5

Epilogue

Is this the end? Is this the end?
In Memoriam A.H.H. XII

So many worlds, so much to do,
So little done, such things to be.
In Memoriam A.H.H. LXXIII

In the earlier chapters we have described the mathematical background – and the mathematical details – of many classical linear and nonlinear water-wave phenomena. In addition, in the later chapters, we have presented many of the important and modern ideas that connect various aspects of soliton theory with the mathematical theory of water waves. However, much that is significant in the practical application of theories to real water waves – turbulence, random depth variations, wind shear, and much else – has been omitted. There are two reasons for this: first, most of these features are quite beyond the scope of an introductory text, and, second, the modelling of these types of phenomena follows a less systematic and well-understood path. Of course, that is not meant to imply that these approaches are unimportant; such studies have received much attention, and with good reason since they are essential in the design of man-made structures and in our endeavours to control nature.

What we have attempted here, in a manner that we hope makes the mathematical ideas transparent, is a description of some of the current approaches to the *theory* of water waves. To this end we have moved from the simplest models of wave propagation over stationary water of constant depth (sometimes including the effects of surface tension), to more involved problems (for example, with 'shear' or variable depth), but then only for gravity waves. It is our intention, in this short concluding chapter, to give an indication of how the effects of viscosity – the friction inherent in any flow of water – manifest themselves in our mathematical description. The approach that we adopt is based on following a rather systematic and precise route, rather than invoking any *ad hoc* modelling of the phenomena. Nevertheless, careful and wise modelling can often provide quick, neat and accurate results, even if this is possible only by a skilled practitioner. Here, we shall restrict our discussion to that of gravity waves (although, in the case of the linear theory, the application to short waves will also provide an estimate for the damping of capillary waves). We shall first examine linear harmonic waves, and obtain a measure of their damping due to the viscosity of the water. Then we shall discuss the attenuation of the solitary wave and, finally, provide two descriptions of the undular bore (which requires some viscous contribution for its existence). In all but one of these calculations we shall consider only one-dimensional (plane) surface waves moving over stationary water of constant depth.

5.1 The governing equations with viscosity

We consider plane waves that propagate in the x-direction, so our governing equations (written in original physical variables) are, from Appendix A (equations (A.2)),

$$\begin{aligned} u_t + uu_x + wu_z &= -\frac{1}{\rho}P_x + \nu(u_{xx} + u_{zz}); \\ w_t + uw_x + ww_z &= -\frac{1}{\rho}P_z - g + \nu(w_{xx} + w_{zz}); \\ u_x + w_z &= 0. \end{aligned}$$

These equations describe an incompressible fluid with a kinematic viscosity, ν. The boundary conditions (given in Appendix B) are chosen to be those relevant to a gravity wave (so $\Gamma = 0$, but see Q5.4) in the absence of any wind shear. Thus equation (B.1), the normal stress condition, gives

$$P - 2\mu\{h_x^2 u_x - h_x(u_z + w_x) + w_z\}/(1 + h_x^2) = P_a,$$

where P_a is the (constant) pressure in the atmosphere and μ is the coefficient of Newtonian viscosity; equation (B.3), one of the two tangential stress conditions, likewise gives

$$2h_x(u_x - w_z) + (h_x^2 - 1)(u_z + w_x) = 0.$$

These conditions apply on the free surface $z = h(x, t)$ (and we note that equation (B.2) is redundant for plane waves moving only in the x-direction). On the bottom (taken as $z = 0$) we use the boundary conditions

$$u = w = 0 \quad \text{on} \quad z = 0,$$

and, finally, we have the familiar kinematic condition

$$w = h_t + uh_x \quad \text{on} \quad z = h.$$

Our first task is to obtain the nondimensional version of these equations; to this end we use the scheme introduced in Section 1.3.1 (and see Q1.35), namely

$$x \to \lambda x, \quad z \to h_0 z, \quad t \to (\lambda/\sqrt{gh_0})t,$$
$$u \to \sqrt{gh_0}u, \quad w \to (h_0\sqrt{gh_0}/\lambda)w$$

with

$$h = h_0 + a\eta \quad \text{and} \quad P = P_a + \rho g(h_0 - z) + \rho g h_0 p.$$

The equations of motion then become

$$\begin{aligned} u_t + uu_x + wu_z &= -p_x + \frac{1}{\delta R}(u_{zz} + \delta^2 u_{xx}); \\ \delta^2(w_t + uw_x + ww_z) &= -p_z + \frac{\delta}{R}(w_{zz} + \delta^2 w_{xx}); \\ u_x + w_z &= 0, \end{aligned}$$

with

$$\left.\begin{aligned} p - \eta - \frac{2\delta}{R}\{w_z - \varepsilon\eta_x(u_z + \delta^2 w_x) \\ + \varepsilon^2\delta^2\eta_x^2 u_x\}/(1 + \varepsilon^2\delta^2\eta_x^2) = 0; \\ (1 - \varepsilon^2\delta^2\eta_x^2)(u_z + \delta^2 w_x) + 2\varepsilon\delta^2(w_z - u_x)\eta_x = 0; \\ w = \varepsilon(\eta_t + u\eta_x) \end{aligned}\right\} \text{on } z = 1 + \varepsilon\eta$$

and

$$u = w = 0 \quad \text{on} \quad z = 0.$$

We have introduced our familiar parameters, $\varepsilon = a/h_0$ and $\delta = h_0/\lambda$, and the Reynolds numbers is $R = h_0\sqrt{gh_0}/\nu$ (which uses only the scale length h_0 in its definition); for many problems in fluid mechanics we are interested in the case of $R \to \infty$.

The small-amplitude limit of these equations, described by $\varepsilon \to 0$, is obtained by employing the further transformation

$$(u, w, p) \to \varepsilon(u, w, p).$$

This gives the set of equations and boundary conditions

$$u_t + \varepsilon(uu_x + wu_z) = -p_x + \frac{1}{\delta R}(u_{zz} + \delta^2 u_{xx}); \tag{5.1}$$

$$\delta^2\{w_t + \varepsilon(uw_x + ww_z)\} = -p_z + \frac{\delta}{R}(w_{zz} + \delta^2 w_{xx}); \tag{5.2}$$

$$u_x + w_z = 0 \tag{5.3}$$

with

$$\left.\begin{aligned} &p - \eta - \frac{2\delta}{R}\{w_z - \varepsilon\eta_x(u_z + \delta^2 w_x) && (5.4)\\ &\qquad + \varepsilon^2\delta^2\eta_x^2 u_x\}/(1 + \varepsilon^2\delta^2\eta_x^2) = 0; \\ &(1 - \varepsilon^2\delta^2\eta_x^2)(u_z + \delta^2 w_x) + 2\varepsilon\delta^2(w_z - u_x)\eta_x = 0; && (5.5)\\ &w = \eta_t + \varepsilon u\eta_x && (5.6)\end{aligned}\right\} \text{ on } z = 1 + \varepsilon\eta$$

and

$$u = w = 0 \quad \text{on} \quad z = 0. \tag{5.7}$$

These equations, or a simple variant of them, will be discussed in the following sections, where we shall describe the construction of appropriate asymptotic solutions. (It is easily seen that these equations recover our earlier versions for one-dimensional motion when we take $R \to \infty$, with equation (5.5) now redundant and (5.7) becoming simply $w = 0$ on $z = 0$.)

5.2 Applications to the propagation of gravity waves

All the problems that we have examined in this text can, in principle at least, be re-examined with the appropriate contribution from the viscous effects included. Many of these problems rapidly become very involved indeed, so we choose to look at a few of the simpler ones (although even these, as we shall see, are considerably more complicated than their inviscid counterparts). In addition we shall also describe, via two different models, a phenomenon that requires some viscous contribution in the equations in order for an appropriate solution to exist. This is the undular bore, a special – and rather weak – version of the bore that was described using a discontinuity in Section 2.7. It turns out to be fairly straightforward to *write down* a model equation which contains the essential characteristics of the undular bore, but it is far from a routine calculation to *derive* such an equation.

5.2.1 Small amplitude harmonic waves

The first problem that we tackle is that of harmonic gravity waves moving on the surface of a stationary viscous fluid; the inviscid problem has been described in Section 2.1. From Section 5.1 we obtain the governing equations, after imposing the small-amplitude limit $\varepsilon \to 0$, in the form

$$u_t = -p_x + \frac{1}{\delta R}(u_{zz} + \delta^2 u_{xx}); \quad \delta^2 w_t = -p_z + \frac{\delta}{R}(w_{zz} + \delta^2 w_{xx}); \tag{5.8}$$

$$u_x + w_z = 0 \tag{5.9}$$

with

$$p - \eta - \frac{2\delta}{R} w_z = 0; \quad u_z + \delta^2 w_x = 0; \quad w = \eta_t \quad \text{on} \quad z = 1 \tag{5.10}$$

and

$$u = w = 0 \quad \text{on} \quad z = 0. \tag{5.11}$$

We consider here the most general *linear* problem, in that we treat the parameters δ and R as fixed (as $\varepsilon \to 0$). The solution that we seek (cf. Section 2.1) is to take the form

$$\eta = E, \quad u = U(z)E, \quad w = W(z)E, \quad p = P(z)E \tag{5.12}$$

with

$$E = \exp\{\mathrm{i}(kx - \omega t\},$$

where k is the (real) wave number; we anticipate that the presence of the terms associated with R will produce an imaginary contribution to the frequency ω. The real part of ω will – as before – give the phase speed of the gravity waves. The solution described in (5.12) has been written, for convenience, with the amplitude of η as $A = 1$ (which could be reinstated as A by writing AE for E throughout); the four expressions in (5.12) must be combined with their complex conjugates in order to produce a real solution-set.

