Eigen values and Eigen Vectors

⑥

Definition:

If A is an $n \times n$ matrix, then a non-zero vector v in $\mathbb{R}^n$ is called eigen vector of A if Av is a scalar multiple of v, that is,
$Av = \lambda v$ --- ① for some scalar λ. The scalar λ is called an eigen value of A and v is called an eigen vector of A corresponding to λ.

Characteristic Matrix:

Let a matrix $A = (a_{ij})_{n \times n}$ and $I_n = I$ be a identity matrix of same order over field F.
Then,

$$\lambda I - A = \lambda \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{pmatrix} \quad \text{--- ①}$$

is said to be characteristic matrix of A.

Characteristic Polynomial

The determinant of matrix $\lambda I - A$, i.e

$$|\lambda I - A| = \begin{vmatrix} \lambda - a_{11} & -a_{12} & -a_{1n} \\ -a_{21} & \lambda - a_{22} & -a_{2n} \\ \cdots & \cdots & \cdots \\ -a_{n1} & -a_{n2} \cdots & \lambda - a_{nn} \end{vmatrix} \quad \text{--- ②}$$

is said to be characteristic Polynomial.

P.T.O

Characteristic Equation:

The Equation $|\lambda I - A| = 0$ is, or

$$\begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \cdots & \cdots & & \cdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix} = 0$$

is said to be characteristic equation.

** * Find the eigen values and eigen vectors of the matrix $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$.

Sol^n: The characteristic matrix of A is

$$\lambda I - A = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \lambda - 3 & 1 \\ -1 & \lambda - 1 \end{pmatrix}$$

Now the characteristic equation of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 3 & 1 \\ -1 & \lambda - 1 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - 3)(\lambda - 1) + 1 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 + 1 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 4 = 0$$

$$\Rightarrow (\lambda - 2)^2 = 0$$

$$\Rightarrow \lambda = 2, 2.$$

This is the eigen value of A, and $\lambda = 2$ is the only one eigen value of A.

Now by definition $v = \begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvectors of A corresponding to λ if and only if v is a non-trivial solution of $\cancel{(\lambda I - A)x} = (\lambda I - A)v = 0$,

P.T.O

that is, of

$$\begin{pmatrix} \lambda-3 & 1 \\ -1 & \lambda-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} (\lambda-3)x + y \\ -x + (\lambda-1)y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \left.\begin{array}{l} (\lambda-3)x + y = 0 \\ -x + (\lambda-1)y = 0 \end{array}\right\} \text{--- (1)}$$

Now, for $\lambda = 2$, (1) $\Rightarrow \left.\begin{array}{l} -x+y=0 \\ -x+y=0 \end{array}\right\} \Rightarrow x-y=0 \text{ --- (2)}$

$\therefore$ The system (2) is consistent and has more than one solution.

Now let $y = a$, then (2) $\Rightarrow x = a, y = a$.

Therefore the eigen vectors of A corresponding to the eigen value $\lambda = 2$ are non-zero vectors of the form $v = \begin{pmatrix} a \\ a \end{pmatrix}$

In particular, let $a = 1$, then $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigen vector corresponding to the eigen value $\lambda = 2$.

#. H.W **

(1) Find the eigen values of the matrix

A.R (a) $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$, A.R (b) $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$. D.AK (c) $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$

Ans.- (c) $\lambda = 1, 4,$ $v_1 = \begin{pmatrix} a \\ -a \end{pmatrix}, v_2 = \begin{pmatrix} b \\ 2b \end{pmatrix}$.

