UNCERTAINTY MODELING

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Introduction

- In most engineering applications, one aims to solve physical problems by converting them into a deterministic mathematical model. This is a rough approximation of reality, as many physical input parameters describing the problem are fixed. In reality however , these parameters, like material properties, boundary conditions etc show some randomness which influences the solution.

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- In order to include this uncertainty in the mathematical model, probabilistic method have been developed. Statistical approaches which require a large sample of random numbers turns out to be costly. PC and SRBM methods are some of the non statistical approaches that address the problem of computational cost and leveraging deterministic codes.

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- Develop robust design solutions.

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  - Spatial uncertainty can be represented by means of random fields (a probabilistic model)
  - The random fields are discretized.
  - Form the stochastic PDE's which results in a system of random algebraic equations

Method to Solve Stochastic PDE’s

Uncertainty analysis techniques

Monte Carlo Simulations

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- Drawback: It requires sufficient number of samples and is computationally costly.

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- Drawback: Only applicable to Gaussian random fields.

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- Reformulation enables application to Non-Gaussian uncertainty models
- Leverages the ideas developed in PC expansions
- With SRBM's accuracy of the projection schemes can be increased by imposing the orthogonality condition.

Stochastic FEM Procedure

An overview

Stochastic FEM in Linear Mechanics

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- In Stochastic FEM, due to introduction of input random variable, the uncertainty is propagated and the basic response quantity is a random vector of nodal displacement U(ø)

Stochastic FEM in Linear Mechanics

- The U(ø) can be expanded as

$$U(ø) = \sum^{\infty}_{i=0} U_i \text{€}_i$$

€ $_i$ i > 0 are multidimensional Hermite polynomials that form an orthogonal basis of L^2 Hilbert space of random variables with finite variance.

Stochastic FEM in Linear Mechanics

- In deterministic case the global stiffness matrix is

  $$K = \text{Đ}_e \int_V B^T \,.\; D \,.\, B \, dv$$

  B is the stain displacement matrix that relates the components of strain to element nodal displacement

  D is the elasticity matrix and

  Đ_e is the assembly procedure over all elements

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- When material properties are described as means of random variable, the D matrix and hence the global stiffness matrix becomes random

Stochastic FEM in Linear Mechanics

- The random global stiffness matrix can be expanded onto the polynomial chaos as follows

$$K = \sum\nolimits^{\infty}_{i=0} K_i \, \text{€}_i$$

Where

$$K_i = \int_V B^T \,.\, E \, D(\text{ø}) \,.\, B \, dv$$

B is deterministic matrix while D(ø) is random.

Stochastic FEM in Linear Mechanics

- So the discretized stochastic equilibrium equation becomes

$$(\sum^{\infty}_{i=0} K_i \text{€}_i) \,.\, (\sum^{\infty}_{i=0} U_i \text{€}_i) = F$$

Assuming the deterministic force vector.

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After truncating the series at M, the residual eq. is

$$E_p = \left(\sum^{\infty}_{i=0} K_i \text{€}_i\right) . \left(\sum^{\infty}_{i=0} U_i \text{€}_i\right) - F$$

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Coefficients U_0, … U_{p-1} are computed using Galerkin method minimising the residual defined above. This is achieved by requiring that this residual be orthogonal to the space spanned by €_i

Stochastic FEM in Linear Mechanics

- This leads to the linear system

$$\begin{bmatrix} K_{0,0} & \cdots & K_{0,p-1} \\ K_{1,0} & \cdots & K_{1,p-1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ K_{p-1,0} & \cdots & K_{p-1,p-1} \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ \cdot \\ \cdot \\ U_{p-1} \end{bmatrix} = \begin{bmatrix} F_0 \\ F_1 \\ \cdot \\ \cdot \\ F_{p-1} \end{bmatrix}$$

Solving the linear system yields the expansion coefficients of the vector of nodal displacements

Thank You