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Aspekty přechodu od kvantové mechaniky ke kvantové teorii pole

Bakalářská práce

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Abstrakt

V této bakalářské/diplomové/rigorózní práci se věnujeme ...

Abstract

In this thesis we study ...

Místo tohoto listu vložte kopii oficiálního (podepsaného) zadání práce.

Poděkování

Na tomto místě bych chtěl(-a) poděkovat ...

Prohlášení

Prohlašuji, že jsem svoji bakalářskou/diplomovou/rigorózní práci vypracoval(-
a) samostatně s využitím informačních zdrojů, které jsou v práci citovány.

Brno 1. května 2018

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Milan Suk

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Nomenclature

- \mathbb{C} množina všech komplexních čísel
- \mathbb{R} množina všech reálných čísel
- \mathbb{Z} množina všech celých čísel

Introduction

The Quantum Mechanics (QM) is a fundamental theory in physics which aims to describe the world at the lowest scales of measured units. It had a great success of explaining a lot of new observations in the very beginning of 20th century, e.g. spectra of atoms, the black-body radiation or photoelectric effect. In 1905, Albert Einstein published his famous article *On the Electrodynamics of Moving Bodies* and proposed the Special theory of relativity (STR). STR corrects the classical mechanics in a way that it can predict a state of a physical system having *relativistic velocities* and setups a new restrictions for mathematical formulations of physical theories.

A logical step is to develop a physical theory which preserves STR and is capable of describing particles in quantum mechanics. This theory is known as Relativistic Quantum Mechanics (RQM) and it has been successful in prediction of interesting phenomena in physics like antimatter or spin. Nevertheless, RQM can't be introduced without formal inconsistencies and it can't deal with varying number of particles. The modern framework which consistently combines QM and STR is the Quantum Field Theory (QFT).

The thesis consists of three main chapters. In the first one we will discuss the formulation of RQM, i.e. we will construct two relativistic quantum mechanical equations - the Klein-Gordon equation and the Dirac equation. In the second chapter we will focus on the failures of these formulations and we show the need for a new theory. In the last chapter we will briefly introduce the new framework of QFT. In the terms of Quantum scalar fields we will try to check whether the new formulation solves the problems described in the second chapter.

Chapter 1

Relativistic quantum mechanics

In this chapter we will introduce the basics of the canonical quantization. We will start by introducing the Poisson brackets and its basic properties. Then we will define a quantization map \mathcal{O} which maps the classical dynamical variable into the operator within a Hilbert space. Using this process we will then formulate the Klein-Gordon equation, an equation for describing a spinless particle in the quantum mechanics. We will see that this equation is second order both in time and space which will result some obstacles. With the hope of solving this issue we will construct the Dirac equation which describes the $\frac{1}{2}$ -spin particle.

1.1 Canonical quantization

Canonical quantization is a process of transition from a classical theory to the quantum one. If a quantum problem has an analog in the classical mechanics we can use the process of quantization to find the formulation in the quantum mechanics using the analogous equations in the classical one. In the classical mechanics a state of the system is represented by coordinates in the *phase space* $(\vec{q}_1, \dots, \vec{q}_n, \vec{p}_1, \dots, \vec{p}_n, t)$. In the quantum mechanics, the analog of q_i and p_i are operators \hat{q}_i and \hat{p}_i acting in the *Hilbert space*.

Definition 1. Given two dynamical variables in the phase space \mathcal{Q} , $f(\vec{q}_1, \dots, \vec{q}_n, \vec{p}_1, \dots, \vec{p}_n, t) \in C^\infty(\mathcal{Q}, \mathcal{R})$ and $g(\vec{q}_1, \dots, \vec{q}_n, \vec{p}_1, \dots, \vec{p}_n, t) \in C^\infty(\mathcal{Q}, \mathcal{R})$, the binary operation $\{f, g\}$ is called the **Poisson bracket** of these functions and takes the form

$$\{f, g\} = \sum_{i=1}^N \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \quad (1.1)$$

