Linear Combination

Definition of Linear Combination:

Let V be a vector space over the field F and let $v_1, \cdots v_n \in V$. then any vector $v \in V$ is called a linear combination of $v_1, v_2, \cdots v_n$ if and only if there exist scalars $\alpha_1, \alpha_2, \cdots \alpha_n$ in F.

such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

$$= \sum_{i=1}^{n} \alpha_i v_i$$

*** Ex-1: Consider the vectors $v_1 = \cancel{(2,3,4)} (1,0,1)$, $v_2 = (0,-1,1)$ and $v_3 = (-1,-1,1)$ in $\mathbb{R}^3$. Show that $v = (2,3,4)$ is a linear combination of v_1, v_2 and v_3.

Soln: In order to show that v is a linear combination of v_1, v_2 and v_3, there must be scalars α_1, α_2, and α_3 in F, such that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

i.e. $(2,3,4) = \alpha_1(1,0,1) + \alpha_2(0,-1,1) + \alpha_3(-1,-1,1)$

$\Rightarrow (2,3,4) = (\alpha_1, 0, \alpha_1) + (0, -\alpha_2, \alpha_2) + (-\alpha_3, -\alpha_3, \alpha_3)$

$\Rightarrow (2,3,4) = (\alpha_1 - \alpha_3, -\alpha_2 - \alpha_3, \alpha_1 + \alpha_2 + \alpha_3)$

Now equating corresponding components and forming linear system we get

$$\left.\begin{aligned} \alpha_1 - \alpha_3 &= 2 \\ -\alpha_2 - \alpha_3 &= 3 \\ \alpha_1 + \alpha_2 + \alpha_3 &= 4 \end{aligned}\right\} - (1)$$

Reduce the system (1) to echelon form by elementary operations.

$$\underset{R_3' \to R_3 - R_1}{\sim} \left.\begin{aligned} \alpha_1 - \alpha_3 &= 2 \\ -\alpha_2 - \alpha_3 &= 3 \\ \alpha_2 + 2\alpha_3 &= 2 \end{aligned}\right\} \underset{R_3' \to R_3 + R_2}{\sim} \left.\begin{aligned} \alpha_1 - \alpha_3 &= 2 \\ -\alpha_2 - \alpha_3 &= 3 \\ \alpha_3 &= 5 \end{aligned}\right\} -- (2)$$

Now from equation (2) we have $\alpha_3 = 5$, substituting $\alpha_3 = 5$ and solving (2) we get

$$\alpha_2 = -8, \text{ and } \alpha_1 = 7.$$

Hence $V = 7V_1 - 8V_2 + 5V_3$

Therefore, V is a linear combination of V_1, V_2, and V_3.

(showed)

H.W.:

* If $v = (2,5,3)$, $v_1 = (1,-3,2)$, $v_2 = (2,-4,-1)$ and $v_3 = (1,-5,7)$ show that v is not a linear combination of v_1, v_2 and v_3.

D.A.K

page - 833

* Ex-2

Express the vector $v = (1, -2, 5)$ as a linear combination of the vectors $v_1 = (1,1,1)$ $v_2 = (1,2,3)$ and $v_3 = (2,-1,1)$.

Soln:

In order to show that v is a linear combination of v_1, v_2 and v_3, there must be scalars x, y, and z in F. such that

$$V = xv_1 + yv_2 + zv_3$$

i.e. $(1,-2,5) = x(1,1,1) + y(1,2,3) + z(2,-1,1)$

$\Rightarrow (1,-2,5) = (x,x,x) + (y,2y,3y) + (2z,-z,z)$

$\Rightarrow (1,-2,5) = (x+y+2z,\ x+2y-z,\ x+3y+z)$

Now equating corresponding components and forming linear system, we get

$$\left.\begin{aligned} x+y+2z &= 1 \\ x+2y-z &= -2 \\ x+3y+z &= 5 \end{aligned}\right\} \quad -(1)$$

Reduce the system (1) to echelon form by elementary operation.

$R_2' \to R_2 - R_1$, $R_3' \to R_3 - R_1$

$$\left.\begin{aligned} x+y+2z &= 1 \\ y-3z &= -3 \\ 2y-z &= 4 \end{aligned}\right\}$$

$R_3' \to R_3' - 2R_2$

$$\left.\begin{aligned} x+y+2z &= 1 \\ y-3z &= -3 \\ 5z &= 10 \end{aligned}\right\} \quad -(2)$$

From equation (2) we have $z = 2$, and solving (2) we get $y = 3$, and $x = -6$

Hence $V = -6v_1 + 3v_2 + 2v_3$

H.W D.A.K Page-833

** Ex-3: Express the vector $U = (3, 9, -4, -2) \in \mathbb{R}^4$ as a linear combination of $u_1 = (1, -2, 0, 3)$, $u_2 = (2, 3, -1, 0)$, $u_3 = (2, -1, 2, 1)$.

