Fuzzy Option Pricing for Jump Diffusion Models using Neuro Volatility Models

***Abstract*—Recently there has been a growing interest in studying fuzzy option pricing using Monte Carlo (MC) methods for diffusion models. In short-term option pricing, the most important parameter is the volatility parameter. The traditional volatility estimator has a larger asymptotic variance. The recently proposed data-driven neuro-volatility models have smaller variances, and they have been used here to obtain direct volatility forecasts. The expected return of options determined by traditional option pricing models does not account for the uncertainty associated with the underlying asset price volatility. Fuzzy set theory is a capable tool to account for ambiguity and vagueness associated with data. Asymmetric nonlinear adaptive fuzzy numbers used in this study allow us to integrate uncertainties associated with data that are distributed with high skewness and kurtosis (non-normal with heavy-tailed). This paper uses fuzzy set theory and data-driven volatility forecasts to study option prices of a call option (*SPX.US.06.16.23.C4000.Index*) on the S&P 500 index. Four modeling approaches have been considered, Black-Scholes (BS) model, option pricing with normal errors, option pricing with t errors, and the Jump-Diffusion (JD) model. Fuzzy α-cuts of option prices are presented and discussed under different parameter values. Our experimental study suggests that the JD model predicts the call option price more accurately compared to BS, normal errors, and t errors using the volatility estimate from the Bayesian setup. The α-cuts of option price using the BS model predict the option price more accurately compared to the other approaches.**

***Index Terms*—Asymmetric Nonlinear Adaptive Fuzzy Numbers, Call Option Price, Jump-Diffusion, Neuro-Volatility Forecasts**

I. Introduction

Financial derivatives that give a right to buy or sell an underlying asset at a specified price and time are known as options. Mainly there are two types of options, call options and put options. Holding a call option gives the right to buy a stated asset, and a put option gives the right to sell the asset to the contract holder. Investors use options to reduce the risk exposure of their investments. Depending on the style of the option, the holder has the choice to execute the agreement on or before its expiration date. The specified price of the option is known as the strike price. When the strike price is lower than the market price, a call option holder has the opportunity to purchase the underlying asset (stock) at a price lower than the market price. The option holder can buy the asset (stock) at a lower price and sell it immediately in the market at a higher market price. The holder of the call option benefits from the price rising if the price difference can cover the holder's costs and commissions.

Modeling option prices is a popular research area in modern finance. Black-Scholes (BS), a popular approach to modeling option prices, assumes log returns of underlying asset prices are normally distributed. A recent study by Thavaneswaran et al. [1] has shown that in many cases, log returns are non-normal and t distributed with heavy tails. Also, there is an indication of a significant correlation considering absolute and squared log return of assets suggesting time-varying volatility models are more appropriate for option value computations.

It can be observed that the variance of asset returns changes over time. Studies such as [2] suggest that Garch coefficient BS models are good choices to model data with time-varying variances. In 2010, Thavaneswaran and Signh [3] proposed a new method for pricing derivatives under the jump-diffusion (JD) model with random volatility by viewing the call price as an expected value of a truncated lognormal distribution. Option pricing models with time-varying volatility require an estimation of the volatility process. In literature, forecasts of conditional volatility are obtained by taking the square root of the forecast of conditional variance. Study [1] has shown that the asymptotic variance of this estimate is larger, and thus, it is an inefficient way to obtain the volatility estimate. In this paper, we use recently proposed data-driven neuro volatility models to obtain direct volatility (as opposed to variance) forecasts [4].

The fuzzy set theory can be used to account for ambiguity and vagueness arising due to financial market uncertainties. Previous studies have considered incorporating fuzzy variables to capture uncertainty in option pricing [5]. Fuzzified asset price, risk-free rate, and asset volatility for options such as real options and European options are used in [6] and [7]. Moreover, recent studies such as [8] and [9] discuss incorporating fuzzified volatility in option pricing.

