# On Quantile Treatment Effects, Rank Similarity, and Multiple IVs

Sukjin Han & Haiqing Xu

U of Bristol & UT Austin

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# Observed vs. Counterfactual Distributions

**Question**:

How relationship between observed vs. counterfactual distributions plays a role in identification of treatment effects under endogeneity?

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e.g., **Condition**:

For arbitrary $A$ and $\tilde{A}$, if $F_{Y_1|\eta\in A} \leq F_{Y_1|\eta\in \tilde{A}}$ then $F_{Y_0|\eta\in A} \leq F_{Y_0|\eta\in \tilde{A}}$

# Observed vs. Counterfactual Distributions

**Condition 1.1** (preservation of stochastic dominance):

For arbitrary $A$ and $\tilde{A}$,

$$\text{if } F_{Y_1|\eta\in A} \leq F_{Y_1|\eta\in\tilde{A}}$$

$$\text{then } F_{Y_0|\eta\in A} \leq F_{Y_0|\eta\in\tilde{A}}$$

This paper...

- proposes this condition (and related ones) as possible source of identification...
- for quantile treatment effect (QTE) and average treatment effect (ATE) for treated and untreated populations, and
- proposes a simple procedure to calculate bounds using linear programming

# Example

$D$: observed indicator of college degree (endogenous)

$Y_1$: hypothetical earnings with college education

$Y_0$: hypothetical earnings without college education

$Y = DY_1 + (1 - D)Y_0$: observed earnings

$Z$: instrument(s) for $D$ (e.g., local earnings, distance to college)

Common parameters of interest are

$$QTE_\tau = Q_{Y_1}(\tau) - Q_{Y_0}(\tau)$$

and

$$ATE = E[Y_1 - Y_0]$$

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...or more fundamentally

$$QTE_\tau(d) = Q_{Y_1|D=d}(\tau) - Q_{Y_0|D=d}(\tau)$$

and

$$ATE(d) = E[Y_1 - Y_0|D = d]$$

# Well-Known Approach: Rank Similarity

Assume

$$Y_d = q(d, U_d)$$

where $q(d, \cdot)$ is strictly increasing and $U_d \sim U[0, 1]$

Assume $D = h(Z, \eta)$ and $Z \perp U_d$

Rank similarity (Chernozhukov & Hansen 05):

$$F_{U_1|\eta} = F_{U_0|\eta}$$

- observationally equivalent to $U_1 = U_0 \equiv U$ (Chernozhukov & Hansen 13)
- reducing dimension of unobs'ed heterogeneity

Under these assumptions, $QTE_\tau$ and $ATE$ are point identified

# Motivation: Relaxing Rank Similarity

Rank similarity (Chernozhukov & Hansen 05):

$$F_{U_1|\eta} = F_{U_0|\eta}$$

Strong assumption with fragile empirical justification

- e.g., wages in US CPS for past decades (Massoumi & Wang 19)

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Recall **Condition 1.1**:

$$\text{For arbitrary } A \text{ and } \tilde{A}, \quad \begin{array}{c} F_{Y_1|\eta\in A} \leq F_{Y_1|\eta\in\tilde{A}} \\ \Longrightarrow \\ F_{Y_0|\eta\in A} \leq F_{Y_0|\eta\in\tilde{A}} \end{array}$$

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Stronger than Condition 1.1 is **Condition 2**:

For arbitrary $A$ and $\tilde{A}$,
$$\begin{gathered} F_{Y_1|\eta\in A} \leq F_{Y_1|\eta\in\tilde{A}} \\ \Longleftrightarrow \\ F_{Y_0|\eta\in A} \leq F_{Y_0|\eta\in\tilde{A}} \end{gathered}$$

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We show rank similarity implies Condition 2 (within their model)

Therefore, we propose Condition 1.1 as substantial relaxation of rank similarity

# Making Use of Condition 1

**Condition 1.1**:

For arbitrary $A$ and $\tilde{A}$, 
$$\begin{array}{c} F_{Y_1|\eta\in A} \leq F_{Y_1|\eta\in\tilde{A}} \\ \Longrightarrow \\ F_{Y_0|\eta\in\tilde{A}} \leq F_{Y_0|\eta\in\tilde{A}} \end{array}$$

Key step: Find $A$ and $\tilde{A}$ such that

$$F_{Y_1|\eta\in A, D=1} \leq F_{Y_1|\eta\in\tilde{A}, D=1}$$

- more likely if $Z$ can produce finer partition of Supp$(\eta)$
- therefore, multiple IVs can be helpful (makes sense given how much we give up by dropping rank similarity)

# What We Partially Identify

**Condition 1.1**:

For arbitrary $A$ and $\tilde{A}$,
$$F_{Y_1|\eta\in A} \leq F_{Y_1|\eta\in\tilde{A}} \Longrightarrow F_{Y_0|\eta\in A} \leq F_{Y_0|\eta\in\tilde{A}}$$

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We provide models that...

