Long Term Bond Markets and Investor Welfare

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Abstract

A simple framework is developed to analyze the important role of the bond market for the welfare of a long horizon small investor when both inflation and the real interest rate are stochastic. Closed form solutions for investor utility in different settings are used to show that the welfare loss due to the absence of long term bond markets increases with the investor's horizon and risk aversion, and with the size and persistence of the shocks to the real interest rate, but decreases with the correlation between the stock return and real interest rate. Using parameter estimates derived from U.S. data, we show that the welfare costs of the lack of a bond market can be substantial.

Long Term Bonds and Investor Welfare

The last 20 years have seen increasing recognition of the importance of private capital markets, and particularly of equity markets, as a mechanism for achieving efficiency, allocating capital, providing appropriate managerial incentives. It is this recognition that in large part lies behind the wave of state industry privatizations that has swept most western economies, and led even the formerly communist economies of China and the former Soviet Union to develop stock markets. Gibbon (1998, 2000) reports that the cumulative value of proceeds raised through privatization programs by governments during the last two decades exceeds $\$1$ trillion, and the amount of such revenue raised each year is now roughly $\$140$ billion.[1] Many countries experienced dramatic increases in market capitalization relative to GDP between 1990 and 1998. For example, China's stock market capitalization rose from 0.5% of its GDP in 1990 to about 25% in 1998.[2]

However, if private capital markets are to be used to allocate capital, then the capital itself must be under private, rather than state, control, and since a major component of a nation's stock of capital corresponds to the life cycle savings of individuals, the development of private capital markets presupposes a pool of private savings that are controlled by individuals. Therefore, the privatization of industry presupposes the privatization of savings. Countries as diverse as Chile and Sweden have modified their state pension systems so that a part of social security contributions are invested privately with some discretion of the individual over the type of fund in their retirement plans; and a similar proposal has been under consideration in the US. More

[1] D'Souza and Megginson (1999) point out that share issue provatization contributes the largest fraction of total proceeds raised by governments. Boutchkova and Megginson (2000) argue that "Although governments usually adopt privation programs to raise revenue, $\cdots$, most also hope that privatizations implemented through public share offerings will develop their national stock markets."

[2] Source: World Bank, "World Development Indicators 2000".

generally, even those countries that have so far maintained pay as you go state social insurance schemes, have seen the importance of these schemes for the provision of retirement income decline relative to that of privately managed savings.[3] The net result of these changes is that the individual in most countries is now much more responsible for the provision of his retirement income than ever before.

This new situation has given rise to several concerns. Several authors[4] have argued that individuals are not sufficiently sophisticated to manage their own savings, and Benartzi (2001) and Benartzi and Thaler (2001a, 2001b) provide empirical evidence pointing to systematic errors in the investment decisions of individuals saving for retirement. On the other hand, Brennan and Torous (1999) have shown, at least for a limited class of decisions, that decision errors may have only second order importance for investor welfare. These papers are concerned with the welfare of investors given the menu of investment alternatives that are available to them, on the assumption that individuals make mistakes because they are imperfectly rational. In this paper on the other hand, we are concerned with the effect oon investor welfare of the menu of securities that is available to them, assuming that the investors make rational decisions. In particular, we study the role of long term bonds in the optimal investment portfolios of long term investors.

The importance of long term bonds in developed economies is illustrated by US where the total value of the US fixed income market was around \$12 trillion,[5] which compares with a total

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[3] Benartzi and Thaler (2001b) report that a recent social security reform in Sweden gives the workers the right to direct 2.5% of their salary to individual accounts.

[4] See, for example, Bernheim (1996) and Beese (1993).

[5] This includes federal bonds, corporate, municipal, and asset backed securities, which are primarily backed by fixed rate mortgages. The stock of federal bonds outstanding, including treasury securites and bonds issued by federal agencies, was \$5.7 trillion in December 1999, of which \$3.1 trillion was accounted for by Treasury securities. Concern has been expressed about the consequences of retiring a significant portion of the federal debt. Source of data: Treasury Bulletin at http://www.fms.treas.gov/bulletin/b40fd.pdf.

US stock market value of around $16 trillion.[6] A significant proportion of the outstanding stock of bonds is held directly by individuals, and 28% of individual portfolios consist of bonds as compared with 45% in equities. The relative importance of bonds is even greater in countries such as France and Germany where public equity markets are less developed:[7] bonds account for 33% of household portfolios in France and 36% in Germany.

In contrast, bond markets in emerging economies are often small and unimportant. For example, in China the outstanding amount of fixed income securities was 2087 billion Yuan ($252 billion) in December 1999 while the value of outstanding equities was 4809 billion Yuan ($581 billion). This stock-bond ratio of 2.3 contrasts with the figure of 1.3 for the US, and many developing economies have even less well developed bond markets than does China. For example, Thailand's outstanding balance of all bonds is only about $28.9 billion at the end of 1999,[8] while the total stock market capitalization is $57 billion at the end of 1999 with a stock-bond ratio of almost two.[9]

In order to assess the welfare benefit for an investor of the availability of a market for long term bonds it is necessary to analyze payoffs from the optimal dynamic investment strategies of long horizon investors under different assumptions about the availability of long term bond investments. While the early work on portfolio strategies for long horizon investors was concerned with optimal allocations between stock and cash and tend to ignore the role of long term bonds,[10]

[6] Source of Data: Center for Research on Security Prices (CRSP).

[7] La Porta et. al (1997) attribute this to the reduced legal protections afforded shareholders by legal systems that are not based on common law. The data is from Allen and Gale (2000).

[8] Source of data: International Finance Corporation Emerging Markets Information Center.

[9] Source of data: International Federation of Stock Exchanges.

[10] See, for example, Latane and Tuttle (1967), Mossin (1968), Hakansson (1970), Merton (1969) and Samuelson (1969), who consider the optimal allocation between stock and cash when the investment opportunity set is constant (i.e., returns are i.i.d.). Merton (1971) analyzed the stock-cash allocation with a stochastic investment opportunity set, and demonstrated that this creates a set of 'hedge' demands in addition to the standard myopic demand. Fischer

several recent papers have analyzed the demand for long term bonds when interest rates are stochastic.[11] These papers have greatly advanced our understanding of the role of long term bonds in dynamic investment choice, but none of them considers the welfare costs to investors of restrictions on the availability of long term fixed income investment. While a full-fledged analysis of the welfare effects of the introduction of long term bonds would require a general equilibrium model and would consider the effect of the introduction of new assets on the equilibrium rates of return on other assets, we adopt a simpler, partial equilibrium, approach and consider the effect of investment opportunities on the welfare of a small investor taking the rates of return on all assets as given and unaffected by the introduction of new assets.

These assumptions, while restrictive, enable us to develop a simple model with which to study the effect of the menu of available investments on the portfolio choice and welfare of a small investor. The model allows for both a stochastic real interest rate and stochastic inflation so that bonds are potentially important instruments for hedging these state variables for long horizon (zero weight) individual investors. We derive the indirect utility function of the investor under different assumptions about the set of available securities and use this to assess the importance of the bond market by comparing investors' certainty equivalent wealth.

We find that the size of the welfare loss due to the absence of long term bonds increases with the investment horizon and the volatility of both the real interest rate and unexpected inflation rate, but decreases with the intensity of real interest rate mean reversion and the correlation between the stock return and real interest rate. For long horizon investors, the welfare loss also

(1975) studies the demand for and equilibrium premium of index bonds when inflation is uncertain and no long term nominal bonds exist.

[11] Such papers, for example, include Brennan, Schwartz and Lagnado (1997), Campbell and Viceira (2000), Liu (1999), Sorensen (1999), Brennan and Xia (1999, 2001) and Wachter (1999).

increases with the investor's level of risk aversion.

To illustrate the magnitude of the welfare loss that would be caused by the absence of a market for long term bonds, we calculate the loss using parameter estimates from Brennan and Xia (2001) for the U.S.. Our results suggest that the loss is potentially quite large. The loss in certainty equivalent return can be as large as 6% per year for an investor with twenty-year investment horizon and a coefficient of relative risk aversion of five. In an economy with quite persistent real interest rate shocks, the welfare loss for an investor with a thirty year horizon is equivalent to that due to a 90% reduction in his initial wealth. The correlation between the stock return and the real interest rate plays a critical role in determining the welfare loss. For an investor with twenty-year horizon, the welfare loss measured in terms of initial wealth varies from about 70% of initial wealth to zero as the correlation varies from zero to one. Because it is optimal to hold highly leveraged positions in bonds to hedge against changes in investment opportunities, we also consider the welfare effect of the bond market when the investor faces short sales constraints. Our results show that the bond market is important even for constrained investors.

We describe the characteristics of the investor investment opportunity set in Section I. In section II, we construct optimal portfolio strategies for investors who can and cannot invest in bonds. In Section III, we provide some illustrative calculations for the unconstrained case based on parameter estimates from U.S. data. Section IV reports results of the constrained case and Section V concludes the paper.

I. The Economy and Possible Investment Opportunities

The economy is assumed to have a single commodity whose price, Π, follows a diffusion process:[12]

$$\frac{d\Pi}{\Pi} = \pi dt + \sigma_{\Pi} dz_{\Pi}, \tag{1}$$

so that the (realized) rate of inflation is locally stochastic. The volatility of the inflation rate, σ_{Π}, is a constant, but the instantaneous expected rate of inflation, or proportional drift of the price level, π, is assumed to follow an Ornstein-Uhlenbeck process:

$$d\pi = \alpha(\bar{\pi} - \pi)dt + \sigma_{\pi} dz_{\pi}. \tag{2}$$

The investor is assumed to be able to invest in a single stock index, whose nominal price is assumed to follow a Geometric Brownian motion with constant risk premium and volatility,

$$\frac{dS}{S} = (R_f + \sigma_S \lambda_S)dt + \sigma_S dz_S, \tag{3}$$

where λ_S is the constant unit risk premium associated with the innovation dz_S, and R_f is the nominal interest rate defined later.

We assume that the real interest rate, r, follows the Ornstein-Uhlenbeck process:

$$dr = \kappa(\bar{r} - r)dt + \sigma_r dz_r, \tag{4}$$

[12] The setup of the model is similar to that of Brennan and Xia (2001)

where $\bar{r}$, the long run mean, σ_r, the volatility, and κ, the mean reversion coefficient, are known constants. If an instantaneously riskless *real* asset existed, then its instantaneous real rate of return would be r. However, we assume that in this economy no instantaneously riskless real asset exists. Note that this assumption is not without empirical relevance. Even countries such as Canada, the U.S. and the U.K., which have issued inflation indexed bonds, have done so for only a limited time and there are typically only a few (long) maturities available with small market capitalization and low liquidity.

