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Linear differential equation with constant co-efficient. when right hand side is zero.

Linear diff^n eq^n with constant co-efficient

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Eq^n of the nth order :— The typical form of linear diff^n eq^n with constant co-efficient of the nth order is —

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = X \longrightarrow (i)$$

or. symbollically $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n) y = X$ where $D = \frac{d}{dx}$

and $a_1, a_2, a_3 \cdots a_n$ & X are function of x only or a constant,
Also the above eq^n is briefly written as $f(D)y = X$
where $f(D) = D^n + a_1 D^{n-1} + \cdots + a_n$
This eq^n may be of two forms namely when the right hand member is zero & when the right hand member is a f^n of x.

* **Eq^n with right hand member zero :—** Let the eq^n be $f(D)y = 0$. Now if $y = y_1, y = y_2, \cdots y = y_n$ be the solution of $f(D)y = 0$ then $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ will be the general solution of $f(D)y = 0$
For convenience let us prove it in case of 2nd order eq^n. i.e. for the eq^n $(D^2 + a_1 D + a_2)y = 0 \longrightarrow (i)$
$\therefore y = y_1, y = y_2$ are the solutions of $(D^2 + a_1 D + a_2) y = 0$

$$\therefore (D^2 + a_1 D + a_2) y_1 = 0 \longrightarrow (ii)$$

$$(D^2 + a_1 D + a_2) y_2 = 0 \longrightarrow (iii)$$

Now multiplying (ii) by c_1 & (iii) by c_2 & adding we have,

$$c_1 D^2 y_1 + c_1 a_1 D y_1 + c_1 a_2 y_1 + c_2 D^2 y_2 + c_2 a_1 D y_2 + c_2 a_2 y_2 = 0$$

$$\Rightarrow D^2(c_1 y_1 + c_2 y_2) + a_1 D(c_1 y_1 + c_2 y_2) + a_2 (c_1 y_1 + c_2 y_2) = 0$$

which shows that $y = c_1 y_1 + c_2 y_2$ will be the general solution of $(D^2 + a_1 D + a_2) y = 0$.

Linear differential equation* 32nd Batch 103/A. M.H. Hall. J.O.

Auxiliary equation (A.E) :— Let $y = e^{mx}$ be a trial solution of $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0$ then it must satisfy the eq^n. Now $\frac{dy}{dx} = me^{mx}$, $\frac{d^2y}{dx^2} = m^2e^{mx}$.

so we have,

$$m^2e^{mx} + a_1 m e^{mx} + a_2 e^{mx} = 0$$

$$\Rightarrow (m^2 + a_1 m + a_2)\, e^{mx} = 0$$

Since $e^{mx} \neq 0$ $\therefore m^2 + a_1 m + a_2 = 0$ this equation is called the auxiliary equation of $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0$.

Since the ~~eqn is a~~ A.E is a quadratic eq^n in m it has two roots and they may be

(i) real and distinct (ii) real & equal (iii) imaginary.

(i) **Auxiliary eq^n having real & distinct roots :—** If m_1, m_2 are real and distinct then $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ is the general solution. since it satisfies the eq^n & contains two independent arbitrary constants equal in number to the order of the equation.

(ii) **Auxiliary eq^n having two equal roots :—** If $m_1 = m_2 = \alpha$ then as earlier the solution is $y = c_1 e^{\alpha x} + c_2 e^{\alpha x}$

$\Rightarrow y = (c_1 + c_2) e^{\alpha x} = c e^{\alpha x}$ where $c_1 + c_2 = c$ which is not the general solution since it involves only one independent constant but the eq^n is of 2nd order. Now let us devise a method for finding the general solution. Since the A.E. has two equal roots each being to α.