- rationalize these conditions
- justify the *assumption-driven* parameters of interest for a policymaker concerning "conservative" populations

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- rationalize these conditions
- justify the *assumption-driven* parameters of interest for a policymaker concerning "conservative" populations

Condition 2 (with "$\Longleftrightarrow$") yields bounds for $QTE_\tau$ and $ATE$

# Previous Related Approaches

IV quantile models (with rank similarity):

- Chernozhukov & Hansen 05, Vuong & Xu 17

Triangular models (in the sense of explicit first stage):

- Control function approach with continuous $D$: Chesher 03, Lee 07, Imbens & Newey 09
- Threshold crossing first stage with binary $D$:
  - ATE (rank similarity): Shaikh & Vytlacil 11, Vytlacil & Yildiz 07
  - MTE: Mogstad et al. 08, Han & Yang 22

Local QTE with binary $D$:

- Abadie, Angrist & Imbens 02 (local parameter, strong homogeneity with multiple IVs (Mogstad et al. 21))

Generalized IV:

- Chesher & Rosen 17 (sharp bounds with IVs)

# This Paper

This paper...

- relaxes rank similarity (without completely abandoning it) and
- constructs informative bounds on QTE and ATE for the treated or untreated
- for binary endogenous treatment
- using discrete IVs, and
- bounds that are simple to calculate

Condition 1.1 differs from other assumptions on the relationship between $Y_1$ and $Y_0$

- e.g., stochastic increasing for distributional treatment effects (with experimental data) (Frandsen & Lefgren 21)

# I. Key Conditions and Bounds on Treatment Effects

# Maintained Assumptions

$D$: observed treatment indicator (endogenous)

$Y_1$: counterfactual outcome of being treated

$Y_0$: counterfactual outcome of not being treated

$Y = DY_1 + (1 - D)Y_0$

$Z$: vector of binary IVs or a multi-valued IV, taking values $\{z_1, ..., z_L\}$

Suppress $X$ for simplicity; easy to incorporate (in the paper)

Assume $D = h(Z, \eta)$ where $\eta \in \mathcal{T}$ has arbitrary dimensions

Define counterfactual treatment $D_z = h(z, \eta)$

**Assumption Z**

For $d \in \{0, 1\}$ and $z \in \{z_1, ..., z_L\}$, (i) $Y_{d,z} = Y_d$;
(ii) $Z \perp (Y_d, D_z)$.

# Preservation of Stochastic Dominance

## Condition 1.1

For arbitrary weight functions $w : \mathcal{T} \to \mathbb{R}_+$ and $\tilde{w} : \mathcal{T} \to \mathbb{R}_+$ s.t. $\int w(t)dt = \int \tilde{w}(t)dt = 1$,

$$\int w(x)F_{Y_1|\eta}(\cdot|t)dt \leq \int \tilde{w}(t)F_{Y_1|\eta}(\cdot|t)dt$$
$$\Longrightarrow$$
$$\int w(t)F_{Y_0|\eta}(\cdot|t)dt \leq \int \tilde{w}(t)F_{Y_0|\eta}(\cdot|t)dt.$$

# Preservation of Stochastic Dominance

Condition 1.1 implies the following:

## Condition 1.1*

For arbitrary nonnegative weight vectors $(w_1, \ldots, w_L)$ and $(\tilde{w}_1, \ldots, \tilde{w}_L)$ s.t. $\sum_{\ell=1}^{L} w_\ell = \sum_{\ell=1}^{L} \tilde{w}_\ell = 1$,

$$\sum_{\ell=1}^{L} w_\ell P[Y_1 \leq \cdot | D = 1, Z = z_\ell] \leq \sum_{\ell=1}^{L} \tilde{w}_\ell P[Y_1 \leq \cdot | D = 1, Z = z_\ell]$$

$$\Longrightarrow$$

$$\sum_{\ell=1}^{L} w_\ell P[Y_0 \leq \cdot | D = 1, Z = z_\ell] \leq \sum_{\ell=1}^{L} \tilde{w}_\ell P[Y_0 \leq \cdot | D = 1, Z = z_\ell].$$

# Preservation of Stochastic Dominance

Under Assumption Z, Condition $1.1'$ is equivalent to:

## Condition 1.1*

For arbitrary positive weight vectors $(w_1, ..., w_L)$ and $(\tilde{w}_1, ..., \tilde{w}_L)$ s.t. $\sum_{\ell=1}^{L} w_\ell = \sum_{\ell=1}^{L} \tilde{w}_\ell = 1$,

$$\sum_{\ell=1}^{L} w_\ell P[Y_1 \leq \cdot | D_{z_\ell} = 1] \leq \sum_{\ell=1}^{L} \tilde{w}_\ell P[Y_1 \leq \cdot | D_{z_\ell} = 1]$$

$$\Longrightarrow$$

$$\sum_{\ell=1}^{L} w_\ell P[Y_0 \leq \cdot | D_{z_\ell} = 1] \leq \sum_{\ell=1}^{L} \tilde{w}_\ell P[Y_0 \leq \cdot | D_{z_\ell} = 1].$$

# Preservation of Stochastic Dominance

Under Assumption Z, Condition 1.1$'$ is equivalent to:

**Condition 1.1***

For arbitrary positive weight vectors $(w_1, \ldots, w_L)$ and $(\tilde{w}_1, \ldots, \tilde{w}_L)$ s.t. $\sum_{\ell=1}^{L} w_\ell = \sum_{\ell=1}^{L} \tilde{w}_\ell = 1$,

$$\sum_{\ell=1}^{L} w_\ell P[Y_1 \leq \cdot | D_{z_\ell} = 1] \leq \sum_{\ell=1}^{L} \tilde{w}_\ell P[Y_1 \leq \cdot | D_{z_\ell} = 1]$$

$$\Longrightarrow$$

$$\sum_{\ell=1}^{L} w_\ell P[Y_0 \leq \cdot | D_{z_\ell} = 1] \leq \sum_{\ell=1}^{L} \tilde{w}_\ell P[Y_0 \leq \cdot | D_{z_\ell} = 1].$$

$\{D_{z_\ell} = 1\}$ (for $z_1, \ldots, z_L$) captures different compliance types

- e.g., when $L = 2$ with LATE monotonicity, then between always-takers and compliers

# Preservation of Stochastic Dominance

## Lemma 0

Under Assumption Z, Condition 1.1 implies Condition 1.1*.

