A Brief Introduction to Characteristic Boundary Conditions

Overview

Many interesting physical phenomena in nature can be described by the propagation of waves. Some of the most prominent examples include the emanation of sound from an instrument, the vibrations of a string, the breaking of waves on the shore, and even the formation of shocks in supersonic flight. Each of these processes can be expressed mathematically, through the form of a conservation law that describes how individual waves act to transport and preserve quantities like mass, momentum, or energy.

In the absence of dissipative effects such as friction, conservation laws for wave propagation problems are *hyperbolic* in nature, meaning all information within the system is transmitted in a purely convective manner. Intricate systems such as the equations of gas dynamics, may have multiple types of waves present, with each one affecting a very specific combination of variables in the system. For example, when an acoustic wave passes through air it induces minute changes in pressure, while leaving the local temperature completely unaltered.

Using the theory of characteristics, which underpins the broader study of hyperbolic equations, it is possible to derive a form of a given conservation law that reveals the behavior associated with each kind of wave present in a system. Furthermore, it can be shown that we can actually manipulate these waves, letting us enforce some desired behavior within the physical problem of interest. This has important consequences when considering cases on finite domains, and leads to elegant and robust boundary condition treatment for problems solved *numerically* on discrete meshes.

Characteristic Form of Hyperbolic Equations

We start with a generic hyperbolic conservation law of the form

$$\mathbf{u}_t + f(\mathbf{u})_x = 0$$

where $\mathbf{u}$ is a vector and $f(\mathbf{u})$ is a flux function. We can rewrite this equation in quasi-linear form, i.e,

$$\mathbf{u}_t + A\mathbf{u}_x = 0$$
$$A = \frac{\partial f(\mathbf{u})}{\partial \mathbf{u}}$$

Furthermore, since the system is hyperbolic it is diagonlizable with strictly real eigenvalues. This leads to the two related eigenvalue problems of

$$AR = R\Lambda \qquad A^T L = L\Lambda$$

where Λ, R, and L contain the eigenvalues, right eigenvectors, and left eigenvectors of A respectively. The left eigenvectors are attained by considering the transpose of A, given by A^T. Physically, each eigenvalue corresponds to the speed at which a specific type of wave is propagated by the system, such as waves traveling at $\pm c$ in the second order wave equation. Right eigenvectors describe what combination of the variables in $\mathbf{u}$ are propagated by each wave. Left eigenvectors describe how much of each variable is included in this combination.

We can rewrite the equation further as

$$\mathbf{u}_t + R\Lambda R^{-1}\mathbf{u}_x = 0 \qquad \text{or} \qquad R^{-1}\mathbf{u}_t + \Lambda R^{-1}\mathbf{u}_x = 0$$

which effectively decouples the original system into a set of individual wave equations. Letting $\mathbf{w} = R^{-1}\mathbf{u}$ be a vector of 'characteristic variables' we can see that the above system is truly just a set of first order wave equations of the form

$$w_{it} + \lambda_i w_{ix} = 0$$

with w_i the i^{th} characteristic variable and λ_i the wave speed. For general non-linear problems we retain the form

$$R^{-1}\mathbf{u}_t + \Lambda R^{-1}\mathbf{u}_x = 0$$

for which individual component equations of this system are given by

$$l_i^T\mathbf{u}_t + \lambda_i l_i^T\mathbf{u}_x = 0$$

with l_i^T being a left eigenvector. We now define a new variable called the 'characteristic wave amplitude variation' as

$$\mathscr{L}_i = \lambda_i l_i^T\mathbf{u}_x$$

or in matrix form

$$\mathscr{L} = \Lambda R^{-1}\mathbf{u}_x$$

with the name stemming from the fact that

$$l_i^T\mathbf{u}_t = -\mathscr{L}_i$$

thus the time variation of a given characteristic wave is governed by some $\mathscr{L}$. Using these definitions, we can rewrite the original governing equation as follows,

$$\mathbf{u}_t + f(\mathbf{u})_x = \mathbf{u}_t + R\mathscr{L} = 0$$

Now we make a distinct observation. If one is able to explicitly manipulate the values of $\mathscr{L}$, in theory specific behaviors could be introduced into the original hyperbolic equation system. The most logical place for this 'manipulation' to occur is at boundary points. Consequently, it is becoming evident that we can actually prescribe boundary condition behavior by modifying the characteristic wave amplitude variations.

