

Estimating Causal Effects of Discrete and Continuous Treatments with Binary Instruments

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Abstract

We propose an instrumental variable framework for identifying and estimating average and quantile effects of discrete and continuous treatments with binary instruments. The basis of our approach is a local copula representation of the joint distribution of the potential outcomes and unobservables determining treatment assignment. This representation allows us to introduce an identifying assumption, so-called *copula invariance*, that restricts the local dependence of the copula with respect to the treatment propensity. We show that copula invariance identifies treatment effects for the entire population and other subpopulations such as the treated. The identification strategy is constructive and leads to straightforward semiparametric estimation procedures based on distribution regression. An application to the effect of sleeping on well-being uncovers interesting patterns of heterogeneity.

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1 Introduction

Endogeneity and heterogeneity are key challenges in causal inference. Endogeneity arises because most treatments and policies of interest are the result of decisions made by economic agents. Heterogeneity also arises naturally as much of the agents’ characteristics is unobserved to the researcher. Accounting for endogeneity and heterogeneity in treatment effects is crucial to answer policy questions, such as how to allocate social resources and combating inequalities. This paper contributes to the literature by proposing a flexible instrumental variable (IV) modeling framework for identifying heterogeneous treatment effects under endogeneity, which yields straightforward semiparametric estimation and inference procedures.

Without additional assumptions, IV strategies cannot point-identify meaningful treatment effects. The literature has proposed different solutions to deal with this challenge that exhibit trade-offs between adding structure to the treatment assignment and potential outcomes. One line of research has restricted the structure and heterogeneity of the potential outcomes while being flexible in treatment assignment mechanisms (e.g., [Chernozhukov and Hansen \(2005\)](#) with binary treatment and [Newey and Powell \(2003\)](#) with continuous treatment). Another line of research has shown the usefulness of restricting the treatment assignment while being flexible regarding how the potential outcomes are formed (e.g., [Imbens and Angrist \(1994\)](#) with binary treatment and [Imbens and Newey \(2009\)](#) with continuous treatment). There are also partial identification solutions that impose less structure on the treatment assignment and potential outcomes (e.g., [Manski \(1990\)](#) and [Balke and Pearl \(1997\)](#) for earlier references, and [Chesher and Rosen \(2020\)](#) for a more recent survey).

We explore an intermediate direction that imposes structure on the relationship between the treatment assignment and potential outcomes to achieve point identification of meaningful heterogeneous treatment effects. The basis of this approach is a local Gaussian representation of the copula of the potential outcomes and unobservables of the treatment assignment. We emphasize that this representation is fully *nonparametric*, that is, it does not require that potential outcomes and treatment unobservables are jointly or marginally Gaussian (see

[Chernozhukov et al., 2020a](#)). Indeed, the bivariate Gaussian structure always holds locally by treating the correlation parameter as an implicit function that equates the bivariate Gaussian distribution with the copula. We use this representation to introduce an assumption that has not been previously considered for identification of treatment effects. This assumption, so-called *copula invariance* (CI), restricts the local dependence of the copula with respect to treatment propensity. We show that, even with a binary IV, copula invariance identifies quantile and average treatment effects (QTE and ATE) of binary and ordered treatments and quantile and average structural functions (QSF and ASF) of continuous treatments for the entire population and other subpopulations such as the treated. The same approach applies without modification to continuous, discrete and mixed continuous-discrete outcomes. When covariates are available, copula invariance is allowed to hold conditional on covariates, which can be viewed as an additional source of heterogeneity in our model.

Our framework is also useful to compare the assumptions of different strategies to identify treatment effects with endogeneity. For example, our approach imposes more restrictions on the dependence structure (i.e., the form of endogeneity), while allowing for richer patterns of effect heterogeneity, compared to [Chernozhukov and Hansen \(2005\)](#) and [Newey and Powell \(2003\)](#), or more heterogeneity of the treatment assignment, compared to [Imbens and Angrist \(1994\)](#) and [Imbens and Newey \(2009\)](#).¹ In this sense, we expand the directions of modeling trade-offs. Another attractive feature of our identification strategy is that it leads to straightforward semiparametric estimation procedures based on distribution regression for both discrete and continuous treatments.

To illustrate the practical usefulness of our approach, we apply it to estimating the distributional effects of sleep on well-being. In this case, sleep time is treated as a continuous treatment. We use the data from the experimental analysis of [Bessone et al. \(2021b\)](#), who studied the effects of randomized interventions to increase sleep time of low-income adults in India. A simple two-stage least squares analysis suggests that sleep has moderate or insignificant average effects on well-being. Using our method, we document interesting patterns of

¹We provide a more detailed comparison with the existing strategies below.

heterogeneity across the distributions of sleep time and well-being that standard analyses focusing on average effects miss.

1.1 Related Literature

The literature on identification of heterogeneous treatment effects with endogeneity is vast. We focus the review on approaches that do not impose distributional assumptions to achieve point identification.

A first strand of literature focused on imposing assumptions in the generation of the potential outcomes, such as rank invariance and rank similarity. Rank invariance imposes that the outcome equation is strictly monotonic in a scalar unobservable such that there is a one-to-one mapping between the potential outcome and unobservable, and the unobservable is the same for all the potential outcomes. This assumption is very convenient for the identification analysis. It yields moment conditions either via invariance of quantile to identify the QTE and ATE in the case of discrete treatments ([Chernozhukov and Hansen \(2005\)](#)) or via invertibility to identify the ASF in the case of continuous treatments ([Newey and Powell \(2003\)](#), [Blundell et al. \(2007\)](#)). [Chernozhukov and Hansen \(2005\)](#)'s rank similarity is slightly weaker than rank similarity as it does not necessarily restrict the unobservable to be the same for all the potential outcomes. However, both assumptions produce the same testable restriction ([Chernozhukov and Hansen \(2013\)](#)). In subsequent work, [Vuong and Xu \(2017\)](#) showed that rank invariance and strict monotonicity are powerful enough to identify individual treatment effects (under suitable regularity conditions) in addition to the QTE and ATE. This literature remains flexible about the treatment selection process. Our CI assumption and rank similarity or invariance are non-nested. The former concerns dependence between potential outcomes and selection unobservables, whereas the latter concerns dependence between potential outcomes. As shown below, CI allows for more general patterns of treatment effect heterogeneity than rank invariance and rank similarity. Also, rank invariance and similarity rely on strict monotonicity on the outcome equation to achieve point identification.

This assumption can only hold for continuous outcomes. Moreover, these approaches rely on completeness conditions on the relationship between the treatment and instrument that rule out, for example, an ordered and continuous treatment when the instrument is binary. CI does not rely on monotonicity nor completeness and therefore can accommodate discrete and mixed discrete-continuous outcomes and ordered and continuous treatments with a binary instrument.

A second strand of literature focuses on assumptions imposed in the treatment assignment, such as a scalar unobservable. [Imbens and Angrist \(1994\)](#) and [Heckman and Vytlacil \(2005\)](#) combined this assumption with monotonicity of potential treatments with respect to a binary instrument to show identification of local and marginal effects of discrete treatments. [Abadie et al. \(2002\)](#) and [Carneiro and Lee \(2009\)](#) extended this approach to the corresponding quantile effects. [Newey et al. \(1999\)](#) and [Imbens and Newey \(2009\)](#) used strict monotonicity of the treatment selection equation with respect to a scalar unobservable and large support of the instrument (ruling out discrete instruments) to identify global treatment effects using a control variable approach. Compared to this strand of the literature, CI restricts the relationship between potential outcomes and treatment assignment, but allows for identifying global treatment effects of discrete and continuous treatments with discrete instruments without relying on monotonicity assumptions on the treatment assignment mechanism.

There are also approaches that combine or modify the assumptions of the previous strands. [Chesher \(2003\)](#) showed identification of the quantile effect of continuous treatments on continuous outcomes with continuous instruments assuming strict monotonicity of the outcome and treatment selection equations with respect to scalar unobservables. Under similar assumptions, [D’Haultfoeulle and Février \(2015\)](#) and [Torgovitsky \(2015\)](#) found that quantiles effects can be identified with discrete instruments. [Newey and Stouli \(2021\)](#) avoided the large support requirement on the instrument of [Imbens and Newey \(2009\)](#) by assuming a parametric structure on the expectation of the treatment conditional on the instrument that enables extrapolation outside the instrument support. Again, these restrictions are different and not nested with CI. In [Appendix A](#), we provide a more detailed comparison of CI with

these and other approaches.

The copula is a powerful tool that has been previously employed in econometrics for identification and estimation. For example, [Chen et al. \(2006\)](#) used a parametric copula to achieve efficient estimation in a class of multivariate distributions. In a semiparametric triangular model with binary dependent variables, [Han and Vytlačil \(2017\)](#) introduced a class of single-parameter copulas to model the dependence structure between the unobservables and established a condition on the copula under which the parameters are identified. They showed that many well-known copulas including the Gaussian copula satisfy the condition. When we restrict our attention to a binary treatment and binary outcome, the current paper’s framework is relevant to [Han and Vytlačil \(2017\)](#). However, while they assumed a parametric copula for the dependence structure, we assume CI. Assuming a Gaussian copula in [Han and Vytlačil \(2017\)](#) can be viewed as an extreme special case of CI. [Han and Lee \(2019\)](#) developed sieve estimation and inference methods based on [Han and Vytlačil \(2017\)](#), and [Han and Lee \(2023\)](#) extended them to semiparametric models for dynamic treatment effects.

[Arellano and Bonhomme \(2017\)](#) and [Chernozhukov et al. \(2020a\)](#) studied IV identification of selection models using assumptions on the copula between the latent outcome and selection unobservable. [Arellano and Bonhomme \(2017\)](#) assumed real analytical copula and continuous instrument. [Chernozhukov et al. \(2020a\)](#) used the local Gaussian representation and copula exclusion, which is a special case of CI, with a binary instrument. Therefore, our framework with binary treatment is closely related to their setup. However, even in the case of binary treatment, the current setting differs from [Chernozhukov et al. \(2020a\)](#) in several dimensions. First, our setting requires two-way sample selection due to the switching of treatment status. Second, we introduce a general selection model that does not follow the typical threshold-crossing structure, which is important to allow for rich selection patterns. Third, because of these features, the identification analysis involves local representation and copula invariance that are specific to treatment status and the value of the IV. More importantly, the use of local representation and CI for ordered and continuous treatments are completely new to this paper. The identification strategies in these two cases are distinct from the binary

case although they rely on the same CI assumption. Finally, in the difference-in-differences setup, [Athey and Imbens \(2006\)](#) showed that average and quantile treatment effects on the treated (and the untreated) can be identified under the time invariance of copula between the counterfactual outcome and treatment, see also [Ghanem et al. \(2023\)](#).

1.2 Organization of the Paper

Section [2.1](#) introduce key variables and parameters of interest, Section [2.2](#) introduces the local Gaussian representation, and Section [2.3](#) posits the main identifying assumptions that will be used throughout the analyses. We devote Sections [3.1–3.3](#) to the identification analyses with binary, ordered, and continuous treatments, respectively. Section [4](#) discusses copula invariance in further detail (e.g., by providing sufficient conditions) and Section [5](#) discusses estimation and inference. Section [6](#) provides the empirical application. The Appendix is organized as follows. Appendix [A](#) compares our identification approach with those in the previous literature. Appendix [B](#) shows how copulas other than Gaussian can also be used for local representation. Appendices [C](#) and [D](#) extend the identification analysis of the main text to understand the role of the support of IVs and of the presence of covariates. Appendix [E](#) presents alternative identification strategies with variants of copula invariance. Appendix [G](#) contains proofs.

1.3 Notation

For scalar random variables X and Y and possibly multivariate random variable Z , $F_{X,Y|Z}$ denotes the joint distribution of X and Y conditional on Z , $F_{X|Z}$ denotes the (marginal) distribution of X conditional on Z , and F_Z denotes the marginal (joint) distribution of Z . We use calligraphic letters to denote support sets of random variables. For example, \mathcal{Z} denotes the support of Z . The symbol \perp denotes (stochastic) independence; for example, $X \perp Y$ means that X is independent of Y . The interior of the set \mathcal{D} is denoted as $\text{int}(\mathcal{D})$.

2 Setup and Assumptions

We consider three classes of models depending on the type of treatment variable: binary, ordered or continuous. Before investigating identification, we introduce the setup, parameters of interest and identifying assumptions that are common to all classes of models.

2.1 Preliminaries

Let $Y \in \mathcal{Y} \subseteq \mathbb{R}$ denote the scalar outcome and $D \in \mathcal{D} \subseteq \mathbb{R}$ denote the scalar treatment. We consider binary, ordered and continuous treatments with $\mathcal{D} = \{0, 1\}$, $\mathcal{D} = \{1, \dots, K\}$ and \mathcal{D} equal to an uncountable set, respectively. The outcome is not restricted, it can be continuous, discrete or mixed continuous-discrete. Let $Z \in \{0, 1\}$ be the binary IV. We focus on a binary instrument as the most challenging case; the analysis readily extends to discrete or continuous Z . Let Y_d denote the potential outcome given $d \in \mathcal{D}$ and D_z the potential treatment given $z \in \{0, 1\}$. They are related to the observed outcome and treatment through $Y = Y_D$ and $D = D_Z$. All the identification analysis is conditional on a vector of covariates $X \in \mathcal{X} \subseteq \mathbb{R}^{d_x}$ for some positive integer d_x . We explicitly incorporate X when we discuss estimation and inference in Section 5.

We consider a general treatment assignment equation:

$$D_z = h(z, V_z), \tag{2.1}$$

where we normalize $V_z \sim U[0, 1]$. We provide examples of the function h for each type of treatment below. By allowing for a different unobservable V_z at each value of z , we essentially permit D to be a function of the *vector* of unobservables (V_0, V_1) . Even this general version of a treatment assignment model is not necessary for our analyses but simplifies the exposition; see Appendix F.

We are interested in identifying the distribution of Y_d , F_{Y_d} , for $d \in \mathcal{D}$, and functionals of

F_{Y_d} , such as quantile and average structural functions. Thus, by using appropriate operators:

$$QSF_\tau(d) \equiv Q_{Y_d}(\tau) = \mathcal{Q}_\tau(F_{Y_d}),$$

$$ASF(d) \equiv E[Y_d] = \mathcal{E}(F_{Y_d}),$$

where $\mathcal{Q}_\tau(F) \equiv \inf\{y \in \mathcal{Y} : F(y) \geq \tau\}$ and $\mathcal{E}(F) \equiv \int_{\mathcal{Y}} [1 - F(y)] dy$. Quantile and average effects can be expressed as $QSF_\tau(d) - QSF_\tau(d')$ and $ASF(d) - ASF(d')$ for a binary or ordered treatment and $\partial QSF_\tau(d)/\partial d$ and $\partial ASF(d)/\partial d$ for a continuous treatment. When the treatment is binary, we may also be interested in the distribution of Y_d in subpopulations such as the treated, $F_{Y_d|D}(\cdot | 1)$, and untreated, $F_{Y_d|D}(\cdot | 0)$, for $d \in \{0, 1\}$, and functionals of these distributions.

2.2 Local Gaussian Representation

Treatment endogeneity can be captured by the joint distribution of the potential outcome and unobservable of the treatment assignment. We use a conditional version of the local Gaussian representation (LGR) to represent such a joint distribution. This representation is the basis of our identification and estimation strategies. Throughout the paper, let $C(u_1, u_2; \rho)$ denote the Gaussian copula with correlation coefficient ρ , that is

$$C(u_1, u_2; \rho) = \Phi_2(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \rho),$$

where $\Phi_2(\cdot, \cdot; \rho)$ is the standard bivariate Gaussian distribution with parameter ρ and Φ is the standard univariate Gaussian distribution.

The following lemma shows that the conditional copula of any bivariate random variable has a local Gaussian representation ([Chernozhukov et al., 2020a](#)).

Lemma 2.1 (LGR). *For any random variables Y , V and Z , the joint distribution of Y and*

V conditional on Z admits the representation:

$$F_{Y,V|Z}(y, v | z) = C(F_{Y|Z}(y | z), F_{V|Z}(v | z); \rho_{Y,V;Z}(y, v; z)), \text{ for all } (y, v, z),$$

where $\rho_{Y,V;Z}(y, v; z)$ is the unique solution in ρ to

$$F_{Y,V|Z}(y, v | z) = C(F_{Y|Z}(y | z), F_{V|Z}(v | z); \rho).$$

In the lemma, note that the solution $\rho_{Y,V;Z}(y, v; z)$ depends on both the dependence structure and the marginals. This distinction becomes important later. Lemma 2.1 can be equivalently stated as the LGR of a copula instead of a distribution; see Section 4.3. Note that Gaussianity is not essential for the local representation. In Appendix B, we provide other copulas that can be used for the representation. Gaussianity, however, is convenient to introduce identifying assumptions, interpret these assumptions using joint normality as a benchmark of comparison, and to develop semiparametric estimators.

2.3 Assumptions

We maintain the following assumptions:

Assumption EX (Independence). For $d \in \mathcal{D}$ and $z \in \{0, 1\}$, $Z \perp\!\!\!\perp Y_d$ and $Z \perp\!\!\!\perp V_z$.

Assumption REL (Relevance). (i) $Z \in \{0, 1\}$; (ii) $0 < \Pr(Z = 1) < 1$; and (iii) for \mathcal{D} being either $\{0, 1\}$ or $\{1, \dots, K\}$, $\Pr(D = d | Z = 1) \neq \Pr(D = d | Z = 0)$ and $0 < \Pr(D = d | Z = z) < 1$ for $(z, d) \in \{0, 1\} \times \mathcal{D}$ and, for uncountable \mathcal{D} , $F_{D|Z}(d | 1) \neq F_{D|Z}(d | 0)$ and $0 < F_{D|Z}(d | z) < 1$ for $(z, d) \in \{0, 1\} \times \text{int}(\mathcal{D})$.

