Report 3

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February 19, 2020

Contents

1 The half line problem

In this section, we deal with this term

$$\partial_x (\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t \left(\eta \int_0^x \eta_t \, \mathrm{d}x'\right)\}\}$$

More generally, we have the following result:

Theorem 1. *For nice enough f defined on $x \geqslant 0$, we have*

$$(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} = \frac{1}{\pi}\int_0^\infty f(y)\left(\frac{1}{x-y} + \frac{1}{x+y}\right) \mathrm{d}y.$$

Before we begin, recall the Riemann-Lebesgue lemma:

Lemma 2 (Theorem 11.6, [1])**.** *Assume that $f \in L(I)$. Then, for each real β, we have*

$$\lim_{\alpha\to\infty} \int_I f(t)\sin(\alpha t + \beta)\, \mathrm{d}t = 0.$$

Proof of Theorem 1. Consider
$$(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\}.$$

For generality, we consider $(\mathcal{F}_s^k)^{-1}\{G(k)\}$, where G is a function of k defined on $k \geqslant 0$. Expanding the integral, we obtain:
$$\begin{aligned}
(\mathcal{F}_s^k)^{-1}\{G(k)\} &= \int_0^\infty \sin(kx)G(k)\,\mathrm{d}k \\
&= \frac{1}{2i}\int_0^\infty (e^{ikx} - e^{-ikx})G(k)\,\mathrm{d}k \\
&= \frac{1}{2i}\left[\int_0^\infty e^{ikx}G(k)\,\mathrm{d}k - \int_0^\infty e^{-ikx}G(k)\,\mathrm{d}k\right] \\
&= \frac{1}{2i}\left[\int_0^\infty e^{ikx}G(k)\,\mathrm{d}k + \int_0^{-\infty} e^{ikx}G(-k)\,\mathrm{d}k\right] && \text{(apply } k \mapsto -k \text{ in the 2nd term)} \\
&= \frac{1}{2i}\left[\int_0^\infty e^{ikx}G(k)\,\mathrm{d}k + \int_{-\infty}^{0} e^{ikx}(-G(-k))\,\mathrm{d}k\right],
\end{aligned}$$
where $-G(-k)$ is an odd extension to $k < 0$. Now, observe the following:
$$\begin{aligned}
\frac{2}{\pi}\int_0^\infty \cos(kx)f(x)\,\mathrm{d}x &= \frac{1}{\pi}\int_0^\infty (e^{ikx} + e^{-ikx})f(x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\int_0^\infty e^{ikx}f(x)\,\mathrm{d}x + \int_0^\infty e^{-ikx}f(x)\,\mathrm{d}x\right] \\
&= \frac{1}{\pi}\left[-\int_0^{-\infty} e^{-ikx}f(-x)\,\mathrm{d}x + \int_0^\infty e^{-ikx}f(x)\,\mathrm{d}x\right] && \text{(apply } x \mapsto -x \text{ in the 1st term)} \\
&= \frac{1}{\pi}\left[\int_{-\infty}^0 e^{-ikx}f(-x)\,\mathrm{d}x + \int_0^\infty e^{-ikx}f(x)\,\mathrm{d}x\right] \\
&= \frac{1}{\pi}\int_{-\infty}^\infty e^{-ikx}F(x)\,\mathrm{d}x,
\end{aligned}$$
where we used an even extension to $x < 0$ and defined
$$F(x) = \begin{cases} f(x) & x > 0 \\ f(-x) & x < 0 \end{cases}.$$
For $k > 0$, we have
$$G(k) = \mathcal{F}_c^k\{f\} = \frac{2}{\pi}\int_0^\infty \cos(kx)f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_{-\infty}^\infty e^{-ikx}F(x)\,\mathrm{d}x. \tag{1}$$

For $k < 0$, we have

$$-G(-k) = -\mathcal{F}_c^{-k}\{f\} = -\frac{2}{\pi}\int_0^\infty \cos(-kx)f(x)\,\mathrm{d}x = -\frac{2}{\pi}\int_0^\infty \cos(kx)f(x)\,\mathrm{d}x = -\frac{1}{\pi}\int_{-\infty}^\infty e^{-ikx}F(x)\,\mathrm{d}x, \tag{2}$$

since cosine is an even function. Thus, using (1) and (2), we obtain

$$\begin{aligned}
(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} &= \frac{1}{2i}\left[\int_0^\infty e^{ikx}\mathcal{F}_c^k\{f\}\,\mathrm{d}k + \int_{-\infty}^0 e^{ikx}(-\mathcal{F}^(- k)_c\{f\})\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty e^{ikx}\int_{-\infty}^\infty e^{-iky}F(y)\,\mathrm{d}y\,\mathrm{d}k - \int_{-\infty}^0 e^{ikx}\int_{-\infty}^\infty e^{-iky}F(y)\,\mathrm{d}y\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty e^{ikx}\int_{-\infty}^\infty e^{ik(x-y)}F(y)\,\mathrm{d}y\,\mathrm{d}k - \int_{-\infty}^0\int_{-\infty}^\infty e^{ik(x-y)}F(y)\,\mathrm{d}y\,\mathrm{d}k\right].
\end{aligned} \tag{3}$$

Let

$$\begin{aligned}
V(k) &= \int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y = -V(-k), \\
U(k) &= \int_{-\infty}^\infty \cos(k(x-y))F(y)\,\mathrm{d}y = U(-k),
\end{aligned}$$

so that V is odd and U is even. This allows to rewrite (3) as:

$$\begin{aligned}
(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} &= \frac{1}{2\pi i}\left[\int_0^\infty e^{ikx}\int_{-\infty}^\infty e^{ik(x-y)}F(y)\,\mathrm{d}y\,\mathrm{d}k - \int_{-\infty}^0\int_{-\infty}^\infty e^{ik(x-y)}F(y)\,\mathrm{d}y\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty U(k) + iV(k)\,\mathrm{d}k - \int_{-\infty}^0 U(k) + iV(k)\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty U(k) + iV(k)\,\mathrm{d}k + \int_\infty^0 U(-k) + iV(-k)\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty U(k) + iV(k)\,\mathrm{d}k + \int_0^\infty -U(-k) + i(-V(-k))\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty U(k) + iV(k)\,\mathrm{d}k + \int_0^\infty -U(k) + iV(k)\,\mathrm{d}k\right] \\
&= \frac{1}{\pi}\int_0^\infty V(k)\,\mathrm{d}k,
\end{aligned}$$

where on the third last line, we flipped the bounds of integration and brought the minus sign inside the integral, and on the second last line, we used that U is even and V is odd. Thus, we obtain

$$(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} = \frac{1}{\pi}\int_0^\infty V(k)\,\mathrm{d}k = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y\,\mathrm{d}k.$$

Note that the integral in k is an improper integral, so

$$\int_0^\infty\int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y\,\mathrm{d}k = \lim_{\alpha\to\infty}\int_0^\alpha\int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y\,\mathrm{d}k.$$

