Report 3

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Contents

1 The half line problem

In this section, we deal with this term

$$\partial_x (\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t \left(\eta \int_0^x \eta_t \, \mathrm{d}x'\right)\}\}$$

More generally, we have the following result:

Theorem 1. *For nice enough f defined on $x \geqslant 0$, we have*

$$(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} = \frac{1}{\pi} \int_0^\infty f(y) \left(\frac{1}{x-y} + \frac{1}{x+y}\right) \mathrm{d}y.$$

Before we begin, recall the Riemann-Lebesgue lemma:

Lemma 2 (Theorem 11.6, [1])**.** *Assume that $f \in L(I)$. Then, for each real β, we have*

$$\lim_{\alpha \to \infty} \int_I f(t) \sin(\alpha t + \beta) \, \mathrm{d}t = 0.$$

Proof of Theorem 1. Consider

$$(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\}.$$

For generality, we consider $(\mathcal{F}_s^k)^{-1}\{G(k)\}$, where G is a function of k defined on $k \geqslant 0$. Expanding the integral, we obtain:

$$\begin{aligned}
(\mathcal{F}_s^k)^{-1}\{G(k)\} &= \int_0^\infty \sin(kx) G(k) \, \mathrm{d}k \\
&= \frac{1}{2i} \int_0^\infty (e^{ikx} - e^{-ikx}) G(k) \, \mathrm{d}k \\
&= \frac{1}{2i} \left[\int_0^\infty e^{ikx} G(k) \, \mathrm{d}k - \int_0^\infty e^{-ikx} G(k) \, \mathrm{d}k \right] \\
&= \frac{1}{2i} \left[\int_0^\infty e^{ikx} G(k) \, \mathrm{d}k + \int_0^{-\infty} e^{ikx} G(-k) \, \mathrm{d}k \right] && \text{(apply } k \mapsto -k \text{ in the 2nd term)} \\
&= \frac{1}{2i} \left[\int_0^\infty e^{ikx} G(k) \, \mathrm{d}k + \int_{-\infty}^0 e^{ikx} (-G(-k)) \, \mathrm{d}k \right],
\end{aligned}$$

where $-G(-k)$ is an odd extension to $k < 0$. Now, observe the following:

$$\begin{aligned}
\frac{2}{\pi} \int_0^\infty \cos(kx) f(x) \, \mathrm{d}x &= \frac{1}{\pi} \int_0^\infty (e^{ikx} + e^{-ikx}) f(x) \, \mathrm{d}x \\
&= \frac{1}{\pi} \left[\int_0^\infty e^{ikx} f(x) \, \mathrm{d}x + \int_0^\infty e^{-ikx} f(x) \, \mathrm{d}x \right] \\
&= \frac{1}{\pi} \left[- \int_0^{-\infty} e^{-ikx} f(-x) \, \mathrm{d}x + \int_0^\infty e^{-ikx} f(x) \, \mathrm{d}x \right] && \text{(apply } x \mapsto -x \text{ in the 1st term)} \\
&= \frac{1}{\pi} \left[\int_{-\infty}^0 e^{-ikx} f(-x) \, \mathrm{d}x + \int_0^\infty e^{-ikx} f(x) \, \mathrm{d}x \right] \\
&= \frac{1}{\pi} \int_{-\infty}^\infty e^{-ikx} F(x) \, \mathrm{d}x,
\end{aligned}$$

where we used an even extension to $x < 0$ and defined

$$F(x) = \begin{cases} f(x) & x > 0 \\ f(-x) & x < 0 \end{cases}.$$

For $k > 0$, we have

$$G(k) = \mathcal{F}_c^k\{f\} = \frac{2}{\pi}\int_0^\infty \cos(kx) f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_{-\infty}^\infty e^{-ikx} F(x)\,\mathrm{d}x. \tag{1}$$

For $k < 0$, we have

$$-G(-k) = -\mathcal{F}_c^{-k}\{f\} = -\frac{2}{\pi}\int_0^\infty \cos(-kx) f(x)\,\mathrm{d}x = -\frac{2}{\pi}\int_0^\infty \cos(kx) f(x)\,\mathrm{d}x = -\frac{1}{\pi}\int_{-\infty}^\infty e^{-ikx} F(x)\,\mathrm{d}x, \tag{2}$$

since cosine is an even function. Thus, using (1) and (2), we obtain

$$\begin{aligned}
(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} &= \frac{1}{2i}\left[\int_0^\infty e^{ikx}\mathcal{F}_c^k\{f\}\,\mathrm{d}k + \int_{-\infty}^0 e^{ikx}(-\mathcal{F}^(- k)_c\{f\})\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty e^{ikx}\int_{-\infty}^\infty e^{-iky}F(y)\,\mathrm{d}y\,\mathrm{d}k - \int_{-\infty}^0 e^{ikx}\int_{-\infty}^\infty e^{-iky}F(y)\,\mathrm{d}y\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty e^{ikx}\int_{-\infty}^\infty e^{ik(x-y)}F(y)\,\mathrm{d}y\,\mathrm{d}k - \int_{-\infty}^0 \int_{-\infty}^\infty e^{ik(x-y)}F(y)\,\mathrm{d}y\,\mathrm{d}k\right].
\end{aligned} \tag{3}$$

Let

$$\begin{aligned}
V(k) &= \int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y = -V(-k), \\
U(k) &= \int_{-\infty}^\infty \cos(k(x-y))F(y)\,\mathrm{d}y = U(-k),
\end{aligned}$$

so that V is odd and U is even. This allows to rewrite (3) as:

$$\begin{aligned}
(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} &= \frac{1}{2\pi i}\left[\int_0^\infty e^{ikx}\int_{-\infty}^\infty e^{ik(x-y)}F(y)\,\mathrm{d}y\,\mathrm{d}k - \int_{-\infty}^0 \int_{-\infty}^\infty e^{ik(x-y)}F(y)\,\mathrm{d}y\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty U(k) + iV(k)\,\mathrm{d}k - \int_{-\infty}^0 U(k) + iV(k)\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty U(k) + iV(k)\,\mathrm{d}k + \int_\infty^0 U(-k) + iV(-k)\,\mathrm{d}k\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\left[\int_0^\infty U(k) + iV(k)\,\mathrm{d}k + \int_0^\infty -U(-k) + i(-V(-k))\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty U(k) + iV(k)\,\mathrm{d}k + \int_0^\infty -U(k) + iV(k)\,\mathrm{d}k\right] \\
&= \frac{1}{\pi}\int_0^\infty V(k)\,\mathrm{d}k,
\end{aligned}
$$

where on the third last line, we flipped the bounds of integration and brought the minus sign inside the integral, and on the second last line, we used that U is even and V is odd. Thus, we obtain

$$
(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} = \frac{1}{\pi}\int_0^\infty V(k)\,\mathrm{d}k = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y\,\mathrm{d}k.
$$

Note that the integral in k is an improper integral, so

$$
\int_0^\infty \int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y\,\mathrm{d}k = \lim_{\alpha\to\infty}\int_0^\alpha \int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y\,\mathrm{d}k.
$$

Now, interchanging the order of integration, we have

$$
\begin{aligned}
\int_0^\alpha \int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y\,\mathrm{d}k &= \int_{-\infty}^\infty F(y)\int_0^\alpha \sin(k(x-y))\,\mathrm{d}k\,\mathrm{d}y \\
&= \int_{-\infty}^\infty F(y)\left[-\frac{\cos(k(x-y))}{x-y}\,\Big|_0^\alpha\right]\mathrm{d}y \\
&= \int_{-\infty}^\infty F(y)\left[\frac{1}{x-y} - \frac{\cos(\alpha(x-y))}{x-y}\right]\mathrm{d}y \\
&= \int_{-\infty}^\infty F(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y.
\end{aligned}
$$

The interchange is justified, since sine is bounded and differentiable on $\mathbb{R}$. Finally, we use the Riemann-Lebesgue lemma to deal with the last term:

$$
\begin{aligned}
\int_{-\infty}^\infty F(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y &= \int_0^\infty f(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y + \int_{-\infty}^0 f(-y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y \\
&= \int_0^\infty f(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y - \int_\infty^0 f(y)\frac{1-\cos(\alpha(x+y))}{x+y}\,\mathrm{d}y \\
&= \int_0^\infty f(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y + \int_0^\infty f(y)\frac{1-\cos(\alpha(x+y))}{x+y}\,\mathrm{d}y
\end{aligned}
$$

$$= \int_0^\infty f(y)\frac{1}{x-y}\,\mathrm{d}y - \int_0^\infty f(y)\frac{\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y + \int_0^\infty f(y)\frac{1}{x+y}\,\mathrm{d}y - \int_0^\infty f(y)\frac{\cos(\alpha(x+y))}{x+y}\,\mathrm{d}y.$$

As $\alpha \to \infty$, the terms

$$\int_0^\infty f(y)\frac{\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y, \qquad \int_0^\infty f(y)\frac{\cos(\alpha(x+y))}{x+y}\,\mathrm{d}y \to 0$$

by the Riemann-Lebesgue lemma with $\beta = \pi/2$, so that

$$\int_{-\infty}^\infty F(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y = \int_0^\infty f(y)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y.$$

Thus,

$$(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y\,\mathrm{d}k = \frac{1}{\pi}\int_0^\infty f(y)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y.$$

The proof is complete. $\square$

Remark 3. Note that the integral

$$\frac{1}{\pi}\int_0^\infty f(y)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y =$$

looks like a convolution-type transform. In fact, the term with $1/(x-y)$ is pretty much the Hilbert transform, but on a half-line.

The theorem yields

$$\partial_x(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\}\} = \partial_x\left(\frac{1}{\pi}\int_0^\infty \partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y\right).$$

For generality, let $f(y) = \partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)$. Note the following:

$$\begin{aligned}\partial_x\left(\frac{1}{\pi}\int_0^\infty f(y)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y\right) &= \frac{1}{\pi}\int_0^\infty f(y)\partial_x\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y \\ &= -\frac{1}{\pi}\int_0^\infty f(y)\left[\frac{1}{(x-y)^2}+\frac{1}{(x+y)^2}\right]\mathrm{d}y,\end{aligned}$$

so that

$$\partial_x(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\}\} = -\frac{1}{\pi}\int_0^\infty \partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)\left[\frac{1}{(x-y)^2}+\frac{1}{(x+y)^2}\right]\mathrm{d}y. \tag{4}$$

As can be seen, the integral (4) is singular whenever $x = y$ or $x = -y$, over y. To deal with this issue, one may need to use a Residue theorem. To conclude, the surface expression on a half-line becomes:

$$\begin{aligned}\eta_{tt} - \eta_{xx} &= \mu^2 \left(\frac{1}{3}\eta_{xxxx} + \partial_x (\mathcal{F}_s^k)^{-1} \{ \mathcal{F}_c^k \{ \partial_t \left(\eta \int_0^x \eta_t \, \mathrm{d}x' \right) \} \} + \frac{1}{2}\partial_x^2 \left(\int_0^x \eta_t \, \mathrm{d}x' \right)^2 \right) \\ &= \mu^2 \left(\frac{1}{3}\eta_{xxxx} - \frac{1}{\pi} \int_0^\infty \partial_t \left(\eta \int_0^y \eta_t \, \mathrm{d}y' \right) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] \mathrm{d}y + \frac{1}{2}\partial_x^2 \left(\int_0^x \eta_t \, \mathrm{d}x' \right)^2 \right).\end{aligned}$$

2 Approximate equations: half-line

In this section, we derive the approximate equations from

$$\eta_{tt} - \eta_{xx} = \varepsilon \left(\frac{1}{3}\eta_{xxxx} + \frac{d}{dx}\frac{1}{\pi} \int_0^\infty \partial_t \left(\eta \int_0^y \eta_t \, \mathrm{d}y' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y + \frac{1}{2}\partial_x^2 \left(\int_0^x \eta_t \, \mathrm{d}x' \right)^2 \right). \tag{5}$$

As we approximate, we assume an expansion of η in ε :

$$\eta = \eta_0 + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2). \tag{6}$$

2.1 First order approximation

Substitution of (6) into equation (5) yields

$$\eta_{0tt} - \eta_{0xx} + \varepsilon(\eta_{1tt} - \eta_{1xx}) = \varepsilon \left[\frac{1}{3}\eta_{0xxxx} + \frac{d}{dx}\frac{1}{\pi} \int_0^\infty \partial_t \left(\eta \int_0^y \eta_t \, \mathrm{d}y' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y + \frac{1}{2}\partial_x^2 \left(\int_{-\infty}^x (\eta_0 + \varepsilon\eta_1)_t \, \mathrm{d}x' \right)^2 \right) + \mathcal{O}(\varepsilon^2). \tag{7}$$

In the leading order $\mathcal{O}(\varepsilon^0)$, equation (7) becomes

$$\eta_{0tt} - \eta_{0xx} = 0. \tag{8}$$

This is the wave equation with velocity 1, whose solution depends on the type of boundary conditions we prescribe for η at $x = 0$. For now, we prescribe

$$\eta_x(0,t) = 0.$$

The general solution is

$$\eta(x,t) = \begin{cases} F(x-t) + G(x+t) & x > t \\ F(t-x) + G(x+t) & x < t \end{cases},$$

where F, G are to be determined.