EX is standard in IV strategies. It is weaker than $Z \perp\!\!\!\perp (\{Y_d\}_{d \in \mathcal{D}}, V_0, V_1)$ or $Z \perp\!\!\!\perp (Y_d, V_z)$ for $(d, z) \in \mathcal{D} \times \{0, 1\}$. Also, in EX, a standard exclusion restriction is implicit in the notation: $Y_d = Y_{d,z}$ almost surely, where $Y_{d,z}$ is the potential outcome given (d, z) . REL(ii)–(iii) are the usual relevance condition for the IV and the boundary condition.

We make the following assumption about the local dependence parameter of the LGR of (Y_d, V_z) conditional on Z :

Assumption CI (Copula Invariance). *For $d \in \mathcal{D}$, $\rho_{Y_d, V_z; Z}(y, v; z)$ is a constant function of (v, z) , that is*

$$\rho_{Y_d, V_z; Z}(y, v; z) = \rho_{Y_d}(y), \quad (y, v, z) \in \mathcal{Y} \times \mathcal{V} \times \{0, 1\}.$$

CI is a high-level condition that (together with the other assumptions we maintain) is sufficient for identification in all the cases that we consider, but it is not necessary. We provide weaker conditions for each case in the following section. Section 4 provides more interpretable conditions for CI and compares CI with alternative identifying assumptions that have been used in the literature such as rank invariance and rank similarity.

3 Identification Analysis

3.1 Binary Treatment

We start by considering the identification of the causal effects of a binary treatment $D \in \mathcal{D} = \{0, 1\}$. To reflect this, we consider a treatment selection equation

$$D_z = h(z, V_z) = 1\{V_z \leq \pi(z)\}, \tag{3.1}$$

with propensity score

$$\Pr[D = 1 \mid Z = z] = \Pr[D_z = 1 \mid Z = z] = \Pr[D_z = 1] = \pi(z),$$

by EX and the normalization $V_z \sim U[0, 1]$. Note that V_1 and V_0 are two distinct unobservables that do not restrict the behavior of D_0 and D_1 . The LATE monotonicity assumption of Imbens and Angrist (1994), imposes either $D_1 \geq D_0$ or $D_0 \geq D_1$ almost surely, which corresponds to $V_1 = V_0$ almost surely (Vytlacil, 2002).

For the identification analysis, consider

$$\begin{aligned}\Pr[Y \leq y, D = 1 \mid Z = z] &= \Pr[Y_1 \leq y, D_z = 1 \mid Z = z] = C(F_{Y_1|Z}(y|z), \pi(z); \rho_{Y_1, V_z; Z}(y, \pi(z); z)) \\ &= C(F_{Y_1}(y), \pi(z); \rho_{Y_1, V_z; Z}(y, \pi(z); z)), \quad (y, z) \in \mathcal{Y} \times \{0, 1\},\end{aligned}\quad (3.2)$$

where the second equality uses equation (3.1) and Lemma 2.1, and the last equality follows from EX. For each $y \in \mathcal{Y}$, this is a system of two equations in three unknowns, $F_{Y_1}(y)$, $\rho_{Y_1, V_z; Z}(y, \pi(0); 0)$ and $\rho_{Y_1, V_z; Z}(y, \pi(1); 1)$.

The following theorem shows that the condition

$$\rho_{Y_1, V_1; Z}(y, \pi(1); 1) = \rho_{Y_1, V_0; Z}(y, \pi(0); 0) \equiv \rho_{Y_1}(y), \quad y \in \mathcal{Y}, \quad (3.3)$$

which is implied by CI, identifies the distribution of Y_1 and the local dependence parameter of the LGR of Y_1 and V_z . A similar argument shows that the distribution of Y_0 and the local dependence parameter of the LGR of Y_0 and V_z are identified.

Theorem 3.1. *Suppose $D_z \in \{0, 1\}$ satisfies (3.1) for $z \in \{0, 1\}$. Under Assumptions EX, REL, and CI, the functions $y \mapsto F_{Y_d}(y)$ and $y \mapsto \rho_{Y_d}(y)$ are identified on $y \in \mathcal{Y}$, for $d \in \{0, 1\}$.*

Remark 3.1 (Identification of $F_{Y_1|D}$ and $F_{Y_0|D}$). *We show identification for the treated, $D = 1$, identification for the untreated follows by a similar argument. The distribution of Y_1 is trivially identified from $F_{Y_1|D}(y \mid 1) = F_{Y|D}(y \mid 1)$. Identification of the distribution of Y_0 follows from*

$$F_{Y_0|D}(y \mid 1) = \frac{F_{Y_0}(y) - (1 - \pi)F_{Y|D}(y \mid 0)}{\pi},$$

where $\pi \equiv \Pr[D = 1]$ and $F_{Y_0}(y)$ is identified by Theorem 3.1.

Remark 3.2 (Random Intention to Treat). *Under random intention to treat or one-sided compliance, we have $D = 0$ whenever $Z = 0$ (i.e. $\Pr(D = 0 \mid Z = 0) = 1$), so that REL is violated. In this case $F_{Y_1}(y)$ is no longer identified because one of the equations in (3.2)*

becomes uninformative as $\pi(0) = 0$. We can still identify the distributions of the potential outcomes for the treated using the same analysis of Remark 3.1 because $F_{Y_0}(y) = F_{Y|Z}(y | 0)$. When we impose a stronger version of CI that $\rho_{Y_1}(y) = \rho_{Y_0}(y)$ (i.e., rank similarity), then we can also identify treatment effects for the entire population; see Section A.1 for details.

3.2 Ordered Treatment

We consider identification of the causal effects of a multivalued ordered treatment $D \in \mathcal{D} = \{1, \dots, K\}$ using a binary instrument $Z \in \{0, 1\}$. There are now K potential outcomes denoted as (Y_1, \dots, Y_K) , which are related to the observed outcome as $Y = \sum_{d \in \mathcal{D}} Y_d 1\{D = d\}$. As before, we denote the potential treatments as (D_0, D_1) .

We assume a threshold-crossing model for the treatment selection equation, which can be viewed as a natural extension of model (3.1) from two to multiple treatment levels,

$$D_z = h(z, V_z) = \begin{cases} 1, & \pi_0(z) < V_z \leq \pi_1(z) \\ 2, & \pi_1(z) < V_z \leq \pi_2(z) \\ \vdots & \vdots \\ K, & \pi_{K-1}(z) < V_z \leq \pi_K(z) \end{cases}, \quad (3.4)$$

where $\pi_0(z) = 0$ and $\pi_K(z) = 1$. Equation (3.4) generalizes the model in Section 7.2 of Heckman and Vytlacil (2007) by allowing for a different impact of the instrument on the different cutoffs; see Remark 3.3.

Under the normalization $V_z \sim U[0, 1]$ and EX, the threshold functions $\pi_d(z)$ are identified by the distribution of the observed treatment conditional on the instrument as $\pi_d(z) =$

$F_{D|Z}(d | z)$ for $d \in \mathcal{D}$. For the identification analysis, consider

$$\begin{aligned} F_{Y,D|Z}(y, d | z) - F_{Y,D|Z}(y, d-1 | z) &= \Pr[Y_d \leq y, \pi_{d-1}(z) < V_z \leq \pi_d(z) | Z = z] \\ &= C(F_{Y_d}(y), \pi_d(z); \rho_{Y_d, V_z; Z}(y, \pi_d(z); z)) \\ &\quad - C(F_{Y_d}(y), \pi_{d-1}(z); \rho_{Y_k, V_z; Z}(y, \pi_{d-1}(z); z)), \quad (y, d, z) \in \mathcal{Y} \times \mathcal{D} \times \{0, 1\}, \end{aligned} \quad (3.5)$$

where the first equality follows from (3.4) and the second equality from EX and Lemma 2.1. For each $d \in \mathcal{D}$ and $y \in \mathcal{Y}$, (3.5) is a system of two equations on five unknowns: $F_{Y_d}(y)$, $\rho_{Y_d, V_0; Z}(y, \pi_{d-1}(0); 0)$, $\rho_{Y_d, V_0; Z}(y, \pi_d(0); 0)$, $\rho_{Y_d, V_1; Z}(y, \pi_{d-1}(1); 1)$, and $\rho_{Y_d, V_1; Z}(y, \pi_d(1); 1)$. For $k \in \{1, K\}$, one of the terms in the right hand side drops out because either $\pi_{d-1}(z) = 0$ or $\pi_d(z) = 1$, yielding a system of two equations on three unknowns. The distribution of the potential outcome and local dependence parameter can be identified using the condition (3.3) from the binary treatment case and by REL(iii) because $\pi_d(z) = \sum_{j=1}^d \Pr(D = j | Z = z)$.

For $k \in \mathcal{D} \setminus \{1, K\}$, condition (3.3) reduces the number of unknowns to three but is not sufficient to identify the unknowns. We impose additionally copula invariance between consecutive treatment levels

$$\rho_{Y_d, V_z; Z}(y, \pi_d(z); z) = \rho_{Y_d, V_z; Z}(y, \pi_{d-1}(z); z) \equiv \rho_{Y_d}(y), \quad (y, z) \in \mathcal{Y} \times \{0, 1\}. \quad (3.6)$$

This condition is also implied by CI and reduces the number of unknowns to two: $F_{Y_d}(y)$ and $\rho_{Y_d}(y)$. The Jacobian of the resulting system of equations, however, does not satisfy the conditions to apply the global univalence results of Gale and Nikaido (1965) even under REL(iii). We show existence and uniqueness of solution using an alternative global univalence result of Ambrosetti and Prodi (1995).² To apply this result we make the following sufficient condition on the distribution of the treatment conditional on the instrument:

Assumption U_{OC} (Uniformity in Ordered Choice). *Either $F_{D|Z}(d | 0) < F_{D|Z}(d | 1)$ for all $d \in \mathcal{D} \setminus \{K\}$ or $F_{D|Z}(d | 0) > F_{D|Z}(d | 1)$ for all $d \in \mathcal{D} \setminus \{K\}$.*

²This results was previously used to show identification by De Paula et al. (2019) a different setting.

U_{OC} does not necessarily follow from $REL(iii)$ and imposes the same ordering between $F_{D|Z}(d | 0)$ and $F_{D|Z}(d | 1)$ for all $d \in \mathcal{D} \setminus \{K\}$. Like $REL(iii)$, U_{OC} can be directly tested from the data. Remark 3.3 shows that Heckman and Vytlacil (2007)'s ordered choice model satisfies U_{OC} . It is interesting to see what type of compliance behavior with respect to D_0 and D_1 is ruled out by this sufficient condition. To explore this, define the compliers and defiers of order $j \in \mathcal{D} \setminus \{K\}$ as

$$C_j \equiv \bigcup_{d=1}^{K-j} \{D_0 = d, D_1 = d + j\},$$

$$B_j \equiv \bigcup_{d=1}^{K-j} \{D_1 = d, D_0 = d + j\}.$$

Assumption EG (Exchangeability). V_0 and V_1 are exchangeable, i.e., $C(v_0, v_1) = C(v_1, v_0)$.

Assumption EG states that the distribution for (V_0, V_1) is symmetric; most known copulas are symmetric. Under this assumption, we can interpret Assumption U_{OC} in terms of compliance behavior:

Lemma 3.1 (Compliance Shares). *Under Assumption EG , $\pi_d(1) > \pi_d(0)$ (resp. $<$) for all $d \in \mathcal{D} \setminus \{K\}$ implies that the share of all complier groups is smaller (resp. larger) than the share of all defier groups, that is, $\Pr[\bigcup_{j=1}^{K-1} C_j] < \Pr[\bigcup_{j=1}^{K-1} B_j]$ (resp. $>$).*

The condition about the share of compliers and defiers is reminiscent of a similar assumption used in De Chaisemartin (2017) in the case of binary treatment. Another simple interpretation of Lemma 3.1 can be made under the restriction $V_0 = V_1$. In this special case, we can easily see that there is no defiers (i.e., $\Pr[D_1 < D_0] = 0$) if and only if $\pi_d(1) < \pi_d(0)$ for all $k \in \mathcal{D} \setminus \{K\}$. In general, when V_z is not restricted, Assumption EG alone does not eliminate compliers or defiers. We summarize the identification result:

Theorem 3.2. *Suppose D_z , $z \in \{0, 1\}$, satisfies (3.4). Under Assumptions EX , REL , CI , and U_{OC} , the functions $y \mapsto F_{Y_d}(y)$ and $y \mapsto \rho_{Y_d}(y)$ are identified on $y \in \mathcal{Y}$, for $d \in \mathcal{D}$.*

Remark 3.3 (Comparison with [Heckman and Vytlacil \(2007\)](#)). *Heckman and Vytlacil (2007, Section 7.2) consider an ordered choice model, where the instrument is restricted to shift all cutoffs by the same amount. Suppose that*

$$D_z = \begin{cases} 1, & -\infty < \mu(z) + V \leq \pi_1 \\ 2, & \pi_1 < \mu(z) + V \leq \pi_2 \\ \vdots & \vdots \\ K, & \pi_{K-1} < \mu(z) + V < \infty \end{cases}. \quad (3.7)$$

where $V \mid Z \sim N(0, 1)$. This model is a special case of the model we consider in this section if we normalize V_z .

3.3 Continuous Treatment

Suppose $D \in \mathcal{D} \subseteq \mathbb{R}$ is an uncountable set and $d \mapsto F_{D|Z}(d \mid z)$ is strictly increasing on \mathcal{D} , for $z \in \{0, 1\}$. Assume the treatment selection equation,

$$D_z = h(z, V_z) = F_{D|Z}^{-1}(V_z \mid z), \quad (3.8)$$

where $V_z \sim U(0, 1)$. For the identification analysis, consider

$$F_{Y|D,Z}(y \mid d, z) = F_{Y_d|D_z,Z}(y \mid d, z) = F_{Y_d|V_z,Z}(y \mid F_{D|Z}(d \mid z), z),$$

where the first equality follows from [EX](#) and the second from equation (3.8) and a change of variable. Let $\mu_{d,y} \equiv \Phi^{-1}(F_{Y_d}(y))$ and $\eta_v \equiv \Phi^{-1}(v)$. By the properties of the conditional

distribution, Lemma 2.1, EX and the properties of the Gaussian copula,

$$F_{Y_d|V_z,Z}(y | v, z) = \frac{(\partial/\partial v)F_{Y_d,V_z|Z}(y, v | z)}{(\partial/\partial v)F_{V_z|Z}(v | z)} = \Phi\left(\frac{\mu_{d,y} - \rho_{Y_d,V_z;Z}(y, v; z)v}{\sqrt{1 - \rho_{Y_d,V_z;Z}(y, v; z)^2}}\right) + \phi_2(\mu_{d,y}, v; \rho_{Y_d,V_z;Z}(y, v; z))(\partial/\partial v)\rho_{Y_d,V_z;Z}(y, v; z).$$

Assume that

$$\rho_{Y_d,V_z;Z}(y, F_{D|Z}(d | 1); 1) = \rho_{Y_d,V_z;Z}(y, F_{D|Z}(d | 0); 0) \equiv \rho_{Y_d}(y), \quad y \in \mathcal{Y},$$

and

$$(\partial/\partial v)\rho_{Y_d,V_z;Z}(y, F_{D|Z}(d | z); z) = 0, \quad z \in \{0, 1\},$$

where the differentiability of $v \mapsto \rho_{Y_d,V_z;Z}(y, v; z)$ follows by continuity of (Y_d, V_z) . Note that the previous conditions are implied by CI. Then, combining the previous equations yields

$$\Phi^{-1}(F_{Y|D,Z}(y | d, z)) = a_{d,y} + b_{d,y}\Phi^{-1}(F_{D|Z}(d | z)), \quad z \in \{0, 1\}, \quad (3.9)$$

where $a_{d,y} = \mu_{d,y}/\sqrt{1 - \rho_{Y_d}(y)^2}$ and $b_{d,y} = -\rho_{Y_d}(y)/\sqrt{1 - \rho_{Y_d}(y)^2}$. Equation (3.9) a linear system of two equations on two unknowns: $a_{d,y}$ and $b_{d,y}$, which has solution

$$\begin{aligned} a_{d,y} &= \frac{\Phi^{-1}(F_{Y|D,Z}(y | d, 0))\Phi^{-1}(F_{D|Z}(d | 1)) - \Phi^{-1}(F_{Y|D,Z}(y | d, 1))\Phi^{-1}(F_{D|Z}(d | 0))}{\Phi^{-1}(F_{D|Z}(d | 1)) - \Phi^{-1}(F_{D|Z}(d | 0))}, \\ b_{d,y} &= \frac{\Phi^{-1}(F_{Y|D,Z}(y | d, 1)) - \Phi^{-1}(F_{Y|D,Z}(y | d, 0))}{\Phi^{-1}(F_{D|Z}(d | 1)) - \Phi^{-1}(F_{D|Z}(d | 0))}, \end{aligned} \quad (3.10)$$

under REL.

The following theorem shows that the distribution of the potential outcomes and the local dependence $\rho_{Y_d}(y)$ are identified.

Theorem 3.3. *Suppose D_z , $z \in \{0, 1\}$, satisfies (3.8). Under Assumptions EX, REL, and*

CI, the functions $y \mapsto F_{Y_d}(y)$ and $y \mapsto \rho_{Y_d}(y)$ are identified on $y \in \mathcal{Y}$, for $d \in \mathcal{D}$ by

$$F_{Y_d}(y) = \Phi \left(\frac{a_{d,y}}{\sqrt{1 + b_{d,y}^2}} \right), \quad \rho_d(y) = \frac{-b_{d,y}}{\sqrt{1 + b_{d,y}^2}},$$

where $a_{d,y}$ and $b_{y,d}$ are defined in (3.10).