Following Vasicek (1977), let λ be a vector whose elements, λ_i, are the constant market prices of risk associated with innovations in the stochastic variables i ($i = S$, r, π and Π), the nominal return of a nominal bond which matures at time T with a nominal payoff of $\$1$, $\frac{dP(t,T)}{P(t,T)}$, is then given by:[13]

$$\frac{dP}{P} = \left[r + \pi - \sigma_\Pi^2 + \sigma_\Pi \lambda_\Pi - B\sigma_r \lambda_r - C\sigma_\pi \lambda_\pi\right] dt - B\sigma_r dz_r - C\sigma_\pi dz_\pi. \tag{5}$$

The instantaneous nominal riskfree interest rate, R_f, is obtained by taking the limit of the return on the nominal bond in equation (5) by letting $(t - T) \to 0$:

$$R_f = r + \pi - \sigma_\Pi^2 + \sigma_\Pi \lambda_\Pi \tag{6}$$

and $-\sigma_\Pi^2 + \sigma_\Pi \lambda_\Pi$ is the risk premium for the nominal instantaneous riskfree asset. The Fisher equation does not hold in this economy unless the market price of unexpected inflation risk, λ_Π, is zero, or there is no unexpected inflation. Using equation (6), the nominal return on the nominal

[13] The detailed derivation of bond price is omitted, but is available upon request.

bond in equation (5) can also be written as:

$$\frac{dP}{P} = [R_f - B\sigma_r\lambda_r - C\sigma_\pi\lambda_\pi]\,dt - B\sigma_r dz_r - C\sigma_\pi dz_\pi, \tag{7}$$

where

$$B(t,T) = \kappa^{-1}\left(1 - e^{\kappa(t-T)}\right), \tag{8}$$

$$C(t,T) = \alpha^{-1}\left(1 - e^{\alpha(t-T)}\right). \tag{9}$$

Equation (7) shows that the nominal risk premium on bonds depends only on their exposures to innovations in the real interest rate and in expected rate of inflation.

There are four sources of risk in the economy due to innovations in stock price, the real interest rate, expected inflation and unexpected inflation. The investor potentially has available three types of securities: the nominal riskless asset, a continuum nominal bonds with maturities from zero to τ^{max} and the stock index. Equation (7) shows that returns on long term nominal bonds load on dz_r and dz_π, and therefore provide a natural hedge for risks associated with r and π. While the risk associated with unexpected inflation, dz_Π, is correlated with the stock and bond returns, the correlation is not perfect and some part of the unexpected inflation cannot be hedged by the available nominal securities. To isolate the unhedgeable component of unexpected inflation, we decompose unexpected inflation into a linear combination of the hedgeable risks of dz_S, dz_r, and dz_π, and an additional orthogonal component, dz_u:

$$\frac{d\Pi}{\Pi} = \pi dt + \sigma_\Pi dz_\Pi = \pi dt + \xi_S dz_S + \xi_r dz_r + \xi_\pi dz_\pi + \xi_u dz_u = \pi dt + \xi' dz + \xi_u dz_u \tag{10}$$

where $\xi \equiv [\xi_S, \xi_r, \xi_\pi]'$, and $dz = [dz_S, dz_r, dz_\pi]'$. By construction, dz_u is independent of all nominal security returns and $\xi_u dz_u$ is the unhedgeable component of unexpected inflation.

II. Optimal Portfolio Choice With and Without Bond Market

In order to assess the importance of the bond market for investors, we consider the portfolio problem of an investor who is concerned with maximizing the expected utility of wealth at some fixed horizon date,[14] T. The importance of the bond market is highlighted by comparing the optimal dynamic portfolio strategies and their associated certainty equivalent wealth[15] for investors with and without the opportunity to invest in bonds.

We start our analysis with the most developed economy which allows the investor to invest in cash, a single stock index, and a continuum of nominal bonds[16] with maturities from zero to τ^*. In the economy with both bond and stock markets, the investor can achieve his optimum by investing in cash, stock, and any two nominal bonds with different maturities. These four securities can perfectly hedge the risks associated with dz_S, dz_r and dz_π, while any additional nominal bonds do not help hedge the orthogonal risk. We then examine the optimal portfolio strategy and welfare of the investor in economies without a bond market or with only an underdeveloped bond market. We solve the investor's optimization problem by dynamic programming. Section A presents the general framework. Section B considers the problem when there is no bond market and section C considers the problem when the investor has only limited bond market.

[14]We study the terminal wealth problem without the consumption over time in order to emphasize the role of maturity.

[15]This is the sure amount at the horizon that the investor would exchange for \$1 of current wealth and the investment opportunities up to the horizon.

[16]We emphasize the role of nominal bonds in the long-horizon investor's optimal portfolio choice. Campbell and Viceira (1999) and Brennan and Xia (2000) provide some discussion of the role of indexed bonds.

A. The General Framework

Consider the problem of an investor with an iso-elastic utility function who is concerned with maximizing the expected utility of real wealth at time T:

$$\max_{\mathbf{x}} E_t \left\{ \frac{(W_T/\Pi_T)^{1-\gamma}}{1-\gamma} \right\}, \tag{11}$$

$$\text{subject to} \quad \frac{dW}{W} = (R_f + x'\Lambda)\, dt + x'\sigma dz, \quad (W > 0) \tag{12}$$

where γ is the coefficient of relative risk aversion.[17] The vector $\Lambda \equiv [\sigma_S \lambda_S,\ -B(t,T_1)\lambda_r - C(t,T_1)\lambda_\pi,\ -B(t,T_2)\lambda_r - C(t,T_2)\lambda_\pi]'$ contains the nominal risk premiums for the stock and two nominal bonds with maturities T_1 and T_2. The matrix of loadings of the three securities on the innovations, dz_S, dz_r and dz_π, is denoted by σ, whose first row is $(\sigma_S, 0, 0)$, and whose second and third rows are $(0, -B(t,T_j)\sigma_r, -C(t,T_j)\sigma_\pi)$ $(j = 1, 2)$. The vector of optimal proportional wealth allocations to the stock and two nominal bonds, $\mathbf{x}^* \equiv ({x_s}^*, {x_1}^*, {x_2}^*)'$, is the choice variable, and the balance of the portfolio, $\mathbf{1} - \mathbf{x}'\mathbf{i}$, is invested in the nominal riskless asset at the rate R_f. Equation (12) describes the dynamics of the nominal wealth process.

Let ρ be the correlation matrix of dz_S, dz_r and dz_π with rows $[1, \rho_{Sr}, \rho_{S\pi}]$, $[\rho_{Sr}, 1, \rho_{r\pi}]$ and

[17] When $\gamma = 1$, the above equation reduces to the case of log utility.

$[\rho_{S\pi}, \rho_{r\pi}, 1]$, then the Bellman equation for this problem is:[18]

$$
\begin{aligned}
\max_x \quad & \left\{ J_t + \frac{1}{2} W^2 \left(\mathbf{x}' \sigma \rho \sigma' \mathbf{x} \right) J_{WW} + W \mathbf{x}' \sigma \rho e_2 \sigma_r J_{Wr} + W \Pi \mathbf{x}' \sigma \rho \xi J_{W\Pi} + \Pi \xi' \rho e_2 \sigma_r J_{r\Pi} \right. \\
+ \quad & W \mathbf{x}' \sigma \rho e_3 \sigma_\pi J_{W\pi} + \Pi \xi' \rho e_3 \sigma_\pi J_{\pi\Pi} + \sigma_r \sigma_\pi \rho_{r\pi} J_{r\pi} + \frac{1}{2} \sigma_\pi^2 J_{\pi\pi} + \alpha(\bar{\pi} - \pi) J_\pi \\
+ \quad & \left. \frac{1}{2} \Pi^2 \left(\xi' \rho \xi + \xi_u^2 \right) J_{\Pi\Pi} + \frac{1}{2} \sigma_r^2 J_{rr} + \kappa(\bar{r} - r) J_r + W(R_f + x'\Lambda) J_W + \Pi \pi J_\Pi \right\} \\
= \quad & 0, \qquad (13)
\end{aligned}
$$

and the corresponding first order condition is given by

$$
x^* = (\Omega)^{-1} \left[-\frac{J_W}{W J_{WW}} \Lambda - \frac{J_{Wr}}{W J_{WW}} \left(\sigma \rho e_2 \sigma_r \right) - \frac{J_{W\Pi}}{W J_{WW}} \left(\sigma \rho \xi \Pi \right) \right], \qquad (14)
$$

where $e_2 = [0, 1, 0]'$, $\Lambda \equiv \sigma\lambda$ is given above, and $\Omega \equiv \sigma\rho\sigma'$ is the (3×3) variance covariance matrix of the nominal security returns. The vector $\sigma\rho e_2\sigma_r$ denotes the covariances between the security returns and the real interest rate and $\sigma\rho\xi$ is the vector of covariances between the security returns and the realized inflation.

Equation (14) expresses the optimal portfolio as the sum of three portfolios in a form that is familiar from Merton (1973). The first portfolio is proportional to the nominal mean-variance tangency portfolio. The amount invested in the tangency portfolio is inversely related to the investor's relative risk aversion. The second portfolio, $\Omega^{-1}\sigma\rho e_2\sigma_r$, is the portfolio that has the largest correlation with innovations in the state variable r and serves as the hedge portfolio for

[18]In addition to wealth, W, there are three stochastic variables in the economy: r, π and Π, so the following Bellman equation is written out in the general form. Because the investor has power utility function and is only concerned with his real wealth W/Π in this problem, the indirect utility function, J, is separable in real wealth, and the real interest rate r is the only state variable governing the real investment opportunity set. Therefore, the expected inflation, π, will drop from the J function in the rest of the paper.

r. The third portfolio, $\Omega^{-1}\sigma\rho\xi$, is the hedge portfolio for the stochastic price level Π.

The investor's optimal portfolio strategy is given in the following theorem which is proved in the appendix. The certainty equivalent wealth (CEW), an expression of which is given in the theorem, is the sure amount of money at the horizon that the investor would exchange for $1 of curent wealth today and the opportunity to invest up to the horizon. Similarly, a certainty equivalent return (CER) is the riskless return required on one dollar of initial investment so that the investor is indifferent between earning this riskless rate and having the opportunity to invest in all available assets.

Theorem 1 *Optimal allocation and investor welfare with cash, stock and two bonds*

(i) The vector of optimal proportions of wealth allocated to the stock and the two nominal bonds is:

$$\mathbf{x}^* \equiv \begin{bmatrix} x_S^* \\ x_{B_1}^* \\ x_{B_2}^* \end{bmatrix} = \frac{1}{\gamma}\left(\sigma'\right)^{-1}\left[\rho^{-1}\begin{pmatrix} \lambda_S \\ \lambda_r \\ \lambda_\pi \end{pmatrix} + \begin{pmatrix} 0 \\ (1-\gamma)B(t,T)\sigma_r \\ 0 \end{pmatrix} + (\gamma-1)\begin{pmatrix} \xi_S \\ \xi_r \\ \xi_\pi \end{pmatrix}\right], \quad (15)$$

where T is the investment horizon, and ξ's are defined in equation (10). Since λ_i are the constant market prices of risk associated with innovations in the **correlated** *stochastic variables i ($i = S$, r, π and Π) with correlation matrix ρ, $\rho^{-1}\lambda$ is the vector of the* **orthogonalized** *constant market prices of risk associated with stock price, the real interest rate and expected inflation.*

(ii) The investor's expected utility at time t under the optimal policy $\mathbf{x}^*$, $J^*(W_t, r_t, \Pi_t, t)$, is

separable in real wealth, $w_t \equiv W_t/\Pi_t$*, and can be written as:*

$$J^*(W, r, \Pi, t) \equiv E_t \left\{ \frac{(w_T^*)^{1-\gamma}}{1-\gamma} \right\} = \frac{(w_t)^{1-\gamma}}{1-\gamma} \exp^{(1-\gamma)[B(t,T)r_t + d(t,T)]} \tag{16}$$

where w_T^* *is the real wealth at period* T *under the optimal policy, and expressions for* $B(t,T)$ *and* $d(t,T)$ *are given in the appendix.*

(iii) The investor's certainty equivalent wealth is given by

$$CEW^* = \exp\{B(t,T)r_t + d(t,T)\} \tag{17}$$

(iv) The investor's continuously compounded certainty equivalent rate of return is

$$CER^* = \ln CEW^*/(T-t) = \frac{B(t,T)}{T-t} r_t + \frac{d(t,T)}{T-t} \tag{18}$$

Part (i) of Theorem 1 gives the optimal allocation to stock and two nominal bonds. We note that the demand for stock in this economy is independent of the stochastic process for the interest rate or the investment horizon. This is because of the availability of the nominal bonds, which the investor can use to hedge the stochastic investment opportunity set, and the stock allocation can then be under a constant opportunity set when the investor has power utility function. If the investor can invest only in stock and cash, the optimal stock allocation will be horizon dependent and include a term that corresponds to the hedge for the stochastic interest rate.

