CURTIS-TITS AND PHAN SYSTEMS IN GAP

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Abstract. In this note, we briefly discuss the implementation of our algorithm in GAP which is about the construction of Curtis-Tits and Phan systems in the classical groups of odd characteristic.

1. Introduction

Let X be a black box group encrypting a black box group of Lie type of odd characteristic, Δ be the extended Dynkin diagram of the corresponding algebraic group. Assume that the nodes in Δ is numbered as $1, 2, \ldots, n$.

We call a set of subgroups $\{\mathsf{K}_1, \mathsf{K}_2, \ldots, \mathsf{K}_n\}$ *an extended Curtis-Tits system* for X if

(1) $\mathsf{X} = \langle \mathsf{K}_1, \mathsf{K}_2, \ldots, \mathsf{K}_n \rangle$.
(2) K_i, $i = 1, 2, \ldots, n$, encrypts $(\mathrm{P})\mathrm{SL}_2(q)$ where q is the size of the underlying field.
(3) $[\mathsf{K}_i, \mathsf{K}_j] = 1$ if the nodes i and j are not connected in Δ.
(4) $\langle \mathsf{K}_i, \mathsf{K}_j \rangle$ encrypts $\mathrm{SL}_3(q)$ if i and j are connected with a single bond.
(5) $\langle \mathsf{K}_i, \mathsf{K}_j \rangle$ encrypts $\mathrm{Sp}_4(q)$ if i and j are connected with a double bond.
(6) $\langle \mathsf{K}_i, \mathsf{K}_j \rangle$ encrypts $\mathrm{G}_2(q)$ or ${}^3D_4(q)$ if i and j are connected with a triple bond.

For more details about the Curtis-Tits Theorem, see [4].

An extended Phan system for X is defined similarly. The only difference between Curtis-Tits and Phan system is that the item (4) above should be

(4′) $\langle \mathsf{K}_i, \mathsf{K}_j \rangle$ encrypts $\mathrm{SU}_3(q^2)$ if i and j are connected with a single bond.

For more details about Phan's theorem, see [9, 10, 1] and Phan's systems for specific classical groups, see [2, 7, 5, 6, 8].

For orthogonal groups of type D_n (untwisted), we need to introduce *an extended pseudo-Phan system*. This system occurs when n is odd, that is, when we start to construct an amalgamation for the groups of type D_n with connected SL_2-subgroups generating SU_3, we end up with a Phan system when n is even and pseudo-Phan system when n is odd. In pseudo-Phan system, the first two nodes, which corresponds to the fork in the Dynkin diagram, K_1 and K_2 are identified and $\langle \mathsf{K}_1, \mathsf{K}_2 \rangle$ encrypts $\mathrm{PSL}_2(q^2)$ and the first three nodes $\langle \mathsf{K}_1, \mathsf{K}_2, \mathsf{K}_3 \rangle$ encrypts $\mathrm{SL}_4(q)$.

The construction of such systems was first presented in the paper [3]. Although the basic ideas are kept the same, we have improved our arguments to remove the hypothesis of having the characteristic of the underlying field or the size of the field given as an input. This GAP code is representing the ideas which differs from the procedures presented in [3] in this respect. In this GAP code, black box classical groups are considered to be given with a set of generators together with an exponent of the group. No other additional oracles or information is assumed. The algorithm, with slight modification, is applicable in more general black box setup.

2. Input and Output

In the present GAP code, we consider black box groups encrypting one of the following classical groups defined over a field of odd characteristic with the size of the underlying field ≥ 7. In the items below,"..." means that all the intermediate subgroups and factor groups are considered.

- $\mathrm{PSL}_n, \ldots, \mathrm{SL}_n$, *group of type* A_{n-1}, $n \geq 3$.
- Sp_{2n}, PSp_{2n}, *group of type* C_n, $n \geq 2$.
- $\mathrm{P}\Omega_{2n+1}, \ldots, \mathrm{SO}_{2n+1}$, *group of type* B_n, $n \geq 2$.
- $\mathrm{P}\Omega^+_{2n}, \ldots, \mathrm{SO}^+_{2n}$, *group of type* D_n, $n \geq 3$.