Let $p(z) \equiv P[D=1|Z=z]$, then $p(z) = P[\eta \in H(z)]$ with $H(z) \equiv \{\eta : 1 = h(z,\eta)\}$

Then, for example,

$$\begin{aligned}
&\sum_\ell w_\ell P[Y_1 \le \cdot | D=1, Z=z_\ell] \\
&= \sum_\ell w_\ell P[Y_1 \le \cdot | \eta \in H(z_\ell)] \\
&= \int \frac{\sum_\ell w_\ell 1[t \in H(z)]}{p(z_\ell)} P[Y_1 \le \cdot | \eta = t] dt
\end{aligned}$$

Take $w(t) = \frac{\sum_\ell w_\ell 1[t \in H(z_\ell)]}{p(z_\ell)}$, which satisfies $\int w(t)dt = 1$ $\square$

# Bounds on $F_{Y_0|D=1}$

Recall $p(z_\ell) \equiv P[D = 1|Z = z_\ell]$

Let $\Gamma_p = \{(\gamma_1, ..., \gamma_L) \in \mathbb{R}^L : \sum_{\ell=1}^L \gamma_\ell = 0 \text{ and } \sum_{\ell=1}^L p(z_\ell)\gamma_\ell = 1\}$

**Theorem 1**

Suppose Assumption Z and Condition 1.1* hold. For $\gamma = (\gamma_1, ..., \gamma_L) \in \Gamma_p$, suppose

$$P[Y \leq \cdot|D = 1] \leq \sum_{\ell=1}^{L} \gamma_\ell P[Y \leq \cdot, D = 1|Z = z_\ell] \tag{1}$$

Then $F_{Y_0|D=1}$ is upper bounded by

$$P[Y_0 \leq \cdot|D = 1] \leq -\sum_{\ell=1}^{L} \gamma_\ell P[Y \leq \cdot, D = 0|Z = z_\ell]$$

Analogous theorem for lower bound

# Bounds on $F_{Y_0|D=1}$: Proof

WLOG, let $\gamma_\ell \leq 0$ for $\ell \leq \ell^*$ and $\gamma_\ell > 0$ for $\ell > \ell^*$

Let $q(z_\ell) \equiv P[Z = z_\ell | D = 1]$

Then (1) can be rewritten as

$$\sum_{\ell=1}^{\ell^*} \frac{q(z_\ell)}{a} P[Y_1 \leq y | D = 1, Z = z_\ell] + \sum_{\ell=\ell^*+1}^{L} \frac{q(z_\ell)}{a} P[Y_1 \leq y | D = 1, Z = z_\ell] \leq \sum_{\ell=1}^{L} \frac{\gamma_\ell p(z_\ell)}{a} P[Y_1 \leq y | D = 1, Z = z_\ell]$$

where $a \equiv 1 - \sum_{\ell=1}^{\ell^*} p(z_\ell)\gamma_\ell$

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Let $q(z_\ell) \equiv P[Z = z_\ell | D = 1]$

Then (1) can be rewritten as

$$\sum_{\ell=1}^{\ell^*} \frac{q(z_\ell) - \gamma_\ell p(z_\ell)}{a} P[Y_1 \leq y | D = 1, Z = z_\ell] + \sum_{\ell=\ell^*+1}^{L} \frac{q(z_\ell)}{a} P[Y_1 \leq y | D = 1, Z = z_\ell] \leq \sum_{\ell=\ell^*+1}^{L} \frac{\gamma_\ell p(z_\ell)}{a} P[Y_1 \leq y | D = 1, Z = z_\ell]$$

where $a \equiv 1 - \sum_{\ell=1}^{\ell^*} \gamma_\ell p(z_\ell)$ and the positive weights sum to 1

# Bounds on $F_{Y_0|D=1}$: Proof

Therefore, by Condition 1.1*,

$$\sum_{\ell=1}^{\ell^*} \frac{q(z_\ell) - \gamma_\ell p(z_\ell)}{a} P[Y_0 \leq y | D = 1, Z = z_\ell]$$

$$+ \sum_{\ell=\ell^*+1}^{L} \frac{q(z_\ell)}{a} P[Y_0 \leq y | D = 1, Z = z_\ell]$$

$$\leq \sum_{\ell=\ell^*+1}^{L} \frac{\gamma_\ell p(z_\ell)}{a} P[Y_0 \leq y | D = 1, Z = z_\ell]$$

# Bounds on $F_{Y_0|D=1}$: Proof

Equivalently, we have

$$
\begin{aligned}
& P[Y_0 \leq y | D = 1] \\
\leq \; & \sum_{\ell=1}^{L} \gamma_\ell P[Y_0 \leq y, D = 1 | Z = z_\ell] \\
= \; & \sum_{\ell=1}^{L} \gamma_\ell \left[ P[Y_0 \leq y | Z = z_\ell] - P[Y_0 \leq y, D = 0 | Z = z_\ell] \right] \\
= \; & P[Y_0 \leq y] \sum_{\ell=1}^{L} \gamma_\ell - \sum_{\ell=1}^{L} \gamma_\ell P[Y \leq y, D = 0 | Z = z_\ell] \\
= \; & - \sum_{\ell=1}^{L} \gamma_\ell P[Y \leq y, D = 0 | Z = z_\ell]
\end{aligned}
$$

