

Chapter 3

*Perhaps make a comment in KdV derivation in previous section
that getting to equation (2.18) requires a lot of work as
all the B.C. must also be expanded.*

Non-local derivation on the whole line

*You might want to consider a transition that bridges the gap from
the previous section to this. Hint at "we are looking for a more efficient
way of deriving asy. models on diff. & domains".*

Recall that the equations of fluid motion are challenging to work with directly, due to the nonlinear boundary conditions and the unknown domain. In addition, these complications lead to difficulties when attempting to deal with questions of existence and well-posedness. To address these issues, reformulations of the problem have been introduced: they result in equivalent problems that are more tractable. While such reformulations can be helpful, they may suffer from other issues. Below, we give a short overview of these formulations, along with explaining the pros and cons of each. Our main goal is to look for a reformulation in the water surface η , and that generalises easily to two-dimensional problem. We chose this criteria with a view towards applications: indeed, in applications, determining the water surface η is the main interest.

For example, for one-dimensional surfaces (no y variable), conformal mappings can be used to eliminate these problems (for an overview, see Dyachenko et al., 1996). However, this approach is limited to one-dimensional surfaces. For both one- and two-dimensional surfaces, other

formulations (such as the Hamiltonian formulation given in Zakharov, 1968 or the Zakharov–Craig–Sulem formulation, Craig and Sulem, 1993) reduce the Euler equations to a system of two equations, in terms of surface variables $q = \phi(x, \eta)$ and η only, by introducing a Dirichlet-to-Neumann operator (DNO). A new non-local formulation is introduced in Ablowitz, Fokas, and Musslimani, 2006, (henceforth referred to as the AFM formulation) that results in a system of two equations for the same variables as in the DNO formulation. Both the DNO and AFM formulations reduce the problem from the full fluid domain to a system of equations that depend on the surface elevation $\eta(x, t)$ and the velocity potential evaluated at the surface, $q(x, t) = \phi(x, \eta)$. However, these formulations involve solving for an additional function $q(x, t)$, which may be of little relevance in applications, and hard to measure in experiments.

A new formulation is introduced in Oliveras and Vasan, 2013, which reduces the water waves problem to [a system of two equations,] in one variable η . This formulation allows to rigorously investigate one- and two-dimensional water waves. The computation of Stokes-wave asymptotic expansions for periodic waves justifies the use of the formulation; indeed, following Oliveras and Vasan, 2013, the computations can be performed with arguably less effort, especially for two-dimensional waves. Our goal is to further justify the use of this formulation, which we call the \mathcal{H} formulation.

In this chapter, we first rewrite the water wave problem by introducing a normal-to-tangential operator. We then perform a perturbation expansion for the operator, and proceed to obtain an expression for the surface elevation. Finally, performing asymptotics and applying time scales

Indicate that
we typically
are interested
in wave height.

this may
be confusing
to the reader w/o
seeing the eqn.
You might just
say "formally
eliminate $q(x, t)$
so the resulting
equation only
depend on η ".

yields the desired approximate equations. We emphasise that it is not our intention here to further the study of the water wave problem per se, but rather to demonstrate the efficacy of the \mathcal{H} formulation for doing asymptotics.

3.1 Water-wave problem on the whole line: non-local formulation

Recall the water wave problem given in (1.1) along with

$$\phi_{xx} + \phi_{zz} = 0 \quad -h < z < \eta(x, t) \quad (3.1a)$$

$$\phi_z = 0 \quad z = -h \quad (3.1b)$$

$$\eta_t + \phi_x \eta_x = \phi_z \quad z = \eta(x, t) \quad (3.1c)$$

$$\phi_t + g\eta + \frac{1}{2}(\phi_x^2 + \phi_z^2) = 0 \quad z = \eta(x, t) \quad (3.1d)$$

and consider the velocity potential evaluated at the surface:

$$q(x, t) = \phi(x, \eta(x, t), t) \quad \text{surface equation.}$$

$$= \phi(x, \eta(x, t), t)$$

We seek to reformulate the problem (3.1). Combining (3.8c) and (3.1d), evaluated at $z = \eta$, we obtain

$$q_t + \frac{1}{2}q_x^2 + g\eta - \frac{1}{2} \frac{(\eta_t + q_x \eta_x)^2}{1 + \eta_x^2} = 0, \quad (3.2)$$

aim to find

which is an equation for two unknowns q, η . We need an equation in one unknown only.

$$\eta(x, t)$$

notation issue

Given the domain $D = \mathbb{R} \times (-h, \eta)$, let

$$\vec{N} = \begin{bmatrix} -\eta_x \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{T} = \begin{bmatrix} 1 \\ \eta_x \end{bmatrix}$$

be vectors normal and tangent to the surface \mathcal{S} , respectively. We introduce an operator that maps the normal derivative at a surface η to the tangential derivative at the surface: $z = \eta(x, t)$

$$\mathcal{H}(\eta, D)\{\nabla\phi \cdot \vec{N}\} = \nabla\phi \cdot \vec{T}, \quad \checkmark \quad (3.3)$$

~~$-i\partial_x$ indicating~~

where $D = -i\nabla$. For convenience, we drop the vector notation. Note that by (3.8c),

$$\nabla\phi \cdot N = \begin{bmatrix} \phi_x \\ \phi_z \end{bmatrix} \cdot \begin{bmatrix} -\eta_x \\ 1 \end{bmatrix} = \phi_z - \phi_x \eta_x = \eta_t,$$

and by chain rule,

$$\nabla\phi \cdot T = \begin{bmatrix} \phi_x \\ \phi_z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \eta_x \end{bmatrix} = \phi_x + \eta_x \phi_z = q_x.$$

skip the matrix notation and make these all inline equation

This allows us to rewrite (3.3) as

$$\mathcal{H}(\eta, D)\{\eta_t\} = q_x. \quad (3.4)$$

and obtain a system. ~~Combine~~ Together, (3.2) and (3.4) form a system of two eqns for the two unknowns η and q .

$$\begin{aligned} q_t + \frac{1}{2} q_x^2 + g\eta - \frac{1}{2} \frac{(\eta_t + q_x \eta_x)^2}{1 + \eta_x^2} &= 0, \\ \mathcal{H}(\eta, D)\{\eta_t\} &= q_x. \end{aligned}$$

eliminate

~~Well~~ Differentiate ~~ing~~ (3.2) with respect to x and (3.4) with respect to t allows us to further reduce to a single equation for η

$$\partial_t(q_x) + \partial_x \left(\frac{1}{2} q_x^2 + g\eta - \frac{1}{2} \frac{(\eta_t + q_x \eta_x)^2}{1 + \eta_x^2} \right) = 0, \quad (3.5)$$

$$\partial_t(\mathcal{H}(\eta, D)\{\eta_t\}) = q_{xt}. \quad (3.6)$$

Substituting (3.4) and (3.6) into (3.5), we obtain

$$\begin{aligned} \partial_t(\mathcal{H}(\eta, D)\{\eta_t\}) \\ + \partial_x \left(\frac{1}{2} (\mathcal{H}(\eta, D)\{\eta_t\})^2 + \varepsilon\eta - \frac{1}{2} \frac{(\eta_t + q_x \mathcal{H}(\eta, D)\{\eta_t\})^2}{1 + \eta_x^2} \right) = 0. \end{aligned} \quad (3.7)$$

Equation (3.7) represents a scalar equation for the water wave surface η .