Now, interchanging the order of integration, we have

$$\begin{aligned}
\int_0^\alpha\int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y\,\mathrm{d}k &= \int_{-\infty}^\infty F(y)\int_0^\alpha \sin(k(x-y))\,\mathrm{d}k\,\mathrm{d}y\\
&= \int_{-\infty}^\infty F(y)\left[-\frac{\cos(k(x-y))}{x-y}\,\Big|_0^\alpha\right]\mathrm{d}y\\
&= \int_{-\infty}^\infty F(y)\left[\frac{1}{x-y}-\frac{\cos(\alpha(x-y))}{x-y}\right]\mathrm{d}y\\
&= \int_{-\infty}^\infty F(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y.
\end{aligned}$$

The interchange is justified, since sine is bounded and differentiable on $\mathbb{R}$. Finally, we use the Riemann-Lebesgue lemma to deal with the last term:

$$\begin{aligned}
\int_{-\infty}^\infty F(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y &= \int_0^\infty f(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y + \int_{-\infty}^0 f(-y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y\\
&= \int_0^\infty f(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y - \int_\infty^0 f(y)\frac{1-\cos(\alpha(x+y))}{x+y}\,\mathrm{d}y\\
&= \int_0^\infty f(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y + \int_0^\infty f(y)\frac{1-\cos(\alpha(x+y))}{x+y}\,\mathrm{d}y\\
&= \int_0^\infty f(y)\frac{1}{x-y}\,\mathrm{d}y - \int_0^\infty f(y)\frac{\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y\\
&\quad+\int_0^\infty f(y)\frac{1}{x+y}\,\mathrm{d}y - \int_0^\infty f(y)\frac{\cos(\alpha(x+y))}{x+y}\,\mathrm{d}y.
\end{aligned}$$

As $\alpha\to\infty$, the terms

$$\int_0^\infty f(y)\frac{\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y, \qquad \int_0^\infty f(y)\frac{\cos(\alpha(x+y))}{x+y}\,\mathrm{d}y \to 0$$

by the Riemann-Lebesgue lemma with $\beta = \pi/2$, so that

$$\int_{-\infty}^{\infty} F(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y = \int_0^{\infty} f(y)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y.$$

Thus,

$$(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty}\sin(k(x-y))F(y)\,\mathrm{d}y\,\mathrm{d}k = \frac{1}{\pi}\int_0^{\infty} f(y)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y.$$

The proof is complete. □

Remark 3. Note that the integral

$$\frac{1}{\pi}\int_0^{\infty} f(y)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y =$$

looks like a convolution-type transform. In fact, the term with $1/(x-y)$ is pretty much the Hilbert transform, but on a half-line.

The theorem yields

$$\partial_x(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\}\} = \partial_x\left(\frac{1}{\pi}\int_0^{\infty}\partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y\right).$$

For generality, let $f(y) = \partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)$. Note the following:

$$\begin{aligned}\partial_x\left(\frac{1}{\pi}\int_0^{\infty} f(y)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y\right) &= \frac{1}{\pi}\int_0^{\infty} f(y)\partial_x\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y\\ &= -\frac{1}{\pi}\int_0^{\infty} f(y)\left[\frac{1}{(x-y)^2}+\frac{1}{(x+y)^2}\right]\mathrm{d}y,\end{aligned}$$

so that

$$\partial_x(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\}\} = -\frac{1}{\pi}\int_0^{\infty}\partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)\left[\frac{1}{(x-y)^2}+\frac{1}{(x+y)^2}\right]\mathrm{d}y. \tag{4}$$

As can be seen, the integral (4) is singular whenever $x = y$ or $x = -y$, over y. To deal with this issue, one may need to use a Residue theorem. To conclude, the surface expression on a half-line becomes:

$$\begin{aligned}\eta_{tt}-\eta_{xx} &= \mu^2\left(\frac{1}{3}\eta_{xxxx}+\partial_x(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\}\}+\frac{1}{2}\partial_x^2\left(\int_0^x \eta_t\,\mathrm{d}x'\right)^2\right)\\ &= \mu^2\left(\frac{1}{3}\eta_{xxxx}-\frac{1}{\pi}\int_0^{\infty}\partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)\left[\frac{1}{(x-y)^2}+\frac{1}{(x+y)^2}\right]\mathrm{d}y+\frac{1}{2}\partial_x^2\left(\int_0^x \eta_t\,\mathrm{d}x'\right)^2\right).\end{aligned}$$

2 Approximate equations: half-line

In this section, we derive the approximate equations from

$$\eta_{tt} - \eta_{xx} = \varepsilon \left(\frac{1}{3}\eta_{xxxx} + \frac{d}{dx}\frac{1}{\pi}\int_0^\infty \partial_t \left(\eta \int_0^y \eta_t \, \mathrm{d}y' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y + \frac{1}{2}\partial_x^2 \left(\int_0^x \eta_t \, \mathrm{d}x' \right)^2 \right). \tag{5}$$

As we approximate, we assume an expansion of η in ε :

$$\eta = \eta_0 + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2). \tag{6}$$

First order approximation

Substitution of (6) into equation (5) yields

$$\eta_{0tt} - \eta_{0xx} + \varepsilon(\eta_{1tt} - \eta_{1xx}) = \varepsilon \left[\frac{1}{3}\eta_{0xxxx} + \frac{d}{dx}\frac{1}{\pi}\int_0^\infty \partial_t \left(\eta \int_0^y \eta_t \, \mathrm{d}y' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y + \frac{1}{2}\partial_x^2 \left(\int_{-\infty}^x (\eta_0 + \varepsilon\eta_1)_t \, \mathrm{d}x' \right)^2 \right) + \mathcal{O}(\varepsilon^2). \tag{7}$$

In the leading order $\mathcal{O}(\varepsilon^0)$, equation (7) becomes

$$\eta_{0tt} - \eta_{0xx} = 0. \tag{8}$$

This is the wave equation with velocity 1, whose solution depends on the type of boundary conditions we prescribe for η at $x = 0$. For now, we prescribe

$$\eta_x(0, t) = 0.$$

The general solution is

$$\eta(x,t) = \begin{cases} F(x-t) + G(x+t) & x > t \\ F(t-x) + G(x+t) & x < t \end{cases},$$

where F, G are to be determined.