2.2 Second order approximation

As in the velocity potential case, we employ multiple scales. First, we find an expression for η_0. We introduce

$$\tau_0 = t, \qquad \tau_1 = \varepsilon t, \qquad \tau_2 = \varepsilon^2 t, \ldots,$$

so that

$$\eta(x,t) = \eta(x, \tau_0, \tau_1, \ldots).$$

With this in mind, the expansion (6) becomes

$$\eta(x, \tau_0, \tau_1, \ldots) = \eta_0(x, \tau_0, \tau_1, \ldots) + \mathcal{O}(\varepsilon^1). \tag{9}$$

Substituting (9) into (5), within $\mathcal{O}(\varepsilon^0)$, we obtain

$$\eta_{0\tau_0\tau_0} - \eta_{0xx} = 0, \tag{10}$$

so that the general solution is

$$\eta_0(x, \tau_0, \tau_1, \ldots) = \begin{cases} F_2(x - \tau_0, \tau_1, \ldots) + G_2(x + \tau_0, \tau_1, \ldots) & x \geqslant \tau_0 \\ F_1(\tau_0 - x, \tau_1, \ldots) + G_1(x + \tau_0, \tau_1, \ldots) & x < \tau_0 \end{cases}.$$

Now, although we have found an expression for η_0, the functions F_i, G_i used are still general functions. To determine F_i, G_i, we proceed to the next order, i.e. $\mathcal{O}(\varepsilon^1)$. We introduce

$$\xi = x - \tau_0 \qquad \zeta = x + \tau_0$$

so that

$$\eta_0(x, \tau_0, \tau_1, \ldots) = \begin{cases} F_1(\xi, \tau_1, \ldots) + G_1(\zeta, \tau_1, \ldots) & x \geqslant t \\ F_2(-\xi, \tau_1, \ldots) + G_2(\zeta, \tau_1, \ldots) & x < t \end{cases},$$

and

$$\partial_x = \partial_\xi \frac{\mathrm{d}\xi}{\mathrm{d}x} + \partial_\zeta \frac{\mathrm{d}\zeta}{\mathrm{d}x} = \partial_\xi + \partial_\zeta,$$
$$\partial_t = \partial_\xi \frac{\mathrm{d}\xi}{\mathrm{d}t} + \partial_\zeta \frac{\mathrm{d}\zeta}{\mathrm{d}t} + \partial_{\tau_1} \frac{\mathrm{d}\tau_1}{\mathrm{d}t} = -\partial_\xi + \partial_\zeta + \varepsilon\partial_{\tau_1}.$$

Remark 4. We emphasize the piecewise nature of solutions, which is why we write that F_1, F_2 as different functions even though they share the same variable ξ. It is very important to be aware which F_i we need to use, as we will demonstrate when dealing with the non-local terms. In addition, we also need to impose some more conditions at $x = \tau_0$, to reinforce some sort of continuity between F_1 and F_2..

2.2.1 The case $x < \tau_0$

We consider the case $x < t$. First, we use

$$\begin{aligned}\eta &= \eta_0 + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2)\\ &= F_1(t-x, \varepsilon t, \ldots) + G_1(x+t, \varepsilon t, \ldots) + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2)\\ &= F_1(-\xi, \tau_1, \ldots) + G_1(\zeta, \tau_1, \ldots) + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2)\\ &= F_1 + G_1 + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2).\end{aligned}$$

For ease of writing, we suppress explicit dependence on variables, though the reader should bear in mind that function $F_1, (G_1)$ depend on $-\xi, (\zeta), \tau_1, \tau_2$, etc. In addition, observe that

$$(\partial_t^2 - \partial_x^2) = \left(-4\partial_\xi\partial_\zeta + 2\varepsilon(\partial_\zeta\partial_{\tau_1} - \partial_\xi\partial_{\tau_1}) + \varepsilon^2\partial_{\tau_1}^2\right),$$

so that the LHS of (5) becomes

$$(\partial_t^2 - \partial_x^2)\eta = \varepsilon\left(-4\eta_{1\xi\zeta} - 2\partial_\xi\partial_{\tau_1}(F_1)_{\tau_1} + 2\partial_\zeta\partial_{\tau_1}G_1\right) + \mathcal{O}(\varepsilon^2). \tag{11}$$

Now, we deal with the RHS of (5). By appropriate substitutions, the terms become:

$$\begin{aligned}\frac{1}{3}\eta_{xxxx} &= \frac{1}{3}(\partial_\xi^4 F_1 + \partial_\zeta^4 G_1 + \mathcal{O}(\varepsilon));\\ \left(\int_0^x \eta_t \,\mathrm{d}x'\right)^2 &= \left(\int_0^x \eta_{0t}\,\mathrm{d}x'\right)^2 + \mathcal{O}(\varepsilon)\\ &= \left(\int_0^x (-\partial_{\xi'} + \partial_{\zeta'} + \varepsilon\partial_{\tau_1})(F_1+G_1)\,\mathrm{d}x'\right)^2 + \mathcal{O}(\varepsilon)\\ &= \left(\int_0^x -\partial_{\xi'}F_1 + \partial_{\zeta'}G_1\,\mathrm{d}x'\right)^2 + \mathcal{O}(\varepsilon)\\ &= \left(\int_0^x -\partial_{\xi'}F_1\,\mathrm{d}x'\right)^2 - 2\left(\int_0^x (\partial_{\xi'}F_1\,\mathrm{d}x'\right)\left(\int_0^x \partial_{\zeta'}G_1\,\mathrm{d}x'\right) + \left(\int_0^x \partial_{\zeta'}G_1\,\mathrm{d}x'\right)^2 + \mathcal{O}(\varepsilon)\\ &= (F_1 - F_1(\tau_0))^2 - 2(F_1 - F_1(\tau_0))(G_1 - G_1(\tau_0)) + (G_1 - G_1(\tau_0))^2 + \mathcal{O}(\varepsilon),\end{aligned}$$

where for the last line we translate $\xi' = x' - t, \zeta' = x' + t$ to obtain

$$\int_0^x -\partial_{\xi'}(F_1(\tau_0 - \xi'))\,\mathrm{d}x' = \int_{-t}^{x-t}(F_1)_{\xi'}(-\xi', \tau_1)\,\mathrm{d}\xi' = \int_{-\tau_0}^{\xi}(F_1)_{\xi'}(-\xi', \tau_1)\,\mathrm{d}\xi' = F_1 - F_1(\tau_0),$$

$$\int_0^x (G_1)_{\zeta'}(x'+\tau_0,\tau_1)\,\mathrm{d}x' = \int_t^{x+t} (G_1)_{\zeta'}(\zeta',\tau_1)\,\mathrm{d}\zeta' = \int_{\tau_0}^{\zeta} (G_1)_{\zeta'}(\zeta',\tau_1)\,\mathrm{d}\zeta' = G_1 - G_1(\tau_0).$$

Note that previously we wrongly assumed that there is some strange term $F(-t)$. But F that we used was rather F_2, which is appropriate when $x \geqslant \tau_0$. In this case, $x < \tau_0$, so we need to use F_1, which provides the right viewpoint. Next, from Proposition 7, we also have

$$\begin{aligned}
(\partial_\xi + \partial_\zeta) & \int_0^\infty \partial_t \left(\eta \int_0^y \eta_t \,\mathrm{d}y'\right)\left[\frac{1}{x-y} + \frac{1}{x+y}\right]\mathrm{d}y \\
&= \partial_\xi \left(\int_{-\tau_0}^0 2F_1\partial_{\xi'}F_1 \frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + \int_0^\infty 2F_1\partial_{\xi'}F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'}G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty 2G_1\partial_{\zeta'}G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta'\right) \\
&+ \partial_\zeta \left(\int_{-\tau_0}^0 2F_1\partial_{\xi'}F_1 \frac{1}{\zeta+\xi'}\,\mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta' + \int_0^\infty 2F_1\partial_{\xi'}F_1\frac{1}{\zeta+\xi'}\,\mathrm{d}\xi' + \int_{2\tau_0}^\infty 2G_1\partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\right) \\
&+ \partial_\xi \left(A\int_{-\tau_0}^0 -\partial_\xi F_1 \frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta' + (A+B)\int_0^\infty -\partial_\xi F_2\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + \int_{2\tau_0}^\infty \partial_\zeta G_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta'\right) \\
&+ \partial_\zeta \left(A\int_{-\tau_0}^0 -\partial_\xi F_1 \frac{1}{\zeta+\xi'}\,\mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta' + (A+B)\int_0^\infty -\partial_\xi F_2\frac{1}{\zeta+\xi'}\,\mathrm{d}\xi' + \int_{2\tau_0}^\infty \partial_\zeta G_2\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\right).
\end{aligned}$$

Finally, note that

$$\begin{aligned}
\frac{1}{2}&(\partial_\xi^2 + 2\partial_\xi\partial_\zeta + \partial_\zeta^2)\left((F_1 - F_1(\tau_0))^2 - 2(F_1-F_1(\tau_0))(G_1-G_1(\tau_0)) + (G_1-G_1(\tau_0))^2\right) \\
&= \frac{1}{2}\partial_\xi^2\left((F_1-F_1(\tau_0))^2 - 2(F_1-F_1(\tau_0))(G_1-G_1(\tau_0))\right) + \frac{1}{2}\partial_\zeta^2\left(-2(F_1-F_1(\tau_0))(G_1-G_1(\tau_0)) + (G_1-G_1(\tau_0))^2\right) \\
&- 2\partial_\xi\partial_\zeta\left((F_1-F_1(\tau_0))(G_1-G_1(\tau_0))\right) \\
&= \partial_\xi\left((F_1-F_1(\tau_0))\partial_\xi F_1 - \partial_\xi F_1(G_1-G_1(\tau_0))\right) + \partial_\zeta\left(-(F_1-F_1(\tau_0))\partial_\zeta G_1 + (G_1-G_1(\tau_0))\partial_\zeta G_1\right) - 2\partial_\xi F_1\partial_\zeta G_1 \\
&= \partial_\xi\left((F_1-F_1(\tau_0)-G_1+G_1(\tau_0))\partial_\xi F_1\right) + \partial_\zeta\left((G_1-G_1(\tau_0)-F_1+F_1(\tau_0))\partial_\zeta G_1\right) - 2\partial_\xi F_1\partial_\zeta G_1 \\
&= \partial_\xi\left((F_1-G_1-A)\partial_\xi F_1\right) + \partial_\zeta\left((G_1-F_1+A)\partial_\zeta G_1\right) - 2\partial_\xi F_1\partial_\zeta G_1,
\end{aligned}$$

where we set $A = F_1(\tau_0) - G_1(\tau_0)$.

Substitution of terms into the RHS of (5) leads to:

$$\begin{aligned}
\frac{1}{3}\eta_{xxxx} &+ \frac{d}{dx}\frac{1}{\pi}\int_0^\infty \partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y + \frac{1}{2}\partial_x^2\left(\int_0^x \eta_t\,\mathrm{d}x'\right)^2 \\
&= \frac{1}{3}(\partial_\xi^4 F_1 + \partial_\zeta^4 G_1) + \partial_\xi\left((F_1-G_1-A)\partial_\xi F_1\right) + \partial_\zeta\left((G_1-F_1+A)\partial_\zeta G_1\right) - 2\partial_\xi F_1\partial_\zeta G_1
\end{aligned}$$

$$
\begin{aligned}
&+\partial_\xi \frac{1}{\pi}\left(\int_{-\tau_0}^{0} 2F_1\partial_{\xi'}F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + \int_0^\infty 2F_2\partial_{\xi'}F_2\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'}G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^{\infty} 2G_2\partial_{\zeta'}G_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta'\right)\\
&+\partial_\zeta \frac{1}{\pi}\left(\int_{-\tau_0}^{0} 2F_1\partial_{\xi'}F_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\xi' + \int_0^\infty 2F_2\partial_{\xi'}F_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^{\infty} 2G_2\partial_{\zeta'}G_2\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\right)\\
&+\partial_\xi\left(A\int_{-\tau_0}^{0} -\partial_\xi F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0}\partial_\zeta G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta' + (A+B)\int_0^\infty -\partial_\xi F_2\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + \int_{2\tau_0}^{\infty}\partial_\zeta G_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta'\right)\\
&+\partial_\zeta\left(A\int_{-\tau_0}^{0} -\partial_\xi F_1\frac{1}{\zeta+\xi'}\,\mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0}\partial_\zeta G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta' + (A+B)\int_0^\infty -\partial_\xi F_2\frac{1}{\zeta+\xi'}\,\mathrm{d}\xi' + \int_{2\tau_0}^{\infty}\partial_\zeta G_2\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\right) + \mathcal{O}(\varepsilon).
\end{aligned}
\tag{12}
$$

