# 9

## *The inverse scattering transform for the Korteweg–de Vries (KdV) equation*

We have seen in the previous chapter that the Korteweg–de Vries (KdV) equation is the result of compatibility between

$$v_{xx} + (\lambda + u(x,t))v = 0 \tag{9.1}$$

and

$$v_t = (\gamma + u_x)v + (4\lambda + 2u)v_x, \tag{9.2}$$

where $\gamma$ is constant. More precisely, the equality of the mixed derivatives $v_{xxt} = v_{txx}$ with $\lambda_t = 0$ ("isospectrality") leads to the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{9.3}$$

In the main part of this chapter we will use the above compatibility relations in order to carry out the inverse scattering transform (IST) for the KdV equation on the infinite interval. This includes a basic description of the direct and inverse problems of the time-independent Schrödinger equation and the time evolution of the scattering data. Special soliton solutions as well as the scattering data for a delta function and box potential are given. Also included are the derivation of the conserved quantities/conservation laws of the KdV equation from IST. Finally, as an extension, the IST associated with the second-order scattering system and associated time (8.33)–(8.34), discussed in the previous chapter, is carried out.

For (9.1), the eigenvalues and the behavior of the eigenfunctions as $|x| \to \infty$ determine what we call the *scattering data* $S(\lambda, t)$ at any time $t$, which depends upon the potential $u(x,t)$. The *direct scattering problem* maps the potential into the scattering data. The *inverse scattering problem* reconstructs the potential $u(x,t)$ from the scattering data $S(\lambda, t)$. The initial value problem for the KdV equation is analyzed as follows. At $t = 0$ we give initial data $u(x, 0)$ which we assumes decays sufficiently rapidly at infinity. The initial data is mapped to

$S(\lambda, t = 0)$ via (9.1). The evolution of the scattering data $S(\lambda, t)$ is determined from (9.2). Then $u(x, t)$ is recovered from inverse scattering.

## 9.1 Direct scattering problem for the time-independent Schrödinger equation

Suppose $\lambda = -k^2$; then the time-independent Schrödinger equation (9.1) becomes

$$v_{xx} + \{u(x) + k^2\}v = 0, \tag{9.4}$$

where, for convenience, we have suppressed the time dependence in $u$. We shall further assume that $u(x)$ decays sufficiently rapidly, which for our purposes means that $u$ lies in the space of functions

$$L^1_n: \quad \int_{-\infty}^{\infty} (1 + |x|^n)|u(x)|\, dx < \infty,$$

with $n = 2$ (Deift and Trubowitz, 1979). We also remark that the space $L^1_1$ was the original space used by Faddeev (1963); this space was subsequently used by Marchenko (1986) and Melin (1985). Associated with (9.4) are two complete sets of eigenfunctions for real $k$ that are bounded for all values of $x$, and that have appropriate analytic extensions. These eigenfunctions are defined by the equation and boundary conditions; that is, we identify four eigenfunctions defined by the following asymptotic boundary conditions:

$$\begin{aligned} &\phi(x; k) \sim e^{-ikx}, \quad \overline{\phi}(x; k) \sim e^{ikx} \quad \text{as} \quad x \to -\infty, \\ &\psi(x; k) \sim e^{ikx}, \quad \overline{\psi}(x; k) \sim e^{-ikx} \quad \text{as} \quad x \to \infty. \end{aligned} \tag{9.5}$$

Therefore $\phi(x; k)$, for example, is that solution of (9.4) that tends to $e^{-ikx}$ as $x \to -\infty$. Note that here the bar does not denote complex conjugate, for which we will use the $*$ notation. From (9.4) and the boundary conditions (9.5), we see that

$$\phi(x; k) = \bar{\phi}(x; -k) = \phi^*(x, -k), \tag{9.6a}$$

$$\psi(x; k) = \bar{\psi}(x; -k) = \psi^*(x, -k). \tag{9.6b}$$

Since (9.4) is a linear second-order ordinary differential equation, by the linear independence of its solutions we obtain the following relationships between the eigenfunctions

$$\phi(x;k) = a(k)\bar{\psi}(x;k) + b(k)\psi(x;k), \tag{9.7a}$$

$$\bar{\phi}(x;k) = -\bar{a}(k)\psi(x;k) + \bar{b}(k)\bar{\psi}(x;k) \tag{9.7b}$$

with $k$-dependent functions: $a(k)$, $\bar{a}(k)$, $b(k)$, $\bar{b}(k)$. These functions form the basis of the scattering data we will need.

The *Wronskian* of two functions $\psi, \phi$ is defined as

$$W(\phi, \psi) = \phi\psi_x - \phi_x\psi$$

and for (9.4), from Abel's theorem, the Wronskian is constant. Hence from $\pm\infty$:

$$W(\psi, \overline{\psi}) = -2ik = -W(\phi, \overline{\phi}).$$

The Wronskian also has the following properties

$$W(\phi(x;k), \psi(x;k)) = -W(\psi(x;k), \phi(x;k)),$$
$$W(c_1\phi(x;k), c_2\psi(x;k)) = c_1c_2\, W(\phi(x;k), \psi(x;k)),$$

with $c_1$, $c_2$ arbitrary constants.

From (9.7) it follows that

$$a(k) = \frac{W(\phi(x;k), \psi(x;k))}{2ik}, \qquad b(k) = -\frac{W(\phi(x;k), \bar{\psi}(x;k))}{2ik} \tag{9.8}$$

and

$$\overline{a}(k) = \frac{W(\overline{\phi}(x;k), \overline{\psi}(x;k))}{2ik}, \qquad \overline{b}(k) = \frac{W(\overline{\phi}(x;k), \psi(x;k))}{2ik}.$$

The formula for $a(k)$ follows directly from $W(\phi(x;k), \psi(x;k))$ and using the first of equations (9.7); the others are similar.

Rather than work with the eigenfunctions $\phi(x;k)$, $\bar{\phi}(x;k)$, $\psi(x;k)$ and $\bar{\psi}(x;k)$, it is more convenient to work with the (modified) eigenfunctions $M(x;k)$, $\bar{M}(x;k)$, $N(x;k)$ and $\bar{N}(x;k)$, defined by

$$M(x;k) = \phi(x;k)\, e^{ikx}, \qquad \bar{M}(x;k) := \bar{\phi}(x;k)\, e^{ikx}, \tag{9.9a}$$

$$N(x;k) = \psi(x;k)\, e^{ikx}, \qquad \bar{N}(x;k) = \bar{\psi}(x;k)\, e^{ikx}; \tag{9.9b}$$

then

$$M(x;k) \sim 1, \qquad \bar{M}(x;k) \sim e^{2ikx}, \qquad \text{as} \quad x \to -\infty, \tag{9.10a}$$

$$N(x;k) \sim e^{2ikx}, \qquad \bar{N}(x;k) \sim 1, \qquad \text{as} \quad x \to \infty. \tag{9.10b}$$

Equations (9.7a) and (9.9) imply that

$$\frac{M(x;k)}{a(k)} = \bar{N}(x;k) + \rho(k)N(x;k),$$
$$\frac{\bar{M}(x;k)}{\bar{a}(k)} = -N(x;k) + \bar{\rho}(k)\bar{N}(x;k), \tag{9.11}$$

where

$$\rho(k) = \frac{b(k)}{a(k)}, \tag{9.12a}$$

$$\bar{\rho}(k) = \frac{\bar{b}(k)}{\bar{a}(k)}; \tag{9.12b}$$

$\tau(k) = 1/a(k)$ and $\rho(k)$ are called the *transmission* and *reflection* coefficients respectively. Equations (9.6b), (9.9b) imply that

$$N(x;k) = \bar{N}(x;-k)\, e^{2ikx}. \tag{9.13}$$

Due to this symmetry relation we will only need two eigenfunctions. Namely, from (9.11) and (9.13) we obtain

$$\frac{M(x;k)}{a(k)} = \bar{N}(x;k) + \rho(k)\, e^{2ikx}\bar{N}(x;-k), \tag{9.14}$$

which will be the fundamental equation. Later we will show that (9.14) is equivalent to what we call a generalized *Riemann–Hilbert boundary value problem* (RHBVP); this RHBVP is central to the inverse problem and is a consequence of the analyticity properties of $M(x;k)$, $\bar{N}(x;k)$ and $a(k)$ that are established in the following lemma.

**Lemma 9.1** (i) *$M(x;k)$ can be analytically extended to the upper half k-plane and tends to unity as $|k| \to \infty$ for $\operatorname{Im} k > 0$;*
(ii) *$\bar{N}(x;k)$ can be analytically extended to the lower half k-plane and tends to unity as $|k| \to \infty$ for $\operatorname{Im} k < 0$.*

These analyticity properties are established by studying the linear integral equations that govern $M(x;k)$ and $\bar{N}(x;k)$. We begin by transforming (9.4) via $v(x;k) = m(x;k)e^{-ikx}$. Then $m(x;k)$ satisfies

$$m_{xx}(x;k) - 2ikm_x(x;k) = -u(x)m(x;k).$$

Then assuming that $m \to 1$, either as $x \to \infty$, or as $x \to -\infty$, we find

$$m(x;k) = 1 + \int_{-\infty}^{\infty} G(x-\xi;k)\, u(\xi)\, m(\xi;k)\, d\xi,$$

where $G(x;k)$ is the Green's function that solves

$$G_{xx} - 2ikG_x = -\delta(x).$$

Using the Fourier transform method, i.e., $G(x;k) = \int_C \hat{G}(p,k)e^{ipx}\,dp$ with $\delta(x) = \int e^{ipx}\,dp$, we find

$$G(x;k) = \frac{1}{2\pi}\int_C \frac{e^{ipx}}{p(p-2k)}\,dp,$$

where $C$ is an appropriate contour. We consider $G_\pm(x;k)$ defined by

$$G_\pm(x;k) = \frac{1}{2\pi}\int_{C_\pm} \frac{e^{ipx}}{p(p-2k)}\,dp,$$

where $C_+$ and $C_-$ are the contours from $-\infty$ to $\infty$ that pass below and above, respectively, both of the singularities at $p = 0$ and $p = 2k$. Hence by contour integration

$$G_+(x;k) = \frac{1}{2ik}(1-e^{2ikx})H(x) \quad = \begin{cases} \dfrac{1}{2ik}(1-e^{2ikx}), & \text{if} \quad x > 0, \\ 0, & \text{if} \quad x < 0, \end{cases}$$

$$G_-(x;k) = -\frac{1}{2ik}(1-e^{2ikx})H(-x) = \begin{cases} 0, & \text{if} \quad x > 0, \\ -\dfrac{1}{2ik}(1-e^{2ikx}), & \text{if} \quad x < 0, \end{cases}$$

where $H(x)$ is the Heaviside function given by

$$H(x) = \begin{cases} 1, & \text{if} \quad x > 0, \\ 0, & \text{if} \quad x < 0. \end{cases}$$

We require that $M(x;k)$ and $\bar{N}(x;k)$ satisfy the boundary conditions (9.10). Since from (9.10a), $M(x;k) \to 1$ as $x \to -\infty$, we associate $M(x;k)$ with the Green's function $G_+(x;k)$, so the integral is from $-\infty$ to $x$, and analogously since $\bar{N}(x;k) \to 1$ as $x \to \infty$, we associate $\bar{N}(x;k)$ with the Green's function $G_-(x;k)$. Therefore it follows that $M(x;k)$ and $\bar{N}(x;k)$ are the solutions of the following integral equations

$$\begin{aligned} M(x;k) &= 1 + \int_{-\infty}^{\infty} G_+(x-\xi;k)\,u(\xi)\,M(\xi;k)\,d\xi \\ &= 1 + \frac{1}{2ik}\int_{-\infty}^{x} \left\{1 - e^{2ik(x-\xi)}\right\} u(\xi)\,M(\xi;k)\,d\xi, \end{aligned} \tag{9.15}$$

$$\begin{aligned} \bar{N}(x;k) &= 1 + \int_{-\infty}^{\infty} G_-(x-\xi;k)\,u(\xi)\,\bar{N}(\xi;k)\,d\xi \\ &= 1 - \frac{1}{2ik}\int_{x}^{\infty} \left\{1 - e^{2ik(x-\xi)}\right\} u(\xi)\,\bar{N}(\xi;k)\,d\xi. \end{aligned} \tag{9.16}$$

These are Volterra integral equations that have unique solutions; their Neumann series converge uniformly with $u$ in the function space $L_2^1$, for each $k$: $M(x;k)\,\mathrm{Im}\,k \geq 0$, and for $\overline{N}(x;k)\,\mathrm{Im}\,k \leq 0$ (Deift and Trubowitz, 1979). One can show that their Neumann series converge uniformly for $u$ in this function space. We note: $G_\pm(x;k)$ is analytic for $\mathrm{Im}\,k \gtrless 0$ and vanishes as $|k| \to \infty$; $M(x;k)$ and $\bar{N}(x;k)$ are analytic for $\mathrm{Im}\,k > 0$ and $\mathrm{Im}\,k < 0$ respectively, and tend to unity as $|k| \to \infty$. In the same way it can be shown that $N(x;k)e^{-2ikx} = \psi(x;k)e^{ikx}$ is analytic for $\mathrm{Im}\,k > 0$ and $\overline{M}(x;k)e^{2ikx} = \overline{\psi}(x;k)e^{-ikx}$ is analytic for $\mathrm{Im}\,k < 0$.

## 9.2 Scattering data

A corollary to the above lemma is that $a(k)$ can be analytically extended to the upper half $k$-plane and tends to unity as $|k| \to \infty$ for $\mathrm{Im}\,k > 0$. We note that Deift and Trubowitz (1979) also show that $a(k)$ is continuous for $\mathrm{Im}\,k \geq 0$. The analytic properties of $a(k)$ may be established either from the Wronskian relationship (9.8), noting that $W(\phi(x;k), \psi(x;k)) = W(M(x;k), N(x;k))$ or by using a suitable integral representation for $a(k)$. To derive this integral representation we define

$$\Delta(x;k) = M(x;k) - a(k)\bar{N}(x;k),$$

and then from (9.15) and (9.16) it follows that

$$\Delta(x;k) = 1 - a(k) + \frac{1}{2ik}\int_{-\infty}^{\infty}\left\{1 - e^{2ik(x-\xi)}\right\}u(\xi)\,M(\xi;k)\,d\xi \\ - \frac{1}{2ik}\int_x^{\infty}\left\{1 - e^{2ik(x-\xi)}\right\}u(\xi)\,\Delta(\xi;k)\,d\xi. \quad (9.17)$$

Also, from (9.14) we have

$$\Delta(x;k) = b(k)\,e^{2ikx}\bar{N}(x;-k);$$

therefore from (9.16)

$$\Delta(x;k) = b(k)\,e^{2ikx} - \frac{1}{2ik}\int_x^{\infty}\left\{1 - e^{2ik(x-\xi)}\right\}u(\xi)\,\Delta(\xi;k)\,d\xi. \quad (9.18)$$

Hence by comparing (9.17) and (9.18), and equating coefficients of 1 and $e^{2ikx}$, we obtain the following integral representations for $a(k)$ and $b(k)$:

$$a(k) = 1 + \frac{1}{2ik}\int_{-\infty}^{\infty} u(\xi)\,M(\xi;k)\,d\xi, \quad (9.19)$$

$$b(k) = -\frac{1}{2ik}\int_{-\infty}^{\infty} u(\xi)\,M(\xi;k)\,e^{-2ik\xi}\,d\xi. \quad (9.20)$$

From these integral representations it follows that $a(k)$ is analytic for $\operatorname{Im} k > 0$, and tends to unity as $|k| \to \infty$, while $b(k)$ cannot, in general, be continued analytically off the real $k$-axis. Furthermore, since $M(x;k) \to 1$ as $\kappa \to \infty$, then it follows from (9.19) that $a(k) \to 1$ as $\kappa$ (for $\operatorname{Im} k > 0$). Similarly it can be shown that $\bar{a}(k)$ can be analytically extended to the lower half $k$-plane and tends to unity as $|k| \to \infty$ for $\operatorname{Im} k < 0$.