Second order approximation

As in the velocity potential case, we employ multiple scales. First, we find an expression for η_0. We introduce

$$\tau_0 = t, \qquad \tau_1 = \varepsilon t, \qquad \tau_2 = \varepsilon^2 t, \ldots,$$

so that

$$\eta(x,t) = \eta(x, \tau_0, \tau_1, \ldots).$$

With this in mind, the expansion (6) becomes

$$\eta(x, \tau_0, \tau_1, \ldots) = \eta_0(x, \tau_0, \tau_1, \ldots) + \mathcal{O}(\varepsilon^1). \tag{9}$$

Substituting (9) into (5), within $\mathcal{O}(\varepsilon^0)$, we obtain

$$\eta_{0\tau_0\tau_0} - \eta_{0xx} = 0, \tag{10}$$

so that the general solution is

$$\eta_0(x, \tau_0, \tau_1, \ldots) = \begin{cases} F_2(x - \tau_0, \tau_1, \ldots) + G_2(x + \tau_0, \tau_1, \ldots) & x \geqslant \tau_0 \\ F_1(\tau_0 - x, \tau_1, \ldots) + G_1(x + \tau_0, \tau_1, \ldots) & x < \tau_0 \end{cases},$$

where we recalled the boundary conditions $\eta_x(0, t) = 0$. Now, although we have found an expression for η_0, the functions F_i, G_i used are still general functions. To determine F_i, G_i, we proceed to the next order, i.e. $\mathcal{O}(\varepsilon^1)$. We introduce

$$\xi = x - \tau_0 \qquad \zeta = x + \tau_0$$

so that

$$\eta_0(x, \tau_0, \tau_1, \ldots) = \begin{cases} F_1(\xi, \tau_1, \ldots) + G_1(\zeta, \tau_1, \ldots) & x \geqslant t \\ F_2(-\xi, \tau_1, \ldots) + G_2(\zeta, \tau_1, \ldots) & x < t \end{cases},$$

and

$$\begin{aligned} \partial_x &= \partial_\xi \frac{\mathrm{d}\xi}{\mathrm{d}x} + \partial_\zeta \frac{\mathrm{d}\zeta}{\mathrm{d}x} = \partial_\xi + \partial_\zeta, \\ \partial_t &= \partial_\xi \frac{\mathrm{d}\xi}{\mathrm{d}t} + \partial_\zeta \frac{\mathrm{d}\zeta}{\mathrm{d}t} + \partial_{\tau_1} \frac{\mathrm{d}\tau_1}{\mathrm{d}t} = -\partial_\xi + \partial_\zeta + \varepsilon \partial_{\tau_1}. \end{aligned}$$

Remark 4. We emphasize the piecewise nature of solutions, which is why we write that F_1, F_2 as different functions even though they share the same variable ξ. It is very important to be aware which F_i we need to use, as we will demonstrate when dealing with the non-local terms. In addition, we also need to impose some more conditions at $x = \tau_0$, to reinforce some sort of continuity between F_1 and F_2..

The case $x < \tau_0$

We consider the case $x < t$. First, we use

$$\begin{aligned} \eta &= \eta_0 + \varepsilon \eta_1 + \mathcal{O}(\varepsilon^2) \\ &= F_1(t - x, \varepsilon t, \ldots) + G_1(x + t, \varepsilon t, \ldots) + \varepsilon \eta_1 + \mathcal{O}(\varepsilon^2) \\ &= F_1(-\xi, \tau_1, \ldots) + G_1(\zeta, \tau_1, \ldots) + \varepsilon \eta_1 + \mathcal{O}(\varepsilon^2) \\ &= F_1 + G_1 + \varepsilon \eta_1 + \mathcal{O}(\varepsilon^2). \end{aligned}$$

For ease of writing, we suppress explicit dependence on variables, though the reader should bear in mind that function $F_1, (G_1)$ depend on $-\xi, (\zeta), \tau_1, \tau_2$, etc. In addition, observe that

$$(\partial_t^2 - \partial_x^2) = \left(-4\partial_\xi\partial_\zeta + 2\varepsilon(\partial_\zeta\partial_{\tau_1} - \partial_\xi\partial_{\tau_1}) + \varepsilon^2\partial_{\tau_1}^2\right),$$

so that the LHS of (5) becomes

$$(\partial_t^2 - \partial_x^2)\eta = \varepsilon\left(-4\eta_{1\xi\zeta} - 2\partial_\xi\partial_{\tau_1}(F_1)_{\tau_1} + 2\partial_\zeta\partial_{\tau_1}G_1\right) + \mathcal{O}(\varepsilon^2). \tag{11}$$

Now, we deal with the RHS of (5). By appropriate substitutions, the terms become:

$$\begin{aligned}
\frac{1}{3}\eta_{xxxx} &= \frac{1}{3}(\partial_\xi^4 F_1 + \partial_\zeta^4 G_1 + \mathcal{O}(\varepsilon));\\
\left(\int_0^x \eta_t \,\mathrm{d}x'\right)^2 &= \left(\int_0^x \eta_{0t}\,\mathrm{d}x'\right)^2 + \mathcal{O}(\varepsilon)\\
&= \left(\int_0^x (-\partial_{\xi'} + \partial_{\zeta'} + \varepsilon\partial_{\tau_1})(F_1 + G_1)\,\mathrm{d}x'\right)^2 + \mathcal{O}(\varepsilon)\\
&= \left(\int_0^x -\partial_{\xi'}F_1 + \partial_{\zeta'}G_1\,\mathrm{d}x'\right)^2 + \mathcal{O}(\varepsilon)\\
&= \left(\int_0^x -\partial_{\xi'}F_1\,\mathrm{d}x'\right)^2 - 2\left(\int_0^x (\partial_{\xi'}F_1\,\mathrm{d}x'\right)\left(\int_0^x \partial_{\zeta'}G_1\,\mathrm{d}x'\right) + \left(\int_0^x \partial_{\zeta'}G_1\,\mathrm{d}x'\right)^2 + \mathcal{O}(\varepsilon)\\
&= (F_1 - F_1(\tau_0))^2 - 2(F_1 - F_1(\tau_0))(G_1 - G_1(\tau_0)) + (G_1 - G_1(\tau_0))^2 + \mathcal{O}(\varepsilon),
\end{aligned}$$

where for the last line we translate $\xi' = x' - t, \zeta' = x' + t$ to obtain

$$\begin{aligned}
\int_0^x -\partial_{\xi'}(F_1(\tau_0 - \xi'))\,\mathrm{d}x' &= \int_{-t}^{x-t}(F_1)_{\xi'}(-\xi', \tau_1)\,\mathrm{d}\xi' = \int_{-\tau_0}^{\xi}(F_1)_{\xi'}(-\xi', \tau_1)\,\mathrm{d}\xi' = F_1 - F_1(\tau_0),\\
\int_0^x (G_1)_{\zeta'}(x' + \tau_0, \tau_1)\,\mathrm{d}x' &= \int_t^{x+t}(G_1)_{\zeta'}(\zeta', \tau_1)\,\mathrm{d}\zeta' = \int_{\tau_0}^{\zeta}(G_1)_{\zeta'}(\zeta', \tau_1)\,\mathrm{d}\zeta' = G_1 - G_1(\tau_0).
\end{aligned}$$