The utility of (3.7) depends on whether we can find a useful representation for the operator $\mathcal{H}(\eta, D)$. In the next section, we proceed to find an equation that the \mathcal{H} operator must satisfy.

3.2 Behaviour of the \mathcal{H} operator

Consider the following boundary value problem:

$$\phi_{xx} + \phi_{zz} = 0 \quad -h < z < \eta(x, t) \quad (3.8a)$$

$$\phi_z = 0 \quad z = -h \quad (3.8b)$$

$$\nabla\phi \cdot N = f(x) \quad z = \eta(x, t) \quad (3.8c)$$

Let φ be harmonic on D . Using (3.8a) and that φ_z is also harmonic on D , we have

$$\varphi_z(\phi_{xx} + \phi_{zz}) - \phi((\varphi_z)_{zz} + (\varphi_z)_{xx}) = 0.$$

Taking the integral over the domain yields

$$\int_{-\infty}^{\infty} \int_{-h}^{\eta(x)} \varphi_z(\phi_{xx} + \phi_{zz}) - \phi((\varphi_z)_{zz} + (\varphi_z)_{xx}) dz dx = 0.$$

An application of Green's theorem gives

$$\int_{\partial D} \varphi_z(\nabla\phi \cdot \mathbf{n}) - \phi(\nabla\varphi_z \cdot \mathbf{n}) ds = 0, \quad (3.9)$$

where ∂D is the boundary of the domain, ds is the area element, and \mathbf{n} is the normal vector. Now, observe that

$$-\nabla\varphi_z \cdot \mathbf{n} = \nabla\varphi_x \cdot \mathbf{t},$$

where \mathbf{t} is the tangential vector. We use this to rewrite (3.9) and obtain the following contour integral:

$$\begin{aligned} 0 &= \int_{\partial D} \varphi_z (\nabla \phi \cdot \mathbf{N}) + \phi (\nabla \varphi_z \cdot \mathbf{t}) \, ds \\ &= \int_{\partial D} \varphi_z (\phi_z \, dx - \phi_x \, dz) + \phi (\varphi_{xx} \, dx + \varphi_{xz} \, dz) \end{aligned}$$

We split the contour into four segments:

I typically use
 $\int_{\partial D} (\cdot) \, ds = \int_{-\infty}^{\infty} (\cdot) \Big|_{z=h} \, ds + \dots$

$$\begin{aligned} \int_{\partial D} &= \int_{-\infty}^{\infty} \left|_{z=-h}^{z=-h} \right. + \int_{-h}^{\eta(x)} \left|_{x \rightarrow \infty}^{x \rightarrow \infty} \right. + \int_{\infty}^{-\infty} \left|_{z=\eta(x)}^{z=\eta(x)} \right. + \int_{\eta(x)}^{-h} \left|_{x \rightarrow -\infty}^{x \rightarrow -\infty} \right. \\ &= \int_{-\infty}^{\infty} \left|_{z=-h}^{z=-h} \right. + \int_{-h}^{\eta(x)} \left|_{x \rightarrow \infty}^{x \rightarrow \infty} \right. - \int_{-\infty}^{\infty} \left|_{z=\eta(x)}^{z=\eta(x)} \right. - \int_{-h}^{\eta(x)} \left|_{x \rightarrow -\infty}^{x \rightarrow -\infty} \right. . \end{aligned}$$

people would
might not like
this notation.

Ask Dave if he has
a preference.

Consider each segment:

sketch the domain and consider
a formal principle value $\lim_{R \rightarrow \infty} \int_{-R}^R \dots \, dx$.

- As $|x| \rightarrow \infty$, we know that ϕ and its gradient vanish, so the integrals vanish.
- At $z = -h$, $dz = 0$, so we have

ok. definitely don't use
this here.

you could simply state
"there are no contributions as $|x| \rightarrow \infty$ ".
~~because~~

$$\begin{aligned} \int_{-\infty}^{\infty} \varphi_z (\phi_z \, dx - \phi_x \, dz) + \phi (\varphi_{xx} \, dx + \varphi_{xz} \, dz) &= \int_{-\infty}^{\infty} \phi \varphi_{xx} \, dx \quad (\text{since } \phi_z = 0 \text{ at } z = -h) \\ &= \phi(x, -h) \varphi_x(x, -h) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \phi_x(x, -h) \varphi_x(x, -h) \, dx \\ &= 0, \end{aligned}$$

where we pick φ such that $\varphi_x(x, -h) = 0$.

*You may be able
to skip this section
and simply jump
to equation
(3.10) where you
should note that
 $\chi(\eta, D)\{f(x)\}$ is
defined as
 $\nabla \phi \cdot \mathbf{T} = \chi(\eta, D)\{f(x)\}$*

//
will resume tomorrow.

- At $z = \eta$, $dz = \eta_x dx$, so we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} \varphi_z (\phi_z - \phi_x \eta_x) + \phi (\varphi_{xx} + \varphi_{xz} \eta_x) dx \\
&= \int_{-\infty}^{\infty} \varphi_z (\phi_z - \phi_x \eta_x) + \phi \frac{d\varphi_x(x, \eta)}{dx} dx \\
&= \int_{-\infty}^{\infty} \varphi_z (\phi_z - \phi_x \eta_x) - \varphi_x \frac{d\phi(x, \eta)}{dx} dx \quad (\text{integration by parts}) \\
&= \int_{-\infty}^{\infty} \varphi_z \begin{pmatrix} \phi_x \\ \phi_z \end{pmatrix} \cdot \begin{pmatrix} -\eta_x \\ 1 \end{pmatrix} - \varphi_x \begin{pmatrix} \phi_x \\ \phi_z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \phi_x \eta_x \end{pmatrix} dx \\
&= \int_{-\infty}^{\infty} \varphi_z \nabla \phi \cdot N - \varphi_x \nabla \phi \cdot T dx \\
&= \int_{-\infty}^{\infty} \varphi_z f(x) - \varphi_x(x, \eta) \mathcal{H}(\eta, D)\{f(x)\} dx \quad (\text{recall (3.8c) and (3.3)}).
\end{aligned}$$