A recent study by Liang et al. [10] proposed a new approach to calculate fuzzy call and put option prices using data-driven exponentially weighted moving average (DD-EWMA) and neuro volatility forecasts (NVF). The paper fuzzyfies volatility forecasts using adaptive nonlinear fuzzy numbers, including trapezoidal fuzzy numbers to incorporate uncertainty associated with the markets. Furthermore, commonly used BS formulas and normal MC simulations underestimate option prices due to the underestimation of the volatility of the log returns. Even though α-cuts of adaptive nonlinear fuzzy numbers allow to incorporate uncertainties, it always produces symmetric α-cuts which fail to model uncertainties at extreme ends of the distributions differently. In this paper, we extend the work of Liang et al. [10] to asymmetric nonlinear adaptive fuzzy numbers for volatility forecasts. With asymmetric nonlinear adaptive fuzzy numbers, we obtained asymmetric

α-cuts of volatility forecasts and observe the changes in the option price. As an empirical study, we apply this analysis to a call option (*SPX.US.06.16.23.C4000.Index*) on the S&P 500 index (Standard and Poor's 500). Furthermore, we apply the fuzzy volatility forecast approach to a JD option pricing model. JD model parameter estimates are obtained using Bayesian Markov Chain Monte Carlo setup in WinBUGS.

The remainder of the paper is organized as follows. In Section II, the theories of the data-driven fuzzy forecast of option price using the nonlinear adaptive fuzzy numbers and NVF are given. Section III provides experiment results. Finally, concluding remarks are given in Section IV.

II. Option Pricing and Neuro Fuzzy Volatility Forecasts

A. Black-Scholes Model

In 1973, Black and Scholes [11] proposed that the change in stock prices follows the stochastic differential equation,

$$\begin{aligned} dS(t) &= S(t)(\mu dt + \sigma dB(t)), \\ dB(t) &\sim N(0, dt) \text{ for } t \in (0, T], \text{ and } S(0) > 0 \end{aligned}$$

where $S(t)$ is the stock price at time t, μ denotes the drift rate of the stock return, $\sigma > 0$ denotes the volatility of the stock return, and $B(t)$ is the standard Brownian motion.

A solution to the above equation can be obtained using Ito's lemma [12] as follows:

$$S(t) = S(0)exp\left(\sigma B(t) + \left(\mu - \frac{\sigma^2}{2}\right)t\right).$$

B. Merton Model

Merton proposed a jump-diffusion model first in 1976 [13] which is capable of capturing change in stock price. Merton suggests that changes in the stock price can be modeled with the stochastic differential equation as

$$\begin{aligned} dS(t) &= S(t)((\mu - \lambda k) + \sigma dB(t)) + S(t-)dC(t), \\ dB(t) &\sim N(0, dt), \\ C(t) &= \sum_{i=1}^{\eta(t)}(D_i - 1), \\ D_i &= S(\tau_i)/S(\tau_i-) > 0, \\ k &= E[D - i - 1], \\ F_i &= lnD_i \sim N(m, \delta^2) \text{ for } t \in (0, T], \\ & i \in \mathbb{N} \text{ and } S(0) > 0 \end{aligned}$$

where $S(t)$ is the stock price at time t and $B(t)$ is a standard Brownian motion. Merton extends the Black-Scholes model by assuming there are jumps in stock prices and jumps are occurring at random times $\tau_1, \tau_2, \ldots, \tau_i > 0\ \forall i$. $\eta(t) = i\ ;\ \tau_i \in (0, t]$ is a counting process assuming it follows a Poisson process with intensity $\lambda > 0$, and λ denotes the number of times jumps occurs. Considering the jumps occur from $S(\tau_i-)$ to $S(\tau_i)$, price ratio, D_i, is given by $S(\tau_i)/S(\tau_i-)$ and relative change in the stock price is given by the $D_i - 1$ if jumps occur. The jump size $C(t)$ is the sum of all relative changes in stock price, and it is a compound Poisson process. D_i are assumed to be identical and independent log-normal random variables with parameters m and $\delta > 0$, and k is the expected value of $D_i - 1$. μ is the drift rate of the stock return and the volatility of the stock return is $\sigma > 0$ assuming no jumps in the path of stock prices.

Similar to BS differential equation, a solution to the JD process differential equation can also be obtained as

$$S(t) = S(0)exp\left(\sigma B(t) + \left(\mu - \lambda k - \frac{\sigma^2}{2}\right)t\right)\prod_{i=1}^{\eta(t)} D_i. \quad (1)$$

C. Neuro-Volatility Forecasting

A neural network (NN) is a powerful tool to predict any nonlinear real function on a bounded domain with high accuracy. The simplest version of NN is a feed-forward NN. It contains an input unit that reads the input variables, followed by an arbitrary number of interconnected hidden layers, followed by an output layer. The transformation between two consecutive layers can be represented by nonlinear functions [4].