Part (ii) shows that the investor's expected utility depends only on current real wealth, the investment horizon and the current value of the real interest rate. Part (iii) and (iv) of the theorem imply that since $B(t,T) > 0$, the certainty equivalent wealth (CEW) and the certainty equivalent return are increasing in the real interest rate, r, which measures the favorableness of future investment opportunities.

B. Optimal Allocation Without Bond Market

If the investor can invest only in stock and cash, the control variable, x, becomes a scalar corresponding to the proportion of wealth invested in the stock, x^*_{NB}, and the variance-covariance matrix, Ω, is replaced by ${\sigma_S}^2$, and the risk premium vector, Λ, becomes the scalar, $\sigma_S \lambda_S$. The following theorem summarizes the results for this problem and the details of proof are given in the appendix:

Theorem 2 *Optimal allocation and investor welfare with only cash and stock*

(i) The optimal allocation to stock, ${x_{NB}}^*$*, is given by*

$$x^*_{NB} = \frac{1}{\gamma \sigma_S^2} \left[\sigma_S \lambda_S + (1-\gamma) B(t,T) \sigma_{Sr} - (1-\gamma) \sigma_{S\Pi} \right], \tag{19}$$

where σ_{Sr} *(*$\sigma_{S\Pi}$*) are the covariances between the stock return and real interest rate* r *(realized inflation,* Π*).*

(ii) The indirect utility function, $J^{NB}(W, r, \Pi, t)$*, is given by:*

$$J^{NB}(W, r, \Pi, t) = J^*(W, r, \Pi, t) \exp^{-\frac{1}{2\gamma} h(t,T,dz)' (\mathbf{I} - e_1 e_1' \rho) h(t,T,dz)}. \tag{20}$$

where J^* *is given in equation (16) and* $h(t, T, dz)$ *given in equation (B3).*

(iii) Let CEW^{NB} denote the certainty equivalent wealth of the investor who does not have access to the bond market. Then the ratio of the CEW of such an investor to that of an investor with full access to both stock and bond markets is

$$
\begin{aligned}
CEWR_1 &\equiv \frac{CEW^{NB}}{CEW^*} = \exp\left\{-\frac{(1-\gamma)^2}{2\gamma}(1-\rho_{Sr}^2)var\left(\int_t^T r_s ds\right)\right. \\
&- \frac{(1-\gamma)\sigma_r}{\gamma\kappa}\left[(\gamma\xi_r-\phi_r)(1-\rho_{Sr}^2)+(\gamma\xi_\pi-\phi_\pi)(\rho_{r\pi}-\rho_{Sr}\rho_{S\pi})\right](T-t-B(t,T)) \\
&- \frac{1}{2\gamma}\left[(\gamma\xi_r-\phi_r)^2(1-\rho_{Sr}^2)+2(\gamma\xi_r-\phi_r)(\gamma\xi_\pi-\phi_\pi)(\rho_{r\pi}-\rho_{Sr}\rho_{S\pi})\right. \\
&+ \left.\left.(\gamma\xi_\pi-\phi_\pi)^2(1-\rho_{S\pi}^2)\right](T-t)\right\} \qquad (21) \\
&= \exp\left\{-\frac{1}{2\gamma}h(t,T,dz)'(\mathbf{I}-e_1e_1'\rho)h(t,T,dz)\right\} < 1. \qquad \textit{for}\ \ \gamma>0 \qquad (22)
\end{aligned}
$$

(iv) The difference between the certainty equivalent returns with and without the bond market is

$$
\begin{aligned}
CERD_1 &\equiv \frac{\ln CEW^S - \ln CEW^*}{T-t} \\
&= -\frac{1}{2\gamma(T-t)}h(t,T,dz)'(\mathbf{I}-e_1e_1'\rho)h(t,T,dz) \\
&< 0 \qquad \textit{for}\ \ \gamma>0 \qquad (23)
\end{aligned}
$$

Now the optimal stock allocation has three components: first, the myopic mean-variance efficient allocation, $\frac{1}{\gamma}\frac{\lambda_S}{\sigma_S}$; second, the allocation to hedge r, $\frac{1-\gamma}{\gamma\sigma_S^2}\sigma_{Sr}B(t,T)$, which depends on the investment horizon via $B(t,T)$ and the covariance between the stock return and r, σ_{Sr}; third, the allocation to hedge inflation, $-\frac{1-\gamma}{\gamma\sigma_S^2}\sigma_{S\Pi}$.

The indirect utility function for the investor without access to the bond market is of the same functional form as the one given in equation (16). It depends on the state variable r in the same way, and the utility differential is a deterministic function of the investment horizon. An explicit expression for the welfare loss is given in part (iii).

The inequality (22) implies that investor utility can be improved by developing a bond market. In order to understand why the bond market is important, note that the first term in equation (21) involves $(1-\rho_{Sr}^2)var\left(\int_t^T r_s ds\right)$: this is the component of the variance of the cumulative interest rate which cannot be hedged by the stock return and would disappear if the stock return were perfectly correlated with r. The welfare loss associated with this term is increasing in the horizon and the volatility of the real interest rate process, and decreasing in the real interest rate mean reversion parameter.[19] The welfare loss captured in the second term arises because the investor can only partially hedge risks in the price level. The welfare loss in the third term is due to the fact that the investor only holds a less efficient myopic portfolio when fewer securities are available.

The welfare loss as measured by the certainty equivalent wealth ratio[20] does not depend on the state of the economy or the volatility of the expected inflation, but does depend on the volatility of unexpected inflation. Therefore, in developing countries without a bond market, it is not the level of inflation that matters, it is the volatility about unexpected inflation that makes the investor worse off.

[19]These results are intuitive while the proof is tedious and omitted.

[20]The certainty equivalent wealth (CEW) is the sure amount of money at the horizon that the investor would exchange for \$1 of curent wealth today and the opportunity to invest up to the horizon.

In summary, the welfare loss arises principally from two sources: first, the optimal mean-variance (myopic) portfolio is less efficient because fewer securities are available; second, the hedge portfolio is less efficient because the stock return is not perfectly correlated with r and Π. The less efficient hedging of risks associated with r amounts for the most of the welfare loss because, although the price level affects the investor's real wealth, r is the only state variable in investors' real opportunity set.

C. Optimal Allocation With an Underdeveloped Bond Market

Suppose next that a bond market exists but that it is relatively underdeveloped so that the investor can only invest in a single bond with maturity $T_1 - t$ in addition to cash and stock. The following theorem summarizes the investor's optimal portfolio and utility:

Theorem 3 *Optimal allocation and investor welfare with cash, stock and a single bond*

(i) The optimal proportion of wealth invested in the stock and the single bond is:

$$x^*_{S,B_1} = \frac{1}{\gamma}\hat{\Omega}^{-1}\left(\begin{bmatrix} \sigma_S\lambda_S \\ \sigma_{B_1}\lambda_{B_1} \end{bmatrix} + (1-\gamma)B(t,T)\begin{bmatrix} \sigma_{rS} \\ \sigma_{rB_1} \end{bmatrix} - (1-\gamma)\begin{bmatrix} \sigma_{\Pi S} \\ \sigma_{\Pi B_1} \end{bmatrix}\right) \tag{24}$$

where $\sigma_{B_1}\lambda_{B_1} = -B_1\sigma_r\lambda_r - C_1\sigma_\pi\lambda_\pi$ *is the risk premium of the bond and* $\hat{\Omega}$ *with rows* $(\sigma_S^2, \sigma_{SB_1})$ *and* $(\sigma_{SB_1}, \sigma_{B_1}^2)$ *is the variance-covariance matrix of the returns of the stock and the single bond.*

(ii) The indirect utility function, $J^{S,B_1}(W, r, \Pi, t)$*, is:*

$$J^{S,B_1}(W, r, \Pi, t) = J^*(W, r, \pi, t)\exp^{-\frac{1-\gamma}{2\gamma}var\left(h(t,T,dz)\sqrt{(\mathbf{I}-\omega'(\omega\rho\omega')^{-1}\omega\rho)}\right)}. \tag{25}$$

(iii) The certainty equivalent wealth ratio for the investor facing underdeveloped bond market versus the optimal one is:

$$CEWR_2 \equiv \frac{CEW^{S,B_1}}{CEW^*} = \exp^{-\frac{1}{2\gamma}h(t,T,dz)'(\mathbf{I}-\omega'(\omega\rho\omega')^{-1}\omega\rho)h(t,T,dz)} < 1, \tag{26}$$

(iv) The certainty equivalent return difference is

$$\begin{aligned} CERD_2 &\equiv \frac{\ln CEW^{S,B_1} - \ln CEW^*}{T-t} \\ &= -\frac{1}{2\gamma(T-t)}h(t,T,dz)'(\mathbf{I}-\omega'(\omega\rho\omega')^{-1}\omega\rho)h(t,T,dz) < 0. \end{aligned} \tag{27}$$

Part (i) is the optimal allocation to stock and the single available bond. Because the investor cannot perfectly hedge his investment opportunity set by using only one bond and cash, stock again provides additional hedge service. The demand for stock to hedge real interest rate risk is given by

$$\frac{1-\gamma}{\gamma}\frac{\sigma_{B_1}^2\sigma_{rS} - \sigma_{SB_1}\sigma_{rB_1}}{\sigma_S^2\sigma_{B_1}^2(1-\rho_{B_1S}^2)}B(t,T),$$

where σ_{rB_1} is the covariance between the available bond return and r and ρ_{BS} is the correlation between the stock and the bond returns. This again is non zero and horizon dependent. The importance of this hedge demand depends on the covariance between the stock return and the state variable r, σ_{Sr}, beyond that already accounted for by the bond, $\sigma_{SB_1}\sigma_{rB_1}$.