∴ the diff^n eq^n can be written in the form:

$$\frac{d^2y}{dx^2} - 2\alpha\frac{dy}{dx} + \alpha^2 y = 0$$

Let $y = e^{\alpha x} v$ where v is a f^n of x be a trial sol^n of this equation.

$$\therefore \frac{dy}{dx} = v\alpha e^{\alpha x} + e^{\alpha x}\frac{dv}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = v\alpha^2 e^{\alpha x} + \alpha e^{\alpha x}\frac{dv}{dx} + \alpha e^{\alpha x}\frac{dv}{dx} + e^{\alpha x}\frac{d^2v}{dx^2}$$

$$= v\alpha^2 e^{\alpha x} + 2\alpha e^{\alpha x}\frac{dv}{dx} + e^{\alpha x}\frac{d^2v}{dx^2}.$$

Putting this values in the eq^n we have.

$$\cancel{\alpha^2 v e^{\alpha x}} + \cancel{2\alpha e^{\alpha x}\frac{dv}{dx}} + e^{\alpha x}\frac{d^2v}{dx^2} - \cancel{2\alpha^2 v e^{\alpha x}} - \cancel{2\alpha e^{\alpha x}\frac{dv}{dx}} + \cancel{\alpha^2 v e^{\alpha x}} = 0$$

$$\Rightarrow e^{\alpha x}\frac{d^2v}{dx^2} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} = 0 \quad \text{since } e^{\alpha x} \neq 0.$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dv}{dx}\right) = 0$$

now integrating we get $\frac{dv}{dx} = c_2$

again integrating, we get $v = c_2 x + c_1$

$$\Rightarrow \frac{y}{e^{\alpha x}} = c_1 + c_2 x$$

$$\Rightarrow y = (c_1 + c_2 x)\, e^{\alpha x}$$

This can be treated as general solution since it ~~satif~~ satisfies the equation and contains two independent arbitrary constant.

Similarly, In case of three roots are equal, i.e, $m_1 = m_2 = m_3$, the general solution is

$$y = (c_1 + c_2 x + c_3 x^2)\, e^{m_1 x}$$

(iii) **Auxiliary eq^n having a pair of complex roots.:—**

If $m_1 = \alpha + i\beta$ & $m_2 = \alpha - i\beta$ the general solution is

$$y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$$

$$= c_1 e^{\alpha x} e^{i\beta x} + c_2 e^{\alpha x} . e^{-i\beta x}$$

$$= e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right)$$

$$= e^{\alpha x}\left[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\right]$$

$$= e^{\alpha x}\left[(c_1+c_2)\cos\beta x + i(c_1-c_2)\sin\beta x\right]$$

$$= e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right]$$ [Where $A = c_1 + c_2$ & $B = i(c_1 - c_2)$]

The above results after suitably adjusting constants may also be written as.

$y = e^{\alpha x}. A\cos(\beta x + B)$ or, $y = e^{\alpha x}. A\sin(\beta x + B)$.

* **Imaginary roots repeated:—** If auxiliary eq^n has two equal pairs of imaginary roots, i.e., if $\alpha + i\beta$ and $\alpha - i\beta$ occur twice, then general solution is obtained as.

$$y = e^{\alpha x}\left[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x\right].$$

* When the right hand side is zero :—

Solve ① $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$

Solⁿ :- Given $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$

$\Rightarrow (D^2 - 2D + 5)y = 0 \longrightarrow ①$

Let $y = e^{mx}$ be the trial solution of ①

$\therefore \frac{dy}{dx} = me^{mx}, \quad \frac{d^2y}{dx^2} = m^2 e^{mx}$

$\therefore$ A.E is $e^{mx}(m^2 - 2m + 5) = 0$

$\Rightarrow m^2 - 2m + 5 = 0 \quad \because e^{mx} \neq 0$

$\Rightarrow m = \frac{2 \pm \sqrt{4-20}}{2}$

$= \frac{2 \pm i4}{2} = 1 \pm 2i$

Hence the general solution is $y = e^{x}(A\cos 2x + B\sin 2$

Solve : ② $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 4y = 0$

Solⁿ :- Given $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 4y = 0$; $(D^2 + 6D + 4)y = 0 \longrightarrow ($

Let $y = e^{mx}$ be the trial solution of ①.

