Nonlinear Waves

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Prolegomenon

These are the lecture notes for Amath 573: solitons and nonlinear waves. These notes are constantly changing, thus it is likely that they contain mistakes of all kinds, both innocent and unforgivable. Please point out these mistakes to me so they may be corrected for the benefit of your successors. If you think that a different phrasing of something would result in better understanding, please let me know.

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Chapter 1

Introduction

The topic of this book is the study of nonlinear dispersive wave equations. The emphasis throughout is on developing understanding of such equations through the use of constructive methods. As such, we will not follow two other approaches to study nonlinear wave equations:

- Nonlinear waves: hyperbolic equations and propagation of shocks. For hyperbolic systems of equations dispersion is not present. It can, and should often, be included to accurately describe the phenomena one is interested in modeling. Such equations can rarely be solved analytically, so one has to resort to numerical or perturbation methods. The first part of the classical monograph of Whitham [69] provides an excellent introduction to what can be done analytically. The recent text by Leveque [48] gives a comprehensive overview of current numerical approaches.

- Nonlinear waves: existence and uniqueness theory, *etc.* This second approach usually involves lots of functional analysis. Evans [22] and Tao [67] are excellent recent references. The outcome of this approach usually does not provide solutions, or approximations of solutions. Rather, existence and uniqueness statements of, and estimates for solutions of the problem at hand are the goal.

Both of the above are valid approaches for good courses on nonlinear waves. Apart from a chapter on the method of characteristics, the first approach will not surface in this text. However, numerical and functional-analytic methods occasionally will. Indeed, often the models one ends up studying demand a hybrid approach: the methods we will study can contribute, but so can numerical methods and function-theoretic approaches. Throughout the text, insights and intuition from dynamical systems are useful. Extending these to an infinite-dimensional setting requires care and often results in lots of cumbersome technicalities. We will use a heuristic approach with this, using the intuition as a guide, while realizing its value may be limited without doing extra work.

1.1 Non-integrable *vs.* integrable equations

The equations we will consider model physical or other applications in certain limits such as long- and short-wave regimes. These equations are typically referred to as nonlinear wave equations. Among nonlinear wave equations, some stand out: these equations are very special, because they can be solved analytically, in some sense. They are referred to as **integrable, solvable, or soliton equations**. They have many properties and large classes of exact solutions, the most remarkable of which are the solitons. We will give a more precise definition later, but for now it suffices to say:

Definition. A soliton is a localized solution of an integrable equation which is robust with respect to all kinds of disturbances.

Thus the concept of a soliton has some kind of stability statement built in. Just like any good definition, we will change it to fit our needs.

Other equations, although not integrable, are called near integrable when they reduce to an integrable equation as one or more parameters appearing in the equation assume special values. The study of such equations allows us to consider integrable equations that are perturbed by including extra physical effects such as dissipation, external potentials, *etc.* Near-integrable equations do not possess soliton solutions, but they often have solutions that behave in many ways like solitons. Such solutions are referred to as solitary waves.

Definition. A solitary wave is a localized solution of a near-integrable equation.

Note that we are not requiring that solitary waves are stable with respect to perturbations. This allows us to use the name "solitary wave" for more solutions than would otherwise be possible. It is clear from the above that our understanding of near-integrable systems derives from that of integrable systems. Therefore, we will spend a significant amount of time on integrable systems. We will also limit ourselves almost exclusively to equations with one independent spatial variable.

Do the equations we are talking about model the application situation we are interested in? Not always, but sometimes they do. More often, they are close to a situation we are interested in. In either case they can tell us a lot about the situation we are trying to understand. In this sense, one of the goals of this book is similar to that of an introductory course on ordinary differential equations: often a student comes away with the deception that all ordinary differential equations can be solved explicitly, since exact solution techniques are emphasized throughout almost the entire course. Similarly, we over-emphasize exact solution methods here, not because they can often be used, but because they lead to a precise understanding of the dynamics of integrable equations. This dynamical behavior is qualitatively similar to that of near-integrable equations which are truly prevalent models for a great variety of applications.

The book emphasizes methods, not applications. Different fields of applications will be drawn upon as examples and for homework problems throughout. We will encounter many applications from fluid mechanics and surface water waves. Other application areas include nonlinear optics, Bose-Einstein condensates, mathematical biology, space and plasma

physics.

1.2 About ordinary differential equations

Consider the autonomous ordinary differential equation (ODE)

$$\frac{dx}{dt} = f(x), \quad x(0) = x_0,$$

written as a first order system. Then $I(x)$ is a **conserved quantity** for this system if

$$\frac{d}{dt} I(x) = \nabla I(x) \cdot f(x) = 0.$$

If so, then $I(x) = I(x(0)) = I(x_0)$ is a constant, equal to its value at the initial time.

Example. Let

$$\begin{cases} x' = y \\ y' = -x \end{cases}.$$

Then $H = x^2 + y^2$ is a conserved quantity for this system. A generic two-dimensional system may move anywhere in the (x, y)-plane, but for this system the motion is constrained to circles:

$$H(x, y) = x^2 + y^2 = H(x_0, y_0).$$

In general, conserved quantities put constraints on the allowable motion of a system. The more conserved quantities a system has, the more constrained its motion is: each conserved quantity determines a hypersurface in the phase space of the system, and the motion is constrained to the intersection of all these surfaces. Integrable equations have a maximal number of conserved quantities: for autonomous ODEs this number is $N - 1$, where N is the dimension of the system. The one remaining degree of freedom allows for motion on one-dimensional subsets, *i.e.* curves, which corresponds to time evolution. For partial differential equations (PDEs) this implies an infinite number of conserved quantities. It should be noted that if we have found an infinite number of conserved quantities for a given PDE, it may be that we found all of them. Or maybe we are missing one, or two. Or half of them. This is an unpleasant problem that we will talk about but we will not solve it in this course. Note that all of these considerations about conserved quantities are far more important in three or more dimensions, since in two dimensions the Poincaré-Bendixson theorem [70] imposes significant restrictions on the flow anyways.

Consider the "typical" situation in many particle systems in physics [26]:

$$m_i x_i'' = F_i(x, x'),$$

where i denotes the i-th particle. The number of particles ranges from 10 or more (when studying the solar system, for instance) to 10^{23} (when studying gas or plasma dynamics). Typically there are only three conserved quantities:

- $M = \sum_i m_i = C_1$ (mass),
- $P = \sum_i m_i x_i' = C_2$ (momentum),
- $E = \frac{1}{2}\sum_i m_i (x_i')^2 + V(x)$ (energy),

where $V(x)$ denotes the total potential energy of the system (assuming the forces are conservative). The different statistical ensembles in statistical mechanics [35] (micro-canonical, canonical, and grand-canonical) correspond to assuming which of M, P, and E are constant, and which are allowed to vary (not in time, but across different systems). A system for which *only* these three quantities are conserved is called **ergodic** (the mathematical definition is far more technical than this, but this will do for us). Poincaré (*e.g.*, see [6]) proved that for a generic N-particle system, these are the only conserved quantities. Thus, a generic N-particle system is ergodic.

This leads us to wonder, **what we are doing**. Why are we wasting our time studying integrable or near-integrable systems? After all such systems are clearly non-generic, so we are unlikely to encounter them in applications.

From this perspective, Poincaré's result is rather discouraging. It was not until the work of Kolmogorov, Arnol'd, and Moser (KAM) ([5, 42, 52], see also [70]) that looking at integrable equations became a sensible activity. One of the results coming out of KAM theory is that systems "near" an integrable system retain many of the features of that integrable system. It is true that systems near integrable systems may be ergodic when considered over long times, and may exhibit chaos. But on shorter time scales, in large regions of phase space, the system behaves in many ways like an integrable system. For instance, the level surfaces of conserved quantities may be deformed, but they don't necessarily fall completely apart. They still inhibit transport: they may not confine regions of phase space, but trajectories near them tend to stay near them for very long times. In that sense they are called sticky. Now, all of the results referred to above are for ODEs. Very few of these have been generalized to PDEs, but we will believe that they may so be generalized.

1.3 Two historically important problems in the development of integrable and near-integrable systems

Historically, the study of two seemingly distinct problems has advanced the study of nonlinear waves significantly. The first is the propagation of waves of permanent form in shallow water, following the famous observation by John Scott Russell recounted below. The second is one of the problems devised by Fermi, Pasta, and Ulam for one of the first electronic computers at the Los Alamos National Laboratory. These problems were subsequently shown to be mathematically intimately connected through the so-called Korteweg-de Vries equation.

1.3.1 A Scotsman on a horse

John Scott Russell was an engineer working on different aspects of wave propagation related to vessels on canals. In 1834, he was on horseback, following a horse-drawn boat on a canal near Edinburgh. His story has been recounted often, but it is too seminal not to include it here. In his own words [60]:

> "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."

After observing this "great wave of translation", Russell started his research on this phenomenon, using various experimental facilities, including a 30ft tank, built in his back yard. His experiments are illustrated beautifully in [60], through various meticulously drawn plates accompanying the paper[1], one of which is shown in Fig. 1.1. Russell's results were at odds with beliefs commonly held at the time. Airy, and to a lesser extent, Stokes argued that a solitary wave of the type described by Russell could not propagate without change in form or speed. In essence, Airy's arguments were based on linear ideas. As will be made clear later, Russell's solitary wave is an inherently nonlinear phenomenon. Stokes's results relied heavily on the use of periodic boundary conditions, which do not apply to the solitary wave. A good overview of this history is presented in [13].

Boussinesq [11] was the first to derive a model that leads to a mathematical description of Russell's wave. This model was inherently nonlinear. In particular he wrote down a formula for the soliton solution. In 1895, Korteweg and de Vries [43], wrote down the equation that now bears their name,

$$u_t = uu_x + u_{xxx},$$

(indices denote partial derivatives; the value of the coefficients in front of each term is arbitrary as we will see later) and calculated its stationary solutions. They gave explicit formulae for the soliton solution, as well as for the more general periodic traveling wave solutions. Their work, in greater detail than that of Boussinesq, explained the phenomena

[1] It should be remarked that it took Russell 10 years to conduct his experiments and report on his findings. This is not a publication record that would impress university administrations nowadays!