Note that previously we wrongly assumed that there is some strange term $F(-t)$. But F that we used was rather F_2, which is appropriate when $x \geqslant \tau_0$. In this case, $x < \tau_0$, so we need to use F_1, which provides the right viewpoint. Finally, from Proposition 5, we also have

$$\int_0^\infty \partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)\left[\frac{1}{x-y} + \frac{1}{x+y}\right]\mathrm{d}y$$

$$= \int_0^{\tau_0} (2F_1\partial_\xi F_1 + 2G_1\partial_\zeta G_1)\left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \int_{\tau_0}^{\infty} (2F_2\partial_\xi F_2 + 2G_2\partial_\zeta G_2)\left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y.$$

Substitution of terms into the RHS of (5) leads to:

$$\begin{aligned}
&\frac{1}{3}\eta_{xxxx} + \frac{d}{dx}\frac{1}{\pi}\int_0^\infty \partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)\left[\frac{1}{x-y} + \frac{1}{x+y}\right]\mathrm{d}y + \frac{1}{2}\partial_x^2\left(\int_0^x \eta_t\,\mathrm{d}x'\right)^2 \\
&= \frac{1}{3}(\partial_\xi^4 F_1 + \partial_\zeta^4 G_1 + \frac{1}{\pi}(\partial_\xi + \partial_\zeta)\left(\int_0^{\tau_0} (2F_1\partial_\xi F_1 + 2G_1\partial_\zeta G_1)\left[\frac{1}{x-y} + \frac{1}{x+y}\right]\mathrm{d}y + \int_{\tau_0}^\infty (2F_2\partial_\xi F_2 + 2G_2\partial_\zeta G_2)\left[\frac{1}{x-y} + \frac{1}{x+y}\right]\mathrm{d}y\right) \\
&+ \frac{1}{2}(\partial_\xi^2 + 2\partial_\xi\partial_\zeta + \partial_\zeta^2)\left((F_1 - F_1(\tau_0))^2 - 2(F_1 - F_1(\tau_0))(G_1 - G_1(\tau_0)) + (G_1 - G_1(\tau_0))^2\right) + \mathcal{O}(\varepsilon) \\
&= \frac{1}{3}(\partial_\xi^4 F_1 + \partial_\zeta^4 G_1 + \frac{1}{\pi}(\partial_\xi + \partial_\zeta)\left(\int_0^{\tau_0} (2F_1\partial_\xi F_1 + 2G_1\partial_\zeta G_1)\left[\frac{1}{x-y} + \frac{1}{x+y}\right]\mathrm{d}y + \int_{\tau_0}^\infty (2F_2\partial_\xi F_2 + 2G_2\partial_\zeta G_2)\left[\frac{1}{x-y} + \frac{1}{x+y}\right]\mathrm{d}y\right) \\
&+ \partial_\xi\left((F_1 - G_1)\partial_\xi F_1\right) + \partial_\zeta\left((G_1 - F_1)\partial_\zeta G_1\right) - 2\partial_\xi F_1\partial_\zeta G_1.
\end{aligned} \tag{12}$$

Combining (11) and (12), in $\mathcal{O}(\varepsilon^1)$ we have

$$\begin{aligned}
-4\eta_{1\xi\zeta} &= 2\partial_\xi\partial_{\tau_1}F_1 - 2\partial_\zeta\partial_{\tau_1}G_1 + \frac{1}{3}(\partial_\xi^4 F_1 + \partial_\zeta^4 G_1) \\
&+ \frac{1}{\pi}(\partial_\xi + \partial_\zeta)\left(\int_0^{\tau_0} (2F_1\partial_\xi F_1 + 2G_1\partial_\zeta G_1)\left[\frac{1}{x-y} + \frac{1}{x+y}\right]\mathrm{d}y + \int_{\tau_0}^\infty (2F_2\partial_\xi F_2 + 2G_2\partial_\zeta G_2)\left[\frac{1}{x-y} + \frac{1}{x+y}\right]\mathrm{d}y\right) \\
&+ \partial_\xi\left((F_1 - G_1)\partial_\xi F_1\right) + \partial_\zeta\left((G_1 - F_1)\partial_\zeta G_1\right) - 2\partial_\xi F_1\partial_\zeta G_1.
\end{aligned} \tag{13}$$

By rearranging appropriately, (13) becomes

$$\begin{aligned}
-4\eta_{1\xi\zeta} &= \partial_\xi(2\partial_{\tau_1}F_1 + \frac{1}{3}\partial_\xi^3 F_1 + F_1\partial_\xi F_1 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2F_1\partial_\xi F_1\left[\frac{1}{x-y} + \frac{1}{x+y}\right]\mathrm{d}y + \int_{\tau_0}^\infty 2F_2\partial_\xi F_2\left[\frac{1}{x-y} + \frac{1}{x+y}\right]\mathrm{d}y\right)) \\
&+ \partial_\zeta(-2\partial_{\tau_1}G_1 + \frac{1}{3}\partial_\zeta^3 G_1 + G_1\partial_\zeta G_1 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2G_1\partial_\zeta G_1\left[\frac{1}{x-y} + \frac{1}{x+y}\right]\mathrm{d}y + \int_{\tau_0}^\infty 2G_2\partial_\zeta G_2\left[\frac{1}{x-y} + \frac{1}{x+y}\right]\mathrm{d}y\right)) \\
&+ \partial_\xi(-G_1\partial_\xi F_1 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2G_1\partial_\zeta G_1\left[\frac{1}{x-y} + \frac{1}{x+y}\right]\mathrm{d}y + \int_{\tau_0}^\infty 2G_2\partial_\zeta G_2\left[\frac{1}{x-y} + \frac{1}{x+y}\right]\mathrm{d}y\right)) \\
&+ \partial_\zeta(-F_1\partial_\zeta G_1 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2F_1\partial_\xi F_1\left[\frac{1}{x-y} + \frac{1}{x+y}\right]\mathrm{d}y + \int_{\tau_0}^\infty 2F_2\partial_\xi F_2\left[\frac{1}{x-y} + \frac{1}{x+y}\right]\mathrm{d}y\right)) - 2\partial_\xi F_1\partial_\zeta G_1.
\end{aligned} \tag{14}$$