$\therefore \frac{dy}{dx} = me^{mx}; \quad \frac{d^2y}{dx^2} = m^2 e^{mx}$.

$\therefore$ A.E is $e^{mx}(m^2 + 6m + 4) = 0$

$\Rightarrow (m^2 + 6m + 4) = 0 \quad \because e^{mx} \neq 0$

$\Rightarrow m = \frac{-6 \pm \sqrt{36-16}}{2} = \frac{-6 \pm \sqrt{20}}{2} = -3 \pm \sqrt{5}$

Hence the G.S is $y = Ae^{(-3+\sqrt{5})x} + Be^{(-3-\sqrt{5})x}$

x

(2) Solve: $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$

Sol^n: Given $(D^2 - 4D + 1)y = 0 \longrightarrow (1)$

Let $y = e^{mx}$ be the trial solution of (1)

$\therefore \frac{dy}{dx} = me^{mx}$; $\frac{d^2y}{dx^2} = m^2e^{mx}$

$\therefore$ A.E is $e^{mx}(m^2 - 4m + 1) = 0$

$\Rightarrow m^2 - 4m + 1 = 0 \quad \because e^{mx} \neq 0$

$\therefore m = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$

Hence the G.S is $y = Ae^{(2+\sqrt{3})x} + Be^{(2-\sqrt{3})x}$

Solve: $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$.

Sol^n: Here A.E, $m = 2 \pm i3$

$\therefore$ G.S is, $y = e^{2x}(A\cos 3x + B\sin 3x)$

Solve: $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$

Sol^n: Given that. $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0 \longrightarrow (1)$

Let $y = e^{mx}$ be a trial sol^n of (1).

$\therefore \frac{dy}{dx} = me^{mx}$, $\frac{d^2y}{dx^2} = m^2e^{mx}$

Now from (1) we have.

$(m^2 - 7m + 12)e^{mx} = 0$

$\Rightarrow m^2 - 7m + 12 = 0 \quad \because e^{mx} \neq 0$

$\Rightarrow m = 4, 3.$

Hence the G.S is $y = c_1e^{4x} + c_2e^{3x}$

Solve:- $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

$$\Rightarrow (D^2 + 2D + 1)y = 0 \longrightarrow (1).$$

Soln:- Let $y = e^{mx}$ be a trial soln of (1)

$\therefore \frac{dy}{dx} = me^{mx}$; $\frac{d^2y}{dx^2} = m^2e^{mx}$.

Now from (1) we get A.E.

$(m^2 + 2m + 1) e^{mx} = 0$

$\Rightarrow m^2 + 2m + 1 = 0$ [since $e^{mx} \neq 0$]

$\Rightarrow \cancel{m = \frac{-2 \pm \sqrt{4 - 4}}{2}}$

$\Rightarrow (m+1)^2 = 0$

$\therefore m = -1, -1$

Hence the G.S is $y = (c_1 + c_2 x) e^{-x}$.

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\#Solve ✓ $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0$

Soln:- Given that $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0 \longrightarrow (1)$

Let $y = e^{mx}$ be a trial soln of (1)

$\therefore \frac{dy}{dx} = me^{mx}$, $\frac{d^2y}{dx^2} = m^2e^{mx}$, $\frac{d^3y}{dx^3} = m^3e^{mx}$, $\frac{d^4y}{dx^4} = m^4e^{mx}$

Now putting these values in (1) we get the A.E.