Remark 3.4 (Comparison with Imbens and Newey (2009) and Torgovitsky (2010)). Unlike Imbens and Newey (2009), this approach does not require an instrument with large support nor rank invariance in the treatment selection equation. i.e., $V_1 = V_0 = V$, but imposes *CI*. Unlike Torgovitsky (2010), *CI* does not impose any structural models for Y and D nor restrictions on the dimension of the structural unobservables. Also note that Imbens and Newey (2009) motivate their control function assumption from the joint independence between Z and (Y_d, V) , whereas we only need “marginal” independence between Z and Y_d and between Z and V_z . See Appendix A for a more detailed comparison.

4 Discussions on Copula Invariance

To further understand *CI*, we provide sets of simple sufficient conditions (Sections 4.1), interpret the condition under normality (Section 4.2), and invoke the implicit function theorem (Section 4.3). For ease of illustration, we focus on binary D in this section.

4.1 Sufficient Conditions

4.1.1 Joint Independence

First, we provide sufficient conditions for Assumptions *EX* and *CI*.

Assumption EX' (Joint Independence). For $d, z \in \{0, 1\}$, $Z \perp\!\!\!\perp (Y_d, V_z)$.

Assumption CI' (Unconditional CI). For $d \in \{0, 1\}$,

$$\rho_{Y_d, V_1}(y, \pi(1)) = \rho_{Y_d, V_0}(y, \pi(0)) \equiv \rho_{Y_d}(y).$$

Proposition 4.1. *EX' and CI' imply EX and CI.*

Note that $\rho_{Y_d, V_z; Z}(\cdot, \cdot; z) = \rho_{Y_d, V_z}(\cdot, \cdot)$ by EX'³ and $\rho_{Y_d, V_z}(y, \pi(z)) = \rho_{Y_d}(y)$ by CI'. Therefore, EX' and CI' are sufficient for CI. A sufficient condition for EX' is $(Y_0, Y_1, V_z) \perp\!\!\!\perp Z$, which is imposed in Imbens and Angrist (1994) and Vytlačil (2002) with $V_z = V$, although it is sufficient for the LATE result to have $(Y_d, V) \perp\!\!\!\perp Z$ for $d \in \{0, 1\}$.

Remark 4.1 (CI'). *We might wonder if CI' imposes any restriction on the dependence between V_0 and V_1 . The following example shows that if Y_d , V_0 , and V_1 are jointly normal with the same marginal distributions, the restriction $F_{Y_d, V_0} = F_{Y_d, V_1}$, which is equivalent to CI', does not restrict F_{V_0, V_1} . Let*

$$\begin{pmatrix} Y_d \\ V_0 \\ V_1 \end{pmatrix} \sim \mathcal{N}_3 \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho_{Y_d, V_0} & \rho_{Y_d, V_1} \\ \rho_{Y_d, V_0} & 1 & \rho_{V_0, V_1} \\ \rho_{Y_d, V_1} & \rho_{V_0, V_1} & 1 \end{bmatrix} \right).$$

Under CI', $\rho_{Y_d, V_0} = \rho_{Y_d, V_1}$, so that (Y_d, V_0) and (Y_d, V_1) have the same distribution. Moreover, the matrix

$$\begin{bmatrix} 1 & \rho_{Y_d, V_0} & \rho_{Y_d, V_0} \\ \rho_{Y_d, V_0} & 1 & \rho_{V_0, V_1} \\ \rho_{Y_d, V_0} & \rho_{V_0, V_1} & 1 \end{bmatrix}$$

is positive definite for any $-1 < \rho_{V_0, V_1} < 1$. In other words, the condition $\rho_{Y_d, V_0} = \rho_{Y_d, V_1}$ does not restrict the distribution of (V_0, V_1) .

4.1.2 Monotonicity

Next, we investigate the interaction between CI and the LATE monotonicity (Imbens and Angrist (1994)). The specification of D_z in (2.1) is weaker than LATE monotonicity due to vector unobservables (V_1, V_0) . Specifically, we can generate any compliance patterns from the

³In fact, not only EX' implies $\rho_{Y_d, V_z; Z}(\cdot, \cdot; z) = \rho_{Y_d, V_z}(\cdot, \cdot)$, but the converse is also true. See Remark 4.3 below.

joint distribution of (V_1, V_0) :

$$\Pr[D_1 = 1, D_0 = 1] = \Pr[V_1 \leq \pi(1), V_0 \leq \pi(0)]$$

$$\Pr[D_1 = 0, D_0 = 1] = \Pr[V_1 > \pi(1), V_0 \leq \pi(0)]$$

$$\Pr[D_1 = 1, D_0 = 0] = \Pr[V_1 \leq \pi(1), V_0 > \pi(0)]$$

$$\Pr[D_1 = 0, D_0 = 0] = \Pr[V_1 > \pi(1), V_0 > \pi(0)]$$

Suppose we maintain [EX'](#) for ease of discussion. Then, the following conditions are sufficient for [CI](#).

Assumption \mathbf{RI}_S (Rank Invariance in Selection). *For $d \in \{0, 1\}$, $V_1 = V_0 = V$.*

Assumption \mathbf{RI}_J (Joint Rank Invariance in Selection). *For $d \in \{0, 1\}$, (Y_d, V_1) and (Y_d, V_0) are identically distributed such that $\rho_{Y_d, V_0}(y, v) = \rho_{Y_d, V_1}(y, v) \equiv \rho_{Y_d, V}(y, v)$.*

Assumption \mathbf{CI}'' (CI in Treatment Propensity). $\rho_{Y_d, V}(y, v) = \rho_{Y_d, V}(y)$.

[\$\mathbf{RI}_S\$](#) is equivalent to LATE monotonicity, and it implies [\$\mathbf{RI}_J\$](#) . [\$\mathbf{RI}_J\$](#) and [\$\mathbf{CI}''\$](#) (or [\$\mathbf{RI}_S\$](#) and [\$\mathbf{CI}''\$](#)) being sufficient for [CI](#) shows how [CI](#) may interfere with treatment compliance patterns.

Proposition 4.2. *Under [EX'](#), [\$\mathbf{RI}_J\$](#) and [\$\mathbf{CI}''\$](#) imply [CI](#). [\$\mathbf{RI}_S\$](#) implies [\$\mathbf{RI}_J\$](#) .*

4.2 Normality

Here we want to further understand [CI](#) by imposing normality on the joint distribution of unobservables. The normality assumption is also illustrative to comparing [CI](#) with the rank similarity (RS) and the rank invariance (RI) assumption of [Chernozhukov and Hansen \(2005\)](#). In general, [CI](#) restricts the relationship between the parameters across conditional distributions whereas RI and RS directly restrict the relationship between the potential outcomes. In this sense, [CI](#) allows for more effect heterogeneity.

For ease of notation, we consider a slightly different version of the treatment selection equation (3.1):

$$D_z = 1\{\tilde{V}_z \leq q(z)\},$$

where $\tilde{V}_z \mid Z \sim N(0, 1)$ is an alternative normalization such that $q(z) \equiv \Phi^{-1}(\pi(z))$.⁴ We maintain EX and REL.

Assumption NM (Normality). $(Y_0, Y_1, \tilde{V}_0, \tilde{V}_1)$ are jointly normal conditional on Z ,

$$\begin{pmatrix} Y_0 \\ Y_1 \\ \tilde{V}_0 \\ \tilde{V}_1 \end{pmatrix} \mid Z = z \sim \mathcal{N}_4 \left(\begin{pmatrix} \mu_0 \\ \mu_1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_0^2 & \sigma_{01}(z) & \sigma_{0V_0}(z) & \sigma_{0V_1}(z) \\ \sigma_{01}(z) & \sigma_1^2 & \sigma_{1V_0}(z) & \sigma_{1V_1}(z) \\ \sigma_{0V_0}(z) & \sigma_{1V_0}(z) & 1 & \rho_{V_0V_1}(z) \\ \sigma_{0V_1}(z) & \sigma_{1V_1}(z) & \rho_{V_0V_1}(z) & 1 \end{pmatrix} \right).$$

In this distribution, some of the parameters do not depend on z because of EX. To compare CI, RI and RS, it is convenient to work with the standardized potential outcomes:

$$\tilde{Y}_0 = \frac{Y_0 - \mu_0}{\sigma_0}, \quad \tilde{Y}_1 = \frac{Y_1 - \mu_1}{\sigma_1}.$$

By NM,

$$\begin{pmatrix} \tilde{Y}_0 \\ \tilde{Y}_1 \\ \tilde{V}_0 \\ \tilde{V}_1 \end{pmatrix} \mid Z = z \sim \mathcal{N}_4 \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_{01}(z) & \rho_{0V_0}(z) & \rho_{0V_1}(z) \\ \rho_{01}(z) & 1 & \rho_{1V_0}(z) & \rho_{1V_1}(z) \\ \rho_{0V_0}(z) & \rho_{1V_0}(z) & 1 & \rho_{V_0V_1}(z) \\ \rho_{0V_1}(z) & \rho_{1V_1}(z) & \rho_{V_0V_1}(z) & 1 \end{pmatrix} \right),$$

where

$$\rho_{dd'}(z) = \frac{\sigma_{dd'}(z)}{\sigma_d \sigma_{d'}}, \quad \rho_{dV_z}(z) = \frac{\sigma_{dV_z}(z)}{\sigma_d}$$

⁴This is because of the normalization $V_z \mid Z \sim U[0, 1]$ and thus $\tilde{V}_z = \Phi^{-1}(V_z) \mid Z \sim N(0, 1)$.

for $d, d', z \in \{0, 1\}$. Then, **CI** imposes that

$$\rho_{dV_1}(1) = \rho_{dV_0}(0), \quad d \in \{0, 1\},$$

that is, the correlation between the potential outcomes and the latent treatment assignment does not depend on the value of the instrument. Note that **CI** still imposes substantial invariance restrictions even under normality.⁵ However, **CI** does not impose any restriction on the correlation between the potential outcomes, $\rho_{01}(z)$.

In terms of the standardized variables, RS can be stated as follows:

Assumption RS (Rank Similarity). *Conditional on $\tilde{V}_z = v$ and $Z = z$, $\tilde{Y}_0 \sim \tilde{Y}_1$.*

By **NM**, $\tilde{Y}_d \mid \tilde{V}_z = v, Z = z \sim \mathcal{N}(\rho_{dV_z}(z)v, 1 - \rho_{dV_z}(z)^2)$ for $d \in \{0, 1\}$. **RS** therefore imposes that $\rho_{0V_z}(z) = \rho_{1V_z}(z)$ for $z \in \{0, 1\}$. Compared to **CI**, **RS** imposes that the correlation of the two potential outcomes with the latent treatment assignment is the same for all the values of the instrument, but does not require that the correlation is independent of the value of the instrument. Unlike **CI**, **RS** implicitly imposes restrictions on $\rho_{01}(z)$; see Appendix A.1. Note that **RS** also restricts selection on gains because

$$\text{Cov}(Y_1 - Y_0, \tilde{V}_z \mid Z = z) = \text{Cov}(\sigma_1 \tilde{Y}_1 - \sigma_0 \tilde{Y}_0, \tilde{V}_z \mid Z = z) = \sigma_1 \rho_{1V_z}(z) - \sigma_0 \rho_{0V_z}(z),$$

which, for example, vanishes if $\sigma_1 = \sigma_0$ under **RS**. **CI**, instead, imposes that the selection on gains is independent of the instrument, that is

$$\text{Cov}(Y_1 - Y_0, \tilde{V}_1 \mid Z = 1) = \text{Cov}(Y_1 - Y_0, \tilde{V}_0 \mid Z = 0).$$

In general, **CI** and **RS** are not nested.

In terms of the standardized variables, RI can be stated as follows:

⁵**CI** does *not* trivially holds under joint normality. This is because even under the independence assumptions $(Y_0, V_z) \perp\!\!\!\perp Z$ and $(Y_1, V_z) \perp\!\!\!\perp Z$ (i.e. **EX'**), we only achieve $\rho_{dV_z}(z) = \rho_{dV_z}$. After imposing independence and RI for the selection, normality becomes sufficient for **CI**.

Assumption RI (Rank Invariance). $\tilde{Y}_0 = \tilde{Y}_1$, *almost surely*.

Under **NM**, **RI** imposes that $\rho_{01}(z) = 1$ and $\rho_{0V_z}(z) = \rho_{1V_z}(z)$. **RI** is therefore more restrictive than **RS** as it imposes the same restrictions plus perfect correlation between the standardized potential outcomes. Appendix A.1 contains further discussions on **RS** and **RI** in comparison to **CI**.

4.3 Local Dependence as Implicit Function

The LGR in Lemma 2.1 can also be expressed in terms of the copula function. In particular,

$$\tilde{C}(u_1, u_2 \mid z) = C(u_1, u_2; \rho(u_1, u_2; z)). \quad (4.1)$$

where $\tilde{C}(u_1, u_2 \mid z)$ is the conditional copula of (Y, V_z) given $Z = z$, that is the joint distribution of $U_1 = F_{Y|Z}(Y \mid Z)$ and $U_2 = F_{V_z|Z}(V_z \mid Z)$ conditional on $Z = z$, and C is the Gaussian copula. Note that $U_2 = V_z$ under **EX** and the normalization $V_z \sim U(0, 1)$.

Note that $\rho(u_1, u_2; z)$ can be viewed as an implicit function in (4.1). For any $z \neq z'$, consider

$$\begin{aligned} \tilde{C}(u_1, u_2 \mid z) - \tilde{C}(u_1, u_2 \mid z') &= C_\rho(u_1, u_2; \tilde{\rho}) \{ \rho(u_1, u_2; z) - \rho(u_1, u_2; z') \} \\ &= \phi(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \tilde{\rho}) \{ \rho(u_1, u_2; z) - \rho(u_1, u_2; z') \}, \end{aligned}$$

where $\tilde{\rho}$ lies between $\rho(u_1, u_2; z)$ and $\rho(u_1, u_2; z')$ and $\phi(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \rho) \neq 0$ for $(u_1, u_2) \in (0, 1)^2$.

To focus on **CI**, we maintain **EX'** and **RI_J**, so that **CI** boils down to **CI''**, $\rho_{Y_d, V}(y, v) = \rho_{Y_d, V}(y)$. To understand **CI''**, consider the LGR of the (unconditional) copula of (Y_d, V) :

$$\tilde{C}(u_1, u_2) = C(u_1, u_2; \rho(u_1, u_2)).$$

CI'' is equivalent to $\rho(u_1, u_2) = \rho(u_1)$. Since \tilde{C} and C are differentiable in almost all (u_1, u_2)

(by the definition of copula), so is ρ by the implicit function theorem. Then, for $\tilde{C}(u_1 | u_2)$ and $C(u_1 | u_2)$ being conditional copulas,

$$\begin{aligned}\tilde{C}(u_1 | u_2) &= C(u_1 | u_2; \rho(u_1, u_2)) + C_\rho(u_1, u_2; \rho(u_1, u_2)) \frac{\partial \rho(u_1, u_2)}{\partial u_2} \\ &= C(u_1 | u_2; \rho(u_1, u_2)) + \phi(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \rho(u_1, u_2)) \frac{\partial \rho(u_1, u_2)}{\partial u_2}.\end{aligned}$$

We can interpret $\phi(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \rho(u_1, u_2)) \frac{\partial \rho(u_1, u_2)}{\partial u_2}$ as the adjustment term that equates the two conditional copulas.⁶ In general, the LGR for the joint distribution does *not* imply the same representation for the conditional distribution. Rewrite the equation to have

$$\frac{\partial \rho(u_1, u_2)}{\partial u_2} = \frac{\tilde{C}(u_1 | u_2) - C(u_1 | u_2; \rho(u_1, u_2))}{\phi(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \rho(u_1, u_2))}$$

for $(u_1, u_2) \in (0, 1)^2$, which captures a (normalized) deviation from local Gaussianity. Given this result, **CI''** w.r.t. u_2 is equivalent to $\tilde{C}(u_1 | u_2) = C(u_1 | u_2; \rho(u_1))$. We thus have the following result:

Proposition 4.3. *Under **EX'** and **RI_J**, **CI** holds if $\tilde{C}(u_1 | u_2) = C(u_1 | u_2; \rho(u_1))$.*

Remark 4.2 (Stochastic Monotonicity). *$\tilde{C}(u_1 | u_2) = C(u_1 | u_2; \rho(u_1))$ implies that $u_2 \mapsto \tilde{C}(u_1 | u_2)$ is monotonic for each u_1 . This restricts the dependence between U_1 and U_2 . For example, if U_1 and U_2 are continuous, then the effect of U_2 on the τ -quantile of U_1 cannot change sign with respect to the value of U_2 , but can change sign with τ . Note that if U_1 and U_2 are jointly normal, then this τ -quantile effect cannot change sign with τ . More generally, the stochastic monotonicity condition holds if, for example, the conditional distribution has a monotone likelihood ratio.*

Remark 4.3. *Under the LGR (4.1), we can show that the equivalence between statistical*

⁶Note that $\phi(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \rho(u_1, u_2)) \rightarrow 0$ as $u_1 \rightarrow 1$ or 0 , which is consistent with $\tilde{C}(u_1 | u_2)$ and $C(u_1 | u_2)$ being CDFs (and similarly in the previous case).

independence and a restriction on the dependence parameter as an implicit function:

$$\begin{aligned} (U_1, U_2) \perp\!\!\!\perp Z &\Leftrightarrow \tilde{C}(u_1, u_2 \mid z) - \tilde{C}(u_1, u_2 \mid z') \neq 0 \quad \text{for any } z \neq z' \text{ and } (u_1, u_2) \in (0, 1)^2 \\ &\Leftrightarrow \rho(u_1, u_2; Z) = \rho(u_1, u_2) \quad \text{almost surely, for any } (u_1, u_2) \in (0, 1)^2. \end{aligned}$$

5 Estimation and Inference

We estimate flexible semiparametric models for the distributions of the potential outcomes for the three types of treatments based on distribution regression (DR). We also show how to construct estimators of treatment effect parameters such as the conditional and unconditional QTE and the ATE using the plug-in rule. In this section, we make the role of the covariates X explicit. For estimation, we assume we have access to a random sample of size n from (Y, D, Z, X) , $\{(Y_i, D_i, Z_i, X_i)\}_{i=1}^n$.