Input: The input of our algorithm is

I1 a set of generators of one of the groups listed above, and
I2 an exponent of the group.

We use the order of the group for the exponent of the input group. Note that as the groups in consideration are in GAP Library, their orders are known.

Output: The algorithm returns the type of the group with its Lie rank together with a list of generators of subgroups which are isomorphic to SL_2 or PSL_2. These subgroups correspond to the nodes in the extended Dynkin diagram forming an extended Curtis-Tits or extended (pseudo) Phan system for the input group. The last entry of the output list is corresponding to the extra node in the extended Dynkin diagram. One of the following output is produced depending on the input.

O1 The group is of type A_{n-1} together with an extended Curtis-Tits system.

O2 The group is of type C_n together with an extended Curtis-Tits or extended Phan system.

O3 The group is of type B_n together with an extended Curtis-Tits system or extended Phan system.

O4 The group is of type D_n together with an extended Curtis-Tits or extended Phan system.

O5 There are two possibilities to conclude from the construction:
1. The group is of type B_n together with an extended Curtis-Tits or extended Phan system, or
2. The group is of type D_{n+1} together with an extended pseudo-Phan system.

In Table 1 below, we present which output can possibly be returned by our algorithm.

Type	n	System	Output
A_{n-1}	-	Curtis-Tits	**O1**
C_n	-	Curtis-Tits or Phan	**O2**
B_n	odd	Curtis-Tits or Phan	**O3**
B_n	even	Curtis-Tits or Phan	**O5**
D_n	even	Curtis-Tits or Phan	**O4**
D_n	odd	Curtis-Tits	**O4**
D_n	odd	pseudo-Phan	**O5**

TABLE 1. The possible outputs depending on the types of groups.

We shall note here some remarks.

(1) The present GAP code can not distinguish a Curtis-Tits system from a Phan system and also the output type **O5** does not tell us which type of group we have. All of these complications in the present code will be overcome completely after we include our SL_2-recognition algorithm in this code.
(2) The code recognises the groups PSL_4 and PSp_4 as orthogonal groups. This is justified as $\mathrm{PSL}_4 \cong \mathrm{P}\Omega_6^+$ and $\mathrm{PSp}_4 \cong \Omega_5$.

(3) Notice that the twisted classical groups $\mathrm{SU}_n, \mathrm{P}\Omega_{2n}^-, \cdots, \mathrm{O}_{2n}^-$ are excluded from the input. These groups together with the exceptional groups of Lie type of odd characteristic will be included in our GAP code later.

3. Examples

We show how our GAP code runs in this section. Our code can be run by the function call CPT*("Generators of a group", "Exponent of the group")*.

We first start with the group SL(10, 7). We choose the field size as small as possible because we will use GAP functions for an isomorphism testing with the groups SL(2, 7) and SL(3, 7).

```
gap> x:=SL(10,7);; S:=GeneratorsOfGroup(x);; exp:=Order(x);;
gap>
gap> l:=CPT(S,exp);;
#I  Commuting product of (P)SL2's is constructed.
#I  A fundamental SL2-subgroup is constructed.
#I  2nd fundamental SL_2-subgroup is constructed.
#I  3rd fundamental SL_2-subgroup is constructed.
#I  4th fundamental SL_2-subgroup is constructed.
#I  5th fundamental SL_2-subgroup is constructed.
#I  6th fundamental SL_2-subgroup is constructed.
#I  7th fundamental SL_2-subgroup is constructed.
#I  8th fundamental SL_2-subgroup is constructed.
#I  9th fundamental SL_2-subgroup is constructed.
#I  The fundamental SL_2-subgroup for the extended Dynkin diagram is constructed.
#I  The group is of type A_9 together with an extended Curtis-Tits system.
gap>
```

The output is a list of 10 entries — the number of nodes in the extended Dynkin diagram.

```
gap> Length(l);
10
gap>
```