NNs differ from traditional time series forecasting models used in finance. It does not require tuning a high number of parameters, and all the parameters do not need to be optimized in a NN to get a universal approximate solution. Thavaneswaran et al. [4] were the first to study volatility and VaR forecasting using a generalized neuro-volatility model. They trained their neuro-volatility model, a feed-forward neural network, on the p lagged values of the centered absolute log returns to predict the next value. In this work, we use the neuro-volatility model where the p-lagged values of the volatility of absolute log returns $(V_{t-1}, \ldots, V_{t-p})$ are used as inputs to predict V_t (Fig. 1).

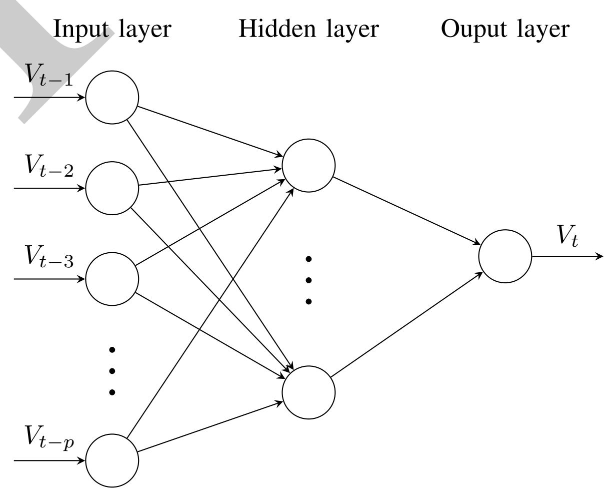

Fig. 1. Illustration of feed-forward neuro-volatility model

D. Asymmetric Nonlinear Adaptive Fuzzy Numbers

Let R represent a set of all real numbers. A fuzzy number $A(x)$, $x \in R$ has the following form

$$A(x) = \begin{cases} g(x) & \text{if} \quad [a,b) \\ 1 & \text{if} \quad [b.c) \\ h(x) & \text{if} \quad [c,d) \\ 0 & \text{otherwise.} \end{cases}$$

Here g and h are both real functions. However, function g is a right continuous and increasing function, whereas, function h is a left continuous and decreasing function. The parameters a, b, c, d take real values satisfying $a < b < c < d$. Further details of fuzzy numbers are discussed in the research work of Bodjanova [14] and Zimmermann [15].

The nonlinear adaptive asymmetric fuzzy number A (Fig. 2) can be denoted as $A = [a, b, c, d]_{m,n}$ and functions g and h are further defined as follows

$$g(x) = \left(\frac{x-a}{b-a}\right)^m,$$

$$h(x) = \left(\frac{d-x}{d-c}\right)^n,$$

where m and n are positive real numbers (m, $n > 0$). When m and n are equal to 1, the notation of the fuzzy number simplifies to $A = [a, b, c, d]$, commonly known as a trapezoidal fuzzy number, which is a linear fuzzy number.

The horizontal representations of fuzzy sets are the α-cuts. The α-cuts of the asymmetric adaptive nonlinear fuzzy numbers $A = (a, b, c, d)_{m,n}$ are given by

$$[a + \alpha^{\frac{1}{m}}(b-a), d - \alpha^{\frac{1}{n}}(d-c)] \tag{2}$$

for all $\alpha \in [0, 1]$.

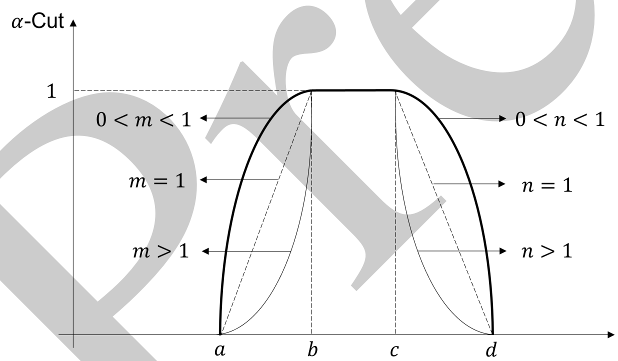

Fig. 2. Nonlinear adaptive asymmetric fuzzy number

The α-cuts of the annualized volatility is calculated using different α, (a, b, c, d), m, and n values and it is given by

$$\overline{\sigma^A(\alpha)} := [\sigma_1^A(\alpha), \sigma_2^A(\alpha)] = [a + \alpha^{\frac{1}{m}}(b-a), d - \alpha^{\frac{1}{n}}(d-c)].$$

For this study, we choose (a, b, c, d) as the 0.05, 0.25, 0.75, and 0.95 quantiles respectively of the volatility forecasts.

