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Vector Space & Subspace

Define Vector space and Subspace.

Vector space :

Definition:

Let K, F be a given field and Let V be a none empty set with rules of addition and scaler multiplication which assigns to any $u,v \epsilon V$ a sum $u+v \epsilon V$ and to any $u \epsilon V$, $f \epsilon F$ a product $fu \epsilon V$.

Then V is called a vector space over F (and the elements of V are called vector) if the following axioms hold.

Axioms are below:

A(1) Addition is commutative :

For all vector's $\mathbf{u,v \epsilon V, u+v = v+u}$.

A(2) Addition is Associative :

For all vector's $\mathbf{u,v,w \epsilon V, (u+v)+w = u+(v+w)}$.

A(3) Existence of 0 (zero Vector's) :

There exists a vector $0 \epsilon V$ such that for all $\mathbf{u \epsilon V, u+0=0+u=u}$.

A(4) Existence of Negative :

For each $u \epsilon v$ there is a vector $-u \epsilon v$ for which,

$$\mathbf{u+(-u)=(-u)+u=0.}$$

M(1) Distributive Law :

For any scalar $\alpha \epsilon F$ and any vector's $u,v \epsilon V$, $\boldsymbol{\alpha(u+v) = \alpha u+\alpha v}$.

M(2) Distributive Law :

For any scalars, $\alpha,\beta \epsilon F$ and any vectors $u \epsilon V$, $\boldsymbol{(\alpha\beta)u = \alpha(\beta u)}$.

M(3) Associative Law :

For any scalars $\alpha\beta \epsilon F$ and any vectors $u \epsilon V$, $\boldsymbol{(\alpha\beta)u = \alpha(\beta u)}$.

M(4) Unitary Law :

For each $u \epsilon V$, $\mathbf{1u = u}$. Where 1 is the unite scalar and $1 \epsilon F$.

Subspace :

Definition : Let W be a subset of a vector space V over a field F. W is called a subspace of V if W is itself a vector space over F with respect to the operations of vector addition and scalar multiplication on V.

Theorem :

Let, V be a vector space, with operations Addition (+) and Multiplication (.) , and Let W be a subset of V .Then W is a subspace of V if and only if the following conditions hold.

Sub0 W is nonempty : The zero vector belongs to W.

Sub1 closure under (+) : If u and v are any vectors in W, then u+v is in W.

Sub2 closure under (.) : If v is any vector in W, and c is any real number, then c.v is in W.

Euclidean Space :

$\mathbf{R}^n$ is the set all real numbers usual addition and multiplication.

($\mathbf{R}^n \longrightarrow$ Euclidean space) **.**

Prove that, For each positive integer n, Euclidean space $\mathbf{R}^n$ is a vector space.

Proof :

We shall have to show that $\mathbf{R}^n$ satisfies all axioms of a vector space.

(i) Let, $u = (u_1,u_2,\dots\dots,u_n)$ and $v = (v_1,v_2,\dots\dots,v_n)$ be in $\mathbf{R}^n$,

Then $u+v = (u_1,u_2,\dots\dots,u_n) + (v_1,v_2,\dots\dots,v_n)$

$= (u_1+v_1,u_2+v_2,\dots\dots,u_n+v_n)$

$= (v_1+u_1,v_2+u_2,\dots\dots,v_n+u_n)$.

So, A(1) is True.

(ii) Let, $u = (u_1,u_2,\dots\dots,u_n)$ and $v = (v_1,v_2,\dots\dots,v_n)$ and $w = (w_1,w_2,\dots\dots,w_n)$ be in $\mathbf{R}^n$. Then, $(u+v) + w = (u_1+v_1,u_2+v_2,\dots\dots,u_n+v_n)+(w_1,w_2,\dots\dots w_n)$

$= (u_1+v_1+w_1,\ u_2+v_2+w_2,\dots\dots,u_n,v_n,w_n)$

$= u +(v+w)$.

So, the axiom A(2) holds.

(iii) Let, $0 = (0,0,\dots\dots,0)$ be in $\mathbf{R}^n$. Then for any $u = (u_1,u_2,\dots\dots,u_n)$ in $\mathbf{R}^n$ we will have $u+0 = (u_1,u_2,\dots\dots,u_n) + (0,0,\dots\dots,0)$

$$= (u_1+0, u_2+0,\dots\dots,u_n+0)$$

$$= (u_1,u_2,\dots\dots, u_n) = u.$$

So, the axiom A(3) holds.

(iv) Let, $u = (u_1,u_2,\dots\dots,u_n)$ and set $-u = (-u_1,-u_2,\dots\dots,-u_n)$
Then, $u+(-u) = (u_1,u_2,\dots\dots,u_n) + (-u_1,-u_2,\dots\dots,-u_n)$

$$= (u_1-u_1, u_2-u_2,\dots\dots, u_n-u_n)$$

$$= (0,0,\dots\dots,0) = 0.$$

So, the axiom A(4) holds.

(V) Let, α be a real number (scalar) and $u = (u_1,u_2,\dots\dots,u_n)$ and $v = (v_1,v_2,\dots\dots,v_n)$ be vectors in $\mathbf{R}^n$. Then,

$$\begin{aligned}\alpha(u+v) &= \alpha\{(u_1,u_2,\dots\dots,u_n) + (v_1,v_2,\dots\dots,v_n)\}\\ &= \alpha(u_1+v_1, u_2+v_2,\dots\dots,u_n+v_n)\\ &= (\alpha u_1+\alpha v_1, \alpha u_2+\alpha v_2,\dots\dots, \alpha u_n+\alpha v_n)\\ &= \alpha(u_1u_2,\dots\dots,u_n) + \alpha(v_1v_2,\dots\dots,v_n)\\ &= \alpha u + \alpha v.\end{aligned}$$

So, the axiom M(1) holds.

(vi) Let, α,β be that the real numbers (scalars) and $u = (u_1,u_2,\dots\dots,u_n)$ be in $\mathbf{R}^n$. Then,

$$\begin{aligned}(\alpha+\beta)u &= \big((\alpha+\beta)u_1, (\alpha+\beta)u_2,\dots\dots, (\alpha+\beta)u_n\big)\\ &= (\alpha u_1+\beta u_1, \alpha u_2+\beta u_2,\dots\dots, \alpha u_n+\beta u_n)\\ &= (\alpha u_1, \alpha u_2,\dots\dots,\alpha u_n) + (\beta u_1,\beta u_2,\dots\dots,\beta u_n)\\ &= \alpha(u_1u_2,\dots\dots,u_n) + \beta(u_1u_2,\dots\dots,u_n)\\ &= \alpha u + \beta u\end{aligned}$$

So, the axiom M(2) is satisfied.

(vii) Let, α,β is real numbers (scalar) and $u = (u_1,u_2,\dots\dots,u_n)$ be in $\mathbf{R}^n$.
Then,

$$\begin{aligned}(\alpha\beta)u &= \alpha\beta(u_1,u_2,\dots\dots,u_n)\\ &= (\alpha\beta u_1, \alpha\beta u_2,\dots\dots, \alpha\beta u_n)\\ &= \alpha(\beta u_1, \beta u_2,\dots\dots, \beta u_n)\\ &= \alpha\big(\beta(u_1,u_2,\dots\dots,u_n)\big)\\ &= \alpha(\beta u)\end{aligned}$$

So, the axiom M(4) is satisfied.

Therefore $\mathbf{R}^n$ is a Vector space . **[Proved]**

Prove that, W is not a subspace of $\mathbf{R}^3$ where w = {(a,b,1) : a,bϵ **R**}

Proof :

Let, V= $\mathbf{R}^3$

W= {(a,b,1) : (a,b)ϵ **R**}

0 = (0,0,0) $\notin$ W since the third component vectors in W is 1.

So, W is not a subspace of $\mathbf{R}^3$. **[Proved]**

State and proof fundamental Theorem of Subspace.

Fundamental Theorem of Subspace :

Statement :

W will be subspace of subset of Vector space v(F) iff (if and only)

(a) W is non-empty. i.e W$\neq$ $\emptyset$
(b) W is closed under addition, i.e $\forall$ u,wϵW = u + wϵW. [$\forall$= $\boldsymbol{For\ all}$]
(c) W is closed under scalar multiplication, i.e $\forall \alpha \epsilon F, \forall w \epsilon W = \alpha w \epsilon W$

Proof :

First suppose W satisfies (a), (b) and (c) we have to show that W is a subspace of V. Now by (a) W is non-empty and by (b) and (c) vector addition and scalars multiplication is well defined in W.

Again the vectors in W belongs to V then following axioms holds in W.

(i) Addition is commutative :

For all vector's **u,vϵV, u+v = v+u.**

(ii) Addition is Associative :

For all vector's **u,v,wϵV, (u+v)+w = u+(v+w).**

(iii) Existence of 0 (zero Vector's) :

There exists a vector 0ϵV such that for all **uϵV,u+0=0+u=u.**

(iv) Existence of Negative :

For each $u \epsilon v$ there is a vector $-u \epsilon v$ for which,

$$u+(-u)=(-u)+u=0.$$

(v) Distributive Law :

For any scalar $\alpha \epsilon F$ and any vector's $u,v \epsilon V$, $\boldsymbol{\alpha(u+v) = \alpha u+\alpha v}$.

(vi) Distributive Law :

For any scalars, $\alpha,\beta \epsilon F$ and any vectors $u \epsilon V$, $\boldsymbol{(\alpha\beta)u = \alpha(\beta u)}$.

(vii) Associative Law :

For any scalars $\alpha\beta \epsilon F$ and any vectors $u \epsilon V$, $\boldsymbol{(\alpha\beta)u = \alpha(\beta u)}$.

(viii) Unitary Law :

For each $u \epsilon V$, $\boldsymbol{1u = u}$. Where 1 is the unite scalar and $1 \epsilon F$.

We will prove (vii) :

By (a) W is non -empty.

Say $u \epsilon W$. Then by (c)

For $0 \epsilon F$, $0u = 0 \epsilon W$

(vii) is proved.

We will prove (viii) :

By (a), W is non-empty.

Say $w \epsilon W$, Then by (c)

For $-1 \epsilon F$, $(-1)u \epsilon W = -u \epsilon W$

And by (b) $u+(-u) = 0, \forall\ u \epsilon W$

(viii) is proved.

Therefore, W is satisfied of all condition of vector V. So, W is a subspace of subset of vector space V(F).

Conversely Let, W is a subspace of Vector Space V(F) then W will be satisfy the condition of (a), (b) and (c). Because of (a), (b) and (c) are the part of conditions of Vector Space.