Each entry is isomorphic to SL(2, 7). We check here for the first two entries.

```
gap> IsomorphismGroups(SL(2,7),Group(l[1]));
CompositionMapping(
[ (2,29,5,8)(3,36,6,15)(4,43,7,22)(9,30,33,12)(10,37,34,19)(11,44,35,26)(13,
    16,31,40)(14,23,32,47)(17,38,41,20)(18,45,42,27)(21,24,39,48)(25,46,49,28)
    , (8,9,11,10,13,14,12)(15,17,19,18,21,16,20)(22,25,27,26,23,24,28)(29,33,
    35,34,31,32,30)(36,41,37,42,39,40,38)(43,49,45,44,47,48,46) ] ->
[ < immutable compressed matrix 10x10 over GF(7) >,
  < immutable compressed matrix 10x10 over GF(7) > ], <action isomorphism> )
gap>
gap>
gap> IsomorphismGroups(SL(2,7),Group(l[2]));
CompositionMapping(
[ (2,29,5,8)(3,36,6,15)(4,43,7,22)(9,30,33,12)(10,37,34,19)(11,44,35,26)(13,
    16,31,40)(14,23,32,47)(17,38,41,20)(18,45,42,27)(21,24,39,48)(25,46,49,28)
    , (8,9,11,10,13,14,12)(15,17,19,18,21,16,20)(22,25,27,26,23,24,28)(29,33,
    35,34,31,32,30)(36,41,37,42,39,40,38)(43,49,45,44,47,48,46) ] ->
[ < immutable compressed matrix 10x10 over GF(7) >,
  < immutable compressed matrix 10x10 over GF(7) > ], <action isomorphism> )
gap>
```

Now, the group generated by the first two entries of the list is isomorphic to SL(3, 7).

```
gap> IsomorphismGroups(SL(3,7),Group(Concatenation(l[1],l[2])));
CompositionMapping(
[ (2,200,297,251,105,152,50)(3,250,53,301,149,202,99)(4,300,103,51,199,252,
    148)(5,56,153,101,249,296,197)(6,100,203,151,299,52,246)(7,150,247,201,55,
    102,295)(8,61,161,111,255,305,205)(10,308,106,60,209,257,156)(11,206,307,
    259,110,155,58)(12,160,256,204,63,108,303)(13,253,59,306,158,210,107)(14,
    109,208,157,302,62,254)(15,118,212,168,312,68,262)(16,166,265,214,64,119,
    311)(18,65,162,117,266,314,213)(19,263,70,310,167,211,115)(20,217,313,260,
    114,165,66)(21,309,116,69,215,261,164)(22,175,269,219,75,125,319)(23,71,
    173,126,272,318,221)(24,223,322,271,120,170,74)(26,122,218,174,317,77,
    270)(27,320,121,73,224,267,172)(28,268,76,316,171,222,123)(29,226,326,276,
    132,182,82)(30,325,133,78,228,279,180)(31,127,230,177,329,81,278)(32,280,
    79,328,176,227,131)(34,179,274,231,80,128,327)( [...] ),
  (2,321,278,288,323,26,104)(3,84,335,302,85,34,154)(4,135,98,64,141,42,
    198)(5,192,107,120,155,44,248)(6,207,164,176,211,10,298)(7,264,221,232,
    267,18,54)(8,258,162,53,80,150,263)(9,163,222,340,241,172,19)(11,49,112,
    165,283,184,115)(12,225,331,270,101,195,70)(13,124,276,130,24,159,167)(14,
    87,95,39,304,168,310)(15,315,218,103,137,200,320)(16,144,110,47,67,219,
    73)(17,220,279,61,256,229,27)(20,281,94,327,151,210,121)(21,181,333,187,
    32,216,224)(22,72,274,153,194,250,83)(23,238,96,244,40,273,275)(25,277,
    336,118,313,286,35)(28,337,109,90,201,261,178)(29,129,330,203,209,300,
    140)(30,57,166,147,251,318,235)(31,289,111,259,48,324,332)(33,334,93,175,
    76,343,37)(36,186,92,247,266,56,191)(38,113,223,156,301,81,292)(41,97,108,
    226,133,58,45)(43,243,106,297,317,100,206)( [...] ) ] ->
[ < immutable compressed matrix 10x10 over GF(7) >,
  < immutable compressed matrix 10x10 over GF(7) > ], <action isomorphism> )
gap>
```

