1

Mathematical preliminaries

For nothing is that errs from law
In Memoriam A.H.H. LXXIII

Science moves, but slowly slowly, creeping on from point to point
Locksley Hall

Before we commence our presentation of the theory of water waves, we require a firm and precise base from which to start. This must be, at the very least, a statement of the relevant governing equations and boundary conditions. However, it is more satisfactory, we believe, to provide some background to these equations, albeit within the confines of an introductory and relatively brief chapter. The intention is therefore to present a derivation of the equations for inviscid fluid mechanics (*Euler's equation* and the equation of *mass conservation*) and a few of their properties. (The corresponding equations for a viscous fluid – primarily the *Navier–Stokes equation* – appear in Appendix A.) Coupled to these general equations is the set of boundary (and initial) conditions which select the water-wave problem from all other possible solutions of the equations. Of particular importance, as we shall see, are the conditions that define and describe the surface of the fluid; these include the *kinematic condition* and the rôles of *pressure* and *surface tension*. Some rather general consequences of coupling the equations and boundary conditions will also be mentioned.

Once we have available the complete prescription of the water-wave problem, based on a particular model (such as for inviscid flow), we may 'normalise' in any manner that is appropriate. It turns out to be very convenient – and is indeed typical of the applied mathematical approach – to introduce a suitable set of *nondimensional variables*. Further, a useful next step (which is particularly significant for our work in Chapters 3 and 4) is to *scale* the variables with respect to the small parameters thrown up by the nondimensionalisation. All this will enable us to characterise, in a rather precise way, the various types of approximation that we shall employ. In the process, we shall give a summary of the equations that represent different approximations of the full water-wave problem.

Throughout, we take the opportunity to present all the relevant equations in both rectangular Cartesian and cylindrical coordinates.

In the final stage of this preliminary discussion we provide a brief overview of some of the ideas that will permeate many of the problems that we shall encounter. This involves a simple introduction to the mathematics of wave propagation, where we describe the important phenomena associated with the *nonlinearity*, *dispersion* and *dissipation* of the wave. Further, much of our work in the newer aspects of water-wave theory will be with small-amplitude waves and with the slow evolution of wave properties; these may occur separately or together. In order to extract useful and relevant solutions in these cases, we shall require the application of asymptotic methods. Here we present an introduction to the use of *asymptotic expansions*, which will include both near-field and far-field asymptotics and the method of multiple scales.

These mathematical preliminaries may cover material already familiar to some readers, in whole or in part. Those with a background in fluid mechanics could ignore Section 1.1, whereas, for example, those who have received a basic course in wave propagation and elementary asymptotics could ignore Section 1.4. In Chapter 2, and thereafter, we start by giving a summary of the equations and boundary conditions that are relevant to each topic under discussion; this, at its simplest level, is all that is necessary to begin those studies.

1.1 The governing equations of fluid mechanics

In these derivations we shall use a vector notation and the methods of the vector calculus. (The tensor calculus is used in the brief derivation of the Navier–Stokes equation given in Appendix A, although the resulting equation is also written there in terms of vectors.) Here we shall derive the equations of mass conservation and motion (Newton's Second Law) in the absence of thermal changes (which are altogether irrelevant in the propagation of water waves). Any energy equation is therefore a consequence of only the motion (through Newton's Second Law) without any contributions from the thermodynamics of the fluid.

The notation that we shall adopt is the conventional one: at any point in the fluid, the velocity of the fluid is $\mathbf{u}(\mathbf{x}, t)$ where $\mathbf{x}$ is the position vector and t is a time coordinate. The density (mass/unit volume) of the fluid is $\rho(\mathbf{x}, t)$ (but for water-wave applications, as we shall mention later, we take $\rho = \text{constant}$); the pressure at any point in the fluid is $P(\mathbf{x}, t)$. If the choice of coordinates is the familiar right-handed rectangular Cartesian system, then we write

$$\mathbf{x} \equiv (x, y, z) \quad \text{and} \quad \mathbf{u} \equiv (u, v, w).$$

We shall assume that $\mathbf{u}$, ρ, and P are continuous functions (in $\mathbf{x}$ and t) – usually called the *continuum hypothesis* – and that they are also suitably differentiable functions.

1.1.1 The equation of mass conservation

Imagine a volume V, which is bounded by the surface S, within (and totally occupied by) the fluid. We treat V as fixed relative to some chosen inertial frame, so that the fluid in motion may cross the imaginary surface S. Given that the density of the fluid is $\rho(\mathbf{x}, t)$, then the rate of change of mass in V is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathrm{V}} \rho \, \mathrm{d}v\right)$$

where $\int_{\mathrm{V}} \mathrm{d}v$ represents the triple integral over V. Now, let $\mathbf{n}$ be the outward unit normal on S (see Figure 1.1) so that the outward velocity component of the fluid across S is $\mathbf{u} \cdot \mathbf{n}$. Thus the net rate at which mass flows *out* of V is

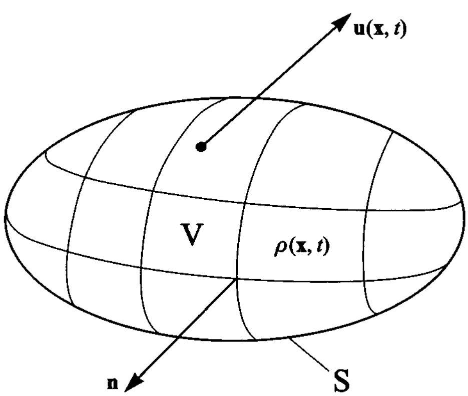

Figure 1.1. The volume V bounded by the surface S; $\rho(x, t)$ is the density of the fluid, $\mathbf{u}(\mathbf{x}, t)$ is the velocity at a point in the fluid and $\mathbf{n}$ is the outward normal on S.

$$\int_S \rho \mathbf{u} \cdot \mathbf{n} \mathrm{d}s,$$

where this is the double integral over S.

Under the fundamental assumption that matter (mass) is neither created nor destroyed anywhere in the fluid, the rate of change of mass in V is brought about only by the rate of mass flowing *into* V across S, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_V \rho \, \mathrm{d}v\right) = -\int_S \rho \mathbf{u} \cdot \mathbf{n} \mathrm{d}s.$$

This equation is rewritten by the application of *Gauss' theorem* (the *divergence theorem*) to the integral on the right, to give

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_V \rho \, \mathrm{d}v\right) + \int_V \nabla \cdot (\rho \mathbf{u}) \mathrm{d}v = 0$$

where ∇ is the familiar *del* operator (used here in the *divergence* of $\rho \mathbf{u}$). Further, since V is fixed in our coordinate system, the only dependence on t is through $\rho(\mathbf{x}, t)$, so we may write

$$\int_V \left\{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u})\right\} \mathrm{d}v = 0. \tag{1.1}$$

(We shall write more about *differentiation under the integral sign* later; see also Q1.30, Q1.31.) Now equation (1.1) is clearly applicable to any V totally occupied by the fluid, so the limits (represented symbolically by V) of the triple integral are therefore arbitrary; the integral is then always zero (for a continuous integrand, which we assume here) only if

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{1.2}$$

This equation, (1.2), is one form of the *equation of mass conservation* (sometimes called the *continuity* equation, referring to the continuity of matter). (The argument that takes us from (1.1) to (1.2) can be rehearsed in the simple example

$$\int_a^b f(x) \mathrm{d}x = 0 \quad \text{for arbitrary } a, b \Rightarrow f(x) = 0;$$

this is left as an exercise.)

It is usual to expand (1.2) as

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{u}) + (\mathbf{u} \cdot \nabla)\rho = 0,$$

and then introduce

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \tag{1.3}$$

the *material* (or *convective*) *derivative*; see Q1.5 and Section 1.2.1. Equation (1.2) therefore becomes

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla \cdot \mathbf{u} = 0, \tag{1.4}$$

from which we see that for an incompressible flow defined by

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0, \tag{1.5}$$

we have

$$\nabla \cdot \mathbf{u} = 0. \tag{1.6}$$

(A function ($\mathbf{u}$) which satisfies equation (1.6), so that the divergence of $\mathbf{u}$ is zero, is said to be *solenoidal*.) Equation (1.5) describes the constancy of ρ on individual fluid particles; we shall, however, interpret incompressibility as meaning $\rho =$ constant everywhere (which is clearly a solution of (1.5), and a very good model for fluids like water). Some of these basic ideas are explored in Q1.7–Q1.9.

1.1.2 The equation of motion: Euler's equation

We now turn our attention to the application of Newton's Second Law to a fluid, but a fluid which is assumed to be *inviscid*; that is, it has zero viscosity (internal friction). (The corresponding equation for a viscous fluid – the Navier–Stokes equation – is described in Appendix A.) Newton's Second Law requires us to balance the rate of change of (linear) momentum of the fluid against the resultant force acting on the fluid. First, therefore, we must find a representation of the forces acting on the fluid.

There are two types of force that are relevant in fluid mechanics: a *body force*, which is more or less the same for all particles and has its source exterior to the fluid, and a *local* (or *short-range*) *force*, which is the force exerted on a fluid element by other elements nearby. The body force which is almost always present is gravity, and this is certainly the case in the study of water waves. We define the general body force to be $\mathbf{F}(\mathbf{x}, t)$ per unit mass; if $\mathbf{F}$ is due solely to the (constant) acceleration of gravity (g) then we would write $\mathbf{F} \equiv (0, 0, -g)$ in both Cartesian and cylindrical coordinates (with z measured positive upwards). The local force is comprised of a pressure contribution together with any viscous forces that are present; in general, of course, this is conveniently represented by the stress tensor in the fluid: see Appendix A. Here we retain only the pressure (P), which produces a *normal* force acting *onto* any element of fluid.

To proceed we define (just as before) an imaginary volume V, bounded by the surface S, which is fixed in our frame of reference and totally occupied by the fluid. The total force (body + local) acting *on* the fluid in V is

$$\int_{\mathrm{V}} \rho \mathbf{F} \mathrm{d}v - \int_{\mathrm{S}} P\mathbf{n} \mathrm{d}s;$$

see Figure 1.2. (We remember that $\mathbf{n}$ is the *outward* unit normal on S.) Applying Gauss' theorem to the second integral (see Q1.2), we obtain the resultant force

$$\int_{\mathrm{V}} (\rho \mathbf{F} - \nabla P) \mathrm{d}v. \tag{1.7}$$

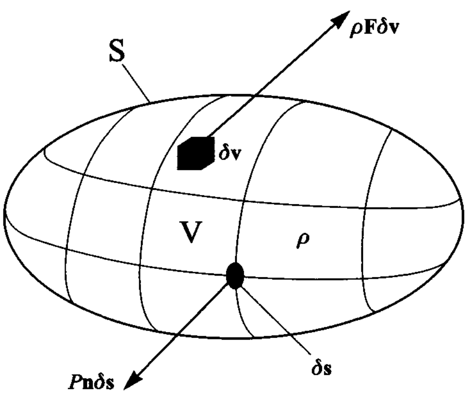

Figure 1.2. The volume V bounded by the surface S; the body force on an element is $\rho \mathbf{F} \delta v$ and the pressure force *on* an element of area is $-P\mathbf{n}\delta s$.

The rate of change of momentum of the fluid in V is simply

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathrm{V}} \rho\,\mathbf{u}\,\mathrm{d}v\right), \tag{1.8}$$

and the rate of flow of momentum across S into V is

$$-\int_{\mathrm{S}} \rho\,\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\mathrm{d}s. \tag{1.9}$$

Now Newton's Second Law for the fluid in V (upon recalling that V is fixed in our coordinate frame) may be expressed as:

rate of change of momentum of fluid in V
= resultant force acting on fluid in V
+ rate of flow of momentum across S into V.

Thus from equations (1.7)–(1.9) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathrm{V}} \rho\,\mathbf{u}\,\mathrm{d}v\right) = \int_{\mathrm{V}} (\rho\mathbf{F} - \nabla P)\mathrm{d}v - \int_{\mathrm{S}} \rho\,\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\mathrm{d}s,$$

which is written more compactly by (a) taking $\mathrm{d}/\mathrm{d}t$ through the integral sign, (b) applying Gauss' theorem to each component of (1.9) (see Q1.3), and, (c), rearranging, to yield

$$\int_{\mathrm{V}} \left\{\frac{\partial}{\partial t}(\rho\,\mathbf{u}) + \rho\,\mathbf{u}(\nabla\cdot\mathbf{u}) + (\mathbf{u}\cdot\nabla)\rho\mathbf{u}\right\}\mathrm{d}v = \int_{\mathrm{V}} (\rho\,\mathbf{F} - \nabla P)\mathrm{d}v. \tag{1.10}$$

We expand the integrand on the left side of this equation as

$$\int_{\mathrm{V}} \left\{\rho\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\frac{\partial\rho}{\partial t} + \rho\,\mathbf{u}(\nabla\cdot\mathbf{u}) + \mathbf{u}(\mathbf{u}\cdot\nabla)\rho + \rho(\mathbf{u}\cdot\nabla)\mathbf{u}\right\}\mathrm{d}v = \int_{\mathrm{V}} \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\mathrm{d}v, \tag{1.11}$$

where we have used the equation of mass conservation, (1.4), and introduced the material derivative, (1.3). It is clear that, with sufficient understanding of the notion of the material derivative (see Q1.4–Q1.6), we could write (1.11) directly: it is the appropriate form of 'mass × acceleration' for all the fluid in V.

The equation (1.10), with (1.11), now becomes

$$\int_{V}\left(\rho\frac{\mathbf{Du}}{\mathrm{d}t}-\rho\mathbf{F}+\nabla P\right)\mathrm{d}v=\mathbf{0}$$

and, as before, for this to be valid for arbitrary V (and a continuous integrand) we must have

$$\frac{\mathbf{Du}}{\mathrm{D}t}=-\frac{1}{\rho}\nabla P+\mathbf{F}, \tag{1.12}$$

when written in its usual form. This is *Euler's equation*, which is the result of applying Newton's Second Law to an inviscid (that is, frictionless) fluid. (Notice that the pressure, P, may be defined relative to an arbitrary constant value without altering equation (1.12).)

It is convenient, particularly in view of our later work, to present the three components of Euler's equation, (1.12), and also the equation of mass conservation, in the two coordinate systems that we shall use. In rectangular Cartesian coordinates, $\mathbf{x}\equiv(x,y,z)$, with $\mathbf{u}\equiv(u,v,w)$ and $\mathbf{F}\equiv(0,0,-g)$, and for constant density, equations (1.12) and (1.6) become, respectively,

$$\left.\begin{aligned}&\frac{\mathrm{D}u}{\mathrm{D}t}=-\frac{1}{\rho}\frac{\partial P}{\partial x},\quad \frac{\mathrm{D}v}{\mathrm{D}t}=-\frac{1}{\rho}\frac{\partial P}{\partial y},\quad \frac{\mathrm{D}w}{\mathrm{D}t}=-\frac{1}{\rho}\frac{\partial P}{\partial z}-g\\ &\text{where}\\ &\frac{\mathrm{D}}{\mathrm{D}t}\equiv\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z},\end{aligned}\right\} \tag{1.13}$$

and

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \tag{1.14}$$

These same equations written in cylindrical coordinates, $\mathbf{x}\equiv(r,\theta,z)$, with $\mathbf{u}\equiv(u,v,w)$ (where the same notation for $\mathbf{u}$ in this system should not cause any confusion: it will be plain which coordinates are being used in a given calculation) are, again with $\mathbf{F}\equiv(0,0,-g)$ and $\rho=$ constant,

$$\left.\begin{aligned}&\frac{\mathrm{D}u}{\mathrm{D}t}-\frac{v^2}{r}=-\frac{1}{\rho}\frac{\partial P}{\partial r},\quad \frac{\mathrm{D}v}{\mathrm{D}t}+\frac{uv}{r}=-\frac{1}{\rho}\frac{1}{r}\frac{\partial P}{\partial \theta},\\ &\frac{\mathrm{D}w}{\mathrm{D}t}=-\frac{1}{\rho}\frac{\partial P}{\partial z}-g\end{aligned}\right\} \tag{1.15}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v}{r}\,\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z},$$

and

$$\frac{1}{r}\,\frac{\partial}{\partial r}(ru) + \frac{1}{r}\,\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0. \tag{1.16}$$

These equations, (1.13–1.16), will form the basis for the developments described in Chapters 2, 3, and 4, when coupled to the appropriate boundary conditions (Section 1.2) and – usually – after suitable simplification (Section 1.3). (The corresponding equations for a viscous fluid are presented in Appendix A, and are used in Chapter 5.)