The functions $a(k)$, $\bar{a}(k)$, $b(k)$, $\bar{b}(k)$ satisfy the condition

$$a(k)\bar{a}(k) + b(k)\bar{b}(k) = -1. \tag{9.21}$$

This follows from the Wronskian and its relations:

$$\begin{aligned} W(\phi(x;k), \bar{\phi}(x;k)) &= W(a(k)\bar{\psi}(x;k) + b(k)\psi(x;k), \\ &\qquad - \bar{a}(k)\psi(x;k) + \bar{b}(k)\bar{\psi}(x;k)) \\ &= \left\{a(k)\bar{a}(k) + b(k)\bar{b}(k)\right\} W(\psi(x;k), \bar{\psi}(x;k)), \end{aligned}$$

with $W(\phi(x;k), \bar{\phi}(x;k)) = 2ik = -W(\psi(x;k), \bar{\psi}(x;k))$. From the definitions of the reflection and transmission coefficients, we see that $\tau(k)$ and $\rho(k)$ satisfy

$$|\rho(k)|^2 + |\tau(k)|^2 = 1.$$

Using (9.4)–(9.7) and their complex conjugates we have the symmetry conditions

$$\begin{aligned} \bar{a}(k) &= -a(-k) = -a^*(k^*), \\ \bar{b}(k) &= b(-k) = b^*(k^*), \end{aligned} \tag{9.22}$$

where $a^*(k^*)$, $b^*(k^*)$ are the complex conjugates of $a(k)$, $b(k)$, respectively.

The zeros of $a(k)$ play an important role in the inverse problem; they are poles in the generalized Riemann–Hilbert problem obtained from (9.14). We note the following: the function $a(k)$ can have a finite number of zeros at $k_1, k_2, \ldots, k_N$, where $k_j = i\kappa_j$, $\kappa_j \in \mathbb{R}$, $j = 1, 2, \ldots, N^{\#}$ (i.e., they all lie on the imaginary axis), in the upper half $k$-plane.

First we see that from equations (9.21), (9.22) that

$$|a(k)|^2 - |b(k)|^2 = 1,$$

for $k \in \mathbb{R}$; hence $a(k) \neq 0$ for $k \in \mathbb{R}$. Recall that $\phi(x;k)$, $\psi(x;k)$ and $\bar{\psi}(x;k)$ are solutions of the Schrödinger equation (9.4) satisfying the boundary conditions

$$\begin{aligned} &\phi(x;k) \sim e^{-ikx}, && \text{as} && x \to -\infty, \\ &\psi(x;k) \sim e^{ikx}, && \text{as} && x \to \infty, \\ &\bar{\psi}(x;k) \sim e^{-ikx}, && \text{as} && x \to \infty, \end{aligned} \tag{9.23}$$

together with the relationship

$$\phi(x;k) = a(k)\bar{\psi}(x;k) + b(k)\psi(x;k). \tag{9.24}$$

From (9.8) it follows that

$$W(\phi(x;k), \psi(x;k)) = \phi(x;k)\psi_x(x;k) - \phi_x(x;k)\psi(x;k) = 2ika(k).$$

Thus at any zero $k_j$ of $a(k)$, we have that $\phi(x;k), \psi(x;k)$ are linearly independent. In order to have a rapidly decaying state; i.e., a bound state where $\phi(x;k_j)$ is in $L^2(\mathbb{R})$, we must have that $\operatorname{Im} k_j > 0$.

The eigenfunction $\phi(x;k)$ and its complex conjugate $\phi^*(x;k^*)$ satisfy the equations

$$\begin{aligned} \phi_{xx} + \left\{u(x) + k^2\right\}\phi &= 0, \\ \phi^*_{xx} + \left\{u(x) + (k^*)^2\right\}\phi^* &= 0, \end{aligned} \tag{9.25}$$

respectively. Hence

$$\frac{\partial}{\partial x} W(\phi, \phi^*) + \left[(k^*)^2 - k^2\right]\phi\phi^* = 0,$$

which implies

$$\left[(k^*)^2 - k^2\right]\int_{-\infty}^{\infty} |\phi(x;k)|^2\, dx = 0 \tag{9.26}$$

since $\phi$ and its derivatives vanish as $x \to \pm\infty$. Further, calling $k_j = \xi_j + \eta_j$, we see by taking the real part of (9.26): $\left(k_j^*\right)^2 - k_j^2 = -2\xi_j\eta_j = 0$, any zero of $a$ must be purely imaginary; i.e., $\operatorname{Re} k_j = \xi_j = 0$.

To show that $a(k)$ has only a finite number of zeros, we note that $a \to 1$ as $|k| \to \infty$; furthermore, it is shown by Deift and Trubowitz (1979) that $a(k)$ is continuous for $\operatorname{Im} k \geq 0$. Hence there can be no cluster points of zeros of $a(k)$ either in the upper half $k$-plane or on the real $k$-axis and so necessarily $a(k)$ can possess only a finite number of zeros.

We can also show that the zeros of $a(k)$ are simple by studying the derivative $da/dk$ with respect to $k$: but we will not do this here (see Deift and Trubowitz, 1979).

Hence, as long as $u \in L^1_2$, $M(x;k)/a(k)$ is a meromorphic function in the upper half $k$-plane with a finite number of simple poles at $k = i\kappa_1, i\kappa_2, \ldots, i\kappa_{N^\#}$. Then set

$$\frac{M(x;k)}{a(k)} = \mu_+(x;k) + \sum_{j=1}^{N^\#} \frac{A_j(x)}{k - i\kappa_j}, \tag{9.27}$$

where $\mu_+(x;k)$ is analytic in the upper half $k$-plane. Integrating (9.27) around $i\kappa_j$ and using (9.14) and the fact that at a discrete eigenvalue $M, N$ are proportional, $M(x;i\kappa_j) = b_j N(x;i\kappa_j)$, shows that

$$A_j(x) = C_j \bar{N}(x;-i\kappa_j)\,\exp(-2\kappa_j x),$$

where $C_j = b_j/a_j'$. Hence from (9.14) we obtain

$$\mu_+(x;k) = \bar{N}(x;k) - \sum_{j=1}^{N^\#} \frac{C_j}{k - i\kappa_j}\,\exp(-2\kappa_j x)\bar{N}(x;-i\kappa_j) \\ + \rho(k)\,\exp(2ikx)\bar{N}(x;-k). \tag{9.28}$$

Furthermore, $\mu_+(x;k) \to 1$ as $\kappa \to 1$, for $\operatorname{Im} k > 0$.

## 9.3 The inverse problem

Equation (9.28) defines a Riemann–Hilbert problem (cf. Ablowitz and Fokas, 2003) in terms of the scattering data $S(\lambda) = \left\{(\kappa_j, C_j)_{j=1}^{N^\#}, \rho(k), a(k)\right\}$. We will use the following.

Consider the $\mathcal{P}^\pm$ projection operator defined by

$$(\mathcal{P}^\pm f)(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(\zeta)}{\zeta - (k \pm i0)}\,d\zeta = \lim_{\epsilon\downarrow 0}\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(\zeta)}{\zeta - (k \pm i\epsilon)}\,d\zeta\right\}. \tag{9.29}$$

Suppose that $f_\pm(k)$ is analytic in the upper/lower half $k$-plane and $f_\pm(k) \to 0$ as $|k| \to \infty$ (for $\operatorname{Im} k \gtrless 0$). Then

$$(\mathcal{P}^\pm f_\mp)(k) = 0,$$
$$(\mathcal{P}^\pm f_\pm)(k) = \pm f_\pm(k).$$

These results follow from contour integration.

To explain the ideas most easily, we first assume that there are no poles; that is, $a(k) \neq 0$. Then (9.28) reduces to (9.14), because $M(x;k)/a(k) = \mu_+$. In (9.14) unity from both $\overline{N}$ and $M/a$ and operating on (9.14) with $\mathcal{P}^-$ and using the analytic properties of $M$, $\bar{N}$ and $a$ yields

$$\mathcal{P}^-\left[\left(\frac{M(x;k)}{a(k)} - 1\right)\right] = \mathcal{P}^-\left[(\bar{N}(x;k) - 1) + \rho(k)e^{2ikx}\bar{N}(x;-k)\right].$$

This implies

$$0 = \left(\bar{N}(x;k) - 1\right) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho(\zeta)\,e^{2i\zeta x}\,\bar{N}(x;-\zeta)}{\zeta - (k - i0)}\,d\zeta.$$

Then employing the symmetry: $N(x;k) = e^{2ikx}\bar{N}(x;-k)$ and $N(x;-k) = e^{-2ikx}\bar{N}(x;k)$, it follows that

$$\begin{aligned}\bar{N}(x;k) &= 1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(\zeta)\,N(x;\zeta)}{\zeta-(k-i0)}\,d\zeta,\\ N(x;k) &= e^{2ikx}\left\{1+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(\zeta)\,N(x;\zeta)}{\zeta+k+i0}\,d\zeta\right\}.\end{aligned} \tag{9.30}$$

As $k \to \infty$, equation (9.30a) shows that

$$\bar{N}(x;k) \sim 1 - \frac{1}{2\pi i}\,\frac{1}{k}\int_{-\infty}^{\infty}\rho(\zeta)\,N(x;\zeta)\,d\zeta, \tag{9.31}$$

From the integral equation for $\bar{N}(x;k)$, equation (9.16), using the Riemann–Lebesgue lemma, we also see that as $\kappa \to \infty$

$$\bar{N}(x;k) \sim 1 - \frac{1}{2ik}\int_{x}^{\infty}u(\xi)\,d\xi. \tag{9.32}$$

Therefore, by comparing (9.31) and (9.32), it follows that the potential can be reconstructed from

$$u(x) = -\frac{\partial}{\partial x}\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}\rho(\zeta)\,N(x;\zeta)\,d\zeta\right\}. \tag{9.33}$$

For the case when $a(k)$ has zeros, one can extend the above result. Suppose that

$$a(i\kappa_j) = 0, \qquad \kappa_j \in \mathbb{R}, \qquad j = 1, 2, \ldots, N^{\#};$$

then define

$$\begin{aligned}M_j(x) &= M(x;i\kappa_j),\\ N_j(x) &= \exp(-2\kappa_j x)\bar{N}(x;-i\kappa_j),\end{aligned}$$

for $j = 1, 2, \ldots, N^{\#}$. In (9.28) subtracting unity from both $\mu_+(x,k)$ and $\overline{N}(x;k)$ and applying the $\mathcal{P}^-$ projection operator (9.29) to (9.28), recalling that $N(x;k) = e^{2ikx}\bar{N}(x;-k)$ and carrying out the contour integration of simple poles, we find the following integral equations

$$N(x;k) = e^{2ikx}\left\{1 - \sum_{j=1}^{N^{\#}}\frac{C_jN_j(x)}{k+i\kappa_j} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(\zeta)\,N(x;\zeta)}{\zeta+k+i0}\,d\zeta\right\}, \tag{9.34}$$

$$N_p(x) = \exp(-2\kappa_p x)\left\{1 + i\sum_{j=1}^{N^{\#}}\frac{C_jN_j(x)}{\kappa_p+\kappa_j} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(\zeta)\,N(x;\zeta)}{\zeta+i\kappa_p}\,d\zeta\right\}, \tag{9.35}$$

for $p = 1, 2, \ldots N^{\#}$, with the potential reconstructed from [following the same method that led to (9.33)]

$$u(x) = \frac{\partial}{\partial x}\left\{2i\sum_{j=1}^{N^{\#}} C_j N_j(x) - \frac{1}{\pi}\int_{-\infty}^{\infty} \rho(k)\, N(x;k)\, dk\right\}. \tag{9.36}$$

Existence and uniqueness of solutions to equations such as (9.34)–(9.36) is given by Beals et al. (1988). We also note that the above integral equations (9.34) and (9.35) can be transformed to Gel'fand–Levitan–Marchenko integral equations, see Section 9.5 below, from which one can also deduce the existence and uniqueness of solutions (cf. Marchenko, 1986).