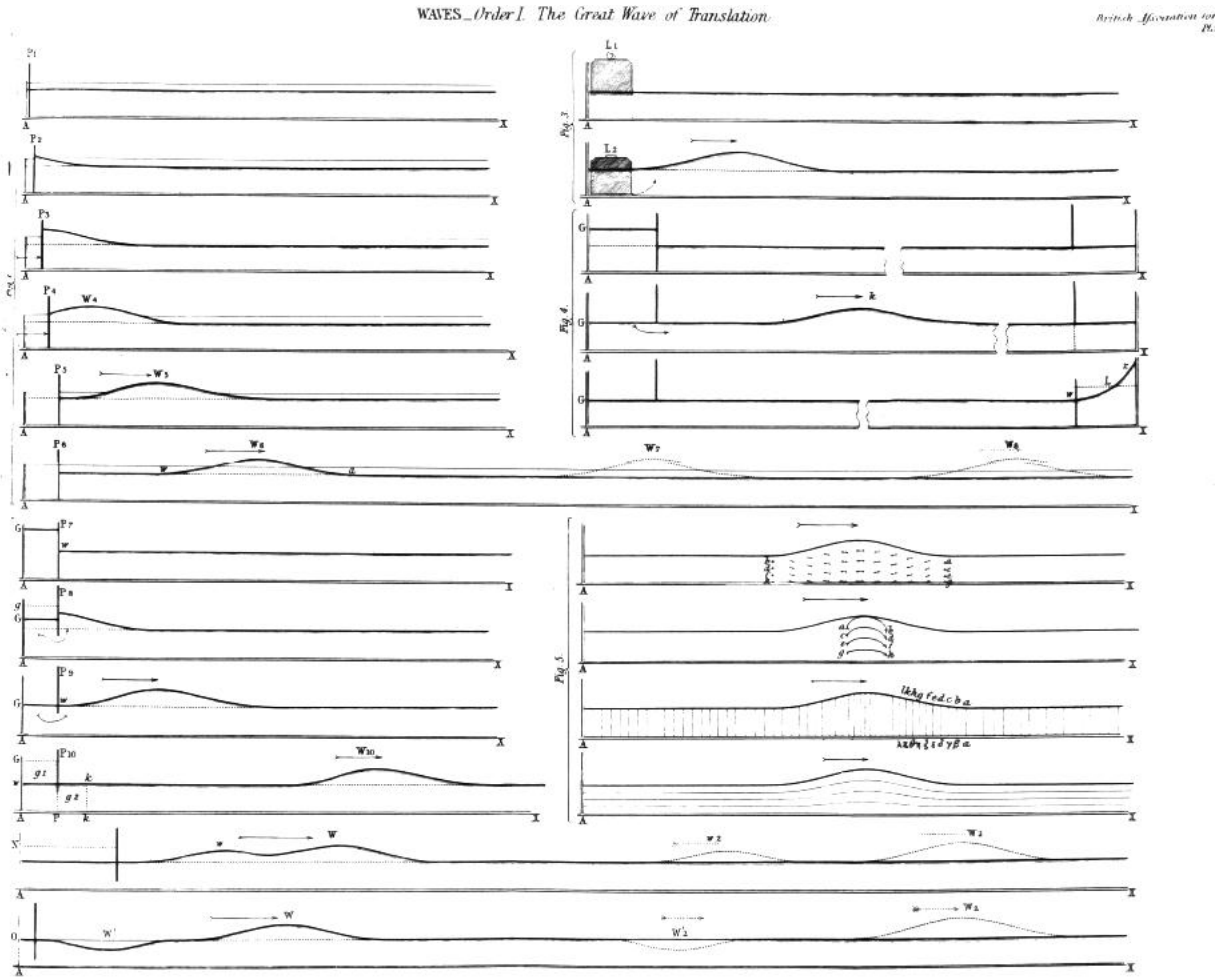


Figure 1.1: Illustrations from [60] on the experimental creation of the "Wave of Translation" in a wave tank.

observed by Russell and explains carefully the different assumptions which lead Airy and Stokes to their own conclusions. For instance, solitary waves of their equation that have higher amplitude travel faster than those that have lower amplitude. This implies that higher-amplitude waves that are initially trailing waves of lower amplitude will catch up with them, resulting in an interaction process. This process is nonlinear, since the KdV equation is nonlinear.

1.3.2 The Fermi-Pasta-Ulam problem

Question: what is the temperature of a metal?

To understand this question, we have to understand the concept of thermal equilibrium.

In statistical mechanics, there is an unproven hypothesis that after a long time the energy of a system is spread equally among all modes, or across all degrees of freedom. This hypothesis is referred to as the equipartition of energy [35].

It was Fermi's idea to test this hypothesis using one of the first computers at Los Alamos (Maniac I, 1955). The model Fermi, Pasta, and Ulam (FPU, [23]) used was that of a one-dimensional chain of N particles of equal mass m, coupled together with nonlinear springs. This is not a good representation of the model required for the crystal structure of a metal, but it serves as a simple example of what is already a complicated phenomenon. Let y_n denote the displacement of the n-th particle from its equilibrium position. Then the equation of motion for y_n is

$$m\frac{d^2y_n}{dt^2} = F_{\text{right}} - F_{\text{left}}, \quad n = 1, 2, \ldots, N,$$

where N is assumed to be large (close to 10^{23}, say 64). There are two main hypotheses FPU wanted to test:

- There is equipartition of energy over all modes,
- This equipartition happens independent of the initial conditions.

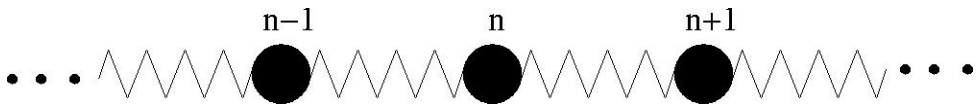


Figure 1.2: The setup of the FPU problem

The case of linear springs

In this case

$$F_{\text{right}} = -k(y_n - y_{n+1}), \quad F_{\text{left}} = -k(y_{n-1} - y_n),$$

which is just Hooke's law [26]. For $n = 1, \ldots, N$ we have

$$\begin{aligned} m\frac{d^2y_n}{dt^2} &= -k(y_n - y_{n+1}) + k(y_{n-1} - y_n) \\ &= k(y_{n+1} - 2y_n + y_{n-1}). \end{aligned}$$

Since these equations are linear with constant coefficients, we look for a solution of the form $y_n = A_n e^{\omega t}$. Substitution gives

$$m\omega^2 A_n = k(A_{n+1} - 2A_n + A_{n-1}).$$

Thus the allowed frequencies ω are determined by imposing that this equation has a nontrivial solution (in other words, there is a solution for which $A_n \neq 0$, for at least one n). This results in a determinant equated to zero. This equation is the dispersion relation for this problem. Once the frequencies are fixed this way, the amplitudes A_n may be determined by linear algebra and using the initial conditions. In this case, it is clear that there is no equipartition of energy: the normal modes of the system correspond to the eigenvectors of our matrix equation. These are orthogonal (hence normal) to one another because our matrix is symmetric. Using this basis of normal modes, the original system of differential equations is uncoupled, and it is clear that the energy in a normal mode does not impact any of the other modes, thus there is no equipartition of energy. Conceptually, this is most easily seen as follows: the solution of an N-dimensional linear system of differential equations $y' = My$ (here M is a square matrix of size $N \times N$) is given by a superposition of N linearly independent modes:

$$y = c_1 y^{(1)}(t) + c_2 y^{(2)}(t) + \ldots + c_N y^{(N)}(t).$$

Suppose that the initial condition $y(0) = y^{(1)}(0)$ is given. Then the solution of the initial-value problem is $y(t) = y^{(1)}(t)$ and there is no way for the modes $y^{(k)}(t)$, $k = 2, \ldots, N$ to ever be excited.

Thus equipartition of energy requires at least some amount of nonlinearity, to couple the modes.

The case of nonlinear springs

The simplest nonlinear term to include in the force law is of course quadratic. Thus we have

$$F_{\text{right}} = k(y_{n+1} - y_n) + \alpha k(y_{n+1} - y_n)^2, \quad F_{\text{left}} = k(y_n - y_{n-1}) + \alpha k(y_n - y_{n-1})^2.$$

FPU solved the coupled ODEs

$$m\frac{d^2 y_n}{dt^2} = k\left[(y_{n+1} - 2y_n + y_{n-1}) + \alpha(y_{n+1} - y_n)^2 - \alpha(y_n - y_{n-1})^2\right]$$

numerically by running them on the Maniac I with $N = 64$. This was one of the very first instances of computional physics. It appears the problem cannot be done analytically any more: due to the nonlinearity all our linear algebra approaches from the linear case cease to work. Starting with all energy in one mode (for which they used a normal mode from the linear problem), they observed no equipartition of energy. Only about 5 of the 64 modes were ever excited. After some time, they noticed that approximately all energy was again in the initial mode. This process then repeats.

What is going on? How do we explain this?

1.4 The work of Zabusky and Kruskal: derivation of the KdV equation

Zabusky and Kruskal [72] attempted to explain the FPU observations. They added several assumptions to the FPU model, in order to simplify it. These assumptions were consistent with the observations of FPU. The first assumption was to consider only long waves, as only low modes were excited. Thus, Zabusky and Kruskal examined the limit case of long wavelength. In this limit, the different particles appear close together and a long wave does not vary that much between two neighboring particles. So they had the idea of taking a continuum limit of the FPU lattice, resulting in a partial differential equation. We have much more training to deal with PDEs than with lattices, so maybe this limit problem will be simpler to deal with.