E. Fuzzy Option Pricing based on Fuzzy Volatility

The method of pricing the European call and put option can be used to find the price of any claim in the modified BS model to incorporate time-varying volatility

$$dS_t = rS_t dt + \sigma_t S_t dW_t$$

and

$$r_t = \log \frac{S_t}{S_{t-1}} = \sigma_t Z_t$$

where $\{S_t\}$ is the stock process, r is the risk-free interest rate, $\{W_t\}$ is a Wiener process, $\{\sigma_t\}$ is the volatility process, and Z_t is the innovation. Equivalently, we have

$$S_t = S_0 e^{\sigma_t W_t + (r - \frac{\sigma_t^2}{2})t}.$$

Let K be the strike price of an option, S_t is the current stock price, the time to expiration is $T - t$, the risk-free rate is r, and the volatility is σ. Assume $\overline{\sigma_A(\alpha)}$ are the α-cuts of the annualized volatility, the call option prices can be calculated as

$$\overline{C(\alpha)} := [C_1(\alpha), C_2(\alpha)] = N(d_1)S_t - N(d_2)Ke^{-r(T-t)}$$

where d_1 and d_2 are calculated based on the α-cuts of the annualized volatility

$$d_1 = \frac{\ln \frac{S_t}{K} + (r + \frac{\overline{\sigma(\alpha)}^2}{2})(T-t)}{\overline{\sigma(\alpha)}\sqrt{T-t}}, d_2 = d_1 - \overline{\sigma(\alpha)}\sqrt{T-t}.$$

If the normality assumption is not valid, the above equation cannot be used and MC simulation based on the data-driven t distribution of the asset log returns is used. Liang et al. [10] have shown that fuzzy call option price $[C_1(\alpha), C_2(\alpha)]$ is calculated as

$$C_1(\alpha) = mean(e^{-r(T-t)}pmax(FP_1 - K, 0))$$
$$C_2(\alpha) = mean(e^{-r(T-t)}pmax(FP_2 - K, 0)$$

where

$$FP_1 := S_t e^{\sigma_1^A(\alpha)\sqrt{T-t}*ST + \left(r - \frac{(\sigma_1^A(\alpha))^2}{2}\right)(T-t)},$$

$$FP_2 := S_t e^{\sigma_2^A(\alpha)\sqrt{T-t}*ST + \left(r - \frac{(\sigma_2^A(\alpha))^2}{2}\right)(T-t)},$$

and $pmax$ represents calculating the average and the maximum over the simulated data.

F. Jump-Diffusion Model

Assuming that the starting stock price is St, we can simulate jump times $\tau_1, \tau_2, \ldots, \tau_i > 0$ between jumps where the stock price evolves according to geometric Brownian motion assuming that Poisson and Wiener processes driving jumps and the continuous part of the stock price are independent.

It follows that, conditional on the times $\tau_1, \tau_2, \ldots, \tau_i$ of the jumps, the solution of the JD process from (1) can be written as

$$\ln S(\tau_{j+1}) = \ln S(\tau_j) + \left(\mu^Q - \frac{\sigma^2}{2}\right)(\tau_{j+1} - \tau_j) + \sigma[W^Q(\tau_{j+1}) - W^Q(\tau_j))] + Y_{j+1}$$

The risk-free drift of the asset price process, μ^Q, is given by

$$\mu^Q = r - q - \lambda^Q m_J^Q$$

where r is the risk-free rate of return, q is the dividend yield of the asset price process, λ^Q is the risk-free intensity of the jump process, and m_J^Q is the risk-free mean relative jump size given by

$$m_J^Q = E\left(e^Y - 1\right) = \exp\left(\mu_J^Q + \frac{\sigma_J^2}{2}\right) - 1$$

and

$$Y \sim N\left(\mu_J^Q, \sigma_J^2\right)$$

Algorithm 1 illustrates how to simulate one step of this recursion.

Algorithm 1 Jump-Diffusion

- Generate R_{j+1} from the exponential distribution with mean $1/\lambda^Q$
- Generate $Z_{j+1} \sim N(0, 1)$
- Generate
$$Y_{j+1} \sim N\left(\mu_J^Q, \sigma_J^2\right)$$
- Set $\tau_{j+1} = \tau_j + R_{j+1}$ and
$$\ln S(\tau_{j+1}) = \ln S(\tau_j) + \left(\mu^Q - \frac{\sigma^2}{2}\right) R_{j+1}$$
$$+\sigma\sqrt{R_{j+1}} Z_{j+1} + Y_{j+1}$$

The exponential random variable R_{j+1} can be generated by setting $R_{j+1} = -\ln(U)/\lambda^Q$ where $U \sim Unif[0, 1]$.

