Report 2

Sultan Aitzhan

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Contents

1 Water-wave problem on the whole line: non-local formulation

Recall the full water-wave problem on a whole line:

$$\phi_{xx} + \phi_{zz} = 0 \qquad -h < z < \eta(x,t) \tag{1a}$$

$$\phi_z = 0 \qquad z = -h \tag{1b}$$

$$\eta_t + \phi_x \eta_x = \phi_z \qquad z = \eta(x,t) \tag{1c}$$

$$\phi_t + g\eta + \frac{1}{2}(\phi_x^2 + \phi_z^2) = 0 \qquad z = \eta(x,t) \tag{1d}$$

where we set the surface tension to be zero. Consider the velocity potential evaluated at the surface:

$$q(x) = \phi(x, \eta(x)).$$

Combining (1c) and (1d), we obtain

$$q_t + \frac{1}{2}q_x^2 + g\eta - \frac{1}{2}\frac{(\eta_t + q_x\eta_x)^2}{1+\eta_x^2} = 0, \tag{2}$$

which is an equation for two unknowns q, η. Now, we introduce an operator that maps the normal derivative at a surface η to the tangential derivative at the surface:

$$\mathcal{H}(\eta, D)\{\nabla\phi \cdot \vec{N}\} = \nabla\phi \cdot \vec{T}, \tag{3}$$

where $D = -i\nabla$. For convenience, we drop the vector notation. Note that

$$\nabla\phi \cdot N = \begin{bmatrix}\phi_x \\ \phi_z\end{bmatrix} \cdot \begin{bmatrix}-\eta_x \\ 1\end{bmatrix} = \phi_z - \phi_x\eta_x = \eta_t$$

and

$$\nabla\phi \cdot T = \begin{bmatrix}\phi_x \\ \phi_z\end{bmatrix} \cdot \begin{bmatrix}1 \\ \eta_x\end{bmatrix} = \phi_x + \eta_x\phi_z = q_x.$$

This allows us to rewrite (3) as

$$\mathcal{H}(\eta, D)\{\eta_t\} = q_x. \tag{4}$$

Looking at the system

$$\begin{aligned} q_t + \frac{1}{2}q_x^2 + g\eta - \frac{1}{2}\frac{(\eta_t + q_x\eta_x)^2}{1+\eta_x^2} &= 0, \\ \mathcal{H}(\eta, D)\{\eta_t\} &= q_x, \end{aligned}$$

we recognise that we can rewrite the full water-wave problem in terms of the $\mathcal{H}$ operator. This is done by differentiating (2) with respect to x and (4) with respect to t :

$$\partial_t(q_x) + \partial_x\left(\frac{1}{2}q_x^2 + g\eta - \frac{1}{2}\frac{(\eta_t + q_x\eta_x)^2}{1+\eta_x^2}\right) = 0, \tag{5}$$

$$\partial_t(\mathcal{H}(\eta, D)\{\eta_t\}) = q_{xt}. \tag{6}$$

Substituting (6) into (5), we obtain

$$\partial_t\left(\mathcal{H}(\eta, D)\{\eta_t\}\right) + \partial_x\left(\frac{1}{2}\left(\mathcal{H}(\eta, D)\{\eta_t\}\right)^2 + \varepsilon\eta - \frac{1}{2}\frac{(\eta_t + \eta_x\mathcal{H}(\eta, D)\{\eta_t\})^2}{1+\eta_x^2}\right) = 0. \tag{7}$$

The utility of (7) depends on whether we can find a useful representation for the operator $\mathcal{H}(\eta, D)$. In the next section, we describe an equation that the $\mathcal{H}$ operator must satisfy.

1.0.1 Behaviour of the $\mathcal{H}$ operator

Consider the following boundary value problem:

$$\phi_{xx} + \phi_{zz} = 0 \qquad -h < z < \eta(x,t) \tag{8a}$$

$$\phi_z = 0 \qquad z = -h \tag{8b}$$

$$\nabla\phi \cdot N = f(x) \qquad z = \eta(x,t) \tag{8c}$$

Let φ be harmonic, so that

$$\varphi_{xx} + \varphi_{zz} = 0.$$

Clearly, φ_z is harmonic, so we have

$$\varphi_z(\phi_{xx} + \phi_{zz}) - \phi((\varphi_z)_{zz} + (\varphi_z)_{xx}) = 0.$$

Taking the integral over the domain yields

$$\int_{-\infty}^{\infty}\int_{-h}^{\eta(x)} \varphi_z(\phi_{xx} + \phi_{zz}) - \phi((\varphi_z)_{zz} + (\varphi_z))\,\mathrm{d}z\,\mathrm{d}x = 0.$$

An application of Green's theorem yields

$$\int_D \varphi_z(\nabla\phi \cdot N) - \phi(\nabla\varphi_z \cdot N)\,\mathrm{d}s = 0,$$

where D is the boundary of the domain, $\mathrm{d}s$ is the area element. Now, observe the following:

$$\begin{aligned} -\nabla\varphi_z \cdot N = -\begin{pmatrix}\varphi_{zx}\\ \varphi_{zz}\end{pmatrix} \cdot \begin{pmatrix} -\dfrac{\mathrm{d}z}{\mathrm{d}s} \\ \dfrac{\mathrm{d}x}{\mathrm{d}s}\end{pmatrix} &= -\begin{pmatrix}\varphi_{zx}\\ -\varphi_{xx}\end{pmatrix} \cdot \begin{pmatrix} -\dfrac{\mathrm{d}z}{\mathrm{d}s} \\ \dfrac{\mathrm{d}x}{\mathrm{d}s}\end{pmatrix} \\ &= \begin{pmatrix}\varphi_{zx}\\ \varphi_{xx}\end{pmatrix} \cdot \begin{pmatrix} \dfrac{\mathrm{d}z}{\mathrm{d}s} \\ \dfrac{\mathrm{d}x}{\mathrm{d}s}\end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \varphi_{xx} \\ \varphi_{xz} \end{pmatrix} \cdot \begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}s} \\ \frac{\mathrm{d}z}{\mathrm{d}s} \end{pmatrix}$$
$$= \nabla\varphi_x \cdot T,$$

from which we rewrite the integral equation:

$$0 = \int_D \varphi_z(\nabla\phi \cdot N) + \phi(\nabla\varphi_z \cdot T)\,\mathrm{d}s.$$

Applying the dot product, we obtain a contour integral:

$$\int_D \varphi_z(\phi_z\,\mathrm{d}x - \phi_x\,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) = 0. \tag{9}$$

We split the contour into four segments:

$$\begin{aligned}\int_D &= \int_{-\infty}^{\infty}\Big|^{z=-h} + \int_{-h}^{\eta(x)}\Big|^{x\to\infty} + \int_{\infty}^{-\infty}\Big|^{\eta(x)} + \int_{\eta(x)}^{-h}\Big|^{x\to-\infty} \\ &= \int_{-\infty}^{\infty}\Big|^{z=-h} + \int_{-h}^{\eta(x)}\Big|^{x\to\infty} - \int_{-\infty}^{\infty}\Big|^{\eta(x)} - \int_{-h}^{\eta(x)}\Big|^{x\to-\infty}.\end{aligned}$$

Consider each segment:

- At $|x| \to \infty$, we know that ϕ and its gradient vanish, so the integral also vanishes on these segments.
- At $z = -h$, $\mathrm{d}z = 0$, so we have

$$\begin{aligned}\int_{-\infty}^{\infty} \varphi_z(\phi_z\,\mathrm{d}x - \phi_x\,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) &= \int_{-\infty}^{\infty} \varphi_z\phi_z + \phi\varphi_{xx}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \phi\varphi_{xx}\,\mathrm{d}x \qquad (\text{since } \phi_z = 0 \text{ at } z = -h) \\ &= \phi(x,-h)\varphi_x(x,-h)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \phi_x(x,-h)\varphi_x(x,-h)\,\mathrm{d}x \\ &= 0.\end{aligned}$$

  where we pick φ such that $\varphi_x(x,-h) = 0$.
- At $z = \eta$, $\mathrm{d}z = \eta_x\,\mathrm{d}x$, so we have

$$\begin{aligned}\int_{-\infty}^{\infty} \varphi_z(\phi_z - \phi_x\varepsilon\phi_x\eta_x) + \phi(\varphi_{xx} + \varphi_{xz}\eta_x)\,\mathrm{d}x &= \int_{-\infty}^{\infty} \varphi_z(\begin{pmatrix}\phi_x \\ \phi_z\end{pmatrix} \cdot \begin{pmatrix}-\eta_x \\ 1\end{pmatrix} + \phi\frac{\mathrm{d}\varphi_x(x,\eta)}{\mathrm{d}x}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \varphi_z\nabla\phi \cdot N + \phi\frac{\mathrm{d}\varphi_x(x,\eta)}{\mathrm{d}x}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \varphi_z\nabla\phi \cdot N - \varphi_x\frac{\mathrm{d}\phi(x,\eta)}{\mathrm{d}x}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \varphi_z\nabla\phi \cdot N - \varphi_x\begin{pmatrix}\phi_x \\ \phi_z\end{pmatrix} \cdot \begin{pmatrix}1 \\ \phi_x\eta_x\end{pmatrix}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \varphi_z\nabla\phi \cdot N - \varphi_x\nabla\phi \cdot T\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \varphi_z f(x) - \varphi_x(x,\eta)\mathcal{H}(\eta, D)\{f(x)\}\,\mathrm{d}x.\end{aligned}$$

Combining segments, we obtain

$$\int_{-\infty}^{\infty} \varphi_z f(x) - \varphi_x(x,\eta)\mathcal{H}(\eta, D)\{f(x)\}\,\mathrm{d}x = 0.$$

Note that we could choose $\varphi(x,z) = e^{-ikx}\sinh(k(z+h))$, so that the integral becomes:

$$\int_{-\infty}^{\infty} e^{-ikx}(k\cosh(k(\eta+h))f(x) + ik\sinh(k(\eta+h))\mathcal{H}(\eta,D)\{f(x)\})\,\mathrm{d}x = 0.$$

Assume that $k \neq 0$. Dividing by k and multiplying by i yields an equation that relates f and the operator $\mathcal{H}$ acting on f :

$$\int_{-\infty}^{\infty} e^{-ikx}(i\cosh(k(\eta+h))f(x) - \sinh(k(\eta+h))\mathcal{H}(\eta,D)\{f(x)\})\,\mathrm{d}x = 0. \tag{10}$$

Remark 1. One has to be careful with dividing by k: if $k=0$, then none of the following work will be valid. Thus, we need to deal with this problem.

Remark 2. A related comment to the above is that the choice of φ is not unique. Indeed, we only required that φ is harmonic, and that $\varphi_x(x,-h) = 0$, which allowed us to cancel the contribution from the bottom. If we choose different φ, then we will end up with different version of (10). This is important to be aware of, since different versions can give an insight into why the case $k=0$ is problematic. Furthermore, if we let $\varphi(x+iz)^n, n \in \mathbb{N}$, then we will end up with conservation laws, which we will exploit later.