1.1.3 Vorticity, streamlines and irrotational flow

A fundamental property of a fluid flow is the *curl* of the velocity field: $\nabla_{\wedge}\mathbf{u}$. This is called the *vorticity*, and it is conventionally represented by the vector $\boldsymbol{\omega}$; the vorticity measures the local spin or rotation of the fluid (that is, the rotational motion – as compared with the translational) of a fluid element (see Q1.12). In consequence, flows, or regions of flows, in which $\boldsymbol{\omega} \equiv \mathbf{0}$ are said to be *irrotational*; such flows can often be analysed by using particularly routine methods. Unfortunately, real flows are very rarely irrotational anywhere, but for many flows the vorticity is very small almost everywhere, and these may therefore be modelled by assuming irrotationality. Nevertheless, many important aspects of fluid flow require $\boldsymbol{\omega} \neq \mathbf{0}$ somewhere, and the study of such flows normally involves a consideration of the dynamics of vorticity and its properties. In water-wave problems, however, classical aspects of vorticity play a rather minor rôle, and so a deep knowledge of vorticity is not a prerequisite for a study of water waves. (Some small exploration of vorticity is offered in the exercises: see Q1.13–Q1.17.)

Now, before we make use of the vorticity vector in Euler's equation, we introduce a very powerful – but related – concept in the study of fluid motion: the *streamline*. Consider the family of (imaginary) curves which everywhere have the velocity vector as their tangent; these curves are the streamlines. If such a curve is described by $\mathbf{x} = \mathbf{x}(s;\,t)$ (at any instant in time), where s is the parameter which maps out the curve, then the streamlines are the solutions of

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} \propto \mathbf{u} \quad \text{or} \quad \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \mathbf{u}(\mathbf{x}, t) \quad (\text{at fixed } t). \tag{1.17}$$

In this second representation, the constant of proportionality has been absorbed into the definition of s. Then, for example, in rectangular Cartesian coordinates this vector equation becomes the three scalar equations

$$\frac{\mathrm{d}x}{\mathrm{d}s} = u, \quad \frac{\mathrm{d}y}{\mathrm{d}s} = v, \quad \frac{\mathrm{d}z}{\mathrm{d}s} = w,$$

or equivalently,

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v} = \frac{\mathrm{d}z}{w}, \tag{1.18}$$

for the streamlines. (The streamline should not be confused with the *path* of a particle; this is defined (see Q1.4 and also Q1.19) by

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{u}(\mathbf{x}, t), \tag{1.19}$$

so particle paths and streamlines coincide, in general, only for steady flow; see Q1.19.) The streamlines provide a particularly effective way of describing a flow field: even a simple sketch of the streamlines for a flow often enables important characteristics to be recognised at a glance. (An associated concept, the *stream function*, is described in Q1.20–Q1.23.)

We now turn to a brief consideration of the results that can be obtained when the vorticity, $\boldsymbol{\omega}$, is introduced into Euler's equation, (1.12),

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla P + \mathbf{F}. \tag{1.20}$$

For our purposes we shall assume that $\rho =$ constant (but see Q1.18), and that the body force is represented by a *conservative* force field: $\mathbf{F} = -\nabla\Omega$ for some *potential function* $\Omega(\mathbf{x}, t)$, where the negative sign is a convenience. (This choice for $\mathbf{F}$ applies to most examples of interest; for our studies we shall use $\Omega = gz$ where g is the (constant) acceleration of gravity and z is measured positive upwards.) Equation (1.20) therefore becomes

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla\left(\frac{P}{\rho} + \Omega\right),$$

which is rewritten by introducing the identity (see Q1.1)

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) - \mathbf{u}\wedge(\nabla\wedge\mathbf{u})$$

where $\nabla\wedge\mathbf{u} = \boldsymbol{\omega}$, the vorticity. Thus we obtain

$$\frac{\partial\mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{P}{\rho} + \Omega\right) = \mathbf{u}\wedge\boldsymbol{\omega}, \tag{1.21}$$

and there are two cases worthy of further examination.

The first is for steady flow, where $\mathbf{u}$, P, and Ω are all independent of time, t. Equation (1.21) therefore becomes

$$\nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{P}{\rho} + \Omega\right) = \mathbf{u}\wedge\boldsymbol{\omega},$$

and a simple geometrical property enables us to make headway with this (apparently) intractable equation. (An alternative approach is to dot both sides with the vectors $\mathbf{u}$ and, separately, $\boldsymbol{\omega}$.) It is a familiar result that ∇f, the *gradient* of f, is a vector orthogonal to the surface $f(\mathbf{x}) = \text{constant}$; thus $\mathbf{u}\wedge\boldsymbol{\omega}$ is perpendicular to the surfaces

$$\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{P}{\rho} + \Omega = \text{constant}. \tag{1.22}$$

But $\mathbf{u}\wedge\boldsymbol{\omega}$ is also perpendicular to both the vectors $\mathbf{u}$ and $\boldsymbol{\omega}$, so the surfaces (1.22) must contain lines which are everywhere parallel to $\mathbf{u}$ and $\boldsymbol{\omega}$. One such set of lines is the family of streamlines, (1.17) and (1.18). Thus equation (1.22), known as *Bernoulli's equation* (or *theorem*), applies on streamlines; it describes the conservation of energy (kinetic + work done by pressure forces + potential) for a steady (and inviscid) flow with vorticity. This is a fundamental and powerful result in the study of elementary flows. (Bernoulli's equation is also valid on the family of lines which has $\boldsymbol{\omega}$ as the tangent to the lines at every point, but these lines are not usually of much interest in this context.)

The second case, of some importance in water-wave problems, is for irrotational but unsteady flow. Now for irrotational flow we have $\boldsymbol{\omega} = \nabla\wedge\mathbf{u} = \mathbf{0}$, and so $\mathbf{u} = \nabla\phi$ for a potential function $\phi(\mathbf{x}, t)$, the *velocity potential*; the study of irrotational flows reduces to the problem of determining ϕ; see Q1.24. Indeed, for irrotational and incompressible flow we have

$$\mathbf{u} = \nabla\phi \quad \text{and} \quad \nabla\cdot\mathbf{u} = 0,$$

so ϕ satisfies *Laplace's equation*

$$\nabla^2\phi = 0. \tag{1.22}$$

Thus the *nonlinear* Euler's equation, (1.12), and the equation of mass conservation, (1.6), have been replaced by a classical *linear* second-order partial differential equation (provided that $\boldsymbol{\omega} = \mathbf{0}$ and $\mathbf{F} = -\nabla\Omega$). If we use $\mathbf{u} = \nabla\phi$ in equation (1.21), with $\boldsymbol{\omega} = \mathbf{0}$, then it follows directly that

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{P}{\rho} + \Omega\right) = \mathbf{0},$$

so

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{P}{\rho} + \Omega = f(t), \tag{1.23}$$

where $f(t)$ is an arbitrary function of integration. (It is always possible to redefine ϕ as $\phi + \int f(t)\mathrm{d}t$ and thereby remove $f(t)$ from equation (1.23); of course, this choice of ϕ does not affect the velocity field since $\nabla(\int f(t)\mathrm{d}t) = \mathbf{0}$.) Equation (1.23) is known by some authors as Bernoulli's equation (cf. equation (1.22)) or, at least more accurately, as the Bernoulli equation for unsteady flow. A less confusing name – unfortunately used rather rarely nowadays – is the *pressure equation*, which we prefer; this helps to avoid the possible problems of interpretation which we mention below. ('Pressure equation' is used to indicate that P is completely determined (to within initial data) once the velocity field is known through ϕ.)

If it is now assumed, in addition, that the flow is steady then equation (1.23) becomes

$$\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{P}{\rho} + \Omega = \text{ constant}, \tag{1.24}$$

which is equation (1.22) – or is it? Equations (1.23) and (1.24) describe the fluid *everywhere*; there is no reference to streamlines, as there is with equation (1.22). Equation (1.24) is associated with the *same* constant throughout the fluid, whereas equation (1.22) assigns *different* constants to *different* streamlines. This important distinction provides a contrast between irrotational and rotational steady flows.

We complete this section by quoting Laplace's equation, which is valid for incompressible, irrotational flow, in both rectangular Cartesian coordinates

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0, \tag{1.25}$$

and cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}=0. \tag{1.26}$$

The corresponding velocities are

$$\mathbf{u}\equiv\left(\frac{\partial\phi}{\partial x},\ \frac{\partial\phi}{\partial y},\ \frac{\partial\phi}{\partial z}\right)$$

in rectangular Cartesian coordinates, and

$$\mathbf{u}\equiv\left(\frac{\partial\phi}{\partial r},\ \frac{1}{r}\frac{\partial\phi}{\partial\theta},\ \frac{\partial\phi}{\partial z}\right)$$

in cylindrical coordinates.

1.2 The boundary conditions for water waves

The boundary conditions that define water-wave problems come in various forms. We first briefly describe these before we examine them in detail. At the surface, called a *free* surface because it is not defined by velocity conditions (as on a *rigid* boundary, for example), the atmosphere exerts stresses on the fluid surface. In general, these stresses will include a viscous component (which is particularly relevant if we wish to model the effects of a surface wind, for example). However, if the fluid may be reasonably modelled as inviscid, then the atmosphere exerts only a pressure on the surface. This pressure is often taken to be a constant – the atmospheric pressure – but it may vary in time and also from point to point on the surface. (The passage of a region of higher/lower pressure could be used to model the movement of a storm or other similar phenomena.) Further, any surface tension effects can also be included at a curved surface (in the presence of a wave, for example) giving rise to the maintenance of a pressure *difference* across the surface. We should comment that our philosophy here is to regard the conditions obtaining at the surface as prescribed. A more complete theory would couple the motion of the water surface and the air above it, but the small density of air compared with that of water makes our approach feasible. Nevertheless, one method – not discussed in this text – for studying ocean waves, for example, is to consider the exchange of momentum and energy between the air and the surface waves.

Another, perhaps less obvious, condition requires a statement that the (moving) surface is a surface of the fluid; that is, it is always composed of fluid particles. This is called the *kinematic condition*, since it does not involve the action of forces; the stress conditions described above (which obviously generate forces at the surface) are called the *dynamic conditions* (which reduce to just one condition for an inviscid fluid).

At the bottom of the fluid we shall assume, throughout our work, that the bed there is impermeable. Then, if the fluid is treated as viscous, we must impose the no-slip condition on this surface (so that fluid particles in contact with the surface move with that surface). Thus, for a fixed rigid boundary, the fluid velocity will be zero here. On the other hand, if the fluid is modelled as inviscid, then the bottom topography becomes a surface of the fluid, so that fluid particles in contact with the bed move in this surface. This therefore mirrors the kinematic condition at the free surface, except that the bottom is prescribed *a priori*. For many of our problems, the bottom surface will be fixed and rigid (but not necessarily a horizontal plane); however, it could move in a prescribed manner if we wished to model a marine earthquake, for example.

In most of our applications, the fluid will be assumed to extend to infinity in all horizontal directions. We might, rarely, encounter a boundary wall which will then provide the same type of boundary condition as the bottom topography.

Finally, we comment that the rôle of initial data is relatively unimportant in the type of water-wave problems that we shall discuss. Of course, the wave must be initiated in some fashion (by a suitable disturbance of the surface), but in most problems we shall assume that this has already occurred. Our main interest will be in following the evolution of the wave in many – and varied – situations.

We now turn to a careful formulation of these boundary conditions, based on the principles that we have just outlined, for an inviscid fluid. The corresponding results for a viscous fluid are briefly described and presented in Appendix B.

1.2.1 The kinematic condition

The free surface, whose determination is usually the primary objective in water-wave problems, will be represented by

$$z = h(\mathbf{x}_\perp, t), \tag{1.27}$$

where $\mathbf{x}_\perp$ denotes the two-vector which is perpendicular to the z-direction. In rectangular Cartesian coordinates we therefore have $\mathbf{x}_\perp \equiv (x, y)$, and in cylindrical coordinates this is $\mathbf{x}_\perp \equiv (r, \theta)$. Now a surface $F(\mathbf{x}, t) = $ constant which moves with the fluid, so that it always contains the same fluid particles, must satisfy

$$\frac{\mathrm{D}F}{\mathrm{D}t} = 0;$$

see Q1.5. The free surface, written in the form

$$z - h(\mathbf{x}_\perp, t) = 0,$$

must therefore satisfy this same condition:

$$\frac{\mathrm{D}}{\mathrm{D}t}\{z - h(\mathbf{x}_\perp, t)\} = 0,$$

the fluid particles being those that move in the surface. This yields, directly,

$$w - \{h_t + (\mathbf{u}_\perp \cdot \nabla_\perp)h\} = 0$$

(where the subscript in t denotes the time derivative), since

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \mathbf{u}_\perp \cdot \nabla_\perp + w\frac{\partial}{\partial z}$$

where $\nabla_\perp$ is the grad operator perpendicular to the direction of the z-coordinate. (This symbol is usually pronounced 'del-perp'.) The velocity vector has been written as $\mathbf{u} \equiv (\mathbf{u}_\perp, w)$, although both $\mathbf{u}_\perp$ and $\nabla_\perp$ are, strictly, unnecessary notations here since $h = h(\mathbf{x}_\perp, t)$ only; we choose to use them in order to make quite clear the structure of the boundary condition. The kinematic condition is therefore

$$w = h_t + (\mathbf{u}_\perp \cdot \nabla_\perp)h \quad \text{on } z = h(\mathbf{x}_\perp, t), \tag{1.28}$$

and the evaluation of $z = h$ is needed to define the velocity field required here. (An alternative derivation of equation (1.28) is discussed in Q1.27.)

1.2.2 The dynamic condition

In the absence of viscous forces, the simplest dynamic condition merely requires that the pressure, P, is prescribed on $z = h(\mathbf{x}_\perp, t)$; the corresponding result for a viscous fluid is given in Appendix B. For most problems studied in the theory of water waves, it is usual to set

$P = P_a =$ constant, the pressure of the atmosphere. Of course, the simplicity of this boundary condition tends to obscure the fact that the evaluation is on the free surface ($z = h$) whose determination is part – often the most significant part – of the solution of the problem.

One special version of the dynamic boundary condition is offered by the case of an incompressible, irrotational, unsteady flow. From the pressure equation, (1.23), with $\Omega = gz$, we have

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} + \frac{P}{\rho} + gz = f(t)$$

everywhere. We consider the problem for which $P = P_a$ on $z = h(\mathbf{x}_\perp, t)$, then continuity of pressure requires that

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} + \frac{P_a}{\rho} + gh = f(t) \quad \text{on } z = h.$$

Further, let us suppose that, somewhere (as $|\mathbf{x}_\perp| \to \infty$, for example), the fluid is stationary with $P = P_a$ and $h = h_0 =$ constant; then

$$f(t) = \frac{P_a}{\rho} + gh_0$$

so

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} + g(h - h_0) = 0 \quad \text{on } z = h. \tag{1.29}$$

This equation, (1.29), constitutes one of the simplest descriptions of the surface-pressure condition. This is then one of the boundary conditions to be used in the construction of the relevant solution of Laplace's equation for ϕ.

For a rotational flow we cannot employ the pressure equation, and so we must solve Euler's equation with P given on $z = h$. Indeed, as we shall see, it turns out that there is very little to choose – even for irrotational flow – between solving Euler's equation with $P = P_a$ or Laplace's equation with (1.29), at least in the suitably approximate forms that we usually encounter.

Now we turn to the extension of this dynamic condition (for an inviscid fluid) which accommodates the effects of *surface tension* (which supports a pressure difference across a curved surface). The classical description of surface tension is represented by

$$\text{pressure difference} = \Delta P = \frac{\Gamma}{R}, \tag{1.30}$$

where $1/R$ is the mean curvature

$$\frac{1}{R} = \frac{1}{\kappa_1} + \frac{1}{\kappa_2},$$

and κ_1, κ_2 are the principal radii of curvature. (The quantity $1/R$ is often called the *Gaussian curvature*.) The parameter Γ (> 0) is the coefficient of surface tension (force/unit length), and $\Delta P > 0$ if the surface is convex; see Figure 1.3. The fundamental equation, (1.30), is usually called *Laplace's formula* and, in general, Γ varies with temperature; here we shall treat Γ as a constant. The result of using this equation in the dynamic condition is to replace, for example, $P = P_a =$ constant at the fluid surface by

$$P = P_a - \frac{\Gamma}{R} \quad \text{on} \quad z = h(\mathbf{x}_\perp, t), \tag{1.31}$$

so that the pressure in the fluid at $z = h$ is increased if the surface is concave ($R < 0$).

