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Vector Space & Subspace

Define Vector space and Subspace.

Vector space :

Definition:

Let K, F be a given field and Let V be a none empty set with rules of addition and scaler multiplication which assigns to any u,vϵV a sum u+vϵV and to any uϵV, fϵF a product fuϵV.

Then V is called a vector space over F (and the elements of V are called vector) if the following axioms hold.

Axioms are below:

A(1) Addition is commutative :

For all vector's **u,vϵV, u+v = v+u.**

A(2) Addition is Associative :

For all vector's **u,v,wϵV, (u+v)+w = u+(v+w).**

A(3) Existence of 0 (zero Vector's) :

There exists a vector 0ϵV such that for all **uϵV,u+0=0+u=u.**

A(4) Existence of Negative :

For each uϵv there is a vector -uϵv for which,

u+(-u)=(-u)+u=0.

M(1) Distributive Law :

For any scalar αϵF and any vector's u,vϵV,**α(u+v) =αu+αv.**

M(2) Distributive Law :

For any scalars, α,βϵF and any vectors uϵV, **(αβ)u = α(βu).**

M(3) Associative Law :

For any scalars αβϵF and any vectors uϵV, **(αβ)u = α(βu).**

M(4) Unitary Law :

For each uϵV, **1u = u.** Where 1 is the unite scalar and 1ϵF.

Subspace :

Definition : Let W be a subset of a vector space V over a field F. W is called a subspace of V if W is itself a vector space over F with respect to the operations of vector addition and scalar multiplication on V.

Theorem :

Let, V be a vector space, with operations Addition (+) and Multiplication (.) , and Let W be a subset of V .Then W is a subspace of V if and only if the following conditions hold.

Sub0 W is nonempty : The zero vector belongs to W.

Sub1 closure under (+) : If u and v are any vectors in W, then u+v is in W.

Sub2 closure under (.) : If v is any vector in W, and c is any real number, then c.v is in W.

Euclidean Space :

$\mathbf{R^n}$ is the set all real numbers usual addition and multiplication.
($\mathbf{R^n} \longrightarrow$ Euclidean space) **.**

Prove that, For each positive integer n, Euclidean space $\mathbf{R^n}$ is a vector space.

Proof :

We shall have to show that $\mathbf{R^n}$ satisfies all axioms of a vector space.

(i) Let, $u = (u_1,u_2,\dots \dots,u_n)$ and $v = (v_1,v_2,\dots \dots,v_n)$ be in $\mathbf{R^n}$,

Then $u+v = (u_1,u_2,\dots \dots,u_n) + (v_1,v_2,\dots \dots,v_n)$

$= (u_1+v_1,u_2+v_2,\dots \dots,u_n+v_n)$

$= (v_1+u_1,v_2+u_2,\dots \dots,v_n+u_n)$.

So, A(1) is True.

(ii) Let, $u = (u_1,u_2,\dots \dots,u_n)$ and $v = (v_1,v_2,\dots \dots,v_n)$ and $w = (w_1,w_2,\dots \dots,w_n)$ be in $\mathbf{R^n}$. Then, $(u+v) + w = (u_1+v_1,u_2+v_2,\dots \dots,u_n+v_n)+(w_1,w_2,\dots \dots w_n)$

$= (u_1+v_1+w_1, u_2+v_2+w_2,\dots \dots,u_n,v_n,w_n)$

$= u +(v+w)$.

So, the axiom A(2) holds.

(iii) Let, $0 = (0,0,\ldots\ldots,0)$ be in $\mathbf{R}^n$. Then for any $u = (u_1,u_2,\ldots\ldots,u_n)$ in $\mathbf{R}^n$ we will have $u+0 = (u_1,u_2,\ldots\ldots,u_n) + (0,0,\ldots\ldots,0)$

$$= (u_1+0, u_2+0,\ldots\ldots,u_n+0)$$

$$= (u_1,u_2,\ldots\ldots, u_n) = u.$$

So, the axiom A(3) holds.

(iv) Let, $u = (u_1,u_2,\ldots\ldots,u_n)$ and set $-u = (-u_1,-u_2,\ldots\ldots,-u_n)$

Then, $u+(-u) = (u_1,u_2,\ldots\ldots,u_n) + (-u_1,-u_2,\ldots\ldots,-u_n)$

$$= (u_1-u_1, u_2-u_2,\ldots\ldots, u_n-u_n)$$

$$= (0,0,\ldots\ldots,0) = 0.$$

So, the axiom A(4) holds.