[Hence Proved]

Theorem :

Let W be a subset of V. Then show that, W is a Subspace of V(F) iff.

(i) $0 \epsilon W$, i.e $W \neq \emptyset$

(ii) For all $\alpha, \beta \epsilon F$ and for all $u, w \epsilon W = (\alpha u + \beta w) \epsilon W$.

Proof:

Frist suppose that the subset W satisfies (i) and (ii).

Now by (i),

W is non-empty as $0 \epsilon W$.

Again by (ii),

$\forall\ \alpha, \beta \epsilon F$ and $\forall\ u, w \epsilon W = (\alpha u + \beta w) \epsilon W$

Now Let $\alpha=1$ and $\beta=1$ then, $\alpha u + \beta w = 1.u + 1.w = u + w \epsilon W$ $\quad [\ \alpha u + \beta w \epsilon W\ ]$

W is closed under Vector addition.

Again if $w \epsilon W$ and for $\alpha \epsilon F$. We get , $\alpha w = \alpha w + 0 = \alpha w + 0.w \epsilon W$

So, W is closed under scalar multiplication.

Thus W satisfies the three conditions of the fundamental theorem and therefore W is a subspace of V(F).

Conversely Let, W is a Subspace of Vector Space V(F) then W will be satisfy the (i) and (ii) are the part of condition of Vector Space.

Show that , x-axis and y-axis is Subspace of Vector Space R^2 .

Proof:

Let, set of points on x-axis $U=\{(a,0) : a \epsilon \mathbf{R}\}$ … … … (i)

And set of points on y-axis $V=\{(0,b) : b \epsilon \mathbf{R}\}$ … … …(ii)

Since $\mathbf{R}^2$ is the set of all two dimensional Vector Space.

$\therefore U, V \epsilon \mathbf{R}^2$.

For (1)

Since uϵ $\mathbf{R}^2$ $\therefore$ 0=(0,0)ϵU, U$\neq\emptyset$.

Let, any two vectors u=(a_1,0), v=(a_2,0)ϵU, a_1,$a_2\epsilon\mathbf{R}$ and any two scalar $\alpha,\beta\epsilon$F

$$\therefore \alpha u + \beta v = \alpha(a_1,0) + \beta(a_2,0)$$
$$= (\alpha a_1 + \beta a_2, 0+0)$$
$$=(\alpha a_1 + \beta a_2, 0)\epsilon U$$

$\therefore \alpha a_1\epsilon\mathbf{R}$, $\beta a_2\epsilon\mathbf{R}$

$\therefore \alpha a_1+ \beta a_2\epsilon\mathbf{R}$

Since ($\alpha a_1 + \beta a_2$)$\epsilon\mathbf{R}$ and the second component is 0.

$\therefore$ U is a Subspace of $\mathbf{R}^2$.

For (2)

Since v$\epsilon\mathbf{R}^2$ $\therefore$ 0=(0,0)ϵV, V$\neq\emptyset$.

Let, any two vectors u= (0,b_1),v= (0,b_2)ϵV, b_1b_2 $\epsilon\mathbf{R}$ and any two scalar $\alpha,\beta\epsilon$F

$$\therefore \alpha u + \beta v = \alpha(0,b_1) + \beta(0,b_2)$$
$$= (0+0, \alpha b_1 + \beta b_2,)$$
$$=(0, \alpha b_1 + \beta b_2)\epsilon V$$

$\therefore \alpha b_1\epsilon\mathbf{R}$, $\beta b_2\epsilon\mathbf{R}$

$\therefore \alpha b_1+ \beta b_2\epsilon\mathbf{R}$

Since ($\alpha b_1 + \beta b_2$)$\epsilon\mathbf{R}$ and the first component is 0.

$\therefore$ V is Subspace of $\mathbf{R}^2$. **[Showed]**

Show that, W is a subspace of $\mathbf{R}^3$ where w = {(a,b,c) : a+b+c=0}

Proof:

Let, V= $\mathbf{R}^3$

Now given W={(a,b,c) : a+b+c=0}

$\therefore$ 0=(0,0,0)ϵW, W$\neq\emptyset$

Let any two vectors u=(a_1,b_1,c_1)ϵW, $a_1+b_1+c_1=0$

and v= (a_2,b_2,c_2)ϵW, $a_2+b_2+c_2=0$ and any two scalar $\alpha,\beta\epsilon$F

$\therefore\ \alpha u + \beta v = \alpha(a_1,b_1,c_1) + \beta(a_2,b_2,c_2)$

$=(\alpha a_1 + \beta a_2,\ \alpha b_1 + \beta b_2, \alpha c_1+\beta c_2) \epsilon W$

Since $\alpha a_1 + \beta a_2 + \alpha b_1 + \beta b_2 + \alpha c_1 + \beta c_2$

$= \alpha(a_1+b_1+c_1)+\beta(a_2,b_2,c_2)$

$= \alpha.0 +\beta.0$

$= 0+0 = 0 \in W$

$\therefore$ W is a Subspace of vector $\mathbf{R}^3$. **[Showed]**

Show that, T= {(a,b,c,d) $\epsilon \mathbf{R}^4$: 2a-3b+5c-d=0} is a Subspace of $\mathbf{R}^4$**.**

Proof:

For 0 $\epsilon \mathbf{R}^4$, 0= (0,0,0)ϵT

Since 2.0-3.0+5.0-0 = 0,

Hence T is non-empty.

Suppose that u=(a,b,c,d) and v=(a',b',c',d') are in T, Then 2a-3b+5c-d=0 .

Now for any scalars α,β we have, $\alpha u + \beta v = \alpha(a,b,c,d) + \beta(a',b',c',d')$

$= (\alpha a,\alpha b,\alpha c, \alpha d) + (\beta a', \beta b',\beta c',\beta d')$

$= (\alpha a + \beta a', \alpha b + \beta b',\alpha c+\beta c',\alpha d+\beta d')$

Also we have, $2(\alpha a + \beta a') - 3(\alpha b + \beta b') + 5(\alpha c+\beta c') - (\alpha d+\beta d')$

$= 2\alpha a+2\beta a'-3\alpha b-3\beta b'+5\alpha c+5\beta c'-\alpha d-\beta d'$

$= 2\alpha a-3\alpha b+5\alpha c-\alpha d+2\beta a'-3\beta b'+5\beta c'-\beta d'$

$= \alpha(2a-3b+5c-d) + \beta(2a'-3b'+5c'-d')$

$= \alpha.0 +\beta.0$

$=0$

Thus $\alpha u + \beta v \ \epsilon T$ and So, T is a Subspace of $\mathbf{R}^4$**.** **[Showed]**

Linear Combination

Definition of Linear Combination:

Let V be a vector space over the field F and Let v_1, ……. $v_n \epsilon$ V, then any vector v ϵ V is
called a linear combination of v_1, v_2, …. v_n if and only if there exist scalars $\propto_1, \propto_2$, ……. $\propto_n$ in F.

Such that $V = \propto_1 v_1 + \propto_2 v_2 + \ldots\ldots\ldots + \propto_n v_n$

$= \sum_{i=1}^{n} \propto \text{ivi}$

Example 1: Consider the vectors $v_1 = (1, 0, 1)$, $v_2 = (0, -1, 1)$ and $v_3 = (-1, -1, 1)$ in $I\mathcal{R}^3$. Show that V = (2, 3, 4) is a linear combination of v_1, v_2 and v_3.

Solution:

In order to show that V is a linear combination of v_1, v_2, and v_3, there must be scalars $\propto_1, \propto_2$ and $\propto_3$ in F, Such that $V = \propto_1 v_1 + \propto_2 v_2 + \propto_3 v_3$

i.e. $(2, 3, 4) = \propto_1(1, 0, 1) + \propto_2(0, -1, 1) + \propto_3(-1, -1, 1)$

=> $(2, 3, 4) = \propto_1(\propto_1, 0, \propto_1) + \propto_2(0, -\propto_2, \propto_2) + \propto_3(-\propto_3, -\propto_3, \propto_3)$

=> $(2, 3, 4) = (\propto_1 - \propto_3, - \propto_2 - \propto_3, \propto_1+\propto_2+\propto_3)$

Now equating corresponding components and forming linear system we get

$$\left.\begin{aligned} \propto_1 - \propto_3 &= 2 \\ - \propto_2 - \propto_3 &= 3 \\ \propto_1+\propto_2+\propto_3 &= 4 \end{aligned}\right] \ldots\ldots\ldots \ (1)$$

Reduce the system (1) to echelon form by elementary operations.

$R3' \to \widetilde{R3} - R1$ $\quad \propto_1 - \propto_3 = 2$

$- \propto_2 - \propto_3 = 3$

$\propto_1+\propto_2+\propto_3 = 2$

$R3' \rightarrow R3 + R2$

$$\left.\begin{array}{r} \propto_1 - \propto_3 = 2 \\ -\propto_2 - \propto_3 = 3 \\ \propto_3 = 5 \end{array}\right] \quad \ldots\ldots\ldots (2)$$

Now from equation (2) we have $\propto_3 = 5$, substituting $\propto_3 = 5$ and solving (2) we get $\propto_2 = -8$, and $\propto_1 = 7$.

Hence $V = 7v_1 - 8v_2 + 5v_3$

Therefore, V is a Linear combination of v_1, v_2, v_3.

Example 2: Express the vector V = (1, -2, 5) as a linear combination of the vectors v_1 = (1, 1, 1), v_2 = (1, 2, 3) and v_3 = (2, -1, 1).

Solution:

In order to show that V is a linear combination of v_1, v_2, and v_3, there must be scalars x, y and z in F. Such that $V = xv_1 + yv_2 + zv_3$

i.e. (1, -2, 5) = x (1, 1, 1) + y (1, 2, 3) + z (2, -1, 1)

=> (1, -2, 5) = (x, x, x) + y (y, 2y, 3y) + z (2z, -z, z)

=> (1, -2, 5) = (x + y+ 2z, x +2y –z, x +3y +z)

Now equating corresponding components and forming linear system, we get

$$\left.\begin{array}{l} x+y+2z = 1 \\ x+2y-z = -2 \\ x+3y+z = 5 \end{array}\right] \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Reduce the system (1) to echelon form by elementary operation.