Combining segments, we obtain:

skip to here indicate when you mean that $\mathcal{H}(\eta, D)\{f(x)\} = \nabla \phi \cdot T$

$$\int_{-\infty}^{\infty} \varphi_z f(x) - \varphi_x(x, \eta) \mathcal{H}(\eta, D)\{f(x)\} dx = 0. \quad (3.10)$$

Note that $\varphi(x, z) = e^{-ikx} \sinh(k(z + h))$, $k \in \mathbb{R}$ is one solution of

$$\Delta \varphi = 0, \quad \varphi_z(-h, z) = 0.$$

Then, (3.10) becomes

$$\int_{-\infty}^{\infty} e^{-ikx} (k \cosh(k(\eta + h)) f(x) + ik \sinh(k(\eta + h)) \mathcal{H}(\eta, D)\{f(x)\}) dx = 0.$$

It can be shown that we can take out k in the integral, so that the below holds for all $k \in \mathbb{R}$:

$$\int_{-\infty}^{\infty} e^{-ikx} (i \cosh(k(\eta+h))f(x) - \sinh(k(\eta+h))\mathcal{H}(\eta, D)\{f(x)\}) dx = 0. \quad (3.11)$$

(*) see footnote

The equation (3.11) gives a description of how the operator $\mathcal{H}(\eta, D)$ behaves, in dimensional coordinates.

In summary, introduction of the normal-to-tangential operator $\mathcal{H}(\eta, D)$ allows to reduce the water waves problem (3.1) to a scalar equation for η :

$$\begin{aligned} & \partial_t (\mathcal{H}(\eta, D)\{\eta_t\}) \\ & + \partial_x \left(\frac{1}{2} (\mathcal{H}(\eta, D)\{\eta_t\})^2 + \varepsilon \eta - \frac{1}{2} \frac{(\eta_t + \eta_x \mathcal{H}(\eta, D)\{\eta_t\})^2}{1 + \eta_x^2} \right) = 0, \end{aligned} \quad (3.12)$$

where the operator H is described via

$$\int_{-\infty}^{\infty} e^{-ikx} (i \cosh(k(\eta+h))f(x) - \sinh(k(\eta+h))\mathcal{H}(\eta, D)\{f(x)\}) dx = 0, \quad k \in \mathbb{R}. \quad (3.13)$$

In doing so, we go from solving for two unknowns ϕ and η to solving for one unknown η . Now, the utility of (3.12) depends on whether we can find a useful representation of the operator $\mathcal{H}(\eta, D)$.

3.3 Perturbation expansion of the \mathcal{H} operator

In this section, we derive a perturbation series of the \mathcal{H} operator. First, observe that (3.13) is written in dimensional coordinates. To obtain the

(*) you should include a remark that states that "technically" this is only valid for $k \neq 0$. However, for the case where $f(x) = \eta_t$ as in the water wave problem, then in the limit as $k \rightarrow 0$, (3.11) reduces to $\int_{-\infty}^{\infty} \eta_t dx = 0$ which is known to be true as it is conservation of mass. Cite Benjamin & Olver JFM paper from 80's

non-dimensional version, introduce new variables: we

$$t^* = \frac{t\sqrt{gh}}{L}, \quad x^* = \frac{x}{L}, \quad z^* = \frac{z}{h}, \quad \eta^* = \frac{\eta}{a}, \quad k^* = Lk,$$

rescale functions via

$$\phi = \frac{Lga}{\sqrt{gh}}\phi^*, \quad q^* = \frac{\sqrt{gh}}{agL}q,$$

indicate that
this is the same
scaling used in (2.15)
where $\lambda = \frac{1}{L}$.

skip
and define parameters ε and μ so that

$$\varepsilon = \frac{a}{h}, \quad \mu = \frac{h}{L}, \quad \varepsilon\mu = \frac{a}{L}.$$

Starting with

$$\int_{\partial D} \varphi_z(\phi_z dx - \phi_x dz) + \phi(\varphi_{xx} dx + \varphi_{xz} dz) = 0,$$

one may continue the same procedure to obtain the nondimensional version of (3.11):

$$\int_{-\infty}^{\infty} e^{-ikx} (i \cosh(\mu k(\eta+1))g(x) - \sinh(\mu k(\eta+1))\mathcal{H}(\varepsilon\eta, D)\{g(x)\}) dx = 0, \quad (3.14)$$

where $k \in \mathbb{R}$ and we dropped the starred notation for convenience. In addition, note that g and f are related by

$$g(x^*) = \frac{\sqrt{gh}}{ga} g(x^* L) = \frac{\sqrt{gh}}{ga} f(x).$$

just state this

you might consider moving the nondimensional form to the end of the previous section
Make an important point that (3.14) gives an implicit relationship that can be solved for $\mathcal{H}(\eta, D)\{f(x)\}$ as explored in the next section

New section
Solving for
 $\mathcal{H}(\eta, D)\{f(x)\}$

Short intro Φ stating that following Craig & Sulem, you will find
Since $\varepsilon \ll 1$, we expand the hyperbolic functions as a Taylor series in ε : a formal
series expansion

$$\cosh(\mu k(\eta + 1)) = \cosh(\mu k) + \mu k \varepsilon \eta \sinh(\mu k) + \mathcal{O}(\varepsilon^2),$$

$$\sinh(\mu k(\eta + 1)) = \sinh(\mu k) + \mu k \varepsilon \eta \cosh(\mu k) + \mathcal{O}(\varepsilon^2).$$

remember that
 $\varepsilon \ll 1$.

Now, we note that the idea of a regular perturbation series applies not only to classical functions but also to operators. Therefore, we expand



$$\mathcal{H}(\eta, D)\{g(x)\} = [\mathcal{H}_0 + \varepsilon \mathcal{H}_1 + \mathcal{O}(\varepsilon^2)] (\varepsilon \eta, D)\{g(x)\}.$$

→ Be read Craig/Sulem for some terminology / defining this series.
Equation (3.14) becomes:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-ikx} (i [\cosh(\mu k) + \mu k \varepsilon \eta \sinh(\mu k) + \mathcal{O}(\varepsilon^2)] g(x) \\ - [\sinh(\mu k) + \mu k \varepsilon \eta \cosh(\mu k) + \mathcal{O}(\varepsilon^2)] [\mathcal{H}_0 + \varepsilon \mathcal{H}_1 + \mathcal{O}(\varepsilon^2)] (\varepsilon \eta, D)\{g(x)\}) dx = 0. \end{aligned} \quad (3.15)$$

