School of Science and Technology

B.Sc. in Computer Science and Engineering

Assignment-2

Assignment On: Eigen Values and Eigen Vectors & Linear Dependence and Linear Independence

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Eigen values and Eigen vectors

Definition: If A is a n×n matrix, then a non-zero vector v in $\mathbb{R}^n$ is called Eigen vector of A if Av is a scalar multiple of v, that is, Av = λv …. ….. ….. (1) For some scalar λ. The scalar λ is called an eigen value of A and v is called an eigen vector of A corresponding to λ.

Characteristic Matrix: Let a matrix $A=(a_i)_{n*n}$ and $I_n=I$ be a identity matrix of some order over field F.

Then,

$$\lambda I-A=\lambda\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots\dots & & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}-\begin{pmatrix} a_{11} & a_{12}\dots\dots & a_{13} \\ a_{21} & a_{21}\dots. & a_{21} \\ \dots & & \\ a_{n1} & a_{n1}\dots.. & a_{n1} \end{pmatrix}$$

$$=\begin{pmatrix} \lambda-a_{11} & -a_{12}\dots\dots\dots.. & -a_{1n} \\ -a_{21} & \lambda-a_{22}\dots\dots.. & a_{2n} \\ -a_{n1} & -a_{n2}\dots\dots\dots. & \lambda-a_{nn} \end{pmatrix}$$ is to be characteristic matrix of A

Characteristic Polynomial:

The determiner of matrix λI – A i.e.

$$|\lambda I-A|=\begin{vmatrix} \lambda-a_{11} & -a_{12} & -a_{1n} \\ -a_{21} & \lambda-a_{22} & a_{2n} \\ -a_{n1} & -a_{n2} & \lambda-a_{nn} \end{vmatrix}\dots\dots..(2)$$

is said to be characteristic polynomial.

 # Find the Eigen values and Eigen vectors of the matrix A = $\begin{pmatrix} \mathbf{3} & \mathbf{-1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$

Solution: The characteristic matrix of A is $\lambda I-A=\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} \lambda-3 & 1 \\ -1 & \lambda-1 \end{pmatrix}$

Now the characteristic equation of A is

$$|\lambda I-A|=\begin{vmatrix} \lambda-3 & 1 \\ -1 & \lambda-1 \end{vmatrix}=0$$

⇨ $(\lambda-3)(\lambda-1)+1=0$
⇨ $\lambda^2-4\lambda+3+1=0$
⇨ $\lambda^2-4\lambda+4=0$
⇨ $(\lambda-2)^2=0$
⇨ λ = 2, 2.

This is the Eigen value of A, and λ = 2 is the only one Eigen value of A.

Now by definition V=$\begin{pmatrix} x \\ y \end{pmatrix}$ is an Eigen vector of A corresponding to λ if and only if V is a non-trivial solution of $(\lambda\, \mathrm{I} - \mathrm{A})V = 0$, that is of $\begin{pmatrix} \lambda - 3 & 1 \\ -1 & \lambda - 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

⇨ $\begin{pmatrix} (\lambda - 3)x + y \\ -x(\lambda + 1)y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

⇨ $\begin{pmatrix} (\lambda - 3)x + y = 0 \\ -x(\lambda + 1)y = 0 \end{pmatrix}$ ……………..(1)

Now, for λ = 2, equation (1) => $\begin{pmatrix} -x + y = 0 \\ -x + y = 0 \end{pmatrix}$ => x-y = 0 …………..(2)

∴ the system (2) is consistent and has more than one solution.

Now, Let y= a, then (2) => x = a , y = a. Therefore the Eigen vectors of A corresponding to the Eigen value λ = 2 are non-zero vector of the form V =$\begin{pmatrix} a \\ a \end{pmatrix}$.

In particular, Let, a = 1, then V =$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an Eigen vector corresponding to the Eigen value λ = 2.

