

ASSIGNMENT#02



COMPUTER SCIENCE & ENGINEERING 1ST YEAR 2ND SEMESTER

LINEAR ALGEBRA & DIFFERENTIAL EQUATION COURSE & CODE : MAT-1231

GROUP: 01

TOPIC: LINEAR TRANSFORMATION & BASIS AND DIMENSION

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Basis and Dimension

Basis:

Let, \mathbf{V} be a vector space and $\{\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_n\}$ Is a finite set of vectors in \mathbf{V} .

We call $\{\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_n\}$ a basis for \mathbf{V} if and only if

(I) $\{\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_n\}$ is linearly independent

(II) $\{\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_n\}$ Spans \mathbf{V} .

Example: $\{(1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, 0, \dots, 1)\}$ standard basis.

Dimension:

The number of vectors in any basis of a finite dimensional vector space (\mathbf{V}) is called dimension.

Or, equivalently, the dimension of a vector space is equal to the maximum number of linearly independent vectors contained in it.

Theorem:

Every basis of a finite dimensional vector space has the same number of vectors.

Proof:

Let, $\mathbf{S} = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ and $\mathbf{T} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be two basis of finite dimensional vector space $\mathbf{V(F)}$.

We have to show that $\mathbf{m=n}$.

Since \mathbf{S} and \mathbf{T} the basis of $\mathbf{V(F)}$ then,

(i) both \mathbf{S} and \mathbf{T} are the generator of $\mathbf{V(F)}$

(ii) both \mathbf{S} and \mathbf{T} are linearly independent

Now, if \mathbf{S} is the generator of $\mathbf{V(F)}$ and \mathbf{T} is linearly independent, then by Exchange lemma,

$$\mathbf{m} \geq \mathbf{n} \dots\dots\dots (1)$$

Similarly, if \mathbf{S} is linearly independent and \mathbf{T} is the generator of $\mathbf{V(F)}$, we have from the same lemma,

$$\mathbf{n} \geq \mathbf{m} \dots\dots\dots (2)$$

From equation (1) and (2), we get $\mathbf{m=n}$.

[Proved].

Theorem:

If U and W be the two finite dimensional subspace of $V(F)$,
Then $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$.

Proof:

Since U and W are subspaces, $U \cap W$ will be the subspace of both U and W .

Let $\dim U = m$, $\dim W = n$ and $\dim(U \cap W) = r$. Let $\{v_1, v_2, \dots, v_r\}$ be a basis of $(U \cap W)$.

Now extend the basis of U and W in the form $\{v_1, v_2, \dots, v_r, u_1, u_2, \dots, u_{m-r}\}$ and $\{v_1, v_2, \dots, v_r, w_1, w_2, \dots, w_{n-r}\}$ respectively.

So, the union of the basis of U and W is A (say).

$$A = \{v_1, v_2, \dots, v_r, u_1, u_2, \dots, u_{m-r}, w_1, w_2, \dots, w_{n-r}\} \dots\dots\dots (1)$$

A has exactly $r + m - r + n = m + n - r$ elements which proves the theorem.

Now, we have to show that A is a basis of $U + W$.

Now, since $\{v_i, u_j\}$ generates U and $\{v_i, w_k\}$ generates W then the union $A = \{v_i, u_j, w_k\}$ will generate $(U + W)$.

Now, we have to show that they are linearly independent.

Now, $\forall x_1, x_2, \dots, x_r, y_1, y_2, \dots, y_{m-r}, z_1, z_2, \dots, z_{n-r} \in F$

$$\text{Let, } x_1v_1 + x_2v_2 + \dots + x_rv_r + y_1u_1 + y_2u_2 + \dots + y_{m-r}u_{m-r} + z_1w_1 + z_2w_2 + \dots + z_{n-r}w_{n-r} = 0 \dots\dots\dots (2)$$

$$\Rightarrow x_1v_1 + x_2v_2 + \dots + x_rv_r + y_1u_1 + y_2u_2 + \dots + y_{m-r}u_{m-r} = -z_1w_1 - z_2w_2 - \dots - z_{n-r}w_{n-r} \dots\dots\dots (3)$$

But **L.H.S.** of (3) is a vector of U and **R.H.S.** of (3) is a vector of W . This implies that both belongs to $(U \cap W)$.

Therefore, for any scalars $t_1, t_2, \dots, t_r \in F$

$$-z_1w_1 - z_2w_2 - \dots - z_{n-r}w_{n-r} = t_1v_1 + t_2v_2 + \dots + t_rv_r \\ \Rightarrow t_1v_1 + t_2v_2 + \dots + t_rv_r + z_1w_1 + z_2w_2 + \dots + z_{n-r}w_{n-r} = 0 \dots\dots\dots (4)$$

But $\{v_1, v_2, \dots, v_r, w_1, w_2, \dots, w_{n-r}\}$ is a basis of W and is linearly independent.

$$\Rightarrow z_1 = z_2 = \dots = z_{n-r} = t_1 = t_2 = \dots = t_r = 0$$

Then (2) becomes,

$$x_1 v_1 + x_2 v_2 + \dots + x_r v_r + y_1 u_1 + y_2 u_2 + \dots + y_{m-r} u_{m-r} = 0 \dots\dots\dots (5)$$

But $\{v_1, v_2, \dots, v_r, u_1, u_2, \dots, u_{m-r}\}$ is a basis of U and so linearly independent.

Then $x_1 = x_2 = \dots = x_r = y_1 = y_2 = \dots = y_{m-r} = 0$

$\Rightarrow A = \{v_i, u_j, w_k\}$ is linearly independent

$\Rightarrow A$ is a basis of $U + W$

$\Rightarrow \dim(U + W) = m + n - r$

$\therefore \dim(U + W) = \dim U + \dim W - \dim(U \cap W)$

[Proved]

Problem Solving:

Problem#01:

Let, V be the subspace of \mathbb{R}^3 spanned by the vectors $(1, 2, 1)$, $(0, -1, 0)$ and $(2, 0, 2)$. Find basis and the dimension of V .

Solⁿ :

Form the matrix whose rows are given vectors and reduce the matrix to row echelon form

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

We multiply first row by 2 and then subtract from the third row.

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -4 & 0 \end{bmatrix}$$

We multiply second row by 4 and subtract from the third row

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We multiply second row by 2 and add with the first row

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We multiply second row by -1

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is in echelon form and the non-zero rows in the matrix are **(1, 0, 1)** and **(0, 1, 0)**. These non-zero rows form a basis of the row space and consequently a basis of **V**; that is Basis of **V**={(1, 0, 1), (0, 1, 0)} and dim **U=2**.

[Answer].

Problem #02:

Find the basis and dimension of the vector set,

$$S = \{(-1, 2, -1, 0), (0, 3, 1, 2), (1, 1, -2, 2), (2, 1, 0, -1)\}$$

Solⁿ:

Form the matrix whose rows are given vectors and reduce the matrix to row echelon form.

$$\begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 2 \\ 1 & 1 & -2 & 2 \\ 2 & 1 & 0 & -1 \end{bmatrix}$$

$$\begin{array}{l} R'_3 \rightarrow R_3 + R_1 \\ R'_4 \rightarrow R_4 + 2R_1 \end{array} \sim \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 2 \\ 0 & 3 & -3 & 2 \\ 0 & 5 & -2 & -1 \end{bmatrix}$$

$$\begin{array}{l} R'_3 \rightarrow R_3 - R_2 \\ R'_4 \rightarrow 3R'_4 - 5R_2 \end{array} \sim \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & -11 & -13 \end{bmatrix}$$

$$R'_4 \rightarrow 4R_4 - 11R_3 \sim \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & -0 & -52 \end{bmatrix}$$

This matrix is in echelon form and the non-zero rows in the matrix are, **(-1, 2, -1, 0)**, **(0, 3, 1, 2)**, **(0, 0, -4, 0)** and **(0, 0, 0, -52)**.