At leading order

Within $\mathcal{O}(\varepsilon^0)$: Using (3.15), we obtain

$$\int_{-\infty}^{\infty} e^{-ikx} (i \cosh(\mu k) g(x) - \sinh(\mu k) \mathcal{H}_0 (\varepsilon \eta, D)\{g(x)\}) dx = 0.$$

Dividing by $\sinh(\mu k)$ yields ← make a statement that this is only valid for $k \neq 0$.

$$\int_{-\infty}^{\infty} e^{-ikx} (i \coth(\mu k) g(x) - \mathcal{H}_0 (\varepsilon \eta, D)\{g(x)\}) dx = 0.$$

✳️ Notation-wise, you might consider

$$\mathcal{X}(\varepsilon \eta, D)\{g(x)\} = \sum_{j=0}^{\infty} \mathcal{X}_j(\varepsilon \eta, D)\{g(x)\} \quad \text{where} \quad \mathcal{X}_j(\varepsilon \eta, D) = \varepsilon^j \mathcal{H}(\eta, D)$$

\mathcal{X}_j is homogeneous in η .

Splitting the integrand and recognizing the Fourier transform yields:

$$\begin{aligned}\mathcal{F}_k\{\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\}\} &= \int_{-\infty}^{\infty} e^{-ikx} \mathcal{H}_0(\varepsilon\eta, D)\{g(x)\} dx \\ &= \int_{-\infty}^{\infty} e^{-ikx} i \coth(\mu k) g(x) dx \\ &= i \coth(\mu k) \mathcal{F}_k\{g(x)\}. \end{aligned}$$

faster than $\mathcal{O}(uk)$

Provided that $\mathcal{F}_k\{g(x)\} \neq 0$, then this is defined for all k .
Finally, we invert Fourier transform to obtain

$$\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\} = \mathcal{F}_k^{-1}\{i \coth(\mu k) \mathcal{F}_k\{g(x)\}\}, \quad (3.16)$$

where we write out k 's explicitly to keep track of transforms.

Within $\mathcal{O}(\varepsilon^1)$: from (3.15), we obtain

$$\int_{-\infty}^{\infty} e^{-ikx} (i\mu k \eta \sinh(\mu k) g(x) - [\sinh(\mu k) \mathcal{H}_1 + \mu k \eta \cosh(\mu k) \mathcal{H}_0] (\varepsilon\eta, D)\{g(x)\}) dx = 0.$$

Dividing by $\sinh(\mu k)$ and dropping $(\varepsilon\eta, D)$ for the notational convenience,
we have \hookrightarrow again $k \neq 0 \dots$

$$\int_{-\infty}^{\infty} e^{-ikx} (i\mu k \eta g(x) - [\mathcal{H}_1 + \mu k \eta \coth(\mu k) \mathcal{H}_0]\{g(x)\}) dx = 0.$$

Splitting the integral and recognising the Fourier transform yields:

$$\begin{aligned}\mathcal{F}_k\{\mathcal{H}_1\{g(x)\}\} &= \int_{-\infty}^{\infty} e^{-ikx} \mathcal{H}_1\{g(x)\} dx \\ &= \int_{-\infty}^{\infty} e^{-ikx} (i\mu k \eta g - \mu k \eta \coth(\mu k) \mathcal{H}_0\{g(x)\}) dx \\ &= \mu \mathcal{F}_k\{ik\eta g\} - \mu k \coth(\mu k) \mathcal{F}_k\{\eta \mathcal{H}_0\{g(x)\}\}.\end{aligned}$$

Inverting Fourier transform and using (3.16), we obtain an expression for \mathcal{H}_1 :

$$\mathcal{H}_1\{g(x)\} = \mathcal{F}_k^{-1}\{\mu\eta g\} - \mathcal{F}_k^{-1}\{\mu k \coth(\mu k) \mathcal{F}_k\{\eta \mathcal{H}_0\{g(x)\}\}\}$$

lost a "ik"

$$\begin{aligned} &= \mu \partial_x(\eta g) - \mathcal{F}_k^{-1}\{\mu k \coth(\mu k) \mathcal{F}_k\{\eta \mathcal{H}_0\{g(x)\}\}\} \\ &= \mu \partial_x(\eta g) - \mathcal{F}_k^{-1}\{\mu k \coth(\mu k) \mathcal{F}_k\{\eta \mathcal{F}_l^{-1}\{i \coth(\mu l) \mathcal{F}_l\{g\}\}\}\}. \end{aligned}$$

In sum, we find a representation for the \mathcal{H} operator within two orders:

$$\mathcal{H}(\varepsilon\eta, D)\{g(x)\} = [\mathcal{H}_0 + \varepsilon \mathcal{H}_1](\varepsilon\eta, D)\{g(x)\} + \mathcal{O}(\varepsilon^2),$$

where

$$\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\} = \mathcal{F}_k^{-1}\{i \coth(\mu k) \mathcal{F}_k\{g(x)\}\},$$

$$\mathcal{H}_1(\varepsilon\eta, D)\{g(x)\} = \mu \partial_x(\eta g) - \mathcal{F}_k^{-1}\{\mu k \coth(\mu k) \mathcal{F}_k\{\eta \mathcal{F}_l^{-1}\{i \coth(\mu l) \mathcal{F}_l\{g\}\}\}\}.$$

these
equations
and refer
to them
later.

you should mention that a formal recursion can be found. So that \mathcal{H}_i can be written in terms of \mathcal{H}_j for $i \leq j$.

Recall the expression for \mathcal{H}_0 :

$$\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\} = \mathcal{F}_k^{-1}\{i \coth(\mu k) \mathcal{F}_k\{g(x)\}\}.$$

Expanding $\coth(\mu k)$ via its Laurent series in μk gives

$$\coth(\mu k) = \frac{1}{\mu k} + \frac{\mu k}{3} + \mathcal{O}(\mu^3).$$

It is readily seen that since $\coth(\mu k)$ has a simple pole as $k \rightarrow 0$, so do $\mathcal{H}_0, \mathcal{H}_1$ and \mathcal{H} .

good. tie this back to needing

$\mathcal{F}_k\{g(x)\} \rightarrow 0$ as $k \rightarrow 0$ at a rate

such that $\mathcal{F}_k\{\mathcal{H}_i \varepsilon g(x)\} \rightarrow 0$ as $k \rightarrow 0$.

really drives home that
this part is new.

you show all the derivation for the H formulation but not velocity potential.
You should comment
① why you made this choice
② how the compare.
getting to 2.18 isn't easy - you need to indicate that.