## 9.4 The time dependence of the scattering data

In this section we will find out how the scattering data evolves. The time evolution of the scattering data may be obtained by analyzing the asymptotic behavior of the associated time evolution operator, which as we have seen earlier for the KdV equation is

$$v_t = (u_x + \gamma)v + (4k^2 - 2u)v_x,$$

with $\gamma$ a constant. If we let $v = \phi(x;k)$ and make the transformation

$$\phi(x,t;k) = M(x,t;k)\, e^{-ikx},$$

$M$ then satisfies

$$M_t = (\gamma - 4ik^3 + u_x + 2iku)M + (4k^2 - 2u)M_x. \tag{9.37}$$

Recall that from (9.11)

$$M(x,t;k) = a(k,t)\bar{N}(x,t;k) + b(k,t)N(x,t;-k),$$

where $\rho(k,t) = b(k,t)/a(k,t)$. From (9.10) the asymptotic behavior of $M(x,t;k)$ is given by

$$\begin{aligned} &M(x,t;k) \to 1, && \text{as} \quad x \to -\infty, \\ &M(x,t;k) \to a(k,t) + b(k,t)\, e^{2ikx}, && \text{as} \quad x \to \infty. \end{aligned}$$

By using using the fact that $u \to 0$ rapidly as $x \to \pm\infty$ and the above equation it follows from (9.37) that

$$\begin{gathered} \gamma - 4ik^3 = 0, \quad x \to -\infty \\ a_t + b_t\, e^{2ikx} = 8ik^3 b\, e^{2ikx}, \quad x \to +\infty, \end{gathered}$$

and by equating coefficients of $e^0$, $e^{2ikx}$ we find

$$a_t = 0, \qquad b_t = 8ik^3 b. \tag{9.38}$$

Solving (9.38) yields

$$a(k,t) = a(k,0), \qquad b(k,t) = b(k,0)\exp(8ik^3 t), \tag{9.39}$$

and so

$$\rho(k,t) \equiv \frac{b(k,t)}{a(k,t)} = \rho(k,0)\exp(8ik^3 t).$$

Since $a(k,t)$ does not evolve in time, the discrete eigenvalues, which are the zeros of $a(k)$ that are finite in number, simple, and lie on the imaginary axis, satisfy

$$k_j = i\kappa_j = \text{constant}, \qquad j = 1, 2, \ldots, N^{\#}.$$

Since the eigenvalues are constant in time, we say this is an "isospectral flow". Similarly we find that the time dependence of the $C_j(t)$ is given by

$$C_j(t) = C_j(0)\ \exp\left(8ik_j^3 t\right) = C_j(0)\ \exp\left(8\kappa_j^3 t\right). \tag{9.40}$$

## 9.5 The Gel'fand–Levitan–Marchenko integral equation

The Gel'fand–Levitan–Marchenko integral equation may be derived from the above formulation. We assume a "triangular" kernel $K(x,s;t)$ (see also below) such that

$$N(x,t;k) = e^{2ikx}\left\{1 + \int_x^{\infty} K(x,s;t)\, e^{ik(s-x)}\, ds\right\}, \tag{9.41}$$

where $K(x,s;t)$ is assumed to decay rapidly as $x$ and $s \to \infty$. Substituting (9.41) into (9.34)–(9.35) and taking the Fourier transform, i.e., operating with

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(x-y)}, \qquad \text{for} \quad y > x$$

we find that

$$K(x,y;t) + F(x+y;t) + \int_x^{\infty} K(x,s;t)\, F(s+y;t)\, ds = 0, \qquad y > x, \tag{9.42}$$

where

$$F(x;t) = \sum_{j=1}^{N^{\#}} -iC_j(0)\exp\left(8\kappa_j^3 t - \kappa_j x\right) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho(k,t)\, e^{ikx}\, dk. \tag{9.43}$$

Substituting (9.41) into (9.36) we have

$$u(x,t) = 2\,\frac{\partial}{\partial x}K(x,x;t). \tag{9.44}$$

Thus we have shown that the Gel'fand–Levitan–Marchenko equation (9.42) arises from the integral equations that result from the Riemann–Hilbert boundary value problem.

We can also show that $K(x,s,t)$ satisfies a PDE in $x$, $s$; i.e., it is independent of $k$. Using (9.41), then from

$$N = e^{ikx}\psi$$

we have

$$\psi = e^{ikx} + \int_x^\infty K(x,s,t)e^{iks}\,ds.$$

We wish to use this in

$$\psi_{xx} + (u(x,t)+k^2)\psi = 0 \tag{9.45}$$

so we calculate

$$\psi_x = ike^{ikx} - K(x,x,t)e^{ikx} + \int_x^\infty K_x(x,s,t)e^{iks}\,ds$$

and

$$\begin{aligned}\psi_{xx} = -k^2e^{ikx} &- \frac{d}{dx}K(x,x,t)e^{ikx}\\ &- ikK(x,x,t)e^{ikx} + \int_x^\infty K_{xx}(x,s,t)e^{iks}\,ds - K_x(x,x,t)e^{ikx}.\end{aligned}$$

Part of the third term in (9.45) becomes after integrating by parts

$$\begin{aligned}k^2\int_x^\infty K(x,s,t)e^{iks}\,ds &= -\int_x^\infty K(x,s,t)\frac{\partial^2}{\partial s^2}e^{iks}\,ds\\ &= K(x,x,t)ike^{ikx} + \int_x^\infty \frac{\partial K}{\partial s}\frac{\partial}{\partial s}e^{iks}\,ds\\ &= -ikK(x,x,t)e^{ikx} - \frac{\partial K}{\partial s}(x,x,t)e^{ikx}\\ &\quad - \int_x^\infty \frac{\partial^2 K}{\partial s^2}e^{iks}\,ds.\end{aligned}$$

Then using all of the above in (9.45) gives

$$
\begin{aligned}
-\frac{d}{dx}K(x,x,t)e^{ikx} + \int_x^\infty \frac{\partial^2}{\partial x^2}K(x,s,t)e^{iks}\,ds - \frac{\partial}{\partial x}K(x,s=x,t)e^{ikx} \\
+ ue^{ikx} + \int_x^\infty u(x)K(x,s,t)e^{iks}\,ds - \frac{\partial K}{\partial s}(x,x,t)e^{ikx} \\
- \int_x^\infty \frac{\partial^2 K}{\partial s^2}(x,s,t)e^{iks}\,ds = 0
\end{aligned}
$$

which in turn leads to

$$
\begin{aligned}
\int_x^\infty [(K_{xx} - K_{ss}) - uK(x,s,t)]e^{ikx}\,ds \\
+ \left[-\frac{d}{dx}K(x,x,t) \underbrace{- \frac{\partial}{\partial x}K(x,s=x,t) - \frac{\partial}{\partial s}K(x,x,t)}_{-\frac{d}{dx}K(x,x,t)} + u\right] e^{ikx} = 0
\end{aligned}
$$

which for arbitrary kernels leads to

$$
(K_{xx} - K_{ss}) - u(x)K(x,s,t) = 0, \quad u(x,t) = 2\frac{d}{dx}K(x,x,t), \quad s > x
$$

where $t$ is a parameter and $K(x,s;t)$ decays rapidly as $x$ and $s \to \infty$. Since $K$ satisfies the above system, which is a well-posed Goursat problem (Garabedian, 1984), this indicates the solution exists and is unique. Hence, this shows that $K(x,y,t)$ satisfies a wave-like PDE independent of $k$.

## 9.6 Outline of the inverse scattering transform for the KdV equation

We have seen that associated with the KdV equation (9.3) is a linear scattering problem (9.1) and time-dependent equation (9.2). These equations are compatible if the eigenvalues $\lambda \to i\kappa_j$, $j = 1, \ldots N^\#$, are time-independent, or isospectral, i.e. $\kappa_{j,t} = 0$.

The direct scattering problem maps a decaying potential ($u \in L_2^1$) into the scattering data. The inverse scattering problem reconstructs the potential from the scattering data. The time evolution equation tells us how the scattering data evolves.

The solution of the KdV equation (9.3) proceeds as follows: at time $t = 0$, given $u(x,0)$ we solve the direct scattering problem. This involves solving the integral equations (9.15)–(9.16) and using these results to obtain the scattering data (9.19)–(9.20). The scattering data needed are:

$$S(k,0)=\left(\{\kappa_j, C_j(0)\}_{j=1}^{N^\#}, \rho(k,0)\equiv\frac{b(k,0)}{a(k,0)}\right),$$

where $\kappa_j$, $j=1,2,\ldots,N^\#$, are the locations of the zeros of $a(k,0)$ and $C_j(0)$, $j=1,2,\ldots,N^\#$, are often called the associated "normalization constants" that were also defined above.

Next we determine the time evolution of the scattering data using equations (9.39)–(9.40). This yields the scattering data at any time $t$:

$$S(k,t)=\left(\{\kappa_j, C_j(t)\}_{j=1}^{N^\#}, \rho(k,t)\equiv\frac{b(k,t)}{a(k,t)}\right).$$

For the inverse problem, at time $t$, we solve either the integral equations for $N(x,t)$ and reconstruct $u(x,t)$ via (9.34)–(9.36) or solve the Gel'fand–Levitan–Marchenko equations for $K(x,y,t)$ and reconstruct $u(x,t)$ via (9.42)–(9.44).

This method is conceptually analogous to the Fourier transform method for solving linear equations (discussed in Chapter 2), except however the step of solving the inverse scattering problem requires one to solve a linear integral equation (i.e., a Riemann–Hilbert boundary value problem) that adds complications. Schematically the solution is given by

$$\begin{array}{ccc}
u(x,0) & \xrightarrow{\text{direct scattering}} & S(k,0)=\left(\{\kappa_n, C_j(0)\}_{j=1}^{N^\#}, \rho(k,0), a(k,0)\right)\\
 & & \Big\downarrow t\text{: time evolution}\\
u(x,t) & \xleftarrow{\text{inverse scattering}} & S(k,t)=\left(\{\kappa_n, C_j(t)\}_{j=1}^{N^\#}, \rho(k,t), a(k,t)\right)
\end{array}$$

In the analogy of IST to the Fourier transform method for linear partial differential equations, the direct scattering problem and the scattering data play the role of the Fourier transform and the inverse scattering problem the inverse Fourier transform.

## 9.7 Soliton solutions of the KdV equation

In this section we will describe how the inverse scattering method can be used to find soliton solutions of the Korteweg–de Vries equation (9.3).

### 9.7.1 The Riemann–Hilbert method

Pure soliton solutions may be obtained from (9.34)–(9.36) in the case when $\rho(k,t)=0$ on the real $k$-axis. This is usually called a reflectionless potential. We assume a special form for $\rho(k;t)$, namely

$$\rho(k;t) = \begin{cases} 0, & \text{if } k \text{ real}, \\ \displaystyle\sum_{j=1}^{N^\#} \frac{C_j(0)}{k - i\kappa_j} \exp\left(8\kappa_j^3 t\right), & \text{if } \operatorname{Im} k > 0. \end{cases}$$

Then we obtain a linear algebraic system for the $N_j(x,t)$

$$N_j(x,t) - \sum_{m=1}^{N^\#} \frac{iC_m(0)}{\kappa_m + \kappa_j} \exp\left(-2\kappa_m x + 8\kappa_m^3 t\right) N_m(x,t) = \exp(-2\kappa_j x), \qquad (9.46)$$

where $j = 1, 2, \ldots, N$, $N_j(x,t) := N(x; k = i\kappa_j; t)$, and the potential $u(x,t)$ is given by

$$u(x,t) = 2\frac{\partial}{\partial x}\left\{\sum_{j=1}^{N^\#} \exp\left(8\kappa_j^3 t\right) iC_j(0) N_j(x,t)\right\}.$$

In the case of the one-soliton solution, the number of eigenvalues is $N^\# = 1$ and (9.46) reduces to

$$N_1(x,t) - \frac{iC_1(0)}{2\kappa_1} \exp\left(-2\kappa_1 x + 8\kappa_1^3 t\right) N_1(x,t) = \exp(-2\kappa_1 x),$$

and so

$$N_1(x,t) = \frac{2\kappa_1 \exp(-2\kappa_1 x)}{2\kappa_1 - iC_1(0)\exp\left(-2\kappa_1 x + 8\kappa_1^3 t\right)}.$$

Therefore

$$\begin{aligned} u(x,t) &= 2\frac{\partial}{\partial x}\left\{\exp\left(8\kappa_1^3 t\right) iC_1(0) N_1(x,t)\right\} \\ &= 2\frac{\partial}{\partial x}\left\{\frac{2\kappa_1 iC_1(0)\exp\left(-2\kappa_1 x + 8\kappa_1^3 t\right)}{2\kappa_1 - iC_1(0)\exp\left(-2\kappa_1 x + 8\kappa_1^3 t\right)}\right\} \\ &= 2\frac{\partial}{\partial x}\left\{\frac{2\kappa_1 iC_1(0)}{2\kappa_1 \exp\left(2\kappa_1 x - 8\kappa_1^3 t\right) - iC_1(0)}\right\} \\ &= \frac{8\kappa_1^2}{\left[\exp\left(\kappa_1 x - \kappa_1^3 t - \kappa_1 x_1\right) + \exp\left(-\kappa_1 x + \kappa_1^3 t + \kappa_1 x_1\right)\right]^2} \end{aligned}$$

where we define $-iC_1(0) = 2\kappa_1 \exp(2\kappa_1 x_1)$; hence we obtain the one-soliton solution

$$u(x,t) = 2\kappa_1^2 \operatorname{sech}^2 \left\{ \kappa_1 \left( x - 4\kappa_1^2 t - x_1 \right) \right\}.$$

In the case of the two-solution solution, we have to first solve a linear system

$$N_1(x,t) + \sum_{j=1}^{2} \frac{C_j(0)}{\kappa_j + \kappa_1} \exp\left(-2\kappa_j x + 8\kappa_j^3 t\right) N_j(x,t) = \exp(-2\kappa_1 x),$$
$$N_2(x,t) + \sum_{j=1}^{2} \frac{C_j(0)}{\kappa_j + \kappa_2} \exp\left(-2\kappa_j x + 8\kappa_j^3 t\right) N_j(x,t) = \exp(-2\kappa_2 x),$$

for $N_1(x,t)$ and $N_2(x,t)$. After some algebra, the two-soliton solution of the KdV equation is found to be

$$u(x,t) = \frac{4\left(\kappa_2^2 - \kappa_1^2\right)\left[\left(\kappa_2^2 - \kappa_1^2\right) + \kappa_1^2 \cosh(2\kappa_2\xi_2) + \kappa_2^2 \cosh(2\kappa_1\xi_1)\right]}{\left[(\kappa_2 - \kappa_1)\cosh(\kappa_1\xi_1 + \kappa_2\xi_2) + (\kappa_2 + \kappa_1)\cosh(\kappa_2\xi_2 - \kappa_1\xi_1)\right]^2},$$

where $\xi_i = x - 4\kappa_i^2 t - x_i$, and $C_i(0) = 2\kappa_i \exp(2\kappa_i x_i)$, for $i = 1, 2$.

### 9.7.2 Gel'fand–Levitan–Marchenko approach

Again we suppose that the potential $u(x,0)$ is reflectionless, $\rho(k,0) = 0$, and the time-independent Schrödinger equation (9.4) has $N^{\#}$ discrete eigenvalues $k_n = i\kappa_n$, $n = 1, 2, \ldots, N^{\#}$, such that $0 < \kappa_1 < \kappa_2 < \cdots < \kappa_{N^{\#}}$. To connect with the standard approach, in the kernel $F(x,t)$ in the Gel'fand–Levitan–Marchenko equation (9.42)–(9.43) we take $-iC_n = c_n^2$. In this way the associated eigenfunctions $\psi_n(x)$, $n = 1, 2, \ldots, N^{\#}$, satisfy the normalization condition

$$\int_{-\infty}^{\infty} \psi_n^2(x)\, dx = 1,$$

and the asymptotic limit as $x \to +\infty$, $\lim_{x\to+\infty} \psi_n(x) \exp(\kappa_n x) = c_n(0)$, where $c_j(0)$ are called the Gel'fand–Levitan–Marchenko normalization constants; and $c_n(t) = c_n(0) \exp\left(4\kappa_n^3 t\right)$.