These non-zero rows form a basis of the row space and consequently a basis of \mathbf{S} ; that is Basis of $\mathbf{S} = \{(-1, 2, -1, 0), (0, 3, 1, 2), (0, 0, -4, 0), (0, 0, 0, -52)\}$
And $\dim \mathbf{S} = 4$.

[Answer].

H.W

Problem #01:

Find the basis and dimension of the vector sets,

$$\mathbf{V} = \{(1, -2, 4, 1), (2, -3, 9, -1), (1, 0, 6, -5), (2, -5, 7, 5)\}$$

Solⁿ:

Form the matrix whose rows are given vectors and reduce the matrix to row echelon form.

$$\begin{array}{l} \begin{matrix} R'_2 \rightarrow R_2 - 2R_1 \\ R'_3 \rightarrow R_3 - R_1 \\ R'_4 \rightarrow R_4 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & -2 & 4 & 1 \\ 2 & -3 & 9 & -1 \\ 1 & 0 & 6 & -5 \\ 2 & -5 & 7 & 5 \end{bmatrix} \\ \\ \begin{matrix} R'_3 \rightarrow R_3 - 2R_2 \\ R'_4 \rightarrow R_4 + R_2 \end{matrix} \sim \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 2 & -6 \\ 0 & -1 & -1 & 3 \end{bmatrix} \\ \\ \begin{matrix} R'_3 \rightarrow R_3 - 2R_2 \\ R'_4 \rightarrow R_4 + R_2 \end{matrix} \sim \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

This matrix is in echelon form and the non-zero rows in the matrix are $(1, -2, 4, 1)$ and $(0, 1, 1, -3)$. These non-zero rows form a basis of the row space and consequently a basis of \mathbf{V} ; that is,

Basis of $\mathbf{V} = \{(1, -2, 4, 1), (0, 1, 1, -3)\}$ and $\dim \mathbf{V} = 2$.

[Answer].

Problem #02:

Find the basis and dimension of the vector sets, $W = \{(1, 2, 1), (3, 1, 2), (1, -3, 4)\}$

Solⁿ:

Form the matrix whose rows are given vectors and reduce the matrix to row echelon form.

$$\begin{array}{l}
 \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 1 & -3 & 4 \end{bmatrix} \\
 R'_2 \rightarrow R_2 - 3R_1 \\
 R'_3 \rightarrow R_3 - R_1 \quad \sim \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & -5 & 3 \end{bmatrix} \\
 R'_3 \rightarrow R_3 - R_2 \quad \sim \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & 0 & 4 \end{bmatrix}
 \end{array}$$

This matrix is in echelon form and the non-zero rows in the matrix are $(1, 2, 1)$, $(0, -5, -1)$ and $(0, 0, 4)$. These non-zero rows form a basis of the row space and consequently a basis of W ; that is,

Basis of $W = \{(1, 2, 1), (0, -5, -1), (0, 0, 4)\}$ and $\dim W = 3$.

[Answer].

Problem #03:

Find the basis and dimension of the vector sets,

$S = \{(1, -2, 0, 0, 3), (2, -5, -3, -2, 6), (0, 5, 15, 10, 0), (2, 6, 18, 8, 6)\}$.

Solⁿ:

Form the matrix whose rows are given vectors and reduce the matrix to row echelon form.

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

$$\begin{array}{l} R'_2 \rightarrow R_2 - 2R_1 \\ R'_4 \rightarrow R_4 - 2R_1 \end{array} \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 5 & 15 & 10 & 0 \\ 0 & 10 & 18 & 8 & 0 \end{bmatrix}$$

$$\begin{array}{l} R'_3 \rightarrow R_3 + 5R_2 \\ R'_4 \rightarrow R_4 + 10R_2 \end{array} \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & -12 & 0 \end{bmatrix}$$

$$R_3 \leftrightarrow R_4 \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 0 & -12 & -12 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in echelon form and the non-zero rows in the matrix are **(1, -2, 0, 0, 3)**, **(0, -1, -3, -2, 0)** and **(0, 0, -12, -12, 0)**. These non-zero rows form a basis of the row space and consequently a basis of **S**; that is,

Basis of **S** = **(1, -2, 0, 0, 3), (0, -1, -3, -2, 0), (0, 0, -12, -12, 0)** and **dim S = 3**.

[Answer].

Problem #04:

Find the basis and dimension of the vector sets,

T = {(1, -2, 5, -3), (2, 3, 1, 4), (3, 8, -3, -5)}.

Solⁿ:

Form the matrix whose rows are given vectors and reduce the matrix to row echelon form.

$$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & 4 \\ 3 & 8 & -3 & -5 \end{bmatrix}$$

$$\begin{array}{l} R'_2 \rightarrow R_2 - 2R_1 \\ R'_3 \rightarrow R_3 - 3R_1 \end{array} \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 10 \\ 0 & 14 & -18 & 4 \end{bmatrix}$$

$$R'_3 \rightarrow R_3 - 2R_2 \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 10 \\ 0 & 0 & 0 & -16 \end{bmatrix}$$

This matrix is in echelon form and the non-zero rows in the matrix are **(1, -2, 5, -3)**, **(0, 7, -9, 10)** and **(0, 0, 0, -16)**.

These non-zero rows form a basis of the row space and consequently a basis of **T**; that is,

Basis of **T** = **{(1, -2, 5, -3), (0, 7, -9, 10), (0, 0, 0, -16)}** and **dim T = 3**.

[Answer].

Problem #05:

Find the basis and dimension of the vector sets, **U** = **{(1, 1, 1), (1, 2, 3), (3, 4, 5)}**.

Solⁿ:

Form the matrix whose rows are given vectors and reduce the matrix to row echelon form.

$$\begin{array}{l} \begin{array}{l} R'_2 \rightarrow R_2 - R_1 \\ R'_3 \rightarrow R_3 - 3R_1 \end{array} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix} \\ \\ R'_3 \rightarrow R_3 - R_2 \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \\ \\ \begin{array}{l} R'_3 \rightarrow R_3 - R_2 \end{array} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

This matrix is in echelon form and the non-zero rows in the matrix are **(1, 1, 1)** and **(0, 1, 2)**. These non-zero rows form a basis of the row space and consequently a basis of **U**; that is,

Basis of **U** = **{(1, 1, 1), (0, 1, 2)}** and **dim U = 2**.

[Answer].

Problem Solving:

Problem #01:

Determine a basis and the dimension for the solution space of the homogeneous system.

$$\begin{aligned}x - 3y + z &= 0 \\2x - 6y + 2z &= 0 \\3x - 9y + 3z &= 0\end{aligned}$$

Solⁿ:

Given system of linear equation,

$$\begin{aligned}x - 3y + z &= 0 \\2x - 6y + 2z &= 0 \\3x - 9y + 3z &= 0\end{aligned}$$

Reduce the system to echelon form. We multiply first equation by **2** and **3** and then subtract from the second and third equations respectively. Then we get

$$\begin{aligned}x - 3y + z &= 0 \\0 &= 0 \\0 &= 0\end{aligned}$$

$$\text{i.e. } x - 3y + z = 0$$

the system is in echelon form and has only one non-zero equation in three unknowns.

