LE12: Linear optimization, duality and geometry

In linear optimization, we often consider a primal problem in the form max $c^T x$ subject to the constraints $Ax \leq b$ and $x \geq 0$. The corresponding **dual** problem is then: min $b^T y$ subject $A^T y \geq c$ and $y \geq 0$.

The fundamental **duality theorem** states: If both programs have feasible solutions, then both are solvable, and for the optimal solutions x^* and y^*, the equality $c^T x^* = \boldsymbol{b^T y^*}$ holds. If one of the programs does not have a feasible solution, then the other also does not have an **optimal** solution.

Farkas' Lemma (1902). For example, it states that for the set $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$, it is non-empty if and only if for every y such that $A^T y \geq 0$, then necessarily $\boldsymbol{b^T y \geq 0}$ also holds.

Geometrically speaking, the feasible region P of a linear program is a convex set. A set A is called **convex** if, for any two points $x, y \in A$, it always contains the entire **line** between those points. The smallest convex set encompassing a given set A is called the **convex hull** of A. If the generating set A is finite, the result $\text{conv}(A)$ is called a convex **polyhedron**.

The **corners** of the polyhedron are of particular interest for the simplex method. A point x is a convex polyhedron if and only if it cannot be represented as a **convex** linear combination of other points from P; that is, it does not lie 'between' any two other points in the set. According to Theorem 24.10, a convex polyhedron is identical to the convex hull of its **corners**.

A concrete example of this theory is graph theory. If one seeks a maximal **matching** in a graph G, the relaxation of this integer problem leads to an LP. The dual problem is directly related to the search for a minimal **vertex cover**, where each edge $e = uv$ is a constraint of the form $y_u + y_v \geq 1$ delivers.