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Bakalářská práce

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Aspekty přechodu od kvantové mechaniky ke kvantové teorii pole

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Abstrakt

V této bakalářské/diplomové/rigorózní práci se věnujeme ...

Abstract

In this thesis we study ...

Místo tohoto listu vložte kopii oficiálního (podepsaného) zadání práce.

Poděkování

Na tomto místě bych chtěl(-a) poděkovat ...

Prohlášení

Prohlašuji, že jsem svoji bakalářskou/diplomovou/rigorózní práci vypracoval(-a) samostatně s využitím informačních zdrojů, které jsou v práci citovány.

Brno 1. května 2018

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Milan Suk

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Nomenclature

$\mathbb{C}$	množina všech komplexních čísel
$\mathbb{R}$	množina všech reálných čísel
$\mathbb{Z}$	množina všech celých čísel

Introduction

The Quantum Mechanics (QM) is a fundamental theory in physics which aims to describe the world at the lowest scales of measured units. It had a great success of explaining a lot of new observations in the very beginning of 20th century, e.g. spectra of atoms, the black-body radiation or photoelectric effect. In 1905, Albert Einstein published his famous article *On the Electrodynamics of Moving Bodies* and proposed the Special theory of relativity (STR). STR corrects the classical mechanics in a way that it can predict a state of a physical system having *relativistic velocities* and setups a new restrictions for mathematical formulations of physical theories.

A logical step is to develop a physical theory which preserves STR and is capable of describing particles in quantum mechanics. This theory is known as Relativistic Quantum Mechanics (RQM) and it has been successful in prediction of interesting phenomena in physics like antimatter or spin. Nevertheless, RQM can't be introduced without formal inconsistencies and it cant deal with varying number of particles. The modern framework which consistently combines QM and STR is the Quantum Field Theory (QFT).

The thesis consists of three main chapters. In the first one we will discuss the formulation of RQM, i.e. we will construct two relativistic quantum mechanical eqautions - the Klein-Gordon equation and the Dirac equation. In the second chapter we will focus on the failures of these formulations and we show the need for a new theory. In the last chapter we will briefly introduce the new framework of QFT. In the terms of Quantum scalar fields we will try to check whether the new formulation solves the problems described in the second chapter.

Chapter 1

Relativistic quantum mechanics

In this chapter (TODO)

1.1 Canonical quantization

Canonical quantization is a process of transition from a classical theory to the quantum one, i.e. procedure for quantizing the theory. The term *canonical* comes from (TODO).

Definition 1. *Given two functions $f(\vec{q}, \vec{p}, t)$ and $g(\vec{q}, \vec{p}, t)$, the binary operation $\{f, g\}$ is called the **Poisson bracket** of these functions and takes the form*

$$\{f, g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{1.1}$$

Theorem 1. *The Poisson brackets of canonical coordinates q_i and p_i are*

$$\begin{aligned} \{q_i, q_j\} &= 0 \\ \{p_i, p_j\} &= 0 \\ \{q_i, p_j\} &= \delta_{ij} \end{aligned} \tag{1.2}$$

Proof. The proof of 1 can be easily done from the definition 1. The first two equations are manifestly zero because $\frac{\partial q_i}{\partial p_j} = \frac{\partial p_i}{\partial q_j} = 0$ and the last can one be proved as follows.

$$\begin{aligned} \{q_i, p_j\} = \sum_{k=1}^{N} \left(\frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) = \sum_{k=1}^{N} \frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} &= \begin{cases} 1 & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases} \\ &\implies \{q_i, p_j\} = \delta_{ij} \end{aligned} \tag{1.3}$$

$\square$

Dirac has formulated a technique for generating quantum operators from classical functions. The famous statement goes as follows.

$$\{A, B\} \to \frac{1}{i\hbar}[\hat{A}, \hat{B}] \tag{1.4}$$

For the next discussion let's define $\mathcal{O}$ map which takes a function from the configuration space and maps it to the Hilbert space.