So the system has $3 - 1 = 2$ free variable which are **y** and **z**.

Hence the dimension of the solution space is **2**.

Set (i) **y = 1, z = 0** (ii) **y = 0, z = 1**, to obtain the solution.

Solutions $V_1 = (3, 1, 0)$, $V_2 = (-1, 0, 1)$

Hence the set $\{(3, 1, 0), (-1, 0, 1)\}$ is a basis of the solution space.

[Answer].

H.W

Problem #01:

Find the dimension and basis of the solution space of the following homogenous system

$$\begin{aligned}x + 2y - 4z + 3s - t &= 0 \\x + 2y - 2z + 2s + t &= 0 \\2x + 4y - 2z + 3s + 4t &= 0\end{aligned}$$

Solⁿ:

Given system of linear equation,

$$\begin{aligned}x + 2y - 4z + 3s - t &= 0 \\x + 2y - 2z + 2s + t &= 0 \\2x + 4y - 2z + 3s + 4t &= 0\end{aligned}$$

Reducing the system to echelon form,

$$\begin{aligned}L'_2 &\rightarrow L_2 - L_1 \\L'_3 &\rightarrow L_3 - 2L_1\end{aligned} \quad \sim \quad \begin{cases} x + 2y - 4z + 3s - t = 0 \\ 2z - s + 2t = 0 \\ 6z - 3s + 6t = 0 \end{cases}$$

$$\begin{aligned}L'_3 &\rightarrow L_3 - 3L_2 \\L'_1 &\rightarrow L_1 - 2L_2\end{aligned} \quad \sim \quad \begin{cases} x + 2y + s + 3t = 0 \\ 2z - s + 2t = 0 \\ 0 = 0 \end{cases}$$

The system is in echelon form and has **2** non-zero equations in **5** unknowns. So the system has **5 - 2 = 3** free variable which are **y, z, t**.

Let, **y = 1, z = 0, t = 0** then, **x = -2, s = 0**.

y = 0, z = 1, t = 0 then, **x = -2, s = 2**

y = 0, z = 0, t = 1 then, **x = -5, s = 2**

Therefore, **(x, y, z, s, t) = {(-2, 1, 0, 0, 0), (-2, 0, 1, 2, 0), (-5, 0, 0, 2, 1)}**.

Hence a basis of the solution space is **{(-2, 1, 0, 0, 0), (-2, 0, 1, 2, 0), (-5, 0, 0, 2, 1)}** and dimension of the system is **3**.

[Answer].

Problem #02:

Find the dimension and a basis of the solution space of the following homogeneous system.

$$\begin{aligned}x + y - t &= 0 \\x + 2y + 3z &= 0 \\2x + 3y + 3z + t &= 0\end{aligned}$$

Solⁿ:

Given system of linear equation,

$$\begin{cases} x + y - t = 0 \\ x + 2y + 3z = 0 \\ 2x + 3y + 3z + t = 0 \end{cases}$$

Reducing the system to echelon form we get,

$$\begin{aligned}L'_2 &\rightarrow L_2 - L_1 \\ L'_3 &\rightarrow L_3 - 2L_1\end{aligned} \quad \sim \quad \begin{cases} x + y - t = 0 \\ y + 3z + t = 0 \\ y + 3z + 3t = 0 \end{cases}$$

$$L'_3 \rightarrow L_3 - L_2 \quad \sim \quad \begin{cases} x + y - t = 0 \\ y + 3z + t = 0 \\ 2t = 0 \end{cases}$$

$$\text{i.e.} \quad \sim \quad \begin{cases} x + y - t = 0 \\ y + 3z + t = 0 \\ t = 0 \end{cases}$$

The system is in echelon form and has **3** non-zero equations in **4** unknowns. So the system has **4 - 3 = 1** free variable which is **z**.

Let, **z = 1**, then **x = 3, y = -3, t = 0**.

Therefore, **(x, y, z, t) = (3, -3, 1, 0)**.

Hence a basis of the solution space is **{(3, -3, 1, 0)}**

and the dimension is **1**.

[Answer].

Problem #03:

Find the dimension and basis of the solution space of the following homogenous system

$$\begin{aligned}x + y + z &= 0 \\x + 2y + 3z &= 0 \\3x + 4y + 5z &= 0\end{aligned}$$

Solⁿ:

Given system of linear equation,

$$\begin{cases} x + y + z = 0 \\ x + 2y + 3z = 0 \\ 3x + 4y + 5z = 0 \end{cases}$$

Reducing the system to echelon form we get,

$$\begin{array}{l} L'_2 \rightarrow L_2 - L_1 \\ L'_3 \rightarrow L_3 - 3L_1 \end{array} \quad \sim \quad \begin{cases} x + y + z = 0 \\ y + 2z = 0 \\ y + 2z = 0 \end{cases}$$

$$\text{i.e.} \quad \sim \quad \begin{cases} x + y + z = 0 \\ y + 2z = 0 \end{cases}$$

The system is in echelon form and has **2** non-zero equations in **3** unknowns. So the system has **3 - 2 = 1** free variable which is **z**.

Let, **z = 1**, then **y = -2, x = 1**.

Therefore, **(x, y, z) = (1, -2, 1)**.

Hence a basis of the solution space is **{(1, -2, 1)}** and the dimension is **1**.

[Answer].

Linear Transformation

Linear Transformation:

Linear Transformation let, \mathbf{U} and \mathbf{V} be two vector spaces over the same field \mathbf{F} . A Linear Transformation \mathbf{T} of \mathbf{U} into \mathbf{V} written as,

$\mathbf{T}:\mathbf{U} \longrightarrow \mathbf{V}$, is a Transformation \mathbf{T} of \mathbf{U} into \mathbf{V} such that

1. $\mathbf{T}(\mathbf{u}_1 + \mathbf{u}_2) = \mathbf{T}(\mathbf{u}_1) + \mathbf{T}(\mathbf{u}_2)$ for all $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{U}$
2. $\mathbf{T}(\alpha \mathbf{u}) = \alpha \mathbf{T}(\mathbf{u})$ for all $\mathbf{u} \in \mathbf{U}$ and all $\alpha \in \mathbf{F}$

Kernel of a Linear Transformation / mapping:

Let $\mathbf{T}: \mathbf{V}(\mathbf{F}) \longrightarrow \mathbf{U}(\mathbf{F})$ be a linear transformation, Then Kernel of transformation or KerT is defined by $\text{KerT} = \{\mathbf{v} \in \mathbf{V}(\mathbf{F}) : \mathbf{T}(\mathbf{v}) = \mathbf{0}\}$

Example: Let $\mathbf{T}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation and defined by $\mathbf{T}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{0}, \mathbf{y}, \mathbf{z})$. here $\text{KerT} = \{(\mathbf{x}, \mathbf{0}, \mathbf{0}) : \mathbf{x} \in \mathbb{R}\} = \mathbf{x} \text{ axis}$.