Ans:- $U = -\frac{7}{17}u_1 + \frac{42}{17}u_2 - \frac{13}{17}u_3$.

***

* Express the matrix $E = \begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}$ as a linear combination of $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$.

Soln:

Let $E = xA + yB + zC$ ---(1) where x, y, z are scalar in F

Then (1) $\Rightarrow \begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} = x\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} + y\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} + z\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} x & x \\ 0 & -x \end{pmatrix} + \begin{pmatrix} y & y \\ -y & 0 \end{pmatrix} + \begin{pmatrix} z & -z \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} x+y+z & x+y-z \\ -y & -x \end{pmatrix}$

Now equating corresponding components and forming linear system we get.

$$\left.\begin{aligned} x+y+z &= 3 \\ x+y-z &= -1 \\ -y &= 1 \\ -x &= -2 \end{aligned}\right\} \text{---(2)}$$

Now solving (2) we get $x = 2$, $y = -1$, $z = 2$

$\therefore E = 2A - B + 2C$

i.e. E is a linear combination of A, B and C.

Ex:- ***

Show that the matrix $E = \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}$ can not be express as a linear combination of $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

Soln: Let $E = xA + yB + zC$ --- (1) where, x, y and z are scalar in F.

Then, $\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix} = x\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + y\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} + z\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} x & x \\ 0 & x \end{pmatrix} + \begin{pmatrix} y & y \\ -y & 0 \end{pmatrix} + \begin{pmatrix} z & -z \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} x+y+z & x+y-z \\ -y & x \end{pmatrix}$

Now equating the corresponding component and forming linear system we get.

$$\left.\begin{aligned} x+y+z &= 2 \\ x+y-z &= 1 \\ -y &= -1 \\ x &= -2 \end{aligned}\right\} \text{---} (2)$$

Now solving (2) we get $x = -2$, $y = 1$, and $z = -2$, $z = 3$ which is impossible. So the equation (2) has no solution or inconsistent

Therefore Matrix E can not be express as a linear combination of A, B, and C.

**

*H.W ⇒

Show that the matrix $E = \begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}$ can not be express as a linear combination of $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$.

*** # H.W: Express the matrix $E=\begin{pmatrix}5 & 1\\ -2 & 3\end{pmatrix}$ as a linear ~~matr~~ combination of $A=\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$, $B=\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}$ and $C=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$

Ans: $E = 2A + 3B + 4C$.

*** $\boxed{\#}$ **Generator / linear span:** If S is a non-empty subset of a vector space V, then $L(S)$ is the linear span or Generator of S is the set of all linear combinations of finite sets of elements of S.

*** * **Ex:** Show that the vectors $u=(1,2,3)$, $v=(0,1,2)$ and $w=(0,0,1)$ generate $\mathbb{R}^3$.

or show that $[(1,2,3),(0,1,2),(0,0,1)] = \mathbb{R}^3$.

Proof: we must determinate whether an arbitrary vector $V_3(R) = (a,b,c)$ in $\mathbb{R}^3$ can be expressed as a linear combination $V_3(R) = xu + yv + zw$ of the vectors u, v and w, where x, y and z are scalars. ~~in F~~.

Now Expressing this equation in terms of components gives

$$(a,b,c) = x(1,2,3) + y(0,1,2) + z(0,0,1)$$

$$= (x, 2x, 3x) + (0, y, 2y) + (0, 0, z)$$

$$= (x,\ 2x+y,\ 3x+2y+z)$$

* $\boxed{\text{or} \Rightarrow V_3(R) = x(1,2,3) + y(0,1,2) + z(0,0,1)}$

P.T.O

Equating corresponding components and forming the linear system we get

$$\left.\begin{aligned} x &= a \\ 2x+y &= b \\ 3x+2y+z &= c \end{aligned}\right\} \Rightarrow \begin{aligned} z+2y+3x &= c \\ y+2x &= b \\ x &= a \end{aligned}$$

The above system is in echelon form and is consistent. In fact the system has the solution $x=a$, $y=b-2a$, $z=c-2b+a$

* Thus u, v, and w generat (span) $\mathbb{R}^3$.

~~[or → $V_3(R)$ is a generate of $\mathbb{R}^3$]~~

*[or → Thus the given set is a generator of $V_3(R)$

therefore $[(1,2,3),(0,1,2),(0,0,1)] = \mathbb{R}^3$]

(showed)

***

H.W: (1) show that $\mathbb{R}^3 = [(1,2,1),(2,1,0),(1,-1,2)]$

***

(2) show that $\mathbb{R}^3 = [(1,-2,1),(2,-1,1),(2,3,1)]$