Then this is solvable by a separation of variables (cf. Kay and Moses, 1956; Gardner et al., 1974). Here we take

$$F(x;t) = \sum_{n=1}^{N^{\#}} c_n^2(0) \exp\left(8\kappa_n^3 t - \kappa_n x\right),$$

and so the Gel'fand–Levitan–Marchenko equation becomes

$$K(x,y;t)+\sum_{n=1}^{N^\#}c_n^2(t)\exp\{-\kappa_n(x+y)\}$$
$$+\int_x^\infty K(x,s;t)\sum_{n=1}^{N}c_n^2(t)\exp\{-\kappa_n(s+y)\}\,ds=0. \qquad (9.47)$$

Now we seek a solution of this equation in the form

$$K(x,y;t)=\sum_{n=1}^{N^\#}L_n(x,t)\exp(-\kappa_n y),$$

and then the integral equation (9.47) reduces to a linear algebraic system

$$L_n(x,t)+\sum_{m=1}^{N^\#}\frac{c_n^2(t)}{\kappa_m+\kappa_n}\exp\{-(\kappa_m+\kappa_n)x\}L_m(x,t)+c_n^2(t)\exp(-\kappa_n x)=0,$$

for $n=1,2,\ldots,N^\#$. We can, in principle, solve this linear system. We will go a little further.

If we let $L_n(x,t)=-c_n(t)w_n(x,t)$, for $n=1,2,\ldots,N^\#$, this yields a system that can be written in the convenient form

$$(\mathsf{I}+\mathsf{B})\mathbf{w}=\mathbf{f}, \qquad (9.48)$$

where $\mathbf{w}=(w_1,w_2,\ldots,w_{N^\#})$, $\mathbf{f}=(f_1,f_2,\ldots,f_{N^\#})$ with $f_n(x,t)=c_n(t)\exp(-\kappa_n x)$ for $n=1,2,\ldots,N^\#$, $\mathsf{I}$ is the $N^\#\times N^\#$ identity matrix and $\mathsf{B}$ is a symmetric, $N^\#\times N^\#$ matrix with entries

$$B_{mn}(x,t)=\frac{c_m(t)c_n(t)}{\kappa_m+\kappa_n}\exp\{-(\kappa_m+\kappa_n)x\}, \qquad m,n=1,2,\ldots,N^\#.$$

The system (9.48) has a unique solution when $\mathsf{B}$ is positive definite. If we consider the quadratic form

$$\mathbf{v}^{\mathrm{T}}\mathsf{B}\mathbf{v}=\sum_{m=1}^{N^\#}\sum_{n=1}^{N^\#}\frac{c_m(t)c_n(t)v_m v_n}{\kappa_m+\kappa_n}\exp\{-(\kappa_m+\kappa_n)x\}$$
$$=\int_x^\infty\left[\sum_{n=1}^{N^\#}c_n(t)v_n\exp(-\kappa_n y)\right]^2dy,$$

where $\mathbf{v}=(v_1,v_2,\ldots,v_{N^\#})$, which is positive and vanishes only if $\mathbf{v}=\mathbf{0}$; then $\mathsf{B}$ is positive definite.

If in the kernel, $F(x,t)$, of the Gel'fand–Levitan–Marchenko equation (9.47), we take $N^{\#} = 1$ and define $c_1^2(0) = 2\kappa_1 \exp(2\kappa_1 x_1)$, then we find the one-soliton solution

$$u(x,t) = 2\kappa_1^2 \operatorname{sech}^2 \left\{ \kappa_1 \left( x - 4\kappa_1^2 t - x_1 \right) \right\},$$

which is the same as derived from the RHBVP approach. The general $N^{\#}$-soliton case was analyzed in detail by Gardner et al. (1974).

## 9.8 Special initial potentials

In the previous sections we obtained pure soliton solutions for the KdV equation corresponding to reflectionless potentials. However, more general choices of the initial data $u(x,0)$ give rise to non-zero reflection coefficients; unfortunately, in this case it generally is not possible to solve the inverse equations in explicit form. However, long-time asymptotic information can be obtained (Ablowitz and Segur, 1979; Deift and Zhou, 1992; Deift et al., 1994).

### 9.8.1 Delta function initial profile

As an example consider the delta function initial profile given by

$$u(x;0) = Q\delta(x),$$

where $Q$ is constant and $\delta(x)$ is the Dirac delta function. We use the Schrödinger equation

$$v_{xx} + (k^2 + u(x,0))v = 0$$

with the scattering definitions given by (9.23)–(9.24). So

$$\begin{aligned} \phi(x;k) &= e^{-ikx}, \qquad &&\text{for} \quad x < 0, \\ \psi(x;k) &= e^{ikx}, \qquad &&\text{for} \quad x > 0, \qquad \text{and} \\ \bar{\psi}(x;k) &= e^{-ikx}, \qquad &&\text{for} \quad x > 0. \end{aligned}$$

Then requiring the function $v$ to be continuous and satisfy the jump condition

$$[v_x]_{0-}^{0^+} + Qv(0) = 0,$$

and letting $v(x,k) = \phi(x;k)$ yields the following equations

$$\begin{aligned} a(k,0) + b(k,0) &= 1 \\ a(k,0)(-ik) + b(k,0)(ik) &= -Q - ik. \end{aligned}$$

Hence

$$a(k,0) = \frac{2ik + Q}{2ik} = a(k,t) = \frac{1}{\tau(k,t)}, \quad b(k,0) = \frac{-Q}{2ik}, \quad \rho(k,t) = \frac{-Q}{2ik + Q}.$$

Thus in this case, when $Q > 0$, there is a single discrete eigenvalue $\lambda = -\kappa_1^2$, where $\kappa_1 = Q/2$. The reflection coefficient is non-zero and given by

$$\rho(k,0) = \frac{-\kappa_1}{\kappa_1 + ik},$$

and similarly we find $C_1(0) = i\kappa_1$. The time evolution follows as above; it is given by

$$\kappa_1 = \text{constant}, \; C_1(t) = i\kappa_1 \exp\left(8\kappa_1^3 t\right), \qquad \rho(k,t) = \frac{-\kappa_1 \exp(8ik^3 t)}{\kappa_1 + ik}.$$

### 9.8.2 Box initial profile

Here we outline how to find the data associated with a box profile at $t = 0$:

$$u(x,0) = \begin{cases} H^2, & |x| < L/2 \\ 0, & |x| > L/2 \end{cases}$$

where $H, L > 0$ are constant. Again, we use the Schrödinger equation

$$v_{xx} + (k^2 + u(x,0))v = 0$$

with the scattering functions/coefficients satisfying (9.23)–(9.24).

Since this is a linear, constant coefficient, second-order differential equation we have

$$v(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & x < -L/2 \\ Ce^{i\lambda x} + De^{-i\lambda x}, & |x| < L/2 \\ Ee^{ikx} + Fe^{-ikx}, & x > L/2 \end{cases}$$

where $\lambda \equiv \sqrt{H^2 + k^2}$ and so

$$v'(x) = \begin{cases} Aike^{ikx} - Bike^{-ikx}, & x < -L/2 \\ Ci\lambda e^{i\lambda x} - Di\lambda e^{-i\lambda x}, & |x| < L/2 \\ Eike^{ikx} - Fike^{-ikx}, & x > L/2. \end{cases}$$

Let us first consider the case when $A = 0$ and $B = 1$; then, requiring $v$ and $v_x$ to be continuous at $x = \pm L/2$ implies

$$\begin{aligned} e^{ikL/2} &= Ce^{-i\lambda L/2} + De^{i\lambda L/2}, \\ -ike^{ikL/2} &= Ci\lambda e^{-i\lambda L/2} - Di\lambda e^{i\lambda L/2}, \\ Ee^{ikL/2} + Fe^{-ikL/2} &= Ce^{i\lambda L/2} + De^{-i\lambda L/2}, \\ Eike^{ikL/2} - Fike^{-ikL/2} &= Ci\lambda e^{i\lambda L/2} - Di\lambda e^{-i\lambda L/2}. \end{aligned}$$

Solving for the constants $C, D, E, F$ gives

$$
\begin{aligned}
C &= \frac{\lambda - k}{2\lambda} e^{i(k+\lambda)L/2}, \\
D &= \frac{\lambda + k}{2\lambda} e^{i(k-\lambda)L/2}, \\
E &= -\frac{k^2 - \lambda^2}{4k\lambda}\left(e^{iL\lambda} - e^{-iL\lambda}\right), \\
F &= -\frac{e^{iL(k-\lambda)}}{4k\lambda}\left\{(k-\lambda)^2 e^{2iL\lambda} - (k+\lambda)^2\right\}.
\end{aligned}
$$

We define the function $\phi \sim e^{-ikx}$ as $x \to -\infty$, so

$$
\phi(x) \equiv \begin{cases} e^{-ikx}, \ x < -L/2 \\ e^{ikL/2}\left\{\cos\left[\lambda(L/2 + x)\right] - \frac{ik}{\lambda}\sin\left[\lambda(L/2+x)\right]\right\}, \ |x| < L/2 \\ e^{-ik(x-L)}\cos(\lambda L) \\ \qquad -\frac{i\sin(\lambda L)}{2k\lambda}\left[e^{-ik(x-L)}(\lambda^2 + k^2) - e^{ikx}(\lambda^2 - k^2)\right], \ x > L/2. \end{cases}
$$

Now, let us consider the case when $A = 1$ and $B = 0$; then solving for $C$, $D$, $E$, $F$ by enforcing continuity of $v$ and $v_x$ at $x = \pm L/2$ gives

$$
\begin{aligned}
C &= \frac{\lambda + k}{2\lambda} e^{i(k-\lambda)L/2}, \\
D &= \frac{\lambda - k}{2\lambda} e^{i(k+\lambda)L/2}, \\
E &= e^{-ikL}\left\{\cos(\lambda L) + i\frac{\lambda^2 + k^2}{2k\lambda}\sin(\lambda L)\right\}, \\
F &= i\frac{\lambda^2 - k^2}{2k\lambda}\sin(\lambda L).
\end{aligned}
$$

We then define $\overline{\phi} \sim e^{-ikx}$ as $x \to -\infty$ and find

$$
\overline{\phi}(x) \equiv \begin{cases} e^{ikx}, \ x < -L/2 \\ e^{-ikL/2}\left\{\cos\left[\lambda(L/2 + x)\right] + \frac{ik}{\lambda}\sin\left[\lambda(L/2+x)\right]\right\}, \ |x| < L/2 \\ e^{ik(x-L)}\cos(\lambda L) \\ \qquad +\frac{i\sin(\lambda L)}{2k\lambda}\left[e^{ik(x-L)}(\lambda^2 + k^2) - e^{-ikx}(\lambda^2 - k^2)\right], \ x > L/2. \end{cases}
$$

Recall the eigenfunctions $\psi, \overline{\psi}$ have the asymptotic boundary conditions: $\psi \sim e^{ikx}$ as $x \to \infty$ and $\overline{\psi} \sim e^{-ikx}$ as $x \to \infty$; we write:

$$
\phi(x) = a(k)\overline{\psi}(x) + b(k)\psi(x) \sim a(k)e^{-ikx} + b(k)e^{ikx},
$$

as $x \to \infty$. By comparing coefficients we find that

$$a(k) = e^{ikL}\left\{\cos\left(L\sqrt{H^2+k^2}\right) - i\frac{H^2+2k^2}{2k\sqrt{H^2+k^2}}\sin\left(L\sqrt{H^2+k^2}\right)\right\},$$
$$b(k) = \frac{iH^2}{2k\sqrt{H^2+k^2}}\sin\left(L\sqrt{H^2+k^2}\right).$$

Since the zeros of $a(k)$ are purely imaginary, let $k = i\kappa$ where $\kappa \in \mathbb{R}$. Then

$$a(i\kappa) = e^{-\kappa L}\left\{\cos\left(L\sqrt{H^2-\kappa^2}\right) - \frac{H^2-2\kappa^2}{2\kappa\sqrt{H^2-\kappa^2}}\sin\left(L\sqrt{H^2-\kappa^2}\right)\right\}.$$

Thus, the zeros of $a(i\kappa)$ occur when

$$\tan\left(L\sqrt{H^2-\kappa^2}\right) = \frac{2\kappa\sqrt{H^2-\kappa^2}}{H^2-2\kappa^2}.$$

Note that the right-hand side is a monotonically increasing function from $[0,\infty)$ on $0 \le \kappa < H/\sqrt{2}$ and from $(-\infty, 0]$ on $H/\sqrt{2} < \kappa \le H$, and that $\tan\left(LH\sqrt{1-\kappa^2/H^2}\right)$ has $LH/\pi$ periods for $\kappa$ between 0 and $H$. Thus, the number of zeros of $a(i\kappa)$ depends on the size of $LH$, such that there are $n \in \mathbb{N}$ zeros of $a(i\kappa)$ for $0 < \kappa < H$ when

$$(n-1)\pi < LH \le n\pi.$$

Additionally, using the same reasoning as above, there are up to $n$ zeros for $-H < \kappa < 0$ when $LH < n\pi$. For $|\kappa| > H$, we can write

$$\tanh\left(L\sqrt{\kappa^2-H^2}\right) = \frac{2\kappa\sqrt{\kappa^2-H^2}}{H^2-2\kappa^2},$$

and it is seen that there are no zeros for $\kappa > H$. As earlier, the time dependence of the scattering data can be determined from $a(k,t) = a(k,0), b(k,t) = b(k,0)\exp(8ik^3t)$, etc.

## 9.9 Conserved quantities and conservation laws

As mentioned in Chapter 8, one of the major developments in the study of integrable systems was the recognition that special equations, such as the KdV equation, have an infinite number of conserved quantities/conservation laws. These were derived by Gardner et al. (1967) by making use of the Miura transformation. That derivation can be found in Section 8.3. In this section we use IST to derive these results.

Recall from Lemma 9.1 that $a(k)$ and $M(x;k) = \phi(x;k)\exp(-ikx)$ are analytic in the upper half $k$-plane and that $a(k)$ has a finite number of simple zeros on the imaginary axis: $\{k_m = iK_m\}_{m=1}^N$, in the upper half $k$-plane and $a(k) \to 1$ as $k \to \infty$. Moreover we have shown in Section 9.4 that $a(k)$ is constant in time. We will use this fact to obtain the conserved densities.