Let $B(X_i)$, $B(X_i, Z_i)$, and $B(D_i, X_i, Z_i)$ denote vectors of transformations of X_i , (X_i, Z_i) , and (D_i, X_i, Z_i) , respectively. Let Φ and Φ_2 be the univariate and bivariate Gaussian CDFs, respectively. Define the indicators $I_i(y) \equiv 1\{Y_i \leq y\}$ and $J_i(d) \equiv 1\{D_i \leq d\}$. Let $\bar{\mathcal{D}}$ and $\bar{\mathcal{Y}}$ be two finite grids covering \mathcal{D} and \mathcal{Y} .⁷

5.1 Binary Treatments

We consider a DR model for the conditional potential outcome distributions,

$$F_{Y_d|X}(y|x) = \Phi(B(x)' \beta_d(y)), \quad d \in \{0, 1\}, \quad (5.1)$$

and a Probit model for the propensity score,

$$\pi(z, x) = \Pr[D = 1 \mid Z = z, X = x] = \Phi(B(z, x)' \pi). \quad (5.2)$$

⁷We can set $\bar{\mathcal{Y}} = \mathcal{Y}$ when \mathcal{Y} is finite. We only use $\bar{\mathcal{D}}$ when D is continuous.

We model the local dependence parameter as

$$\rho_{Y_d|X}(y; x) = \rho(B(x)' \gamma_d(y)), \quad d \in \{0, 1\}, \quad (5.3)$$

where $\rho(u) = \tanh(u) \in [-1, 1]$, the Fisher transformation.⁸ Define $\theta_d(y) \equiv (\beta_d(y), \gamma_d(y))$.

Together, (5.1), (5.2), and (5.3) imply the bivariate DR model

$$\begin{aligned} \Pr(Y \leq y, D = 1 \mid Z = z) &= \Phi_2(B(x)' \beta_1(y), B(z, x)' \pi; \rho(B(x)' \gamma_1(y))), \\ \Pr(Y \leq y, D = 0 \mid Z = z) &= \Phi_2(B(x)' \beta_0(y), -B(z, x)' \pi; -\rho(B(x)' \gamma_0(y))), \end{aligned}$$

where we have used the symmetry properties of the bivariate Gaussian distribution.

We propose computationally tractable two-step maximum likelihood estimators, building on Chernozhukov et al. (2020a).

Algorithm 5.1 (Estimation of Binary Treatment Model). *We compute the estimator in 2 stages:*

1. *Treatment equation: estimate π using a Probit regression:*

$$\hat{\pi} = \arg \max_c \sum_{i=1}^n [D_i \log \Phi(B(X_i, Z_i)' c) + (1 - D_i) \log(1 - \Phi(B(X_i, Z_i)' c))]$$

2. *Outcome equation: for $y \in \bar{\mathcal{Y}}$ and $d \in \{0, 1\}$, $\hat{F}_{Y_d|X}(y|x) = \Phi(B(x)' \hat{\beta}_d(y))$, where*

$$\begin{aligned} \hat{\theta}_1(y) &= \arg \max_{t=(b,g)} \sum_{i=1}^n D_i [I_i(y) \log \Phi_2(X_i' b, B(X_i, Z_i)' \hat{\pi}, \rho(X_i' g)) \\ &\quad + (1 - I_i(y)) \log \Phi_2(-B(X_i)' b, B(X_i, Z_i)' \hat{\pi}, \rho(B(X_i)' g))], \\ \hat{\theta}_0(y) &= \arg \max_{t=(b,g)} \sum_{i=1}^n (1 - D_i) [I_i(y) \log \Phi_2(B(X_i)' b, -B(X_i, Z_i)' \hat{\pi}, -\rho(B(X_i)' g)) \\ &\quad + (1 - I_i(y)) \log \Phi_2(-B(X_i)' b, -B(X_i, Z_i)' \hat{\pi}, -\rho(B(X_i)' g))]. \end{aligned}$$

⁸To simplify the notation, we use the same vector of transformations in models (5.1) and (5.3). This is not essential, and one can use different specifications in both models.

Rearrange the estimates $y \mapsto \hat{F}_{Y_d|X}(y | x)$ on $\bar{\mathcal{Y}}$ if needed.

Remark 5.1 (Computation). *The first stage of Algorithm 5.1 is a conventional Probit regression, and estimation can proceed using existing software. The second stage is computationally more expensive since it involves a nonlinear smooth optimization problem. This optimization problem can be solved using standard algorithms such as Newton-Raphson.*

5.2 Ordered Treatments

We model all the components using flexible generalized linear and DR models:

$$\begin{aligned} F_{Y_d|X}(y | x) &= \Phi(B(x)' \beta_d(y)), \\ \rho_{Y_d;X}(y | x) &= \rho(B(x)' \gamma_d(y)), \\ \pi_d(x, z) &= F_{D|Z,X}(d | z, x) = \Phi(B(z, x)' \pi(d)), \end{aligned}$$

where $\rho(u) = \tanh(u)$.

Algorithm 5.2 (Estimation of Ordered Treatment Model). *We compute the estimator in 2 stages:*

1. *Treatment equation: set $\hat{\pi}_0(z, x) = 0$ and $\hat{\pi}_K(z, x) = 1$ for all (z, x) . For $d \in \{1, \dots, K-1\}$, $\hat{\pi}_d(z, x) = \Phi(B(z, x)' \hat{\pi}(d))$, where*

$$\hat{\pi}(d) \in \arg \max_p \sum_{i=1}^n [J_i(d) \log \Phi(B(Z_i, X_i)' p) + (1 - J_i(d)) \log \Phi(-B(Z_i, X_i)' p)].$$

*Rearrange the estimates $d \mapsto \hat{\pi}_d(z, x)$ on \mathcal{D} if needed. This rearrangement is important to avoid having logarithms of negative numbers in the second stage.*⁹

2. *Outcome equation: for $y \in \bar{\mathcal{Y}}$ and $d \in \bar{\mathcal{D}}$, $\hat{F}_{Y_d|X}(y | x) = \Phi(B(x)' \hat{\beta}_d(y))$, where*

$$\hat{\theta}_d(y) = (\hat{\beta}_d(y), \hat{\gamma}_d(y)) \in \arg \max_{b, g} \sum_{i=1}^n 1\{D_i = d\} [I_i(y) \log g_{d,i}(b, g) + (1 - I_i(y)) \log \bar{g}_{d,i}(b, g)],$$

⁹In the second stage, $g_{d,i}(b, g) > 0$ and $\bar{g}_{d,i}(b, g) > 0$ a.s. if $\hat{\pi}_d(Z_i, X_i) > \hat{\pi}_{d-1}(Z_i, X_i)$ a.s.

where

$$g_{d,i}(b, g) \equiv \Phi_2(B(X_i)'b, \hat{\pi}_d(Z_i, X_i), \rho(B(X_i)'g)) - \Phi_2(B(X_i)'b, \hat{\pi}_{d-1}(Z_i, X_i), \rho(B(X_i)'g)),$$

and

$$\bar{g}_{d,i}(b, g) \equiv \hat{\pi}_d(Z_i, X_i) - \hat{\pi}_{d-1}(Z_i, X_i) - g_{d,i}(b, g).$$

Rearrange the estimates $y \mapsto \hat{F}_{Y_d|X}(y | x)$ on $\bar{\mathcal{Y}}$ if needed.

Remark 5.2 (Computation). *As in the previous section, the first stage is a sequence of Probit regressions that can be solved using standard software. The second stage is a nonlinear smooth optimization problem that can be solved using standard algorithms such as Newton-Raphson.*

5.3 Continuous Treatments

We construct plug-in estimators based on the closed-form solutions in Section 3.3. We consider DR models for $F_{Y|D,Z,X}$ and $F_{D|Z,X}$,

$$F_{Y|D,Z,X}(y | d, z, x) = \Phi(B(d, z, x)'\beta(y)), \quad (5.4)$$

$$F_{D|Z,X}(d | z, x) = \Phi(B(z, x)'\pi(d)). \quad (5.5)$$

Algorithm 5.3 (Estimation of Continuous Treatment Model). *We compute the estimator in 2 stages:*

1. *Observable conditional distributions: for $y \in \bar{\mathcal{Y}}$ and $d \in \bar{\mathcal{D}}$, $\hat{F}_{Y|D,Z,X}(y|d, z, x) = \Phi(B(d, z, x)'\hat{\beta}(y))$ and $\hat{F}_{D|Z,X}(d|z, x) = \Phi(B(z, x)'\hat{\pi}(d))$, where*

$$\begin{aligned} \hat{\beta}(y) &= \arg \max_b \sum_{i=1}^n [I_i(y) \log \Phi(B(D_i, Z_i, X_i)'b) + (1 - I_i(y)) \log(1 - \Phi(B(D_i, Z_i, X_i)'b))] \\ \hat{\pi}(d) &= \arg \max_p \sum_{i=1}^n [J_i(d) \log \Phi(B(Z_i, X_i)'p) + (1 - J_i(d)) \log(1 - \Phi(B(Z_i, X_i)'p))] \end{aligned}$$

2. *Potential outcome distributions: for $y \in \bar{\mathcal{Y}}$ and $d \in \bar{\mathcal{D}}$, $\hat{F}_{Y_d|X}(y|x) = \Phi(\hat{\mu}_{d,y;x})$, where*

$$\hat{\mu}_{d,y;x} = \hat{a}_{d,y;x} / \sqrt{1 + \hat{b}_{d,y;x}^2} \text{ and}$$

$$\begin{aligned} \hat{a}_{d,y;x} &= \frac{(B(d, 0, x)' \hat{\beta}(y))(B(1, x)' \hat{\pi}(d)) - (B(d, 1, x)' \hat{\beta}(y))(B(0, x)' \hat{\pi}(d))}{B(1, x)' \hat{\pi}(d) - B(0, x)' \hat{\pi}(d)}, \\ \hat{b}_{d,y;x} &= \frac{B(d, 1, x)' \hat{\beta}(y) - B(d, 0, x)' \hat{\beta}(y)}{B(1, x)' \hat{\pi}(d) - B(0, x)' \hat{\pi}(d)}. \end{aligned}$$

Rearrange the estimates $y \mapsto \hat{F}_{Y_d|X}(y | x)$ on $\bar{\mathcal{Y}}$ if needed.

5.4 Estimation of Unconditional Quantile and Average Structural Function

Finally, we can estimate the marginal distribution of the potential outcomes, quantile structural function (QSF) and average structural function (ASF) by plugging in the estimators obtained above. To give a unified treatment to all the treatment cases, we set $\bar{\mathcal{D}} = \mathcal{D}$ when D is binary or ordered.

Algorithm 5.4 (Estimation of F_{Y_d} , QSF and ASF). *Estimation proceeds in two steps.*

1. *Unconditional distribution: for $y \in \bar{\mathcal{Y}}$ and $d \in \bar{\mathcal{D}}$,*

$$\hat{F}_{Y_d}(y) = \frac{1}{n} \sum_{i=1}^n \hat{F}_{Y_d|X}(y | X_i)$$

For $y \in \mathcal{Y} \setminus \bar{\mathcal{Y}}$ and $d \in \bar{\mathcal{D}}$,

$$\hat{F}_{Y_d}(y) = \max\{\hat{F}_{Y_d}(\bar{y}) : \bar{y} < y, \bar{y} \in \bar{\mathcal{Y}}\}.$$
¹⁰

2. *Quantile and average structural function:*

$$\widehat{QSF}_\tau(d) = \mathcal{Q}_\tau(\hat{F}_{Y_d}),$$

$$\widehat{ASF}(d) = \mathcal{E}(\hat{F}_{Y_d}).$$

¹⁰In practice, one can also use linear extrapolation when D is continuous.

The quantile and average effects can be obtained accordingly.

5.5 Inference

The target parameters in Section 5.4 are function-valued. Inference on these parameters can be performed using resampling methods. To provide a unified framework, we denote the functional parameters by $u \mapsto \delta_u, u \in \mathcal{U}$, where $\mathcal{U} \subset \mathcal{Y} \times \mathcal{D} \times \mathcal{T}$, where $\mathcal{T} \subset (c, 1 - c)$ for $c > 0$. In practice, we approximate \mathcal{U} using a fine grid $\bar{\mathcal{U}}$. We denote the estimator of δ_u obtained from Algorithms 5.1, 5.2, 5.3, and 5.4 as $\hat{\delta}_u$.

We focus on constructing uniform confidence bands, $CB_{(1-\alpha)}(\delta_u)$, satisfying

$$\lim_{n \rightarrow \infty} \Pr[\delta_u \in CB_{(1-\alpha)}(\delta_u), \text{ for all } u \in \mathcal{U}] = 1 - \alpha.$$

Uniform confidence bands can be used to test a variety of hypotheses of interest, such as the hypotheses of no effect or constant effects when applied to treatment effect parameters, or stochastic dominance.

Under standard regularity conditions, $\sqrt{n}(\hat{\delta}_u - \delta_u)$ converges in distribution to a mean-zero Gaussian process $G_\delta(u)$ in $\ell^\infty(\mathcal{U})$ and the bootstrap is valid. This follows from standard arguments for DR coefficients (e.g., Chernozhukov et al., 2013, 2020a) and the functional delta method.

The following algorithm provides a generic bootstrap construction of $CB_{(1-\alpha)}(\delta_u)$. When δ_u is equal to the QSF or quantile effect functions, the algorithm only applies when the outcomes are continuous. When the outcomes are discrete or mixed discrete-continuous, the estimators of the QSF or quantile effect functions are not be asymptotically Gaussian in general. In this case, uniform confidence bands for the QSF and quantile effect functions can be constructed as described in Chernozhukov et al. (2020b).

Algorithm 5.5 (Uniform Confidence Bands for Functional Parameters¹¹).

¹¹See, for example, Chernozhukov et al. (2013, 2020a,b) for similar algorithms.

1. For $u \in \bar{\mathcal{U}}$, obtain B bootstrap draws of the estimator $\hat{\delta}_u$, $\{\hat{\delta}_u^{(b)} : 1 \leq b \leq B\}$.
2. For $u \in \bar{\mathcal{U}}$, compute the standard error,

$$SE(\hat{\delta}_u) = (\hat{Q}_\delta(0.75, u) - \hat{Q}_\delta(0.25, u)) / (\Phi^{-1}(0.75) - \Phi^{-1}(0.25)),$$

where $\hat{Q}_\delta(\tau, u)$ is the τ -quantile of $\{\hat{\delta}_u^{(b)} : 1 \leq b \leq B\}$.

3. Compute the critical value as

$$cv(1 - \alpha) = (1 - \alpha)\text{-quantile of } \left\{ \max_{u \in \bar{\mathcal{U}}} \frac{|\hat{\delta}_u^{(b)} - \hat{\delta}_u|}{SE(\hat{\delta}_u)} : 1 \leq b \leq B \right\}.$$

4. Compute the $(1 - \alpha)$ uniform confidence band as

$$CB_{(1-\alpha)}(\delta_u) = [\hat{\delta}_u \pm cv(1 - \alpha)SE(\hat{\delta}_u)], \quad u \in \bar{\mathcal{U}}.$$

The estimation algorithms for binary and ordered treatments (Algorithms 5.1 and 5.2) involve nonlinear optimization problems. Therefore, following Chernozhukov et al. (2020a), we recommend using the multiplier bootstrap in Step 1 of Algorithm 5.5. The multiplier bootstrap is a computationally efficient resampling procedure based on the influence function of the estimator that avoids re-estimating the parameters in Algorithms 5.1 and 5.2 in each of the B bootstrap iterations.

The estimation approach for continuous treatments in Algorithm 5.3 does not involve solving a nonlinear optimization problem in the second step and is computationally less expensive than Algorithms 5.1 and 5.2. Therefore, the standard empirical bootstrap is a natural alternative to the multiplier bootstrap in Step 1 when the sample size is small or moderate, as for example in Section 6.

6 Empirical Application

We illustrate our method by estimating the distributional effects of sleep on well-being. We use the data from the experimental analysis of [Bessone et al. \(2021b\)](#), who analyzed the effects of randomized interventions to increase sleep of low-income adults in India.¹² Specifically, they considered two main treatments (see their Section III for details): (i) *devices + encouragement* (information on the benefits of and tips to improve sleep, encouragements and a sleep tracker, and devices to improve sleep environment) and (ii) *devices + incentives* (same as (i) and payments for each minute of sleep increase). In addition, they cross-randomized a *nap treatment* (the opportunity to nap during the day).