$R3' \rightarrow R3 - R1$ $\quad x + y + 2z = 1$

$R2' \rightarrow R2 - R1$ $\quad y - 3z = -3$

$2y - z = 4$

$R3' \to \widetilde{R3} - 2R2$

$$\left.\begin{aligned} x + y + 2z &= 1 \\ y - 3z &= -3 \\ 5z &= 10 \end{aligned}\right] \quad \text{..........} \ (2)$$

From equation (2) we have z =2 and solving (2) we get y=3 and x = 6
Hence V = $-6v_1 + 3v_2 + 2v_3$

Example 3: Express the matrix E = $\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}$ as a linear combination of A = $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$,
B = $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ and C = $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$.

Solution:

Let E = xA + yB + zC .... (1), where x, y, z are scalar in F.

Then, $\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} = x\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} + y\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} + z\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} x & x \\ 0 & -x \end{pmatrix} + \begin{pmatrix} y & y \\ -y & 0 \end{pmatrix} + \begin{pmatrix} z & -z \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} x+y+z & x+y-z \\ -y & -x \end{pmatrix}$

Now equating corresponding components and forming linear system we get,

$$\left.\begin{aligned} x + y + z &= 3 \\ x + y - z &= -1 \\ -y &= 1 \\ -x &= -2 \end{aligned}\right] \quad \text{.............} \ (2)$$

Now solving (2) we get x = 2, y = -1, z = 2

∴ E = 2A – B + 2C

i.e. E is a linear combination of A, B, and C.

H.W

1. Show that the matrix E = $\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}$ cannot be express as a linear combination of
A = $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, B = $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ and C = $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$.

Solution:

Let, E = xA + yB + zC ……….. (1) where x, y, z are scalar in F.

Then, $\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} = x\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + y\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} + z\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

=> $\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} x & x \\ 0 & -x \end{pmatrix} + \begin{pmatrix} y & y \\ -y & 0 \end{pmatrix} + \begin{pmatrix} z & -z \\ 0 & 0 \end{pmatrix}$

=> $\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} x+y+z & x+y-z \\ -y & x \end{pmatrix}$

Now equating the corresponding component and forming linear system we get,

$$\left.\begin{aligned} x + y + z &= 3 \\ x + y - z &= -1 \\ -y &= 1 \\ x &= -2 \end{aligned}\right] \quad \ldots\ldots\ldots\ldots\ (2)$$

Now solving (2) we get x = -2, y = -1, z = - 2 and z = 6, which is impossible.

So, the equation (2) has no solution or inconstant. Therefore matrix E cannot be express as a linear combination of A, B and C.

2. Express the matrix E = $\begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix}$ as a linear combination of A = $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, B = $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and C = $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Solution:

Let, E = xA + yB + zC .... (1) Where x, y, z are scalar in F.

Then, $\begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix} = x\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + y\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} + z\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

=> $\begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} x & 0 \\ x & 0 \end{pmatrix} + \begin{pmatrix} y & -y \\ 0 & y \end{pmatrix} + \begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix}$

=> $\begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} x+y & -y+z \\ x-z & y \end{pmatrix}$

Now equating the corresponding component and forming linear system we get,

$$\left.\begin{aligned} x + y &= 5 \\ -y + z &= 1 \\ x - z &= -2 \\ y &= 3 \end{aligned}\right] \quad \text{..............(2)}$$

Now solving (2) we get x = 2, y = 3, z = 4

∴ E = 2A + 3B + 4C

i.e. E is a linear combination of A, B, and C.

***Show that the matrix E $=\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}$ can not be express of a linear combination of A $=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ B$=\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ and C$=\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

Solution: Let E=xA+yB+zC…(1) where, x,y and z are scaler in F.

Then $\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix} = x\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + y\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} + z\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

Or, $\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}, = \begin{pmatrix} x & x \\ 0 & x \end{pmatrix} + \begin{pmatrix} y & y \\ -y & 0 \end{pmatrix} + \begin{pmatrix} z & -z \\ 0 & 0 \end{pmatrix}$

Or, $\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} x+y+z & x+y-z \\ -y & x \end{pmatrix}$

Now equation the corresponding component and forming linear system we get,

$$\begin{cases} x+y+z=2 \\ x+y-z=1 \\ -y=-1 \\ x=-2 \end{cases} \quad \text{equation ....(2)}$$

Now solving (2) we get x=-2, y=1, and z=-2, z=3 which is impossible. So the equation (2) has no solution or inconsistent. Therefore Matrix E cannot be express as a linear combination of A,B, and C.

Generator/Linear Span: If S is a non-empty subset of a vector space V, then L(S) is the linear span or generator of S in the set of all linear combination of finite sets of elements of S.

***Show that the vectors u=(1,2,3), v=(0,1,2) and w=(0,0,1) generate $\mathbf{R^3}$. Or show that [(1,2,3),(0,1,2),(0,0,1)= $\mathbf{R^{3}}$.

Solution: We must determine whether an arbitrary vector $V_3(R)$ =(a,b,c) in $\mathbf{R^3}$ can be expressed as a linear combination $V_3(R)$ = xu+yv+zw of the vectors u,v and w , where x,y and z are scalars . Now expressing this equation in terms of components gives

(a+b+c)= x(1,2,3)+y(0,1,2)+z(0,0,1)

=(x,2x,3x) + (0,y,2y) + (0,0,1)

= (x, 2x+y, 3x+2y+z)

Equation corresponding components and forming the linear system we get,

x=0		Z+2y+3x=c
2x+y=b	=>	y+2x=b
3x+2y+z=c		x=a

The above system is echelon form and is consistant . In fact the system has the solution x=a, y= b-2a, z= c-2b+a

Thus, u,v and w generate (Span) $\mathbf{R^3}$.

ASSIGNMENT#02

COMPUTER SCIENCE & ENGINEERING

1ST YEAR 2ND SEMESTER

LINEAR ALGEBRA & DIFFERENTIAL EQUATION

COURSE & CODE : MAT-1231

GROUP: 01

TOPIC: LINEAR TRANSFORMATION & BASIS AND DIMENSION

DATE OF SUBMISSION : 07-08-2020

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Basis and Dimension

Basis:

Let, **V** be a vector space and $\{\boldsymbol{V_1}, \boldsymbol{V_2}, \dots, \mathbf{V}_{\boldsymbol{n}}\}$ Is a finite set of vectors in **V**.

We call $\{\boldsymbol{V_1}, \boldsymbol{V_2}, \dots, \mathbf{V}_{\boldsymbol{n}}\}$ a basis for **V** if and only if

(I) $\{\boldsymbol{V_1}, \boldsymbol{V_2}, \dots, \mathbf{V}_{\boldsymbol{n}}\}$is linearly independent

(II) $\{\boldsymbol{V_1}, \boldsymbol{V_2}, \dots, \mathbf{V}_{\boldsymbol{n}}\}$ Spans **V**.

Example: $\{(\mathbf{1}, \mathbf{0}, \mathbf{0}, \dots, \mathbf{0}), (\mathbf{0}, \mathbf{1}, \mathbf{0}, \dots, \mathbf{0}), \dots, (\mathbf{0}, \mathbf{0}, \mathbf{0}, \dots \mathbf{1})\}$ standard basis.

Dimension:

The number of vectors in any basis of a finite dimensional vector space (**V**) is called dimension.

Or, equivalently, the dimension of a vector space is equal to the maximum number of linearly independent vectors contained in it.

Theorem:

Every basis of a finite dimensional vector space has the same number of vectors.

Proof:

Let, ***S* = {** $\boldsymbol{u_1, u_2, \dots\dots, u_m}$ **}** and ***T* = {** $\boldsymbol{v_1, v_2, \dots\dots, v_n}$ **}** be two basis of finite dimensional vector space ***V(F)***.

We have to show that **m=n**.

Since ***S*** and ***T*** the basis of ***V(F)*** then,

(i) both ***S*** and ***T*** are the generator of ***V(F)***

(ii) both ***S*** and ***T*** are linearly independent

Now, if ***S*** is the generator of ***V(F)*** and ***T*** is linearly independent, then by Exchange lemma,

$$\mathbf{m \geq n} \quad \dots\dots\dots\dots\dots \quad \mathbf{(1)}$$

Similarly, if ***S*** is linearly independent an ***T*** is the generator of ***V(F)***, we have from the same lemma,

$$\mathbf{n \geq m} \quad \dots\dots\dots\dots\dots \quad \mathbf{(2)}$$

From equation **(1)** and **(2)**, we get **m=n**.

[Proved].

Theorem:

If U and W be the two finite dimensional subspace of $V(F)$,
Then $\dim (U + W) = \dim U + \dim W - \dim (U \cap W)$.

Proof:

Since U and W are subspaces, $U \cap W$ will be the subspace of both U and W.

Let $\dim U = m$, $\dim W = n$ and $\dim (U \cap W) = r$. Let $\{ v_1, v_2, \ldots\ldots, v_r \}$ be a basis of $(U \cap W)$.

Now extend the basis of U and W in the form $\{v_1, v_2, \ldots\ldots, v_r, u_1, u_2, \ldots\ldots, u_{m-r}\}$ and $\{v_1, v_2, \ldots\ldots, v_r, w_1, w_2, \ldots\ldots, w_{n-r}\}$ respectively.

So, the union of the basis of U and W is A (say).

$$A = \{v_1, v_2, \ldots\ldots, v_r, u_1, u_2, \ldots\ldots, u_{m-r}, w_1, w_2, \ldots\ldots, w_{n-r}\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

A has exactly $r + m - r + n = m + n - r$ elements which proves the theorem.

Now, we have to show that A is a basis of $U + W$.

Now, since $\{ v_i, u_j \}$ generates U and $\{v_i, w_k\}$ generates W then the union $A = \{v_i, u_j, w_k\}$ will generate $(U + W)$.

Now, we have to show that they are linearly independent.

Now, $\forall\ x_1, x_2 \ldots\ldots, x_r, y_1, y_2, \ldots\ldots, y_{m-r}, z_1, z_2, \ldots\ldots, z_{n-r} \in F$

Let,
$$x_1v_1 + x_2v_2 + \ldots.. + x_rv_r + y_1u_1 + y_2u_2 + \ldots.. + y_{m-r}u_{m-r} + z_1w_1 + z_2w_2 + \ldots.. + z_{n-r}w_{n-r} = 0 \quad \ldots\ldots\ldots\ldots (2)$$

$$\Rightarrow x_1v_1 + x_2v_2 + \ldots.. + x_rv_r + y_1u_1 + y_2u_2 + \ldots.. + y_{m-r}u_{m-r} = - z_1w_1 - z_2w_2 - \ldots.. - z_{n-r}w_{n-r} \quad \ldots\ldots\ldots\ldots (3)$$

But **L.H.S.** of **(3)** is a vector of U and **R.H.S.** of **(3)** is a vector of W. This implies that both belongs to $(U \cap W)$.