First, let

$$\begin{cases} x_n = nh, \\ y_n = y(x_n), \end{cases}$$

where h is the lattice spacing. Then

$$\begin{aligned} y_{n+1} = y(x_{n+1}) = y(x_n + h) = y_n + hy_x(x_n) + \frac{h^2}{2}y_{xx}(x_n) + \frac{h^3}{6}y_{xxx} + \dots, \\ y_{n-1} = y(x_{n-1}) = y(x_n - h) = y_n - hy_x(x_n) + \frac{h^2}{2}y_{xx}(x_n) - \frac{h^3}{6}y_{xxx} + \dots, \end{aligned}$$

and

$$y_{n+1} - 2y_n + y_{n-1} = h^2 y_{xx}(x_n) + \frac{h^4}{12}y_{xxxx}(x_n) + \theta(h^6).$$

Similarly, the nonlinear term gives

$$\begin{aligned} \alpha[(y_{n+1} - y_n)^2 &- (y_n - y_{n-1})^2] \\ &= \alpha(y_{n+1} - 2y_n + y_{n-1})(y_{n+1} - y_{n-1}) \\ &= \alpha\left(h^2 y_{xx}(x_n) + \frac{h^4}{12}y_{xxxx}(x_n) + \dots\right)\left(2hy_x(x_n) + \frac{h^3}{3}y_{xxx}(x_n) + \dots\right) \\ &= 2\alpha h^3 y_x(x_n)y_{xx}(x_n) + \theta(\alpha h^5). \end{aligned}$$

the FPU problem becomes

$$my_{tt} = k\left(h^2 y_{xx} + \frac{h^4}{12}y_{xxxx} + 2\alpha h^3 y_x y_{xx} + \theta(\alpha h^5)\right),$$

where everything is evaluated at x_n. Thus

$$y_{tt} = \frac{kh^2}{m}\left(y_{xx} + \frac{h^2}{12}y_{xxxx} + 2\alpha h y_x y_{xx} + \theta(\alpha h^3)\right). \tag{1.1}$$

The dominant terms of this PDE are

$$y_{tt} = \frac{kh^2}{m}y_{xx},$$

which is a wave equation with d'Alembert solution

$$y = f(x - vt) + g(x + vt), \tag{1.2}$$

where $v^2 = kh^2/m$. The first term $f(x - vt)$ represents a right-going wave, whereas the second term $g(x + vt)$ represents a left-going wave. Given that the terms we ignored are smaller than the ones we kept, we expect the solution of our original PDE (1.1) to be of the form (1.2), with small corrections.

Let us focus on the right-going wave. We put ourselves in a frame of reference moving along with the right-going wave, while introducing an additional variable to measure variations in this coordinate frame. Let

$$\begin{cases} X = x - vt = x - \sqrt{\dfrac{kh^2}{m}}t \\ T = \alpha h v t \end{cases}.$$

Note that T is a "slow" time: it takes a large change in t to achieve a small change in T, since both α (amount of nonlinearity) and h (lattice spacing) are supposed to be small. The above rules induce transformation rules on the derivatives:

$$\begin{cases} \dfrac{\partial}{\partial x} = \dfrac{\partial X}{\partial x}\dfrac{\partial}{\partial X} + \dfrac{\partial T}{\partial x}\dfrac{\partial}{\partial T} = \dfrac{\partial}{\partial X} \\ \dfrac{\partial}{\partial t} = \dfrac{\partial X}{\partial t}\dfrac{\partial}{\partial X} + \dfrac{\partial T}{\partial t}\dfrac{\partial}{\partial T} = -v\dfrac{\partial}{\partial X} + \alpha h v\dfrac{\partial}{\partial T} \end{cases}.$$

The FPU problem (1.1) in these new variables becomes

$$\begin{aligned} v^2\left(\frac{\partial^2 y}{\partial X^2} + \frac{h^2}{12}\frac{\partial^4 y}{\partial X^4} + 2\alpha h\frac{\partial y}{\partial X}\frac{\partial^2 y}{\partial X^2}\right) &= v^2\left(-\frac{\partial}{\partial X} + \alpha h\frac{\partial}{\partial T}\right)\left(-\frac{\partial}{\partial X} + \alpha h\frac{\partial}{\partial T}\right)y \\ &= v^2\left(-\frac{\partial}{\partial X} + \alpha h\frac{\partial}{\partial T}\right)\left(-\frac{\partial y}{\partial X} + \alpha h\frac{\partial y}{\partial T}\right) \\ &= v^2\left(\frac{\partial^2 y}{\partial X^2} - 2\alpha h\frac{\partial^2 y}{\partial X \partial T} + \alpha^2 h^2\frac{\partial^2 y}{\partial T^2}\right). \end{aligned}$$

Simplifying this gives (the wave equation term drops out, as expected)

$$\frac{\partial^2 y}{\partial T \partial X} + \frac{\partial y}{\partial X}\frac{\partial^2 y}{\partial X^2} + \frac{h}{24\alpha}\frac{\partial^4 y}{\partial X^4} - \frac{\alpha h}{2}\frac{\partial^2 y}{\partial T^2} = 0.$$

Remembering that both α and h are supposed to be small, we may ignore the last term, as being significantly smaller than the others. Further, defining $\delta^2 = h/(24\alpha)$ (for historical reasons; note that $h/(24\alpha)$ is not necessarily positive) and setting $\phi = \partial y/\partial X$ we obtain

$$\frac{\partial \phi}{\partial T} + \phi \frac{\partial \phi}{\partial X} + \delta^2 \frac{\partial^3 \phi}{\partial X^3} = 0,$$

the KdV equation[2]!

1.5 The work of Zabusky and Kruskal: properties of the KdV equation

The KdV equation was first written down by Boussinesq in 1877 [10], but it is named after Korteweg and deVries who derived the equation in 1895 [43] in the same context as Boussinesq, that of long waves in shallow water in narrow canals. As already mentioned, Korteweg and deVries discussed the stationary solutions of their equation. They noted the existence of a large class of solutions expressed in terms of elliptic functions, but also of the solitary wave solution. They found that this solution is expressed in terms of hyperbolic functions:

$$\phi = 12k^2\delta^2 \mathrm{sech}^2 k(x - 4k^2\delta^2 t + \varphi), \tag{1.3}$$

where k and φ are free constants.

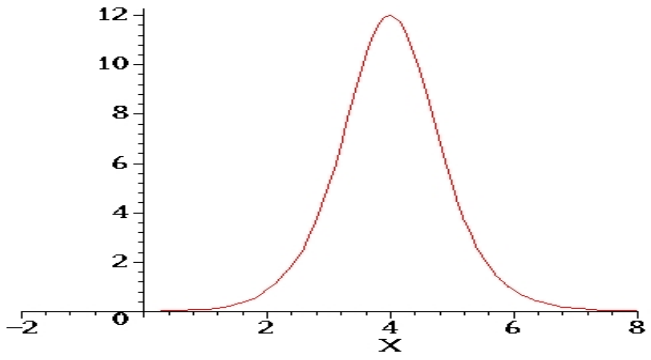


Figure 1.3: The soliton solution (1.3) of the KdV equation with $k = 1$, $\delta^2 = 1$, at $t = 1$. The soliton moves to the right with constant speed $4k^2\delta^2$.

[2]As mentioned above, the value of the particular coefficients is irrelevant: indeed by scaling the dependent variable $u(x,t)$ and the independent variables x and t, any value of the coefficients may be attained. In terms of the application where the equation is derived, the scaling of the variables simply means we are measuring quantities in different units.

Zabusky and Kruskal [72] solved the KdV equation numerically with periodic boundary conditions and initial condition $u(x, t = 0) = \cos x$, on $x \in [0, 2\pi]$. The results of their numerics is displayed in Fig. 1.4. Their initial condition is given by the dotted line. They noticed the following:

1. First the solution starts to steepen, until it almost reaches a point of breaking (*i.e.*, a near-vertical tangent), shown in dashes.

2. Before the breaking occurs, wiggles occur past (*i.e.*, to the left) the shock-like part. These wiggles grow and separate from the front of the shock. They form solitary entities which travel at separate speeds. This is shown by the solid line.

3. These solitary waves interact with each other, due to the periodic boundary conditions. This interaction is "clean", meaning there is not much of a remnant of the interaction, apart from a shift in position of the two contributing solitary waves.

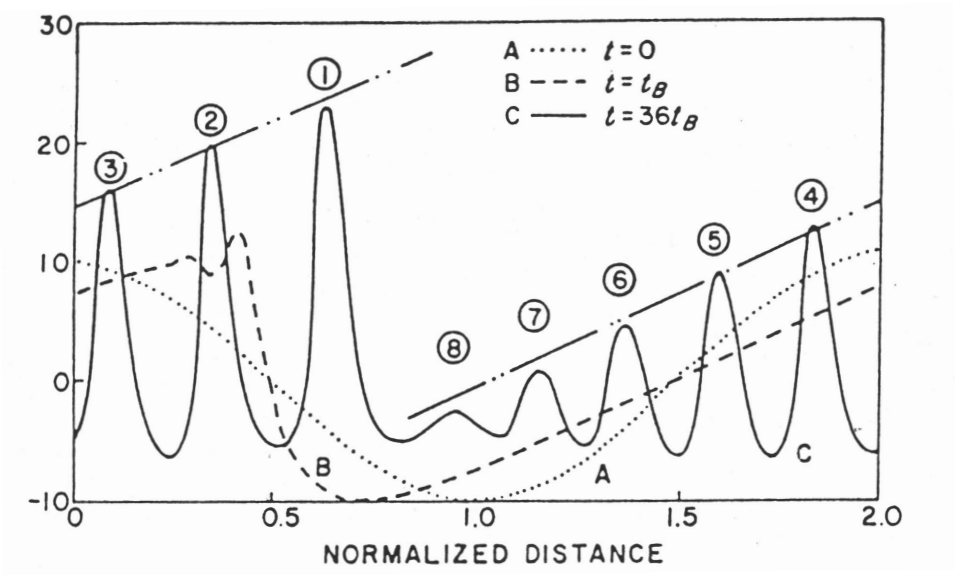


Figure 1.4: The numerical results of Kruskal and Zabusky (1965).

Zabusky and Kruskal found that each one of these solitary waves could be matched with the solitary wave solution (1.3) of the KdV equation, for different values of k (δ is fixed by the equation). Because these solitary waves interact in many ways like particles undergoing an elastic collision (*i.e.*, they keep their identity but they are in different positions after the collision), Zabusky and Kruskal gave them an elementary-particle-like name: **solitons**.

Assuming that the observations and conclusions of Zabusky and Kruskal were correct, this offers at least somewhat of a qualitative explanation of the FPU problem observations: the initial condition gives rise to a certain number of solitons. Since these interact in what

is an almost linear manner (they do not exchange energy, for instance), no more modes get excited than there are solitons created from the initial conditions. In the original FPU experiment, this was about five or six. Also, since periodic boundary conditions are used, it does not take too long before all solitons are arranged so that they (up to a shift in x) recreate the initial condition. Clearly all of these arguments are very qualitative and require further justification, but they allow for a glimpse of what is actually going on.

1.6 Balancing dispersion and steepening

Let us analyze the KdV equation a little further. We will study the dynamics of small-amplitude and near-shock solutions in more detail in future chapters. Here we look at the effect the different terms of the KdV equation contribute to the dynamics.