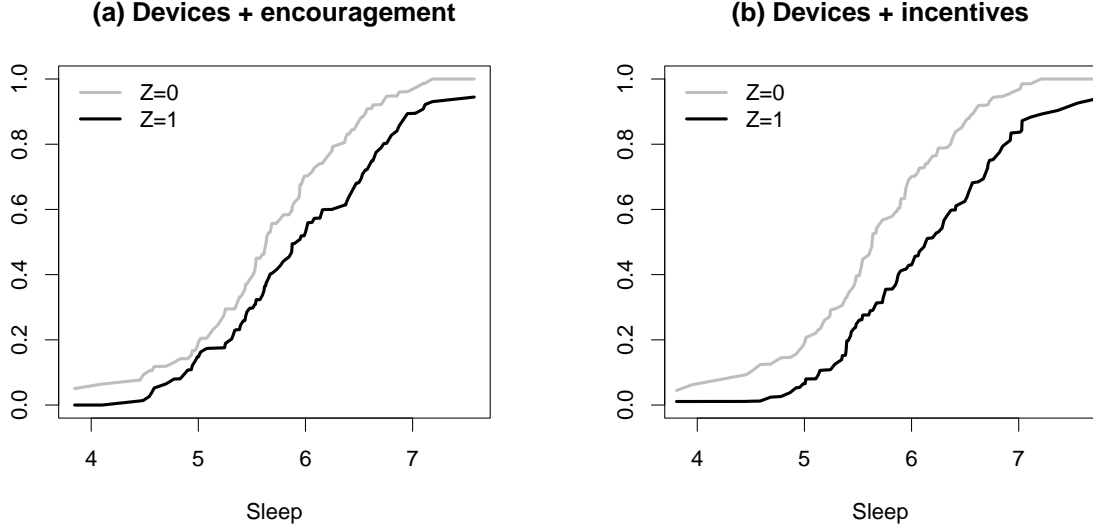
The outcome of interest (Y) is an overall index of individual well-being. The treatment (D) is the number of hours of sleep per night. Following [Bessone et al. \(2021b, Table A.XVII\)](#) and the recent reanalysis by [Dong and Lee \(2023\)](#), we use the randomly assigned experimental treatments as instruments for sleep. We focus on the two main experimental treatments, devices + encouragement (Z_1) and devices + incentives (Z_2) and restrict the sample to individuals who did not receive the nap treatment, as in [Dong and Lee \(2023\)](#). Both instruments are equal to one if the individuals received the experimental treatment and zero otherwise. The vector of covariates (X) includes controls for gender, three age indicators, and the baseline well-being index, as in [Bessone et al. \(2021b, Table A.XVII\)](#).

We start by analyzing the (distributional) first-stage relationship between D and Z_1 and D and Z_2 . Figure 1 plots $n^{-1} \sum_{i=1}^n \hat{F}_{D|Z,X}(\cdot|1, X_i)$ and $n^{-1} \sum_{i=1}^n \hat{F}_{D|Z,X}(\cdot|0, X_i)$ for both instruments. It shows that both instruments induce a shift in the distribution of D , with the shift being more pronounced for Z_2 .

The treatment takes on many values and is treated as continuous in this application, and we use the estimators for continuous treatments in Section 5.3. Figure 2 plots estimates of the QTE as a function of the quantile level τ , $\mathcal{Q}(\hat{F}_{Y_{d''}}) - \mathcal{Q}(\hat{F}_{Y_{d'}})$, including 90% pointwise confidence intervals (CIs) computed using the empirical bootstrap. We set d'' and d' to be

¹²We downloaded the data from the Harvard Dataverse replication package ([Bessone et al., 2021a](#)).

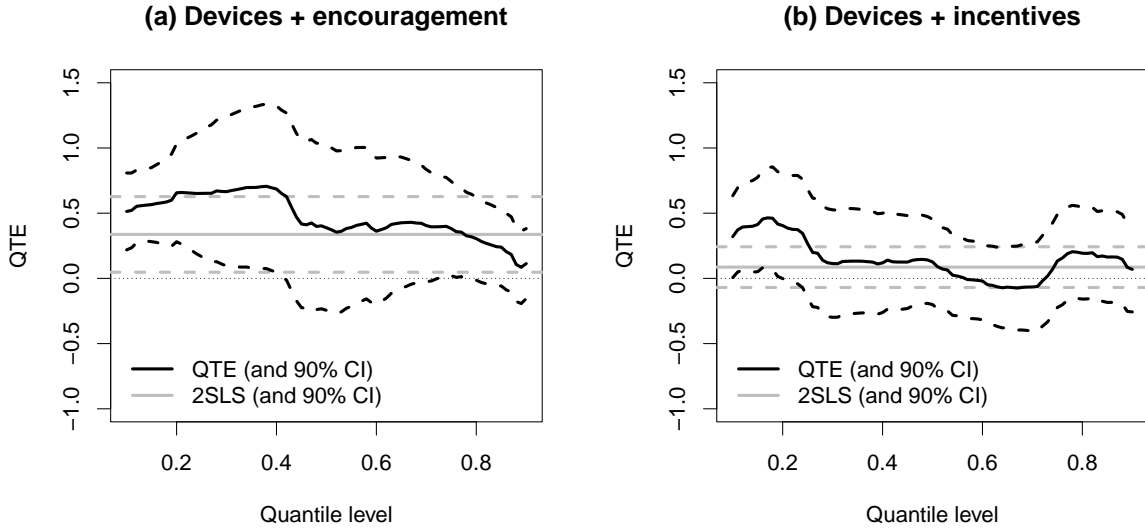
Figure 1: Distributional first stage



Notes: The sample sizes are $n = 152$ in Figure 1(a) and $n = 151$ in Figure 1(b). All specifications control for gender, three age indicators, and the baseline well-being index.

equal to the 75% and 25% quantile of the empirical distribution of D , respectively. We report the results separately for each instrument (setting the other instrument to zero). For comparison, we also report two-stage least squares (2SLS) estimates using the same set of covariates with 90% confidence intervals.

Figure 2: Quantile treatment effects

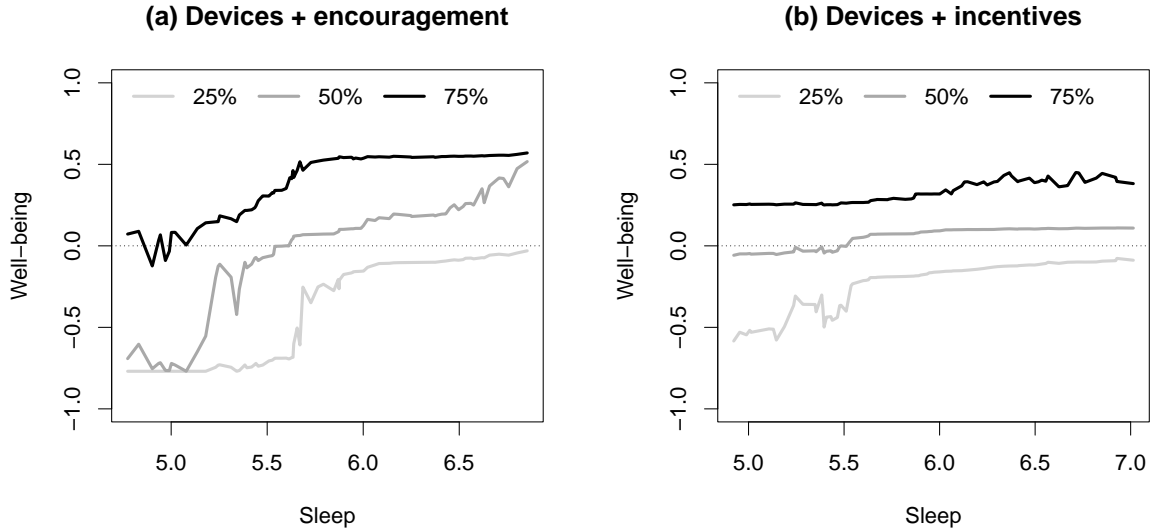


Notes: The sample sizes are $n = 152$ in Figure 2(a) and $n = 151$ in Figure 2(b). Pointwise CIs for the QTE are computed using the empirical bootstrap with 10,000 repetitions. All specifications control for gender, three age indicators, and the baseline well-being index.

Figure 2 reveals interesting effect heterogeneity across the distribution of well-being. For both instruments, the QTEs are the largest and significant at the 90% level over a range of quantile levels below the median and smaller and insignificant at the upper tail. The 2SLS estimates based on Z_1 are larger than those based on Z_2 . The estimates based on Z_1 are significant at the 90% level, while those based on Z_2 are not.

We treat sleep as a continuous random variable. It is therefore also interesting to investigate the effect heterogeneity across the distribution of sleep. Figure 3 displays the estimated QSF, $\widehat{QSF}_\tau(\cdot)$, for $\tau \in \{0.25, 0.5, 0.75\}$. The estimated QSFs display an increasing overall pattern for both instruments and all three quantile levels. However, there are some notable nonmonotonicities and interesting patterns of heterogeneity, especially for values of sleep at the lower tail of the sleep distribution.

Figure 3: Quantile structural function



Notes: The sample sizes are $n = 152$ in Figure 2(a) and $n = 151$ in Figure 2(b). All specifications control for gender, three age indicators, and the baseline well-being index.

The analysis in this section showcases the value-added that our method can bring to standard empirical analyses focusing on average effects. While a simple two-stage least squares analysis suggests that sleep has moderate or insignificant average effects on well-being, the proposed approach uncovers interesting patterns of heterogeneity in the effect of sleep on well-being.

Appendix to “Treatment Endogeneity and Local Copula Invariance”

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A Comparisons to Previous Studies

Here we compare the proposed LGR-based identification approach to the literature and show that it can unify and complement existing methods. To simplify the exposition, we will abstract from covariates.

A.1 Rank Similarity and Rank Invariance in Chernozhukov and Hansen (2005) and Vuong and Xu (2017)

The instrumental variables quantile regression (IVQR) model of Chernozhukov and Hansen (2005) provides an alternative set of conditions under which the QSF_τ is point identified, provided that the outcome is continuous. The IVQR model is based on the Skorohod representation

$$Y_d = Q_{Y_d}(U_d), \quad U_d \sim U[0, 1], \quad d \in \{0, 1\}.$$

Chernozhukov and Hansen (2005) consider a general selection mechanism, $D = \delta(Z, V)$, where V can be vector-valued and the instrument is assumed to satisfy $U_d \perp\!\!\!\perp Z$ (which is implied by Assumption EX). The key assumption of the IVQR model is RS, $U_1 \stackrel{d}{=} U_0 \mid Z, V$. RS weakens the classical RI assumption, which requires $U_1 = U_0$ almost surely. The IVQR model yields the following conditional moment restriction (Chernozhukov and Hansen, 2005, Theorem 1),

$$\tau = \Pr[Y_1 \leq Q_{Y_1}(\tau), D_z = 1 \mid Z = z] + \Pr[Y_0 \leq Q_{Y_0}(\tau), D_z = 0 \mid Z = z], \quad z \in \{0, 1\}. \quad (\text{A.1})$$

Under the general selection model (2.1) and the independence assumption, equation (A.1) can be rewritten as

$$\begin{aligned} \tau &= \Pr[Y_1 \leq Q_{Y_1}(\tau), V_z \leq \pi(z) \mid Z = z] + \Pr[Y_0 \leq Q_{Y_0}(\tau), V_z > \pi(z) \mid Z = z] \\ &= \Pr[Y_1 \leq Q_{Y_1}(\tau), V_z \leq \pi(z) \mid Z = z] + \tau - \Pr[Y_0 \leq Q_{Y_0}(\tau), V_z \leq \pi(z) \mid Z = z]. \end{aligned}$$

Thus, (A.1) can alternatively be written as

$$\Pr[Y_1 \leq Q_{Y_1}(\tau), V_z \leq \pi(z) | Z = z] = \Pr[Y_0 \leq Q_{Y_0}(\tau), V_z \leq \pi(z) | Z = z], \quad z \in \{0, 1\}.$$

Using Lemma 2.1, we can further rewrite it as

$$C(\tau, \pi(z), \rho_{Y_1, V_z; Z}(Q_{Y_1}(\tau), \pi(z); z)) = C(\tau, \pi(z), \rho_{Y_0, V_z; Z}(Q_{Y_0}(\tau), \pi(z); z)), \quad z \in \{0, 1\}.$$

This shows that the IVQR model also relies on a copula invariance assumption,

$$\rho_{Y_1, V_z; Z}(Q_{Y_1}(\tau), \pi(z); z) = \rho_{Y_0, V_z; Z}(Q_{Y_0}(\tau), \pi(z); z), \quad z \in \{0, 1\}. \quad (\text{A.2})$$

The IVQR model imposes a restriction across potential outcomes, whereas CI is a restriction between the outcome and the unobservable component in the selection equation. To make this more explicit, it is instructive to rewrite equation (A.2) using the “counterfactual mapping” $\phi(y) \equiv Q_{Y_1}(F_{Y_0}(y))$ of [Vuong and Xu \(2017\)](#) as

$$\rho_{Y_1, V_z; Z}(\phi(y), \pi(z); z) = \rho_{Y_0, V_z; Z}(y, \pi(z); z), \quad z \in \{0, 1\}.$$

Note that the copula invariance assumption underlying the IVQR model restricts treatment effect heterogeneity, unlike CI.

The conditional moment restriction (A.1) and the implied copula invariance assumption are not sufficient for point identification. [Chernozhukov and Hansen \(2005\)](#) provide a set of sufficient conditions for point identification that require the Jacobian of (A.1) to be of full rank and continuous. For the case where D and Z are binary, [Vuong and Xu \(2017\)](#) provide weaker conditions by directly analyzing identification of the counterfactual mapping ϕ , which satisfies $Y_1 = \phi(Y_0)$ under continuity of the outcome and RI. Their key identification

condition is piecewise strict monotonicity of

$$\Delta_d(\cdot) = (-1)^d(\Pr[Y \leq \cdot, D = d|Z = 0] - \Pr[Y \leq \cdot, D = d|Z = 1]).$$

The approach of [Vuong and Xu \(2017\)](#) is extended to the case of ordered treatments in [Han and Xu \(2023\)](#).

In sum, both the IVQR model and our approach rely on copula invariance assumptions to identify causal effects for the overall population using instruments. Relative to the IVQR model, the proposed identification approach has two main advantages. First, it allows for unrestricted treatment effect heterogeneity (see Section 4.2). Second, it does not rely on continuity of the outcome to achieve point identification, and it naturally accommodates discrete and mixed discrete-continuous outcomes.

Remark A.1 (RS Restricts Treatment Effect Heterogeneity). *Consider the setting of Section 4.2 where we impose the classical LATE assumptions, $\tilde{V}_1 = \tilde{V}_0 = \tilde{V}$ and $(Y_1, Y_0, \tilde{V}) \perp\!\!\!\perp Z$ ([Imbens and Angrist, 1994](#); [Vytlačil, 2002](#)). Under these assumptions, by [NM](#),*

$$\begin{pmatrix} \tilde{Y}_0 \\ \tilde{Y}_1 \\ V \end{pmatrix} \mid Z = z \sim \mathcal{N}_3 \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_{01} & \rho_{0V} \\ \rho_{01} & 1 & \rho_{1V} \\ \rho_{0V} & \rho_{1V} & 1 \end{pmatrix} \right).$$

[CI](#) holds by construction. [CI](#) therefore does not restrict treatment effect heterogeneity since ρ_{01} and hence the relationship between \tilde{Y}_1 and \tilde{Y}_0 is completely unrestricted. [RS](#) imposes that $\rho_{0V} = \rho_{1V} = \rho_V$, which restrict treatment effect heterogeneity. Thus, the requirement that

$$\Sigma \equiv \begin{pmatrix} 1 & \rho_{01} & \rho_V \\ \rho_{01} & 1 & \rho_V \\ \rho_V & \rho_V & 1 \end{pmatrix}$$

is a variance-covariance matrix and hence positive semi-definite poses implicit restrictions

on ρ_{01} . Specifically, all three eigenvalues of Σ ,

$$\lambda_1 = 1 - \rho_{01}, \quad \lambda_2 = \frac{1}{2} \left(-\sqrt{\rho_{01}^2 + 8\rho_V^2} + \rho_{01} + 2 \right), \quad \text{and } \lambda_3 = \frac{1}{2} \left(\sqrt{\rho_{01}^2 + 8\rho_V^2} + \rho_{01} + 2 \right),$$

need to be positive. While λ_1 and λ_3 are always positive, the requirement that $\lambda_2 \geq 0$ is equivalent to $\rho_{01} \geq 2\rho_V^2 - 1$. This shows that [RS](#) imposes restrictions on ρ_{01} and thus on effect heterogeneity. These restrictions crucially depend on the degree of endogeneity, measured by ρ_V . There are two polar cases: (i) if the treatment is exogenous so that $\rho_V = 0$, then [RS](#) does not imply any restrictions on ρ_{01} ; (ii) if the endogeneity is “maximal” so that $\rho_V = 1$, then $\rho_{01} = 1$ (as under [RI](#)). In other words, the stronger the endogeneity the “closer” [RS](#) becomes to [RI](#).

A.2 LATE Monotonicity in [Imbens and Angrist \(1994\)](#)

The LATE framework of [Imbens and Angrist \(1994\)](#) provides conditions for identifying causal effects for the subpopulation of compliers. The compliers are the individuals who react to the instrument so that $D_1 \geq D_0$ almost surely. Compared to the assumptions in [Section 3.1](#), the LATE framework restricts the selection model by setting $V_1 = V_0 = V$ almost surely, and relies a stronger joint independence assumption, $(Y_0, Y_1, V) \perp\!\!\!\perp Z$, but does not impose any copula invariance assumption.