(V) Let, α be a real number (scalar) and $u = (u_1,u_2,\ldots\ldots,u_n)$ and $v = (v_1,v_2,\ldots\ldots,v_n)$ be vectors in $\mathbf{R}^n$. Then,

$\alpha (u+v) = \alpha\{ (u_1,u_2,\ldots\ldots,u_n) + (v_1,v_2,\ldots\ldots,v_n) \}$

$= \alpha(u_1+v_1, u_2+v_2,\ldots\ldots,u_n+v_n)$

$= (\alpha u_1+ \alpha v_1, \alpha u_2+ \alpha v_2,\ldots\ldots, \alpha u_n+ \alpha v_n)$

$= \alpha(u_1u_2,\ldots\ldots,u_n)+ \alpha (v_1v_2,\ldots\ldots,v_n)$

$= \alpha u + \alpha v$. So, the axiom M(1) holds.

(vi) Let, α,β be that the real numbers (scalars) and $u = (u_1,u_2,\ldots\ldots,u_n)$ be in $\mathbf{R}^n$. Then, $(\alpha+\beta)u = \left((\alpha+\beta)u_1 , (\alpha+\beta)u_2,\ldots\ldots, (\alpha+\beta)u_n\right)$

$= (\alpha u_1+\beta u_1, \alpha u_2+\beta u_2,\ldots\ldots, \alpha u_n+\beta u_n)$

$= (\alpha u_1, \alpha u_2,\ldots\ldots,\alpha u_n) +(\beta u_1,\beta u_2,\ldots\ldots,\beta u_n)$

$= \alpha (u_1u_2,\ldots\ldots,u_n) + \beta (u_1u_2,\ldots\ldots,u_n)$

$= \alpha u + \beta u$

So, the axiom M(2) is satisfied.

(vii) Let, α,β is real numbers (scalar) and $u = (u_1,u_2,\ldots\ldots,u_n)$ be in $\mathbf{R}^n$.

Then, $(\alpha\beta)u = \alpha\beta (u_1,u_2,\ldots\ldots,u_n)$

$= (\alpha\beta u_1, \alpha\beta u_2,\ldots\ldots, \alpha\beta u_n)$

$= \alpha (\beta u_1, \beta u_2,\ldots\ldots, \beta u_n)$

$= \alpha \left(\beta (u_1,u_2,\ldots\ldots,u_n) \right)$

$= \alpha (\beta u)$

So, the axiom M(4) is satisfied.

Therefore $\mathbf{R}^n$ is a Vector space . **[Proved]**

Prove that, W is not a subspace of $\mathbf{R}^3$ where w = {(a,b,1) : a,bϵ **R**}

Proof :

Let, V= $\mathbf{R}^3$

W= {(a,b,1) : (a,b)ϵ **R**}

0 = (0,0,0) $\notin$ W since the third component vectors in W is 1.

So, W is not a subspace of $\mathbf{R}^3$. **[Proved]**

State and proof fundamental Theorem of Subspace.

Fundamental Theorem of Subspace :

Statement :

W will be subspace of subset of Vector space v(F) iff (if and only)

(a) W is non-empty. i.e W$\neq \emptyset$
(b) W is closed under addition, i.e $\forall$ u,wϵW = u + wϵW. [$\forall$= ***For all***]
(c) W is closed under scalar multiplication, i.e $\forall \alpha \epsilon F, \forall w \epsilon W = \alpha w \epsilon W$

Proof :

First suppose W satisfies (a), (b) and (c) we have to show that W is a subspace of V. Now by (a) W is non-empty and by (b) and (c) vector addition and scalars multiplication is well defined in W.

Again the vectors in W belongs to V then following axioms holds in W.

(i) Addition is commutative :

For all vector's **u,vϵV, u+v = v+u.**

(ii) Addition is Associative :

For all vector's **u,v,wϵV, (u+v)+w = u+(v+w).**

(iii) Existence of 0 (zero Vector's) :

There exists a vector 0ϵV such that for all **uϵV,u+0=0+u=u.**

(iv) **Existence of Negative :**

For each uϵv there is a vector -uϵv for which,

$u+(-u)=(-u)+u=0$.

