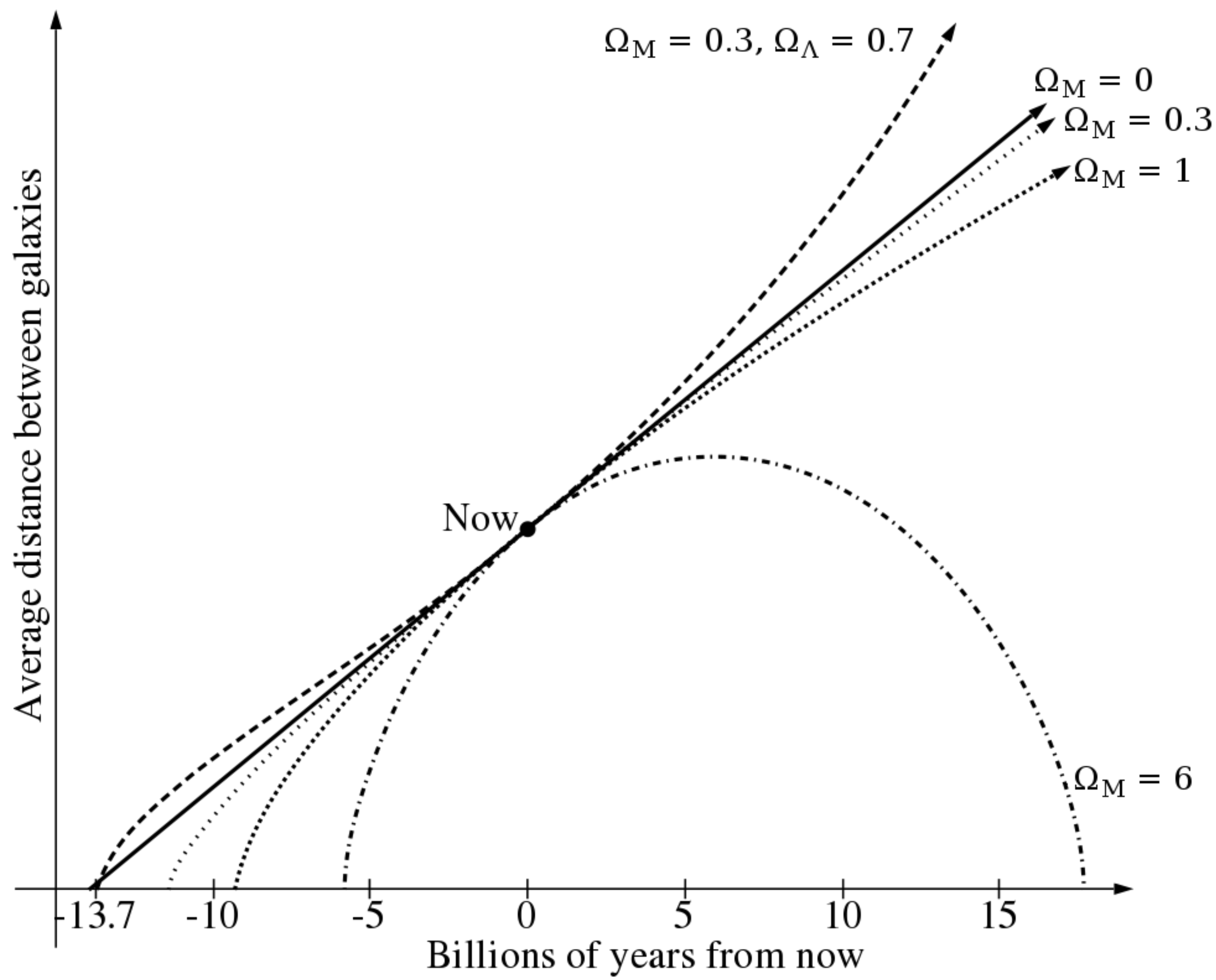


The History of Cosmic Expansion

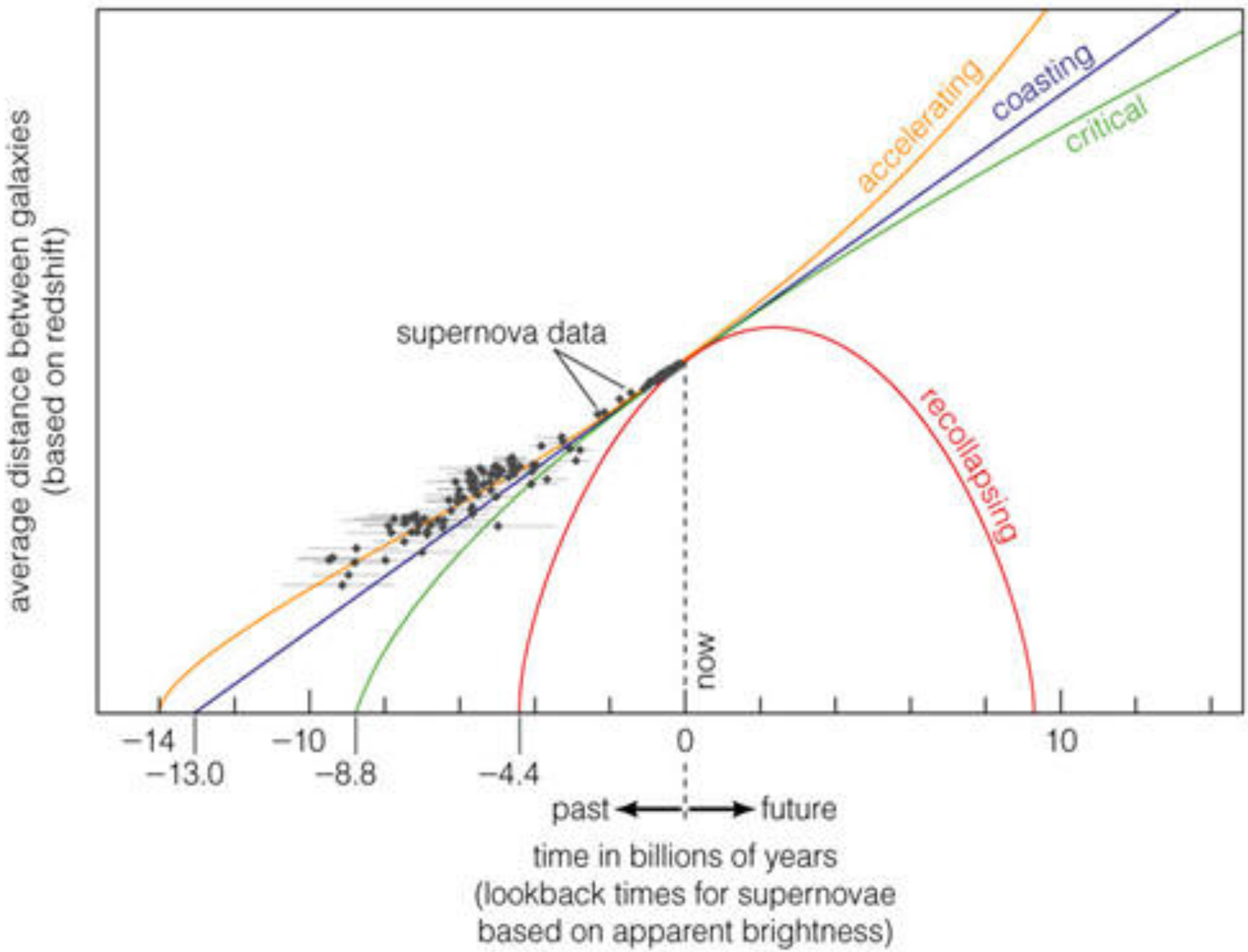


Expansion History (and Future) of the Universe: Determined by its Physical Energy Content