The choice described by equations (5.12) is substituted into equations (5.8) and (5.9) to give

$$-\mathrm{i}\omega U = -\mathrm{i}kP + \frac{1}{\delta R}(U'' - \delta^2 k^2 U); \quad -\mathrm{i}\omega\delta^2 W = -P' + \frac{\delta}{R}(W'' - \delta^2 k^2 W); \tag{5.13}$$

$$\mathrm{i}kU + W' = 0. \tag{5.14}$$

The boundary conditions, (5.10), yield

$$P - 1 - \frac{2\delta}{R}W' = 0; \quad U' + \mathrm{i}k\delta^2 W = 0; \quad W = -\mathrm{i}\omega \quad \text{on} \quad z = 1 \quad (5.15)$$

and from (5.11) we obtain

$$U = W = 0 \quad \text{on} \quad z = 0. \quad (5.16)$$

The simplest manoeuvre that leads to a suitable single equation (for $W(z)$), is to substitute for U from (5.14) into the first equation of (5.13), and to differentiate this equation once with respect to z. Then P' can be eliminated between the pair of equations resulting from (5.13), to give

$$\omega(W'' - \delta^2 k^2 W) = \frac{\mathrm{i}}{\delta R}(W^{\mathrm{iv}} - 2\delta^2 k^2 W'' + \delta^4 k^4 W). \quad (5.17)$$

The boundary conditions (5.15), after using (5.14) to eliminate U and the first of (5.13) to eliminate P, give

$$\left.\begin{aligned} k^2 - \mathrm{i}\omega W' - \frac{1}{\delta R}(W''' - 3\delta^2 k^2 W') = 0; \\ W'' + \delta^2 k^2 W = 0; \quad W = -\mathrm{i}\omega \end{aligned}\right\} \text{on } z = 1, \quad (5.18)$$

and from (5.16) we have simply

$$W = W' = 0 \quad \text{on} \quad z = 0. \quad (5.19)$$

Equation (5.17), which is a linear equation for $W(z)$ with constant coefficients, has solutions of the form $W = \exp(\lambda z)$, where

$$\begin{aligned} \omega(\lambda^2 - \delta^2 k^2) &= \frac{\mathrm{i}}{\delta R}(\lambda^4 - 2\delta^2 k^2 \lambda^2 + \delta^4 k^4) \\ &= \frac{\mathrm{i}}{\delta R}(\lambda^2 - \delta^2 k^2)^2. \end{aligned}$$

Thus

$$\lambda = \pm\delta k \quad \text{or} \quad \lambda^2 = \delta^2 k^2 - \mathrm{i}\omega\delta R,$$

so the general solution can be written as

$$W(z) = A\sinh\delta kz + B\cosh\delta kz + C\sinh\mu z + D\cosh\mu z$$

for arbitrary constants A, B, C and D, with

$$\mu^2 = \delta^2 k^2 - \mathrm{i}\omega\delta R. \quad (5.20)$$

The boundary conditions on $z = 0$, (5.19), require that $D = -B$ and that $C = -A\delta k/\mu$, so

$$W = A(\sinh \delta kz - \frac{\delta k}{\mu} \sinh \mu z) + B(\cosh \delta kz - \cosh \mu z).$$

This expression for W is used in the boundary conditions on $z = 1$, (5.18); the second of these gives

$$A\left\{2\delta^2 k^2 \sinh \delta k - \frac{\delta k}{\mu}(\delta^2 k^2 + \mu^2) \sinh \mu\right\}$$
$$= -B\{2\delta^2 k^2 \cosh \delta k - (\delta^2 k^2 + \mu^2) \cosh \mu\},$$

and the third yields simply

$$A(\sinh \delta k - \frac{\delta k}{\mu} \sinh \mu) + B(\cosh \delta k - \cosh \mu) = -\mathrm{i}\omega.$$

These two equations are solved for A and B, and then, finally, the complete expression for W is used in the first boundary condition in (5.18). After $\mathrm{i}\omega$ is eliminated by using (5.20), we obtain the dispersion relation between δk, μ and R:

$$\delta k(\delta k \cosh \delta k \sinh \mu - \mu \sinh \delta k \cosh \mu)$$
$$+ \frac{1}{R^2}\{4\mu\delta^2 k^2(\mu^2 + \delta^2 k^2) + 4\mu\delta^3 k^3(\mu \sinh \delta k \sinh \mu - \delta k \cosh \delta k \cosh \mu)$$
$$+ (\mu^2 + \delta^2 k^2)^2(\delta k \sinh \delta k \sinh \mu - \mu \cosh \delta k \cosh \mu)\} = 0; \quad (5.21)$$

the details of this straightforward but rather lengthy calculation are left to the reader. (Expression (5.21), written using a slightly different notation, can be found in Kakutani & Matsuuchi (1975).) The interpretation of equation (5.21) is that it determines the complex frequency, ω (via (5.20)), for given (real) values of δk and R.

The involved nature of the dispersion relation is quite evident; indeed, even a numerical study of it is far from routine. We shall quote a few relevant observations about the (asymptotic) solutions for ω, the details of which are to be found in the exercises (Q5.1–Q5.3); the essential character of the complex frequency is presented in Figure 5.1. Here we produce a representation of both the real and imaginary parts of (ω/k), for various R, where these curves are based on the asymptotic behaviours of the solution of the dispersion relation. Figure 5.1 is intended to give only an idea of the variation of (ω/k) with δk, rather than accurate numerical estimates. What we see is that the real part of (ω/k), which is the speed of the harmonic wave, is very nearly unity for all δk not too small and R

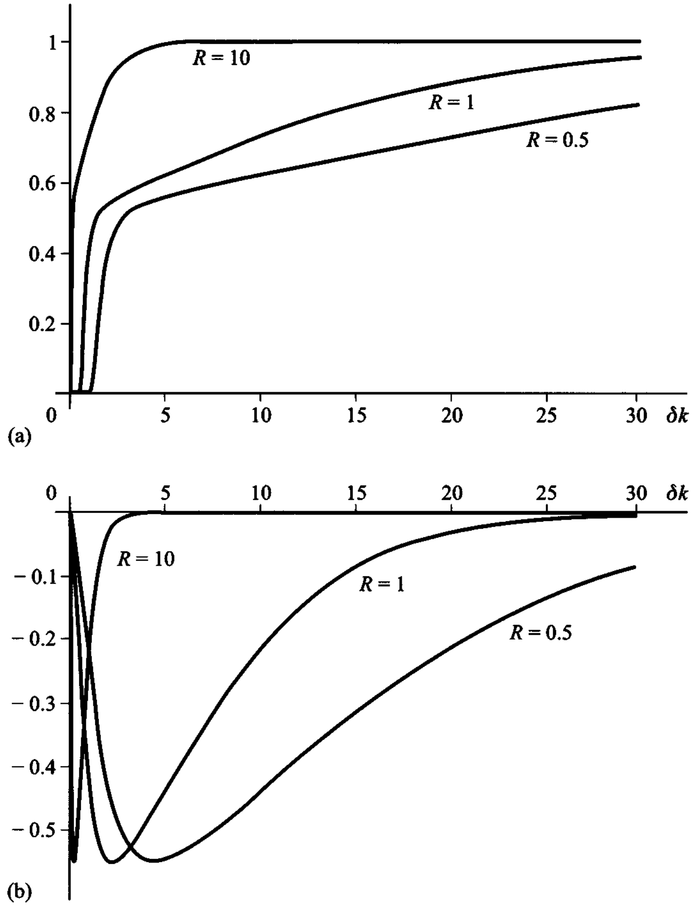

Figure 5.1. Plots of (a) the real part of ω/k and (b) the imaginary part of ω/k, for the values of Reynolds number $R = 0.5$, 1, 10, based on the asymptotic behaviours of the dispersion relation, (5.21).

increasing – and even for moderate R; see also equation (5.22). On the other hand, the damping of the wave (and it is always damped, since $\mathscr{I}m(\omega) < 0 \ \forall \delta k \neq 0$) varies quite significantly with δk, although this variation is restricted to a narrow band as R increases. For large R and δk not too small (actually the critical size is $\delta k = \mathrm{O}(1/R)$; see Q5.2), the damping is very small indeed, which again is evident in equation (5.22) below.

For conventional gravity waves, the Reynolds number (R) is typically quite large: anywhere from about 10^3 upwards, and for deep water this could be much larger. Thus the approximation of interest to us is described by $R \to \infty$; under this limiting process we find (Q5.1) that

$$\omega \sim k\left\{\sqrt{\frac{\tanh \delta k}{\delta k}} - \frac{(1+\mathrm{i})}{2\sqrt{2R}}\,\frac{(\delta k)^{1/4}}{\cosh^{5/4}\delta k \sinh^{3/4}\delta k}\right\}, \qquad (5.22)$$

where we have chosen the waves to be right-running (and hence the positive square root is taken). The leading term is our very familiar result for the propagation speed of (inviscid) gravity waves, first given in equation (2.13). The viscous contribution in (5.22), which is provided by the term in $1/\sqrt{R}$, possesses both real and imaginary parts and therefore affects the speed of the wave as well as its attenuation. The decay of the harmonic wave, in this approximation, is controlled by the negative exponent proportional to

$$(\delta k)^{5/4}\operatorname{sech}^{5/4}\delta k \operatorname{cosech}^{3/4}\delta k; \qquad (5.23)$$

this function is plotted in Figure 5.2. It is clear that long waves, described by $\delta k \to 0$, are very weakly damped, but that shorter waves (δk increasing) have much higher damping rates. (The exponential decay of the expression in (5.23), as δk increases indefinitely, is not to be relied

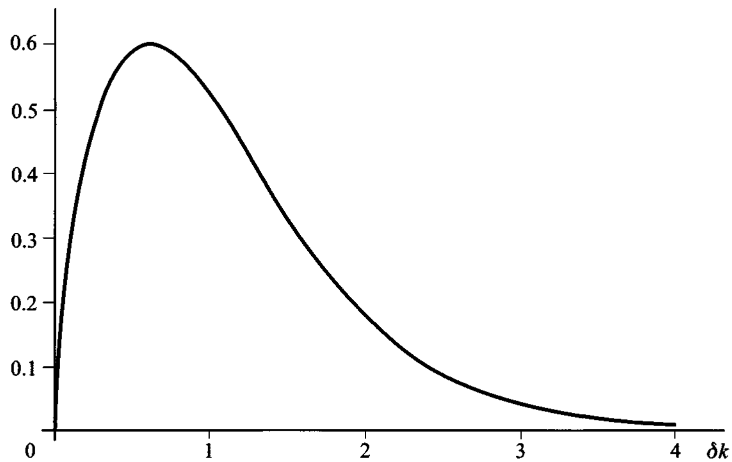

Figure 5.2. The function, (5.23), which provides the dominant contribution, for large Reynolds number, to the damping of harmonic waves.

upon, since the argument underpinning (5.22) was $R \to \infty$ at δk fixed; a different asymptotic structure appears for $\delta k \to \infty$ and, indeed, probably we should then include the surface tension contribution; see Q5.4.) The damping rate of shorter, as compared with longer, waves provides the explanation for the limited distances over which capillary waves are seen to survive, as compared with gravity waves (as we commented in Section 2.1.2). Other approximations and interpretations of the dispersion relation, (5.21), can be found in the exercises at the end of this chapter.

5.2.2 *Attenuation of the solitary wave*

In Section 2.9 we quoted Russell's description of his chase, on horseback, of a solitary wave; his evidence, and much that has been collected since his time, indicates that the solitary wave is only very weakly affected by viscosity. We shall study the way in which the viscous effects, as described by the Navier–Stokes equations, provide a slow evolution of the solitary wave. This we shall do using the *method of multiple scales*, the scales being associated with the propagation of the wave, the nonlinear evolution of the wave, and the evolution on a viscous scale.