 # Find all Eigen values and corresponding Eigen vectors of the matrix.

$$A = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$$

Solution: The characteristic matrix of A is $\lambda \mathrm{I} - \mathrm{A} = \begin{pmatrix} \lambda + 3 & -1 & 1 \\ 7 & \lambda - 5 & 1 \\ 6 & -6 & \lambda + 2 \end{pmatrix}$

Now the characteristic equation is $|\, \lambda \mathrm{I} - \mathrm{A}\, | = \begin{bmatrix} \lambda + 3 & -1 & 1 \\ 7 & \lambda - 5 & 1 \\ 6 & -6 & \lambda + 2 \end{bmatrix}$

⇨ $(\lambda + 3)(\lambda^2 - 3\lambda - 4) + (7\lambda = 8)(-6\lambda - 12) = 0$
⇨ $(\lambda^3 - 13\lambda - 12) + (7\lambda = 8)(-6\lambda - 12) = 0$
⇨ $\lambda^3 - 12\lambda - 16 = 0$
⇨ $(\lambda + 2)^2(\lambda - 4) = 0$
⇨ Λ = -2, -2, 4

Which is the Eigen values of A.

Now, by definition V = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is an Eigen vector of A corresponding to the Eigen value λ if and only if V is a non- trivial solution of (λI − A)V=0 ………… (1)

Now equation 1 becomes $\begin{pmatrix} \lambda + 3 & -1 & 1 \\ 7 & \lambda - 5 & 1 \\ 6 & -6 & \lambda + 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$(\lambda + 3)x - y + z = 0$$
⇨ $7x + (\lambda - 5)y + z = 0 \}$ …………..(2)
$$6x - 6y + (\lambda + 2)z = 0$$

Now, when λ = $\lambda_1 = -2$ then Equation 2 becomes

$$\begin{matrix} x-y+z=0 \\ 7x-7y+z=0 \\ 6x-6y=0 \end{matrix} \quad \underset{\sim}{R_2' = R_2 - R_1} \quad \begin{matrix} x-y+z=0 \\ x-y+z=0 \\ 6x-6y=0 \end{matrix} \quad \sim \begin{matrix} x-y+z=0 \\ x-y=0 \end{matrix} \quad \text{.............. (3)}$$

Therefore the system is consistent and has more than one solution.

Now let y=1, then (3) => x = 1, y = 1 , z = 0.

Therefore for $\lambda_1 = -2$, the corresponding Eigen vector is $v_1 = v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and for that Eigen value corresponding all Eigen vectors are

$kv_1 = k\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} k \\ k \\ 0 \end{pmatrix}$ where k ϵ $\mathbb{R}$ $(k \neq 0)$

Again when λ = $\lambda_2 = 4$ then equation (2) becomes

$$\left.\begin{matrix} 7x-y+z=0 \\ 7x-y+z=0 \\ 6x-6y+6z=0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} 7x-y+z=0 \\ x-y+z=0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} 7x-y+z=0 \\ x=0 \end{matrix}\right\} \text{............(4).}$$

Therefore the system is consistent and has more than one solution.

Now let z=1, then (4) => x = 0, y = 1, z = 1. Therefore for $\lambda_2 = 4$, the corresponding Eigen Vector is $V_2 = V = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and all the Eigen vectors corresponding the Eigen value $\lambda_2 = 4$ are $kv_2 = k\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ k \\ k \end{pmatrix}$, where k ϵ $\mathbb{R}$ $(k \neq 0)$.

Cayley- Hamilton Theorem:

Every square matrix satisfies its own characteristics equation, i.e. if the characteristic equation of the nth order matrix A is

f (λ) = $\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2}+$ …. ….. …….+ $+a_{n-1}\lambda+a_n$=0

then Cayley-Hamilton theorem states that,

f(A)= $A^n + a_nA^{n-1} + a_2A^{n-2}+$ …. ….. …….+ $+a_{n-1}A+a_nI$=0, where I is the nth order unit matrix and 0 is the nth order zero matrix.