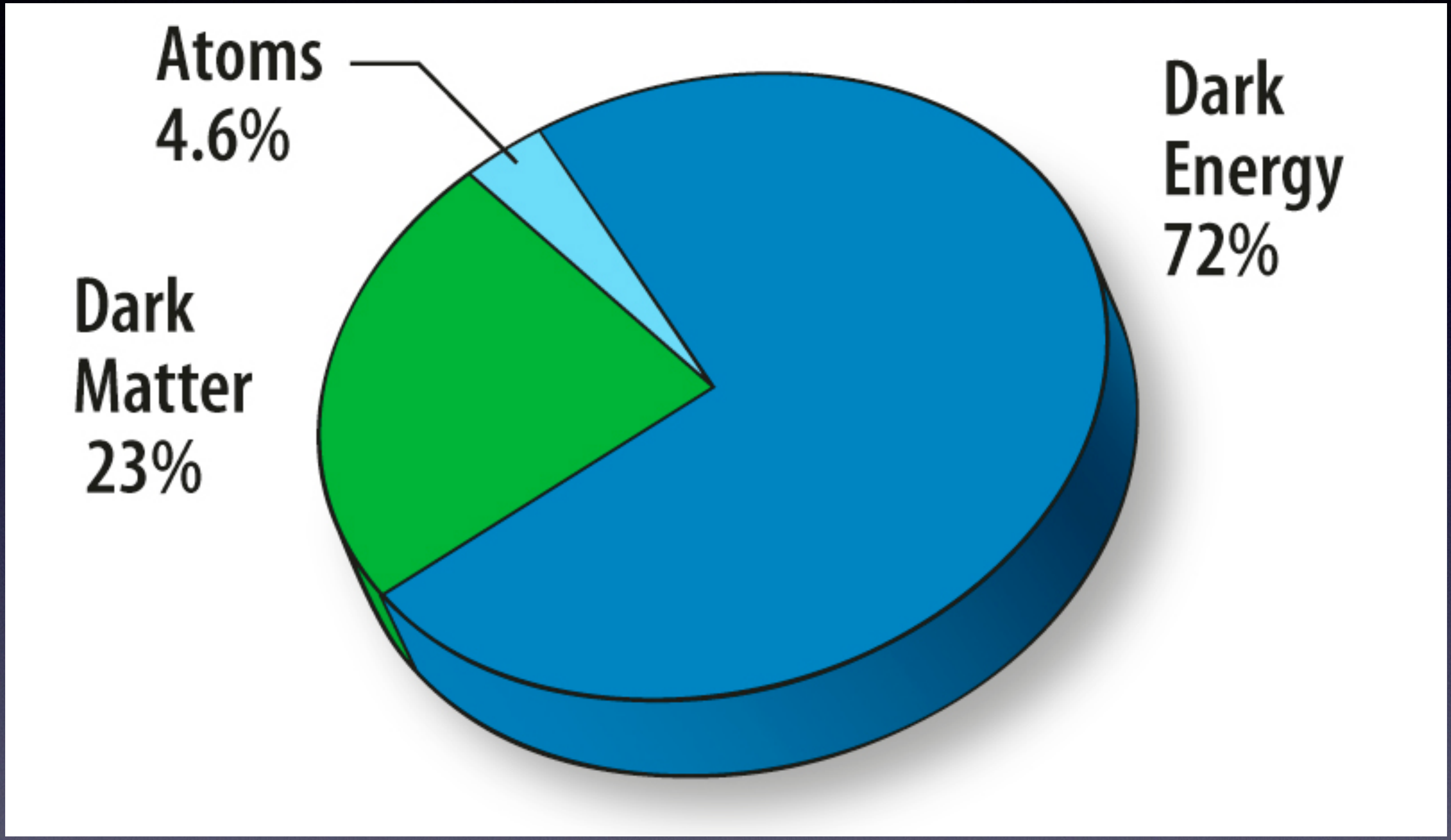
Ω_M = Matter Density; Ω_Λ = Dark Energy Density



Supernovae Trace the History of Cosmic Expansion



Cosmological Energy Content

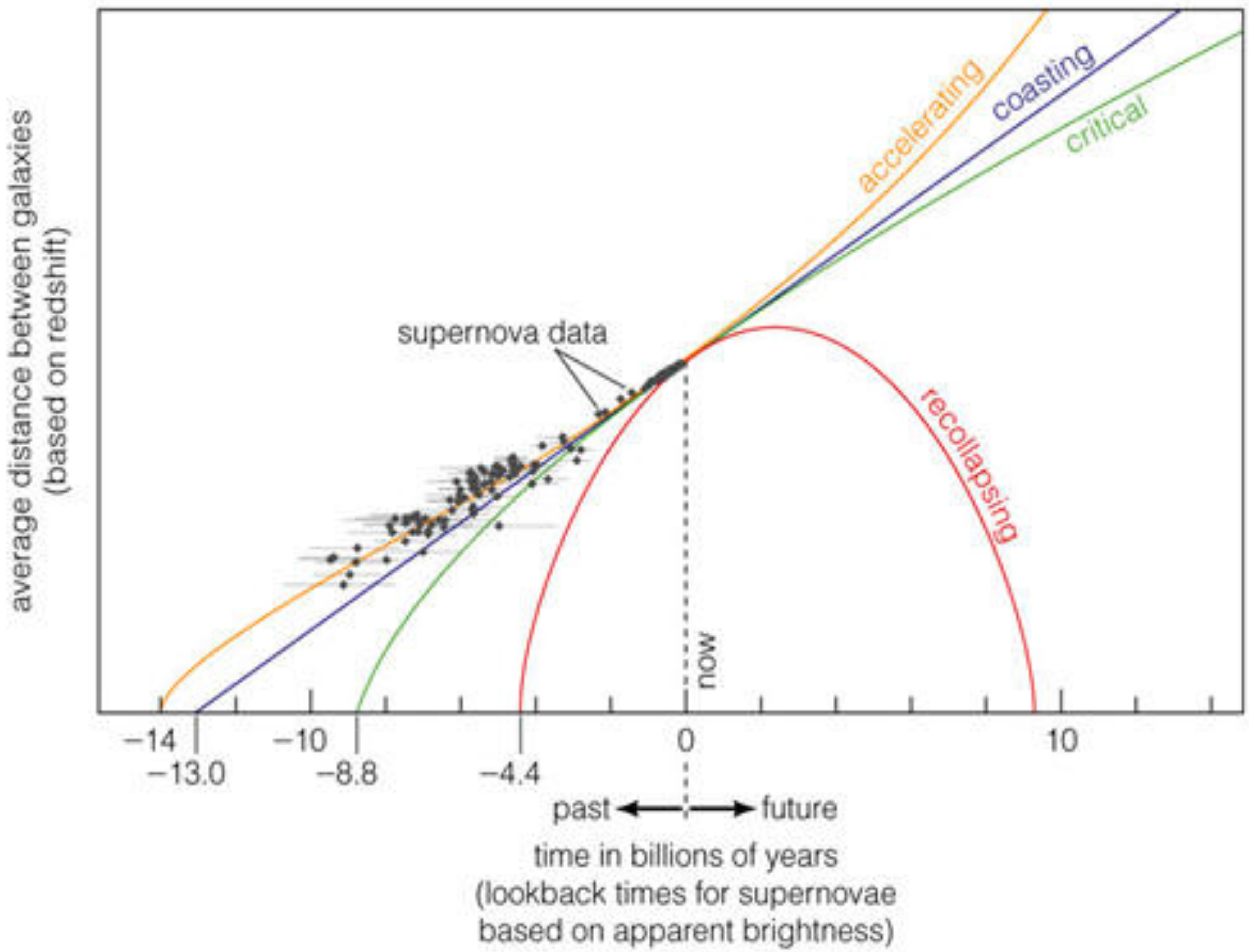


What is Dark Energy?