Under the LATE assumptions, the marginal potential outcome distributions for the compliers are identified (e.g., [Abadie, 2002](#), Lemma 2.1):

$$F_{Y_1|D_1 \geq D_0}(y) = \frac{E[1\{Y \leq y\}D|Z = 1] - E[1\{Y \leq y\}D|Z = 0]}{E[D|Z = 1] - E[D|Z = 0]}$$

$$F_{Y_0|D_1 \geq D_0}(y) = \frac{E[1\{Y \leq y\}(1 - D)|Z = 1] - E[1\{Y \leq y\}(1 - D)|Z = 0]}{E[1 - D|Z = 1] - E[1 - D|Z = 0]}$$

The main drawback of the LATE framework is its lack of external validity: it only identifies effects for a subpopulation that depends on the instrument. Different approaches exist for extrapolating from the LATE to externally valid global treatment effects. Examples include

approaches based on structural models (e.g., [Heckman et al., 2003](#)), covariates (e.g., [Angrist and Fernández-Val, 2013](#)), rank similarity (e.g., [Wüthrich, 2020](#)), and the smoothness of the marginal treatment effect function (e.g., [Mogstad et al., 2018](#); [Han and Yang, 2023](#)).

The identification results in Section 3.1 apply under the LATE assumptions augmented with Assumption CI. Therefore, the proposed method provides a new approach to external validity based on copula invariance assumptions. It has important practical advantages relative to the existing methods referenced above. It avoids parametric structural assumptions, does not require covariates, accommodates discrete outcomes, and does not resort to partial identification.

For ordered treatments, Remark 3.3 discusses [Heckman and Vytlacil \(2007\)](#), which is a special case of our model. [Vytlacil \(2006\)](#) considers the ordered selection model with random thresholds:

$$D_z = \begin{cases} 1, & -\infty < v(z) \leq V_1 \\ 2, & V_1 < v(z) \leq V_2 \\ \vdots & \vdots \\ K, & V_{K-1}(z) < v(z) \leq \infty \end{cases}$$

where V_1, \dots, V_{K-1} are random thresholds such that $V_1 \leq \dots \leq V_{K-1}$ and assumes $Z \perp (V_1, \dots, V_{K-1}, Y_1, \dots, Y_K)$. Under the normalization $V_z \sim N(0, 1)$, our model has a similar representation:

$$D_z = \begin{cases} 1, & -\infty < v(z) \leq V_1(z) \\ 2, & V_1(z) < v(z) \leq V_2(z) \\ \vdots & \vdots \\ K, & V_{K-1}(z) < v(z) \leq \infty \end{cases}$$

upon setting $V_k(z) = -V_z + \pi_k(z) + v(z)$. This representation looks more general because

the random thresholds can vary with z . To get the same representation, we can impose in our model that $V_0 = V_1 = V$ and $\pi_k(z) = \pi_k - v(z)$, and set $V_k = -V + \pi_k$, but this version of the model is more restrictive than Vytlačil's model because it imposes restrictions on the joint distribution of V_1, \dots, V_{K-1} . More generally, our identification analysis carries over with suitable adjustments if we allow for threshold-specific unobservables, that is

$$D_z = \begin{cases} 1, & -\infty \leq \tilde{V}_{1,z} \leq \pi_1(z) \\ 2, & \pi_1(z) < \tilde{V}_{2,z} \leq \pi_2(z) \\ \vdots & \vdots \\ K, & \pi_{K-1}(z) < \tilde{V}_{K,z} \leq \infty \end{cases}. \quad (\text{A.3})$$

where $\tilde{V}_{1,z} \leq \dots \leq \tilde{V}_{K,z}$. This model is more general than Vytlačil's model. It is equivalent when $V_{k,1} = V_{k,0}$ for all k and $\pi_k(z) = \pi_k - v(z)$, upon setting $V_k = -\tilde{V}_k + \pi_k$.

Remark A.2. *We continue Remark 3.3 and discuss Heckman and Vytlačil (2007)'s model in comparison to ours using the notation introduced in the proof of Theorem 3.2. The cutoffs of the two models are related as $\pi_\ell(z) = \pi_\ell - \mu(z)$. Under this model, we have*

$$r_\ell(z) = \frac{\pi_\ell - \mu(z) - \rho_{d,y} F_{d,y}}{\sqrt{1 - \rho_{d,y}^2}}$$

which is particularly easy-to-interpret because

$$r_d(z) - r_{d-1}(z) = \frac{\pi_d - \pi_{d-1}}{\sqrt{1 - \rho_{d,y}^2}}.$$

On the other hand, in the general model with the normalization $V_z \mid Z \sim N(0, 1)$, we have

$$r_d(z) - r_{d-1}(z) = \frac{\pi_d(z) - \pi_{d-1}(z)}{\sqrt{1 - \rho_{d,y}^2}}.$$

Note that in the simplified model, the full-rank condition holds by construction. Specifically,

since $r_d(z) - r_{d-1}(z)$ does not depend on z , we can write $r_d(z) = r_{d-1}(z) + c$ for $c > 0$. Therefore, we can rewrite $\lambda(z)$ as

$$\frac{\phi(r_{d-1}(z) + c) - \phi(r_{d-1}(z))}{\Phi(r_{d-1}(z) + c) - \Phi(r_{d-1}(z))}.$$

This function is monotonically decreasing in $r_{d-1}(z)$. Thus, as long as $\mu(1) \neq \mu(0)$ so that $r_{d-1}(1) \neq r_{d-1}(0)$, the full rank condition holds.

A.3 Control Function Approach in Imbens and Newey (2009)

For a continuous treatment, Imbens and Newey (2009) consider identification based on a control function approach. A simple version of their model consists of a structural outcome equation, $Y = g(D, \varepsilon)$, and a reduced form treatment assignment equation, $D = h(Z, V)$, where V is scalar and $v \mapsto h(\cdot, v)$ is strictly monotone. The main idea is to use $V = F_{D|Z}(D | Z)$ as a control function that satisfies $D \perp\!\!\!\perp \varepsilon | V$. Under the assumption that $(\varepsilon, V) \perp\!\!\!\perp Z$, the latter holds. The key to their identification approach is a common support assumption that the support of V conditional on D equals the support of V , which potentially requires a large support of Z . This assumption and the control function assumption yield

$$E[Y | D = d, V = v] = \int g(d, e) dF_{\varepsilon|D,V}(e | d, v) = \int g(d, e) dF_{\varepsilon|V}(e | v).$$

In terms of our framework, while they require $V_1 = V_0 = V$ almost surely in (2.1) and strict monotonicity with respect to V . We also do not require Z to have a large support and allow for binary Z . On the other hand, we assume CI as a trade-off.

To avoid the large support assumption of Imbens and Newey (2009), Newey and Stouli (2021) impose parametric structure for extrapolation. Their main assumptions are $E[Y | D, V] = \beta' p(D) \otimes q(V)$ where $p(\cdot)$ and $q(\cdot)$ are known and $E[(p(D) \otimes q(V))(p(D) \otimes q(V))']$ is positive definite. Then binary IV is enough to identify the average structural function. They also generalize the result for the case where either $p(\cdot)$ or $q(\cdot)$ is known. While their

approach imposes parametric structure on the conditional mean, we impose copula invariance, restricting the dependence structure of the unobservables.

A.4 Conditional Copula Invariance in [Torgovitsky \(2010\)](#)

For a continuous treatment, [Torgovitsky \(2010\)](#) considers identification based on a conditional copula invariance assumption.¹³ He assumes that Y and D are continuous and considers the model, $Y = m(D, U)$, where $u \mapsto m(d, u)$ is strictly increasing for every d , which implies RI. The (possibly binary) instrument Z is assumed to be marginally independent of U , $U \perp\!\!\!\perp Z$, which is implied by [EX](#). Moreover, he imposes a mild local dependence assumption between D and Z .

The key condition of [Torgovitsky \(2010\)](#) is the conditional copula invariance assumption. Let $R \equiv F_{D|Z}(D | Z)$ and consider $\Pr[U \leq u, D \leq Q_{D|Z}(r | z) | Z = z]$. Then the assumption requires that the copula of $(U, D) | Z = 1$ is equal to the copula of $(U, D) | Z = 0$ (focusing on binary Z). Under $U \perp\!\!\!\perp Z$, this can be written as

$$\tilde{C}(F_U(u), r; 1) = \tilde{C}(F_U(u), r; 0). \quad (\text{A.4})$$

Using Lemma [2.1](#), these two copulas have the following LGR: for $z \in \{0, 1\}$,

$$\tilde{C}(F_U(u), r; z) = C(F_U(u), r; \rho_{U,D;Z}(F_U(u), r; z)).$$

Hence, the conditional copula invariance assumption ([A.4](#)) can be written as

$$\rho_{U,D;Z}(F_U(u), r; 1) = \rho_{U,D;Z}(F_U(u), r; 0). \quad (\text{A.5})$$

Comparing equation ([A.5](#)) to [CI](#), we can see that both copula invariance assumptions restrict the dependence of the joint distribution of (Y_d, D) on Z by requiring the correlation parameter

¹³Portions of [Torgovitsky \(2010\)](#) were published in [Torgovitsky \(2015\)](#). Since the role of the conditional copula invariance assumption is only discussed in [Torgovitsky \(2010\)](#), we focus on comparing our method and assumptions to this paper.

not to depend on $Z = z$. Since [Torgovitsky \(2010\)](#) maintains the RI assumption, restricting the copula of (U, D) is sufficient. Our identification strategy does not depend on RI such that we need to impose copula invariance restrictions for both potential outcomes. As a trade-off of not assuming RI, we require copula invariance that the correlation parameter is not a function of the reduced-form parameter.

Overall, our identification results complement [Torgovitsky \(2010\)](#) by showing that copula invariance assumptions also are useful with binary treatments. While the underlying copula invariance assumptions are related, our identification strategy fundamentally differs from [Torgovitsky \(2010\)](#). It accommodates binary and ordered treatments and does not rely in RI assumptions.

B Local Representation with Other Copulas

It may be the case that a joint distribution can be locally represented using a copula that is not necessarily Gaussian. We ask what other copulas can be used for local representation. In showing the full rank of Jacobian for the original identification strategy, the following quantity typically arises once the Jacobian is transformed using elementary operations:

$$\frac{C_\rho(F_{d,y}, \pi(z); \rho)}{C_1(F_{d,y}, \pi(z); \rho)} - \frac{C_\rho(F_{d,y}, \pi(z'); \rho)}{C_1(F_{d,y}, \pi(z'); \rho)},$$

where C_ρ and C_1 are the derivatives w.r.t. ρ and the first argument, respectively. Therefore, any copula that satisfies

$$\frac{C_\rho(F_{d,y}, \pi(z); \rho)}{C_1(F_{d,y}, \pi(z); \rho)} \neq \frac{C_\rho(F_{d,y}, \pi(z'); \rho)}{C_1(F_{d,y}, \pi(z'); \rho)} \quad (\text{B.1})$$

for $\pi(z) \neq \pi(z')$ will yield the full rank Jacobian. [Han and Vytlačil \(2017\)](#) show that any single-parameter copula that follows the ordering of stochastic increasingness w.r.t. ρ satisfies (B.1). Therefore, among them, a comprehensive copula can be a candidate for the local representation. The following Archimedean copulas are such copulas:

Example 1 (Clayton copula).

$$C(u_1, u_2; \rho) = \max\{u_1^{-\rho} + u_2^{-\rho} - 1, 0\}^{-1/\rho}, \quad \rho \in [-1, \infty) \setminus \{0\}$$

and $C \rightarrow C_L$ when $\rho \rightarrow -1$, $C \rightarrow C_U$ when $\rho \rightarrow \infty$, and $C \rightarrow C_I$ when $\rho \rightarrow 0$.

Example 2 (Frank copula).

$$C(u_1, u_2; \rho) = -\frac{1}{\rho} \ln \left(1 + \frac{(e^{-\rho u_1} - 1)(e^{-\rho u_2} - 1)}{e^{-\rho} - 1} \right), \quad \rho \in (-\infty, \infty) \setminus \{0\}$$

and $C \rightarrow C_L$ when $\rho \rightarrow -\infty$, $C \rightarrow C_U$ when $\rho \rightarrow \infty$, and $C \rightarrow C_I$ when $\rho \rightarrow 0$.

C Identification Power of IV Support

When the instrument takes more than two values, we may be able to relax Assumption [CI](#).

We illustrate this focusing on the case of binary D . Let $\mathbf{Z} \equiv (Z_1, \dots, Z_K)$ be the vector of binary IVs, that is, $Z_k \in \{0, 1\}$ for $k = 1, \dots, K$. This vector might arise from having multiple binary instruments or constructing indicators from a multivalued instrument. We can prove that [REL](#) and the following copula invariance are sufficient for full rank of the corresponding Jacobian. Define a selection equation $D_{\mathbf{z}} = 1\{V_{\mathbf{z}} \leq \pi(\mathbf{z})\}$ where $\pi(\mathbf{z}) \equiv \Pr[D = 1 \mid \mathbf{Z} = \mathbf{z}]$.

We make the following assumptions.

Assumption EX2. For $d, z_k \in \{0, 1\}$ for all k , $\mathbf{Z} \perp\!\!\!\perp Y_d$ and $\mathbf{Z} \perp\!\!\!\perp V_{\mathbf{z}}$.

Assumption CI2 (Partial Copula Invariance). For $d \in \{0, 1\}$, $\rho_{d,y}(0, \dots, 0, 1) = \rho_{d,y}(0, \dots, 0, 0) \equiv \rho_{d,y}^0$, where $\rho_{d,y}(\mathbf{z}) \equiv \rho_{Y_d, V_{\mathbf{z}}}(y, \pi(\mathbf{z}))$.

Assumption REL2. (i) $\mathbf{Z} \in \{0, 1\}^K$; (ii) $0 < \Pr(\mathbf{Z} = \mathbf{z}) < 1$ and $0 < \Pr(D = d \mid \mathbf{Z} = \mathbf{z}) < 1$, for $d \in \mathcal{D}$ and $\mathbf{z} \in \{(0, \dots, 0, 0), (0, \dots, 0, 1)\}$; and (iii) $\Pr(D = d \mid \mathbf{Z} = (0, \dots, 0, 0)) \neq \Pr(D = d \mid \mathbf{Z} = (0, \dots, 0, 1))$ for $d \in \mathcal{D}$.

[EX2](#) is the analog of [EX](#) in the multiple instrument case. [CI2](#) can be justified if, conditional on $Z_k = 0$ for $k = 1, \dots, K - 1$ (i.e., the status quo), Z_K does not shift the joint

distribution of (Y_d, V_z) . In general, with the vector of IVs, there will always be only one more parameter than the number of identifying equations, which is 2^K . [CI2](#) reduces this additional parameter. For illustration, let $K = 2$, that is, consider two binary IVs, Z_1 and Z_2 , in $\{0, 1\}$. Then, the resulting equations for $D = 1$ are

$$\begin{aligned} F_{Y|D,\mathbf{Z}}(y \mid 1, (1, 1))\pi(1, 1) &= C(F_{Y_1}(y), \pi(1, 1); \rho_{1,y}(1, 1)), \\ F_{Y|D,\mathbf{Z}}(y \mid 1, (1, 0))\pi(1, 0) &= C(F_{Y_1}(y), \pi(1, 0); \rho_{1,y}(1, 0)), \\ F_{Y|D,\mathbf{Z}}(y \mid 1, (0, 1))\pi(0, 1) &= C(F_{Y_1}(y), \pi(0, 1); \rho_{1,y}^0), \\ F_{Y|D,\mathbf{Z}}(y \mid 1, (0, 0))\pi(0, 0) &= C(F_{Y_1}(y), \pi(0, 0); \rho_{1,y}^0), \end{aligned}$$

where there are four unknowns: $(F_{Y_1}(y), \rho_{1,y}(1, 1), \rho_{1,y}(1, 0), \rho_{1,y}^0)$. Then we can show that the corresponding Jacobian has full rank as long as

$$\frac{C_\rho(F_{Y_1}(y), \pi(0, 1); \rho_{1,y}^0)}{C_1(F_{Y_1}(y), \pi(0, 1); \rho_{1,y}^0)} \neq \frac{C_\rho(F_{Y_1}(y), \pi(0, 0); \rho_{1,y}^0)}{C_1(F_{Y_1}(y), \pi(0, 0); \rho_{1,y}^0)},$$

which is guaranteed since $\pi(0, 1) \neq \pi(0, 0)$ by [REL2](#), and Lemma 4.1 in [Han and Vytlacil \(2017\)](#).¹⁴ For general K , we can prove that the Jacobian of the corresponding system of equations for $D = 1$ has full rank as long as

$$\frac{C_\rho(F_{Y_1}(y), \pi(0, \dots, 0, 1); \rho_{1,y}^0)}{C_1(F_{Y_1}(y), \pi(0, \dots, 0, 1); \rho_{1,y}^0)} \neq \frac{C_\rho(F_{Y_1}(y), \pi(0, \dots, 0, 0); \rho_{1,y}^0)}{C_1(F_{Y_1}(y), \pi(0, \dots, 0, 0); \rho_{1,y}^0)},$$

which is guaranteed by $\pi(0, \dots, 0, 1) \neq \pi(0, \dots, 0, 0)$. Then we identify $2^K \times 1$ vector $(F_{Y_1}(y), \{\rho_{1,y}(\mathbf{z}) : \mathbf{z}_{-K} \neq \mathbf{0}\}, \rho_{1,y}^0)$. A desirable aspect is that no matter how large the system is (i.e., how large 2^K is), the proof of full rank always amounts to checking the ratio of copula derivatives between the two groups defined by the last instrument Z_K given $\mathbf{Z}_{-K} = (0, \dots, 0)$, the status

¹⁴Note that in proving full rank of the Jacobian, only $\pi(0, 1) \neq \pi(0, 0)$ is used. However, $\pi(z_1, z_2) \neq \pi(z'_1, z'_2)$ for any $(z_1, z_2) \neq (z'_1, z'_2)$ are fully utilized to generate the four equations above, because otherwise we have less equations than unknown. [\[IFV: I do not think this is needed. \$F_{Y_1}\(y\)\$ and \$\rho_{1,y}^0\$ are identified from the last 2 equations using the same argument as in the binary instrument case. \$\rho_{1,y}\(1, 1\)\$ and \$\rho_{1,y}\(1, 0\)\$ are then identified from the first 2 equations without additional restrictions\]](#)

quo.

