CSCI 301 M4 Homework

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Collaboration statement: By submitting this assignment, I am attesting that this homework is in full compliance with the course's https://wwu.instructure.com/courses/1340003/pages/academic-dishonesty-guidelines Homework Collaboration Policy and with all the other relevant academic honesty policies of the course and university. I discussed this homework with no one and wrote this solution without input from anyone else.

1. **Proposition** Suppose $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.
Proof. For the sake of contradiction, suppose n^2 is odd and n is not odd.
Then n^2 is odd, and n is even.
Since n is even, there is an integer c for which $n = 2c$.
Then $n^2 = (2c)^2 = 4c^2 = 2(2c^2) = 2(d)$ where d represents $2c^2$.
So n^2 is even, by definition of an even integer.
Thus, n^2 is odd and n^2 is not even, a contradiction.

2. **Proposition** $\sqrt[3]{2}$ is irrational.
Proof. Suppose for the sake of contraction that is is not true that $\sqrt[3]{2}$ is irrational.
Then $\sqrt[3]{2}$ is rational, so there are integers a and b for which $\sqrt[3]{2} = \frac{a}{b}$. Fully reduced this means that a and b are not even (Chapter 6, Book of Proofs, page 139, example 6.1).
Then let's cube both sides of the equation.
So $2 = \frac{a^3}{b^3}$ and **therefore** $2a^3 = b^3$.
We can then take b to be an even integer since any integer multiplied by 2 is even.
If we let $b = 2n$ where n is an even integer.
Thus $2a^3 = 2m^3$. $2a^3 = 8m^3$. $a^3 = 4m^3$.
Since a is a multiple of 4, a is also even.
We can further refine the thought that a is even if it is a multiple of 4 via **Direct Proof**.
Suppose That any integer x or y of either parity when multiplied by 4 is even.
Case 1
Thus Let x represent $2k$ by definition of an even integer.
Then $2k * 4 = 8k = 2(4k)$.
Let $4k = m$, where m represents $4k$.
Therefore we are left with $2m$ which is our definition of an even integer.
Case 2
Thus Let y represent $2l + 1$ by definition of an odd integer.
Then $(2l + 1) * 4 = 8l + 4 = 2(4l + 2)$.
Let $2(4l + 2) = 2n$, where n represents $4k + 2$.
Therefore we are left with $2n$ which is our definition of an even integer.
Therefore by both cases 1 and 2 we have shown that integers of either parity multiplied by 4 result in an even number, by definition of an even number.
Which, supports the proof that a is even as it is a multiple of 4.
Therefore we have both a and b being even resulting in $\sqrt[3]{2}$ being rational and $\sqrt[3]{2}$ being irrational in our initial proposition, a contradiction.

3. **Proposition** If n is a positive integer, then $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
Proof. We will prove this with mathematical induction.
Inductive Hypothesis $n = k$: $1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6}$.
Observe the first positive integer $k = 1$, this statement is $\frac{1(1+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = \frac{6}{6} = 1$, thus the equation holds true for our base case.
Say $k = k+1$. We use direct proof to show that $1^2+2^2+3^2+\ldots+k^2+(k+1)^2 = \frac{k+1(k+2)(2k+3)}{6}$. Suppose $1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$. Then

$$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2$$

.

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$
$$= \frac{k(k+1)(2k+1)6(k+1)^2}{6}$$
$$= (k+1)\frac{2k^2+k+6k+6}{6}$$
$$= (k+1)\frac{2k^2+7k+6}{6}$$
$$= \frac{k+1(k+2)(2k+3)}{6}$$

Thus $1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{k+1(k+2)(2k+3)}{6}$. This proves $k \to k+1$. It follows by induction that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for every $n \in \mathbb{Z}$.

4. **Proposition** If $n \in \mathbb{Z}$, then $(1)(2) + (2)(3) + (3)(4) + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$.
Proof. We will prove this with mathematical induction.
Inductive Hypothesis $n = k$: $(1)(2) + (2)(3) + (3)(4) + \ldots + k(k+1) = \frac{k(k+1)(k+2)}{3}$
Observe the first positive integer $k = 1$, this statement is $\frac{1(1+1)(1+2)}{3} = \frac{1(2)(3)}{3} = \frac{6}{3} = 2$, thus the equation holds true for our base case $n = 1$ as $1 * 2 = 2$.
Say $k = k+1$. We use direct proof to show that $n = k$: $(1)(2)+(2)(3)+(3)(4)+\ldots+k(k+1)+(k+1)(k+2) = \frac{k+1(k+2)(k+3)}{3}$. Suppose $(1)(2) + (2)(3) + (3)(4) + \ldots + k(k+1) = \frac{k(k+1)(k+2)}{3}$. Then

$$(1)(2) + (2)(3) + (3)(4) + \ldots + k(k+1) + (k+1)(k+2)$$
$$= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$$
$$= \frac{k(k+1)(k+2)+3(k+1)(k+2)}{3}$$
$$= \frac{(k^2+n)(k+2)+(3k+3)(k+2)}{3}$$
$$= \frac{(k^3+2k^2+k^2+2k)+(3k^2+3k+6k+6)}{3}$$
$$= \frac{k^3+6k^2+11k+6}{3}$$
$$= \frac{(k+1)(k+2)(k+3)}{3}$$

Thus $(1)(2) + (2)(3) + (3)(4) + \ldots + k(k+1) + (k+1)(k+2) = \frac{k+1(k+2)(k+3)}{3}$. This proves $k \to k+1$. It follows by induction that $(1)(2) + (2)(3) + (3)(4) + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ for every $n \in \mathbb{Z}$.

5. **Proposition** If $n \in \mathbb{Z}$, then $3^1 + 3^2 + 3^3 + \ldots + 3^n = \frac{3^{n+1}-3}{2}$.

Proof. We will prove this with mathematical induction.

Inductive Hypothesis $n = k$: $3^1 + 3^2 + 3^3 + \ldots + 3^k = \frac{3^{k+1}-3}{2}$.

Observe the first positive integer $k = 1$, this statement is $\frac{3^{1+1}-3}{2} = \frac{3^2-3}{2} = \frac{6}{2} = 3$, thus the equation holds true for our base case $n = 1$ as $3^1 = 3$.

Say $k = k+1$, we will use direct proof to show that $P(k) \to P(k+1)$: $3^1+3^2+3^3+\ldots+3^k+3^{k+1} = \frac{3^{k+1+1}-3}{2}$.

Suppose $3^1 + 3^2 + 3^3 + \ldots + 3^k = \frac{3^{k+1}-3}{2}$. Then

$$3^1 + 3^2 + 3^3 + \ldots + 3^k + 3^{k+1}$$
$$= \frac{3^{k+1}-3}{2} + 3^{k+1} - 3$$
$$= \frac{3^{k+1}-3+2*3^{k+1}}{2}$$
$$= \frac{3^{k+1}(3+2)-3}{2}$$
$$= \frac{3^{k+1+1}-3}{2}$$

Thus $3^1 + 3^2 + 3^3 + \ldots + 3^k + 3^{k+1} = \frac{3^{k+1+1}-3}{2}$. This proves $k \to k+1$. It follows by induction that $3^1 + 3^2 + 3^3 + \ldots + 3^n = \frac{3^{n+1}-3}{2}$ for every $n \in \mathbb{Z}$.