Theorem 1. The Poisson brackets of canonical coordinates q_i and p_i are

$$\begin{aligned} \{q_i, q_j\} &= 0 \\ \{p_i, p_j\} &= 0 \\ \{q_i, p_j\} &= \delta_{ij} \end{aligned} \quad (1.2)$$

Proof. The proof of 1 can be easily done from the definition 1. The first two equations are manifestly zero because $\frac{\partial q_i}{\partial p_j} = \frac{\partial p_i}{\partial q_j} = 0$ and the last one can be proved as follows.

$$\{q_i, p_j\} = \sum_{k=1}^N \left(\frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) = \sum_{k=1}^N \frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} = \begin{cases} 1 & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases} \quad (1.3)$$

$$\implies \{q_i, p_j\} = \delta_{ij}$$

□

Paul Dirac has formulated a technique for generating operators in H from dynamical variables in $C^\infty(\mathcal{Q}, \mathcal{R})$. The famous statement goes as follows.

$$\{A, B\} \rightarrow \frac{1}{i\hbar} [\hat{A}, \hat{B}] \quad (1.4)$$

That means that if we have dynamical variables A and B we can get their quantum counterparts \hat{A} and \hat{B} and the defining property is the equation 1.4. Let's define the properties of the \mathcal{O} map (see [7]) and then study some consequences.

Definition 2. Let \mathcal{Q} be a configuration space and H the corresponding Hilbert space. The quantization map is

$$\mathcal{O} : C^\infty(\mathcal{Q}, \mathcal{R}) \rightarrow Op(H) \quad (1.5)$$

having the properties:

1. \mathcal{O} is linear,
2. $\mathcal{O}(f)$ is self-adjoint, $f \in C^\infty(\mathcal{Q}, \mathcal{R})$,
3. the relation between commutator and the Poisson bracket is

$$\mathcal{O}(\{A, B\}) = \frac{1}{i\hbar} [\hat{A}, \hat{B}] \quad (1.6)$$

4. a complete set $\{f_1, \dots, f_n\}$ is mapped to a complete set $\{\mathcal{O}(f_1), \dots, \mathcal{O}(f_n)\}$,
5. an identity $Id_{\mathcal{Q}}$ is mapped to the identity $Id_H = \mathcal{O}(Id_{\mathcal{Q}})$

It is convinient to use $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$ and 1.6 to show that $\mathcal{O}(q_i) = \hat{q}_i$. Firstly, we will rewrite q using the Poisson bracket. The obvious choice is $q_i = \{\frac{q_i^2}{2}, p_i\}$. And from the definition 1.6 we see that $\mathcal{O}(\{\frac{q_i^2}{2}, p_i\}) = \frac{1}{i\hbar} [\hat{q}_i^2, \hat{p}_i]$. Using the identity for commutation relation $[AB, C] = [A, B]C + B[A, C]$ ¹ the commutator on the right-hand side can be written as follows.

¹Proof by directly using the definition of the commutator: $[AB, C] = ABC - CAB = ABC - ACB + ACB - CAB = A[B, C] + [A, C]B$

$$\mathcal{O}(x_i) = \frac{1}{i\hbar} \left[\frac{\hat{x}_i^2}{2}, \hat{p}_i \right] = \frac{1}{2i\hbar} (\hat{x}_i [\hat{x}_i, \hat{p}_i] + [\hat{x}_i, \hat{p}_i] \hat{x}_i) = \frac{1}{2i\hbar} (\hat{x}_i i\hbar + i\hbar \hat{x}_i) = \hat{x}_i$$

$\mathcal{O}(p) = \hat{p}$ is very similar.

$$\mathcal{O}(p_i) = \frac{1}{i\hbar} \left[\hat{x}_i, \frac{\hat{p}_i^2}{2} \right] = \frac{1}{2i\hbar} ([\hat{x}_i, \hat{p}_i] \hat{p}_i + \hat{p}_i [\hat{x}_i, \hat{p}_i]) = \frac{1}{2i\hbar} (i\hbar \hat{p}_i + \hat{p}_i i\hbar) = \hat{p}_i$$