$(m^4 - m^3 - 9m^2 - 11m - 4) e^{mx} = 0$

$\Rightarrow m^4 - m^3 - 9m^2 - 11m - 4 = 0$ Since $e^{mx} \neq 0$

$\Rightarrow m^4 + m^3 - 2m^3 - 2m^2 - 7m^2 - 7m - 4m - 4 = 0$

$\Rightarrow m^3(m+1) - 2m^2(m+1) - 7m(m+1) - 4(m+1) = 0$

$\Rightarrow (m+1)(m^3-2m^2-7m-4)=0$

$\Rightarrow (m+1)(m^3+m^2-3m^2-3m-4m-4)=0$

$\Rightarrow (m+1)\{m^2(m+1)-3m(m+1)-4\cancel{m}(m+1)\}=0$

$\Rightarrow (m+1)^2(m^2-3m-4)=0$

$\Rightarrow (m+1)^2(m^2+m-4m-4)=0$

$\Rightarrow (m+1)^2\{m(m+1)-4(m+1)\}=0$

$\Rightarrow (m+1)^3(m-4)=0$

$\therefore m=-1,-1,-1,4.$

Hence the general solution is

$$y=(c_1+c_2x+c_3x^2)e^{-x}+c_4e^{4x}.$$

Solve: $\dfrac{d^2y}{dx^2}-9\dfrac{dy}{dx}+18y=0$

Solⁿ: Given that $\dfrac{d^2y}{dx^2}-9\dfrac{dy}{dx}+18y=0 \longrightarrow (1)$

Let $y=e^{mx}$ be a trial solⁿ of (1).

$\therefore \dfrac{dy}{dx}=me^{mx}$; $\dfrac{d^2y}{dx^2}=m^2e^{mx}$. $\longrightarrow \#$

Now putting these values into (1) we get.

$(m^2-9m+18)e^{mx}=0$

$\Rightarrow m^2-9m+18=0$ since $e^{mx}\neq 0$

$\Rightarrow (m-6)(m-3)=0$

$\therefore m=6,3.$

Hence the G.S is $y=c_1e^{6x}+c_2e^{3x}$

Solve: $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 19\frac{dy}{dx} + 20y = 0$

Solⁿ: Given that $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 19\frac{dy}{dx} + 20y = 0 \longrightarrow (1)$

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Now from (1) we get the A.E. is

$(m^3 - 2m^2 - 19m + 20)e^{mx} = 0$

$\Rightarrow m^3 - 2m^2 - 19m + 20 = 0 \quad [\because e^{mx} \neq 0]$

$\Rightarrow m^3 - m^2 - m^2 + m - 20m + 20 = 0$

$\Rightarrow m^2(m-1) - m(m-1) - 20(m-1) = 0$

$\Rightarrow (m-1)(m^2 - m - 20) = 0$

$\Rightarrow (m-1)(m-5)(m+4) = 0$

$\therefore m = 1, 5, -4$

Hence the required G.S is $y = c_1 e^{x} + c_2 e^{5x} + c_3 e^{-4x}$.

Solve: $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0$

Solⁿ:

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A.E is $(m^2 - 6m + 25)e^{mx} = 0$

$\Rightarrow m^2 - 6m + 25 = 0 \quad [\because e^{mx} \neq 0]$

$\Rightarrow m = \frac{6 \pm \sqrt{36 - 100}}{2}$

$= \frac{6 \pm \sqrt{64 i^2}}{2} = \frac{6 \pm 8i}{2}$

$= 3 \pm 4i$

Hence the G.S is $y = e^{3x}(A\cos 4x + B\sin 4x)$

Solve: $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - \frac{dy}{dx} + y = 0$

Sol^n: Given that $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - \frac{dy}{dx} + y = 0 \longrightarrow (1)$

Let $y = e^{mx}$ be a trial sol^n of (1) then,

$\frac{dy}{dx} = me^{mx}$, $\frac{d^2y}{dx^2} = m^2e^{mx}$, $\frac{d^3y}{dx^3} = m^3e^{mx}$

$\frac{d^4y}{dx^4} = m^4e^{mx}$. Putting these values into (1) we get

$(m^4 - m^3 - m + 1)e^{mx} = 0$

$\Rightarrow m^4 - m^3 - m + 1 = 0 \quad [\because e^{mx} \neq 0]$

$\Rightarrow m^3(m-1) - 1(m-1) = 0$

$\Rightarrow (m-1)(m^3-1) = 0$

$\Rightarrow (m-1)(m-1)(m^2+m+1) = 0$

$\therefore m = 1, 1, \ -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

Hence the G.S is.