We start with the equations given in Section 5.1, (5.1)–(5.7), but introduce the transformation which describes the scales on which a KdV-type balance occurs, as $\varepsilon \to 0$ for *arbitrary* δ; these are (cf. equations (3.10), (3.11))

$$x \to \frac{\delta}{\varepsilon^{1/2}} x, \quad t \to \frac{\delta}{\varepsilon^{1/2}} t, \quad w \to \frac{\varepsilon^{1/2}}{\delta} w.$$

The equations are therefore

$$\begin{aligned} u_t + \varepsilon(uu_x + wu_z) &= -p_x + \frac{1}{R\sqrt{\varepsilon}}(u_{zz} + \varepsilon u_{xx}); \\ \varepsilon\{w_t + \varepsilon(uw_x + ww_z)\} &= -p_z + \frac{\sqrt{\varepsilon}}{R}(w_{zz} + \varepsilon w_{xx}); \\ u_x + w_z &= 0, \end{aligned}$$

with

$$\left.\begin{aligned} p - \eta - 2\frac{\sqrt{\varepsilon}}{R}\{w_z - \varepsilon\eta_x(u_z + \varepsilon w_x) + \varepsilon^3\eta_x^2 u_x\}/(1 + \varepsilon^3\eta_x^2) = 0; \\ (1 - \varepsilon^3\eta_x^2)(u_z + \varepsilon w_x) + 2\varepsilon^2(w_z - u_x)\eta_x = 0; \\ w = \eta_t + \varepsilon u\eta_x, \end{aligned}\right\} \text{ on } z = 1 + \varepsilon\eta$$

and

$$u = w = 0 \quad \text{on} \quad z = 0.$$

Now, in the derivation of the KdV equation, we introduced the variables

$$\xi = x - t, \qquad \tau = \varepsilon t;$$

see equations (3.17) and (3.18). Here we follow essentially the same route, but include evolution on a suitable (slow) viscous scale and also allow the nonlinear contribution to the speed of the wave to vary on this same scale. (We have already seen that the speed of harmonic waves is altered by the presence of a viscous ingredient; see equation (5.22).) Thus, anticipating a KdV-type of equation with independent variables τ and ξ in the absence of viscosity, we introduce a slow evolution of this system in the form

$$\tau = \varepsilon t, \quad T = \Delta\tau = \varepsilon\Delta t, \quad \xi = x - t - \frac{1}{\Delta}\int_0^T c(T')\mathrm{d}T', \tag{5.24}$$

where we shall treat ξ, τ and T as independent variables (the method of multiple scales), and where Δ is yet to be chosen. Different problems require different choices of Δ, in terms of

$$\varepsilon\ (\to 0) \quad \text{and} \quad r = 1/R\sqrt{\varepsilon}\ (\to 0),$$

which we treat as independent parameters. Under this transformation our governing equations become

$$\varepsilon u_\tau + \varepsilon\Delta u_T - (1 + \varepsilon c)u_\xi + \varepsilon(uu_\xi + wu_z) = -p_\xi + r(u_{zz} + \varepsilon u_{\xi\xi}); \tag{5.25}$$

$$\varepsilon\{\varepsilon w_\tau + \varepsilon\Delta w_T - (1 + \varepsilon c)w_\xi + \varepsilon(uw_\xi + ww_z)\} = -p_z + \varepsilon r(w_{zz} + \varepsilon w_{\xi\xi}); \tag{5.26}$$

$$u_\xi + w_z = 0, \tag{5.27}$$

with

$$\left.\begin{aligned} p - \eta - 2\varepsilon r\{w_z - \varepsilon\eta_\xi(u_z + \varepsilon w_\xi) + \varepsilon^3\eta_\xi^2 u_\xi\}/(1 + \varepsilon^3\eta_\xi^2) = 0; \\ (1 - \varepsilon^3\eta_\xi^2)(u_z + \varepsilon w_\xi) + 2\varepsilon^2(w_z - u_\xi)\eta_\xi = 0; \\ w = \varepsilon\eta_\tau + \varepsilon\Delta\eta_T - (1 + \varepsilon c)\eta_\xi + \varepsilon u\eta_\xi, \end{aligned}\right\} \text{on } z = 1 + \varepsilon\eta \tag{5.28}$$

and

$$u = w = 0 \quad \text{on} \quad z = 0. \tag{5.29}$$

We seek an asymptotic solution of these equations in the form

$$q \sim \sum_{n=0}^{\infty} \varepsilon^n q_n(\xi, \tau, T, z; \Delta, r), \quad \eta \sim \sum_{n=0}^{\infty} \varepsilon^n \eta_n(\xi, \tau, T; \Delta, r),$$

for $\varepsilon \to 0$, where q (and correspondingly q_n) represents u, w and p. Each function q_n and η_n ($n = 0, 1, 2, \ldots$) is, in turn, regarded as possessing an appropriate asymptotic representation as $r \to 0$, $\Delta \to 0$; this is equivalent to seeking a multiple asymptotic expansion in terms of, for example, the asymptotic sequence $\{\varepsilon^n r^m\}$, $n = 0, 1, 2, \ldots$, for a suitable set of values of m and some chosen $\Delta(\varepsilon, r)$. Further, special problems can always be posed for any choice $\Delta = \Delta(\varepsilon)$ and $r = r(\varepsilon)$; that is, $R = R(\varepsilon)$. On physical grounds, such a procedure could be criticised since ε and R are clearly *independent* parameters; however, some of the mathematical problems that are generated in this way enable us to obtain some insight into the structure of these equations and their solutions. We shall comment on this again later, but we note here that an *ab initio* choice of $R = R(\varepsilon)$ reduces the problem to an expansion in one parameter – say ε – only. To proceed, the method of solution that we follow here is, in its general outline, that employed for the derivation of the Korteweg–de Vries equation (as described in Section 3.2.1).

The leading-order equations, as $\varepsilon \to 0$, obtained from equations (5.25)–(5.27), are

$$-u_{0\xi} = -p_{0\xi} + r u_{0zz}; \quad p_{0z} = 0; \quad u_{0\xi} + w_{0z} = 0. \tag{5.30}$$

The boundary conditions, from (5.28) and (5.29), yield

$$p_0 = \eta_0; \quad u_{0z} = 0; \quad w_0 = -\eta_{0\xi} \quad \text{on} \quad z = 1 \tag{5.31}$$

and

$$u_0 = w_0 = 0 \quad \text{on} \quad z = 0. \tag{5.32}$$

It is clear that equations (5.30), for $r \to 0$, possess a solution which admits a *boundary layer*, presumably near $z = 0$ in the light of the no-slip boundary condition on $z = 0$; see Q5.5 and Q5.6. (We might expect a boundary layer to be required also near $z = 1$, in order to accommodate the shear stress condition there. However, as we shall see, the problem of no wind shear does not give rise to a surface boundary layer at the order of approximation to which we shall be working.) We therefore seek, in the first instance, a solution of equations (5.30)–(5.32), in the limit r $\to 0$ but valid away from the boundary layer near $z = 0$. This first approximation in r is denoted by an additional zero suffix, so we obtain

$$u_{00\xi} = p_{00\xi}; \quad p_{00z} = 0; \quad u_{00\xi} + w_{00z} = 0,$$

with

$$p_{00} = \eta_{00}; \quad u_{00z} = 0; \quad w_{00} = -\eta_{00\xi} \quad \text{on} \quad z = 1.$$

This set produces the familiar solution (see Section 3.2.1)

$$p_{00} = \eta_{00}; \quad u_{00} = \eta_{00}; \quad w_{00} = -z\eta_{00\xi}, \quad 0 < z \leq 1, \tag{5.33}$$

which satisfies the shear stress condition ($u_{00z} = 0$) on $z = 1$, but which cannot satisfy the bottom boundary condition ($u_{00} = 0$ on $z = 0$). Solution (5.33), when previously derived for the KdV equation, was valid for $0 \leq z \leq 1$; here it is not valid near $z = 0$, although both p_{00} and w_{00} (at this order) would appear to be uniformly valid on [0,1]. In fact solution (5.33) satisfies the *full* equations valid away from the boundary layer; see Q5.7.

The equations that define the O(ε) problem, from equations (5.25)–(5.29), are

$$u_{0\tau} + \Delta u_{0T} - u_{1\xi} - cu_{0\xi} + u_0 u_{0\xi} + w_0 u_{0z} = -p_{1\xi} + r(u_{1zz} + u_{0\xi\xi});$$
$$-w_{0\xi} = -p_{1z} + rw_{0zz}; \quad u_{1\xi} + w_{1z} = 0,$$

with

$$\left.\begin{array}{l} p_1 + \eta_0 p_{0z} - \eta_1 - 2rw_{0z} = 0; \quad u_{1z} + w_{0\xi} = 0 \\ w_1 + \eta_0 w_{0z} = \eta_{0\tau} + \Delta\eta_{0T} - \eta_{1\xi} - c\eta_{0\xi} + u_0\eta_{0\xi} \end{array}\right\} \text{ on } z = 1$$

and

$$u_1 = w_1 = 0 \quad \text{on} \quad z = 0.$$

This time we start by retaining the terms in r and Δ (cf. Q5.7), but we use the solution previously found, which is valid outside the boundary layer; in particular, we note that

$$u_{0z} = 0; \quad p_{0z} = 0; \quad w_{0z} = -\eta_{0\xi},$$

since $(u_0, p_0, \eta_0) \sim (u_{00}, p_{00}, \eta_{00})$ to all (algebraic) orders in r, as we mentioned above. Thus we obtain the equations

$$\eta_{0\tau} + \Delta\eta_{0T} - u_{1\xi} - c\eta_{0\xi} + \eta_{0\xi} + \eta_0\eta_{0\xi} = -p_{1\xi} + r(u_{1zz} + \eta_{0\xi\xi}); \tag{5.34}$$

$$p_{1z} = -z\eta_{0\xi}; \quad u_{1\xi} + w_{1z} = 0 \tag{5.35}$$

with

$$\left.\begin{aligned} p_1 = \eta_1 - 2r\eta_{0\xi}; \quad u_{1z} - \eta_{0\xi\xi} = 0; \\ w_1 - \eta_0\eta_{0\xi} = \eta_{0\tau} + \Delta\eta_{0T} - \eta_{1\xi} - c\eta_{0\xi} + \eta_0\eta_{0\xi} \end{aligned}\right\} \text{ on } z = 1, \qquad (5.36)$$

for the problem outside the boundary layer. We obtain directly from equations (5.35) and (5.36) (cf. equation (3.26))

$$p_1 = \frac{1}{2}(1 - z^2)\eta_{0\xi\xi} + \eta_1 - 2r\eta_{0\xi}$$

and then from equation (5.34)