Moreover, the group generated by the first and the third entries of the list is isomorphic to $\mathrm{SL}(2,7) \times \mathrm{SL}(2,7)$.

```
gap> D:=DirectProduct(SL(2,7),SL(2,7));;
gap> IsomorphismGroups(D,Group(Concatenation(l[1],l[3])));
CompositionMapping(
[ (1,3,2,4)(5,11,6,12)(7,16,8,15)(9,14,10,13)(17,22,18,24)(19,32,20,31)(21,30,
    23,29)(25,28,26,27), (3,6,10,13,7,16,12)(4,5,9,14,8,15,11)(19,24,28,29,25,
    32,23)(20,22,27,30,26,31,21) ] ->
[ < immutable compressed matrix 10x10 over GF(7) >,
  < immutable compressed matrix 10x10 over GF(7) > ], CompositionMapping(
[ (1,2,4,8,14,20)(3,6,11,15,9,5)(7,13,21,22,28,34)(10,17,24,18,25,31)(12,19,
    26,16,23,29)(27,33,39,41,45,48)(30,36,43,37,40,47)(32,38,44,35,42,46),
  (1,3,7)(2,5,10)(4,9,16)(6,12,20)(8,15,22)(11,18,14)(13,21,27)(17,24,30)(19,
    26,32)(23,29,35)(25,31,37)(28,34,41)(33,40,42)(36,38,45)(39,46,47)(43,48,
    44), (49,50,52,56,62,69)(51,54,60,63,71,78)(53,58,66,70,77,84)(55,61,68,
    72,65,59)(57,64,73,76,83,91)(67,75,81,85,87,94)(74,80,86,92,90,88)(79,82,
    89,96,95,93), (49,51,55)(50,53,59)(52,57,65)(54,60,67)(56,63,72)(58,66,
    74)(61,69,76)(62,70,68)(64,73,79)(71,78,85)(75,82,90)(77,84,92)(80,87,
    95)(81,88,89)(83,91,96)(86,93,94) ] ->
[ (1,2)(3,4)(5,7,9,6,8,10)(11,13,15,12,14,16),
  (1,3,5)(2,4,6)(7,9,11)(8,10,12), (17,18)(19,21,25,20,23,26)(22,24)(27,29,31,
    28,30,32), (17,19,22)(18,20,24)(21,25,27)(23,26,28) ],
 <action isomorphism> ) )
gap>
```

The following is an example for symplectic groups.

```
gap> x:=Sp(10,7);; S:=GeneratorsOfGroup(x);; exp:=Order(x);;
gap> l:=CPT(S,exp);;
#I  Commuting product of (P)SL2's is constructed.
#I  A fundamental SL2-subgroup is constructed.
#I  The group is of symplectic type.
#I  1st short root fundamental SL2 is constructed.
#I  2nd short root fundamental SL2 is constructed.
#I  3rd short root fundamental SL2 is constructed.
#I  4th short root fundamental SL2 is constructed.
#I  The extra node in the extended Dynkin diagram is constructed.
#I  The group is of type C_5 together with an extended Curtis-Tits or extended Phan system.
gap>
```

This time the output list has 6 entries — the number of nodes in the extended Dynkin Diagram.

```
gap> Length(l);
6
gap>
```