Therefore, for any scalars $t_1, t_2, \ldots\ldots\ldots, t_r \in F$

$$- z_1w_1 - z_2w_2 - \ldots\ldots - z_{n-r}w_{n-r} = t_1v_1 + t_2v_2 + \ldots\ldots + t_rv_r$$

$$\Rightarrow t_1v_1 + t_2v_2 + \ldots\ldots + t_rv_r + z_1w_1 + z_2w_2 + \ldots.. + z_{n-r}w_{n-r} = 0 \quad \ldots\ldots\ldots\ldots (4)$$

But $\{v_1, v_2, \ldots\ldots, v_r, w_1, w_2, \ldots\ldots, w_{n-r}\}$ is a basis of W and is linearly independent.

$$\Rightarrow z_1 = z_2 = \ldots\ldots = z_{n-r} = t_1 = t_2 = \ldots\ldots = t_r = 0$$

Then **(2)** becomes,

$$\boldsymbol{x_1v_1 + x_2v_2 + \ldots\ldots + x_rv_r + y_1u_1 + y_2u_2 + \ldots\ldots + y_{m-r}u_{m-r} = 0} \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(5)}$$

But $\{v_1, v_2, \ldots\ldots, v_r, u_1, u_2, \ldots\ldots, u_{m-r}\}$ is a basis of U and so linearly independent.

Then $x_1 = x_2 = \ldots\ldots = x_r = y_1 = y_2 = \ldots\ldots = y_{m-r} = 0$

$\Rightarrow$ $A = \{v_i, u_j, w_k\}$ is linearly independent

$\Rightarrow$ A is a basis of $U + W$

$\Rightarrow$ **dim** $(U + W) = m + n - r$

$\therefore$ **dim** $(U + W) =$ **dim** U + **dim** W – **dim** $(U \cap W)$

[Proved]

Problem Solving:

Problem#01:

Let, **V** be the subspace of $\mathbb{R}^3$spanned by the vectors $(\mathbf{1}, \mathbf{2}, \mathbf{1}), (\mathbf{0}, -\mathbf{1}, \mathbf{0})$ and $(\mathbf{2}, \mathbf{0}, \mathbf{2})$. Find basis and the dimension of **V**.

Sol^n :

Form the matrix whose rows are given vectors and reduce the matrix to roe echelon form

$$\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & -\mathbf{1} & \mathbf{0} \\ \mathbf{2} & \mathbf{0} & \mathbf{2} \end{bmatrix}$$

We multiply first row by **2** and then subtract from the third row.

$$\sim \quad \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{4} & \mathbf{0} \end{bmatrix}$$

We multiply second row by **4** and subtract from the third row

$$\sim \quad \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

We multiply second row by **2** and add with the first row

$$\sim \quad \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

We multiply second row by **-1**

$$\sim \quad \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

This matrix is in echelon form and the non-zero rows in the matrix are **(1, 0, 1)** and **(0, 1, 0)**. These non-zero rows form a basis of the row space and consequently a basis of **V**; that is Basis of **V**=$\{(\mathbf{1},\mathbf{0},\mathbf{1}),(\mathbf{0},\mathbf{1},\mathbf{0})\}$ and dim **U=2**.

[Answer].

Problem #02:

Find the basis and dimension of the vector set,

$\boldsymbol{S} = \{(-\mathbf{1},\mathbf{2},-\mathbf{1},\mathbf{0}),(\mathbf{0},\mathbf{3},\mathbf{1},\mathbf{2}),(\mathbf{1},\mathbf{1},-\mathbf{2},\mathbf{2}),(\mathbf{2},\mathbf{1},\mathbf{0},-\mathbf{1})\}$

Sol^n:

Form the matrix whose rows are given vectors and reduce the matrix to row echelon form.

$$\begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 2 \\ 1 & 1 & -2 & 2 \\ 2 & 1 & 0 & -1 \end{bmatrix}$$

$$\begin{matrix} R_3' \to R_3 + R_1 \\ R_4' \to R_4 + 2R_1 \end{matrix} \sim \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 2 \\ 0 & 3 & -3 & 2 \\ 0 & 5 & -2 & -1 \end{bmatrix}$$

$$\begin{matrix} R_3' \to R_3 - R_2 \\ R_4' \to 3R_4' - 5R_2 \end{matrix} \sim \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & -11 & -13 \end{bmatrix}$$

$$R_4' \to 4R_4 - 11R_3 \sim \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & -0 & -52 \end{bmatrix}$$

This matrix is in echelon form and the non-zero rows in the matrix are,

$(-\mathbf{1},\mathbf{2},-\mathbf{1},\mathbf{0})$, $(\mathbf{0},\mathbf{3},\mathbf{1},\mathbf{2})$, $(\mathbf{0},\mathbf{0},-\mathbf{4},\mathbf{0})$ *and* $(\mathbf{0},\mathbf{0},\mathbf{0},-\mathbf{52})$.

These non-zero rows form a basis of the row space and consequently a basis of **S**; that is Basis of $\boldsymbol{S} = \{(-\mathbf{1}, \mathbf{2}, -\mathbf{1}, \mathbf{0}), (\mathbf{0}, \mathbf{3}, \mathbf{1}, \mathbf{2}), (\mathbf{0}, \mathbf{0}, -\mathbf{4}, \mathbf{0}), (\mathbf{0}, \mathbf{0}, \mathbf{0}, -\mathbf{52})\}$
And **dim**$\boldsymbol{S} = \mathbf{4}$.

[Answer].

H.W

Problem #01:

Find the basis and dimension of the vector sets,
***V* = {(1, -2, 4, 1), (2, -3, 9, -1), (1, 0, 6, -5), (2, -5, 7, 5)}**

Sol^n:

Form the matrix whose rows are given vectors and reduce the matrix to row echelon form.

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 2 & -3 & 9 & -1 \\ 1 & 0 & 6 & -5 \\ 2 & -5 & 7 & 5 \end{bmatrix}$$

$$\begin{matrix} R_2' \to R_2 - 2R_1 \\ R_3' \to R_3 - R_1 \\ R_4' \to R_4 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 2 & -6 \\ 0 & -1 & -1 & 3 \end{bmatrix}$$

$$\begin{matrix} R_3' \to R_3 - 2R_2 \\ R_4' \to R_4 + R_2 \end{matrix} \sim \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in echelon form and the non-zero rows in the matrix are **(1, -2, 4, 1)** and **(0, 1, 1, -3)**. These non-zero rows form a basis of the row space and consequently a basis of ***V***; that is,
Basis of ***V*** = **{(1, -2, 4, 1), (0, 1, 1, -3)}** and dim ***V* = 2**.

[Answer].

Problem #02:

Find the basis and dimension of the vector sets, ***W* = {(1, 2, 1), (3, 1, 2), (1, -3, 4)}**

Sol^n:

Form the matrix whose rows are given vectors and reduce the matrix to row echelon form.

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 1 & -3 & 4 \end{bmatrix}$$

$$\begin{matrix} R_2' \to R_2 - 3R_1 \\ R_3' \to R_3 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & -5 & 3 \end{bmatrix}$$

$$R_3' \to R_3 - R_2 \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

This matrix is in echelon form and the non-zero rows in the matrix are **(1, 2, 1), (0, -5, -1)** and **(0, 0, 4)**. These non-zero rows form a basis of the row space and consequently a basis of ***W***; that is,
Basis of ***W* = {(1, 2, 1), (0, -5, -1), (0, 0, 4)}** and dim ***W* = 3**.

[Answer].

Problem #03:

Find the basis and dimension of the vector sets,
***S* = {(1, -2, 0, 0, 3), (2, -5, -3, -2, 6), (0, 5, 15, 10, 0), (2, 6, 18, 8, 6)}**.

Sol^n:

Form the matrix whose rows are given vectors and reduce the matrix to row echelon form.

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

$$R_2' \to R_2 - 2R_1 \quad R_4' \to R_4 - 2R_1 \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 5 & 15 & 10 & 0 \\ 0 & 10 & 18 & 8 & 0 \end{bmatrix}$$

$$R_3' \to R_3 + 5R_2 \quad R_4' \to R_4 + 10R_2 \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & -12 & 0 \end{bmatrix}$$

$$R_3 \leftrightarrow R_4 \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 0 & -12 & -12 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in echelon form and the non-zero rows in the matrix are **(1, -2, 0, 0, 3), (0, -1, -3, -2, 0)** and **(0, 0, -12, -12, 0)**. These non-zero rows form a basis of the row space and consequently a basis of ***S***; that is,
Basis of ***S*** **= (1, -2, 0, 0, 3), (0, -1, -3, -2, 0), (0, 0, -12, -12, 0)}** and dim ***S*** **= 3**.

***[Answer]*.**

Problem #04:

Find the basis and dimension of the vector sets,
T **= {(1, -2, 5, -3), (2, 3, 1, 4), (3, 8, -3, -5)}**.

Sol^n:

Form the matrix whose rows are given vectors and reduce the matrix to row echelon form.

$$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & 4 \\ 3 & 8 & -3 & -5 \end{bmatrix}$$

$$R_2' \to R_2 - 2R_1 \quad R_3' \to R_3 - 3R_1 \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 10 \\ 0 & 14 & -18 & 4 \end{bmatrix}$$

$$R_3' \to R_3 - 2R_2 \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 10 \\ 0 & 0 & 0 & -16 \end{bmatrix}$$

This matrix is in echelon form and the non-zero rows in the matrix are **(1, -2, 5, -3), (0, 7, -9, 10)** and **(0, 0, 0, -16)**.
These non-zero rows form a basis of the row space and consequently a basis of ***T***; that is,
Basis of ***T*** = **{(1, -2, 5, -3), (0, 7, -9, 10), (0, 0, 0, -16)}** and dim ***T*** **= 3**.

[Answer].

Problem #05:

Find the basis and dimension of the vector sets, ***U* = {(1, 1, 1), (1, 2, 3), (3, 4, 5)}**.

Sol^n:

Form the matrix whose rows are given vectors and reduce the matrix to row echelon form.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}$$

$$\begin{matrix} R_2' \to R_2 - R_1 \\ R_3' \to R_3 - 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$R_3' \to R_3 - R_2 \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is in echelon form and the non-zero rows in the matrix are **(1, 1, 1)** and **(0, 1, 2)**. These non-zero rows form a basis of the row space and consequently a basis of ***U***; that is,
Basis of ***U* = {(1, 1, 1), (0, 1, 2)}** and dim ***U* = 2**.