Parts (iii) and (iv) show that restricting the investor to only a single bond reduces his welfare as measured by his certainty equivalent wealth or certainty equivalent return. The reason for the welfare loss is again due to the inadequate hedge instrument available to the investor. Thus, developing a complete bond market helps investors to hedge interest rate and inflation risks effectively and thereby improves welfare.

III. Illustrative Calculations and Results

In this section we provide some illustrative calculations of the magnitude of the welfare loss when an investor is completely or partially prevented from investing in the bond market. Because both state variables, r and π, are unobservable, the model parameters are estimated using a Kalman filter. Data on inflation, stock return and constant maturity zero coupon government bond yields are not readily available,[21] so our illustrative calculations are based on parameter estimates from U.S. data.

Brennan and Xia (2001) estimate the same model using monthly data on CPI inflation and eleven constant maturity U.S. Treasury discount bond yields with maturities of 1, 3, 6, and 9 months, and 1, 2, 3, 4, 5, 7 and 10 years for the period from January 1970 to December 1995.[22] Campbell and Viceira (1998) estimate a similar discrete time model using a Kalman filter and a slightly different data set. They use quarterly nominal zero-coupon yields at maturies 3 months, 1, 3, and 10 years from 1952 to 1996, and provide estimates for the whole sample and a sub-sample. Their estimates are transformed to our notation and reported in Table I. We note that the estimates depend on the sample period: for example, estimates of κ vary from 0.06 to 0.63 for

[21] We generally can use the nominal risk free rate and the realized inflation rate to filter out the state variables, r and π, and estimate the parameters. However, we are unable to obtain an estimate of the risk premiums for real interest rate and expected inflation risk using just the nominal risk free rate and realized inflation rate. Since we provide analytical solutions to the welfare loss, the shortcoming of using only U.S. data is somewhat mitigated.

[22] Brennan and Xia also estimate the model using annual data on nominal risk free rate and inflation, and get that $\kappa = 0.107$ and $\sigma_r = 0.013$.

different sample period and estimates of σ_r range from 1% to 3.3% per year. In general, the two sets of estimates are consistent with each other.

Using parameter estimates from Brennan and Xia (2001), we calculate the certainty equivalent return differences between a complete bond market and no bond market, $CERD_1$, and the certainty equivalent return differences between a complete bond market and an underdeveloped bond market, $CERD_2$, given in equations (23) and (27). The results for both $\kappa = 0.631$ and $\kappa = 0.105$ are reported in Table II. First, the certainty equivalent return difference is quite large, even for a myopic investor. This reflects the fact that the reduced menu of available securities leads to a less efficient mean-variance optimal portfolio. The certainty equivalent return shortfall varies from 0.15% per year for $\gamma = 15$ to 1.35% per year for $\gamma = 1.5$. Secondly, while the magnitude of the shortfall decreases with the intensity of mean reversion of the real interest rate, it remains substantial even when the intensity is high. Thirdly, the annual shortfall increases with the investment horizon, and it is as high as 1.12% per year for $T = 20$ years, $\kappa = 0.631$ and $\gamma = 5$. Fourthly, even the availability of a single bond maturity contributes to investor welfare in a material way. For example, the above mentioned investor with access to a single bond has a certainty equivalent return shortfall of only 0.74% per year, as compared to the 1.12% shortfall when he has no access to the bond market at all. When $\kappa = 0.105$, the improvement is even more substantial. For example, the same investor has an annual shortfall of 6.09% per year without any bond investment and the shortfall is reduced to 1.57% when he can invest in a single bond. Finally, the welfare loss is (generally) increasing with the level of risk aversion except for investors with short horizons.

Figure 1 plots the certainty equivalent wealth ratios for no bond market and single bond,

$CEWR_1$ and $CEWR_2$, given in equations (22) and (26), as functions of the investment horizon. The ratios are always below unity and decrease with the investment horizon. For long horizon investors, $CEWR_1$ is only about 0.1, meaning that the investor welfare loss due to the absence of a bond market is approximately equivalent to a 90% reduction in the initial wealth. For investors with access to a single bond, the ratio is higher, but still only slightly above 0.5. Figure 2 plots the certainty equivalent wealth ratios as functions of the speed of mean reversion in the real interest rate. The ratios are again all below unity, but increase with the speed of mean reversion. When $\kappa = 0.95$, the two ratios are approximately 0.8 and 0.9, respectively. Therefore, even though the shocks to the real interest rate only have a half life less than three quarters, the investor still has a welfare loss of about 20% and 10% of his wealth in the respective cases of no bond market and an underdeveloped bond market.

As we discussed earlier, the investor welfare loss comes from two sources: the less efficient mean-variance portfolio and the reduced effectiveness in hedging the real interest rate and the price level. The correlation between stock returns and r, ρ_{Sr}, directly affects how effectively the investor can hedge r and indirectly affects how effectively[23] the investor can hedge Π. Thus, we expect a change in ρ_{Sr} to have a major effect in the certainty equivalent wealth ratios. Figure 3 plots the certainty equivalent wealth ratios as ρ_{Sr} varies between negative one and one. The ratio is close to one, meaning that the absence of long term bonds imposes virtually no cost on investors, when the stock is a good hedge against the real interest rate risk. It is at a minimum when the correlation is zero. At that point, $CEWR_1$ is only 0.28 so that the absence of a bond market is equivalent to a reduction of 70% on the investors initial wealth! The estimated

[23]This is because $\sigma_\Pi dz_\Pi$ is decomposed into four terms: $\xi_S dz_S$, $\xi_r dz_r$, $\xi_\pi dz_\pi$ and $\xi_u dz_u$, and the covariance between stock returns and price level depends on the correlation ρ_{Sr} and $\rho_{S\pi}$.

correlation is -0.13, which yields a certainty equivalent wealth ratio of 0.3.

Since the correlation between stock returns and π, $\rho_{S\pi}$, affects only indirectly how effectively the investor can hedge the price level, the effect of $\rho_{S\pi}$ on the certainty equivalent wealth ratios is smaller than that of ρ_{Sr}. Figure 4 plots the ratios as $\rho_{S\pi}$ is varied. $CEWR_1$, the certainty equivalent wealth ratio for an investor who is precluded from holding any bonds, is virtually independent of the correlation because the effect of $\rho_{S\pi}$ is overshadowed by the effect of ρ_{Sr} which is fixed at -0.13. $CEWR_2$, the certainty equivalent wealth ratio for an investor who is restricted to holding a single bond, shows the same pattern as we observe in Figure 3. It is U-shaped and close to unity when the correlation is either negative one or one. In this case, because the investor can more effectively hedge r using the single bond, the effect of ρ_{Sr} becomes less important and the effect of $\rho_{S\pi}$ becomes noticeable. Thus, it is the correlation between the stock return and the real interest rate that plays an important role.

Figure 5 plots the certainty equivalent wealth ratios as the volatility of unexpected inflation varies from zero to 5%, assuming that unexpected inflation is perfectly correlated with the innovations in expected inflation. $CEWR_1$ (no bond market) is almost flat at around 0.3, indicating that the volatility of unexpected inflation has very small effect on the welfare loss of such an investor. Even when the volatility is zero, the welfare loss is still quite substantial. This is because the predominant factor affecting the investor's welfare is how well the investor can hedge real interest rate risk, and the effect of the unexpected inflation volatility is overshadowed by it. For an investor with access to a single bond, the CEW ratio decreases or the welfare loss increases with the volatility. When the investor has some limited access to the bond market, a higher volatility of unexpected inflation makes the investor worse off. For such an investor, the effect of unexpected inflation volatility on investor welfare is of the same order of importance as the effect of how effectively the investor hedges the remaining real interest rate risk using stock.

Finally, we assess the impact of real interest rate volatility on welfare loss. Figure 6 plots the certainty equivalent wealth ratios when the volatility of real interest rate varies from zero to 7%. $CEWR_1$ quickly goes to zero when real interest rate volality increases: the welfare loss is equivalent to about 10% reduction of his initial wealth when $\sigma_r = 0$, but it quickly goes to a reduction of almost 100% of his initial wealth when σ_r is 5%. $CEWR_2$ also decreases with σ_r, but it levels off when σ_r is around 2.5%. $CEWR_2$ is bounded from below at around 0.7, so the investor with access to a single bond has at most a welfare loss of about 70% of his initial wealth.

IV. The Effect of Portfolio Constraints

We have shown that the welfare effect of introducing a bond market is quite significant for long horizon risk averse investors. The certainty equivalent return difference can be as large as 25% per year when no bond is available and close to 9% when a single bond is available. These results are calculated assuming that the investor does not face any short sales constraints. Inspecting the optimal portfolio allocations when bonds are available reveals that the investor holds highly leveraged positions. For example, if the investor invests in cash, stock, three-year and ten-year bonds, the investor borrows the three-year bond, and the short positions are as high as 300% of his wealth. The long position in the ten-year bond is more than ten times his wealth. Such highly leveraged positions are obviously not feasible in reality. Therefore, the results reported in the previous section serve as an upper bound of the welfare effect of the bond market. In this section, we show the effect of imposing short sales and leverage constraints. Given that some short positions are allowed in many markets, the results in this section serve as a lower bound of the importance of bond market.

Table III reports the annual certainty equivalent return when short sales and leverages are prohibited. The parameters are the same as for Table II. First, the role of the bond market remains quite important. Even with very high mean reversion of the real interest rate, the loss for an investor without access to the bond market is about $3-5\%$ per year. The certainty equivalent wealth difference ranges from 4% to as high as 14% when the shocks to real interest rate are quite persistent ($\kappa = 0.105$). Thus, the absence of good instruments to hedge real interest rate and expected inflation risk has a substantial effect on investor welfare. Second, introducing a single bond into the economy can now eliminate almost all the welfare loss. For most investors with reasonable level of risk aversion and length of investment horizons, the annual certainty equivalent return difference is less than 100 basis points. This indicates that that the single bond is a much better hedge for investment opportunity set than stocks in this economy. Although the annual certainty equivalent return difference is small, it should be interpreted together with the stringent portfolio constraints. For many investors who can take some short positions, the annual certainty equivalent return can be larger.

Imposing portfolio constraints significantly reduces the welfare importance of the bond market since investors tend to hold highly leveraged positions when they have full access to bonds. However, the investor is made materially better off when only one bond is introduced, indicating the important hedge role long term bonds is to long horizon investors. Further improvement is smaller but still meaningful when more bonds are introduced.

V. Conclusion

We have analyzed the investment problems of risk averse long horizon investors when there is either no, or only limited, access to the bond market. This problem was analyzed in a simple setting in which the investor faces a stochastic real interest rate, stochastic inflation and stochastic expected inflation. When investors cannot invest in long term (nominal) bonds, they will use stock as an alternative hedge instrument to hedge the stochastic investment opportunity set, and they suffer a welfare loss relative to their counterparts with full access to the (nominal) bond market.

The investor's welfare loss is represented in two ways. First, we calculate the certainty equivalent wealth ratio for investors with restricted access to the bond market versus those with full access to the (nominal) bond market. This ratio is always less than one. Second, we calculate the certainty equivalent return difference for investors with and without access to the bond market. We have shown that the welfare loss increases with the investment horizon and the volatility of the real interest rate process and decreases with the intensity of mean reversion of the real interest rate. For long investment horizons, the welfare loss also increases with investor's level of risk aversion.