3.4 Deriving an expression for surface elevation

We proceed to derive approximate equations for the surface η . Recall the scalar equation (3.7):

$$\partial_t (\mathcal{H}(\eta, D)\{\eta_t\}) + \partial_x \left(\frac{1}{2} (\mathcal{H}(\eta, D)\{\eta_t\})^2 + \varepsilon \eta - \frac{1}{2} \frac{(\eta_t + \eta_x \mathcal{H}(\eta, D)\{\eta_t\})^2}{1 + \eta_x^2} \right) = 0. \quad (3.17)$$

In nondimensional var

The non-dimensional version of (3.17) is given by

$$\begin{aligned} \partial_t (\mathcal{H}(\varepsilon \eta, D)\{\varepsilon \mu \eta_t\}) + \partial_x \left(\frac{1}{2} \left(\mathcal{H}(\varepsilon \eta, D)\{\varepsilon \mu \eta_t\} \right)^2 \right. \\ \left. + \varepsilon \eta - \frac{1}{2} \varepsilon^2 \mu^2 \frac{(\eta_t + \eta_x \mathcal{H}(\varepsilon \eta, D)\{\varepsilon \mu \eta_t\})^2}{1 + \varepsilon^2 \mu^2 \eta_x^2} \right) = 0. \end{aligned} \quad (3.18)$$

Also, recall our maximal balance assumption $\varepsilon = \mu^2$ and recall the first order and second order expansions for $\coth(\mu k)$:

$$\coth(\mu k) = \frac{1}{\mu k} + \mathcal{O}(\mu) = \frac{1}{\mu k} + \frac{\mu k}{3} + \mathcal{O}(\mu^3).$$

At Within $\mathcal{O}(\mu^0)$: In the leading order, the equation (3.18) becomes

$$\partial_t (\mathcal{H}_0(\varepsilon \eta, D)\{\varepsilon \mu \eta_t\}) + \varepsilon \partial_x \eta = 0.$$

Bringing ∂_t inside \mathcal{H}_0 and substituting the expression for \mathcal{H}_0 , we obtain:

skip to here.

$$\mathcal{F}_k^{-1} \{ i \coth(\mu k) \mathcal{F}_k \{\varepsilon \mu \eta_t\} \} + \varepsilon \partial_x \eta = 0. \quad + \mathcal{O}(\varepsilon^2)$$

Inverting the Fourier transform and multiplying by $k/(i\varepsilon)$ yields

$$\mu k \coth(\mu k) \widehat{\eta}_{ttk} + k^2 \widehat{\eta}_k = 0. \quad + \mathcal{O}(\varepsilon)$$

* earlier, you equate powers of ε . Here, it's μ^2 . Choose one and be consistent.

Expanding $\coth(\mu k)$ in the leading order gives *and noting $u^2 = \varepsilon$,*

$$\widehat{\eta}_{ttk} + k^2 \widehat{\eta}_k = 0. \quad + O(\varepsilon)$$

Inverting the Fourier transform, we have

$$\eta_{tt} + (-i\partial_x)^2 \eta = 0, \quad + O(\varepsilon)$$

which is

$$\eta_{tt} - \eta_{xx} = 0. \quad + O(\varepsilon)$$

This is the wave equation, as we desired. *If we consider higher order terms,*

Within $O(\varepsilon^2)$: The non-dimensional equation (3.18) becomes

$$\partial_t (\mathcal{H}_0\{\varepsilon\mu\eta_t\} + \varepsilon\mathcal{H}_1\{\varepsilon\mu\eta_t\}) + \partial_x \left(\frac{1}{2} (\mathcal{H}_0\{\varepsilon\mu\eta_t\})^2 + \varepsilon\eta \right) = 0. \quad (3.19)$$

Using (refer to \mathcal{H}_0 and \mathcal{H}_1 definitions) #)

$$\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\} = \varepsilon\mu\mathcal{F}_k^{-1}\{i\coth(\mu k)\widehat{\eta}_{tk}\};$$

$$\mathcal{H}_1(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\} = \varepsilon\mu^2(\eta\eta_t)_x - \varepsilon\mathcal{F}_k^{-1}\{\mu k\coth(\mu k)\mathcal{F}_k\{\eta\mathcal{F}_l^{-1}\{i\mu\coth(\mu l)\widehat{\eta}_{tl}\}\}\}.$$

Observe that

$$\frac{1}{2} (\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\})^2 = \frac{1}{2} (\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\})^2 = \frac{\varepsilon^2}{2} \left(\mathcal{F}_k^{-1}\{i\mu\coth(\mu j)\widehat{\eta}_{tk}\} \right)^2,$$

skip

and

$$\begin{aligned} \partial_t \left(\left[\mathcal{H}_0(\varepsilon\eta, D) + \varepsilon\mathcal{H}_1(\varepsilon\eta, D) \right] \{\varepsilon\mu\eta_t\} \right) \\ = \varepsilon\mu\mathcal{F}_k^{-1}\{i\coth(\mu k)\widehat{\eta}_{ttk}\} + \varepsilon^2\mu^2(\eta\eta_t)_{tx} \\ - \varepsilon^2\mathcal{F}_k^{-1}\{\mu k\coth(\mu k)\mathcal{F}_k\{\partial_t [\eta\mathcal{F}_l^{-1}\{i\mu\coth(\mu l)\widehat{\eta}_{tl}\}]\}\}. \end{aligned}$$

skip to here

The single equation (3.19) becomes

$$\begin{aligned} \varepsilon\mu\mathcal{F}_k^{-1}\{i\coth(\mu k)\widehat{\eta}_{ttk}\} + \varepsilon^2\mu^2(\eta\eta_t)_{tx} \\ - \varepsilon^2\mathcal{F}_k^{-1}\{\mu k\coth(\mu k)\mathcal{F}_k\{\partial_t [\eta\mathcal{F}_l^{-1}\{i\mu\coth(\mu l)\widehat{\eta}_{tl}\}]\}\} \\ + \frac{\varepsilon^2}{2}\partial_x \left(\mathcal{F}_j^{-1}\{i\mu\coth(\mu j)\widehat{\eta}_{tj}\} \right)^2 + \varepsilon\partial_x\eta = 0. \end{aligned}$$

+ O(ε^3)

Application of Fourier transform yields and expanding the coth's,

$$\begin{aligned} \varepsilon\mu i\coth(\mu k)\widehat{\eta}_{ttk} + \varepsilon^2\mu^2 ik(\eta\eta_t)_t - \varepsilon^2\mu k\coth(\mu k)\mathcal{F}_k\{\partial_t [\eta\mathcal{F}_l^{-1}\{i\mu\coth(\mu l)\widehat{\eta}_{tl}\}]\} \\ + \frac{\varepsilon^2}{2}ik\mathcal{F}_k\left\{\left(\mathcal{F}_j^{-1}\{i\mu\coth(\mu j)\widehat{\eta}_{tj}\}\right)^2\right\} + \varepsilon ik\widehat{\eta}_k = 0. \end{aligned}$$