Solutions with small amplitude

For solutions with small amplitude, the term $\phi\phi_x$ is smaller than the two others, and we may consider

$$\phi_T + \delta^2\phi_{XXX} = 0. \tag{1.4}$$

This is the linear KdV equation. The last term is dispersive: it results in different wave numbers contributing to the solution to propagate with different speeds. On the whole line $-\infty < x < \infty$, this results in contributions from different wave numbers to separate in space. As a consequence, the solution tends to spread out, resulting in an overall decrease in the amplitude of the solution. If we are using a finite domain, this is not possible, as there is nowhere for the solution to spread out to. A typical time evolution for initial data specified on the whole line is displayed in Fig. 1.5.

Solutions with large amplitude, δ small

In the original numerical experiments of Zabusky and Kruskal [72], $\delta^2 = 1/12$. For solutions with sufficiently large amplitude, if δ is small, we may neglect the dispersive term. Then we need to consider the equation

$$\phi_T + \phi\phi_X = 0. \tag{1.5}$$

This is sometimes called Burgers' equation, or the dissipationless form of Burgers' equation (the full Burgers' equation is $\phi_T + \phi\phi_X = \nu\phi_{xx}$, where ν is the dissipation coefficient). Using the method of characteristics, we may show that profiles that have fronts will steepen, *i.e.*, the fronts become more pronounced, as shown in Fig. 1.6. The method of characteristics allows us to say that the above equation is a first-order wave equation, or transport equation but with variable velocity ϕ.

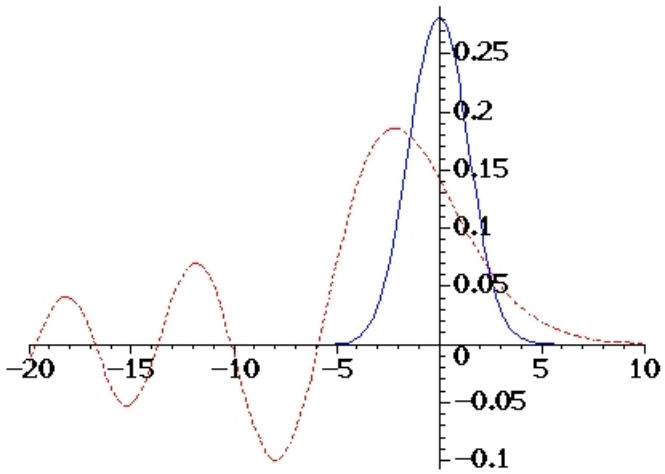


Figure 1.5: The effect of dispersion on Gaussian initial data. The solid blue line is a Gaussian initial condition. The dashed red line is the time evolution using the linear dispersive equation (1.4).

Thus for this equation, steep fronts get steeper and they eventually develop a shock. However, just before this shock is formed, the solution is very steep and therefore the dispersive term $\delta^2\phi_{XXX}$ will have an impact, even for small δ^2. This term tends to smear out any solution, as we argued above. In the case of a steepening solution, it prevents the formation of a shock. Thus we may reach the following heuristic conclusion:

The KdV equation provides a balance between steepening and dispersion.

This conclusion is accurate for many integrable equations and provides some useful intuition about what makes these equations special. Equations such as the KdV equation typically incorporate many different effects and these all have to balance in particular ways to account for soliton formation. At this point it is fair to say that we have not solved the FPU paradox at any level whatsoever, but at least we have connected the problem with another problem that appears to be special.

1.7 An overview of the current textbooks and research monographs

This section provides a limited and biased overview of books currently available that cover multiple aspects of nonlinear wave equations. Not included are works like [14] or [56], which are limited to (usually more in-depth) coverage of one or a few aspects of nonlinear waves.

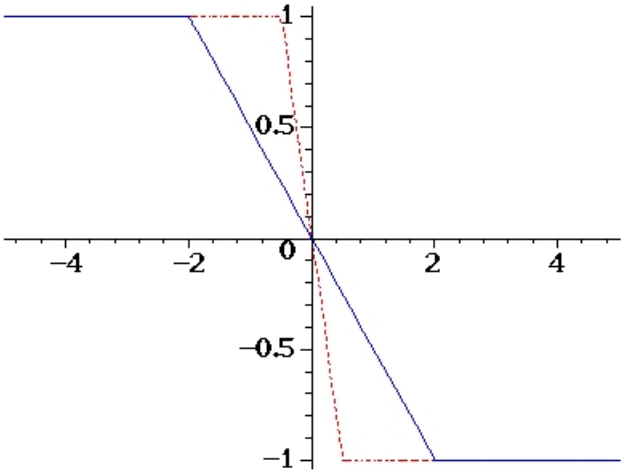


Figure 1.6: The effect of steepening on piecewise linear initial data. The solid blue line is the initial condition, the dashed red line shows the time evolution according to (1.5), approaching a shock.

Specific emphasis is on topics that are not or barely discussed in this text. The books are discussed in chronological order.

- **Whitham, *Linear and Nonlinear Waves* (1974) [69]**. This is the undisputed classic on analytic methods for hyperbolic systems (part I) and dispersive equations (part II). There is almost no coverage of integrable systems, the renaissance of which was just in its infancy in 1974. The book is missing more modern methods, but its treatment of the classical methods, including those created by the author, is excellent.

- **Ablowitz & Segur, *Solitons and the Inverse Scattering Transform* (1981) [4]**. Written by two experts in the field of nonlinear waves, this research monograph covers many aspects of integrable equations well. By now, some approaches are a bit dated, but the exposition remains excellent. Good introductions to differential-difference systems and pole dynamics. A recommended classic.

- ***Integrable systems: selected papers* (1981) [54]**. This is a collection of previously published translated papers outlining some Russian contributions to the field. The papers are mostly reviews. Although not easy reading, the individual papers are, almost without exception, classics.

- **Novikov, Manakov, Pitaevski, & Zakharov, *Theory of Solitons* (1984) [55]**. Although suffering from poor typesetting, this is the best introduction to the many of the contributions of the Moscow school to soliton theory. It contains extensive material

on the periodic problem. This material is somewhat dated now, but some of the topics treated can simply not be found in any other book.

- **Newell, *Solitons in mathematics and physics* (1985) [53]**. These are lecture notes from a CBMS conference, and it shows: the material is quite accessible, and the coverage is wide. This is a good book to start learning about integrable systems. Starting with applications, the author covers different solution techniques, ending with an introduction to the algebraic theory of integrable systems.

- **Drazin & Johnson, *Solitons: an introduction* (1989) [20]**. This is often referred to as the undergraduate introduction to soliton theory. To some extent this is correct, and one could argue it is one of the few textbooks on this list, as opposed to the many research monographs. The exposition is easy to follow for most of the text, but I find the treatment of inverse scattering limited and confusing.

- **Fordy, *Soliton theory: a survey of results* (1990) [24]**. Not as much a book as a collection of review papers. I have always liked it. The quality of the individual papers is high, especially those in the second half of the book. There are excellent contributions on Hamiltonian systems, the periodic problem, Painlevé analysis and the prolongation method.

- **Infeld and Rowlands, *Nonlinear waves, solitons and chaos* (First edition, 1990) [36]**. An excellent introduction to many techniques for nonlinear wave equations, not just integrable ones. The emphasis is more on the physics than the mathematics. The silly chapter on chaos is added for marketability, as the authors admit.

- **Ablowitz & Clarkson, *Solitons, nonlinear evolution equations and inverse scattering* (1991) [2]**. In some ways a sequel to [4] with a more modern treatment on inverse scattering, including inverse scattering in more than one spatial dimension. Also included is an excellent introduction to Painlevé analysis.

- **Knobel, *An introduction to the mathematical theory of waves* (2000) [41]**. This is a delightful little book suitable for undergraduates. It introduces important wave phenomena, such as dispersion and breaking. Its focus is on the seminal wave equations, some of which are integrable. No specific techniques from integrability are used however.

- **Dauxois & Peyrard, *Physics of solitons* (2010) [17]**. A new text from a physics perspective. The mathematics is very loose here, but there are some excellent application examples, including interesting ones using the Sine-Gordon equation.

- **Yang, *Nonlinear waves in integrable and nonintegrable systems* (2010) [71]**. A new book, to appear in November 2010. I have been able to look at its content only for a limited time, but it appears promising. There is an appealing mix of integrable and nonintegrable systems, and of analytical and numerical methods.

- **Ablowitz, *Nonlinear dispersive waves. Asymptotic analysis and solitons* (2010) [1]**. This textbook will be published soon. It contains a good mix of techniques for both integrable and non-integrable equations. The treatment is application based, with many applications drawn from nonlinear optics.

1.8 Exercises

1. The Korteweg-deVries (KdV) equation

$$u_t = uu_x + u_{xxx}$$

is often written with different coefficients. By using a scaling transformation on all variables (dependent and independent), show that the choice of the coefficients is irrelevant: by choosing a suitable scaling, we can use any coefficients we please. Can you say the same for the modified KdV (mKdV) equation

$$u_t = u^2u_x + u_{xxx}?$$

2. The Nonlinear Schrödinger (NLS) equation is written as

$$ia_t = -a_{xx} + \sigma|a|^2a,$$

where $\sigma = \pm 1$. The equation is often written with different coefficients. By using a scaling transformation on all variables (dependent and independent), show that the choice of the coefficients is almost irrelevant: by choosing a suitable scaling, we can use any coefficients we please, except for the sign of the ratio of the coefficients of the two terms on the right-hand side.

3. (Use maple or mathematica for this problem.) Consider the KdV equation $u_t + uu_x + u_{xxx} = 0$. Show that

$$u = 12\partial_x^2 \ln\left(1 + e^{k_1x - k_1^3t + \alpha}\right)$$

is a one-soliton solution of the equation (*i.e.*, rewrite it in the familiar sech^2 form).

Now check that

$$u = 12\partial_x^2 \ln\left(1 + e^{k_1x - k_1^3t + \alpha} + e^{k_2x - k_2^3t + \beta} + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 e^{k_1x - k_1^3t + \alpha + k_2x - k_2^3t + \beta}\right)$$

is also a solution of the equation. It is a two-soliton solution, as we will verify later. By changing t, we can see how the two solitons interact. With $\alpha = 0$ and $\beta = 1$, examine the following three regions of parameter space: (a) $k_1/k_2 > \sqrt{3}$, (b) $\sqrt{3} > k_1/k_2 > \sqrt{(3+\sqrt{5})/2}$, (c) $k_1/k_2 < \sqrt{(3+\sqrt{5})/2}$. Discuss the different types of collisions. Here "examine" and "discuss" are to be interpreted in an experimental sense: play around with this solution and observe what happens.