To gauge the importance of bond markets, we calibrate our model to U.S. data and provide some illustrative calculations. We find that welfare losses are quite substantial for risk averse long horizon investors. For an investor with a twenty-year investment horizon and risk aversion parameter of five, the certainty equivalent return loss is over one percent per year when there is no access to the bond market and is still 0.74% even when there is only a single bond maturity. This is even more significant when we note that this happens in an economy with $\kappa = 0.631$, which implies that shocks to the real interest rate have only a half life of five quarter. The loss is much larger in an economy with persistent real interest rate shocks: the certainty equivalent return difference is as large as 6% and 1.6% per year respectively.

Because investors tend to hold highly leveraged positions to hedge their investment opportunity set, we also calculate the certainty equivalent return differences when investors are constrained to non negative portfolio allocations. The certainty equivalent return difference is still large for an investor with no access to the bond market, but introducing a single bond substantially reduces the welfare loss. Even in the latter case, the certainty equivalent return difference can still be as high as 81 basis points per year.

The simple partial equilibrium analysis in this paper shows that the availability of long term fixed income instruments can have a very significant effect in improving investor welfare. This points to the need to develop bond markets in economies when they do not exist already. Future work will address the role of bond markets in a general equilibrium seeting, in which the introduction of new securities can affect the prices of existing securities.

Appendix

A. Proof of Theorem 1

Conjecture that the optimized indirect utility function of the investor is of the following form

$$J(W, r, \Pi, t) = \frac{(W/\Pi)^{1-\gamma}}{1-\gamma} \exp^{(1-\gamma)[c(t,T)r_t + d(t,T)]} \tag{A1}$$

where $c(t, T)$ and $d(t, T)$ will be determined by method of undetermined coefficient. Calculating J_W, J_{WW}, $J_{W\Pi}$ and J_{Wr} and substituting them into equation (14), we get the first order condition

$$\begin{aligned} x^* &= \frac{1}{\gamma}\Omega^{-1}\Lambda + \frac{(1-\gamma)c(t,T)}{\gamma}\left(\Omega^{-1}\sigma\rho e_2\sigma_r\right) - \frac{1-\gamma}{\gamma}\left(\Omega^{-1}\sigma\rho\xi\right) && \text{(A2)} \\ &= \frac{1}{\gamma}\Omega^{-1}\Lambda + \left(1 - \frac{1}{\gamma}\right)\Omega^{-1}\sigma\rho\left(\xi_S, \xi_r - c(t,T)\sigma_r, \xi_\pi\right)' && \text{(A3)} \end{aligned}$$

Substituting equation (A2) into the Bellman equation (13), we have the following partial differential equation:

$$\begin{aligned} 0 &= (1 - \kappa c(t,T) + \partial c/\partial t)\, r + \partial d/\partial t + \kappa\bar{r}c(t,T) + \frac{1}{2}(1-\gamma)\sigma_r^2 c^2(t,T) \\ &- \frac{\gamma-2}{2}\left(\xi'\rho\xi + \xi_u^2\right) - (1-\gamma)\sigma_r c(t,T)\xi'\rho e_2 + \frac{1}{2}\gamma(x^*)'\sigma\rho\sigma(x^*) - \xi'\lambda - \xi_u\lambda_u. \quad \text{(A4)} \end{aligned}$$

This can be further simplified into the next two ordinary differential equations:

$$\begin{aligned} \partial c/\partial t &= \kappa c(t,T) - 1 && \text{(A5)} \\ \partial d/\partial t &= -\kappa \bar{r} c(t,T) - \frac{1}{2}(1-\gamma)\sigma_r^2 c^2(t,T) + \frac{\gamma - 2}{2}\left(\xi' \rho \xi + \xi_u^2\right) \\ &+ (1-\gamma)\sigma_r c(t,T)\xi' \rho e_2 - \frac{1}{2}\gamma (x^*)' \sigma \rho \sigma (x^*) + \xi' \lambda + \xi_u \lambda_u && \text{(A6)} \end{aligned}$$

with terminal conditions $c(T,T) = 0$ and $d(T,T) = 0$. Solving this system of ordinary differential equations gives us

$$\begin{aligned} c(t,T) &= \frac{1 - e^{-\kappa(T-t)}}{\kappa} = B(t,T), && \text{(A7)} \\ d(t,T) &= \frac{\phi' \rho \phi (T-t)}{2\gamma} + \left[\bar{r} - \frac{(1-\gamma)\sigma_r \phi' \rho e_2}{\gamma \kappa}\right]\left[(T-t) - B(t,T)\right] \\ &+ \frac{(1-\gamma)\sigma_r^2}{4\gamma\kappa^3}\left[2\kappa(T-t) - 3 + 4e^{\kappa(t-T)} - e^{2\kappa(t-T)}\right] \\ &+ \left[\phi_u \xi_u - \frac{\gamma}{2}\xi_u^2\right](T-t), && \text{(A8)} \end{aligned}$$

where $\phi = \xi - \rho^{-1}\lambda$. Substitute the solutions of $c(t,T)$ and $d(t,T)$ into the first order condition, we can derive the optimal proportion wealth invested in stock and the two nominal bonds:

$$\mathbf{x}^* = \frac{1}{\gamma}\left(\sigma'\right)^{-1}\left[\gamma \xi - \phi + (1-\gamma) B(t,T) \sigma_r e_2\right]. \tag{A9}$$

Therefore, the conjecture of the optimal indirect utility function is confirmed, and the investor's expected utility at time t under the optimal policy $\mathbf{x}^*$, $J^*(W_t, r_t, \Pi_t, t)$, is separable in real wealth, $w_t \equiv W_t/\Pi_t$, and can be written as:

$$J^*(W, r, \Pi, t) \equiv E_t \left\{ \frac{(w_T^*)^{1-\gamma}}{1-\gamma} \right\} = \frac{(w_t)^{1-\gamma}}{1-\gamma} \psi^*(r, t, T) \tag{A10}$$

where w_T^* is the real wealth at period T under the optimal policy, and $\psi_1(r, t, T)$ represents the contribution to the investor's expected utility of the remaining investment opportunities up to the horizon:

$$\psi^*(r, t, T) = \exp^{(1-\gamma)[B(t,T)r_t + d(t,T)]} . \tag{A11}$$

The certainty equivalent wealth is defined as the sure amount given to the investor at the end of the investment horizon so that the investor is indifferent between receiving that amount and receiving $\$1$ at the begining to invest in bonds and stock. It is given by

$$\frac{CEW^{1-\gamma}}{1-\gamma} = \frac{(\$1)^{1-\gamma}}{1-\gamma} \exp^{(1-\gamma)[B(t,T)r_t + d(t,T)]} . \tag{A12}$$

B. Proof of Theorem 2

Substitute equation (19) into (A6), we get that

$$
\begin{aligned}
& d_S(t,T) = d(t,T) - \frac{(1-\gamma)^2}{2\gamma}(1-\rho_{Sr}^2)var\left(\int_t^T r_s ds\right) \\
- \quad & \frac{(1-\gamma)\sigma_r}{\gamma\kappa}\left[(\gamma\xi_r-\phi_r)(1-\rho_{Sr}^2)+(\gamma\xi_\pi-\phi_\pi)(\rho_{r\pi}-\rho_{Sr}\rho_{S\pi})\right](T-t-B(t,T)) \\
- \quad & \frac{1}{2\gamma}\left[(\gamma\xi_r-\phi_r)^2(1-\rho_{Sr}^2)+2(\gamma\xi_r-\phi_r)(\gamma\xi_\pi-\phi_\pi)(\rho_{r\pi}-\rho_{Sr}\rho_{S\pi})\right. \\
+ \quad & \left.(\gamma\xi_\pi-\phi_\pi)^2(1-\rho_{S\pi}^2)\right](T-t) \qquad \text{(B1)} \\
= \quad & d(t,T)-\frac{1}{2\gamma}h(t,T,dz)'(\mathbf{I}-e_1e_1'\rho)h(t,T,dz) \qquad \text{(B2)}
\end{aligned}
$$

where $\mathbf{I}$ is a (3×3) identity matrix, $e_1=(1,0,0)'$, $h(t,T,dz)$ is a function of the investment horizon and the stochastic innovations of the economy:

$$
h(t,T)=(1-\gamma)\int_t^T\frac{1-e^{\kappa(s-T)}}{\kappa}e_2dz(s)-(\gamma\xi-\phi)\int_t^T dz(s). \qquad \text{(B3)}
$$

The second term in equation (B2) is positive because it is of a quadratic form and the matrix $(\mathbf{I}-e_1e_1'\rho)$ is semi-positive definite.

C. Proof of Theorem 3

The optimal allocation to stock and that bond is still given by equation (A2) but with different parameters. The variance-covariance matrix is now given by $\hat{\Omega}\equiv\omega\rho\omega'$ and ω is a (2×3) matrix with rows $(\sigma_S,0,0)$ and $(0,-B_1\sigma_r,-C_1\sigma_\pi)$. The vector of risk premia is now $[\sigma_S\lambda_S,-B(t,T_1)\sigma_r\lambda_r-C(t,T_1)\sigma_\pi\lambda_\pi]'$. Solving equation (A2) with the new parameters gives us equation (24).

Substitute equation (24) into (A6), we get that

$$d_{S,B_1}(t,T) \quad = \quad d(t,T) - \frac{1}{2\gamma} h(t,T,dz)'(\mathbf{I} - \omega'(\omega\rho\omega')^{-1}\omega\rho)h(t,T,dz). \tag{C1}$$

We can verify that the matrix $(\mathbf{I} - \omega'(\omega\rho\omega')^{-1}\omega\rho)$ is semi-positive definite, and the second term (after the minus sign) in the above equation is positive. Therefore, the indirect utility function is derived by substituting the expression for $d_{S,B_1}(t,T)$. The CEW and CED are derived by using their definition and the function for J^{S,B_1}

Table I
Estimates of Model Parameters

This table reproduces the estimation result from Brennan and Xia (2001) (BX) Table I and Campbell and Viceira (2000) (CV) Table I. Results in the second and third columns report Maximum Likelihood BX parameter estimates for the joint process of real interest rate, expected rate of inflation and stock returns estimated by Kalman Filter using monthly yields of eleven U.S. constant maturity treasury bonds, CPI data and CRSP value-weighted stock returns for the period from January 1970 to December 1995. In addition, Brennan and Xia also estimate the model using annual data on nominal risk free rate and inflation, and get that $\kappa = 0.107$ and $\bar{\sigma} = 0.013$, which are not reported. The last four columns report results from CV which estimates a similar discrete time model using a similar procedure and quarterly nominal zero-coupon yields at maturies 3 months, 1, 3, and 10 years from 1952 to 1996. CV est. 1 is based on data from the first quarter of 1952 to the third quarter of 1996 while CV est. 2 is based on data from first quarter of 1983 to the third quarter of 1996. CV estimates are transformed from CV Table I to conform to our notation.