Divide by $i\varepsilon$ and recall $\varepsilon = \mu^2$ to obtain

$$\begin{aligned} \mu\coth(\mu k)\widehat{\eta}_{ttk} + \mu^4 k(\eta\eta_t)_t - \mu^3 k\coth(\mu k)\mathcal{F}\{\partial_t [\eta\mathcal{F}^{-1}\{\mu\coth(\mu l)\widehat{\eta}_{tl}\}]_l\}_k \\ + \frac{\mu^2}{2}k\mathcal{F}\left\{\left(\mathcal{F}^{-1}\{i\mu\coth(\mu j)\widehat{\eta}_{tj}\}_j\right)^2\right\}_k + k\widehat{\eta}_k = 0. \end{aligned}$$

~~Expanding $\coth(\mu k)$ -like terms yields~~

$$\left(\frac{1}{k} + \frac{\mu^2 k}{3} \right) \widehat{\eta_{ttk}} + \mu^4 k (\eta \eta_t)_t - \mu^2 k \left(\frac{1}{k} + \frac{\mu^2 k}{3} \right) \mathcal{F}_k \left\{ \partial_t \left[\eta \mathcal{F}_l^{-1} \left\{ \left(\frac{1}{l} + \frac{\mu^2 l}{3} \right) \widehat{\eta_{tl}} \right\}_l \right] \right\} \\ - \frac{\mu^2}{2} k \mathcal{F}_k \left\{ \left(\mathcal{F}_j^{-1} \left\{ \left(\frac{1}{j} + \frac{\mu^2 j}{3} \right) \widehat{\eta_{tj}} \right\} \right)^2 \right\} + k \widehat{\eta_k} = 0. \quad (3.20)$$

~~Keeping only terms~~

Within $\mathcal{O}(\mu^2)$, the equation (3.20) becomes

$$\left(\frac{1}{k} + \frac{\mu^2 k}{3} \right) \widehat{\eta_{ttk}} - \mu^2 \mathcal{F}_k \left\{ \partial_t \left[\eta \mathcal{F}_l^{-1} \left\{ \frac{1}{l} \widehat{\eta_{tl}} \right\} \right] \right\} - \frac{\mu^2}{2} k \mathcal{F}_k \left\{ \left(\mathcal{F}_j^{-1} \left\{ \frac{1}{j} \widehat{\eta_{tj}} \right\} \right)^2 \right\} + k \widehat{\eta_k} = 0,$$

where we keep only ^{up to} ~~through~~ $\mathcal{O}(\mu^2)$
or rearranging and multiplying by k , we have

$$\widehat{\eta_{ttk}} + k^2 \widehat{\eta_k} + \mu^2 \left(\frac{k^2}{3} \widehat{\eta_{ttk}} - k \mathcal{F}_k \left\{ \partial_t \left[\eta \mathcal{F}_l^{-1} \left\{ \frac{1}{l} \widehat{\eta_{tl}} \right\} \right] \right\} - \frac{1}{2} k^2 \mathcal{F}_k \left\{ \left(\mathcal{F}_j^{-1} \left\{ \frac{1}{j} \widehat{\eta_{tj}} \right\} \right)^2 \right\} \right) = 0.$$

Finally, inverting the Fourier transform yields:

$$\eta_{tt} - \eta_{xx} + \mu^2 \left(-\frac{\partial_x^2}{3} \eta_{tt} + i \partial_x \left(\partial_t \left[\eta \mathcal{F}_l^{-1} \left\{ \frac{1}{l} \widehat{\eta_{tl}} \right\} \right] \right) + \frac{1}{2} \partial_x^2 \left(\mathcal{F}_j^{-1} \left\{ \frac{1}{j} \widehat{\eta_{tj}} \right\} \right)^2 \right) = 0, \quad + \mathcal{O}(\mu^4)$$

or more conveniently,

$$\eta_{tt} - \eta_{xx} = \mu^2 \left(\frac{\partial_x^2}{3} \eta_{tt} - i \partial_x \left(\partial_t \left[\eta \mathcal{F}_l^{-1} \left\{ \frac{1}{l} \widehat{\eta_{tl}} \right\} \right] \right) - \frac{1}{2} \partial_x^2 \left(\mathcal{F}_j^{-1} \left\{ \frac{1}{j} \widehat{\eta_{tj}} \right\} \right)^2 \right). \quad (3.21)$$

We seek to simplify (3.21). First, integration by parts gives ~~allows us to rewrite~~

$$\begin{aligned}\frac{1}{l} \hat{\eta}_{tt} &= \frac{1}{l} \frac{2}{\pi} \int_{-\infty}^{\infty} e^{-ilx} \eta_t dx \\ &= \frac{1}{l} \frac{2}{\pi} e^{-ilx} \int_{-\infty}^x \eta_t(x', t) dx' \Big|_{-\infty}^{\infty} + i \frac{2}{\pi} \int_{-\infty}^{\infty} e^{-ilx} \int_{-\infty}^x \eta_t(x', t) dx' dx \\ &= i \frac{2}{\pi} \int_{-\infty}^{\infty} e^{-ilx} \int_{-\infty}^x \eta_t(x', t) dx' dx \\ &= i \mathcal{F}_t \left\{ \int_{-\infty}^x \eta_t(x', t) dx' \right\},\end{aligned}$$

~~so that~~

$$\mathcal{F}_l^{-1} \left\{ \frac{1}{l} \hat{\eta}_{tt} \right\} = \mathcal{F}_l^{-1} \left\{ i \mathcal{F}_t \left\{ \int_{-\infty}^x \eta_t(x', t) dx' \right\} \right\} = i \int_{-\infty}^x \eta_t(x', t) dx', \quad (3.22)$$

where we applied the Fourier inversion theorem. Using (3.22) and that $\eta_{tt} = \eta_{xx} + \mathcal{O}(\mu^2)$, the equation (3.21) becomes

$$\begin{aligned}\eta_{tt} - \eta_{xx} &= \mu^2 \left(\frac{1}{3} \eta_{xxxx} + \partial_x \partial_t \left[\eta \left(\int_{-\infty}^x \eta_t dx' \right) \right] + \frac{1}{2} \partial_x^2 \left(\int_{-\infty}^x \eta_t dx' \right)^2 \right) \\ &= \mu^2 \left(\frac{1}{3} \eta_{xxxx} + \partial_x \left[\eta_t \left(\int_{-\infty}^x \eta_t dx' \right) + \eta \eta_x \right] + \frac{1}{2} \partial_x^2 \left(\int_{-\infty}^x \eta_t dx' \right)^2 \right) \\ &= \varepsilon \left[\frac{1}{3} \eta_{xxxx} + \partial_x^2 \left(\frac{\eta^2}{2} + \left(\int_{-\infty}^x \eta_t dx' \right)^2 \right) \right] + \mathcal{O}(\varepsilon^2)\end{aligned} \quad (3.23)$$

just jump to here.
indicate many steps are omitted for brevity

In summary, the second order approximation of a scalar equation for η resulted into equation (3.23).