It is clear that a complication in this formulation involves the precise description required for the curvature, $1/R$. Fairly elementary geometrical considerations lead, for the choice of rectangular Cartesian coordinates with $h = h(x, y, t)$, to

$$\frac{1}{R} = \frac{(1 + h_y^2)h_{xx} + (1 + h_x^2)h_{yy} - 2h_x h_y h_{xy}}{(1 + h_x^2 + h_y^2)^{3/2}}, \tag{1.32}$$

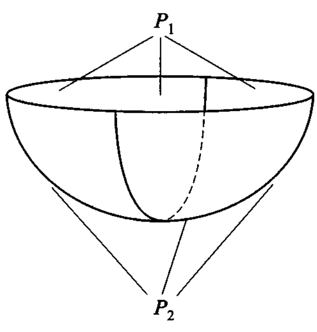

Figure 1.3. A convex surface with an 'internal' pressure P_1 and an 'external' pressure P_2, where $P_1 > P_2$.

where subscripts denote partial derivatives. A simple special case of this result recovers the well-known expression for curvature in only one direction:

$$\frac{1}{R} = \frac{h_{xx}}{(1+h_x^2)^{3/2}}, \quad h = h(x, t). \tag{1.33}$$

The corresponding representation in cylindrical coordinates, where $h = h(r, \theta, t)$, is

$$\frac{1}{R} = \frac{\left(1+\frac{1}{r^2}h_\theta^2\right)h_{rr} + \frac{1}{r^2}\left\{(1+h_r^2)(h_{\theta\theta}+rh_r) - 2(h_{r\theta} - \frac{1}{r}h_\theta)h_r h_\theta\right\}}{\left(1+h_r^2+\frac{1}{r^2}h_\theta^2\right)^{3/2}}. \tag{1.34}$$

1.2.3 The bottom condition

For an inviscid fluid, the bottom constitutes – like the free surface – a boundary which is defined as a surface moving with the fluid. Let us represent the (impermeable) bed of the flow by

$$z = b(\mathbf{x}_\perp, t);$$

for this to be a fluid surface then

$$\frac{\mathrm{D}}{\mathrm{D}t}\{z - b(\mathbf{x}_\perp, t)\} = 0.$$

Thus

$$w = b_t + (\mathbf{u}_\perp \cdot \nabla_\perp)b \quad \text{on} \quad z = b, \tag{1.35}$$

where $b(\mathbf{x}_\perp, t)$ will be prescribed in our problems. However, it should be mentioned that there are classes of problem (which we shall not discuss) where b is not known *a priori*; this situation can arise in the study of sediment movement, for example. Most of the calculations that we shall encounter in our work will involve a stationary bottom condition, so that equation (1.35) becomes

$$w = (\mathbf{u}_\perp \cdot \nabla_\perp)b \quad \text{on} \quad z = b. \tag{1.36}$$

(In the case of one-dimensional propagation, where $b = b(x)$ with $\mathbf{x}_\perp \equiv (x, 0)$ and $\mathbf{u}_\perp \equiv (u, 0)$, this reduces to the simple condition

$$w = u\frac{\mathrm{d}b}{\mathrm{d}x} \quad \text{on} \quad z = b(x),$$

which is readily understood from elementary considerations.)

1.2.4 An integrated mass conservation condition

Now that we have written down the general conditions that describe the kinematics of the motion at both the free surface and the bottom, we show how they can be combined with the equation of mass conservation. This produces a conservation condition for the whole motion, which will prove a useful result in some of our later work. First, the equation of mass conservation, (1.6), is written as

$$\nabla_\perp \cdot \mathbf{u}_\perp + w_z = 0,$$

which is then integrated in z over the depth of the fluid; that is, from $z = b(\mathbf{x}_\perp, t)$ to $z = h(\mathbf{x}_\perp, t)$. This yields

$$\int_b^h \nabla_\perp \cdot \mathbf{u}_\perp \mathrm{d}z + [w]_b^h = 0,$$

and then the conditions defining w on the bottom and the surface, (1.35) and (1.28), are introduced to give

$$\int_b^h \nabla_\perp \cdot \mathbf{u}_\perp \mathrm{d}z + h_t + (\mathbf{u}_{\perp \mathrm{s}} \cdot \nabla_\perp)h - \{b_t + (\mathbf{u}_{\perp \mathrm{b}} \cdot \nabla_\perp)b\} = 0. \qquad (1.37)$$

The subscripts s and b denote evaluations on the surface ($z = h$) and the bottom ($z = b$), respectively.

To proceed, it is necessary to interchange the differential and integral operations in the first term. This is accomplished by a careful application of the rule for 'differentiating under the integral sign'; see Q1.30. Here, this term becomes

$$\nabla_\perp \cdot \int_b^h \mathbf{u}_\perp \mathrm{d}z - (\mathbf{u}_{\perp \mathrm{s}} \cdot \nabla_\perp)h + (\mathbf{u}_{\perp \mathrm{b}} \cdot \nabla_\perp)b,$$

and so equation (1.37) can be written as

$$(h - b)_t + \nabla_\perp \cdot \int_b^h \mathbf{u}_\perp \mathrm{d}z = 0.$$

This equation is conveniently expressed as

$$d_t + \nabla_\perp \cdot \bar{\mathbf{u}}_\perp = 0, \qquad (1.38)$$

where $d = h - b$ is the (local) depth of the water, and

$$\bar{\mathbf{u}}_\perp = \int_b^h \mathbf{u}_\perp \mathrm{d}z, \tag{1.39}$$

so that $\bar{\mathbf{u}}_\perp / d$ is an average of the horizontal vector components describing the motion of the fluid. As a simple application of equation (1.38), consider motion in only one horizontal direction: let $\mathbf{u}_\perp \equiv (u, 0)$, for example; then we obtain

$$d_t + \bar{u}_x = 0.$$

If, further, we suppose that there is no motion at infinity (that is, $\bar{u} \to 0$ as $|x| \to \infty$), and that $b = b(x)$ with $h(x, t) = h_0 + H(x, t)$ where $H \to 0$ as $|x| \to \infty$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{-\infty}^{\infty} H(x, t)\mathrm{d}x\right\} = 0 \quad \text{or} \quad \int_{-\infty}^{\infty} H(x, t)\mathrm{d}x = \text{constant}. \tag{1.40}$$

This latter condition means that, for all time and for all surface waves represented by $H(x, t)$, the mass of fluid associated with the wave (assumed finite here) is conserved – an otherwise obvious result. It is clear that this conclusion is true, no matter the solution for $H(x, t)$; indeed, it may prove impossible to obtain the form of $H(x, t)$ except in special cases, and then only approximately, but (1.40) will still hold precisely.

1.2.5 An energy equation and its integral

We have already introduced an energy equation – Bernoulli's equation, (1.22) – but we shall now present a more general result. This does not require the restriction to steady flow, for example, nor to the alternative choice of irrotational flow (which led to the pressure equation, (1.23)). The new equation is, in a sense, a global energy equation; it describes the consequences on general fluid motion of using Newton's Second Law: that is, Euler's equation, (1.12). Once we have derived this equation, we shall apply it to our water-wave problem by integrating it over the depth of the fluid (exactly as we did for the mass conservation equation in Section 1.2.4).

We start with equation (1.21),

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{P}{\rho} + \Omega\right) = \mathbf{u}\wedge\boldsymbol{\omega}, \tag{1.41}$$

which is derived from Euler's equation for an incompressible fluid ($\rho =$ constant) and a conservative body force, $F = -\nabla\Omega$; we shall assume that $\Omega = \Omega(\mathbf{x})$, which applies to most situations of practical interest. To proceed, we take the scalar product of equation (1.41) with $\mathbf{u}$ to give

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) + (\mathbf{u}\cdot\nabla)\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{P}{\rho} + \Omega\right) = 0, \tag{1.42}$$

since $\mathbf{u}\cdot(\mathbf{u}\wedge\boldsymbol{\omega}) = 0$ (two of the vectors are parallel). Because the fluid is incompressible, we have $\nabla\cdot\mathbf{u} = 0$; we choose to add to equation (1.42) the expression

$$\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{P}{\rho} + \Omega\right)(\nabla\cdot\mathbf{u}) \quad (= 0)$$

and hence we obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) + \nabla\cdot\left\{\mathbf{u}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{P}{\rho} + \Omega\right)\right\} = 0;$$

see Q1.1(a) for the relevant differential identity. It is convenient to add a further zero contribution, namely $\partial\Omega/\partial t$, to give

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \Omega\right) + \nabla\cdot\left\{\mathbf{u}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{P}{\rho} + \Omega\right)\right\} = 0,$$

which is often rewritten (by multiplying throughout by ρ) as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u} + \rho\Omega\right) + \nabla\cdot\left\{\mathbf{u}\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u} + P + \rho\Omega\right)\right\} = 0. \tag{1.43}$$

This is an energy equation; we recognise the kinetic energy per unit volume ($\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}$) and the corresponding potential energy ($\rho\Omega$; for example, $\rho g z$). The equation represents the balance between the rate of change of the total (mechanical) energy and the energy flow carried by the velocity field, together with the contribution from the rate of working of the pressure forces. Clearly this energy equation is a general result in the theory of inviscid (and incompressible) fluids; we now apply it to the study of water waves.

Following the development presented in Section 1.2.4, we write equation (1.43), with $\Omega = gz$, in the form

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+\rho gz\right)+\nabla_\perp\left\{\mathbf{u}_\perp\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+P+\rho gz\right)\right\}$$
$$+\frac{\partial}{\partial z}\left\{w\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+P+\rho gz\right)\right\}=0$$

and then integrate over z, from $z=b(\mathbf{x}_\perp,t)$ to $z=h(\mathbf{x}_\perp,t)$; this yields

$$\int_b^h\left\{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+\rho gz\right)+\nabla_\perp\cdot\left[\mathbf{u}_\perp\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+P+\rho gz\right)\right]\right\}$$
$$+\left[w\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+P+\rho gz\right)\right]_b^h=0.$$

The evaluations at the surface (s), and the bottom (b), from equations (1.28) and (1.35), then give

$$\int_b^h\left\{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+\rho gz)+\nabla_\perp\cdot\left[\mathbf{u}_\perp\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+P+\rho gz\right)\right]\right\}\mathrm{d}z\right.$$
$$+\{h_t+(\mathbf{u}_{\perp\mathrm{s}}\cdot\nabla_\perp)h\}\left(\frac{1}{2}\rho\mathbf{u}_\mathrm{s}\cdot\mathbf{u}_\mathrm{s}+P_\mathrm{s}+\rho gh\right)$$
$$-\{b_t+(\mathbf{u}_{\perp\mathrm{b}}\cdot\nabla_\perp)b\}\left(\frac{1}{2}\rho\mathbf{u}_\mathrm{b}\cdot\mathbf{u}_\mathrm{b}+P_\mathrm{b}+\rho gb\right)=0. \qquad (1.44)$$

As before, it is necessary to interchange the differential and integral operations (see Q1.30); the first of these integrals (involving $\partial/\partial t$) gives

$$\frac{\partial}{\partial t}\left\{\int_b^h\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+\rho gz\right)\mathrm{d}z\right\}-\left(\frac{1}{2}\rho\mathbf{u}_\mathrm{s}\cdot\mathbf{u}_\mathrm{s}+\rho gh\right)h_t$$
$$+\left(\frac{1}{2}\rho\mathbf{u}_\mathrm{b}\cdot\mathbf{u}_\mathrm{b}+\rho gb\right)b_t. \qquad (1.45)$$

The second integral (in $\nabla_\perp$) similarly becomes

$$\nabla_\perp\cdot\int_b^h\mathbf{u}_\perp\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+P+\rho gz\right)\mathrm{d}z-\left(\frac{1}{2}\rho\mathbf{u}_\mathrm{s}\cdot\mathbf{u}_\mathrm{s}+P_\mathrm{s}+\rho gh\right)(\mathbf{u}_{\perp\mathrm{s}}\cdot\nabla_\perp)h$$
$$+\left(\frac{1}{2}\rho\mathbf{u}_\mathrm{b}\cdot\mathbf{u}_\mathrm{b}+P_\mathrm{b}+\rho gb)(\mathbf{u}_{\perp\mathrm{b}}\cdot\nabla_\perp\right)b. \qquad (1.46)$$

Upon using (1.45) and (1.46) in equation (1.44), this equation reduces to

$$\frac{\partial}{\partial t}\left\{\int_b^h \left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+\rho gz\right)\mathrm{d}z\right\}+\nabla_\perp\cdot\int_b^h \mathbf{u}_\perp\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+P+\rho gz\right)\mathrm{d}z$$
$$+P_\mathrm{s}h_t-P_\mathrm{b}b_t=0. \qquad (1.47)$$

It is conventional to write

$$\mathscr{E}=\int_b^h \left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+\rho gz\right)\mathrm{d}z \qquad (1.48)$$

and

$$\mathscr{F}=\int_b^h \mathbf{u}_\perp\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+P+\rho gz\right)\mathrm{d}z, \qquad (1.49)$$

where $\mathscr{E}$ is the energy in the flow, per unit horizontal area, and $\mathscr{F}$ is the horizontal energy flux vector. The energy equation, (1.47), therefore becomes

$$\mathscr{E}_t+\nabla_\perp\cdot\mathscr{F}+\mathscr{P}=0, \qquad (1.50)$$

where $\mathscr{P}=P_\mathrm{s}h_t-P_\mathrm{b}b_t$ is the net energy input due to the pressure forces doing work on the upper and lower boundaries of the fluid. In the case of a stationary bottom boundary, then $b_t=0$; further, if the pressure in the fluid at the surface (P_s) is constant, then we may assign $P_\mathrm{s}=0$; consequently $\mathscr{P}=0$ and so

$$\mathscr{E}_t+\nabla_\perp\cdot\mathscr{F}=0. \qquad (1.51)$$

(If $P_\mathrm{s}=P_\mathrm{a}=$ constant, then we may redefine P in the fluid to be $P+P_\mathrm{a}$: the governing equations are unaltered. With this choice the surface pressure is now $P_\mathrm{s}=0$, but then the form of P used in (1.49) must be adjusted to accommodate this choice unless the P used in (1.49), and earlier, is measured relative to P_a. Of course, $P_\mathrm{s}=0$ is only possible if the coefficient of surface tension is set to zero; in general, the surface tension forces do work on the moving free surface.)

The energy equations presented here, particularly (1.51), can be used to describe the energy associated with a wave motion by averaging over a wavelength; see Section 2.1.2.

1.3 Nondimensionalisation and scaling

The governing equations and boundary conditions that have been described define a class of water-wave problems. For most of the discussions in this text, we shall be concerned with gravity waves propagating on the surface of an inviscid fluid. (This means that we shall often ignore the effects of surface tension, for example.) The arguments that suggest that such simplifying assumptions lead to problems worthy of consideration will be rehearsed later. However, we take this opportunity to emphasise that the main thrust of our work will be towards an understanding of the equations (and boundary conditions), and what they imply for wave propagation. It is not our purpose to provide an engineering or physical appraisal of the usefulness of these theories as they apply to the many and varied types of water waves that are encountered in nature. The importance of these considerations should not be underestimated though; they are paramount in the design of ships, offshore platforms, breakwaters, and dams, in the prediction and avoidance of catastrophes following earthquakes or storms, and a host of other areas of significance to mankind. Nevertheless, we shall extend our methods to some more obviously relevant and practical applications, such as flows with shear (rotational flows) and propagation over variable depth.

It is clear that our field of discussion will be somewhat restricted, but even so we shall still face immensely difficult mathematical problems that we wish to overcome. The most natural way forward is to develop a suitable – but systematic – approximation procedure. To this end we need to characterise problems in terms of the sizes of various fundamental parameters. These parameters are introduced by defining a set of nondimensional variables.

1.3.1 Nondimensionalisation

The nondimensionalisation that we adopt makes use of the length scales, time scales, etc., that naturally appear in the problem; this is altogether the obvious (and conventional) choice. First we introduce the appropriate length scales: we take h_0 to be a typical depth of the water and λ as the typical wavelength of the surface wave. (These and the other scales are depicted in Figure 1.4.) In order to define a time scale, we require a suitable velocity scale. Now, many of the problems that we shall consider involve the propagation of long waves, and the speed of these waves (as we shall demonstrate later) is approximately $\sqrt{gh_0}$; we make this choice

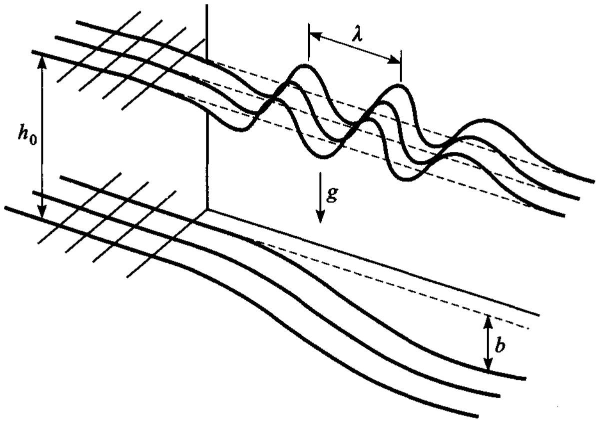

Figure 1.4. The scales for the water-wave problem: h_0 is the undisturbed or typical depth, λ is a typical wavelength, b is the bottom surface, and g is the acceleration of gravity.

for the speed scale. This choice is still useful even if we do not study, specifically, long gravity waves.