Definition 2. *Let Q be a configuration space and H a Hilbert space. $End(V)$ is an endomorphism of space V, i.e. a set of linear operators $\{L : V \to V \mid L \text{ linear}\}$.*

$$\begin{aligned} \mathcal{O} &: Q \to End(H) \\ \mathcal{O}(\{f, g\}) &= \frac{1}{i\hbar}[\hat{A}, \hat{B}] \end{aligned} \tag{1.5}$$

Where $\hat{A}$ is the operator corresponding the the classical value f and B corresponds to g.

Now we will use the fact that $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$ to show that $\mathcal{O}(q_i) = \hat{q}_i$. Firstly, we will rewrite q using the Poisson bracket. The obvious choice is $q_i = \{\frac{q_i^2}{2}, p_i\}$. And from the definition 2 we see that $\mathcal{O}(\{\frac{q_i^2}{2}, p_i\}) = \frac{1}{i\hbar}[\frac{\hat{q}_i^2}{2}, \hat{p}_i]$. Using the identity for commutation relation $[AB, C] = [A, B]C + B[A, C]$[1] the commutator on the right-hand side can be written as follows.

$$\mathcal{O}(x_i) = \frac{1}{i\hbar}\left[\frac{\hat{x}_i^2}{2}, \hat{p}_i\right] = \frac{1}{2i\hbar}\big(\hat{x}_i[\hat{x}_i, \hat{p}_i] + [\hat{x}_i, \hat{p}_i]\hat{x}_i\big) = \frac{1}{2i\hbar}\big(\hat{x}_i i\hbar + i\hbar\hat{x}_i\big) = \hat{x}_i$$

The proof of $\mathcal{O}(p) = \hat{p}$ is very similar.

$$\mathcal{O}(p_i) = \frac{1}{i\hbar}\left[\hat{x}_i, \frac{\hat{p}_i^2}{2}\right] = \frac{1}{2i\hbar}\big([\hat{x}_i, \hat{p}_i]\hat{p}_i + \hat{p}_i[\hat{x}_i, \hat{p}_i]\big) = \frac{1}{2i\hbar}\big(i\hbar\hat{p}_i + \hat{p}_i i\hbar\big) = \hat{p}_i$$

In the *x-representation* the explicit form of $\hat{x}$ and $\hat{p}$ is $\hat{x} = x$ and $\hat{p_x} = -i\hbar\frac{\partial}{\partial x}$. TODO: study $\mathcal{O}(\{f(\vec{q}, \vec{p})\})$ and $\mathcal{O}(\{f + g, h\})$. How to get $\mathcal{O}(E)$?

1.2 Klein-Gordon equation

Klein-Gordon equation is relativistic quantum mechanical equation named after Oskar Klein and Walter Gordon and it describes relativistic spinless particles. It can be found using quantization of the relativistic energy equation. We need to apply the $\mathcal{O}$ map on each side while taking into account that $\mathcal{O}(E^2) = -\hbar^2\frac{\partial^2}{\partial t^2}$ and $\mathcal{O}(\vec{p}^2) = -\hbar^2\Delta$.

[1] Proof by directly using the definition of the commutator: $[AB, C] = ABC - CAB = ABC - ACB + ACB - CAB = A[B, C] + [A, C]B$

$$E^2 = \vec{p}^2 c^2 + m^2 c^4$$
$$-\hbar^2 \frac{\partial^2}{\partial t^2}\psi = -\hbar^2 c^2 \Delta\psi + m^2 c^4 \psi$$

Let's rewrite the equation in a form so we can see it is *lorentz invariant*. We use the fact that *4-gradient* transforms the same way as any other 4-vector and than express the Klein-Gordon equation using 4-gradient dot products.

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + \frac{m^2 c^2}{\hbar^2}\right)\psi = 0 \tag{1.6}$$

Using the metric $\eta^{\mu\nu} = diag(1, -1, -1, -1)$ and the *4-gradient* $\partial_\mu = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ it is possible to write the term $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$ as follows.

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_\mu\partial^\nu$$

1.3 Dirac equation

Chapter 2

Failures of the Relativistic Quantum Mechanics

In this chapter we will discuss

2.1 Probability current

In non-relativistic case we can find the continuity equation by

2.2 Energy solutions

2.3 Causality

Chapter 3

Quantum field theory

3.1 Scalar field

In non-relativistic case we can find the continuity equation by

3.2 Second quantization

3.3 Causality problem

Conclusion

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