Remark 4. At the end of Section 2.4, we mentioned secular terms in the next order. We now show directly by examining the dispersion relation (3.23). Assume a plane wave solution of the form $\tilde{\eta}(x, t) = \exp(i(kx -$

there is something about this argument that still bothers me. Typically, I see it more related to ill-posedness. I know you saw it in Ablowitz. I'll check there in the AM. But it's not a bad idea to reconfirm.

ωt). Substituting $\tilde{\eta}$ into the linearised equation

$$\tilde{\eta}_{tt} - \tilde{\eta}_{xx} = \varepsilon \frac{1}{3} \tilde{\eta}_{xxxx},$$

in line

leads to the following relation

$$-\omega^2 + k^2 = \varepsilon \frac{k^4}{3}.$$

(in line)

$$\omega = \sqrt{k^2 - \varepsilon \frac{k^4}{3}}$$

$|k|$

Substituting the negative root of ω into the wave solution gives

$$\omega = k \sqrt{1 - \varepsilon \frac{k^2}{3}}$$

explain the large k limit

$$\eta(x, t) \approx \exp(ikx) \exp\left(\sqrt{\frac{\varepsilon}{3}} k^2 t\right).$$

As is seen, this solution is unbounded in time as $k \rightarrow \infty$. This phenomenon suggests that (3.33) contains secularity and therefore warrants an application of time scales.

Can you explain how the multiple time scales would address this?

3.5 Derivation of wave and KdV equations

would be helpful for reader unfam. with multiple time scales.

We derive the approximate equations from

$$\eta_{tt} - \eta_{xx} = \varepsilon \left[\frac{1}{3} \eta_{xxxx} + \partial_x^2 \left(\frac{\eta^2}{2} + \left(\int_{-\infty}^x \eta_t dx' \right)^2 \right) \right]. \quad (3.24)$$

Homework

We assume an expansion of η in ε :

$$\eta = \eta_0 + \varepsilon \eta_1 + \mathcal{O}(\varepsilon^2). \quad (3.25)$$

$$\text{Ansatz: } \eta = \pm \sqrt{k^2 - \varepsilon \frac{k^2}{3}}$$

$$\begin{aligned} \eta &= \pm \sqrt{k^2 - \varepsilon \frac{k^2}{3}} \\ &= \pm \sqrt{k^2} \sqrt{1 - \varepsilon \frac{k^2}{3}} \end{aligned}$$

~~First order approximation~~ ing (for parallelism)

Substitution of (3.25) into equation (3.24) yields

$$\eta_{0tt} - \eta_{0xx} + \varepsilon(\eta_{1tt} - \eta_{1xx}) = \varepsilon \left[\frac{1}{3} \eta_{0xxxx} + \partial_x^2 \left(\frac{(\eta_0 + \varepsilon \eta_1)^2}{2} + \left(\int_{-\infty}^x (\eta_0 + \varepsilon \eta_1)_t dx' \right)^2 \right) \right] + \mathcal{O}(\varepsilon^2). \quad (3.26)$$

In the leading order $\mathcal{O}(\varepsilon^0)$, equation (3.26) becomes

$$\eta_{0tt} - \eta_{0xx} = 0.$$

This is the wave equation with ~~speed~~ velocity 1, and whose general solution is

$$\eta_0 = F(x-t) + G(x+t),$$

where F, G are general functions.

~~Second order approximation~~

As was discussed in Remark (4), anticipating the secular terms, we introduce slow time scales

$$\tau_0 = t, \quad \tau_1 = \varepsilon t.$$

so that

$$\eta(x, t) = \eta(x, \tau_0, \tau_1).$$

The expansion (3.25) becomes

$$\begin{aligned} \eta(x, \tau_0, \tau_1) &= \eta_0(x, \tau_0, \tau_1) + \mathcal{O}(\varepsilon^1), \\ &\eta_0(x, \tau_0, \tau_1) + \varepsilon \eta_1(x, \tau_0, \tau_1) + \mathcal{O}(\varepsilon^2) \end{aligned} \quad (3.27)$$

*skip due to multiple scales.
State "anticipating
the need for multiple
time scales" or
something so that
effect.*

Substituting (3.27) into (3.24), within $\mathcal{O}(\varepsilon^0)$, we obtain

$$\eta_{0\tau_0\tau_0} - \eta_{0xx} = 0, \quad (3.28)$$

so that the general solution is

$$\eta_0(x, \tau_0, \tau_1) = F(x - \tau_0, \tau_1) + G(x + \tau_0, \tau_1).$$

Although we have found an expression for η_0 , the functions F, G used are still general functions. To determine them, we proceed to the next order, where we can understand the dependence of F, G on the slow time scale τ_1 . In addition, we introduce left-going and right-going variables

$$\xi = x - \tau_0 \quad \zeta = x + \tau_0.$$

These new variables imply

$$\begin{aligned}\partial_x &= \partial_\xi \frac{d\xi}{dx} + \partial_\zeta \frac{d\zeta}{dx} = \partial_\xi + \partial_\zeta, \\ \partial_t &= \partial_\xi \frac{d\xi}{dt} + \partial_\zeta \frac{d\zeta}{dt} + \partial_{\tau_1} \frac{d\tau_1}{dt} = -\partial_\xi + \partial_\zeta + \varepsilon \partial_{\tau_1}.\end{aligned}$$

We can rewrite (3.27) as follows:

$$\begin{aligned}\eta &= \eta_0 + \varepsilon \eta_1 + \mathcal{O}(\varepsilon^2) \\ &= F(\xi, \tau_1) + G(\zeta, \tau_1) + \varepsilon \eta_1 + \mathcal{O}(\varepsilon^2).\end{aligned}$$

For ease of writing, we suppress explicit dependence on variables, though the reader should bear in mind that function F (G) depend on ξ (ζ), τ_1 .