G. Parameter Estimation under Bayesian Setup

We consider a time-discretization of the jump-Diffusion model, and it implies that at most a single jump can occur over each time interval

$$Y_{t+1} = \mu^* + \sigma(W_{t+1} - W_t) + J_{t+1}\xi_{t+1}$$

where

$$\mu^* = \mu - q - \lambda m_J - \frac{\sigma^2}{2},$$

$$P(J_t = 1) = \lambda \in (0, 1),$$

and the jumps retain their structure (q is the dividend yield of the asset price process).

The parameters and state variable vectors are given by $\Theta = \{\mu, \sigma, \lambda, \mu_J, \sigma_J\}$ and $X = \{J_t, \xi_t\}, t = 1, \ldots, T$.

We use the Bayesian MCMC algorithm to estimate the parameters. Our MCMC algorithm samples from

$$p(\Theta, X|Y) = p(\Theta, J, \xi|Y)$$

where J and ξ are vectors containing the time series of jump times and sizes. Also,

$$p(Y_t|\Theta, J_t, \xi_t) = N(\mu^* + \xi_t J_t, \sigma^2)$$

$$p(\xi_t|\Theta) \propto N(\mu_J, \sigma_J^2)$$

$$p(J_t|\Theta) = Bern(\lambda)$$

In MCMC option pricing, we assume that jump intensity under measure Q is the same as under P. In principle, it must be adjusted for the price of jump risk, θ_J, as $\lambda^Q = \lambda^P(1-\theta_J)$.

III. Experiments

We consider adjusted closing price data for S&P 500 from April 1st, 2010, to March 1st, 2022. The study focuses on a specific call option '*SPX.US.06.16.23.C4000.Index*' on the S&P 500 index, which matures on June 16th, 2023. The observed date for the call option is February 15th, 2023. The strike price for the call option is listed as USD 4000, and the risk-free rate for this study is the average treasury bill rate (T-bill rate) from December 31st, 2009, to March 1st, 2023. All the data are obtained from Bloomberg[1].

Summary statistics of log returns and observed volatility of S&P 500 are obtained and summarized in Table I. A study by Thavaneswaran et al. [1] introduce the idea of sign correlation and explains how it can be used to determine the appropriate distribution of given data. The calculated sample sign correlation value here is 0.6554 (less than 0.7979). Thus, t distribution with degrees of freedom (d.f.) 3.1719 is appropriate to model S&P 500 log returns. Also, we observe that log returns are skewed with high excess kurtosis indicating the distribution of log returns is non-normal and heavy-tailed. The table also provides the auto-correlation function values of log returns, the absolute value of log returns, and the squared values of log returns. Since the values are not equal to zero, the series is significantly autocorrelated, indicating volatility clustering. Similar observations can also be made on observed volatility. Specifically, note the high skewness and kurtosis of observed volatility. The distribution of Observed volatilities is non-normal with heavy-tailed. This emphasizes the importance of accounting uncertainties in modeling with asymmetric nonlinear adaptive fuzzy numbers. The observed volatility for the study period is given in Fig. 3. It can be seen that the volatility changes over time, and a significant peak can be observed during a peak time of the COVID-19 pandemic (April and May 2020).

TABLE I
Summary Statistics of S&P 500 (January 2010 to March 2022)

	Log-returns	Observed Volatility
Mean	0.0005	0.0112
Standard Deviation	0.0112	0.0130
Skewness	-0.7252	3.7498
Kurtosis	13.0925	28.3041
Sign Correlation	0.6554	0.6947
Degrees of Freedom (t)	3.1719	3.7270
ACF (log returns)	-0.1254	0.3401
ACF (absolute log returns)	0.3348	0.3401
ACF (squared log returns)	0.4584	0.4558

[1]https://www.bloomberg.com/canada

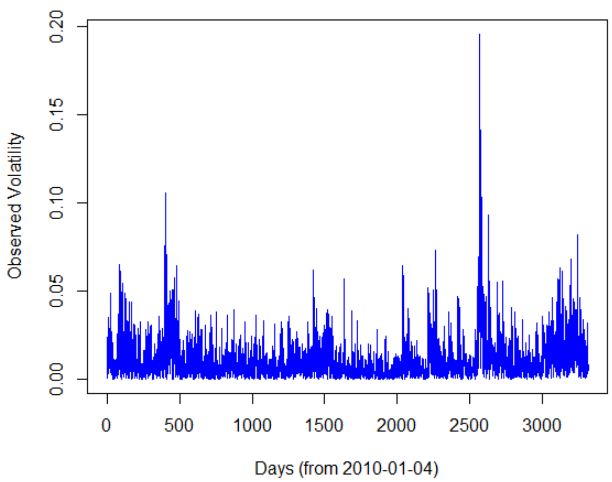

Fig. 3. Observed Volatility - S&P 500 of log-returns (January 2010 to March 2022)

The actual call option price of '*SPX.US.06.16.23.C4000.Index*' at maturity (June 16, 2023) is USD 283.65. We fit all four models (BS, Normal MC, t (d.f. 3.17) MC, and JD MC) to estimate the call option price, and the results are summarized in Table II. JD model provides an option price estimate (USD 281.75) close to the actual price (USD 283.65). Whereas BS and Normal MC underestimates call option price and t MC with 3.17 d.f. overestimates the call option price. However, these point estimates failed to provide any information about the uncertainty associated with the call option price. Thus, we obtain asymmetric α-cuts of the call option price below to observe possible extreme prices.

TABLE II
CALL OPTION PRICE ESTIMATES

Approach/Model	Call Option Price Estimate (USD)
Black-Scholes	242.80
Normal MC	238.31
t (3.17) MC	318.51
Jump-Diffusion MC	281.75

Tables III, IV, and V summarize α-cuts of NVF for S&P 500 with 63 overlapping rolling windows (covering three months of data). The tables provide the lower and upper bounds of the α-cuts considering different values of parameters m, n, and α. As expected from Fig. 2, α-cuts get narrower as α increases. For all three cases of m ($m = 0.5, 1.0, 2.0$), lower bounds remain unchanged and upper bounds decrease as n increases. For given values of α and n, lower bounds increase, and upper bounds remain unchanged as m increases. Thus, as m, n, and α increase, α-cuts obtained from NVF are narrower, and as they decrease, we can observe volatility bounds for extreme cases.

TABLE III
α-CUTS OF σ_A USING NVF FOR S&P 500 ($m = 0.50$)

α	$n = 0.50$ LB	$n = 0.50$ UB	$n = 1.0$ LB	$n = 1.0$ UB	$n = 2.0$ LB	$n = 2.0$ UB
0.0	0.0096	0.0168	0.0096	0.0168	0.0096	0.0168
0.1	0.0097	0.0168	0.0097	0.0166	0.0097	0.0160
0.2	0.0097	0.0167	0.0097	0.0163	0.0097	0.0157
0.3	0.0098	0.0166	0.0098	0.0161	0.0098	0.0155
0.4	0.0100	0.0164	0.0100	0.0158	0.0100	0.0153
0.5	0.0102	0.0162	0.0102	0.0156	0.0102	0.0151
0.6	0.0105	0.0159	0.0105	0.0154	0.0105	0.0149
0.7	0.0108	0.0156	0.0108	0.0151	0.0108	0.0148
0.8	0.0111	0.0153	0.0111	0.0149	0.0111	0.0146
0.9	0.0115	0.0149	0.0115	0.0146	0.0115	0.0145
1.0	0.0119	0.0144	0.0119	0.0144	0.0119	0.0144

LB - Lower Bound, UB - Upper Bound

TABLE IV
α-CUTS OF σ_A USING NVF FOR S&P 500 ($m = 1.0$)

α	$n = 0.50$ LB	$n = 0.50$ UB	$n = 1.0$ LB	$n = 1.0$ UB	$n = 2.0$ LB	$n = 2.0$ UB
0.0	0.0096	0.0168	0.0096	0.0168	0.0096	0.0168
0.1	0.0099	0.0168	0.0099	0.0166	0.0099	0.0160
0.2	0.0101	0.0167	0.0101	0.0163	0.0101	0.0157
0.3	0.0103	0.0166	0.0103	0.0161	0.0103	0.0155
0.4	0.0105	0.0164	0.0105	0.0158	0.0105	0.0153
0.5	0.0108	0.0162	0.0108	0.0156	0.0108	0.0151
0.6	0.0110	0.0159	0.0110	0.0154	0.0110	0.0149
0.7	0.0112	0.0156	0.0112	0.0151	0.0112	0.0148
0.8	0.0115	0.0153	0.0115	0.0149	0.0115	0.0146
0.9	0.0117	0.0149	0.0117	0.0146	0.0117	0.0145
1.0	0.0119	0.0144	0.0119	0.0144	0.0119	0.0144