Image of linear transformation:

Let $\mathbf{T}: \mathbf{V}(\mathbf{F}) \rightarrow \mathbf{U}(\mathbf{F})$ be a linear transformation Then image of transformation or ImT is defined by

$\text{ImT} = \{\mathbf{u} \in \mathbf{U}(\mathbf{F}) : \mathbf{T}(\mathbf{v}) = \mathbf{u}, \mathbf{v} \in \mathbf{V}(\mathbf{F})\}$

Example: Let $\mathbf{T}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation and defined by $\mathbf{T}(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{0})$. Hear $\text{ImT} = \{(\mathbf{x}, \mathbf{0}) : \mathbf{x} \in \mathbb{R}\} = \mathbf{x} \text{ axis}$.

Problem Solving:

Problem#01:

Show that the following transformation defined a linear operator on \mathbb{R}^3 ,
 $\mathbf{T}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{x} + \mathbf{y}, -\mathbf{x} - \mathbf{y}, \mathbf{z})$.

Solⁿ:

Let $\mathbf{U} = (\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1)$ and $\mathbf{V} = (\mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2)$

$$\begin{aligned}\text{Then } \mathbf{U} + \mathbf{V} &= (x_1, y_1, z_1) + (x_2, y_2, z_2) \\ &= (x_1 + x_2, y_1 + y_2, z_1 + z_2)\end{aligned}$$

$$\text{And } \alpha \mathbf{U} = \alpha(x_1, y_1, z_1) = (\alpha x_1, \alpha y_1, \alpha z_1) \text{ Where } \alpha \in \mathbf{F}$$

$$\text{Thus } \mathbf{T}(\mathbf{U}) = \mathbf{T}(x_1, y_1, z_1) = (x_1 + y_1, -x_1 - y_1, z_1)$$

$$\mathbf{T}(\mathbf{V}) = \mathbf{T}(x_2, y_2, z_2) = (x_2 + y_2, -x_2 - y_2, z_2)$$

$$\begin{aligned}\mathbf{T}(\mathbf{U} + \mathbf{V}) &= \mathbf{T}(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= \{ (x_1 + x_2) + (y_1 + y_2), -(x_1 + x_2) - (y_1 + y_2), (z_1 + z_2) \} \\ &= (x_1 + y_1, -x_1 - y_1, z_1) + (x_2 + y_2, -x_2 - y_2, z_2) \\ &= \mathbf{T}(\mathbf{U}) + \mathbf{T}(\mathbf{V})\end{aligned}$$

Also for any $\alpha \in \mathbf{F}$

$$\begin{aligned}\mathbf{T}(\alpha \mathbf{U}) &= \mathbf{T}(\alpha x_1, \alpha y_1, \alpha z_1) \\ &= (\alpha x_1 + \alpha y_1, -\alpha x_1 - \alpha y_1, \alpha z_1) \\ &= \alpha(x_1 + y_1, -x_1 - y_1, z_1) \\ &= \alpha \mathbf{T}(\mathbf{U})\end{aligned}$$

Since \mathbf{U} , \mathbf{V} and α are arbitrary So, \mathbf{T} is a linear operator.

[Showed].

Problem#02:

Let $\mathbf{T}: \mathbb{R}^3 \rightarrow \mathbb{R}^5$ be a defined as $\mathbf{T}(x_1, x_2, x_3) = (x_1 - x_2, 0, x_1 - x_3, x_2, 0)$. Show that \mathbf{T} is a linear transformation.

Solⁿ:

$$\text{Let } \mathbf{U} = (x_1, x_2, x_3) \text{ And } \mathbf{V} = (x'_1, x'_2, x'_3)$$

$$\begin{aligned}\text{Then } \mathbf{T}(\mathbf{U}) + \mathbf{T}(\mathbf{V}) &= \mathbf{T}(x_1, x_2, x_3) + \mathbf{T}(x'_1, x'_2, x'_3) \\ &= (x_1 - x_2, 0, x_1 - x_3, x_2, 0) + (x'_1 - x'_2, 0, x'_1 - x'_3, x'_2, 0) \\ &= (x_1 + x'_1 - x_2 - x'_2, 0, x_1 + x'_1 - x_3 - x'_3, x_2 + x'_2, 0) \dots\dots\dots (1)\end{aligned}$$

$$\begin{aligned}\text{Again } \mathbf{T}(\mathbf{U} + \mathbf{V}) &= \mathbf{T}(x_1 + x'_1, x_2 + x'_2, x_3 + x'_3) \\ &= (x_1 + x'_1 - x_2 - x'_2, 0, x_1 + x'_1 - x_3 - x'_3, x_2 + x'_2, 0) \dots\dots\dots (2)\end{aligned}$$

Now from (1) and (2) We get

$$\mathbf{T}(\mathbf{U}+\mathbf{V}) = \mathbf{T}(\mathbf{U})+\mathbf{T}(\mathbf{V})$$

Again for any scalar $\alpha \in \mathbf{F}$

$$\begin{aligned}\mathbf{T}(\alpha\mathbf{U}) &= \mathbf{T}(\alpha x_1, \alpha x_2, \alpha x_3) \\ &= (\alpha x_1 - \alpha x_2, 0, \alpha x_1 - \alpha x_3, \alpha x_2, 0) \\ &= \alpha (x_1 - x_2, 0, x_1 - x_3, x_2, 0) \\ &= \alpha \mathbf{T}(\mathbf{U})\end{aligned}$$

Since \mathbf{U} , \mathbf{V} and α are arbitrary So, \mathbf{T} is a linear operator.

[Shown].

H.W

Problem #01:

Let $\mathbf{T}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a defined as $\mathbf{T}(x, y) = (x + 2y, 2x - y)$. Show that \mathbf{T} is a linear transformation.

Solⁿ:

Let, $\mathbf{U} = (x_1, y_1)$ and $\mathbf{V} = (x_2, y_2)$

$$\begin{aligned}\text{Then, } \mathbf{T}(\mathbf{U}) + \mathbf{T}(\mathbf{V}) &= \mathbf{T}(x_1, y_1) + \mathbf{T}(x_2, y_2) \\ &= (x_1 + 2y_1, 2x_1 - y_1) + (x_2 + 2y_2, 2x_2 - y_2) \\ &= (x_1 + 2y_1 + x_2 + 2y_2, 2x_1 - y_1 + 2x_2 - y_2) \\ &\dots\dots\dots (1)\end{aligned}$$

$$\begin{aligned}\mathbf{T}(\mathbf{U}+\mathbf{V}) &= \mathbf{T}(x_1 + x_2, y_1 + y_2) \\ &= \{(x_1 + x_2) + 2(y_1 + y_2), 2(x_1 + x_2) - (y_1 + y_2)\} \\ &= (x_1 + 2y_1 + x_2 + 2y_2, 2x_1 - y_1 + 2x_2 - y_2) \\ &\dots\dots\dots (2)\end{aligned}$$

Now from eq. (1) & (2) we get,

$$\mathbf{T}(\mathbf{U}+\mathbf{V}) = \mathbf{T}(\mathbf{U})+\mathbf{T}(\mathbf{V})$$

Again for any scalar $\alpha \in F$

$$\begin{aligned}T(\alpha U) &= T(\alpha x_1, \alpha y_1) \\&= (\alpha x_1 + 2\alpha y_1, 2\alpha x_1 - \alpha y_1) \\&= \alpha(x_1 + 2y_1, 2x_1 - y_1) \\&= \alpha T(U)\end{aligned}$$

Since U, V and α are arbitrary So, T is a linear operator.

[Showed].