Combining (11) and (12), in $\mathcal{O}(\varepsilon^1)$ we have

$$
\begin{aligned}
-4\eta_{1\xi\zeta} &= 2\partial_\xi\partial_{\tau_1}F_1 - 2\partial_\zeta\partial_{\tau_1}G_1 + \frac{1}{3}(\partial_\xi^4 F_1 + \partial_\zeta^4 G_1) + \partial_\xi\left((F_1-G_1-A)\partial_\xi F_1\right) + \partial_\zeta\left((G_1-F_1+A)\partial_\zeta G_1\right) - 2\partial_\xi F_1\partial_\zeta G_1\\
&+\partial_\xi \frac{1}{\pi}\left(\int_{-\tau_0}^{0} 2F_1\partial_{\xi'}F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + \int_0^\infty 2F_2\partial_{\xi'}F_2\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'}G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^{\infty} 2G_2\partial_{\zeta'}G_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta'\right)\\
&+\partial_\zeta \frac{1}{\pi}\left(\int_{-\tau_0}^{0} 2F_1\partial_{\xi'}F_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\xi' + \int_0^\infty 2F_2\partial_{\xi'}F_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^{\infty} 2G_2\partial_{\zeta'}G_2\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\right)\\
&+\partial_\xi \frac{1}{\pi}\left(A\int_{-\tau_0}^{0} -\partial_\xi' F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + A\int_{\tau_0}^{2\tau_0}\partial_\zeta' G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta' + (A+B)\int_0^\infty -\partial_\xi' F_2\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + (A+B)\int_{2\tau_0}^{\infty}\partial_\zeta' G_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta'\right)\\
&+\partial_\zeta \frac{1}{\pi}\left(A\int_{-\tau_0}^{0} -\partial_\xi' F_1\frac{1}{\zeta+\xi'}\,\mathrm{d}\xi' + A\int_{\tau_0}^{2\tau_0}\partial_\zeta' G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta' + (A+B)\int_0^\infty -\partial_\xi' F_2\frac{1}{\zeta+\xi'}\,\mathrm{d}\xi' + (A+B)\int_{2\tau_0}^{\infty}\partial_\zeta' G_2\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\right)
\end{aligned}
\tag{13}
$$

By rearranging appropriately, (13) becomes

$$\begin{aligned}
-4\eta_{1\xi\zeta} &= \partial_\xi(2\partial_{\tau_1}F_1 + \frac{1}{3}\partial_\xi^3 F_1 + (F_1 - A)\partial_\xi F_1 + \frac{1}{\pi}\left(\int_{-\tau_0}^{0} 2F_1\partial_{\xi'}F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + \int_0^\infty 2F_2\partial_{\xi'}F_2\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + A\int_{-\tau_0}^{0} -\partial_{\xi'}F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + (A+B)\int_0^\infty -\partial_{\xi'}F_2\frac{1}{\xi-\xi'}\,\mathrm{d}\xi'\right))\\
&+ \partial_\zeta(-2\partial_{\tau_1}G_1 + \frac{1}{3}\partial_\zeta^3 G_1 + (G_1 + A)\partial_\zeta G_1 + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty 2G_2\partial_{\zeta'}G_2\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta' + A\int_{\tau_0}^{2\tau_0} \partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta' + (A+B)\int_{2\tau_0}^\infty \partial_{\zeta'}G_2\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\right))\\
&+ \partial_\xi(-G_1\partial_\xi F_1 + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'}G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty 2G_2\partial_{\zeta'}G_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta' + A\int_{\tau_0}^{2\tau_0} \partial_{\zeta'}G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta' + (A+B)\int_{2\tau_0}^\infty \partial_{\zeta'}G_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta'\right))\\
&+ \partial_\zeta(-F_1\partial_\zeta G_1 + \frac{1}{\pi}\left(\int_{-\tau_0}^{0} 2F_1\partial_{\xi'}F_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\xi' + \int_0^\infty 2F_2\partial_{\xi'}F_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\xi' + A\int_{-\tau_0}^{0} -\partial_{\xi'}F_1\frac{1}{\zeta+\xi'}\,\mathrm{d}\xi' + (A+B)\int_0^\infty -\partial_{\xi'}F_2\frac{1}{\zeta+\xi'}\,\mathrm{d}\xi'\right))\\
&- 2\partial_\xi F_1\partial_\zeta G_1.
\end{aligned} \tag{14}$$

Integration of (14) with respect to ζ yields

$$\begin{aligned}
-4\eta_{1\xi} &= \zeta\partial_\xi(2\partial_{\tau_1}F_1 + \frac{1}{3}\partial_\xi^3 F_1 + (F_1 - A)\partial_\xi F_1 + \frac{1}{\pi}\left(\int_{-\tau_0}^{0}(2F_1 - A)\partial_{\xi'}F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + \int_0^\infty(2F_2 - (A+B))\partial_{\xi'}F_2\frac{1}{\xi-\xi'}\,\mathrm{d}\xi'\right))\\
&+ (-2\partial_{\tau_1}G_1 + \frac{1}{3}\partial_\zeta^3 G_1 + (G_1 + A)\partial_\zeta G_1 + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0}(2G_1 + A)\partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty(2G_2 + (A+B))\partial_{\zeta'}G_2\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\right))\\
&+ \partial_\xi\int(G_1\partial_\xi F_1 + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0}(2G_1 + A)\partial_{\zeta'}G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty(2G_2 + (A+B))\partial_{\zeta'}G_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta'\right))\,\mathrm{d}\zeta\\
&+ (-F_1\partial_\zeta G_1 + \frac{1}{\pi}\left(\int_{-\tau_0}^{0}(2F_1 - A)\partial_{\xi'}F_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\xi' + \int_0^\infty(2F_2 - (A+B))\partial_{\xi'}F_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\xi'\right)) - 2\partial_\xi F_1 G_1.
\end{aligned} \tag{15}$$

and further integration with respect to ξ leads to

$$\begin{aligned}
-4\eta_1 &= \zeta(2\partial_{\tau_1}F_1 + \frac{1}{3}\partial_\xi^3 F_1 + (F_1 - A)\partial_\xi F_1 + \frac{1}{\pi}\left(\int_{-\tau_0}^{0}(2F_1 - A)\partial_{\xi'}F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + \int_0^\infty(2F_2 - (A+B))\partial_{\xi'}F_2\frac{1}{\xi-\xi'}\,\mathrm{d}\xi'\right))\\
&+ \xi(-2\partial_{\tau_1}G_1 + \frac{1}{3}\partial_\zeta^3 G_1 + (G_1 + A)\partial_\zeta G_1 + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0}(2G_1 + A)\partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty(2G_2 + (A+B))\partial_{\zeta'}G_2\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\right))\\
&+ \int(G_1\partial_\xi F_1 + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0}(2G_1 + A)\partial_{\zeta'}G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty(2G_2 + (A+B))\partial_{\zeta'}G_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta'\right))\,\mathrm{d}\zeta\\
&+ \int(-F_1\partial_\zeta G_1 + \frac{1}{\pi}\left(\int_{-\tau_0}^{0}(2F_1 - A)\partial_{\xi'}F_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\xi' + \int_0^\infty(2F_2 - (A+B))\partial_{\xi'}F_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\xi'\right))\,\mathrm{d}\xi - 2F_1 G_1.
\end{aligned}$$

Since η_1 must be bounded, we must have

$$2\partial_{\tau_1} F_1 + \frac{1}{3}\partial_\xi^3 F_1 + (F_1 - A)\partial_\xi F_1 + \frac{1}{\pi}\left(\int_{-\tau_0}^{0} (2F_1 - A)\partial_{\xi'} F_1 \frac{1}{\xi - \xi'}\,\mathrm{d}\xi' + \int_0^\infty (2F_2 - (A+B))\partial_{\xi'} F_2 \frac{1}{\xi - \xi'}\,\mathrm{d}\xi'\right) = 0, \tag{16}$$

$$-2\partial_{\tau_1} G_1 + \frac{1}{3}\partial_\zeta^3 G_1 + (G_1 + A)\partial_\zeta G_1 + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0} (2G_1 + A)\partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty (2G_2 + (A+B))\partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'}\,\mathrm{d}\zeta'\right) = 0. \tag{17}$$

In other words, we have obtained two KdV-like equations, (16) and (17), whose solutions F_1, G_1 describe behaviour of the surface elevation in the leading order, when $x < \tau_0$.

2.2.2 The case $x \geqslant \tau_0$

On the domain $x \geqslant \tau_0$, we use

$$\begin{aligned}
\eta &= \eta_0 + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2)\\
&= F_2(x - t, \varepsilon t, \ldots) + G_2(x + t, \varepsilon t, \ldots) + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2)\\
&= F_2(\xi, \tau_1, \ldots) + G_2(\zeta, \tau_1, \ldots) + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2)\\
&= F_2 + G_2 + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

For ease of writing, we suppress explicit dependence on variables, though the reader should bear in mind that function $F_2, (G_2)$ depend on $\xi, (\zeta), \tau_1, \tau_2$, etc. In addition, D'Alembert operator becomes

$$(\partial_t^2 - \partial_x^2) = \left(-4\partial_\xi\partial_\zeta + 2\varepsilon(\partial_\zeta\partial_{\tau_1} - \partial_\xi\partial_{\tau_1}) + \varepsilon^2\partial_{\tau_1}^2\right),$$

so that the LHS of (5) becomes

$$(\partial_t^2 - \partial_x^2)\eta = \varepsilon\left(-4\eta_{1\xi\zeta} - 2\partial_\xi\partial_{\tau_1} F_1 + 2\partial_\zeta\partial_{\tau_1} G_1\right) + \mathcal{O}(\varepsilon^2). \tag{18}$$

Now, we deal with the RHS of (5). By appropriate substitutions, the terms become:

$$\begin{aligned}
\frac{1}{3}\eta_{xxxx} &= \frac{1}{3}(\partial_\xi^4 F_2 + \partial_\zeta^4(G_2) + \mathcal{O}(\varepsilon));\\
\left(\int_0^x \eta_t\,\mathrm{d}x'\right)^2 &= \left(\int_0^{\tau_0} \eta_t\,\mathrm{d}x' + \int_{\tau_0}^x \eta_t\,\mathrm{d}x'\right)^2\\
&= \left(\int_0^{\tau_0} \eta_{0t}\,\mathrm{d}x' + \int_{\tau_0}^x \eta_{0t}\,\mathrm{d}x'\right)^2 + \mathcal{O}(\varepsilon)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_0^{\tau_0} (-\partial_{\xi'} + \partial_{\zeta'} + \varepsilon \partial_{\tau_1})(F_1 + G_1)\, \mathrm{d}x' + \int_{\tau_0}^{x} (-\partial_{\xi'} + \partial_{\zeta'} + \varepsilon \partial_{\tau_1})(F_2 + G_2)\, \mathrm{d}x' \right)^2 + \mathcal{O}(\varepsilon) \\
&= \left(\int_0^{\tau_0} (-\partial_{\xi'} + \partial_{\zeta'})(F_1 + G_1)\, \mathrm{d}x' + \int_{\tau_0}^{x} (-\partial_{\xi'} + \partial_{\zeta'})(F_2 + G_2)\, \mathrm{d}x' \right)^2 + \mathcal{O}(\varepsilon), \\
&= (-F_2 + G_2 + A + B)^2 + \mathcal{O}(\varepsilon),
\end{aligned}
$$

where for the last line we have simplified as follows:

$$
\begin{aligned}
\int_0^{\tau_0} -\partial_{\xi'}(F_1(\tau_0 - \xi'))\, \mathrm{d}x' &= -\int_{-\tau_0}^{0} \partial_{\xi'} F_1(-\xi', \tau_1)\, \mathrm{d}\xi' = -F_1(0) + F_1(\tau_0), \\
\int_0^{\tau_0} \partial_{\zeta'} G_1(x' + \tau_0, \tau_1)\, \mathrm{d}x' &= \int_{\tau_0}^{2\tau_0} \partial_{\zeta'} G_1(\zeta', \tau_1)\, \mathrm{d}\zeta' = G_1(2\tau_0) - G_1(\tau_0) \\
\int_{\tau_0}^{x} -\partial_{\xi'}(F_2(\tau_0 - \xi'))\, \mathrm{d}x' &= -\int_0^{x-\tau_0} \partial_{\xi'} F_2(\xi', \tau_1)\, \mathrm{d}\xi' = -F_2 + F_2(0), \\
\int_{\tau_0}^{x} \partial_{\zeta'} G_2(x' + \tau_0, \tau_1)\, \mathrm{d}x' &= \int_{2\tau_0}^{x+\tau_0} \partial_{\zeta'} G_2(\zeta', \tau_1)\, \mathrm{d}\zeta' = G_2 - G_2(2\tau_0).
\end{aligned}
$$

Addition of terms yields

$$
\begin{aligned}
\int_0^{\tau_0} (-\partial_{\xi'} + \partial_{\zeta'})(F_1 + G_1)\, \mathrm{d}x' + \int_{\tau_0}^{x} (-\partial_{\xi'} + \partial_{\zeta'})(F_2 + G_2)\, \mathrm{d}x' &= -F_1(0) + F_1(\tau_0) + G_1(2\tau_0) - G_1(\tau_0) - F_2 + F_2(0) + G_2 - G_2(2\tau_0) \\
&= -F_2 + G_2 - F_1(0) + F_2(0) + G_1(2\tau_0) - G_2(2\tau_0) + F_1(\tau_0) - G_1(\tau_0) \\
&= -F_2 + G_2 + A + B,
\end{aligned}
$$

where we write

$$
F_1(\tau_0) - G_1(\tau_0) = A \qquad - F_1(0) + F_2(0) + G_1(2\tau_0) - G_2(2\tau_0) = B.
$$