Integration of (14) with respect to ζ yields

$$\begin{aligned}-4\eta_{1\xi} &= \zeta\partial_\xi(2\partial_{\tau_1}F_1 + \frac{1}{3}\partial_\xi^3 F_1 + F_1\partial_\xi F_1 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2F_1\partial_\xi F_1 \left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y + \int_{\tau_0}^{\infty} 2F_2\partial_\xi F_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right))\\
&\quad + (-2\partial_{\tau_1}G_1 + \frac{1}{3}\partial_\zeta^3 G_1 + G_1\partial_\zeta G_1 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2G_1\partial_\zeta G_1 \left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y + \int_{\tau_0}^{\infty} 2G_2\partial_\zeta G_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right))\\
&\quad + \partial_\xi\int(G_1\partial_\xi F_1 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2G_1\partial_\zeta G_1 \left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y + \int_{\tau_0}^{\infty} 2G_2\partial_\zeta G_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right))\,\mathrm{d}\zeta\\
&\quad + (-F_1\partial_\zeta G_1 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2F_1\partial_\xi F_1 \left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y + \int_{\tau_0}^{\infty} 2F_2\partial_\xi F_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right)) - 2\partial_\xi F_1 G_1.\end{aligned} \tag{15}$$

and further integration with respect to ξ leads to

$$\begin{aligned}-4\eta_{1} &= \zeta(2\partial_{\tau_1}F_1 + \frac{1}{3}\partial_\xi^3 F_1 + F_1\partial_\xi F_1 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2F_1\partial_\xi F_1 \left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y + \int_{\tau_0}^{\infty} 2F_2\partial_\xi F_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right))\\
&\quad + \xi(-2\partial_{\tau_1}G_1 + \frac{1}{3}\partial_\zeta^3 G_1 + G_1\partial_\zeta G_1 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2G_1\partial_\zeta G_1 \left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y + \int_{\tau_0}^{\infty} 2G_2\partial_\zeta G_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right))\\
&\quad + \int(G_1\partial_\xi F_1 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2G_1\partial_\zeta G_1 \left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y + \int_{\tau_0}^{\infty} 2G_2\partial_\zeta G_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right))\,\mathrm{d}\zeta\\
&\quad + \int(-F_1\partial_\zeta G_1 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2F_1\partial_\xi F_1 \left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y + \int_{\tau_0}^{\infty} 2F_2\partial_\xi F_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right))\,\mathrm{d}\xi - 2F_1 G_1.\end{aligned}$$

Since η_1 must be bounded, we must have

$$2\partial_{\tau_1}F_1 + \frac{1}{3}\partial_\xi^3 F_1 + F_1\partial_\xi F_1 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2F_1\partial_\xi F_1 \left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y + \int_{\tau_0}^{\infty} 2F_2\partial_\xi F_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right) = 0, \tag{16}$$

$$-2\partial_{\tau_1}G_1 + \frac{1}{3}\partial_\zeta^3 G_1 + G_1\partial_\zeta G_1 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2G_1\partial_\zeta G_1 \left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y + \int_{\tau_0}^{\infty} 2G_2\partial_\zeta G_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right) = 0. \tag{17}$$

In other words, we have obtained two KdV-like equations, (16) and (17), whose solutions F_1, G_1 describe behaviour of the surface elevation in the leading order, when $x < \tau_0$.

The case $x \geqslant \tau_0$

On the domain $x \geqslant \tau_0$, we use

$$\eta = \eta_0 + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2)$$

$$
\begin{aligned}
&= F_2(x-t, \varepsilon t, \ldots) + G_2(x+t, \varepsilon t, \ldots) + \varepsilon \eta_1 + \mathcal{O}(\varepsilon^2) \\
&= F_2(\xi, \tau_1, \ldots) + G_2(\zeta, \tau_1, \ldots) + \varepsilon \eta_1 + \mathcal{O}(\varepsilon^2) \\
&= F_2 + G_2 + \varepsilon \eta_1 + \mathcal{O}(\varepsilon^2).
\end{aligned}
$$

For ease of writing, we suppress explicit dependence on variables, though the reader should bear in mind that function $F_2, (G_2)$ depend on $\xi, (\zeta), \tau_1, \tau_2$, etc. In addition, D'Alembert operator becomes

$$
(\partial_t^2 - \partial_x^2) = \left(-4\partial_\xi \partial_\zeta + 2\varepsilon(\partial_\zeta \partial_{\tau_1} - \partial_\xi \partial_{\tau_1}) + \varepsilon^2 \partial_{\tau_1}^2\right),
$$

so that the LHS of (5) becomes

$$
(\partial_t^2 - \partial_x^2)\eta = \varepsilon\left(-4\eta_{1\xi\zeta} - 2\partial_\xi \partial_{\tau_1} F_1 + 2\partial_\zeta \partial_{\tau_1} G_1\right) + \mathcal{O}(\varepsilon^2). \tag{18}
$$

Now, we deal with the RHS of (5). By appropriate substitutions, the terms become:

$$
\begin{aligned}
\frac{1}{3}\eta_{xxxx} &= \frac{1}{3}(\partial_\xi^4 F_2 + \partial_\zeta^4 (G_2) + \mathcal{O}(\varepsilon)); \\
\left(\int_0^x \eta_t \, \mathrm{d}x'\right)^2 &= \left(\int_0^{\tau_0} \eta_t \, \mathrm{d}x' + \int_{\tau_0}^x \eta_t \, \mathrm{d}x'\right)^2 \\
&= \left(\int_0^{\tau_0} \eta_{0t} \, \mathrm{d}x' + \int_{\tau_0}^x \eta_{0t} \, \mathrm{d}x'\right)^2 + \mathcal{O}(\varepsilon) \\
&= \left(\int_0^{\tau_0} (-\partial_{\xi'} + \partial_{\zeta'} + \varepsilon\partial_{\tau_1})(F_1 + G_1) \, \mathrm{d}x' + \int_{\tau_0}^x (-\partial_{\xi'} + \partial_{\zeta'} + \varepsilon\partial_{\tau_1})(F_2 + G_2) \, \mathrm{d}x'\right)^2 + \mathcal{O}(\varepsilon) \\
&= \left(\int_0^{\tau_0} (-\partial_{\xi'} + \partial_{\zeta'})(F_1 + G_1) \, \mathrm{d}x' + \int_{\tau_0}^x (-\partial_{\xi'} + \partial_{\zeta'})(F_2 + G_2) \, \mathrm{d}x'\right)^2 + \mathcal{O}(\varepsilon), \\
&= F_2^2 - 2F_2 G_2 + G_2^2 + \mathcal{O}(\varepsilon),
\end{aligned}
$$

where for the last line we have simplified as follows:

$$
\begin{aligned}
\int_0^{\tau_0} -\partial_{\xi'}(F_1(\tau_0 - \xi')) \, \mathrm{d}x' &= -\int_{-\tau_0}^0 \partial_{\xi'} F_1(-\xi', \tau_1) \, \mathrm{d}\xi' = -F_1(0) + F_1(\tau_0), \\
\int_0^{\tau_0} \partial_{\zeta'} G_1(x' + \tau_0, \tau_1) \, \mathrm{d}x' &= \int_{\tau_0}^{2\tau_0} \partial_{\zeta'} G_1(\zeta', \tau_1) \, \mathrm{d}\zeta' = G_1(2\tau_0) - G_1(\tau_0) \\
\int_{\tau_0}^x -\partial_{\xi'}(F_2(\tau_0 - \xi')) \, \mathrm{d}x' &= -\int_0^{x-\tau_0} \partial_{\xi'} F_2(\xi', \tau_1) \, \mathrm{d}\xi' = -F_2 + F_2(0),
\end{aligned}
$$

$$\int_{\tau_0}^{x} \partial_{\zeta'} G_2(x'+\tau_0,\tau_1)\,\mathrm{d}x' = \int_{2\tau_0}^{x+\tau_0} \partial_{\zeta'} G_2(\zeta',\tau_1)\,\mathrm{d}\zeta' = G_2 - G_2(2\tau_0).$$