The characteristic speed, $\sqrt{gh_0}$, and the wavelength, λ, define a typical time associated with horizontal propagation, which is what interests us here; this is $\lambda/\sqrt{gh_0}$. We use $\sqrt{gh_0}$ to define the scale of the horizontal velocity components, but the vertical component (w) is treated differently. So that the equation of mass conservation makes good sense – and to be consistent with the boundary conditions – we must take this scale to be $h_0\sqrt{gh_0}/\lambda$. (One way to see this is to consider two-dimensional motion, for example

$$u_x + w_z = 0,$$

and then introduce the stream function, $\psi(x, z, t)$ (see Q1.20, Q1.34), so that

$$u = \psi_z \quad \text{and} \quad w = -\psi_x;$$

the scale of ψ is therefore $h_0\sqrt{gh_0}$, and that for w follows directly.)

The surface wave itself leads to the introduction of a further parameter: a typical (perhaps the maximum) amplitude of the wave. This is most conveniently done by writing the surface, $z = h(\mathbf{x}_\perp, t)$, as

$$h = h_0 + a\eta(\mathbf{x}_\perp, t) \tag{1.52}$$

where a is this typical amplitude; the function η is therefore nondimensional. We are now able to define the set of nondimensional variables, which we first do for rectangular Cartesian coordinates. (The cylindrical version is very similar.) Rather than introduce a new notation for all our variables, we choose – where convenient – to write, for example, $x \to \lambda x$. This is to be read that x is replaced by λx, so that hereafter the symbol x will denote a nondimensional variable. With this understanding, we define

$$x \to \lambda x, \quad y \to \lambda y, \quad z \to h_0 z, \quad t \to (\lambda/\sqrt{gh_0})t, \tag{1.53}$$

$$u \to \sqrt{gh_0}u, \quad v \to \sqrt{gh_0}v, \quad w \to (h_0\sqrt{gh_0}/\lambda)w \tag{1.54}$$

with

$$h = h_0 + a\eta \quad \text{and} \quad b \to h_0 b. \tag{1.55}$$

Finally, the pressure is rewritten as

$$P = P_a + \rho g(h_0 - z) + \rho g h_0 p \tag{1.56}$$

where P_a is the (constant) pressure of the atmosphere, $\rho g(h_0 - z)$ the hydrostatic pressure distribution (see Q1.11) and the pressure scale, $\rho g h_0$, is based on the pressure at depth $z = h_0$. The pressure variable p introduced here, therefore measures the deviation from the hydrostatic pressure distribution; we shall find that $p \neq 0$ during the passage of a wave.

The Euler equation in component form, (1.13), and the equation of mass conservation, (1.14), now become

$$\left.\begin{aligned} &\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x}, \quad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{\partial p}{\partial y}, \quad \delta^2\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z}, \\ &\text{where} \\ &\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}, \end{aligned}\right\} \tag{1.57}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{1.58}$$

These equations are written exclusively in terms of nondimensional variables, where $\delta = h_0/\lambda$ is the *long wavelength* or *shallowness* parameter; we shall have much to write about δ later.

The corresponding nondimensionalisation for cylindrical coordinates is precisely that given in (1.53)–(1.56), but with the transformations on x and y replaced by

$$r \to \lambda r. \tag{1.59}$$

The governing equations, (1.15) and (1.16), therefore become

$$\left.\begin{aligned}
&\frac{\mathrm{D}u}{\mathrm{D}t} - \frac{v^2}{r} = -\frac{\partial p}{\partial r}, \quad \frac{\mathrm{D}v}{\mathrm{D}t} + \frac{uv}{r} = -\frac{1}{r}\frac{\partial p}{\partial \theta}, \quad \delta^2 \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z},\\
&\text{where}\\
&\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}
\end{aligned}\right\} \tag{1.60}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0, \tag{1.61}$$

all expressed in nondimensional variables; δ is defined exactly as above: $\delta = h_0/\lambda$. Finally, in both versions, the upper and lower surfaces of the fluid are represented by

$$z = 1 + \varepsilon\eta \quad \text{and} \quad z = b, \tag{1.62}$$

respectively. Here we have introduced the second important parameter in water-wave theory: $\varepsilon = a/h_0$, the *amplitude* parameter.

Now we turn to the boundary conditions, which are treated in precisely the same fashion. Thus we see that the surface kinematic condition, (1.28), becomes

$$w = \varepsilon\{\eta_t + (\mathbf{u}_\perp \cdot \nabla)\eta\} \quad \text{on} \quad z = 1 + \varepsilon\eta, \tag{1.63}$$

in nondimensional variables. Similarly, the most general dynamic condition that we shall use in most of our work, (1.31) with (1.32) (for $h = h(x, y, t)$), yields

$$p - \varepsilon\eta = -\varepsilon\left(\frac{\Gamma}{\rho g \lambda^2}\right)\left\{\frac{(1+\varepsilon^2\delta^2\eta_y^2)\eta_{xx} + (1+\varepsilon^2\delta^2\eta_x^2)\eta_{yy} - 2\varepsilon^2\delta^2\eta_x\eta_y\eta_{xy}}{(1+\varepsilon^2\delta^2\eta_x^2 + \varepsilon^2\delta^2\eta_y^2)^{3/2}}\right\}$$
$$\text{on } z = 1 + \varepsilon\eta, \tag{1.64}$$

where we write $\Gamma/(\rho g \lambda^2) = \delta^2 W$ with $W = \Gamma/(\rho g h_0^2)$, a *Weber number*. (It is usual to define this with respect to the appropriate $(\text{speed})^2 = gh_0$, and the corresponding depth scale; sometimes, to avoid confusion, we shall write W_e for W.) This nondimensional parameter is used to measure the size of the surface tension contribution. A corresponding result is obtained in cylindrical coordinates, with $h = h(r, \theta, t)$; see Q1.36. An alternative dynamic condition is provided by the pressure equation, (1.29), for irrotational flow; this is discussed in Q1.37. Finally, the bottom boundary condition, (1.35), yields the unchanged form

$$w = b_t + (\mathbf{u}_\perp \cdot \nabla)b \quad \text{on} \quad z = b, \tag{1.65}$$

in nondimensional variables.

1.3.2 Scaling of the variables

We have described the nondimensionalisation of the governing equations, but another equally important transformation is also required. An examination of the surface boundary conditions, (1.63) and (1.64), yields the observation that both w and p (on $z = 1 + \varepsilon\eta$) are essentially proportional to ε; that is, proportional to the wave amplitude. This makes good sense, particularly as $\varepsilon \to 0$, for then $w \to 0$ and $p \to 0$: there is no disturbance of the free surface – it becomes a horizontal surface on which $w = 0 = p$. Thus we define a set of scaled variables, chosen to be consistent with the boundary conditions and governing equations; we write (again avoiding the introduction of a new notation)

$$p \to \varepsilon p, \quad w \to \varepsilon w, \quad (u, v) \to \varepsilon(u, v) \quad (\text{or } \mathbf{u}_\perp \to \varepsilon \mathbf{u}_\perp). \tag{1.66}$$

(The original, physical, variables are easily recovered from (1.66) and (1.53)–(1.56); for example, if w is the scaled variable from (1.66), then $\varepsilon(h_0\sqrt{gh_0}/\lambda)w$ is the original w.)

The equations (1.57) and (1.58) become

$$\left.\begin{aligned} &\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x}, \quad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{\partial p}{\partial y}, \quad \delta^2\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z}, \\ &\text{where} \\ &\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \varepsilon\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right) \\ &\text{and} \\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \end{aligned}\right\} \tag{1.67}$$

The equations in cylindrical coordinates, (1.60) and (1.61), are, correspondingly,

$$\left.\begin{aligned}
&\frac{\mathrm{D}u}{\mathrm{D}t}-\frac{\varepsilon v^2}{r}=-\frac{\partial p}{\partial r}, \quad \frac{\mathrm{D}v}{\mathrm{D}t}+\frac{\varepsilon uv}{r}=-\frac{1}{r}\frac{\partial p}{\partial \theta}, \quad \delta^2\frac{\mathrm{D}w}{\mathrm{D}t}=-\frac{\partial p}{\partial z},\\
&\text{where}\\
&\qquad\frac{\mathrm{D}}{\mathrm{D}t}\equiv\frac{\partial}{\partial t}+\varepsilon\left(u\frac{\partial}{\partial r}+\frac{v}{r}\frac{\partial}{\partial \theta}+w\frac{\partial}{\partial z}\right)\\
&\text{and}\\
&\qquad\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z}=0.
\end{aligned}\right\} \tag{1.68}$$

The surface boundary conditions, (1.63) and (1.64), are now written as

$$\left.\begin{aligned}
w&=\eta_t+\varepsilon(\mathbf{u}_\perp\cdot\nabla_\perp)\eta\\
p&=\eta-\delta^2 W\left\{\frac{(1+\varepsilon^2\delta^2\eta_y^2)\eta_{xx}+(1+\varepsilon^2\delta^2\eta_x^2)\eta_{yy}-2\varepsilon^2\delta^2\eta_x\eta_y\eta_{xy}}{(1+\varepsilon^2\delta^2\eta_x^2+\varepsilon^2\delta^2\eta_y^2)^{3/2}}\right\}
\end{aligned}\right\} \tag{1.69}$$

both on $z=1+\varepsilon\eta$, and on the bottom (1.65) becomes

$$w=\varepsilon^{-1}b_t+(\mathbf{u}_\perp\cdot\nabla_\perp)b \quad \text{on} \quad z=b. \tag{1.70}$$

For this last boundary condition we shall consider problems for which b_t is proportional to ε (or smaller); indeed, for almost all our discussions the bottom boundary will be stationary, so $b_t\equiv 0$. (The scaled dynamic conditions in cylindrical coordinates, and for irrotational flow, are given in Q1.36 and Q1.37, respectively.)

As we shall discuss in due course, scaling is not restricted to the dependent variables. Much of our later work (particularly in Chapters 3 and 4) relies on seeking solutions in appropriate scaled regions of space and time. So, for example, we might be interested in the solution when the depth variation is slow (for example, $b=b(\varepsilon\mathbf{x}_\perp)$), and then the transformation (scaling) $\mathbf{x}_\perp\to\varepsilon\mathbf{x}_\perp$ is likely to be required. This, and related ideas, will be described more fully in the brief introduction to asymptotics and multiple scales (Section 1.4), and when we need to develop the techniques needed to solve specific problems.

1.3.3 Approximate equations

The significance and usefulness of the nondimensionalisation and scaling presented above will now be made clear. The parameters, ε and δ, are used to define, in a rather precise manner, various approximate versions of the governing equations and boundary conditions. Similar ideas apply to the other parameters (such as W and R, the Weber and Reynolds numbers, respectively); we shall comment on these as it becomes necessary.

The two most commonly used – and useful – approximations are

(a) $\varepsilon \to 0$: the linearised problem;
(b) $\delta \to 0$: the long-wave (or shallow-water) problem.

The first of these, case (a), requires that the amplitude of the surface wave be small; then, in a first approximation, the equations become linear. For example, in rectangular Cartesian coordinates, equations (1.67), (1.69), and (1.70) simplify to

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}, \quad \frac{\partial v}{\partial t}=-\frac{\partial p}{\partial y}, \quad \delta^2\frac{\partial w}{\partial t}=-\frac{\partial p}{\partial z}, \quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,\\
&\text{with}\\
&\qquad w=\eta_t \quad \text{and} \quad p=\eta-\delta^2 W(\eta_{xx}+\eta_{yy}) \quad \text{on} \quad z=1\\
&\text{and}\\
&\qquad\qquad w=(\mathbf{u}_\perp\cdot\nabla_\perp)b \quad \text{on} \quad z=b(<1).
\end{aligned}\right\} \tag{1.71}$$

In these equations we have chosen $b_t \equiv 0$, and treated δ and W as fixed parameters as $\varepsilon \to 0$, as they clearly are. We note that, in particular, the evaluation on the (unknown) free surface has become an evaluation on the known surface, $z=1$, even though the unknown free surface, η, still appears in the equations. The linear equations expressed in cylindrical coordinates take a similar form (from equations (1.68) and (1.70) and Q1.36). (The corresponding equations for irrotational flow are obtained in Q1.38.)

For case (b), the waves are long; that is, of long wavelength (or the water is shallow), in the sense that $\delta = h_0/\lambda$ is small. (Both descriptions are commonly used; we shall more often use the former – long waves – rather than the latter.) This time we keep ε and W fixed, and (with $b_t \equiv 0$) the approximation $\delta \to 0$ yields the problem

$$\left.\begin{aligned}
&\frac{\mathrm{D}u}{\mathrm{D}t}=-\frac{\partial p}{\partial x}, \quad \frac{\mathrm{D}v}{\mathrm{D}t}=-\frac{\partial p}{\partial y}, \quad \frac{\partial p}{\partial z}=0, \quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,\\
&\text{where}\\
&\qquad\qquad \frac{\mathrm{D}}{\mathrm{D}t}\equiv\frac{\partial}{\partial t}+\varepsilon\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)
\end{aligned}\right\} \tag{1.72}$$

with

$$w = \eta_t + \varepsilon(\mathbf{u}_\perp \cdot \nabla_\perp)\eta \quad \text{and} \quad p = \eta \quad \text{on} \quad z = 1 + \varepsilon\eta$$

and

$$w = (\mathbf{u}_\perp \cdot \nabla_\perp)b \quad \text{on} \quad z = b.$$

The equations which describe small amplitude *and* long waves (so $\varepsilon \to 0$ and $\delta \to 0$), are clearly consistent with both sets (1.71) and (1.72): the resulting equations are those of (1.71), but with

$$\frac{\partial p}{\partial z} = 0; \quad p = \eta \quad \text{on} \quad z = 1, \tag{1.73}$$

or (1.72) with $\varepsilon = 0$.

The solutions of these various approximate equations will form the basis for many of our descriptions in the selection of classical water-wave problems presented in Chapter 2.

1.4 The elements of wave propagation and asymptotic expansions

In this final section we describe the basic ideas that provide the essential background to any discussion of wave propagation. We shall present a brief overview of the mathematical description of elementary wave propagation: *d'Alembert's solution* of the wave equation, and the important properties of *dispersion*, *dissipation* and *nonlinearity*. Then we shall outline the concept of an asymptotic expansion, and show how this can be used to obtain appropriate asymptotic solutions of wave-like equations. This will introduce the important technique of rescaling the variables with respect to the (small) parameter(s) in the problem.

1.4.1 Elementary ideas in the theory of wave propagation

Wave propagation theories, at their simplest, usually involve the application of fundamental physical principles (to the motion of a stretched string, for example), leading to the classical one-dimensional *wave equation*

$$u_{tt} - c^2 u_{xx} = 0. \tag{1.74}$$

The function $u(x, t)$ represents the amplitude of the wave, c (> 0) is a constant, and the subscripts denote partial derivatives. This equation has the general solution

$$u(x, t) = f(x - ct) + g(x + ct), \tag{1.75}$$

written in terms of the *characteristic variables* $(x \pm ct)$; this solution, (1.75), is commonly described as *d'Alembert's solution*, where f and g are arbitrary functions. If, as is usual, x is a spatial coordinate and t a time coordinate, then c is a speed, so the solution represents right (f) and left (g) propagating waves. The functions f and g can be determined, for example, from suitable initial data, such as u and u_t prescribed at $t = 0$ (the *Cauchy problem*); see Q1.39.

The two wave components, f and g, propagate at constant speed (c) with unchanging form; they do not interact with themselves nor with each other. This is equivalent to the statement that the governing equation is *linear* which, of course, is precisely the form of (1.74). Each component is a separate and independent linear wave.

Now, for most of our work on water waves, we shall describe waves that propagate only in one direction (which usually will be to the right). One simple way to do this is simply to set $g \equiv 0$; an alternative is to suppose that the initial data is on *bounded* (or *compact*) *support*. Then, after an appropriate finite time, the two components (f and g) will move apart and no longer overlap (see Q1.40). In either event, it is then possible to follow just the one component. An equivalent approach is to restrict the discussion, *ab initio*, to waves propagating in one direction only; this is accomplished by working with the equation

$$u_t + cu_x = 0, \tag{1.76}$$

which has the general solution

$$u(x, t) = f(x - ct). \tag{1.77}$$

This is then completely determined, given the function $u(x, 0) = f(x)$.