Determination of an inverse matrix of a non-singular matrix by Cayley Hamilton theorem:

Let A be a non-singular matrix of n order and the characteristic polynomial is

f (λ) = $\lambda^n + a_{n-1}\lambda^{n-1} +$ …. ….. …….+ $+a_1\lambda+a_0$ …………………(1)

then $A^{-1} = -\frac{1}{a_0}(A^{n-1} + a_{n-1}A^{n-2} + \cdots \dots \dots \dots \dots a_1I)$

Using Cayley- Hamilton theorem find A^{-1} and A^{-2} of the matrix A= $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Solution:

Here the characteristic of A is λI – A =$\begin{pmatrix} \lambda-1 & -1 & -1 \\ -1 & \lambda+1 & -1 \\ -1 & -1 & \lambda+1 \end{pmatrix}$ …………………..(1)

Now the characteristic polynomial of A is

f (λ) = | λI – A | = $\begin{bmatrix} \lambda-1 & -1 & -1 \\ -1 & \lambda+1 & 1 \\ -1 & -1 & \lambda+1 \end{bmatrix}$

= $(\lambda-1)\{(\lambda+1)^2-1\}+(-\lambda-1-1)-1(1+\lambda+1)$

= $(\lambda-1)(\lambda^2+2\lambda)$- λ-2-λ-2

= $\lambda^3-2\lambda^2-\lambda^2-2\lambda-2\lambda-4$

= $\lambda^3+\lambda^2-4\lambda-4$ ………………..(2)

Now using cayley- Hamilton theorem we get

f(A)=0

∴ $A^3+A^2-4A-4=0$

$\Rightarrow\ A^{-1}(A3 + A2 - 4A - 4) = A^{-1}0$

$\Rightarrow\ A^2 + A - 4I - 4A^{-1}=0$

$\Rightarrow\ 4A^{-1} = A^2 + A - 4I$

$\Rightarrow\ A^{-1} = \frac{1}{4}(A^2 + A - 4I)$ ……………..(3)

But $A^2 = A \times A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

$\therefore$ (3) => $A^{-1} = \frac{1}{4}\left\{\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}\right\}$

$= \frac{1}{4}\begin{pmatrix} 3+1-4 & 1+1-0 & 1+1-0 \\ 1+1-0 & 3-1-4 & -1+1-0 \\ 1+1-0 & -1+1-0 & 3-1-4 \end{pmatrix}$

$= \frac{1}{4}\begin{pmatrix} 0 & 2 & 2 \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ Ans.

Again multiplying both sides of (3) by A^{-1}, we get

$A^{-2} = \frac{1}{4}\left\{\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 2 & 2 \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{pmatrix}\right\}$

$= \frac{1}{4}\begin{pmatrix} 1+1 & 1-2 & 1-2 \\ 1-2 & -1+1+2 & 1 \\ 1-2 & 1 & -1+1+2 \end{pmatrix}$

$= \frac{1}{4}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$ Ans.

Linear Dependence and Linear Independence

Linear Dependence: Let v be a vector space over the field F. The vector $v_1, v_2, \ldots v_n \varepsilon v$ are said to be linearly dependent over F or simply dependent if there exists a non-trivial combination of them equal to the zero vector 0.

That is $\alpha_1 v_1 + \alpha_2 v_2 + \ldots.. + \alpha_n v_n = 0$

Where $\alpha_1 \neq 0$ for at least one i.

- ❖ Show that the three vectors (1,3,2),(1,-7,-8),(2,1,-1) are linearly dependent?

  Sol^n :

  Set a linear combination of the given vectors equal to zero by using unknown scalar x,y,z:

  $X(1,3,2)+y(1,-7,-8)+z(2,1,-1)=(0,0,0)$

  $\Rightarrow (x,3x,2x)+(y,-7y,-8y)+(2z,z,-7)=(0,0,0)$

  $\Rightarrow (x+y+2z,3x-7y+z,2x-8y-z)=(0,0,0)$

  Equation corresponding components and forming the linear system, we get

$$\left.\begin{array}{c} x+y+2z=0 \\ 3x-7y+z=0 \\ 2x-8y-z=0 \end{array}\right\}$$

$$\begin{array}{l} L_2' \to L_2-3L_1 \\ L_3' \to L_3-2L_1 \end{array} \qquad \left\{\begin{array}{r} x+y+2z=0 \\ -10y-5z=0 \\ -10y-5z=0 \end{array}\right.$$

$$\left\{\begin{array}{l} x+y+2z=0 \\ 2y+z=0 \end{array}\right.$$

  The system is in echelon form and has only two non zero equation in three unknowns, hence the system has non-zero solution. Thus the original vectors are linearly dependent.