Addition of terms yields

$$\begin{aligned}\int_0^{\tau_0} (-\partial_{\xi'}+\partial_{\zeta'})(F_1+G_1)\,\mathrm{d}x' + \int_{\tau_0}^{x} (-\partial_{\xi'}+\partial_{\zeta'})(F_2+G_2)\,\mathrm{d}x' &= -F_1(0)+F_1(\tau_0)+G_1(2\tau_0)-G_1(\tau_0)-F_2+F_2(0)+G_2-G_2(2\tau_0)\\ &= -F_2+G_2-F_1(0)+F_2(0)+G_1(2\tau_0)-G_2(2\tau_0)+F_1(\tau_0)-G_1(\tau_0)\\ &= -F_2+G_2,\end{aligned}$$

where we recall the interface conditions $F_1(0)=F_2(0), G_1(2\tau_0)=G_2(2\tau_0)$ and the boundary conditions $\eta_x(0,t)=0$. The latter yields that

$$\eta_x(0,t) = \partial_x(F_1(\tau_0-x)+G_1(\tau_0+x))\Big|^{x=0} = -F_1'(\tau_0)+G_1'(\tau_0) = 0 \implies F_1'(\tau_0)=G_1'(\tau_0) \implies F_1(\tau_0)=G_1(\tau_0),$$

where we set the scalar of integration to be 0. Finally, by Proposition 5, we also have

$$\begin{aligned}&\int_0^{\infty} \partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y\\ &\quad= \int_0^{\tau_0} (2F_1\partial_\xi F_1+2G_1\partial_\zeta G_1)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y + \int_{\tau_0}^{\infty} (2F_2\partial_\xi F_2+2G_2\partial_\zeta G_2)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y.\end{aligned}$$

Substitution of terms into the RHS of (5) leads to:

$$\begin{aligned}&\frac{1}{3}\eta_{xxxx} + \frac{d}{dx}\frac{1}{\pi}\int_0^{\infty} \partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y + \frac{1}{2}\partial_x^2\left(\int_0^x \eta_t\,\mathrm{d}x'\right)^2\\ &\quad= \frac{1}{3}(\partial_\xi^4 F_2+\partial_\zeta^4 G_2 + \frac{1}{\pi}(\partial_\xi+\partial_\zeta)\left(\int_0^{\tau_0} (2F_1\partial_\xi F_1+2G_1\partial_\zeta G_1)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y + \int_{\tau_0}^{\infty} (2F_2\partial_\xi F_2+2G_2\partial_\zeta G_2)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y\right)\\ &\quad+ \frac{1}{2}(\partial_\xi^2+2\partial_\xi\partial_\zeta+\partial_\zeta^2)\left(F_2^2-2F_2G_2+G_2^2\right)+\mathcal{O}(\varepsilon)\\ &\quad= \frac{1}{3}(\partial_\xi^4 F_2+\partial_\zeta^4 G_2 + \frac{1}{\pi}(\partial_\xi+\partial_\zeta)\left(\int_0^{\tau_0} (2F_1\partial_\xi F_1+2G_1\partial_\zeta G_1)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y + \int_{\tau_0}^{\infty} (2F_2\partial_\xi F_2+2G_2\partial_\zeta G_2)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y\right)\\ &\quad+ \partial_\xi\left((F_2-G_2)\partial_\xi F_2\right)+\partial_\zeta\left((G_2-F_2)\partial_\zeta G_2\right)-2\partial_\xi F_2\partial_\zeta G_2.\end{aligned} \tag{19}$$

Combining (18) and (19), in $\mathcal{O}(\varepsilon^1)$ we have

$$\begin{aligned}
-4\eta_{1\xi\zeta} &= 2\partial_\xi\partial_{\tau_1}F_2 - 2\partial_\zeta\partial_{\tau_1}G_2 + \frac{1}{3}(\partial_\xi^4 F_2 + \partial_\zeta^4 G_2)\\
&+ \frac{1}{\pi}(\partial_\xi + \partial_\zeta)\left(\int_0^{\tau_0}(2F_1\partial_\xi F_1 + 2G_1\partial_\zeta G_1)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y + \int_{\tau_0}^\infty (2F_2\partial_\xi F_2 + 2G_2\partial_\zeta G_2)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right)\\
&+ \partial_\xi\left((F_2-G_2)\partial_\xi F_2\right) + \partial_\zeta\left((G_2-F_2)\partial_\zeta G_2\right) - 2\partial_\xi F_2\partial_\zeta G_2.
\end{aligned} \tag{20}$$

By rearranging appropriately, (20) becomes

$$\begin{aligned}
-4\eta_{1\xi\zeta} &= \partial_\xi(2\partial_{\tau_1}F_2 + \frac{1}{3}\partial_\xi^3 F_2 + F_2\partial_\xi F_2 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2F_1\partial_\xi F_1\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y + \int_{\tau_0}^\infty 2F_2\partial_\xi F_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right))\\
&+ \partial_\zeta(-2\partial_{\tau_1}G_2 + \frac{1}{3}\partial_\zeta^3 G_2 + G_2\partial_\zeta G_2 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2G_1\partial_\zeta G_1\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y + \int_{\tau_0}^\infty 2G_2\partial_\zeta G_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right))\\
&+ \partial_\xi(-G_2\partial_\xi F_2 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2G_1\partial_\zeta G_1\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y + \int_{\tau_0}^\infty 2G_2\partial_\zeta G_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right))\\
&+ \partial_\zeta(-F_2\partial_\zeta G_2 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2F_1\partial_\xi F_1\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y + \int_{\tau_0}^\infty 2F_2\partial_\xi F_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right)) - 2\partial_\xi F_2\partial_\zeta G_2.
\end{aligned} \tag{21}$$

