HW 1

Nicole Sullivan

```
knitr::opts_chunk$set(echo = T, message = F, warning = F)
```

```
# DO THESE FIRST BEFORE LOADING RETICULATE PKG
# create a new environment
#reticulate::conda_create("hw1")
# install numpy
#reticulate::conda_install("hw1", "numpy")
# reticulate::conda_install("hw1", "scipy")

reticulate::use_condaenv("hw1")
```

```
library(tidyverse)
library(reticulate) # to use Python

# Plotting aesthetics
loaded_font <- 'Didact Gothic'
text_color <- 'white'

# Hex codes
hex_purple <- "#5E72E4" #primary
hex_blue_lt <- "#5DCEF0" #info
hex_green <- "#63CF89" #success
hex_pink <- "#EA445B" #danger
hex_orange <- "#EC603E" #warning
hex_blue_dk <- "#172B4D" #default
hex_grey <- "#51535e"
hex_blue_deep <- "#0f151c"

proj_theme <- theme(plot.background = element_rect(fill = "#1e2936", color = "transparent"),
        plot.margin = margin(t = "1.5", r = "1.5", b = "1.5", l = "1.5", unit = "cm"),
        panel.background = element_rect(fill = "#1e2936"),
        panel.grid.major = element_blank(),
        panel.grid.minor = element_blank(),
        plot.title = element_text(family = loaded_font, color = text_color, hjust = 0.5, face = "bold", s
ize = 25),
        plot.caption = element_text(family = loaded_font, color = text_color, size = 9),
        axis.title = element_text(family = loaded_font, size = 15, color = text_color),
        axis.text = element_text(family = loaded_font, color = text_color, size = 10),
        strip.background = element_rect(fill = "#0f151c"),
        strip.text = element_text(color = "#a1aab5", family = loaded_font, face = "bold", size = 18),
        legend.background = element_rect(fill = "transparent"),
        legend.title = element_text(family = loaded_font, color = text_color),
        legend.text = element_text(family = loaded_font, color = text_color),
        legend.position = "bottom",
        legend.key = element_rect(fill = NA))
```

1

> Find the Maximum Likelihood Estimation (MLE) of θ in the following probabilistic density functions. In each case, consider a random sample of size n. Show your calculation.

1a

> $f(x|\theta) = xe^{-x2}, x \geq 0$

$$L(\theta|x) = \prod p(x_i|\theta)$$

Take the log:

$$= \sum logp(x_i|\theta)$$

Plug in the likelihood:

$$= \sum log\left(\frac{x}{\theta^2}e^{\frac{-x^2}{2\theta^2}}\right)$$

Distribute the logs:

$$= \sum(\log\frac{x}{\theta^2} - \frac{x^2}{2\theta^2})$$

Distribute the summation:

$$= \sum\log\frac{x}{\theta^2} - \frac{\sum x^2}{2\theta^2}$$

Use log rules to rewrite:

$$= \sum logx - \sum log\theta^2 - \frac{\sum x^2}{2\theta^2}$$

Now, we take the derivative w.r.t. θ:

$$\frac{\partial}{\partial\theta} = 0 - \frac{2N}{\theta} - -\frac{\sum x^2}{\theta^3}$$

$$= -\frac{2N}{\theta} + \frac{\sum x^2}{\theta^3}$$

Set equal to zero to maximize:

$$0 = -\frac{2N}{\theta} + \frac{\sum x^2}{\theta^3}$$

$$\frac{2N}{\theta} = \frac{\sum x^2}{\theta^3}$$

$$2N\theta^2 = \sum x^2$$

$$\hat{\theta} = \sqrt{\frac{\sum x^2}{2N}}$$

1b

> $f(x|\theta,\alpha,\beta) = \alpha\theta^{-\alpha\beta}x^\beta e^{-(\frac{x}{\theta})^\beta}$

Plug in likelihood and take logs:

$$\sum log(\alpha\theta^{-\alpha\beta}) + \sum logx^\beta - \sum\left(\frac{x}{\theta}^\beta\right)$$

$$\sum log(\alpha\theta^{-\alpha\beta}) + \sum logx^\beta - \frac{\sum x^\beta}{\theta^\beta}$$