Implications for Boundary Conditions

The variation of characteristic waves at domain boundaries can be used to gain a physical understanding of boundary condition behavior. For this to occur, we need to consider a problem bounded over some finite domain. An easy example is a one-dimensional hyperbolic problem over the interval $x \in \mathcal{D}$, with $\mathcal{D} = [0,\ L]$. To further simplify our investigations into boundary behavior, we focus exclusively on the boundary at $x = L$. As previously mentioned, a hyperbolic system of n variables has n characteristic waves, each wave propagating at a speed given by an eigenvalue λ. Based on the speed of each wave, different situations arise at the boundary $x = L$.

We first consider the case sketched in Fig. 1 where all eigenvalues are positive, i.e, $\lambda_i > 0 \;\; i = 1, 2, 3, ...\ n$. Physically this corresponds to all waves traveling with a positive speed, meaning they each propagate in the $+x$ direction. At the domain boundary, this means all information *comes from within the domain itself*, so the behavior of the boundary is determined *exclusively* by right running characteristic waves . A physical example of this would be supersonic flow exiting a rocket nozzle. In this situation the local flow speed u is greater than the local speed of sound c, meaning all characteristic wave speeds of the governing Euler equations are positive ($0 < u - c < u < u + c$). In this case no information can travel into the nozzle from downstream, and the state of the nozzle exit is determined entirely by upstream conditions.

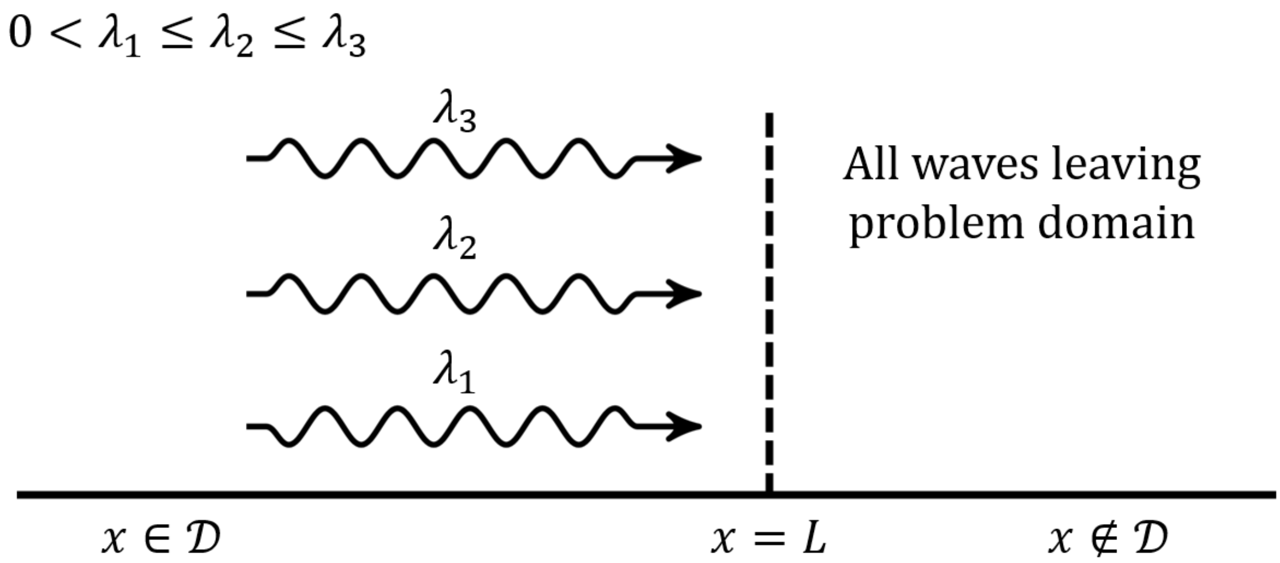

Figure 1: Situation where all waves at boundary have a positive speed, with $\lambda_i > 0 \;\forall\; i$