[Answer].

Problem Solving:

Problem #01:

Determine a basis and the dimension for the solution space of the homogeneous system.

$$\begin{aligned} x - 3y + z &= 0 \\ 2x - 6y + 2z &= 0 \\ 3x - 9y + 3z &= 0 \end{aligned}$$

Sol^n:

Given system of linear equation,

$$\begin{aligned} x - 3y + z &= 0 \\ 2x - 6y + 2z &= 0 \\ 3x - 9y + 3z &= 0 \end{aligned}$$

Reduce the system to echelon form. We multiply first equation by **2** and **3** and then subtract from the second and third equations respectively. Then we get

$$\begin{aligned} x - 3y + z &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

$$\textit{i.e.}\ x - 3y + z = 0$$

the system is in echelon form and has only one non-zero equation in three unknowns.
So the system has $3 - 1 = 2$ free variable which are y and z.
Hence the dimension of the solution space is 2.
Set $(i)\ y = 1,\ z = 0\ (ii)\ y = 0,\ z = 1$, to obtain the solution.
Solutions $V_1 = (3, 1, 0),\ V_2 = (-1, 0, 1)$
Hence the set $\{(3, 1, 0),\ (-1, 0, 1)\}$ is a basis of the solution space.

[Answer].

H.W

Problem #01:

Find the dimension and basis of the solution space of the following homogenous system

$$\begin{gathered} x + 2y - 4z + 3s - t = 0 \\ x + 2y - 2z + 2s + t = 0 \\ 2x + 4y - 2z + 3s + 4t = 0 \end{gathered}$$

Sol^n:

Given system of linear equation,

$$\begin{gathered} x + 2y - 4z + 3s - t = 0 \\ x + 2y - 2z + 2s + t = 0 \\ 2x + 4y - 2z + 3s + 4t = 0 \end{gathered}$$

Reducing the system to echelon form,

$$\begin{matrix} L_2' \to L_2 - L_1 \\ L_3' \to L_3 - 2L_1 \end{matrix} \quad \sim \quad \begin{cases} x + 2y - 4z + 3s - t = 0 \\ \quad 2z - s + 2t = 0 \\ \quad 6z - 3s + 6t = 0 \end{cases}$$

$$\begin{matrix} L_3' \to L_3 - 3L_2 \\ L_1' \to L_1 - 2L_2 \end{matrix} \quad \sim \quad \begin{cases} x + 2y + s + 3t = 0 \\ \quad 2z - s + 2t = 0 \\ \quad 0 = 0 \end{cases}$$

The system is in echelon form and has **2** non-zero equations in **5** unknowns. So the system has **5 – 2 = 3** free variable which are ***y, z, t***.

Let, **y= 1, z= 0, t=0** then**, x= -2, s= 0.**

y= 0, z= 1, t=0 then, **x= -2, s= 2**

y= 0, z= 0, t=1 then, **x=-5, s=2**

Therefore, **(x, y, z, s, t) = {(-2, 1, 0, 0, 0), (-2, 0, 1, 2, 0), (-5, 0, 0, 2, 1)}**.
Hence a basis of the solution space is **{(-2, 1, 0, 0, 0), (-2, 0, 1, 2, 0), (-5, 0, 0, 2, 1)}** and dimension of the system is **3.**

[Answer].

Problem #02:

Find the dimension and a basis of the solution space of the following homogeneous system.

$$\begin{gathered} x+y-t=0 \\ x+2y+3z=0 \\ 2x+3y+3z+t=0 \end{gathered}$$

Sol^n:

Given system of linear equation,

$$\begin{cases} x+y-t=0 \\ x+2y+3z=0 \\ 2x+3y+3z+t=0 \end{cases}$$

Reducing the system to echelon form we get,

$$\begin{matrix} L_2' \to L_2 - L_1 \\ L_3' \to L_3 - 2L_1 \end{matrix} \quad \sim \begin{cases} x+y-t=0 \\ y+3z+t=0 \\ y+3z+3t=0 \end{cases}$$

$$L_3' \to L_3 - L_2 \quad \sim \begin{cases} x+y-t=0 \\ y+3z+t=0 \\ 2t=0 \end{cases}$$

$$i.e. \sim \begin{cases} x+y-t=0 \\ y+3z+t=0 \\ t=0 \end{cases}$$

The system is in echelon form and has **3** non-zero equations in **4** unknowns. So the system has **4 – 3 = 1** free variable which is ***z***.

Let, **z = 1**, then **x = 3**, **y = -3**, **t = 0**.

Therefore, **(x, y, z, t) = (3, -3, 1, 0)**.

Hence a basis of the solution space is **{(3, -3, 1, 0)}** and the dimension is **1**.

[Answer].

Problem #03:

Find the dimension and basis of the solution space of the following homogenous system

$$\begin{aligned} x + y + z &= 0 \\ x + 2y + 3z &= 0 \\ 3x + 4y + 5z &= 0 \end{aligned}$$

Sol^n:

Given system of linear equation,

$$\begin{cases} x + y + z = 0 \\ x + 2y + 3z = 0 \\ 3x + 4y + 5z = 0 \end{cases}$$

Reducing the system to echelon form we get,

$$\begin{matrix} L_2' \to L_2 - L_1 \\ L_3' \to L_3 - 3L_1 \end{matrix} \quad \sim \quad \begin{cases} x + y + z = 0 \\ \quad y + 2z = 0 \\ \quad y + 2z = 0 \end{cases}$$

$$\textit{i.e.} \quad \sim \quad \begin{cases} x + y + z = 0 \\ \quad y + 2z = 0 \end{cases}$$

The system is in echelon form and has **2** non-zero equations in **3** unknowns. So the system has **3 – 2 = 1** free variable which is ***z***.

Let, **z = 1**, then **y = -2, x = 1**.

Therefore, **(x, y, z) = (1, -2, 1)**.

Hence a basis of the solution space is **{(1, -2, 1)}** and the dimension is **1**.

[Answer].

Linear Transformation

Linear Transformation:

Linear Transformation let, **U** and **V** be two vector spaces over the same field **F**. A Linear Transformation **T** of **U** into **V** written as,

T:U $\longrightarrow$ **V** , is a Transformation **T** of **U** into **V** such that

1. $\mathbf{T(u_1 + u_2) = T(u_1) + T(u_2)}$ **for all** $\mathbf{u_1, u_2 \in U}$
2. $\mathbf{T(\alpha u) = \alpha T(u)}$ **for all** $\mathbf{u \in U}$ **and all** $\boldsymbol{\alpha} \in \mathbf{F}$

Kernel of a Linear Transformation / mapping:

Let **T: V(F)** $\longrightarrow$ **U(F)** be a linear transformation , Then Kernel of transformation or Ker**T** is defined by Ker**T = {v ∈ V(F) : T(v)= 0}**

Example: Let **T:** $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation and defined by **T(x, y, z) = (0, y, z)**. here Ker**T = {(x, 0, 0): x∈ℝ} = x axis** .

Image of linear transformation:

Let **T: V(F) →U(F)** be a linear transformation Then image of transformation or **ImT** is defined by

Im**T = { u ∈ U(F) : T(v) = u,v ∈ V(F) }**

Example: Let **T:** $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation and defined by **T(x, y) = (x, 0)**. Hear Im**T = { (x, 0) : x∈ ℝ } =x axis**.

Problem Solving:

Problem#01:

Show that the following transformation defined a linear operator on $\mathbb{R}^3$,

T(x, y, z)= (x+y, -x-y, z).

Sol^n:

Let **U=** $(\mathbf{x_1, y_1, z_1})$and **V=**$(\mathbf{x_2, y_2, z_2})$

Then **U+V** = $(x_1, y_1, z_1) + (x_2, y_2, z_2)$

= $(x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$

And **αU** = $\alpha(x_1,\ y_1,\ z_1)$ = $(\alpha x_1,\ \alpha y_1,\ \alpha z_1)$ Where **α ∈ F**

Thus **T(U)** = **T** $(x_1,\ y_1,\ z_1)$ = $(x_1 + y_1,\ -x_1 - y_1,\ z_1)$

T(V) = **T** $(x_2,\ y_2,\ z_2)$ = $(x_2 + y_2,\ -x_2 - y_2,\ z_2)$

T (U+V) = **T** $(x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$

= $\{ (x_1 + x_2) + (y_1 + y_2),\ -(x_1 + x_2) - (y_1 + y_2),\ (z_1 + z_2)\}$

= $(x_1 + y_1,\ -x_1 - y_1,\ z_1) + (x_2 + y_2,\ -x_2 - y_2,\ z_2)$

= **T(U) + T(V)**

Also for any **α ∈ F**

T (αU)= **T** $(\alpha x_1,\ \alpha y_1,\ \alpha z_1)$

= $(\alpha x_1 + \alpha y_1, -\alpha x_1 - \alpha y_1,\ \alpha z_1)$

= **α** $(x_1 + y_1,\ -x_1 - y_1,\ z_1)$

=**αT(U)**

Since **U**, **V** and **α** are arbitrary So, **T** is a linear operator.

[Showed].

Problem#02:

Let **T:** $\mathbb{R}^3 \rightarrow \mathbb{R}^5$ be a defined as **T**$(x_1,\ x_2,\ x_3)$ = $(x_1 - x_2,\ 0,\ x_1 - x_3,\ x_2,\ 0)$. Show that **T** is a linear transformation.

Sol^n:

Let **U**= $(x_1,\ x_2,\ x_3)$ And **V**= $(x_1',\ x_2',\ x_3')$

Then **T(U)+T(V)** = $T(x_1, x_2, x_3) + T(x_1', x_2', x_3')$

= $(x_1 - x_2,\ 0,\ x_1 - x_3,\ x_2, 0)$ + $(x_1' - x_2',\ 0,\ x_1' - x_3',\ x_2', 0)$

= $(x_1 + x_1' - x_2 - x_2',\ 0,\ x_1 + x_1' - x_3 - x_3',\ x_2 + x_2', 0)$ .... **(1)**

Again **T(U+V)** = **T**$(x_1 + x_1',\ x_2 + x_2',\ x_3 + x_3')$

= $(x_1 + x_1' - x_2 - x_2',\ 0,\ x_1 + x_1' - x_3 - x_3',\ x_2 + x_2',\ 0)$ .... **(2)**

Now from **(1)** and **(2)** We get

$$\mathbf{T(U+V) = T(U)+T(V)}$$

Again for any scalar $\boldsymbol{\alpha} \in \mathbf{F}$

$\mathbf{T(\alpha U)} = \mathbf{T}\ (\alpha x_1,\ \alpha x_2,\ \alpha x_3)$

$= (\alpha x_1 - \alpha x_2,\ 0,\ \alpha x_1 - \alpha x_3,\ \alpha x_2, 0)$

$= \alpha\ \ (x_1 - x_2, 0,\ x_1 - x_3,\ x_2, 0)$

$= \alpha T(U)$

Since **U**, **V** and **α** are arbitrary So, **T** is a linear operator.