TODO: Groenewold theorem, Deformation quantization and geometric quantization note. ■

From now, we will use the explicit form of the operators in the *x-representation*. ■

$$\begin{aligned} \mathcal{O}(p_i) &= -i\hbar \frac{\partial}{\partial x^i} \\ \mathcal{O}(E) &= i\hbar \frac{\partial}{\partial t} \end{aligned} \tag{1.7}$$

1.2 Klein-Gordon equation

Klein-Gordon equation is relativistic quantum mechanical equation named after Oskar Klein and Walter Gordon and it describes relativistic spinless particles. It can be found using quantization of the relativistic energy equation. We need to apply the \mathcal{O} map on each side while taking into account that $\mathcal{O}(E^2) = -\hbar^2 \frac{\partial^2}{\partial t^2}$ and $\mathcal{O}(\vec{p}^2) = -\hbar^2 \Delta$.

$$\begin{aligned} E^2 &= \vec{p}^2 c^2 + m^2 c^4 \\ -\hbar^2 \frac{\partial^2}{\partial t^2} \psi &= -\hbar^2 c^2 \Delta \psi + m^2 c^4 \psi \end{aligned}$$

After a little rearranging and setting $\frac{m^2 c^2}{\hbar^2} = \mu^2$, we will finally get the famous form 1.8.

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta + \mu^2 \right) \psi = 0 \tag{1.8}$$

Let's rewrite the equation in a form so we can see it is *lorentz invariant*. We use the fact that *4-gradient* transforms the same way as any other 4-vector and then express the Klein-Gordon equation using 4-gradient dot products. Using the metric $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and the *4-gradient* $\partial_\mu = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ it is possible to write the term $\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta$ as follows.

$$\begin{aligned} \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} &= \eta^{\mu\nu} \partial_\mu \partial_\nu = \partial_\nu \partial^\nu \\ \left(\partial_\nu \partial^\nu + \mu^2 \right) \psi &= 0 \end{aligned} \tag{1.9}$$

1.3 Dirac equation

Dirac tried to find an equation that is first order in both space and time by finding the *square root* of the Klein-Gordon equation. Let's start with the equation 1.8.

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta + \mu^2\right)\psi = 0$$

Chapter 2

Failures of the Relativistic Quantum Mechanics

In this chapter we will discuss the problems of relativistic quantum mechanics. We will start with the problem of probability density. We will show that the probability density is positive-definite for Dirac Equation but is not in the case of the Klein-Gordon equation. Then we will TODO:

2.1 Probability density

In the non-relativistic case we can find the continuity equation by taking the complex conjugate of the Schrodinger's equation.

$$\begin{aligned}i\hbar \frac{\partial}{\partial t} \psi &= \hat{H} \psi \\ -i\hbar \frac{\partial}{\partial t} \psi^* &= (\hat{H} \psi)^*\end{aligned}$$

Then we multiply the first equation by ψ^* and the second one by ψ .

$$\begin{aligned}i\hbar \psi^* \frac{\partial}{\partial t} \psi &= \psi^* \hat{H} \psi \\ i\hbar \psi \frac{\partial}{\partial t} \psi^* &= \psi (\hat{H} \psi)^*\end{aligned}$$

If we subtract these two equations from each other and use the formula for derivative $(a \cdot b)' = a' \cdot b + a \cdot b'$ we will get a form from which it will be easy to identify the continuity equation $\frac{\partial \rho}{\partial t} + \text{div} \vec{j} = 0$.

$$i\hbar \frac{\partial |\psi|^2}{\partial t} = \psi^* \hat{H} \psi - \psi (\hat{H} \psi)^*$$

We assume the potential V in $\hat{H} = \frac{\hat{p}^2}{2m} + V$ to be real-valued thus the right-hand side will simplify to

$$\psi^* \frac{\hbar^2}{2m} \nabla^2 \psi - \psi \frac{\hbar^2}{2m} \nabla^2 \psi^* = \frac{\hbar^2}{2m} \left(\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^* \right)$$

and the desired form is

$$\frac{\partial |\psi|^2}{\partial t} + \text{div} \left[\frac{\hbar}{2mi} \left(\psi \nabla \psi^* - \psi^* \nabla \psi \right) \right] = 0$$