The second case to consider is sketched in Fig. 2 where at $x = L$ there is a single wave with a negative speed. Physically this corresponds to a waves traveling with a negative speed and propagating in the $-x$ direction. At the domain boundary, this means some information *comes from outside the domain*, so the behavior of the boundary is determined by right running and left running characteristic waves. A natural occurrence of this would be subsonic flow ($u < c$) exiting a nozzle, where the characteristic wave speeds are both positive ($0 < u < u + c$) and negative ($u - c < 0$). In this situation information must travel upstream into the nozzle, and the state of the nozzle exit is now determined by both upstream *and* downstream conditions.

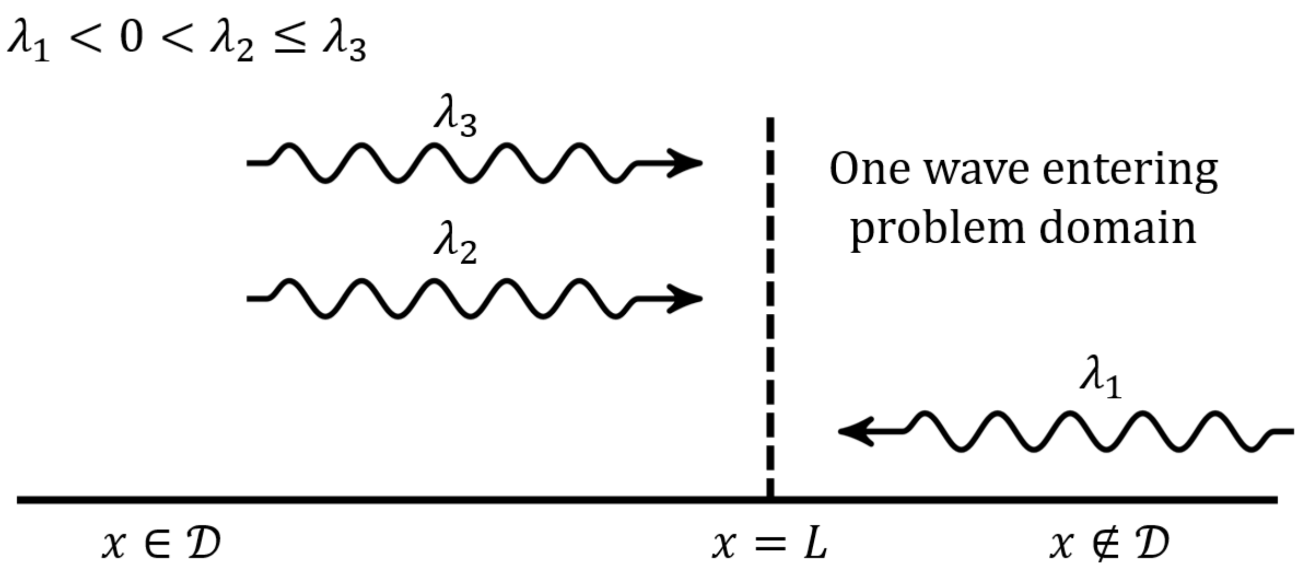

Figure 2: Situation where a wave at the boundary has a negative speed, with a single $\lambda_i < 0$

In this second case however, we have run into an issue. We know theoretically that our boundary behavior is determined by downstream conditions, however our domain is truncated at $x = L$. Consequently, we do not explicitly know how the incoming wave behaves and our solution lacks vital information. In such cases the problem is said to be ill-posed. Luckily, a workaround exists if one is able to manually impose conditions on the incoming wave!