[Showed].

H.W

Problem #01:

Let $\mathbf{T}$: $\mathbb{R}^2 \ \rightarrow\ \mathbb{R}^2$ be a defined as $\mathbf{T}(x, y) = (x + 2y,\ 2x - y)$. Show that **T** is a linear transformation.

Sol^n:

Let, $U = (x_1, y_1)$ and $V = (x_2, y_2)$

Then, $\mathbf{T(U) + T(V)} = \mathbf{T}(x_1, y_1)\ +\ \mathbf{T}(x_2, y_2)$

$= ,(x_1 + 2y_1,\ 2x_1 - y_1) + (x_2 + 2y_2,\ 2x_2 - y_2)$

$= \ (x_1 + 2y_1\ + x_2 + 2y_2,\ 2x_1 - y_1\ + 2x_2 - y_2)$

.... **(1)**

$\mathbf{T(U+V)}\ = \mathbf{T}(x_1 + x_2, y_1 + y_2)$

$= \{(x_1 + x_2) + 2(y_1 + y_2), 2(x_1 + x_2) - (y_1 + y_2)\}$

$= (x_1 + 2y_1\ + x_2 + 2y_2,\ 2x_1 - y_1\ + 2x_2 - y_2)$

.... **(2)**

Now from eq. **(1)** & **(2)** we get,

$$\mathbf{T(U+V) = T(U)+T(V)}$$

Again for any scalar $\boldsymbol{\alpha} \in \mathbf{F}$

$$\begin{aligned}
\mathbf{T}(\alpha\mathbf{U}) &= \mathbf{T}\,(\alpha x_1,\ \alpha y_1) \\
&= (\alpha x_1 + 2\alpha y_1,\ 2\alpha x_1 - \alpha y_1) \\
&= \alpha\,(x_1 + 2y_1,\ 2x_1 - y_1) \\
&= \alpha\,T(U)
\end{aligned}$$

Since **U**, **V** and **α** are arbitrary So, ***T*** is a linear operator.

[Showed].

Problem #02:

Let **T:** $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a defined as $\mathbf{T}(x,\ y,\ z) = (x + 2y,\ y - z,\ x + 2z)$. Show that **T** is a linear transformation.

Sol^n:

Let $\mathbf{U} = (x_1,\ y_1,\ z_1)$ and $\mathbf{V} = (x_2,\ y_2,\ z_2)$

$$\begin{aligned}
\text{Then } \mathbf{U}+\mathbf{V} &= (x_1, y_1, z_1) + (x_2, y_2, z_2) \\
&= (x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)
\end{aligned}$$

And $\boldsymbol{\alpha}\mathbf{U} = \boldsymbol{\alpha}(x_1,\ y_1,\ z_1) = (\alpha x_1,\ \alpha y_1,\ \alpha z_1)$ Where $\boldsymbol{\alpha} \in \mathbf{F}$

Thus, $\mathbf{T}(\mathbf{U}) = \mathbf{T}\,(x_1,\ y_1,\ z_1) = (x_1 + 2y_1,\ y_1 - z_1,\ x_1 + 2z_1)$

$\mathbf{T}(\mathbf{V}) = \mathbf{T}\,(x_2,\ y_2,\ z_2) = (x_2 + 2y_2,\ y_2 - z_2,\ x_2 + 2z_2)$

$$\begin{aligned}
\mathbf{T}\,(\mathbf{U}+\mathbf{V}) &= \mathbf{T}\,(x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2) \\
&= \{\,(x_1 + x_2) + 2(y_1 + y_2),\ (y_1 + y_2) - (z_1 + z_2),\ (x_1 + x_2) + 2(z_1 + z_2)\} \\
&= (x_1 + 2y_1,\ y_1 - z_1,\ x_1 + 2z_1) + (x_2 + 2y_2,\ y_2 - z_2,\ x_2 + 2z_2) \\
&= \mathbf{T}(\mathbf{U}) + \mathbf{T}(\mathbf{V})
\end{aligned}$$

Also for any $\boldsymbol{\alpha} \in \mathbf{F}$

$$\begin{aligned}
\mathbf{T}\,(\alpha\mathbf{U}) &= \mathbf{T}\,(\alpha x_1,\ \alpha y_1,\ \alpha z_1) \\
&= (\alpha x_1 + 2\alpha y_1,\ \alpha y_1 - \alpha z_1,\ \alpha x_1 + 2\alpha z_1) \\
&= \alpha\,(x_1 + 2y_1,\ y_1 - z_1,\ x_1 + 2z_1) \\
&= \alpha\mathbf{T}(\mathbf{U})
\end{aligned}$$

Since **U**, **V** and **α** are arbitrary So, **T** is a linear operator.

[Showed].

Problem #03:

Let **T:** $\mathbb{R}^3 \rightarrow \mathbb{R}^5$ be a defined as **T**$(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$ = $(\mathbf{x_1} - \mathbf{x_2}, \mathbf{0}, \mathbf{x_1} - \mathbf{x_3}, \mathbf{x_2}, \mathbf{0})$. Show that **T** is a linear transformation.

Sol^n^:

Let **U**= $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$ and **V**= $(\mathbf{x_1'}, \mathbf{x_2'}, \mathbf{x_3'})$

Then **T(U)+T(V)** = **T**$(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}) + \mathbf{T}(\mathbf{x_1'}, \mathbf{x_2'}, \mathbf{x_3'})$

$= (\mathbf{x_1} - \mathbf{x_2}, \mathbf{0}, \mathbf{x_1} - \mathbf{x_3}, \mathbf{x_2}, \mathbf{0}) + (\mathbf{x_1'} - \mathbf{x_2'}, \mathbf{0}, \mathbf{x_1'} - \mathbf{x_3'}, \mathbf{x_2'}, \mathbf{0})$

$= (\mathbf{x_1} + \mathbf{x_1'} - \mathbf{x_2} - \mathbf{x_2'}, \mathbf{0}, \mathbf{x_1} + \mathbf{x_1'} - \mathbf{x_3} - \mathbf{x_3'}, \mathbf{x_2} + \mathbf{x_2'}, \mathbf{0})$ .... **(1)**

Again **T(U+V)** = **T**$(\mathbf{x_1} + \mathbf{x_1'}, \mathbf{x_2} + \mathbf{x_2'}, \mathbf{x_3} + \mathbf{x_3'})$

$= (\mathbf{x_1} + \mathbf{x_1'} - \mathbf{x_2} - \mathbf{x_2'}, \mathbf{0}, \mathbf{x_1} + \mathbf{x_1'} - \mathbf{x_3} - \mathbf{x_3'}, \mathbf{x_2} + \mathbf{x_2'}, \mathbf{0})$ .... **(2)**

Now from **(1)** and **(2)** We get,

T(U+V) = T(U)+T(V)

Again for any scalar $\boldsymbol{\alpha} \in \mathbf{F}$

T(αU)= **T** $(\boldsymbol{\alpha}\mathbf{x_1}, \boldsymbol{\alpha}\mathbf{x_2}, \boldsymbol{\alpha}\mathbf{x_3})$

$= (\boldsymbol{\alpha}\mathbf{x_1} - \boldsymbol{\alpha}\mathbf{x_2}, \mathbf{0}, \boldsymbol{\alpha}\mathbf{x_1} - \boldsymbol{\alpha}\mathbf{x_3}, \boldsymbol{\alpha}\mathbf{x_2}, \mathbf{0})$

$= \boldsymbol{\alpha} \ (\boldsymbol{x_1} - \boldsymbol{x_2}, \mathbf{0}, \boldsymbol{x_1} - \boldsymbol{x_3}, \boldsymbol{x_2}, \mathbf{0})$

$= \boldsymbol{\alpha} \, \boldsymbol{T}(\boldsymbol{U})$

Since **U**, **V** and **α** are arbitrary So, **T** is a linear operator.

[Showed].

Problem #04:

Let **T:** $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a defined as **T**$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ = $(\boldsymbol{x} + \boldsymbol{y}, \boldsymbol{y} + \boldsymbol{z}, \boldsymbol{z} + \boldsymbol{x})$. Show that **T** is a linear transformation.

Sol^n^:

Let **U**= $(\mathbf{x_1}, \mathbf{y_1}, \mathbf{z_1})$and **V**=$(\mathbf{x_2}, \mathbf{y_2}, \mathbf{z_2})$

Then **U+V** = $(\mathbf{x_1}, \mathbf{y_1}, \mathbf{z_1}) + (\mathbf{x_2}, \mathbf{y_2}, \mathbf{z_2})$

$= (\mathbf{x_1} + \mathbf{x_2}, \mathbf{y_1} + \mathbf{y_2}, \mathbf{z_1} + \mathbf{z_2})$

And **αU** = $\alpha(x_1, y_1, z_1)$ = $(\alpha x_1, \alpha y_1, \alpha z_1)$ Where **α ∈ F**

Thus, **T(U)** = **T** (x_1, y_1, z_1) = $(x_1 + y_1, y_1 + z_1, z_1 + x_1)$

T(V) = **T** (x_2, y_2, z_2) = $(x_2 + y_2, y_2 + z_2, z_2 + x_2)$

T (U+V) = **T** $(x_1 + x_2, y_1 + y_2, z_1 + z_2)$
= $\{ (x_1 + x_2) + (y_1 + y_2), (y_1 + y_2) + (z_1 + z_2), (z_1 + z_2) + (x_1 + x_2)\}$
= $(x_1 + y_1, y_1 + z_1, z_1 + x_1) + (x_2 + y_2, y_2 + z_2, z_2 + x_2)$
= **T(U) + T(V)**

Also for any **α ∈ F**
T (αU)= **T** $(\alpha x_1, \alpha y_1, \alpha z_1)$
= $(\alpha x_1 + \alpha y_1, \alpha y_1 + \alpha z_1, \alpha z_1 + \alpha x_1)$
= **α** $(x_1 + y_1, y_1 + z_1, z_1 + x_1)$
= **αT(U)**

Since **U**, **V** and **α** are arbitrary So, **T** is a linear operator.