1.0.2 Nondimensional, nonlocal formulation

We derive the non-dimensional version of the above work. Let

$$t^\star = \frac{t\sqrt{gh}}{L}, \qquad x^\star = \frac{x}{L}, \qquad z^\star = \frac{z}{h}, \qquad \eta^\star = \frac{\eta}{a}, \qquad k^\star = Lk, \qquad \phi = \frac{Lga}{\sqrt{gh}}\phi^\star, \qquad q^\star = \frac{\sqrt{gh}}{agL}q, \tag{11}$$

and define parameters ε and μ so that

$$\varepsilon = \frac{a}{h}, \qquad \mu = \frac{h}{L}, \qquad \varepsilon\mu = \frac{a}{L}.$$

Let $\varphi(x,z)$ be harmonic, and recall the equation (9)

$$\int_D \varphi_z(\phi_z\,\mathrm{d}x - \phi_x\,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) = 0,$$

which we obtained in the previous section. In non-dimensional coordinates,

$$\mathrm{d}x = L\,\mathrm{d}x^\star, \qquad \mathrm{d}z = h\,\mathrm{d}z^\star, \qquad \phi_z = \frac{L}{h}\frac{ga}{\sqrt{gh}}\phi^\star_{z^\star}, \qquad \phi_x = \frac{ga}{\sqrt{gh}}\phi^\star_{x^\star}.$$

Moreover, we leave φ the same but rescale its variables, which should be contrasted with that we rescaled both the function ϕ and its variables:

$$\varphi_x = \frac{1}{L}\varphi_{x^\star}, \qquad \varphi_z = \frac{1}{h}\varphi_{z^\star}, \qquad \varphi_{xx} = \frac{1}{L^2}\varphi_{x^\star x^\star}, \qquad \varphi_{xz} = \frac{1}{Lh}\varphi_{x^\star z^\star}.$$

Then, equation (9) becomes

$$\int_D \varphi_z(\frac{1}{\mu^2}\phi_z\,\mathrm{d}x - \phi_x\,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) = 0, \tag{12}$$

where we have dropped starred notation. Now, split the contour into the following segments

$$\begin{aligned}\int_D &= \int_{-\infty}^{\infty}\Big|^{z=-1} + \int_{-1}^{\varepsilon\eta(x)}\Big|^{x\to\infty} + \int_{\infty}^{-\infty}\Big|^{z=\varepsilon\eta(x)} + \int_{\varepsilon\eta(x)}^{-1}\Big|^{x\to-\infty} \\ &= \int_{-\infty}^{\infty}\Big|^{z=-1} + \int_{-1}^{\varepsilon\eta(x)}\Big|^{x\to\infty} - \int_{-\infty}^{\infty}\Big|^{z=\varepsilon\eta(x)} - \int_{-1}^{\varepsilon\eta(x)}\Big|^{x\to-\infty}.\end{aligned}$$

Consider integral on each of the segments (WRITE OUT COMPLETELY):

- As $|x| \to \infty$, we know that ϕ and its gradient vanish, so the integral also vanishes on these segments.
- At $z = -1$, $\mathrm{d}z = 0$, so we have

$$\begin{aligned}\int_{-\infty}^{\infty} \varphi_z(\frac{1}{\mu^2}\phi_z\,\mathrm{d}x - \phi_x\,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) &= \int_{-\infty}^{\infty} \frac{1}{\mu^2}\varphi_z\phi_z + \phi\varphi_{xx}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \phi\varphi_{xx}\,\mathrm{d}x \qquad (\text{since } \phi_z = 0 \text{ at } z = -1) \\ &= \phi(x,-h)\varphi_x(x,-h)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \phi_x(x,-h)\varphi_x(x,-h)\,\mathrm{d}x \\ &= 0,\end{aligned}$$

where we pick φ such that $\varphi_x(x,-h) = 0$.
- At $z = \varepsilon\eta$, $\mathrm{d}z = \varepsilon\eta_x\,\mathrm{d}x$. Moreover, introduce

$$\tilde{\nabla} = \begin{pmatrix} \phi_x \\ \frac{1}{\mu}\phi_z \end{pmatrix} \qquad \tilde{N} = \begin{pmatrix} -\varepsilon\phi_x\eta_x \\ 1 \end{pmatrix}$$

we have

$$\begin{aligned}\int_{-\infty}^{\infty} \varphi_z(\frac{1}{\mu^2}\phi_z - \phi_x\varepsilon\phi_x\eta_x) + \phi(\varphi_{xx} + \varepsilon\varphi_{xz}\eta_x)\,\mathrm{d}x &= \int_{-\infty}^{\infty} \frac{1}{\mu}\varphi_z \begin{pmatrix} \phi_x \\ \frac{1}{\mu}\phi_z \end{pmatrix} \cdot \begin{pmatrix} -\varepsilon\phi_x\eta_x \\ 1 \end{pmatrix} + \phi\frac{\mathrm{d}\varphi_x(x,\varepsilon\eta)}{\mathrm{d}x}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{1}{\mu}\varphi_z\tilde{\nabla}\phi\cdot\tilde{N} + \phi\frac{\mathrm{d}\varphi_x(x,\varepsilon\eta)}{\mathrm{d}x}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{1}{\mu}\varphi_z\tilde{\nabla}\phi\cdot\tilde{N} - \varphi_x\frac{\mathrm{d}\phi(x,\varepsilon\eta)}{\mathrm{d}x}\,\mathrm{d}x + \varphi\phi(x,\varepsilon\eta)\Big|_{-\infty}^{\infty} \\ &= \int_{-\infty}^{\infty} \frac{1}{\mu}\varphi_z\tilde{\nabla}\phi\cdot\tilde{N} - \varphi_x\frac{\mathrm{d}\phi(x,\varepsilon\eta)}{\mathrm{d}x}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{1}{\mu}\varphi_z\tilde{\nabla}\phi\cdot\tilde{N} - \varphi_x\begin{pmatrix} \phi_x \\ \frac{1}{\mu}\phi_z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \varepsilon\phi_x\eta_x \end{pmatrix}\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{1}{\mu}\varphi_z\tilde{\nabla}\phi\cdot\tilde{N} - \varphi_x\tilde{\nabla}\phi\cdot\tilde{T}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{1}{\mu}\varphi_z g(x) - \varphi_x(x,\varepsilon\eta)\mathcal{H}(\varepsilon\eta, D)\{g(x)\}\,\mathrm{d}x.\end{aligned}$$

Observe that

$$\nabla\phi\cdot N = \frac{ga}{\sqrt{gh}}\tilde{\nabla}\phi\cdot\tilde{N} = \frac{ga}{\sqrt{gh}}g(x^\star) = f(x^\star L) = f(x),$$

so that

$$g(x^\star) = \frac{\sqrt{gh}}{ga}f(x^\star L) = \frac{\sqrt{gh}}{ga}f(x).$$

Combining segments, we obtain

$$\int_{-\infty}^{\infty} \frac{1}{\mu}\varphi_z g(x) - \varphi_x(x,\varepsilon\eta)\mathcal{H}(\varepsilon\eta, D)\{g(x)\}\,\mathrm{d}x = 0.$$

As before, we choose $\varphi(x,z) = e^{-ikx}\sinh(k(z+h)) = e^{-ik^\star x^\star}\sinh(\mu k^\star(z^\star+1)) = e^{-ikx}\sinh(\mu k(z+1))$ so that the integral becomes:

$$\int_{-\infty}^{\infty} e^{-ikx}(k\cosh(\mu k(\eta+1))g(x) + ik\sinh(\mu k(\eta+1))\mathcal{H}(\varepsilon\eta, D)\{g(x)\})\,\mathrm{d}x = 0.$$

Assume that $k \neq 0$. Dividing by k and multiplying by i yields an equation that relates g and the operator $\mathcal{H}$ acting on g :

$$\int_{-\infty}^{\infty} e^{-ikx}(i\cosh(\mu k(\eta+1))g(x) - \sinh(\mu k(\eta+1))\mathcal{H}(\varepsilon\eta, D)\{g(x)\})\,\mathrm{d}x = 0. \tag{13}$$

1.0.3 Representation of the $\mathcal{H}$ operator

In this section, we derive a representation of the $\mathcal{H}$ operator in the leading two terms. To begin, consider (13) and expand in ε :

$$\begin{aligned}\cosh(\mu k(\eta+1)) &= \cosh(\mu k) + \mu k\varepsilon\eta\sinh(\mu k) + \dots,\\ \sinh(\mu k(\eta+1)) &= \sinh(\mu k) + \mu k\varepsilon\eta\cosh(\mu k) + \dots,\\ \mathcal{H}(\eta, D)\{g(x)\} &= [\mathcal{H}_0 + \varepsilon\mathcal{H}_1 + \dots](\varepsilon\eta, D)\{g(x)\}.\end{aligned}$$

Equation (13) becomes:

$$\begin{aligned}&\int_{-\infty}^{\infty} e^{-ikx}(i[\cosh(\mu k) + \mu k\varepsilon\eta\sinh(\mu k) + \dots]g(x)\\ &\qquad - [\sinh(\mu k) + \mu k\varepsilon\eta\cosh(\mu k) + \dots][\mathcal{H}_0 + \varepsilon\mathcal{H}_1 + \dots](\varepsilon\eta, D)\{g(x)\})\,\mathrm{d}x = 0.\end{aligned}$$

Within $\mathcal{O}(\varepsilon^0)$: from expansions above, we have

$$\int_{-\infty}^{\infty} e^{-ikx}(i\cosh(\mu k)g(x) - \sinh(\mu k)\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\})\,\mathrm{d}x = 0.$$

Dividing by $\sinh(\mu k)$, we obtain

$$\int_{-\infty}^{\infty} e^{-ikx}(i\coth(\mu k)g(x) - \mathcal{H}_0(\varepsilon\eta, D)\{g(x)\})\,\mathrm{d}x = 0.$$

Splitting the integrand and recognizing Fourier transform yields:

$$\mathcal{F}\{\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\}\}_k = \int_{-\infty}^{\infty} e^{-ikx}\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\}\,\mathrm{d}x = \int_{-\infty}^{\infty} e^{-ikx} i\coth(\mu k)g(x)\,\mathrm{d}x = i\coth(\mu k)\mathcal{F}\{g(x)\}_k.$$

Finally, we invert Fourier transform to obtain

$$\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\} = \mathcal{F}^{-1}\{i\coth(\mu k)\mathcal{F}\{g(x)\}_k\}_k,$$

where we write out k's explicitly to keep track of transforms.
Within $\mathcal{O}(\varepsilon^1)$: from expansions above, we have

$$\int_{-\infty}^{\infty} e^{-ikx}(i\mu k\eta\sinh(\mu k)g(x) - [\sinh(\mu k)\mathcal{H}_1 + \mu k\eta\cosh(\mu k)\mathcal{H}_0](\varepsilon\eta, D)\{g(x)\})\,\mathrm{d}x = 0.$$

Dividing by $\sinh(\mu k)$ and dropping $(\varepsilon\eta, D)$ for convenience, we have

$$\int_{-\infty}^{\infty} e^{-ikx}(i\mu k\eta g(x) - [\mathcal{H}_1 + \mu k\eta\coth(\mu k)\mathcal{H}_0]\{g(x)\})\,\mathrm{d}x = 0.$$

Rearranging and recognising Fourier transform yields:

$$\begin{aligned}\mathcal{F}\{\mathcal{H}_1\{g(x)\}\}_k &= \int_{-\infty}^{\infty} e^{-ikx}\mathcal{H}_1\{g(x)\})\,\mathrm{d}x\\ &= \int_{-\infty}^{\infty} e^{-ikx}(i\mu k\eta g - \mu k\eta\coth(\mu k)\mathcal{H}_0\{g(x)\})\,\mathrm{d}x\\ &= \mu\mathcal{F}\{ik\eta g\}_k - \mu k\coth(\mu k)\mathcal{F}\{\eta\mathcal{H}_0\{g(x)\}\}\}_k.\end{aligned}$$

Inverting Fourier transform, we obtain an expression for $\mathcal{H}_1$:

$$\begin{aligned}\mathcal{H}_1\{g(x)\} &= \mathcal{F}^{-1}\{\mu\eta g\}_k - \mathcal{F}^{-1}\{\mu k\coth(\mu k)\mathcal{F}\{\eta\mathcal{H}_0\{g(x)\}\})\}_k\}\\ &= \mu\partial_x(\eta g) - \mathcal{F}^{-1}\{\mu k\coth(\mu k)\mathcal{F}\{\eta\mathcal{H}_0\{g(x)\}\})\}_k\}_k\\ &= \mu\partial_x(\eta g) - \mathcal{F}^{-1}\{\mu k\coth(\mu k)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\coth(\mu l)\mathcal{F}\{g\}_l\}_l\}_k\}_k.\end{aligned}$$

In sum, we obtain

$$\begin{aligned}\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\} &= \mathcal{F}^{-1}\{i\coth(\mu k)\mathcal{F}\{g(x)\}_k\}_k,\\ \mathcal{H}_1(\varepsilon\eta, D)\{g(x)\} &= \mu\partial_x(\eta g) - \mathcal{F}^{-1}\{\mu k\coth(\mu k)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\coth(\mu l)\mathcal{F}\{g\}_l\}_l\}_k\}_k.\end{aligned}$$

1.0.4 Deriving an expression for surface elevation

In this section, we would like to derive an expression for η. We can do this because the scalar equation (7) is written in terms of η. The non-dimensional version of (7) is given by

$$\partial_t \left(\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right) + \partial_x \left(\frac{1}{2} \left(\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 + \varepsilon\eta - \frac{1}{2}\varepsilon^2\mu^2 \frac{(\eta_t + \eta_x \mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\})^2}{1+\varepsilon^2\mu^2\eta_x^2} \right) = 0. \tag{14}$$

Within $\mathcal{O}(\mu^0)$. In the leading order, the equation (14) becomes

$$\partial_t \left(\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right) + \varepsilon\partial_x\eta = 0.$$

Substituting an expression for $\mathcal{H}_0$, we obtain:

$$\mathcal{F}^{-1}\{i\coth(\mu k)\mathcal{F}\{\varepsilon\mu\eta_{tt}\}_k\}_k + \varepsilon\partial_x\eta = 0,$$

where we brought the time derivative inside the transform. Inverting the Fourier transform and multiplying by $\frac{k}{i\varepsilon}$ yields

$$\mu k\coth(\mu k)\widehat{\eta_{tt}}_k + k^2\widehat{\eta}_k = 0.$$

Recall

$$\coth(\mu k) \approx \frac{1}{\mu k} + \mathcal{O}(\mu),$$

so that

$$\widehat{\eta_{tt}}_k + k^2\widehat{\eta}_k = 0.$$

Inverting the Fourier transform, we have

$$\eta_{tt} + (-i\partial_x)^2\eta = 0,$$

which is

$$\eta_{tt} - \eta_{xx} = 0.$$

This is the wave equation, as we desired.