Parameter	BX est.	s.e.	CV est. 1	s.e.	CV est. 2	s.e.
Stock Return Process: $\frac{dS}{S} = (R_f + \lambda_S \sigma_S)dt + \sigma_S dz_S$						
σ_S	0.158	(n.a.)		n.a.		
λ_S	0.343	(0.057)		n.a.		
Real Interest Rate: $dr = \kappa(\bar{r} - r)dt + \sigma_r dz_r$						
$\bar{r}$	0.012	(0.002)	0.014	(n.a.)	0.029	(n.a.)
κ	0.631	(0.003)	0.563	(0.026)	0.056	(0.017)
σ_r	0.026	(0.004)	0.010	(n.a.)	0.033	(n.a.)
λ_r	-0.209	(0.077)				
Expected Inflation: $d\pi = \alpha(\bar{\pi} - \pi)dt + \sigma_\pi dz_\pi$						
$\bar{\pi}$	0.054	(0.023)	0.038	(n.a.)	0.035	(n.a.)
α	0.027	(0.009)	0.003	(0.005)	0.604	(0.100)
σ_π	0.014	(0.005)	0.005	(n.a.)	0.008	(n.a.)
λ_π	-0.105	(0.005)				
Realized Inflation: $d\Pi/\Pi = \pi dt + \sigma_\Pi dz_\Pi$						
σ_Π	0.013	(0.005)	0.016	(n.a.)	0.015	(n.a.)
Parameters of the Pricing Kernel Process: Φ						
ϕ_S	-0.333	(n.a.)		n.a.		
ϕ_r	0.170	(n.a.)		n.a.		
ϕ_π	0.120	(n.a.)				
ρ_{Sr}	-0.129	(n.a.)		n.a.		
$\rho_{S\pi}$	-0.024	(n.a.)		n.a.		
$\rho_{r\pi}$	-0.061	(0.002)	0.048	(n.a.)	-0.298	(n.a.)

Table II

Annual Certainty Equivalent Return Difference without Portfolio Constraints

This table reports the annual certainty equivalent return difference: $CERD_1$ and $CERD_2$ as defined in equations (23) and (27). Certainty equivalent return is the sure annual rate of return such that the investor would be indifferent between receiving this rate of return and having \$1 and the investment opportunities up to the horizon. Panel A reports the ratios when $\kappa = 0.631$ and Panel B reports the ratio for $\kappa = 0.107$ while the other parameters are the same and are given in the second column of Table I. In the case of underdeveloped bond market, we assume that the single available bond has a maturity of ten years.

	Risk Aversion Parameter, γ									
	1.5	3.0	5.0	7.0	15.0	1.5	3.0	5.0	7.0	15.0
Horizon	$CER^S - CER^*$ (% per year)					$CER^{S,B_1} - CER^*$ (% per year)				
Panel A: $\kappa = 0.631$										
1 month	-1.35	-0.68	-0.42	-0.30	-0.15	-0.52	-0.27	-0.16	-0.12	-0.06
1 Year	-1.40	-0.79	-0.56	-0.47	-0.39	-0.56	-0.35	-0.28	-0.25	-0.26
5 Years	-1.50	-1.04	-0.92	-0.92	-1.16	-0.64	-0.54	-0.57	-0.62	-0.93
10 Years	-1.53	-1.12	-1.05	-1.09	-1.48	-0.66	-0.61	-0.67	-0.77	-1.21
20 Years	-1.55	-1.17	-1.12	-1.18	-1.66	-0.68	-0.65	-0.74	-0.85	-1.37
30 Years	-1.56	-1.19	-1.15	-1.22	-1.72	-0.68	-0.66	-0.76	-0.88	-1.42
Panel B: $\kappa = 0.105$										
1 month	-1.35	-0.68	-0.42	-0.30	-0.15	0.00	0.00	0.00	0.00	0.00
1 Year	-1.41	-0.82	-0.59	-0.51	-0.45	0.00	-0.01	-0.02	-0.03	-0.06
5 Years	-1.66	-1.51	-1.71	-2.02	-3.44	-0.02	-0.12	-0.26	-0.41	-1.01
10 Years	-1.93	-2.39	-3.34	-4.36	-8.62	-0.05	-0.32	-0.72	-1.14	-2.83
20 Years	-2.31	-3.81	-6.09	-8.44	-17.99	-0.10	-0.68	-1.57	-2.50	-6.26
30 Years	-2.53	-4.72	-7.91	-11.18	-24.39	-0.14	-0.93	-2.16	-3.44	-8.63

Table III

Annual Certainty Equivalent Return Difference with Portfolio Constraints

This table reports the annual certainty equivalent return difference when the investor is constrained to have portfolio constraints of between zero and one. Certainty equivalent return is the sure annual rate of return such that the investor would be indifferent between receiving this rate of return and having $1 and the investment opportunities up to the horizon. Panel A reports the ratios when $\kappa = 0.631$ and Panel B reports the ratio for $\kappa = 0.107$ while the other parameters are the same and are given in the second column of Table I. In the case of underdeveloped bond market, we assume that the single available bond has a maturity of ten years. The results are calculated numerically by solving the Bellman equation. When $\gamma = 15$ and the horizon is over ten years, the program fails to converge, and we report 'n.a.' in those places. When the investor faces short sales constraints, the certainty equivalant wealth will depend on the value of r, and we assume that $r = 2\%$.

	Risk Aversion Parameter, γ									
	1.5	3.0	5.0	7.0	15.0	1.5	3.0	5.0	7.0	15.0
Horizon	$CER^S - CER^*$ (% per year)					$CER^{S,B_1} - CER^*$ (% per year)				
	Panel A: $\kappa = 0.631$									
1 month	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
1 Year	-3.85	-3.92	-4.93	-3.96	-2.99	-0.00	-0.00	-0.97	-0.00	-0.00
5 Years	-4.81	-4.96	-4.51	-3.99	-3.72	-0.00	-0.00	-0.34	-0.35	-0.54
10 Years	-5.23	-5.66	-4.53	-4.28	-4.10	-0.12	-0.07	-0.29	-0.46	-0.66
20 Years	-5.42	-5.13	-4.52	-4.31	-4.19	-0.21	-0.04	-0.58	-0.81	-1.53
30 Years	-5.49	-5.12	-4.56	-4.32	-4.23	-0.11	-0.02	-0.28	-0.43	-0.80
	Panel B: $\kappa = 0.105$									
1 month	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
1 Year	-3.85	-3.92	-3.92	-3.96	-3.96	-0.00	-0.00	-0.00	-0.00	-0.00
5 Years	-5.71	-5.96	-5.43	-5.37	-6.41	-0.44	-0.00	-0.17	-0.17	-0.18
10 Years	-6.51	-6.57	-6.80	-7.46	-13.95	-0.38	-0.00	-0.07	-0.15	-0.60
20 Years	-7.05	-7.69	-8.93	-10.70	n.a.	-0.41	-0.06	-0.03	-0.13	n.a.
30 Years	-7.30	-8.29	-10.12	-12.97	n.a.	-0.43	-0.08	-0.02	-0.07	n.a.

Figure 1.
Certainty Equivalent Wealth Ratio as a Function of Investment Horizon
($\gamma = 5$ and $\kappa = 0.105$)

This figure plots the certainty equivalent wealth ratio as a function of the investment horizon when the investor does not have access to the bond market or can only invest in one bond with maturity of ten years. The other parameter values are from Brennan and Xia (2001) reported in Table I. The certainty equivalent wealth is the sure amount at the horizon so that the investor is indifferent between having that amount and having \$1 at time 0 and the investment opportunity set up to the horizon.

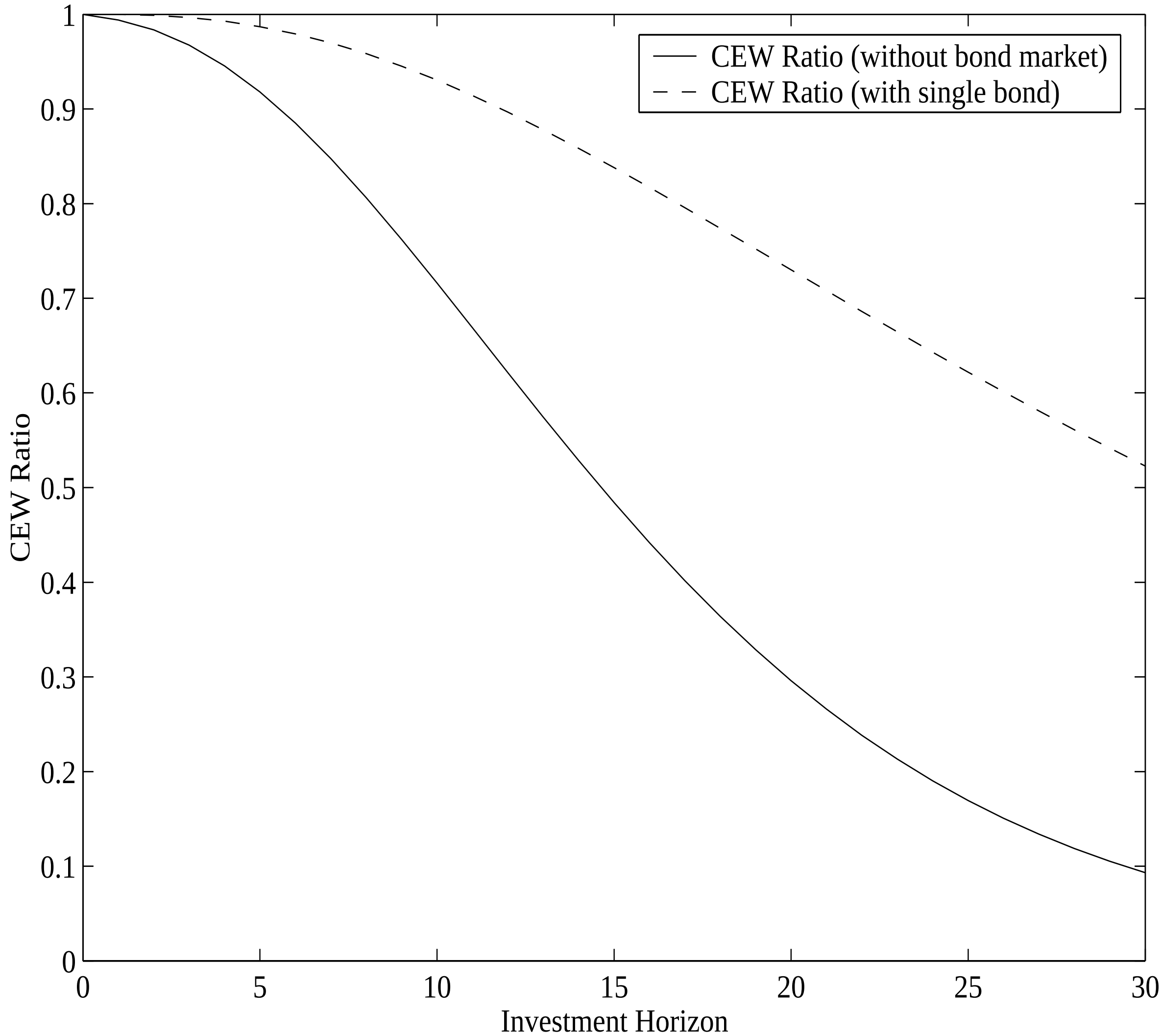


Figure 2.
Certainty Equivalent Wealth Ratio as a Function of the Intensity of Mean Reversion
($\gamma = 5$ and $T = 20$)

This figure plots the certainty equivalent wealth ratio as a function of the speed of mean reversion for the real interest rate when the investor does not have access to the bond market or can only invest in one bond with maturity of ten years. The other parameter values are from Brennan and Xia (2001) reported in Table I. The certainty equivalent wealth is the sure amount at the horizon so that the investor is indifferent between having that amount and having \$1 and the investment opportunity set up to the horizon.