We can relate $\log a(k)$ to the potential $u$ and its derivatives. From (9.5) and (9.8) we have

$$a(k) = \frac{1}{2ik} W(\phi, \psi)$$
$$= \frac{1}{2ik}(\phi\psi_x - \phi_x\psi) = \lim_{x\to+\infty} \frac{1}{2ik}\left(\phi ike^{ikx} - \phi_x e^{ikx}\right).$$

Letting

$$\phi = e^{p-ikx}, \quad p \to 0 \text{ as } x \to -\infty \tag{9.49}$$

yields

$$a(k) = \lim_{x\to+\infty} e^p\left(1 - \frac{p_x}{2ik}\right)$$

or

$$\log a = \lim_{x\to+\infty} p + \lim_{x\to+\infty} \log\left(1 - \frac{p_x}{2ik}\right). \tag{9.50}$$

Substituting (9.49) into

$$\phi_{xx} + (k^2 + u)\phi = 0$$

and calling $\mu = p_x = ik + \dfrac{\phi_x}{\phi}$ gives

$$\mu_x + \mu^2 - 2ik\mu + u = 0 \tag{9.51}$$

which is a Ricatti equation for $\mu$ in terms of $u$. This equation has a series solution, in terms of inverse powers of $k$, of the form

$$\mu = \sum_{n=1}^{\infty} \frac{\mu_n}{(2ik)^n} = \frac{\mu_1}{2ik} + \frac{\mu_2}{(2ik)^2} + \frac{\mu_3}{(2ik)^3} + \cdots, \tag{9.52}$$

which yields a recursion relation for the $\mu_n$. From (9.51) the first three terms are

$$\mu_1 = u, \quad \mu_2 = u_x, \quad \mu_3 = u^2 + u_{xx} \tag{9.53}$$

and in general $\mu_n$ satisfies

$$\mu_{n,x} + \sum_{q=1}^{n-1} \mu_q\mu_{n-q} - \mu_{n+1} = 0 \tag{9.54}$$

which further yields $\mu_4 = (2u^2 + u_{xx})_x$, $\mu_5 = \mu_{4,x} + (u^2)_{xx} + 2u^3 - u_x^2$, and so on. From (9.53)–(9.54) we see that all of $\mu_n$ are polynomial functions of $u$ and its derivatives, i.e., $\mu_n = \mu_n(u, u_x, u_{xx}, \dots)$ and $\mu_n \to 0$ as $x \to \infty$. Hence,

$$\lim_{x\to+\infty} \log\left(1 - \frac{p_x}{2ik}\right) = 0.$$

Then from (9.50) and (9.52),

$$\begin{aligned}\log a &= \lim_{x\to+\infty} \int_{-\infty}^{x} p_x\, dx = \int_{-\infty}^{\infty} \mu\, dx \\ &= \sum_{n=1}^{\infty} \frac{1}{(2ik)^n} \int_{-\infty}^{\infty} \mu_n(u, u_x, u_{xx}, \dots)\, dx = \sum_{n=1}^{\infty} \frac{C_n}{(2ik)^2},\end{aligned} \tag{9.55}$$

which implies we have an infinite number of conserved quantities

$$C_n = \int_{-\infty}^{\infty} \mu_n(u, u_x, u_{xx}, \dots)\, dx.$$

The first three non-trivial conserved quantities are, from (9.53)–(9.54) ($C_{2n}$ are trivial)

$$C_1 = \int_{-\infty}^{\infty} u\, dx, \quad C_3 = \int_{-\infty}^{\infty} u^2\, dx, \quad C_5 = \int_{-\infty}^{\infty} \left(2u^3 - u_x^2\right) dx.$$

We can also use the large $k$ expansion to retrieve the infinite number of conservation laws

$$\frac{\partial T_n}{\partial t} + \frac{\partial F_n}{\partial x} = 0, \quad n = 1, 2, 3, \dots$$

associated with the KdV equation. As discussed earlier there is an associated linear dependence with the Schrödinger scattering problem; that is,

$$\phi_t = (u_x + 4ik^3)\phi + (4k^2 - 2u)\phi_x. \tag{9.56}$$

We note that the fixed boundary condition $\phi \sim \exp(-ikx)$ as $x \to -\infty$ is consistent with (9.56). Substituting $\phi = \exp(p - ikx)$, $\mu = p_x$ into (9.56) and taking a derivative with respect to $x$ yields after some manipulation

$$\mu_t = \frac{\partial}{\partial x}[2iku + u_x + (4k^2 - 2u)\mu].$$

Then the expansion (9.52) gives

$$\frac{\partial \mu_n}{\partial t} + \frac{\partial}{\partial x}(\mu_{n+2} + 2u\mu_n) = 0, \quad n \geq 1$$

with the $\mu$ obtained from (9.53)–(9.54). Substitution of $\mu_n$ yields the infinite number of conservation laws. The first two non-trivial conservation laws are

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}(u_{xx} + 3u^2) = 0$$
$$\frac{\partial}{\partial t}(u^2 + u_{xx}) + \frac{\partial}{\partial x}\left[u_{xxxx} + 4(u^2)_{xx} + 4u^3 - 3u_x^2\right] = 0.$$

Finally, we note that the conserved densities can also be computed in terms of the scattering data. This requires some complex analysis.

Consider the function

$$\alpha(k) = a(k)\prod_{m=1}^{N^\#}\frac{k-k_m^*}{k-k_m}, \tag{9.57}$$

where $k_m^*$ is the complex conjugate of $k_m$. This function is analytic in the upper half $k$-plane with no zeros and $\alpha(k) \to 1$ as $k \to \infty$. By the Schwarz reflection principle (see Ablowitz and Fokas, 2003) the complex conjugate of (9.57)

$$\alpha(k)^* = a(k)^*\prod_{m=1}^{N^\#}\frac{k-k_m}{k-k_m^*} \tag{9.58}$$

is analytic in the lower half $k$-plane with no zeros. Then by Cauchy's integral theorem for $\mathrm{Im}\{k\} > 0$

$$\log\alpha(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log\alpha(z)}{z-k}\,dz$$
$$0 = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log\alpha(z)^*}{z-k}\,dz.$$

Adding the above equations for $\mathrm{Im}\{k\} > 0$ and using (9.57) and (9.58) gives

$$\log a(k) = -\frac{1}{2\pi ik}\int_{-\infty}^{\infty}\frac{\log aa^*}{1-z/k}\,dz + \sum_{m=1}^{N^\#}\log\left(\frac{1-k_m/k}{1-k_m^*/k}\right).$$

Then expanding for large $k$ gives

$$\log a(k) = -\frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{1}{k^n}\int_{-\infty}^{\infty}z^{n-1}\log\left(aa^*(z)\right)\,dz$$
$$+\sum_{n=1}^{\infty}\frac{1}{k^n}\sum_{m=1}^{N^\#}\frac{((k_m^*)^n - k_m^n)}{n} = \sum_{n=1}^{\infty}\frac{C_n}{(2ik)^n}.$$

The right-hand side of the above equation is also obtained from (9.55); thus $C_n$ are the conserved densities now written in terms of scattering data. Recall,

$$1 - \left|\frac{b}{a}(k)\right|^2 = 1 - |\rho(k)|^2 = \frac{1}{|a(k)|^2}.$$

Thus we can write the non-trivial conserved densities in terms of the data obtained from $\log a(k)$:

$$C_{2n+1} = (2i)^{2n+1}\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty} z^{2n}\log(1-|\rho|^2)\,dz + \sum_{m=1}^{N^\#}\frac{(k_m^*)^{2n+1} - k_m^{2n+1}}{2n+1}\right],$$

for $n \geq 1$, or noting that $k_m = i\kappa_m$, $\kappa_m > 0$

$$C_{2n+1} = \frac{1}{\pi}\int_{-\infty}^{\infty}(-1)^n(2z)^{2n}\log(1-|\rho(z)|^2)\,dz + \frac{2}{2n+1}\sum_{m=1}^{N^\#}(2\kappa_m)^{2n+1},$$

which gives the conserved densities in terms of the scattering data $\{\rho(z), \kappa_m|_{m=1}^N\}$.

## 9.10 Outline of the IST for a general evolution system – including the nonlinear Schrödinger equation with vanishing boundary conditions

In the previous sections of this chapter we concentrated on the IST for the KdV equation with vanishing boundary values. In this section we will discuss the main steps in the IST associated with the following compatible linear $2\times 2$ system (Lax pair):

$$v_x = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix} v, \tag{9.59a}$$

and

$$v_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} v, \tag{9.59b}$$

where $v$ is a two-component vector, $v(x,t) = \left(v^{(1)}(x,t), v^{(2)}(x,t)\right)^{\mathrm{T}}$.[1] In Chapter 8 we described how the compatibility of the above general linear system yielded a class of nonlinear evolution equations. This class includes as special cases the nonlinear Schrödinger (NLS), modified Korteweg–de Vries (mKdV)

[1] Unless otherwise specified, superscripts $^{(\ell)}$ with $\ell = 1, 2$ denote the $\ell$th component of the corresponding two-component vector and $^{\mathrm{T}}$ denotes matrix transpose.

and sine–, sinh–Gordon equations. The equality of the mixed derivatives of $v$, i.e., $v_{xt} = v_{tx}$, is equivalent to the statement that $q$ and $r$ satisfy these evolution equations, if $k$, the scattering parameter or eigenvalue, is independent of $x$ and $t$. As before, we refer to the equation with the $x$ derivative, (9.59a), as the scattering problem: its solutions are termed eigenfunctions (with respect to the eigenvalue $k$) and the equation with the $t$ derivative, (9.59b), is called the time equation.

For example, the particular case of the NLS equation results when we take the time-dependent system to be

$$v_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} v = \begin{pmatrix} 2ik^2 + iqr & 2kq + iq_x \\ 2kr + ir_x & -2ik^2 - iqr \end{pmatrix} v, \tag{9.60}$$

which by compatibility of (9.59a)–(9.60) yields

$$iq_t = q_{xx} - 2rq^2 \tag{9.61a}$$

$$-ir_t = r_{xx} - 2qr^2. \tag{9.61b}$$

Then the reduction $r = \sigma q^*$ with $\sigma = \mp 1$, gives, as a special case, the NLS equation

$$iq_t = q_{xx} \pm 2|q|^2 q. \tag{9.62}$$

We say the NLS equation is focusing or defocusing corresponding to $\sigma = \mp 1$.

In what follows we will assume that the potentials, i.e., the solutions of the nonlinear evolution equations $q(x, t), r(x, t)$, decay sufficiently rapidly as $x \to \pm\infty$; at least $q, r \in L^1$. The reader can also see the references mentioned earlier in this chapter and in Chapter 8. Here we will closely follow the methods and notation employed by Ablowitz et al. (2004b).

We can rewrite the scattering problem (9.59a) as

$$v_x = (ik\mathbf{J} + \mathbf{Q})\, v$$

where

$$\mathbf{J} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}.$$

and $\mathbf{I}$ is the $2 \times 2$ identity matrix.

### 9.10.1 Direct scattering problem

#### Eigenfunctions and integral equations

When the potentials $q, r \to 0$ sufficiently rapidly as $x \to \pm\infty$ the eigenfunctions are asymptotic to the solutions of

$$v_x = \begin{pmatrix} -ik & 0 \\ 0 & ik \end{pmatrix} v$$

when $|x|$ is large. We single out the solutions of (9.59a) (eigenfunctions) defined by the following boundary conditions:

$$\phi(x;k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, \qquad \overline{\phi}(x;k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx} \qquad \text{as } x \to -\infty \tag{9.63a}$$

$$\psi(x;k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}, \qquad \overline{\psi}(x;k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx} \qquad \text{as } x \to +\infty. \tag{9.63b}$$

Note that here and in the following the bar does not denote complex conjugate, for which we will use the $^*$ notation. It is convenient to introduce eigenfunctions with constant boundary conditions by defining

$$M(x;k) = e^{ikx}\phi(x;k), \qquad \overline{M}(x;k) = e^{-ikx}\overline{\phi}(x;k)$$
$$N(x;k) = e^{-ikx}\psi(x;k), \qquad \overline{N}(x;k) = e^{ikx}\psi(x;k).$$

In terms of the above notation, the eigenfunctions, also called *Jost solutions*, $M(x;k)$ and $\overline{N}(x;k)$, are solutions of the differential equation

$$\partial_x \chi(x;k) = ik\,(\mathbf{J} + \mathbf{I})\,\chi(x;k) + (\mathbf{Q}\chi)\,(x;k), \tag{9.64}$$

while the eigenfunctions $N(x;k)$ and $\overline{M}(x;k)$ satisfy

$$\partial_x \overline{\chi}(x;k) = ik\,(\mathbf{J} - \mathbf{I})\,\overline{\chi}(x;k) + (\mathbf{Q}\overline{\chi})\,(x;k), \tag{9.65}$$

with the constant boundary conditions

$$M(x;k) \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \overline{M}(x;k) \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{as } x \to -\infty \tag{9.66a}$$

$$N(x;k) \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \overline{N}(x;k) \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{as } x \to +\infty. \tag{9.66b}$$

Solutions of the differential equations (9.64) and (9.65) can be represented by means of the following integral equations

$$\chi(x;k) = w + \int_{-\infty}^{+\infty} \mathbf{G}(x-x';k)\mathbf{Q}(x')\chi(x';k)\,dx'$$
$$\overline{\chi}(x;k) = \overline{w} + \int_{-\infty}^{+\infty} \overline{\mathbf{G}}(x-x';k)\mathbf{Q}(x')\overline{\chi}(x';k)\,dx'$$

where $w = (1,0)^{\mathrm{T}}$, $\overline{w} = (0,1)^{\mathrm{T}}$ and the (matrix) Green's functions $\mathbf{G}(x;k)$ and $\overline{\mathbf{G}}(x;k)$ satisfy the differential equations

$$[\mathbf{I}\partial_x - ik\,(\mathbf{J}+\mathbf{I})]\,\mathbf{G}(x;k) = \delta(x),$$
$$[\mathbf{I}\partial_x - ik\,(\mathbf{J}-\mathbf{I})]\,\overline{\mathbf{G}}(x;k) = \delta(x)\,,$$

where $\delta(x)$ is the Dirac delta (generalized) function. The Green's functions are not unique, and, as we will show below, the choice of the Green's functions and the choice of the inhomogeneous terms $w$, $\overline{w}$ together uniquely determine the eigenfunctions and their analytic properties in $k$.