The first and the last entry in the output list are corresponding to long root SL_2-subgroups in the extended Dynkin diagram of the group. Therefore, the first and the last two entries in the output list generate $\mathrm{Sp}(4,7)$.

```
gap> IsomorphismGroups(Sp(4,7),Group(Concatenation(l[1],l[2])));
CompositionMapping( [ (1,476,469,137,251,488,6)(2,579,433,192,342,587,5)(7,474,622,352,545,51,
    451)(8,577,542,263,418,67,561)(9,484,688,757,44,419,636)(10,120,25,548,478,236,379)(11,584,
    752,701,60,423,729)(12,89,33,428,581,185,291)(13,287,575,157,28,27,524)(14,608,163,23,640,
    369,604)(15,551,550,481,152,733,266)(16,601,186,37,410,331,629)(17,376,472,211,36,35,
    613)(18,517,216,31,694,280,511)(19,435,430,444,206,641,355)(20,508,237,53,327,412,633)(21,
    180,712,96,288,198,521)(22,403,523,302,150,553,774)(24,326,265,408,314,252,140)(26,658,374,
    365,606,228,730)(29,232,695,127,244,144,503)(30,316,612,390,204,437,795)(32,409,354,324,401,
    343,194)(34,649,285,276,513,175,637)(38,361,375,741,306,574,155)(39,230,489,466,405,370,
    535)(42,779,149,190,567,492,338)(45,315,161,424,681,357,447)(46,445,494,87,784,135,
    368)( [...] ), (1,250,618,9,2,321,534,11)(3,26,749,82,4,34,684,113)(5,571,581,453,6,466,478,
    562)(7,150,339,480,8,204,246,506)(10,300,59,507,12,388,43,540)(13,274,40,674,17,363,56,
    625)(14,447,769,156,18,558,783,210)(15,312,456,719,19,399,564,692)(16,426,550,352,20,546,
    430,263)(21,48,410,404,29,64,327,317)(22,173,551,175,30,226,435,228)(23,155,398,629,31,209,
    311,633)(24,791,122,223,32,790,91,170)(25,324,409,358,33,408,326,269)(27,716,585,357,35,635,
    486,268)(28,509,688,687,36,602,752,751)(37,224,766,724,53,171,767,715)(38,310,788,117,54,
    397,771,86)(39,262,573,491,55,351,470,590)(41,402,271,604,57,315,360,511)(42,165,778,519,58,
    218,784,609)(44,493,369,514,60,591,280,535)(45,141,287,610,61,195,376,520)(46,529,709,233,
    62,425,645,181)(47,596,600,140,63,499,504,194)( [...] ) ] ->
[ < immutable compressed matrix 10x10 over GF(7) >, < immutable compressed matrix 10x10 over GF(
    7) > ], <action isomorphism> )
gap>
```

```
gap> IsomorphismGroups(Sp(4,7),Group(Concatenation(l[5],l[6])));
CompositionMapping( [ (1,342,488,476,378,192,5)(2,251,587,579,290,137,6)(7,510,27,521,688,734,
    16)(8,603,35,503,752,647,20)(9,474,198,186,575,426,107)(11,577,144,237,472,546,76)(13,484,
    601,364,712,625,545)(14,723,41,682,610,486,240)(15,661,357,25,649,393,702)(17,584,508,275,
    695,674,418)(18,707,57,630,520,585,335)(19,737,268,33,658,305,758)(21,287,83,451,331,388,
    44)(22,699,171,39,446,797,40)(23,753,651,397,160,628,380)(24,131,273,675,441,464,763)(26,
    430,291,90,402,747,147)(28,37,180,51,636,416,99)(29,376,114,561,412,300,60)(30,679,224,55,
    557,782,56)(31,689,632,310,213,728,292)(32,188,362,745,554,569,792)(34,550,379,121,315,678,
    201)(36,53,232,67,729,468,130)(38,439,772,667,262,731,159)(43,663,432,725,773,314,182)(46,
    771,165,241,751,526,170)(48,403,482,657,125,768,489)(50,770,514,390,363,705,691)( [...] ),
  (1,604,287,659,2,511,376,736)(3,765,718,704,4,764,717,657)(5,501,789,272,6,598,780,361)(7,
    719,231,656,8,692,179,683)(9,471,286,42,11,574,375,58)(10,63,92,734,12,47,123,647)(13,590,
    797,433,17,491,782,469)(14,64,740,380,18,48,667,292)(15,294,480,334,19,382,506,239)(16,689,
    498,581,20,753,458,478)(21,534,473,158,29,618,576,212)(22,275,367,283,30,364,278,372)(23,
    735,778,41,31,652,784,57)(24,539,665,191,32,620,672,135)(25,723,716,307,33,707,635,395)(26,
    341,399,53,34,248,312,37)(27,420,390,769,35,482,302,783)(28,189,73,767,36,132,104,766)(38,
    243,88,593,54,256,119,495)(39,712,152,379,55,695,206,291)(40,418,309,406,56,545,396,
    319)(43,144,296,269,59,198,384,358)(44,229,630,250,60,176,682,321)(45,499,494,456,61,596,
    592,564)(46,503,714,71,62,521,642,102)( [...] ) ] ->
[ < immutable compressed matrix 10x10 over GF(7) >, < immutable compressed matrix 10x
    10 over GF(7) > ], <action isomorphism> )
gap>
```

