Appendix B

The boundary conditions for a viscous fluid

The inclusion of viscosity in the modelling of the fluid requires that, at the free surface, the stresses there must be known (given) and, at the bottom, that there is no slip between the fluid and the bottom boundary. The surface stresses are resolved to produce the normal stress and any two (independent) tangential stresses. The normal stress is prescribed, predominantly, by the ambient pressure above the surface, but it may also contain a contribution from the surface tension (see Section 1.2.2). The tangential stresses describe the shearing action of the air at the surface, and therefore may be significant in the analysis of the motion of the surface which interacts with a surface wind. The bottom condition is the far simpler (and familiar) one which states that, for a viscous fluid, the fluid in contact with a solid boundary must move with that boundary.

The appropriate stress conditions are derived by considering the equilibrium of an element of the surface under the action of the forces generated by the stresses. The normal and shear stresses in the fluid (see Appendix A) produce forces that are resolved normal and tangential to the free surface, although the details of this calculation will not be reproduced here. It is sufficient for our purposes (and for general reference) to quote the results – in both rectangular Cartesian and cylindrical coordinates – for the three surface stresses.

First, in rectangular Cartesian coordinates, $\mathbf{x} \equiv (x, y, z)$ and $\mathbf{u} \equiv (u, v, w)$, with the free surface given by $z = h(x, y, t)$, we obtain the normal stress condition:

$$P - 2\mu\{h_x^2 u_x + h_y^2 v_y - h_x(u_z + w_x) - h_y(v_z + w_y) \\ + h_x h_y(u_y + v_x) + w_z\}/(1 + h_x^2 + h_y^2) = P_a - \Gamma/R \qquad \text{(B.1)}$$

on $z = h$ (and $\Gamma = 0$ in the absence of surface tension; $1/R$ is defined in equation (1.32)). The two tangential stress conditions (both taken to be zero; that is, no wind) are written as

$$h_x(v_z + w_y) - h_y(u_z + w_x) + 2h_x h_y(u_x - v_y) - (h_x^2 - h_y^2)(u_y + v_x) = 0; \qquad \text{(B.2)}$$

$$2h_x^2(u_x - w_z) + 2h_y^2(v_y - w_z) + 2h_x h_y(u_y + v_x) \\ + (h_x^2 + h_y^2 - 1)\{h_x(u_z + w_x) + h_y(v_z + w_y)\} = 0, \qquad \text{(B.3)}$$

both evaluated on $z = h$. The corresponding surface conditions, written now in cylindrical coordinates, $\mathbf{x} \equiv (r, \theta, z)$ and $\mathbf{u} \equiv (u, v, w)$, are

$$P - 2\mu\left\{h_r^2 u_r + \frac{1}{r^2}h_\theta^2\left(\frac{1}{r}v_\theta + \frac{u}{r}\right) - h_r(u_z + w_r) - \frac{1}{r}h_\theta\left(v_z + \frac{1}{r}w_\theta\right)\right.$$
$$\left. + \frac{1}{r}h_\theta h_r\left[\frac{1}{r}u_\theta + r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right] + w_z\right\}\Big/\left(1 + h_r^2 + \frac{1}{r^2}h_\theta^2\right) = P_a - \Gamma/R \quad \text{(B.4)}$$

on $z = h(r, \theta, t)$, and the expression for $1/R$ is given in equation (1.34);

$$h_r\left(v_z + \frac{1}{r}w_\theta\right) - \frac{1}{r}h_\theta(u_z + w_r) + \frac{2}{r}h_\theta h_r\left\{r\frac{\partial}{\partial r}\left(\frac{u}{r}\right) - \frac{1}{r}v_\theta\right\}$$
$$- \left(h_r^2 - \frac{1}{r^2}h_\theta^2\right)\left\{\frac{1}{r}u_\theta + r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right\} = 0, \quad \text{(B.5)}$$

$$2h_r^2(u_r - w_z) + \frac{2}{r^2}h_\theta^2\left(\frac{1}{r}v_\theta + \frac{u}{r} - w_z\right) + \frac{2}{r}h_\theta h_r\left\{\frac{1}{r}u_\theta + r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right\}$$
$$+ \left(h_r^2 + \frac{1}{r^2}h_\theta^2 - 1\right)\left\{h_r\left(u_z + w_r\right) + \frac{1}{r}h_\theta\left(v_z + \frac{1}{r}w_\theta\right)\right\} = 0, \quad \text{(B.6)}$$

both on $z = h$.

The bottom boundary condition is far more easily expressed. Let the bottom boundary, $z = b(\mathbf{x}_\perp, t)$, translate with the velocity $\mathbf{u}_\perp = \mathbf{U}_\perp \equiv (U, V)$, then the viscous boundary condition is

$$\left.\begin{aligned} w &= b_t + (\mathbf{U}_\perp \cdot \nabla_\perp)b \\ \text{with} \quad \mathbf{u}_\perp &= \mathbf{U}_\perp \end{aligned}\right\} \quad \text{on } z = b. \quad \text{(B.7)}$$

Of course, if this boundary is stationary then $\mathbf{U}_\perp = \mathbf{0}$, and then if $b_t = 0$ we recover the most elementary bottom condition:

$$u = v = w = 0 \quad \text{on} \quad z = b. \quad \text{(B.8)}$$

Finally, it is clear that all the above boundary conditions reduce to those for an inviscid fluid described in Section 1.2. For $\mu = 0$, equations (B.1) and (B.4) both become equation (1.31); equations (B.2, B.3, B.5, B.6) are redundant, and equations (B.7) are just equation (1.35) (after setting $\mathbf{U}_\perp = \mathbf{u}_\perp$ where $\mathbf{u}_\perp$ is evaluated in the fluid on $z = b$).