Dark Energy Equation of state $P = w\rho$

Is $w + 1 = 0$? (Cosmological Constant: $w = -1$)

Supernovae Trace the History of Cosmic Expansion

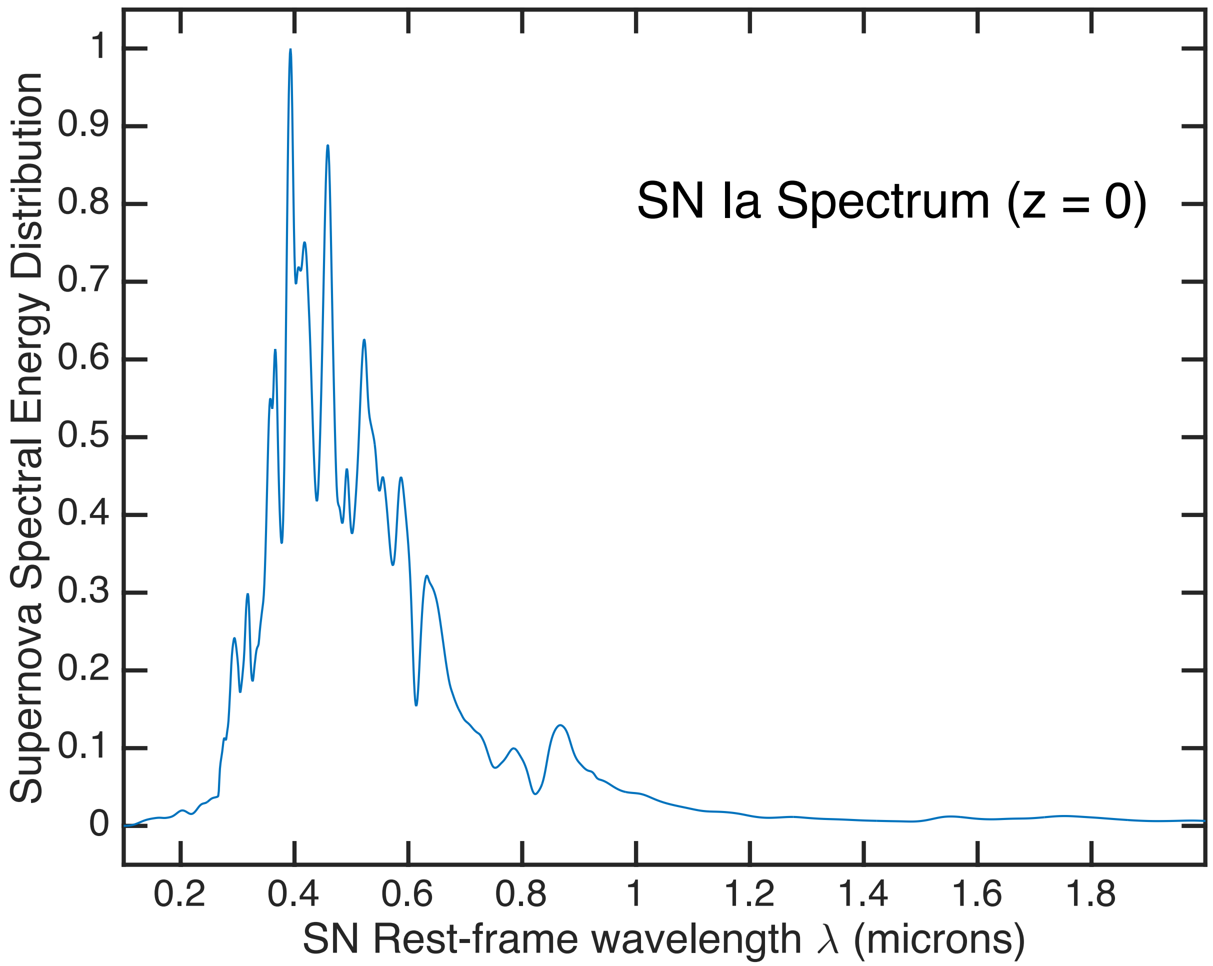


But we don't actually measure these things!

"Lookback" Time → Distance (μ)

Relative Size of Universe → Redshift (z)

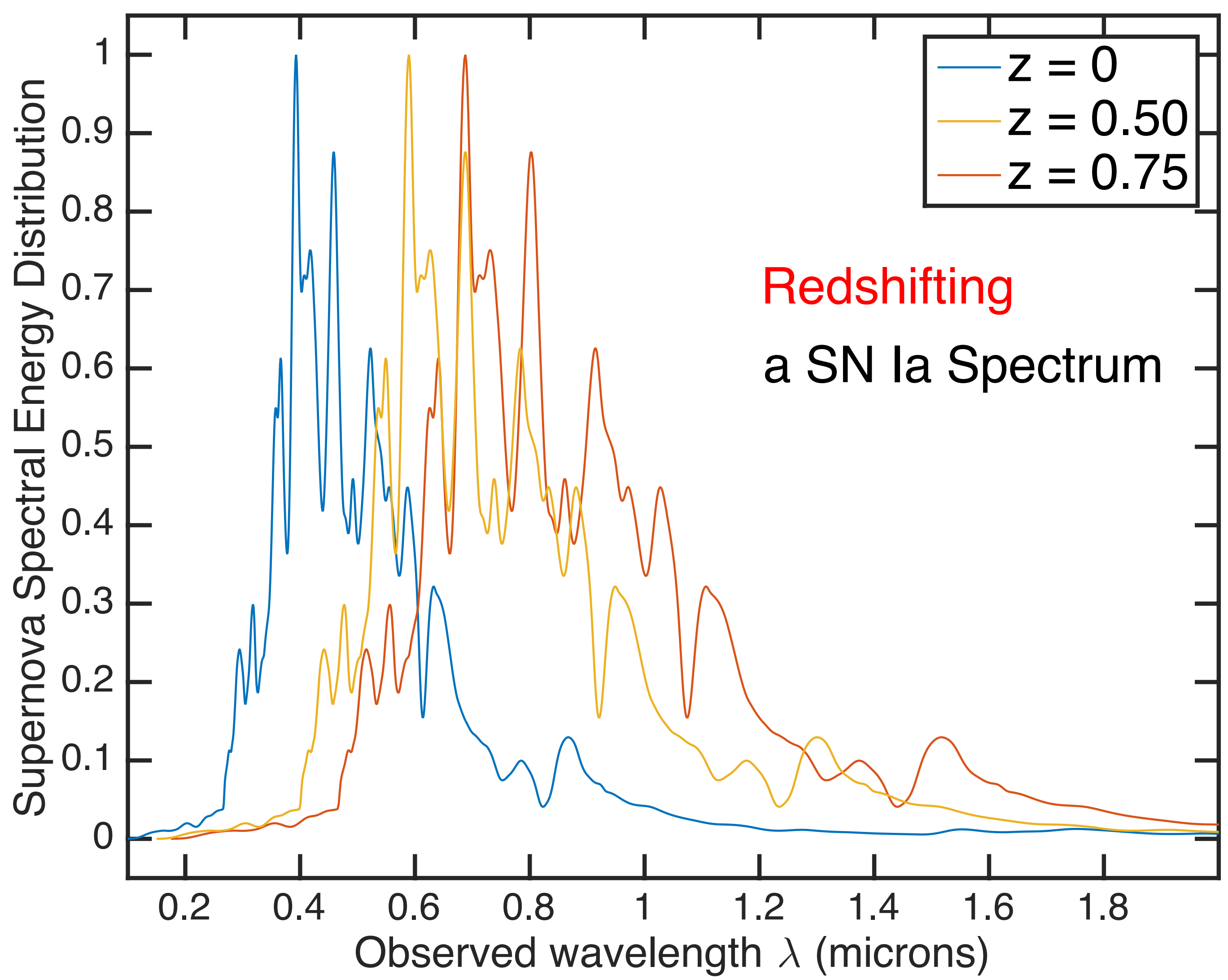
Expansion of the Universe: Redshifts (z)



Expansion of Universe over time "stretches" out wavelengths of light

Measure of cosmic expansion between observer and SN event

Spectral Lines are observed at longer wavelengths than originally emitted by the supernova: redshift (z)