Wave propagation equations, at least when derived from more complete physical models, are unlikely to be as simple as (1.74) or (1.76). More careful analyses, but with the restriction to unidirectional propagation, might lead to the linear equations

$$u_t + u_x + u_{xxx} = 0 \tag{1.78}$$

or

$$u_t + u_x - u_{xx} = 0. \tag{1.79}$$

(In these two equations, the coefficients have been normalised; this is always possible by redefining $x \to \alpha x$, $t \to \beta t$, for suitable constants

α, β.) One very familiar method for solving linear partial differential equations is to seek the *harmonic solution*

$$u(x, t) = e^{i(kx-\omega t)}, \tag{1.80}$$

where k is a real parameter. (A real solution for u can be constructed by taking the real or imaginary part, or by forming $A \exp\{i(kx - \omega t)\}$ + complex conjugate, where $A(k)$ is complex valued.) Upon substitution of (1.80) into (1.78) and (1.79), it follows that (1.80) is a solution of (1.78) if

$$\omega = k - k^3, \tag{1.81}$$

and of (1.79) if

$$\omega = k - ik^2. \tag{1.82}$$

In the case of (1.81), we see that

$$kx - \omega t = k\{x - (1 - k^2)t\},$$

so that the speed of propagation,

$$\frac{\omega}{k} = 1 - k^2, \tag{1.83}$$

is a function of k. Thus waves with different *wave number*, k, travel at different speeds (which, in this example, might be to the left or the right, depending on whether $k^2 > 1$ or $k^2 < 1$, respectively). This property of a wave is known as *dispersion*, and the wave is said to be *dispersive*; equation (1.78) is the simplest (unidirectional) dispersive wave equation and (1.81) is its *dispersion relation*. A solution of this equation, which is the sum of two components, each associated with different values of k, exhibits the property that each component will move at its own speed given by (1.83). Thus, if the solution is initially on compact support, the two components will move apart, or *disperse*. The separate components do *not* change shape, although the observed sum does give the appearance of a changing profile.

The speed, ω/k, is called the *phase speed* of the wave; this describes the motion of each individual component. However, as we shall discuss later, another speed, defined by $d\omega/dk$, describes the motion of a *group* of waves. This is called the *group speed* and, as we shall explain later, it is the speed at which energy is propagated.

A similar discussion for equation (1.79) yields

$$u(x, t) = \exp\{ik(x - t) - k^2 t\};$$

this describes a wave which propagates at a speed of unity (to the right) for all k, but which decays as $t \to +\infty$ (for $k \neq 0$). This phenomenon of a decaying wave is called *dissipation*; it usually arises from a physical system that incorporates some frictional behaviour, such as fluid viscosity. The values of the coefficients in equation (1.79) are unimportant, but the relative sign of the terms u_t and u_{xx} is; if this changes then the wave amplitude will grow without bound as $t \to +\infty$. (Further one-dimensional linear wave equations of this type can be constructed, with combinations of even and/or odd derivatives in x; see Q1.41.)

Finally, for us, a significant property of many of the waves that we shall encounter is that they are *nonlinear*. The simplest model equations usually involve linearisation (and so, perhaps, might lead to equations (1.78) or (1.79)), but a more careful analysis will often lead to a nonlinear equation, such as

$$u_t + (1+u)u_x = 0. \tag{1.84}$$

The general solution of this equation is obtained directly from the *method of characteristics*:

$$u = \text{constant on lines } \frac{\mathrm{d}x}{\mathrm{d}t} = 1 + u.$$

Thus, supposing that we are given $u(x, 0) = f(x)$, the solution of (1.84) is

$$u(x, t) = f\{x - (1+u)t\} \tag{1.85}$$

which, for general f, provides an *implicit* relation for $u(x, t)$. Only when f is particularly simple is it possible to solve for u explicitly; see Q1.43. Nevertheless, the solution can always be represented geometrically by using the information carried along the characteristic lines. Thus any point on a wave profile, at which u takes the value u_0, will propagate at the constant speed $1 + u_0$. Consequently, points of larger u_0 move faster than those of smaller u_0; this implies that a wave profile will change shape, as represented in Figure 1.5. This might result in a profile which becomes multivalued after a finite time, as our figure shows; this corresponds to the intersection of the characteristic lines. When this happens, it is usual to regard the solution as unacceptable, because we normally expect the solution-function to be single-valued (in x for any t). The solution can be made single-valued by the insertion of a discontinuity (or jump) which separates the characteristic lines and does not allow them to intersect; this is shown in Figure 1.6. (A discontinuous function is not, strictly, a proper solution of the equation (1.84), but it should be a

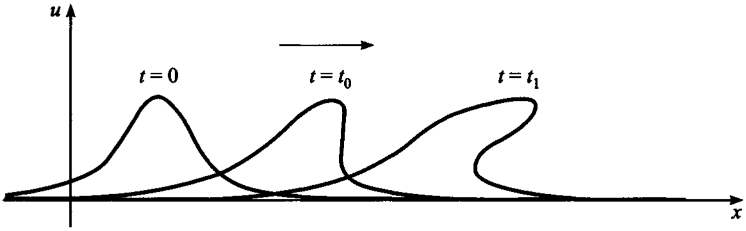

Figure 1.5. A breaking wave, according to equation (1.85); at $t = t_0$ the wave is about to break and at $t = t_1$ ($> t_0$) the wave has broken.

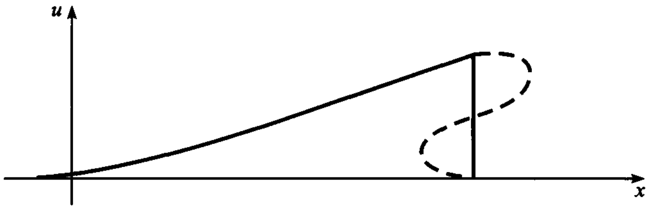

Figure 1.6. The insertion of a discontinuity (or jump) to make the solution single-valued.

solution of the underlying integral equation; see, for example, equations (1.1) and (1.2). A fuller discussion of discontinuous solutions will be given in Section 2.7.)

The form of the wave, for $t > t_0$ in Figure 1.5, is reminiscent of a wave breaking on a beach; indeed, this type of solution of a nonlinear equation is often called a *breaking wave*. However, this similarity is altogether superficial; waves that approach a beach, and then break, are described by a much more involved theory (which essentially requires the full water-wave equations). A related problem is described in Section 2.8.

1.4.2 Asymptotic expansions

Finally, we introduce the ideas that form the basis for handling the equations and problems that we encounter in water-wave theory, at least in the initial stages of much of the work. The technique that we adopt involves the construction of *asymptotic expansions*. This branch of mathematics has a reasonably long history, and one that has not been divorced from controversy (mainly over the interpretation of divergent series, which often appear in this work). The first systematic approach, to both the definition and use of asymptotic expansions, is due to Poincaré; we shall follow his lead.

First we require a little bit of notation: we write

$$f(x) = \mathrm{o}(g(x)); \quad f(x) = \mathrm{O}(g(x)); \quad f(x) \sim g(x),$$

as $x \to x_0$, if

$$\lim_{x \to x_0} [f(x)/g(x)]$$

is zero, finite non-zero, or unity, respectively. These are usually read as 'little oh', 'big oh' and 'varies as' (or 'asymptotically equal to'), respectively; the function $f(x)$ is the given function under discussion, and $g(x)$ is a suitable *gauge* function. We can then write in this notation

$$\begin{aligned} f(x) &= \frac{1}{2+x^2} = \mathrm{o}(x^{-1}) && \text{as } |x| \to \infty; \\ f(x) &= \frac{1}{2+x^2} = \mathrm{O}(1) && \text{as } x \to 0; \\ f(x) &= \sin 2x \sim 2x && \text{as } x \to 0, \end{aligned}$$

for example. It should be noted that the limit in which the behaviour occurs must be included in the statement of the behaviour.

This description of a function (in a limit) is now extended: we write

$$f(x) - \sum_{n=0}^{N-1} g_n(x) \sim g_N(x) \quad \text{as } x \to x_0,$$

for every $N \geq 1$, where $f(x) \sim g_0(x)$ as $x \to x_0$. It is then usual (and convenient) to express this property in the form

$$f(x) \sim \sum_{n=0}^{\infty} g_n(x) \quad \text{as } x \to x_0, \tag{1.86}$$

where N has been taken to infinity here; this 'series' is called an *asymptotic expansion* of $f(x)$, as $x \to x_0$. Of course, this is only a shorthand notation and does not imply any convergence (or otherwise) of the series in (1.86). In practice, asymptotic expansions are rarely taken beyond a few terms, but it must be possible – in principle – to find them all. This representation is merely a compact way of describing a sequence of limiting processes (as $x \to x_0$) on the functions $\{f(x)/g_0(x)\}$, $[\{f(x) - g_0(x)\}/g_1(x)]$, etc. However, the functions that we shall be working with involve one (or more) parameters; this is now introduced into our definition.

The asymptotic expansions that we require are defined with respect to a parameter, ε say, as $\varepsilon \to 0$, at fixed x. The asymptotic expansion of $f(x; \varepsilon)$ is then written as

$$f(x;\varepsilon) \sim \sum_{n=0}^{\infty} f_n(x;\varepsilon) \quad \text{as} \quad \varepsilon \to 0 \text{ at fixed } x \text{ (or } x = \mathrm{O}(1)), \tag{1.87}$$

where $f_{n+1}(x; \varepsilon) = \mathrm{o}\{f_n(x; \varepsilon)\}$ (as $\varepsilon \to 0$) for every $n \geq 0$. If this asymptotic expansion, (1.87), is defined for all x in the domain of the function, $f(x; \varepsilon)$, the expansion is said to be *uniformly valid*. However, if there is some x (in the domain), and some n for which

$$f_{n+1}(x;\varepsilon) \neq \mathrm{o}\{f_n(x;\varepsilon)\}$$

as $\varepsilon \to 0$, then the asymptotic expansion is said to *break down*, or to be *non-uniform*. Here we have written each term in the expansion as $f_n(x; \varepsilon)$, but quite often this occurs in the much simpler *separable* form: $f_n(x; \varepsilon) = \varepsilon^n a_n(x)$. Happily, this is usually the situation for our problems in water waves.

Briefly, we describe these ideas by considering the example

$$f(x;\varepsilon) = (1 + \varepsilon x + \mathrm{e}^{-x/\varepsilon})^{-1}, \quad 1 \leq x \leq 2, \tag{1.88}$$

for $\varepsilon \to 0^+$. For any $x = \mathrm{O}(1)$, in the given domain, we may therefore write

$$f(x;\varepsilon) \sim 1 - \varepsilon x + \varepsilon^2 x^2 \tag{1.89}$$

or

$$f(x;\varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n (-x)^n \tag{1.90}$$

or even

$$f(x;\varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n (-x)^n - \mathrm{e}^{-x/\varepsilon}. \tag{1.91}$$

(In this last case, it should be remembered that $\mathrm{e}^{-x/\varepsilon} = \mathrm{o}(\varepsilon^n)$ as $\varepsilon \to 0^+$, for every n, if $x > 0$.) Now let us take the same function, (1.88), but define the domain as $0 \leq x \leq 2$; the asymptotic expansions, (1.89)–(1.91), are clearly not uniformly valid when $x = \mathrm{O}(\varepsilon)$, for then $\mathrm{e}^{-x/\varepsilon} = \mathrm{O}(1)$. This choice of x is usually expressed by writing

$$x = \varepsilon X, \quad X = \mathrm{O}(1) \quad \text{as } \varepsilon \to 0;$$

the original function then becomes

$$f(\varepsilon X;\varepsilon) \equiv F(X;\varepsilon) = (1+\varepsilon^2 X + \mathrm{e}^{-X})^{-1},$$

so

$$F(X;\varepsilon) \sim \frac{1}{1+\mathrm{e}^{-X}} - \frac{\varepsilon^2 X}{1+\mathrm{e}^{-X}} \quad \text{as } \varepsilon \to 0. \tag{1.92}$$

Finally, if we further extend the domain to $0 \le x < \infty$, the asymptotic expansions, (1.89)–(1.91), are now not uniformly valid also for $x = \mathrm{O}(\varepsilon^{-1})$. For an x of this magnitude, we define

$$x = \chi/\varepsilon, \ \ \chi = \mathrm{O}(1) \quad \text{as } \varepsilon \to 0,$$

and then

$$f(\chi/\varepsilon;\varepsilon) \equiv \mathscr{F}(\chi;\varepsilon) = (1+\chi+\mathrm{e}^{-\chi/\varepsilon^2})^{-1}$$

which gives

$$\mathscr{F}(\chi;\varepsilon) \sim \frac{1}{1+\chi} \quad \text{as } \varepsilon \to 0. \tag{1.93}$$

These various asymptotic expansions also satisfy the *matching principle*. To demonstrate this, we consider the asymptotic expansions (1.89), (1.92), and (1.93). Thus

$$f \sim 1 - \varepsilon x + \varepsilon^2 x^2 = 1 - \varepsilon^2 X + \varepsilon^4 X^2 \sim 1 - \varepsilon^2 X \quad \text{as } \varepsilon \to 0^+, \ X = \mathrm{O}(1),$$

matches with

$$F \sim \frac{1}{1+\mathrm{e}^{-X}} - \frac{\varepsilon^2 X}{1+\mathrm{e}^{-X}} = \sim 1 - \varepsilon x \quad \text{as } \varepsilon \to 0^+, \ x = \mathrm{O}(1).$$

Similarly,

$$f \sim 1 - \varepsilon x + \varepsilon^2 x^2 = 1 - \chi + \chi^2 \sim 1 - \chi + \chi^2 \quad \text{as } \varepsilon \to 0^+, \ \chi = \mathrm{O}(1),$$

matches with

$$\mathscr{F} \sim \frac{1}{1+\chi} = \frac{1}{1+\varepsilon x} \sim 1 - \varepsilon x + \varepsilon^2 x^2 \quad \text{as } \varepsilon \to 0^+, \ x = \mathrm{O}(1).$$

Simple asymptotic expansions, and the matching principle, are briefly explored in Q1.45, Q1.46; the reader who requires a more expansive and comprehensive discussion of asymptotic expansions, the matching principle, etc., should consult the texts mentioned at the end of this chapter.

The application of asymptotic methods to the solution of differential equations, at least in the context of wave-like problems, is reasonably straightforward and requires no deep knowledge of the subject. The process is initiated – almost always – by assuming that a solution exists (for O(1) values of the independent variables) as a suitable asymptotic expansion with respect to the relevant small parameter. The form of this expansion is governed by the way in which the parameter appears in the equation and, perhaps, also how it appears in the boundary/initial conditions. Usually, a rather simple iterative construction will suggest how this expansion proceeds. In order to explain and describe how these ideas are relevant in theories of wave propagation (and, therefore, to our study of water waves), we consider the partial differential equation

$$u_{tt} - u_{xx} = \varepsilon(u^2 + u_{xx})_{xx}. \tag{1.94}$$

The small parameter, ε, in this equation (which here represents the characteristics of both small amplitude and long waves) suggests that we seek a solution in the form

$$u(x, t; \varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n u_n(x, t) \quad \text{as } \varepsilon \to 0, \tag{1.95}$$

for $x = \mathrm{O}(1)$, $t = \mathrm{O}(1)$. We shall suppose that equation (1.94) is to be solved in $t \geq 0$ and for $-\infty < x < \infty$, with appropriate initial data being prescribed on $t = 0$ (that is, the *Cauchy problem*). The expansion (1.95) is then a solution of equation (1.94) if

$$u_{0tt} - u_{0xx} = 0; \quad u_{1tt} - u_{1xx} = (u_0^2 + u_{0xx})_{xx},$$

and so on. To obtain these, we simply collect together like powers of ε and set to zero each coefficient of ε^n.

We see immediately that the general solution of u_0 (d'Alembert's solution) is

$$u_0(x, t) = f(x - t) + g(x + t),$$

and we will suppose that the initial data is such as to generate only the right-going wave; for example

$$u(x, 0; \varepsilon) = f(x), \quad u_t(x, 0; \varepsilon) = -f'(x). \tag{1.96}$$

(This choice is not strictly necessary, even for our purposes; we could prescribe initial data on compact support as we have mentioned before (with $x = \mathrm{O}(1)$) and then, for large enough time (as we use below), the right- and left-going waves move apart and we may elect to follow just one component; see Q1.40.)