The α-cuts of call option price for S&P 500 using BS, normal MC, t MC with 3.17 d.f., and JD MC are summarized in Tables VI, VII, VIII, and IX, respectively. Note that α-cuts of call option prices are obtained with NVF. For α-cuts using BS and normal errors, the actual price is within the bounds for selected α, m, and n values. However, α-cuts are obtained using t errors, and the JD model does not contain the actual call price.

IV. CONCLUSIONS

Fuzzy option pricing is one of the popular areas of research in computational finance. Popular approaches in option pricing, such as Black-Scholes, assume the normality of asset log returns. Recent studies have shown log returns are non-normal and t distributed with heavy-tail. Our experimental study suggests that the observed volatility of S&P 500 log returns is distributed with high skewness and kurtosis, and symmetric fuzzy numbers fail to account for high skewness and kurtosis in data. The driving idea, unlike the existing MC option pricing with normality assumption, is that incorporate this high ambiguity and vagueness in option pricing using asymmetric nonlinear adaptive fuzzy numbers and α-cuts of neuro volatility forecasts. Jump-Diffusion models and other three modeling approaches (BS, option pricing with normal errors, and option pricing with t errors) have been considered to obtain option pricing estimates, and they are compared against the actual price of a call option (*SPX.US.06.16.23.C4000.Index*) on S&P 500. Results show that the α-cuts of option price using the

TABLE V
α-CUTS OF σ_A USING NVF FOR S&P 500 ($m = 2.0$)

α	$n = 0.50$ LB	$n = 0.50$ UB	$n = 1.0$ LB	$n = 1.0$ UB	$n = 2.0$ LB	$n = 2.0$ UB
0.0	0.0096	0.0168	0.0096	0.0168	0.0096	0.0168
0.1	0.0104	0.0168	0.0104	0.0166	0.0104	0.0160
0.2	0.0107	0.0167	0.0107	0.0163	0.0107	0.0157
0.3	0.0109	0.0166	0.0109	0.0161	0.0109	0.0155
0.4	0.0111	0.0164	0.0111	0.0158	0.0111	0.0153
0.5	0.0112	0.0162	0.0112	0.0156	0.0112	0.0151
0.6	0.0114	0.0159	0.0114	0.0154	0.0114	0.0149
0.7	0.0115	0.0156	0.0115	0.0151	0.0115	0.0148
0.8	0.0117	0.0153	0.0117	0.0149	0.0117	0.0146
0.9	0.0118	0.0149	0.0118	0.0146	0.0118	0.0145
1.0	0.0119	0.0144	0.0119	0.0144	0.0119	0.0144

TABLE VI
α-CUTS OF S&P 500 CALL OPTION PRICE USING BS ($m = 0.50, 1.0, 2.0$)

m	α	$n = 0.50$ LB	$n = 0.50$ UB	$n = 1.0$ LB	$n = 1.0$ UB	$n = 2.0$ LB	$n = 2.0$ UB
0.50	0.1	264.39	386.20	264.39	382.42	264.39	373.35
	0.3	267.41	382.84	267.41	374.03	267.41	363.65
	0.5	273.47	376.13	273.47	365.65	273.47	356.98
	0.7	282.60	366.07	282.60	357.28	282.60	351.57
	0.9	294.86	352.68	294.86	348.92	294.86	346.89
1.0	0.1	267.78	386.20	267.78	382.42	267.78	373.35
	0.3	275.36	382.84	275.36	374.03	275.36	363.65
	0.5	282.98	376.13	282.98	365.65	282.98	356.98
	0.7	290.64	366.07	290.64	357.28	290.64	351.57
	0.9	298.32	352.68	298.32	348.92	298.32	346.89
2.0	0.1	275.98	386.20	275.98	382.42	275.98	373.35
	0.3	284.81	382.84	284.81	374.03	284.81	363.65
	0.5	290.91	376.13	290.91	365.65	290.91	356.98
	0.7	295.88	366.07	295.88	357.28	295.88	351.57
	0.9	300.20	352.68	300.20	348.92	300.20	346.89