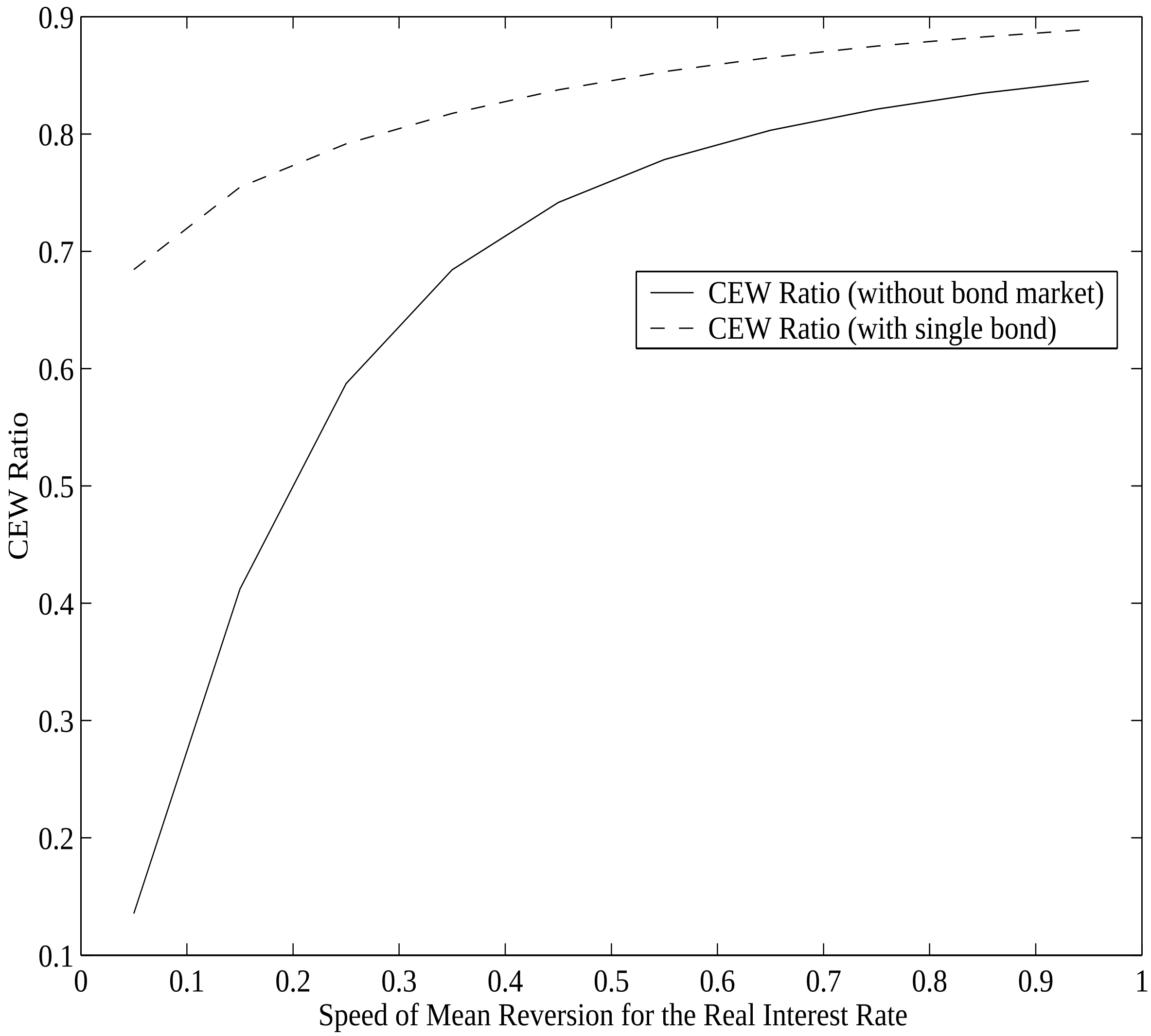


Figure 3.
Certainty Equivalent Wealth Ratio as a Function of the Correlation Between Stock Returns and Real Interest Rate
($T = 20$, $\gamma = 5$ and $\kappa = 0.105$)

This figure plots the certainty equivalent wealth ratio as a function of the correlation between stock returns and real interest rate (ρ_{Sr}) when the investor does not have access to the bond market or can only invest in one bond with maturity of ten years. The other parameter values are from Brennan and Xia (2001) reported in Table I. The certainty equivalent wealth is the sure amount at the horizon so that the investor is indifferent between having that amount and having \$1 and the investment opportunity set up to the horizon.

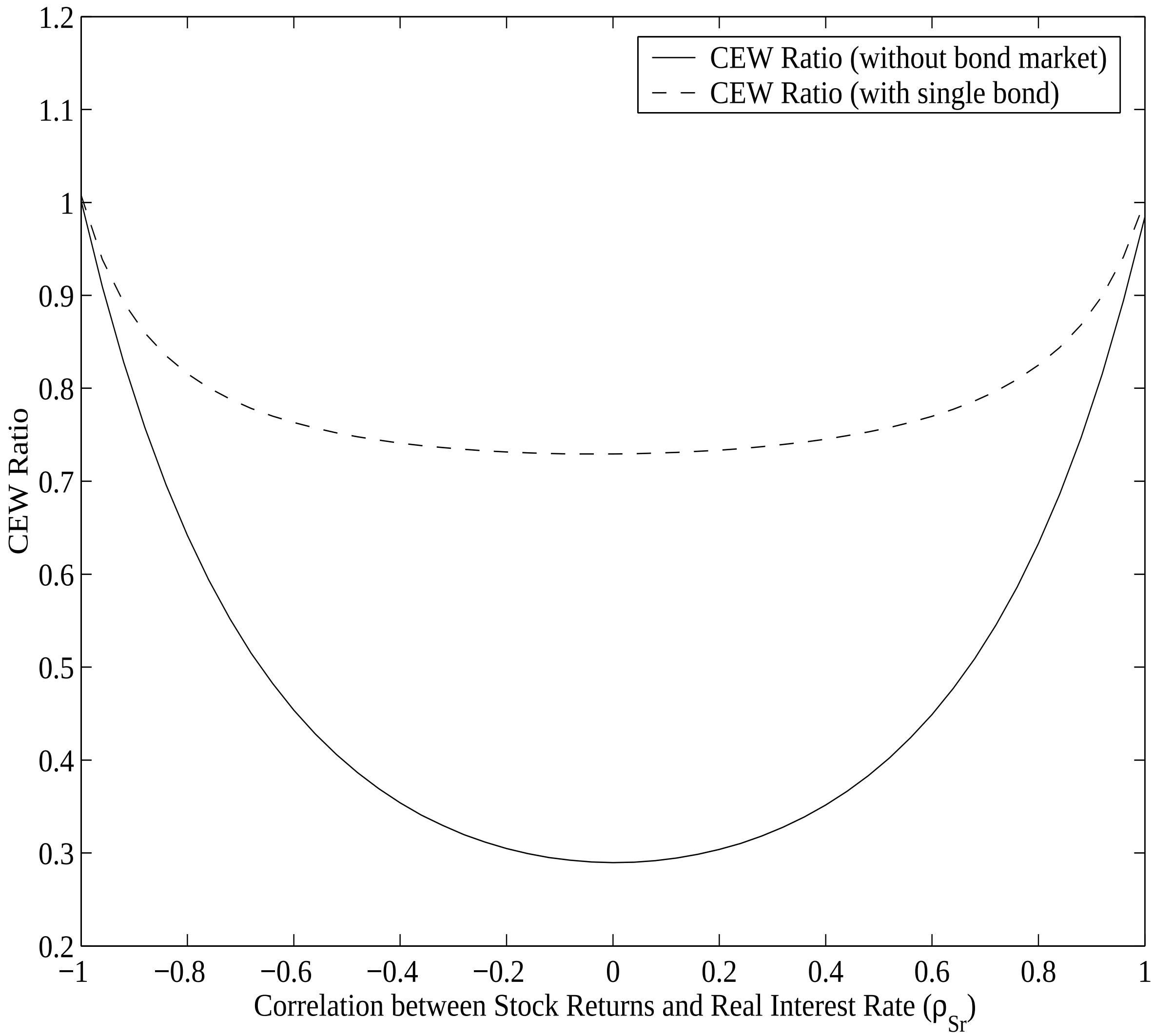


Figure 4.
Certainty Equivalent Wealth Ratio as a Function of the Correlation Between Stock Returns and Expected Inflation
($T = 20$, $\gamma = 5$ and $\kappa = 0.105$)

This figure plots the certainty equivalent wealth ratio as a function of the correlation between stock returns and expected inflation rate ($\rho_{S\pi}$) when the investor does not have access to the bond market or can only invest in one bond with maturity of ten years. The other parameter values are from Brennan and Xia (2001) reported in Table I. The certainty equivalent wealth is the sure amount at the horizon so that the investor is indifferent between having that amount and having \$1 and the investment opportunity set up to the horizon.