Now, with $u_0 = f(x-t)$, we see that

$$u_{1tt} - u_{1xx} = (f^2 + f'')'', \tag{1.97}$$

where the prime denotes the derivative with respect to $(x-t)$. To proceed, it is convenient to introduce the characteristic variables for this equation,

$$\xi = x - t, \quad \zeta = x + t,$$

so that equation (1.97) becomes

$$-4u_{1\xi\zeta} = (f^2 + f'')'',$$

and hence

$$u_1(\xi, \zeta) = -\frac{1}{4}\zeta(f^2 + f'')' + A(\xi) + B(\zeta),$$

where $f = f(\xi)$. The arbitrary functions, A and B, are determined from the initial data: if we use that choice given above, (1.96), then we require (for $u_1(x, t)$)

$$u_1(x, 0) = 0, \quad u_{1t}(x, 0) = 0$$

(since these data, (1.96), are independent of ε), so

$$u_1(\xi, \zeta) = \frac{1}{4}[(\xi - \zeta)\{f^2(\xi) + f''(\xi)\}' + f^2(\zeta) + f''(\zeta) - f^2(\xi) - f''(\xi)]$$

or

$$u_1(x, t) = -\frac{1}{2}tF'(x-t) + \frac{1}{4}\{F(x+t) - F(x-t)\},$$

where $F = f^2 + f''$. The asymptotic expansion, so far, is therefore

$$u(x, t; \varepsilon) \sim f(x-t) - \frac{\varepsilon}{4}\{2tF'(x-t) + F(x-t) - F(x+t)\}. \tag{1.98}$$

For $f(x)$ on compact support (and suitably differentiable), or at least for $f(x) \to 0$ (sufficiently rapidly) as $|x| \to \infty$, it is clear that the asymptotic expansion (1.98) is not uniformly valid for $\varepsilon t = \mathrm{O}(1)$. Further, for our stated condition on $f(x)$, we need consider only $\xi = \mathrm{O}(1)$ and thus we now examine the solution of equation (1.94) for

$$\xi = x - t = \mathrm{O}(1), \quad \tau = \varepsilon t = \mathrm{O}(1) \quad \text{as } \varepsilon \to 0. \tag{1.99}$$

(The asymptotic expansion, (1.98), will also be non-uniform at any values of ξ for which the first, second, or third derivatives of $f(\xi)$ are undefined; we do not normally countenance this possibility in these types of problem. From the above, we see that (1.98) is non-uniform in t no matter how well-behaved $f(\xi)$ might be – and we note that f = constant is of no practical interest!)

In wave-like problems, the region where a large time (or distance) variable is used (like τ in (1.99)) is usually called the *far-field*; the corresponding region for $t = O(1)$ is then referred to as the *near-field*. We note that, for $\xi = x - t = O(1)$, then $t = O(\varepsilon^{-1})$ implies that $x = O(\varepsilon^{-1})$; this relationship between the various asymptotic regions is made clear in Figure 1.7.

The transformation (1.99), applied to equation (1.94), makes use of the identities

$$\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial \xi} \quad \text{and} \quad \frac{\partial}{\partial t} \equiv \varepsilon \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \xi};$$

then the equation for $u(x, t; \varepsilon) \equiv U(\xi, \tau; \varepsilon)$ becomes

$$\varepsilon U_{\tau\tau} - 2U_{\tau\xi} = (U^2 + U_{\xi\xi})_{\xi\xi}. \tag{1.100}$$

An asymptotic solution of this equation is sought in the form

$$U(\xi, \tau; \varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n U_n(\xi, \tau), \quad \varepsilon \to 0, \tag{1.101}$$

for $\xi = O(1)$, $\tau = O(1)$, and then U_0 will satisfy the equation

$$2U_{0\tau\xi} + (U_0^2 + U_{0\xi\xi})_{\xi\xi} = 0,$$

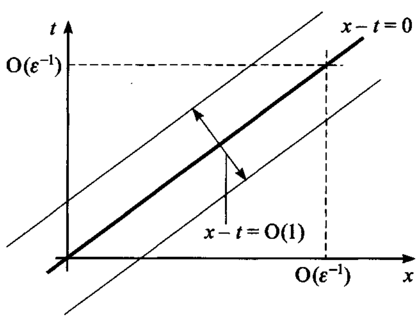

Figure 1.7. A schematic representation of the far-field, where $x = O(\varepsilon^{-1})$, $t = O(\varepsilon^{-1})$, with $x - t = O(1)$; the wavefront is $x - t = 0$.

or

$$2U_{0\tau} + 2U_0 U_{0\xi} + U_{0\xi\xi\xi} = 0, \tag{1.102}$$

where we have invoked decay conditions as $|\xi| \to \infty$. This equation, (1.102), is a third-order nonlinear partial differential equation, which is one variant of a very famous equation: the *Korteweg–de Vries equation*, of which we shall write much (Chapter 3). It turns out that we can formulate the solution of this equation which satisfies (the matching condition)

$$U_0 \to f(\xi) \quad \text{as } \tau \to 0,$$

which corresponds to the initial-value problem for equation (1.102); this solution exists provided $f(\xi)$ decays sufficiently rapidly as $|\xi| \to \infty$. (The method of solution required here is at the heart of *inverse scattering transform* – or *soliton* – theory.) The solution thus obtained, for U_0, constitutes a one-term uniformly valid asymptotic expansion for $\tau \geq 0$ and $\tau = O(1)$ (as $\varepsilon \to 0$). The next term in this expansion satisfies the equation

$$2U_{1\tau\xi} + 2(U_0 U_1)_{\xi\xi} + U_{1\xi\xi\xi\xi} = U_{0\tau\tau}$$

or

$$2U_{1\tau} + 2(U_0 U_1)_{\xi} + U_{1\xi\xi\xi} = -(U_0^2 + U_{0\xi\xi})_{\tau},$$

where we have used equation (1.102) for $U_{0\tau}$, and again imposed decay conditions as $|\xi| \to \infty$. The analysis hereafter is not particularly straightforward; the solution for U_1 is obtained by writing $U_1 = U_{0\xi} V(\xi, \tau)$, which can then be examined to see if the asymptotic expansion (1.101) is uniformly valid as $\tau \to \infty$. This involves very detailed and lengthy discussions, particularly if the general term (U_n) is to be included; such an analysis is altogether beyond the scope of our investigations. Suffice it to record that, for an $f(x)$ which is smooth enough and which decays rapidly (exponentially, for example) at infinity, the far-field expansion in problems of this type is usually uniformly valid. (For some problems, though, it is necessary to write the characteristic variable itself as an asymptotic expansion, a technique related to the familiar method known as the *method of strained coordinates*. That this might be required is easily seen if we attempt to find a representation of the exact characteristics of the original equation. Some of these ideas are touched on in the exercises; see Q1.47–Q1.49, Q1.53.)

Finally, we describe one other type of asymptotic formulation which is often used in wave-like problems; this is based on the *method of multiple scales*. As before, we explain the salient features by developing the ideas for a particular equation (which will be typical of some of our problems in water-wave theory). We consider the equation

$$u_{tt} - u_{xx} - u + \varepsilon(uu_x)_x = 0; \tag{1.103}$$

for $\varepsilon = 0$ this equation has a travelling-wave solution, expressed as a harmonic wave (see (1.80)),

$$u = Ae^{\mathrm{i}(kx-\omega t)} + \text{c.c.}, \tag{1.104}$$

which c.c. denotes the complex conjugate. This solution, (1.104), for an arbitrary complex constant A, leads to the dispersion relation (for $\varepsilon = 0$)

$$\omega^2 = k^2 - 1,$$

which possess real solutions for ω only if $|k| \geq 1$. We shall suppose that $k > 1$, and then there are two possible waves with speeds

$$c_p = \frac{\omega}{k} = \pm\sqrt{1 - k^{-2}}, \tag{1.105}$$

where the subscript p is used to denote the *phase speed*. Now, for a given k and one choice of c_p, we seek a harmonic-wave solution of equation (1.103) which *evolves slowly* on suitable scales. For these problems, a little investigation (or some experience) suggests that we should introduce *slow* variables

$$\zeta = \varepsilon(x - c_g t), \quad \tau = \varepsilon^2 t,$$

where the speed c_g is, in general, not equal to the phase speed, c_p, and is unknown at this stage. In addition, upon writing

$$\xi = x - c_p t,$$

the original equation, (1.103), is transformed according to

$$\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial \xi} + \varepsilon\frac{\partial}{\partial \zeta}; \quad \frac{\partial}{\partial t} \equiv -c_p\frac{\partial}{\partial \xi} - \varepsilon c_g\frac{\partial}{\partial \zeta} + \varepsilon^2\frac{\partial}{\partial \tau},$$

to yield (with $u(x, t; \varepsilon) \equiv U(\xi, \zeta, \tau; \varepsilon)$)

$$\begin{aligned}(c_p^2 - 1)U_{\xi\xi} - U + 2\varepsilon(c_p c_g - 1)U_{\xi\zeta} + \varepsilon^2\{(c_g^2 - 1)U_{\zeta\zeta} - 2c_p U_{\xi\tau}\} \\ + \varepsilon(UU_\xi)_\xi + \varepsilon^2\{(UU_\xi)_\zeta + (UU_\zeta)_\xi\} = \mathrm{O}(\varepsilon^3),\end{aligned}$$

where terms only as far as $\mathrm{O}(\varepsilon^2)$ have been written down. Thus the function $u(x, t; \varepsilon)$ is now treated as a function of the variables (ξ, ζ, τ):

this is the method of multiple scales (the scales here being O(1), O(ε^{-1}), O(ε^{-2}), respectively).

We seek a solution in the form of the asymptotic expansion

$$U(\xi, \zeta, \tau; \varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n U_n(\xi, \zeta, \tau) \quad \text{as } \varepsilon \to 0, \tag{1.106}$$

for ξ, ζ, τ all O(1). Thus

$$(c_p^2 - 1)U_{0\xi\xi} - U_0 = 0,$$

and we take the solution

$$U_0 = A_{01}(\zeta, \tau)\mathrm{e}^{ik\xi} + \text{c.c.} \quad \text{with } c_p^2 = 1 - \frac{1}{k^2}\ (k > 1),$$

where the first subscript in A_{01} denotes the term ε^0, and the second is associated with the choice

$$E^1 = \mathrm{e}^{ik\xi}.$$

At the next order, ε^1, we obtain the equation

$$\begin{aligned}(c_p^2 - 1)U_{1\xi\xi} - U_1 &= 2(1 - c_p c_g)U_{0\xi\zeta} + (U_0 U_{0\xi})_\xi \\ &= 2(1 - c_p c_g)(\mathrm{i}kA_{01\zeta}E + \text{c.c.}) - (2k^2 A_{01}^2 E^2 + \text{c.c.}),\end{aligned}$$

and U_1 is a harmonic function only if

$$c_p c_g = 1, \tag{1.107}$$

which determines c_g. If this choice for c_g is *not* made, then U_1 will include a particular integral proportional to ξE which would lead to a non-uniformity in the asymptotic expansion, (1.106), as $|\xi| \to \infty$; terms like ξE are usually called *secular*, whereas uniformity in ξ is guaranteed only if terms *periodic* (harmonic) in ξ are allowed in U_1. The speed, c_g, which describes the motion of the amplitude A_{01}, is the *group speed* for this wave. To see this we start with the definition (see Section 1.4.1)

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{\mathrm{d}}{\mathrm{d}k}(kc_p),$$

and so we have

$$c_g = c_p + k\frac{\mathrm{d}c_p}{\mathrm{d}k} \quad \text{or} \quad c_p c_g = c_p^2 + \frac{1}{2}k\frac{\mathrm{d}c_p^2}{\mathrm{d}k}$$

which, from (1.105), yields

$$c_p c_g = 1 - \frac{1}{k^2} + k\left(\frac{1}{k^3}\right) = 1,$$

as required (see (1.107)). The solution for U_1 may therefore be written as

$$\begin{aligned} U_1 &= A_{11}(\zeta, \tau)E + \frac{2k^2 A_{01}^2}{4k^2(1 - c_p^2) - 1}E^2 + \text{c.c.} \\ &= A_{11}E + \frac{2}{3}k^2 A_{01}^2 E^2 + \text{c.c.} \end{aligned}$$

where A_{11} is (so far) unknown; this gives a correction (of $\mathrm{O}(\varepsilon)$) to the amplitude of the fundamental, E. We note that U_1 includes a higher harmonic, E^2 (and its complex conjugate, E^{-2}).

To proceed, the equation for U_2 is obtained, which, with (1.107) incorporated, is

$$(c_p^2 - 1)U_{2\xi\xi} - U_2 = (1 - c_g^2)U_{0\zeta\zeta} + 2c_p U_{0\xi\tau} - (U_0^2)_{\xi\zeta} - (U_0 U_1)_{\xi\xi}. \quad (1.108)$$

Again, we impose the condition that U_2 is to contain only terms periodic in ξ; to this end, any terms in E^1 which appear in the forcing terms in equation (1.108) must be removed. Such terms can arise only from

$$\begin{aligned} &(1 - c_g^2)U_{0\zeta\zeta} + 2c_p U_{0\xi\tau} - (U_0 U_1)_{\xi\xi} \\ &= (1 - c_g^2)(A_{01\zeta\zeta}E + \text{c.c.}) + 2c_p(A_{01\tau}ikE + \text{c.c.}) \\ &\quad - \frac{\partial^2}{\partial\xi^2}\left\{(A_{01}E + \bar{A}_{01}E^{-1})\left(\frac{2}{3}k^2 A_{01}^2 E^2 + \frac{2}{3}k^2 \bar{A}_{01}^2 E^{-2} + A_{11}E + \bar{A}_{11}E^{-1}\right)\right\}, \end{aligned}$$

where the overbar denotes the complex conjugate. In this expression, the coefficient of E which is to be set to zero (and, of course, its conjugate for terms E^{-1}) is

$$2ikc_p A_{01\tau} + (1 - c_g^2)A_{01\zeta\zeta} + \frac{2}{3}k^4 A_{01}|A_{01}|^2 = 0; \qquad (1.109)$$

all other terms generate higher harmonics in U_2, which is acceptable for uniform validity as $|\xi| \to \infty$. The equation which describes the evolution of the amplitude of the leading term, equation (1.109), is one version of another important and well-known equation: it is the *Nonlinear Schrödinger equation*, which we shall describe more fully later (Chapter 4). Other derivations of this type of equation are discussed in Q1.50 and Q1.54.

Further reading

This chapter, although it aims to provide a minimal base from which to explore the theory of water waves, cannot develop all the relevant topics to any depth. The following, therefore, referenced by the section numbers used in the chapter, is intended to present some useful – but not essential – additional reading.

1.1 There are many texts – and many good texts – on fluid mechanics; readers may have their favourites, but we list a few that can be recommended. A wide-ranging and well-written text is Batchelor (1967); more recent texts are Paterson (1983) and Acheson (1990), this latter including an introduction to waves in fluids. A more descriptive approach is provided by Lighthill (1986), and there are the classical texts: Lamb (1932), Schlichting (1960), Rosenhead (1964) and Landau & Lifschitz (1959).

1.2, 1.3 We shall provide many references to research papers and texts later, but two texts that can be mentioned at this stage are Stoker (1957) and Crapper (1984). A more general discussion of waves in fluids is given by Lighthill (1978).

1.4.1 For an excellent introduction to the theory of waves (including water waves), see Whitham (1974). An exploration of the concept of group velocity is given by Lighthill (1965). Of course, there is an extensive literature on the theory of partial differential equations; we mention as pre-eminent Garabedian (1964), and Bateman (1932) is also excellent, but good introductory texts are Haberman (1987), Sneddon (1957) and Weinberger (1965); two compact but wide-ranging texts are Vladimirov (1984) and Webster (1966). Finally, two excellent texts on general mathematical methods, including much work on partial differential equations, are Courant & Hilbert (1953) and Jeffreys & Jeffreys (1956).

1.4.2 The classical text, for applications to fluid mechanics, is van Dyke (1964). Introductory texts that cover a wide spectrum of applications, including examples on wave propagation, are Kevorkian & Cole (1985), Hinch (1991) and Bush (1992). More formal approaches to this material are given by Eckhaus (1979) and Smith (1985). The properties of divergent series are described in the excellent text by Hardy (1949), and their everyday use is described by Dingle (1973).

Exercises

Q1.1 *Some differential identities.* Given that $\phi(\mathbf{x})$ is a scalar function, and $\mathbf{u}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ are vector-valued functions, show that

(a) $\nabla \cdot (\phi \mathbf{u}) = (\mathbf{u} \cdot \nabla)\phi + \phi \nabla \cdot \mathbf{u}$;

(b) $\nabla \wedge (\phi \mathbf{u}) = (\nabla \phi) \wedge \mathbf{u} + \phi(\nabla \wedge \mathbf{u})$;

(c) $\mathbf{u} \wedge (\nabla \wedge \mathbf{u}) = \nabla(\frac{1}{2}\mathbf{u} \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\mathbf{u}$;

(d) $\nabla \wedge (\mathbf{u} \wedge \mathbf{v}) = \mathbf{u}(\nabla \cdot \mathbf{v}) - (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} - \mathbf{v}(\nabla \cdot \mathbf{u})$;

[A subscript notation, used together with the summation convention, is a very compact way to obtain these identities.]