Within $\mathcal{O}(\mu^2)$. In the second leading order, the non-dimensional equation (14) becomes

$$\partial_t \left(\mathcal{H}_0\{\varepsilon\mu\eta_t\} + \varepsilon\mathcal{H}_1\{\varepsilon\mu\eta_t\}\right) + \partial_x \left(\frac{1}{2}(\mathcal{H}_0\{\varepsilon\mu\eta_t\})^2 + \varepsilon\eta\right) = 0.$$

Note

$$\begin{aligned}
\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\} &= \varepsilon\mu\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}_k\}_k;\\
\mathcal{H}_1(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\} &= \varepsilon\mu^2(\eta\eta_t)_x - \varepsilon\mathcal{F}^{-1}\{\mu k\coth(\mu k)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\mu\coth(\mu k)\widehat{\eta_t}_l\}_l\}_k\}_k.
\end{aligned}$$

Then,

$$\frac{1}{2}\left(\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 = \frac{1}{2}\left(\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 = \frac{\varepsilon^2}{2}\left(\mathcal{F}^{-1}\{i\mu\coth(\mu j)\widehat{\eta_t}_j\}_j\right)^2,$$

and

$$\begin{aligned}
\partial_t \left(\left[\mathcal{H}_0(\varepsilon\eta, D) + \varepsilon\mathcal{H}_1(\varepsilon\eta, D)\right]\{\varepsilon\mu\eta_t\}\right) &= \varepsilon\mu\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_{tt}}_k\}_k + \varepsilon^2\mu^2(\eta\eta_t)_{tx}\\
&\quad - \varepsilon^2\mathcal{F}^{-1}\{\mu k\coth(\mu k)\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{i\mu\coth(\mu l)\widehat{\eta_t}_l\}_l\right]\}_k\}_k.
\end{aligned}$$

The single equation becomes

$$\begin{aligned}
\varepsilon\mu\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_{tt}}_k\}_k + \varepsilon^2\mu^2(\eta\eta_t)_{tx} - \varepsilon^2\mathcal{F}^{-1}\{\mu k\coth(\mu k)\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{i\mu\coth(\mu l)\widehat{\eta_t}_l\}_l\right]\}_k\}_k\\
+ \frac{\varepsilon^2}{2}\partial_x\left(\mathcal{F}^{-1}\{i\mu\coth(\mu j)\widehat{\eta_t}_j\}_j\right)^2 + \varepsilon\partial_x\eta = 0.
\end{aligned}$$

Application of Fourier transform yields

$$\varepsilon\mu i\coth(\mu k)\widehat{\eta_{tt}}_k+\varepsilon^2\mu^2 ik(\eta\eta_t)_t-\varepsilon^2\mu k\coth(\mu k)\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{i\mu\coth(\mu l)\widehat{\eta_t}_l\}_l\right]\}_k \\ +\frac{\varepsilon^2}{2}ik\mathcal{F}\{\left(\mathcal{F}^{-1}\{i\mu\coth(\mu j)\widehat{\eta_t}_j\}_j\right)^2\}_k+\varepsilon ik\widehat{\eta}_k=0.$$

Divide by $i\varepsilon$:

$$\mu\coth(\mu k)\widehat{\eta_{tt}}_k+\varepsilon\mu^2 k(\eta\eta_t)_t-\varepsilon\mu k\coth(\mu k)\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{\mu\coth(\mu l)\widehat{\eta_t}_l\}_l\right]\}_k \\ +\frac{\varepsilon}{2}k\mathcal{F}\{\left(\mathcal{F}^{-1}\{i\mu\coth(\mu j)\widehat{\eta_t}_j\}_j\right)^2\}_k+k\widehat{\eta}_k=0.$$

Let $\varepsilon=\mu^2$ and recall an expansion:

$$\coth(\mu k)\approx\frac{1}{\mu k}+\frac{\mu k}{3}+\mathcal{O}(\mu^3).$$

Substitution of the expansion yields:

$$\left(\frac{1}{k}+\frac{\mu^2 k}{3}\right)\widehat{\eta_{tt}}_k+\mu^4 k(\eta\eta_t)_t-\mu^2 k\left(\frac{1}{k}+\frac{\mu^2 k}{3}\right)\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{\left(\frac{1}{l}+\frac{\mu^2 l}{3}\right)\widehat{\eta_t}_l\}_l\right]\}_k \\ -\frac{\mu^2}{2}k\mathcal{F}\{\left(\mathcal{F}^{-1}\{\left(\frac{1}{j}+\frac{\mu^2 j}{3}\right)\widehat{\eta_t}_j\}_j\right)^2\}_k+k\widehat{\eta}_k=0.$$

Within $\mathcal{O}(\mu^4)$, the equation becomes

$$\left(\frac{1}{k}+\frac{\mu^2 k}{3}\right)\widehat{\eta_{tt}}_k-\mu^2\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{\frac{1}{l}\widehat{\eta_t}_l\}_l\right]\}_k-\frac{\mu^2}{2}k\mathcal{F}\{\left(\mathcal{F}^{-1}\{\frac{1}{j}\widehat{\eta_t}_j\}_j\right)^2\}_k+k\widehat{\eta}_k=0,$$

or re-arranging and multiplying by k, we have

$$\widehat{\eta_{tt}}_k+k^2\widehat{\eta}_k+\mu^2\left(\frac{k^2}{3}\widehat{\eta_{tt}}_k-k\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{\frac{1}{l}\widehat{\eta_t}_l\}_l\right]\}_k-\frac{1}{2}k^2\mathcal{F}\{\left(\mathcal{F}^{-1}\{\frac{1}{j}\widehat{\eta_t}_j\}_j\right)^2\}_k\right)=0.$$

Finally, inverting Fourier transform yields:

$$\eta_{tt}-\eta_{xx}+\mu^2\left(-\frac{\partial_x^2}{3}\eta_{tt}+i\partial_x\left(\partial_t\left[\eta\mathcal{F}^{-1}\{\frac{1}{l}\widehat{\eta_t}_l\}_l\right]\right)+\frac{1}{2}\partial_x^2\left(\mathcal{F}^{-1}\{\frac{1}{j}\widehat{\eta_t}_j\}_j\right)^2\right)=0,$$

or more conveniently,

$$\eta_{tt}-\eta_{xx}=\mu^2\left(\frac{\partial_x^2}{3}\eta_{tt}-i\partial_x\left(\partial_t\left[\eta\mathcal{F}^{-1}\{\frac{1}{l}\widehat{\eta_t}_l\}_l\right]\right)-\frac{1}{2}\partial_x^2\left(\mathcal{F}^{-1}\{\frac{1}{j}\widehat{\eta_t}_j\}_j\right)^2\right). \tag{15}$$

Observe the following:

$$\begin{aligned}\frac{1}{l}\widehat{\eta_t}_l&=\frac{1}{l}\frac{2}{\pi}\int_{-\infty}^{\infty}e^{-ilx}\eta_t\,\mathrm{d}x\\&=\frac{1}{l}\frac{2}{\pi}e^{-ilx}\int_{-\infty}^{x}\eta_t(x',t)\,\mathrm{d}x'\Big|_{-\infty}^{\infty}+i\frac{2}{\pi}\int_{-\infty}^{\infty}e^{-ilx}\int_{-\infty}^{x}\eta_t(x',t)\,\mathrm{d}x'\,\mathrm{d}x\\&=i\frac{2}{\pi}\int_{-\infty}^{\infty}e^{-ilx}\int_{-\infty}^{x}\eta_t(x',t)\,\mathrm{d}x'\,\mathrm{d}x\\&=i\mathcal{F}\{\int_{-\infty}^{x}\eta_t(x',t)\,\mathrm{d}x'\}_l.\end{aligned}$$

so that

$$\mathcal{F}^{-1}\{\frac{1}{l}\widehat{\eta_t}_l\}_l=\mathcal{F}^{-1}\{i\mathcal{F}\{\int_{-\infty}^{x}\eta_t(x',t)\,\mathrm{d}x'\}_l\}_l=i\int_{-\infty}^{x}\eta_t(x',t)\,\mathrm{d}x',$$

where we applied the Fourier inversion theorem. Moreover, we also have

$$\eta_{tt} = \eta_{xx} + \mathcal{O}(\mu^2).$$

Using these two facts, equation (15) becomes

$$\begin{aligned}
\eta_{tt} - \eta_{xx} &= \mu^2 \left(\frac{1}{3}\eta_{xxxx} + \partial_x \partial_t \left[\eta \left(\int_{-\infty}^{x} \eta_t \, \mathrm{d}x' \right) \right] + \frac{1}{2}\partial_x^2 \left(\int_{-\infty}^{x} \eta_t \, \mathrm{d}x' \right)^2 \right) \\
&= \mu^2 \left(\frac{1}{3}\eta_{xxxx} + \partial_x \left[\eta_t \left(\int_{-\infty}^{x} \eta_t \, \mathrm{d}x' \right) + \eta\eta_x \right] + \frac{1}{2}\partial_x^2 \left(\int_{-\infty}^{x} \eta_t \, \mathrm{d}x' \right)^2 \right) \\
&= \varepsilon \left[\frac{1}{3}\eta_{xxxx} + \partial_x^2 \left(\frac{\eta^2}{2} + \left(\int_{-\infty}^{x} \eta_t \, \mathrm{d}x' \right)^2 \right) \right].
\end{aligned}$$

For direct comparison, this equation is the same as the one in [1, p. 111], the unnumbered equation between (5.20) and (5.21). It remains to derive the wave and KdV equations.

1.0.5 Derivation of wave and KdV equations

In this section, we derive the approximate equations from

$$\eta_{tt} - \eta_{xx} = \varepsilon \left[\frac{1}{3}\eta_{xxxx} + \partial_x^2 \left(\frac{\eta^2}{2} + \left(\int_{-\infty}^{x} \eta_t \, \mathrm{d}x' \right)^2 \right) \right]. \tag{16}$$

As we approximate, we assume an expansion of η in ε :

$$\eta = \eta_0 + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2). \tag{17}$$

First order approximation

Substitution of (17) into equation (16) yields

$$\eta_{0tt} - \eta_{0xx} + \varepsilon(\eta_{1tt} - \eta_{1xx}) = \varepsilon \left[\frac{1}{3}\eta_{0xxxx} + \partial_x^2 \left(\frac{(\eta_0 + \varepsilon\eta_1)^2}{2} + \left(\int_{-\infty}^{x} (\eta_0 + \varepsilon\eta_1)_t \, \mathrm{d}x' \right)^2 \right) \right] + \mathcal{O}(\varepsilon^2). \tag{18}$$

In the leading order $\mathcal{O}(\varepsilon^0)$, equation (18) becomes

$$\eta_{0tt} - \eta_{0xx} = 0. \tag{19}$$

This is the wave equation with velocity 1, and whose general solution is

$$\eta_0 = F(x - t) + G(x + t),$$

where F, G are some functions.