[Showed].

Problem Solving:

Problem#01:

Let **T:** $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation when **T**$(1, 1)$ = **3** and **T**$(0, 1)$ = **-2** then find **T** (a, b).

Sol^n:

Here **{ (1, 1) , (0, 1) }** is a basis of $\mathbb{R}^2$.

Let, **(a, b) = x(1, 1)+y(0, 1) = (x, x+y)**
$\Longrightarrow$ **x =a, x+y = b**
$\therefore$ **x = a, y = b – a**

Now Since **T** is Linear transformation,
T(a, b) = x T(1, 1) + y T(0, 1) = 3x – 2y = 3a – 2(b-a) = 5a-2b

[Answer].

Problem#02:

Let **T:** $V_2(R) \rightarrow V_3(R)$ be a linear transformation Where **T(1,2) = (3, -1, 5)** and **T(0,1) = (2, 1, -1)** Then find **T(a, b)**.

Sol^n:

Here **{(1, 2), (0, 1)}** is a basis of $V_2(R)$

Now let,

$$(a, b) = x(1, 2) + y(0, 1) = (x, 2x+y)$$
$$\Longrightarrow x = a, \quad 2x + y = b$$
$$\therefore x = a, \quad y = b-2a$$

Now using the condition of given linear transformation we get,

$$\begin{aligned} T(a, b) &= x\,T(1, 2) + y\,T(0, 1) \\ &= x(3, -1, 5) + y(2, 1, -1) \\ &= (3x, -x, 5x) + (2y, y, -y) \\ &= (3x+2y, -x+y, 5x-y) \\ &= (2b-a, b-3a, 7a-b) \end{aligned}$$

[Answer].

H.W

Problem#01:

Let **T:** $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear mapping, Where **T(0, 1) = (0, 0)** and **T(1, 1) = (1, 1)** Then find **T(a, b)**.

Sol^n:

Here. **{(0, 1), (1, 1)}** is a basis of $\mathbb{R}^2$

Let, $(a, b) = x(0, 1) + y(1, 1) = (y, x+y)$

$$\Longrightarrow y=a, \quad x+y=b$$
$$\therefore y=a, \quad x=b-a$$

Now, using the condition of given linear transformation we get,

T(a, b) = x. T(0, 1) + y. T(1, 1)

= x(0, 0) + y(1, 1)

= 0 + (y, y))

= (a, a)

***[Answer]*.**

Matrix Representation of a Linear Transformation

Definition:

Let **T: V(F) →V(F)** be a linear operator , let $\{e_1, e_2, \dots\dots\dots\dots.. e_n\}$ be a set of basis of vector space **V(F)**. Then **T(e_1), T(e_2) ...., T(e_n) ∈ V (F)**.

Where each vector **T(e_1) , T(e_2) ...., T(e_n)** will be express as a linear combination of the set $\{e_1, e_2, \dots\dots\dots\dots.. e_n\}$.

Then we get

$$\left.\begin{aligned} T(e_1) &= a_{11}e_1 + a_{12}e_2 + \cdots \dots\dots\dots + a_{1n}e_n \\ T(e_2) &= a_{22}e_1 + a_{22}e_2 + \cdots \dots\dots\dots\dots\dots + a_{2n}e_n \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ T(e_n) &= a_{n1}e_1 + a_{n2}e_2 + \cdots \dots\dots\dots\dots\dots + a_{nn}e_n \end{aligned}\right\} \; , a_{ij} \in F$$

The Transpose matrix of The co-efficient matrix of the above equation is said to be a Matrix Representation of **T**, and it is Denoted by $[\,T\,]_e$ or Shortly ***[T].***

$$[\,T\,]_e = [\,T\,] = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix}^t = \begin{bmatrix} a_{11} & a_{21} & a_{n1} \\ a_{12} & a_{22} & a_{n2} \\ \dots & \dots & \dots \\ a_{1n} & a_{2n} & a_{nn} \end{bmatrix}$$

Problem Solving:

Problem#01:

Let **T:** $\mathbb{R}^3 \to \mathbb{R}^3$ is a linear operator and define by **T(x, y, z) = (x+y, y, z)**. Then find Matrix representation with respect to the standard basis of $\mathbb{R}^3$.

Sol^n:

Let $\{(e_1 = (1, 0, 0)\,,\; e_2 = (0, 1, 0),\; e_3 = (0, 0, 1)\}$ be a standard basis of $\mathbb{R}^3$.

Then from **T(x, y, z) = (x+y, y, z)** we get,

$T(e_1) = T(1, 0, 0) = (1, 0, 0) = \mathbf{1}.e_1 + \mathbf{0}.e_2 + \mathbf{0}.e_3$

$T(e_2) = T(0, 1, 0) = (1, 1, 0) = \mathbf{1}.e_1 + \mathbf{1}.e_2 + \mathbf{0}.e_3$

$T(e_3) = T(0, 0, 1) = (0, 0, 1) = \mathbf{0}.e_1 + \mathbf{0}.e_2 + \mathbf{1}.e_3$

Therefore the matrix representation of **T** is

$$[\,T\,]_e = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^t = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[Answer].

Problem#02:

Let **T:** $\mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation which define by **T(x, y, z, t) = (x-y+z, x+2y, z-t)**. Then find Matrix representation of **T** with respect to the standard basis of $\mathbb{R}^4$ *and* $\mathbb{R}^3$.

Sol^n:

Let $\{(\boldsymbol{e_1} = (\mathbf{1,0,0,0}),\ \boldsymbol{e_2} = (\mathbf{0,1,0,0}),\ \boldsymbol{e_3} = (\mathbf{0,0,1,0}),\ \boldsymbol{e_3} = (\mathbf{0,0,0,1})\}$ and $\{(\boldsymbol{f_1} = (\mathbf{1,0,0}),\ \boldsymbol{f_2} = (\mathbf{0,1,0}),\ \boldsymbol{f_3} = (\mathbf{0,0,1})\}$ are standard basis of $\mathbb{R}^4$ and $\mathbb{R}^3$ respectively.

Then,

T$(\boldsymbol{e_1}) = \boldsymbol{T}(\mathbf{1,0,0,0}) = (\mathbf{1,\ 1,\ 0}) = \mathbf{1.}\,\boldsymbol{f_1} + \mathbf{1.}\,\boldsymbol{f_2} + \mathbf{0.}\,\boldsymbol{f_3}$

T$(\boldsymbol{e_2}) = \boldsymbol{T}(\mathbf{0,1,0,0}) = (\mathbf{-1,\ 2,\ 0}) = \mathbf{-1.}\,\boldsymbol{f_1} + \mathbf{2.}\,\boldsymbol{f_2} + \mathbf{0.}\,\boldsymbol{f_3}$

T$(\boldsymbol{e_3}) = \boldsymbol{T}(\mathbf{0,0,1,0}) = (\mathbf{1,\ 0,\ 1}) = \mathbf{1.}\,\boldsymbol{f_1} + \mathbf{0.}\,\boldsymbol{f_2} + \mathbf{1.}\,\boldsymbol{f_3}$

T$(\boldsymbol{e_4}) = \boldsymbol{T}(\mathbf{0,0,0,1}) = (\mathbf{0,\ 0,\ -1}) = \mathbf{0.}\,\boldsymbol{f_1} + \mathbf{0.}\,\boldsymbol{f_2} - \mathbf{1.}\,\boldsymbol{f_3}$

$$[\,T\,]_e^f = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}^t = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

[Answer].

Problem#03:

Let **T: V₃(R) →V₃(R)** be a linear operator which define by **T(x, y, z) = (2y+z, x-4y, 3x)**. Then find Matrix representation for the **basis**$\{(\boldsymbol{f_1} = (\mathbf{1,1,1}),\ \boldsymbol{f_2} = (\mathbf{1,1,0}),\ \boldsymbol{f_3} = (\mathbf{1,0,0})\}$.

Sol^n:

Let **(a, b, c)** ∈ **V₃**$(\mathbb{R}^3)$

Then, **(a, b, c) = x f_1 + y f_2 + z f_3 , Where x, y, z** $\in \mathbb{R}$
= x (1, 1, 1) + y (1, 1, 0) + z (1, 0, 0)
= (x+y+z, x+y, x)
$\Rightarrow$ **x+y+z = a, x+y = b, x = c**
$\therefore$ **x = c, y = b-c, z = a – b**

Then, **(a, b, c) = c f_1 + (b-c) f_2 + (a-b) f_3** .... **(1)**

Again given, **T(x, y, z) = (2y+z, x-4y, 3x)** .... **(2)**

Now from **(1)** and **(2)** we get
$\mathbf{T}(f_1) = T(1,1,1) = (3,-3,3) = \mathbf{3}\, f_1$ **- 6** f_2 **+ 6** f_3
$\mathbf{T}(f_2) = T(1,1,0) = (2,-3,3) = \mathbf{3}\, f_1$ **- 6** f_2 **+ 5** f_3
$\mathbf{T}(f_3) = T(1,0,0) = (0,1,3) = \mathbf{3}\, f_1$ **- 2** f_2 **-** f_3

Therefore the matrix representation of T is

$$[\,T\,]_f = \begin{bmatrix} 3 & -6 & 6 \\ 3 & -6 & 5 \\ 3 & -2 & -1 \end{bmatrix}^t = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$$

[Answer].

H.W

Problem#01:

Let **T:** $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a linear operator and define by **T (x,y,z) = (2x+y, x-y, z).** Then find Matrix representation with respect to the standard basis of $\mathbb{R}^3$**.**

Sol^n:

Let, $\{(\mathbf{e_1} = (1,0,0),\ \mathbf{e_2} = (0,1,0),\ \mathbf{e_3} = (0,0,1)\}$ be a standard basis of $\mathbb{R}^3$.

Then, from **T(x, y, z) = (2x+y, x-y, z)** we get,
$\mathbf{T}(\mathbf{e_1}) = \mathbf{T}(1,0,0) = (2,1,0) = \mathbf{2.\,e_1 + 1.\,e_2 + 0.\,e_3}$
$\mathbf{T}(\mathbf{e_2}) = \mathbf{T}(0,1,0\,) = (1,-1,0) = \mathbf{1.\,e_1 - 1.\,e_2 + 0.\,e_3}$

$\mathbf{T(e_3) = T(0,0,1) = (0,0,1) = 0.\,e_1 - 0.\,e_2 + 1.\,e_3}$

Therefore the matrix representation of **T** is,

$$[\,\mathbf{T}\,]_e \;=\; \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^t = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[Answer].