Now we will do the same for the Klein-Gordon equation 1.8. Let's do the same procedure here and find the complex conjugate, multiply both equations by ψ and ψ^* and then subtract them. After that let's focus on the term which contains the time derivative and thus will represent the *probability density*.

$$\frac{\partial \rho}{\partial t} \sim \frac{1}{2i} \left[\psi^* \frac{\partial^2}{\partial t^2} \psi - \psi \frac{\partial^2}{\partial t^2} \psi^* \right]$$

Now we can find what ρ is proportional to.

$$\rho \sim \frac{1}{2i} \left[\psi^* \frac{\partial}{\partial t} \psi - \psi \frac{\partial}{\partial t} \psi^* \right] \quad (2.1)$$

Let's divide the wave function ψ into the real and imaginary part.

$$\begin{aligned} \psi &= i\psi_1 + \psi_2 \\ \rho &\sim \left[\psi_2 \frac{\partial}{\partial t} \psi_1 - \psi_1 \frac{\partial}{\partial t} \psi_2 \right] \end{aligned}$$

Now it is easy to see the problem. In the non-relativistic case the probability density was simply $|\psi|^2$ which is non-negative for any $\psi : \mathbb{R} \rightarrow \mathbb{C}$. In the case of Klein-Gordon equation, we can see the probability density is zero whenever the wave function is either exclusively real or imaginary. Moreover, since both $\frac{\partial \psi_1}{\partial t}$ and $\frac{\partial \psi_2}{\partial t}$ are arbitrary, the value can be negative. This is in contradiction with the interpretation of ρ as the *probability density of measuring the system within a given state ψ* . To solve this problem, Dirac introduced his equation. Let's check the probability density using the procedure of deriving the continuity equation from the Dirac equation [4]. Let's start with the Dirac equation of the form as follows.

$$(i\hbar\gamma^\mu \partial_\mu - mc)\psi = 0 \quad (2.2)$$

For further discussion let's introduce some useful theorems.

Theorem 2. Let γ^μ 's be Dirac matrices [5] defined by the properties (see [6])

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_4 \quad (2.3)$$

where $\eta^{\mu\nu} = \text{diag}\{+1, -1, -1, -1\}$, and

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0 \quad (2.4)$$

then following equations hold for $\mu \in \{0, 1, 2, 3\}$, $i \in \{1, 2, 3\}$.

$$\gamma^0 \gamma^i = -\gamma^i \gamma^0 \quad (2.5)$$

$$\begin{aligned}(\gamma^0)^\dagger &= \gamma^0 \\ (\gamma^i)^\dagger &= -\gamma^i\end{aligned}\tag{2.6}$$

Proof. The proof of 2.5 is a direct application of the definition 2.3.

$$\begin{aligned}\{\gamma^0, \gamma^i\} &= \gamma^0\gamma^i + \gamma^i\gamma^0 = 2\eta^{0i}I_4 \\ \gamma^0\gamma^i + \gamma^i\gamma^0 &= 0 \\ \gamma^0\gamma^i &= -\gamma^i\gamma^0\end{aligned}$$

To proof 2.6 we just need to know that $\gamma^0\gamma^0 = I_4$ which can be shown to be true because we constructed the matrices to be so or by checking the definition 2.3: $\{\gamma^0, \gamma^0\} = 2\gamma^0\gamma^0 = 2\eta^{00}I_4$. Then we use the defining property 2.4 and the equation 2.5.

$$\begin{aligned}(\gamma^0)^\dagger &= \gamma^0\gamma^0\gamma^0 = \gamma^0I_4 = \gamma^0 \\ (\gamma^i)^\dagger &= \gamma^0\gamma^i\gamma^0 = -\gamma^0\gamma^0\gamma^i = -I_4\gamma^i = -\gamma^i\end{aligned}$$

□

We will use a similar trick we used within the Schrodinger equation or the Klein-Gordon equation. We take the Hermitian adjoint of the left-hand side of the Dirac equation 2.2.