Next, by Proposition 7, we also have

$$
\begin{aligned}
&(\partial_\xi + \partial_\zeta) \int_0^\infty \partial_t \left(\eta \int_0^y \eta_t \, \mathrm{d}y' \right) \left[\frac{1}{x - y} + \frac{1}{x + y} \right] \mathrm{d}y \\
&\quad = \partial_\xi \left(\int_{-\tau_0}^{0} 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'}\, \mathrm{d}\xi' + \int_0^\infty 2F_2 \partial_{\xi'} F_2 \frac{1}{\xi - \xi'}\, \mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'}\, \mathrm{d}\zeta' + \int_{2\tau_0}^{\infty} 2G_2 \partial_{\zeta'} G_2 \frac{1}{\xi + \zeta'}\, \mathrm{d}\zeta' \right) \\
&\quad + \partial_\zeta \left(\int_{-\tau_0}^{0} 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi + \zeta'}\, \mathrm{d}\xi' + \int_0^\infty 2F_2 \partial_{\xi'} F_2 \frac{1}{\xi + \zeta'}\, \mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'}\, \mathrm{d}\zeta' + \int_{2\tau_0}^{\infty} 2G_2 \partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'}\, \mathrm{d}\zeta' \right)
\end{aligned}
$$

$$+ \partial_\xi \left(A \int_{-\tau_0}^{0} -\partial_{\xi'} F_1 \frac{1}{\xi - \xi'} \,\mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} \,\mathrm{d}\zeta' + (A+B) \int_0^\infty -\partial_{\xi'} F_2 \frac{1}{\xi - \xi'} \,\mathrm{d}\xi' + \int_{2\tau_0}^\infty \partial_{\zeta'} G_2 \frac{1}{\xi + \zeta'} \,\mathrm{d}\zeta' \right)$$
$$+ \partial_\zeta \left(A \int_{-\tau_0}^{0} -\partial_{\xi'} F_1 \frac{1}{\zeta + \xi'} \,\mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} \,\mathrm{d}\zeta' + (A+B) \int_0^\infty -\partial_{\xi'} F_2 \frac{1}{\zeta + \xi'} \,\mathrm{d}\xi' + \int_{2\tau_0}^\infty \partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'} \,\mathrm{d}\zeta' \right) + \mathcal{O}(\varepsilon).$$

Finally, note that

$$\begin{aligned} \frac{1}{2}(\partial_\xi^2 + 2\partial_\xi\partial_\zeta + \partial_\zeta^2)(-F_2 + G_2 + A + B)^2 &= \frac{1}{2}\partial_\xi^2(-F_2 + G_2 + A + B)^2 + \frac{1}{2}\partial_\zeta^2(-F_2 + G_2 + A + B)^2 + \partial_\xi\partial_\zeta(-F_2 + G_2 + A + B)^2 \\ &= \partial_\xi((-F_2 + G_2 + A + B)\partial_\xi(-F_2)) + \partial_\zeta((-F_2 + G_2 + A + B)\partial_\zeta G_2) - 2\partial_\xi F_2 \partial_\zeta G_2. \end{aligned}$$

Substitution of terms into the RHS of (5) leads to:

$$\begin{aligned} &\frac{1}{3}\eta_{xxxx} + \frac{d}{dx}\frac{1}{\pi}\int_0^\infty \partial_t \left(\eta \int_0^y \eta_t \,\mathrm{d}y' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y + \frac{1}{2}\partial_x^2 \left(\int_0^x \eta_t \,\mathrm{d}x' \right)^2 \\ &= \frac{1}{3}(\partial_\xi^4 F_2 + \partial_\zeta^4 G_2) + \partial_\xi((-F_2 + G_2 + A + B)\partial_\xi(-F_2)) + \partial_\zeta((-F_2 + G_2 + A + B)\partial_\zeta G_2) - 2\partial_\xi F_2 \partial_\zeta G_2 \\ &+ \partial_\xi \frac{1}{\pi} \left(\int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'} \,\mathrm{d}\xi' + \int_0^\infty 2F_2 \partial_{\xi'} F_2 \frac{1}{\xi - \xi'} \,\mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} \,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty 2G_2 \partial_{\zeta'} G_2 \frac{1}{\xi + \zeta'} \,\mathrm{d}\zeta' \right) \\ &+ \partial_\zeta \frac{1}{\pi} \left(\int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi + \zeta'} \,\mathrm{d}\xi' + \int_0^\infty 2F_2 \partial_{\xi'} F_2 \frac{1}{\xi + \zeta'} \,\mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} \,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty 2G_2 \partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'} \,\mathrm{d}\zeta' \right) \\ &+ \partial_\xi \frac{1}{\pi} \left(A \int_{-\tau_0}^0 -\partial_{\xi'} F_1 \frac{1}{\xi - \xi'} \,\mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} \,\mathrm{d}\zeta' + (A+B) \int_0^\infty -\partial_{\xi'} F_2 \frac{1}{\xi - \xi'} \,\mathrm{d}\xi' + \int_{2\tau_0}^\infty \partial_{\zeta'} G_2 \frac{1}{\xi + \zeta'} \,\mathrm{d}\zeta' \right) \\ &+ \partial_\zeta \frac{1}{\pi} \left(A \int_{-\tau_0}^0 -\partial_{\xi'} F_1 \frac{1}{\zeta + \xi'} \,\mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} \,\mathrm{d}\zeta' + (A+B) \int_0^\infty -\partial_{\xi'} F_2 \frac{1}{\zeta + \xi'} \,\mathrm{d}\xi' + \int_{2\tau_0}^\infty \partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'} \,\mathrm{d}\zeta' \right) + \mathcal{O}(\varepsilon) \end{aligned} \tag{19}$$

Combining (18) and (19), in $\mathcal{O}(\varepsilon^1)$ we have

$$
\begin{aligned}
-4\eta_{1\xi\zeta} &= 2\partial_\xi\partial_{\tau_1}F_2 - 2\partial_\zeta\partial_{\tau_1}G_2 + \frac{1}{3}(\partial_\xi^4 F_2 + \partial_\zeta^4 G_2) + \partial_\xi((-F_2+G_2+A+B)\partial_\xi(-F_2)) + \partial_\zeta((-F_2+G_2+A+B)\partial_\zeta G_2) - 2\partial_\xi F_2\partial_\zeta G_2 \\
&+ \partial_\xi\frac{1}{\pi}\left(\int_{-\tau_0}^{0} 2F_1\partial_{\xi'}F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + \int_0^\infty 2F_2\partial_{\xi'}F_2\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'}G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^{\infty} 2G_2\partial_{\zeta'}G_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta'\right) \\
&+ \partial_\zeta\frac{1}{\pi}\left(\int_{-\tau_0}^{0} 2F_1\partial_{\xi'}F_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\xi' + \int_0^\infty 2F_2\partial_{\xi'}F_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^{\infty} 2G_2\partial_{\zeta'}G_2\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\right) \\
&+ \partial_\xi\frac{1}{\pi}\left(A\int_{-\tau_0}^{0} -\partial_{\xi'}F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + A\int_{\tau_0}^{2\tau_0} \partial_{\zeta'}G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta' + (A+B)\int_0^\infty -\partial_{\xi'}F_2\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + (A+B)\int_{2\tau_0}^{\infty}\partial_{\zeta'}G_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta'\right) \\
&+ \partial_\zeta\frac{1}{\pi}\left(A\int_{-\tau_0}^{0} -\partial_{\xi'}F_1\frac{1}{\zeta+\xi'}\,\mathrm{d}\xi' + A\int_{\tau_0}^{2\tau_0} \partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta' + (A+B)\int_0^\infty -\partial_{\xi'}F_2\frac{1}{\zeta+\xi'}\,\mathrm{d}\xi' + (A+B)\int_{2\tau_0}^{\infty}\partial_{\zeta'}G_2\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\right)
\end{aligned}
$$

By rearranging appropriately, (20) becomes

$$
\begin{aligned}
-4\eta_{1\xi\zeta} &= \partial_\xi(2\partial_{\tau_1}F_2 + \frac{1}{3}\partial_\xi^3F_2 + F_2\partial_\xi F_2 + \frac{1}{\pi}\left(\int_{-\tau_0}^{0}(2F_1-A)\partial_{\xi'}F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + \int_0^\infty(2F_2-(A+B))\partial_{\xi'}F_2\frac{1}{\xi-\xi'}\,\mathrm{d}\xi'\right)) \\
&+ \partial_\zeta(-2\partial_{\tau_1}G_2 + \frac{1}{3}\partial_\zeta^3G_2 + G_2\partial_\zeta G_2 + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0}(2G_1+A)\partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty(2G_2+A+B)\partial_{\zeta'}G_2\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\right)) \\
&+ \partial_\xi(-G_2\partial_\xi F_2 + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0}(2G_1+A)\partial_{\zeta'}G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty(2G_2+(A+B))\partial_{\zeta'}G_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta'\right)) \\
&+ \partial_\zeta(-F_2\partial_\zeta G_2 + \frac{1}{\pi}\left(\int_{-\tau_0}^{0}(2F_1-A)\partial_{\xi'}F_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\xi' + \int_0^\infty(2F_2-(A+B))\partial_{\xi'}F_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\xi'\right)) - 2\partial_\xi F_2\partial_\zeta G_2.
\end{aligned}
\tag{20}
$$

Integration of (14) with respect to ζ yields

$$
\begin{aligned}
-4\eta_{1\xi} &= \zeta\partial_\xi(2\partial_{\tau_1}F_2 + \frac{1}{3}\partial_\xi^3F_2 + (F_2-A-B)\partial_\xi F_2 + \frac{1}{\pi}\left(\int_{-\tau_0}^{0}(2F_1-A)\partial_{\xi'}F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + \int_0^\infty(2F_2-(A+B))\partial_{\xi'}F_2\frac{1}{\xi-\xi'}\,\mathrm{d}\xi'\right)) \\
&+ (-2\partial_{\tau_1}G_2 + \frac{1}{3}\partial_\zeta^3G_2 + (G_2+A+B)\partial_\zeta G_2 + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0}(2G_1+A)\partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty(2G_2+A+B)\partial_{\zeta'}G_2\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\right)) \\
&+ \partial_\xi\int(-G_2\partial_\xi F_2 + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0}(2G_1+A)\partial_{\zeta'}G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty(2G_2+(A+B))\partial_{\zeta'}G_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta'\right))\,\mathrm{d}\zeta \\
&+ (-F_2\partial_\zeta G_2 + \frac{1}{\pi}\left(\int_{-\tau_0}^{0}(2F_1-A)\partial_{\xi'}F_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\xi' + \int_0^\infty(2F_2-(A+B))\partial_{\xi'}F_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\xi'\right)) - 2\partial_\xi F_2G_2.
\end{aligned}
\tag{21}
$$

and further integration with respect to ξ leads to

$$\begin{aligned}
-4\eta_1 &= \zeta(2\partial_{\tau_1}F_2 + \frac{1}{3}\partial_\xi^3 F_2 + (F_2 - A - B)\partial_\xi F_2 + \frac{1}{\pi}\left(\int_{-\tau_0}^{0}(2F_1 - A)\partial_{\xi'}F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + \int_0^\infty (2F_2 - (A+B))\partial_{\xi'}F_2\frac{1}{\xi-\xi'}\,\mathrm{d}\xi'\right)) \\
&+ \xi(-2\partial_{\tau_1}G_2 + \frac{1}{3}\partial_\zeta^3 G_2 + (G_2 + A + B)\partial_\zeta G_2 + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0}(2G_1 + A)\partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty (2G_2 + A + B)\partial_{\zeta'}G_2\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\right)) \\
&+ \int(-G_2\partial_\xi F_2 + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0}(2G_1 + A)\partial_{\zeta'}G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty (2G_2 + (A+B))\partial_{\zeta'}G_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta'\right))\,\mathrm{d}\zeta \\
&+ \int(-F_2\partial_\zeta G_2 + \frac{1}{\pi}\left(\int_{-\tau_0}^{0}(2F_1 - A)\partial_{\xi'}F_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\xi' + \int_0^\infty (2F_2 - (A+B))\partial_{\xi'}F_2\frac{1}{\xi+\zeta'}\,\mathrm{d}\xi'\right))\,\mathrm{d}\xi - 2F_2G_2.
\end{aligned}$$

Since η_1 must be bounded, we must have

$$2\partial_{\tau_1}F_2 + \frac{1}{3}\partial_\xi^3 F_2 + (F_2 - A - B)\partial_\xi F_2 + \frac{1}{\pi}\left(\int_{-\tau_0}^{0}(2F_1 - A)\partial_{\xi'}F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' + \int_0^\infty (2F_2 - (A+B))\partial_{\xi'}F_2\frac{1}{\xi-\xi'}\,\mathrm{d}\xi'\right) = 0, \tag{22}$$

$$-2\partial_{\tau_1}G_2 + \frac{1}{3}\partial_\zeta^3 G_2 + (G_2 + A + B)\partial_\zeta G_2 + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0}(2G_1 + A)\partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty (2G_2 + A + B)\partial_{\zeta'}G_2\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\right) = 0. \tag{23}$$

In other words, we have obtained two KdV-like equations, (22) and (23), whose solutions F_2, G_2 describe behaviour of the surface elevation in the leading order, when $x \geqslant \tau_0$.