$$\begin{aligned} &\eta_{0\tau} + \Delta\eta_{0T} - c\eta_{0\xi} + \eta_0\eta_{0\xi} + w_{1z} \\ &\qquad = -\eta_{1\xi} + 2r\eta_{0\xi\xi} - \frac{1}{2}(1 - z^2)\eta_{0\xi\xi\xi} + r(u_{1zz} + \eta_{0\xi\xi}) \end{aligned}$$

so that

$$\begin{aligned} w_1 = (c\eta_{0\xi} - \eta_{0\tau} - \Delta\eta_{0T} - \eta_0\eta_{0\xi} + 3r\eta_{0\xi\xi} - \frac{1}{2}\eta_{0\xi\xi\xi} - \eta_{1\xi})z \\ + \frac{1}{6}z^3\eta_{0\xi\xi\xi} + ru_{1z} + f_0(\xi, \tau, T; \Delta, r) \quad (5.37) \end{aligned}$$

where f_0 is an arbitrary function of integration. Finally, the kinematic condition on $z = 1$, in (5.36), yields

$$\begin{aligned} &c\eta_{0\xi} - \eta_{0\tau} - \Delta\eta_{0T} - \eta_0\eta_{0\xi} + 4r\eta_{0\xi\xi} - \frac{1}{3}\eta_{0\xi\xi\xi} - \eta_{1\xi} + f_0 - \eta_0\eta_{0\xi} \\ &\qquad = \eta_{0\tau} + \Delta\eta_{0T} - \eta_{1\xi} - c\eta_{0\xi} + \eta_0\eta_{0\xi}, \end{aligned}$$

or

$$2(\eta_{0\tau} + \Delta_{0T} - c\eta_{0\xi}) + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} + 4r\eta_{0\xi\xi} = f_0, \qquad (5.38)$$

which is to be compared with our conventional KdV equation (3.28):

$$2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = 0.$$

It is left as a simple exercise to confirm that the surface shear stress condition (in (5.36)) is automatically satisfied since, away from $z = 0$, $u_{1zz} + \eta_{0\xi\xi} = 0$ (to within exponentially small terms as $r \to 0$); see Q5.8.

The final stage of this calculation involves the construction of the solution in the boundary layer and hence, via matching, the determination of the function $f_0(\xi, \tau, T; \Delta, r)$. Once this is done we may return to our KdV-type equation, (5.38), and consider the size and rôle of the various new terms that have appeared. The boundary layer, as is evident from equations (5.30), is in the region defined by $z = \mathrm{O}(r^{1/2})$, $r \to 0$, and then $w_0 = \mathrm{O}(r^{1/2})$ (from the equation of mass conservation in (5.30)). Thus we introduce new variables

$$z = r^{1/2}Z, \quad w_0 = r^{1/2}W_0(\xi, \tau, T, Z; r) \tag{5.39}$$

and, correspondingly,

$$u_0 = U_0(\xi, \tau, T, Z; r), \quad p_0 = P_0(\xi, \tau, T, Z; r);$$

of course, η_0 is unchanged since it is not a function of z. Equations (5.30) and (5.32) therefore become

$$-U_{0\xi} = -P_{0\xi} + U_{0ZZ}; \quad P_{0Z} = 0; \quad U_{0\xi} + W_{0Z} = 0 \tag{5.40}$$

with

$$U_0 = W_0 = 0 \quad \text{on} \quad Z = 0.$$

We see immediately that

$$P_0 = \eta_0, \quad Z \geq 0,$$

in order to match to the solution $p_0 = \eta_0$ (which merely restates the uniform validity that we have previously noted). The equation for U_0 then becomes

$$\left.\begin{aligned} &U_{0ZZ} + U_{0\xi} = \eta_{0\xi}, \\ \text{with} \quad & \\ &U_0 = 0 \text{ on } Z = 0; \quad U_0 \to \eta_0 \text{ as } Z \to \infty, \end{aligned}\right\} \tag{5.41}$$

this latter condition ensuring that U_0 and $u_0\,(= \eta_0)$ match.

The problem posed in (5.41) is conveniently reformulated by writing

$$U_0 = \eta_0 + \mathscr{U}_0 \quad \text{and} \quad \zeta = -\xi$$

to give

$$\mathscr{U}_{0ZZ} = \mathscr{U}_{0\zeta}$$

with

$$\mathscr{U}_0 = -\eta_0 \text{ on } Z = 0; \quad \mathscr{U}_0 \to 0 \text{ as } Z \to \infty.$$

When we set

$$\eta_0(\xi, \tau, T) = \eta_0(-\zeta, \tau, T) = -H_0(\zeta, \tau, T),$$

the solution (following Duhamel's method, Q5.9) can be written as

$$\mathscr{U}_0 = \frac{2}{\sqrt{\pi}} \int_0^\infty H_0\left(\zeta - \frac{Z^2}{4y^2}, \tau, T\right) \exp(-y^2)\,\mathrm{d}y, \tag{5.42}$$

so

$$U_0 = \eta_0 - \frac{2}{\sqrt{\pi}} \int_0^\infty \eta_0\left(\xi + \frac{Z^2}{4y^2}, \tau, T\right) \exp(-y^2)\,\mathrm{d}y. \tag{5.43}$$

So that we can match, we require the solution for W_0 which, from equations (5.40) and (5.43) becomes

$$W_0 = -Z\eta_{0\xi} + \frac{2}{\sqrt{\pi}} \int_0^Z \left\{ \int_0^\infty \eta_{0\xi}\left(\xi + \frac{Z^2}{4y^2}, \tau, T\right) \exp(-y^2)\,\mathrm{d}y \right\} \mathrm{d}Z, \tag{5.44}$$

satisfying $W_0 = 0$ on $Z = 0$. The matching is then between (5.44) as $Z \to \infty$, and

$$w \sim w_0 + \varepsilon w_1 \tag{5.45}$$

as $z \to 0$; in particular, written in boundary-layer variables, $z = r^{1/2} Z$ and $w = r^{1/2} W$, (5.45) becomes (from (5.33) and (5.37))

$$W \sim -Z\eta_{0\xi} + \frac{\varepsilon}{r^{1/2}} f_0$$

so matching to (5.44), as $Z \to \infty$, requires

$$\frac{\varepsilon}{r^{1/2}} f_0 = \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \eta_{0\xi}\left(\xi + \frac{Z^2}{4y^2}, \tau, T\right) \exp(-y^2)\,\mathrm{d}y\,\mathrm{d}Z. \tag{5.46}$$

(Notice that the first term, $-Z\eta_{0\xi}$, automatically matches, confirming the uniform validity of the solution w_0.) The appearance of ε in the definition of f_0, through (5.46), is not consistent with our formulation (since we have already expanded in terms of ε^n). This is simply telling us that a *precise* balance of terms will require a choice $r = r(\varepsilon)$, and then the calculation repeated with this choice in place; we shall write more of this shortly. It is left as an exercise (Q5.10) to demonstrate that, from equation (5.46), we may write

$$f_0 = \frac{1}{\varepsilon} \sqrt{\frac{r}{\pi}} \int_\xi^\infty \eta_{0\xi'}(\xi', \tau, T) \frac{\mathrm{d}\xi'}{\sqrt{\xi' - \xi}}. \tag{5.47}$$

Thus, finally, we have a KdV-type equation which incorporates the dominant effects of (laminar) viscosity, provided $r \to 0$ (that is, if the Reynolds number is large enough); our equation is, from (5.38),

$$2(\eta_{0\tau} + \Delta\eta_{0T} - c\eta_{0\xi}) + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = \frac{1}{\varepsilon}\sqrt{\frac{r}{\pi}}\int_{\xi}^{\infty} \eta_{0\xi'}\frac{\mathrm{d}\xi'}{\sqrt{\xi' - \xi}}. \quad (5.48)$$

Here, we have retained only the dominant terms associated with $\Delta\,(\to 0)$ and $r\,(\to 0)$; clearly the term $4r\eta_{0\xi\xi}$ (in equation (5.38)) is much smaller, as $r \to 0$, than that associated with $\sqrt{r}/\varepsilon$ (although the term $\eta_{0\xi\xi}$ will figure in a later calculation). A number of different and important choices can be made that describe diverse problems, each leading to an appropriate balance of terms; we shall return to equation (5.48) in the next section, but let us here examine the problem we first posed: the slow modulation of the solitary wave.

The solitary wave, in the absence of any modulation, is a steady solution of the Korteweg–de Vries equation (that is, $\eta_0 = \eta_0(\xi)$ only), so

$$-2c\eta_{0\xi} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = 0,$$

with the solution

$$\eta_0 = 2c\,\mathrm{sech}^2\left(\xi\sqrt{\frac{3c}{2}}\right).$$

However, we now incorporate a slow modulation of this solution, on the scale T, by virtue of the weak viscous contribution. Thus we choose $\Delta = \sqrt{r}/\varepsilon$, and hence obtain

$$-2c\eta_{0\xi} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = \Delta\left\{-2\eta_{0T} + \frac{1}{\sqrt{\pi}}\int_{\xi}^{\infty} \eta_{0\xi'}\frac{\mathrm{d}\xi'}{\sqrt{\xi' - \xi}}\right\}, \quad (5.49)$$

with $\Delta \to 0$ and where

$$\eta_0 \sim 2c\,\mathrm{sech}^2\left(\xi\sqrt{\frac{3c}{2}}\right), \quad c = c(T). \quad (5.50)$$

The model that we have in mind here is represented in Figure 5.3. The solitary wave is moving into stationary water and as it does so a (thin) boundary layer is initiated near the front of the wave. This boundary layer then grows behind the solitary wave; in the frame at rest relative to

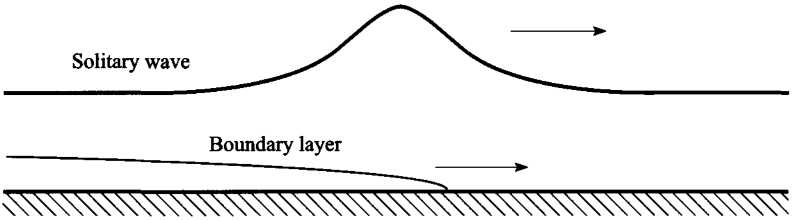

Figure 5.3. Sketch of a solitary wave moving into stationary water with a viscous boundary layer (on the bottom) growing back from the front of the wave.

the wave, the flow is from right to left and the boundary layer is stationary in this frame, and growing to the left.