$$\begin{aligned}[i\hbar\gamma^\mu\partial_\mu\psi - mc\psi]^\dagger &= \\ &= [i\hbar\gamma^0\partial_0\psi + \gamma^1\partial_1\psi + \gamma^2\partial_2\psi + \gamma^3\partial_3\psi - mc\psi]^\dagger = \\ &= -i\hbar[(\gamma^0\partial_0\psi)^\dagger + (\gamma^1\partial_1\psi)^\dagger + (\gamma^2\partial_2\psi)^\dagger + (\gamma^3\partial_3\psi)^\dagger] - mc\psi^\dagger = \\ &= -i\hbar[\partial_0\psi^\dagger(\gamma^0)^\dagger + \partial_1\psi^\dagger(\gamma^1)^\dagger + \partial_2\psi^\dagger(\gamma^2)^\dagger + \partial_3\psi^\dagger(\gamma^3)^\dagger] - mc\psi^\dagger\end{aligned}$$

By using the equation 2.6 we evaluate all the gamma matrices with a dagger.

$$-i\hbar[\partial_0\psi^\dagger\gamma^0 + \partial_1\psi^\dagger(-\gamma^1) + \partial_2\psi^\dagger(-\gamma^2) + \partial_3\psi^\dagger(-\gamma^3)] - mc\psi^\dagger = 0\tag{2.7}$$

Now, the trick is to introduce $\bar{\psi} := \psi^\dagger\gamma^0$, multiply the equation 2.7 by γ^0 from the right and then rewrite all the $\psi^\dagger\gamma^0$ terms with $\bar{\psi}$ using the equation 2.5.

$$\begin{aligned}-i\hbar[\partial_0\psi^\dagger\gamma^0\gamma^0 + \partial_1\psi^\dagger(-\gamma^1\gamma^0) + \partial_2\psi^\dagger(-\gamma^2\gamma^0) + \partial_3\psi^\dagger(-\gamma^3\gamma^0)] - mc\psi^\dagger\gamma^0 &= 0 \\ -i\hbar[\partial_0\psi^\dagger\gamma^0\gamma^0 + \partial_1\psi^\dagger(\gamma^0\gamma^1) + \partial_2\psi^\dagger(\gamma^0\gamma^2) + \partial_3\psi^\dagger(\gamma^0\gamma^3)] - mc\psi^\dagger\gamma^0 &= 0 \\ -i\hbar[\partial_0\bar{\psi}\gamma^0 + \partial_1\bar{\psi}\gamma^1 + \partial_2\bar{\psi}\gamma^2 + \partial_3\bar{\psi}\gamma^3] - mc\bar{\psi} &= 0\end{aligned}$$

And finally, we will multiply this equation by ψ from the left.

$$i\hbar\partial_\mu\bar{\psi}\gamma^\mu\psi + mc\bar{\psi}\psi = 0\tag{2.8}$$

To get rid of the $mc\bar{\psi}\psi$ term, let's take the original equation 2.2 and multiply it by $\bar{\psi}$ from the right. Then we will simply add these two equations

$$\begin{aligned} i\hbar\partial_\mu\bar{\psi}\gamma^\mu\psi + mc\bar{\psi}\psi = 0 \wedge i\hbar\bar{\psi}\gamma^\mu\partial_\mu\psi - mc\bar{\psi}\psi = 0 &\implies \\ \partial_\mu\bar{\psi}\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi = 0 & \\ \partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0 &\quad (2.9) \end{aligned}$$

From the continuity equation 2.9 we see that the probability density should be $\rho = \bar{\psi}\gamma^0\psi$. Which is

$$\rho = \bar{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi$$

and thus always a positive quantity! TODO: discussion on the results.

2.2 Ground state

2.3 Causality

Now we would like to investigate the problem of causality in the quantum mechanics. We're going to study the transition amplitude of a free particle to propagate from one space-time point to another one. We will assume the standard definition of the causality in physics. TODO: discussion on the causality definition.