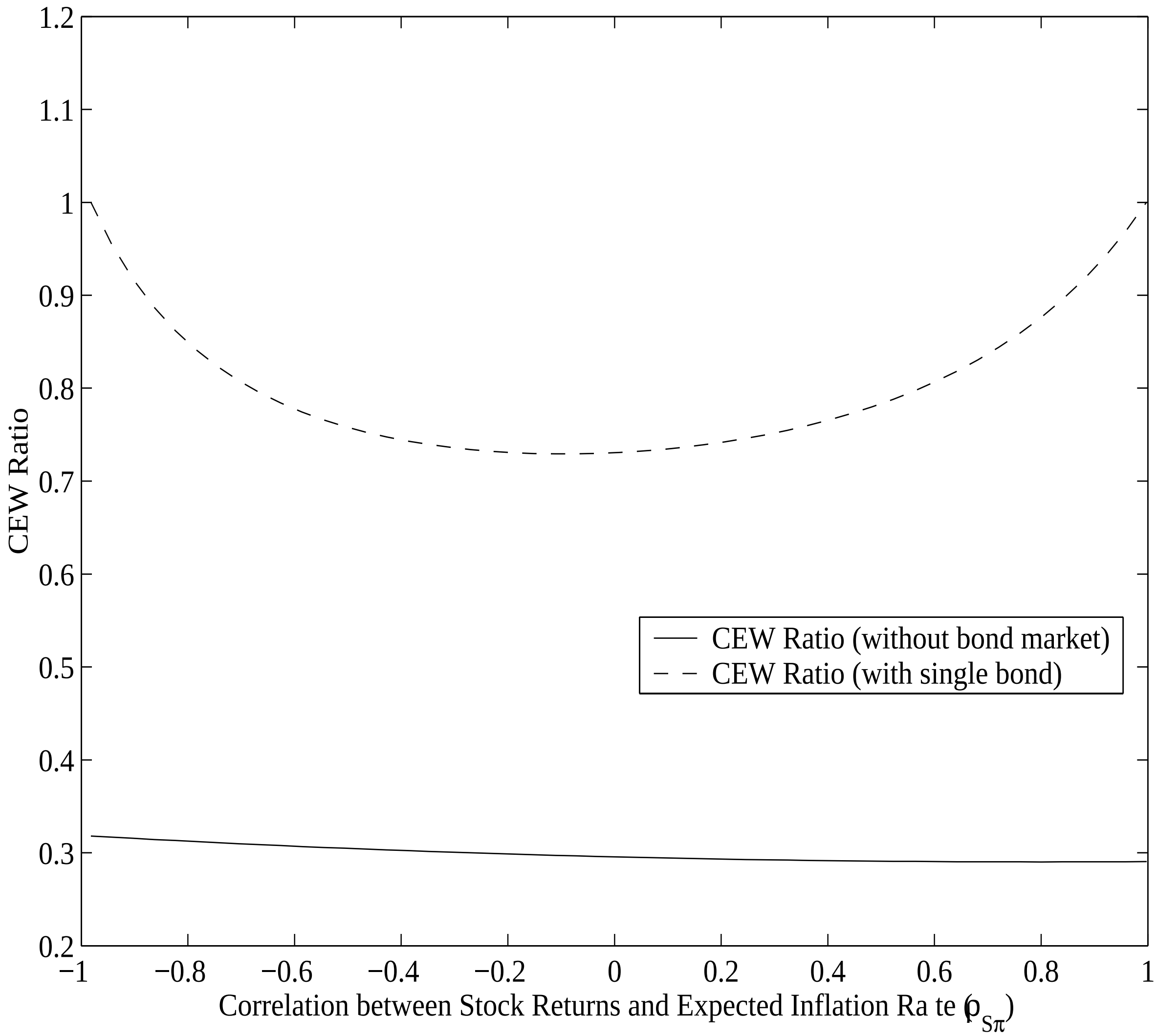


Figure 5.
Certainty Equivalent Wealth Ratio as a Function of the Volatility of Unexpected Inflation
($T = 20$, $\gamma = 5$ and $\kappa = 0.105$)

This figure plots the certainty equivalent wealth ratio as a function of the volatility of unexpected inflation when it is perfectly correlated with expected inflation, i.e., $\sigma_\Pi = \xi_\pi$. The solid line represents the certainty equivalent wealth ratio when the investor does not have access to the bond market, and the dashed line plots the CEW ratio when the investor can invest in a single bond with maturity of ten years. The other parameter values are from Brennan and Xia (2001) reported in Table I. The certainty equivalent wealth is the sure amount at the horizon so that the investor is indifferent between having that amount and having \$1 and the investment opportunity set up to the horizon.

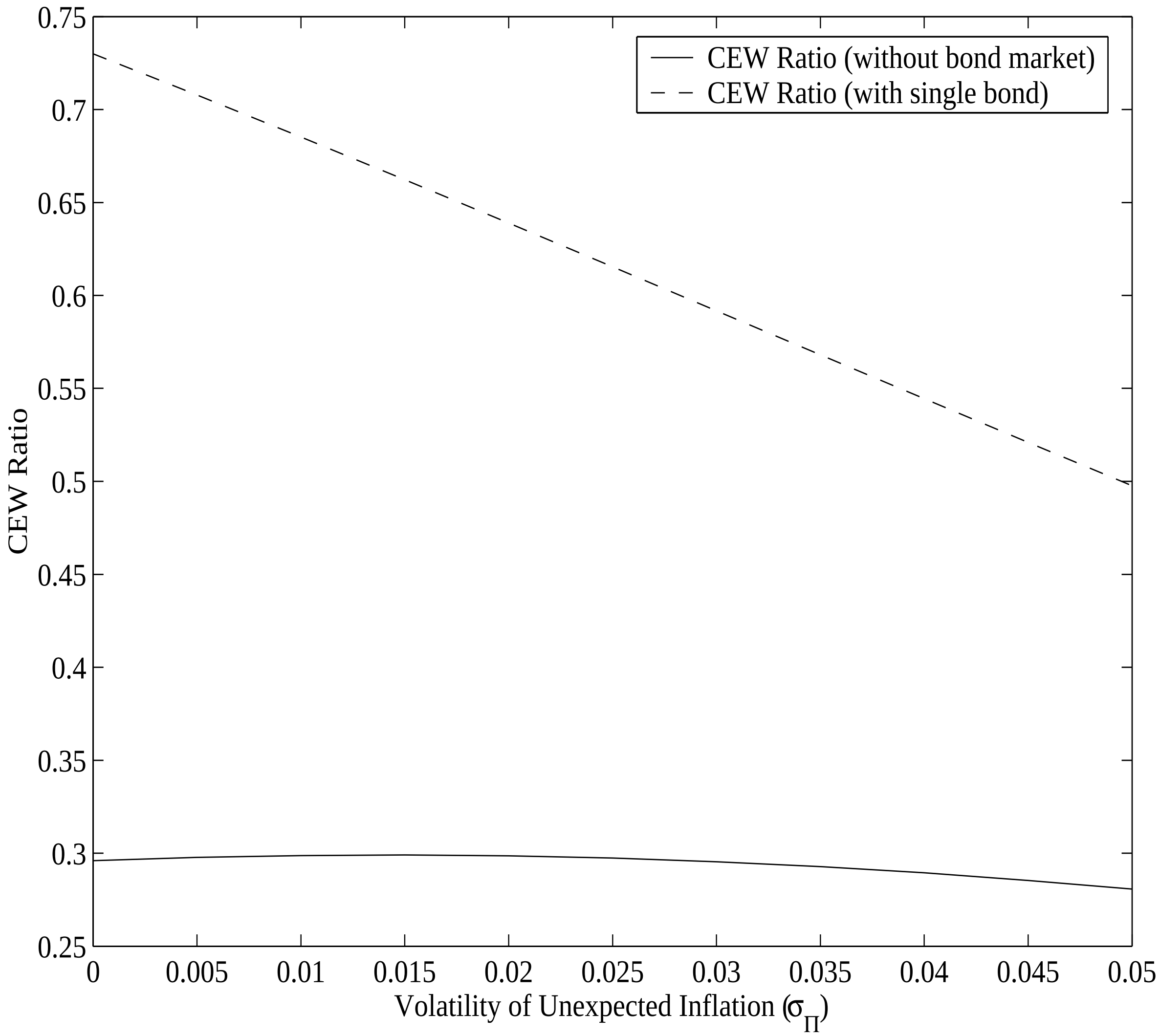
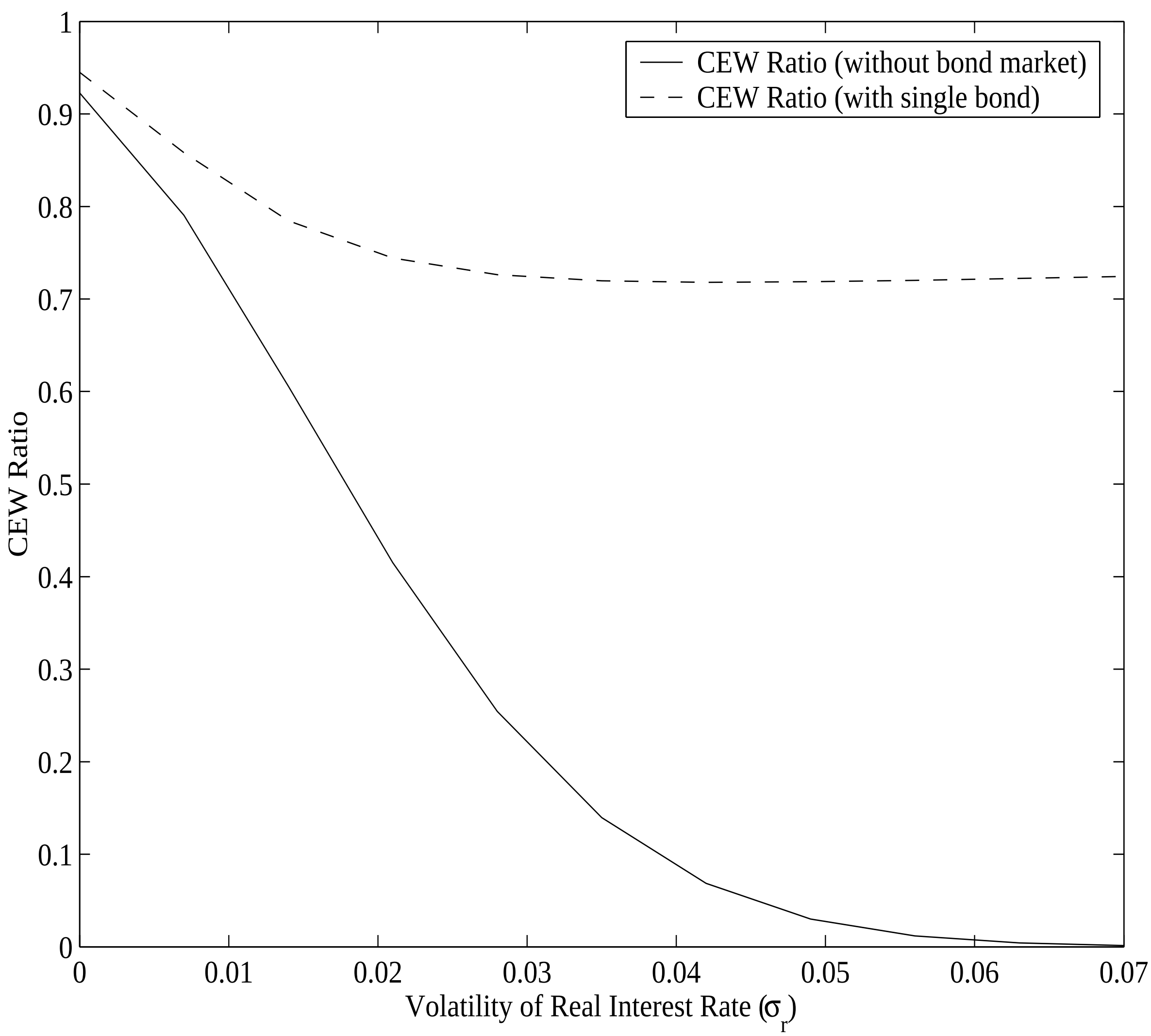


Figure 6.
Certainty Equivalent Wealth Ratio as a Function of the Volatility of Real Interest Rate
($T = 20$, $\gamma = 5$ and $\kappa = 0.105$)

This figure plots the certainty equivalent wealth ratio as a function of the volatility of real interest rate, σ. The solid line represents the certainty equivalent wealth ratio when the investor does not have access to the bond market, and the dashed line plots the CEW ratio when the investor can invest in a single bond with maturity of ten years. The other parameter values are from Brennan and Xia (2001) reported in Table I. The certainty equivalent wealth is the sure amount at the horizon so that the investor is indifferent between having that amount and having \$1 and the investment opportunity set up to the horizon.



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