Q1.2 *Two integral identities.* A volume V is bounded by the surface S on which there is defined the outward normal unit vector, $\mathbf{n}$. Given that $\phi(\mathbf{x})$ is a scalar function, use Gauss' theorem to show that

$$\int_V \nabla \phi \, \mathrm{d}v = \int_S \phi \mathbf{n} \, \mathrm{d}s,$$

and, for the vector function $\mathbf{u}$, that

$$\int_V \nabla \wedge \mathbf{u} \, \mathrm{d}v = \int_S \mathbf{n} \wedge \mathbf{u} \, \mathrm{d}s.$$

[It is convenient to introduce suitable arbitrary constant vectors into Gauss' theorem.]

Q1.3 *Another integral identity.* By considering, separately, each component of the vector $\mathbf{A}$, show that

$$\int_S \mathbf{A}(\mathbf{u} \cdot \mathbf{n}) \mathrm{d}s = \int_V \{(\mathbf{u} \cdot \nabla)\mathbf{A} + \mathbf{A}(\nabla \cdot \mathbf{u})\} \mathrm{d}v.$$

Q1.4 *Acceleration of a fluid particle.* The velocity vector which describes the motion of a particle (point) in a fluid is $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, so that the particle follows the path on which

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{U}(t) = \mathbf{u}\{\mathbf{x}(t), t\}.$$

Write $\mathbf{x} \equiv (x, y, z)$ and $\mathbf{u} \equiv (u, v, w)$ (in rectangular Cartesian coordinates), and hence show that the acceleration of the particle is

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \equiv \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t},$$

the material derivative.

Q1.5 *Material derivative.*

(a) A fluid moves so that its velocity is $\mathbf{u} \equiv (2xt, -yt, -zt)$, written in rectangular Cartesian coordinates. Show that the surface

$$F(x, y, z, t) = x^2 \exp(-2t^2) + (y^2 + 2z^2)\exp(t^2) = \text{constant}$$

moves with the fluid (so that it always contains the same fluid particles; that is, $\mathrm{D}F/\mathrm{D}t = 0$).

(b) Repeat (a) for

$$\mathbf{u} \equiv \left(-\frac{x}{t}, -\frac{y}{2t}, \frac{3z}{2t}\right) \quad \text{and} \quad F = t^2x^2 + ty^2 - \frac{z^2}{t^3}.$$

Q1.6 *Eulerian vs. Lagrangian description.* The Eulerian description of the motion is represented by $\mathbf{u}(\mathbf{x}, t)$: the velocity at any point and at any time. The Lagrangian description follows a given particle (point) in the fluid; the Lagrangian velocity is $\mathbf{u}(\mathbf{x}_0, t)$, where $\mathbf{x} = \mathbf{x}_0$ at $t = 0$ labels the particle.

A particle moves so that

$$\mathbf{x} \equiv \{x_0 \exp(2t^2), y_0 \exp(-t^2), z_0 \exp(-t^2)\},$$

written in rectangular Cartesian coordinates, where $\mathbf{x} = \mathbf{x}_0 \equiv (x_0, y_0, z_0)$ at $t = 0$.

(a) Find the velocity of the particle in terms of $\mathbf{x}_0$ and t (the Lagrangian description), and show that it can be written as

$$\mathbf{u} \equiv (4xt, -2yt, -2zt),$$

the Eulerian description.

(b) Now obtain the acceleration of the particle from the Lagrangian description.

(c) Also write down the Eulerian acceleration, $\partial\mathbf{u}/\partial t$, where $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$.

(d) Show that the Lagrangian acceleration (that is, following a particle) is recovered from

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \quad \text{where} \quad \mathbf{u} = \mathbf{u}(\mathbf{x}, t).$$

Q1.7 *Incompressible flows.* Show that the velocity vectors introduced in Q1.5 (a), (b) and Q1.6 (a) all satisfy the condition for an incompressible flow, namely $\nabla \cdot \mathbf{u} = 0$.

Q1.8 *Steady, incompressible flow.* Show that the particle which moves according to

$$\mathbf{x} \equiv (x_0 e^{\alpha t}, y_0 e^{\beta t}, z_0 e^{\gamma t}),$$

written in rectangular Cartesian coordinates, where $\mathbf{x} \equiv (x_0, y_0, z_0)$ at $t = 0$ and α, β and γ are constants, is a steady flow (that is, $\mathbf{u} = \mathbf{u}(\mathbf{x})$). Find the condition which ensures that $\nabla \cdot \mathbf{u} = 0$.

Q1.9 *Another incompressible flow.* A velocity field is given by

$$\mathbf{u} = f(r)\mathbf{x}, \quad r = |\mathbf{x}| = \sqrt{x^2 + y^2 + z^2},$$

written in rectangular Cartesian coordinates, where $f(r)$ is a scalar function. Find the most general form of $f(r)$ so that $\mathbf{u}$ represents an incompressible flow.

Q1.10 *A solution of Euler's equation.* Written in rectangular Cartesian coordinates, the velocity vector for a flow is

$$\mathbf{u} \equiv (xt, yt, -2zt) \quad \text{where} \quad \mathbf{x} \equiv (x, y, z);$$

show that $\nabla \cdot \mathbf{u} = 0$. Given, further, that the density is constant and that the body force is $\mathbf{F} \equiv (0, 0, -g)$, where g is a constant, find the pressure, $P(\mathbf{x}, t)$, in the fluid which satisfies $P = P_0(t)$ at $\mathbf{x} = \mathbf{0}$.

Q1.11 *Hydrostatic pressure law.* Consider a stationary fluid ($\mathbf{u} \equiv \mathbf{0}$) with $\rho =$ constant, and take $\mathbf{F} \equiv (0, 0, -g)$ with $g =$ constant. Find $P(z)$ which satisfies $P = P_a$ on $z = h_0$, where z is measured positive upwards. What is the pressure on $z = 0$?

Q1.12 *Vorticity.* Consider an imaginary circular disc, of radius R, whose arbitrary orientation is described by the unit vector, $\mathbf{n}$, perpendicular to the plane of the disc. Define the component, in the direction $\mathbf{n}$, of the angular velocity, $\mathbf{\Omega}$, at a point in the fluid by

$$\mathbf{\Omega} \cdot \mathbf{n} = \lim_{R \to 0} \left\{ \frac{1}{2\pi R^2} \oint_C \mathbf{u} \cdot d\boldsymbol{\ell} \right\},$$

where C denotes the boundary (rim) of the disc. Use Stokes' theorem, and the arbitrariness of $\mathbf{n}$, to show that

$$\mathbf{\Omega} = \frac{1}{2}\boldsymbol{\omega},$$

where $\boldsymbol{\omega} = \nabla \wedge \mathbf{u}$ is the vorticity in the fluid at $R = 0$.

[This definition is based on a description applicable to the rotation of solid bodies. Confirm this by considering $\mathbf{u} = \mathbf{U} + \boldsymbol{\Omega} \wedge \mathbf{r}$, where $\mathbf{U}$ is the translational velocity of the body, $\boldsymbol{\Omega}$ is its angular velocity and $\mathbf{r}$ is the position vector of a point relative to a point on the axis of rotation.]

Q1.13 *A simple flow with vorticity.* Written in rectangular Cartesian coordinates, with $\mathbf{F} \equiv (0, 0, -g)$ where g is constant, show that $\mathbf{u} \equiv (U(z), 0, 0)$ with $P = P_0 - \rho g z$ (P_0 and ρ constants) is an exact solution of Euler's equation and the equation of mass conservation. What is the vorticity for this flow? Repeat this calculation for $U = U(y)$.

[A classical example is $U(z) = U_0 + (U_1 - U_0)H(z)$ where U_0 and U_1 are constants, and $H(z)$ is the *Heaviside step function*; this is called a *vortex sheet*.]

Q1.14 *Helmholtz's equation.* Given that $\rho =$ constant and $\mathbf{F} = -\nabla\Omega$, take the curl of Euler's equation to show that

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}.$$

Hence, for a flow that varies in only two spatial dimensions, show that $\boldsymbol{\omega} \cdot \nabla \equiv 0$ and so $\boldsymbol{\omega} = \mathbf{constant}$ on particles. (The vorticity then remains 'trapped' perpendicular to the plane of the flow; cf. Q1.13.)

Q1.15 *Helmholtz's equation for compressible flow.* Show, for a compressible flow (which satisfies the general equation of mass conservation, (1.4)) with $\mathbf{F} = -\nabla\Omega$, that

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left\{\left(\frac{\boldsymbol{\omega}}{\rho}\right) \cdot \nabla\right\}\mathbf{u} - \frac{1}{\rho}\left\{\nabla\left(\frac{1}{\rho}\right)\right\} \wedge \nabla P;$$

cf. Q1.14. Hence, given that the fluid is barotropic (see Q1.18) so that $P = P(\rho)$, show that this equation is that given in Q1.14 with $\boldsymbol{\omega}$ replaced by $\boldsymbol{\omega}/\rho$.

Q1.16 *Vorticity in cylindrical coordinates.* Given that the velocity vector for a flow is $\mathbf{u} \equiv (\theta u, u, \theta u)$, written in cylindrical coordinates (r, θ, z), find the vorticity when $u = u(r)$.

[The vorticity vector for (u, v, w) in cylindrical coordinates is

$$\left(\frac{1}{r}w_\theta - v_z,\ u_z - w_r,\ \frac{1}{r}(rv)_r - \frac{1}{r}u_\theta\right).]$$

Q1.17 *Rankine's vortex*. Find the vorticity for the velocity field

$$\mathbf{u} = \begin{cases} (0, \frac{1}{2}\omega r, 0), & 0 \le r \le a \\ (0, \frac{1}{2}\omega a^2/r, 0), & r > a, \end{cases}$$

written in cylindrical coordinates (see Q1.16), where ω is a constant. Confirm that this $\mathbf{u}$ describes an incompressible flow. With $\rho =$ constant and $\mathbf{F} = -\nabla\Omega$, use Euler's equation to find an expression for $(P/\rho) + \Omega$ that is continuous on $r = a$ and which satisfies $\{(P/\rho) + \Omega\} \to P_0/\rho$ as $r \to \infty$. What condition on P_0 ensures that $(P/\rho) + \Omega > 0$? (This condition is particularly relevant if $\Omega = 0$.)

Q1.18 *Barotropic fluid.* Given that a fluid is described by $P = P(\rho)$, show that

$$\frac{1}{\rho}\nabla P = \nabla\left(\int \frac{\mathrm{d}P}{\rho}\right).$$

[This generalises $\nabla(P/\rho)$ as used in equation (1.21); a barotropic fluid (Greek: βαρος, weight) is one in which lines of constant density coincide with lines of constant pressure.]

Q1.19 *Particle paths and streamlines.* For these flows, expressed in rectangular Cartesian coordinates, find the particle paths that pass through (x_0, y_0, z_0) at $t = 0$. In each case, also find the general equations describing the streamlines. Verify that each flow is incompressible.

(a) $\mathbf{u} \equiv (cx, -cy, 0)$; (b) $\mathbf{u} \equiv (2xt, -2yt, 0)$;
(c) $\mathbf{u} \equiv (x - t, -y, 0)$; (d) $\mathbf{u} \equiv \{cx^2, cy^2, -2c(x + y)z\}$,

where c is a constant.

Q1.20 *Stream functions.* The stream function, $\psi(x, y, t)$, satisfies the equation of mass conservation for incompressible flow

$$u_x + v_y = 0$$

with $u = \psi_y$ and $v = -\psi_x$. For each of the velocity fields given in Q1.19 (a), (b), and (c), find the stream function.

Q1.21 *The stream function.* For the two-dimensional flow field, $\mathbf{u} \equiv (u, v)$ with $\mathbf{x} \equiv (x, y)$, use the definition of the streamline (equation (1.17)) to show that $\psi =$ constant (at fixed t) on streamlines; see Q1.20.

Q1.22 *Stream function in polar coordinates.* Following Q1.20, define a stream function, $\psi(r, \theta, t)$, for the equation of mass conservation

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{1}{r}\frac{\partial v}{\partial \theta}=0.$$

Hence find the stream function for the flow with speed $U(t)$ along the x-axis; that is, along $\theta = 0$.

Q1.23 *Stream function in cylindrical polars.* Define a stream function for the equations of mass conservation

(a) $\dfrac{1}{r}\dfrac{\partial}{\partial r}(ru)+\dfrac{\partial w}{\partial z}=0;$ (b) $\dfrac{1}{r}\dfrac{\partial v}{\partial \theta}+\dfrac{\partial w}{\partial z}=0;$

see Q1.22.

Q1.24 *Irrotational flow.* Show that these velocity fields describe irrotational flows, and find the velocity potential in each case:
(a) $\mathbf{u}=(\mathbf{a}\cdot\mathbf{x})\mathbf{b}+(\mathbf{b}\cdot\mathbf{x})\mathbf{a}$ ($\mathbf{a}, \mathbf{b}$ arbitrary constant vectors);

(b) $\mathbf{u}\equiv\left\{\dfrac{-2xyz}{(x^2+y^2)^2},\ \dfrac{(x^2-y^2)z}{(x^2+y^2)^2},\ \dfrac{y}{x^2+y^2}\right\}$

in rectangular Cartesian coordinates.

Q1.25 *Complex potential.* An incompressible, irrotational flow in two dimensions, with $\mathbf{u}\equiv(u, v)$ and $\mathbf{x}\equiv(x, y)$, leads to the introduction of the stream function, ψ, and velocity potential, ϕ; see Q1.29 and Section 1.1.3. Show that ϕ and ψ satisfy the *Cauchy–Riemann relations*, and hence that there exists a function $w(z)=\phi+\mathrm{i}\psi$ (with $z=x+\mathrm{i}y$), the complex potential. Also demonstrate that

$$\frac{\mathrm{d}w}{\mathrm{d}z}=u-\mathrm{i}v,$$

the *complex velocity*. What flow is represented by the function

$$w(z)=U(t)\mathrm{e}^{\mathrm{i}\alpha}z,$$

where α (a constant) and $U(t)$ are both real?

Q1.26 *Vector potential.* Introduce the stream function, $\psi(x, y, t)$, for the incompressible flow field $\mathbf{u}\equiv(u, v)$ with $\mathbf{x}\equiv(x, y)$; see Q1.20. Define the vector potential $\boldsymbol{\Psi}\equiv(0, 0, \psi)$ and hence show that

$$\nabla\wedge\boldsymbol{\Psi}=\mathbf{u}.$$

Q1.27 *Kinematic condition.* Fluid particles move on the path, $\mathbf{x}=\mathbf{x}(t)$, in the free surface

$$z(t)=h\{\mathbf{x}_\perp(t), t\}.$$

Differentiate this equation with respect to t, and hence show that

$$w = h_t + (\mathbf{u}_\perp \cdot \nabla_\perp)h \quad \text{on} \quad z = h(\mathbf{x}_\perp, t).$$

Q1.28 *A two-dimensional bubble.* An incompressible fluid, $\rho =$ constant, is at rest on a flat horizontal surface and the surface tension causes it to form into a bubble. The pressure in the fluid satisfies the equation of hydrostatic equilibrium, and the free surface is in contact with the atmosphere at pressure $P = P_a =$ constant. Assuming that the 'bubble' exists only in the two-dimensional (x, z)-plane, for $-x_0 \leq x \leq x_0$ and $0 \leq z \leq h(x)$, write down the equation for $h(x)$.

Q1.29 *An axisymmetric bubble.* See Q1.28: now assume that the bubble is defined for $0 \leq r \leq r_0$, $0 \leq z \leq h(r)$, with $0 \leq \theta \leq 2\pi$, expressed in cylindrical coordinates. Write down the equation for $h(r)$. With the notation $\mathrm{h}(0) = \mathrm{h}_0$, define $R = r/r_0$ and $H(R) = h/h_0$; hence write your equation in terms of $H(R)$. Given that $\varepsilon = h_0/r_0 \ll 1$, show that an approximate solution exists which (for suitable parameter values) satisfies

$$H(0) = 1, \quad H'(0) = 0, \quad H(1) = 0,$$

provided $\alpha_0 < \alpha < \alpha_1$ where $\alpha = \rho g r_0^2/\Gamma$ (which uses the standard notation). Here, $\sqrt{\alpha_0}$ (> 0) is the first zero of the Bessel function J_0, and $\sqrt{\alpha_1}$ (> 0) is the second zero of J_1. Find $H'(1)$ and sketch the shape of the bubble.

