* Define vector space and subspace - ①   Vector space ①

* Vector Space:

Definition: Let ~~k~~ F be a given field and let V be a nonempty set with rules of addition and scalar multiplication which assigns to any $u, v \in V$ a sum $u+v \in V$ and to any $u \in V$, $f \in F$ a Product $fu \in V$. Then V is called a vector space over F (and the elements of V are called vector) if the following axioms hold.

A(1) Addition is Commutative
For all vectors $u, v \in V$, $u+v = v+u$

A (2) Addition is associative
For all vectors $u, v, w \in V$, $(u+v)+w = u+(v+w)$

A (3) Existence of 0 (zero vector)
There exists a vector $0 \in V$ such that for all $u \in V$, $u+0 = 0+u = u$.

A (4) Existence of negative
For each $u \in V$ there is a vector $-u \in V$ for which $u+(-u) = (-u)+u = 0$

M(1) For any scalar $\alpha \in F$ and any vectors $u, v \in V$
$\alpha(u+v) = \alpha u + \alpha v$

M(2) For any scalars ~~α~~ $\alpha, \beta \in F$ and any vector $u \in V$
$(\alpha+\beta)u = \alpha u + \beta u$,

M(3) For any scalars $\alpha, \beta \in F$ and any vector u
$(\alpha\beta)u = \alpha(\beta u)$.

M (4) For each $u \in V$, $1u = u$
where 1 is the unite ~~sela~~ scalar and $1 \in F$.

* Subspace: Let W be a subset of a vector space V over a field F. W is called a subspace of V if W is itself a vector space over F with respect to the operations of vector addition and scalar multiplication on V.

SUMON

★ Euclidean space: – R^n is the set all real numbers usual addition and multiplication. ($R^n \to$ Euclidian space)

Q: Prove that for each positive integer n, Euclidean space R^n is a vector space.

Proof: we shall have to show that R^n satisfies all axioms of a vector space

(i) Let $u = (u_1, u_2, \cdots u_n)$ and $v = (v_1, v_2, \cdots, v_n)$ be in R^n. Then $u+v = (u_1, u_2, \cdots u_n) + (v_1, v_2, \cdots v_n)$

$= (u_1+v_1, u_2+v_2, \cdots, u_n+v_n)$

$= (v_1+u_1, v_2+u_2, \cdots, v_n+u_n)$.

So $A(1)$ is true.

(ii) Let $u = (u_1, u_2, \cdots, u_n)$, $v = (v_1, v_2, \cdots, v_n)$ and $w = (w_1, w_2, \cdots w_n)$ be in R^n. Then

$(u+v)+w = (u_1+v_1, u_2+v_2, \cdots u_n+v_n) + (w_1, w_2, \cdots, w_n)$

$= (u_1+v_1+w_1, u_2+v_2+w_2, \cdots, u_n+v_n+w_n)$

$= (u_1, u_2, \cdots, u_n) + (v_1+w_1, v_2+w_2, \cdots, v_n+w_n)$

$= u+(v+w)$. So axiom $A(2)$ holds.

(iii) Let $0 = (0, 0, \cdots, 0)$ be in R^n. Then for any $u = (u_1, u_2, \cdots, u_n)$ in R^n we will have

$u + 0 = \cancel{(u_1+0, u_2+0,} (u_1, u_2, \cdots, u_n) + (0, 0, \cdots, 0)$

$= (u_1+0, u_2+0, \cdots, u_n+0)$

$= (u_1, u_2, \cdots, u_n) = u$. So the axiom $A(3)$ holds.

(iv) Let $u = (u_1, u_2, \dots u_n)$ and set
$-u = (-u_1, -u_2, \dots, -u_n)$. Then
$$u + (-u) = (u_1, u_2, \dots, u_n) + (-u_1, -u_2, \dots -u_n)$$
$$= (u_1 - u_1, u_2 - u_2, \dots u_n - u_n)$$
$$= (0, 0, \dots, 0) = 0.$$ So axiom A(4) holds.