$\square$

# Bounds on $F_{Y_0|D=1}$

Finally, want to collect all $\gamma$ that satisfy (1):

**Corollary 1**

Suppose Assumption Z and Condition 1.1* hold. Then,

$$F^{UB}_{Y_0|D=1}(y) = \min_{\gamma \in \Gamma_p : (1) \text{ holds}} -\sum_{\ell=1}^{L} \gamma_\ell P[Y \leq y, D = 0 | Z = z_\ell]$$

Symmetric condition and bound can be derived for $F^{LB}_{Y_0|D=1}(\cdot)$

Then,

$$F^{LB}_{Y_0|D=1}(\cdot) \leq F_{Y_0|D=1}(\cdot) \leq F^{UB}_{Y_0|D=1}(\cdot)$$

# Bounds on $QTE_\tau(1)$

Note that

$$QTE_\tau(1) = Q_{Y_1|D=1}(\tau) - Q_{Y_0|D=1}(\tau) = Q_{Y|D=1}(\tau) - Q_{Y_0|D=1}(\tau)$$

Worst case bounds for quantile (Manski 94, Blundell et al. 07):

$$Q^{LB}_{Y_0|D=1}(\tau) \leq Q_{Y_0|D=1}(\tau) \leq Q^{UB}_{Y_0|D=1}(\tau)$$

with $Q^{LB}_{Y_0|D=1}(\tau) = F^{UB}_{Y_0|D=1}(\tau)^{-1}$ and $Q^{UB}_{Y_0|D=1}(\tau) = F^{LB}_{Y_0|D=1}(\tau)^{-1}$

# More on Theorem 1

Need to find $\gamma \in \Gamma_p$ that satisfies

$$P[Y \leq \cdot | D = 1] \leq \sum_{\ell=1}^{L} \gamma_\ell P[Y \leq \cdot, D = 1 | Z = z_\ell] \qquad (1)$$

- relates to finding $(w, \tilde{w})$ that satisfy "if" part of Condition 1.1*:

$$\sum_{\ell=1}^{L} w_\ell P[Y_1 \leq \cdot | D = 1, Z = z_\ell] \leq \sum_{\ell=1}^{L} \tilde{w}_\ell P[Y_1 \leq \cdot | D = 1, Z = z_\ell]$$

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When $L = 2$ then $\gamma = \left(\frac{1}{p(z_1) - p(z_2)}, -\frac{1}{p(z_1) - p(z_2)}\right)$, and under LATE monotonicity, (1) is equiv. to

$$Y_1 | \{\text{always-takers}\} \prec_{FOSD} Y_1 | \{\text{compliers}\}$$

# Why Multi-Valued IVs

$$\{\gamma \in \Gamma_p : \gamma \text{ satisfies } (1)\}$$

The size of this set determines the width of our bound

- a larger set $\Rightarrow$ narrower bounds
- more values $Z$ takes $\Rightarrow$ ...
  - greater degree of freedom in $\Gamma_p$ and
  - (1) is more likely to hold

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Multi-valued IV or multiple IVs are common in practice

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Then (1) establishes FOSD btw the mixtures of $F_{Y_1}$ conditional on various always-takers and compliers groups

# Bounds on $QTE_\tau(0)$ and $QTE$

## Condition 1.0

For arbitrary weight functions $w : \mathcal{T} \to \mathbb{R}_+$ and $\tilde{w} : \mathcal{T} \to \mathbb{R}_+$ s.t. $\int w(t)dt = \int \tilde{w}(t)dt = 1$,

$$\int w(x) F_{Y_1|\eta}(\cdot|t)dt \leq \int \tilde{w}(t) F_{Y_1|\eta}(\cdot|t)dt$$

$$\Longleftarrow$$

$$\int w(t) F_{Y_0|\eta}(\cdot|t)dt \leq \int \tilde{w}(t) F_{Y_0|\eta}(\cdot|t)dt.$$

With Condition 1.0 ("$\Longleftarrow$"), we can derive bounds on $QTE_\tau(0)$

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With Condition 1.0 ("$\Longleftarrow$"), we can derive bounds on $QTE_\tau(0)$

With Condition 2 ("$\Longleftrightarrow$"), we can derive bounds on $QTE_\tau$

# Bounds on $QTE_\tau(0)$ and $QTE$

## Condition 1.0

For arbitrary weight functions $w : \mathcal{T} \to \mathbb{R}_+$ and $\tilde{w} : \mathcal{T} \to \mathbb{R}_+$ s.t. $\int w(t)dt = \int \tilde{w}(t)dt = 1$,

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$$\Longleftarrow$$

$$\int w(t) F_{Y_0|\eta}(\cdot|t)dt \leq \int \tilde{w}(t) F_{Y_0|\eta}(\cdot|t)dt.$$

With Condition 1.0 ("$\Longleftarrow$"), we can derive bounds on $QTE_\tau(0)$

With Condition 2 ("$\Longleftrightarrow$"), we can derive bounds on $QTE_\tau$

We also provide a condition for bounds on $ATE(d)$ and $ATE$ (in the paper)