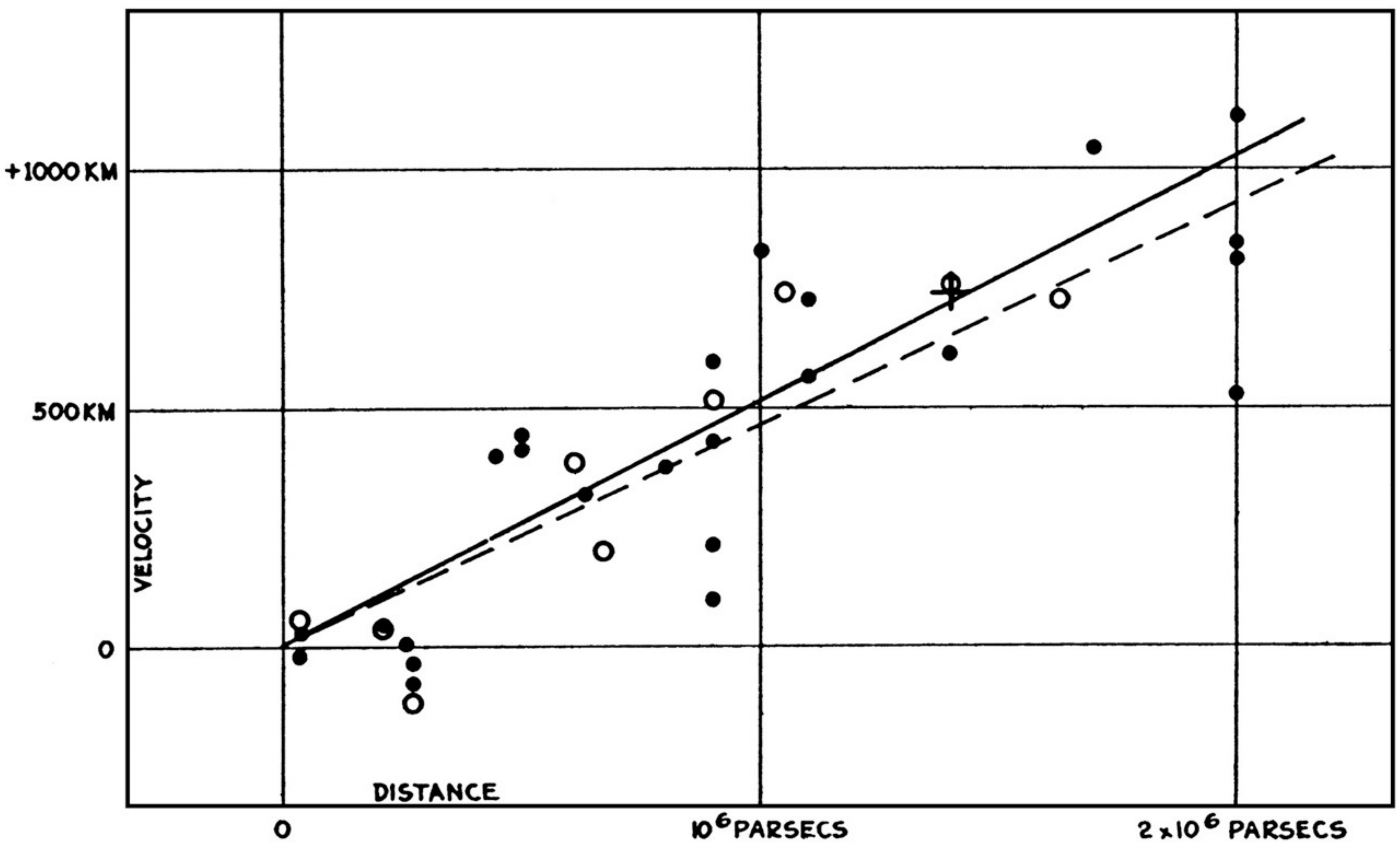


Determining Astronomical Distances using Standard Candles

1. Estimate or model Luminosity L of a Class of Astronomical Objects
2. Measure the apparent brightness or flux F
3. Derive the distance D to Object using Inverse Square Law: $F = L / (4\pi D^2)$
4. Optical Astronomer's units: $\mu = m - M$

m = apparent magnitude [log apparent brightness flux],
M = absolute magnitude [log Luminosity],
μ = distance modulus [log distance].

The Expanding Universe:
Galaxies are moving apart! Hubble's Law (1929)



Einstein & Hubble

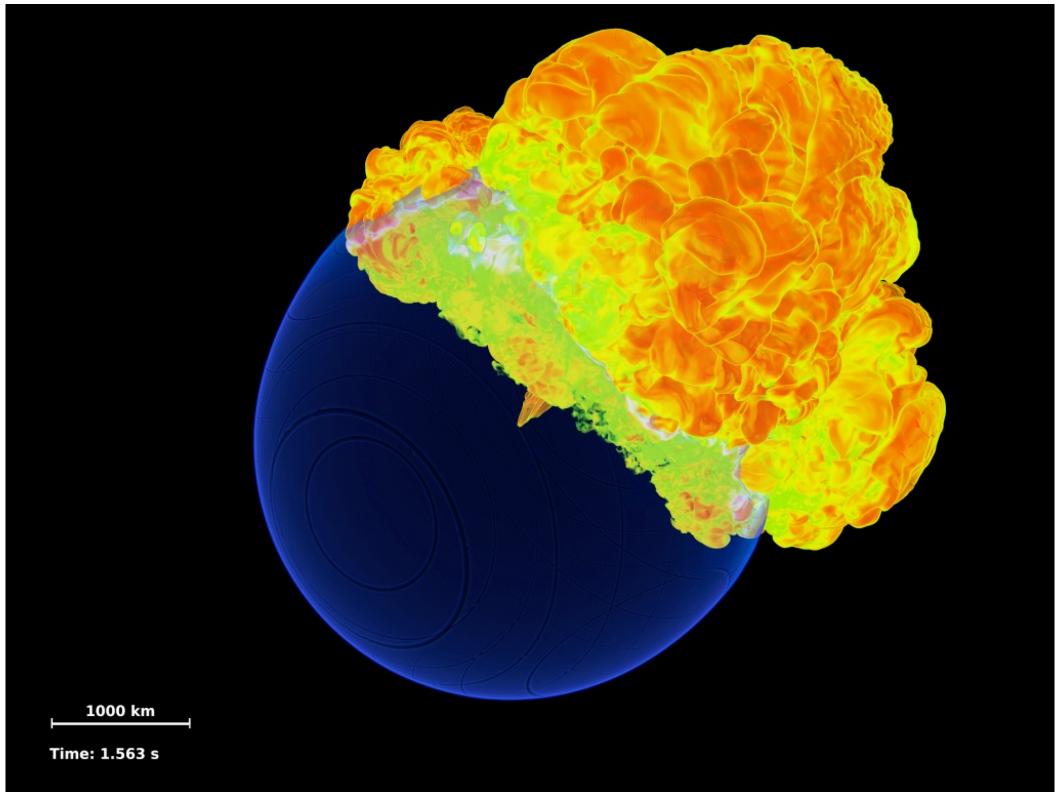
Distance $\propto$ Velocity (Redshift)

But what is the rate of change of the expansion?

(the deceleration parameter) Need better distances!