*Actually write this out.
This is a little confusing*

Observe that ~~the operator $\partial_t^2 \partial_x^2$ can~~

$$(\partial_t^2 - \partial_x^2) = (-4\partial_\zeta \partial_{\tau_1} + 2\varepsilon(\partial_\zeta \partial_{\tau_1} - \partial_\zeta \partial_{\tau_1}) + \varepsilon^2 \partial_{\tau_1}^2), \quad (\text{inline})$$

~~details~~

so that the LHS of (3.24) becomes

~~skip~~

$$(\partial_t^2 - \partial_x^2)\eta = \varepsilon(-4\eta_{1,\zeta\zeta} - 2F_{\tau_1,\zeta} + 2G_{\tau_1,\zeta}) + \mathcal{O}(\varepsilon^2). \quad (3.29)$$

Now, we deal with the RHS of (3.24). By appropriate substitutions, the terms become:

$$\varepsilon(-4\eta_{1,\zeta\zeta} - 2F_{\tau_1,\zeta} + 2G_{\tau_1,\zeta}) =$$

$$\frac{1}{3}\eta_{xxxx} = \frac{1}{3}(F_{\zeta\zeta\zeta\zeta} + G_{\zeta\zeta\zeta\zeta}) + \mathcal{O}(\varepsilon);$$

$$\frac{1}{2}\eta^2 = \frac{1}{2}(F^2 + 2FG + G^2) + \mathcal{O}(\varepsilon),$$

and

$$\begin{aligned} \left(\int_{-\infty}^x \eta_t dx' \right)^2 &= \left(\int_{-\infty}^x \eta_{0t} dx' \right)^2 + \mathcal{O}(\varepsilon) \\ &= \left(\int_{-\infty}^x -F_\zeta + G_\zeta dx' \right)^2 + \mathcal{O}(\varepsilon) \\ &= \left(\int_{-\infty}^x F_\zeta dx' \right)^2 - 2 \left(\int_{-\infty}^x F_\zeta dx' \right) \left(\int_{-\infty}^x G_\zeta dx' \right) \\ &\quad + \left(\int_{-\infty}^x G_\zeta dx' \right)^2 + \mathcal{O}(\varepsilon) \\ &= F^2 - 2FG + G^2 + \mathcal{O}(\varepsilon), \end{aligned}$$

*you can skip
this part of
the detail*

where we assume that F, G vanish as $\xi, \zeta \rightarrow -\infty$. The RHS of (3.24) becomes

$$\begin{aligned} & \varepsilon \left[\frac{1}{3} \eta_{xxxx} + \partial_x^2 \left(\frac{\eta^2}{2} + \left(\int_{-\infty}^x \eta_t dx' \right)^2 \right) \right] \\ &= \varepsilon \left[\frac{1}{3} (F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + (\partial_\xi^2 + 2\partial_\xi\partial_\zeta + \partial_\zeta^2) \left(\frac{3}{2} F^2 + \frac{3}{2} G^2 - FG \right) \right] + \mathcal{O}(\varepsilon^2). \end{aligned} \quad (3.30)$$

Combining (3.29) and (3.30), in $\mathcal{O}(\varepsilon^1)$ we have

$(\partial_\xi + \partial_\zeta)^2$ to save space.

$$-4\eta_{1\xi\zeta} = 2F_{\tau_1\xi} - 2G_{\tau_1\zeta} + \frac{1}{3} (F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + (\partial_\xi^2 + 2\partial_\xi\partial_\zeta + \partial_\zeta^2) \left(\frac{3}{2} F^2 + \frac{3}{2} G^2 - FG \right), \quad (3.31)$$

where we assume that F, G vanish as

In the last term of (3.31), differentiation yields:

$\xi, \zeta \rightarrow -\infty$.

$$(\partial_\xi^2 + 2\partial_\xi\partial_\zeta + \partial_\zeta^2) \left(\frac{3}{2} F^2 + \frac{3}{2} G^2 - FG \right) = \partial_\xi (3FF_\xi - GF_\xi) + \partial_\zeta (3GG_\zeta - FG_\zeta) - 2F_\xi G_\zeta,$$

so that (3.31) becomes

$$\begin{aligned} -4\eta_{1\xi\zeta} &= 2F_{\tau_1\xi} - 2G_{\tau_1\zeta} + \frac{1}{3} (F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + \partial_\xi (3FF_\xi - GF_\xi) + \partial_\zeta (3GG_\zeta - FG_\zeta) - 2F_\xi G_\zeta \\ &= \partial_\xi (2F_{\tau_1} + \frac{1}{3} F_{\xi\xi\xi} + 3FF_\xi) + \partial_\zeta (-2G_{\tau_1} + \frac{1}{3} G_{\zeta\zeta\zeta} + 3GG_\zeta) - (GF_\xi + FG_\zeta) - 2F_\xi G_\zeta. \end{aligned} \quad (3.32)$$

Integration of (3.32) with respect to ζ yields

$$-4\eta_{1\xi} = \partial_\xi (2F_{\tau_1} + \frac{1}{3} F_{\xi\xi\xi} + 3FF_\xi) \zeta + (-2G_{\tau_1} + \frac{1}{3} G_{\zeta\zeta\zeta} + 3GG_\zeta) - \left(F_\xi \int G d\zeta + GF \right),$$

+ $f'(\xi)$

and further integration with respect to ξ leads to

$$-4\eta_1 = \underbrace{(2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + 3FF_{\xi})\zeta}_{\text{secular terms}} + \underbrace{(-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + 3GG_{\zeta})\xi}_{\text{secular terms}} - \left(F \int G d\zeta + G \int F d\xi \right). \quad (3.33)$$

From the equation (3.33), we see that the ξ, ζ terms are the secular terms we wish to remove. Otherwise, η_1 is unbounded in time. Therefore, we must have

$$2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + 3FF_{\xi} = 0 \quad (3.34)$$

$$2G_{\tau_1} - \frac{1}{3}G_{\zeta\zeta\zeta} - 3GG_{\zeta} = 0. \quad (3.35)$$

The equations (3.34) and (3.35) are KdV equations, which allow us to determine the behaviour of F, G on a slow time scale τ_1 .

In conclusion, asymptotic analysis of the non-local formulation in shallow water limit gives rise to two KdV equations, (3.34) and (3.35), for the right-going and left-going waves. Given the right decay initial conditions, we can solve these PDEs by means of the inverse scattering transform (see Ablowitz, 2011, Chapter 9). Keeping the leading-order terms, we have an approximate solution for the surface variable

$$\eta \approx \eta_0 = \underline{F(x-t, \varepsilon t) + G(x+t, \varepsilon t)}. \quad \text{where}$$

The derivation is complete. $\ast\ast\ast$

try using underbrace. In the writing, be clearer than "the 5,6 terms"

* You need to comment on the $-F \int G d\zeta + G \int F d\xi$ term and why it's not important.

*** Not a bad idea to comment about the similarities/differences between the velocity potential derivation and this derivation.