We have already seen that the time variation of the characteristic waves in a given problem is governed by vector $\mathscr{L}$. At boundaries where $\lambda_i < 0$, if we were to explicitly set an associated value of $\mathscr{L}_i$, we could directly influence the physics occurring at the boundary $x = L$.

For example, say we are studying acoustic waves propagating in a one-dimensional subsonic flow. At $x = L$, we do not necessarily know what the incoming acoustic wave at speed $\lambda_1 = u - c$ looks like, so we need to make some assumptions. One option is to set $\mathscr{L}_1 = 0$, giving

$$l_1^T \mathbf{u}_t + \mathscr{L}_1 = l_1^T \mathbf{u}_t + 0 = 0$$
$$\therefore \quad l_1^T \mathbf{u}_t = 0$$

Mathematically, this renders the value of $l_1^T\mathbf{u}$ constant in time, with the value of the wave being fixed by the initial conditions. Physically, this condition dictates that the incoming acoustic wave has no impact on the solution, and it is as if the wave never enters the domain at all. Such a condition is called a 'non-reflecting boundary condition' and aims to entirely remove the effects of incoming waves from the solution. Another common practice is to set the value of the incoming acoustic wave variation to the opposite of the outgoing acoustic wave, *i.e*, $\mathscr{L}_1 = -\mathscr{L}_3$. By fixing the wave behavior in this manner, whatever variations induced by right running acoustic waves are exactly canceled by an incoming, left running acoustic wave. This has the effect of totally reflecting pressure disturbances at the boundary and creates a constant pressure outlet.

Other mathematical conditions can be created by manipulating characteristic wave amplitudes as well, and can be done so for any hyperbolic system of equations. These include Neumann, Dirichlet, and non-reflecting boundaries, with the physical interpretation of each condition being unique to the conservation law being considered.

Application for Numerical Simulations

When attempting to find solutions to conservation laws, we often have to resort to simulation techniques and attain numerical approximations to the true solution. In these computational settings, solutions are typically sought after on a truncated, finite domain that is representative of the actual problem at hand. For example, when trying to study supersonic flow over a wing, it is simpler and more computationally efficient to consider a discretized domain over a 2D airfoil cross section than to simulate the entire wing and the aircraft it is attached to. One consequence of this however, is at the edges of our truncated domain we need to enforce appropriate boundary conditions. This can be readily accomplished using characteristic type boundary conditions, which help make our simulations well-posed.

However, these conditions must be properly implemented in a given code. A brief overview of how to implement characteristic boundary treatment in a numerical setting is given below. For ease, we assume a finite difference scheme is used on a discrete mesh, though the process can be modified for other classes of numerical procedure.

- At interior mesh points, compute spatial flux derivatives as usual using $f(\mathbf{u})_x$
- At boundary points compute the wave amplitude variations $\mathscr{L} = R^{-1}\Lambda\mathbf{u}_x$
  - Ensure a proper 1-sided derivative scheme is used for $\mathbf{u}_x$
- For waves leaving the domain at boundaries, leave $\mathscr{L}_i$ unaltered
- For waves entering the domain at boundaries, modify $\mathscr{L}_i$ as needed
- At boundary points, supplant the term $f(\mathbf{u})_x$ with $R\mathscr{L}$
  - $\mathscr{L}$ has been modified to enforce desired boundary behavior
- Integrate the discretized conservation law in time at all interior *and* boundary points
- Repeat for each time step

A Wave Equation Example

This overview concludes with a brief example of how to implement characteristic boundary conditions on the second order wave equation. Consider the following equation on a compact domain:

$$
\begin{gathered}
u_{tt} + c^2 u_{xx} = 0 \\
u(x,0) = u_0(x) \\
u_t(x,0) = v_0(x) \\
x \in [0, \quad L]
\end{gathered}
$$