# II. Structural Models and Policymaker’s Problems

# Sufficient Conditions: A Structural Model

To further interpret Conditions 1.1 (and 1.0), we propose a model that rationalizes it ("conditional on $Z$" suppressed throughout)

**Model 1**

$$Y_d = q(d, U_d) \qquad \text{for } d \in \{0, 1\}$$

(i) $q(d, \cdot)$ is continuous and monotone increasing

(ii) conditional on $\eta$, $U_d \overset{d}{=} U + \xi_d$ where $\xi_d \perp (\eta, U)$

(iii) $\xi_0$ is (weakly) more noisy than $\xi_1$, i.e., $\xi_0 \overset{d}{=} \xi_1 + V$ for some $V$ independent of $\xi_1$

i.e., $U_0 \overset{d}{=} U_1 + V$

This model nests that in Chernozhukov and Hansen (2005)

- by taking $\xi_d = 0$ for all $d$, $U_0 \overset{d}{=} U_1 \overset{d}{=} U$ conditional on $\eta$

# Sufficient Conditions: A Structural Model

### Condition 1.1

For arbitrary weight functions $w : \mathcal{T} \to \mathbb{R}_+$ and $\tilde{w} : \mathcal{T} \to \mathbb{R}_+$ s.t. $\int w(t)dt = \int \tilde{w}(t)dt = 1$,

$$\int w(x) F_{Y_1|\eta}(\cdot|t)dt \leq \int \tilde{w}(t) F_{Y_1|\eta}(\cdot|t)dt$$
$$\Longrightarrow$$
$$\int w(t) F_{Y_0|\eta}(\cdot|t)dt \leq \int \tilde{w}(t) F_{Y_0|\eta}(\cdot|t)dt.$$

### Lemma 1

Model 1 implies Condition 1.1

Model 1 with $\xi_0$ being *less* noisy than $\xi_1$ implies Condition 1.0

# Example 1: Auction

$Y$: bid (which subsequently forms revenue)

$D$: participating in auction with different format

- $D = 1$ if online vs. 0 if offline

$U_d \overset{d}{=} U + \xi_d$: valuation of the item

- $U$: common valuation (correlated with $D$)
- $\xi_d$: format specific random shocks, $\xi_d \perp (\eta, U)$
  - bidders may have limited info on certain features of auction that affect valuation
  - e.g., they know the distribution of $\xi_d$ but not its realization

What justifies $var(\xi_0) > var(\xi_1)$?

- in offline auction, bidders may be more emotionally affected by others, which makes their bids more variable

# Example 2: Insurance

$Y$ : health outcome

$D$ : getting insurance

$U_d \overset{d}{=} U + \xi_d$: health conditions

- $U$: health conditions known to participant (and thus correlated with $D$)
- $\xi_d$: health conditions not fully known a priori

$var(\xi_0) > var(\xi_1)$: insurance by definition may ensure a certain level of health conditions

# Example 3.1: Vaccination

$Y$ : health outcome

$D$ : getting vaccination (of established vaccine)

$U_d \overset{d}{=} U + \xi_d$: underlying health conditions

- $U$: health conditions known to participant (and thus correlated with $D$)
- $\xi_d$: vaccination status specific health conditions, which is not fully known a priori

$var(\xi_0) > var(\xi_1)$: when not vaccinated, there is risk of dying, while vaccination ensures certain level of immunity

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$var(\xi_0) > var(\xi_1)$: when not vaccinated, there is risk of dying, while vaccination ensures certain level of immunity

This scenario justifies Condition 1.1 $\Rightarrow$ bounds on $QTE_\tau(1)$

# Example 3.0: Frontier Medical Trial

A contrasting example would be risky medical trial

$Y$: health outcome

$D$: participating in medical trial

$U_d \overset{d}{=} U + \xi_d$: underlying health conditions

- $U$: health conditions known to participant (and thus correlated with $D$)
- $\xi_d$: health conditions not fully known a priori

$var(\xi_0) < var(\xi_1)$: with newly developed medicine, there is high risk of unknown side effects

This scenario justifies Condition 1.0 $\Rightarrow$ bounds on $QTE_\tau(0)$

# Policymaker's Problem

**Assumption**: The policymaker concerns risk averse individuals, which are the majority

Under this assumption, the policymaker wants to understand treatment effects for the target individuals in order to provide "insurance"

- literally insurance or policy that serves as insurance (e.g., vaccination, subsidies)

Our procedure provides a statistical tool for such a policymaker

# Policymaker's Problem

Under Model 1, our procedure has the ability to bound treatment effect for individuals with $D = d$ such that $var(\xi_d) < var(\xi_{1-d})$

This is a unique feature of our setting:

- the plausibility of assumptions dictates the parameter of interest
- i.e., "assumption-driven" treatment parameters

# Sufficient Condition: Model 1 with Rank Similarity

## Condition 2

For arbitrary weight functions $w : \mathcal{T} \to \mathbb{R}_+$ and $\tilde{w} : \mathcal{T} \to \mathbb{R}_+$ s.t. $\int w(t)dt = \int \tilde{w}(t)dt = 1$,

$$\int w(x)F_{Y_1|\eta}(\cdot|t)dt \leq \int \tilde{w}(t)F_{Y_1|\eta}(\cdot|t)dt$$

$$\Longleftrightarrow$$

$$\int w(t)F_{Y_0|\eta}(\cdot|t)dt \leq \int \tilde{w}(t)F_{Y_0|\eta}(\cdot|t)dt.$$

## Lemma 2

Model 1 with $F_{U_0|\eta} = F_{U_1|\eta}$ (rank similarity) implies Condition 2.