Q1.30 *Differentiation under the integral sign.* Given

$$I(x) = \int_{a(x)}^{b(x)} f(x, y)\,\mathrm{d}y,$$

show that

$$\frac{\mathrm{d}I}{\mathrm{d}x} = \int_a^b f_x(x, y)\,\mathrm{d}y + f(x, b)\frac{\mathrm{d}b}{\mathrm{d}x} - f(x, a)\frac{\mathrm{d}a}{\mathrm{d}x},$$

where the integral of f_x, and a' and b', are assumed to exist. Verify that this formula recovers a familiar and elementary result in the case: $f = f(y)$, $b(x) = x$, $a(x) =$ constant.

[You may find it helpful to introduce the *primitive* of $f(x, y)$ at fixed x: $g(x, y) = \int f(x, y)\,\mathrm{d}y$.]

Q1.31 *Differentiation under the integral sign: examples.* Use the formula given in Q1.30 to

(a) find an expression for $\mathrm{d}I/\mathrm{d}x$, where

$$I(x) = \int_{x}^{x^2} \frac{\mathrm{e}^{xy}}{y}\,\mathrm{d}y, \quad x > 0;$$

(b) show that

$$\phi(x, t) = \frac{1}{t^3} \int_{-t}^{t} (t^2 - y^2) g(x + y)\,\mathrm{d}y,$$

where g is a twice differentiable function, is a solution of the partial differential equation

$$\phi_{xx} - \phi_{tt} - \frac{n}{t}\phi_t = 0,$$

for a certain value of the positive integer, n, which should be determined. [Hint: integrate your expression for ϕ_{xx} by parts, twice.]

Q1.32 *An energy equation.* An incompressible, inviscid flow with $\mathbf{F} = -\nabla\Omega$ is described by Euler's equation. Take the scalar product of this equation with the velocity vector, $\mathbf{u}$. Integrate the resulting equation over the volume V, which is fixed in space, and hence show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathrm{V}} \frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}v = -\int_{\mathrm{S}} \left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u} + P + \rho\Omega\right)\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}s,$$

where S bounds V.

Q1.33 *Energy and a uniqueness theorem.*

(a) The kinetic energy of the fluid occupying the volume V is

$$T = \frac{1}{2}\int_{\mathrm{V}} \rho\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}v;$$

see Q1.32. For an incompressible, irrotational flow, show that

$$T = \frac{1}{2}\int_{\mathrm{S}} \rho\phi\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}s, \quad \mathbf{u} = \nabla\phi.$$

(b) The result in (a) can be used to provide a uniqueness theorem. Suppose that there are two possible flows, $\mathbf{u}_1 = \nabla\phi_1$ and $\mathbf{u}_2 = \nabla\phi_2$ both satisfying the same given conditions on S. Write $\mathbf{U} = \mathbf{u}_1 - \mathbf{u}_2$ and $\Phi = \phi_1 - \phi_2$, and show that

$$I = \int_V \mathbf{U} \cdot \mathbf{U}\,\mathrm{d}v = \int_S \Phi\mathbf{U} \cdot \mathbf{n}\,\mathrm{d}s,$$

and hence that $I = 0$ if either ϕ or $\mathbf{u}$ is prescribed on S. With these boundary conditions, deduce that $\mathbf{U} \equiv \mathbf{0}$, so $\mathbf{u}_1 \equiv \mathbf{u}_2$: the velocity field is unique.

Q1.34 *Elementary nondimensionalisation.* A two-dimensional flow, with $\mathbf{u} \equiv (u, w)$ and $\mathbf{x} \equiv (x, z)$, is both incompressible and irrotational. Nondimensionalise according to

$$u \to cu, \quad x \to \lambda x, \quad z \to hz,$$

and hence obtain the nondimensionalisation of the stream function, ψ, of the velocity potential, ϕ, and of w. Write down the nondimensional version of $w = \phi_z$.

Q1.35 *A Reynolds number.* Use the scheme described in Section 1.3 to nondimensionalise the Navier–Stokes equation (Appendix A), and hence obtain the nondimensional parameter which incorporates the viscosity, μ, and is based on the scales associated with the horizontal motion.

[The reciprocal of this parameter is called the *Reynolds number*; knowledge of its size is of fundamental importance in the study of fluid mechanics.]

Q1.36 *Dynamic condition in cylindrical coordinates.* Use the nondimensionalisation described in Section 1.3.1 to obtain the corresponding boundary condition written in cylindrical coordinates; cf. equation (1.64). Further, by suitably scaling the pressure in terms of ε, and by using an appropriate definition of the Weber number, rewrite this condition and then approximate it for $\varepsilon \to 0$.

Q1.37 *Nondimensionalisation of the pressure equation.* Use the nondimensionalisation described in Section 1.3.1, followed by the scaling adopted in Section 1.3.2, to obtain the appropriate form of the pressure boundary condition, equation (1.29). (This will require a nondimensionalisation and scaling of the velocity potential, ϕ; see Q1.34.)

Q1.38 *Irrotational flow: approximations.* Write down the non-dimensional, scaled equations for irrotational flow in the absence of surface tension (see Q1.37) and for a bottom boundary which is independent of time. Hence obtain the approximate form of these equations in the limit of small amplitude, $\varepsilon \to 0$. Also write down the corresponding approximate boundary condition when surface tension is included.

Q1.39 *The classical wave equation.* Use the method of characteristics to derive d'Alembert's solution of the wave equation

$$u_{tt} - c^2 u_{xx} = 0,$$

and hence obtain that solution which satisfies $u(x, 0) = p(x)$ and $u_t(x, 0) = q(x)$, $-\infty < x < \infty$.

Q1.40 *Data on compact support.* See Q1.39; now suppose that both $p(x)$ and $q(x)$ are zero for $x < 0$ and $x > x_0$ (> 0). Describe the form of the solution for $t > x_0/(2c)$.

Q1.41 *Dispersion relation.* Discuss the nature of the solution of the equation

$$u_t + u_x + u_{xxx} - u_{xx} = 0,$$

on the basis of its dispersion relation.

Q1.42 *Dispersion relations compared.* Compare the dispersion relations for the two equations

$$u_t + u_x + u_{xxx} = 0; \quad u_t + u_x - u_{xxt} = 0,$$

particularly for long waves ($k \to 0$) and short waves ($k \to \infty$).

Q1.43 *Nonlinear wave equation.* Obtain, explicitly, the solution of the equation

$$u_t + (1 + u)u_x = 0,$$

which satisfies

$$u(x, 0) = \begin{cases} \alpha x, & 0 \le x < 1 \\ \alpha(2 - x), & 1 \le x \le 2 \\ 0, & \text{otherwise}, \end{cases}$$

where α is a positive constant. Also, by using the characteristics, sketch this solution at various times, $t \ge 0$, and include $t = 1/\alpha$.

Q1.44 *An implicit solution.* Find the (implicit) solution of the equation

$$u_t + uu_x = 0,$$

which satisfies $u(x, 0) = \cos \pi x$. Show that $u(x, t)$ first has a point where u_x is infinite at time $t = \pi^{-1}$. What happens to this solution if it is allowed to develop beyond $t = \pi^{-1}$?

Q1.45 *Asymptotic expansions I.*

(a) Obtain the first two terms in the asymptotic expansion of

$$f(x; \varepsilon) = \left(1 + \varepsilon x - \frac{\varepsilon}{\varepsilon + x} + \mathrm{e}^{-x/\varepsilon}\right)^{-1}, \quad x \geq 0, \varepsilon > 0,$$

for $x = \mathrm{O}(1)$ as $\varepsilon \to 0$. Also obtain the leading order terms in the expansions valid for (i) $x = \mathrm{O}(\varepsilon)$, (ii) $x = \mathrm{O}(\varepsilon^{-1})$. Show that your expansions satisfy the matching principle.

(b) Obtain the first three terms in an asymptotic expansion of

$$f(x; \varepsilon) = (1 + \varepsilon x + \varepsilon^2 x^4)^{-1/2}, \quad x \geq 0, \varepsilon > 0,$$

for $x = \mathrm{O}(1)$ as $\varepsilon \to 0$. Show that your expansion is not uniformly valid as $x \to \infty$. In the *two* further asymptotic expansions that are required, find the first two terms in each and confirm that they satisfy the matching principle.

Q1.46 *Asymptotic expansions II.* The function

$$f(x; \varepsilon) = (1 - \varepsilon x - \varepsilon^4 x^3 - \mathrm{e}^{-x/\varepsilon})^{1/2} \quad \varepsilon > 0,$$

is real for $0 \leq x \leq x_0(\varepsilon)$. Construct asymptotic expansions of $f(x; \varepsilon)$, as $\varepsilon \to 0$, as follows:

(a) $x = \mathrm{O}(1)$: first two terms algebraic in ε, first exponentially small;

(b) $x = \mathrm{O}(\varepsilon)$: first two terms;

(c) $x = \mathrm{O}(\varepsilon^{-1})$: first two terms.

Show that your expansions satisfy the matching principle and, from your expansion obtained in (c), deduce that $x_0(\varepsilon) \sim \varepsilon^{-1} - 1$ as $\varepsilon \to 0^+$.

Q1.47 *Long–distance scale.* For the propagation equation

$$u_{tt} - u_{xx} = \varepsilon(u^2 + u_{xx})_{xx},$$

introduce the characteristic variable $\xi = x - t$, and the long-distance variable $X = \varepsilon x$, and hence obtain the appropriate Korteweg–de Vries equation which describes the first approximation to u (as $\varepsilon \to 0$) in the far-field. (You may assume that $u \to 0$ as $|\xi| \to \infty$.)

Q1.48 *Left-going wave.* See Q1.47; for this equation, find the Korteweg–de Vries equation (as the first approximation for $\varepsilon \to 0$) in the far-field defined by $\zeta = x + t = \mathrm{O}(1)$ and $\tau = \varepsilon t = \mathrm{O}(1)$.

Q1.49 *Left- and right-going waves.* See Q1.47 (and also Q1.48); introduce the characteristic variables $\xi = x - t$, $\zeta = x + t$, and the long-time variable $\tau = \varepsilon t$. Seek a solution

$$u \sim f(\xi, \tau) + g(\zeta, \tau) + \varepsilon u_1(\xi, \zeta, \tau),$$

for ξ, ζ and τ all O(1) as $\varepsilon \to 0$, in which f and g separately satisfy appropriate Korteweg–de Vries equations. (You may assume that $f(g)$ decays as $|\xi| \to \infty$ (as $|\zeta| \to \infty$).) What, then, is the solution for u_1?

Q1.50 *Nonlinear Schrödinger equation.* A wave is described by the equation

$$u_{tt} - u_{xx} - u = \varepsilon\{(u_x)^2 - uu_{xx}\}.$$

Use the method of multiple scales with

$$\xi = x - c_p t, \quad \zeta = \varepsilon(x - c_g t), \quad \tau = \varepsilon^2 t,$$

and seek an asymptotic solution in the form

$$u \sim \sum_{n=0}^{\infty} \varepsilon^n \sum_{m=0}^{n+1} A_{nm}(\zeta, \tau) E^m + \text{c.c.},$$

as $\varepsilon \to 0$, which is uniformly valid as $|\xi| \to \infty$. Here, $E = \exp(\mathrm{i}k\xi)$ and k (> 1) is a given (real) number. Find $c_p(k)$ and $c_g(k)$ (and confirm that $c_g = \mathrm{d}(kc_p)/\mathrm{d}k$), and show that

$$2\mathrm{i}kc_p \frac{\partial A_{01}}{\partial \tau} + (1 - c_g^2)\frac{\partial^2 A_{01}}{\partial \zeta^2} - 8k^4 A_{01}|A_{01}|^2 = 0.$$

Q1.51 *Wave hierarchies I.* A wave, which satisfies $u \to 0$ as $x \to +\infty$, is described by the multiwave speed equation

$$\left\{\frac{\partial}{\partial t} + (c_1 + \varepsilon^2 u)\frac{\partial}{\partial x} + \varepsilon^2 \frac{\partial u}{\partial x}\right\}\left\{\frac{\partial}{\partial t} + (c_2 + \varepsilon^2 u)\frac{\partial}{\partial x}\right\}u + \varepsilon^2\left\{\frac{\partial}{\partial t} + (c + \varepsilon u)\frac{\partial}{\partial x}\right\}u = 0,$$

where c_1, c_2 and c are constants. Show that, if $c_1 < c < c_2$, then on the time scale ε^{-2} the wave moving at speed c_1 decays exponentially in time, to leading order as $\varepsilon \to 0^+$. (To accomplish this, you will find it convenient to introduce

$\xi = x - c_1 t = O(1)$, $\tau = \varepsilon^2 t = O(1)$.) Now show that this same property is exhibited by the wave moving at speed c_2.

Q1.52 *Wave hierarchies II.* See Q1.51; show that, on the time scale ε^{-4}, the wave moving at speed c has diffused over a distance $O(\varepsilon^{-3})$ about its wavefront. In particular, show that this wave is described by an equation of the form

$$\phi_T + \phi\phi_X = \lambda\phi_{XX},$$

to leading order as $\varepsilon \to 0^+$. (Similar to Q1.51, it is useful to introduce $X = \varepsilon^3(x - ct) = O(1)$, $T = \varepsilon^4 t = O(1)$.) Determine the constant λ, and confirm that $\lambda > 0$ provided $c_1 < c < c_2$. Find the solution of this leading-order problem which describes a steady wave and which satisfies $u \to 0$ as $X \to \infty$, $u \to 1$ as $X \to -\infty$.

[This equation, here expressed in terms of ϕ, is a famous and important equation: it is the *Burgers equation*, which can be linearised by the *Hopf–Cole transformation* $\phi = -2\lambda\partial(\ln\theta)/\partial X$.]

Q1.53 *A nonlinear wave equation.* A wave motion is described by the equation

$$\left(\frac{\partial}{\partial t} + \varepsilon u \frac{\partial}{\partial x}\right)^2 u - \frac{\partial^2 u}{\partial x^2} = \varepsilon \frac{\partial^4 u}{\partial x^4}.$$

Introduce $\xi = x - t$ and $\tau = \varepsilon t$, and hence show that the leading approximation (as $\varepsilon \to 0$) satisfies the equation

$$2u_{\tau\xi} + 2uu_{\xi\xi} + u_\xi^2 + u_{\xi\xi\xi\xi} = 0, \qquad (*)$$

where $\xi = O(1)$, $\tau = O(1)$.

Now introduce *two* characteristic variables

$$\xi \sim x - t + \varepsilon f(x + t, \tau); \quad \eta \sim x + t + \varepsilon g(x - t, \tau),$$

and seek a solution

$$u = F(\xi, \tau) + G(\eta, \tau) + o(\varepsilon)$$

as $\varepsilon \to 0$, where F satisfies $(*)$ and G satisfies the corresponding equation for left-running waves. Confirm that the results are consistent when only left- or right-running waves alone are present.

Q1.54 *Bretherton's equation.* A weakly nonlinear dispersive wave is described by

$$u_{tt} + u_{xx} + u_{xxxx} + u = \varepsilon u^3.$$

Introduce the variables $X = \varepsilon x$, $T = \varepsilon t$ and θ, where

$$\theta_x = k(X, T), \quad \theta_t = -\omega(X, T),$$

and seek an asymptotic solution

$$u \sim \sum_{n=0}^{\infty} \varepsilon^n U_n(\theta, X, T), \quad \varepsilon \to 0,$$

which is uniformly valid as $|\theta| \to \infty$. Write

$$U_0 = A(X, T)e^{i\theta} + \text{c.c.},$$

and obtain the equation for A which ensures that U_1 is periodic in θ. Introduce the dispersion relation, relating ω and k, and hence show that

$$A_T + \omega'(k)A_X = \frac{3i}{2\omega}A|A|^2 - \frac{1}{2}k_X\omega''(k)A,$$

and then re-express this by writing $A = \alpha e^{i\beta}$ (for α, β real).

[This model equation for the weakly nonlinear interaction of dispersive waves was introduced by Bretherton (1964).]

Q1.55 *Steady travelling waves*. Seek a solution of each of these equations in the form $u(x, t) = f(x - ct)$, where c is a constant, satisfying the boundary conditions given:

(a) $u_t - 6uu_x + u_{xxx} = 0$ with $u, u_x, u_{xx} \to 0$ as $|x| \to \infty$;

(b) $u_t + uu_x = u_{xx}$ with $u \to 0$ as $x \to \infty$, $u \to u_0$ (> 0) as $x \to -\infty$.

[The solution to (a) is the *solitary wave* of the Korteweg–de Vries equation, and (b) gives the *Taylor shock profile* of the Burgers equation.]