TABLE VII
α-CUTS OF S&P 500 CALL OPTION PRICES WITH NORMAL ERRORS ($m = 0.50, 1.0, 2.0$)

m	α	$n = 0.50$ LB	$n = 0.50$ UB	$n = 1.0$ LB	$n = 1.0$ UB	$n = 2.0$ LB	$n = 2.0$ UB
0.50	0.1	263.97	386.41	264.85	384.49	259.82	382.74
	0.3	259.22	382.08	264.03	376.03	267.00	356.29
	0.5	273.05	366.17	275.97	373.35	273.59	357.50
	0.7	277.37	371.93	280.89	346.74	285.13	353.27
	0.9	291.45	361.96	291.74	340.78	292.23	340.52
1.0	0.1	264.43	383.36	271.09	380.67	264.31	383.90
	0.3	274.03	387.16	277.19	376.34	273.47	360.54
	0.5	280.47	383.17	286.32	363.35	281.10	352.28
	0.7	290.64	364.12	291.17	350.74	288.57	353.06
	0.9	298.36	345.21	293.26	349.42	301.38	345.94
2.0	0.1	279.46	389.78	275.94	381.87	268.67	372.23
	0.3	289.53	388.14	285.15	380.88	282.63	361.99
	0.5	291.32	382.05	287.53	362.97	296.39	349.92
	0.7	294.93	373.56	304.91	348.55	296.95	352.05
	0.9	306.74	356.31	305.77	357.85	301.00	350.77

BS model predict the option price more accurately compared to the other approaches.

TABLE VIII
α-CUTS OF S&P 500 CALL OPTION PRICES WITH t ERRORS ($m = 0.50, 1.0, 2.0$)

m	α	$n = 0.50$ LB	$n = 0.50$ UB	$n = 1.0$ LB	$n = 1.0$ UB	$n = 2.0$ LB	$n = 2.0$ UB
0.50	0.1	341.13	585.24	341.06	554.23	343.59	539.65
	0.3	356.40	564.46	358.23	535.20	349.63	532.61
	0.5	375.22	564.22	358.45	525.99	356.96	516.26
	0.7	396.28	537.37	384.24	551.02	387.73	524.50
	0.9	391.65	548.49	399.67	505.92	412.27	481.31
1.0	0.1	354.38	586.46	357.98	564.43	349.23	559.00
	0.3	368.57	583.47	356.64	562.73	373.67	532.61
	0.5	372.65	640.40	411.58	550.78	378.51	504.25
	0.7	392.58	531.69	410.97	546.33	387.05	511.39
	0.9	422.12	519.39	404.96	511.84	403.47	513.83
2.0	0.1	368.56	546.92	368.06	590.42	372.05	535.94
	0.3	391.93	545.28	405.12	550.30	365.93	519.69
	0.5	383.67	550.25	398.31	538.81	410.06	521.81
	0.7	398.26	542.22	411.17	505.51	413.52	475.66
	0.9	426.29	500.06	404.50	509.37	428.63	491.35

TABLE IX
α-CUTS OF S&P 500 CALL OPTION PRICES USING JD MODEL ($m = 0.50, 1.0, 2.0$)

m	α	$n = 0.50$ LB	$n = 0.50$ UB	$n = 1.0$ LB	$n = 1.0$ UB	$n = 2.0$ LB	$n = 2.0$ UB
0.50	0.1	319.20	506.29	317.72	504.65	305.96	480.29
	0.3	321.63	493.36	317.20	480.86	315.03	466.22
	0.5	325.39	481.32	338.52	466.58	332.66	447.25
	0.7	340.28	463.46	348.95	456.15	343.40	437.32
	0.9	365.03	436.76	358.96	447.92	357.26	435.89
1.0	0.1	322.40	499.59	323.55	493.84	321.40	481.52
	0.3	330.95	494.50	333.92	478.98	343.79	465.42
	0.5	347.74	490.51	348.66	476.50	343.47	463.82
	0.7	354.86	476.26	362.19	451.47	356.83	440.52
	0.9	359.17	456.08	364.41	438.95	367.80	448.17
2.0	0.1	333.96	501.27	336.33	512.63	331.47	482.66
	0.3	349.31	492.14	343.41	485.58	348.51	484.42
	0.5	362.03	485.40	356.68	473.29	358.65	461.40
	0.7	355.51	462.86	366.35	451.44	355.40	439.98
	0.9	377.30	441.80	368.82	453.38	366.88	440.07

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