Next, we take the derivative w.r.t. θ (α and β are hyperparameters):

$$\frac{\partial}{\partial\theta} = \frac{N\alpha}{\theta^{-\alpha\beta-1}} \cdot \alpha\beta + 0 + \beta\sum x^\beta\theta^{-\beta-1}$$

$$= \frac{-N\alpha^2\beta}{\theta^{-\alpha\beta-1}} + \beta\sum x^\beta\theta^{-\beta-1}$$

Now, set equal to zero.

$$0 = \frac{-N\alpha^2\beta}{\theta^{-\alpha\beta-1}} + \beta\sum x^\beta\theta^{-\beta-1}$$

$$\frac{N\alpha^2\beta}{\theta^{-\alpha\beta-1}} = \beta\sum x^\beta\theta^{-\beta-1}$$

$$\frac{N\alpha^2\beta}{\theta^{-\alpha\beta-1}} = \frac{\beta\sum x^\beta}{\theta^{\beta+1}}$$

$$\frac{N\alpha^2\beta\theta^{\beta+1}}{\theta^{-\alpha\beta-1}} = \beta\sum x^\beta$$

$$\theta^{(\beta+1)-(-\alpha\beta-1)} = \frac{\sum x^\beta}{N\alpha^2}$$

$$\theta^{\beta-\alpha\beta+2} = \frac{\sum x^\beta}{N\alpha^2}$$

$$\hat{\theta} = \left(\frac{x^\beta}{N\alpha^2}\right)^{\frac{1}{\beta-\alpha\beta+2}}$$

1c

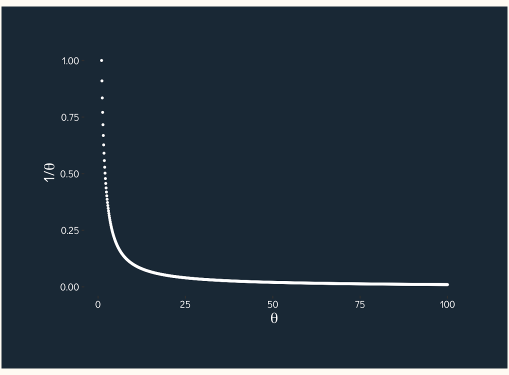
> $f(x|\theta) = \frac{1}{\theta}, 0 \leq x \leq \theta, \theta > 0$

This function has no minimum or maximum, as it is monotonically decreasing.

To illustrate, we can also draw (or plot) the function:

```
import numpy as np

theta = np.linspace(1, 100, 1000)

ll_1c_results = []

for i in theta:

    ll_1c_results.append(1/i)
```

```
ll_1c <- as.matrix(py$ll_1c_results)

ggplot() +
  aes(seq(1, 100, length.out = 1000), as.numeric(ll_1c[,1])) +
  geom_point(color = "white", size = 0.7) +
  labs(x = expression(theta),
       y = expression(paste("1/", theta))) +
  proj_theme
```



2

> We want to build a pattern classifier with continuous attribute using Bayes' Theorem. The object to be classified has one feature, x in the range 0 ≤ x < 6.

2a

> Assuming equal priors, P (C1) = P (C2) = 0.5, classify an object with the attribute value x = 2.5.

To classify, I'll determine the value of the discriminant for each class for this value of x. Then I'll classify the observation as the class with the highest discriminant.

Class 1:

$$P(x|C_1) \times P(C_1) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12} = 0.08\bar{3}$$

Class 2:

$$P(x|C_2) \times P(C_2) = \frac{1}{4}(2.5 - 1) \times 0.5$$

$$= \frac{1}{4}(1.5) \times \frac{1}{2} = \frac{1}{4} \times \frac{3}{2} \times \frac{1}{2} = \frac{3}{16} = 0.1875$$

$p(C_2|x) > p(C_1|x)$ ∴ I'll classify as C_2.

2b

> Assuming unequal priors, P(C1) = 0.7,P(C2) = 0.3, classify an object with the attribute value x = 4.

Class 1:

$$p(x|C_1) \times p(C_1) = \frac{1}{6} \times 0.7 = 0.11\bar{6}$$

Class 2:

$$p(x|C_2) \times p(C_2) = \frac{1}{4}(5 - 4) \times 0.3$$

$$= \frac{1}{4} \times 0.3 = 0.075$$

$p(C_1|x) > p(C_2|x)$ ∴ I'll classify as C_1.