2.2.3 Conclusion

In summary, we have started out with a general solution of the wave equation on the right half-line:

$$\eta_0(x,\tau_0,\tau_1,\dots) = \begin{cases} F_2(x-\tau_0,\tau_1,\dots) + G_2(x+\tau_0,\tau_1,\dots) & x \geqslant \tau_0 \\ F_1(\tau_0 - x,\tau_1,\dots) + G_1(x+\tau_0,\tau_1,\dots) & x < \tau_0 \end{cases},$$

and obtained a system of 4 equations in four unknowns F_1, F_2, G_1, G_2 :

$$\begin{aligned}
&2\partial_{\tau_1} F_1 + \frac{1}{3}\partial_\xi^3 F_1 + (F_1 - A)\partial_\xi F_1 + \frac{1}{\pi}\left(\int_{-\tau_0}^{0} (2F_1 - A)\partial_{\xi'} F_1 \frac{1}{\xi - \xi'}\,\mathrm{d}\xi' + \int_0^\infty (2F_2 - (A+B))\partial_{\xi'} F_2 \frac{1}{\xi - \xi'}\,\mathrm{d}\xi'\right) = 0, \qquad x < \tau_0;\\
&2\partial_{\tau_1} F_2 + \frac{1}{3}\partial_\xi^3 F_2 + (F_2 - A - B)\partial_\xi F_2 + \frac{1}{\pi}\left(\int_{-\tau_0}^{0} (2F_1 - A)\partial_{\xi'} F_1 \frac{1}{\xi - \xi'}\,\mathrm{d}\xi' + \int_0^\infty (2F_2 - (A+B))\partial_{\xi'} F_2 \frac{1}{\xi - \xi'}\,\mathrm{d}\xi'\right) = 0, \qquad x \geqslant \tau_0;\\
&-2\partial_{\tau_1} G_1 + \frac{1}{3}\partial_\zeta^3 G_1 + (G_1 + A)\partial_\zeta G_1 + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0} (2G_1 + A)\partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty (2G_2 + (A+B))\partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'}\,\mathrm{d}\zeta'\right) = 0, \qquad x < \tau_0;\\
&-2\partial_{\tau_1} G_2 + \frac{1}{3}\partial_\zeta^3 G_2 + (G_2 + A + B)\partial_\zeta G_2 + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0} (2G_1 + A)\partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty (2G_2 + A + B)\partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'}\,\mathrm{d}\zeta'\right) = 0, \qquad x \geqslant \tau_0,
\end{aligned} \tag{24}$$

where

$$A = F_1(\tau_0) - G_1(\tau_0), \qquad B = F_2(0) - F_1(0) + G_1(2\tau_0) - G_2(2\tau_0).$$

Remark 5. Observe that one can decouple this system by further forcing

$$F_1(0) = F_2(0), \qquad F_2(0) - F_1(0) = G_2(2\tau_0) - G_1(2\tau_0)$$

in which case, $A, B = 0$, and we obtain 2 systems of 2 equations: one in F_1, F_2 :

$$\begin{aligned}
&2\partial_{\tau_1} F_1 + \frac{1}{3}\partial_\xi^3 F_1 + F_1\partial_\xi F_1 + \frac{1}{\pi}\left(\int_{-\tau_0}^{0} 2F_1\partial_{\xi'} F_1 \frac{1}{\xi - \xi'}\,\mathrm{d}\xi' + \int_0^\infty 2F_2\partial_{\xi'} F_2 \frac{1}{\xi - \xi'}\,\mathrm{d}\xi'\right) = 0, \qquad x < \tau_0;\\
&2\partial_{\tau_1} F_2 + \frac{1}{3}\partial_\xi^3 F_2 + F_2\partial_\xi F_2 + \frac{1}{\pi}\left(\int_{-\tau_0}^{0} 2F_1\partial_{\xi'} F_1 \frac{1}{\xi - \xi'}\,\mathrm{d}\xi' + \int_0^\infty 2F_2\partial_{\xi'} F_2 \frac{1}{\xi - \xi'}\,\mathrm{d}\xi'\right) = 0, \qquad x \geqslant \tau_0;
\end{aligned} \tag{25}$$

and another one in G_1, G_2 :

$$\begin{aligned}
&-2\partial_{\tau_1} G_1 + \frac{1}{3}\partial_\zeta^3 G_1 + G_1\partial_\zeta G_1 + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty 2G_2\partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'}\,\mathrm{d}\zeta'\right) = 0, \qquad x < \tau_0;\\
&-2\partial_{\tau_1} G_2 + \frac{1}{3}\partial_\zeta^3 G_2 + G_2\partial_\zeta G_2 + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty 2G_2\partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'}\,\mathrm{d}\zeta'\right) = 0, \qquad x \geqslant \tau_0.
\end{aligned} \tag{26}$$

Note that all equations look like KdV except for the integral terms.

2.2.4 Analysis of the system

In this section, we analyse the system (24). First, we switch back to x, t coordinates and for simplicity consider the case $x < \tau_0$.Recalling the changes of variables, we have

$$\begin{aligned} x-\tau_0 &= \xi, \implies \partial_\xi = \partial_x \\ x+\tau_0 &= \zeta, \implies \partial_\zeta = \partial_x \\ y-\tau_0 &= \xi', \implies \mathrm{d}\xi' = \mathrm{d}y \\ y+\tau_0 &= \zeta', \implies \mathrm{d}\zeta' = \mathrm{d}y \end{aligned}$$

Also, we have that $\partial_t = -\partial_\xi + \partial_\zeta + \varepsilon\partial_{\tau_1}$, which implies $\partial_{\tau_1} = \frac{1}{\varepsilon}(\partial_t + \partial_\xi - \partial_\zeta)$.

Addition

Consider

$$2\partial_{\tau_1}F_1 + \frac{1}{3}\partial_x^3 F_1 + F_1\partial_x F_1 - A\partial_\xi F_1 + \frac{1}{\pi}\left(\int_0^{\tau_0} 2F_1\partial_{x'}F_1 - A\partial_{\xi'}F_1\frac{1}{x-x'}\,\mathrm{d}x + \int_{\tau_0}^{\infty}(2F_2\partial_{x'}F_2 - (A+B)\partial_{x'}F_2)\frac{1}{x-x'}\,\mathrm{d}x'\right) = 0, \qquad x<\tau_0; \tag{27}$$

$$-2\partial_{\tau_1}G_1 + \frac{1}{3}\partial_x^3 G_1 + G_1\partial_x G_1 + A\partial_\zeta G_1 + \frac{1}{\pi}\left(\int_0^{\tau_0}(2G_1\partial_{x'}G_1 + A\partial_{\zeta'}G_1)\frac{1}{x-x'}\,\mathrm{d}x' + \int_{\tau_0}^{\infty}(2G_2\partial_{x'}G_2 + (A+B)\partial_{\zeta'}G_2)\frac{1}{x-x'}\,\mathrm{d}x'\right) = 0, \qquad x<\tau_0; \tag{28}$$

Addition of (27) and (28) yields:

$$\begin{aligned} 2\partial_{\tau_1}(F_1 - G_1) + \frac{1}{3}\partial_x^3(F_1+G_1) + F_1\partial_x F_1 + G_1\partial_x G_1 + A(-\partial_\xi F_1 + \partial_\zeta G_1) + \frac{1}{\pi}\Bigg(\int_0^{\tau_0}(2F_1\partial_{x'}F_1 + 2G_1\partial_{x'}G_1 - A\partial_{\xi'}F_1 + A\partial_{\zeta'}G_1)\frac{1}{x-x'}\,\mathrm{d}x \\ + \int_{\tau_0}^{\infty}(2F_2\partial_{x'}F_2 + 2G_2\partial_{x'}G_2 - (A+B)\partial_{x'}F_2 + (A+B)\partial_{\zeta'}G_2)\frac{1}{x-x'}\,\mathrm{d}x'\Bigg)) = 0. \end{aligned}$$

Collecting derivatives, we obtain:

$$\begin{aligned} 2\partial_{\tau_1}(F_1 - G_1) + \frac{1}{3}\partial_x^3\eta_0 + \frac{1}{2}\partial_x(F_1^2 + G_1^2) + A(-\partial_\xi + \partial_\zeta)(F_1+G_1) + \frac{1}{\pi}\Bigg(\int_0^{\tau_0}\partial_{x'}(F_1^2+G_1^2) + A(-\partial_{\xi'} + \partial_{\zeta'})(F_1+G_1)\frac{1}{x-x'}\,\mathrm{d}x \\ + \int_{\tau_0}^{\infty}(\partial_{x'}(F_2^2+G_2^2) + (A+B)(-\partial_{x'} + \partial_{\zeta'})(F_2+G_2)\frac{1}{x-x'}\,\mathrm{d}x'\Bigg)) = 0. \end{aligned}$$

Using that $a^2+b^2 = (a+b)^2 - 2ab, F_1 + G_1 = \eta_0$ and $\partial_{\tau_0} = -\partial_\xi + \partial_\zeta$, we rewrite

$$2\partial_{\tau_1}(F_1 - G_1) + \frac{1}{3}\partial_x^3\eta_0 + \frac{1}{2}\partial_x((F_1+G_1)^2 - 2F_1G_1) + A\partial_{\tau_0}\eta_0 + \frac{1}{\pi}\Bigg(\int_0^{\tau_0}\partial_{x'}((F_1+G_1)^2 - 2F_1G_1) + A\partial_{\tau_0}\eta_0\frac{1}{x-x'}\,\mathrm{d}x$$

$$+\int_{\tau_0}^{\infty}(\partial_{x'}((F_2+G_2)^2-2F_2G_2)+(A+B)\partial_{\tau_0}\eta_0)\frac{1}{x-x'}\,\mathrm{d}x'\Big)\Big)=0,$$

so that

$$\begin{aligned}2\partial_{\tau_1}(F_1-G_1)+\frac{1}{3}\partial_x^3\eta_0+\frac{1}{2}\partial_x(\eta_0^2-2F_1G_1)+A\partial_{\tau_0}\eta_0+\frac{1}{\pi}\Big(\int_0^{\tau_0}\partial_{x'}(\eta_0^2-2F_1G_1)+A\partial_{\tau_0}\eta_0\frac{1}{x-x'}\,\mathrm{d}x\\+\int_{\tau_0}^{\infty}(\partial_{x'}(\eta_0^2-2F_2G_2)+(A+B)\partial_{\tau_0}\eta_0)\frac{1}{x-x'}\,\mathrm{d}x'\Big)\Big)=0.\end{aligned}\tag{29}$$

Finally, note that

$$\partial_{\tau_0}\eta_0=-\partial_\xi F_i+\partial_\zeta G_i \implies \partial_{\tau_0}\int\eta_0\,\mathrm{d}x=\int\partial_{\tau_0}\eta_0\,\mathrm{d}x=\int-\partial_\xi F_i+\partial_\zeta G_i\,\mathrm{d}x=-F_i+G_i.$$

Therefore, we rewrite (29) to have

$$-2\partial_{\tau_1}\partial_{\tau_0}\int\eta_0\,\mathrm{d}x+\frac{1}{3}\partial_x^3\eta_0+\frac{1}{2}\partial_x(\eta_0^2-2F_1G_1)+A\partial_{\tau_0}\eta_0+\frac{1}{\pi}\Big(\int_0^{\tau_0}\partial_{x'}(\eta_0^2-2F_1G_1)+A\partial_{\tau_0}\eta_0\frac{1}{x-x'}\,\mathrm{d}x+\int_{\tau_0}^{\infty}(\partial_{x'}(\eta_0^2-2F_2G_2)+(A+B)\partial_{\tau_0}\eta_0)\frac{1}{x-x'}\,\mathrm{d}x'\Big)\Big)=0.$$

Subtraction

Now, switch coordinates in (27) and (28) again to so that we can work with:

$$2\partial_{\tau_1}F_1+\frac{1}{3}\partial_\xi^3F_1+F_1\partial_\xi F_1-A\partial_xF_1+\frac{1}{\pi}\Big(\int_0^{\tau_0}2F_1\partial_{\xi'}F_1-A\partial_{x'}F_1\frac{1}{x-x'}\,\mathrm{d}x+\int_{\tau_0}^{\infty}(2F_2\partial_{\xi'}F_2-(A+B)\partial_{\xi'}F_2)\frac{1}{x-x'}\,\mathrm{d}x'\Big)=0,\qquad x<\tau_0;\tag{30}$$

$$-2\partial_{\tau_1}G_1+\frac{1}{3}\partial_\zeta^3G_1+G_1\partial_\zeta G_1+A\partial_xG_1+\frac{1}{\pi}\Big(\int_0^{\tau_0}(2G_1\partial_{\zeta'}G_1+A\partial_{x'}G_1)\frac{1}{x-x'}\,\mathrm{d}x'+\int_{\tau_0}^{\infty}(2G_2\partial_{\zeta'}G_2+(A+B)\partial_{x'}G_2)\frac{1}{x-x'}\,\mathrm{d}x'\Big)=0,\qquad x<\tau_0;\tag{31}$$

Subtraction of (30) from (31) gives:

$$\begin{aligned}0=-2\partial_{\tau_1}(G_1+F_1)+\frac{1}{3}(\partial_\zeta^3G_1-\partial_\xi^3F_1)+G_1\partial_\zeta G_1+A\partial_xG_1-F_1\partial_\xi F_1+A\partial_xF_1+\frac{1}{\pi}\Big(\int_0^{\tau_0}(2G_1\partial_{\zeta'}G_1+A\partial_{x'}G_1-2F_1\partial_{\xi'}F_1+A\partial_{x'}F_1)\frac{1}{x-x'}\,\mathrm{d}x'\\+\int_{\tau_0}^{\infty}(2G_2\partial_{\zeta'}G_2+(A+B)\partial_{x'}G_2-2F_2\partial_{\xi'}F_2+(A+B)\partial_{x'}F_2)\frac{1}{x-x'}\,\mathrm{d}x'\Big).\end{aligned}$$

