Chapter 3 - Proofs

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3.1. Proof Strategies

1. **Theorem.** *Suppose n is an integer larger than 1 and n is not prime. Then* $2^n - 1$ *is not prime.*
(a) Hypotheses:
n is an integer larger than 1
n is not prime
Conclusion:
$2^n - 1$ is not prime
When $n = 6$ both the hypotheses are true, because $6 > 1$ and 6 is not prime $(2 * 3 = 6)$.
The theorem says that $2^6 - 1$ is not prime, and this is correct since $2^6 - 1 = 63$ and 63 is not prime $(9 * 7 = 63)$.

(b) For the case $n = 15$, $n > 1$ and n is not prime $(15 = 3 * 5)$, so the hypotheses are true. The conclusion is also true because $2^{15} - 1 = 32767$ and 32767 is not prime $(32767 = 151 * 31 * 7)$.

(c) For the case $n = 1$, $11 > 1$ and 11 is prime. Because one of the hypotheses is false, nothing can be concluded from the theorem in this case.

2. **Theorem.** *Suppose that* $b^2 > 4ac$. *Then the quadratic equation* $ax^2+bx+c = 0$ *has exactly two real solutions.*
(a) Hypothesis: $b^2 > 4ac$ Conclusion: The quadratic equation $ax^2 + bx + c = 0$ has exactly two real solutions.

(b) To give an instance of the theorem, values of a,b and c must be specified, but not x. This is because the values of a, b and c must be known to decide if the hypothesis is true. The value of x is unspecified because the theorem applies to all possible values of x.

(c)$a = 2, b = 5, c = 3$
$5^2 > 4 * 2 * 3 \equiv 25 > 24$ which is true, so the hypothesis holds.
$2x^2 + 5x + 3 = 0$
$2 * 3 = 6$

$2+3=5$
$2x^2+2x+3x+3$
$2x(x+1)+3(x+1)$
$(2x+3)(x+1)=0$
$x=-1$ or $x=-3/2$ so there are exactly two real solutions.
The conclusion is correct.

(d)$a=2, b=4, c=3$
$4^2>4*2*3 \equiv 16>24$ which is false. Since the hypothesis is false, nothing can be concluded from the theorem in this case.

3. **Incorrect Theorem.** *Suppose n is a natural number larger than 2, and n is not a prime number. Then* $2n+13$ *is not a prime number.*
Hypotheses: n is a natural number larger than 2
n is not a prime number
Conclusion: $2n+13$ is not a prime number
Counterexample: $n=8$
$8>2$ and 8 is not prime, so the hypotheses are true.
$2*8+13=29$ which is prime, so the conclusion is false.
The hypotheses are true, but the conclusion is false, so the theorem is incorrect.

4. *Proof.* Suppose $0<a<b$. Then $b-a>0$.
$(b-a)(b+a)>0*(b+a)$
$b^2+ab-ab-a^2>0$
$b^2-a^2>0$
Since $b^2-a^2>0$, it follows that $a^2<b^2$. Therefore if $0<a<b$ then $a^2<b^2$.

5. Suppose a and b are real numbers. $a<b<0 \Rightarrow a^2>b^2$
A negative number multiplied by a negative is positive, and both a and b are negative.
Multiply by a: $a^2>ab$
Multiply by b: $ab>b^2$
$a^2>ab>b^2$, therefore if $a<b<0$ then $a^2>b^2$.

6. Suppose a and b are real numbers. If $0<a<b$ then $1/b<1/a$.
Divide both sides by a:
$a<b \equiv a/a<b/a \equiv 1<b/a$
Then divide both sides by b:
$1/b<b/(ab) \equiv 1/b<1/a$
Therefore if $0<a<b$ then $1/b<1/a$.