2c

> Consider a decision function $\phi(x)$ of the form $\phi(x)$ = (|x – 3|) – α with one free parameter α in the range 0 ≤ α ≤ 2. You classify a given input x as class 2 if and only if ϕ(x) < 0, or equivalently 3 - α < x < 3 + α, otherwise you choose x as class 1. Assume equal priors, P(C1) = P(C2) = 0.5, what is the optimal decision boundary - that is, what is the value of α which minimizes the probability of misclassification? What is the resulting probability of misclassification with this optimal value for α? (Hint: take advantage of the symmetry around x = 3.)

We basically treat this as a minimization problem: the optimal α will minimize **total misclassification**. To get total misclassification, we need to sum:

- $P(C_2|C_1)$: the probability of classifying as C_2 when the observation is actually C_1, and
- $P(C_1|C_2)$: the probability of classifying as C_1 when the observation is really C_2.

Or, more succinctly put:

$$\text{Total misclassification rate} = P(C_2|C_1) + P(C_1|C_2)$$

where:

- $P(C_2|C_1) = \int_{C_2}^{\infty} f_1(x)dx$
- $P(C_1|C_2) = \int_{C_1}^{\infty} f_2(x)dx$

or, in plain English, the misclassification rate for, say, C_1 is the total area under the overlap of the C_2 and C_1 *in the* C_2 *classification range*.

That means that:

- $P(C_2|C_1) = \int_{3-\alpha}^{3+\alpha} \frac{1}{6}dx$
- $P(C_1|C_2) = \int_{0}^{3-\alpha} \frac{1}{4}(x-1)dx + \int_{3+\alpha}^{6} \frac{1}{4}(5-x)dx$

and:

$$TMR = \int_{3-\alpha}^{3+\alpha} \frac{1}{6}dx + \int_{0}^{3-\alpha} \frac{1}{4}(x-1)dx + \int_{3+\alpha}^{6} \frac{1}{4}(5-x)dx$$

Integrating w.r.t x we get:

$$TMR = \left[\frac{1}{6}x\right]_{3-\alpha}^{3+\alpha} + \left[\frac{1}{4}x^2 + \frac{1}{4}x\right]_{0}^{3-\alpha} + \left[\frac{1}{4}x^2 - \frac{5}{4}x\right]_{3+\alpha}^{6}$$

and then evaluating the integral gives:

$$TMR = \frac{1}{8}\alpha^2 + \frac{3}{8} + \frac{1}{24}(-4\alpha + 3(\alpha-3)(\alpha-1))$$

To minimize misclassification, we take the derivative w.r.t. α:

$$\frac{d}{d\alpha}TMR = \frac{\alpha}{2} - \frac{2}{3}$$

and then set equal to zero:

$$0 = \frac{\alpha}{2} - \frac{2}{3}$$

and solve for α:

$$\alpha = \frac{4}{3}$$

Therefore, we know that the value of α that minimizes misclassification is $\frac{4}{3}$.

To determine the rate of misclassification that will result from this decision boundary, we just need to plug $\frac{4}{3}$ back into our original equation for TMR:

$$\frac{1}{8}\left(\frac{4}{3}\right)^2 + \frac{3}{8} + \frac{1}{24}(-4 \cdot \frac{4}{3} + 3(\frac{4}{3} - 3)(\frac{4}{3} - 1))$$

which can easily be evaluated with code:

```
alpha <- 4/3

(1/8) * alpha^2 + (3/8) + ( (1/24) * (-4 * alpha + 3 *(alpha - 3) * (alpha -1)))
```

```
## [1] 0.3055556
```

or, there's about a **31% total misclassification rate** when using $\alpha = \frac{4}{3}$.

Just for fun, we can also confirm these results by integrating/minimizing programmatically.