Collecting derivatives and reversing product rule yields

$$\begin{aligned}0=-2\partial_{\tau_1}(G_1+F_1)+\frac{1}{3}(\partial_\zeta^3-\partial_\xi^3)(G_1+F_1)+\frac{1}{2}(\partial_\zeta G_1^2-\partial_\xi F_1^2)+A\partial_x(F_1+G_1)+\frac{1}{\pi}\Big(\int_0^{\tau_0}(\partial_{\zeta'}G_1^2-\partial_{\xi'}F_1^2+A\partial_{x'}(G_1+F_1))\frac{1}{x-x'}\,\mathrm{d}x'\\+\int_{\tau_0}^{\infty}(\partial_{\zeta'}G_2^2-\partial_{\xi'}F_2^2+(A+B)\partial_{x'}(G_2+F_2))\frac{1}{x-x'}\,\mathrm{d}x'\Big).\end{aligned}\tag{32}$$

Finally, observe that $\partial_{\zeta'} + \partial_{\xi'} = \partial_{x'}$ and $\partial_{\zeta'} - \partial_{\xi'} = \partial_{\tau_0}$, so that

$$\begin{aligned}\partial_{\zeta'} G_i^2 - \partial_{\xi'} F_i^2 = (\partial_{\zeta'} + \partial_{\xi'})(G_i^2 - F_i^2) = (\partial_{\zeta'} + \partial_{\xi'})((G_i - F_i)(G_i + F_i)) &= \partial_{x'}\eta_0(G_i - F_i) + \eta_0(\partial_{\zeta'} + \partial_{\xi'})(G_i - F_i) \\ &= \partial_{x'}\eta_0(G_i - F_i) + \eta_0(\partial_{\zeta'} - \partial_{\xi'})(G_i + F_i) \\ &= \partial_{x'}\eta_0(G_i - F_i) + \eta_0\partial_{\tau_0}\eta_0,\end{aligned}$$

and

$$\begin{aligned}(\partial_\zeta^3 - \partial_\xi^3)\eta_0 = (\partial_\zeta - \partial_\xi)(\partial_\zeta^2 + \partial_\zeta\partial_\xi + \partial_\xi^2)\eta_0 &= (\partial_\zeta - \partial_\xi)(\partial_\zeta^2 + 2\partial_\zeta\partial_\xi + \partial_\xi^2)\eta_0 - (\partial_\zeta - \partial_\xi)\partial_\zeta\partial_\xi\eta_0 \\ &= (\partial_{\tau_0})(\partial_\zeta + \partial_\xi)^2\eta_0 - (\partial_\zeta - \partial_\xi)\partial_\zeta\partial_\xi(F + G) \\ &= (\partial_{\tau_0})(\partial_x)^2\eta_0 = \partial_{\tau_0 xx}\eta_0.\end{aligned}$$

With this in mind, we rewrite (32) to obtain

$$\begin{aligned}0 = -2\partial_{\tau_1}\eta_0 + \frac{1}{3}\partial_{\tau_0 xx}\eta_0 + \frac{1}{2}(\partial_x\eta_0(G_1 - F_1)) + \frac{1}{2}\eta_0\partial_{\tau_0}\eta_0 + A\partial_x\eta_0 + \frac{1}{\pi}\bigg(\int_0^{\tau_0} (\partial_{x'}\eta_0(G_1 - F_1) + \eta_0\partial_{\tau_0}\eta_0 + A\partial_{x'}\eta_0)\frac{1}{x - x'}\,\mathrm{d}x' \\ + \int_{\tau_0}^{\infty} (\partial_{x'}\eta_0(G_2 - F_2) + \eta_0\partial_{\tau_0}\eta_0 + (A + B)\partial_{x'}\eta_0)\frac{1}{x - x'}\,\mathrm{d}x'\bigg)\end{aligned}$$

Recall once more that

$$\partial_{\tau_0}\int \eta_0\,\mathrm{d}x = -F_i + G_i.$$

Substitution yields:

$$\begin{aligned}0 &= -2\partial_{\tau_1}\eta_0 + \frac{1}{3}\partial_{\tau_0 xx}\eta_0 + \frac{1}{2}(\partial_x\eta_0\partial_{\tau_0}\int \eta_0\,\mathrm{d}x) + \frac{1}{2}\eta_0\partial_{\tau_0}\eta_0 + A\partial_x\eta_0 + \frac{1}{\pi}\bigg(\int_0^{\tau_0} (\partial_{x'}\eta_0\partial_{\tau_0}\int \eta_0\,\mathrm{d}x + \eta_0\partial_{\tau_0}\eta_0 + A\partial_{x'}\eta_0)\frac{1}{x - x'}\,\mathrm{d}x' \\ &\qquad + \int_{\tau_0}^{\infty} (\partial_{x'}\eta_0\partial_{\tau_0}\int \eta_0\,\mathrm{d}x + \eta_0\partial_{\tau_0}\eta_0 + (A + B)\partial_{x'}\eta_0)\frac{1}{x - x'}\,\mathrm{d}x'\bigg) \\ &= -2\partial_{\tau_1}\eta_0 + \frac{1}{3}\partial_{\tau_0 xx}\eta_0 + \frac{1}{2}\partial_x\eta_0(A + \partial_{\tau_0}\int \eta_0\,\mathrm{d}x) + \frac{1}{2}\eta_0\partial_{\tau_0}\eta_0 + \frac{1}{\pi}\bigg(\int_0^{\tau_0} (\partial_{x'}\eta_0(A + \partial_{\tau_0}\int \eta_0\,\mathrm{d}x) + \eta_0\partial_{\tau_0}\eta_0)\frac{1}{x - x'}\,\mathrm{d}x' \\ &\qquad + \int_{\tau_0}^{\infty} (\partial_{x'}\eta_0(A + B + \partial_{\tau_0}\int \eta_0\,\mathrm{d}x) + \eta_0\partial_{\tau_0}\eta_0)\frac{1}{x - x'}\,\mathrm{d}x'\bigg)\end{aligned}$$

Summary

In summary, on $x < \tau_0$, we obtain two equations in x coordinates: addition yields

$$-2\partial_{\tau_1}\partial_{\tau_0}\int \eta_0\,\mathrm{d}x + \frac{1}{3}\partial_x^3\eta_0 + \frac{1}{2}\partial_x(\eta_0^2 - 2F_1G_1) + A\partial_{\tau_0}\eta_0 + \frac{1}{\pi}\left(\int_0^{\tau_0} \partial_{x'}(\eta_0^2 - 2F_1G_1) + A\partial_{\tau_0}\eta_0 \frac{1}{x-x'}\,\mathrm{d}x + \int_{\tau_0}^{\infty} (\partial_{x'}(\eta_0^2 - 2F_2G_2) + (A+B)\partial_{\tau_0}\eta_0)\frac{1}{x-x'}\,\mathrm{d}x'\right)) = 0,$$

and subtraction yields

$$\begin{aligned}-2\partial_{\tau_1}\eta_0 + \frac{1}{3}\partial_{\tau_0 xx}\eta_0 + \frac{1}{2}\partial_x\eta_0(A + \partial_{\tau_0}\int\eta_0\,\mathrm{d}x) + \frac{1}{2}\eta_0\partial_{\tau_0}\eta_0 + \frac{1}{\pi}\Bigg(\int_0^{\tau_0}(\partial_{x'}\eta_0(A+\partial_{\tau_0}\int\eta_0\,\mathrm{d}x) + \eta_0\partial_{\tau_0}\eta_0)\frac{1}{x-x'}\,\mathrm{d}x' \\ + \int_{\tau_0}^{\infty}(\partial_{x'}\eta_0(A+B+\partial_{\tau_0}\int\eta_0\,\mathrm{d}x) + \eta_0\partial_{\tau_0}\eta_0)\frac{1}{x-x'}\,\mathrm{d}x'\Bigg) = 0.\end{aligned}$$

2.2.5 The Hilbert Transform term

Proposition 6. *We have*

$$\begin{aligned}&\int_0^{\infty}\partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y \\ &\quad = \int_0^{\tau_0}(2F_1\partial_\xi F_1 + 2G_1\partial_\zeta G_1)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y + \int_{\tau_0}^{\infty}(2F_2\partial_\xi F_2 + 2G_2\partial_\zeta G_2)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y \\ &\quad + A\int_0^{\tau_0}(-\partial_\xi F_1 + \partial_\zeta G_1)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y + (A+B)\int_{\tau_0}^{\infty}(-\partial_\xi F_2 + \partial_\zeta G_2)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y,\end{aligned}$$

where

$$\begin{aligned}A &= F_1(\tau_0)) - G_1(\tau_0), \\ B &= F_2(0) - F_1(0) + G_1(2\tau_0) - G_2(2\tau_0).\end{aligned}$$

Proof. Note:

$$\begin{aligned}\int_0^{\infty}\partial_t\left(\eta\int_0^y\eta_t\,\mathrm{d}y'\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y &= \int_0^{\infty}(-\partial_\xi + \partial_\zeta + \varepsilon\partial_{\tau_1})\left((\eta_0+\varepsilon\eta_1)\int_0^y(\eta_0+\varepsilon\eta_1)_t\,\mathrm{d}y'\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y + \mathcal{O}(\varepsilon^2) \\ &= \int_0^{\infty}(-\partial_\xi + \partial_\zeta)\left((\eta_0+\varepsilon\eta_1)\int_0^y(\eta_0+\varepsilon\eta_1)_t\,\mathrm{d}y'\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y + \mathcal{O}(\varepsilon)\end{aligned}$$

$$= \int_0^\infty (-\partial_\xi + \partial_\zeta) \left(\eta_0 \int_0^y (\eta_0)_t \, \mathrm{d}y' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y + \mathcal{O}(\varepsilon)$$
$$= \int_0^\infty (-\partial_\xi + \partial_\zeta) \left(\eta_0 \int_0^y (-\partial_\xi + \partial_\zeta + \varepsilon \partial_{\tau_1}) \eta_0 \, \mathrm{d}y' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y + \mathcal{O}(\varepsilon)$$
$$= \int_0^\infty (-\partial_\xi + \partial_\zeta) \left(\eta_0 \int_0^y (-\partial_\xi + \partial_\zeta) \eta_0 \, \mathrm{d}y' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y + \mathcal{O}(\varepsilon).$$

Now, recalling that η_0 is piecewise, we split the integral:

$$\int_0^{\tau_0} (-\partial_\xi + \partial_\zeta) \left(\eta_0 \int_0^y (-\partial_\xi + \partial_\zeta) \eta_0 \, \mathrm{d}y' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y, \tag{33}$$

and

$$\int_{\tau_0}^\infty (-\partial_\xi + \partial_\zeta) \left(\eta_0 \int_0^y (-\partial_\xi + \partial_\zeta) \eta_0 \, \mathrm{d}y' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y. \tag{34}$$

We deal with (33):

$$\begin{aligned}
\int_0^{\tau_0} (-\partial_\xi + \partial_\zeta) \left(\eta_0 \int_0^y (-\partial_\xi + \partial_\zeta) \eta_0 \, \mathrm{d}y' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y &= \int_0^{\tau_0} (-\partial_\xi + \partial_\zeta) \left((F_1 + G_1) \int_0^y (-\partial_\xi + \partial_\zeta)(F_1 + G_1) \, \mathrm{d}y' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y \\
&= \int_0^{\tau_0} (-\partial_\xi + \partial_\zeta) \left((F_1 + G_1) \int_0^y (-\partial_\xi F_1 + \partial_\zeta G_1) \, \mathrm{d}y' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y \\
&= \int_0^{\tau_0} (-\partial_\xi + \partial_\zeta) \left((F_1 + G_1)(-(F_1 - F_1(\tau_0)) + G_1 - G_1(\tau_0)) \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y \\
&= \int_0^{\tau_0} (-\partial_\xi + \partial_\zeta) \left((F_1 + G_1)(-F_1 + G_1 + F_1(\tau_0)) - G_1(\tau_0)) \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y \\
&= \int_0^{\tau_0} \big((-\partial_\xi + \partial_\zeta)(-F_1^2 + G_1^2) + (-\partial_\xi + \partial_\zeta)(F_1 + G_1)(F_1(\tau_0)) - G_1(\tau_0)) \big) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y \\
&= \int_0^{\tau_0} \left(2F_1 \partial_\xi F_1 + 2G_1 \partial_\zeta G_1 + A(-\partial_\xi F_1 + \partial_\zeta G_1) \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y,
\end{aligned}$$

where we can set $F_1(\tau_0)) - G_1(\tau_0) = A$ by imposing a free end condition $\eta_x(0,t) = 0$. Now, we deal with (34):

$$\int_{\tau_0}^\infty (-\partial_\xi + \partial_\zeta) \left(\eta_0 \int_0^y (-\partial_\xi + \partial_\zeta) \eta_0 \, \mathrm{d}y' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y$$

$$
\begin{aligned}
&= \int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta)\left((F_2+G_2)\left(\int_0^{\tau_0}(-\partial_\xi+\partial_\zeta)(F_1+G_1)\,\mathrm{d}y' + \int_{\tau_0}^{y}(-\partial_\xi+\partial_\zeta)(F_2+G_2)\,\mathrm{d}y'\right)\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y \\
&= \int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta)\left((F_2+G_2)\left((-F_1(0)+F_1(\tau_0)+G_1(2\tau_0)-G_1(\tau_0)-F_2+F_2(0)+G_2-G_2(2\tau_0)\right)\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y \\
&= \int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta)\left((F_2+G_2)\left(-F_2+G_2+F_2(0)-F_1(0)+F_1(\tau_0)+G_1(2\tau_0)-G_2(2\tau_0)-G_1(\tau_0)\right)\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y \\
&= \int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta)\left((F_2+G_2)(-F_2+G_2)\right) + (-\partial_\xi + \partial_\zeta)\left((F_2+G_2)(F_2(0)-F_1(0)+F_1(\tau_0)+G_1(2\tau_0)-G_2(2\tau_0)-G_1(\tau_0)\right)\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y,
\end{aligned}
$$

where for the second and third lines we translate $\xi' = x'-t, \zeta' = x'+t$ to obtain

$$
\begin{aligned}
\int_t^y -\partial_{\xi'}(F_2(\xi'-\tau_0))\,\mathrm{d}x' &= \int_0^{y-t}(F_2)_{\xi'}(\xi',\tau_1)\,\mathrm{d}\xi' = \int_0^{\xi}(F_2)_{\xi'}(\xi',\tau_1)\,\mathrm{d}\xi' = F_2(\xi,\tau_1)-F_2(0,\tau_1), \\
\int_t^y \partial_{\zeta'}(G_2)(\xi'+\tau_0,\tau_1)\,\mathrm{d}x' &= \int_{2t}^{y+t}(G_2)_{\zeta'}(\zeta',\tau_1)\,\mathrm{d}\zeta' = \int_{2\tau_0}^{\zeta}(G_2)_{\zeta'}(\zeta',\tau_1)\,\mathrm{d}\zeta' = G_2(\zeta,\tau_1)-G_2(2\tau_0,\tau_1).
\end{aligned}
$$