Type Ia Supernovae (SN Ia) are Almost Standard Candles

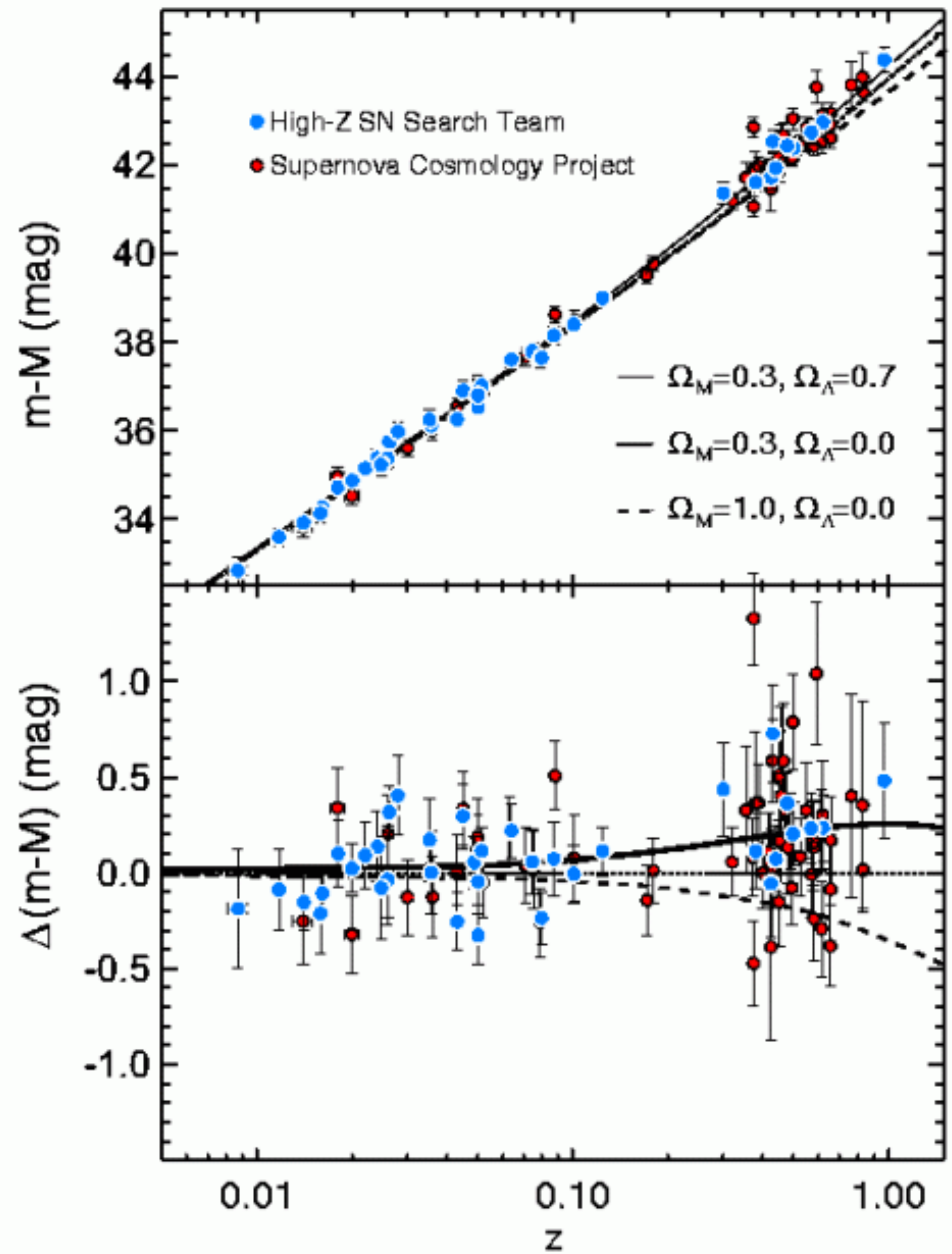
- Progenitor: C/O White Dwarf Star accreting mass leads to instability
- Thermonuclear Explosion: Deflagration/Detonation
- Nickel to Cobalt to Iron Decay + radiative transfer powers the light curve
- General Idea, but Theoretical Astrophysics simulations cannot quantitatively reproduce realistic observations (use empirical models)



Credit: FLASH Center

SN Ia Hubble Diagram (Distance Moduli vs. z):

The Universe is accelerating ($\Omega_\Lambda > 0$)!