$$y = (c_1 + c_2x)e^x + e^{-x/2}\left[c_3 \cos\frac{\sqrt{3}}{2}x + c_4 \sin\frac{\sqrt{3}}{2}x\right]$$

Solve: $(D^2 - 2D + 5)^2 y = 0 \longrightarrow (1)$

Sol^n:

A.E is $(m^2 - 2m + 5)^2 = 0 \quad [\because e^{mx} \neq 0]$

$\Rightarrow m = \frac{2 \pm \sqrt{4-20}}{2}, \ \frac{2 \pm \sqrt{4-20}}{2}$

$= 1 \pm 2i, \ 1 \pm 2i$

Hence the G.S is

$$y = e^x[(A + Bx)\cos 2x + (C + Dx)\sin 2x].$$

Solve: $\frac{d^4y}{dx^4} + 4y = 0 \longrightarrow (1)$.

Solⁿ: #

A.E $\Rightarrow (m^4+4) = 0 \quad [\because e^{mx} \neq 0]$

$\Rightarrow (m^2+2)^2 - 2.2m^2 = 0$

$\Rightarrow (m^2+2)^2 - (2m)^2 = 0$

$\Rightarrow (m^2-2m+2)(m^2+2m+2) = 0$

$\therefore m^2 - 2m + 2 = 0$ & $m^2 + 2m + 2 = 0$

$m = \frac{2 \pm \sqrt{4-8}}{2}$ $\therefore m = \frac{-2 \pm \sqrt{4-8}}{2}$

$= 1 \pm i$ $= -1 \pm i$

Hence the G.S is.

$y = e^{x}(c_1 \cos x + c_2 \sin x) + e^{-x}(c_3 \cos x + c_4 \sin x)$

Solve: $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0$; $y(0) = -3$, $y'(0) = -1$. $\longrightarrow (1)$.

Solⁿ: Given $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0$

Let $y = e^{mx}$ be the trial solⁿ of the given equation

$\therefore \frac{dy}{dx} = me^{mx}$; $\frac{d^2y}{dx^2} = m^2 e^{mx}$

Putting these values we get.

A.E, is $(m^2 - 6m + 25) = 0 \quad (\because e^{mx} \neq 0)$

$m = \frac{6 \pm \sqrt{36-100}}{2} = 3 \pm i4$

$\therefore$ G.S $= e^{3x}(A \cos 4x + B \sin 4x)$

When $x=0$, $y(0) = 1(A\cos 0 + 0)$

$-3 = A.$

Again $y(x) = e^{3x}(A\cos 4x + B\sin 4x)$

$y'(x) = 3e^{3x}(A\cos 4x + B\sin 4x) + e^{3x}(-A4\sin 4x + 4B\cos 4x)$

$\therefore y'(0) = 3.1(A.1+0) + 1(0+4.B1)$

$-1 = 3A + 4B$

$4B = -1 - 3A = -1 + 9 = 8$

$\therefore B = 2$

$\therefore y(x) = e^{3x}(-3\cos 4x + 2\sin 4x).$

Solve: $\frac{d^2s}{dt^2} + 8\frac{ds}{dt} + 25s = 0$, when $t=0$, $s=4$ & $\frac{ds}{dt} = -16$

Soln: Let $s = e^{mt}$ $m = -4 \pm i3.$

$\therefore$ G.S is $s = e^{-4t}(A\cos 3t + B\sin 3t)$

when $t=0$, $s = 1(A+0)$

$\Rightarrow 4 = A$

$\therefore A = 4$

Again. $\frac{ds}{dt} = -4e^{-4t}(A\cos 3t + B\sin 3t) + e^{-4t}(-3A\sin 3t + 3B\cos 3t)$

when $t=0$, $\frac{ds}{dt} = -4(A) + 1(3B)$

$\Rightarrow -16 = -4.4 + 3B$

$\therefore B = 0$

$\therefore s = e^{-4t}(4\cos 3t)$

** **Theorem :—** If $y = y_1$, $y = y_2$, ... $y = y_n$ are linearly independent solutions of

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n)\, y = 0,$$

then $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ is the general or complete solution of the diffⁿ equation, where, $c_1, c_2, \ldots, c_n$ are n arbitrary constants.