Definition 3. Suppose we understand what is meant by a cause C and an effect E . If C happens before E and E is in the future lightcone from the perspective of C then there might be a causality between C and E . Or formulated more usefully for the computation - if $X_C = (0, 0)$ is the C 's space-time event and $X_E = (x_E, ct_E)$ is the E 's space-time event then X_C can cause the X_E if $t_E > |x_E|$.

We will follow the same calculation like in [3]. In the correspondence with the fact that the space translation is generated by the momentum operator \hat{p} , the time evaluation is generated by the energy operator \hat{H} . So if we know the state of the system $|x_0\rangle = |x(t=0)\rangle$, the state in the future will be $|x_0\rangle = |x(t=t')\rangle = \hat{U}(0, t') |x_1\rangle = \exp\left\{-\frac{i}{\hbar}\hat{H}t'\right\} |x_0\rangle$. The amplitude of the system going from the state $|x_0\rangle$ to $|x_1\rangle$ is

$$\langle x_1 | \exp\left\{-\frac{i}{\hbar}\hat{H}t'\right\} | x_0 \rangle \quad (2.10)$$

which we will evaluate using the identity \hat{I} expressed in the continuous basis $\{|p\rangle\}$ and the expression for $\langle p|x\rangle = \exp\left\{-\frac{i}{\hbar}px\right\}$. We will assume a single particle without an external potential thus the Hamiltonian is simply $\hat{H} = \frac{\hat{p}^2}{2m}$.

$$\begin{aligned}
A_1 &= \langle x_1 | e^{-\frac{i}{\hbar} \frac{\hat{p}^2}{2m} t'} | x_0 \rangle = \int_{\mathbb{R}} dp \langle x_1 | e^{-\frac{i}{\hbar} \frac{\hat{p}^2}{2m} t'} | p \rangle \langle p | x_0 \rangle = \\
&= \int_{\mathbb{R}} \frac{dp}{2\pi\hbar} e^{-\frac{i}{\hbar} \frac{p^2}{2m} t'} \langle x_1 | p \rangle \langle p | x_0 \rangle = \int_{\mathbb{R}} dp e^{-\frac{i}{\hbar} \frac{p^2}{2m} t'} e^{\frac{i}{\hbar} p(x_1 - x_0)} = \\
&= \int_{\mathbb{R}} \frac{dp}{2\pi\hbar} e^{-\frac{i}{\hbar} \left[\frac{p^2}{2m} t' - p(x_1 - x_0) \right]} = \left| \frac{\left[\frac{p^2}{2m} t' - p(x_1 - x_0) \right]}{\frac{t'}{2m} \left[p - \frac{m}{t'}(x_1 - x_0) \right]^2 - \frac{m}{2t'}(x_1 - x_0)^2} \right| = \\
&= e^{\frac{im}{2\hbar t'}(x_1 - x_0)^2} \int_{\mathbb{R}} \frac{dp}{2\pi\hbar} e^{-\frac{it'}{2\hbar m} \left[p - \frac{m}{t'}(x_1 - x_0) \right]^2}
\end{aligned}$$

In the integral we can substitute $p \rightarrow p + \frac{m}{t'}(x_1 - x_0)$ so that the boundaries remain unchanged. Now the problem is that with the pure imaginary exponent we have an oscillatory function with no other term to guarantee the integral converges. As was shown in [8], we can actually use the trick of *rotating* the p in the complex plane by $\frac{\pi}{4}$ and get a known form of a gaussian integral multiplied by the phase factor.

$$\left| p = e^{-\frac{i\pi}{4}} r dp = e^{-\frac{i\pi}{4}} dr \right| = e^{\frac{i\pi}{4}} e^{\frac{im}{2\hbar t'}(x_1 - x_0)^2} \int_{\mathbb{R}} \frac{dp}{2\pi\hbar} e^{-\frac{t'}{2\hbar m} r^2}$$

$$A_1 = \sqrt{\frac{\hbar m}{2\pi i t'}} e^{\frac{im}{2\hbar t'}(x_1 - x_0)^2} \quad (2.11)$$