Problem#02:

Let **T:** $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear operator and define by **T(x, y) = (4x–2y, 2x+y)**. Then find Matrix representation **T** with respect to the standard basis of $\mathbb{R}^2$.

Sol^n:

Let, $\{(\mathbf{e_1} = (\mathbf{1,0})\,,\ \mathbf{e_2} = (\mathbf{0,1})\}$ be a standard basis of $\mathbb{R}^2$.

Then, from **T(x, y) = (4x–2y, 2x+y)** we get

$\mathbf{T(e_1) = T(1,0) = (4,2) = 4.\,e_1 + 2.\,e_2}$

$\mathbf{T(e_2) = T(0,1) = (-2,1) = -2.\,e_1 + 1.\,e_2}$

Therefore the matrix representation of **T** is,

$$[\,\mathbf{T}\,]_e \;=\; \begin{bmatrix} 4 & 2 \\ -2 & 1 \end{bmatrix}^t = \begin{bmatrix} 4 & -2 \\ 2 & 1 \end{bmatrix}$$

[Answer].

Problem#03:

Let **T:** $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a linear transformation which is define by

$\mathbf{T\,(x_1, x_2, x_3) = (x_1 - x_2 + x_3,\ 2x_1 - x_2 + 3x_3,\ -x_1 + 2x_2 + x_3\,)}$.

Then find Matrix representation with respect to the standard basis of $\mathbb{R}^3$.

Sol^n:

Let $\{(\mathbf{e_1} = (\mathbf{1,0,0}),\ \mathbf{e_2} = (\mathbf{0,1,0}),\ \mathbf{e_3} = (\mathbf{0,0,1})\}$ be a standard basis of $\mathbb{R}^3$.

Then from $\mathbf{T(x_1,\ x_2,\ x_3) = (x_1 - x_2 + x_3,\ 2x_1 - x_2 + 3x_3,\ -x_1 + 2x_2 + x_3)}$ we get

$\mathbf{T(e_1) = T(1,0,0) = (1,2,\ -1) = 1.\,e_1 + 2.\,e_2 - 1.\,e_3}$

$\mathbf{T(e_2) = T(0,1,0) = (-1,\ -1,2) = -1.\,e_1 - 1.\,e_2 + 2.\,e_3}$

$\mathbf{T(e_3) = T(0,0,1) = (1,3,1) = 1.\, e_1 + 3.\, e_2 + 1.\, e_3}$

Therefore the matrix representation of **T** is

$$[\,\mathbf{T}\,]_e \quad = \quad \begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 2 \\ 1 & 3 & 1 \end{bmatrix}^t = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 3 \\ -1 & 2 & 1 \end{bmatrix}$$

[Answer].

Problem#04:

Let **T:** $\mathbb{R}^4 \rightarrow \mathbb{R}^3$ is a linear transformation which define by **T(x, y, z, t) = (x-y+z, x+y, y-t)**. Then find Matrix representation of **T** with respect to the standard basis of $\mathbb{R}^4$ and $\mathbb{R}^3$.

Sol^n:

Let, $\{(\mathbf{e_1} = (\mathbf{1,0,0,0}),\ \mathbf{e_2} = (\mathbf{0,1,0,0}),\ \mathbf{e_3} = (\mathbf{0,0,1,0}),\ \mathbf{e_3} = (\mathbf{0,0,0,1})\}$ and $\{(\mathbf{f_1} = (\mathbf{1,0,0}),\ \mathbf{f_2} = (\mathbf{0,1,0}),\ \mathbf{f_3} = (\mathbf{0,0,1})\}$ are standard basis of $\mathbb{R}^4$ and $\mathbb{R}^3$ respectively.

Then from **T(x, y, z, t) = (x-y+z, x+y, y-t)** we get,

$\mathbf{T(e_1) = T(1,0,0,0) = (1,1,0) = 1.\, f_1 + 1.\, f_2 + 0.\, f_3}$

$\mathbf{T(e_2) = T(0,1,0,0) = (-1,1,1) = -1.\, f_1 + 1.\, f_2 + 1.\, f_3}$

$\mathbf{T(e_3) = T(0,0,1,0) = (1,0,0) = 1.\, f_1 + 0.\, f_2 + 0.\, f_3}$

$\mathbf{T(e_4) = T(0,0,0,1) = (0,0,\ -1) = 0.\, f_1 + 0.\, f_2 - 1.\, f_3}$

Therefore the matrix representation of **T** is,

$$[\,\mathbf{T}\,]_e^f \quad = \quad \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}^t = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

[Answer].

Problem#05:

Let **T:** $\mathbb{R}^3 \rightarrow \mathbb{R}^4$ is a linear transformation which define by

T (x, y, z) = (x+y, x+z, x—y, x—z). Then find Matrix representation of **T** with respect to the standard basis of $\mathbb{R}^3$ and $\mathbb{R}^4$.

Sol^n:

Let, $\{(\mathbf{e_1} = (\mathbf{1}, \mathbf{0}, \mathbf{0}),\ \mathbf{e_2} = (\mathbf{0}, \mathbf{1}, \mathbf{0}),\ \mathbf{e_3} = (\mathbf{0}, \mathbf{0}, \mathbf{1})\}$ and

$\{(\mathbf{f_1} = (\mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}),\ \mathbf{f_2} = (\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}), \mathbf{f_3} = (\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}), \mathbf{f_4} = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1})\}$ are standard basis of $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively.

Then from **T(x, y, z) = (x+y, x+z, x—y, x—z)** we get,

$\mathbf{T(e_1) = T(1,\ 0,\ 0) = (1,\ 1,\ 1,\ 1) = 1.\,f_1 + 1.\,f_2 + 1.\,f_3 + 1.\,f_4}$

$\mathbf{T(e_2) = T(0,\ 1,\ 0) = (1,\ 0,\ -1,\ 0) = 1.\,f_1 + 0.\,f_2 - 1.\,f_3 + 0.\,f_4}$

$\mathbf{T(e_3) = T(0,\ 0,\ 1) = (0,\ 1,\ 0,\ -1) = 0.\,f_1 + 1.\,f_2 + 0.\,f_3 - 1.\,f_4}$

Therefore the matrix representation of **T** is,

$$[\,\mathbf{T}\,]_{\mathbf{e}}^{\mathbf{f}} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{-1} \end{bmatrix}^{t} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{-1} \end{bmatrix}$$

[Answer].

Problem#06:

Find the Matrix representation of **T** for the given basis $\{(\mathbf{f_1} = (\mathbf{1},\ \mathbf{1},\ \mathbf{0}),\ \mathbf{f_2} = (\mathbf{1},\ \mathbf{0},\ \mathbf{1}),\ \mathbf{f_3} = (\mathbf{0},\ \mathbf{1},\ \mathbf{1})\}$ where **T** is defined by

T: $\mathbb{R}^3 \rightarrow \mathbb{R}^3$; **T(x, y, z) = (x+y, y+z, z+x)**.

Sol^n:

Let, **(a, b ,c)** ∈ $\mathbb{R}^3$

Then **(a, b, c) = x.** $\mathbf{f_1}$ **+ y.** $\mathbf{f_2}$ **+ z.** $\mathbf{f_3}$

= x (1, 1, 0) + y (1, 0, 1) + z (0, 1, 1)

= (x+y, x+z, y+z)

$\Rightarrow$ **x+y = a** .... **(1)** **x+z = b** .... **(2)** **y+z = c** .... **(3)**

Now,

(1)+(2) $\Rightarrow$

2x+y+z = a+b

$\Rightarrow$**2x+c = a+b**

$\Rightarrow$ **x=** $\frac{(a+b-c)}{2}$

From eq. **(1)**

$$y = a - x$$
$$= a - \frac{a+b-c}{2}$$
$$\therefore y = \frac{a-b+c}{2}$$

From eq. **(2)**

$$z = b - x$$
$$= b - \frac{a+b-c}{2}$$
$$\therefore z = \frac{b-a+c}{2}$$

Then, **(a, b, c)** = $\frac{(a+b-c)}{2}.\, f_1 + \frac{a-b+c}{2}.\, f_2 + \frac{b-a+c}{2}.\, f_3$ .... **(4)**

Again given , **T(x, y, z) = (x+y, y+z, z+x)** .... **(5)**

Now from,

$T(f_1) = T(1,1,0) = (2,1,1) = 1.\, f_1 + 1.\, f_2 + 0.\, f_3$

$T(f_2) = T(1,0,1) = (1,1,2) = 0.\, f_1 + 1.\, f_2 + 1.\, f_3$

$T(f_3) = T(0,1,1) = (1,2,1) = 1.\, f_1 + 0.\, f_2 + 1.\, f_3$

Therefore the matrix representation of **T** is

$$[\,T\,]_f = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}^t = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

[Answer].

Problem#07:

Let **T:** $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a linear operator and which is define by **T(x, y, z) = (x+2y, x, y-z)**. Find the Matrix representation of **T** for the given basis $\{(f_1 = (1,1,1)\,, f_2 = (0,1,1), f_3 = (0,0,1)\}$.

Sol^n:

Let, **(a, b, c)** $\in \mathbb{R}^3$

Then **(a, b, c) = x.** $\mathbf{f_1}$ **+ y.** $\mathbf{f_2}$ **+ z.** $\mathbf{f_3}$

= x (1, 1, 1) + y (0, 1, 1) + z (0, 0, 1)

= (x, x+y, x+y+z)

$\Rightarrow$ **x = a, x+y = b, x+y+z = c**

$\therefore$ **x = a, y = b-a, z = c-b**

Then, **(a, b, c) = a.** $\mathbf{f_1}$ **+ (b-a).** $\mathbf{f_2}$ **+(c − b).** $\mathbf{f_3}$ .... **(1)**

Again given, **T(x, y, z) = (x+2y, x, y-z)** .... **(2)**

Now from **(1**) and **(2)**

$\mathbf{T(f_1) = T(1,1,1) = (3,1,0) = 3.\, f_1 - 2.\, f_2 + 0.\, f_3}$

$\mathbf{T(f_2) = T(0,1,1) = (2,0,0) = 2.\, f_1 - 2.\, f_2 + 0.\, f_3}$

$\mathbf{T(f_3) = T(0,0,1) = (0,0,-1) = 0.\, f_1 + 0.\, f_2 - 1.\, f_3}$

Therefore the matrix representation of **T** is

$$[\,T\,]_f = \begin{bmatrix} 3 & -2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}^t = \begin{bmatrix} 3 & 2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

[Answer].

THANK YOU