Linear Dependence and Linear Independence

③

Linear Dependence: Let V be a vector space over the field F. The vectors $v_1, v_2, \cdots v_n \in V$ are said to be linearly dependent over F or simply dependent if there exists a non-trivial combination of them equal to the zero vector 0.

That is $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$
where $\alpha_i \neq 0$ for at least one i.

1st ~~Linear~~ ***

* Show that the three vectors $(1,3,2)$, $(1,-7,-8)$, $(2,1,-1)$ are linearly dependent.

Solⁿ:

~~Let~~ Set a linear combination of the given vectors equal to zero by using unknown scalars x, y, z:

$$x(1,3,2) + y(1,-7,-8) + z(2,1,-1) = (0,0,0)$$

$$\Rightarrow (x, 3x, 2x) + (y, -7y, -8y) + (2z, z, -z) = (0,0,0)$$

$$\Rightarrow (x+y+2z,\ 3x-7y+z,\ 2x-8y-z) = (0,0,0)$$

Equating corresponding components and forming the linear system, we get

$$\left.\begin{aligned} x+y+2z &= 0 \\ 3x-7y+z &= 0 \\ 2x-8y-z &= 0 \end{aligned}\right\}$$

$L_2' \to L_2 - 3L_1$
$L_3' \to L_3 - 2L_1$

$$\left\{\begin{aligned} x+y+2z &= 0 \\ -10y-5z &= 0 \\ -10y-5z &= 0 \end{aligned}\right. \sim \left\{\begin{aligned} x+y+2z &= 0 \\ 2y+z &= 0 \end{aligned}\right.$$

P.T.O

The system is in echelon form and has only two non-zero equation in three unknowns, hence the system has non-zero solution, Thus the original vectors are linearly dependent.

2nd * Show that the set of vectors $\{(2,1,2), (0,1,-1), (4,3,3)\}$ is linearly dependent.

Proof: Form ~~From~~ the matrix whose rows are the given vectors and reduce the matrix to echelon form by using the elementary row operations:

$$\begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 4 & 3 & 3 \end{bmatrix}$$

we multiply first row by 2 and then subtract from the third row,

$$\sim \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

we subtract second row ~~by 2 and then~~ from the third row.

$$\sim \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix is in echelon form and has a zero row: hence the vectors are linearly dependent.

Linear independence:

Let V be a vector space over the field F. The vectors $v_1, v_2, \dots v_n \in V$ are said to be linearly independent over F or simply independent if the only linear combination of them equal to 0 (zero) is the trivial one. ~~In this case~~

i.e.,

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0 \text{ if and only if } \alpha_1 = \alpha_2 = \cdots \alpha_n = 0.$$

* Show that the set of vectors $\{(1,1,-1), (2,1,0), (-1,1,2)$ is linearly independent.

Proof: Set a linear combination of the given vectors equal to the zero vector using unknown scalar x, y, z:

$$x(1,1,-1) + y(2,1,0) + z(-1,1,2) = (0,0,0)$$

$$\Rightarrow (x, x, -x) + (2y, y, 0) + (-z, z, 2z) = (0,0,0)$$

$$\Rightarrow (x+2y-z,\ x+y+z,\ -x+2z) = (0,0,0)$$

Equating corresponding components and forming the linear system, we get:

$$\left.\begin{aligned} x+2y-z &= 0 \\ x+y+z &= 0 \\ -x+2z &= 0 \end{aligned}\right\} \xrightarrow[L_3+L_1]{L_2-L_1} \left\{\begin{aligned} x+2y-z &= 0 \\ -y+2z &= 0 \\ 2y &= 0 \end{aligned}\right.$$

In echelon form there are exactly three equations in three unknowns; hence the system has only the zero solution $x=0, y=0, z=0$. Accordingly, the vectors are linearly independent.

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* Show that the vectors $(2,-1,4)$, $(3,6,2)$ and $(2,10,-4)$ are linearly independent.

Proof: ~~From~~ Form the matrix whose rows are the given vectors and reduce the matrix to echelon form by elementary row operations.

$$\begin{bmatrix} 2 & -1 & 4 \\ 3 & 6 & 2 \\ 2 & 10 & -4 \end{bmatrix}$$

we devide third row by 2 and then interchange with the first row.

$$\sim \begin{bmatrix} 1 & 5 & -2 \\ 3 & 6 & 2 \\ 2 & -1 & 4 \end{bmatrix}$$

we multiply first row by 3 and 2 and then subtract from the second and third rows respectively.

$$\sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -9 & 8 \\ 0 & -11 & 8 \end{bmatrix}$$

we multiply second row by $\frac{11}{9}$ and then subtract from the third row.

$$\sim \begin{bmatrix} 1 & 5 & -2 \\ 0 & -9 & 8 \\ 0 & 0 & -\frac{16}{9} \end{bmatrix}$$

Since the echelon matrix has no zero-row.

$\therefore$ the vectors are linearly independent

* H.W:

① Test the dependency of the following vector sets:

(a) $\{(1,2,-3), (2,0,-1), (7,6,-11)\}$ — D

(b) $\{(2,0,-1), (1,1,0), (0,-1,1)\}$ — I

(c) $\{(3,0,1,-1), (2,-1,0,1), (1,1,1,-2)\}$ → D

(d) $\{(1,-2,-1), (1,-4,3), (0,2,4)\}$ — I

(e) $\{(1,-4,2), (3,-5,1), (-1,1,1)\}$ — I

(f) $\{(1,1,-2), (1,2,3), (4,5,-3)\}$ — D

(g) $\{(1,0,0), (0,1,0), (1,1,0)\}$ — D

(a - c) → A.R

(d - g) → D.A.K.

** Linear span/Generate: Let $V \in \mathbb{R}^n$ be a vector space and $\{v_1, v_2, \ldots v_n\}$ be a vector set, Now if $V = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ where $\alpha_1, \alpha_2, \ldots \alpha_n \in F$ Then we can say that vector set is the Generator of the vector space $\mathbb{R}^n$, and its called linear span.

* Non zero row and column are linearly independent *