Proof :— Given that

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n)\, y = 0 \longrightarrow (I)$$

Let us denote the given eqⁿ (I) by $f(D)\, y = 0$,

where $f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n$.

Since $y = y_1$, $y = y_2$, ... $y = y_n$ are solutions of the eqⁿ,

$$\therefore f(D) y_1 = 0,\ f(D) y_2 = 0, \ldots, f(D) y_n = 0 \longrightarrow (II)$$

Now putting $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ in (I), we have

$$D^n(c_1 y_1 + c_2 y_2 + \cdots + c_n y_n) + a_1 D^{n-1}(c_1 y_1 + c_2 y_2 + \cdots + c_n y_n)$$
$$+ \cdots + a_n (c_1 y_1 + c_2 y_2 + \cdots + c_n y_n) = 0$$

$$\Rightarrow c_1(D^n y_1 + a_1 D^{n-1} y_1 + \cdots + a_n) + c_2(D^n y_2 + a_1 D^{n-1} y_2 + \cdots + a_n)$$
$$+ \cdots + c_n(D^n y_n + a_1 D^{n-1} y_n + \cdots + a_n) = 0$$

$$\Rightarrow c_1 f(D) y_1 + c_2 f(D) y_2 + \cdots + c_n f(D) y_n = 0$$

or, $c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_n \cdot 0 = 0$ [by (II)]

Since (I) is satisfied by $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$, it is a solⁿ of (I). Also since it contains n arbitrary constants, it is the general or complete solution of the eqn.

Solve:- $\frac{d^2x}{dt^2} - 4x = 0$, when $t=0$, then $x=10$ & $\frac{dx}{dt} = 0$.

Solⁿ:- Let $x = e^{mt}$ $\therefore m = \pm 2$

$\therefore x = Ae^{2t} + Be^{-2t} \longrightarrow (i)$

when $t=0$, $x = A + B$

$10 = A+B \longrightarrow (ii)$

again $\frac{dx}{dt} = A2e^{2t} + Be^{-2t}(-2)$

$0 = A - B \longrightarrow (iii)$

from (ii) & (iii) $A = B = 5$.

$\therefore$ ~~G.S~~ $x = 5(e^{2t} + e^{-2t})$.

Solve:- $\frac{d^4y}{dx^4} - a^4y = 0$

Solⁿ:- Given $\frac{d^4y}{dx^4} - a^4y = 0$; $(D^4 - a^4)y = 0 \longrightarrow (i)$

Let $y = e^{mx}$ be the trial solⁿ of (i)

$\therefore$ A.E is $(m^4 - a^4)e^{mx} = 0$

$\Rightarrow m^4 - a^4 = 0$ $[\because e^{mx} \neq 0]$

$\Rightarrow (m^2+a^2)(m^2-a^2) = 0$

$\therefore m = \pm a, \pm ai$

Hence the general solution is

$y = c_1 e^{ax} + c_2 e^{-ax} + (c_3 \cos ax + c_4 \sin ax)$

Solve:- $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$

Sol^n:- # A.E is $m^2 + am + b = 0$ $\because e^{mx} \neq 0$

$\Rightarrow m = \frac{-a \pm \sqrt{a^2-4b}}{2}$

$\therefore$ G.S is, $y = Ae^{\left(\frac{-a+\sqrt{a^2-4b}}{2}\right)x} + Be^{\left(\frac{-a-\sqrt{a^2-4b}}{2}\right)x}$

Solve:- $\frac{d^4y}{dx^4} + m^4 y = 0$

Sol^n:- # A.E is $D^4 + m^4 = 0$ $[\because e^{mx} \neq 0]$.