Integration of (14) with respect to ζ yields

$$\begin{aligned}
-4\eta_{1\xi} &= \zeta\partial_\xi(2\partial_{\tau_1}F_2 + \frac{1}{3}\partial_\xi^3 F_2 + F_2\partial_\xi F_2 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2F_1\partial_\xi F_1\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y + \int_{\tau_0}^\infty 2F_2\partial_\xi F_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right))\\
&+ (-2\partial_{\tau_1}G_2 + \frac{1}{3}\partial_\zeta^3 G_2 + G_2\partial_\zeta G_2 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2G_1\partial_\zeta G_1\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y + \int_{\tau_0}^\infty 2G_2\partial_\zeta G_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right))\\
&+ \partial_\xi\int(G_2\partial_\xi F_2 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2G_1\partial_\zeta G_1\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y + \int_{\tau_0}^\infty 2G_2\partial_\zeta G_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right))\,\mathrm{d}\zeta\\
&+ (-F_2\partial_\zeta G_2 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2F_1\partial_\xi F_1\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y + \int_{\tau_0}^\infty 2F_2\partial_\xi F_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right)) - 2\partial_\xi F_2 G_2.
\end{aligned} \tag{22}$$

and further integration with respect to ξ leads to

$$-4\eta_1 = \zeta(2\partial_{\tau_1}F_2 + \frac{1}{3}\partial_\xi^3 F_2 + F_2\partial_\xi F_2 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2F_1\partial_\xi F_1\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y + \int_{\tau_0}^\infty 2F_2\partial_\xi F_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right))$$

$$+\xi(-2\partial_{\tau_1}G_2+\frac{1}{3}\partial_\zeta^3G_2+G_2\partial_\zeta G_2+\frac{1}{\pi}\left(\int_0^{\tau_0}2G_1\partial_\zeta G_1\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y+\int_{\tau_0}^{\infty}2G_2\partial_\zeta G_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right))$$
$$+\int(G_2\partial_\xi F_2+\frac{1}{\pi}\left(\int_0^{\tau_0}2G_1\partial_\zeta G_1\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y+\int_{\tau_0}^{\infty}2G_2\partial_\zeta G_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right))\,\mathrm{d}\zeta$$
$$+\int(-F_2\partial_\zeta G_2+\frac{1}{\pi}\left(\int_0^{\tau_0}2F_1\partial_\xi F_1\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y+\int_{\tau_0}^{\infty}2F_2\partial_\xi F_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right))\,\mathrm{d}\xi-2F_2G_2.$$

Since η_1 must be bounded, we must have

$$2\partial_{\tau_1}F_2+\frac{1}{3}\partial_\xi^3F_2+F_2\partial_\xi F_2+\frac{1}{\pi}\left(\int_0^{\tau_0}2F_1\partial_\xi F_1\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y+\int_{\tau_0}^{\infty}2F_2\partial_\xi F_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right)=0, \tag{23}$$
$$-2\partial_{\tau_1}G_2+\frac{1}{3}\partial_\zeta^3G_2+G_2\partial_\zeta G_2+\frac{1}{\pi}\left(\int_0^{\tau_0}2G_1\partial_\zeta G_1\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y+\int_{\tau_0}^{\infty}2G_2\partial_\zeta G_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right)=0. \tag{24}$$

In other words, we have obtained two KdV-like equations, (23) and (24), whose solutions F_2, G_2 describe behaviour of the surface elevation in the leading order, when $x \geqslant \tau_0$.

Conclusion

In summary, we have obtained 2 systems of 2 equations: one in F_1, F_2 :

$$\begin{aligned}2\partial_{\tau_1}F_1+\frac{1}{3}\partial_\xi^3F_1+F_1\partial_\xi F_1+\frac{1}{\pi}\left(\int_0^{\tau_0}2F_1\partial_\xi F_1\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y+\int_{\tau_0}^{\infty}2F_2\partial_\xi F_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right)=0;\\2\partial_{\tau_1}F_2+\frac{1}{3}\partial_\xi^3F_2+F_2\partial_\xi F_2+\frac{1}{\pi}\left(\int_0^{\tau_0}2F_1\partial_\xi F_1\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y+\int_{\tau_0}^{\infty}2F_2\partial_\xi F_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right)=0,\end{aligned} \tag{25}$$

and another one in G_1, G_2 :

$$\begin{aligned}-2\partial_{\tau_1}G_1+\frac{1}{3}\partial_\zeta^3G_1+G_1\partial_\zeta G_1+\frac{1}{\pi}\left(\int_0^{\tau_0}2G_1\partial_\zeta G_1\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y+\int_{\tau_0}^{\infty}2G_2\partial_\zeta G_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right)=0;\\-2\partial_{\tau_1}G_2+\frac{1}{3}\partial_\zeta^3G_2+G_2\partial_\zeta G_2+\frac{1}{\pi}\left(\int_0^{\tau_0}2G_1\partial_\zeta G_1\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y+\int_{\tau_0}^{\infty}2G_2\partial_\zeta G_2\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right)=0.\end{aligned} \tag{26}$$

It should be noted that even though two systems (25) and (26) seem to be unrelated, this separation was due to the boundary conditions that we imposed at the interface

$$F_1(0)=F_2(0), G_1(2\tau_0)=G_2(2\tau_0),$$

and the free boundary condition at the midpoint

$$\eta_x(0,t)=-F_1'(\tau_0)+G_1'(\tau_0)=0.$$

The Hilbert Transform term

Proposition 5. *We have*

$$\begin{aligned}&\int_0^\infty \partial_t \left(\eta \int_0^y \eta_t \,\mathrm{d}y'\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y \\ &\quad = \int_0^{\tau_0} (2F_1\partial_\xi F_1 + 2G_1\partial_\zeta G_1) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \int_{\tau_0}^\infty (2F_2\partial_\xi F_2 + 2G_2\partial_\zeta G_2) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y.\end{aligned}$$

Proof. Note:

$$\begin{aligned}\int_0^\infty \partial_t \left(\eta \int_0^y \eta_t \,\mathrm{d}y'\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y &= \int_0^\infty (-\partial_\xi + \partial_\zeta + \varepsilon\partial_{\tau_1}) \left((\eta_0 + \varepsilon\eta_1) \int_0^y (\eta_0 + \varepsilon\eta_1)_t \,\mathrm{d}y'\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \mathcal{O}(\varepsilon^2) \\ &= \int_0^\infty (-\partial_\xi + \partial_\zeta) \left((\eta_0 + \varepsilon\eta_1) \int_0^y (\eta_0 + \varepsilon\eta_1)_t \,\mathrm{d}y'\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \mathcal{O}(\varepsilon) \\ &= \int_0^\infty (-\partial_\xi + \partial_\zeta) \left(\eta_0 \int_0^y (\eta_0)_t \,\mathrm{d}y'\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \mathcal{O}(\varepsilon) \\ &= \int_0^\infty (-\partial_\xi + \partial_\zeta) \left(\eta_0 \int_0^y (-\partial_\xi + \partial_\zeta + \varepsilon\partial_{\tau_1})\eta_0 \,\mathrm{d}y'\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \mathcal{O}(\varepsilon) \\ &= \int_0^\infty (-\partial_\xi + \partial_\zeta) \left(\eta_0 \int_0^y (-\partial_\xi + \partial_\zeta)\eta_0 \,\mathrm{d}y'\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \mathcal{O}(\varepsilon).\end{aligned}$$

Now, recalling that η_0 is piecewise, we split the integral:

$$\int_0^{\tau_0} (-\partial_\xi + \partial_\zeta) \left(\eta_0 \int_0^y (-\partial_\xi + \partial_\zeta)\eta_0 \,\mathrm{d}y'\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y, \tag{27}$$

and

$$\int_{\tau_0}^\infty (-\partial_\xi + \partial_\zeta) \left(\eta_0 \int_0^y (-\partial_\xi + \partial_\zeta)\eta_0 \,\mathrm{d}y'\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y. \tag{28}$$

