B222-E141030 - Sensor systems

Lab 4 – Kalman filter

1. Introduction

This lab concerns the problem of applying a Kalman filter to a variety of measurements. The measurements are numerical data that has been recorded over time in order to Provide samples to filter. There are two measurement sets, one that was recorded in one Dimension and another recorded in two dimensions. In practice the Kalman filter would be Used actively to filter the measurements in real time.
The Kalman filter is used to more accurately determine the desired state variables. In practice there will be some level of noise that we want to remove. The Kalman filter allows us to weight the accuracy of our predictions and measurements to produce the best results.

2. Methods

The Kalman filter follows a cycle of prediction and update. In this cycle the next state and state covariance is predicted. Then measurements are obtained from the sensor that is being actively used. After the measurements are obtained the Kalman gain is calculated. The Kalman gain represents the weights that are placed on both the predictions and measurements. The state and state covariance is then updated. This process is then looped over the next time step. These steps can be modeled in the follow series of equations:

- Predict next state $$X_{t,t-1} = \Phi X_{t-1,t-1} \quad (1)$$
- Predict next state covariance $$S_{t,t-1} = \Phi S_{t-1,t-1} \Phi^T + Q \quad (2)$$
- Obtain measurement(s)Y_t.
- Calculate the Kalman gain (weights) $$K_t = S_{t,t-1} M^T [M S_{t,t-1} M^T + R]^{-1} \quad (3)$$
- Update state $$X_{t,t} = X_{t,t-1} + K_t (Y_t - M X_{t,t-1}) \quad (4)$$
- Update state covariance $$S_{t,t} = [I - K_t M] S_{t,t-1} \quad (5)$$
- Loop (t is incremented by Δt)

It is important to note that in this series of equations initial values need to be defined. These values include the state at time 0, the state covariance at time 0, the dynamic noise covariance and the measurement noise covariance. It is the job of the designer to determine how these values should be defined based on available information. The designer must also define the state transition matrix,$\emptyset$ and the observation matrix, M. The transition matrix is determined by observing how each of the state variables contribute to each state transition equation. Similarly, the observation matrix is determined by observing how each of the state variables contribute to the observation equations.

3. 1D Kalman Filter

For the one dimensional measurements that were being filtered for this lab a constant velocity model was used. The state variables were determined to be both position and velocity and were represented in the following matrix:

$$X_t = \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix} \tag{6}$$

In matrix 6, x_t represents position and $x_{\dot{t}}$ represents the velocity.

For this model at time 0 it was chosen that both the position and velocity would be set to 0. Since a constant velocity model was chosen the state transition equations could be modeled as follows.

$$x_{t+1} = x_t + T\dot{x}_t \tag{7}$$

$$\dot{x}_{t+1} = \dot{x}_t \tag{8}$$

These equations then allowed for the state transition matrix to be produced. This matrix was defined by matrix 9 below.

$$\Phi = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \tag{9}$$

For this example the value T was set to equal one. T was set equal to one because the time step for each iteration of new measurements was defined to be one second. It was then important to dene the state estimate covariance, St. The state estimate covariance represents the level of uncertainty that there is in the state variables shown in matrix 6.

Specifically, the diagonal of matrix 10 represents the uncertainty of the position and velocity estimates. The elements of the diagonal represent the covariances of the position and velocity estimates.

$$S_t = \begin{bmatrix} \sigma_x^2 & \sigma_{x,\dot{x}} \\ \sigma_{x,\dot{x}} & \sigma_{\dot{x}}^2 \end{bmatrix} \tag{10}$$

For this model at time 0 the state covariance matrix was defined to be the identity matrix. This placed an initial uncertainty on the state variables, but not their covariances. Following this step the observation equations and variables were redefined. For this model it was assumed that the position was being sensed. The observation variable was defined in the following matrix:

$$Y_t = [\tilde{x}_t] \tag{11}$$

The observation equation was also simply defined as the following:

$$\tilde{x}_t = x_t \tag{12}$$

These definitions allowed for the observation matrix, M to be defined by the 1x2 matrix below.

$$M = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{13}$$

The final two variables that needed to be initialized were the dynamic noise covariance and the measurement noise covariance, represented by Q and R respectively. These matrices would directly impact how the filtering would behave. Depending on the ratio between the two noises the filter would either more closely follow the measurements or the state transition equations. In this lab the noises were tuned to adjust the behavior of the filter and observe the results. The generic form of both the Q and R matrix used for this 1D model are as follows:

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_a^2 \end{bmatrix} \tag{14}$$

$$R = [\sigma_n^2] \tag{15}$$

In matrix 15 the singular value represents the variance of the measurement noise of position. There is no dynamic noise on the position portion of the state transition equations. Thus matrix 14 is only composed of the variance for the velocity portion of the dynamic noise.

Both the dynamic noise and measurement noise matrices were not explicitly defined in this section because they can be shown in an embedded form in the Q and R matrices.