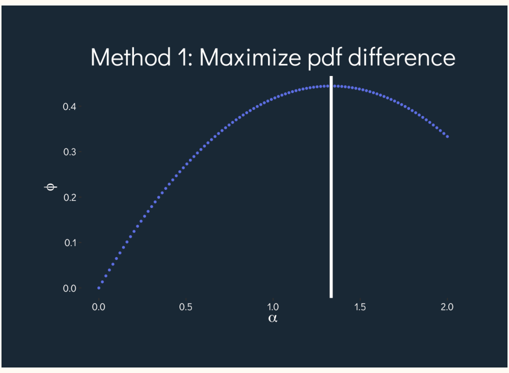
Method 1: Minimize the overlap between the pdf's

The first method just uses intuition: to minimize misclassification, we just need to minimize the overlap of the pdf's of the two classes, or essentially, maximize the difference between the two:

$$argmax(\int P(x|C_1) - \int P(x|C_2))$$

```
from scipy.integrate import quad

def c1(x):

    if x >= 0 and x < 6:
        return 1/6
    else:
        return 0

def c2(x):

    if x >= 1 and x < 3:
        return 0.25 * (x - 1)

    elif x >= 3 and x < 5:
        return 0.25 * (5 - x)

    else:
        return 0

def phi(alpha):

    lwr = 3 - alpha
    upr = 3 + alpha

    c2_area = quad(c2, a = lwr, b = upr)[0]
    c1_area = quad(c1, a = lwr, b = upr)[0]

    return c2_area - c1_area

phi_result = []
alpha_vals = np.linspace(0, 2, 100)

for i in alpha_vals:

    phi_result.append(phi(i))


max_area = alpha_vals[np.array(phi_result).argmax()]
```

```
ggplot() +
  aes(as.numeric(py$alpha_vals), as.numeric(py$phi_result)) +
  geom_point(size = 0.7, color = hex_purple) +
  geom_vline(xintercept = py$max_area, size = 1.4, color = 'white') +
  labs(title = "Method 1: Maximize pdf difference",
       x = expression(alpha),
       y = expression(phi)) +
  proj_theme
```



Our misclassification rate is minimized at $\alpha = \frac{4}{3}$, or when $\frac{5}{3} < x < \frac{14}{3}$.

```
c2_misclass = quad(c2, a = 1, b = 5/3)[0] + quad(c2, a = 13/3, b = 5)[0]
pc2_misclass = c2_misclass * 0.5 # 0.5 prior prob
pc1_misclass = quad(c1, a = 5/3, b = 15/3)[0] * 0.5 # 0.5 prior prob
```

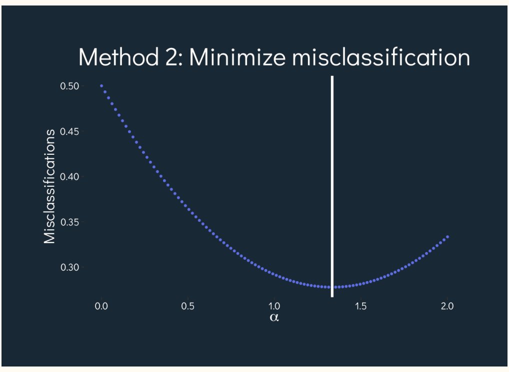
Method 2: integrate to find total misclassification for each α & minimize

```

def p_misclass(alpha):

    lwr = 3 - alpha
    upr = 3 + alpha

    c1_misclass = quad(c1, a = lwr, b = upr)[0]
    c2_misclass = quad(c2, a = 0, b = lwr)[0] + quad(c2, a = upr, b = 6)[0]

    return c1_misclass + c2_misclass
```

```

misclass_result = []
alpha_vals = np.linspace(0, 2, 100)

for i in alpha_vals:

    misclass_result.append(p_misclass(i))
```

```

min_misclass = alpha_vals[np.array(misclass_result).argmin()]
# min misclass
# np.array(misclass_result).argmin() * 0.5 # x prior
```

```
ggplot() +
  aes(as.numeric(py$alpha_vals), as.numeric(py$misclass_result) * 0.5) +
  geom_point(size = 0.7, color = hex_purple) +
  geom_vline(xintercept = py$min_misclass, color = 'white', size = 1.2) +
  labs(title = "Method 2: Minimize misclassification",
       x = expression(alpha),
       y = "Misclassifications") +
  proj_theme
```



We again find the optimal α is $\frac{4}{3}$!