We deal with (27):

$$\begin{aligned}\int_0^{\tau_0} (-\partial_\xi + \partial_\zeta) \left(\eta_0 \int_0^y (-\partial_\xi + \partial_\zeta)\eta_0 \,\mathrm{d}y'\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y &= \int_0^{\tau_0} (-\partial_\xi + \partial_\zeta) \left((F_1 + G_1) \int_0^y (-\partial_\xi + \partial_\zeta)(F_1 + G_1) \,\mathrm{d}y'\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y \\ &= \int_0^{\tau_0} (-\partial_\xi + \partial_\zeta) \left((F_1 + G_1) \int_0^y (-\partial_\xi F_1 + \partial_\zeta G_1) \,\mathrm{d}y'\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\tau_0} (-\partial_\xi + \partial_\zeta)\left((F_1+G_1)(-(F_1 - F_1(\tau_0)) + G_1 - G_1(\tau_0))\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y \\
&= \int_0^{\tau_0} (-\partial_\xi + \partial_\zeta)\left((F_1+G_1)(-F_1 + G_1 + F_1(\tau_0)) - G_1(\tau_0))\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y \\
&= \int_0^{\tau_0} \big((-\partial_\xi + \partial_\zeta)(-F_1^2 + G_1^2) + (-\partial_\xi + \partial_\zeta)(F_1+G_1)(F_1(\tau_0)) - G_1(\tau_0))\big) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y \\
&= \int_0^{\tau_0} (2F_1\partial_\xi F_1 + 2G_1\partial_\zeta G_1) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y,
\end{aligned}
$$

where we can set $F_1(\tau_0)) - G_1(\tau_0) = 0$ by imposing a free end condition $\eta_x(0,t) = 0$. Now, we deal with (28):

$$
\begin{aligned}
&\int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta)\left(\eta_0 \int_0^y (-\partial_\xi + \partial_\zeta)\eta_0 \,\mathrm{d}y'\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y \\
&\quad= \int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta)\left((F_2+G_2)\left(\int_0^{\tau_0} (-\partial_\xi + \partial_\zeta)(F_1+G_1)\,\mathrm{d}y' + \int_{\tau_0}^{y} (-\partial_\xi + \partial_\zeta)(F_2+G_2)\,\mathrm{d}y'\right)\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y \\
&\quad= \int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta)\left((F_2+G_2)\left((-F_1(0) + F_1(\tau_0) + G_1(2\tau_0) - G_1(\tau_0) - F_2 + F_2(0) + G_2 - G_2(2\tau_0)\right)\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y \\
&\quad= \int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta)\left((F_2+G_2)\left(-F_2 + G_2 + F_2(0) - F_1(0) + F_1(\tau_0) + G_1(2\tau_0) - G_2(2\tau_0) - G_1(\tau_0)\right)\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y \\
&\quad= \int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta)\left((F_2+G_2)(-F_2+G_2)\right) + (-\partial_\xi + \partial_\zeta)\left((F_2+G_2)(F_2(0) - F_1(0) + F_1(\tau_0) + G_1(2\tau_0) - G_2(2\tau_0) - G_1(\tau_0)\right)\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y.
\end{aligned}
$$

where for the last line we translate $\xi' = x' - t, \zeta' = x' + t$ to obtain

$$
\begin{aligned}
\int_t^y -\partial_{\xi'}(F_2(\xi' - \tau_0))\,\mathrm{d}x' &= \int_0^{y-t} (F_2)_{\xi'}(\xi', \tau_1)\,\mathrm{d}\xi' = \int_0^{\xi} (F_2)_{\xi'}(\xi', \tau_1)\,\mathrm{d}\xi' = F_2(\xi, \tau_1) - F_2(0, \tau_1), \\
\int_t^y \partial_{\zeta'}(G_2)(\xi' + \tau_0, \tau_1)\,\mathrm{d}x' &= \int_{2t}^{y+t} (G_2)_{\zeta'}(\zeta', \tau_1)\,\mathrm{d}\zeta' = \int_{2\tau_0}^{\zeta} (G_2)_{\zeta'}(\zeta', \tau_1)\,\mathrm{d}\zeta' = G_2(\zeta, \tau_1) - G_2(2\tau_0, \tau_1).
\end{aligned}
$$

We have that

$$
\begin{aligned}
\int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta)\left((F_2+G_2)(-F_2+G_2)\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y &= \int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta)(-F_2^2 + G_2^2) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y \\
&= \int_{\tau_0}^{\infty} (\partial_\xi(F_2^2) + \partial_\zeta(G_2^2)) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y
\end{aligned}
$$

$$= \int_{\tau_0}^{\infty} (2F_2\partial_\xi F_2 + 2G_2\partial_\zeta G_2) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y.$$

Looking at the term

$$\int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta)\left((F_2+G_2)(F_2(0) - F_1(0) + F_1(\tau_0) + G_1(2\tau_0) - G_2(2\tau_0) - G_1(\tau_0))\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y, \tag{29}$$

we observe interaction between F_i, G_i at the interface $x = \tau_0$. If we impose continuity, then $F_2(0) = F_1(0), G_2(2\tau_0) = G_1(2\tau_0)$, which leaves us with $F_1(\tau_0) - G_1(\tau_0)$. As before, we can eliminate this term by imposing a free end condition $\eta_x(0,t) = 0$. Therefore, the term (29) vanishes due to boundary conditions. So, we obtain

$$\begin{aligned}\int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta)\left(\eta_0 \int_0^y (-\partial_\xi + \partial_\zeta)\eta_0 \,\mathrm{d}y'\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y &= \int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta)\left((F_2+G_2)(-F_2+G_2)\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y \\ &= \int_{\tau_0}^{\infty} (2F_2\partial_\xi F_2 + 2G_2\partial_\zeta G_2) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y,\end{aligned}$$

so that

$$\begin{aligned}&\int_0^{\infty} \partial_t\left(\eta \int_0^y \eta_t \,\mathrm{d}y'\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y \\ &= \int_0^{\tau_0} (-\partial_\xi + \partial_\zeta)\left(\eta_0 \int_0^y (-\partial_\xi + \partial_\zeta)\eta_0 \,\mathrm{d}y'\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta)\left(\eta_0 \int_0^y (-\partial_\xi + \partial_\zeta)\eta_0 \,\mathrm{d}y'\right) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y \\ &= \int_0^{\tau_0} (2F_1\partial_\xi F_1 + 2G_1\partial_\zeta G_1) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \int_{\tau_0}^{\infty} (2F_2\partial_\xi F_2 + 2G_2\partial_\zeta G_2) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y.\end{aligned}$$

The proof is complete. □

References

[1] Tom M. Apostol, *Mathematical analysis*, Pearson, 1974.