4. **Cole-Hopf transformation**. Show that every non-zero solution of the heat equation $\theta_t = \nu\theta_{xx}$ gives rise to a solution of the dissipative Burgers' equation $u_t + uu_x = \nu u_{xx}$, through the mapping $u = -2\nu\theta_x/\theta$.

5. From the previous problem, you know that every solution of the heat equation $\theta_t = \nu\theta_{xx}$ gives rise to a solution of the dissipative Burgers' equation $u_t + uu_x = \nu u_{xx}$, through the mapping $u = -2\nu\theta_x/\theta$.

   (a) Check that $\theta = 1 + \alpha e^{-kx+\nu k^2 t}$ is a solution of the heat equation. What solution of Burgers' equation does it correspond to? Describe this solution qualitatively (velocity, amplitude, steepness, *etc*) in terms of its parameters.

   (b) Check that $\theta = 1 + \alpha e^{-k_1 x+\nu k_1^2 t} + \beta e^{-k_2 x+\nu k_2^2 t}$ is a solution of the heat equation. What solution of Burgers' equation does it correspond to? Describe the dynamics of this solution, *i.e.*, how does it change in time?

Part I

General nonlinear wave equations

Chapter 2

Linear evolution equations. The dispersion relationship.

In order to understand nonlinear problems, first we have to understand linear problems well. For convenience, we only discuss one-dimensional problems here. The main method for solving the type of linear problems we are interested in is the Fourier transform, or Fourier series, depending on the boundary conditions. Using this method, we attempt to construct the solution of the equation as a linear superposition of modes of the form

$$e^{ikx-i\omega t}.$$

Typically, the equation will impose a condition expressing ω as a function of k: $\omega = \omega(k)$. This is the dispersion relationship.

If we are solving an equation on the whole line $x \in \mathbb{R}$, there are no restrictions on k and a superposition of all possible modes results in an integral over all $k \in \mathbb{R}$ (the Fourier integral). On the other hand, if we are solving an equation on a finite interval with periodic boundary conditions, there are restrictions on k, to ensure the periodicity of the modes is compatible with the length of the interval. In this case the possible k values are quantized and a superposition of all allowed modes gives rise to a (usually infinite) sum (the Fourier series). In fancy terms: it is the same in either case, but the measure of allowed wave numbers k is continuous in the first case and discrete in the second case. In what follows we focus on the $x \in \mathbb{R}$ case.

2.1 The dispersion relationship and the general solution

A linear evolution equation with constant coefficients is a linear partial differential equation that may be solved for its t derivative:

$$u_t = F(u, u_x, u_{xx}, \ldots), \tag{2.1}$$

where F is linear and homogeneous in its arguments, and has constant coefficients:

$$F(u, u_x, u_{xx}, \ldots) = a_0 u + a_1 u_x + a_2 u_{xx} + \ldots,$$

where the coefficients a_j, $j = 0, 1, 2, \ldots$ are constant. Let us consider this equation with vanishing boundary conditions at infinity:

$$|u| \to 0 \quad \text{as} \quad x \to \pm\infty,$$

and similar conditions on the spatial derivatives of u, as necessary. Allowing only first derivatives in t is not a restriction, as u and F may be vectors. In what follows, we consider the scalar case first. We remark on the general case below.

Let us look for the dispersion relationship. To this end, we consider basic modes of the form

$$u = e^{ikx - i\omega t}.$$

Then

$$u_t = -i\omega u, \quad u_x = iku, \quad \ldots, \quad u_{nx} = (ik)^n u.$$

Using the linearity and homogeneity of F, we obtain $-i\omega u = F(u, iku, \ldots) = uF(1, ik, (ik)^2, \ldots)$. In order to have solutions other than $u = 0$ we require

$$\omega = iF(1, ik, (ik)^2, \ldots) = \omega(k).$$

This is the linear dispersion relationship. For any given wave number k it gives the allowed frequency ω.

Thus the modes are of the form

$$e^{ikx - i\omega(k)t}.$$

The most general solution of the evolution equation we may obtain from this is a linear superposition of all of these:

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(k) e^{ikx - i\omega(k)t} dk, \tag{2.2}$$

where the front factor of $1/2\pi$ was chosen for convenience. Note that we have restricted this superposition to real values of k so that only modes that are bounded in x are included.

Example. Consider the linear KdV equation $u_t = u_{xxx}$. We easily find the dispersion relation: $-i\omega = (ik)^3 \Rightarrow \omega = k^3$, so that we have found the solution

$$u = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(k) e^{ikx - ik^3 t} dk.$$

Remark. In the above we have assumed that u and F are scalar. It is an easy exercise to work out more general cases.

1. **The general vector case.** Suppose that u is a vector of dimension $N > 1$. We obtain

$$-i\omega u = F(1, ik, (ik)^2, \ldots)u \quad \Rightarrow \quad (F(1, ik, (ik)^2, \ldots) + i\omega I)u = 0,$$

where $F(1, ik, (ik)^2, \ldots)$ is a matrix of size $N \times N$. The dispersion relation is given by

$$\det(F(1, ik, (ik)^2, \ldots) + i\omega I) = 0,$$

and the different branches of the dispersion relation $\omega_j(k)$, $j = 1, \ldots, N$ are the eigenvalues of the matrix $-iF(1, ik, (ik)^2, \ldots)$. Corresponding to each branch is an eigenvector $A_j(k)$ of $-iF(1, ik, (ik)^2, \ldots)$. The combination

$$A_j(k)e^{ikx - i\omega_j(k)t}$$

is referred to as a mode of the linear evolution equation. In general, a linear evolution equation has as many modes as the dimension of the system. The most general solution for our linear evolution system using these modes is given by

$$u = \frac{1}{2\pi}\sum_{j=1}^{N}\int_{-\infty}^{\infty} a_j(k)A_j(k)e^{ikx - i\omega_j(k)t}dk.$$

Example. Consider the "free"[1] linear Schrödinger equation $iu_t = -u_{xx}$. We can find the dispersion relation immediately, following what we did for scalar problems, to find $\omega = k^2$. If we let $u = q + ip$, where q and p are real-valued functions, by equating real and imaginary parts we find that $q_t = -p_{xx}$, $p_t = q_{xx}$. Writing this as a system, we have

$$\frac{\partial}{\partial t}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & -\partial_x^2 \\ \partial_x^2 & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}.$$

To find the dispersion relation, we examine

$$\begin{pmatrix} i\omega & k^2 \\ -k^2 & i\omega \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} = 0,$$

from which $\omega^2 = k^4$, so that $\omega_1 = k^2$, $\omega_2 = -k^2$. Finding the eigenvectors, we have the modes

$$\begin{pmatrix} i \\ 1 \end{pmatrix}e^{ikx - k^2t} \quad \text{and} \quad \begin{pmatrix} -i \\ 1 \end{pmatrix}e^{ikx + k^2t}.$$

[1] Because there is no external potential present.

Superimposing all these solutions, we have

$$\begin{pmatrix} q \\ p \end{pmatrix} = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(k)\begin{pmatrix} i \\ 1 \end{pmatrix} e^{ikx-ik^2t}dk + \frac{1}{2\pi}\int_{-\infty}^{\infty} a^*(k)\begin{pmatrix} -i \\ 1 \end{pmatrix} e^{-ikx+ik^2t}dk,$$

where $a^*(k)$ denotes the complex conjugate of $a(k)$. We have replaced k by $-k$ in the second term, and we have chosen the expansion coefficient to be the complex conjugate of the first one, to ensure reality of the solution. It may seem counterintuitive that by writing the Schrödinger equation in real vector form, we obtain two modes with frequencies k^2 and $-k^2$, whereas the Schrödinger equation in its complex form only gives rise to $\omega = k^2$, at least at first glance. The discrepancy is due to the fact we did not include the complex conjugate equation $-iu_t^* = -u_{xx}^*$. Of course, using the real form both the information from the equation and its complex conjugate are incorporated.

2. **Higher-order scalar equations.** Equations with higher-order time derivatives can always be reduced to first-order-in-time systems, but they occur so frequently and are so easily dealt with, that we discuss them briefly in their own right here. As before, time derivatives are replaced by $-i\omega$ and x derivatives by ik. This leads to a polynomial equation in ω and k, the dispersion relation. The degree of this equation as a function of ω equals the highest time derivative in the original equation. The different branches of the dispersion relation give rise to the different modes of the scalar equation.

   Example. Consider the wave equation $u_{tt} - c^2u_{xx} = 0$, where c is a positive constant. The dispersion relation is found easily: $(-i\omega)^2 - c^2(ik)^2 = 0$ so that $\omega_{1,2} = \pm ck$. The superposition of all modes gives

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a_1(k)e^{ikx-ickt}dk + \frac{1}{2\pi}\int_{-\infty}^{\infty} a_2(k)e^{ikx+ickt}dk.$$

   Clearly, as in the previous example, $a_2(k)$ is related to $a_1(k)$ to ensure reality of $u(x,t)$.

3. **More general equations.** The concept of a dispersion relation can be extended for equations that are not necessarily in evolution form. In fact, it is shown in [69] how an equation with an arbitrary dispersion relation (not necessarily polynomial) may be constructed. In principle, the ideas remain the same: time derivatives are replaced by $-i\omega$ and x derivatives by ik, and a functional relationship between ω and k is derived.

   Example. Consider the linearized Benjamin-Bona-Mahoney (BBM) [8] equation:

$$u_t + u_x - u_{xxt} = 0.$$

   Formally replacing $\partial_t \to -i\omega$, $\partial_x \to ik$, we obtain

$$-i\omega + ik - (ik)^2(-i\omega) = 0 \quad \Rightarrow \quad \omega = \frac{-k}{1+k^2},$$

and a solution using superposition is obtained as before.