Second order approximation

As in the velocity potential case, we employ multiple scales. First, we find an expression for η_0. We introduce

$$\tau_0 = t, \qquad \tau_1 = \varepsilon t, \qquad \tau_2 = \varepsilon^2 t, \ldots,$$

so that

$$\eta(x, t) = \eta(x, \tau_0, \tau_1, \ldots).$$

With this in mind, the expansion (17) becomes

$$\eta(x, \tau_0, \tau_1, \ldots) = \eta_0(x, \tau_0, \tau_1, \ldots) + \mathcal{O}(\varepsilon^1). \tag{20}$$

Substituting (20) into (16), within $\mathcal{O}(\varepsilon^0)$, we obtain

$$\eta_{0\tau_0\tau_0} - \eta_{0xx} = 0, \tag{21}$$

so that the general solution is

$$\eta_0(x, \tau_0, \tau_1, \ldots) = F(x - \tau_0, \tau_1, \ldots) + G(x + \tau_0, \tau_1, \ldots).$$

Now, although we have found an expression for η_0, the functions F, G used are still general functions. To determine F, G, we proceed to the next order, i.e. $\mathcal{O}(\varepsilon^1)$. We introduce

$$\xi = x - \tau_0 \qquad \zeta = x + \tau_0$$

so that

$$\begin{aligned}
\partial_x &= \partial_\xi \frac{\mathrm{d}\xi}{\mathrm{d}x} + \partial_\zeta \frac{\mathrm{d}\zeta}{\mathrm{d}x} = \partial_\xi + \partial_\zeta, \\
\partial_t &= \partial_\xi \frac{\mathrm{d}\xi}{\mathrm{d}t} + \partial_\zeta \frac{\mathrm{d}\zeta}{\mathrm{d}t} + \partial_{\tau_1} \frac{\mathrm{d}\tau_1}{\mathrm{d}t} = -\partial_\xi + \partial_\zeta + \varepsilon \partial_{\tau_1}.
\end{aligned}$$

We can rewrite (20) as follows

$$\begin{aligned}
\eta &= \eta_0 + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2) \\
&= F(x - t, \varepsilon t, \ldots) + G(x + t, \varepsilon t, \ldots) + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2) \\
&= F(\xi, \tau_1, \ldots) + G(\zeta, \tau_1, \ldots) + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2) \\
&= F + G + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

For ease of writing, we suppressed explicit dependence on variables, though the reader should bear in mind that function F (G) depend on ξ $(\zeta), \tau_1, \tau_2$, etc. In addition, observe that

$$\begin{aligned}
(\partial_t^2 - \partial_x^2) &= \left((-\partial_\xi + \partial_\zeta + \varepsilon\partial_{\tau_1})^2 - (\partial_\xi + \partial_\zeta)^2\right) \\
&= \left(\partial_\xi^2 - 2\partial_\xi\partial_\zeta + \partial_\zeta^2 + 2\varepsilon(\partial_\zeta\partial_{\tau_1} - \partial_\xi\partial_{\tau_1}) + \varepsilon^2\partial_{\tau_1}^2 - \partial_\xi^2 - 2\partial_\xi\partial_\zeta - \partial_\zeta^2\right) \\
&= \left(-4\partial_\xi\partial_\zeta + 2\varepsilon(\partial_\zeta\partial_{\tau_1} - \partial_\xi\partial_{\tau_1}) + \varepsilon^2\partial_{\tau_1}^2\right),
\end{aligned}$$

so that the LHS of (16) becomes

$$\begin{aligned}
(\partial_t^2 - \partial_x^2)\eta &= \left(-4\partial_\xi\partial_\zeta + 2\varepsilon(\partial_\zeta\partial_{\tau_1} - \partial_\xi\partial_{\tau_1}) + \varepsilon^2\partial_{\tau_1}^2\right)(F + G + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2)) \\
&= -4\partial_\xi\partial_\zeta(F + G + \varepsilon\eta_1) + 2\varepsilon(\partial_\zeta\partial_{\tau_1} - \partial_\xi\partial_{\tau_1})(F + G) + \mathcal{O}(\varepsilon^2) \\
&= \varepsilon\left(-4\eta_{1\xi\zeta} - 2F_{\tau_1\xi} + 2G_{\tau_1\zeta}\right) + \mathcal{O}(\varepsilon^2).
\end{aligned} \tag{22}$$

Now, we deal with the RHS of (16). By appropriate substitutions, the terms become:

$$\begin{aligned}
\frac{1}{3}\eta_{xxxx} &= \frac{1}{3}(\partial_x^2)^2\eta \\
&= \frac{1}{3}(\partial_\xi^2 + 2\partial_\xi\partial_\zeta + \partial_\zeta^2)^2\eta \\
&= \frac{1}{3}(\partial_\xi^4 + \partial_\zeta^4 + 4\partial_\xi^3\partial_\zeta + 2\partial_\xi\partial_\zeta^3 + 6\partial_\xi\partial_\zeta)(F + G + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2)) \\
&= \frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta} + \varepsilon(\partial_\xi + \partial_\zeta)^4\eta_1 + \mathcal{O}(\varepsilon^2)) \\
&= \frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta} + \mathcal{O}(\varepsilon)); \\
\frac{1}{2}\eta^2 &= \frac{1}{2}\left(F + G + \varepsilon\eta_1\right)^2 \\
&= \frac{1}{2}\left((F + G)^2 + 2\varepsilon(F + G)\eta_1 + \varepsilon^2\eta_1^2\right) \\
&= \frac{1}{2}(F^2 + 2FG + G^2) + \varepsilon(F + G)\eta_1 + \mathcal{O}(\varepsilon^2) \\
&= \frac{1}{2}(F^2 + 2FG + G^2) + \mathcal{O}(\varepsilon); \\
\left(\int_{-\infty}^{x} \eta_t \,\mathrm{d}x'\right)^2 &= \left(\int_{-\infty}^{x} \eta_{0t} \,\mathrm{d}x' + \varepsilon \int_{-\infty}^{x} \eta_{1t} \,\mathrm{d}x'\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_{-\infty}^{x} \eta_{0t}\,\mathrm{d}x' + \varepsilon\int_{-\infty}^{x} \eta_{1t}\,\mathrm{d}x'\right)^2 \\
&= \left(\int_{-\infty}^{x} \eta_{0t}\,\mathrm{d}x'\right)^2 + \mathcal{O}(\varepsilon) \\
&= \left(\int_{-\infty}^{x} (-\partial_\xi + \partial_\zeta + \varepsilon\partial_{\tau_1})(F+G)\,\mathrm{d}x'\right)^2 + \mathcal{O}(\varepsilon) \\
&= \left(\int_{-\infty}^{x} -F_\xi + G_\zeta\,\mathrm{d}x' + \varepsilon\int_{-\infty}^{x} \partial_{\tau_1}(F+G)\,\mathrm{d}x'\right)^2 + \mathcal{O}(\varepsilon) \\
&= \left(\int_{-\infty}^{x} -F_\xi + G_\zeta\,\mathrm{d}x'\right)^2 + \mathcal{O}(\varepsilon) \\
&= \left(\int_{-\infty}^{x} F_\xi\,\mathrm{d}x'\right)^2 - 2\left(\int_{-\infty}^{x} F_\xi\,\mathrm{d}x'\right)\left(\int_{-\infty}^{x} G_\zeta\,\mathrm{d}x'\right) + \left(\int_{-\infty}^{x} G_\zeta\,\mathrm{d}x'\right)^2 + \mathcal{O}(\varepsilon) \\
&= F^2 - 2FG + G^2 + \mathcal{O}(\varepsilon),
\end{aligned}
$$

where for the last line we translate $\xi' = x' - t, \zeta' = x' + t$ to obtain

$$
\begin{aligned}
\int_{-\infty}^{x} F_\xi\,\mathrm{d}x' = \lim_{a\to-\infty}\int_a^x F_{\xi'}(x'-t,\tau_1)\,\mathrm{d}x' &= \lim_{a\to-\infty}\int_{a-t}^{x-t} F_{\xi'}(\xi',\tau_1)\,\mathrm{d}\xi' \\
&= \lim_{a\to-\infty}\int_{a-t}^{\xi} F_{\xi'}(\xi',\tau_1)\,\mathrm{d}\xi' \\
&= \int_{-\infty}^{\xi} F_{\xi'}(\xi',\tau_1)\,\mathrm{d}\xi' = F(\xi,\tau_1), \\
\int_{-\infty}^{x} G'_\zeta\,\mathrm{d}x' = \lim_{a\to-\infty}\int_a^x F_{\zeta'}(x'-t,\tau_1)\,\mathrm{d}x' &= \lim_{a\to-\infty}\int_{a+t}^{x+t} G_{\zeta'}(\zeta',\tau_1)\,\mathrm{d}\zeta' \\
&= \lim_{a\to-\infty}\int_{a-t}^{\zeta} G_{\zeta'}(\zeta',\tau_1)\,\mathrm{d}\zeta' \\
&= \int_{-\infty}^{\zeta} G_{\zeta'}(\zeta',\tau_1)\,\mathrm{d}\zeta' = G(\zeta,\tau_1).
\end{aligned}
$$

Note we assumed F, G vanish as $\xi, \zeta \to -\infty$. Substitution of terms into the RHS of (16) leads to:

$$
\begin{aligned}
&\varepsilon\left[\frac{1}{3}\eta_{xxxx} + \partial_x^2\left(\frac{\eta^2}{2} + \left(\int_{-\infty}^{x} \eta_t\,\mathrm{d}x'\right)^2\right)\right] \\
&= \varepsilon\left[\frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + (\partial_\xi^2 + 2\partial_\xi\partial_\zeta + \partial_\zeta^2)\left(\frac{1}{2}(F^2 + 2FG + G^2) + F^2 - 2FG + G^2\right)\right] + \mathcal{O}(\varepsilon^2) \\
&= \varepsilon\left[\frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + (\partial_\xi^2 + 2\partial_\xi\partial_\zeta + \partial_\zeta^2)\left(\frac{3}{2}F^2 + \frac{3}{2}G^2 - FG\right)\right] + \mathcal{O}(\varepsilon^2). \qquad (23)
\end{aligned}
$$

Combining (22) and (23), in $\mathcal{O}(\varepsilon^1)$ we have

$$
-4\eta_{1\xi\zeta} = 2F_{\tau_1\xi} - 2G_{\tau_1\zeta} + \frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + (\partial_\xi^2 + 2\partial_\xi\partial_\zeta + \partial_\zeta^2)\left(\frac{3}{2}F^2 + \frac{3}{2}G^2 - FG\right). \qquad (24)
$$

In the last term of (24), differentiation yields:

$$
(\partial_\xi^2 + 2\partial_\xi\partial_\zeta + \partial_\zeta^2)\left(\frac{3}{2}F^2 + \frac{3}{2}G^2 - FG\right) = \partial_\xi(3FF_\xi - GF_\xi) + \partial_\zeta(3GG_\zeta - FG_\zeta) - 2F_\xi G_\zeta,
$$

so that equation (24) becomes

$$
-4\eta_{1\xi\zeta} = 2F_{\tau_1\xi} - 2G_{\tau_1\zeta} + \frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + \partial_\xi(3FF_\xi - GF_\xi) + \partial_\zeta(3GG_\zeta - FG_\zeta) - 2F_\xi G_\zeta
$$

$$= \partial_\xi(2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + 3FF_\xi) + \partial_\zeta(-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + 3GG_\zeta) - (GF_\xi + FG_\zeta). \tag{25}$$

Integration of (25) with respect to ζ yields

$$-4\eta_{1\xi} = \partial_\xi(2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + 3FF_\xi)\zeta + (-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + 3GG_\zeta) - \left(F_\xi \int G \,\mathrm{d}\zeta + GF\right),$$

and further integration with respect to ξ leads to

$$-4\eta_1 = (2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + 3FF_\xi)\zeta + (-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + 3GG_\zeta)\xi - \left(F \int G \,\mathrm{d}\zeta + G \int F \,\mathrm{d}\xi\right).$$

Since η_1 must be bounded, we must have

$$2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + 3FF_\xi = 0 \tag{26}$$

$$2G_{\tau_1} - \frac{1}{3}G_{\zeta\zeta\zeta} - 3GG_\zeta = 0. \tag{27}$$

In other words, we have obtained two KdV equations, (26) and (27), whose solutions describe behaviour of the surface elevation in the leading order. The derivation is complete.

2 Water-wave problem on the whole line: non-local formulation

The (tentative) half-line problem is given by the following system:

$$\phi_{xx} + \phi_{zz} = 0, \qquad -h < z < \eta(x,t), \tag{28a}$$
$$\phi_z = 0, \qquad z = -h, \tag{28b}$$
$$\phi_x = 0, \qquad x = 0, \tag{28c}$$
$$\eta_t + \phi_x\eta_x = \phi_z, \qquad z = \eta(x,t), \tag{28d}$$
$$\phi_t + g\eta + \frac{1}{2}(\phi_x^2 + \phi_z^2) = 0, \qquad z = \eta(x,t), \tag{28e}$$
$$\phi_z(0,\eta,t) = \eta_t(0,t), \qquad (x,z) = (0,\eta). \tag{28f}$$
$$|\phi| \to 0, |\eta| \to 0, \qquad x \to \infty, \tag{28g}$$

Introducing the nondimensional variables as before yields the non-dimensional problem:

$$\varepsilon\phi_{xx} + \phi_{zz} = 0 \qquad -1 < z < \varepsilon\eta \tag{29a}$$
$$\phi_z = 0 \qquad z = -1 \tag{29b}$$
$$\phi_x = 0 \qquad x = 0 \tag{29c}$$
$$\varepsilon\eta_t + \varepsilon^2\phi_x\eta_x = \phi_z \qquad z = \varepsilon\eta \tag{29d}$$
$$\phi_t + \eta + \frac{1}{2}(\varepsilon\phi_x^2 + \phi_z^2) = 0 \qquad z = \varepsilon\eta \tag{29e}$$
$$\phi_z(0,\varepsilon\eta,t) = \varepsilon\eta_t(0,t) \qquad (x,z) = (0,\varepsilon\eta), \tag{29f}$$

and the conditions on decay of ϕ and η remain the same, except there is only the right side. We seek a non-local formulation of the problem.