Ans.- (b) $\lambda = 1, 2,$
$v_1 =$
$v_2 =$

Ans.- (a) $\lambda = 1, 5$
$v_1 = \begin{pmatrix} a \\ a \end{pmatrix}$
$v_2 = \begin{pmatrix} -3b \\ b \end{pmatrix}$

***

* Find all eigenvalues and corresponding eigenvectors of the matrix.

$$A = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$$

Solution: The characteristics matrix of A is

$$\lambda I - A = \begin{pmatrix} \lambda+3 & -1 & 1 \\ 7 & \lambda-5 & 1 \\ 6 & -6 & \lambda+2 \end{pmatrix}$$

Now the characteristic equation is

$$|\lambda I - A| = \begin{vmatrix} \lambda+3 & -1 & 1 \\ 7 & \lambda-5 & 1 \\ 6 & -6 & \lambda+2 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+3)(\lambda^2-3\lambda-4) + (7\lambda+8) + (-6\lambda-12) = 0$$

$$\Rightarrow (\lambda^3-13\lambda-12) + (7\lambda+8) + (-6\lambda-12) = 0$$

$$\Rightarrow \lambda^3 - 12\lambda - 16 = 0$$

$$\Rightarrow (\lambda+2)^2(\lambda-4) = 0$$

$$\Rightarrow \lambda = -2, -2, 4$$

which are the eigen values of A.

Now by definition $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is an eigenvector of A corresponding to the eigen value λ if and only if v is a non-trivial solution of $(\lambda I - A)v = 0$ --- ①

Now ① $\Rightarrow$

$$\begin{pmatrix} \lambda+3 & -1 & 1 \\ 7 & \lambda-5 & 1 \\ 6 & -6 & \lambda+2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{l} (\lambda+3)x - y + z = 0 \\ 7x + (\lambda-5)y + z = 0 \\ 6x - 6y + (\lambda+2)z = 0 \end{array}\right\} \text{---- ②}$$

P.T.O

Now, when $\lambda = \lambda_1 = -2$ then

$$② \Rightarrow \left.\begin{aligned} x-y+z&=0 \\ 7x-7y+z&=0 \\ 6x-6y&=0 \end{aligned}\right\} \xrightarrow{R_2 \to R_2 - R_1} \left.\begin{aligned} x-y+z&=0 \\ x-y+z&=0 \\ 6x-6y&=0 \end{aligned}\right\} \sim \left.\begin{aligned} x-y+z&=0 \\ x-y&=0 \end{aligned}\right\} \cdots ③$$

There fore the system is consistent and has more than one solution

Now let $y=1$.

Then $③ \Rightarrow x=1,\ y=1,\ z=0$.

There fore for $\lambda_1 = -2$, the corresponding eigen vectors is

$V_1 = V = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and for that eigen value

Corresponding all the eigenvectors are

$kV_1 = k\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} k \\ k \\ 0 \end{pmatrix}$ where $k \in \mathbb{R}\ (k \neq 0)$

Again when $\lambda = \lambda_2 = 4$ then

$$② \Rightarrow \left.\begin{aligned} 7x-y+z&=0 \\ 7x-y+z&=0 \\ 6x-6y+6z&=0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 7x-y+z&=0 \\ x-y+z&=0 \end{aligned}\right\}$$

$$\Rightarrow \left.\begin{aligned} 7x-y+z&=0 \\ x&=0 \end{aligned}\right\} \cdots ④$$

There fore the system is consistent and has more than one solution.

Now let $z=1$, then $④ \Rightarrow x=0,\ y=1,\ z=1$. therefore for $\lambda_2 = 4$, the corresponding ~~vect~~ eigen vector is

$V_2 = V = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

And all the eigen vectors corresponding the eigen value $\lambda_2 = 4$ are $kV_2 = k\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ k \\ k \end{pmatrix}$,

where $k \in \mathbb{R},\ (k \neq 0)$.

H.W.

Find all eigen values and the corresponding eigenvectors of the matrix.