The most direct way to obtain the details of the modulation is to invoke the condition

$$\eta_0 \to 0 \quad \text{as} \quad |\xi| \to \infty$$

and to form the integral over all ξ of equation (5.49), to give

$$[-2c\eta_0 + \frac{3}{2}\eta_0^2 + \frac{1}{3}\eta_{0\xi\xi}]_{-\infty}^{\infty} = \Delta\left\{-2\frac{\mathrm{d}}{\mathrm{d}T}\int_{-\infty}^{\infty}\eta_0 \mathrm{d}\xi + \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\int_{\xi}^{\infty}\eta_{0\xi'}\frac{\mathrm{d}\xi'}{\sqrt{\xi'-\xi}}\mathrm{d}\xi\right\}$$

and then

$$2\frac{\mathrm{d}}{\mathrm{d}T}\int_{-\infty}^{\infty}\eta_0 \mathrm{d}\xi = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\int_{\xi}^{\infty}\eta_{0\xi'}\frac{\mathrm{d}\xi'}{\sqrt{\xi'-\xi}}\mathrm{d}\xi.$$

This identity provides an equation for $c(T)$, which can be obtained by introducing (5.50) to give

$$2\frac{\mathrm{d}}{\mathrm{d}T}\left\{2c\int_{-\infty}^{\infty}\operatorname{sech}^2\left(\xi\sqrt{\frac{3c}{2}}\right)\mathrm{d}\xi\right\} = -4c\sqrt{\frac{3c}{2\pi}}\int_{-\infty}^{\infty}\int_{\xi}^{\infty}\operatorname{sech}^2\left(\xi'\sqrt{\frac{3c}{2}}\right)\tanh\left(\xi'\sqrt{\frac{3c}{2}}\right)\frac{\mathrm{d}\xi'}{\sqrt{\xi'-\xi}}\mathrm{d}\xi.$$

which can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}T}\left\{c\sqrt{\frac{2}{3c}}\int_{-\infty}^{\infty}\operatorname{sech}^2 y\,\mathrm{d}y\right\}$$
$$= -\frac{c}{\sqrt{\pi}}\left(\frac{2}{3c}\right)^{1/4}\int_{-\infty}^{\infty}\int_{y}^{\infty}\operatorname{sech}^2 y'\tanh y'\frac{\mathrm{d}y'}{\sqrt{y'-y}}\mathrm{d}y. \quad (5.51)$$

The precise values of the constants that appear here are not particularly significant; the form of equation (5.51) is simply

$$\frac{\mathrm{d}}{\mathrm{d}T}(\sqrt{c}) = -2\mu c^{3/4} \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}T}(c^{-1/4}) = \mu,$$

where $\mu\,(>0)$ is a constant (whose value turns out to be approximately 0.08), and hence

$$\frac{c}{c_0} = (1+\mu c_0^{1/4}T)^{-4}, \quad T \geq 0, \quad (5.52)$$

where $c = c_0$ at $T = 0$. Equation (5.52) is our main result here (first obtained by Keulegan (1948)); it describes the attenuation of the amplitude of the solitary wave, and there is some experimental evidence to suggest that its general form is not too wide of the mark. Certainly we must not expect close agreement, mainly because our simple theory does not even attempt to represent a (probably) turbulent flow moving over a rough bed. Further discussion of results of this type can be found in some of the references at the end of this chapter.

We shall return to the KdV equation, with its viscous contribution as represented in (5.48), in the next section, but first we must describe another phenomenon in water waves: the undular bore.

5.2.3 Undular bore – model I

In Section 2.7 we introduced and discussed the hydraulic jump, as well as its counterpart, which moves relative to the physical frame: the bore. These phenomena were modelled as a discontinuity, although in reality there is usually a fairly narrow region over which the flow properties change markedly. This transition is observed to occur through a region of highly turbulent motion, which takes the form of a continually breaking wave. However, a river flow can sometimes support a change of flow properties that is far more gradual, without – or almost without – any sign of extensive turbulence. This happens if the change in levels is not too great; then it is often observed that behind the *smooth* transition there is a train of waves. This phenomenon is called the *undular bore*; see Figure 5.4. The interpretation of what is seen is that, rather than a considerable dissipation of energy at the front (as in the bore), the undular bore structure allows all (or most) of the energy loss to occur by transporting the energy away in the wave motion. We can expect this to occur when the amount of energy to be lost is quite small – so we have a 'weak' bore; a model for the energy loss can then be provided by a fairly small amount of (laminar) viscous dissipation. This is the essential idea behind the model for the undular bore that we describe here and, under slightly different assumptions, in the next section. Furthermore, we anticipate that the surface wave itself is a nonlinear object, so the oscillatory part of the profile is also likely to be nonlinear: for example, a cnoidal wave (discussed in Q2.67). Thus we look for a KdV-type of equation, which incorporates some appropriate viscous contribution – but this is precisely what we did in the previous section.

The calculation that produces our governing equation is not repeated here. It is precisely that described in Section 5.2.2, except that now we do not require the modulational ingredient (which was required in order to discuss the evolution of the solitary wave). Thus we dispense with the scale T and with $c(T)$, which were introduced in equations (5.24): we use only ξ and τ. Further, because our aim here is not to develop a slow modulation, the most convenient approach is to make the special choice $r = \mathrm{O}(\varepsilon^2)$ (so that the Reynolds number is such that $R^{-1} = \mathrm{O}(\varepsilon^{5/2})$). The problem now involves the single parameter ε, for $\varepsilon \to 0$, and it is then a simple exercise to confirm that our previous calculation goes through, resulting in the equation for the surface wave:

$$2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = \frac{1}{\sqrt{\pi\mathscr{R}}}\int_{\xi}^{\infty} \eta_{0\xi'}(\xi', \tau)\,\frac{\mathrm{d}\xi'}{\sqrt{\xi' - \xi}}. \tag{5.53}$$

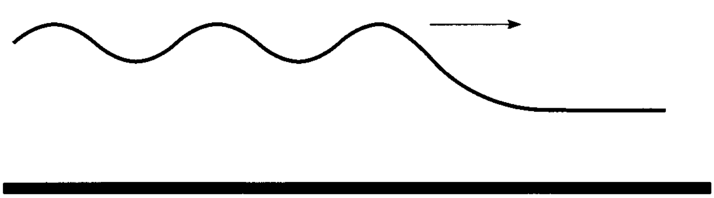

Figure 5.4. A sketch of the undular bore.

We have written $r = \varepsilon^2/\mathscr{R}$ (that is, $R = \varepsilon^{-5/2}\mathscr{R}$), and otherwise we have quoted from equation (5.48). Equation (5.53), or variants of it, have been obtained by Ott & Sudan (1970), Byatt-Smith (1971) and Kakutani & Matsuuchi (1975); the work of Byatt-Smith, in particular, is directed towards a description of the undular bore.

The equation for $\eta_0(\xi, \tau)$ represents the action of a thin viscous boundary layer – remember that $R^{-1} = O(\varepsilon^{5/2})$ as $\varepsilon \to 0$ – which grows from near the wavefront; this is the mechanism which provides the dissipation of energy. Now, provided we restrict attention to regions not too far behind the front, we may seek steady solutions of equation (5.53). Clearly, far enough behind the front, the boundary layer will have grown sufficiently large that it can no longer be treated as thin: the boundary layer will then interact with, and disrupt, the surface wave. When this happens we shall not be able to sustain a steady solution. With this caveat in mind, Byatt-Smith (1971) discusses the nature of the steady solution given by

$$-2c\eta_0' + 3\eta_0\eta_0' + \frac{1}{3}\eta_0''' = \frac{1}{\sqrt{\pi\mathscr{R}}}\int_{\xi}^{\infty} \eta_0'(\xi' - c\tau)\frac{d\xi'}{\sqrt{\xi' - \xi}},$$

where $\eta_0 = \eta_0(\xi - c\tau)$ and the prime on η_0 denotes the derivative with respect to $(\xi - c\tau)$. It is convenient to rewrite the integral with

$$\xi' = \xi + \zeta'$$

and then to set $\xi - c\tau = \zeta$; this yields

$$-2c\eta_0' + 3\eta_0\eta_0' + \frac{1}{3}\eta_0''' = \frac{1}{\sqrt{\pi\mathscr{R}}}\int_{0}^{\infty} \eta_0'(\zeta + \zeta')\frac{d\zeta'}{\sqrt{\zeta'}},$$

which is integrated once with respect to ζ, with the condition

$$\eta_0 \to 0 \quad \text{as} \quad \zeta \to +\infty.$$

Thus we obtain a nonlinear, ordinary integro-differential equation

$$-2c\eta_0 + \frac{3}{2}\eta_0^2 + \frac{1}{3}\eta_0'' = \frac{1}{\sqrt{\pi\mathscr{R}}}\int_{0}^{\infty} \eta_0(\zeta + \zeta')\frac{d\zeta'}{\sqrt{\zeta'}}, \tag{5.54}$$

which describes steady solutions $\eta_0(\zeta)$, for various $\mathscr{R}$ and c. Equation (5.54) was integrated numerically by Byatt-Smith, for quite large values of $\mathscr{R}$ ($= 10^5/\pi$ and $10^6/\pi$) and two values of c. (Our equation (5.54), although not identical to that derived by Byatt-Smith, is precisely equivalent to it.) An example of the form of solution (5.54) is shown in Figure 5.5, which makes clear that the essential character of the undular bore is recovered. A detailed numerical integration of this equation shows that: (a) the amplitude of the waves increases as c increases (completely consistent with the nonlinear character of solutions of the KdV equation); (b) the period of the oscillation increases as $\mathscr{R}$ increases.

There can be no doubt that equation (5.53), and then equation (5.54) for steady solutions, embody a mechanism which would appear to provide a perfectly reasonable model of the undular bore. However, there are some features of this approach to the problem which, although not of great significance taken individually, add up to a slightly unsatisfactory description. This model uses a well-structured and continually evolving boundary layer on $z = 0$, but a more realistic river flow is likely to have such a boundary layer completely disrupted (by an uneven bottom, for example). Indeed, we might expect that some dissipation – perhaps the major contribution – occurs near the front and that any further energy loss in the flow behind is insignificant; the excess is still propagated away. The boundary layer, as we have already commented, necessarily produces an unsteady profile (which, certainly, is not an important consideration for large $\mathscr{R}$). Nevertheless, a model that admits a completely steady solution (on the scales employed) would be a slight improvement. Finally, the equation itself, (5.53) or (5.54), is a nonlinear

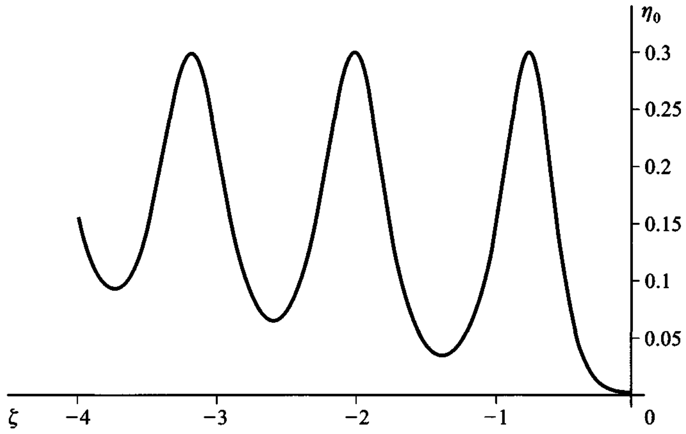

Figure 5.5. A numerical solution of equation (5.54), for $R = 10^4/\pi$ and $c = 0.1$.

integro-differential equation which is therefore not readily analysed; a simpler equation (which still embodies the relevant physics) would be an advantage. Thus, if we can find a model that addresses most of these points, we will have produced a useful alternative description of the undular bore. Of course, in the context of the theory of water waves, this might also prove to be an instructive mathematical exercise.