For the orthogonal groups of type B_n, the first entry is isomorphic to $\mathrm{PSL}(2,q)$ – a short root SL_2-subgroup. The rest of the entries are isomorphic to $\mathrm{SL}(2,q)$. We give an example for the group $\mathrm{SO}(13,7)$.

```
gap> x:=SO(13,7);; S:=GeneratorsOfGroup(x);; exp:=Order(x);;
gap> l:=CPT(S,exp);;
#I  Commuting product of (P)SL2's is constructed.
#I  A fundamental SL2-subgroup is constructed.
#I  The group is of orthogonal type.
#I  2nd fundamental SL_2-subgroup is constructed.
#I  3rd fundamental SL_2-subgroup is constructed.
#I  4th fundamental SL_2-subgroup is constructed.
#I  5th fundamental SL_2-subgroup is constructed.
#I  6th fundamental SL_2-subgroup is constructed.
#I  There are two possibilities to conclude from the construction:
#I  1. The group is of type B_6 together with an extended Curtis-Tits or extended Phan system, or
#I  2. The group is of type D_7 together with an extended pseudo-Phan system.
gap>
```

```
gap> IsomorphismGroups(PSL(2,7),Group(l[1]));
[ (1,2)(3,8)(4,7)(5,6), (1,2,3,5)(4,8,7,6) ] ->
[ < immutable compressed matrix 13x13 over GF(7) >, < immutable compressed matrix 13x
    13 over GF(7) > ]
gap>
```

For the orthogonal group of type D_n. We show two examples when we have outputs **O4** and **O5**.

```
gap> x:=SO(1,14,7);; S:=GeneratorsOfGroup(x);; exp:=Order(x);;
gap> l:=CPT(S,exp);;
#I  Commuting product of (P)SL2's is constructed.
#I  A fundamental SL2-subgroup is constructed.
#I  The group is of orthogonal type.
#I  2nd fundamental SL_2-subgroup is constructed.
#I  3rd fundamental SL_2-subgroup is constructed.
#I  4th fundamental SL_2-subgroup is constructed.
#I  5th fundamental SL_2-subgroup is constructed.
#I  6th fundamental SL_2-subgroup is constructed.
#I  7th fundamental SL_2-subgroup is constructed.
#I  The group is of type D_7 together with an extended Curtis-Tits system.
gap>
```

```
gap> x:=SO(1,14,7);; S:=GeneratorsOfGroup(x);; exp:=Order(x);;
gap> l:=CPT(S,exp);;
#I  Commuting product of (P)SL2's is constructed.
#I  A fundamental SL2-subgroup is constructed.
#I  The group is of orthogonal type.
#I  2nd fundamental SL_2-subgroup is constructed.
#I  3rd fundamental SL_2-subgroup is constructed.
#I  4th fundamental SL_2-subgroup is constructed.
#I  5th fundamental SL_2-subgroup is constructed.
#I  6th fundamental SL_2-subgroup is constructed.
#I  There are two possibilities to conclude from the construction:
#I  1. The group is of type B_6 together with an extended Curtis-Tits or extended Phan system, or
#I  2. The group is of type D_7 together with an extended pseudo-Phan system.
gap>
```

When we have an output of type **O5** for orthogonal groups of type D_n, the first entry is isomorphic to $\mathrm{PSL}(2,q^2)$.

```
gap> IsomorphismGroups(PSL(2,49),Group(l[1]));
[ (2,9)(3,36)(4,10)(5,18)(6,13)(7,40)(8,42)(11,20)(12,15)(14,31)(16,44)(17,29)(19,32)(21,
    27)(22,41)(23,30)(24,34)(25,45)(26,38)(28,46)(33,49)(35,43)(39,50)(47,48),
  (1,8,5,3,7,4,2)(9,44,10,46,19,39,38)(11,23,34,41,30,32,35)(12,22,17,31,43,26,14)(13,49,15,50,
    21,20,40)(16,28,42,37,47,45,33)(18,25,36,48,24,27,29) ] ->
[ < immutable compressed matrix 14x14 over GF(7) >, < immutable compressed matrix 14x
    14 over GF(7) > ]
gap>
```

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