2.0.1 Dimensional, non-local formulation for the half-line

First, we begin with a dimensional system. As previously, let φ be harmonic. Then, after some manipulation, we have the following contour integral (in dimensional variables):

$$\int_D \varphi_z(\phi_z \,\mathrm{d}x - \phi_x \,\mathrm{d}z) + \phi(\varphi_{xx} \,\mathrm{d}x + \varphi_{xz} \,\mathrm{d}z) = 0 \tag{30}$$

Break the contour D in the following segments:

$$\begin{aligned}\int_D &= \int_{-\infty}^{\infty}\Big|^{z=-h} + \int_{-h}^{\eta(x)}\Big|^{x=\infty} + \int_{\infty}^{-\infty}\Big|^{z=\eta(x)} + \int_{\eta}^{-h}\Big|^{x=0} \\ &= \int_{-\infty}^{\infty}\Big|^{z=-h} + \int_{-h}^{\eta(x)}\Big|^{x=\infty} - \int_{-\infty}^{\infty}\Big|^{z=\eta(x)} - \int_{-h}^{\eta}\Big|^{x=0}\end{aligned}$$

We consider each of the segments:

- At $x = \infty$, the integral vanishes due to behaviour of ϕ and its gradient.
- At $x = 0$, $\mathrm{d}x = 0$, so we have

$$\begin{aligned}\int_{-h}^{\eta} -\varphi_z\phi_x + \phi\varphi_{xz}\,\mathrm{d}z &= \int_{-h}^{\eta} \phi\varphi_{xz} - \varphi_z\phi_x\,\mathrm{d}z \\ &= \int_{-h}^{\eta} \phi\varphi_{xz}\,\mathrm{d}z \qquad (\text{since } \phi_x = 0 \text{ at } x = 0) \\ &= \phi\varphi_x\Big|_{-h}^{\eta} - \int_{-h}^{\eta} \phi_z\varphi_x\,\mathrm{d}z \\ &= \phi(0,\eta)\varphi_x(0,\eta) - \phi(0,-h)\varphi_x(0,-h) - \int_{-h}^{\eta} \phi_z(0,z)\varphi_x(0,z)\,\mathrm{d}z.\end{aligned}$$

- At $z = -h$, $\mathrm{d}z = 0$, so we have

$$\begin{aligned}\int_0^{\infty} \varphi_z\phi_z + \phi\varphi_{xx}\,\mathrm{d}x &= \int_0^{\infty} \phi\varphi_{xx}\,\mathrm{d}x \qquad (\text{since } \phi_z(x,-h) = 0) \\ &= \phi(x,-h)\varphi_x(x,-h)\;\Big|_0^{\infty} - \int_0^{\infty} \phi_x(x,-h)\varphi_x(x,-h)\,\mathrm{d}x \\ &= -\phi(0,-h)\varphi_x(0,-h) - \int_0^{\infty} \phi_x(x,-h)\varphi_x(x,-h)\,\mathrm{d}x.\end{aligned}$$

- At $z = \eta(x)$, $\mathrm{d}z = \eta_x\,\mathrm{d}x$, so we have

$$\begin{aligned}\int_0^{\infty} \varphi_z(\phi_x - \phi_x\eta_x) + \phi(\varphi_{xx} + \varphi_{xz}\eta_x)\,\mathrm{d}x &= \int_0^{\infty} \varphi_z\frac{\partial\phi}{\partial N} + \phi\frac{\mathrm{d}\varphi_x(x,\eta(x))}{\mathrm{d}x}\,\mathrm{d}x \\ &= \int_0^{\infty} \varphi_z\frac{\partial\phi}{\partial N}\,\mathrm{d}x + \phi(x,\eta)\varphi_x(x,\eta)\Big|_0^{\infty} - \int_0^{\infty} \phi_x(x,\eta)\varphi_x(x,\eta)\,\mathrm{d}x \\ &= \int_0^{\infty} \varphi_z(x,\eta)\frac{\partial\phi}{\partial N}(x,\eta)\,\mathrm{d}x - \phi(0,\eta)\varphi_x(0,\eta) - \int_0^{\infty} \phi_x(x,\eta)\varphi_x(x,\eta)\,\mathrm{d}x.\end{aligned}$$

Combine the segments:

$$\begin{aligned}0 &= \int_D \varphi_z(\phi_x\,\mathrm{d}x - \phi_x\,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) \\ &= \left\{\int_{-\infty}^{\infty}\Big|^{z=-h} + \int_{-h}^{\eta(x)}\Big|^{x=\infty} - \int_{-\infty}^{\infty}\Big|^{z=\eta(x)} - \int_{-h}^{\eta}\Big|^{x=0}\right\} \varphi_z(\phi_x\,\mathrm{d}x - \phi_x\,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) \\ &= -\phi(0,-h)\varphi_x(0,-h) - \int_0^{\infty} \phi_x(x,-h)\varphi_x(x,-h)\,\mathrm{d}x \\ &- \int_0^{\infty} \varphi_z(x,\eta)\frac{\partial\phi}{\partial N}(x,\eta)\,\mathrm{d}x + \phi(0,\eta)\varphi_x(0,\eta) + \int_0^{\infty} \phi_x(x,\eta)\varphi_x(x,\eta)\,\mathrm{d}x \\ &- \phi(0,\eta)\varphi_x(0,\eta) + \phi(0,-h)\varphi_x(0,-h) + \int_{-h}^{\eta} \phi_z(0,z)\varphi_x(0,z)\,\mathrm{d}z \\ &= -\int_0^{\infty} \phi_x(x,-h)\varphi_x(x,-h)\,\mathrm{d}x - \int_0^{\infty} \varphi_z(x,\eta)\frac{\partial\phi}{\partial N}(x,\eta)\,\mathrm{d}x + \int_0^{\infty} \phi_x(x,\eta)\varphi_x(x,\eta)\,\mathrm{d}x + \int_{-h}^{\eta} \phi_z(0,z)\varphi_x(0,z)\,\mathrm{d}z\end{aligned}$$

$$= \int_0^\infty \phi_x(x,\eta)\varphi_x(x,\eta) - \phi_x(x,-h)\varphi_x(x,-h) - \varphi_z(x,\eta)\frac{\partial\phi}{\partial N}(x,\eta)\,\mathrm{d}x + \int_{-h}^{\eta} \phi_z(0,z)\varphi_x(0,z)\,\mathrm{d}z$$

Force $\varphi_x(0,z)=0$, so that we are left with:

$$\int_0^\infty \varphi_z(x,\eta)\frac{\partial\phi}{\partial N}(x,\eta) + \phi_x(x,-h)\varphi_x(x,-h) - \phi_x(x,\eta)\varphi_x(x,\eta)\,\mathrm{d}x = 0. \tag{31}$$

Let $\varphi = \cos(kx)\sinh(k(z+h))$, and note that $\phi_x(x,\eta) = \dfrac{\partial\phi}{\partial T}$ is the tangential derivative at $z=\eta$. The equation (31) becomes:

$$\int_0^\infty k\cos(kx)\cosh(k(\eta+h))\frac{\partial\phi}{\partial N}(x,\eta) + k\sin(kx)\sinh(k(\eta+h))\frac{\partial\phi}{\partial T}(x,\eta)\,\mathrm{d}x = 0, \tag{32}$$

since $\varphi_x(x,-h) = -k\sin(kx)\sinh(k(h-h)) = 0$. Let $\dfrac{\partial\phi}{\partial N}(x,\eta) = f(x)$, $\dfrac{\partial\phi}{\partial T}(x,\eta) = \mathcal{H}(\eta,D)\{f(x)\}$ and assume $k\neq 0$, so that we obtain

$$\int_0^\infty \cos(kx)\cosh(k(\eta+h))f(x) + \sin(kx)\sinh(k(\eta+h))\mathcal{H}(\eta,D)\{f(x)\}\,\mathrm{d}x = 0. \tag{33}$$

Observe that (37) is kinda like the real part of the original equation but on a half-line (maybe the other half-line is the imaginary part).

2.0.2 Non-local, non-dimensional formulation on a half-line

As previously, let ϕ be harmonic. Then, after some manipulation, we have the following contour integral (in non-dimensional variables):

$$\int_D \varphi_z\left(\frac{1}{\mu^2}\phi_z\,\mathrm{d}x - \phi_x\,\mathrm{d}z\right) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) = 0. \tag{34}$$

Break the contour D in the following segments:

$$\{x\to\infty, -1<z<\varepsilon\eta\}, \qquad \{x=0, -1<z<\varepsilon\eta\}, \qquad \{z=-1, 0<x<\infty\}, \qquad \{z=\varepsilon\eta, 0<x<\infty\}.$$

We consider each of the segments.

- At $x\to\infty$, the integral vanishes due to behaviour of ϕ and its gradient.
- At $x=0$, $\mathrm{d}x=0$, so we have

$$\int_{-1}^{\varepsilon\eta} -\varphi_z\phi_x + \phi\varphi_{xz}\,\mathrm{d}z = \phi(0,\varepsilon\eta)\varphi_x(0,\varepsilon\eta) - \phi(0,-1)\varphi_x(0,-1) - \int_{-1}^{\varepsilon\eta}\phi_z(0,z)\varphi_x(0,z)\,\mathrm{d}z.$$

- At $z=-1$, $\mathrm{d}z=0$, so we have

$$\int_0^\infty \varphi_z\phi_z + \phi\varphi_{xx}\,\mathrm{d}x = -\phi(0,-1)\varphi_x(0,-1) - \int_0^\infty \phi_x(x,-1)\varphi_x(x,-1)\,\mathrm{d}x.$$

- At $z=\varepsilon\eta(x)$, $\mathrm{d}z = \varepsilon\eta_x\,\mathrm{d}x$, so we have

$$\begin{aligned}
&\int_0^\infty \frac{1}{\mu^2}\varphi_z(\phi_x - \varepsilon\phi_x\eta_x) + \phi(\varphi_{xx} + \varepsilon\varphi_{xz}\eta_x)\,\mathrm{d}x \\
&\qquad = \int_0^\infty \frac{1}{\mu}\varphi_z\tilde{\nabla}\phi\cdot\tilde{N} + \frac{\mathrm{d}\phi_x(x,\varepsilon\eta(x))}{\mathrm{d}x}\,\mathrm{d}x \\
&\qquad = \int_0^\infty \frac{1}{\mu}\varphi_z\tilde{\nabla}\phi\cdot\tilde{N}\,\mathrm{d}x + \phi(x,\varepsilon\eta)\varphi_x(x,\varepsilon\eta)\mid_0^\infty - \int_0^\infty \phi_x(x,\varepsilon\eta)\varphi_x(x,\varepsilon\eta)\,\mathrm{d}x \\
&\qquad = \int_0^\infty \frac{1}{\mu}\varphi_z\tilde{\nabla}\phi\cdot\tilde{N}\,\mathrm{d}x - \phi(0,\varepsilon\eta)\varphi_x(0,\varepsilon\eta) - \int_0^\infty \phi_x(x,\varepsilon\eta)\varphi_x(x,\varepsilon\eta)\,\mathrm{d}x
\end{aligned}$$

Combine the segments:

$$
\begin{aligned}
0 &= \int_D \varphi_z(\frac{1}{\mu^2}\phi_x \,\mathrm{d}x - \phi_x \,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) \\
&= \phi(0,\varepsilon\eta)\varphi_x(0,\varepsilon\eta) - \phi(0,-1)\varphi_x(0,-1) - \int_{-1}^{\varepsilon\eta} \phi_z(0,z)\varphi_x(0,z)\,\mathrm{d}z \\
&- \phi(0,-1)\varphi_x(0,-1) - \int_0^\infty \phi_x(x,-1)\varphi_x(x,-1)\,\mathrm{d}x \\
&+ \int_0^\infty \frac{1}{\mu}\varphi_z \tilde{\nabla}\phi\cdot\tilde{N}\,\mathrm{d}x - \phi(0,\varepsilon\eta)\varphi_x(0,\varepsilon\eta) - \int_0^\infty \phi_x(x,\varepsilon\eta)\varphi_x(x,\varepsilon\eta)\,\mathrm{d}x \\
&= -2\phi(0,-h)\varphi_x(0,-1) - \int_{-1}^{\varepsilon\eta} \phi_z(0,z)\varphi_x(0,z)\,\mathrm{d}z \\
&+ \int_0^\infty \frac{1}{\mu}\varphi_z \tilde{\nabla}\phi\cdot\tilde{N}\,\mathrm{d}x - \int_0^\infty \phi_x(x,\varepsilon\eta)\varphi_x(x,\varepsilon\eta)\,\mathrm{d}x - \int_0^\infty \phi_x(x,-1)\varphi_x(x,-1)\,\mathrm{d}x.
\end{aligned}
$$