The Accelerating Universe
2011 Nobel Prize in Physics

Photo: U. Montan

Saul Perlmutter

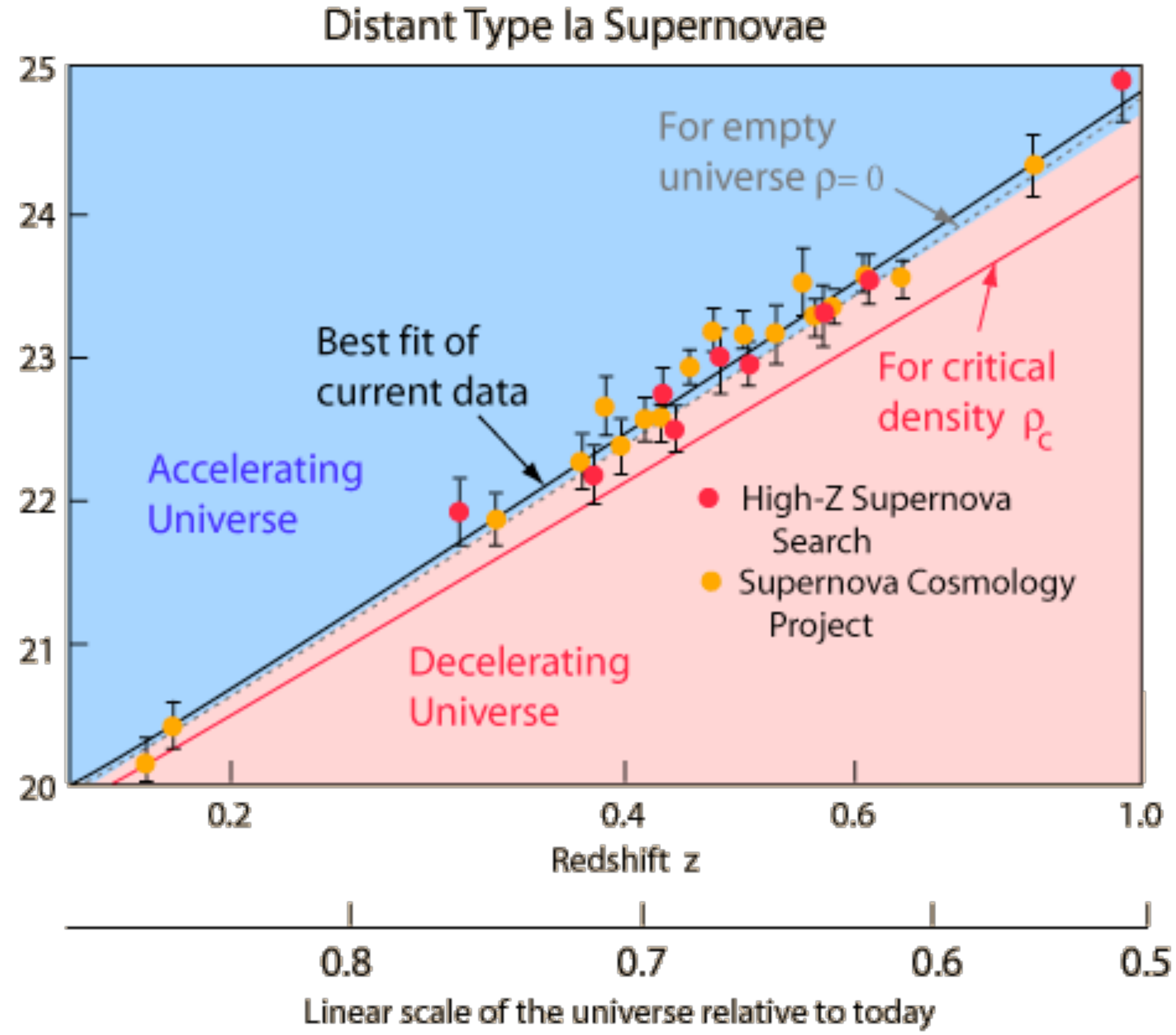
Photo: U. Montan

Brian P. Schmidt

Photo: U. Montan

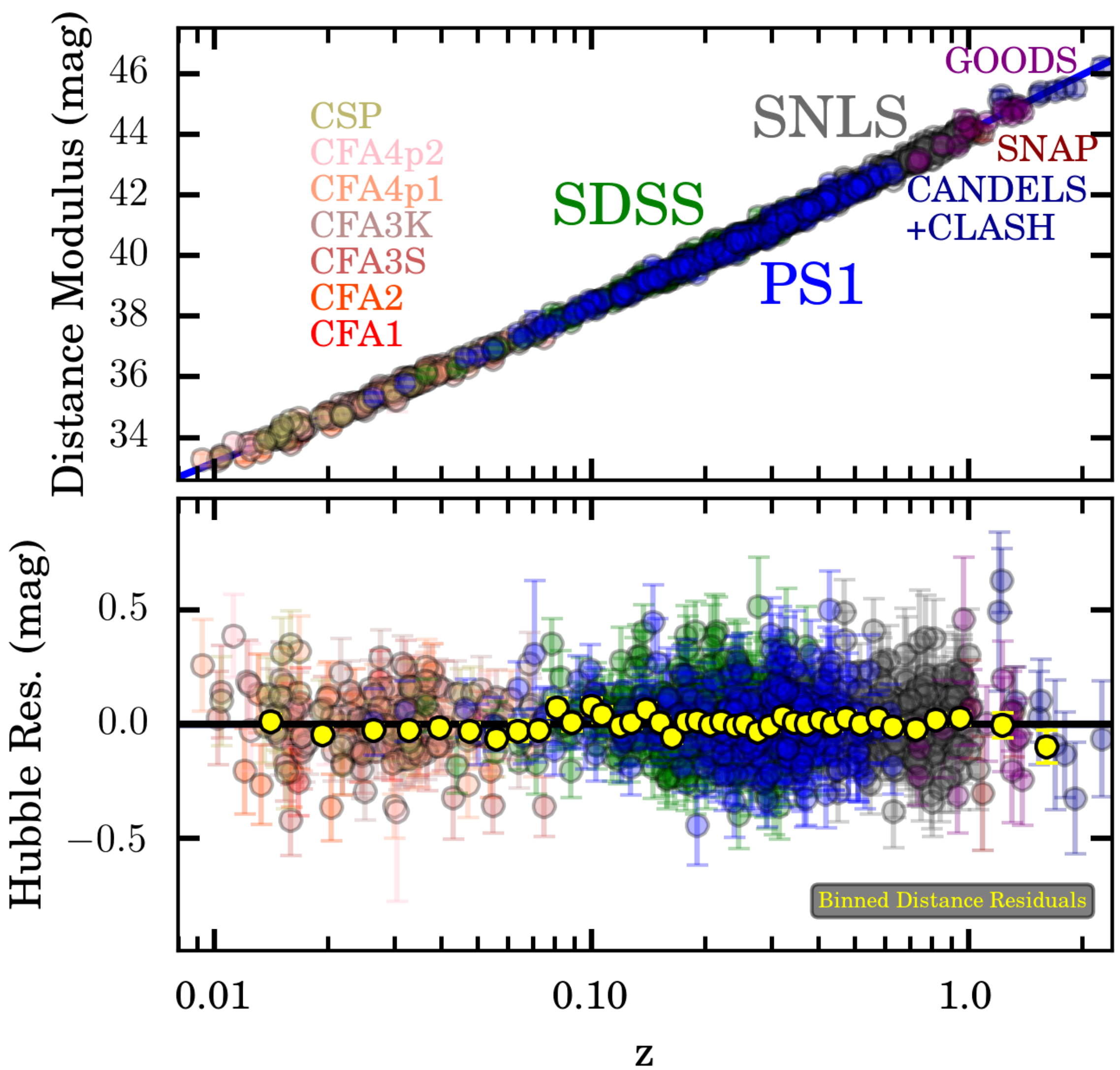
Adam G. Riess

The Nobel Prize in Physics 2011 was divided, one half awarded to Saul Perlmutter, the other half jointly to Brian P. Schmidt and Adam G. Riess *"for the discovery of the accelerating expansion of the Universe through observations of distant supernovae"*.