(v) **Distributive Law :**

For any scalar $\alpha\epsilon F$ and any vector's u,vϵV, **$\alpha(u+v) = \alpha u+\alpha v$.**

(vi) **Distributive Law :**

For any scalars, $\alpha,\beta\epsilon F$ and any vectors uϵV, **$(\alpha\beta)u = \alpha(\beta u)$.**

(vii) **Associative Law :**

For any scalars $\alpha\beta\epsilon F$ and any vectors uϵV, **$(\alpha\beta)u = \alpha(\beta u)$.**

(viii) **Unitary Law :**

For each uϵV, **1u = u.** Where 1 is the unite scalar and 1ϵF.

We will prove (vii) :

By (a) W is non -empty.

Say uϵW . Then by (c)

For 0ϵF, 0u = 0ϵW

(vii) is proved.

We will prove (viii) :

By (a), W is non-empty.

Say wϵW, Then by (c)

For -1ϵF, (-1)uϵW = -uϵW

And by (b) u+(-u) = 0, $\forall$ uϵW

(viii) is proved.

Therefore, W is satisfied of all condition of vector V. So, W is a subspace of subset of vector space V(F).

Conversely Let, W is a subspace of Vector Space V(F) then W will be satisfy the condition of (a), (b) and (c). Because of (a), (b) and (c) are the part of conditions of Vector Space.

[Hence Proved]

Theorem :

Let W be a subset of V. Then show that, W is a Subspace of V(F) iff.

(i) $0 \epsilon W$, i.e $W \neq \emptyset$

(ii) For all $\alpha, \beta \epsilon F$ and for all $u, w \epsilon W = (\alpha u + \beta w) \epsilon W$.

Proof:

Frist suppose that the subset W satisfies (i) and (ii).

Now by (i),

W is non-empty as $0 \epsilon W$.

Again by (ii),

$\forall\ \alpha, \beta \epsilon F$ and $\forall\ u, w \epsilon W = (\alpha u + \beta w) \epsilon W$

Now Let $\alpha=1$ and $\beta=1$ then, $\alpha u + \beta w = 1.u + 1.w = u + w \epsilon W$ $\quad [\ \alpha u + \beta w \epsilon W\ ]$

W is closed under Vector addition.

Again if $w \epsilon W$ and for $\alpha \epsilon F$. We get , $\alpha w = \alpha w + 0 = \alpha w + 0.w \epsilon W$

So, W is closed under scalar multiplication.

Thus W satisfies the three conditions of the fundamental theorem and therefore W is a subspace of V(F).

Conversely Let, W is a Subspace of Vector Space V(F) then W will be satisfy the (i) and (ii) are the part of condition of Vector Space.

Show that , x-axis and y-axis is Subspace of Vector Space R^2 .

Proof:

Let, set of points on x-axis $U=\{(a,0) : a \epsilon \mathbf{R}\}$ … … … (i)

And set of points on y-axis $V=\{(0,b) : b \epsilon \mathbf{R}\}$ … … …(ii)

Since $\mathbf{R}^2$ is the set of all two dimensional Vector Space.

$\therefore U, V \epsilon \mathbf{R}^2$.

For (1)

Since uϵ $\mathbf{R}^2$ $\therefore$ 0=(0,0)ϵU, U≠∅.

Let, any two vectors u=(a_1,0), v=(a_2,0)ϵU, a_1,$a_2$$\epsilon$**R** and any two scalar α,βϵF

$\therefore$ αu + βv = α(a_1,0) + β(a_2,0)

= (αa_1 + βa_2, 0+0)

=(αa_1 + βa_2, 0)ϵU

$\therefore$ α$a_1$$\epsilon$**R** , β$a_2$$\epsilon$**R**

$\therefore$ αa_1+ β$a_2$$\epsilon$**R**

Since (αa_1 + βa_2)ϵ**R** and the second component is 0.

$\therefore$ U is a Subspace of $\mathbf{R}^2$.

For (2)

Since v$\epsilon$$\mathbf{R}^2$ $\therefore$ 0=(0,0)ϵV, V≠∅.