(v) Let α be a real number (scalar) and ~~u = (u1, u2, u~~
$u = (u_1, u_2, \dots u_n)$ and $v = (v_1, v_2, \dots, v_n)$ be vectors in $\mathbb{R}^n$. Then.
$$\alpha(u+v) = \alpha(u_1+v_1, u_2+v_2, \dots, u_n+v_n)$$
$$= (\alpha u_1 + \alpha v_1, \alpha u_2 + \alpha v_2, \dots, \alpha u_n + \alpha v_n)$$
$$= (\alpha u_1, \alpha u_2, \dots, \alpha u_n + \alpha v_1, \alpha v_2, \dots, \alpha v_n)$$
$$= \alpha(u_1, u_2, \dots, u_n) + \alpha(v_1, v_2, \dots, v_n)$$
$$= \alpha u + \alpha v$$
So that axiom M(1) holds.

(vi) Let α, β be that the real numbers (scalars) and $u = (u_1, u_2, \dots, u_n)$ be in $\mathbb{R}^n$. Then
$$(\alpha+\beta)u = ((\alpha+\beta)u_1, (\alpha+\beta)u_2, \dots, (\alpha+\beta)u_n)$$
$$= (\alpha u_1 + \beta u_1, \alpha u_2 + \beta u_2, \dots, \alpha u_n + \beta u_n)$$
$$\cancel{= (\alpha u_1 + \alpha u_2 + \dots, +}$$
$$= (\alpha u_1, \alpha u_2, \dots, \alpha u_n) + (\beta u_1, \beta u_2, \dots, \beta u_n),$$
$$= \alpha(u_1, u_2, \dots, u_n) + \beta(u_1, u_2, \dots, u_n)$$
$$= \alpha u + \beta u$$ So axiom M(2) is satisfied

(vii) Let α, β be real numbers (scalar), and $u = (u_1, u_2, \ldots u_n)$ be in $\mathbb{R}^n$. Then

$$(\alpha\beta)u = \alpha\beta(u_1, u_2, \ldots, u_n) = (\alpha\beta u_1, \alpha\beta u_2, \ldots, \alpha\beta u_n)$$
$$= \alpha(\beta u_1, \beta u_2, \ldots, \beta u_n)$$
$$= \alpha(\beta(u_1, u_2, \ldots, u_n))$$
$$= \alpha(\beta u)$$

So axiom M(3) holds.

(viii) Let 1 be the unit scalar and $u = (u_1, u_2, \ldots u_n)$ be in $\mathbb{R}^n$. Then

$$1u = 1(u_1, u_2, \ldots, u_n) = (1u_1, 1u_2, \ldots, 1u_n)$$
$$= (u_1, u_2, \ldots u_n) = u$$

So axiom M(4) is satisfied.

There fore $\mathbb{R}^n$ is a vector space.

=x= (Hence Proved)

* Prove that W is not a subspace of ~~$\mathbb{R}$~~ $\mathbb{R}^3$ where $W = \{(a, b, 1) : a, b \in \mathbb{R}\}$

Proof: Let $V = \mathbb{R}^3$

$W = \{(a, b, 1) : (a, b) \in \mathbb{R}\}$

$\therefore 0 = (0, 0, 0) \notin W$ since the third ~~com~~ Componant vectors in W is 1.

$\therefore W$ is not a subspace of $\mathbb{R}^3$.

(Proved)

Q. State and Proof fundamental thm of Subspace

Fundamental theorem of Subspace ?

Statement: W will be subspace of subset of vector space $V(F)$ iff .

W→

ⓐ W is non-empty, i.e $W \neq \phi$

ⓑ W is closed under addition, i.e $\forall u, w \in W \Rightarrow u+w \in W$

ⓒ W is closed under scalar multiplication, i.e $\forall \alpha \in F, \forall w \in W \Rightarrow \alpha w \in W$.

Proof: First suppose W satisfies ⓐ, ⓑ and ⓒ we have to show that W is a subspace of V.

Now by ⓐ W is non-empty and by ⓑ and ⓒ vector addition and scalars multiplication is well defined in W.