5.2.4 Undular bore – model II

It is surprisingly simple to *write down* an equation that should contain the essential features seen in the undular bore. This equation is to admit solutions that describe a smooth transition from one depth to another (like the Burgers equation), together with an oscillatory (dispersive) wave; see Q1.55 and Q2.67. Such an equation might take the form

$$u_t + uu_x + u_{xxx} = u_{xx}, \tag{5.55}$$

where we have set all coefficients to unity. But, no matter how attractive this appears, it must be treated as useful only if it can be shown to arise (from the relevant governing equations) under some consistent limiting process. This is what we shall now demonstrate, and then we present a brief discussion of the solutions of the resulting equation (which *is* essentially (5.55)).

The governing equations that we start from here are those given in Section 5.1 (and then as transformed in Section 5.2.2 to remove the parameter δ) but with an important addition. The undisturbed flow is no longer stationary; it is a fully developed Poiseuille channel flow moving under gravity, and so we introduce gravity components $(g\sin\alpha, -g\cos\alpha)$ and replace εu by $U(z) + \varepsilon u$; see Figure 5.6. The resulting equations then become

$$u_t + (U + \varepsilon u)u_x + (U' + \varepsilon u_z)w = -p_x + \beta + \frac{1}{R\varepsilon\sqrt{\varepsilon}}(U'' + \varepsilon u_{zz} + \varepsilon^2 u_{xx}); \tag{5.56}$$

$$\varepsilon\{w_t + (U + \varepsilon u)w_x + \varepsilon w w_z\} = -p_z + \frac{\sqrt{\varepsilon}}{R}(w_{zz} + \varepsilon w_{xx}); \tag{5.57}$$

$$u_x + w_z = 0, \tag{5.58}$$

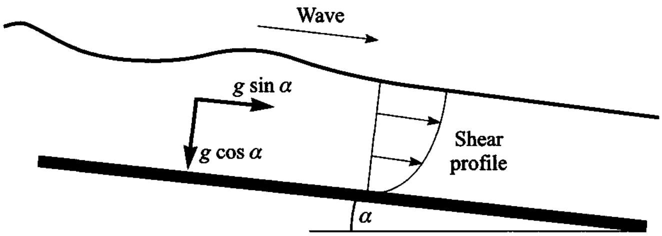

Figure 5.6. A sketch of the fully developed (Poiseuille) velocity profile for the flow moving under gravity, and the surface wave.

with

$$\left.\begin{aligned} p - \eta - \frac{2\sqrt{\varepsilon}}{R}\{w_z - \eta_x(U' + \varepsilon u_z + \varepsilon^2 w_x) \\ + \varepsilon^3 \eta_x^2 u_x\}/(1 + \varepsilon^3 \eta_x^2) = 0; \\ (1 - \varepsilon^3 \eta_x^2)(U' + \varepsilon u_z + \varepsilon^2 w_x) + 2\varepsilon^3(w_z - u_x)\eta_x = 0; \\ w = \eta_t + (U + \varepsilon u)\eta_x, \end{aligned}\right\} \quad \text{on } z = 1 + \varepsilon\eta \tag{5.59}$$

and

$$U + \varepsilon u = 0, \quad w = 0 \quad \text{on} \quad z = 0. \tag{5.60}$$

The constant β is defined as

$$\beta = \frac{\tan\alpha}{\varepsilon\sqrt{\varepsilon}}, \tag{5.61}$$

and this expresses the required component of gravity down the channel which is needed in order to maintain the flow $U(z)$. In the absence of any surface wave, the equations (5.56)–(5.60) become simply

$$\beta + \frac{U''}{R\varepsilon\sqrt{\varepsilon}} = 0; \quad U'(1) = 0; \quad U(0) = 0,$$

so we have

$$U(z) = U_0(2z - z^2), \tag{5.62}$$

the Poiseuille profile, where

$$2U_0 = R\beta\varepsilon\sqrt{\varepsilon} = R\tan\alpha,$$

and we treat U_0 (which is essentially a Froude number for this flow) as O(1). This choice of U_0 obviously gives $U(z) = \mathrm{O}(1)$, and provides the balance between R and α which is required to produce the fully developed flow at leading order.

For sufficiently large Reynolds number, R, the equations (5.56)–(5.60) admit solutions which represent waves that move at speeds determined by the Burns condition, when $U(z)$ is given by (5.62), and also nonlinear waves that move over this shear flow; see Sections 3.4.1 and 3.4.2. However, our governing equations here contain a viscous contribution, and this enables another type of wave to exist. To see how this arises, let us initially transform

$$X = \Delta x, \quad T = \Delta t, \quad w = \Delta W, \tag{5.63}$$

and take the limiting process $\Delta \to 0$, at fixed ε and R, with $U(z)$ given by (5.62). (The wave that we are about to describe exists even for $\Delta = O(1)$, but it is easier to see the appropriate balance with $\Delta \to 0$.) The leading equations, as $\Delta \to 0$, are then

$$\left.\begin{array}{l} u_{zz} = 0; \quad p_z = 0; \quad u_X + W_z = 0, \\ \text{with} \\ p = \eta; \quad u_z + U''\eta = 0; \quad W = \eta_T + U\eta_X \quad \text{on} \quad z = 1 \\ \text{and} \\ u = W = 0 \quad \text{on} \quad z = 0, \end{array}\right\} \tag{5.64}$$

where we have written $U'(1+\varepsilon\eta) = U'(1) + \varepsilon\eta U''(1)\ldots$. The solution of the set (5.64) is immediately

$$p = \eta, \quad u = 2U_0\eta z, \quad W = -U_0\eta_X z^2$$

with

$$\eta_T + 2U_0\eta_X = 0; \quad \text{that is,} \quad \eta = \eta(X - 2U_0T).$$

Thus there exists a surface wave which moves to the right at a speed $2U_0$, which is twice the surface speed (U_0) of the underlying Poiseuille flow.

In consequence, the linear equations for the surface wave allow three possible waves: two waves whose speed is determined by the Burns condition (see Q3.45(b)), one of which satisfies $c_1 > 0$ and the other gives $c_2 < 0$, and one which moves at a speed $2U_0$ (> 0). It can be shown (following on from Q3.46(b)) that

$$c_2 < 2U_0 < c_1$$

and, further, that these three waves form a wave hierarchy of the type discussed in Q1.51 and Q1.52. In particular, it follows that the waves which move at the Burns speeds (c_1, c_2) decay – they are called *dynamic* waves – leaving the main disturbance to move at the speed $2U_0$ (which is called the *kinematic* wave). A discussion of kinematic and dynamic waves can be found in Lighthill & Whitham (1955), Whitham (1959, 1974); the application of these ideas in the current context, and to the undular bore, is given in Johnson (1972). It is sufficient for our purposes here to investigate more fully the nature of the propagation at the speed $2U_0$, on the assumption that the dynamic waves decay and therefore, eventually, play no rôle.

The model that we are employing represents a flowing river – so it is more realistic – with a surface wave that propagates forwards ('downhill') at a speed greater than the surface speed of the undisturbed flow. The wave moves into undisturbed conditions ahead, and we wish to determine how this wave evolves and whether a change in depth (with undulations) is possible. The approach that we adopt is the familiar one of following the wave (which moves at the speed $2U_0$), constructing its evolution on a suitable long time scale and, here, also making an appropriate choice of the Reynolds number.

To this end we introduce

$$\xi = x - 2U_0 t, \quad \tau = \varepsilon t \tag{5.65}$$

and choose

$$R = \sqrt{\varepsilon}\mathscr{R}, \tag{5.66}$$

although other scales exist, involving appropriate combinations of ε, δ and R. The choice made here is the simplest that produces the required balance of terms. The equations and boundary conditions, (5.56)–(5.60), then become

$$\begin{aligned} &\varepsilon u_\tau + (U - 2U_0 + \varepsilon u)u_\xi + (U' + \varepsilon u_z)w \\ &\qquad = -p_\xi + \left(\beta + \frac{U''}{\varepsilon^2\mathscr{R}}\right) + \frac{1}{\varepsilon\mathscr{R}}(u_{zz} + \varepsilon u_{\xi\xi}); \end{aligned} \tag{5.67}$$

$$\varepsilon\{\varepsilon w_\tau + (U - 2U_0 + \varepsilon u)w_\xi + \varepsilon w w_z\} = -p_z + \frac{1}{\mathscr{R}}(w_{zz} + \varepsilon w_{\xi\xi}); \tag{5.68}$$

$$u_\xi + w_z = 0, \tag{5.69}$$

with

$$\left.\begin{aligned} p-\eta-\frac{2}{\mathscr{R}}\{w_z-\eta_\xi(U'+\varepsilon u_z+\varepsilon^2 w_\xi)\\ +\varepsilon^3\eta_\xi^2 u_\xi\}/(1+\varepsilon^3\eta_\xi^2)=0;\\ (1-\varepsilon^3\eta_\xi^2)(U'+\varepsilon u_z+\varepsilon^2 w_\xi)+2\varepsilon^3(w_z-u_\xi)\eta_\xi=0;\\ w=\varepsilon\eta_\tau+(U-2U_0+\varepsilon u)\eta_\xi, \end{aligned}\right\} \text{ on } z=1+\varepsilon\eta \tag{5.70}$$

and

$$U+\varepsilon u=0, \quad w=0 \quad \text{on} \quad z=0, \tag{5.71}$$

where $U(z)$ is to satisfy (5.62). We seek an asymptotic solution of these equations in the form

$$q\sim\sum_{n=0}^{\infty}\varepsilon^n q_n(\xi,\tau,z), \quad \eta\sim\sum_{n=0}^{\infty}\varepsilon^n\eta_n(\xi,\tau), \quad \varepsilon\to 0,$$

where q (and correspondingly q_n) represent u, w and p; the new Reynolds number, $\mathscr{R}$, is then held fixed as $\varepsilon\to 0$. The leading order problem from equations (5.67)–(5.71) is directly

$$u_{0zz}=0; \quad p_{0z}=\frac{1}{\mathscr{R}}w_{0zz}; \quad u_{0\xi}+w_{0z}=0,$$

with

$$p_0=\eta_0+\frac{2}{\mathscr{R}}w_{0z}; \quad u_{0z}+\eta_0 U''=0; \quad w_0=(U-2U_0)\eta_{0\xi} \quad \text{on} \quad z=1,$$

and

$$u_0=w_0=0 \quad \text{on} \quad z=0,$$

which are essentially equations (5.64). The only difference arises in the way in which p_0 is determined here, but since p_0 is found *after* u_0, w_0 and η_0 are fixed, this is not critical. Indeed, we see that

$$u_0=2U_0\eta_0 z, \quad w_0=-U_0\eta_{0\xi}z^2, \quad p_0=\eta_0-\frac{2U_0}{\mathscr{R}}(1+z)\eta_{0\xi}, \tag{5.72}$$

where $\eta_0(\xi,\tau)$ is an arbitrary function (at this order).