CI2 may become innocuous with large K , because within a finer cell (defined by \mathbf{Z}), individuals tend to be homogeneous and thus share the same joint distribution of (Y_1, V_z) , justifying the copula invariance. The trade-off is that in this case instruments may be weak (i.e., $\pi(0, \dots, 0, 1) \approx \pi(0, \dots, 0, 0)$) for the same reason. Therefore, a large K may not necessarily be preferred. Finally, note that Z_k being binary is not essential and we can have discrete or continuous Z_k with different supports across instruments.

D Identification with Covariates

In this section, we extend our main identification analyses to the case where covariates X present. We focus on binary and continuous D . The case with ordered D is analogous to that with binary D . Consider (potentially endogenous) covariate $X \in \mathcal{X}$.

Assumption EX3 (Conditional Independence). *For $d \in \mathcal{D}$ and $z \in \{0, 1\}$, $Z \perp\!\!\!\perp Y_d \mid X$ and $Z \perp\!\!\!\perp V_z \mid X$.*

Assumption REL3 (Relevance). *(i) $Z \in \{0, 1\}$; (ii) $0 < \Pr(Z = 1 \mid X) < 1$, almost surely; and (iii) for $\mathcal{D} = \{0, 1\}$, $\Pr(D = 1 \mid Z = 1, X) \neq \Pr(D = 1 \mid Z = 0, X)$ and $0 < \Pr(D = 1 \mid Z = z, X) < 1$ almost surely, for $z \in \{0, 1\}$ and, for uncountable \mathcal{D} , $F_{D|Z,X}(d \mid 1, X) \neq F_{D|Z,X}(d \mid 0, X)$ and $0 < F_{D|Z,X}(d \mid z, X) < 1$ almost surely, for $(z, d) \in \{0, 1\} \times \text{int}(\mathcal{D})$.*

Assumption CI3 (Conditional Copula Invariance). *For $d \in \mathcal{D}$, $\rho_{Y_d, V_z; Z, X}(y, v; z, x)$ is a constant function of (v, z) , that is*

$$\rho_{Y_d, V_z; Z, X}(y, v; z, X) = \rho_{Y_d; X}(y; X), \text{ almost surely, } (y, v, z) \in \mathcal{Y} \times \mathcal{V} \times \{0, 1\}.$$

Note that copula invariance is allowed to hold conditional of covariates. Therefore, we allow for observed heterogeneity in the dependence structure. We show the identifiability of

$F_{d,y}(x) \equiv F_{Y_d|X}(y | x)$, from which we can construct conditional parameters:

$$QSF_\tau(d; x) \equiv Q_{Y_d|X}(\tau|x) = \mathcal{Q}_\tau(F_{Y_d|X}(\cdot|x)),$$

$$ASF(d; x) \equiv E[Y_d|X = x] = \mathcal{E}(F_{Y_d|X}(\cdot|x)).$$

Marginal QSF_τ and ASF are also identified from

$$F_{Y_d}(y) = \int F_{Y_d|X}(y | x) dF_X(x),$$

where F_X is the distribution of X .

D.1 Binary Treatment

Define a selection equation $D_z = 1\{V_z \leq \pi(z, X)\}$, where $\pi(z, x) \equiv \Pr[D = 1 | Z = z, X = x]$. Consider

$$\begin{aligned} \Pr[Y \leq y, D = 1 | Z = z, X = x] &= \Pr[Y_1 \leq y, V_z \leq \pi(z, x) | Z = z, X = x] \\ &= C(F_{Y_1|X}(y | x), \pi(z, x); \rho_{Y_1;X}(y; x)), \quad z \in \{0, 1\}, \end{aligned}$$

where the last equation is by [EX3](#) and [CI3](#). Now, let $F_{d,y}(x) \equiv F_{Y_d|X}(y | x)$, and $\rho_{d,y}(x) \equiv \rho_{Y_d}(y; x)$. Then, we have the system of two equations

$$\Pr[Y \leq y, D = 1 | Z = 1, X = x] = C(F_{1,y}(x), \pi(1, x); \rho_{1,y}(x)),$$

$$\Pr[Y \leq y, D = 1 | Z = 0, X = x] = C(F_{1,y}(x), \pi(0, x); \rho_{1,y}(x)),$$

with two unknowns for every $x \in \mathcal{X}$: $(F_{1,y}(x), \rho_{1,y}(x))$. This system has full rank if

$$\frac{C_\rho(F_{1,y}(x), \pi(1, x); \rho_{1,y}(x))}{C_1(F_{1,y}(x), \pi(1, x); \rho_{1,y}(x))} \neq \frac{C_\rho(F_{1,y}(x), \pi(0, x); \rho_{1,y}(x))}{C_1(F_{1,y}(x), \pi(0, x); \rho_{1,y}(x))}$$

for all x , which is guaranteed by [REL3](#).

Remark D.1. *CI3 and CI2 are complementary. Which one to impose depends on the plausibility in given applications. On the one hand, CI3 imposes invariance for every subgroup defined by $X = x$, whereas CI2 imposes invariance for a single subgroup defined by $\mathbf{Z}_{-K} = (0, \dots, 0)$. On the other hand, CI2 imposes stronger exclusion restrictions.*

D.2 Continuous Treatment

Let $F_{d,y}(x) \equiv F_{Y_d|X}(y | x)$ and $\pi_{z,d}(x) \equiv F_{D_z|X}(d | x)$. Assume that $d \mapsto F_{D_z|X}(d | X)$ is strictly increasing almost surely.

For the generalized selection $D_z = h(z, X, V_z)$, we let $V_z \equiv F_{D_z|X}(D_z | X)$ and $h(z, X, \cdot) \equiv F_{D_z|X}^{-1}(\cdot | X)$, so that $\pi_{z,\cdot}(x) = h^{-1}(z, x, \cdot)$. Also note that $V_z \perp\!\!\!\perp Z$ by Assumption EX3 and $v \mapsto h(z, x, v)$ is strictly increasing. Then, by EX3,

$$F_{Y_d, D_z|Z, X}(y, d | z, x) = C(F_{d,y}(x), \pi_{z,d}(x); \rho_{Y_d, V_z; Z, X}(y, \pi_{z,d}(x); z, x)).$$

By Assumption CI3, $\rho_{Y_d, V_z; Z, X}(y, \pi_{z,d}(x); z, x) \equiv \rho_{Y_d}(y; x) \equiv \rho_{d,y}(x)$. Then,

$$\begin{aligned} F_{Y|D, Z, X}(y | d, z, x) &= F_{Y_d|D_z, Z, X}(y | d, z, x) = \frac{(\partial/\partial d)F_{Y_d, D_z|Z, X}(y, d | z, x)}{(\partial/\partial d)F_{D_z|Z, X}(d | z, x)} \\ &\equiv \Phi(a_{d,y}(x) + b_{d,y}(x)\Phi^{-1}(\pi_{z,d}(x))) \end{aligned}$$

where $a_{d,y}(x) \equiv \Phi^{-1}(F_{d,y}(x))/\sqrt{1 - \rho_{d,y}^2(x)}$ and $b_{d,y}(x) \equiv -\rho_{d,y}(x)/\sqrt{1 - \rho_{d,y}^2(x)}$. The argument from here is the same as in the case without covariates.

E Alternative Identification Strategies

For the case of binary D , we show there can be alternative identification strategies using a version of copula invariance. The analysis can be extended to the ordered treatment case. Here we assume EX and REL and the treatment selection equation $D_z = 1[V_z \leq p(z)]$ where $V_z \sim U[0, 1]$. We consider strategies that use a subpopulation defined by each treatment level

separately and strategies that combine the two subpopulations.

E.1 Restrictions Within Treatment Levels

We focus here on the level of $D = 1$. A similar analysis follows for $D = 0$. For $y \in \mathcal{Y}$, consider the LGR of the observed probabilities, that is

$$\Pr(Y_1 \leq y, D = 1 \mid Z = z) = C(F_{Y_1}(y), \pi(z); \rho_1(y, \pi(z); z)), \quad z \in \{0, 1\},$$

where $\rho_d(y, \pi(z); z) \equiv \rho_{Y_d, V_z; Z}(y, \pi(z); z)$. The identification problem is that we have two probabilities to identify three parameters: $F_{Y_1}(y)$, $\rho_1(y, \pi(0); 0)$ and $\rho_1(y, \pi(1); 1)$. So far, we have reduced the number of parameters by imposing the condition:

$$\rho_1(y, \pi(0); 0) = \rho_1(y, \pi(1); 1).$$

This restriction is imposed separately for each value of y . However, it is also possible to impose restrictions across values of y . Assume that there exists $y' \in \mathcal{Y}$ be such that $F_{Y_1}(y) \neq F_{Y_1}(y')$ and

$$\rho_1(y, \pi(z); z) = \rho_1(y', \pi(z); z), \quad z \in \{0, 1\}. \quad (\text{E.1})$$

This condition leads to the following system of four equations with four unknowns:

$$\begin{aligned} \Pr(Y_1 \leq y, D = 1 \mid Z = z) &= C(F_{Y_1}(y), \pi(z); \rho_1(y, \pi(z); z)), \quad z \in \{0, 1\}, \\ \Pr(Y_1 \leq y', D = 1 \mid Z = z) &= C(F_{Y_1}(y'), \pi(z); \rho_1(y, \pi(z); z)), \quad z \in \{0, 1\}. \end{aligned}$$

Then, it is possible to find conditions under which the solution to this system exists and is unique. This condition is appealing in that it does not imposes restrictions across levels of z .

Let C_j denote the partial derivative of the Gaussian copula C with respect to the j th

argument, $j \in \{1, 2, 3\}$. The Jacobian of the system of equations is

$$J(y, y') = \begin{pmatrix} C_1(y, 1) & 0 & C_3(y, 1) & 0 \\ C_1(y, 0) & 0 & 0 & C_3(y, 0) \\ 0 & C_1(y', 1) & C_3(y', 1) & 0 \\ 0 & C_1(y', 0) & 0 & C_3(y', 0) \end{pmatrix},$$

where $C_j(k, z) := C_j(F_{Y_1}(k), \pi(z); \rho_1(k, \pi(z); z)) > 0$ for $j \in \{1, 3\}$, $k \in \{y, y'\}$ and $z \in \{0, 1\}$.

By the Laplace expansion, the Jacobian determinant is

$$\det(J(y, y')) = C_1(y, 0)C_1(y', 1)C_3(y, 1)C_3(y', 0) - C_1(y, 1)C_1(y', 0)C_3(y', 1)C_3(y, 0),$$

which does not vanish if

$$\frac{C_3(y, 1) C_3(y', 0)}{C_1(y, 1) C_1(y', 0)} \neq \frac{C_3(y, 0) C_3(y', 1)}{C_1(y, 0) C_1(y', 1)}.$$

Let

$$\lambda(k, z) = \frac{\phi(u(k, z))}{\Phi(u(k, z))}, \quad u(k, z) := \frac{\Phi^{-1}(\pi(z)) - \rho_1(k, \pi(z); z)\Phi^{-1}(F_{Y_1}(k))}{\sqrt{1 - \rho_1(k, \pi(z); z)^2}},$$

where ϕ and Φ are the standard normal PDF and CDF, respectively. Then, using that $C_3(k, z) = \lambda(k, z)C_1(k, z)$, the previous condition can be expressed as

$$\frac{\lambda(y, 1)}{\lambda(y, 0)} \neq \frac{\lambda(y', 1)}{\lambda(y', 0)} \quad \text{or} \quad \frac{\lambda(y, 1)}{\lambda(y', 1)} \neq \frac{\lambda(y, 0)}{\lambda(y', 0)}, \quad (\text{E.2})$$

that is, the change in the conditional inverse Mills ratio from $z = 0$ to $z = 1$ is different at y and y' , or the change in the conditional inverse Mills ration from y to y' is different at $z = 0$ and $z = 1$. For example, if the identification condition holds locally for $y' = y + dy$, then the condition becomes

$$\frac{\partial \log \lambda(y, 1)}{\partial y} \neq \frac{\partial \log \lambda(y, 0)}{\partial y}.$$

Theorem E.1. *Suppose $D \in \{0, 1\}$ satisfies (3.1). Suppose Assumptions [EX](#) and [REL](#) holds. Given $y \in \mathcal{Y}$, suppose that there exists $y' \in \mathcal{Y}$ such that (E.1) and (E.2) hold. Then, $F_{Y_d}(y)$ and $\rho_{Y_d}(y)$ are identified for $d \in \{0, 1\}$.*

In general, let d_y be the number of values of Y that we use to construct the system of equations. Then, we have $2d_y$ equations and $3d_y$ unknowns. Therefore, we need to reduce d_y parameters by whichever combinations of copula invariance (E.1) and the alternative assumptions.

E.2 Restrictions Between Treatment Levels

Alternative to the previous subsection, we can impose restrictions involving parameters for different treatment levels. This strategy is based on the system of equations

$$\begin{aligned}\Pr(Y_1 \leq y, D = 1 \mid Z = z) &= C(F_{Y_1}(y), \pi(z); \rho_1(y, \pi(z); z)), \quad z \in \{0, 1\}, \\ \Pr(Y_0 \leq y, D = 0 \mid Z = z) &= F_{Y_0}(y) - C(F_{Y_0}(y), \pi(z); \rho_0(y, \pi(z); z)), \quad z \in \{0, 1\},\end{aligned}$$

where $\rho_d(y, \pi(z); z) \equiv \rho_d(F_{Y_d}(y), \pi(z); z)$ for clarification. Assume RS analogous to that in [Chernozhukov and Hansen \(2005\)](#):

$$\rho_0(y, \pi(z); z) = \rho_1(\phi(y), \pi(z); z), \quad z \in \{0, 1\}, \tag{E.3}$$

where $\phi(y)$ is such that $F_{Y_0}(y) = F_{Y_1}(\phi(y))$, leading to a system of four equations with four unknowns

$$\begin{aligned}\Pr(Y_1 \leq \phi(y), D = 1 \mid Z = z) &= C(F_{Y_0}(y), \pi(z); \rho_0(y, \pi(z); z)), \quad z \in \{0, 1\}, \\ \Pr(Y_0 \leq y, D = 0 \mid Z = z) &= F_{Y_0}(y) - C(F_{Y_0}(y), \pi(z); \rho_0(y, \pi(z); z)), \quad z \in \{0, 1\}.\end{aligned}$$

Adding the previous equations yields

$$\Pr(Y_1 \leq \phi(y), D = 1 \mid Z = z) + \Pr(Y_0 \leq y, D = 0 \mid Z = z) = F_{Y_0}(y), \quad z \in \{0, 1\},$$

so that $\phi(y)$ can be identified from

$$\begin{aligned} \Pr(Y_1 \leq \phi(y), D = 1 \mid Z = 1) + \Pr(Y_0 \leq y, D = 0 \mid Z = 1) \\ = \Pr(Y_1 \leq \phi(y), D = 1 \mid Z = 0) + \Pr(Y_0 \leq y, D = 0 \mid Z = 0), \end{aligned} \quad (\text{E.4})$$

provided that this equation has unique solution; see Section [A.1](#) for related discussions. This is the same condition used in [Vuong and Xu \(2017\)](#), but we do not rely on absolute continuity to arrive to this equation. Another feature of our approach is that we identify the local dependence parameter. If Y_0 and Y_1 are discrete, however, this equation might not have solution. If $\phi(y)$ is identified, then $F_{Y_0}(y)$ and $\rho_0(y, p(0); 0)$ are identified from

$$\begin{aligned} \Pr(Y_1 \leq \phi(y), D = 1 \mid Z = 0) &= C(F_{Y_0}(y), \pi(0); \rho_0(y, \pi(0); 0)), \\ \Pr(Y_0 \leq y, D = 0 \mid Z = 0) &= F_{Y_0}(y) - C(F_{Y_0}(y), \pi(0); \rho_0(y, \pi(0); 0)), \end{aligned}$$

because the Jacobian of this system

$$J(y) = \begin{pmatrix} C_1(y, 0) & C_3(y, 0) \\ 1 - C_1(y, 0) & -C_3(y, 0) \end{pmatrix}$$

has negative determinant everywhere:

$$\det(J(y)) = -C_3(y, 0) < 0.$$

Finally, $\rho_0(y, \pi(1); 1)$ is identified from

$$\Pr(Y_1 \leq \phi(y), D = 1 \mid Z = 1) = C(F_{Y_0}(y), \pi(1); \rho_0(y, \pi(1); 1)),$$

because the right hand side is monotonically increasing in $\rho_0(y, \pi(1); 1)$.