We have that

$$
\begin{aligned}
\int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta)\left((F_2+G_2)(-F_2+G_2)\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y &= \int_{\tau_0}^{\infty}(-\partial_\xi+\partial_\zeta)(-F_2^2+G_2^2)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y \\
&= \int_{\tau_0}^{\infty}(\partial_\xi(F_2^2)+\partial_\zeta(G_2^2))\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y \\
&= \int_{\tau_0}^{\infty}(2F_2\partial_\xi F_2 + 2G_2\partial_\zeta G_2)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y.
\end{aligned}
$$

Let

$$F_2(0)-F_1(0)+G_1(2\tau_0)-G_2(2\tau_0) = B,$$

so that

$$F_2(0)-F_1(0)+G_1(2\tau_0)-G_2(2\tau_0)+F_1(\tau_0)-G_1(\tau_0) = A+B.$$

We then see that:

$$
\begin{aligned}
&\int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta)\left((F_2+G_2)(F_2(0)-F_1(0)+F_1(\tau_0)+G_1(2\tau_0)-G_2(2\tau_0)-G_1(\tau_0)\right)\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y \\
&= (A+B)\int_{\tau_0}^{\infty}(-\partial_\xi+\partial_\zeta)(F_2+G_2)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y
\end{aligned}
$$

$$= (A+B)\int_{\tau_0}^{\infty}(-\partial_\xi F_2+\partial_\zeta G_2)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y. \tag{35}$$

we observe interaction between F_i, G_i at the interface $x=\tau_0$. Note that if we impose continuity, then $F_2(0)=F_1(0), G_2(2\tau_0)=G_1(2\tau_0)$, which leaves us with $F_1(\tau_0)-G_1(\tau_0)$. As before, we can eliminate this term by imposing a free end condition $\eta_x(0,t)=0$, and restricting the scalar of integration to be 0. In this case, $A+B=0$ and so the term (35) vanishes due to boundary conditions. More generally, we obtain

$$\begin{aligned}
&\int_{\tau_0}^{\infty}(-\partial_\xi+\partial_\zeta)\left(\eta_0\int_0^y(-\partial_\xi+\partial_\zeta)\eta_0\,\mathrm{d}y'\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y\\
&\quad=\int_{\tau_0}^{\infty}(-\partial_\xi+\partial_\zeta)\left((F_2+G_2)(-F_2+G_2)\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y+(A+B)\int_{\tau_0}^{\infty}(-\partial_\xi F_2+\partial_\zeta G_2)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y\\
&\quad=\int_{\tau_0}^{\infty}(2F_2\partial_\xi F_2+2G_2\partial_\zeta G_2)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y+(A+B)\int_{\tau_0}^{\infty}(-\partial_\xi F_2+\partial_\zeta G_2)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y,
\end{aligned}$$

so that

$$\begin{aligned}
&\int_0^{\infty}\partial_t\left(\eta\int_0^y\eta_t\,\mathrm{d}y'\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y\\
&\quad=\int_0^{\tau_0}(-\partial_\xi+\partial_\zeta)\left(\eta_0\int_0^y(-\partial_\xi+\partial_\zeta)\eta_0\,\mathrm{d}y'\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y+\int_{\tau_0}^{\infty}(-\partial_\xi+\partial_\zeta)\left(\eta_0\int_0^y(-\partial_\xi+\partial_\zeta)\eta_0\,\mathrm{d}y'\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y\\
&\quad=\int_0^{\tau_0}(2F_1\partial_\xi F_1+2G_1\partial_\zeta G_1)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y+\int_{\tau_0}^{\infty}(2F_2\partial_\xi F_2+2G_2\partial_\zeta G_2)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y\\
&\quad+A\int_0^{\tau_0}(-\partial_\xi F_1+\partial_\zeta G_1)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y+(A+B)\int_{\tau_0}^{\infty}(-\partial_\xi F_2+\partial_\zeta G_2)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y.
\end{aligned}$$

The proof is complete. □

Proposition 7. *Let*

$$\begin{aligned}
A &= F_1(\tau_0))-G_1(\tau_0),\\
B &= F_2(0)-F_1(0)+G_1(2\tau_0)-G_2(2\tau_0).
\end{aligned}$$

Then, we have

$$(\partial_\xi+\partial_\zeta)\int_0^{\infty}\partial_t\left(\eta\int_0^y\eta_t\,\mathrm{d}y'\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y$$

$$
\begin{aligned}
&= \partial_\xi \left(\int_{-\tau_0}^{0} 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'} \, \mathrm{d}\xi' + \int_0^\infty 2F_2 \partial_{\xi'} F_2 \frac{1}{\xi - \xi'} \, \mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} \, \mathrm{d}\zeta' + \int_{2\tau_0}^{\infty} 2G_2 \partial_{\zeta'} G_2 \frac{1}{\xi + \zeta'} \, \mathrm{d}\zeta' \right) \\
&+ \partial_\zeta \left(\int_{-\tau_0}^{0} 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi + \zeta'} \, \mathrm{d}\xi' + \int_0^\infty 2F_2 \partial_{\xi'} F_2 \frac{1}{\xi + \zeta'} \, \mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} \, \mathrm{d}\zeta' + \int_{2\tau_0}^{\infty} 2G_2 \partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'} \, \mathrm{d}\zeta' \right) \\
&+ \partial_\xi \left(A \int_{-\tau_0}^{0} -\partial_{\xi'} F_1 \frac{1}{\xi - \xi'} \, \mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} \, \mathrm{d}\zeta' + (A+B) \int_0^\infty -\partial_{\xi'} F_2 \frac{1}{\xi - \xi'} \, \mathrm{d}\xi' + \int_{2\tau_0}^{\infty} \partial_{\zeta'} G_2 \frac{1}{\xi + \zeta'} \, \mathrm{d}\zeta' \right) \\
&+ \partial_\zeta \left(A \int_{-\tau_0}^{0} -\partial_{\xi'} F_1 \frac{1}{\zeta + \xi'} \, \mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} \, \mathrm{d}\zeta' + (A+B) \int_0^\infty -\partial_{\xi'} F_2 \frac{1}{\zeta + \xi'} \, \mathrm{d}\xi' + \int_{2\tau_0}^{\infty} \partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'} \, \mathrm{d}\zeta' \right)
\end{aligned}
$$

Proof. By Proposition 6, we have

$$
\begin{aligned}
&(\partial_\xi + \partial_\zeta) \int_0^\infty \partial_t \left(\eta \int_0^y \eta_t \, \mathrm{d}y' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y \\
&\quad = (\partial_\xi + \partial_\zeta) \int_0^{\tau_0} (2F_1 \partial_\xi F_1 + 2G_1 \partial_\zeta G_1) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y + (\partial_\xi + \partial_\zeta) \int_{\tau_0}^{\infty} (2F_2 \partial_\xi F_2 + 2G_2 \partial_\zeta G_2) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y \\
&\quad + A(\partial_\xi + \partial_\zeta) \int_0^{\tau_0} (-\partial_\xi F_1 + \partial_\zeta G_1) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y + (A+B)(\partial_\xi + \partial_\zeta) \int_{\tau_0}^{\infty} (-\partial_\xi F_2 + \partial_\zeta G_2) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y,
\end{aligned}
$$

Before we start, recall the following changes of variables; they will be used throughout the proof:

$$x - \tau_0 = \xi, \tag{36}$$

$$x + \tau_0 = \zeta, \tag{37}$$

$$y - \tau_0 = \xi', \tag{38}$$

$$y + \tau_0 = \zeta'. \tag{39}$$

Consider

$$(\partial_\xi + \partial_\zeta) \int_0^{\tau_0} (2F_1 \partial_\xi F_1 + 2G_1 \partial_\zeta G_1) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y.$$

First, observe that

$$
\begin{aligned}
\int_0^{\tau_0} 2F_1(\tau_0 - y) \partial_{\xi'} F_1(\tau_0 - y) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y &= \int_{-\tau_0}^{0} 2F_1(-\xi') \partial_{\xi'} F_1(-\xi') \left[\frac{1}{x - \xi' - t} + \frac{1}{x + \xi' + t} \right] \mathrm{d}\xi' && \text{(use (38))} \\
&= \int_{-\tau_0}^{0} 2F_1(-\xi') \partial_{\xi'} F_1(-\xi') \left[\frac{1}{\xi + \tau_0 - \xi' - \tau_0} + \frac{1}{\zeta - \tau_0 + \xi' + \tau_0} \right] \mathrm{d}\xi' && \text{(use (36) and (37))}
\end{aligned}
$$

$$= \int_{-\tau_0}^{0} 2F_1 \partial_{\xi'} F_1 \left[\frac{1}{\xi - \xi'} + \frac{1}{\zeta + \xi'}\right] \mathrm{d}\xi',$$

and

$$\begin{aligned}
\int_0^{\tau_0} 2G_1(y+\tau_0)\partial_{\zeta'} G_1(y+\tau_0) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y &= \int_{\tau_0}^{2\tau_0} 2G_1(\zeta')\partial_{\zeta'} G_1(\zeta') \left[\frac{1}{x-\zeta'-\tau_0} + \frac{1}{x+\zeta'+\tau_0}\right] \mathrm{d}\zeta' && \text{(use (39))}\\
&= \int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'} G_1 \left[\frac{1}{\zeta-\tau_0-\zeta'+\tau_0} + \frac{1}{\xi+\tau_0+\zeta'-\tau_0}\right] \mathrm{d}\zeta' && \text{(use (37)) and (36)}\\
&= \int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'} G_1 \left[\frac{1}{\zeta-\zeta'} + \frac{1}{\xi+\zeta'}\right] \mathrm{d}\zeta'.
\end{aligned}$$

This yields:

$$\begin{aligned}
(\partial_\xi + \partial_\zeta) \int_0^{\tau_0} (2F_1\partial_\xi F_1 + 2G_1\partial_\zeta G_1) \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y &= (\partial_\xi + \partial_\zeta) \int_{-\tau_0}^{0} 2F_1\partial_{\xi'} F_1 \left[\frac{1}{\xi-\xi'} + \frac{1}{\zeta+\xi'}\right] \mathrm{d}\xi' + (\partial_\xi + \partial_\zeta) \int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'} G_1 \left[\frac{1}{\zeta-\zeta'} + \frac{1}{\xi+\zeta'}\right] \mathrm{d}\zeta'\\
&= \partial_\xi \int_{-\tau_0}^{0} 2F_1\partial_{\xi'} F_1 \frac{1}{\xi-\xi'} \mathrm{d}\xi' + \partial_\zeta \int_{-\tau_0}^{0} 2F_1\partial_{\xi'} F_1 \frac{1}{\zeta+\xi'} \mathrm{d}\xi'\\
&\quad + \partial_\xi \int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'} G_1 \frac{1}{\xi+\zeta'} \mathrm{d}\zeta' + \partial_\zeta \int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'} G_1 \frac{1}{\zeta-\zeta'} \mathrm{d}\zeta'\\
&= \partial_\xi \left(\int_{-\tau_0}^{0} 2F_1\partial_{\xi'} F_1 \frac{1}{\xi-\xi'} \mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'} G_1 \frac{1}{\xi+\zeta'} \mathrm{d}\zeta'\right)\\
&\quad + \partial_\zeta \left(\int_{-\tau_0}^{0} 2F_1\partial_{\xi'} F_1 \frac{1}{\zeta+\xi'} \mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'} G_1 \frac{1}{\zeta-\zeta'} \mathrm{d}\zeta'\right). \qquad (40)
\end{aligned}$$