A.R ***(a) $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{bmatrix}$. Ans:- $\lambda = 1, 2, -2$; $v_1 = \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} -\frac{k}{5} \\ \frac{4k}{5} \\ k \end{pmatrix}$, $v_3 = \begin{pmatrix} k \\ 0 \\ k \end{pmatrix}$

D.A.K ** (b) $A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$. Ans:- $\lambda = \underline{2, 2}, 6$; $v_1 = \begin{pmatrix} k \\ -k \\ 0 \end{pmatrix}$. $v_2 = \begin{pmatrix} k \\ 0 \\ -k \end{pmatrix}$, $v_3 = \begin{pmatrix} k \\ 2k \\ k \end{pmatrix}$

Hints:- $x+y+z=0$
$y=-1, z=0$
$y=0, z=-1$

**** **Cayley-Hamilton Theorem:** Every square matrix satisfies its own characteristic equation, i.e if the characteristic equation of the nth order matrix A is

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0$$

then Cayley-Hamilton theorem states that

$$f(A) = A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_{n-1}A + a_n I = 0,$$ where

I is the nth order unit matrix and 0 is the nth order zero matrix.

** **Determination of an inverse matrix of a non-singular matrix by Cayley Hamilton theorem:** *

Let A be a non-singular matrix of n order and the characteristic polynomial is

$$f(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 \quad \cdots\cdots (1)$$

Then ~~*~~ * $$A^{-1} = -\frac{1}{a_0}\left(A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I\right)$$

P.T.O

***

* Using Cayley-Hamilton theorem find A^{-1} and A^{-2} of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

Solⁿ:

Here the characteristic matrix of A is

$$\lambda I - A = \begin{pmatrix} \lambda-1 & -1 & -1 \\ -1 & \lambda+1 & -1 \\ -1 & -1 & \lambda+1 \end{pmatrix} \cdots\cdots (1)$$

Now the characteristic polynomial of A is

$$f(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda-1 & -1 & -1 \\ -1 & \lambda+1 & 1 \\ -1 & -1 & \lambda+1 \end{vmatrix}$$

$$= (\lambda-1)\{(\lambda+1)^2 - 1\} + (-\lambda-1-1) - 1(1+\lambda+1)$$

$$= (\lambda-1)(\lambda^2+2\lambda) - \lambda - 2 - \lambda - 2$$

$$= \lambda^3 + 2\lambda^2 - \lambda^2 - 2\lambda - 2\lambda - 4$$

$$= \lambda^3 + \lambda^2 - 4\lambda - 4 \cdots\cdots (2)$$

Now using Cayley-Hamilton theorem we get

$$f(A) = 0$$

$$\therefore A^3 + A^2 - 4A - 4 = 0$$

$$\Rightarrow A^{-1}(A^3 + A^2 - 4A - 4) = A^{-1}0$$

$$\Rightarrow A^2 + A - 4I - 4A^{-1} = 0$$

$$\Rightarrow 4A^{-1} = A^2 + A - 4I$$

$$\Rightarrow A^{-1} = \frac{1}{4}(A^2 + A - 4I) \cdots\cdots (3)$$

But $A^2 = A \times A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

P.T.O

$$\therefore (3) \Rightarrow A^{-1} = \frac{1}{4}\left\{\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}\right\}$$

$$= \frac{1}{4}\begin{pmatrix} 3+1-4 & 1+1\mp 0 & 1+1\mp 0 \\ 1+1\mp 0 & 3-1-4 & -1+1\mp 0 \\ 1+1\mp 0 & -1+1\mp 0 & 3-1\mp 4 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 0 & 2 & 2 \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$ Ans.

Again multiplying ~~by A^{-1} in~~ both sides ~~by~~ of ③ by A^{-1}, we get

$$A^{-2} = \frac{1}{4}(A + I - 4A^{-1})$$

$$= \frac{1}{4}\left\{\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 2 & 2 \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{pmatrix}\right\}$$

$$= \frac{1}{4}\begin{pmatrix} 1+1 & 1-2 & 1-2 \\ 1-2 & -1+1+2 & 1 \\ 1-2 & 1 & -1+1+2 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$ Ans.

***

* H.W. Using Cayley-Hamilton theorem find the A^{-1} and A^{-2} of the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

* $\boxed{f(A) = A^3 - 3A^2 - 5A + I}$, $A^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -6 \\ -2 & -1 & 5 \end{bmatrix}$, $A^{-2} = ?$