Again the vectors in W belongs to V then following axioms hold in W.

i) $(u+v)+w = (u+(v+w)$ $\forall u, v, w \in W$.

ii) $u+v = v+u$.

iii) $\alpha(u+v) = \alpha u + \alpha v$ | $\alpha \in F$, $\alpha' \in F$

iv) $(\alpha+\alpha')u = \alpha u + \alpha' u$

v) $(\alpha\alpha')u = \alpha(\alpha' u)$

vi) $1 \in F$ and $u \in W \Rightarrow 1.u = u$,

Now we need to Prove:

vii) $\forall\, 0 \in W$, where $0+u = u+0 = u$, $\forall u \in W$

and viii) $\forall\, u \in W \Rightarrow -u \in W \Rightarrow u+(-u) = (-u)+u = 0$

P.T.O

We will prove (vii): By (a) W is non-empty.
Say $u \in W$, then by (c)
for $0 \in F \Rightarrow 0.u = 0 \in W$
and by (b) $0 + u = u \in W$
$\therefore$ (vii) is proved

We will prove (viii): By (a). W is non-empty
Say $u \in W$, by (c)
for $-1 \in F \Rightarrow (-1) u \in W \Rightarrow -u \in W$
and by (b)
$u + (-u) = 0, \ \forall u \in W$
$\therefore$ (viii) is proved.
Therefore W is satisfied of all condition of vector V. So W is a subspace of subset of vector Space V(F)

Conversely let W is a subspace of vector Space V(F) then W will be satisfy the condition of (a) (b) and (c). Because of (a), (b) and (c) are the part of conditions of vector space.

(Hence Proved)

Sinn
30.10.17

Thm. Let W be a subset of V. then show that W is a subspace of $V(F)$ iff

(i) $0 \in W$, i.e $W \neq \phi$

(ii) For all $\alpha, \beta \in F$ and for all $u, w \in W$

$$\Rightarrow \alpha u + \beta w \in W.$$

Proof: First suppose ~~that~~ the subset W satisfies (i) and (ii).

Now, by (i)

W is non-empty as $0 \in W$.

Again by (ii)

$\Rightarrow \forall \alpha, \beta \in F$ and $\forall u, w \in W$

$\Rightarrow \alpha u + \beta w \in W$

Now Let $\alpha = 1$, and $\beta = 1$ then

$$\alpha u + \beta w = 1.u + 1.w = u + w \in W \; [\because \alpha u + \beta w \in W]$$

$\therefore$ W is closed under vector addition

Again if $w \in W$ and for $\alpha \in F$, we get

$$\alpha w = \alpha w + 0 = \alpha w + 0.w \in W$$

$\therefore$ W is closed under scalar multiplication

Thus W satisfieds the three conditions of the fundamental thm. and therefore W is a subspace of $V(F)$.

Conversely let W is a subspace of vector space $V(F)$ then W will be satisfie the condition of (i) and (ii), Because of (i) and (ii) are the part of condition of vector space.

* Show that x axis and y axis is subspace of vector space $\mathbb{R}^2$.

Proof: [1] Let set of Points on x-axis $U = \{(a,0) : a \in \mathbb{R}\}$ ... (1)
and set of points on y-axis $V = \{(0,b) : b \in \mathbb{R}\}$ ... (2)

Since $\mathbb{R}^2$ is the set of all two dimentional vector space $\therefore U, V \in \mathbb{R}^2$.

For (1)

Since $U \in \mathbb{R}^2$, $0 = (0,0) \in U \Rightarrow U \neq \phi$

Let any two vectors $u = (a_1, 0)$, $v = (a_2, 0) \in U$, ~~and~~ $a_1, a_2 \in \mathbb{R}$, and any two scalar $\alpha, \beta \in F$

$$\therefore \alpha u + \beta v = \alpha(a_1, 0) + \beta(a_2, 0)$$
$$= (\alpha a_1 + \beta a_2, 0+0)$$
$$= (\alpha a_1 + \beta a_2, 0) \in U$$

$\because \alpha a_1 \in \mathbb{R}$
$\beta a_2 \in \mathbb{R}$
$\therefore \alpha a_1 + \beta a_2 \in \mathbb{R}$

Since $\alpha a_1 + \beta a_2 \in \mathbb{R}$ and the second Component is 0

$\therefore U$ is a subspace of $\mathbb{R}^2$.

