Linear Transformation

Linear Transformation: Let, U and V be two vector spaces over the same field F. A Linear transformation T of U into V, written as $T: U \to V$, is a transformation T of U into V such that

(i) $T(u_1+u_2) = T(u_1)+T(u_2)$ for all $u_1, u_2 \in U$

(ii) $T(\alpha u) = \alpha T(u)$ for all $u \in U$ and all $\alpha \in F$.

* Kernel of a Linear transformation/mapping:

Let. $T: V(F) \to U(F)$ be a linear transformation Then kernel of transformation or ker T is defined by $\ker T = \{v \in V(F) : T(v) = 0\}$

Ex: Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation and defined by $T(x,y,z) = (0,y,z)$, here $\ker T = \{(x,0,0) : x \in \mathbb{R}\} = x$ axis.

* Image of linear Transformation:

Let $T: V(F) \to U(F)$ be a linear transformation Then Image of transformation or Im T is defined by $\operatorname{Im} T = \{u \in U(F) : T(v) = u, v \in V(F)\}$

R_T range of T.

Ex: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ ~~be~~ be a linear transformation and defined by : $T(x,y) = (x,0)$.

Here $\operatorname{Im} T = \{(x,0) : x \in \mathbb{R}\} = x$ axis.

***
* Show that the following transformation defines a linear operator on $\mathbb{R}^3$.

$$T(x, y, z) = (x+y, -x-y, z).$$

Proof:

Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$

Then $u+v = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (x_1+x_2, y_1+y_2, z_1+z_2)$

and $\alpha u = \alpha(x_1, y_1, z_1) = (\alpha x_1, \alpha y_1, \alpha z_1)$ where $\alpha \in F$

Thus $T(u) = T(x_1, y_1, z_1) = (x_1+y_1, -x_1-y_1, z_1)$

$T(v) = T(x_2, y_2, z_2) = (x_2+y_2, -x_2-y_2, z_2)$

$T(u+v) = T(x_1+x_2, y_1+y_2, z_1+z_2)$

$= ((x_1+x_2)+(y_1+y_2), -(x_1+x_2)-(y_1+y_2), (z_1+z_2))$

$= (x_1+y_1, -x_1-y_1, z_1) + (x_2+y_2, -x_2-y_2, z_2)$

$= T(u) + T(v)$

Also for any $\alpha \in F$

$T(\alpha u) = T(\alpha x_1, \alpha y_1, \alpha z_1)$

$= (\alpha x_1 + \alpha y_1, -\alpha x_1 - \alpha y_1, \alpha z_1)$

$= \alpha(x_1+y_1, -x_1-y_1, z_1)$

$= \alpha T(u)$

Since u, v and α are arbitrary, T is a linear operator.

H.W. ** (i) $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (x+2y, 2x-y)$. (A.R)

399 *** (ii) $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(x, y, z) = (x+2y, y-z, x+2z)$. D.A.K

*** (iii) $T: \mathbb{R}^3 \to \mathbb{R}^5$, $T(x_1, x_2, x_3) = (x_1-x_2, 0, x_1-x_3, x_2, 0)$. "

** (iv) $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(x, y, z) = (x+y, y+z, z+x)$

* Let $T: \mathbb{R}^3 \to \mathbb{R}^5$ be defined as $T(x_1, x_2, x_3) = (x_1 - x_2, 0, x_1 - x_3, x_2, 0)$. Show that T is a linear transformation

Soln:

Let $u = (x_1, x_2, x_3)$ And $v = (x_1', x_2', x_3')$

Then $T(u) + T(v) = T(x_1, x_2, x_3) + T(x_1', x_2', x_3')$

$= (x_1 - x_2, 0, x_1 - x_3, x_2, 0) + (x_1' - x_2', 0, x_1' - x_3', x_2', 0)$

$= (x_1 - x_2 + x_1' - x_2', 0, x_1 - x_3 + x_1' - x_3', x_2 + x_2', 0)$

$= (x_1 + x_1' - x_2 - x_2', 0, x_1 + x_1' - x_3 - x_3' + x_2 + x_2', 0)$ ---- ①

Again $T(u+v) = T(x_1 + x_1', x_2 + x_2', x_3 + x_3')$

$= (x_1 + x_1' - x_2 - x_2', 0, x_1 + x_1' - x_3 - x_3' + x_2 + x_2', 0)$ ---- ②

Now from ① and ② we get

$T(u+v) = T(u) + T(v)$.