At the next order we obtain, from equation (5.67),

$$(U-2U_0)u_{0\xi}+U'w_0=-p_{0\xi}+\frac{1}{\mathscr{R}}(u_{1zz}+u_{0\xi\xi});$$

from (5.69) we get simply

$$u_{1\xi} + w_{1z} = 0.$$

The boundary conditions yield

$$\left.\begin{aligned} \eta_1 U'' + u_{1z} + w_{0\xi} = 0; \\ w_1 + \eta_0 w_{0z} = \eta_{0\tau} + (\eta_0 U' + u_0)\eta_{0\xi} + (U - 2U_0)\eta_{1\xi} \end{aligned}\right\} \text{ on } z = 1$$

and

$$u_1 = w_1 = 0 \quad \text{on} \quad z = 0,$$

where we have omitted the boundary condition on the pressure at the surface, and equation (5.68), at this order; these enable p_1 to be determined, but this is not required in order to find the equation for $\eta_0(\xi, \tau)$. (This has happened because p essentially uncouples from the other functions, as we alluded to above; the construction of p_1 is left as an exercise in Q5.14.) It is altogether straightforward to show that

$$u_1 = \frac{\mathscr{R}U_0^2}{6}(z^4 - 4z^3)\eta_{0\xi} + \frac{\mathscr{R}z^2}{2}\eta_{0\xi} - \frac{U_0}{3}(3z^2 + 2z^3)\eta_{0\xi\xi} + Az, \qquad (5.73)$$

where $A(\xi, \tau)$ is an arbitrary function of integration; similarly

$$w_1 = -\frac{\mathscr{R}U_0^2}{30}(z^5 - 5z^4)\eta_{0\xi\xi} - \frac{\mathscr{R}}{6}z^3\eta_{0\xi\xi} + \frac{U_0}{6}(2z^3 + z^4)\eta_{0\xi\xi\xi} - \frac{z^2}{2}A_\xi. \quad (5.74)$$

The surface boundary conditions yield, first,

$$A = 2U_0\eta_1 + \mathscr{R}\left(\frac{4}{3}U_0^2 - 1\right)\eta_{0\xi} + 5U_0\eta_{0\xi\xi},$$

and then

$$\eta_{0\tau} + 4U_0\eta_0\eta_{0\xi} + 2U_0\eta_{0\xi\xi\xi} = \frac{\mathscr{R}}{3}\left(1 - \frac{8}{5}U_0^2\right)\eta_{0\xi\xi}, \qquad (5.75)$$

the required *Korteweg–de Vries–Burgers* (*KVB*) *equation.*

This is the equation that we seek, but its construction is very different from the corresponding equation based on boundary-layer arguments. That equation, (5.53), recovers the classical KdV equation as the Reynolds number is increased indefinitely – a comforting property; our new equation, (5.75), is *dominated* by the effects of viscosity. We require viscous stresses to balance gravity and so provide the ambient flow, and also a special limiting process ($\varepsilon \to 0$, $\mathscr{R}$ fixed) in order to generate the appropriate internal viscous dissipation (represented by the term $\eta_{0\xi\xi}$).

The removal of the viscous contribution involves $U_0 \to 0$ and $\mathscr{R} \to \infty$, which clearly destroys the character of our KVB equation: we cannot recover the KdV equation in the way we might have expected (but it does arise if we let $\mathscr{R} \to 0$). Nevertheless, we have succeeded in our intention to find a limiting process that balances KdV nonlinearity and dispersion against Burgers nonlinearity and dissipation.

The KVB equation, (5.75), possesses a number of interesting features. First, it and our model admit a steady solution; second, the damping (or dissipative) term

$$\frac{\mathscr{R}}{3}\left(1-\frac{8}{5}U_0^2\right)\eta_{0\xi\xi},$$

has a negative coefficient if

$$U_0^2 > \frac{5}{8}. \tag{5.76}$$

This condition implies an energy input and, presumably, we must anticipate that our model is no longer valid. In fact the speed of the surface wave is, to leading order as $\varepsilon \to 0$, $2U_0$, and all speeds have been non-dimensionalised with respect to $\sqrt{g_0 h_0}$, $g_0 = g\cos\alpha$; thus $2U_0$ is the Froude number of the wave. Thus, when we write $F = 2U_0$, condition (5.76) becomes

$$F^2 > \frac{5}{2} \quad \text{or} \quad F \gtrsim 1.58.$$

Now it is commonly observed that bores with F larger than about 1.2 have turbulent, breaking fronts; on the other hand, if F is less than this (but, of course, $F > 1$; see Section 2.7), we typically observe the undular bore. This suggests that our model has captured an important phenomenon (even though the values do not quite correspond); indeed, Dressler (1949) has shown that the condition $F^2 \geq 5/2$ heralds the formation of *roll waves*, which, locally, have the appearance of turbulent bores.

This brief discussion of the rôle of laminar viscosity in water-wave theory is brought to a close as we present a few observations on the steady solutions of the KVB equation, (5.75). We seek a solution in the form $\eta_0(\xi - c\tau)$, to give

$$-c\eta_0' + 4U_0\eta_0\eta_0' + 2U_0\eta_0''' = \frac{\mathscr{R}}{3}(1-\frac{8}{5}U_0^2)\eta_0''$$

which, after one integration in $\zeta = \xi - c\tau$ and imposing the condition $\eta_0 \to 0$ as $\zeta \to +\infty$, yields

$$-c\eta_0 + 2U_0\eta_0^2 + 2U_0\eta_0'' = \frac{\mathscr{R}}{3}(1 - \frac{8}{5}U_0^2)\eta_0'.$$

This equation is conveniently normalised by introducing the transformation

$$\eta_0 \to \frac{c}{2U_0}\eta_0, \quad \zeta \to \sqrt{\frac{2U_0}{c}}\zeta,$$

to give

$$\eta_0^2 - \eta_0 + \eta_0'' = \lambda\eta_0', \tag{5.77}$$

where

$$\lambda = \frac{1}{3}\mathscr{R}(1 - \frac{8}{5}U_0^2)/\sqrt{2U_0c}.$$

In the form (5.77), we then have

$$\eta_0 \to 1 \quad \text{as} \quad \zeta \to -\infty,$$

if solutions exist for which this is possible, which certainly requires $\lambda > 0$ i.e. $U_0^2 < 5/8$. It is an elementary exercise (see Q5.15) to seek the asymptotic behaviours

$$\eta_0 \sim a\exp(-\alpha\zeta), \quad \zeta \to +\infty$$

and

$$\eta_0 \sim 1 - b\exp(\beta\zeta), \quad \zeta \to -\infty,$$

and to find that

$$\alpha = \frac{1}{2}\left\{\sqrt{\lambda^2 + 4} - \lambda\right\}, \quad \beta = \frac{1}{2}\left\{\lambda \pm \sqrt{\lambda^2 - 4}\right\}.$$

The choice of the sign in α ensures that $\eta_0 \to 0$ as $\zeta \to +\infty$; in β, either sign is possible (and it is clear here that we must have $\lambda > 0$), but the profile as $\zeta \to -\infty$ may be either monotonic ($\lambda \geq 2$) or oscillatory ($0 < \lambda < 2$). Thus equation (5.77) will allow either a monotonic transition through the jump (a non-turbulent classical jump) or an oscillation about $\eta_0 = 1$ (the undular bore). The interpretation is quite simply that larger λ means larger dissipation (mainly in the neighbourhood of $\zeta = 0$), so no wave is required to transport the excess energy away. For smaller λ, the wave is needed to carry the surplus energy away from the front.

A number of interesting properties of the steady KVB equation can be explored and exploited; see Q5.16–Q5.19. We conclude by presenting two solutions of the KVB equation, (5.77), in Figure 5.7; these are based on

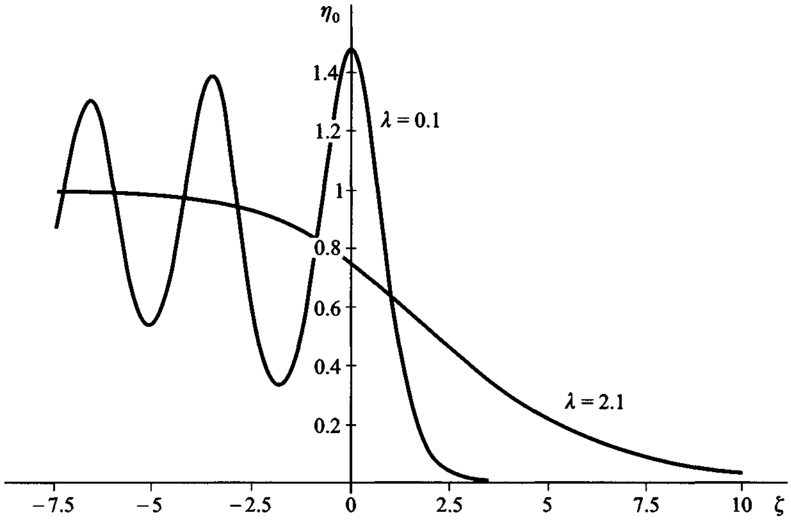

Figure 5.7. Two solutions of the steady state Korteweg–de Vries–Burgers (KVB) equation, (5.77), for $\lambda = 0.1, 2.1$.

numerical solutions of the equation. One is oscillatory ($\lambda = 0.1$), and is therefore a representation of the undular bore, and the other is a monotonic profile ($\lambda = 2.1$). The undular bore displays a front that is reminiscent of the solitary wave, and the oscillations can be described by an evolving cnoidal wave; both these properties can be formalised by examining the limit $\lambda \to 0$ (as considered in Johnson (1970)). The monotonic profile, on the other hand, takes the form of a distorted *tanh* curve, where the distortion is progressively less pronounced as $\lambda \to \infty$; see Q5.18.