Force $\phi_x(0,z)=0$, so that we are left with:

$$
\int_0^\infty \frac{1}{\mu}\varphi_z \tilde{\nabla}\phi\cdot\tilde{N}\,\mathrm{d}x - \phi_x(x,\varepsilon\eta)\varphi_x(x,\varepsilon\eta)\,\mathrm{d}x - \phi_x(x,-1)\varphi_x(x,-1)\,\mathrm{d}x \tag{35}
$$

Let $\varphi = \cos(kx)\sinh(\mu k(z+1))$, and note that $\phi_x(x,\varepsilon\eta) = \dfrac{\partial\phi}{\partial T}$ is the tangential derivative at $z=\varepsilon\eta$. The above becomes:

$$
\int_0^\infty k\cos(kx)\cosh(\mu k(\eta+1))\tilde{\nabla}\phi\cdot\tilde{N} + k\sin(kx)\sinh(\mu k(\eta+1))\tilde{\nabla}\phi\cdot\tilde{T}\,\mathrm{d}x = 0, \tag{36}
$$

since $\varphi_x(x,-1) = -k\sin(kx)\sinh(k(1-1)) = 0$. Let

$$
\tilde{\nabla}\phi\cdot\tilde{T} = \frac{\partial\phi}{\partial N}(x,\varepsilon\eta) = f(x), \frac{\partial\phi}{\partial T}(x,\varepsilon\eta) = \mathcal{H}(\varepsilon\eta,D)\{f(x)\}
$$

and assume $k\neq 0$, so that we obtain

$$
\int_0^\infty \cos(kx)\cosh(\mu k(\eta+1))f(x) + \sin(kx)\sinh(\mu k(\eta+1))\mathcal{H}(\varepsilon\eta,D)\{f(x)\}\,\mathrm{d}x = 0. \tag{37}
$$

2.0.3 Perturbation expansion for $\mathcal{H}$

Suppose

$$
\mathcal{H}(\varepsilon\eta,D)\{f(x)\} = \sum_{j=0}^\infty \varepsilon^j \mathcal{H}_0(\varepsilon\eta,D)\{f(x)\}.
$$

For notational convenience, we assume throughout that the H operator is evaluated at (ε, D), so that we drop this term in writing. Expand in ε :

$$
\begin{aligned}
\cosh(\mu k(\varepsilon\eta+1)) &= \cosh(\mu k) + \varepsilon\mu k\eta\sinh(\mu k) + \frac{(\varepsilon\mu k\eta)^2}{2}\cosh(\mu k) + \ldots, \\
\sinh(\mu k(\varepsilon\eta+1) &= \sinh(\mu k) + \varepsilon\mu k\eta\cosh(\mu k) + \frac{(\varepsilon\mu k\eta)^2}{2}\sinh(\mu k) + \ldots,
\end{aligned}
$$

so that (37) becomes

$$
\begin{aligned}
\int_0^\infty &\cos(kx)(\cosh(\mu k) + \varepsilon\mu k\eta\sinh(\mu k) + \ldots)f(x) \\
&+ \sin(kx)(\sinh(\mu k) + \varepsilon\mu k\eta\cosh(\mu k) + \ldots)(\mathcal{H}_0 + \varepsilon\mathcal{H}_1 + \ldots)\{f(x)\}\,\mathrm{d}x = 0.
\end{aligned} \tag{38}
$$

Within $\mathcal{O}(\varepsilon^0)$, we obtain

$$
\int_0^\infty \cos(kx)\cosh(\mu k)f(x) + \sin(kx)\sinh(\mu k)\mathcal{H}_0\{f(x)\}\,\mathrm{d}x = 0.
$$

Let $\mathcal{F}_c^k$ indicate the Fourier cosine transform, and similarly for the Fourier sine transform. Then, we have

$$\begin{aligned}
\mathcal{F}_s^k\{\mathcal{H}_0\{f(x)\}\} &= -\mathcal{F}_c^k\{\coth(\mu k)f(x)\} \\
&\implies \mathcal{H}_0\{f(x)\} = -(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\coth(\mu k)f(x)\}\} \\
&\implies \mathcal{H}_0\{f(x)\} = -\int_0^\infty \sin(kx)\coth(\mu k)\left(\frac{2}{\pi}\int_0^\infty \cos(kx)f(x)\,\mathrm{d}x\right)\mathrm{d}k \\
&\implies \mathcal{H}_0\{f(x)\} = -\int_0^\infty \sin(kx)\coth(\mu k)\widehat{f_c^k}\,\mathrm{d}k = -\{\mathcal{F}_s^k\}^{-1}\{\coth(\mu k)\widehat{f_c^k}\}.
\end{aligned}$$

Within $\mathcal{O}(\varepsilon^1)$, the equation (38) is

$$\int_0^\infty \cos(kx)\mu k\eta f(x) + \sin(kx)(\mathcal{H}_1\{f(x)\} + \mu k\eta\coth(\mu k)\mathcal{H}_0\{f(x)\})\,\mathrm{d}x = 0.$$

Then,

$$\begin{aligned}
\int_0^\infty \sin(kx)\mathcal{H}_1\{f(x)\}\,\mathrm{d}x &= -\mu k\left[\int_0^\infty \cos(kx)\eta f(x)\,\mathrm{d}x + \coth(\mu k)\int_0^\infty \sin(kx)\eta\mathcal{H}_0\{f(x)\}\,\mathrm{d}x\right] \\
&= -\mu k\left[\widehat{(\eta f(x))}_c^k + \coth(\mu k)\widehat{(\eta\mathcal{H}_0\{f(x)\})}_c^k\right],
\end{aligned}$$

where Fourier sine transform is inverted to obtain

$$\mathcal{H}_1\{f(x)\} = -\{\mathcal{F}_s^k\}^{-1}\{\mu k\widehat{(\eta f(x))}_c^k + \mu k\coth(\mu k)\widehat{(\eta\mathcal{H}_0\{f(x)\})}_c^k\}.$$

In sum, we obtain

$$\begin{aligned}
\mathcal{H}_0(\varepsilon\eta, D)\{f(x)\} &= -\{\mathcal{F}_s^k\}^{-1}\{\coth(\mu k)\widehat{f_c^k}\}, \\
\mathcal{H}_1(\varepsilon\eta, D)\{f(x)\} &= -\{\mathcal{F}_s^k\}^{-1}\{\mu k\widehat{(\eta f(x))}_c^k + \mu k\coth(\mu k)\widehat{(\eta\mathcal{H}_0\{f(x)\})}_c^k\}.
\end{aligned}$$

2.0.4 Approximation Procedure

First, we'd like to approximate. Recall the non-dimensional single equation:

$$\partial_t\left(\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right) + \partial_x\left(\frac{1}{2}\left(\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 + \varepsilon\eta - \frac{1}{2}\varepsilon\mu^2\frac{(\eta_t + \eta_x\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\})^2}{1+\varepsilon^2\mu^2\eta_x^2}\right) = 0. \tag{39}$$

Within $\mathcal{O}(\mu^0)$:

we have $\mathcal{H} \approx \mathcal{H}_0$, and the single equation becomes:

$$\partial_t\left(\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right) + \varepsilon\partial_x\eta = 0. \tag{40}$$

Note

$$\begin{aligned}
\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\} &= -\int_0^\infty \sin(kx)\coth(\mu k)\widehat{(\varepsilon\mu\eta_t)}_c^k\,\mathrm{d}k \\
&= -\varepsilon\int_0^\infty \sin(kx)\mu\coth(\mu k)\widehat{(\eta_t)}_c^k\,\mathrm{d}k \\
&= -\varepsilon\int_0^\infty \sin(kx)\left(\frac{1}{k} + \frac{\mu^2 k}{3} + \dots\right)\widehat{(\eta_t)}_c^k\,\mathrm{d}k \\
&\approx -\varepsilon\int_0^\infty \sin(kx)\frac{1}{k}\widehat{(\eta_t)}_c^k\,\mathrm{d}k.
\end{aligned}$$

Substituting into (40) yields

$$\begin{aligned}
\partial_t(-\varepsilon\int_0^\infty \sin(kx)\frac{1}{k}\widehat{(\eta_t)}_c^k\,\mathrm{d}k) + \varepsilon\eta_x = 0 &\implies -\int_0^\infty \sin(kx)\frac{1}{k}\widehat{(\eta_{tt})}_c^k\,\mathrm{d}k + \eta_x = 0 \\
&\implies -\frac{1}{k}\widehat{(\eta_{tt})}_c^k - k\widehat{\eta_s^k} = 0
\end{aligned}$$

Within $\mathcal{O}(\mu^2)$:

We have $\mathcal{H} \approx \mathcal{H}_0 + \varepsilon\mathcal{H}_1$, and the single equation becomes:

$$\partial_t \left(\mathcal{H}_0\{\varepsilon\mu\eta_t\} + \varepsilon\mathcal{H}_1\{\varepsilon\mu\eta_t\}\right) + \partial_x \left(\frac{1}{2}(\mathcal{H}_0\{\varepsilon\mu\eta_t\})^2 + \varepsilon\eta\right) = 0. \tag{41}$$

Note

$$\begin{aligned}
\mathcal{H}_0\{\varepsilon\mu\eta_t\} &= -\{\mathcal{F}_s^k\}^{-1}\{\coth(\mu k)\widehat{\varepsilon\mu(\eta_t)}_c^k\} \\
&= -\varepsilon\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_t)}_c^k\}, \\
\mathcal{H}_1\{\varepsilon\mu\eta_t\} &= -\{\mathcal{F}_s^k\}^{-1}\{k\mu\widehat{(\eta\varepsilon\mu\eta_t)}_c^k + k\mu\coth(\mu k)\widehat{(\eta\mathcal{H}_0\{\varepsilon\mu\eta_t\})}_c^k\} \\
&= -\varepsilon\mu\{\mathcal{F}_s^k\}^{-1}\{\mu k\widehat{(\eta\eta_t)}_c^k + \mu k\coth(\mu k)\widehat{(\eta\mathcal{H}_0\{\eta_t\})}_c^k\}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\partial_t \left(\mathcal{H}_0\{\varepsilon\mu\eta_t\} + \varepsilon\mathcal{H}_1\{\varepsilon\mu\eta_t\}\right) \\
&= -\partial_t \left(\varepsilon\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_t)}_c^k\} + \varepsilon^2\mu\{\mathcal{F}_s^k\}^{-1}\{\mu k\widehat{(\eta\eta_t)}_c^k + \mu k\coth(\mu k)\widehat{(\eta\mathcal{H}_0\{\eta_t\})}_c^k\}\right) \\
&= -\varepsilon \left(\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_{tt})}_c^k\} + \varepsilon\mu\{\mathcal{F}_s^k\}^{-1}\{\mu k\partial_t\widehat{(\eta\eta_t)}_c^k + \mu k\coth(\mu k)\partial_t\widehat{(\eta\mathcal{H}_0\{\eta_t\})}_c^k\}\right)
\end{aligned}$$

Moreover,

$$\begin{aligned}
\frac{1}{2}(\mathcal{H}_0\{\varepsilon\mu\eta_t\})^2 &= \frac{1}{2}(-\varepsilon\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_t)}_c^k\})^2 \\
&= \frac{\varepsilon^2}{2}(\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_t)}_c^k\})^2.
\end{aligned}$$

When combining the terms, the equation (41) becomes

$$\begin{aligned}
&\partial_t(\mathcal{H}_0\{\varepsilon\mu\eta_t\} + \varepsilon\mathcal{H}_1\{\varepsilon\mu\eta_t\}) + \partial_x \left(\frac{1}{2}(\mathcal{H}_0\{\varepsilon\mu\eta_t\})^2 + \varepsilon\eta\right) = 0 \\
\implies &-\varepsilon(\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_{tt})}_c^k\} + \varepsilon\mu\{\mathcal{F}_s^k\}^{-1}\{\mu k\partial_t\widehat{(\eta\eta_t)}_c^k + \mu k\coth(\mu k)\partial_t\widehat{(\eta\mathcal{H}_0\{\eta_t\})}_c^k\}) \\
&+ \frac{\varepsilon^2}{2}\partial_x(\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_t)}_c^k\})^2 + \varepsilon\eta_x = 0. \\
\implies &-(\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_{tt})}_c^k\} + \varepsilon\mu\{\mathcal{F}_s^k\}^{-1}\{\mu k\partial_t\widehat{(\eta\eta_t)}_c^k + \mu k\coth(\mu k)\partial_t\widehat{(\eta\mathcal{H}_0\{\eta_t\})}_c^k\}) \\
&+ \frac{\varepsilon}{2}\partial_x(\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_t)}_c^k\})^2 + \eta_x = 0.
\end{aligned}$$