The representation (2.2) is only useful if it allows us to solve the evolution equation for a sufficiently large set of initial conditions. Let $u(x,0) = u_0(x)$ be a given initial condition with $u_0(x) \in L_2(-\infty,\infty)$, *i.e.*,

$$\int_{-\infty}^{\infty} |u_0(x)|^2 dx < \infty.$$

In this case the initial-value problem

$$\begin{cases} u_t = F(u, u_x, \ldots) \\ u(x,0) = u_0(x), \quad u \to 0 \quad \text{as} \quad x \to \pm\infty \end{cases}$$

has a unique solution: we have

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(k) e^{ikx - i\omega(k)t} dk.$$

At $t = 0$ this becomes

$$u_0(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(k) e^{ikx} dk,$$

which implies that $a(k)$ is the Fourier transform of $u_0(x)$, provided $u_0(x)$ has a Fourier transform, which is guaranteed by the condition $u_0(x) \in L_2(-\infty,\infty)$, and

$$u_0(x) = \mathcal{F}^{-1}[a(k)](x) \quad \Rightarrow \quad a(k) = \mathcal{F}[u_0(x)](k) = \int_{-\infty}^{\infty} u_0(x) e^{-ikx} dx.$$

We say that the set of modes is complete in the space of initial conditions we are using, $L_2(-\infty,\infty)$ in this case. In other words: every initial condition in $L_2(-\infty,\infty)$ can be written as a linear combination of the fundamental modes.

2.2 Consequences of the dispersion relationship

It follows from the previous section that the scalar linear evolution equation may be rewritten as $u_t = -i\omega(-i\partial_x)u$. In algebraic terms, this implies that the dispersion relationship is the symbol of the scalar linear evolution equation. This algebraic viewpoint is particularly handy in quantum mechanics, especially in the Wigner formalism [65].

In any case, having established that the solution of the initial-value problem

$$\begin{cases} u_t = F(u, u_x, \ldots) \\ u(x,0) = u_0(x), \quad u \to 0 \quad \text{as} \quad x \to \pm\infty \end{cases} \tag{2.3}$$

is given by

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(k)e^{ikx-i\omega(k)t}dk,$$

with

$$a(k) = \int_{-\infty}^{\infty} u_0(x)e^{-ikx}dx,$$

what may we conclude from this? In other words: is this a useful result? We wrote down a representation for the solution of the initial-value problem, but unless we are able to obtain qualitative or quantitative information about the solution from it, we have merely transformed our original problem into a new problem.

We will show below that a lot of important information about the equation and its solutions is contained in the dispersion relation. The following are immediate:

- If $\operatorname{Im}\omega(k) > 0$ for some real k, then the corresponding mode grows exponentially in time, thus there is sensitive dependence on the initial data, and the problem is unstable. The most unstable mode (the one you might expect to see in an experimental setting where many perturbations are present) is the one that maximizes $\operatorname{Im}\omega(k)$.

- If $\operatorname{Im}\omega(k) \to \infty$ for any value of k, finite or infinite, then the problem is ill-posed: small perturbations in the initial conditions lead to unbounded growth for any finite time t, no matter how small.

- If $\operatorname{Im}\omega(k) < 0$ for all real k, the problem is dissipative and asymptotically stable.

- By Parseval's relation

$$\begin{aligned}\int_{-\infty}^{\infty} |u|^2 dx &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathcal{F}[u(x)](k)|^2 dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |a(k)|^2 e^{2\operatorname{Im}\omega(k)t}dk.\end{aligned}$$

  Thus $\int_{-\infty}^{\infty}|u|^2dx$ (we call this quantity the energy) is conserved (*i.e.*, independent of time) if and only if $\operatorname{Im}\omega(k) \equiv 0$ for all real k. Then the dispersion relationship is real-valued on the real line. There is no decay or growth.

If there is a term $\partial_x^n u$ in the differential equation (2.1) (assuming it has real coefficients), it contributes a term $-i(ik)^n = -i^{n+1}k^n$ to $\omega(k)$. This term is real if n is odd. If n is even, this term is imaginary and the term results in either dissipation or growth.

2.3 Asymptotics of the solution

The dominant behavior of the solution of the initial-value problem (2.3) is clear if there are real k values for which $\mathrm{Im}\,\omega(k) \neq 0$, based on what we have discussed. Let us assume that $\mathrm{Im}\,\omega(k) = 0$ for all real k. Also, let us assume that the differential equation we are dealing with is not $u_t = cu_x$, for some c. This is the one-dimensional transport equation. Its solution is a simple translation of the initial data. Let us assume we are dealing with a more complicated equation. Our solution is given by

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(k)e^{ikx-i\omega(k)t}dk,$$

with

$$a(k) = \int_{-\infty}^{\infty} u_0(x)e^{-ikx}dx.$$

Despite having no growth or decay under the above assumptions, it is possible for this solution to have an asymptotic state, as the finite amount of initial energy can spread over an infinite region. If we were considering periodic boundary conditions, an asymptotic state would not be possible.

Every individual mode travels with the phase velocity

$$c_p = c_p(k) = \omega(k)/k,$$

and the contribution of wave number k to the integral is

$$a(k)e^{ik(x-c_p(k)t)}.$$

We assumed above that we were not simply dealing with the one-dimensional transport equation, so $\omega(k) \neq ck$, where $c = c_p$, a constant. If c_p is not constant, the problem is more complicated than mere translation of initial data. We say the problem is dispersive if

$$\frac{\partial^2\omega}{\partial k^2} \neq 0.$$

Then different modes travel at different phase speeds.

To examine the asymptotics of the kind of integral given above, two methods are used frequently: (a) **the method of stationary phase** (developed by Kelvin), and (b) **the method of steepest descent** [3]. The latter is more general, but also more complicated. Before we consider the method of stationary phase in more detail, let us look at the integral. We have

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(k)e^{i\phi(k)t}dk,$$

with $\phi(k)$ given by

$$\phi(k) = \frac{x}{t}k - \omega(k).$$

We will consider the long-time asymptotics of $u(x,t)$, along rays of constant x/t. Thus the result we will obtain will give us the asymptotic state of $u(x,t)$ for $t \to \infty$, while keeping x/t constant.

For large t, the modes corresponding to nearby wave numbers k and $k + \delta k$ have very different exponents, leading to destructive interference. This idea is similar to the ideas leading to the Riemann-Lebesgue lemma. To avoid this destructive interference we need $\phi'(k) = 0$, so that $\phi(k)$ is nearly constant and the two exponents are similar. Thus, for a given ray through the origin of the (x,t)-plane, we find that the dominant modes will be those for which $\phi'(k) = 0$. For dispersive equations, different wave numbers dominate along different rays. We have

$$\phi(k) = \frac{x}{t}k - \omega(k) \quad \Rightarrow \quad \phi'(k) = \frac{x}{t} - \frac{d\omega}{dk}.$$

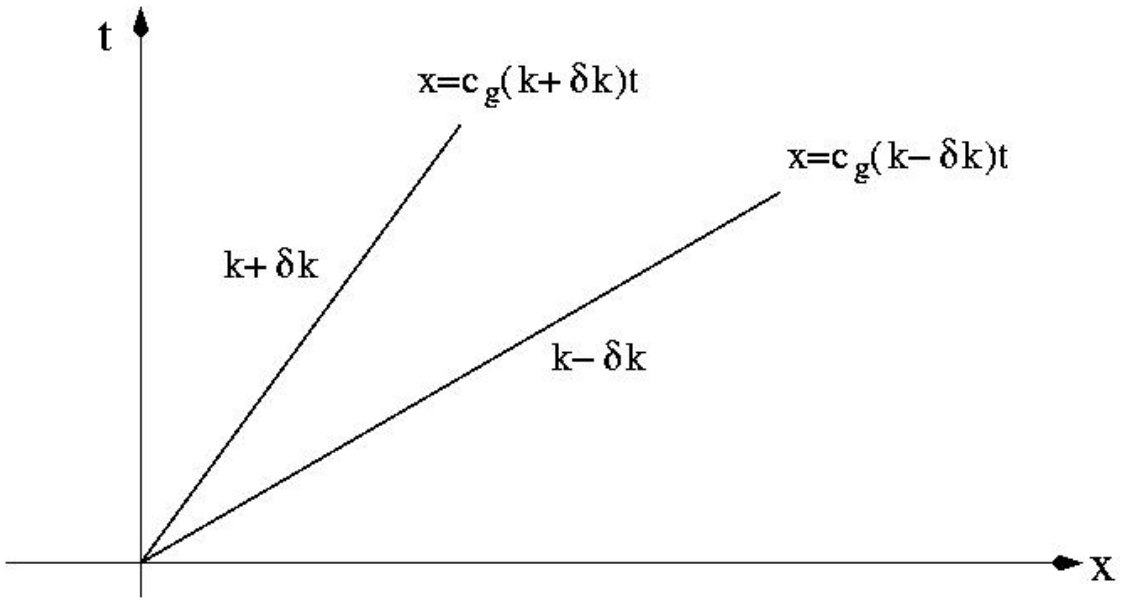


Figure 2.1: Different wave numbers $k - \delta k$ and $k + \delta k$ dominate along different rays, along which the information propagates with different velocities $c_g(k - \delta k)$ and $c_g(k + \delta k)$.

Define the group velocity $c_g = c_g(k)$ to be

$$c_g = \frac{d\omega}{dk}.$$

Thus a particular wave number k dominates along a ray for which $x/t = c_g(k)$. This demonstrates the importance of the group velocity.

Thus the wave numbers $k - \delta k$ and $k + \delta k$ dominate along different rays through the origin, as indicated in Fig. 2.1. The energy contribution from the modes with wave numbers between $k - \delta k$ and $k + \delta k$ is

$$\frac{1}{2\pi} \int_{k-\delta k}^{k+\delta k} |a(\kappa)|^2 d\kappa.$$

This contribution is spreading over a region that is linearly increasing in time, thus we expect

$$|u(x,t)|^2 \approx 1/t,$$

because of conservation of energy, by Parseval's identity. Thus

$$|u(x,t)| \approx t^{-1/2}, \quad \text{or} \quad |u(x,t)| = \theta(t^{-1/2}).$$

The dispersive character of the solution is manifested in the spreading of the solution over a larger region, as $t \to \infty$. Because energy is conserved for dispersive evolution equations, this results in a decrease of the amplitude.

The preceding heuristic arguments turn out to be correct, unless $\phi''(k) = 0$ where $\phi'(k) = 0$. In this case, the region where a certain wave number dominates is larger and nearby wave numbers spread slower. Overall this results in a slower decay of the solution amplitude.

2.4 The method of stationary phase

In this section we apply the method of stationary phase to make the ideas from the previous section more precise. Thus we examine the long-time asymptotics of the expression

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(k) e^{i\phi(k)t} dk, \tag{2.4}$$

with $\phi = kx/t - \omega(k)$ along rays of constant x/t. Stationary phase points occur when $\phi'(k) = 0$, from which

$$x = c_g(k)t,$$

with $c_g(k) = \omega'(k)$, the group velocity. The asymptotic form of (2.4) has a contribution from every solution k_0 of $\phi'(k) = 0$. In what follows, we assume there is a unique solution, and the asymptotic form consists of a single term.

We expand $\phi(k)$ near k_0:

$$\begin{aligned} \phi(k) = & \quad \phi(k_0) + (k - k_0)\phi'(k_0) + \frac{1}{2}(k - k_0)^2 \phi''(k_0) + \mathcal{O}(|k - k_0|^3) \\ = & \quad \phi(k_0) + \frac{1}{2}(k - k_0)^2 \phi''(k_0) + \mathcal{O}(|k - k_0|^3), \end{aligned} \tag{2.5}$$

using the stationarity condition. The dominant contribution from waves with wave number k_0 to (2.4) is

$$\begin{aligned}\frac{1}{2\pi}\int_{k_0-\delta}^{k_0+\delta} a(k)e^{i\phi(k)t}dk &\approx \frac{1}{2\pi}a(k_0)\int_{k_0-\delta}^{k_0+\delta} e^{i\left[\phi(k_0)+\frac{1}{2}(k-k_0)^2\phi''(k_0)\right]t}dk \\ &\approx \frac{1}{2\pi}a(k_0)e^{i\phi(k_0)t}\int_{k_0-\delta}^{k_0+\delta} e^{i\frac{1}{2}(k-k_0)^2\phi''(k_0)t}dk \\ &= \frac{1}{\pi}a(k_0)e^{i\phi(k_0)t}\int_{k_0}^{k_0+\delta} e^{i\frac{1}{2}(k-k_0)^2\phi''(k_0)t}dk,\end{aligned}$$

where δ is a small fixed number. In the last step, we used that the integrand is even around k_0. The integral is worked out by using the substitution $\kappa^2 = \frac{1}{2}(k-k_0)^2|\phi''(k_0)|t$. This results in

$$\begin{aligned}\frac{1}{2\pi}\int_{k_0-\delta}^{k_0+\delta} a(k)e^{i\phi(k)t}dk &\approx \frac{1}{\pi}a(k_0)e^{i\phi(k_0)t}\sqrt{\frac{2}{|\phi''(k_0)|t}}\int_0^{\delta\sqrt{|\phi''(k_0)|t/2}} e^{i\kappa^2\mathrm{sgn}(\phi''(k_0))}d\kappa \\ &\approx \frac{1}{\pi}a(k_0)e^{i\phi(k_0)t}\sqrt{\frac{2}{|\phi''(k_0)|t}}\int_0^{\infty} e^{i\,\kappa^2\,\mathrm{sgn}(\phi''(k_0))}d\kappa \\ &\approx \frac{1}{\pi}a(k_0)e^{i\phi(k_0)t}\sqrt{\frac{2}{|\phi''(k_0)|t}}\frac{\sqrt{\pi}}{2}e^{i\,\pi\,\mathrm{sgn}(\phi''(k_0))/4},\end{aligned}$$

where $\mathrm{sgn}(x)$ denotes the sign of the real quantity x:

$$\mathrm{sgn}(x) = \begin{cases} -1 & x<0 \\ 0 & x=0 \\ 1 & x>0 \end{cases}.$$

Also, we have replaced the upper bound by ∞ since we are interested in the large t behavior, thus the quantity $\sqrt{t}$ in the upper bound is large. Lastly, we have used the Fresnel integral [3]

$$\int_0^{\infty} e^{i\kappa^2}d\kappa = \frac{\sqrt{\pi}}{2}e^{i\pi/4}.$$

The final expression for the stationary-phase approximation to $u(x,t)$ using the contribution from the stationary point k_0 is given by

$$u(x,t) = \frac{a(k_0)}{\sqrt{2\pi t|\phi''(k_0)|}}e^{i\phi(k_0)t+i\,\pi\,\mathrm{sgn}(\phi''(k_0))/4}. \tag{2.6}$$

As stated before, if there is more than one stationary point, the contributions from all these points should be added up to get the final approximate expression for $u(x,t)$. This

formula confirms our earlier heuristic argument that the amplitude of the solution decays proportional to $t^{-1/2}$. Of course, the above argument is contingent on $\phi''(k_0) \neq 0$. If this were not the case, the expansion (2.5) shows that higher-order terms in $k - k_0$ need to be included to capture the dominant contribution. Our result (2.6) shows this restriction as well, as the expression is not defined when $\phi''(k_0) = 0$.

Example. Consider the linear KdV equation

$$u_t + u_{xxx} = 0.$$

Its dispersion relationship is $\omega = -k^3$. Thus the general solution of this equation in $L_2(-\infty, \infty)$ is

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(k) e^{ikx + ik^3 t} dk. \tag{2.7}$$

We have

$$\phi(k) = \frac{x}{t} k + k^3 \quad \Rightarrow \quad \phi'(k) = \frac{x}{t} + 3k^2 \quad \Rightarrow \quad \phi''(k) = 6k.$$

Stationary points occur when $k^2 = -x/(3t)$. This results in real wave number values only if $x/t < 0$, hence $x < 0$ since we are interested in the behavior as $t \to \infty$. We conclude that on the positive real line there are no real stationary points and the solution decays, due to destructive interference. Using the method of steepest descent, one shows that this decay is exponential. It is possible to guess this: if the initial condition is analytic, we may use repeated integration by parts to demonstrate that the decay of the solution is faster than any power. This is confirmed by Fig. 2.2, where the expression (2.7) with $a(k) = \operatorname{sech}(k)$ and $x/t = 1$ has been evaluated for 20 different values of t.

Let $x/t < 0$. The stationary points are given by

$$k_\pm = \pm \sqrt{-\frac{x}{3t}}.$$

We see that there are two stationary points, k_+ and k_-. This will result in two contributions to the asymptotic form of the integral. First, we consider the contribution from k_+. Since $k_+ > 0$, $\phi''(k_+) > 0$. Using (2.6), we get

$$\frac{a(k_+)}{2\sqrt{3\pi t k_+}} e^{i\phi(k_+)t + i\,\pi/4}. \tag{2.8}$$

Similarly, we get for the contribution from $k_- = -k_+$:

$$\frac{a(k_-)}{2\sqrt{-3\pi t k_-}} e^{i\phi(k_-)t - i\,\pi/4} = \frac{a(k_-)}{2\sqrt{3\pi t k_+}} e^{-i\phi(k_+)t - i\,\pi/4}, \tag{2.9}$$

since $\phi(k_-) = -\phi(k_+)$. Adding the two contributions (2.8) and (2.9), we obtain the asymptotic expression

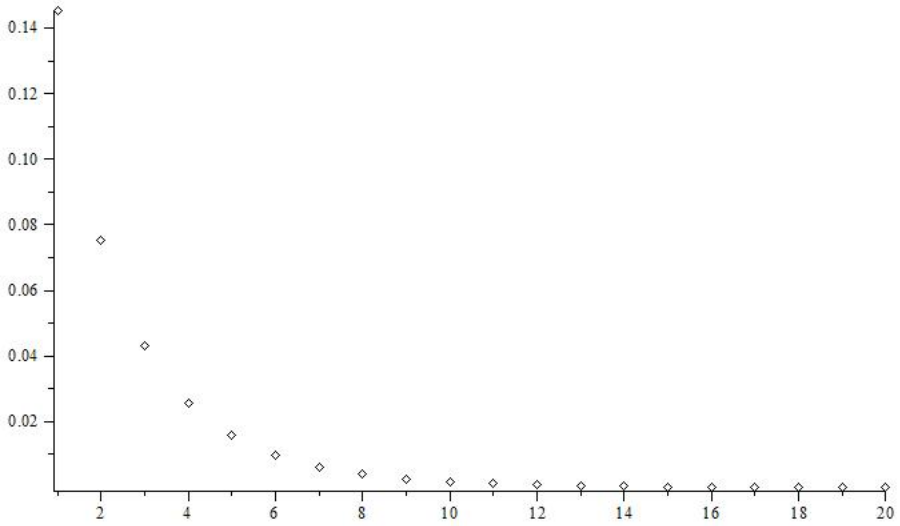


Figure 2.2: The graph of the integrand of (2.7), with $a(k) = \operatorname{sech}(k)$ for $x/t = 1$, evaluated at $t = 1, 2, \ldots, 20$, illustrating the decay of the solution for positive values of x/t, due to the absence of any stationary points.

$$u(x,t) \approx \frac{a(k_+)}{2\sqrt{3\pi t k_+}} e^{i\phi(k_+)t + i\,\pi/4} + \frac{a(k_-)}{2\sqrt{3\pi t k_+}} e^{-i\phi(k_+)t - i\,\pi/4}.$$

Note that the second term is the complex conjugate of the first one, since $a(k_-) = a(-k_+) = a^*(k_+)$. Thus

$$u(x,t) \approx \operatorname{Re}\left(\frac{a(k_+)}{\sqrt{3\pi k_+ t}} e^{i\phi(k_+)t + i\pi/4} \right). \tag{2.10}$$

This expression is valid for large t, along rays of constant $x/t < 0$. This asymptotic result is easy to evaluate, and as such much more convenient than the integral form (2.7) of the solution. The numerical evaluation of such improper integrals with oscillatory integrands is nontrivial (see Fig. 2.3), and gets increasingly problematic as t increases, while the approximation we have obtained continues to improve, as shown in Fig. 2.4.