Let, any two vectors u= (0,b_1),v= (0,b_2)ϵV, b_1b_2 ϵ**R** and any two scalar α,βϵF

$\therefore$ αu + βv = α(0,b_1) + β(0,b_2)

= (0+0, αb_1 + βb_2,)

=(0, αb_1 + βb_2)ϵV

$\therefore$ α$b_1$$\epsilon$**R** , β$b_2$$\epsilon$**R**

$\therefore$ αb_1+ β$b_2$$\epsilon$**R**

Since (αb_1 + βb_2)ϵ**R** and the first component is 0.

$\therefore$ V is Subspace of $\mathbf{R}^2$. **[Showed]**

Show that, W is a subspace of $\mathbf{R}^3$ where w = {(a,b,c) : a+b+c=0}

Proof:

Let, V= $\mathbf{R}^3$

Now given W={(a,b,c) : a+b+c=0}

$\therefore$ 0=(0,0,0)ϵW, W≠∅

Let any two vectors u=(a_1,b_1,c_1)ϵW, a_1+b_1+c_1=0

and v= (a_2,b_2,c_2)ϵW, a_2+b_2+c_2=0 and any two scalar α,βϵF

$\therefore\ \alpha u + \beta v = \alpha(a_1,b_1,c_1) + \beta(a_2,b_2,c_2)$

$=(\alpha a_1 + \beta a_2,\ \alpha b_1 + \beta b_2, \alpha c_1+\beta c_2) \in W$

Since $\alpha a_1 + \beta a_2 + \alpha b_1 + \beta b_2 + \alpha c_1 + \beta c_2$

$= \alpha(a_1+b_1+c_1)+\beta(a_2,b_2,c_2)$

$= \alpha.0 +\beta.0$

$= 0+0 = 0 \in W$

$\therefore$ W is a Subspace of vector $\mathbf{R}^3$. **[Showed]**

Show that, $T= \{(a,b,c,d) \in \mathbf{R}^4 : 2a-3b+5c-d=0\}$ is a Subspace of $\mathbf{R}^4$.

Proof:

For $0 \in \mathbf{R}^4$, $0= (0,0,0) \in T$

Since $2.0-3.0+5.0-0 = 0$,

Hence T is non-empty.

Suppose that $u=(a,b,c,d)$ and $v=(a',b',c',d')$ are in T, Then $2a-3b+5c-d=0$.

Now for any scalars α,β we have, $\alpha u + \beta v = \alpha(a,b,c,d) + \beta(a',b',c',d')$

$= (\alpha a,\alpha b,\alpha c, \alpha d) + (\beta a', \beta b',\beta c',\beta d')$

$= (\alpha a + \beta a', \alpha b + \beta b',\alpha c+\beta c',\alpha d+\beta d')$

Also we have, $2(\alpha a + \beta a') - 3(\alpha b + \beta b') + 5(\alpha c+\beta c') - (\alpha d+\beta d')$

$= 2\alpha a+2\beta a'-3\alpha b-3\beta b'+5\alpha c+5\beta c'-\alpha d-\beta d'$

$= 2\alpha a-3\alpha b+5\alpha c-\alpha d+2\beta a'-3\beta b'+5\beta c'-\beta d'$

$= \alpha(2a-3b+5c-d) + \beta(2a'-3b'+5c'-d')$

$= \alpha.0 +\beta.0$

$=0$

Thus $\alpha u + \beta v \in T$ and So, T is a Subspace of $\mathbf{R}^4$. **[Showed]**

Linear Combination

Definition of Linear Combination:

Let V be a vector space over the field F and Let v_1, ……. $v_n \in V$, then any vector v $\in V$ is
called a linear combination of v_1, v_2, …. v_n if and only if there exist scalars $\propto_1, \propto_2$, ……. $\propto_n$ in F.

Such that $V = \propto_1 v_1 + \propto_2 v_2 + \ldots\ldots\ldots + \propto_n v_n$

$= \sum_{i=1}^{n} \propto ivi$

Example 1: Consider the vectors $v_1 = (1, 0, 1)$, $v_2 = (0, -1, 1)$ and $v_3 = (-1, -1, 1)$ in $I\mathcal{R}^3$. Show that $V = (2, 3, 4)$ is a linear combination of v_1, v_2 and v_3.