Apply Fourier sine transform:

$$\begin{aligned}
-(\mu\coth(\mu k)\widehat{(\eta_{tt})}_c^k + \varepsilon\mu^2 k\partial_t\widehat{(\eta\eta_t)}_c^k + \varepsilon\mu^2 k\coth(\mu k)\partial_t\widehat{(\eta\mathcal{H}_0\{\eta_t\})}_c^k) \\
+ \frac{\varepsilon}{2}\mathcal{F}_s^k\{\partial_x(\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_t)}_c^k\})^2\} + \widehat{(\eta_x)}_s^k = 0,
\end{aligned}$$

(but then $\mathcal{F}_s^k\{\eta_x\} = -k\widehat{\eta}_c^k$.) Suppose we do not divide by $\coth(\mu k)$. Then, letting $\varepsilon = \mu^2$ and expanding $\mathcal{H}_0$ in

$$\begin{aligned}
-(\mu\coth(\mu k)\widehat{(\eta_{tt})}_c^k + \varepsilon\mu^2 k\partial_t\widehat{(\eta\eta_t)}_c^k + \varepsilon\mu^2 k\coth(\mu k)\partial_t\widehat{(\eta\mathcal{H}_0\{\eta_t\})}_c^k) \\
+ \frac{\varepsilon}{2}\mathcal{F}_s^k\{\partial_x(\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_t)}_c^k\})^2\} + \widehat{(\eta_x)}_s^k = 0,
\end{aligned}$$

we obtain

$$-\mu\coth(\mu k)\widehat{(\eta_{tt})}_c^k - \mu^4 k\partial_t\widehat{(\eta\eta_t)}_c^k + \mu^3 k\coth(\mu k)\partial_t\mathcal{F}_c^k\{\left(\eta\{\mathcal{F}_s^l\}^{-1}\{\mu\coth(\mu l)\widehat{(\eta_t)}_c^l\}\right)\}$$

$$+\frac{\mu^2}{2}\mathcal{F}_s^k\{\partial_x(\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_t)_c^k}\})^2\}+\widehat{(\eta_x)_s^k}=0.$$

Expanding $\coth(\mu k)$ yields:

$$-\mu\left(\frac{1}{\mu k}+\frac{\mu k}{3}\right)\widehat{(\eta_{tt})_c^k}-\mu^4 k\partial_t\widehat{(\eta\eta_t)_c^k}+\mu^3 k\left(\frac{1}{\mu k}+\frac{\mu k}{3}\right)\partial_t\mathcal{F}_c^k\{\left(\eta\{\mathcal{F}_s^l\}^{-1}\{\mu\left(\frac{1}{\mu l}+\frac{\mu l}{3}\right)\widehat{(\eta_t)_c^l}\}\right)\}$$
$$+\frac{\mu^2}{2}\mathcal{F}_s^k\{\partial_x(\{\mathcal{F}_s^k\}^{-1}\{\mu\left(\frac{1}{\mu k}+\frac{\mu k}{3}\right)\widehat{(\eta_t)_c^k}\})^2\}+\widehat{(\eta_x)_s^k}=0,$$

and so

$$-\left(\frac{1}{k}+\frac{\mu^2 k}{3}\right)\widehat{(\eta_{tt})_c^k}-\mu^4 k\partial_t\widehat{(\eta\eta_t)}_c^k+\left(\mu^2+\frac{\mu^4k^2}{3}\right)\partial_t\mathcal{F}_c^k\{\left(\eta\{\mathcal{F}_s^l\}^{-1}\{\left(\frac{1}{l}+\frac{\mu^2 l}{3}\right)\widehat{(\eta_t)_c^l}\}\right)\}$$
$$+\frac{\mu^2}{2}\mathcal{F}_s^k\{\partial_x(\{\mathcal{F}_s^k\}^{-1}\{\left(\frac{1}{k}+\frac{\mu^2 k}{3}\right)\widehat{(\eta_t)_c^k}\})^2\}+\widehat{(\eta_x)_s^k}=0.$$

Removing the terms of order $\mathcal{O}(\mu^4)$ and rearranging, we obtain:

$$-\frac{1}{k}\widehat{(\eta_{tt})_c^k}+\widehat{(\eta_x)_s^k}+\mu^2\left(\partial_t\mathcal{F}_c^k\{\left(\eta\{\mathcal{F}_s^l\}^{-1}\{\frac{1}{l}\widehat{(\eta_t)_c^l}\}\right)\}+\frac{1}{2}\mathcal{F}_s^k\{\partial_x(\{\mathcal{F}_s^k\}^{-1}\{\frac{1}{k}\widehat{(\eta_t)_c^k}\})^2\}-\frac{k}{3}\widehat{(\eta_{tt})_c^k}\right)=0. \tag{42}$$

Now, we would like to manipulate (42) so that we can apply inverse Fourier sine transform. First, note

$$\begin{aligned}
-\frac{1}{k}\frac{1}{2\pi}\widehat{(\eta_{tt})_c^k}&=-\frac{1}{k}\frac{1}{2\pi}\int_0^\infty\cos(kx)\eta_{tt}\,\mathrm{d}x\\
&=-\frac{1}{2\pi}\frac{\cos(kx)}{k}\left(\int_0^x\eta_{tt}\,\mathrm{d}x'\right)\Big|_0^\infty-\frac{1}{2\pi}\int_0^\infty\sin(kx)\left(\int_0^x\eta_{tt}\,\mathrm{d}x'\right)\mathrm{d}x\\
&=-\mathcal{F}_s^k\{\int_0^x\eta_{tt}\,\mathrm{d}x'\},
\end{aligned}$$

where the final line follows sine $\int_0^\infty\eta_{tt}\,\mathrm{d}x'=0$ is a conservation law (TO BE VERIFIED). Second, observe that

$$\begin{aligned}
\frac{1}{l}\widehat{(\eta_t)_c^l}&=\frac{1}{l}\frac{1}{2\pi}\int_0^\infty\cos(lx)\eta_t\,\mathrm{d}x\\
&=\frac{1}{2\pi}\frac{\cos(lx)}{l}\left(\int_0^x\eta_t\,\mathrm{d}x'\right)\Big|_0^\infty+\frac{1}{2\pi}\int_0^\infty\sin(lx)\left(\int_0^x\eta_t\,\mathrm{d}x'\right)\mathrm{d}x\\
&=\mathcal{F}_s^l\{\int_0^x\eta_t\,\mathrm{d}x'\},
\end{aligned}$$

where similarly the last line follows since $\int_0^\infty\eta_t\,\mathrm{d}x'=0$, a conservation law (TO BE VERIFIED). This identity yields:

$$\begin{aligned}
\partial_t\mathcal{F}_c^k\{\eta\{\mathcal{F}_s^l\}^{-1}\{\frac{1}{l}\widehat{(\eta_t)_c^l}\}&=\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x\eta_t\,\mathrm{d}x'\right)\},\\
\frac{1}{2}\mathcal{F}_s^k\{\partial_x(\{\mathcal{F}_s^k\}^{-1}\{\frac{1}{k}\widehat{(\eta_t)_c^k}\})^2\}&=\frac{1}{2}\mathcal{F}_s^k\{\partial_x\left(\int_0^x\eta_t\,\mathrm{d}x'\right)^2\}.
\end{aligned}$$

Thirdly, we have

$$\begin{aligned}
-\frac{k}{3}\widehat{(\eta_{tt})_c^k}&=-\frac{k}{3}\frac{1}{2\pi}\int_0^\infty\cos(kx)\eta_{tt}\,\mathrm{d}x\\
&=-\frac{k}{3}\frac{1}{2\pi}\frac{\sin(kx)}{k}\eta_{tt}\Big|_0^\infty+\frac{1}{3}\frac{1}{2\pi}\int_0^\infty\sin(kx)\eta_{ttx}\,\mathrm{d}x
\end{aligned}$$

$$= \frac{1}{3}\mathcal{F}_s^k\{\eta_{ttx}\},$$

where the last line follows by the assumption $\lim_{x\to\infty}\eta_{tt} = 0$. With these manipulations in mind, the equation (42) becomes

$$-\mathcal{F}_s^k\{\int_0^x \eta_{tt}\,\mathrm{d}x'\} + \widehat{(\eta_x)}_s^k + \mu^2\left(\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\} + \frac{1}{2}\mathcal{F}_s^k\{\partial_x\left(\int_0^x \eta_t\,\mathrm{d}x'\right)^2\} + \frac{1}{3}\mathcal{F}_s^k\{\eta_{ttx}\}\right) = 0. \tag{43}$$

Inverting the Fourier sine transform, we obtain

$$-\int_0^x \eta_{tt}\,\mathrm{d}x' + \eta_x + \mu^2\left((\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\}\} + \frac{1}{2}\partial_x\left(\int_0^x \eta_t\,\mathrm{d}x'\right)^2 + \frac{1}{3}\eta_{ttx}\right) = 0. \tag{44}$$

Take the derivative with respect to x :

$$-\eta_{tt} + \eta_{xx} + \mu^2\left(\partial_x(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\}\} + \frac{1}{2}\partial_x^2\left(\int_0^x \eta_t\,\mathrm{d}x'\right)^2 + \frac{1}{3}\eta_{ttxx}\right) = 0. \tag{45}$$

Rearranging and using $\eta_{tt} = \eta_{xx} + \mathcal{O}(\mu^2)$, we obtain the equation

$$\eta_{tt} - \eta_{xx} = \mu^2\left(\frac{1}{3}\eta_{xxxx} + \partial_x(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\}\} + \frac{1}{2}\partial_x^2\left(\int_0^x \eta_t\,\mathrm{d}x'\right)^2\right). \tag{46}$$

Clearly, the presence of the term with mixed transforms complicates things: we cannot apply integration by parts like we did for other terms, because doing so results in a multiple of k in the new term, which is exactly what we want to avoid. For comparison, the whole line equation for the surface is given by

$$\eta_{tt} - \eta_{xx} = \mu^2\left(\frac{1}{3}\eta_{xxxx} + \partial_x\left[\eta_t\left(\int_{-\infty}^x \eta_t\,\mathrm{d}x'\right) + \eta\eta_x\right] + \frac{1}{2}\partial_x^2\left(\int_{-\infty}^x \eta_t\,\mathrm{d}x'\right)^2\right).$$

other stuff

Consider the term

$$\varepsilon\mu^2\frac{(\eta_t + \eta_x\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\})^2}{1+\varepsilon^2\mu^2\eta_x^2}.$$

Note that $|\varepsilon\mu\eta_x| < 1$, so

$$\frac{1}{1+\varepsilon^2\mu^2\eta_x^2} = \frac{1}{1-(-\varepsilon^2\mu^2\eta_x^2)} \approx 1 + \varepsilon^2\mu^2\eta_x^2,$$

by geometric series argument. Furthermore,

$$\begin{aligned}(\eta_t + \eta_x\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\})^2 &\approx \eta_t^2 + 2\eta_t\eta_x\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\\ &\approx \eta_t^2 + 2\eta_t\eta_x\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\},\end{aligned}$$

so we can assume

$$\varepsilon\mu^2\frac{(\eta_t + \eta_x\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\})^2}{1+\varepsilon^2\mu^2\eta_x^2} \approx \varepsilon\mu^2(\eta_t^2 + 2\eta_t\eta_x\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\})$$

Then,

$$\begin{aligned}\frac{1}{2}\left(\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 &\approx \frac{1}{2}\left(\left[\mathcal{H}_0(\varepsilon\eta, D) + \varepsilon\mathcal{H}_1(\varepsilon\eta, D)\right]\{\varepsilon\mu\eta_t\}\right)^2\\ &= \frac{1}{2}\left(\left[\mathcal{H}_0(\varepsilon\eta, D)^2 + 2\varepsilon\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\mathcal{H}_1(\varepsilon\eta, D)\right]\{\varepsilon\mu\eta_t\}\right)\\ &= \frac{1}{2}\left(\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 + \varepsilon\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\mathcal{H}_1(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 + \varepsilon^3\mu^2\mathcal{H}_0(\varepsilon\eta, D)\{\eta_t\}\mathcal{H}_1(\varepsilon\eta, D)\{\eta_t\} \\
&= \frac{(\varepsilon\mu)^2}{2}\left[\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]^2 \\
&+ \varepsilon^3\mu^2\left[\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]\left(\mu(\eta f)_x - \mathcal{F}^{-1}\{\mu k\coth(\mu k)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\}\}\right).
\end{aligned}
$$