Again for any scalar $\alpha \in F$

$T(\alpha u) = T(\alpha x_1, \alpha x_2, \alpha x_3)$

$= (\alpha x_1 - \alpha x_2, 0, \alpha x_1 - \alpha x_3, \alpha x_2, 0)$

$= \alpha(x_1 - x_2, 0, x_1 - x_3, x_2, 0)$

$= \alpha T(u)$

Therefore T is a linear transformation.

* Let, $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, when $T(1,1) = 3$ and $T(0,1) = -2$, Then find $T(a,b)$.

Soln:

Here $\{(1,1), (0,1)\}$ is a basis of $\mathbb{R}^2$.

Let $(a,b) = x(1,1) + y(0,1) = (x, x+y)$

$\Rightarrow x = a, \; x+y = b$

$\therefore x = a, \; y = b-a.$

Now since T is linear transformation,

$$\therefore T(a,b) = xT(1,1) + yT(0,1) = 3x - 2y$$
$$= 3a - 2(b-a)$$
$$= 5a - 2b \text{ (Ans..)}$$

* Let $T: V_2(R) \to V_3(R)$ be a linear transformation where $T(1,2) = (3,-1,5)$ and $T(0,1) = (2,1,-1)$ Then find $T(a,b)$.

Soln:

Here $\{(1,2), (0,1)\}$ is a bassis of $V_2(R)$

Now let,

$$(a,b) = x(1,2) + y(0,1) = (x, 2x+y)$$

$\Rightarrow x = a, \; 2x + y = b$

$\Rightarrow x = a, \; y = b - 2a.$

Now using the condition of given linear transformation we get,

$$T(a,b) = xT(1,2) + yT(0,1)$$
$$= x(3,-1,5) + y(2,1,-1)$$
$$= (3x, -x, 5x) + (2y, y, -y)$$
$$= (3x+2y, -x+y, 5x-y)$$
$$= (2b-a, b-3a, 7a-b). \text{ Ans.}$$

H.W. * Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear mapping, where $T(0,1) = (0,0)$, $T(1,1) = (1,1)$, Then find $T(a,b)$.

Matrix Representation of a Linear Transformation

Defination: Let $T: V(F) \to V(F)$ be a linear operator. Let $\{e_1, e_2, \cdots e_n\}$ be a set of basis of vector space $V(F)$. Then $T(e_1), T(e_2) \cdots T(e_n) \in V(F)$. where each vector $T(e_1), T(e_2) \cdots T(e_n)$ will be express as a linear combination of the set $\{e_1, e_2, \cdots e_n\}$. Then we get

$$\left.\begin{aligned} T(e_1) &= a_{11} e_1 + a_{12} e_2 + \cdots + a_{1n} e_n \\ T(e_2) &= a_{21} e_1 + a_{22} e_2 + \cdots + a_{2n} e_n \\ \cdots & \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ T(e_n) &= a_{n1} e_1 + a_{n2} e_2 + \cdots + a_{nn} e_n \end{aligned}\right\}, \quad a_{ij} \in F$$

The Transpose Matrix of the coefficient matrix of the above equations is ~~called~~ said to be a Matrix representation of T, and it is denoted by $[T]_e$ or shortly $[T]$.

$$[T]_e = [T] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}^t = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & & a_{n2} \\ \cdots & \cdots & & \cdots \\ a_{1n} & a_{2n} & & a_{nn} \end{bmatrix}$$

* Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator and defined by $T(x, y, z) = (x+y, y, z)$. Then find Matrix representation with respect to the standared basis of $\mathbb{R}^3$.

Soln

Let $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ be standared basis of $\mathbb{R}^3$.

Then from $T(x, y, z) = (x+y, y, z)$ we get

$$T(e_1) = T(1, 0, 0) = (1, 0, 0) = 1 e_1 + 0 . e_2 + 0 . e_3$$
$$T(e_2) = T(0, 1, 0) = (1, 1, 0) = 1 . e_2 + 1 . e_2 + 0 . e_3$$
$$T(e_3) = T(0, 0, 1) = (0, 0, 1) = 0 . e_1 + 0 . e_2 + 1 . e_3$$

Therefore the matrix representation of T is

$$[T]_e = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^t = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Ans

* Let, $T: \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation which is defined by $T(x, y, z, t) = (x-y+z, x+2y, z-t)$. Then find matrix representation of T w.r. to standared basis of $\mathbb{R}^4$ and $\mathbb{R}^3$.