Solution:

In order to show that V is a linear combination of v_1, v_2, and v_3, there must be scalars $\propto_1, \propto_2$ and $\propto_3$ in F, Such that $V = \propto_1 v_1 + \propto_2 v_2 + \propto_3 v_3$

i.e. $(2, 3, 4) = \propto_1(1, 0, 1) + \propto_2(0, -1, 1) + \propto_3(-1, -1, 1)$

$\Rightarrow (2, 3, 4) = \propto_1(\propto_1, 0, \propto_1) + \propto_2(0, -\propto_2, \propto_2) + \propto_3(-\propto_3, -\propto_3, \propto_3)$

$\Rightarrow (2, 3, 4) = (\propto_1 - \propto_3, - \propto_2 - \propto_3, \propto_1 + \propto_2 + \propto_3)$

Now equating corresponding components and forming linear system we get

$$\left.\begin{array}{l} \propto_1 - \propto_3 = 2 \\ - \propto_2 - \propto_3 = 3 \\ \propto_1 + \propto_2 + \propto_3 = 4 \end{array}\right] \ldots\ldots\ldots \quad (1)$$

Reduce the system (1) to echelon form by elementary operations.

$R3' \rightarrow R3 - R1$ $\quad \propto_1 - \propto_3 = 2$

$- \propto_2 - \propto_3 = 3$

$\propto_1 + \propto_2 + \propto_3 = 2$

$$R3' \rightarrow \widetilde{R3} + R2 \qquad \left.\begin{array}{r} \propto_1 - \propto_3 = 2 \\ -\propto_2 - \propto_3 = 3 \\ \propto_3 = 5 \end{array}\right] \text{.......... (2)}$$

Now from equation (2) we have $\propto_3 = 5$, substituting $\propto_3 = 5$ and solving (2) we get $\propto_2 = -8$, and $\propto_1 = 7$.

Hence $V = 7v_1 – 8v_2 + 5v_3$

Therefore, V is a Linear combination of v_1, v_2, v_3.

Example 2: Express the vector V = (1, -2, 5) as a linear combination of the vectors v_1 = (1, 1, 1), v_2 = (1, 2, 3) and v_3 = (2, -1, 1).

Solution:

In order to show that V is a linear combination of v_1, v_2, and v_3, there must be scalars x, y and z in F. Such that $V = xv_1 + yv_2 + zv_3$

i.e. (1, -2, 5) = x (1, 1, 1) + y (1, 2, 3) + z (2, -1, 1)

=> (1, -2, 5) = (x, x, x) + y (y, 2y, 3y) + z (2z, -z, z)

=> (1, -2, 5) = (x + y+ 2z, x +2y –z, x +3y +z)

Now equating corresponding components and forming linear system, we get

$$\left.\begin{array}{l} x+y+2z = 1 \\ x+2y-z = -2 \\ x+3y+z = 5 \end{array}\right] \text{................ (1)}$$

Reduce the system (1) to echelon form by elementary operation.

$$\begin{array}{ll} R3' \rightarrow R3 - R1 & x + y + 2z = 1 \\ R2' \rightarrow \widetilde{R2} - R1 & \quad y - 3z = -3 \\ & 2y - z = 4 \end{array}$$

$R3' \rightarrow \widetilde{R3} - 2R2$ x + y + 2z = 1
y – 3z = -3 ……….. (2)
5z = 10

From equation (2) we have z =2 and solving (2) we get y=3 and x = 6
Hence V = -6v_1 + 3v_2 + 2v_3

Example 3: Express the matrix E = $\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}$ as a linear combination of A = $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$,
B = $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ and C = $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$.

Solution:

Let E = xA + yB + zC ………… (1), where x, y, z are scalar in F.

Then, $\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} = x\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} + y\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} + z\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

⇨ $\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} x & x \\ 0 & -x \end{pmatrix} + \begin{pmatrix} y & y \\ -y & 0 \end{pmatrix} + \begin{pmatrix} z & -z \\ 0 & 0 \end{pmatrix}$

⇨ $\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} x+y+z & x+y-z \\ -y & -x \end{pmatrix}$

Now equating corresponding components and forming linear system we get,

x + y + z = 3
x + y – z = -1
-y = 1 ………….. (2)
-x = -2

Now solving (2) we get x = 2, y = -1, z = 2

∴ E = 2A – B + 2C

i.e. E is a linear combination of A, B, and C.

H.W

1. Show that the matrix E = $\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}$ cannot be express as a linear combination of

A = $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, B = $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ and C = $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$.