In the leading two orders, we have

$$
\partial_t\left(\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right) + \partial_x\left(\frac{1}{2}\left(\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 + \varepsilon\eta - \frac{1}{2}\varepsilon\mu^2\frac{(\eta_t + \eta_x\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\})^2}{1+\varepsilon^2\mu^2\eta_x^2}\right) = 0 \implies
$$

$$
\begin{aligned}
&\partial_t([\mathcal{H}_0(\varepsilon\eta, D) + \varepsilon\mathcal{H}_1(\varepsilon\eta, D)]\{\varepsilon\mu\eta_t\} + \partial_x\Bigg(\frac{1}{2}\left(\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 + \varepsilon\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\mathcal{H}_1(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\} \\
&\qquad + \varepsilon\eta - \frac{1}{2}\varepsilon\mu^2\eta_t^2\Bigg) = 0 \\
&\partial_t([\mathcal{H}_0(\varepsilon\eta, D) + \varepsilon\mathcal{H}_1(\varepsilon\eta, D)]\{\varepsilon\mu\eta_t\} + \partial_x\Bigg(\frac{1}{2}\left(\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 + \varepsilon^3\mu^2\mathcal{H}_0(\varepsilon\eta, D)\{\eta_t\}\mathcal{H}_1(\varepsilon\eta, D)\{\eta_t\} \\
&\qquad + \varepsilon\eta - \frac{1}{2}\varepsilon\mu^2\eta_t^2\Bigg) = 0.
\end{aligned}
$$

Consider each term:

$$
\begin{aligned}
\partial_t\left([\mathcal{H}_0(\varepsilon\eta, D) + \varepsilon\mathcal{H}_1(\varepsilon\eta, D)]\{\varepsilon\mu\eta_t\}\right) &= \varepsilon\mu\partial_t\left([\mathcal{H}_0(\varepsilon\eta, D) + \varepsilon\mathcal{H}_1(\varepsilon\eta, D)]\{\eta_t\}\right) \\
&= \varepsilon\mu\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_{tt}}\} + \varepsilon^2\mu^2(\eta\eta_t)_{tx} \\
&- \varepsilon^2\mu\mathcal{F}^{-1}\{\mu k\coth(\mu k)\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]\}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\varepsilon\mu\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_{tt}}\} + \varepsilon^2\mu^2(\eta\eta_t)_{tx} - \varepsilon^2\mu\mathcal{F}^{-1}\{\mu k\coth(\mu k)\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]\} \\
&\qquad + \frac{(\varepsilon\mu)^2}{2}\partial_x\left[\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]^2 \\
&\qquad + \varepsilon^3\mu^2\partial_x\left[\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]\left[\mu(\eta f)_x - \mathcal{F}^{-1}\{\mu l\coth(\mu l)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\coth(\mu j)\widehat{\eta_t}\}_k\}\right] \\
&\qquad + \varepsilon\partial_x\eta - \frac{1}{2}\varepsilon\mu^2\partial_x(\eta_t^2) = 0.
\end{aligned}
$$

Invert Fourier transform:

$$
\begin{aligned}
&\varepsilon\mu i\coth(\mu k)\widehat{\eta_{tt}} + \varepsilon^2\mu^2 ik\widehat{(\eta\eta_t)_t} - \varepsilon^2\mu(\mu k\coth(\mu k)\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right])\} \\
&\qquad + \frac{(\varepsilon\mu)^2}{2}ik\mathcal{F}\{\left[\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]^2\} \\
&\qquad + \varepsilon^3\mu^2 ik\mathcal{F}\left(\left[\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]\left[\mu(\eta f)_x - \mathcal{F}^{-1}\{\mu l\coth(\mu l)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\coth(\mu j)\widehat{\eta_t}\}_k\}\right]\right) \\
&\qquad + \varepsilon ik\widehat{\eta} - \frac{1}{2}\varepsilon\mu^2 ik\widehat{\eta_t^2} = 0.
\end{aligned}
$$

Divide by $i\varepsilon$:

$$
\begin{aligned}
&\mu\coth(\mu k)\widehat{\eta_{tt}} + \varepsilon\mu^2 k\widehat{(\eta\eta_t)_t} - \varepsilon\mu(\mu k\coth(\mu k)\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{\coth(\mu k)\widehat{\eta_t}\}\}\right]) \\
&\qquad + \frac{\varepsilon\mu^2}{2}k\mathcal{F}\{\left[\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]^2\} \\
&\qquad + \varepsilon^2\mu^2 k\mathcal{F}\left(\left[\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]\left[\mu(\eta f)_x - \mathcal{F}^{-1}\{\mu l\coth(\mu l)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\coth(\mu j)\widehat{\eta_t}\}_k\}\right]\right) \\
&\qquad + k\widehat{\eta} - \frac{1}{2}\mu^2 k\widehat{\eta_t^2} = 0.
\end{aligned}
$$

Divide by $\coth(\mu k)$:

$$
\mu\widehat{\eta_{tt}} + \varepsilon\mu^2 k\tanh(\mu k)\widehat{(\eta\eta_t)_t} - \varepsilon\mu(\mu k\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{\coth(\mu k)\widehat{\eta_t}\}\right]\})
$$

$$
\begin{aligned}
&+\frac{\varepsilon\mu^2}{2}k\tanh(\mu k)\mathcal{F}\{\left[\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]^2\}\\
&+\varepsilon^2\mu^2 k\tanh(\mu k)\mathcal{F}\left(\left[\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]\left[\mu(\eta f)_x-\mathcal{F}^{-1}\{\mu l\coth(\mu l)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\coth(\mu j)\widehat{\eta_t}\}_k\}\right]\right)\\
&+\tanh(\mu k)k\widehat{\eta}-\tanh(\mu k)\frac{1}{2}\mu^2 k\widehat{\eta_t^2}=0.
\end{aligned}
$$

Let $\varepsilon=\mu^2$ and recall expansions:

$$
\tanh(\mu k)\approx \mu k-\frac{(\mu k)^3}{3}+\mathcal{O}(\mu^5),\qquad \coth(\mu k)\approx\frac{1}{\mu k}+\frac{\mu k}{3}+\mathcal{O}(\mu^3).
$$

Substitute the expansions appropriately:

$$
\begin{aligned}
&\mu\widehat{\eta_{tt}}+\mu^4 k\left(\mu k-\frac{(\mu k)^3}{3}\right)\widehat{(\eta\eta_t)_t}-i\mu^3 k(\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{\left(\frac{1}{k}+\frac{\mu^2 k}{3}\right)\widehat{\eta_t}\}\right])\\
&\quad-\frac{\varepsilon\mu^2}{2}k\left(\mu k-\frac{(\mu k)^3}{3}\right)\mathcal{F}\{\left[\mathcal{F}^{-1}\{\left(\frac{1}{\mu l}+\frac{\mu l}{3}\right)\widehat{\eta_t}\}\right]^2\}\\
&\quad+\varepsilon^2\mu^3 k^2\left(1-\frac{(\mu k)^2}{3}\right)\mathcal{F}\left(\left[\mathcal{F}^{-1}\{i\left(\frac{1}{\mu k}+\frac{\mu k}{3}\right)\widehat{\eta_t}\}\right]\left[\mu(\eta f)_x-\mathcal{F}^{-1}\{\left(1+\frac{(\mu l)^2}{3}\right)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\left(\frac{1}{\mu j}+\frac{\mu j}{3}\right)\widehat{\eta_t}\}\}\right]\right)\\
&\quad+\left(\mu k-\frac{(\mu k)^3}{3}\right)k\widehat{\eta}-\left(\mu k-\frac{(\mu k)^3}{3}\right)\frac{1}{2}\mu^2 k\widehat{\eta_t^2}=0.
\end{aligned}
$$

Consider the term on the third line: we divide by μ, and eliminate terms within $\mathcal{O}(\mu^4)$:

$$
\begin{aligned}
\varepsilon^2\mu^3 k^2\left(1-\frac{(\mu k)^2}{3}\right)\mathcal{F}\left(\left[\mathcal{F}^{-1}\{i\left(\frac{1}{\mu k}+\frac{\mu k}{3}\right)\widehat{\eta_t}\}\right]\left[\mu(\eta f)_x-\mathcal{F}^{-1}\{\left(1+\frac{(\mu l)^2}{3}\right)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\left(\frac{1}{\mu j}+\frac{\mu j}{3}\right)\widehat{\eta_t}\}\}\right]\right)&=\\
\varepsilon^2\mu k^2\left(1-\frac{(\mu k)^2}{3}\right)\mathcal{F}\left(\left[\mathcal{F}^{-1}\{i\left(\frac{\mu}{\mu k}+\frac{\mu^2 k}{3}\right)\widehat{\eta_t}\}\right]\left[\mu^2(\eta f)_x-\mathcal{F}^{-1}\{\left(1+\frac{(\mu l)^2}{3}\right)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\left(\frac{\mu}{\mu j}+\frac{\mu^2 j}{3}\right)\widehat{\eta_t}\}\}\right]\right)&=\\
\varepsilon^2\mu k^2\left(1-\frac{(\mu k)^2}{3}\right)\mathcal{F}\left(\left[\mathcal{F}^{-1}\{i\left(\frac{1}{k}+\frac{\mu^2 k}{3}\right)\widehat{\eta_t}\}\right]\left[\mu^2(\eta f)_x-\mathcal{F}^{-1}\{\left(1+\frac{(\mu l)^2}{3}\right)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\left(\frac{1}{j}+\frac{\mu^2 j}{3}\right)\widehat{\eta_t}\}\}\right]\right)&\approx\\
\varepsilon^2\mu k^2\mathcal{F}\left(\eta\left[\mathcal{F}^{-1}\{\frac{1}{k}\widehat{\eta_t}\}\right]\left[\mathcal{F}^{-1}\{\frac{1}{j}\widehat{\eta_t}\}\right]\right)&,
\end{aligned}
$$

divide by μ to obtain

$$
\varepsilon^2 k^2\mathcal{F}\left(\eta\left[\mathcal{F}^{-1}\{\frac{1}{k}\widehat{\eta_t}\}\right]\left[\mathcal{F}^{-1}\{\frac{1}{j}\widehat{\eta_t}\}\right]\right)=\varepsilon^2 k^2\mathcal{F}\left(\eta\left[\int_{-\infty}^{x}\eta_t\,\mathrm{d}x'\right]^2\right),
$$

Divide through by μ, eliminate terms of order $\mathcal{O}(\mu^4)$ and rearrange:

$$
\widehat{\eta_{tt}}+k^2\widehat{\eta}+\mu^2\left(-ik(\mathcal{F}\{\partial_t\left[\eta\int_{-\infty}^{x}\eta_t\,\mathrm{d}x\}\right])-\frac{1}{2}k^2\mathcal{F}\{\left[\mathcal{F}^{-1}\{\frac{1}{l}\widehat{\eta_t}\}\right]^2\}-\frac{k^4}{3}\widehat{\eta}-\frac{k^2}{2}\widehat{\eta_t^2}\right)=0.
$$

Finally, invert Fourier transform:

$$
\eta_{tt}-\eta_{xx}+\mu^2\left(-\partial_x\partial_t\left[\eta\int_{-\infty}^{x}\eta_t\,\mathrm{d}x\}\right]+\frac{1}{2}\partial_x^2\left[\mathcal{F}^{-1}\{\frac{1}{l}\widehat{\eta_t}\}\right]^2-\frac{1}{3}\eta_{xxxx}+\frac{1}{2}\partial_x^2\eta_t^2\right)=0,
$$

or more conveniently,

$$
\eta_{tt}-\eta_{xx}=\mu^2\ \left(\frac{1}{3}\eta_{xxxx}-\frac{1}{2}\partial_x^2\eta_t^2+\partial_x\partial_t\left[\eta\int_{-\infty}^{x}\eta_t\,\mathrm{d}x\}\right]+\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\int_{-\infty}^{x}\eta_t\,\mathrm{d}x'\right]^2\right)
$$

For direct comparison, in [1, p. 110], the corresponding equation is the equation (5.20)

$$
\eta_{tt}-\eta_{xx}=\mu^2\ \left(\frac{1}{3}\eta_{xxxx}+\eta_{xt}\int_{-\infty}^{x}\eta_t\,\mathrm{d}x'+\eta_x^2+\eta_t^2+\eta\eta_{xx}+\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\int_{-\infty}^{x}\eta_t\,\mathrm{d}x'\right]^2\right),
$$

References

[1] Mark J. Ablowitz, *Nonlinear dispersive waves: Asymptotic analysis and solitons*, Cambridge University Press, 2011.