# Necessary and Sufficient Condition: Rank Linearity

**Lemma 2**

Model 1 with $F_{U_0|\eta} = F_{U_1|\eta}$ (rank similarity) implies Condition 2.

Converse is *not* true!

Counter-example:

**Rank Linearity**

$Y_d = q(d, U_d)$ (monotonic $q(d, \cdot)$) with

$$F_{Y_0|\eta}(\cdot|t) = \lambda(\cdot)F_{Y_1|\eta}(\psi(\cdot)|t)$$

where $\psi(\cdot)$ is one-to-one and onto mapping and $\lambda(\cdot)$ is consistent with $F_{Y_d|\eta}$ being a proper CDF

Rank linearity implies Condition 2 but is *weaker* than rank similarity

# Necessary and Sufficient Condition: Rank Linearity

## Rank Linearity

$Y_d = q(d, U_d)$ (monotonic $q(d, \cdot)$) with

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Rank linearity implies Condition 2 but is *weaker* than rank similarity:

$$F_{U_0|\eta}(q^{-1}(0, y)|t) = \lambda(y) F_{U_1|\eta}(q^{-1}(1, \psi(y))|t)$$

and choose $\lambda(y) = 1$ and $\psi(y) = q(1, q^{-1}(0, y))$

# Necessary and Sufficient Condition: Rank Linearity

**Rank Linearity**

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and choose $\lambda(y) = 1$ and $\psi(y) = q(1, q^{-1}(0, y))$

In fact, rank linearity may be equivalent to Condition 2

# Necessary and Sufficient Condition: Rank Linearity

Suppose $Y_d \in \left\{y_{d,1}, \cdots, y_{d,k_d}\right\}$ and $\eta \in \{t_1, \cdots, t_{k_\eta}\}$

## Lemma 3

For any $\tilde{F}_d$ on $\left\{y_{d,1}, \cdots, y_{d,k_d}\right\}$, suppose there always exists a nonnegative sequence $\left\{c_{d,1}, \cdots, c_{d,k_\eta}\right\}$ s.t.

$$\tilde{F}_d(\cdot) = \sum_{j=1}^{k_\eta} c_{d,j} F_{Y_d|\eta}(\cdot|t_j). \tag{2}$$

Then, Condition 2 holds if and only if (i) $k_0 = k_1$ and (ii) for some one-to-one and onto mapping $\psi(\cdot)$ and $\lambda(\cdot) > 0$,

$$F_{Y_0|\eta}(\cdot|t_j) = \lambda(\cdot) F_{Y_1|\eta}(\psi(\cdot)|t_j), \qquad \text{for } j = 1, \cdots, k_\eta.$$

# Necessary and Sufficient Condition: Rank Linearity

Suppose $Y_d \in \{y_{d,1}, \cdots, y_{d,k_d}\}$ and $\eta \in \{t_1, \cdots, t_{k_\eta}\}$

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$$F_{Y_0|\eta}(\cdot|t_j) = \lambda(\cdot) F_{Y_1|\eta}(\psi(\cdot)|t_j), \qquad \text{for } j = 1, \cdots, k_\eta.$$

We conjecture an analogous result with continuous $Y_d$ and $\eta$ (in progress)

# Necessary and Sufficient Condition: Proof

We only prove necessity

Recall Condition 2:

$$\sum_{j=1}^{k_\eta} w_j F_{Y_1|\eta}(\cdot|t_j) \leq \sum_{j=1}^{k_\eta} \tilde{w}_j F_{Y_1|\eta}(\cdot|t_j)$$

$$\Longleftrightarrow$$

$$\sum_{j=1}^{k_\eta} w_j F_{Y_0|\eta}(\cdot|t_j) \leq \sum_{j=1}^{k_\eta} \tilde{w}_j F_{Y_0|\eta}(\cdot|t_j)$$

# Necessary and Sufficient Condition: Proof

We only prove necessity

Recall Condition 2:

$$\sum_{j=1}^{k_\eta} \delta_j F_{Y_1|\eta}(\cdot|t_j) \leq 0$$

$$\iff$$

$$\sum_{j=1}^{k_\eta} \delta_j F_{Y_0|\eta}(\cdot|t_j) \leq 0$$

# Necessary and Sufficient Condition: Proof

We only prove necessity

Recall Condition 2:

$$\sum_{j=1}^{k_\eta} \delta_j F_{Y_1|\eta}(\cdot|t_j) \leq 0$$

$$\iff$$

$$\sum_{j=1}^{k_\eta} \delta_j F_{Y_0|\eta}(\cdot|t_j) \leq 0$$

Define cone:

$$\Delta_d \equiv \{\delta : \sum_{j=1}^{k_\eta} \delta_j F_{Y_d|\eta}(\cdot|t_j) \leq 0, \sum_{j=1}^{k_\eta} \delta_j = 0\}$$

Then, by Condition 2, $\Delta_1 = \Delta_0$

# Necessary and Sufficient Condition: Proof

Define polar cone:

$$\Delta_d^* \equiv \{F_d \in \mathbb{R}^{k_\eta} | F_d{}'\delta \leq 0, \ \forall \delta \in \Delta_d\}$$

