COMPLETE FORMULA SHEET EC516 (Fall 2018)

Unit Step

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & otherwise \end{cases}$$

Unit Impulse

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & otherwise \end{cases}$$

Complex Exponentials and Sinusoids

$$e^{j\omega n} = \cos(\omega n) + j\sin(\omega n) \quad \cos(\omega n) = (1/2)(e^{j\omega n} + e^{-j\omega n}) \quad \sin(\omega n) = (1/2j)(e^{j\omega n} - e^{-j\omega n})$$

Impulse Train

$$\sum_{m=-\infty}^{\infty} \delta[n - Nm] = \frac{1}{N}\sum_{k=0}^{N-1} e^{j\frac{2\pi k n}{N}}$$

DT Convolution: $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$ **CT Convolution**: $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$

FSF: $\sum_{k=0}^{N-1} \alpha^k = \begin{cases} \frac{1-\alpha^N}{1-\alpha} & ;\alpha \neq 1 \\ N & ;\alpha = 1 \end{cases}$ **ISF**: $\sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha} \quad ;|\alpha| < 1$

CTFT: $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$; **Inverse CTFT**: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$

DTFT: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$; **Inverse DTFT**: $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$

Basic DTFT Properties

$$x[n-n_0] \Leftrightarrow e^{-j\omega n_0}X(e^{j\omega}) \qquad e^{j\omega_0 n}x[n] \Leftrightarrow X(e^{j(\omega-\omega_0)}) \qquad x^*[n] \Leftrightarrow X^*(e^{-j\omega})$$

$$x[-n] \Leftrightarrow X(e^{-j\omega}) \qquad x[n]*h[n] \Leftrightarrow X(e^{j\omega})H(e^{j\omega})$$

$$x[n]\times h[n] \Leftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})H(e^{j(\omega-\theta)})\,d\theta$$

$$x[n] \quad real \quad \Rightarrow \quad |X(e^{j\omega})| \quad even, \quad \angle X(e^{j\omega}) \quad odd$$

Common DTFT Pairs

$$\delta[n-n_0] \Leftrightarrow e^{-j\omega n_0} \qquad u[n]-u[n-N] \Leftrightarrow \frac{\sin(\omega N/2)}{\sin(\omega/2)} e^{-j\omega(N-1)/2}$$

$$\sum_{k=-\infty}^{\infty} \delta[n-kN] \Leftrightarrow \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega-(2\pi k/N)) \qquad \frac{\sin \omega_0 n}{\pi n} \Leftrightarrow \begin{cases} 1 & 0 \le |\omega| \le \omega_0 \\ 0 & \omega_0 < |\omega| \le \pi \end{cases}$$

Common CTFT Pairs

$$\delta(t-t_0) \Leftrightarrow e^{-j\omega t_0} \qquad u(t+T)-u(t-T) \Leftrightarrow 2\sin(\omega T)/\omega$$

$$\sin(\omega_0 t)/\pi t \Leftrightarrow u(\omega+\omega_0)-u(\omega-\omega_0) \qquad \sum_{k=-\infty}^{\infty} \delta(t-kT) \Leftrightarrow \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega-(2\pi k/T))$$

Expander by M :

$$y[n] = \begin{cases} x[n/M] & for \quad n = kM \\ 0 & otherwise \end{cases} \qquad Y(e^{j\omega}) = X(e^{j\omega M})$$

Compressor by M:

$$y[n] = x[nM] \qquad Y(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X(e^{(j\frac{\omega}{M} - \frac{2\pi k}{M})})$$

N-point DFT:

$$X[k]_N = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N} \qquad 0 \le k \le N-1$$

Inverse N-point DFT:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]_N e^{j2\pi kn/N} \qquad 0 \le n \le N-1$$

Properties of DFT:

$$x[(n-n_0)_N] \Leftrightarrow e^{-jk\frac{2\pi}{N}n_0} X[k]_N \qquad x[(-n)_N] \Leftrightarrow X[(-k)_N]_N \qquad x^*[n] \Leftrightarrow X^*[(-k)_N]_N$$

$$e^{j\frac{2\pi k_0}{N}n} x[n] \Leftrightarrow X[(k-k_0)_N]_N \qquad \sum_{k=0}^{N-1} x[k] h[(n-k)_N] \Leftrightarrow X[k]_N \times H[k]_N$$

General Form of Difference Equation:

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{m=0}^{M} b_m x[n-m]$$

z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Properties of z-transform:

$x[n-n_0] \Leftrightarrow z^{-n_0} X(z)$ $\quad$ $x[-n] \Leftrightarrow X(z^{-1})$ $\quad$ $x^*[n] \Leftrightarrow X^*(z^*)$

$x[n] * h[n] \Leftrightarrow X(z) \times H(z)$

Common z-transform pair: $a^n u[n] \Leftrightarrow \dfrac{1}{1 - az^{-1}}$

All-Pole Model: $\dfrac{G}{1 + \sum_{k=1}^{P} a_k z^{-k}}$;

Pade Matching: Match first P+1 points of impulse response of model to first P+1 points of data.