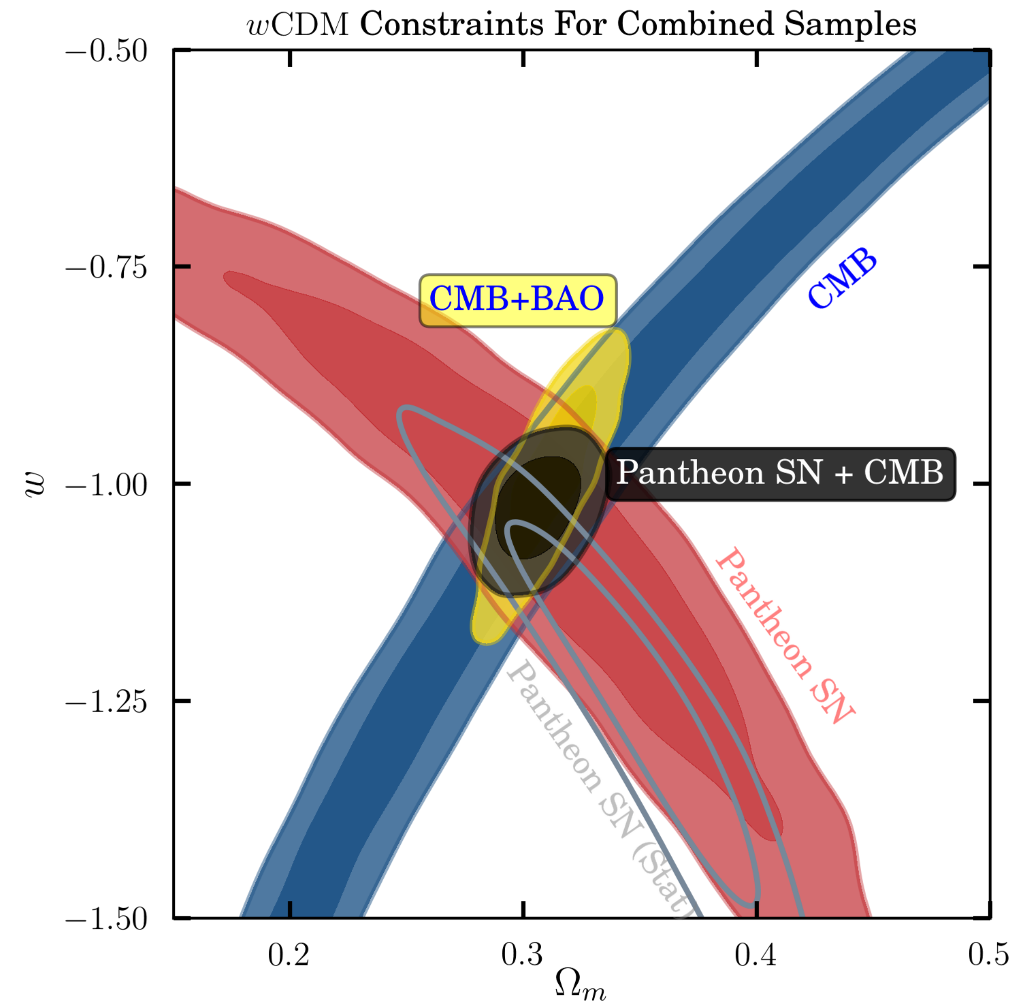
20 Years Later…

Hubble Diagram
Modern SN Ia Surveys



Pantheon Compilation
(1049 spec-confirmed SNe Ia)
Scolnic et al. 2018

Cosmological Constraints



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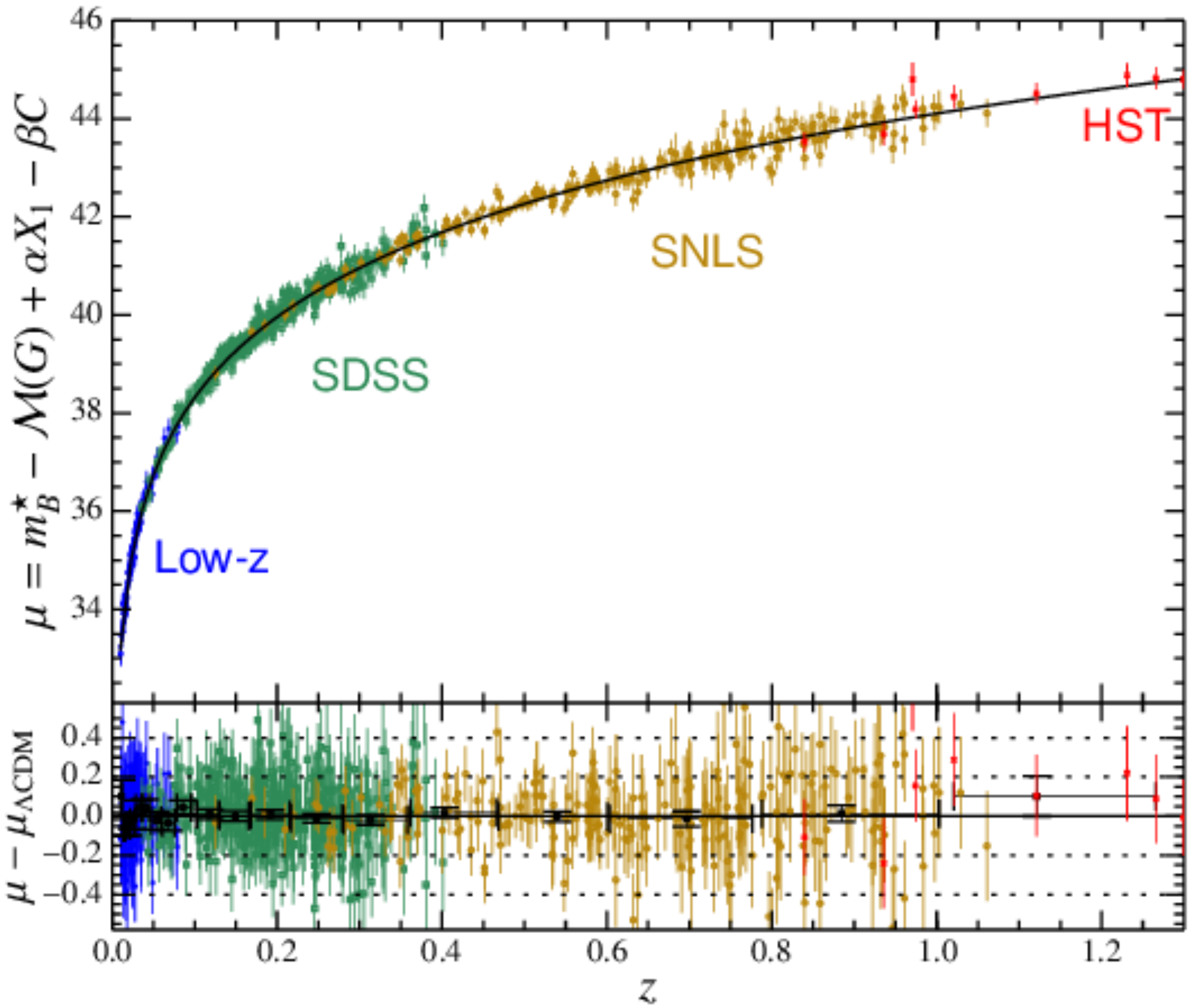
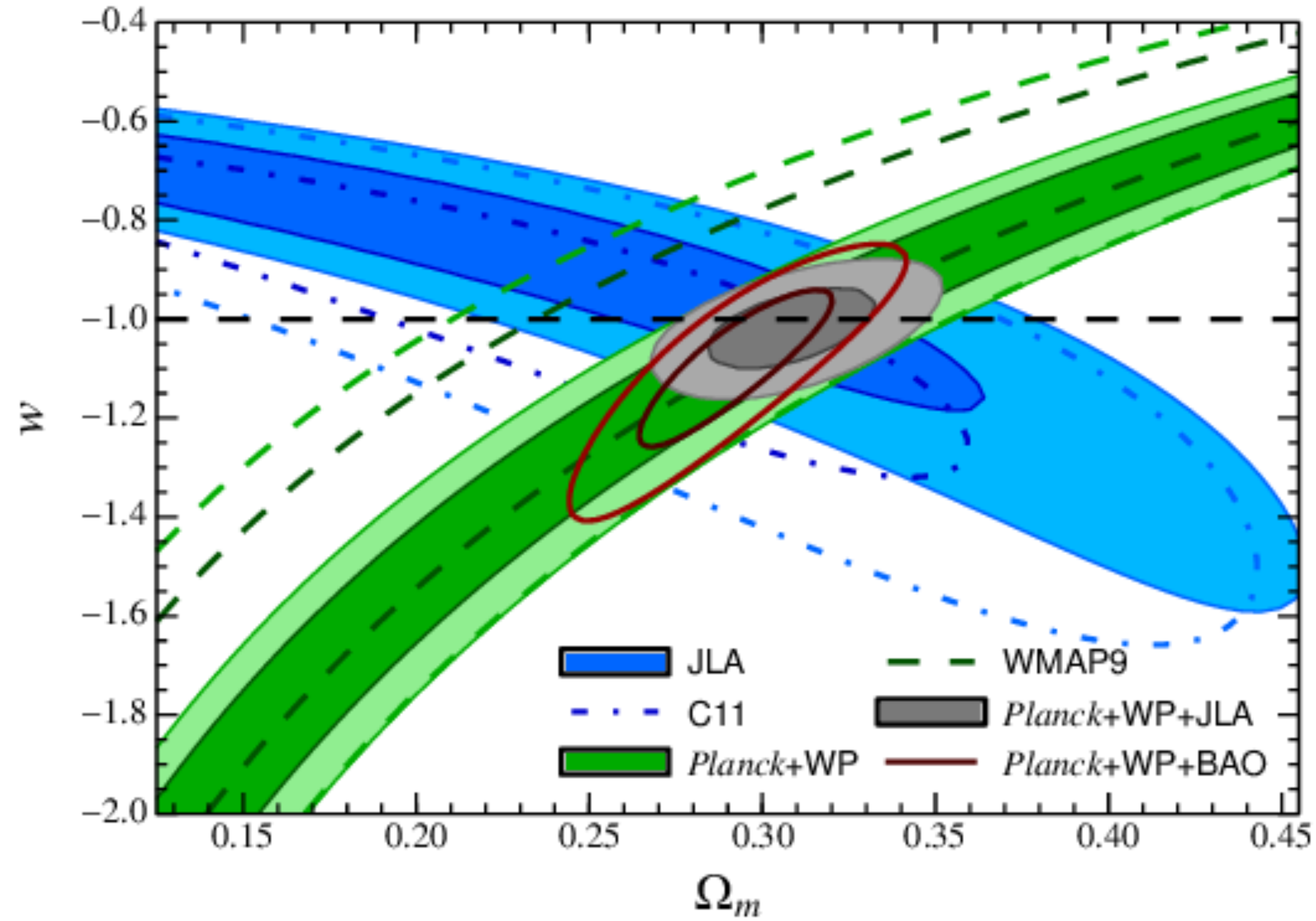
Type Ia SN Cosmology

Hubble Diagram
Modern SN Ia Surveys



Joint Lightcurve Analysis
(JLA, Betoule et al. 2014)

Cosmological Constraints



$w = -1.027 \pm 0.055$ (stat+sys)