Then $\Delta_1^* = \Delta_0^*$ (and thus $k_1 = k_0$)

## Necessary and Sufficient Condition: Proof

Define polar cone:

$$\Delta_d^* \equiv \{F_d \in \mathbb{R}^{k_\eta} | F_d{}'\delta \leq 0, \ \forall \delta \in \Delta_d\}$$

Then $\Delta_1^* = \Delta_0^*$ (and thus $k_1 = k_0$)

Note $\Delta_d^*$ is a convex cone whose extreme ray is generated by

$$\left\{ \left( F_{Y_d|\eta}(y|t_1), \cdots, F_{Y_d|\eta}(y|t_{k_\eta}) \right)' : y = y_{d,1}, \cdots, y_{d,k_d} \right\}$$

which are linearly indep. by (2)

# Necessary and Sufficient Condition: Proof

Define polar cone:

$$\Delta_d^* \equiv \{F_d \in \mathbb{R}^{k_\eta} | {F_d}'\delta \leq 0, \ \forall \delta \in \Delta_d\}$$

Then $\Delta_1^* = \Delta_0^*$ (and thus $k_1 = k_0$)

Note $\Delta_d^*$ is a convex cone whose extreme ray is generated by

$$\left\{ \Big( F_{Y_d|\eta}(y|t_1), \cdots, F_{Y_d|\eta}(y|t_{k_\eta}) \Big)' : y = y_{d,1}, \cdots, y_{d,k_d} \right\}$$

which are linearly indep. by (2)

Therefore, any extreme ray in $\Delta_0^*$ must be proportional to some extreme ray in $\Delta_1^*$: For any $y$

$$\Big( F_{Y_0|\eta}(y|t_1), \cdots, F_{Y_0|\eta}(y|t_{k_\eta}) \Big) = \lambda \times \Big( F_{Y_1|\eta}(y'|t_1), \cdots, F_{Y_1|\eta}(y'|t_{k_\eta}) \Big)$$

for some $\lambda > 0$ and $y'$ $\square$

# III. Systematic Calculation of Bounds

# Computation of Bounds

Recall

$$F^{UB}_{Y_0|D=1}(\cdot) = \min_{\gamma \in \Gamma_p : (1) \text{ holds}} -\sum_{\ell=1}^{L} \gamma_\ell P[Y \leq \cdot, D = 0 | Z = z_\ell]$$

where

$$P[Y \leq \cdot | D = 1] \leq \sum_{\ell=1}^{L} \gamma_\ell P[Y \leq \cdot, D = 1 | Z = z_\ell] \qquad (1)$$

and $\Gamma_p = \{(\gamma_1, ..., \gamma_L) \in \mathbb{R}^L : \sum_{\ell=1}^{L} \gamma_\ell = 0 \text{ and } \sum_{\ell=1}^{L} p(z_\ell)\gamma_\ell = 1\}$

# Computation of Bounds: Semi-Infinite Program

Simplifying notation:

$$p_{y,d|z} \equiv \{P[Y \leq y, D = d|Z = z_\ell]\}_{\ell=1}^{L}$$
$$p_{y|1} \equiv P[Y \leq y|D = 1]$$

Consider the following semi-infinite program:

$$F^{UB}_{Y_0|D=1}(\bar{y}) = \min_{\gamma \in \Gamma_p} -p'_{\bar{y},0|z}\gamma$$
$$s.t. \quad p_{y|1} - p'_{y,1|z}\gamma \leq 0, \quad \forall y \in \mathcal{Y} \qquad (1)$$

- feasible as long as $\exists$ such $\gamma$
- i.e., (1) is testable from data
- if $Y$ is discrete, then we already have linear program (but not in general)

# Computation of Bounds: Linear Program I

The semi-infinite program:

$$
\begin{aligned}
F^{UB}_{Y_0|D=1}(\bar{y}) = \min_{\gamma \in \Gamma_p} & -p'_{\bar{y},0|z}\gamma \\
s.t. \quad & p_{y|1} - p'_{y,1|z}\gamma \leq 0, \quad \forall y \in \mathcal{Y}
\end{aligned}
$$

In practice, with i.i.d. $\{Y_i, D_i, Z_i\}_{i=1}^n$, we solve linear program:

$$
\begin{aligned}
\widehat{F}^{UB}_{Y_0|D=1}(\bar{y}) = \min_{\gamma \in \Gamma_{\hat{p}}} & -\hat{p}_{\bar{y},0|z}{}'\gamma \\
s.t. \quad & \hat{p}_{Y_i|1} - \hat{p}_{Y_i,1|z}{}'\gamma \leq 0, \quad \forall i = 1, ..., n
\end{aligned}
$$

The optimization can be unstable

# Computation of Bounds: Linear Program II

The semi-infinite program:

$$F^{UB}_{Y_0|D=1}(\bar{y}) = \min_{\gamma \in \Gamma_p} -p'_{\bar{y},0|z}\gamma$$

$$s.t. \quad p_{y|1} - p'_{y,1|z}\gamma \leq 0, \quad \forall y \in \mathcal{Y}$$