Problem #02:

Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a defined as $T(x, y, z) = (x + 2y, y - z, x + 2z)$. Show that T is a linear transformation.

Solⁿ:

Let $U = (x_1, y_1, z_1)$ and $V = (x_2, y_2, z_2)$

$$\begin{aligned}\text{Then } U+V &= (x_1, y_1, z_1) + (x_2, y_2, z_2) \\&= (x_1 + x_2, y_1 + y_2, z_1 + z_2)\end{aligned}$$

And $\alpha U = \alpha(x_1, y_1, z_1) = (\alpha x_1, \alpha y_1, \alpha z_1)$ Where $\alpha \in F$

Thus, $T(U) = T(x_1, y_1, z_1) = (x_1 + 2y_1, y_1 - z_1, x_1 + 2z_1)$

$$T(V) = T(x_2, y_2, z_2) = (x_2 + 2y_2, y_2 - z_2, x_2 + 2z_2)$$

$$\begin{aligned}T(U+V) &= T(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\&= \{(x_1 + x_2) + 2(y_1 + y_2), (y_1 + y_2) - (z_1 + z_2), (x_1 + x_2) + 2(z_1 + z_2)\} \\&= (x_1 + 2y_1, y_1 - z_1, x_1 + 2z_1) + (x_2 + 2y_2, y_2 - z_2, x_2 + 2z_2) \\&= T(U) + T(V)\end{aligned}$$

Also for any $\alpha \in F$

$$\begin{aligned}T(\alpha U) &= T(\alpha x_1, \alpha y_1, \alpha z_1) \\&= (\alpha x_1 + 2\alpha y_1, \alpha y_1 - \alpha z_1, \alpha x_1 + 2\alpha z_1) \\&= \alpha(x_1 + 2y_1, y_1 - z_1, x_1 + 2z_1) \\&= \alpha T(U)\end{aligned}$$

Since U, V and α are arbitrary So, T is a linear operator.

[Showed].

Problem #03:

Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^5$ be defined as $T(x_1, x_2, x_3) = (x_1 - x_2, 0, x_1 - x_3, x_2, 0)$. Show that T is a linear transformation.

Solⁿ:

Let $U = (x_1, x_2, x_3)$ and $V = (x'_1, x'_2, x'_3)$

$$\begin{aligned} \text{Then } T(U) + T(V) &= T(x_1, x_2, x_3) + T(x'_1, x'_2, x'_3) \\ &= (x_1 - x_2, 0, x_1 - x_3, x_2, 0) + (x'_1 - x'_2, 0, x'_1 - x'_3, x'_2, 0) \\ &= (x_1 + x'_1 - x_2 - x'_2, 0, x_1 + x'_1 - x_3 - x'_3, x_2 + x'_2, 0) \dots\dots\dots (1) \end{aligned}$$

$$\begin{aligned} \text{Again } T(U+V) &= T(x_1 + x'_1, x_2 + x'_2, x_3 + x'_3) \\ &= (x_1 + x'_1 - x_2 - x'_2, 0, x_1 + x'_1 - x_3 - x'_3, x_2 + x'_2, 0) \dots\dots\dots (2) \end{aligned}$$

Now from (1) and (2) We get,

$$T(U+V) = T(U) + T(V)$$

Again for any scalar $\alpha \in F$

$$\begin{aligned} T(\alpha U) &= T(\alpha x_1, \alpha x_2, \alpha x_3) \\ &= (\alpha x_1 - \alpha x_2, 0, \alpha x_1 - \alpha x_3, \alpha x_2, 0) \\ &= \alpha (x_1 - x_2, 0, x_1 - x_3, x_2, 0) \\ &= \alpha T(U) \end{aligned}$$

Since U, V and α are arbitrary So, T is a linear operator.

[Shown].

Problem #04:

Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined as $T(x, y, z) = (x + y, y + z, z + x)$. Show that T is a linear transformation.

Solⁿ:

Let $U = (x_1, y_1, z_1)$ and $V = (x_2, y_2, z_2)$

$$\begin{aligned} \text{Then } U+V &= (x_1, y_1, z_1) + (x_2, y_2, z_2) \\ &= (x_1 + x_2, y_1 + y_2, z_1 + z_2) \end{aligned}$$

And $\alpha \mathbf{U} = \alpha(x_1, y_1, z_1) = (\alpha x_1, \alpha y_1, \alpha z_1)$ Where $\alpha \in F$

Thus, $\mathbf{T}(\mathbf{U}) = \mathbf{T}(x_1, y_1, z_1) = (x_1 + y_1, y_1 + z_1, z_1 + x_1)$

$$\mathbf{T}(\mathbf{V}) = \mathbf{T}(x_2, y_2, z_2) = (x_2 + y_2, y_2 + z_2, z_2 + x_2)$$

$$\begin{aligned}\mathbf{T}(\mathbf{U} + \mathbf{V}) &= \mathbf{T}(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= \{(x_1 + x_2) + (y_1 + y_2), (y_1 + y_2) + (z_1 + z_2), (z_1 + z_2) + (x_1 + x_2)\} \\ &= (x_1 + y_1, y_1 + z_1, z_1 + x_1) + (x_2 + y_2, y_2 + z_2, z_2 + x_2) \\ &= \mathbf{T}(\mathbf{U}) + \mathbf{T}(\mathbf{V})\end{aligned}$$

Also for any $\alpha \in F$

$$\begin{aligned}\mathbf{T}(\alpha \mathbf{U}) &= \mathbf{T}(\alpha x_1, \alpha y_1, \alpha z_1) \\ &= (\alpha x_1 + \alpha y_1, \alpha y_1 + \alpha z_1, \alpha z_1 + \alpha x_1) \\ &= \alpha(x_1 + y_1, y_1 + z_1, z_1 + x_1) \\ &= \alpha \mathbf{T}(\mathbf{U})\end{aligned}$$

Since \mathbf{U} , \mathbf{V} and α are arbitrary So, \mathbf{T} is a linear operator.

[Shown].

Problem Solving:

Problem#01:

Let $\mathbf{T}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation when $\mathbf{T}(1, 1) = 3$ and $\mathbf{T}(0, 1) = -2$ then find $\mathbf{T}(a, b)$.

Solⁿ:

Here $\{(1, 1), (0, 1)\}$ is a basis of \mathbb{R}^2 .

$$\text{Let, } (a, b) = x(1, 1) + y(0, 1) = (x, x+y)$$

$$\Rightarrow x = a, x+y = b$$

$$\therefore x = a, y = b - a$$

Now Since \mathbf{T} is Linear transformation,

$$\mathbf{T}(a, b) = x \mathbf{T}(1, 1) + y \mathbf{T}(0, 1) = 3x - 2y = 3a - 2(b-a) = 5a - 2b$$

[Answer].

Problem#02:

Let $T: V_2(\mathbb{R}) \rightarrow V_3(\mathbb{R})$ be a linear transformation Where $T(1,2) = (3, -1, 5)$ and $T(0,1) = (2, 1, -1)$ Then find $T(a, b)$.