Conventional Linear Formula (Standardiseable)

$$M_s \sim N(M_0 + \alpha \cdot x + \beta \cdot c, \sigma_{\text{int}}^2)$$

For us: Simplify:
Let's say Supernovae are *standard* candles

$$M_s \sim N(M_0, \sigma_{\text{int}}^2)$$ Population Distribution

$$m_s = M_s + \mu(z_s; \boldsymbol{\theta})$$ (Log) Inverse Square Law

$$\boldsymbol{\theta} = (H_0, \Omega_M, \Omega_L, w)$$ Cosmological Parameters

Theoretical Distance Modulus (for flat Universe; see lecture 15 for more general case)

where the true distance modulus at the observed redshift z_s is

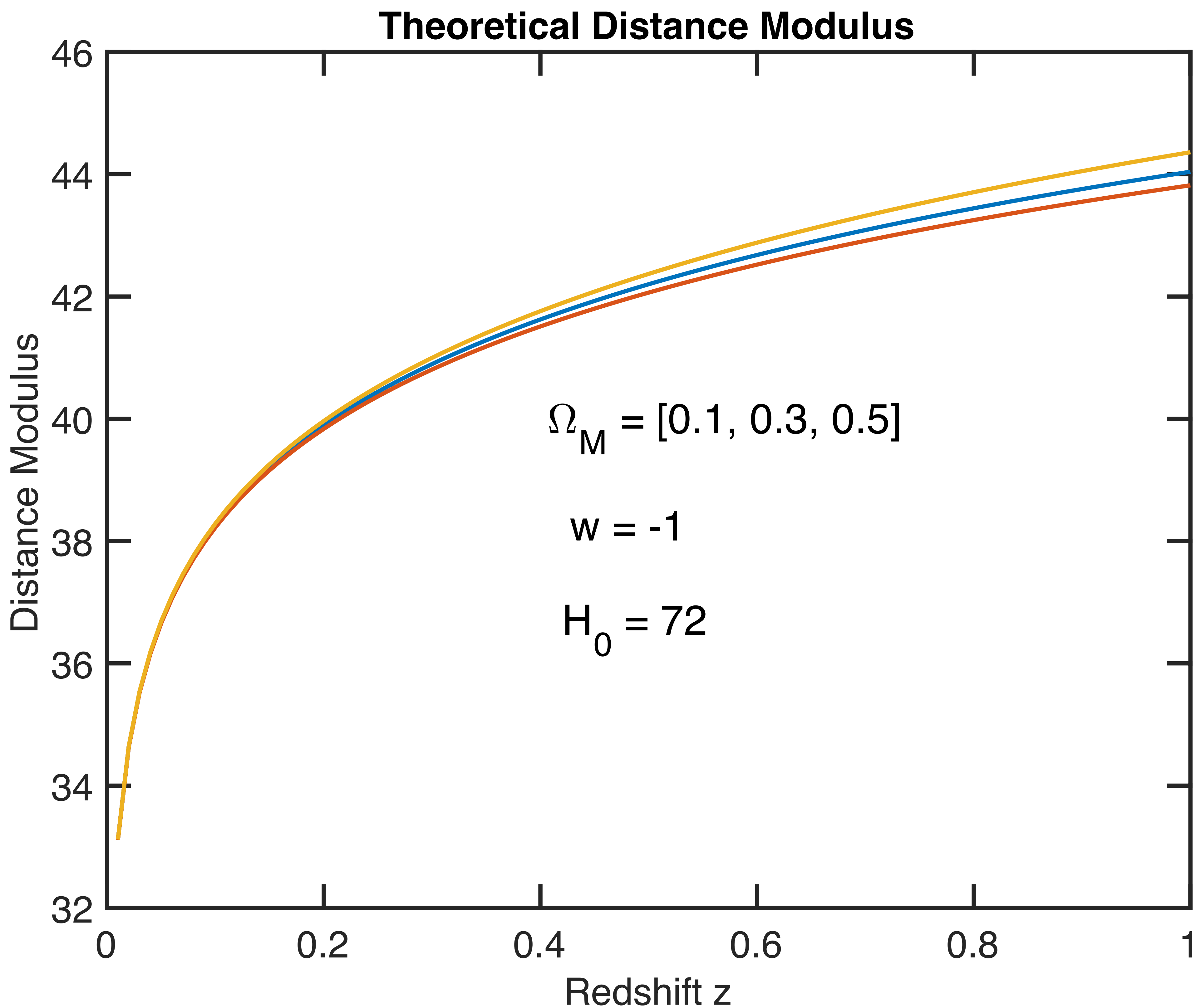
$$\mu(z_s; H_0, w, \Omega_M) = 25 + 5\log_{10}\left[\frac{c}{H_0}\tilde{d}(z_s; w, \Omega_M)\ \ \mathrm{Mpc}^{-1}\right] \tag{4}$$

where Mpc is a mega-parsec (a unit of distance), c is the speed of light, H_0 is the Hubble constant, and (w, Ω_M) are other cosmological parameters, and

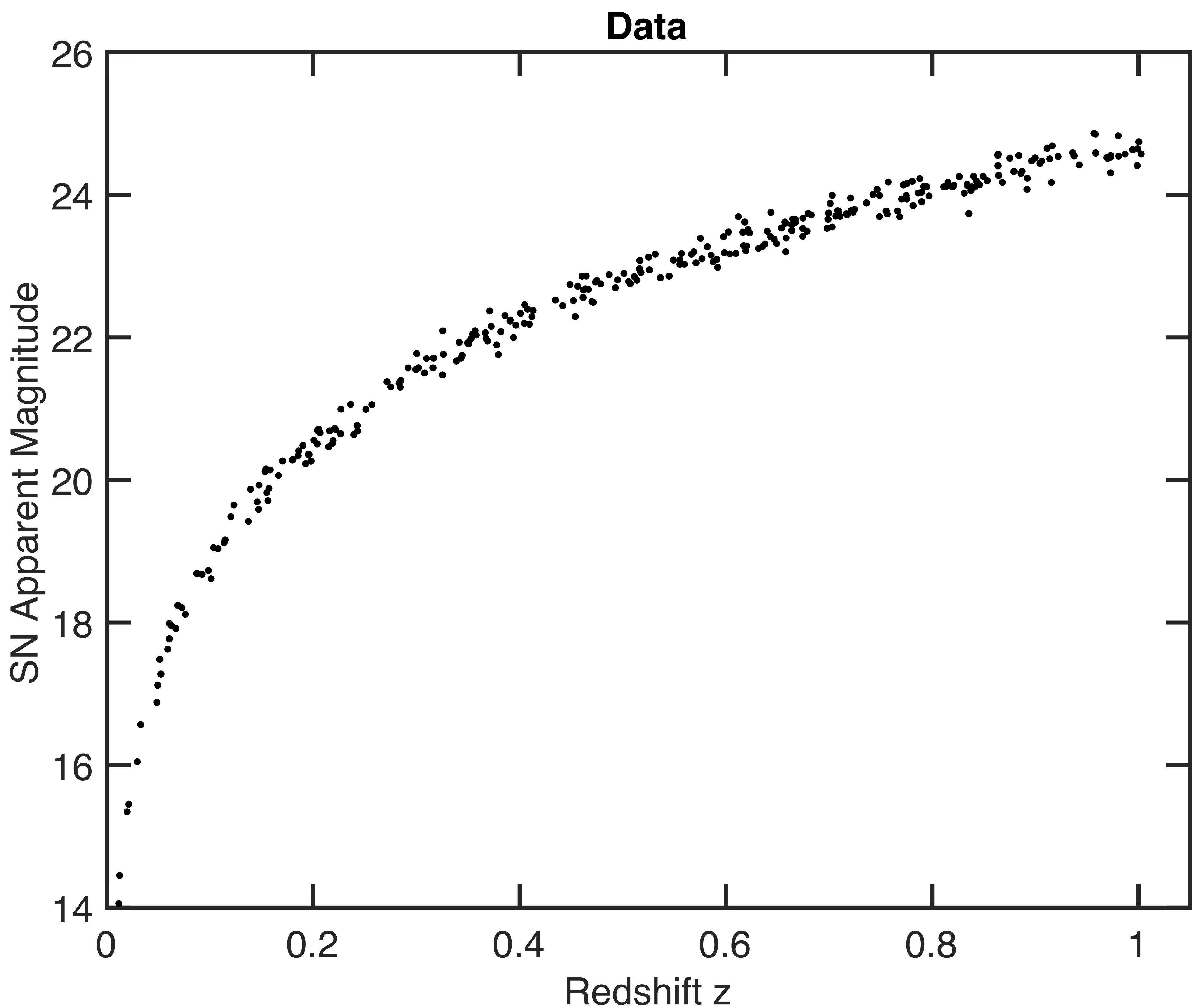
$$\tilde{d}(z; w, \Omega_M) = (1+z)\int_0^z \frac{dz'}{\sqrt{\Omega_M(1+z')^3 + (1-\Omega_M)(1+z')^{3(1+w)}}} \tag{5}$$

(derived from standard cosmology)

Theoretical Distance Modulus



Idealised supernova dataset



First assume w = -1

Write down model & posterior