By a similar argument, one may show that

$$\int_{\tau_0}^{\infty} 2F_2\partial_\xi F_2 \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y = \int_0^{\infty} 2F_2\partial_\xi F_2 \left[\frac{1}{\xi-\xi'} + \frac{1}{\zeta+\xi'}\right] \mathrm{d}\xi',$$

and

$$\int_{\tau_0}^{\infty} 2G_2\partial_\zeta G_2 \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y = \int_{2\tau_0}^{\infty} 2G_2\partial_\zeta G_2 \left[\frac{1}{\zeta-\zeta'} + \frac{1}{\xi+\zeta'}\right] \mathrm{d}\zeta',$$

so that

$$(\partial_\xi+\partial_\zeta)\int_{\tau_0}^{\infty}(2F_2\partial_\xi F_2+2G_2\partial_\zeta G_2)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y=(\partial_\xi+\partial_\zeta)\int_0^{\infty}2F_2\partial_\xi F_2\left[\frac{1}{\xi-\xi'}+\frac{1}{\zeta+\xi'}\right]\mathrm{d}\xi'+(\partial_\xi+\partial_\zeta)\int_{2\tau_0}^{\infty}2G_2\partial_\zeta G_2\left[\frac{1}{\zeta-\zeta'}+\frac{1}{\xi+\zeta'}\right]\mathrm{d}\zeta'$$
$$\begin{aligned}
&=\partial_\xi\int_0^{\infty}2F_1\partial_{\xi'}F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi'+\partial_\zeta\int_0^{\infty}2F_1\partial_{\xi'}F_1\frac{1}{\zeta+\xi'}\,\mathrm{d}\xi'\\
&+\partial_\xi\int_{2\tau_0}^{\infty}2G_1\partial_{\zeta'}G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta'+\partial_\zeta\int_{2\tau_0}^{\infty}2G_1\partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\\
&=\partial_\xi\left(\int_0^{\infty}2F_1\partial_{\xi'}F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi'+\int_{2\tau_0}^{\infty}2G_1\partial_{\zeta'}G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta'\right)\\
&+\partial_\zeta\left(\int_0^{\infty}2F_1\partial_{\xi'}F_1\frac{1}{\zeta+\xi'}\,\mathrm{d}\xi'+\int_{2\tau_0}^{\infty}2G_1\partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\right).
\end{aligned}\tag{41}$$

Finally, bringing (40) and (41) yields

$$\begin{aligned}
&(\partial_\xi+\partial_\zeta)\int_0^{\tau_0}(2F_1\partial_\xi F_1+2G_1\partial_\zeta G_1)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y+(\partial_\xi+\partial_\zeta)\int_{\tau_0}^{\infty}(2F_2\partial_\xi F_2+2G_2\partial_\zeta G_2)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y\\
&=\partial_\xi\left(\int_{-\tau_0}^{0}2F_1\partial_{\xi'}F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi'+\int_{\tau_0}^{2\tau_0}2G_1\partial_{\zeta'}G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta'\right)+\partial_\zeta\left(\int_{-\tau_0}^{0}2F_1\partial_{\xi'}F_1\frac{1}{\zeta+\xi'}\,\mathrm{d}\xi'+\int_{\tau_0}^{2\tau_0}2G_1\partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\right)\\
&+\partial_\xi\left(\int_0^{\infty}2F_1\partial_{\xi'}F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi'+\int_{2\tau_0}^{\infty}2G_1\partial_{\zeta'}G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta'\right)+\partial_\zeta\left(\int_0^{\infty}2F_1\partial_{\xi'}F_1\frac{1}{\zeta+\xi'}\,\mathrm{d}\xi'+\int_{2\tau_0}^{\infty}2G_1\partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\right)\\
&=\partial_\xi\left(\int_{-\tau_0}^{0}2F_1\partial_{\xi'}F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi'+\int_0^{\infty}2F_1\partial_{\xi'}F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi'+\int_{\tau_0}^{2\tau_0}2G_1\partial_{\zeta'}G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta'+\int_{2\tau_0}^{\infty}2G_1\partial_{\zeta'}G_1\frac{1}{\xi+\zeta'}\,\mathrm{d}\zeta'\right)\\
&+\partial_\zeta\left(\int_{-\tau_0}^{0}2F_1\partial_{\xi'}F_1\frac{1}{\zeta+\xi'}\,\mathrm{d}\xi'+\int_{\tau_0}^{2\tau_0}2G_1\partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'+\int_0^{\infty}2F_1\partial_{\xi'}F_1\frac{1}{\zeta+\xi'}\,\mathrm{d}\xi'+\int_{2\tau_0}^{\infty}2G_1\partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\right),
\end{aligned}$$

which gives the first part of the identity. For the second part, it is straightforward that

$$\int_0^{\tau_0}(-\partial_\xi F_1+\partial_\zeta G_1)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y=\int_{-\tau_0}^{0}-\partial_\xi F_1\left[\frac{1}{\xi-\xi'}+\frac{1}{\zeta+\xi'}\right]\mathrm{d}\xi'+\int_{\tau_0}^{2\tau_0}\partial_\zeta G_1\left[\frac{1}{\zeta-\zeta'}+\frac{1}{\xi+\zeta'}\right]\mathrm{d}\zeta'$$

so that

$$(\partial_\xi+\partial_\zeta)\int_0^{\tau_0}(-\partial_\xi F_1+\partial_\zeta G_1)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y=(\partial_\xi+\partial_\zeta)\int_{-\tau_0}^{0}-\partial_\xi F_1\left[\frac{1}{\xi-\xi'}+\frac{1}{\zeta+\xi'}\right]\mathrm{d}\xi'+(\partial_\xi+\partial_\zeta)\int_{\tau_0}^{2\tau_0}\partial_\zeta G_1\left[\frac{1}{\zeta-\zeta'}+\frac{1}{\xi+\zeta'}\right]\mathrm{d}\zeta'$$

$$= \partial_\xi \left(\int_{-\tau_0}^{0} -\partial_\xi F_1 \frac{1}{\xi - \xi'} \, \mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \frac{1}{\xi + \zeta'} \, \mathrm{d}\zeta' \right) + \partial_\zeta \left(\int_{-\tau_0}^{0} -\partial_\xi F_1 \frac{1}{\zeta + \xi'} \, \mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \frac{1}{\zeta - \zeta'} \, \mathrm{d}\zeta' \right).$$

Similarly,

$$\int_{\tau_0}^{\infty} (-\partial_\xi F_2 + \partial_\zeta G_2) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y = \int_0^{\infty} -\partial_\xi F_2 \left[\frac{1}{\xi - \xi'} + \frac{1}{\zeta + \xi'} \right] \mathrm{d}\xi' + \int_{2\tau_0}^{\infty} \partial_\zeta G_2 \left[\frac{1}{\zeta - \zeta'} + \frac{1}{\xi + \zeta'} \right] \mathrm{d}\zeta'$$

so that

$$\begin{aligned}
(\partial_\xi + \partial_\zeta) \int_{\tau_0}^{\infty} (-\partial_\xi F_2 + \partial_\zeta G_2) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y &= (\partial_\xi + \partial_\zeta) \left(\int_0^{\infty} -\partial_\xi F_2 \left[\frac{1}{\xi - \xi'} + \frac{1}{\zeta + \xi'} \right] \mathrm{d}\xi' \right) + (\partial_\xi + \partial_\zeta) \left(\int_{2\tau_0}^{\infty} \partial_\zeta G_2 \left[\frac{1}{\zeta - \zeta'} + \frac{1}{\xi + \zeta'} \right] \mathrm{d}\zeta' \right) \\
&= \partial_\xi \left(\int_0^{\infty} -\partial_\xi F_2 \frac{1}{\xi - \xi'} \, \mathrm{d}\xi' + \int_{2\tau_0}^{\infty} \partial_\zeta G_2 \frac{1}{\xi + \zeta'} \, \mathrm{d}\zeta' \right) + \partial_\zeta \left(\int_0^{\infty} -\partial_\xi F_2 \frac{1}{\zeta + \xi'} \, \mathrm{d}\xi' + \int_{2\tau_0}^{\infty} \partial_\zeta G_2 \frac{1}{\zeta - \zeta'} \, \mathrm{d}\zeta' \right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
A(\partial_\xi + \partial_\zeta) \int_0^{\tau_0} &(-\partial_\xi F_1 + \partial_\zeta G_1) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y + (A+B)(\partial_\xi + \partial_\zeta) \int_{\tau_0}^{\infty} (-\partial_\xi F_2 + \partial_\zeta G_2) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y \\
&= A\partial_\xi \left(\int_{-\tau_0}^{0} -\partial_\xi F_1 \frac{1}{\xi - \xi'} \, \mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \frac{1}{\xi + \zeta'} \, \mathrm{d}\zeta' \right) + A\partial_\zeta \left(\int_{-\tau_0}^{0} -\partial_\xi F_1 \frac{1}{\zeta + \xi'} \, \mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \frac{1}{\zeta - \zeta'} \, \mathrm{d}\zeta' \right) \\
&+ (A+B)\partial_\xi \left(\int_0^{\infty} -\partial_\xi F_2 \frac{1}{\xi - \xi'} \, \mathrm{d}\xi' + \int_{2\tau_0}^{\infty} \partial_\zeta G_2 \frac{1}{\xi + \zeta'} \, \mathrm{d}\zeta' \right) + (A+B)\partial_\zeta \left(\int_0^{\infty} -\partial_\xi F_2 \frac{1}{\zeta + \xi'} \, \mathrm{d}\xi' + \int_{2\tau_0}^{\infty} \partial_\zeta G_2 \frac{1}{\zeta - \zeta'} \, \mathrm{d}\zeta' \right) \\
&= \partial_\xi \left(A \int_{-\tau_0}^{0} -\partial_\xi F_1 \frac{1}{\xi - \xi'} \, \mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \frac{1}{\xi + \zeta'} \, \mathrm{d}\zeta' + (A+B) \int_0^{\infty} -\partial_\xi F_2 \frac{1}{\xi - \xi'} \, \mathrm{d}\xi' + \int_{2\tau_0}^{\infty} \partial_\zeta G_2 \frac{1}{\xi + \zeta'} \, \mathrm{d}\zeta' \right) \\
&+ \partial_\zeta \left(A \int_{-\tau_0}^{0} -\partial_\xi F_1 \frac{1}{\zeta + \xi'} \, \mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \frac{1}{\zeta - \zeta'} \, \mathrm{d}\zeta' + (A+B) \int_0^{\infty} -\partial_\xi F_2 \frac{1}{\zeta + \xi'} \, \mathrm{d}\xi' + \int_{2\tau_0}^{\infty} \partial_\zeta G_2 \frac{1}{\zeta - \zeta'} \, \mathrm{d}\zeta' \right),
\end{aligned}$$

which yields the second part of identity. Therefore,

$$\begin{aligned}
\partial_\xi + \partial_\zeta) \int_0^{\infty} &\partial_t \left(\eta \int_0^y \eta_t \, \mathrm{d}y' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y \\
&= (\partial_\xi + \partial_\zeta) \int_0^{\tau_0} (2F_1 \partial_\xi F_1 + 2G_1 \partial_\zeta G_1) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y + (\partial_\xi + \partial_\zeta) \int_{\tau_0}^{\infty} (2F_2 \partial_\xi F_2 + 2G_2 \partial_\zeta G_2) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y \\
&+ A(\partial_\xi + \partial_\zeta) \int_0^{\tau_0} (-\partial_\xi F_1 + \partial_\zeta G_1) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y + (A+B)(\partial_\xi + \partial_\zeta) \int_{\tau_0}^{\infty} (-\partial_\xi F_2 + \partial_\zeta G_2) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&= \partial_\xi \left(\int_{-\tau_0}^{0} 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'} \, \mathrm{d}\xi' + \int_{0}^{\infty} 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'} \, \mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} \, \mathrm{d}\zeta' + \int_{2\tau_0}^{\infty} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} \, \mathrm{d}\zeta' \right) \\
&+ \partial_\zeta \left(\int_{-\tau_0}^{0} 2F_1 \partial_{\xi'} F_1 \frac{1}{\zeta + \xi'} \, \mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} \, \mathrm{d}\zeta' + \int_{0}^{\infty} 2F_1 \partial_{\xi'} F_1 \frac{1}{\zeta + \xi'} \, \mathrm{d}\xi' + \int_{2\tau_0}^{\infty} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} \, \mathrm{d}\zeta' \right) \\
&+ \partial_\xi \left(A \int_{-\tau_0}^{0} -\partial_\xi F_1 \frac{1}{\xi - \xi'} \, \mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \frac{1}{\xi + \zeta'} \, \mathrm{d}\zeta' + (A+B) \int_{0}^{\infty} -\partial_\xi F_2 \frac{1}{\xi - \xi'} \, \mathrm{d}\xi' + \int_{2\tau_0}^{\infty} \partial_\zeta G_2 \frac{1}{\xi + \zeta'} \, \mathrm{d}\zeta' \right) \\
&+ \partial_\zeta \left(A \int_{-\tau_0}^{0} -\partial_\xi F_1 \frac{1}{\zeta + \xi'} \, \mathrm{d}\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \frac{1}{\zeta - \zeta'} \, \mathrm{d}\zeta' + (A+B) \int_{0}^{\infty} -\partial_\xi F_2 \frac{1}{\zeta + \xi'} \, \mathrm{d}\xi' + \int_{2\tau_0}^{\infty} \partial_\zeta G_2 \frac{1}{\zeta - \zeta'} \, \mathrm{d}\zeta' \right).
\end{aligned}
$$

This gives the desired identity. □

References

[1] Tom M. Apostol, *Mathematical analysis*, Pearson, 1974.