Solⁿ:

Here $\{(1, 2), (0, 1)\}$ is a basis of $V_2(\mathbb{R})$

Now let,

$$(a, b) = x(1, 2) + y(0, 1) = (x, 2x+y)$$

$$\Rightarrow x = a, \quad 2x + y = b$$

$$\therefore x = a, \quad y = b-2a$$

Now using the condition of given linear transformation we get,

$$\begin{aligned} T(a, b) &= x T(1, 2) + y T(0, 1) \\ &= x (3, -1, 5) + y (2, 1, -1) \\ &= (3x, -x, 5x) + (2y, y, -y) \\ &= (3x+2y, -x+y, 5x-y) \\ &= (2b-a, b-3a, 7a-b) \end{aligned}$$

[Answer].

H.W

Problem#01:

Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear mapping, Where $T(0, 1) = (0, 0)$ and $T(1, 1) = (1, 1)$ Then find $T(a, b)$.

Solⁿ:

Here. $\{(0, 1), (1, 1)\}$ is a basis of \mathbb{R}^2

$$\text{Let, } (a, b) = x(0, 1) + y(1, 1) = (y, x+y)$$

$$\Rightarrow y=a, \quad x+y=b$$

$$\therefore y=a, \quad x=b-a$$

Now, using the condition of given linear transformation we get,

$$\mathbf{T(a, b)} = \mathbf{x. T(0, 1) + y. T(1, 1)}$$

$$= \mathbf{x(0, 0) + y(1, 1)}$$

$$= \mathbf{0 + (y, y)}$$

$$= \mathbf{(a, a)}$$

[Answer].

$$\mathbf{T}(e_3) = T(0, 0, 1) = (0, 0, 1) = \mathbf{0}.e_1 + \mathbf{0}.e_2 + \mathbf{1}.e_3$$

Therefore the matrix representation of \mathbf{T} is

$$[T]_e = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^t = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[Answer].

Problem#02:

Let $\mathbf{T}: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ is a linear transformation which define by

$\mathbf{T}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}) = (\mathbf{x}-\mathbf{y}+\mathbf{z}, \mathbf{x}+2\mathbf{y}, \mathbf{z}-\mathbf{t})$. Then find Matrix representation of \mathbf{T} with respect to the standard basis of \mathbb{R}^4 and \mathbb{R}^3 .

Solⁿ:

Let $\{(\mathbf{e}_1 = (1, 0, 0, 0), \mathbf{e}_2 = (0, 1, 0, 0), \mathbf{e}_3 = (0, 0, 1, 0), \mathbf{e}_4 = (0, 0, 0, 1))\}$ and $\{(\mathbf{f}_1 = (1, 0, 0), \mathbf{f}_2 = (0, 1, 0), \mathbf{f}_3 = (0, 0, 1))\}$ are standard basis of \mathbb{R}^4 and \mathbb{R}^3 respectively.

Then,

$$\mathbf{T}(\mathbf{e}_1) = T(1, 0, 0, 0) = (1, 1, 0) = 1. \mathbf{f}_1 + 1. \mathbf{f}_2 + 0. \mathbf{f}_3$$

$$\mathbf{T}(\mathbf{e}_2) = T(0, 1, 0, 0) = (-1, 2, 0) = -1. \mathbf{f}_1 + 2. \mathbf{f}_2 + 0. \mathbf{f}_3$$

$$\mathbf{T}(\mathbf{e}_3) = T(0, 0, 1, 0) = (1, 0, 1) = 1. \mathbf{f}_1 + 0. \mathbf{f}_2 + 1. \mathbf{f}_3$$

$$\mathbf{T}(\mathbf{e}_4) = T(0, 0, 0, 1) = (0, 0, -1) = 0. \mathbf{f}_1 + 0. \mathbf{f}_2 - 1. \mathbf{f}_3$$

$$[T]_e^f = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}^t = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

[Answer].

Problem#03:

Let $\mathbf{T}: \mathbf{V}_3(\mathbb{R}) \rightarrow \mathbf{V}_3(\mathbb{R})$ be a linear operator which define by $\mathbf{T}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (2\mathbf{y}+\mathbf{z}, \mathbf{x}-4\mathbf{y}, 3\mathbf{x})$.

Then find Matrix representation for the

basis $\{(\mathbf{f}_1 = (1, 1, 1), \mathbf{f}_2 = (1, 1, 0), \mathbf{f}_3 = (1, 0, 0))\}$.

Solⁿ:

Let $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{V}_3(\mathbb{R}^3)$

$$\begin{aligned}
\text{Then, } (\mathbf{a}, \mathbf{b}, \mathbf{c}) &= x \mathbf{f}_1 + y \mathbf{f}_2 + z \mathbf{f}_3, \text{ Where } x, y, z \in \mathbb{R} \\
&= x (1, 1, 1) + y (1, 1, 0) + z (1, 0, 0) \\
&= (x+y+z, x+y, x) \\
\Rightarrow x+y+z &= a, \quad x+y = b, \quad x = c \\
\therefore x &= c, \quad y = b-c, \quad z = a-b
\end{aligned}$$

$$\text{Then, } (\mathbf{a}, \mathbf{b}, \mathbf{c}) = c \mathbf{f}_1 + (b-c) \mathbf{f}_2 + (a-b) \mathbf{f}_3 \quad \dots\dots\dots (1)$$

$$\text{Again given, } T(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (2\mathbf{y}+\mathbf{z}, \mathbf{x}-4\mathbf{y}, 3\mathbf{x}) \quad \dots\dots\dots (2)$$

Now from (1) and (2) we get

$$\begin{aligned}
T(\mathbf{f}_1) &= T(1, 1, 1) = (3, -3, 3) = 3 \mathbf{f}_1 - 6 \mathbf{f}_2 + 6 \mathbf{f}_3 \\
T(\mathbf{f}_2) &= T(1, 1, 0) = (2, -3, 3) = 3 \mathbf{f}_1 - 6 \mathbf{f}_2 + 5 \mathbf{f}_3 \\
T(\mathbf{f}_3) &= T(1, 0, 0) = (0, 1, 3) = 3 \mathbf{f}_1 - 2 \mathbf{f}_2 - \mathbf{f}_3
\end{aligned}$$

Therefore the matrix representation of T is

$$[T]_f = \begin{bmatrix} 3 & -6 & 6 \\ 3 & -6 & 5 \\ 3 & -2 & -1 \end{bmatrix}^t = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$$

[Answer].

H.W

Problem#01:

Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a linear operator and define by $T(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (2\mathbf{x}+\mathbf{y}, \mathbf{x}-\mathbf{y}, \mathbf{z})$. Then find Matrix representation with respect to the standard basis of \mathbb{R}^3 .

Solⁿ:

Let, $\{\mathbf{e}_1 = (1, 0, 0), \mathbf{e}_2 = (0, 1, 0), \mathbf{e}_3 = (0, 0, 1)\}$ be a standard basis of \mathbb{R}^3 .

Then, from $T(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (2\mathbf{x}+\mathbf{y}, \mathbf{x}-\mathbf{y}, \mathbf{z})$ we get,

$$\begin{aligned}
T(\mathbf{e}_1) &= T(1, 0, 0) = (2, 1, 0) = 2 \cdot \mathbf{e}_1 + 1 \cdot \mathbf{e}_2 + 0 \cdot \mathbf{e}_3 \\
T(\mathbf{e}_2) &= T(0, 1, 0) = (1, -1, 0) = 1 \cdot \mathbf{e}_1 - 1 \cdot \mathbf{e}_2 + 0 \cdot \mathbf{e}_3
\end{aligned}$$

$$\mathbf{T}(\mathbf{e}_3) = \mathbf{T}(0, 0, 1) = (0, 0, 1) = 0 \cdot \mathbf{e}_1 - 0 \cdot \mathbf{e}_2 + 1 \cdot \mathbf{e}_3$$

Therefore the matrix representation of \mathbf{T} is,

$$[\mathbf{T}]_{\mathbf{e}} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^t = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[Answer].