Solution:

Let, E = xA + yB + zC ……….. (1) where x, y, z are scalar in F.

Then, $\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} = x\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + y\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} + z\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

=> $\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} x & x \\ 0 & -x \end{pmatrix} + \begin{pmatrix} y & y \\ -y & 0 \end{pmatrix} + \begin{pmatrix} z & -z \\ 0 & 0 \end{pmatrix}$

=> $\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} x+y+z & x+y-z \\ -y & x \end{pmatrix}$

Now equating the corresponding component and forming linear system we get,

$$\left.\begin{aligned} x + y + z &= 3 \\ x + y - z &= -1 \\ -y &= 1 \\ x &= -2 \end{aligned}\right] \quad \text{…………. (2)}$$

Now solving (2) we get x = -2, y = -1, z = - 2 and z = 6, which is impossible.

So, the equation (2) has no solution or inconstant. Therefore matrix E cannot be express as a linear combination of A, B and C.

2. Express the matrix E = $\begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix}$ as a linear combination of A = $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, B = $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and C = $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Solution:

Let, E = xA + yB + zC ……….. (1) Where x, y, z are scalar in F.

Then, $\begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix} = x\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + y\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} + z\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

=> $\begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} x & 0 \\ x & 0 \end{pmatrix} + \begin{pmatrix} y & -y \\ 0 & y \end{pmatrix} + \begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix}$

=> $\begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} x+y & -y+z \\ x-z & y \end{pmatrix}$

Now equating the corresponding component and forming linear system we

get,

$$\left.\begin{aligned} x + y &= 5 \\ -y + z &= 1 \\ x - z &= -2 \end{aligned}\right] \quad \text{.............(2)}$$

y = 3

Now solving (2) we get x = 2, y = 3, z = 4

∴ E = 2A + 3B + 4C

i.e. E is a linear combination of A, B, and C.

***Show that the matrix E $=\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}$ can not be express of a linear combination of A $=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ B$=\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ and C$=\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

Solution: Let E=xA+yB+zC…(1) where, x,y and z are scaler in F.

Then $\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix} = x\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + y\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} + z\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

Or, $\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}, = \begin{pmatrix} x & x \\ 0 & x \end{pmatrix} + \begin{pmatrix} y & y \\ -y & 0 \end{pmatrix} + \begin{pmatrix} z & -z \\ 0 & 0 \end{pmatrix}$

Or, $\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} x+y+z & x+y-z \\ -y & x \end{pmatrix}$

Now equation the corresponding component and forming linear system we get,

$$\begin{cases} x+y+z=2 \\ x+y-z=1 \\ -y=-1 \\ x=-2 \end{cases} \quad \text{equation …….(2)}$$

Now solving (2) we get x=-2, y=1, and z=-2, z=3 which is impossible. So the equation (2) has no solution or inconsistent. Therefore Matrix E cannot be express as a linear combination of A,B, and C.

Generator/Linear Span: If S is a non-empty subset of a vector space V, then L(S) is the linear span or generator of S in the set of all linear combination of finite sets of elements of S.

***Show that the vectors u=(1,2,3), v=(0,1,2) and w=(0,0,1) generate $\mathbf{R^3}$. Or show that [(1,2,3),(0,1,2),(0,0,1)= $\mathbf{R^3}$.

Solution: We must determine whether an arbitrary vector $V_3(R)$ =(a,b,c) in $\mathbf{R^3}$ can be expressed as a linear combination $V_3(R)$ = xu+yv+zw of the vectors u,v and w , where x,y and z are scalars . Now expressing this equation in terms of components gives

$$
\begin{aligned}
(a+b+c) &= x(1,2,3)+y(0,1,2)+z(0,0,1) \\
&= (x,2x,3x) + (0,y,2y) + (0,0,1) \\
&= (x, 2x+y, 3x+2y+z)
\end{aligned}
$$

Equation corresponding components and forming the linear system we get,

$$
\begin{array}{lcl}
x=0 & & Z+2y+3x=c \\
2x+y=b & \Rightarrow & y+2x=b \\
3x+2y+z=c & & x=a
\end{array}
$$

The above system is echelon form and is consistant . In fact the system has the solution x=a, y= b-2a, z= c-2b+a

Thus, u,v and w generate (Span) $\mathbf{R^3}$.