Direct Least Squares Modeling: Minimize $E_D = \sum_{n=0}^{\infty} (h[n] - x[n])^2$, where $h[n]$ is the impulse response of the model and $x[n]$ is the data.

Indirect Least Squares Modeling: Minimize $E_I = \sum_{n=0}^{\infty} (x[n] + \sum_{k=1}^{P} a_k x[n-k])^2$, where $x[n]$ is the data and the a_k are the parameters of the all-pole model. The parameter G in the all-pole model found by letting its square be equal to the minimized value of E_I.

Discrete Cosine Transforms: Given an N-point real-valued signal $x[n]$,

Type I: DTFT of $x[n] + x[-n]$ sampled at $2\pi k / (2N-2)$ for $k = 0, 1, \ldots, N-1$

Type 2: Multiply DTFT of $x[n]+x[-n-1]$ by $e^{-j\omega/2}$ and sample it at $2\pi k/(2N-1)$ for $k=0,1,\ldots,N-1$

Time-Dependent Fourier Transform

$$X_w[n,\omega)=\sum_{m=-\infty}^{\infty} w[m]x[n+m]e^{-j\omega m}=x[n]*w[-n]e^{j\omega n} \qquad X_w[n,k]=X_w[nL,2\pi k/M)$$

Inverse Relations:

$$x[n]=\frac{1}{2\pi w[0]}\int_{-\pi}^{\pi} X_w[n,\omega)\,d\omega \qquad x[n]=\frac{1}{2\pi W(e^{j0})}\sum_{m=-\infty}^{\infty}\int_{-\pi}^{\pi} X_w[m,\omega)e^{j\omega(n-m)}\,d\omega$$

Synthesis Equations:

$$x[n]=\frac{1}{Mw[0]}\sum_{k=0}^{M-1} X_w[n,k] \qquad x[n]=\frac{1}{M}\sum_{i=-\infty}^{\infty} f[n-iL]\sum_{k=0}^{M-1} X_w[i,k]e^{j\frac{2\pi k}{M}(n-iL)}$$

FBS Condition: L=1 and $w[nM]=0$ for $n\neq 0$

GFBS Condition: $\sum_{i=-\infty}^{\infty} f[n-iL]w[n-iL-pM]=\delta[p]$

Two-Dimensional Fourier Transform:

$$X(e^{j\omega_1},e^{j\omega_2})=\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} x[n_1,n_2]e^{-j(\omega_1 n_1+\omega_2 n_2)}$$

Uniform DFT Filterbank:

It is a bank of *N* filters, all with the same input signal. The *k*th filter in the filterbank has system function $H(e^{-j2\pi k/N}z)$ for some prototype system function $H(z)$. If $H(z)=1+z^{-1}+z^{-2}+\ldots+z^{-(N-1)}$ then the Uniform DFT filterbank becomes just a DFT filterbank.

Polyphase Decomposition:

$$H(z)=E_0(z^M)+z^{-1}E_1(z^M)+\ldots+z^{-(M-1)}E_{M-1}(z^M)\,;\; E_k(z)\leftrightarrow h[nM+k]$$

Complex Cepstrum:

If $\hat{x}[n]$ is the complex cepstrum of signal $x[n]$, then $\hat{x}[n]$ has the z-transform $\hat{X}(z) = \log(X(z))$.

Restricted z-transform model for defining a complex cepstrum:

$$X(z) = G\frac{\prod_{k=1}^{M_i}(1-a_k z^{-1})\prod_{l=1}^{M_o}(1-b_l z)}{\prod_{r=1}^{N_i}(1-c_r z^{-1})\prod_{s=1}^{N_o}(1-d_s z)}$$

where $G > 0, \quad |a_k| < 1, \quad |b_l| < 1, \quad |c_r| < 1, \quad |d_s| < 1$.

Series Expansion of Natural Logarithm: $\log(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}$

Minimum Phase Signal: Signal is causal and all poles and zeros are inside the unit circle.

Maximum Phase Signal: Signal is anti-causal and all poles and zeros are outside the unit circle.