Problem#02:

Let $\mathbf{T}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear operator and define by $\mathbf{T}(\mathbf{x}, \mathbf{y}) = (4\mathbf{x}-2\mathbf{y}, 2\mathbf{x}+\mathbf{y})$. Then find Matrix representation \mathbf{T} with respect to the standard basis of \mathbb{R}^2 .

Solⁿ:

Let, $\{(\mathbf{e}_1 = (1, 0), \mathbf{e}_2 = (0, 1))\}$ be a standard basis of \mathbb{R}^2 .

Then, from $\mathbf{T}(\mathbf{x}, \mathbf{y}) = (4\mathbf{x}-2\mathbf{y}, 2\mathbf{x}+\mathbf{y})$ we get

$$\mathbf{T}(\mathbf{e}_1) = \mathbf{T}(1, 0) = (4, 2) = 4 \cdot \mathbf{e}_1 + 2 \cdot \mathbf{e}_2$$

$$\mathbf{T}(\mathbf{e}_2) = \mathbf{T}(0, 1) = (-2, 1) = -2 \cdot \mathbf{e}_1 + 1 \cdot \mathbf{e}_2$$

Therefore the matrix representation of \mathbf{T} is,

$$[\mathbf{T}]_{\mathbf{e}} = \begin{bmatrix} 4 & 2 \\ -2 & 1 \end{bmatrix}^t = \begin{bmatrix} 4 & -2 \\ 2 & 1 \end{bmatrix}$$

[Answer].

Problem#03:

Let $\mathbf{T}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a linear transformation which is define by

$$\mathbf{T}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = (\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3, 2\mathbf{x}_1 - \mathbf{x}_2 + 3\mathbf{x}_3, -\mathbf{x}_1 + 2\mathbf{x}_2 + \mathbf{x}_3).$$

Then find Matrix representation with respect to the standard basis of \mathbb{R}^3 .

Solⁿ:

Let $\{(\mathbf{e}_1 = (1, 0, 0), \mathbf{e}_2 = (0, 1, 0), \mathbf{e}_3 = (0, 0, 1))\}$ be a standard basis of \mathbb{R}^3 .

Then from $\mathbf{T}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = (\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3, 2\mathbf{x}_1 - \mathbf{x}_2 + 3\mathbf{x}_3, -\mathbf{x}_1 + 2\mathbf{x}_2 + \mathbf{x}_3)$ we get

$$\mathbf{T}(\mathbf{e}_1) = \mathbf{T}(1, 0, 0) = (1, 2, -1) = 1 \cdot \mathbf{e}_1 + 2 \cdot \mathbf{e}_2 - 1 \cdot \mathbf{e}_3$$

$$\mathbf{T}(\mathbf{e}_2) = \mathbf{T}(0, 1, 0) = (-1, -1, 2) = -1 \cdot \mathbf{e}_1 - 1 \cdot \mathbf{e}_2 + 2 \cdot \mathbf{e}_3$$

$$\mathbf{T}(\mathbf{e}_3) = \mathbf{T}(0, 0, 1) = (1, 3, 1) = \mathbf{1} \cdot \mathbf{e}_1 + \mathbf{3} \cdot \mathbf{e}_2 + \mathbf{1} \cdot \mathbf{e}_3$$

Therefore the matrix representation of \mathbf{T} is

$$[\mathbf{T}]_{\mathbf{e}} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & -\mathbf{1} \\ -\mathbf{1} & -\mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{3} & \mathbf{1} \end{bmatrix}^t = \begin{bmatrix} \mathbf{1} & -\mathbf{1} & \mathbf{1} \\ \mathbf{2} & -\mathbf{1} & \mathbf{3} \\ -\mathbf{1} & \mathbf{2} & \mathbf{1} \end{bmatrix}$$

[Answer].

Problem#04:

Let $\mathbf{T}: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ is a linear transformation which define by

$\mathbf{T}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}) = (\mathbf{x}-\mathbf{y}+\mathbf{z}, \mathbf{x}+\mathbf{y}, \mathbf{y}-\mathbf{t})$. Then find Matrix representation of \mathbf{T} with respect to the standard basis of \mathbb{R}^4 and \mathbb{R}^3 .

Solⁿ:

Let, $\{(\mathbf{e}_1 = (1, 0, 0, 0), \mathbf{e}_2 = (0, 1, 0, 0), \mathbf{e}_3 = (0, 0, 1, 0), \mathbf{e}_4 = (0, 0, 0, 1))\}$ and $\{(\mathbf{f}_1 = (1, 0, 0), \mathbf{f}_2 = (0, 1, 0), \mathbf{f}_3 = (0, 0, 1))\}$ are standard basis of \mathbb{R}^4 and \mathbb{R}^3 respectively.

Then from $\mathbf{T}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}) = (\mathbf{x}-\mathbf{y}+\mathbf{z}, \mathbf{x}+\mathbf{y}, \mathbf{y}-\mathbf{t})$ we get,

$$\mathbf{T}(\mathbf{e}_1) = \mathbf{T}(1, 0, 0, 0) = (1, 1, 0) = \mathbf{1} \cdot \mathbf{f}_1 + \mathbf{1} \cdot \mathbf{f}_2 + \mathbf{0} \cdot \mathbf{f}_3$$

$$\mathbf{T}(\mathbf{e}_2) = \mathbf{T}(0, 1, 0, 0) = (-1, 1, 1) = -\mathbf{1} \cdot \mathbf{f}_1 + \mathbf{1} \cdot \mathbf{f}_2 + \mathbf{1} \cdot \mathbf{f}_3$$

$$\mathbf{T}(\mathbf{e}_3) = \mathbf{T}(0, 0, 1, 0) = (1, 0, 0) = \mathbf{1} \cdot \mathbf{f}_1 + \mathbf{0} \cdot \mathbf{f}_2 + \mathbf{0} \cdot \mathbf{f}_3$$

$$\mathbf{T}(\mathbf{e}_4) = \mathbf{T}(0, 0, 0, 1) = (0, 0, -1) = \mathbf{0} \cdot \mathbf{f}_1 + \mathbf{0} \cdot \mathbf{f}_2 - \mathbf{1} \cdot \mathbf{f}_3$$

Therefore the matrix representation of \mathbf{T} is,

$$[\mathbf{T}]_{\mathbf{e}}^{\mathbf{f}} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} \\ -\mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{1} \end{bmatrix}^t = \begin{bmatrix} \mathbf{1} & -\mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & -\mathbf{1} \end{bmatrix}$$

[Answer].

Problem#05:

Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ is a linear transformation which define by

$T(x, y, z) = (x+y, x+z, x-y, x-z)$. Then find Matrix representation of T with respect to the standard basis of \mathbb{R}^3 and \mathbb{R}^4 .

Solⁿ:

Let, $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ and

$\{f_1 = (1, 0, 0, 0), f_2 = (0, 1, 0, 0), f_3 = (0, 0, 1, 0), f_4 = (0, 0, 0, 1)\}$ are standard basis of \mathbb{R}^3 and \mathbb{R}^4 respectively.