Theorem E.2. *Suppose $D \in \{0, 1\}$ satisfies (3.1). Suppose Assumptions [EX](#) and [REL](#) hold. Also, suppose that (E.3) holds and (E.4) has a unique solution in ϕ . Then, $F_{Y_d}(y)$ and $\rho_{Y_d}(y)$ are identified for $d \in \{0, 1\}$ and $y \in \mathcal{Y}$.*

[Vuong and Xu \(2017\)](#) assume continuity and RI on the potential outcomes such that $Y_1 = \phi(Y_0)$ a.s. for a strictly monotone transformation $y \mapsto \phi(y)$. Using the equations above with $y' = \phi(y)$ and taking differences between $z = 1$ and $z = 0$ yields

$$\begin{aligned} \Pr(Y_0 \leq y, D = 0 \mid Z = 0) - \Pr(Y_0 \leq y, D = 0 \mid Z = 1) \\ = \Pr(Y_1 \leq \phi(y), D = 1 \mid Z = 1) - \Pr(Y_1 \leq \phi(y), D = 1 \mid Z = 0). \end{aligned}$$

This equation identifies the mapping $y \mapsto \phi(y)$ under monotonicity and support conditions. Another possibility is to impose LATE monotonicity assumptions on the treatment selection equation and identify ϕ from the distributions of the potential outcomes for the compliers. Note that we can also combine the restrictions within and between treatment levels.

F Alternative Selection Equation

When D is binary, alternative to (3.1), we may consider

$$D_z = 1\{D_z^* \leq 0\}, \tag{F.1}$$

where D_1^* and D_0^* are two distinct r.v.'s. Let $D^* \equiv D_Z^*$ so that $D = 1\{D^* \leq 0\}$, which is consistent with the notation in [Chernozhukov et al. \(2020a\)](#). Note that (3.1) is a special case

of (F.1) where $V_z \equiv D_z^* + \pi(z)$ so that $D_z^* = V_z - \pi(z)$ and is normalized to $V_z \sim U[0, 1]$. The alternative LGR using D_z^* still requires a similar version of CI due to the following:

$$\begin{aligned} F_{Y|D,Z}(y \mid D = 1, Z = z)\pi(z) &= \Pr[Y_1 \leq y, D_z^* \leq 0 \mid Z = z] \\ &= C(F_{Y_1|Z}(y \mid z), F_{D_z^*|Z}(0 \mid z); \rho_{Y_1, D_z^*; Z}(y, 0; z)) \\ &= C(F_{Y_1}(y), F_{D_z^*|Z}(0 \mid z); \rho_{Y_1}(y)), \end{aligned}$$

where the last equation is by EX and the following CI:

$$\rho_{Y_1, D_1^*; Z}(y, 0; 1) = \rho_{Y_1, D_0^*; Z}(y, 0; 0). \quad (\text{F.2})$$

This CI looks weaker than the original CI under (3.1). This is not the case, however, which becomes clear if we use the original correlation coefficient function from Lemma 2.1:

$$\rho_{U_1, U_{21}; Z}(F_{Y_1}(y), F_{D_1^*|Z}(0 \mid 1); 1) = \rho_{U_1, U_{20}; Z}(F_{Y_1}(y), F_{D_0^*|Z}(0 \mid 0); 0),$$

where $U_1 \equiv F_{Y_1}(Y_1)$ and $U_{2z} \equiv F_{D_z^*|Z}(D_z^*|Z)$. Note that we require ρ being a constant function of the second argument (in addition to other invariance requirements). This is still the case even with a stronger version of independence $(Y_d, D_z^*) \perp\!\!\!\perp Z$, in which we require

$$\rho_{U_1, U_{21}}(F_{Y_1}(y), F_{D_1^*}(0)) = \rho_{U_1, U_{20}}(F_{Y_1}(y), F_{D_0^*}(0)), \quad (\text{F.3})$$

where $U_1 \equiv F_{Y_1}(Y_1)$ and $U_{2z} \equiv F_{D_z^*}(D_z^*)$.

Similar discussions can be made in the case of ordered and continuous treatments. For example, with continuous D , we can alternatively introduce the following copula invariance without introducing V_z : (i) $\rho_{Y_d, D_z; Z}(y, d; z) = \rho_{Y_d, D_z}(y, d)$ and (ii) $\rho_{Y_d, D_z}(y, d) = \rho_{Y_d, D_z}(y)$. Note that (i) holds if and only if $(Y_d, D_z) \perp\!\!\!\perp Z$ and (ii) is a CI that resembles (F.3).

G Proofs

G.1 Proof of Theorem 3.1

Note that $\pi(z)$ is identified as a reduced-form parameter. Let $F_{d,y} \equiv F_{Y_d}(y)$ and $\rho_{d,y} \equiv \rho_{Y_d}(y)$ be the structural parameters of interest. Consider the following mapping between the structural and reduced-form parameters:

$$F_{Y|D,Z}(y|1,0)\pi(0) = C(F_{1,y}, \pi(0); \rho_{1,y}), \quad (\text{G.1})$$

$$F_{Y|D,Z}(y|1,1)\pi(1) = C(F_{1,y}, \pi(1); \rho_{1,y}), \quad (\text{G.2})$$

or $\pi_y = G(\theta_y)$ where $\theta_y \equiv (F_{1,y}, \rho_{1,y})'$ and $\pi_y \equiv (F_{Y|D,Z}(y|1,0)\pi(0), F_{Y|D,Z}(y|1,1)\pi(1))'$. Let C_1 , C_2 and C_ρ denote the derivative of copula $C(u_1, u_2; \rho)$ with respect to u_1 , u_2 and ρ , respectively. Consider the Jacobian of the system of nonlinear equations (G.1)–(G.2):

$$J = \frac{\partial G}{\partial \theta_y} = \begin{bmatrix} C_1(F_{1,y}, \pi(0); \rho_{1,y}) & C_\rho(F_{1,y}, \pi(0); \rho_{1,y}) \\ C_1(F_{1,y}, \pi(1); \rho_{1,y}) & C_\rho(F_{1,y}, \pi(1); \rho_{1,y}) \end{bmatrix}.$$

The matrix has full rank if and only if

$$\frac{C_\rho(F_{1,y}, \pi(1); \rho_{1,y})}{C_1(F_{1,y}, \pi(1); \rho_{1,y})} \neq \frac{C_\rho(F_{1,y}, \pi(0); \rho_{1,y})}{C_1(F_{1,y}, \pi(0); \rho_{1,y})}, \quad (\text{G.3})$$

which is true by Assumption REL and Lemma 4.1 in Han and Vytlačil (2017) as Gaussian copula satisfies the stochastically increasing ordering condition (Assumption 6 in Han and Vytlačil (2017)). Therefore, the matrix is a P-matrix (except at the boundary of the parameter space), and thus one can apply Gale and Nikaido (1965)'s global univalence theorem, which identifies θ_y .¹⁵

¹⁵In cases where we combine more equations, the principle minors of the resulting Jacobian may be zero. In that case, Hadamard's global inverse function theorem can be applied instead. According to Hadamard's theorem (Hadamard, 1906), the solution of $\pi_y = G(\theta_y)$ is unique if (i) G is proper, (ii) the Jacobian of G vanishes nowhere, and (iii) $G(\Theta_y)$ is simply connected. Condition (i) trivially holds with our definition of G . Since the parameter space $\Theta_y = [0, 1] \times [-1, 1]$ is simply connected and G is continuous, Condition

Analogously, we have

$$\begin{aligned}
F_{Y|D,Z}(y|D=0, Z=z)(1-\pi(z)) &= \Pr[Y_0 \leq y|Z=z] - \Pr[Y_0 \leq y, V_z \leq \pi(z)|Z=z] \\
&= \Pr[Y_0 \leq y|Z=z] - C(F_{Y_0|Z}(y|z), \pi(z); \rho_{Y_0, V_z; Z}(y, \pi(z); z)) \\
&= \Pr[Y_0 \leq y] - C(F_{Y_0}(y), \pi(z); \rho_{Y_0}(y))
\end{aligned}$$

and

$$F_{y|0,0} \cdot (1 - \pi(0)) = F_{Y_0}(y) - C(F_{Y_0}(y), \pi(0); \rho_{0,y}), \quad (\text{G.4})$$

$$F_{y|0,1} \cdot (1 - \pi(1)) = F_{Y_0}(y) - C(F_{Y_0}(y), \pi(1); \rho_{0,y}), \quad (\text{G.5})$$

and the mapping has a unique solution for $\tilde{\theta}_y \equiv (F_{Y_0}(y), \rho_{0,y})'$ by a similar argument as above.

G.2 Proof of Theorem 3.2

Recall from the text that we additionally impose copula invariance between a pair of levels:

$$\rho_{Y_d, V_z; Z}(y, \pi_d(z); z) = \rho_{Y_d, V_z; Z}(y, \pi_{d-1}(z); z) \equiv \rho_{Y_d}(y) \equiv \rho_{d,y}. \quad (\text{G.6})$$

Now, following [Ambrosetti and Prodi \(1995\)](#), we show that (i) the system has a unique solution when $\rho_{d,y} = 0$, (ii) the function that defines the system is continuous and proper with a range that is a connected set, and (iii) it is locally invertible. Similar to the argument in Footnote 15, (ii) can be easily shown. Note that (i) is trivially true. Therefore, we are remained to prove (iii) by showing the full rank of the following Jacobian with $F_{d,y} \equiv F_{Y_d}(y)$:

$$J_d = \begin{bmatrix} C_1(F_{d,y}, \pi_d(0); \rho_{d,y}) - C_1(F_{d,y}, \pi_{d-1}(0); \rho_{d,y}) & C_\rho(F_{d,y}, \pi_d(0); \rho_{d,y}) - C_\rho(F_{d,y}, \pi_{d-1}(0); \rho_{d,y}) \\ C_1(F_{d,y}, \pi_d(1); \rho_{d,y}) - C_1(F_{d,y}, \pi_{d-1}(1); \rho_{d,y}) & C_\rho(F_{d,y}, \pi_d(1); \rho_{d,y}) - C_\rho(F_{d,y}, \pi_{d-1}(1); \rho_{d,y}) \end{bmatrix}.$$

(iii) holds if the Jacobian of G is positive or negative semi-definite on Θ_y because simple connectedness is preserved under a monotone map. We can show that the Jacobian is semidefinite and has full rank, which prove Conditions (iii) and (ii), respectively, and hence the uniqueness of the solution.

This Jacobian has full rank if and only if

$$\frac{C_\rho(F_{d,y}, \pi_d(1); \rho_{d,y}) - C_\rho(F_{d,y}, \pi_{d-1}(1); \rho_{d,y})}{C_1(F_{d,y}, \pi_d(1); \rho_{d,y}) - C_1(F_{d,y}, \pi_{d-1}(1); \rho_{d,y})} \neq \frac{C_\rho(F_{d,y}, \pi_d(0); \rho_{d,y}) - C_\rho(F_{d,y}, \pi_{d-1}(0); \rho_{d,y})}{C_1(F_{d,y}, \pi_d(0); \rho_{d,y}) - C_1(F_{d,y}, \pi_{d-1}(0); \rho_{d,y})}. \quad (\text{G.7})$$

Showing this is more involved than showing (G.3) with the binary treatment, because the equality can arise due to two points on the indifference curve. Nonetheless, the full-rank condition (G.7) can be expressed as $\lambda(0) \neq \lambda(1)$, where

$$\lambda(z) \equiv \frac{\phi(r_d(z)) - \phi(r_{d-1}(z))}{\Phi(r_d(z)) - \Phi(r_{d-1}(z))}, \quad r_\ell(z) \equiv \frac{\Phi^{-1}(\pi_\ell(z)) - \rho_{d,y} F_{d,y}}{\sqrt{1 - \rho_{d,y}^2}},$$

and $\Phi(\cdot)$ and $\phi(\cdot)$ are univariate Gaussian CDF and PDF, respectively. To interpret this condition, we note that it can be related to the mean of truncated Gaussian random variable (r.v.): for $A \sim N(\mu, \sigma^2)$,

$$E(A \mid l < A < u) = \mu - \sigma \frac{\phi\left(\frac{u-\mu}{\sigma}\right) - \phi\left(\frac{l-\mu}{\sigma}\right)}{\Phi\left(\frac{u-\mu}{\sigma}\right) - \Phi\left(\frac{l-\mu}{\sigma}\right)}.$$

This formula simplifies to the standard inverse Mills ratio under one-sided truncation. Therefore, the full-rank condition $\lambda(0) \neq \lambda(1)$ can be equivalently expressed as

$$E[A \mid \pi_{d-1}(0) < \Phi(A) < \pi_d(0)] \neq E[A \mid \pi_{d-1}(1) < \Phi(A) < \pi_d(1)]$$

with $\mu = \rho_{d,y} F_{d,y}$ and $\sigma^2 = 1 - \rho_{d,y}^2$. For example, this holds when threshold functions are such that $\pi_{d-1}(0) < \pi_{d-1}(1)$ and $\pi_d(0) < \pi_d(1)$. By transitivity, Assumption U_{OC} guarantees this.

G.3 Proof of Lemma 3.1

Before presenting a formal proof, it is helpful to consider an illustrative example with $K = 4$.

In this case we have three complier groups and three defier groups:

$$C_1 \equiv \{D_0 = 1, D_1 = 2\} \cup \{D_0 = 2, D_1 = 3\} \cup \{D_0 = 3, D_1 = 4\},$$

$$C_2 \equiv \{D_0 = 1, D_1 = 3\} \cup \{D_0 = 2, D_1 = 4\},$$

$$C_3 \equiv \{D_0 = 1, D_1 = 4\},$$

$$B_1 \equiv \{D_1 = 1, D_0 = 2\} \cup \{D_1 = 2, D_0 = 3\} \cup \{D_1 = 3, D_0 = 4\},$$

$$B_2 \equiv \{D_1 = 1, D_0 = 3\} \cup \{D_1 = 2, D_0 = 4\},$$

$$B_3 \equiv \{D_1 = 1, D_0 = 4\}.$$

Note that the union of

$$C_1 \equiv \{D_0 = 1, D_1 = 2\} \cup \{D_0 = 2, D_1 = 3\} \cup \{D_0 = 3, D_1 = 4\},$$

$$C_2 \equiv \{D_0 = 1, D_1 = 3\} \cup \{D_0 = 2, D_1 = 4\},$$

$$C_3 \equiv \{D_0 = 1, D_1 = 4\}$$

is identical to the double-counting union of

$$C_1 \equiv \{D_0 = 1, D_1 = 2\} \cup \{D_0 = 2, D_1 = 3\} \cup \{D_0 = 3, D_1 = 4\},$$

$$C_2 \equiv \{D_0 = 1, D_1 = 3\} \cup \{D_0 = 2, D_1 = 4\},$$

$$C_3 \equiv \{D_0 = 1, D_1 = 4\},$$

$$C_2 \equiv \{D_0 = 1, D_1 = 3\} \cup \{D_0 = 2, D_1 = 4\},$$

$$C_3 \equiv \{D_0 = 1, D_1 = 4\} \cup \{D_0 = 1, D_1 = 4\}.$$

Then, by taking the union in each column of above expression, we have

$$\begin{aligned}
\Pr \left[\bigcup_{j=1}^3 C_j \right] &= \Pr[\{0 < V_0 \leq \pi_1(0), \pi_1(1) < V_1 \leq 1\} \\
&\quad \cup \{0 < V_0 \leq \pi_2(0), \pi_2(1) < V_1 \leq 1\} \\
&\quad \cup \{0 < V_0 \leq \pi_3(0), \pi_3(1) < V_1 \leq 1\}] \\
&= \Pr[\{0 < V_1 \leq \pi_1(0), \pi_1(1) < V_0 \leq 1\} \\
&\quad \cup \{0 < V_1 \leq \pi_2(0), \pi_2(1) < V_0 \leq 1\} \\
&\quad \cup \{0 < V_1 \leq \pi_3(0), \pi_3(1) < V_0 \leq 1\}] \\
&< \Pr[\{0 < V_1 \leq \pi_1(1), \pi_1(0) < V_0 \leq 1\} \\
&\quad \cup \{0 < V_1 \leq \pi_2(1), \pi_2(0) < V_0 \leq 1\} \\
&\quad \cup \{0 < V_1 \leq \pi_3(1), \pi_3(0) < V_0 \leq 1\}] \\
&= \Pr \left[\bigcup_{j=1}^3 B_j \right],
\end{aligned}$$

where the second equality is by Assumption [EG](#) and the inequality is by $\pi_d(1) > \pi_d(0)$ for all $d = 1, 2, 3$.

Now, the following is the formal proof of the lemma. Let $\pi_0(z) = 0$ and $\pi_K(z) = 1$ for all

z . Then,

$$\begin{aligned}
\Pr \left[\bigcup_{j=1}^{K-1} C_j \right] &= \Pr \left[\bigcup_{j=1}^{K-1} \bigcup_{s=0}^{s+j+1=K} \{ \pi_s(0) < V_0 \leq \pi_{s+1}(0), \pi_{s+j}(1) < V_1 \leq \pi_{s+j+1}(1) \} \right] \\
&= \Pr \left[\bigcup_{j=1}^{K-1} \{ 0 < V_0 \leq \pi_j(0), \pi_j(1) < V_1 \leq 1 \} \right] \\
&= \Pr \left[\bigcup_{j=1}^{K-1} \{ 0 < V_1 \leq \pi_j(0), \pi_j(1) < V_0 \leq 1 \} \right] \\
&< \Pr \left[\bigcup_{j=1}^{K-1} \{ 0 < V_1 \leq \pi_j(1), \pi_j(0) < V_0 \leq 1 \} \right] \\
&= \Pr \left[\bigcup_{j=1}^{K-1} B_j \right],
\end{aligned}$$

where the second equality is from the derivation similar to the case of $K = 4$, the third equality is by Assumption [EG](#), and the inequality is by $\pi_d(1) > \pi_d(0)$ for all $d \in \mathcal{D} \setminus \{K\}$. The proof of the opposite direction of inequality is symmetric.

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