$\Rightarrow (D^2+m^2)^2 - 2m^2D^2 = 0$

$\Rightarrow (D^2 + \sqrt{2}mD + m^2)(D^2 - \sqrt{2}mD + m^2) = 0$

$\therefore D = \frac{-\sqrt{2}m \pm \sqrt{2m^2 - 4m^2}}{2}$ $\boxed{\text{when } D^2 + \sqrt{2}mD + m^2 = 0}$

$= \frac{-\sqrt{2}m \pm \sqrt{2}mi}{2} = \frac{-m \pm mi}{\sqrt{2}}$

again $D = \frac{\sqrt{2}m \pm \sqrt{2m^2 - 4m^2}}{2} = \frac{\sqrt{2}m \pm \sqrt{2}mi}{2}$

$= \frac{m \pm mi}{\sqrt{2}}$

$\therefore$ Hence the required G.S is.

$y = e^{\frac{1}{\sqrt{2}}mx}\left(c_1 \cos\frac{1}{\sqrt{2}}mx + c_2 \sin\frac{1}{\sqrt{2}}mx\right)$

$+ e^{-\frac{1}{\sqrt{2}}mx}\left(c_3 \cos\frac{1}{\sqrt{2}}mx + c_4 \sin\frac{1}{\sqrt{2}}mx\right)$

Solve :- $\dfrac{d^2y}{dx^2} - 2k\dfrac{dy}{dx} + k^2y = 0$

Soln :- # A.E is $m^2 - 2km + k^2 = 0 \quad \because e^{mx} \neq 0$

$\Rightarrow (m-k)^2 = 0$

$\therefore m = k, k$

$\therefore$ G.S is $y = (c_1 + c_2x)\, e^{kx}$.

Solve :- $(D^2+1)^3 (D^2+D+1)^2 y = 0.$

Soln :- Here the auxiliary equation is.

$(m^2+1)^3 (m^2+m+1)^2 = 0$

which gives.

$m^2+1 = 0$ (thrice)

$\Rightarrow m^2 = -1$

$\Rightarrow m = \pm i$ (thrice)

and $m^2+m+1 = 0$ twice.

$\therefore m = \dfrac{-1 \pm \sqrt{1-4}}{2}$

$= -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ (twice).

Hence the required solution is.

$y = (c_1 + c_2x + c_3x^2)\cos x + (c_4 + c_5x + c_6x^2)\sin x$

$+ e^{-x/2}\left[(c_7 + c_8x)\cos\frac{\sqrt{3}}{2}x + (c_9 + c_{10}x)\sin\frac{\sqrt{3}}{2}x\right]$

where $c_1, c_2, ---- c_{10}$ are arbitrary constant.

(Solved)

Solve :- $(D^3+6D^2+11D+6)y=0$.

Solⁿ :-

#

A.E is $m^3+6m^2+11m+6=0$ [Since $e^{mx}\neq 0$]

$\Rightarrow m^3+m^2+5m^2+5m+6m+6=0$

$\Rightarrow m^2(m+1)+5m(m+1)+6(m+1)=0$

$\Rightarrow (m+1)\ (m^2+3m+2m+6)=0$

$\Rightarrow (m+1)\ (m+3)\ (m+2)=0$

$\therefore\ m=-1,-2,-3.$

Hence the required solution is.

$y=c_1e^{-x}+c_2e^{-2x}+c_3e^{-3x}$

(Solved)

Solve :- $(D^3-D^2-D+1)y=0$

Solⁿ :-

#

$\therefore$ A.E is $D^3-D^2-D+1=0$ since $e^{mx}\neq 0$.