Dual program:

$$F^{UB,\dagger}_{Y_0|D=1}(\bar{y}) = \sup_{\Lambda \succeq 0, \lambda \in \mathbb{R}^2} \int_{\mathcal{Y}} p_{y|1} d\Lambda(y) - \begin{bmatrix} 0 & 1 \end{bmatrix} \lambda$$

$$s.t. \qquad \begin{bmatrix} 1 & p \end{bmatrix} \lambda - \int_{\mathcal{Y}} p_{y,1|z} d\Lambda(y) - p_{\bar{y},0|z} = 0$$

- $\Lambda$ is a nonnegative measure

Strong duality may hold (in progress)

# Computation of Bounds: Linear Program II

Approximate $\lambda(y) \equiv d\Lambda(y)/dy$ using Bernstein polynomial:

$$\lambda(y) \approx \sum_{j=1}^{J} \theta_j b_j(y)$$

Then, results in linear program:

$$F^{UB,\dagger\dagger}_{Y_0|D=1}(\bar{y}) = \max_{\theta \in \mathbb{R}^J_+, \lambda \in \mathbb{R}^2} \theta' b^1 - \begin{bmatrix} 0 & 1 \end{bmatrix} \lambda$$

$$s.t. \qquad \begin{bmatrix} 1 & p \end{bmatrix} \lambda - B_1' \theta - \hat{p}_{\bar{y},0|z} = 0$$

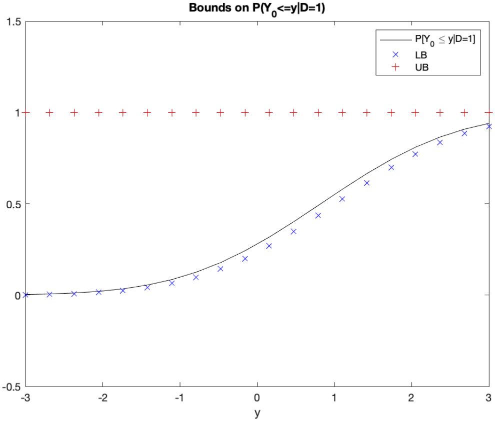
- $\theta \equiv (\theta_1, ..., \theta_J)'$
- $b^d \equiv (b^d_1, ..., b^d_J)'$ with $b^d_j \equiv \int_{\mathcal{Y}} b_j(y) \hat{p}_{y|d} dy$
- $\boldsymbol{b}_{d,j} \equiv (b_{d,j,1}, ..., b_{d,j,L})'$ with $b_{d,j,\ell} \equiv \int_{\mathcal{Y}} b_j(y) \hat{p}_{y,d|z_\ell} dy$
- $B_d \equiv \begin{bmatrix} \boldsymbol{b}_{d,1} & \cdots & \boldsymbol{b}_{d,J} \end{bmatrix}$

# IV. Numerical Studies

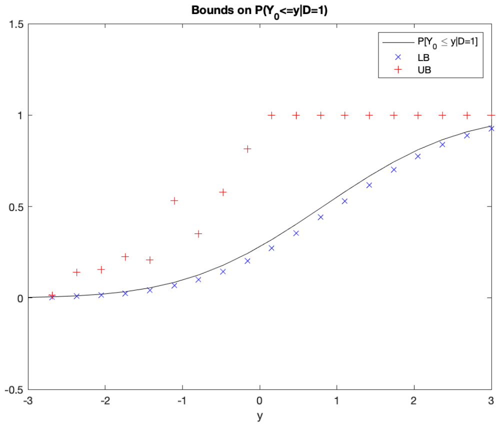
# Numerical Illustration: Design

- $Y_d = q(d, U_d) = 1 - d + (d+1)U_d$
  - $Y_1 = 2U_1$ and $Y_0 = 1 + U_0$
- $(U, \eta) \sim BVN((0,0)', \Sigma)$
- $V \sim N(0, \sigma_V^2)$ and $\xi_1 \sim N(0, \sigma_V^2)$
- $\xi_0 = \xi_1 + V$
- $U_d = U + \xi_d$
- $Z \sim Bin(L-1, p)/(L-1) \in [0, 1]$
  - $L$ is the number of values $Z$ takes
  - $Z$ is normalized
- $D = 1\{\pi_0 + \pi_1 Z \geq \eta\}$
- $Y = DY_1 + (1-D)Y_0$

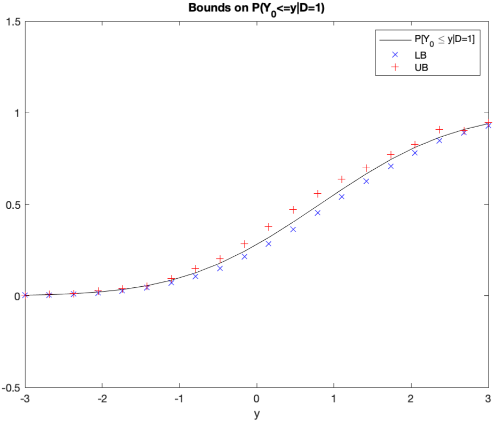
# Bounds on $P[Y_0 \leq y|D = 1]$ when $L = 2$



# Bounds on $P[Y_0 \leq y|D = 1]$ when $L = 5$



# Bounds on $P[Y_0 \leq y|D = 1]$ when $L = 6$



# V. Summary and Conclusions

# Summary

The paper...

- proposes a way to weaken rank similarity and
- shows how to construct informative bounds on QTE and ATE
- for treated or untreated (e.g., risk-averse) populations
- using multi-valued IVs

Calculation of the bounds are simple

Inference is an open question

Thank You! ☺