Then from $T(x, y, z) = (x+y, x+z, x-y, x-z)$ we get,

$$T(e_1) = T(1, 0, 0) = (1, 1, 1, 1) = 1.f_1 + 1.f_2 + 1.f_3 + 1.f_4$$

$$T(e_2) = T(0, 1, 0) = (1, 0, -1, 0) = 1.f_1 + 0.f_2 - 1.f_3 + 0.f_4$$

$$T(e_3) = T(0, 0, 1) = (0, 1, 0, -1) = 0.f_1 + 1.f_2 + 0.f_3 - 1.f_4$$

Therefore the matrix representation of T is,

$$[T]_{e,f} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}^t = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

[Answer].

Problem#06:

Find the Matrix representation of T for the given

basis $\{f_1 = (1, 1, 0), f_2 = (1, 0, 1), f_3 = (0, 1, 1)\}$ where T is defined by

$T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$; $T(x, y, z) = (x+y, y+z, z+x)$.

Solⁿ:

Let, $(a, b, c) \in \mathbb{R}^3$

Then $(a, b, c) = x.f_1 + y.f_2 + z.f_3$

$$= x(1, 1, 0) + y(1, 0, 1) + z(0, 1, 1)$$

$$= (x+y, x+z, y+z)$$

$$\Rightarrow x+y = a \quad \text{..... (1)} \qquad x+z = b \quad \text{..... (2)} \qquad y+z = c \quad \text{..... (3)}$$

Now,

$$(1)+(2) \Rightarrow$$

$$2\mathbf{x}+\mathbf{y}+\mathbf{z} = \mathbf{a}+\mathbf{b}$$

$$\Rightarrow 2\mathbf{x}+\mathbf{c} = \mathbf{a}+\mathbf{b}$$

$$\Rightarrow \mathbf{x} = \frac{(\mathbf{a}+\mathbf{b}-\mathbf{c})}{2}$$

From eq. (1)

$$\begin{aligned}\mathbf{y} &= \mathbf{a}-\mathbf{x} \\ &= \mathbf{a}-\frac{\mathbf{a}+\mathbf{b}-\mathbf{c}}{2} \\ \therefore \mathbf{y} &= \frac{\mathbf{a}-\mathbf{b}+\mathbf{c}}{2}\end{aligned}$$

From eq. (2)

$$\begin{aligned}\mathbf{z} &= \mathbf{b}-\mathbf{x} \\ &= \mathbf{b}-\frac{\mathbf{a}+\mathbf{b}-\mathbf{c}}{2} \\ \therefore \mathbf{z} &= \frac{\mathbf{b}-\mathbf{a}+\mathbf{c}}{2}\end{aligned}$$

$$\text{Then, } (\mathbf{a}, \mathbf{b}, \mathbf{c}) = \frac{(\mathbf{a}+\mathbf{b}-\mathbf{c})}{2} \cdot \mathbf{f}_1 + \frac{\mathbf{a}-\mathbf{b}+\mathbf{c}}{2} \cdot \mathbf{f}_2 + \frac{\mathbf{b}-\mathbf{a}+\mathbf{c}}{2} \cdot \mathbf{f}_3 \dots\dots\dots (4)$$

$$\text{Again given, } \mathbf{T}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{x}+\mathbf{y}, \mathbf{y}+\mathbf{z}, \mathbf{z}+\mathbf{x}) \dots\dots\dots (5)$$

Now from,

$$\mathbf{T}(\mathbf{f}_1) = \mathbf{T}(1, 1, 0) = (2, 1, 1) = 1 \cdot \mathbf{f}_1 + 1 \cdot \mathbf{f}_2 + 0 \cdot \mathbf{f}_3$$

$$\mathbf{T}(\mathbf{f}_2) = \mathbf{T}(1, 0, 1) = (1, 1, 2) = 0 \cdot \mathbf{f}_1 + 1 \cdot \mathbf{f}_2 + 1 \cdot \mathbf{f}_3$$

$$\mathbf{T}(\mathbf{f}_3) = \mathbf{T}(0, 1, 1) = (1, 2, 1) = 1 \cdot \mathbf{f}_1 + 0 \cdot \mathbf{f}_2 + 1 \cdot \mathbf{f}_3$$

Therefore the matrix representation of \mathbf{T} is

$$[\mathbf{T}]_{\mathbf{f}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}^t = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{[Answer].}$$

Problem#07:

Let $\mathbf{T}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a linear operator and which is define by

$\mathbf{T}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{x}+2\mathbf{y}, \mathbf{x}, \mathbf{y}-\mathbf{z})$. Find the Matrix representation of \mathbf{T} for the given basis $\{(\mathbf{f}_1 = (1, 1, 1), \mathbf{f}_2 = (0, 1, 1), \mathbf{f}_3 = (0, 0, 1)\}$.

Solⁿ:

Let, $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbb{R}^3$

$$\begin{aligned}\text{Then } (\mathbf{a}, \mathbf{b}, \mathbf{c}) &= \mathbf{x} \cdot \mathbf{f}_1 + \mathbf{y} \cdot \mathbf{f}_2 + \mathbf{z} \cdot \mathbf{f}_3 \\ &= \mathbf{x} (1, 1, 1) + \mathbf{y} (0, 1, 1) + \mathbf{z} (0, 0, 1) \\ &= (\mathbf{x}, \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}+\mathbf{z})\end{aligned}$$

$$\begin{aligned}\Rightarrow \mathbf{x} &= \mathbf{a}, & \mathbf{x}+\mathbf{y} &= \mathbf{b}, & \mathbf{x}+\mathbf{y}+\mathbf{z} &= \mathbf{c} \\ \therefore \mathbf{x} &= \mathbf{a}, & \mathbf{y} &= \mathbf{b}-\mathbf{a}, & \mathbf{z} &= \mathbf{c}-\mathbf{b}\end{aligned}$$

$$\text{Then, } (\mathbf{a}, \mathbf{b}, \mathbf{c}) = \mathbf{a} \cdot \mathbf{f}_1 + (\mathbf{b}-\mathbf{a}) \cdot \mathbf{f}_2 + (\mathbf{c}-\mathbf{b}) \cdot \mathbf{f}_3 \dots\dots\dots (1)$$

$$\text{Again given, } \mathbf{T}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{x}+2\mathbf{y}, \mathbf{x}, \mathbf{y}-\mathbf{z}) \dots\dots\dots (2)$$

Now from (1) and (2)

$$\mathbf{T}(\mathbf{f}_1) = \mathbf{T}(1, 1, 1) = (3, 1, 0) = 3 \cdot \mathbf{f}_1 - 2 \cdot \mathbf{f}_2 + 0 \cdot \mathbf{f}_3$$

$$\mathbf{T}(\mathbf{f}_2) = \mathbf{T}(0, 1, 1) = (2, 0, 0) = 2 \cdot \mathbf{f}_1 - 2 \cdot \mathbf{f}_2 + 0 \cdot \mathbf{f}_3$$

$$\mathbf{T}(\mathbf{f}_3) = \mathbf{T}(0, 0, 1) = (0, 0, -1) = 0 \cdot \mathbf{f}_1 + 0 \cdot \mathbf{f}_2 - 1 \cdot \mathbf{f}_3$$

Therefore the matrix representation of \mathbf{T} is

$$[\mathbf{T}]_{\mathbf{f}} = \begin{bmatrix} 3 & -2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}^t = \begin{bmatrix} 3 & 2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

[Answer].

THANK YOU