From the final result 2.11 we see that the probability density of measuring the transition $(x_0, 0) \rightarrow (x_1, t')$ will be

$$P_1 \sim \frac{1}{t'}$$

thus for an arbitrary choice of $(x_1 - x_0)$ there is no fixed time t' . This results in a possibility for the particle being outside of the lightcone with a non-zero probability. This result clearly violates the causality. But the conclusion is not unexpected in the case of the non-relativistic free particle. We used the definition within the relativity theory to formulate the condition for the causality but the Hamiltonian \hat{H} doesn't even include the c constant. We could try to get a proper result by assuming the relativistic energy 2.12. As we have shown, a quantization of such an expression lead us to the Klein-Gordon equation which has the Lorentz symmetry and thus is a good candidate for the right transition amplitude.

$$E^2 = \vec{p}^2 c^2 + m^2 c^4 \quad (2.12)$$

The hamiltonian for such a case is $\hat{H} = c\sqrt{\hat{p}^2 + (mc)^2}$. The new amplitude A_2 is then given by

$$\begin{aligned}
A_2 &= \langle x_1 | e^{-\frac{i}{\hbar} c \sqrt{\hat{p}^2 + (mc)^2} t'} | x_0 \rangle = \int_{\mathbb{R}} dp \langle x_1 | e^{-\frac{i}{\hbar} c \sqrt{p^2 + (mc)^2} t'} | p \rangle \langle p | x_0 \rangle = \\
&= \int_{\mathbb{R}} \frac{dp}{2\pi\hbar} e^{-\frac{i}{\hbar} c \sqrt{p^2 + (mc)^2} t'} e^{\frac{i}{\hbar} p(x_1 - x_0)}
\end{aligned}$$

Let's analyze this integral in two ways. Firstly, let's try to solve the problem numerically to get the idea of how the integral behaves. The problem is similar we had to deal with within the previous integral. The integrated function is oscillatory and thus the convergence is not expected in general. Numerically, we'll try to deal with that problem by choosing suitable boundaries. The best choice seems to be $p \in (-mc, mc)$ since it setups the desired condition on the *velocity* to be smaller than the speed of light. Then we only need to split the integral into a real and imaginary part.

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{dp}{2\pi\hbar} e^{-\frac{i}{\hbar} c \sqrt{p^2 + (mc)^2} t'} e^{\frac{i}{\hbar} p(x_1 - x_0)} = \\
&= \left| \begin{array}{l} k_{t'}(p) = -\frac{1}{\hbar} c \sqrt{p^2 + (mc)^2} t' \\ l_{x_1}(p) = \frac{1}{\hbar} p(x_1 - x_0) \end{array} \right| \rightarrow \\
&\rightarrow \int_{-mc}^{mc} \frac{dp}{2\pi\hbar} \left[\sin(k) \sin(l) - \sin(k) \sin(l) + i(\sin(k) \cos(l) + \cos(k) \sin(l)) \right]
\end{aligned}$$

Now, we can integrate the function numerically using the Python¹ program 1 because we integrate the real part and the imaginary part separately. The graph contains the final probability amplitude.

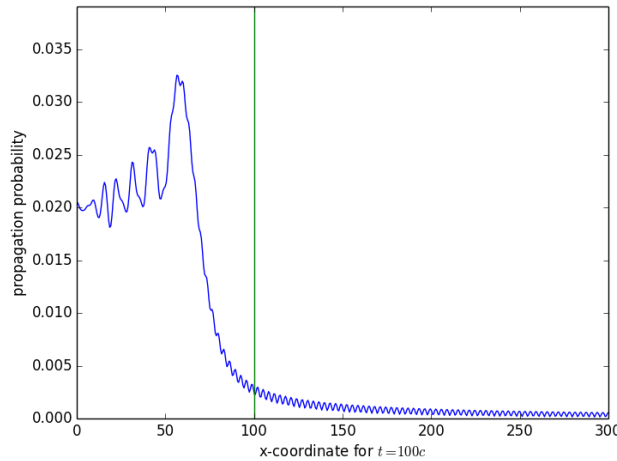


Figure 2.1: Propagation amplitude for a free relativistic particle

¹The script uses *scipy*, *matplotlib* and *numpy* packages. The used interpreter was *cpython*, version 2.7