- ❖ Show that the set of vectors {(2,1,2),(0,1,-1),(4,3,3)} is linearly dependent?

Proof: From the matrix whose rows are the given vectors and reduce the matrix to echelon form by using the elementary row operations:

$$\begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 4 & 3 & 3 \end{bmatrix}$$

we multiply first row by 2 and then subtract from the third row,

$$\sim \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

we subtract second row from the third row

$$\sim \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix is in echelon form and has a zero row, hence the vectors are linearly dependent.

Linear Independence: Let v be a vector space over the field F. The vectors $v_1, v_2, \ldots. v_n \varepsilon v$ are said to be linearly independent over F or simply independent if the only linear combination of them equal to 0(zero) is the trivial one.

i.e, $\alpha_1 v_1 + \alpha_1 v_2 + \ldots\ldots + \alpha_n v_n = 0$ if and only if $\alpha_1 = \alpha_2 = \ldots\ldots \alpha_n = 0$.

- Show that the set of vectors {(1,1,-1),(2,1,0),(-1,1,1) is linearly independent.

Proof:

Set a linear combination of the given vectors equal to the zero vector using unknown scalar x,y,z:

X(1,1,-1)+y(2,1,0)+z(-1,1,1)=(0,0,0)

⇨(x,x,-x)+(2y,y,0)+(-z,z,z)=(0,0,0)

⇨(x+2y-z,x+y+z,-x+z)=(0,0,0)

Equating corresponding components and forming the linear system, we get.

$$\left.\begin{aligned} x+2y-z&=0 \\ x+y+z&=0 \\ -x+z&=0 \end{aligned}\right\}$$

$$\begin{matrix} \sim \\ L_2-L_1 \\ L_3+L_1 \end{matrix} \quad \left\{\begin{aligned} x+2y-z&=0 \\ -y+2z&=0 \\ 2y&=0 \end{aligned}\right.$$

In echelon form there are exactly three equations in three unknown, hence the system has only the zero solution x=0,y=0,z=0

Accordingly the vectors are linearly independent.

❖ Show that the vectors (2,-1,4),(3,6,2) and (2,10,-4) are linearly independent.

Proof: From the matrix whose rows are the given vectors and reduce the matrix to echelon form by elementary row operations.

$$\begin{bmatrix} 2 & -1 & 4 \\ 3 & 6 & 2 \\ 2 & 10 & -4 \end{bmatrix}$$

We divide third row by 2 and then interchange with the first row.

$$\begin{bmatrix} 1 & 5 & -2 \\ 3 & 6 & 2 \\ 2 & -1 & 4 \end{bmatrix}$$

We Multiply first row by 3 and 2 and then subtract from the second and third rows respectively.

$$\begin{bmatrix} 1 & 5 & -3 \\ 0 & -9 & 8 \\ 0 & -11 & 8 \end{bmatrix}$$

we multiply second row by $\frac{11}{9}$ and then subtract from the third row.

$$\begin{bmatrix} 1 & 5 & -2 \\ 0 & -9 & 8 \\ 0 & 0 & -\frac{16}{9} \end{bmatrix}$$

since the echelon matrix has no zero row.

$\therefore$ The vectors are linearly independent.

❖ *Linear spans/ Generate:*

Let $v \varepsilon R^n$ be a vector space and $\{v_1, v_2, \ldots v_n\}$ be a vector set. Now if $V = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ where $\alpha_1, \alpha_2 \ldots \alpha_n \varepsilon F$. Then we can say that vector set is the Generator of the vector space R^n and its called linear spans.

✶ Non Zero row and column are linearly independent ✶