*** **Basis:** Let V be a vector space and $\{v_1, v_2, \cdots, v_n\}$ is a finite set of vectors in V.

We call $\{v_1, v_2, \cdots, v_n\}$ a basis for V if and only if

(i) $\{v_1, v_2, \cdots, v_n\}$ is linearly independent.

(ii) $\{v_1, v_2, \cdots, v_n\}$ spans V.

example: $\{(1, 0, 0, \cdots 0), (0, 1, 0, \cdots 0), \cdots (0, 0, 0, \cdots 1)\}$
standard basis.

*** **Dimension:** The number of vectors in any basis of a finite dimensional vector space(V) is called Dimension.

Or, equivalently, the dimension of a vector space is equal to the maximum number of linearly independent vectors contained in it.

*** Let U be the subspace of $\mathbb{R}^3$ spanned by the vectors $(1, 2, 1)$, $(0, -1, 0)$ and $(2, 0, 2)$. Find basis and the dimension of U.

Soln: Form the matrix whose rows are given vectors and reduce the matrix to row-echelon form.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

We multiply first row by 2 and then subtract from the third row.

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -4 & 0 \end{bmatrix}$$

we multiply second row by 4 and subtract from the third row.

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we multiply second row by 2 and add with the first row.

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we multiply second row by (-1).

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is in echelon form and the non-zero rows in the matrix are $(1,0,1)$ and $(0,1,0)$. These non-zero rows form a basis of the row space. and consequently a basis of U; that is

Basis of $U = \{(1,0,1), (0,1,0)\}$ and $\dim U = 2$.

* Find the Basis and Dimension of the vector set
$S = \{(-1, 2, -1, 0), (0, 3, 1, 2), (1, 1, -2, 2), (2, 1, 0, -1)\}$

Sol:

Form the matrix whose rows are given vectors and reduce the matrix to row-echelon form.

$$\begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 2 \\ 1 & 1 & -2 & 2 \\ 2 & 1 & 0 & -1 \end{bmatrix}$$

$R_3' \to R_3 + R_1$
$R_4' \to R_4 + 2R_1$

$$\begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 2 \\ 0 & 3 & -3 & 2 \\ 0 & 5 & -2 & -1 \end{bmatrix}$$

$R_3' \to R_3 - R_2$
$3R_4 - 5R_2$

$$\begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & -11 & -13 \end{bmatrix}$$

$R_4' \to 4R_4 - 11R_3$

$$\begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -52 \end{bmatrix}$$

This matrix is in echelon form and the non-zero rows in the matrix are $(-1, 2, -1, 0)$, $(0, 3, 1, 2)$, $(0, 0, -4, 0)$ and $(0, 0, 0, -52)$. These non-zero rows form a basis of the row space and consequently a basis of S; that is

Basis of $S = \{(-1, 2, -1, 0), (0, 3, 1, 2), (0, 0, -4, 0), (0, 0, 0, -52)\}$

and $\dim S = 4$.

H.W:

① Find the basis and dimension of the vector Sets.

DAK ***
ⓐ $V = \{(1,-2,4,1), (2,-3,9,-1), (1,0,6,-5), (2,-5,7,5)\}$ ②

***
DAK ⓑ $W = \{(1,2,1), (3,1,2), (1,-3,4)\}$. ③

**
A.R ⓒ $S = \{(1,-2,0,0,3), (2,-5,-3,-2,6), (0,5,15,10,0), (2,6,18,8,6)\}$ ③

***
D.A.K ⓓ $T = \{(1,-2,5,-3), (2,3,1,4), (3,8,-3,-5)\}$ ②

***
ⓔ $U = \{(1,1,1), (1,2,3), (3,4,5)\}$ ②

***

* Determine a basis and the dimension for the solution space of the homogeneous system.

$$x - 3y + z = 0$$
$$2x - 6y + 2z = 0$$
$$3x - 9y + 3z = 0$$

Solution:

$$\left.\begin{aligned} x - 3y + z &= 0 \\ 2x - 6y + 2z &= 0 \\ 3x - 9y + 3z &= 0 \end{aligned}\right\} \quad \cdots (1)$$

Reduce the system to echelon form. we multiply first equation by 2 and 3 and then subtract from the second and third equations respectively. Then we get

$$\left.\begin{aligned} x - 3y + z &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}\right\}$$

i.e $x - 3y + z = 0$

The system is in echelon form and has only one non-zero equation in three unknowns. So the system has $3-1=2$ free variable which are y and z, Hence the dimension of the solution Space is 2.

Set (i) $y=1, z=0$ (ii) $y=0, z=1$, to obtain the solution.

solutions $v_1 = (3, 1, 0)$, $v_2 = (-1, 0, 1)$.

Hence the set $\{(3,1,0), (-1,0,1)\}$ is a basis of the solution Space.

**
* Find the dimension and a basis of the solution space of the following homogeneous system.

A.R (a)

$$x + 2y - 4z + 3s - t = 0$$
$$x + 2y - 2z + 2s + t = 0$$
$$2x + 4y - 2z + 3s + 4t = 0$$

Ans:- dim = 3, Basis: $\{(-2, 1, 0, 0, 0), (5, 0, -2, 1, 0), (-7, 0, 2, 0, 1)\}$

D.A.K *** (b)

$\cancel{x + y - t = 0}$

$$x + y - t = 0$$
$$x + 2y + 3z = 0$$
$$2x + 3y + 3z + t = 0$$

Ans. dim = 2, Basis: ·

D.A.K *** (c)

$\cancel{x + 2x - z = 0}$

$\cancel{-x + 3y + z = 0}$

$$x + y + z = 0$$
$$x + 2y + 3z = 0$$
$$3x + 4y + 5z = 0$$

Ans. dim = 2. Basis: