Approximate Equations in the Shallow Water Regime of the Water-Wave Problem on Unbounded Domains

Sultan AITZHAN

Capstone Final Report for BSc (Honours) in

Mathematical, Computational and Statistical Sciences

Supervised by: Prof. Katie Oliveras, Prof. Dave Smith

AY 2019/2020

Yale-NUS College Capstone Project

DECLARATION & CONSENT

1. I declare that the product of this Project, the Thesis, is the end result of my own work and that due acknowledgement has been given in the bibliography and references to ALL sources be they printed, electronic, or personal, in accordance with the academic regulations of Yale-NUS College.

2. I acknowledge that the Thesis is subject to the policies relating to Yale-NUS College Intellectual Property (Yale-NUS HR 039).

ACCESS LEVEL

3. I agree, in consultation with my supervisor(s), that the Thesis be given the access level specified below: [check one only]

o Unrestricted access
Make the Thesis immediately available for worldwide access.

o Access restricted to Yale-NUS College for a limited period
Make the Thesis immediately available for Yale-NUS College access only from _____________ (mm/yyyy) to _______________ (mm/yyyy), up to a maximum of 2 years for the following reason(s): (please specify; attach a separate sheet if necessary):
____.

After this period, the Thesis will be made available for worldwide access.

o Other restrictions: (please specify if any part of your thesis should be restricted)
____
____

____
Name & Residential College of Student

____________________________________ ___________________________
Signature of Student Date

____________________________________ ___________________________
Name & Signature of Supervisor Date

Acknowledgements

First and foremost, I would like to thank Prof Katie Oliveras for taking me in as her student and supervising this project. In addition, I extend my gratitude to Prof Dave Smith for his continued enthusiasm and patience in my growth as a mathematics student. The guidance and encouragement provided by these two people make doing mathematics so much more enjoyable.

Along the way, I must acknowledge the numerous mathematics faculty for their support: Prof Spagnuolo, Prof Berrick, Prof Wertz, Prof Heinecke, Prof Stamps, Prof Yu, Prof Dinh and Prof Rodin. I am also grateful to my friends Grace, Sveta, Chengpei, Elaine, Vlad and Ayan, for giving me reasons to laugh every day (amongst many other things). Finally, I thank my family for their never-ending belief in me, and my grandmother, for being the first inspiration to pursue mathematics.

Above all, Alhamdulillah.

YALE-NUS COLLEGE

Abstract

B.Sc (Hons)

Approximate equations in the shallow water regime of the water wave problem on unbounded domains

by Sultan AITZHAN

A free boundary, water-wave problem is studied for an irrotational, inviscid, and incompressible fluid. Specifically, we describe a derivation of approximate equations in the shallow water limit using a non-local formulation, introduced in Oliveras and Vasan, 2013 via a normal-to-tangential operator, in two related settings. One is the classical, whole-line case, and another is a half-line case, which physically is represented by the presence of a wall in the middle, so that the water is flowing to one side. In both settings, non-local formulations yield expressions for the surface elevation, from which the appropriate wave and KdV equations, as well as new approximations are obtained.

Contents

List of Figures

List of Abbreviations

RHS	**R**ight **H**and **S**ide of an equation
LHS	**L**eft **H**and **S**ide of an equation
ODE	**O**rdinary **D**ifferential **E**quation
PDE	**P**artial **D**ifferential **E**quation
BVP	**B**oundary **V**alue **P**roblem
IVP	**I**nitial **V**alue **P**roblem
KdV equation	**K**orteweg **d**e **V**ries equation
DNO	**D**irichlet-to-**N**eumann **O**perator
AFM	**A**blowit-**F**okas-**M**usslimani

Frequently Used Notation

$\mathbb{R}$	Set of real numbers
$\mathbf{v}$	Vector in $\mathbb{R}^3$
$\mathbf{v}^*$	Complex conjugate of $\mathbf{v}$
$\\|\mathbf{v}\\|$	(Euclidean) Norm of $\mathbf{v}$
$v(\mathbf{x})$	Scalar field, i.e. map from $\mathbb{R}^3$ to $\mathbb{R}$
$\mathbf{v}(\mathbf{x})$	Vector field, i.e. map from $\mathbb{R}^3$ to $\mathbb{R}^3$
$\frac{\mathrm{d}f}{\mathrm{d}x}$ or f'	Derivative of f with respect to x
$\frac{\partial f}{\partial x_i}$ or $\partial_{x_i} f$ or f_{x_i}	Partial derivative of f with respect to variable x_i
∇	Gradient operator $\nabla = (\partial_x, \partial_y, \partial_z)^T$, where T is transpose
Δ	Laplace operator $\Delta = \nabla \cdot \nabla$
ε	Parameter for nonlinearity (Chapter 3)
μ	Parameter for amplitude (Chapter 3)
$f(x) = \mathcal{O}(g(x))$	There is $M > 0$ such that for sufficiently large x, $$\|f(x)\| \leqslant M\|g(x)\|$$
$f(x) \sim g(x)$	Same as $f(x) = \mathcal{O}(g(x))$
$f(x) \approx g(x)$	$f(x)$ is approximately equal to $g(x)$
$f(x) \simeq g(x)$	$f(x)$ behaves like $g(x)$ when x is large
$f(x) \ll g(x)$	$f(x)$ is much less than $g(x)$
$f(x) \gg g(x)$	$f(x)$ is much greater than $g(x)$

$\mathcal{F}_k\{f(x)\}$ or $\widehat{f}_k$: Fourier transform of $f(x)$, given by

$$\mathcal{F}_k\{f(x)\} = \int_{-\infty}^{\infty} \exp(-ikx) f(x)\, \mathrm{d}x$$

$\mathcal{F}_k^s\{f(x)\}$ or $\widehat{f}_c^k$: Fourier Sine transform of $f(x)$, given by

$$\mathcal{F}_k^s\{f(x)\} = \int_0^{\infty} \sin(kx) f(x)\, \mathrm{d}x$$

$\mathcal{F}_k^c\{f(x)\}$ or $\widehat{f}_c^k$: Fourier Cosine transform of $f(x)$, given by

$$\mathcal{F}_k^c\{f(x)\} = \int_0^{\infty} \cos(kx) f(x)\, \mathrm{d}x$$

$\mathcal{F}_k^{-1}\{F(k)\}$: Inverse Fourier transform of $f(k)$, given by

$$\mathcal{F}_k^{-1}\{F(k)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ikx) F(k)\, \mathrm{d}k$$

$\{\mathcal{F}_k^s\}^{-1}\{F(k)\}$: Inverse Sine Fourier transform of $f(k)$, given by

$$\{\mathcal{F}_k^s\}^{-1}\{F(k)\} = \frac{2}{\pi} \int_0^{\infty} \sin(kx) F(k)\, \mathrm{d}k$$

$\{\mathcal{F}_k^c\}^{-1}\{F(k)\}$: Inverse Fourier Cosine transform of $F(k)$, given by

$$\{\mathcal{F}_k^c\}^{-1}\{F(k)\} = \frac{2}{\pi} \int_0^{\infty} \cos(kx) F(k)\, \mathrm{d}k$$

Chapter 1

Introduction

Often, mathematical modelling of the real-world phenomena results in ordinary and partial differential equations, whose solutions describe the phenomena. Depending on the equations, mathematicians may or may not have the tools to obtain solutions. Fortunately, there are techniques that allow one to analyse differential equations without directly solving them. One such tool is asymptotic analysis, which leads to simplified equations that are similar to original equations. As such, solutions of the simplified equations model the real-world phenomenon, subject to some error estimates.

In particular, one problem that is amenable to asymptotic methods is the *water-wave problem*, which describes the motion of water and its surface under certain conditions. Assuming an irrotational, incompressible, and inviscid fluid, and a domain with flat bottom, the equations of fluid motion are given by

$$\phi_{xx} + \phi_{zz} = 0, \qquad -h < z < \eta(x,t), \tag{1.1a}$$

$$\phi_z = 0, \qquad z = -h, \tag{1.1b}$$

$$\phi_t + g\eta + \frac{1}{2}(\phi_x^2 + \phi_z^2) = 0, \qquad z = \eta(x,t), \tag{1.1c}$$

$$\eta_t + \phi_x \eta_x = \phi_z, \qquad z = \eta(x,t), \tag{1.1d}$$

where $\phi(x,z,t)$ is the unknown velocity potential, $\nabla\phi$ is the fluid velocity, and $\eta(x,t)$ is the unknown surface elevation. In addition, z is the vertical coordinate, x is the horizontal direction, and g is acceleration due to gravity. We let $x \in \mathbb{R}$, so that (1.1) is the water-wave problem defined *on the whole line*. Although nonlinear partial differential equations (PDEs) (1.1d) and (1.1c) are hard to solve on their own, what makes the problem (1.1) truly difficult is the need to solve the Laplace's equation (1.1a) on a domain with an unknown shape.

To make the equations of motion more tractable, one can reformulate the problem and apply tools of asymptotics. Of particular interest is the work of Ablowitz, Fokas, and Musslimani, 2006 (AFM formulation). In this paper, authors rewrite (1.1) as a system of two equations, for the surface variable $q(x,t) = \phi(x, \eta(x,t))$, i.e. the velocity potential evaluated at the surface. Taking advantage of the new formulation, asymptotic reductions are performed in various physical conditions.

One such interesting physical condition is the shallow water regime, which is defined by small-amplitude waves that have a small depth relative to their wavelength. In this regime, asymptotic methods reveal that the fluid motion is governed by the following approximate equations: the *wave* equation,

$$q_{tt} - q_{xx} = 0, \tag{1.2}$$

and two *Korteweg de Vries (KdV)* equations,

$$\begin{aligned} F_T + \frac{1}{3}F_{\xi\xi\xi} + 3FF_\xi &= 0, \\ G_T + \frac{1}{3}G_{\zeta\zeta\zeta} + 3GG_\zeta &= 0, \end{aligned} \tag{1.3}$$

where $\xi = x - t, \zeta = x + t$, and $q(x,t) = F(\xi, T) + G(\zeta, T)$. Physically, F and G can be interpreted as right-going and left going waves, respectively. The model given by (1.2) and (1.3) is called the *KdV model on the whole line*. We observe that on *the half-line*, when waves are bounded from one side by a wall, a rigorous derivation of the corresponding model remains unknown.

In this capstone project, we consider an alternative formulation of (1.1), as presented in Oliveras and Vasan, 2013. Although slightly different from the AFM formulation, it is contended that this formulation is well-suited for studying asymptotics. We further advocate the efficacy of this formulation by deriving the KdV model. As a brief outline, in Chapter 2, we introduce the reader to asymptotic and perturbative methods needed for the derivation. In Chapter 3, we explain the physical assumptions of the problem and describe the shallow-water regime. In Chapter 4, we reformulate the problem and derive the approximate equations. In Chapter 5, we describe an application of the formulation to a water wave problem on *the half line*, while attempting to rigorously derive a half-line model.

Chapter 2

Asymptotic and perturbative methods

In this chapter, we introduce perturbation theory and the most relevant technique for this project, multiple scale analysis. The focus is on the illustration of ideas through examples, rather than rigorous justification and proofs. Examples are adapted from Chapters 7 and 11 of Bender and Orszag, 1999.

Perturbation theory is a collection of techniques used for obtaining approximate solutions to problems typically involving some small parameter ε. The main idea is to represent the unknown variable f as a *perturbation series*, which is a formal power series $f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \ldots$. Substituting this expression into the original problem decomposes what is a difficult problem into many simpler ones. Solving for the first n terms in the series yields an approximate solution $f \approx f_0 + \varepsilon f_1 + \ldots + \varepsilon^n f_n$. Note that with this approach, perturbation theory is most useful when the first few terms reveal the important features of the solution, and the remaining terms give successively small corrections. Perturbative techniques are applied in numerous settings, including finding roots of polynomials and solving initial value problems for differential equations.

Since in this project we deal with differential equations, we illustrate the method with an ordinary differential equation (ODE).

Example 1. Consider the following boundary value problem (BVP):

$$y'' + y = \frac{\cos x}{1+y}, \qquad y(0) = y(\pi/2) = 1. \tag{2.1}$$

We introduce a small parameter $\varepsilon > 0$ into the problem

$$y'' + y = \frac{\cos x}{1+\varepsilon y} = \cos x(1 - \varepsilon y + \mathcal{O}(\varepsilon^2)),$$

where we use geometric series and introduce asymptotic notation in the last equality (see the table of notation for Big-O notation). Expanding $y = y_0 + \varepsilon y_1 + \mathcal{O}(\varepsilon^2)$ and substituting into the above equation gives

$$(y_0 + \varepsilon y_1)'' + y_0 + \varepsilon y_1 = \cos x(1 - \varepsilon y_0) + \mathcal{O}(\varepsilon^2).$$

By ordering the above in powers of ε, we simplify the original problem into the following two problems

$$\mathcal{O}(\varepsilon^0): \qquad y_0'' + y_0 = \cos x, \qquad y_0(0) = y_0(\pi/2) = 1, \tag{2.2}$$

$$\mathcal{O}(\varepsilon^1): \qquad y_1'' + y_1 = -y_0 \cos x, \qquad y_1(0) = y_1(\pi/2) = 0. \tag{2.3}$$

Solving (2.2) and (2.3) recursively yields y_0 and y_1 :

$$\begin{aligned} y_0(x) &= \frac{4\cos x - (\pi - 2(x+2))\sin x}{4}, \\ y_1(x) &= \frac{2(-9 + 5\cos 2x + 14\sin x) + \cos x(8 - 3(-4 + \pi - 2x)\sin x)}{36}. \end{aligned}$$

An approximate solution becomes $y \approx y_0 + \varepsilon y_1 \to y_0 + y_1$, as $\varepsilon \to 1$. This yields an approximate solution of BVP (2.1). Note that as $\varepsilon \to 1$, the geometric series argument is no longer justified; yet, we still find an approximate solution whose graph resembles the shape of the exact solution (see Figure 2.1).

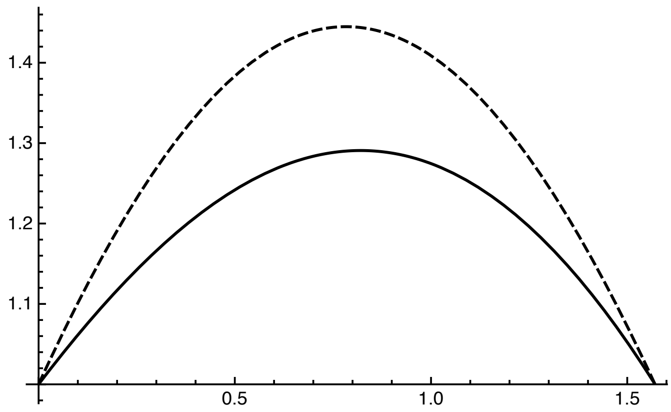


FIGURE 2.1: The exact solution $y(t)$ (line) of the BVP (2.1) vs. an approximate solution $y(t) = y_0 + y_1$ (dashed), where y_0 solves (2.2) and y_1 solves (2.3).

Although the idea is simple, problems may contain subtleties that require careful examination before applying perturbation series.

Example 2. Consider the following initial value problem (IVP) for the weakly nonlinear Duffing oscillator

$$y'' + y + \varepsilon y^3 = 0, \qquad y(0) = 1 \qquad y'(0) = 0. \tag{2.4}$$

Application of the regular perturbation series yields

$$y(t) \approx y_0 + \varepsilon y_1 = \cos t + \varepsilon \left(\frac{1}{32} \cos 3t - \frac{1}{32} \cos t + \frac{3}{8} t \sin t \right), \tag{2.5}$$

which converges as $\varepsilon \to 0$ for fixed t. Note that the convergence is only pointwise, not uniform. Indeed, for values $t \sim 1/\varepsilon$ or larger, the presence of the *secular* term $t \sin t$ implies that y_1 is unbounded in t. However, solutions of the Duffing oscillator are known to be bounded, so y_1 must be bounded. In particular, this suggests that the usual perturbation expansion of y is not sufficient, and that the secularity is an outcome of this misfortune.

2.1 Multiple scale analysis

When ordinary perturbative methods fail to give a uniformly accurate approximation, the method of *multiple scales* is useful. The idea is to introduce a new variable $\tau = \varepsilon t$. Physically, τ represents a longer scale than t, since τ is not negligible when $t \sim 1/\varepsilon$ or larger. Even though $y(t)$ is a function of t alone, through introduction of τ, $y(t)$ becomes a function of t and τ, i.e $y(t, \tau)$. As such, the multiple scales seeks solutions as functions of both t and τ, treating these variables independently. Although an artifice, such treatment in two variables eliminates secularities.

We illustrate the method on the Duffing oscillator in Example 2. Formally, we write $y(t) = Y_0(t, \tau) + \varepsilon Y_1(t, \tau) + \mathcal{O}(\varepsilon^2)$. Using chain rule, we substitute this expansion into (2.4) and collect powers of ε to obtain

$$\mathcal{O}(\varepsilon^0): \quad \frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0, \tag{2.6}$$

$$\mathcal{O}(\varepsilon^1): \quad \frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -Y_0^3 - 2\frac{\partial^2 Y_0}{\partial \tau \partial t}. \tag{2.7}$$

The general solution of (2.6) is $Y_0(t, \tau) = B(\tau)e^{it} + B^*(\tau)e^{-it}$, where $B(\tau)$ is a complex function. We determine $B(\tau)$ by requiring that secular terms do not appear in $Y_1(t, \tau)$. Substituting Y_0 into (2.7) gives

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = \left(-3B^2 B^* - 2i\frac{\mathrm{d}B}{\mathrm{d}\tau}\right) e^{it} + \left(-3B(B^*)^2 + 2i\frac{\mathrm{d}B^*}{\mathrm{d}\tau}\right) e^{-it} \\ - B^3 e^{3it} - (B^*)^3 e^{-3it}. \tag{2.8}$$

Notice that e^{it} and e^{-it} appear in the solution of (2.6), which is also the homogeneous version of (2.7). Therefore, unless the coefficients of e^{it} and e^{-it} vanish, Y_1 grows linearly in t. To preclude secularity, we must have

$$-3B^2 B^* - 2i\frac{\mathrm{d}B}{\mathrm{d}\tau} = 0, \qquad \text{and} \qquad -3B(B^*)^2 + 2i\frac{\mathrm{d}B^*}{\mathrm{d}\tau} = 0.$$

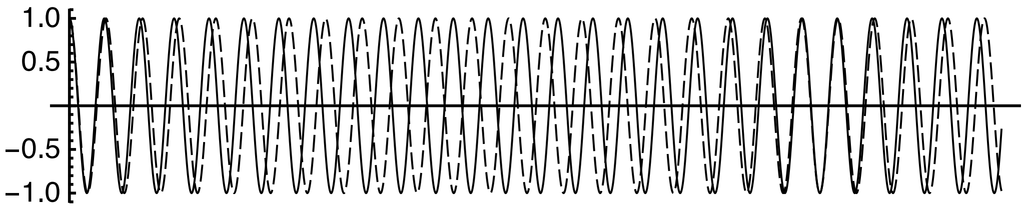
The two equations are complex conjugates of each other, so $B(\tau)$ is not overdetermined. Solving for $B(\tau)$ along with the given initial conditions yields $B(\tau) = \frac{1}{2}e^{i3\tau/8}$. Thus, we obtain $Y_0(t, \tau) = \cos\left(t + \frac{3}{8}\tau\right)$. Finally, using that $\tau = \varepsilon t$ gives an approximate solution

$$y(t) = \cos\left[t\left(1 + \varepsilon\frac{3}{8}\right)\right] + \mathcal{O}(\varepsilon), \qquad \varepsilon \to 0, \quad \varepsilon t = \mathcal{O}(1). \tag{2.9}$$

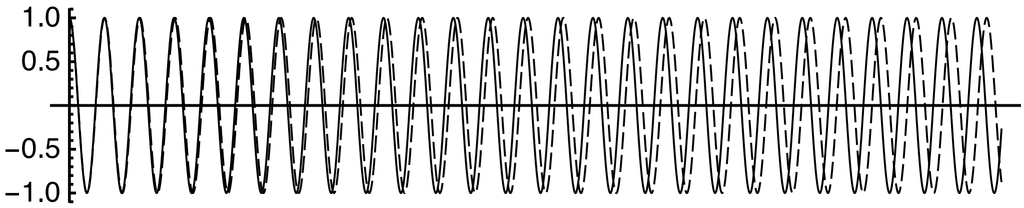
To conclude, we note that while (2.5) approximates y well for $0 \leqslant t \ll \mathcal{O}(1/\varepsilon)$, (2.9) approximates y over a much larger range (see Figure 2.2).

The choice of scales $\tau = \varepsilon t$ tends to be example-specific. In general, one may choose $\tau = f(t)$, where f can be any function. For example, for an IVP $\{y'' + y - \varepsilon t y = 0,\ y(0) = 1,\ y'(0) = 0\}$, one uses $\tau = \sqrt{\varepsilon}t$, and for $y'' + \omega^2(\varepsilon t)y = 0$, one uses $\tau = \int^t \omega(\varepsilon s)\,\mathrm{d}s$.

Remark 3. It is important to understand the need for multiple scales. For the Duffing oscillator IVP, we could solve the problem numerically, or approximate the solution via multiple scales. A question arises: why use



(A) Exact $y(t)$ (line) vs. $y(t) = \cos t$ (dashed)



(B) Exact $y(t)$ (line) vs. $y(t) = \cos\left[t(1+\varepsilon\frac{3}{8})\right]$ (dashed)

FIGURE 2.2: Superposed solutions of the Duffing oscillator IVP when $\varepsilon = 0.1$, for $t \in [0, 160]$. Note that $\cos t$ is not valid for large values of t; at $t = 160$, $\cos t$ is one cycle out of phase with the exact solution $y(t)$ (Plot 2.2a). However, multiple scales solution approximates $y(t)$ closely, even for large values of t (Plot 2.2b).

multiple scales when differential equations can be solved numerically?

Note that many real-world phenomena are expressed in terms of PDEs, which are much harder to solve numerically than ODEs. Due to many differences in the physical phenomena, there is no unified analytic and numerical treatment. In addition, developing numerical schemes can be tricky, since issues such as stability, error, as well as the physical conditions need to be carefully addressed. Furthermore, numerically solving PDEs can be time-consuming and require significant computational power, especially if high precision is required. The latter is particularly important for real-time predicting. This is another reason to prefer multiple scales. When appropriately applied, multiple scales and perturbation methods turn the original, difficult problem into many simpler problems. These problems are much easier to solve either analytically or numerically, and provide further insight into the mathematics and physics of the problem.

Chapter 3

Water-wave problem

The theory of water waves has been a source of intriguing and difficult mathematical problems. Derived by Euler in 1757, the incompressible Euler equations with a free boundary, also known as the full water-wave problem, are widely regarded as the governing equations for water waves and have been the subject of extensive research. At first, the essential character of studying water waves was in bringing together equations of fluid mechanics, illustrating wave propagation, and highlighting the role of boundary conditions. However, during the last 60 years, the complexity of mathematical theories for water waves has exploded. The emergence of *soliton* theory, itself conceived in the context of water waves, has transformed many mathematical aspects of nonlinear wave propagation. Today, the study of water waves is an important part of applied mathematics, serving as a diverse intersection in which many seemingly disparate parts of mathematics come together.

That water waves are a physical phenomenon has an added advantage: often, they may be analysed by direct observation. Indeed, mathematical results provide a description that can be tested whenever an expanse of water is at hand: a river, the ocean, or simply the household sink. In particular, such results may have wide-ranging implications in engineering and climate modelling. Conversely, numerical simulations by themselves have led to many advances in water waves. The complexity and variety of water-wave phenomena inform many applications but also invite other sciences to contribute, thereby making the field suitable for multidisciplinary collaborations.

In this chapter, we delve into the water-wave problem. First, we explain the problem in greater detail and present the shallow water regime. Using asymptotic tools, we then describe and justify the derivation of the KdV model.

3.1 Describing the water-wave problem

Conservation of mass (3.1) and conservation of momentum (3.2) are the two core principles that provide the relevant equations of fluid dynamics. The resulting equations are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{3.1}$$

$$\rho \left[v \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = \mathbf{F} - \nabla P + v_\star \Delta \mathbf{v}, \tag{3.2}$$

where $\rho = \rho(\mathbf{x}, t)$ denotes the fluid mass density, $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ is the fluid velocity, $P(\mathbf{x}, t)$ refers to pressure, $\mathbf{F}(\mathbf{x})$ is an external force, and $v_\star$ is the viscosity due to frictional forces. Derivations of (3.1) and (3.2) can be found in Johnson, 1997, Chapter 3. Assuming that the fluid is inviscid, incompressible, and irrotational, one can follow Section 5.1 of Ablowitz, 2011 to obtain the system

$$\begin{aligned}
\phi_{xx} + \phi_{zz} &= 0, & -h < z < \eta(x,t), && \text{(3.3a)}\\
\phi_z &= 0, & z = -h, && \text{(3.3b)}\\
\phi_t + g\eta + \frac{1}{2}(\phi_x^2 + \phi_z^2) &= 0, & z = \eta(x,t), && \text{(3.3c)}\\
\eta_t + \phi_x\eta_x &= \phi_z, & z = \eta(x,t), && \text{(3.3d)}
\end{aligned}$$

where $\phi(x,z,t)$ is a scalar field such that $\mathbf{v} = \nabla\phi$ and $\eta(x,t)$ is the surface elevation. In addition, z is the vertical coordinate, x is the horizontal direction, and g is acceleration due to gravity. See Figure 3.1 for a visual representation of the domain, which we denote $S = \mathbb{R} \times (-h, \eta)$. In deriving (3.3), the following assumptions are made:

- The problem has 1 horizontal dimension $x \in \mathbb{R}$.
- The fluid velocity $\mathbf{v}$ tends to equilibrium as $|x| \to \infty$.
- The external force is the buoyancy due to gravity, i.e. $\mathbf{F} = -\nabla(\rho_0 g z)$.
- The pressure vanishes on the surface, $P = 0$ at $z = \eta(x,t)$.
- The fluid density is constant, i.e. $\rho(\mathbf{x}, t) = \rho_0$.

As the problem is expressed in terms of ϕ, the scalar potential of velocity $\mathbf{v}$, (3.3) is called the *velocity potential* formulation of the water-wave problem.

We now describe the physical relevance of (3.3), by explaining each equation:

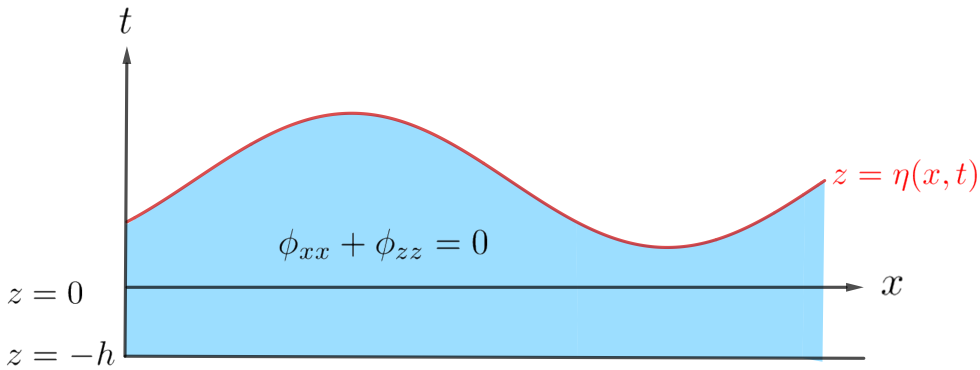


FIGURE 3.1: Schematic for the domain of the water wave problem (3.3).

(3.3a): Assuming that the fluid is irrotational means that the curl vanishes: $\nabla \times \mathbf{v} = 0$. Thus, there is a scalar field ϕ such that $\nabla\phi = \mathbf{v}$. Conservation of mass (3.1) then becomes $\Delta\phi = 0$. In other words, the fluid inside the domain S is incompressible and irrotational.

(3.3b): This equation is an assumption that the bottom is a flat and impermeable surface, so that the fluid cannot escape through the bottom. Since ϕ_z is the vertical velocity, (3.3b) means that there is no flow through the bottom. We call (3.3b) *the bottom condition*.

(3.3c): This equation is the conservation of momentum (3.2) applied at the surface $z = \eta(x,t)$. Since the conservation of momentum is a statement about the balance of external forces and fluid at the fluid's surface, this equation describes the dynamics of the velocity potential on the boundary. We term (3.3c) as *the dynamic boundary condition*.

(3.3d): This equation represents the condition that η is the surface of the fluid. In other words, the surface $z = \eta(x,t)$ is always composed of fluid particles that remain on the surface. We call (3.3d) *the kinematic* *boundary condition*, since it describes the geometry and shape of the surface. The condition should be contrasted with the dynamic condition, which is about the interaction of forces acting at the surface.

Remark 4. While the wave motion is expected to be initiated in some fashion, we are mainly interested in evolution of wave motion. As such, initial conditions are not mentioned explicitly.

3.2 Shallow water regime

For now, (3.3) admits numerous types of water waves: short waves, long waves, intermediate waves. This is because we have yet to specify how wavelength relates to the water depth. Thus, we first examine the *dispersion relation*, which describes the relation between wave velocity and wavelength. This will allow us to focus on the shallow water regime.

We consider small-amplitude waves, or equivalently, we assume that $|\eta| \ll 1$ and $\|\nabla\phi\| \ll 1$. The dispersion relation is obtained by linearising the problem (3.3) around $z = 0$. More concretely, let $\phi_s(x, z, t) = \tilde{\phi}(k, z, t)\exp(ikx)$, and $\eta_s(x, t) = \tilde{\eta}(k, t)\exp(ikx)$; we think of ϕ_s and η_s as special form solutions ϕ and η. We then may follow Section 5.2 of Ablowitz, 2011, to obtain the following ODE:

$$\frac{\partial^2 \tilde{\eta}}{\partial t^2} + gk\tanh(kh)\tilde{\eta} = 0.$$

Assuming that $\tilde{\eta}(k, t) = \tilde{\eta}(k, 0)\exp(-i\omega t)$, the above equation yields the dispersion relation $\omega^2 = gk\tanh(kh)$. Here, ω is a frequency, k is a wave number, and g is gravity.

With this in mind, we focus on the shallow water regime, characterised by small-amplitude waves that have long wavelength relative to the water depth. For shallow water, the wavelength $\frac{2\pi}{k}$ is much bigger than the depth h, so $kh \ll 1$. Expansion of $\tanh(kh)$ in kh allows to rewrite the dispersion relation as

$$\omega^2 = gk(kh - \frac{(kh)^3}{3} + \mathcal{O}((kh)^5)) \simeq ghk^2.$$

Thus, small-amplitude waves in shallow water have frequency $\omega = \pm\sqrt{gh}|k|$, or equivalently, velocity $c_0 = \sqrt{gh}$. Now, using the dispersion relation, we can *rescale* the problem (3.3) to model shallow water waves with velocity $c_0 = \sqrt{gh}$. In nature, tsunamis and tidal waves are examples of this regime.

In addition, the problem (3.3) does not specify the physical dimensions. Since dimensions of the problem are directly related to the units of variables such as wavelength, time, or height, it can be difficult to decide which terms are negligible when performing asymptotic reductions. The process of *non-dimensionalisation* removes the physical dimensions, allowing us to work with "pure" numbers. Letting primes denote the dimensionless variables, we introduce

$$z = hz' \qquad x = \lambda x' \qquad t = \frac{\lambda}{c_0}t' \qquad \eta = a\eta' \qquad \phi = \frac{\lambda g a}{c_0}\phi', \tag{3.4}$$

where $c_0 = \sqrt{gh}$ is the shallow water speed, λ is the wavelength and a is the maximum amplitude of initial data. See Sections 1.3.2-1.3.3 of Lannes, 2013 for a detailed discussion of (3.4).

We define parameters $\varepsilon = \frac{a}{h}, \mu = \frac{h}{\lambda}$. Physically, ε is an amplitude of a wave relative to depth, while μ is a ratio of depth to a typical wavelength. Alternatively, we understand that ε measures nonlinearity and μ measures dispersion.

Note that we have yet to make any assumptions about ε and μ, nor is any relationship between the two parameters prescribed. We make the following assumptions:

- Assume $\mu \ll 1$. Recall that μ is a ratio of depth to wavelength, and in shallow water regime, we expect that depth is much smaller compared to wavelength.

- To obtain interesting approximate equations, we should balance the parameters by connecting them to each other. This is *the principle of maximal balance* (see Kruskal, 1963). We choose $\varepsilon = \mu^2$, which reflects the balance of weak nonlinearity and weak dispersion.

- From the maximal balance principle, we have $\varepsilon \ll 1$. Physically, we consider water waves whose amplitude is small relative to depth.

Transforming (3.3) via chain rule and dropping the primed notation yields

$$
\begin{aligned}
\varepsilon\phi_{xx} + \phi_{zz} &= 0, & -1 < z < \varepsilon\eta, && \text{(3.5a)}\\
\phi_z &= 0, & z = -1, && \text{(3.5b)}\\
\phi_t + \frac{1}{2}\left(\varepsilon\phi_x^2 + \phi_z^2\right) + \eta &= 0, & z = \varepsilon\eta(x,t), && \text{(3.5c)}\\
\varepsilon\left[\eta_t + \varepsilon\phi_x\eta_x\right] &= \phi_z, & z = \varepsilon\eta(x,t). && \text{(3.5d)}
\end{aligned}
$$

The problem (3.5) is a "normalised" problem that models small-amplitude shallow water waves.

Remark 5. Note that there is no reason not to balance in other ways, say $\varepsilon = \sqrt{\mu}$. There are many options: some lead to interesting equations, while others do not. Indeed, it is this assumption in the procedure that determines the relevance of the model derived below.

3.3 Deriving the KdV model

The presence of ε in (3.5) indicates that we can further simplify the problem using ideas developed in Chapter 2. Expand $\phi(x,z,t) = \phi_0(x,z,t) + \varepsilon\phi_1(x,z,t) + \mathcal{O}(\varepsilon^2)$. Substituting the series into (3.5a) and using (3.5b) yields

$$\phi = A - \frac{\varepsilon}{2} A_{xx}(z+1)^2 + \frac{\varepsilon^2}{4!} A_{xxxx}(z+1)^4 + \mathcal{O}(\varepsilon^3) \tag{3.6}$$

valid in $-1 < z < \varepsilon\eta$, where $\phi_0 = A(x,t)$. Substituting (3.6) into (3.5c) and (3.5d), along with appropriate manipulations gives

$$A_{tt} - A_{xx} = \varepsilon \left(\frac{A_{xxxx}}{3} - 2A_x A_{xt} - A_{xx} A_t \right), \tag{3.7}$$

valid up to $\mathcal{O}(\varepsilon)$. Obtaining (3.7) is an especially lengthy calculation, since the dynamic and kinematic conditions must be expanded carefully.

Now, we sketch out the derivation of the KdV model. Expand $A = A_0 + \varepsilon A_1 + \mathcal{O}(\varepsilon^2)$; substituting this expansion into (3.7), we have

$$\mathcal{O}(\varepsilon^0): \quad A_{0tt} - A_{0xx} = 0. \tag{3.8}$$

This is the wave equation, whose general solution is $A_0 = F(x-t) + G(x+t)$, for some functions F, G.

We would like to determine F, G. First, we observe that in parallel to the Duffing oscillator, (3.7) contains secularities in t when solving in the next order. This can partly be shown via the dispersion relation, and therefore warrants an introduction of multiple time scales: $\tau_0 = t, \tau_1 = \varepsilon t$. We also let $\xi = x - \tau_0$ and $\zeta = x + \tau_0$, so that $A_0 = F(\xi, \tau_1) + G(\zeta, \tau_1)$. Via appropriate calculations, from (3.7) one obtains

$$2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + 3FF_{\xi} = 0 \tag{3.9}$$

$$2G_{\tau_1} - \frac{1}{3}G_{\zeta\zeta\zeta} - 3GG_{\zeta} = 0. \tag{3.10}$$

Hence, we obtain two KdV equations, (3.9) and (3.10), which determine the dependence of F, G on τ_1. Solving the KdV equations via initial conditions, we determine $\phi_0 = A_0$, and therefore ϕ in the leading order.

The wave and KdV equations are well-known PDEs. The wave equation arises as a model in numerous fields of physics, such as electrodynamics, plasma physics, and general relativity. The KdV equation appears whenever long waves propagate over dispersive media, be it in the fields of fluid mechanics, nonlinear optics, or Bose-Einstein condensates. Because of how they occur independently of applications, the two equations have been studied extensively. Furthermore, the KdV equation is special: despite being nonlinear, it can be solved exactly in certain domains, using advanced techniques such as the inverse scattering transform (see Chapter 9 of Ablowitz, 2011).

In conclusion, the leading order solution of the water wave problem for small-amplitude, shallow water waves is described by the wave equation (3.8) and two KdV equations (3.9), (3.10). This is the result we seek to obtain on the whole line using a different formulation of the problem, as well as a new extension to the half-line.

3.4 Are asymptotic methods reliable?

Given the emphasis on asymptotic and perturbative methods, one is interested whether these methods provide "reliable" solutions. Formally, how can we justify that solutions of asymptotic equations converge to solutions of the original problem? In the context of the water-wave problem (3.3), is the KdV model, provided by the shallow water approximation, "reasonable"? Of course, in asking these questions, one needs to specify the meaning of “reliable" and “reasonable". Following Lannes, 2013, the validity of the KdV model can be understood from the following questions:

1. Do the solutions of (3.3) exist on the required time scale?
2. Do the solutions of the KdV model exist on the same time scale?
3. Are the asymptotic solutions close to the actual solutions with the corresponding initial data? If so, how close?

If the answer to all three questions is positive, then the asymptotic model is *fully justified*. Indeed, the KdV model is fully justified (see Lannes, 2013, p. 297-298). Actual proofs of the answers require advanced mathematics: working in Sobolev spaces and applying Poincaré and Gagliardo-Nirenberg inequalities (see Chapter 7 and Appendix C in Lannes, 2013). Since the rigorous mathematical justification of the KdV model is beyond the scope of the project, we choose not to discuss this topic.

Chapter 4

Non-local derivation on the whole line

As mentioned in Chapter 3, a direct derivation of the KdV model is subject to lengthy calculations and careful bookkeeping. If we are to change domains (say, the half-line), then additional care must be taken to ensure that the model is correct. As such, we look for a more efficient way of deriving asymptotic models valid on various domains.

Recall that the water-wave problem is challenging to work with directly, due to the nonlinear boundary conditions and the domain with an unknown shape. To address these issues, reformulations of the problem are introduced, which result in equivalent problems that are more tractable. Below, we give a short overview of these formulations.

For one-dimensional surfaces (one horizontal variable), conformal mappings can be used to eliminate some of the issues (for an overview, see Dyachenko et al., 1996). However, this approach is limited to one-dimensional surfaces. For both one- and two-dimensional surfaces, formulations such as the Hamiltonian formulation given in Zakharov, 1968 or Craig and Sulem, 1993 reduce the problem to a system of two equations in terms of surface variables $q(x,t) = \phi(x,\eta(x,t),t)$ and $\eta(x,t)$. This is achieved by introducing a Dirichlet-to-Neumann operator (DNO formulation). Another non-local formulation introduced in Ablowitz, Fokas, and Musslimani, 2006 (AFM formulation) results in a system of two equations for the same variables as in the DNO formulation. However, both the DNO and AFM formulations involve solving for $q(x,t)$, which may be of little relevance in applications and is typically hard to measure in experiments. We are generally interested in the wave height $\eta(x,t)$.

A new formulation is introduced in Oliveras and Vasan, 2013. In the work, the authors formally eliminate $q(x,t)$, reducing the water-wave problem to a system of two equations in one variable $\eta(x,t)$. This formulation allows rigorous investigations of one- and two-dimensional water waves. Computation of Stokes-wave asymptotic expansions for periodic waves justifies the use of the formulation; indeed, following Oliveras and Vasan, 2013, computations can be performed with arguably less effort, especially for two-dimensional waves. Our goal is to further justify the use of this formulation, which we call the $\mathcal{H}$ formulation.

In this chapter, we first rewrite the problem by introducing the normal-to-tangential $\mathcal{H}$ operator. We then perform an expansion for the operator and proceed to obtain an expression for the surface elevation. Finally, asymptotic reductions and multiple time scales yield the desired approximate equations. We emphasise that our intention in this chapter is to demonstrate the efficacy of the $\mathcal{H}$ formulation for doing asymptotics. For efficiency of presentation, many steps in calculations are omitted.

4.1 Non-local formulation on the whole line

We seek to reformulate the problem (3.3). Consider the velocity potential evaluated at the surface $q(x,t) = \phi(x,\eta(x,t),t)$. Combining (3.3d) and (3.3c) evaluated at $z = \eta$, we obtain

$$q_t + \frac{1}{2}q_x^2 + g\eta - \frac{1}{2}\frac{(\eta_t + q_x\eta_x)^2}{1+\eta_x^2} = 0, \tag{4.1}$$

which is an equation for two unknowns $q(x,t)$ and $\eta(x,t)$. We aim to find an equation in $\eta(x,t)$ only.

Let $\mathbf{N} = [-\eta_x, 1]^T$ and $\mathbf{T} = [1, \eta_x]^T$ be vectors normal and tangent to the surface $z = \eta(x,t)$, respectively. We introduce an operator $\mathcal{H}(\eta, D)$ that maps the normal derivative at the surface $z = \eta$ to the tangential derivative at this surface:

$$\mathcal{H}(\eta, D)\{\nabla\phi \cdot \mathbf{N}\} = \nabla\phi \cdot \mathbf{T}, \tag{4.2}$$

where $D = -i\partial_x$. Note that by (3.3d), $\nabla\phi \cdot \mathbf{N} = \phi_z - \phi_x\eta_x = \eta_t$, and by chain rule, $\nabla\phi \cdot \mathbf{T} = \phi_x + \eta_x\phi_z = q_x$. This lets us rewrite (4.2) as

$$\mathcal{H}(\eta, D)\{\eta_t\} = q_x. \tag{4.3}$$

Together, (4.1) and (4.3) form a system of two equations for two unknowns $q(x,t)$, and $\eta(x,t)$. Differentiating (4.1) with respect to x and (4.3) with respect to t reduces the system to a single equation for $\eta(x,t)$:

$$\begin{aligned} &\partial_t\left(\mathcal{H}(\eta, D)\{\eta_t\}\right) \\ &\quad + \partial_x\left(\frac{1}{2}\left(\mathcal{H}(\eta, D)\{\eta_t\}\right)^2 + \varepsilon\eta - \frac{1}{2}\frac{(\eta_t + \eta_x\mathcal{H}(\eta, D)\{\eta_t\})^2}{1+\eta_x^2}\right) = 0. \end{aligned} \tag{4.4}$$

Equation (4.4) represents a scalar equation for $\eta(x,t)$, incorporating the dynamic and kinematic boundary conditions. The utility of (4.4) depends on whether we can find a useful representation for the operator $\mathcal{H}(\eta, D)$.

4.2 Behaviour of the $\mathcal{H}$ operator

In the previous section, we obtain a scalar equation (4.4) in terms of the surface variable $\eta(x,t)$ and the operator $\mathcal{H}$. In this section, we derive another, nonlocal equation for η and $\mathcal{H}$, thereby completing the system.

Consider the following boundary value problem:

$$\phi_{xx} + \phi_{zz} = 0, \qquad -h < z < \eta(x,t), \tag{4.5a}$$

$$\phi_z = 0, \qquad z = -h, \tag{4.5b}$$

$$\nabla\phi \cdot \mathbf{N} = f(x), \qquad z = \eta(x,t), \tag{4.5c}$$

where $f(x)$ is a smooth function. Let φ be harmonic on $S = \mathbb{R} \times (-h, \eta)$; using (4.5a) and that φ_z is also harmonic on S, we have

$$\varphi_z(\phi_{xx} + \phi_{zz}) - \phi((\varphi_z)_{zz} + (\varphi_z)_{xx}) = 0.$$

Taking the integral over the domain yields

$$\int_{-\infty}^{\infty}\int_{-h}^{\eta} \varphi_z(\phi_{xx} + \phi_{zz}) - \phi((\varphi_z)_{zz} + (\varphi_z)_{xx})\,\mathrm{d}z\,\mathrm{d}x = 0.$$

An application of Green's theorem gives

$$\int_{\partial S} \varphi_z(\nabla\phi \cdot \mathbf{n}) - \phi(\nabla\varphi_z \cdot \mathbf{n})\,\mathrm{d}s = 0, \tag{4.6}$$

where ∂S is the boundary of S, $\mathrm{d}s$ is an area element, and $\mathbf{n}$ is the vector outward normal to the boundary. Now, observe that $-\nabla\varphi_z \cdot \mathbf{n} = \nabla\varphi_x \cdot \mathbf{t}$, where $\mathbf{t}$ is the vector tangential to the boundary of the domain. Using this to rewrite (4.6), we obtain the following contour integral:

$$\begin{aligned} 0 &= \int_{\partial S} \varphi_z(\nabla\phi \cdot \mathbf{n}) + \phi(\nabla\varphi_z \cdot \mathbf{t})\,\mathrm{d}s \\ &= \int_{\partial S} \varphi_z(\phi_z\,\mathrm{d}x - \phi_x\,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z). \end{aligned} \tag{4.7}$$

Splitting the contour and rewriting the improper integral as a limit yields

$$\begin{aligned} &\int_{\partial S} (\cdot)\,\mathrm{d}s \\ &\quad= \left\{ \int_{-\infty}^{\infty} (\cdot)\Big|_{z=-h} + \lim_{R\to\infty} \int_{-h}^{\eta} (\cdot)\Big|_{x=R} + \int_{\infty}^{-\infty} (\cdot)\Big|_{z=\eta} + \lim_{R\to\infty} \int_{\eta}^{-h} (\cdot)\Big|_{x=-R} \right\} \mathrm{d}s, \end{aligned}$$

where $(\cdot)$ stands for the integrand in (4.7).

We consider each segment. As $R \to \infty$, we require that ϕ and its gradient vanish, so these integrals vanish. At $z = -h$, we can pick φ such that $\varphi_x(x, -h) = 0$. The bottom condition and integration by parts then reveal that the contribution at $z = -h$ vanishes as well. Finally, at $z = \eta(x,t)$, integration parts and recognising normal and tangential derivatives yield

$$\int_{-\infty}^{\infty} \varphi_z(\phi_z - \phi_x\eta_x) + \phi(\varphi_{xx} + \varphi_{xz}\eta_x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \varphi_z\nabla\phi \cdot \mathbf{N} - \varphi_x\nabla\phi \cdot \mathbf{T}\,\mathrm{d}x.$$

Finally, recalling (4.5c) and (4.2), we reduce the contour integral (4.7) to

$$\int_{-\infty}^{\infty} \varphi_z f(x) - \varphi_x(x,\eta)\mathcal{H}(\eta, D)\{f(x)\}\,\mathrm{d}x = 0. \tag{4.8}$$

Note that $\varphi(x,z) = e^{-ikx}\sinh(k(z+h))$, $k \in \mathbb{R}$ is one solution of the problem $\Delta\varphi = 0$, $\varphi_z(-h,z) = 0$. Substituting $e^{-ikx}\sinh(k(z+h))$ into (4.8) yields

$$\int_{-\infty}^{\infty} e^{-ikx}(k\cosh(k(\eta+h))f(x) + ik\sinh(k(\eta+h))\mathcal{H}(\eta,D)\{f(x)\})\,\mathrm{d}x = 0. \tag{4.9}$$

Taking out k in the integral gives

$$\int_{-\infty}^{\infty} e^{-ikx}(i\cosh(k(\eta+h))f(x) - \sinh(k(\eta+h))\mathcal{H}(\eta,D)\{f(x)\})\,\mathrm{d}x = 0, \tag{4.10}$$

valid for all $k \in \mathbb{R}$. Equation (4.10) gives a description for the operator $\mathcal{H}(\eta,D)$ in dimensional coordinates.

Remark 6. We observe that (4.10) actually holds only for $k \neq 0$. However, for the water wave problem, $f(x) = \eta_t$. As $k \to 0$, (4.10) then reduces to

$$\int_{-\infty}^{\infty} \eta_t\,\mathrm{d}x = 0.$$

This equation is known to be true and represents the conservation of mass. See Benjamin and Olver, 1982 for details.

As mentioned in Chapter 3, to derive asymptotic models we need to work in non-dimensional variables. Using the same rescaling and nondimensionalisation as in (3.4), via the same procedure one obtains

$$\int_{-\infty}^{\infty} e^{-ikx}(i\cosh(\mu k(\eta+1))\tilde{f}(x) - \sinh(\mu k(\eta+1))\mathcal{H}(\varepsilon\eta,D)\{\tilde{f}(x)\})\,\mathrm{d}x = 0, \tag{4.11}$$

where $k \in \mathbb{R}$ and the primed notation is dropped for convenience. In addition, note that $\tilde{f}$ and f are related by $\tilde{f}(x') = \dfrac{\sqrt{gh}}{ga}f(x)$.

In summary, introduction of the normal-to-tangential operator $\mathcal{H}(\eta, D)$ reduces the problem (3.3) to the scalar equation (4.4) for η, where the operator $\mathcal{H}$ is described via (4.10). This is the non-local formulation presented in Oliveras and Vasan, 2013.

4.3 Expansion of the $\mathcal{H}$ operator

As the relation in (4.11) is implicit, it is difficult to solve for the operator $\mathcal{H}(\eta, D)$ directly. Therefore, following Craig and Sulem, 1993, we find a formal series expansion for the operator via perturbative methods. Since $\varepsilon \ll 1$, we expand the hyperbolic functions as a Taylor series in ε :

$$\begin{aligned} \cosh(\mu k(\eta+1)) &= \cosh(\mu k) + \mu k \varepsilon \eta \sinh(\mu k) + \mathcal{O}(\varepsilon^2), \\ \sinh(\mu k(\eta+1)) &= \sinh(\mu k) + \mu k \varepsilon \eta \cosh(\mu k) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Now, we note that the idea of formally expanding as a perturbation series can be extended to operators. The formal expansion is given by

$$\mathcal{H}(\varepsilon\eta, D)\{g(x)\} = \sum_{j=0}^{\infty} \mathcal{H}_j(\varepsilon\eta, D)\{g(x)\},$$

where $\mathcal{H}_j$ is homogeneous of degree j, i.e. $\mathcal{H}_j(\varepsilon\eta, D) = \varepsilon^j \mathcal{H}_j(\eta, D)$. Let $g(x) = \tilde{f}(x)$ so that the scalar equation (4.11) becomes:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-ikx} \Big(& i \left[\cosh(\mu k) + \mu k \varepsilon \eta \sinh(\mu k) + \mathcal{O}(\varepsilon^2) \right] g(x) \\ & - \left[\sinh(\mu k) + \mu k \varepsilon \eta \cosh(\mu k) + \mathcal{O}(\varepsilon^2) \right] \left[\mathcal{H}_0 + \varepsilon \mathcal{H}_1 + \mathcal{O}(\varepsilon^2) \right] (\varepsilon\eta, D)\{g(x)\} \Big) \, \mathrm{d}x = 0, \end{aligned} \tag{4.12}$$

At leading order $\mathcal{O}(\varepsilon^0)$: Using (4.12), we obtain

$$\int_{-\infty}^{\infty} e^{-ikx}(i\cosh(\mu k)g(x) - \sinh(\mu k)\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\})\,\mathrm{d}x = 0.$$

For $k \neq 0$, dividing by $\sinh(\mu k)$ yields

$$\int_{-\infty}^{\infty} e^{-ikx}(i\coth(\mu k)g(x) - \mathcal{H}_0(\varepsilon\eta, D)\{g(x)\})\,\mathrm{d}x = 0.$$

Splitting the integrand and recognising the Fourier transform yields:

$$\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\} = \mathcal{F}_k^{-1}\left\{i\coth(\mu k)\mathcal{F}_k\left\{g(x)\right\}\right\}. \tag{4.13}$$

If $\mathcal{F}_k\left\{g(x)\right\} \to 0$ faster than $\mathcal{O}(\mu k)$ as $k \to 0$, (4.13) is defined for all $k \in \mathbb{R}$ (see Remark 7).

In the next order $\mathcal{O}(\varepsilon^1)$: From (4.12), we obtain

$$\int_{-\infty}^{\infty} e^{-ikx}(i\mu k\eta\sinh(\mu k)g - \left[\sinh(\mu k)\mathcal{H}_1 + \mu k\eta\cosh(\mu k)\mathcal{H}_0\right]\{g\})\,\mathrm{d}x = 0,$$

where we drop $(\varepsilon\eta, D)$ for ease of notation. Dividing by $\sinh(\mu k)$, we have

$$\int_{-\infty}^{\infty} e^{-ikx}(i\mu k\eta g(x) - \left[\mathcal{H}_1 + \mu k\eta\coth(\mu k)\mathcal{H}_0\right]\{g(x)\})\,\mathrm{d}x = 0, \quad k \neq 0.$$

Splitting the integral and recognising the Fourier transform yields

$$\begin{aligned}\mathcal{H}_1\{g(x)\} &= \mathcal{F}_k^{-1}\left\{ik\mathcal{F}_k\left\{\mu\eta g\right\}\right\} - \mathcal{F}_k^{-1}\left\{\mu k\coth(\mu k)\mathcal{F}_k\left\{\eta\mathcal{H}_0\{g\}\right)\right\}\right\} \\ &= \mu\partial_x(\eta g) - \mathcal{F}_k^{-1}\left\{\mu k\coth(\mu k)\mathcal{F}_k\left\{\eta\mathcal{F}_l^{-1}\left\{i\coth(\mu l)\mathcal{F}_l\left\{g\right\}\right\}\right\}\right\},\end{aligned} \tag{4.14}$$

where we write out spectral parameters k, l to keep track of transforms. In sum, by expanding and collecting like powers of ε, we find

$$\mathcal{H}(\varepsilon\eta, D)\{g(x)\} = [\mathcal{H}_0 + \varepsilon\mathcal{H}_1](\varepsilon\eta, D)\{g(x)\} + \mathcal{O}(\varepsilon^2),$$

where $\mathcal{H}_0$ is given by (4.13) and $\mathcal{H}_1$ is given by (4.14). Following this procedure, a formal recursion formula can be obtained, so that each $\mathcal{H}_j$ can be written in terms of $\mathcal{H}_i$, for $i = 0, 1, \dots, j-1$. For our needs, the first two terms are sufficient.

Remark 7. Recall that $\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\} = \mathcal{F}_k^{-1}\{i \coth(\mu k)\mathcal{F}_k\{g(x)\}\}$. Expanding $\coth(\mu k)$ via its Laurent series in μk gives

$$\coth(\mu k) = \frac{1}{\mu k} + \frac{\mu k}{3} + \mathcal{O}(\mu^3).$$

Since $\coth(\mu k)$ has a simple pole at $k = 0$, so do $\mathcal{H}_0, \mathcal{H}_1$ and $\mathcal{H}$. As such, so long as $\mathcal{F}_k\{g(x)\} \to 0$ as $k \to 0$ at a rate faster than $\mathcal{O}(\mu k)$, then $\lim_{k\to 0} \mathcal{F}_k\{\mathcal{H}_0\}$ exists and is finite. With this condition, $\mathcal{H}_0$ and $\mathcal{H}_1$ are defined for all $k \in \mathbb{R}$.

4.4 Deriving an expression for surface elevation

We proceed to derive a second order approximation for η, using (4.4) and the expansion for $\mathcal{H}$. The non-dimensional version of (4.4) is given by

$$\begin{aligned} \partial_t\Big(\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\Big) + \partial_x\bigg(\frac{1}{2}\Big(\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\Big)^2 \\ + \varepsilon\eta - \frac{1}{2}\varepsilon^2\mu^2\frac{(\eta_t + \eta_x\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\})^2}{1 + \varepsilon^2\mu^2\eta_x^2}\bigg) = 0. \end{aligned} \tag{4.15}$$

Recall $\varepsilon = \mu^2$; keeping the terms up to the next order, (4.15) becomes

$$\partial_t\left(\mathcal{H}_0\{\varepsilon\mu\eta_t\} + \varepsilon\mathcal{H}_1\{\varepsilon\mu\eta_t\}\right) + \partial_x\left(\frac{1}{2}(\mathcal{H}_0\{\varepsilon\mu\eta_t\})^2 + \varepsilon\eta\right) = 0. \tag{4.16}$$

Using (4.13) and (4.14), we rewrite (4.16) to obtain

$$\begin{aligned}&\varepsilon\mu\mathcal{F}_k^{-1}\left\{i\coth(\mu k)\widehat{\eta_{tt}}_k\right\} + \varepsilon^2\mu^2(\eta\eta_t)_{tx} + \frac{\varepsilon^2}{2}\partial_x\left(\mathcal{F}_j^{-1}\left\{i\mu\coth(\mu j)\widehat{\eta_t}_j\right\}\right)^2 + \varepsilon\partial_x\eta\\&\quad - \varepsilon^2\mathcal{F}_k^{-1}\left\{\mu k\coth(\mu k)\mathcal{F}_k\left\{\partial_t\left[\eta\mathcal{F}_l^{-1}\left\{i\mu\coth(\mu l)\widehat{\eta_t}_l\right\}\right]\right\}\right\} = 0\end{aligned} \tag{4.17}$$

Applying the Fourier transform, expanding $\coth(\mu k)$-like terms and keeping the terms up to $\mathcal{O}(\mu^2)$ gives

$$\left(\frac{1}{k} + \frac{\mu^2 k}{3}\right)\widehat{\eta_{tt}}_k - \mu^2\mathcal{F}_k\left\{\partial_t\left[\eta\mathcal{F}_l^{-1}\left\{\frac{1}{l}\widehat{\eta_t}_l\right\}\right]\right\} - \frac{\mu^2}{2}k\mathcal{F}_k\left\{\left(\mathcal{F}_j^{-1}\left\{\frac{1}{j}\widehat{\eta_t}_j\right\}\right)^2\right\} + k\widehat{\eta}_k = 0.$$

Inverting the Fourier transform yields

$$\eta_{tt} - \eta_{xx} = \mu^2\left(\frac{\partial_x^2}{3}\eta_{tt} - i\partial_x\left(\partial_t\left[\eta\mathcal{F}_l^{-1}\left\{\frac{1}{l}\widehat{\eta_t}_l\right\}\right]\right) - \frac{1}{2}\partial_x^2\left(\mathcal{F}_j^{-1}\left\{\frac{1}{j}\widehat{\eta_t}_j\right\}\right)^2\right). \tag{4.18}$$

We seek to simplify (4.18). First, integration by parts allows us to write

$$\mathcal{F}_l^{-1}\left\{\frac{1}{l}\widehat{\eta_t}_l\right\} = i\int_{-\infty}^{x}\eta_t(x', t)\,\mathrm{d}x'. \tag{4.19}$$

Using (4.19), $\varepsilon = \mu^2$, and that $\eta_{tt} = \eta_{xx} + \mathcal{O}(\mu^2)$, (4.18) becomes

$$\eta_{tt} - \eta_{xx} = \varepsilon\left[\frac{1}{3}\eta_{xxxx} + \partial_x^2\left(\frac{\eta^2}{2} + \left(\int_{-\infty}^{x}\eta_t\,\mathrm{d}x'\right)^2\right)\right]. \tag{4.20}$$

In sum, we obtain (4.20), which is an expression for the surface elevation η in the next order.

Remark 8. In Section 2.4, we mention secularities in the next order. We show that this is not unexpected via the dispersion relation of (4.20). Assuming a wave solution $\tilde{\eta}(x,t) = \exp(i(kx - \omega t))$ and substituting it into the linearised equation leads to $-\omega^2 + k^2 = \varepsilon k^4/3$. For large k, $\omega \sim \pm i\sqrt{\varepsilon}k^2/\sqrt{3}$. Substituting the negative root of ω into $\tilde{\eta}$ gives $\eta(x,t) \approx \exp(ikx)\exp\left(\sqrt{\frac{\varepsilon}{3}}k^2 t\right)$. As $k \to \infty$, η is unbounded in time. This is unphysical, since in reality the wave height is always bounded. This suggests that *the linear part* of (4.20) contains secularities. By itself, this does not warrant multiple scales, as we do not account for nonlinear effects. Still, this information is useful to keep in mind as we proceed.

Remark 9. Why is the derivation of (4.20) included whereas that of (3.7) omitted? First, it is more efficient to derive (4.20) than (3.7). In the velocity potential case, the expansion (3.6) is substituted into (3.5c) to obtain an expression for η in terms of A. Then, (3.6) and the expression for η are substituted into (3.5d) to obtain (3.7) The derivation becomes rather long, requiring careful and tedious computations.

In our derivation, we begin with the scalar equation (4.4) and the non-local equation (4.11). Substituting (4.13) and (4.14) into (4.4), along with asymptotic reductions, we arrive at the expression (4.20). There is much less algebra involved in the derivation, compared to alternate methods. Asymptotic reductions are fairly immediate, since it is easy to estimate orders of $\mathcal{H}_0$ and $\mathcal{H}_1$, due to presence of $\coth \mu k$ in each expression.

Second, the derivation of (3.7) is a known result found in some texts on nonlinear waves (see Chapter 5 of Ablowitz, 2011). However, our derivation is a new result, illustrating for the first time how the non-local $\mathcal{H}$ formulation can be used to derive a well-known asymptotic model.

4.5 Derivation of wave and KdV equations

We derive the approximate equations from (4.20). Anticipating secular terms, we introduce time scales $\tau_0 = t, \tau_1 = \varepsilon t$, so that $\eta(x,t) = \eta(x,\tau_0,\tau_1)$ and expand $\eta = \eta_0 + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2)$. Substituting the series into (4.20), within $\mathcal{O}(\varepsilon^0)$, we obtain the wave equation

$$\eta_{0\tau_0\tau_0} - \eta_{0xx} = 0, \tag{4.21}$$

whose general solution is $\eta_0(x,\tau_0,\tau_1) = F(x-\tau_0,\tau_1) + G(x+\tau_0,\tau_1)$. While the dependence of F and G on τ_0 is determined via the initial conditions, the dependence on τ_1 remains unknown and is obtained by proceeding to the next order.

We introduce left-going and right-going variables $\xi = x - \tau_0$, $\zeta = x + \tau_0$. By chain rule, new variables imply $\partial_x = \partial_\xi + \partial_\zeta$, and $\partial_t = -\partial_\xi + \partial_\zeta + \varepsilon\partial_{\tau_1}$. For ease of notation, we suppress explicit dependence on variables; the reader should remember that F depends on ξ, τ_1, and G depends on ζ, τ_1.

Through the change of variables, the LHS of (4.20) becomes

$$(\partial_t^2 - \partial_x^2)\eta = \varepsilon\left(-4\eta_{1\xi\zeta} - 2F_{\tau_1\xi} + 2G_{\tau_1\zeta}\right) + \mathcal{O}(\varepsilon^2), \tag{4.22}$$

while the RHS of (4.20) becomes

$$\begin{aligned}
&\frac{1}{3}\eta_{xxxx} + \partial_x^2\left(\frac{\eta^2}{2} + \left(\int_{-\infty}^{x}\eta_t\,\mathrm{d}x'\right)^2\right)\\
&= \varepsilon\left(\frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + (\partial_\xi + \partial_\zeta)^2\left(\frac{3}{2}(F^2+G^2) - FG\right)\right) + \mathcal{O}(\varepsilon^2),
\end{aligned} \tag{4.23}$$

where we assume that F, G vanish as $\xi, \zeta \to -\infty$. Combining (4.22) and (4.23), in $\mathcal{O}(\varepsilon^1)$ we rewrite (4.20) to obtain

$$-4\eta_{1\xi\zeta} = 2(F_{\tau_1\xi} - G_{\tau_1\zeta}) + \frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + (\partial_\xi + \partial_\zeta)^2\left(\frac{3}{2}F^2 + \frac{3}{2}G^2 - FG\right). \tag{4.24}$$

In the last term of (4.24), differentiation yields:

$$(\partial_\xi + \partial_\zeta)^2\left(\frac{3}{2}F^2 + \frac{3}{2}G^2 - FG\right) = \partial_\xi(3FF_\xi - GF_\xi) + \partial_\zeta(3GG_\zeta - FG_\zeta) - 2F_\xi G_\zeta,$$

so that (4.24) becomes

$$\begin{aligned} -4\eta_{1\xi\zeta} = {} & \partial_\xi(2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + 3FF_\xi) + \partial_\zeta(-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + 3GG_\zeta) \\ & - (GF_\xi + FG_\zeta) - 2F_\xi G_\zeta. \end{aligned} \tag{4.25}$$

Integration of (4.25) with respect to ζ and ξ yield

$$\begin{aligned} -4\eta_1 = {} & \underbrace{(2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + 3FF_\xi)\zeta}_{\to\infty \text{ as } \zeta\to\infty} + \underbrace{(-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + 3GG_\zeta)\xi}_{\to\infty \text{ as } \xi\to\infty} \\ & - \left(F\int G\,\mathrm{d}\zeta + G\int F\,\mathrm{d}\xi\right) - 2FG. \end{aligned} \tag{4.26}$$

Since η_1 is required to be bounded, the RHS of (4.26) must be bounded as well. Assuming that $F, G, \int_{-\infty}^{\infty} G\,\mathrm{d}\zeta, \int_{-\infty}^{\infty} F\,\mathrm{d}\xi$ are bounded, the only problematic terms are the coefficients of ξ and ζ in (4.26). Unless the under-braced terms vanish, η_1 grows without bound as $\xi, \zeta \to \infty$. Hence, these are the secular terms we wish to remove. Then, we must have

$$2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + 3FF_\xi = 0 \tag{4.27}$$

$$2G_{\tau_1} - \frac{1}{3}G_{\zeta\zeta\zeta} - 3GG_\zeta = 0. \tag{4.28}$$

Equations (4.27) and (4.28) are the KdV equations, which allow us to determine F, G on a slow time scale τ_1.

We interpret the KdV model, given by (4.21), (4.27) and (4.28), as follows. The wave equation (4.21) prescribes a linear interaction of two wave trains F and G on a fast time scale τ_0. The KdV equations (4.27) and (4.28) govern the evolution on a slow time scale τ_1, when each wave train interacts with itself. In particular, that F and G can be decoupled into separate equations means that the left- and right-going waves do not affect each other long enough to interact strongly on the time scale τ_1.

In conclusion, asymptotic analysis of the non-local formulation in shallow water limit gives rise to two KdV equations, (4.27) and (4.28). Given appropriate initial conditions, we can solve these PDEs by means of the inverse scattering transform (see Ablowitz, 2011, Chapter 9). Keeping the leading-order terms, we have approximated wave height $\eta \approx \eta_0 = F(x-t, \varepsilon t) + G(x+t, \varepsilon t)$, where F and G's dependence on $x-t$ and $x+t$ is determined via the initial conditions, and dependence on εt is determined via (4.27) and (4.28). The derivation is complete.

4.6 On KdV derivation

Although we approximate solutions of the water-wave problem, the derivation of the KdV equations deserves a special attention. Here, we obtain these equations when removing secular terms. In literature, another approach is given in Benjamin, Bona, and Mahony, 1972. The authors begin by considering an advection equation,

$$u_t + c_0 u_x = 0, \tag{4.29}$$

which is a model for small-amplitude long waves, propagating in the positive x direction with speed c_0. This model has limited utility, since nonlinear and dispersive effects accumulate and cause the model to lose its validity over large times. One can correct for these effects by considering each effect separately, by introducing a small parameter $\varepsilon \ll 1$. Nonlinearity is accounted by adding $c_0 \varepsilon u u_x$ to (4.29) and dispersion is accounted by adding $c_0 \varepsilon u_{xxx}$ to (4.29).

Adding the terms results in the respective first-order approximations allowing for weakly nonlinear and dispersive effects. The authors then argue that an approximation accounting for both effects can be anticipated by simply combining the ε terms:

$$u_t + u_x + c_0 \varepsilon (u u_x + u_{xxx}) = 0. \tag{4.30}$$

Introducing dimensionless variables and applying Galilean transformations yield the usual form of the KdV equation: $u_t + u u_x + u_{xxx} = 0$.

The derivation is elegant, and certainly much shorter than the one presented in the previous section. However, addition of the ε terms to get (4.30) presupposes a certain balance between nonlinearity and dispersion. There is no reason to assume this choice of balance; indeed, for a self-consistent theory we must account for the nonlinear and dispersive effects simultaneously. Thus, the resulting derivation can be considered ad-hoc, not relying on the direct derivation from the physical model.

Chapter 5

Water waves on the half-line

5.1 Non-local formulation on the half-line

In Chapter 4, we use the $\mathcal{H}$ formulation to obtain the expected, well-known results on the whole line. In this chapter, we use the formulation to study a slightly different problem: the water-wave problem on the half-line. To the best of our knowledge, this is the first time this problem is studied from a formal derivation perspective. Physically, the problem is motivated by the presence of a wall or an impenetrable barrier at $x = 0$. This requires imposing several conditions on both η, ϕ at $x = 0$. As such, we consider the problem (3.3) on the horizontal domain $x \in [0, \infty)$, along with the boundary conditions listed below:

$$\phi_x = 0, \qquad x = 0, \tag{5.1}$$

$$\phi_z(0, \eta, t) = \eta_t(0, t), \qquad (x, z) = (0, \eta), \tag{5.2}$$

In particular, (5.1) implies that the fluid does not leak through the barrier at $x = 0$, and (5.2) governs an interaction between the fluid and the surface at $x = 0$. Our objective is to derive approximate equations analogous to those derived in Chapter 4. While the wave equation is expected, there is no reason to look forward to the KdV equations. Indeed, a literature review reveals that a KdV-like equation has not been derived on the half-line, in the way that we derive the KdV equation on the whole line from the full water-wave problem. Here, we use the $\mathcal{H}$ formulation to derive an asymptotic model on the half-line, highlight the main differences, and discuss the difficulties that arise.

To begin, the scalar equation (4.4) relating η and $\mathcal{H}$ remains the same, while the non-local, dimensionless equation (4.11) becomes

$$\int_0^\infty \cos(kx)\cosh(\mu k(\eta+1))f(x)+\sin(kx)\sinh(\mu k(\eta+1))\mathcal{H}(\varepsilon\eta,D)\{f(x)\}\,\mathrm{d}x=0.$$

It is worth noting that taking the real part of (4.11) and restricting integrals to $[0,\infty)$ yields the half-line, non-local equation.

By the same procedure, the first two terms in the expansion of $\mathcal{H}$ operator, (4.13) and (4.14), are given by Fourier cosine and sine transforms (defined in front matter):

$$\begin{aligned}
\mathcal{H}_0(\varepsilon\eta,D)\{f(x)\} &= -\{\mathcal{F}_k^s\}^{-1}\left\{\coth(\mu k)\widehat{f_c^k}\right\},\\
\mathcal{H}_1(\varepsilon\eta,D)\{f(x)\} &= -\{\mathcal{F}_k^s\}^{-1}\left\{\mu k\widehat{(\eta f(x))}_c^k+\mu k\coth(\mu k)\widehat{(\eta\mathcal{H}_0\{f(x)\})}_c^k\right\}.
\end{aligned}$$

The notable difference from the whole-line is the presence of Fourier cosine and sine transforms, in place of Fourier transform.

We proceed as before in Section 4.4. Within $\mathcal{O}(\mu^2)$, the scalar equation results into an equivalent of (4.20):

$$\eta_{tt} - \eta_{xx} = \mu^2 \left(\frac{1}{3}\eta_{xxxx} + \partial_x \left\{\mathcal{F}_k^s\right\}^{-1} \left\{ \mathcal{F}_k^c \left\{ \partial_t \left(\eta \int_0^x \eta_t \, \mathrm{d}x' \right) \right\} \right\} + \frac{1}{2}\partial_x^2 \left(\int_0^x \eta_t \, \mathrm{d}x' \right)^2 \right). \tag{5.3}$$

The difference between the equations on two domains is the presence of the inverse sine transform of the cosine transform. In particular, this term can be shown to be the Hilbert transform (see Theorem 1 in Aitzhan, 2020b for the details).

Now, we seek to derive an asymptotic half-line model. Anticipating secularities, as before we introduce the time scales $\tau_0 = t, \tau_1 = \varepsilon t$ and expand $\eta = \eta_0 + \varepsilon\eta_1$. Within $\mathcal{O}(\varepsilon^0)$, (5.3) becomes $\eta_{0\tau_0\tau_0} - \eta_{0xx} = 0$, which is the wave equation. The general solution on the half-line is

$$\eta_0(x, \tau_0, \tau_1) = \begin{cases} F_2(x - \tau_0, \tau_1) + G_2(x + \tau_0, \tau_1) & x \geqslant \tau_0 \\ F_1(\tau_0 - x, \tau_1) + G_1(x + \tau_0, \tau_1) & x < \tau_0 \end{cases}, \tag{5.4}$$

where F_i and G_i, for $i = 1, 2$ are some general functions. We emphasise that even though F_1 and F_2 are both left-going waves, they have different domains, and hence are different functions. Similarly, G_1 and G_2 are different functions, though they are both right-going waves. Finally, that η_0 is piecewise is the direct consequence of the boundary condition at $x = 0$.

As before, initial conditions reveal the dependence of $F_i, G_i, i = 1, 2$ on a time scale τ_0 but the dependence on τ_1 is unknown. Following the same procedure as in Section 4.5, from (5.3) we obtain two half-line versions of (4.26), one valid on $x \geqslant \tau_0$ and another valid on $x < \tau_0$. Since the

Hilbert-like transform requires additional care, derivations of approximate equations become complicated and are omitted for brevity.

Removal of secular terms on the two domains yields a system of 4 equations in four unknowns F_1, F_2, G_1, G_2. For $\xi < 0$, we have

$$\begin{aligned}
&2\partial_{\tau_1} F_1 + \frac{1}{3}\partial_\xi^3 F_1 + (F_1 - A)\partial_\xi F_1 \\
&\quad + \frac{1}{\pi}\left(\int_{-\tau_0}^{0} (2F_1 - A)\partial_{\xi'} F_1 \frac{1}{\xi - \xi'}\,\mathrm{d}\xi' + \int_0^\infty (2F_2 - (A+B))\partial_{\xi'} F_2 \frac{1}{\xi - \xi'}\,\mathrm{d}\xi'\right) = 0, \\
&-2\partial_{\tau_1} G_1 + \frac{1}{3}\partial_\zeta^3 G_1 + (G_1 + A)\partial_\zeta G_1 \\
&\quad + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0} (2G_1 + A)\partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty (2G_2 + (A+B))\partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'}\,\mathrm{d}\zeta'\right) = 0.
\end{aligned} \tag{5.5a}$$

For $\xi \geqslant 0$, we have

$$\begin{aligned}
&2\partial_{\tau_1} F_2 + \frac{1}{3}\partial_\xi^3 F_2 + (F_2 - A - B)\partial_\xi F_2 \\
&\quad + \frac{1}{\pi}\left(\int_{-\tau_0}^{0} (2F_1 - A)\partial_{\xi'} F_1 \frac{1}{\xi - \xi'}\,\mathrm{d}\xi' + \int_0^\infty (2F_2 - (A+B))\partial_{\xi'} F_2 \frac{1}{\xi - \xi'}\,\mathrm{d}\xi'\right) = 0, \\
&-2\partial_{\tau_1} G_2 + \frac{1}{3}\partial_\zeta^3 G_2 + (G_2 + A + B)\partial_\zeta G_2 \\
&\quad + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0} (2G_1 + A)\partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'}\,\mathrm{d}\zeta' + \int_{2\tau_0}^\infty (2G_2 + A + B)\partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'}\,\mathrm{d}\zeta'\right) = 0,
\end{aligned} \tag{5.5b}$$

where $A = F_1(\tau_0) - G_1(\tau_0)$, and $B = F_2(0) - F_1(0) + G_1(2\tau_0) - G_2(2\tau_0)$. Roughly speaking, A comes from nonlocal terms evaluated at $x = 0$ and B arises from nonlocal terms evaluated at $x = \tau_0$.

That the approximate equations are more complicated is expected. Each equation of (5.5) is only slightly similar to KdV: the time derivative is preserved but dispersive and nonlinear terms are different. Note that F_2 is defined on $x - t \geqslant 0$ and F_1 is defined on $x - t < 0$. As such, an (xt)-diagram has two regions divided by $x = t$ (see Figure 5.1). Physically, the left-going wave F_1 is reflected by the barrier at $x = 0$, inducing certain dynamics on the surface η in the *interaction* region $x - t < 0$. A priori, these dynamics are unknown; understanding dynamics in this region is part of the problem. Interestingly, note that if $A = B = 0$, then dispersion is preserved and a part of nonlinearity is affected by the Hilbert transform.

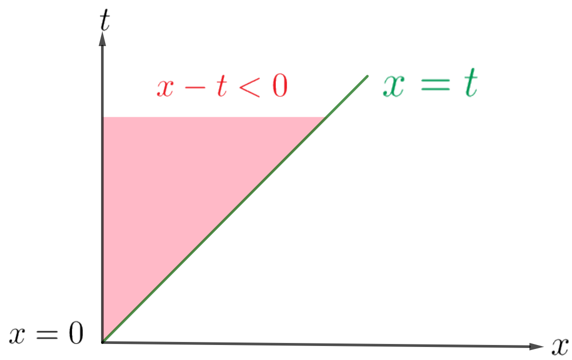


FIGURE 5.1: The interaction region in red, for the half-line water-wave problem.

A careful look at (5.5) reveals that unlike the whole line case, approximate equations are still dependent on the time scale τ_0. This is an issue, as the purpose behind multiple time scales is to separate the dependence on different time scales. As of now, it is not clear why this issue appears. One possible reason is that the linear time scales $\tau_0 = t, \tau_1 = \varepsilon t$ should be replaced with different time scales to obtain complete separation.

In sum, although we do not obtain approximate equations on the half-line, we see the utility of the $\mathcal{H}$ formulation in aiding to understand the physical and mathematical difficulties of the half-line problem. This should not be taken for granted. For example, asymptotic expansions via the velocity potential formulation lead to 4 decoupled KdV equations for $F_i, G_i, i = 1, 2$, which does not agree with the system (5.5) (see Aitzhan, 2020a). As such, the $\mathcal{H}$ formulation shows that the half-line problem has several subtleties, not readily seen via other formulations.

5.2 Concluding remarks

In this project, we examine the shallow water limit of the one dimensional water-wave problem, using the $\mathcal{H}$ formulation as developed in Oliveras and Vasan, 2013. This capstone project presents two contributions to the field of nonlinear waves and raises two future directions.

First, using the $\mathcal{H}$ formulation, we show that water-wave problem reduces to the well-known KdV model. Since we can easily estimate the order of the normal-to-tangential $\mathcal{H}$ operator, the derivation is especially straightforward. One subtle issue that we sidestep is the *equivalence* of the $\mathcal{H}$ formulation to the velocity potential formulation (3.3). While it is clear that solutions of (3.3) solve the $\mathcal{H}$ formulation, the opposite is not so obvious. Without such equivalence, we cannot investigate another aspect of the problem, namely the stability of travelling wave solutions. Therefore, we would like to prove the equivalence, to establish the relevance of the $\mathcal{H}$ formulation with regards to other aspects of the water-wave problem.

Second, we demonstrate the utility of the $\mathcal{H}$ formulation by analysing the water-wave problem on the half-line. Although we do not obtain a KdV-like model, (5.3) presents the distinct features and the associated difficulties of the problem. Furthermore, (5.3) itself is a new, formally derived asymptotic model for bidirectional waves on the half-line. To the best of our knowledge, the decoupled asymptotic model on the half-line has yet to be derived, and achieving this task may provide additional insights into the physics and mathematics of the problem. In particular, the asymptotic model might describe the surface dynamics in the interaction region $x - t < 0$. This direction is the focus of ongoing research.

Bibliography

Ablowitz, M. J. (2011). *Nonlinear dispersive waves: asymptotic analysis and solitons*. Cambridge University Press.

Ablowitz, M. J., A.S. Fokas, and Z.H. Musslimani (2006). "On a new nonlocal formulation of water waves". In: *Journal of Fluid Mechanics*. DOI: 10.1017/S0022112006001091.

Aitzhan, S (2020a). *Report 1*. github.com/sultanaitzhan/Capstone.

— (2020b). *Report 3*. github.com/sultanaitzhan/Capstone.

Bender, C. M. and S. A Orszag (1999). *Advanced mathematical methods for scientists and engineers: asymptotic methods and perturbation theory*. Springer.

Benjamin, T. B. and P. J. Olver (1982). "Hamiltonian structure, symmetries and conservation laws for water waves". In: *Journal of Fluid Mechanics*. DOI: 10.1017/S0022112082003292.

Benjamin, T.B., J. L. Bona, and J.J. Mahony (1972). "Model equations for long waves in nonlinear dispersive systems". In: *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*. DOI: 10.1098/rsta.1972.0032.

Cohen, I. M. and P. K. Kundu (2008). *Fluid mechanics*. Academic Press.

Craig, W. and C. Sulem (1993). "Numerical simulation of gravity waves". In: *Journal of Computational Physics*. DOI: 10.1006/jcph.1993.1164.

Dyachenko, A. I. et al. (1996). "Analytical description of the free surface dynamics of an ideal fluid (canonical formalism and conformal mapping)". In: *Physics Letters A*. DOI: 10.1016/0375-9601(96)00417-3.

Johnson, R. S. (1997). *A modern introduction to the mathematical theory of water waves*. Cambridge University Press.

Kruskal, M.D. (1963). "Asymptotology". In: *Proceedings of the Conference on Mathematical Models in the Physical Sciences*. Ed. by S. Drobot. Englewood Cliffs, New Jersey.: Prentice-Hall, pp. 17–48.

Lannes, D. (2013). *The water waves problem: mathematical analysis and asymptotics*. American Mathematical Society.

Oliveras, K. and V. Vasan (2013). "A new equation describing travelling water waves". In: *Journal of Fluid Mechanics*. DOI: 10.1017/jfm.2012.591.

Zakharov, V. E. (1968). "Stability of periodic waves of finite amplitude on the surface of a deep fluid". In: *Journal of Applied Mechanics and Technical Physics*. DOI: 10.1007/BF00913182.