Listing 1: Numerical integration of the relativistic free particle propagation amplitude

```

from numpy import sqrt, cos, sin, linspace, exp, pi
import scipy.integrate as integrate
import matplotlib.pyplot as plt

hbar, c, m = 1, 1, 1
TIME = 100

def function(x, t):
    def real(p):
        k = - 1 / hbar * c * (sqrt(p ** 2 + (m * c)**2)) * t
        l = 1 / hbar * p * x
        return 1 / (2 * pi * hbar) * (
            cos(k) * cos(l) - sin(k) * sin(l)
        )

    def imaginary(p):
        k = - 1 / hbar * c * (sqrt(p ** 2 + (m * c)**2)) * t
        l = 1 / hbar * p * x
        return 1 / (2 * pi * hbar) * (
            sin(k) * cos(l) + cos(k) * sin(l)
        )

    return real, imaginary

x_list = linspace(0, 3 * TIME, 500)
y_list = []

for x in x_list:
    real, imaginary = function(x, TIME)

    r = integrate.quad(real, -c * m, c * m)
    i = integrate.quad(imaginary, -c * m, c * m)
    y_list.append(sqrt(r[0] ** 2 + i[0] ** 2) / 2)

plt.xlabel("x-coordinate for $t = {}c$".format(TIME))
plt.ylabel("propagation probability")
plt.plot(x_list, y_list)
plt.plot([TIME, TIME], [0, max(y_list) * 1.2])
plt.gca().set_ylim([0, max(y_list) * 1.2])
plt.show()

```

In the graph 2.1 we also plotted the vertical line which marks the point from which the event is not considered causal by our definition and the probability density should be zero there. Although the value in the graph is reaching the zero very quickly it is very small but non-zero in the region outside of the lightcone (i.e. on the right side of the vertical marking line). So we encounter the same problem as in the calculation with the non-relativistic particle. The problem of the numerical approach is we have to deal with the non-zero error of the computation. To make sure the integral is non-zero in the region outside of the lightcone we will investigate the integrand analytically using the *steepest descent method*. Let $S(p)$ be the exponent part of the integrated function and $\Delta x = x_1 - x_0$. During the computation we will assume $(\Delta x)^2 > (ct')^2$ to take into account that we investigate the region where the amplitude should be zero.

$$S(p) = -\frac{i}{\hbar}c\sqrt{p^2 + (mc)^2}t' + \frac{i}{\hbar}p\Delta x$$

Now we should find the saddle points of S .

$$\frac{\partial S(p)}{\partial p} = 0$$

$$p^2 = \frac{(mc\Delta x)^2}{(ct')^2 - (\Delta x)^2}, \quad p_{\pm} = \pm \frac{imc\Delta x}{\sqrt{(\Delta x)^2 - (ct')^2}}$$

Now we will investigate whether the values p_+ , p_- are saddle points by finding the Hessian $\frac{\partial^2 S(p)}{\partial p^2}$.

$$\frac{\partial^2 S(p)}{\partial p^2} = i \frac{((\Delta x)^2 - (ct')^2)^{\frac{3}{2}}}{m(ct')^2}$$

TODO: why to pick the + solution? Thus we can conclude the amplitude A_2 is

$$A_2 \sim e^{S(p_+)} = e^{-\frac{mc}{\hbar}\sqrt{(\Delta x)^2 - (ct')^2}} \quad (2.13)$$

The expression 2.13 is non-zero for an arbitrary choice of Δx and t' thus it is non-zero also in the region outside of the lightcone. Thus we see we deal with the same problem again. The difference is that now we deal with an unexplainable problem. We used the relativistic energy and still the theory allowed the particle to travel faster than the speed of light with non-zero probability!

Chapter 3

Quantum field theory

3.1 Scalar field

In non-relativistic case we can find the continuity equation by

3.2 Second quantization

3.3 Causality problem

Conclusion

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