By using the Fourier transform method, one can represent the Green's functions in the form

$$\mathbf{G}(x;k) = \frac{1}{2\pi i}\int_C \begin{pmatrix} 1/p & 0 \\ 0 & 1/(p-2k) \end{pmatrix} e^{ipx}dp$$
$$\overline{\mathbf{G}}(x;k) = \frac{1}{2\pi i}\int_{\overline{C}} \begin{pmatrix} 1/(p+2k) & 0 \\ 0 & 1/p \end{pmatrix} e^{ipx}\,dp$$

where $C$ and $\overline{C}$ will be chosen as appropriate contour deformations of the real $p$-axis. It is natural to consider $\mathbf{G}_\pm(x;k)$ and $\overline{\mathbf{G}}_\pm(x;k)$ defined by

$$\mathbf{G}_\pm(x;k) = \frac{1}{2\pi i}\int_{C_\pm} \begin{pmatrix} 1/p & 0 \\ 0 & 1/(p-2k) \end{pmatrix} e^{ipx}dp$$
$$\overline{\mathbf{G}}_\pm(x;k) = \frac{1}{2\pi i}\int_{\overline{C}_\pm} \begin{pmatrix} 1/(p+2k) & 0 \\ 0 & 1/p \end{pmatrix} e^{ipx}\,dp$$

where $C_\pm$ and $\overline{C}_\pm$ are the contours from $-\infty$ to $+\infty$ that pass below (+ functions) and above ($-$ functions) both singularities at $p = 0$ and $p = 2k$, or respectively, $p = 0$ and $p = -2k$. Contour integration then gives

$$\mathbf{G}_\pm(x;k) = \pm\theta(\pm x)\begin{pmatrix} 1 & 0 \\ 0 & e^{2ikx} \end{pmatrix},$$
$$\overline{\mathbf{G}}_\pm(x;k) = \mp\theta(\mp x)\begin{pmatrix} e^{-2ikx} & 0 \\ 0 & 1 \end{pmatrix},$$

where $\theta(x)$ is the Heaviside function ($\theta(x) = 1$ if $x > 0$ and $\theta(x) = 0$ if $x < 0$). The "+" functions are analytic and bounded in the upper half-plane of $k$ and the "−" functions are analytic in the lower half-plane. By taking into account the boundary conditions (9.66), we obtain the following integral equations for the eigenfunctions:

$$M(x;k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{+\infty} \mathbf{G}_+(x-x';k)\mathbf{Q}(x')M(x';k)dx' \tag{9.67a}$$

$$N(x;k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{-\infty}^{+\infty} \overline{\mathbf{G}}_+(x-x';k)\mathbf{Q}(x')N(x';k)dx' \tag{9.67b}$$

$$\overline{M}(x;k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{-\infty}^{+\infty} \overline{\mathbf{G}}_-(x-x';k)\mathbf{Q}(x')\overline{M}(x';k)dx' \tag{9.67c}$$

$$\overline{N}(x;k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{+\infty} \mathbf{G}_-(x-x';k)\mathbf{Q}(x')\overline{N}(x';k)dx'. \tag{9.67d}$$

Equations (9.67) are Volterra integral equations, whose solutions can be sought in the form of Neumann series iterates. In the following lemma we will show that if $q, r \in L^1(\mathbb{R})$, the Neumann series associated to the integral equations for $M$ and $N$ converge absolutely and uniformly (in $x$ and $k$) in the upper $k$-plane, while the Neumann series of the integral equations for $\overline{M}$ and $\overline{N}$ converge absolutely and uniformly (in $x$ and $k$) in the lower $k$-plane. This implies that the eigenfunctions $M(x;k)$ and $N(x;k)$ are analytic functions of $k$ for $\operatorname{Im} k > 0$ and continuous for $\operatorname{Im} k \geq 0$, while $\overline{M}(x;k)$ and $\overline{N}(x;k)$ are analytic functions of $k$ for $\operatorname{Im} k < 0$ and continuous for $\operatorname{Im} k \leq 0$.

**Lemma 9.2** *If $q, r \in L^1(\mathbb{R})$, then $M(x;k)$, $N(x;k)$ defined by* (9.67a) *and* (9.67b) *are analytic functions of $k$ for* $\operatorname{Im} k > 0$ *and continuous for* $\operatorname{Im} k \geq 0$, *while $\overline{M}(x;k)$ and $\overline{N}(x;k)$ defined by* (9.67c) *and* (9.67d) *are analytic functions of $k$ for* $\operatorname{Im} k < 0$ *and continuous for* $\operatorname{Im} k \leq 0$. *Moreover, the solutions of the corresponding integral equations are unique in the space of continuous functions.*

The proof can be found in Ablowitz et al. (2004b).

Note that simply requiring $q, r \in L^1(\mathbb{R})$ does not yield analyticity of the eigenfunctions on the real $k$-axis, for which more stringent conditions must be imposed. Having $q, r$ vanishing faster than any exponential as $|x| \to \infty$ implies that all four eigenfunctions are entire functions of $k \in \mathbb{C}$.

From the integral equations (9.67) we can compute the asymptotic expansion at large $k$ (in the proper half-plane) of the eigenfunction. Integration by parts yields

$$M(x;k) = \begin{pmatrix} 1 - \frac{1}{2ik}\int_{-\infty}^{x} q(x')r(x')dx' \\ -\frac{1}{2ik}r(x) \end{pmatrix} + O(1/k^2) \tag{9.68a}$$

$$\overline{N}(x;k) = \begin{pmatrix} 1 + \frac{1}{2ik}\int_{x}^{+\infty} q(x')r(x')dx' \\ -\frac{1}{2ik}r(x) \end{pmatrix} + O(1/k^2) \tag{9.68b}$$

$$N(x;k) = \begin{pmatrix} \frac{1}{2ik}q(x) \\ 1 - \frac{1}{2ik}\int_{x}^{+\infty} q(x')r(x')dx' \end{pmatrix} + O(1/k^2) \tag{9.68c}$$

$$\overline{M}(x;k) = \begin{pmatrix} \frac{1}{2ik}q(x) \\ 1 + \frac{1}{2ik}\int_{-\infty}^{x} q(x')r(x')dx' \end{pmatrix} + O(1/k^2). \tag{9.68d}$$

### 9.10.2 Scattering data

The two eigenfunctions with fixed boundary conditions as $x \to -\infty$ are linearly independent, and the same holds for the two eigenfunctions with fixed boundary conditions as $x \to +\infty$. Indeed, let $u(x,k) = \left(u^{(1)}(x,k), u^{(2)}(x,k)\right)^{\mathrm{T}}$ and $v(x,k) = \left(v^{(1)}(x,k), v^{(2)}(x,k)\right)^{\mathrm{T}}$ be any two solutions of (9.59a) and let us define the Wronskian of $u$ and $v$, $W(u,v)$, as

$$W(u,v) = u^{(1)}v^{(2)} - u^{(2)}v^{(1)}.$$

Using (9.59a) it can be verified that

$$\frac{d}{dx}W(u,v) = 0$$

where $W(u,v)$ is a constant. Therefore, from the prescribed asymptotic behavior for the eigenfunctions, it follows that

$$W\left(\phi,\overline{\phi}\right) = \lim_{x\to-\infty} W\left(\phi(x;k),\overline{\phi}(x;k)\right) = 1$$
$$W\left(\psi,\overline{\psi}\right) = \lim_{x\to+\infty} W\left(\psi(x;k),\overline{\psi}(x;k)\right) = -1,$$

which shows that the solutions $\phi$ and $\overline{\phi}$ are linearly independent, as are $\psi$ and $\overline{\psi}$. As a consequence, we can express $\phi(x;k)$ and $\overline{\phi}(x;k)$ as linear combinations of $\psi(x;k)$ and $\overline{\psi}(x;k)$, or vice versa, with the coefficients of these linear combinations depending on $k$ only. Hence, the relations

$$\phi(x;k) = b(k)\psi(x;k) + a(k)\overline{\psi}(x;k) \tag{9.69a}$$

$$\overline{\phi}(x;k) = \overline{a}(k)\psi(x;k) + \overline{b}(k)\overline{\psi}(x;k) \tag{9.69b}$$

hold for any $k \in \mathbb{C}$ such that the four eigenfunctions exist. In particular, (9.69) hold for $\operatorname{Im} k = 0$ and define the scattering coefficients $a(k)$, $\overline{a}(k)$, $b(k)$, $\overline{b}(k)$ for

$k \in \mathbb{R}$. Comparing the asymptotics of $W(\phi, \overline{\phi})$ as $x \to \pm\infty$ with (9.69) shows that the scattering data satisfy the following equation:

$$a(k)\overline{a}(k) - b(k)\overline{b}(k) = 1 \qquad \forall k \in \mathbb{R}.$$

The scattering coefficients can, in turn, be represented as Wronskians of the eigenfunctions. Indeed, from (9.69) it follows that

$$a(k) = W(\phi, \psi), \qquad \overline{a}(k) = W(\overline{\psi}, \overline{\phi}) \tag{9.70a}$$

$$b(k) = W(\overline{\psi}, \phi), \qquad \overline{b}(k) = W\left(\overline{\phi}, \psi\right). \tag{9.70b}$$

Therefore, if $q$, $r \in L^1(\mathbb{R})$, Lemma 9.2 and equations (9.70) imply that $a(k)$ admit analytic continuation in the upper half $k$-plane, while $\overline{a}(k)$ can be analytically continued in the lower half $k$-plane. In general, $b(k)$ and $\overline{b}(k)$ cannot be extended off the real $k$-axis. It is also possible to obtain integral representations for the scattering coefficients in terms of the eigenfunctions, but we will not do so here.

Equation (9.69) can be written as

$$\mu(x, k) = \overline{N}(x; k) + \rho(k)e^{2ikx}N(x; k) \tag{9.71a}$$

$$\overline{\mu}(x, k) = N(x; k) + \overline{\rho}(k)e^{-2ikx}\overline{N}(x; k), \tag{9.71b}$$

where we introduced

$$\mu(x, k) = M(x; k)/a(k), \qquad \overline{\mu}(x, k) = \overline{M}(x; k)/\overline{a}(k) \tag{9.72}$$

and the *reflection coefficients*

$$\rho(k) = b(k)/a(k), \qquad \overline{\rho}(k) = \overline{b}(k)/\overline{a}(k). \tag{9.73}$$

Note that from the representation of the scattering data as Wronskians of the eigenfunctions and from the asymptotic expansions (9.68), it follows that

$$a(k) = 1 + O(1/k), \qquad \overline{a}(k) = 1 + O(1/k)$$

in the proper half-plane, while $b(k), \overline{b}(k)$ are $O(1/k)$ as $|k| \to \infty$ on the real axis. We further assume that $a(k)$, $\overline{a}(k)$ are continuous for real $k$. Since $a(k)$ is analytic for $\operatorname{Im} k > 0$, continuous for $\operatorname{Im} k = 0$ and $a(k) \to 1$ for $|k| \to \infty$, there cannot be a cluster point of zeros for $\operatorname{Im} k \geq 0$. Hence the number of zeros is finite. Similarly for $\overline{a}(k)$.

## Proper eigenvalues and norming constants

A *proper* (or discrete) *eigenvalue* of the scattering problem (9.59a) is a (complex) value of $k$ corresponding to a bounded solution $v(x, k)$ such that $v(x, k) \to 0$ as $x \to \pm\infty$; usually one requires $v \in L^2(\mathbb{R})$ with respect to $x$. We also call such solutions *bound states*. When $q = r^*$ in (9.59a), the scattering problem is self-adjoint and there are no eigenvalues/bound states. So the only possibility for eigenvalues is when $q = -r^*$. We will see that these eigenvalues correspond to solitons that decay rapidly at infinity; sometimes called "bright" solitons. The so-called "dark" solitons, which tend to constant states at infinity, are contained in an extended theory (cf. Zakharov and Shabat, 1973; Prinari et al., 2006).

Suppose that $k_j = \xi_j + i\eta_j$, with $\eta_j > 0$, is such that $a(k_j) = 0$. Then from (9.70a) it follows that $W(\phi(x;k_j), \psi(x;k_j)) = 0$ and therefore $\phi_j(x) := \phi(x;k_j)$ and $\psi_j(x) = \psi(x;k_j)$ are linearly dependent; that is, there exists a complex constant $c_j$ such that

$$\phi_j(x) = c_j\psi_j(x).$$

Hence, by (9.63a) and (9.63b) it follows that

$$\begin{aligned} \phi_j(x) &\sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\eta_j x - i\xi_j x} && \text{as } x \to -\infty \\ \phi_j(x) = c_j\psi_j(x) &\sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\eta_j x + i\xi_j x} && \text{as } x \to +\infty. \end{aligned}$$

Since we have strong decay for large $|x|$, it follows that $k_j$ is a proper eigenvalue. On the other hand, if $a(k) \neq 0$ for $\operatorname{Im} k > 0$, then solutions of the scattering problem blow up in one or both directions. We conclude that the proper eigenvalues in the region $\operatorname{Im} k > 0$ are precisely the zeros of the scattering coefficient $a(k)$. Similarly, the eigenvalues in the region $\operatorname{Im} k < 0$ are given by the zeros of $\overline{a}(k)$, and these zeros $\overline{k}_j = \overline{\xi}_j + i\overline{\eta}_j$ are such that $\overline{\phi}(x;\overline{k}_j) = \overline{c}_j\overline{\psi}(x;\overline{k}_j)$ for some complex constant $\overline{c}_j$. The coefficients $\left\{c_j\right\}_{j=1}^{J}$ and $\left\{\overline{c}_j\right\}_{j=1}^{\overline{J}}$ are often called *norming constants*. In terms of the eigenfunctions, the norming constants are defined by

$$M_j(x) = c_j\, e^{2ik_jx} N_j(x)\,, \qquad \overline{M}_j(x) = \overline{c}_j\, e^{-2i\overline{k}_jx}\overline{N}_j(x)\,, \tag{9.74}$$

where $M_j(x) = M(x;k_j)$, $\overline{M}_j(x) = \overline{M}(x;\overline{k}_j)$ and similarly for $N_j(x)$ and $\overline{N}_j(x)$. As we will see in the following, discrete eigenvalues and associated norming constants are part of the scattering data (i.e., the data necessary to uniquely solve the inverse problem and reconstruct $q(x,t)$ and $r(x,t)$).

Note that if the potentials $q, r$ are rapidly decaying, such that (9.69) can be extended off the real axis, then

$$\begin{aligned} c_j &= b(k_j) && \text{for } j = 1, \ldots, J \\ \overline{c}_j &= \overline{b}(\overline{k}_j) && \text{for } j = 1, \ldots, \overline{J}. \end{aligned}$$

**Symmetry reductions**

The NLS equation is a special case of the system (9.61) under the symmetry reduction $r = \pm q^*$. This symmetry in the potentials induces a symmetry between the eigenfunctions analytic in the upper $k$-plane and the ones analytic in the lower $k$-plane. In turn, this symmetry of the eigenfunctions induces symmetries in the scattering data.