For (2) since $V \in \mathbb{R}^2$

$$\therefore 0 = (0,0) \in V \Rightarrow V \neq \phi$$

Let any two vectors $u = (0, b_1)$, $v = (0, b_2) \in V$, $b_1, b_2 \in \mathbb{R}$ and any two scalar $\alpha, \beta \in F$

$$\alpha u + \beta v = \alpha(0, b_1) + \beta(0, b_2)$$
$$= (0+0, \alpha b_1 + \beta b_2)$$
$$= (0, \alpha b_1 + \beta b_2) \in V$$

$\because \alpha b_1 \in \mathbb{R}$
$\beta b_2 \in \mathbb{R}$
$\therefore \alpha b_1 + \beta b_2 \in \mathbb{R}$

Since $\alpha b_1 + \beta b_2 \in \mathbb{R}$ and the first Component is 0.

$\therefore V$ is subspace of $\mathbb{R}^2$.

(showed)

* Show that W is a subspace of $\mathbb{R}^3$ where $W = \{(a, b, c) : a+b+c=0\}$

Proof: Let $V = \mathbb{R}^3$

Now given $W = \{(a, b, c) : a+b+c=0\}$

$\therefore 0 = (0, 0, 0) \in W \Rightarrow W \neq \phi$

Let any two vectors $u = (a_1, b_1, c_1) \in W$, $a_1+b_1+c_1=0$

and $v = (a_2, b_2, c_2) \in W$, $a_2+b_2+c_2=0$

and any two scalar $\alpha, \beta \in F$

$\therefore \alpha u + \beta v = \alpha(a_1, b_1, c_1) + \beta(a_2, b_2, c_2)$

$= (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2) \in W$

Since $\alpha a_1 + \beta a_2 + \alpha b_1 + \beta b_2 + \alpha c_1 + \beta c_2$

$= \alpha(a_1+b_1+c_1) + \beta(a_2+b_2+c_2)$

$= \alpha . 0 + \beta . 0$

$= 0 + 0$

$= 0 \in W$

$\therefore$ W is a subspace of vector $\mathbb{R}^3$.

(Showed)

* Show that $T = \{(a, b, c, d) \in \mathbb{R}^4 : 2a - 3b + 5c - d = 0\}$ is a subspace of $\mathbb{R}^4$.

Proof: For $0 \in \mathbb{R}^4$, $0 = (0, 0, 0, 0) \in T$

Since $2 \cdot 0 - 3 \cdot 0 + 5 \cdot 0 - 0 = 0$

Hence T is non-empty

Suppose that $u = (a, b, c, d)$ and $v = (a', b', c', d')$ are in T, then $2a - 3b + 5c - d = 0$ and $2a' - 3b' + 5c' - d' = 0$

Now for any scalars α, β

we have
$$\alpha u + \beta v = \alpha(a, b, c, d) + \beta(a', b', c', d')$$
$$= (\alpha a, \alpha b, \alpha c, \alpha d) + (\beta a', \beta b', \beta c', \beta d')$$
$$= (\alpha a + \beta a', \alpha b + \beta b', \alpha c + \beta c', \alpha d + \beta d')$$

Also we have
$$2(\alpha a + \beta a') - 3(\alpha b + \beta b') + 5(\alpha c + \beta c') - (\alpha d + \beta d')$$
$$= \alpha(2a - 3b + 5c - d) + \beta(2a' - 3b' + 5c' - d')$$
$$= \alpha \cdot 0 + \beta \cdot 0$$
$$= 0$$

Thus $\alpha u + \beta v \in T$ and so T is a subspace of $\mathbb{R}^4$.

(Showed)