4. 2D Kalman Filter

For the second set of sample measurements a 2D constant velocity model was used. The 2D constant velocity model is an extension of the 1D constant velocity model with additional state variables. The state variables were determined to be both position and velocity and were represented in the following matrix:

$$X_t = \begin{bmatrix} x_t \\ \dot{x}_t \\ y_t \\ \dot{y}_t \end{bmatrix} \tag{16}$$

In matrix 16, x_t and xy_t represent position and both x_ t and y_t represent the velocities. For this model at time 0 it was chosen that both the position values would be set to the initial measurement and velocity values would be set to 0. Since a constant velocity model was chosen the state transition equations could be modeled the same as previously shown with two additional equations.

$$\begin{aligned} x_{t+1} &= x_t + T\dot{x}_t \\ \dot{x}_{t+1} &= \dot{x}_t \\ y_{t+1} &= y_t + T\dot{y}_t \\ \dot{y}_{t+1} &= \dot{y}_t \end{aligned} \tag{17}$$

These equations then allowed for the state transition matrix to be produced. This matrix can defined by matrix 18 below.

$$\Phi = \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

For this example the value T was also set to equal one. The state covariance matrix was again defined for the 2D model. The matrix is now size 4x4 because there are four state variables and for state transition equations. The state covariance matrix still follows the same properties that were mentioned above.

$$S_t = \begin{bmatrix} \sigma^2_{x_t} & \sigma_{x_t,y_t} & \sigma_{x_t,\dot{x}_t} & \sigma_{x_t,\dot{y}_t} \\ \sigma_{x_t,y_t} & \sigma^2_{y_t} & \sigma_{y_t,\dot{x}_t} & \sigma_{y_t,\dot{y}_t} \\ \sigma_{x_t,\dot{x}_t} & \sigma_{y_t,\dot{x}_t} & \sigma^2_{\dot{x}_t} & \sigma_{\dot{x}_t,\dot{y}_t} \\ \sigma_{x_t,\dot{y}_t} & \sigma_{y_t,\dot{y}_t} & \sigma_{\dot{x}_t,\dot{y}_t} & \sigma^2_{\dot{y}_t} \end{bmatrix} \tag{19}$$

For this model at time 0 the state covariance matrix was again defined to be the identity matrix. This placed an initial uncertainty on the state variables, but not their covariances.

Following this step the observation equations and variables were defined. For this model is was assumed that the position was being sensed in both the x and y direction. The observation variables were defined in the following matrix:

$$Y_t = \begin{bmatrix} \tilde{x}_t \\ \tilde{y}_t \end{bmatrix} \tag{20}$$

The observation equations were also simply defined as the following:

$$\begin{aligned}\tilde{x}_t &= x_t \\ \tilde{y}_t &= y_t\end{aligned} \tag{21}$$

The equations are defined like this because the measurement data being received is position data directly. These definitions allowed for the observation matrix, M to be defined by the 2x4 matrix below.

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{22}$$

The dynamic noise and measurement covariances were again the final two matrices defined in the setup of the 2D Kalman filter. Both the noises were again tuned to adjust the behavior of the filter and determine what the best ratio was. The generic form of both the Q and R matrix used for this 2D model are as follows:

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{a_1}^2 & \sigma_{a_1,a_2} \\ 0 & 0 & \sigma_{a_1,a_2} & \sigma_{a_2}^2 \end{bmatrix} \tag{23}$$

$$R = \begin{bmatrix} \sigma_{n_1}^2 & \sigma_{n_1,n_2} \\ \sigma_{n_1,n_2} & \sigma_{n_2}^2 \end{bmatrix} \tag{24}$$

Table 1: Summary of three noise ratios observed for 1D measurements.

Ratio Case	Q (σ_a^2)	R (σ_n^2)
1	10	1
2	10^{-12}	10^{7}
3	10^{-7}	1

In matrix 24 the values on the diagonal represent the variances of the measurement noise of the positions, whereas off the diagonal represents their covariance. There is again no dynamic noise on the position portion of the state transition equations. Thus matrix 23 is only composed of the variances and covariances for the velocity portion of the dynamic noise. Both the dynamic noise and measurement noise matrices were not explicitly defined in this section because they can be shown in an embedded form in both the Q and R matrices respectively.

5. Tasks

In this lab, each student is to develop code to operate a Kalman filter. The code can be developed in Matlab, python, or any high level language. No graphics display is required, but plots of results are required. The lab should be done in 2 steps.

First, the data "1D-data.txt" should be run through a constant velocity 1D model. Show the filter result for **<u>three</u>** different ratios of dynamic noise to measurement noise. Discuss the differences between the outputs.

Ratio Case	$Q\ (\sigma_a^2)$	$R\ (\sigma_n^2)$
1	10	1
2	10^{-12}	10^{7}
3	10^{-7}	1

Noise ratios for 1D measurements.

Kalman variables code as the following:

```
phi = [1 1;0 1]; %transition matrix
X = [0;0]; %Initial state definition (t-1,t-1)
S = [1 0;0 1]; %Initial state estimate covariance (t-1,t-1)
Q = [0 0; 0 0.0000001]; % Dynamic noise (prediction accuracy)
R = 1; % Measurement Noise (larger value means weight measurements less)
M = [1 0]; % observation matrix
I = [1 0;0 1]; % 2x2 identity matrix
```

Second, the filter should then be applied on UWB tracking data" 2D-UWB-data.txt".I want to see a graph plotting the measurements and estimated state. I also want to read a brief write-up describing the problem, the data source, and the tuning of the filter to get your results.

<u>(Here the results are only needed for one ratio of dynamic noise to measurement noise).</u>

Kalman variables code as the following:

```
% Define kalman variables
phi = [1 0 1 0;0 1 0 1;0 0 1 0;0 0 0 1]; %transition matrix
X = [measurments(1,1);measurments(1,2);0;0]; %Initial state definition (t-1,t-1)
S = [1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1]; %Initial state estimate covariance (t-1,t-
1)
Q = [0 0 0 0;0 0 0 0;0 0 0.01 0.0001;0 0 0.0001 0.01]; % Dynamic noise
(prediction accuracy)
R = [10 0.0001;0.0001 10]; % Measurement Noise (larger value means weight
measurements less)
M = [1 0 0 0;0 1 0 0]; % observation matrix
I = [1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1]; % 2x2 identity matrix
```