Soln:

Let, $\{e_1 = (1, 0, 0, 0), e_2 = (0, 1, 0, 0), e_3 = (0, 0, 1, 0), e_4 = (0, 0, 0, 1)\}$ and $\{f_1 = (1, 0, 0), f_2 = (0, 1, 0), f_3 = (0, 0, 1)\}$ are standared basis of $\mathbb{R}^4$ and $\mathbb{R}^3$ respectively.

Then
$$T(e_1) = (1, 1, 0) = 1 . f_1 + 1 . f_2 + 0 . f_3$$
$$T(e_2) = (-1, 2, 0) = -1 . f_1 + 2 . f_2 + 0 . f_3$$
$$T(e_3) = (1, 0, 1) = 1 . f_1 + 0 . f_2 + 1 . f_3$$
$$T(e_4) = (0, 0, -1) = 0 . f_1 + 0 . f_2 - 1 . f_3$$

P.T.O

$$\therefore [T]_e^f = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}^t = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Ans.

* Let $T: V_3(R) \to V_3(R)$ be a linear operator which is defined by $T(x, y, z) = (2y+z, x-4y, 3x)$. Then find the matrix representation for the basis $\{f_1 = (1,1,1), f_2 = (1,1,0), f_3 = (1,0,0)\}$.

Soln. Let $(a, b, c) \in V_3(\mathbb{R})$, then

$(a, b, c) = x f_1 + y f_2 + z f_3$, where $x, y, z \in \mathbb{R}$

$= x(1,1,1) + y(1,1,0) + z(1,0,0)$

$= (x+y+z, x+y, x)$

$\Rightarrow x+y+z = a, \quad x+y = b, \quad x = c$

$\Rightarrow x = c, \quad y = b-c, \quad z = a-b$

Then $(a, b, c) = c f_1 + (b-c) f_2 + (a-b) f_3 \quad \cdots$ (1)

Again given $T(x, y, z) = (2y+z, x-4y, 3x) \quad \cdots$ (2)

Now from (1) and (2) we get

$T(f_1) = T(1,1,1) = (3, -3, 3) = 3f_1 - 6f_2 + 6f_3$

$T(f_2) = T(1,1,0) = (2, -3, 3) = 3f_1 - 6f_2 + 5f_3$

$T(f_3) = T(1,0,0) = (0, 1, 3) = 3f_1 - 2f_2 - f_3$

Therefore $[T]_f = \begin{bmatrix} 3 & -6 & 6 \\ 3 & -6 & 5 \\ 3 & -2 & -1 \end{bmatrix}^t$

$$= \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$$

Ans.

H.W. Find matrix representation of T

(1) for the given linear operator and standard basis.

(a) $T: \mathbb{R}^3 \to \mathbb{R}^3$; $T(x, y, z) = (2x+y, x-y, z)$.

(b) $T: \mathbb{R}^2 \to \mathbb{R}^2$; $T(x, y) = (4x-2y, 2x+y)$.

(c) $T: \mathbb{R}^3 \to \mathbb{R}^3$; $T(x_1, x_2, x_3) = (x_1 - x_2 + x_3, 2x_1 - x_2 + 3x_3, -x_1 + 2x_2 + x_3)$.

(d) $T: \mathbb{R}^4 \to \mathbb{R}^3$; $T(x, y, z, t) = (x-y+z, x+y, y-t)$

(e) $T: \mathbb{R}^3 \to \mathbb{R}^4$; $T(x, y, z) = (x+y, x+z, x-y, x-z)$

(2) Find matrix representation of T for the given basis. $\{f_1 = (1,1,0), f_2 = (1,0,1), f_3 = (0,1,1)\}$ where T is defined by $T: \mathbb{R}^3 \to \mathbb{R}^3$;

$T(x, y, z) = (x+y, y+z, z+x)$

(3) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator and which is defined by $T(x, y, z) = (x+2y, x, y-z)$; Find matrix representation of T ~~for~~ w.r.to the given basis $\{f_1 = (1,1,1), f_2 = (0,1,1), f_3 = (0,0,1)\}$

Suman
13.04.14