Further reading

The rôle of viscosity in fluid mechanics in general, and in the theory of water waves in particular, is a very large and important subject. The fundamental effects that are encountered in the study of fluids are best addressed through standard texts on fluid mechanics (given, for example, at the end of Chapter 1). However, in addition to the references already given (including those relevant references contained therein), the reader is directed to Lighthill (1978), Craik (1988) and Mei (1989) for some useful and fairly up-to-date material on viscous dissipation in wave propagation. The text by Debnath (1994) also touches on some of these ideas.

Exercises

Q5.1 *Dispersion relation: large R.* Consider the dispersion relation, (5.21), in the limit $R \to \infty$ at fixed δk, and hence obtain result (5.22):

$$\omega \sim k\left\{\sqrt{\frac{\tanh \delta k}{\delta k}} - \frac{(1+\mathrm{i})}{2\sqrt{2R}}\,\frac{(\delta k)^{1/4}}{\cosh^{5/4}\delta k \sinh^{3/4}\delta k}\right\}.$$

Q5.2 *Dispersion relation: $\delta k = O(1/R)$.* Show that the result obtained in Q5.1 is not uniformly valid, as $\delta k \to 0$, where $\delta k R = \mathrm{O}(1)$. Hence obtain the equations that describe the leading approximation to the dispersion relation, (5.21), in the limit $R \to \infty$ at $\delta k R$ fixed.

[Note: these equations cannot be solved in closed form.]

Q5.3 *Dispersion relation: small δk.* Consider the dispersion relation, (5.21), in the limit $\delta k \to 0$ at fixed R, and hence show that

$$\omega \sim -\mathrm{i}\delta k^2 R/3$$

(where the real part of ω turns out to be exponentially small).

[This case, which is essentially a high-viscosity limit, shows that (at this order) there is *no propagation*, only decay. As an additional exercise, you may show that the expression obtained in Q5.2 agrees, for $\delta k \to \infty$, with that given in Q5.1 as $\delta k \to 0$, and that there is agreement between Q5.2 and Q5.3 for $\delta k \to 0$.]

Q5.4 *Dispersion relation with surface tension.* Repeat the calculation presented in Section 5.2.1, but with the surface pressure condition adjusted to accommodate the effects of surface tension, namely

$$p - \eta - \frac{2\delta}{R} w_z = -\delta^2 W_{\mathrm{e}} \eta_{xx} \quad \text{on} \quad z = 1;$$

cf. Section 2.1. Hence obtain the dispersion relation, with surface tension, which corresponds to that given in equation (5.21).

Q5.5 *A model boundary layer problem.* Obtain the solution of the ordinary differential equation

$$\varepsilon y'' + (1+\varepsilon)y' + y = 0, \quad 0 \le x \le 1,$$

that satisfies

$$y(0; \varepsilon) = 0, \quad y(1; \varepsilon) = 1,$$

where $\varepsilon > 0$ is a constant. Describe the character of this solution, for $\varepsilon \to 0$, in the two cases

(a) x away from $x = 0$; (b) $x = \varepsilon X$, $X = \mathrm{O}(1)$.

[The region near the boundary, measured by $x = \mathrm{O}(\varepsilon)$ is where the *boundary layer* exists; in this narrow region the solution adjusts from the value e (approximately, as $\varepsilon \to 0$) to 0 (on $x = 0$).]

Q5.6 *Asymptotic approach to a boundary layer problem.* Solve the problem given in Q5.5 by seeking two asymptotic solutions, in the form

(a) $y \sim \sum_{n=0}^{\infty} \varepsilon^n y_n(x), \quad \varepsilon \to 0,$

valid for x away from $x = 0$, and satisfying the condition on $x = 1$;

(b) set $x = \varepsilon X$ and write

$$y \sim \sum_{n=0}^{\infty} \varepsilon^n Y_n(X), \quad \varepsilon \to 0,$$

satisfying the condition on $x = 0$ (that is, on $X = 0$).

Now match the solutions obtained in (a) and (b), thereby uniquely determining the solution in (b).

[You need find only the first terms in each expansion, but a second could be found as well, if you are so minded.]

Q5.7 *Inviscid solution of the viscous equations.* Confirm that the solution given in equations (5.33) satisfies all the equations and boundary conditions in (5.30)–(5.32), with the exception of the no-slip condition on $z = 0$.

Q5.8 *Surface shear-stress condition.* Show that, away from the boundary layer (formed as $r \to 0$), a solution exists in which $u_{1zz} + \eta_{0\xi\xi} = 0$ (a term in equation (5.34)). Hence obtain an expression for $u_{1\xi z}$ and use this to confirm that the surface shear-stress condition,

$$u_{1z} - \eta_{0\xi\xi} = 0 \quad \text{on} \quad z = 1,$$

is satisfied.

[This is the counterpart, at $\mathrm{O}(\varepsilon)$, of the solution discussed in Q5.7.]

Q5.9 *Heat/diffusion equation: Duhamel's method.* The calculation of the relevant solution of the equation

$$u_t = u_{xx}, \quad t \geq 0, \quad x \geq 0,$$

is constructed in two stages.

(a) Obtain the solution for $u(x, t)$ which satisfies

$$u(x, 0) = 0, \ x > 0; \quad u(0, t) = 1, \quad t > 0, \tag{$*$}$$

in the form

$$u = U(x, t) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{x/2\sqrt{t}} \exp(-y^2)\,\mathrm{d}y.$$

(b) The effect of raising the 'temperature' on $x = 0$ to 1 at a time $t = t'\,(> 0)$, and then reducing it to zero at $t = t' + h\,(h > 0)$, is represented by the solution

$$u = U(x, t - t') - U(x, t - t' - h).$$

Over a very short time interval this is, approximately, $h(\partial U/\partial t)$ evaluated at time $t - t'$. If, during this interval, the temperature is actually $f(t')$, the resulting temperature over all times is then

$$\hat{u} = \int_{-\infty}^{t} \left(\frac{\partial U}{\partial t}\right)\bigg|_{t-t'} f(t')\,\mathrm{d}t';$$

this is Duhamel's result. Obtain an expression for $\partial U/\partial t$, and hence write down $\hat{u}$; confirm that $u = \hat{u}$ is, indeed, a solution of equation ($*$).

Hence rewrite this solution so that it takes the form quoted in equation (5.42).

[In (a) introduce the *similarity variable*, $x/2\sqrt{t}$. The solution obtained in (b) gives the temperature (in $x > 0$, $t > 0$) when the end, $x = 0$, is set at the variable temperature $f(t)$. Here we have used the interpretation of u as temperature, so ($*$) is called the heat conduction equation in this context; it is also often called the diffusion equation – here the diffusion of heat.]

Q5.10 *An integral identity.* Show that

$$2\int_0^\infty\int_0^\infty f\left(x+\frac{z^2}{4y^2}\right)\exp(-y^2)\,\mathrm{d}y\,\mathrm{d}z=\int_x^\infty f(x')\frac{\mathrm{d}x'}{\sqrt{x'-x}},$$

and evaluate this integral for the choice $f(x)=\exp(-\alpha x)$, where $\alpha\,(>0)$ is a real constant.

Q5.11 *Modulation of the solitary wave.* Follow the procedure described in equation (5.49) *et seq.*, but start by multiplying this equation by η_0. Hence obtain the corresponding expression for $c(T)$.

[This derivation uses the 'conserved' density η_0^2, rather than η_0 as given in the text. You might wish to obtain numerical estimates for the integrals that appear in these two formulae; the two expressions for $c(T)$ should, of course, be identical.]

Q5.12 *Propagation of the modulated solitary wave.* The (nondimensional) speed of the solitary wave, in the characteristic frame, is

$$c(T)=c_0(1+\alpha T)^{-4},\quad T\geq 0,$$

where $\alpha\,(>0)$ is a constant. Obtain an expression for the characteristic variable (ξ) associated with the modulated solitary wave; see equations (5.24).

Q5.13 *Asymptotic behaviour of the bore.* Obtain an asymptotic solution of the equation

$$-2c\eta+\frac{3}{2}\eta^2+\frac{1}{3}\eta''=\lambda\int_0^\infty \eta(x+x')\frac{\mathrm{d}x'}{\sqrt{x'}}$$

(see equation (5.54)), in the form

$$\eta\sim a\mathrm{e}^{-\alpha x}+b\mathrm{e}^{-2\alpha x},\quad x\to+\infty.$$

Determine the relations between c, λ, a, b and α, and compare this behaviour with equation (2.165) *et seq.* and Q2.63.

[The special case considered in Q5.10 will prove useful here. Note that a and α could be related if the front of the wave were to be like a solitary wave.]

Q5.14 *Undular bore: perturbation pressure at $O(\varepsilon)$.* Obtain the pressure term p_1 from equations (5.67)–(5.71), making use of the results given in equations (5.72)–(5.74).

Q5.15 *Asymptotic behaviours of the KVB equation.* Seek solutions of the steady KVB equation

$$\eta^2 - \eta + \eta'' = \lambda\eta' \quad (\lambda > 0, \text{ constant})$$

in the forms

(a) $\eta \sim a\exp(-\alpha\zeta), \quad \zeta \to +\infty;$

(b) $\eta \sim 1 - b\exp(\beta\zeta), \zeta \to -\infty,$

and hence determine α and β.

Q5.16 *Steady state KVB equation: phase plane I.* Discuss the equation given in Q5.15 in the phase plane; that is, in the (η, p) plane where $p = \eta'$. Show that there are two singular points, one at $(0, 0)$ and the other at $(1, 0)$; determine their natures for various λ. In particular, include the cases: $\lambda = 0$; $0 < \lambda < 2$.

Q5.17 *Steady state KVB equation: phase plane II.* Repeat the calculation of Q5.16, but this time write $p = P/\lambda$ and include the cases $1/\lambda = 0$; $\lambda \geq 2$.

Q5.18 *KVB equation: near-Taylor profile.* Introduce the transformation $\zeta = \lambda X$ into the equation in Q5.15, and hence obtain an asymptotic solution in the form

$$\eta \sim \eta_0(X) + \lambda^{-2}\eta_1(X), \quad \lambda \to \infty.$$

[Asymptotic solutions can also be obtained for $\lambda \to 0^+$; see Johnson (1970).]

Q5.19 *KVB equation: special solution.* Show that the steady state KVB equation in Q5.15 has the exact solution

$$\eta = \frac{1}{2}\left\{1 - \tanh(\zeta/2\sqrt{6})\right\} + \frac{1}{4}\operatorname{sech}^2(\zeta/2\sqrt{6}),$$

when $\lambda = 5/\sqrt{6}$.