Indeed, if $v(x,k) = \left(v^{(1)}(x,k), v^{(2)}(x,k)\right)^{\mathrm{T}}$ satisfies (9.59a) and $r = \mp q^*$, then $\hat{v}(x,k) = \left(v^{(2)}(x,k^*), \mp v^{(1)}(x,k^*)\right)^{\dagger}$ (where $^{\dagger}$ denotes conjugate transpose) also satisfies the same (9.59a). Taking into account the boundary conditions (9.63), we get

$$\overline{\psi}(x;k) = \begin{pmatrix} \psi^{(2)}(x;k^*) \\ \mp\psi^{(1)}(x;k^*) \end{pmatrix}^*, \qquad \overline{\phi}(x;k) = \begin{pmatrix} \mp\phi^{(2)}(x;k^*) \\ \phi^{(1)}(x;k^*) \end{pmatrix}^*,$$

$$\overline{N}(x;k) = \begin{pmatrix} N^{(2)}(x;k^*) \\ \mp N^{(1)}(x;k^*) \end{pmatrix}^*, \qquad \overline{M}(x;k) = \begin{pmatrix} \mp M^{(2)}(x;k^*) \\ M^{(1)}(x;k^*) \end{pmatrix}^*.$$

Then, from the Wronskian representations for the scattering data (9.70) there follows

$$\overline{a}(k) = a^*(k^*), \qquad \overline{b}(k) = \mp b^*(k^*),$$

which implies that $k_j$ is a zero of $a(k)$ in the upper half $k$-plane if and only if $k_j^*$ is a zero of $\overline{a}(k)$ in the lower $k$-plane and vice versa. This means that the zeros of $a(k)$ and $\overline{a}(k)$ come in pairs and the number of zeros of each is the same, i.e., $\overline{J} = J$. Hence when there are eigenvalues associated with the potential $q$, with $q = -r^*$, we have that

$$\overline{k}_j = k_j^*, \qquad \overline{c}_j = \mp c_j^* \qquad j = 1, \dots, J.$$

On the other hand, if we have the symmetry $r = \mp q$, with $q$ real, then $\hat{v}(x,k) = \left(v^{(2)}(x,-k), \mp v^{(1)}(x,-k)\right)^{\mathrm{T}}$ also satisfies the same equation (9.59a). Taking into account the boundary conditions (9.63), we get

$$\overline{\psi}(x;k) = \begin{pmatrix} \psi^{(2)}(x;-k) \\ \mp\psi^{(1)}(x;-k) \end{pmatrix}, \qquad \overline{\phi}(x;k) = \begin{pmatrix} \mp\phi^{(2)}(x;-k) \\ \phi^{(1)}(x;-k) \end{pmatrix},$$

$$\overline{N}(x;k) = \begin{pmatrix} N^{(2)}(x;-k) \\ \mp N^{(1)}(x;-k) \end{pmatrix}, \qquad \overline{M}(x;k) = \begin{pmatrix} \mp M^{(2)}(x;-k) \\ M^{(1)}(x;-k) \end{pmatrix}.$$

Then, from the Wronskian representations for the scattering data (9.70), there follows

$$\overline{a}(k) = a(-k), \qquad \overline{b}(k) = \mp b(-k), \tag{9.75}$$

which implies that $k_j$ is a zero of $a(k)$ in the upper half $k$-plane if and only if $-k_j$ is a zero of $\overline{a}(k)$ in the lower $k$-plane, and vice versa. As a consequence,

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when $r = -q$, with $q$ real, the zeros of $a(k)$ and $\overline{a}(k)$ are paired, their number is the same: $\overline{J} = J$ and

$$\overline{k}_j = -k_j, \qquad \overline{c}_j = -c_j \qquad j = 1, \ldots, J. \tag{9.76}$$

Thus if $r = -q$, with $q$ real, both of the above symmetry conditions must hold and when $k_j$ is an eigenvalue so is $-k_j^*$; i.e., either the eigenvalues come in pairs, $\{k_j, -k_j^*\}$, or they lie on the imaginary axis.

### 9.10.3 Inverse scattering problem

The inverse problem consists of constructing a map from the scattering data, that is:

(i) the reflection coefficients $\rho(k)$ and $\overline{\rho}(k)$ for $k \in \mathbb{R}$, defined by (9.73);

(ii) the discrete eigenvalues $\left\{k_j\right\}_{j=1}^{J}$ (zeros of the scattering coefficient $a(k)$ in the upper half plane of $k$) and $\left\{\overline{k}_j\right\}_{j=1}^{\overline{J}}$ (zeros of the scattering coefficient $\overline{a}(k)$ in the lower half plane of $k$);

(iii) the norming constants $\left\{c_j\right\}_{j=1}^{J}$ and $\left\{\overline{c}_j\right\}_{j=1}^{\overline{J}}$, cf. (9.74); back to the potentials $q$ and $r$.

First, we use these data to reconstruct the eigenfunctions (for instance, $N(x;k)$ and $\overline{N}(x;k)$), and then we recover the potentials from the large $k$ asymptotics of the eigenfunctions, cf. equations (9.68). Note that the inverse problem is solved at fixed $t$, and therefore the explicit time dependence is omitted. In fact, in the inverse problem both $x$ and $t$ are treated as parameters.

#### Riemann–Hilbert approach

In the previous section, we showed that the eigenfunctions $N(x;k)$ and $\overline{N}(x;k)$ exist and are analytic in the regions $\operatorname{Im} k > 0$ and $\operatorname{Im} k < 0$, respectively, if $q, r \in L^1(\mathbb{R})$. Similarly, under the same conditions on the potentials, the functions $\mu(x,k)$ and $\overline{\mu}(x,k)$ introduced in (9.72) are meromorphic in the regions $\operatorname{Im} k > 0$ and $\operatorname{Im} k < 0$, respectively, with poles at the zeros of $a(k)$ and $\overline{a}(k)$. Therefore, in the inverse problem we assume these analyticity properties for the unknown eigenfunctions ($N(x;k)$ and $\overline{N}(x;k)$) or modified eigenfunctions ($\mu(x,k)$ and $\overline{\mu}(x,k)$). With these assumptions, (9.71) can be considered as the "jump" conditions of a Riemann–Hilbert problem. To recover the sectionally meromorphic functions from the scattering data, we will convert the Riemann–Hilbert problem into a system of linear integral equations.

Suppose that $a(k)$ and $\overline{a}(k)$ have a finite number of simple zeros in the regions $\operatorname{Im} k > 0$ and $\operatorname{Im} k < 0$, respectively, which we denote as $\left\{k_j,\ \operatorname{Im} k_j > 0\right\}_{j=1}^{J}$ and

$\left\{\overline{k}_j,\ \mathrm{Im}\,\overline{k}_j < 0\right\}_{j=1}^{\overline{J}}$. We will also assume that $a(\xi) \neq 0$ and $\overline{a}(\xi) \neq 0$ for $\xi \in \mathbb{R}$ (i.e., $a$ and $\overline{a}$ have no zeros on the real axis) and $a(k)$ is continuous for $\mathrm{Im}\,k \geq 0$.

Let $f(\zeta)$, $\zeta \in \mathbb{R}$, be an integrable function and consider the projection operators

$$P^{\pm}[f](k) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(\zeta)}{\zeta - (k \pm i0)}\,d\zeta.$$

If $f_+$ (resp. $f_-$) is analytic in the upper (resp. lower) $k$-plane and $f_\pm(k) \to 0$ as $|k| \to \infty$ for $\mathrm{Im}\,k \gtrless 0$, then

$$P^{\pm}\left[f_\pm\right] = \pm f_\pm, \qquad P^{\pm}\left[f_\mp\right] = 0$$

[$P^{\pm}$ are referred to as projection operators into the upper/lower half $k$-planes]. Let us now apply the projector $P^-$ to both sides of (9.71a) and $P^+$ to both sides of (9.71b). Note that $\mu(x,k) = M(x;k)/a(k)$ in (9.71a) does not decay for large $k$; rather it tends to $(1,0)^{\mathrm{T}}$. In addition it has poles at the zeros of $a(k)$. We subtract these contributions from both sides of (9.71a) and then take the projector; similar statements apply to $\overline{\mu}(x,k)$ in (9.71b). So, taking into account the analyticity properties of $N$, $\overline{N}$, $\mu$, $\overline{\mu}$ and the asymptotics (9.68) and using (9.74), we obtain

$$\overline{N}(x;k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{J} \frac{C_j e^{2ik_j x}}{k - k_j} N_j(x) + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho(\zeta) e^{2i\zeta x}}{\zeta - (k - i0)} N(x;\zeta)\,d\zeta \tag{9.77a}$$

$$N(x;k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{j=1}^{\overline{J}} \frac{\overline{C}_j e^{-2i\overline{k}_j x}}{k - \overline{k}_j} \overline{N}_j(x) - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\overline{\rho}(\zeta) e^{-2i\zeta x}}{\zeta - (k + i0)} \overline{N}(x;\zeta)\,d\zeta, \tag{9.77b}$$

where $N_j(x) = N(x;k_j)$, $\overline{N}_j(x) = \overline{N}(x;\overline{k}_j)$ and we introduced

$$C_j = \frac{c_j}{a'(k_j)} \qquad \text{for } j = 1,\ldots,J$$

$$\overline{C}_j = \frac{\overline{c}_j}{\overline{a}'(\overline{k}_j)} \qquad \text{for } j = 1,\ldots,\overline{J}$$

with $'$ denoting the derivative of $a(k)$ and $\overline{a}(k)$ with respect to $k$. We see that the equations defining the inverse problem for $N(x;k)$ and $\overline{N}(x;k)$ depend on the extra terms $\left\{N_j(x)\right\}_{j=1}^{J}$ and $\left\{\overline{N}_j(x)\right\}_{j=1}^{\overline{J}}$. In order to close the system, we evaluate (9.77a) at $k = k_j$ for $j = 1,\ldots,J$ and (9.77b) at $k = \overline{k}_j$ for $j = 1,\ldots,\overline{J}$, thus obtaining

$$\overline{N}_\ell(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{J} \frac{C_j e^{2ik_j x}}{\overline{k}_\ell - k_j} N_j(x) + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\rho(\zeta) e^{2i\zeta x}}{\zeta - \overline{k}_\ell} N(x;\zeta)\, d\zeta \tag{9.78a}$$

$$N_j(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{m=1}^{\overline{J}} \frac{\overline{C}_m e^{-2i\overline{k}_m x}}{k_j - \overline{k}_m} \overline{N}_j(x) - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\overline{\rho}(\zeta) e^{-2i\zeta x}}{\zeta - k_j} \overline{N}(x;\zeta)\, d\zeta\,. \tag{9.78b}$$

Equations (9.77) and (9.78) together constitute a linear algebraic – integral system of equations that, in principle, solve the inverse problem for the eigenfunctions $N(x;k)$ and $\overline{N}(x;k)$.

By comparing the asymptotic expansions at large $k$ of the right-hand sides of (9.77) with the expansions (9.68), we obtain

$$r(x) = -2i \sum_{j=1}^{J} e^{2ik_j x} C_j N_j^{(2)}(x) + \frac{1}{\pi} \int_{-\infty}^{+\infty} \rho(\zeta) e^{2i\zeta x} N^{(2)}(x;\zeta)\, d\zeta \tag{9.79a}$$

$$q(x) = 2i \sum_{j=1}^{\overline{J}} e^{-2i\overline{k}_j x} \overline{C}_j \overline{N}_j^{(1)}(x) + \frac{1}{\pi} \int_{-\infty}^{+\infty} \overline{\rho}(\zeta) e^{-2i\zeta x} \overline{N}^{(1)}(x;\zeta) d\zeta \tag{9.79b}$$

which reconstruct the potentials in terms of the scattering data and thus complete the formulation of the inverse problem (as before, the superscript $^{(\ell)}$ denotes the $\ell$-component of the corresponding vector).

We mention that the issue of establishing existence and uniqueness of solutions for the equations of the inverse problem is usually carried out by converting the inverse problem into a set of Gelfand–Levitan–Marchenko integral equations – which is given next.

### Gel'fand–Levitan–Marchenko integral equations

As an alternative inverse procedure we provide a reconstruction for the potentials by developing the Gel'fand–Levitan–Marchenko (GLM) integral equations, instead of using the projection operators (cf. Zakharov and Shabat, 1972; Ablowitz and Segur, 1981). To do this we represent the eigenfunctions in terms of triangular kernels

$$N(x;k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_x^{+\infty} K(x,s) e^{-ik(x-s)}\, ds, \quad s > x, \quad \operatorname{Im} k > 0 \tag{9.80}$$

$$\overline{N}(x;k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_x^{+\infty} \overline{K}(x,s) e^{ik(x-s)}\, ds, \quad s > x, \quad \operatorname{Im} k < 0. \tag{9.81}$$

Applying the operator $\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{-ik(x-y)}$ for $y > x$ to (9.77a), we find

$$\overline{K}(x, y) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F(x + y) + \int_x^{+\infty} K(x, s)F(s + y)\, ds = 0 \tag{9.82}$$

where

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \rho(\xi)e^{i\xi x}\, d\xi - i\sum_{j=1}^{J} C_j e^{ik_j x}.$$

Analogously, operating on (9.77b) with $\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{ik(x-y)}$ for $y > x$ gives

$$K(x, y) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \overline{F}(x + y) + \int_x^{+\infty} \overline{K}(x, s)\overline{F}(s + y)\, ds = 0 \tag{9.83}$$

where

$$\overline{F}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{\rho}(\xi)e^{-i\xi x}\, d\xi + i\sum_{j=1}^{\overline{J}} \overline{C}_j e^{-i\overline{k}_j x}.$$

Equations (9.82) and (9.83) constitute the Gel'fand–Levitan–Marchenko equations.

Inserting the representations (9.80)–(9.81) for the eigenfunctions into (9.79) we obtain the reconstruction of the potentials in terms of the kernels of the GLM equations, i.e.,

$$q(x) = -2K^{(1)}(x, x), \qquad r(x) = -2\overline{K}^{(2)}(x, x) \tag{9.84}$$

where, as usual, $K^{(j)}$ and $\overline{K}^{(j)}$ for $j = 1, 2$ denote the $j$th component of the vectors $K$ and $\overline{K}$ respectively.

If the symmetry $r = \mp q^*$ holds, then, taking into account (9.75)–(9.76), one can verify that

$$\overline{F}(x) = \mp F^*(x)$$

and consequently

$$\overline{K}(x, y) = \begin{pmatrix} K^{(2)}(x, y) \\ \mp K^{(1)}(x, y) \end{pmatrix}^*.$$

In this case (9.82)–(9.83) solving the inverse problem reduce to

$$K^{(1)}(x, y) = \pm F^*(x + y) \mp \int_x^{+\infty} ds \int_x^{+\infty} ds'\, K^{(1)}(x, s')F(s + s')F^*(y + s)$$

and the potentials are reconstructed by means of the first of (9.84). We also note that when $r = \mp q$ with $q$ real, then $F(x)$ and $K^{(1)}(x, y)$ are real.

### 9.10.4 Time evolution

We will now show how to determine the time dependence of the scattering data. Then, by the inverse transform we establish the solution $q(x,t)$ and $r(x,t)$.