We can invoke a change of variables

$$
w = u_x \qquad v = u_t
$$

and rewrite 2nd order wave equation as system of first order equations

$$
\begin{bmatrix} w \\ v \end{bmatrix}_t + \begin{bmatrix} -v \\ -c^2 w \end{bmatrix}_x = 0
$$

where we have used

$$
w_t = u_{xt} = u_{tx} = v_x
$$

Eigenvectors

To apply characteristic conditions, left and right eigenvectors of the flux Jacobian $A = \frac{\partial f}{\partial u}$ are needed. Taking

$$
A = \begin{bmatrix} \frac{\partial f_1}{\partial w} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial w} & \frac{\partial f_2}{\partial v} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -c^2 & 0 \end{bmatrix}
$$

The eigenstructure of A can be identified in a straightforward manner, giving

$$
\Lambda = \begin{bmatrix} c & 0 \\ 0 & -c \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 1 \\ -c & c \end{bmatrix} \qquad L = \begin{bmatrix} -c & c \\ 1 & 1 \end{bmatrix}
$$

with the eigenvalues dictating wave speeds of $\pm c$ and the left and right eigenvectors satisfying the needed orthogonality property. Thus the characteristic wave amplitude variations,$\mathscr{L}$, are given by

$$
\lambda_1 = c, \qquad \mathscr{L}_1 = \lambda_1 l_1^T \begin{bmatrix} w \\ v \end{bmatrix}_x = c[-c \ \ 1] \begin{bmatrix} w \\ v \end{bmatrix}_x = -c^2 w_x + c v_x
$$

$$
\lambda_2 = -c, \qquad \mathscr{L}_2 = \lambda_2 l_2^T \begin{bmatrix} w \\ v \end{bmatrix}_x = c[c \ \ 1] \begin{bmatrix} w \\ v \end{bmatrix}_x = c^2 w_x + c v_x
$$

Boundary Conditions

We can rewrite the system of first order PDEs according to

$$
\mathbf{u}_t + f(\mathbf{u})_x = \mathbf{u}_t + R\mathscr{L}
$$

which for the second order wave equation gives

$$
\begin{bmatrix} w \\ v \end{bmatrix}_t + \begin{bmatrix} -v \\ -c^2 w \end{bmatrix}_x = \begin{bmatrix} w \\ v \end{bmatrix}_t + \begin{bmatrix} 1 & 1 \\ -c & c \end{bmatrix} \begin{bmatrix} \mathscr{L}_1 \\ \mathscr{L}_2 \end{bmatrix} = 0
$$

Thus we can come up with the following

$$
\begin{gathered}
w_t + (\mathscr{L}_1 + \mathscr{L}_2) = 0 \\
v_t + (-\mathscr{L}_1 + \mathscr{L}_2) = 0
\end{gathered}
$$

We can then use these expressions, along with the definitions of $v = u_t$ and $w = u_x$, to enforce various types of boundary conditions. In general, using a characteristic approach to boundary conditions makes boundary behavior depend on the initial conditions $w_0 = u(x, 0)$ and $v_0 = u_t(x, 0)$, which is to be expected from the purely advective nature of the problem. Assuming for ease $u(x, 0) = u_t(x, 0) = 0$, and that we excite the wave with some prescribed forcing function $f(x, t)$, we can derive the following conditions:

Inlet	(x=0)	$\lambda_1 = c$ into domain
$\mathscr{L}_1 = \mathscr{L}_2$	Dirichlet	
$\mathscr{L}_1 = -\mathscr{L}_2$	Neumann	
$\mathscr{L}_1 = 0$	Non-reflecting	
Outlet	(x=L)	$\lambda_2 = -c$ into domain
$\mathscr{L}_2 = \mathscr{L}_1$	Dirichlet	
$\mathscr{L}_2 = -\mathscr{L}_1$	Neumann	
$\mathscr{L}_2 = 0$	Non-reflecting	

Once the solution for the system $[w \;\; v]^T$ has been solved, either numerically or analytically, the original solution $u(x, t)$ is obtained from

$$u(x, t) = u_0(x) + \int_{t_0}^{t} v(x, \tau) d\tau$$