$\Rightarrow D^3-D^2-D+1=0$

$\Rightarrow D^2(D-1)-1(D-1)=0$

$\Rightarrow (D-1)\ (D+1)\ (D-1)=0$

$\therefore\ D=1,1,-1.$

Hence the required solution is.

$y=(c_1+c_2x)e^{x}+c_3e^{-x}$.

where c_1, c_2, c_3 are arbitrary constant.

(Solved)

V.V.I **Solve:** $\frac{d^4y}{dx^4} + 4\frac{d^3y}{dx^3} + 8\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 4y = 0 \longrightarrow (1)$

Solⁿ: Let $y = e^{mx}$ be a trial solⁿ of (1)

$\therefore \frac{dy}{dx} = me^{mx}$; $\frac{d^2y}{dx^2} = m^2e^{mx}$; $\frac{d^3y}{dx^3} = m^3e^{mx}$; $\frac{d^4y}{dx^4} = m^4e^{mx}$.

$\therefore$ A.E is $e^{mx}(m^4 + 4m^3 + 8m^2 + 8m + 4) = 0$

$\Rightarrow m^4 + 4m^3 + 8m^2 + 8m + 4 = 0$, [Since $e^{mx} \neq 0$]

$\Rightarrow (m^2 + 2m + 2)^2 = 0$ ← মুখস্থ রাখতে হবে।

$\Rightarrow m^2 + 2m + 2 = 0$

$\Rightarrow m = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm \sqrt{-1}$

$\therefore m = -1 \pm i$ (twice).

$\therefore$ G.S $y = [(A + Bx)\cos x + (C + Dx)\sin x]\, e^{-x}$

This is the required solution

(Solved)

V.V.I **Solve:** $(D^4 - 4D^3 + 3D^2 + 4D - 4)y = e^{2x} \longrightarrow (1)$

Solⁿ: Let $y = e^{mx}$ be the trial solⁿ of (1).

#

$\therefore$ A.E is $m^4 - 4m^3 + 3m^2 + 4m - 4 = 0$; [Since $e^{mx} \neq 0$]

$\Rightarrow m^4 - m^3 - 3m^3 + 3m^2 + 4m - 4 = 0$

$\Rightarrow m^3(m-1) - 3m^2(m-1) + 4(m-1) = 0$

$\Rightarrow (m-1)(m^3 - 3m^2 + 4) = 0$

$\Rightarrow (m-1)(m+1)(m^2 - 4m + 4) = 0$

$\Rightarrow (m-1)(m+1)(m-2)^2 = 0$

$\therefore m = 1, -1, 2, 2.$

$\therefore y_c = (c_1 + c_2 x)e^{2x} + c_3 e^{x} + c_4 e^{-x}$

$$P.I = \frac{1}{(D-1)(D+1)(D-2)^2} e^{2x}$$

$$= e^{2x} \frac{1}{(D+1)(D+3)D^2} \cdot 1 \qquad \frac{1}{3} \cdot \frac{1}{D^2} \cdot 1$$

$$= e^{2x} \frac{1}{D^4 + 4D^3 + 3D^2} \cdot 1$$

$$= e^{2x} \frac{1}{3D^2\left(1 + \frac{D^2+4D}{3}\right)} \cdot 1$$

$$= e^{2x} \frac{1}{3D^2} \left\{1 + \frac{D^2+4D}{3}\right\}^{-1} \cdot 1$$

$$= e^{2x} \frac{1}{3D^2} \left(1 + \left(\frac{D^2+4D}{3}\right)^2 + \cdots\right) \cdot 1$$

$$= e^{2x} \frac{1}{3D^2} (1)$$

$$= \frac{1}{3} e^{2x} \frac{x^2}{2}$$

$\therefore y_p = \frac{1}{6} x^2 e^{2x}$

Hence the required general solution is.

$y = (c_1 + c_2 x)e^{2x} + c_3 e^{x} + c_4 e^{-x} + \frac{1}{6} x^2 e^{2x}$

(Solved)

———— X ————