The operator equation (9.59b) determines the evolution of the eigenfunctions, which can be written as

$$\partial_t v = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} v \tag{9.85}$$

where $B, C \to 0$ as $x \to \pm\infty$ (since $q, r \in L^1(\mathbb{R})$). Then the time-dependent eigenfunctions must asymptotically satisfy the differential equation

$$\partial_t v = \begin{pmatrix} A_0 & 0 \\ 0 & -A_0 \end{pmatrix} v \qquad \text{as} \quad x \to \pm\infty \tag{9.86}$$

with

$$A_0 = \lim_{|x|\to\infty} A(x,k).$$

The system (9.86) has solutions that are linear combinations of

$$v^+ = \begin{pmatrix} e^{A_0 t} \\ 0 \end{pmatrix}, \qquad v^- = \begin{pmatrix} 0 \\ e^{-A_0 t} \end{pmatrix}.$$

However, such solutions are not compatible with the fixed boundary conditions of the eigenfunctions, i.e., equations (9.63a)–(9.63b). Therefore, we define time-dependent functions

$$\Phi(x,t;k) = e^{A_0 t}\phi(x,t;k), \qquad \overline{\Phi}(x,t;k) = e^{-A_0 t}\overline{\phi}(x,t;k)$$
$$\Psi(x,t;k) = e^{-A_0 t}\psi(x,t;k), \qquad \overline{\Psi}(x,t;k) = e^{A_0 t}\overline{\psi}(x,t;k)$$

to be solutions of the differential equation (9.85). Then the evolution for $\phi$ and $\overline{\phi}$ becomes

$$\partial_t \phi = \begin{pmatrix} A - A_0 & B \\ C & -A - A_0 \end{pmatrix} \phi, \quad \partial_t \overline{\phi} = \begin{pmatrix} A + A_0 & B \\ C & -A + A_0 \end{pmatrix} \overline{\phi}, \tag{9.87}$$

so that, taking into account (9.69) and evaluating (9.87) as $x \to +\infty$, we obtain

$$\partial_t a = 0, \qquad \partial_t \overline{a} = 0$$
$$\partial_t b = -2A_0 b, \qquad \partial_t \overline{b} = 2A_0 \overline{b}$$

or, explicitly,

$$a(k,t) = a(k,0), \qquad \overline{a}(k,t) = \overline{a}(k,0) \tag{9.88a}$$
$$b(k,t) = b(k,0)e^{-2A_0(k)t}, \qquad \overline{b}(k,t) = \overline{b}(k,0)e^{2A_0(k)t}. \tag{9.88b}$$

From (9.88a) it follows that the discrete eigenvalues (i.e., the zeros of $a$ and $\overline{a}$) are constant as the solution evolves. Not only the number of eigenvalues, but also their locations are fixed. Thus, the eigenvalues are time-independent discrete states of the evolution. In fact, this time invariance is the underlying mechanism of the elastic soliton interaction for the integrable soliton equations. On the other hand, the evolution of the reflection coefficients (9.73) is given by

$$\rho(k,t) = \rho(k,0)e^{-2A_0(k)t}, \qquad \overline{\rho}(k,t) = \overline{\rho}(k,0)e^{2A_0(k)t}$$

and this also gives the evolution of the norming constants:

$$C_j(t) = C_j(0)e^{-2A_0(k_j)t}, \qquad \overline{C}_j(t) = \overline{C}_j(0)e^{2A_0(\overline{k}_j)t}. \tag{9.89}$$

The expressions for the evolution of the scattering data allow one to solve the initial value problem for *all* of the solutions in the class associated with the scattering problem (9.59a). Namely we can solve (see the previous chapter for details) all the equations in the class given by the general evolution operator

$$\begin{pmatrix} r \\ -q \end{pmatrix}_t + 2A_0(L)\begin{pmatrix} r \\ q \end{pmatrix} = 0, \tag{9.90}$$

where $A_0(k) = \lim_{|x|\to\infty} A(x,t,k)$ (here $A_0(k)$ may be the ratio of two entire functions), and $L$ is the integro-differential operator given by

$$L = \frac{1}{2i}\begin{pmatrix} \partial_x - 2r(I_-q) & 2r(I_-r) \\ -2q(I_-q) & -\partial_x + 2q(I_-r) \end{pmatrix},$$

where $\partial_x \equiv \partial/\partial x$ and

$$(I_-f)(x) \equiv \int_{-\infty}^{x} f(y)\,dy. \tag{9.91}$$

Note that $L$ operates on $(r,q)$, and $I_-$ operates both on the functions immediately to its right and also on the functions to which $L$ is applied.

Special cases are listed below:

- When $A_0 = 2ik^2, r = \mp q^*$ we obtain the solution of the NLS equation

  $$iq_t = q_{xx} \pm 2|q|^2 q$$

  i.e., (9.62).
- If $A_0 = -4ik^3$ when $r = \mp q$, $q$ real we find the solution of the modified KdV (mKdV) equation,

  $$q_t \pm 6q^2 q_x + q_{xxx} = 0. \tag{9.92}$$

- If on the other hand $r = \mp q^*$ the solution of the complex mKdV equation results:

$$q_t \pm 6|q|^2 q_x + q_{xxx} = 0.$$

- Finally, if $A_0 = \frac{i}{4k}$ and
  (i) $q = -r = -\frac{1}{2}u_x$, we obtain the solution of the sine – Gordon equation

$$u_{xt} = \sin u; \tag{9.93}$$

  (ii) or, if $q = r = \frac{1}{2}u_x$, then we can find the solution of the sinh–Gordon equation

$$u_{xt} = \sinh u.$$

In summary the solution procedure is as follows:

(i) The scattering data are calculated from the initial data $q(x) = q(x, 0)$ and $r(x) = r(x, 0)$ according to the direct scattering method described in Section 9.10.1.

(ii) The scattering data at a later time $t > 0$ are determined from (9.88)–(9.89).

(iii) The solutions $q(x, t)$ and $r(x, t)$ are recovered from the scattering data using inverse scattering using the reconstruction formulas via the Riemann–Hilbert formulation (9.77)–(9.79) or Gel-fand–Levitan–Marchenko equations (9.82)–(9.83).

If we further require symmetry, $r = \pm q^*$, as discussed in the above section on symmetry reductions, we obtain the solution of the reduced evolution equation.

### 9.10.5 Soliton solutions

In the case where the scattering data comprise proper eigenvalues but $\rho(k) = \overline{\rho}(k) \equiv 0$ for all $k \in \mathbb{R}$ (corresponding to the so-called reflectionless solutions), the algebraic-integral system (9.77) and (9.78) reduces to a linear algebraic system, namely

$$\overline{N}_l(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{J} \frac{C_j e^{2ik_j x} N_j(x)}{\overline{k}_l - k_j}$$

$$N_j(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{m=1}^{\overline{J}} \frac{\overline{C}_m e^{-2i\overline{k}_m x} \overline{N}_m(x)}{k_j - \overline{k}_m},$$

that can be solved in closed form. The one-soliton solution, in particular, is obtained for $J = \overline{J} = 1$ (i.e., one single discrete eigenvalue $k_1 = \xi + i\eta$ and corresponding norming constant $C_1$). In the relevant physical case, when the symmetry $r = -q^*$ holds, using (9.75) and (9.76) in the above system, we get

$$N_1^{(1)}(x) = -\frac{C_1}{k_1 - k_1^*} e^{-2ik_1^* x} \left[ 1 - \frac{|C_1|^2 \, e^{2i\left(k_1 - k_1^*\right)x}}{\left(k_1 - k_1^*\right)^2} \right]^{-1},$$

$$N_1^{(2)}(x) = \left[ 1 - \frac{|C_1|^2 \, e^{2i\left(k_1 - k_1^*\right)x}}{\left(k_1 - k_1^*\right)^2} \right]^{-1},$$

where, as before, $N_1(x) = \left(N_1^{(1)}(x), N_1^{(2)}(x)\right)^{\mathrm{T}}$. Then, if we set

$$k_1 = \xi + i\eta, \qquad e^{2\delta} = \frac{|C_1|}{2\eta},$$

it follows from (9.79) that

$$q(x) = -2i\eta \frac{C_1^*}{|C_1|} e^{-2i\xi x} \operatorname{sech}(2\eta x - 2\delta).$$

Taking into account the time dependence of $C_1$ as given by (9.89), we find

$$q(x) = 2\eta e^{-2i\xi x + 2i \operatorname{Im} A_0(k_1)t - i\psi_0} \operatorname{sech}\left[2\left(\eta(x - x_0) + \operatorname{Re} A_0(k_1)t\right)\right],$$

where $C_1(0) = 2\eta e^{2\eta x_0 + i(\psi_0 + \pi/2)}$.

Thus when we take

(a) $A_0(k_1) = 2ik_1^2 = -4\xi\eta + 2i(\xi^2 - \eta^2)$ where $k_1 = \xi + i\eta, r = -q^*$ we get the well-known bright soliton solution of the NLS equation

$$q(x,t) = 2\eta e^{-2i\xi x + 4i\left(\xi^2 - \eta^2\right)t - i\psi_0} \operatorname{sech}\left[2\eta\left(x - 4\xi t - x_0\right)\right].$$

(b) $A_0(k_1) = -4ik_1^3 = -4\eta^3$ where $k_1 = i\eta, r = -q$, real; recall that due to symmetry all eigenvalues must come in pairs $\left\{k_j, -k_j^*\right\}$ hence with only one eigenvalue $\operatorname{Re} k_1 = \xi = 0$. Then we get the bright soliton solution of the mKdV equation (9.92):

$$q(x,t) = 2\eta \operatorname{sech}\left[2\eta\left(x - 4\eta^2 t - x_0\right)\right].$$

(c) $A_0(k_1) = \frac{i}{4k} = \frac{1}{4\eta}$ where $k_1 = i\eta, r = -q = \frac{u_x}{2}$, real; again due to symmetry $\operatorname{Re} k_1 = \xi = 0$. Then we get the soliton-kink solution of the sine–Gordon equation (9.93):

$$q(x,t) = -\frac{u_x}{2} = -2\eta \operatorname{sech}\left[2\eta\left(x + \frac{1}{4\eta}t - x_0\right)\right],$$

or in terms of $u$

$$u(x,t) = 4\tan^{-1}\exp\left[2\eta\left(x + \frac{1}{4\eta}t - x_0\right)\right].$$

## Exercises

9.1 Use (9.15) and (9.16) to find the Neumann series for $M(x;k)$ and $\overline{N}(x;k)$. Show these series converge uniformly when $u(x)$ decays appropriately (e.g., $u \in L_2^1$).

9.2 Using the results in Exercise 9.1 establish that $a(k)$ is analytic for $\operatorname{Im} k > 0$ and continuous for $\operatorname{Im} k = 0$.

9.3 Find a one-soliton solution to an equation associated with the time-independent Schrödinger equation (9.4) when

$$b(k,t) = b(k,0)e^{-32ik^5 t}, \qquad a(k,t) = a(k,0),$$
$$c_j(t) = c_j(0)e^{32k_j^5 t}, \qquad k_j = i\kappa_j.$$

Use the concepts of Chapter 8 to determine the nonlinear evolution equation this soliton solution solves. (Hint: use (8.51)–(8.52).)

9.4 Use (8.51)–(8.52) to deduce the time-dependence of the scattering data for a nonlinear evolution equation associated with (9.4) whose linear dispersion relation is $\omega(k)$.

9.5 From (9.53)–(9.54) find the next two conserved quantities $C_7$, $C_9$ associated with the KdV equation. Determine the conservation laws.

9.6 Use (9.67) to establish that $M(x;k)$, $N(x;k)$ are analytic for $\operatorname{Im} k > 0$, $\overline{M}(x;k)$, $\overline{N}(x;k)$ for $\operatorname{Im} k < 0$ where $q, r \in L^1(\mathbb{R})$.

9.7 Suppose $A_0(k) = 8ik^4$, associated with (9.85) and (9.90) with $r = -q^*$. Find the time dependence of the scattering data and the one-soliton solution. Use the results of Chapter 8 to find the nonlinear evolution equation that this soliton solution satisfies.

9.8 Consider the equation

$$(u_t + 6uu_x + u_{xxx})_x - 3u_{yy} = 0.$$

(a) Show that the equation has the rational solution

$$u(x,t) = 4\frac{p^2y^2 - X^2 + 1/p^2}{p^2y^2 + X^2 + 1/p^2}$$

where $X = x + 1/p - 3p^2 t$ and $p$ is a real constant.

(b) Make the transformation $y \to iy$ to formally derive the following KP equation

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0$$

and thus show that the above solution is a singular solution of the KP.

9.9 Suppose $K(x,z;t)$ satisfies the Marchenko equation

$$K(x,z;t) + F(x,z;t) + \int_X^\infty K(x,y;t)F(y,z;t)\,dy = 0,$$

where $F$ is a solution of the pair of equations

$$F_{xx} - F_{zz} = (x-z)F$$
$$3tF_t - F + F_{xxx} + F_{zzz} = xF_x + zF_z.$$

(a) Show that

$$u_t + \frac{u}{2t} + 6uu_x + u_{xxx} = 0,$$

where $u(x,t) = \frac{2}{(12t)^{2/3}}\frac{\partial}{\partial X}K(X,X;t)$ with $X = \frac{x}{(12t)^{1/3}}$.

(b) Show that a solution for $F$ is

$$F(x,z;t) = \int_{-\infty}^\infty f(yt^{1/3})\,\mathrm{Ai}(x+y)\,\mathrm{Ai}(y+z)\,dy$$

where $f$ is an arbitrary function and $\mathrm{Ai}(x)$ is the Airy function.

Hint: See Ablowitz and Segur (1981) on how to operate on $K$.

9.10 Show that the KP equation

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0$$

with $\sigma = \pm 1$ can be derived from the compatibility of the system

$$\frac{\partial^2 v}{\partial x^2} + \sigma\frac{\partial v}{\partial y} + uv = 0$$
$$v_t + 4\frac{\partial^3 v}{\partial x^3} + 6u\frac{\partial v}{\partial x} + \left(3u_x - 3\sigma\int_\infty^\infty u_y\,dx + \alpha\right)v = 0$$

where $\alpha$ is constant. Note: there is no "eigenvalue" in this equation. The scattering parameter is inserted when carrying out the inverse scattering (see Ablowitz and Clarkson, 1991).

9.11 Consider the following time-independent Schrödinger equation

$$v_{xx} + (k^2 + Q\,\mathrm{sech}^2\,x)v = 0$$

where $Q$ is constant.

(a) Make the transformation $v(x) = \Psi(\xi)$, with $\xi = \tanh x$ (hence $-1 < \xi < 1$ corresponds to $-\infty < x < \infty$) to find

$$(1-\xi^2)\frac{d^2\Psi}{d\xi^2} - 2\xi\frac{d\Psi}{d\xi} + \left(Q + \frac{k^2}{1-\xi^2}\right)\Psi = 0$$

which is the *associated Legendre equation*, cf. Abramowitz and Stegun (1972).

(b) Show that there are eigenvalues $k = i\kappa$ when

$$\kappa = \left[\left(Q + \frac{1}{4}\right)^{1/2} - m - \frac{1}{2}\right] > 0,$$

where $m = 0, 1, 2, \ldots$ and therefore a necessary condition for eigenvalues is $Q + \frac{1}{4} > \frac{1}{4}$.

(c) Show that when $Q = N(N+1), N = 1, 2, 3, \ldots$, the reflection coefficient vanishes, the discrete eigenvalues are given by $\kappa_n = n, n = 1, 2, 3, \ldots, N$ and the discrete eigenfunctions are proportional to the associated Legendre function $P_N^n(\tanh x)$.

9.12 Show that the solution to the Gel'fand–Levitan–Marchenko (GLM) equation (9.42)–(9.43) with pure continuous spectra is unique. Hint: one method is to show the homogeneous equation has only the zero solution, hence by the Fredholm alternative, the solution to the GLM equation is unique. Then in the homogeneous equation let $K(x, y) = 0$, $y < x$ and take the Fourier transform.

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