2.5 Pattern Formation

The dispersion relation of an equation has consequences beyond allowing us to solve the corresponding linear equation. When studying a nonlinear PDE, the first action to take is to linearize the equation for small solutions, in effect dropping all nonlinear terms. The dispersion relation of the resulting linear equation provides good information about the

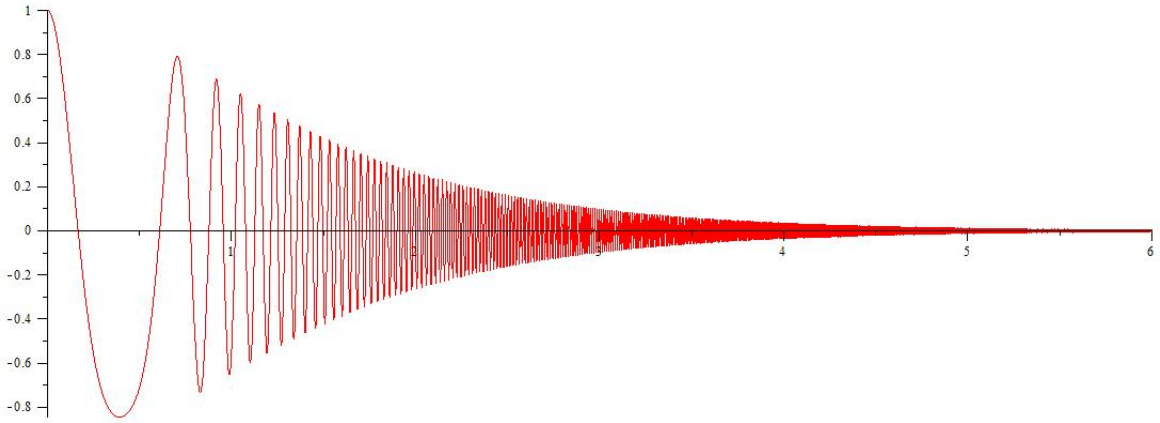


Figure 2.3: The graph of the integrand of (2.7) as a function of k, with $a(k) = \text{sech}(k)$ for $x/t = -1$, $t = 20$. Clearly, the numerical integration of this integrand is costly.

dynamics of small solutions of the nonlinear equation. We illustrate this using an example.

Example. The Kuramoto=Sivashinsky (KS) Equation. Consider the equation

$$u_t + uu_x + u_{xx} + au_{xxxx} = 0, \tag{2.11}$$

where $a > 0$ is a parameter. This equation arises in many applications, such as flame-front dynamics in combustion theory, reaction-diffusion system dynamics and cellular instabilities [46, 63]. Often it is used as one of the simplest mathematical models describing spatio-temporal chaos, pattern formation, and turbulence [61]. Sometimes, another equation is written for the KS equation, namely that obtained by letting $u = w_x$ and integrating with respect to x once. The KS equation is nonlinear, but it also contains two linear terms that have opposite effects, as we will see below.

Let us investigate the dynamics of small-amplitude solutions. To start, we ignore the effect of the nonlinear terms. That leaves us with

$$u_t + u_{xx} + au_{xxxx} = 0, \tag{2.12}$$

which may be solved using Fourier analysis. The dispersion relation is determined by

$$-i\omega = k^2 - ak^4.$$

This is a very different dispersion relation than those we have encountered thus far. specifically, this one does not result in real values of ω for real values of k. In fact, ω is imaginary for all real k. Thus, the corresponding Fourier modes will either grow or decay. such a mode is proportional to

$$e^{ikx - i\omega t} = e^{ikx + (k^2 - ak^4)t}.$$

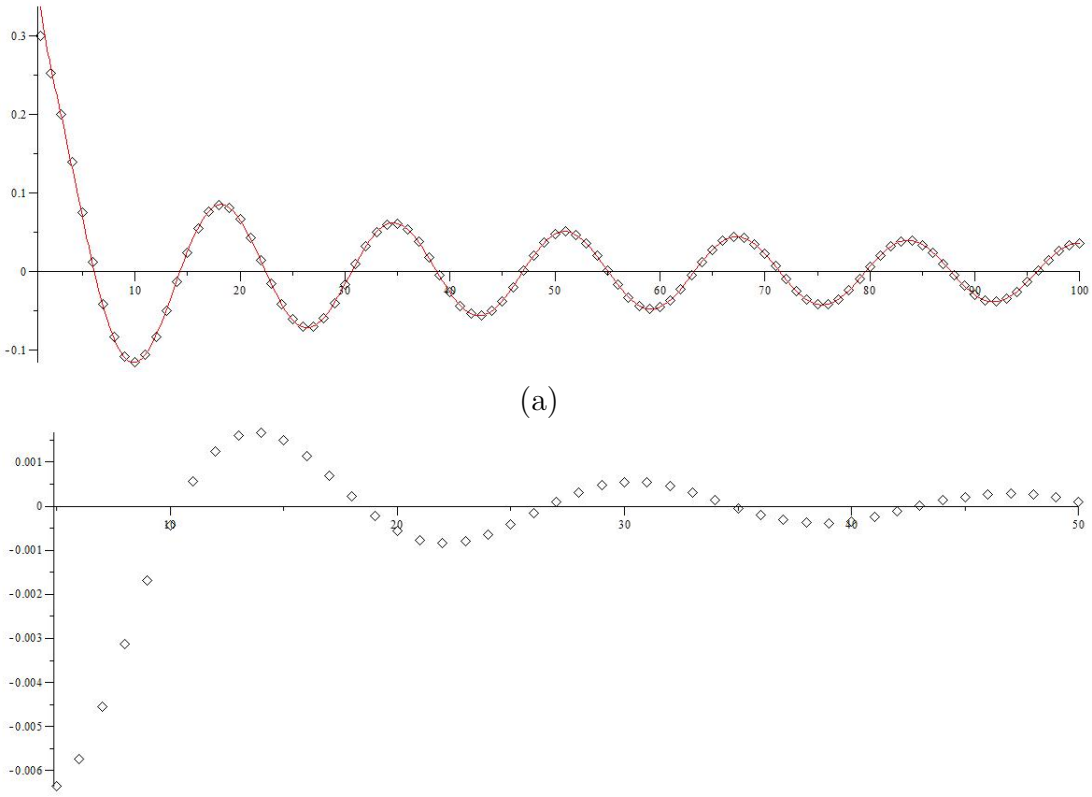


Figure 2.4: The stationary phase approximation (2.10) of (2.7) as a function of t, with $a(k) = \mathrm{sech}(k)$ for $x/t = -1$, displayed in Figure (a). The approximation appears excellent, except for values of, say, $t < 5$. This is made more quantitative in Figure (b) which shows the error of the stationary phase approximation (2.10) by subtracting from this approximation the numerically computed integral given by (2.7), for 50 different values of t.

It is immediate from this that modes for which $-i\omega(k)$ is positive will grow, while those for which $-i\omega(k)$ is negative will decay. The graph of $-i\omega(k)$ as a function of k is shown in Fig. 2.5. the effect of the second derivative is the k^2 behavior at the origin, while the fourth derivative causes the quartic drop off for large values of $|k|$. Thus, modes with small values of $|k|$ (*i.e.*, long waves) grow in time, whereas modes with large wavenumbers $|k|$ (short waves) decay over time. The cut-off is at $k = \pm\sqrt{1/a}$. The maximal growth rate occurs for $\pm k_0 = \pm\sqrt{1/2a}$.

Suppose that we start with an initial condition that consists of small-amplitude white noise, *i.e.*, our initial condition is a superposition of all Fourier wave numbers, and the Fourier amplitude $a(k)$ takes on small random values. The short-wave modes with $|k| > \sqrt{1/a}$ experience immediate damping, so that it appears the solution gets smoother as a function of time. In the mean time, the long-wave modes are growing in time, soon achieving amplitudes where the nonlinear terms cannot be ignored any longer. Since the largest growth rate occurs

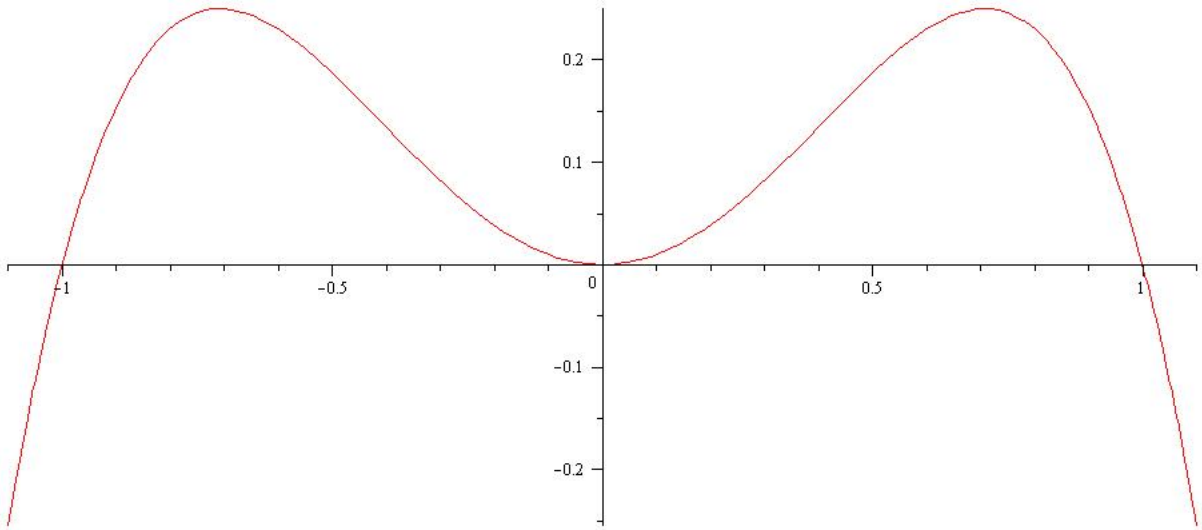


Figure 2.5: The graph of $-i\omega(k)$ for the linearized KS equation (2.12) with $a = 1$, as a function of k.

for the waves with $k = \pm k_0$, these waves experience the most growth. All other modes appear exponentially small compared to these modes. If there is a discrete set of modes because the equation is posed on a finite interval, the modes with k closest to $\pm k_0$ will dominate the small-amplitude dynamics.

The effect of the nonlinear term can be understood as well. Suppose that a mode e^{ikx} is present in the signal $u(x,t)$. Since $u(x,t)$ is real values, the mode e^{-ikx} is also present. The quadratic nonlinear term with its causes these contributions to interact with others, but also with themselves. Thus, immediately, the zero mode, and the modes e^{2ikx}, e^{-2ikx} are generated (if they were not already present) or affected (if they were already present). Next, the modes $e^{\pm 3ikx}$ and $e^{\pm 4ikx}$ are generated or affected. And so on. This simplistic reasoning leads to two conclusions.

1. Long waves grow, due to the presence of the band of unstable modes around the origin, see Fig. 2.5. However, this growth is limited by the nonlinearity which moves energy from wave numbers $\pm k$ to wave numbers $\pm 2k$, $\pm 3k$, *etc.*

2. This energy transfer from long waves to short waves is terminated by the fourth derivative term, which causes short waves to decay in time.

We conclude that the KS equation has a natural period selection mechanism built in: even if the initial condition is not periodic as a function of x, the dynamics of the equation tends to prefer signals that are periodic of period $2\pi/|k_0|$, as these waves experience the largest growth rate, which is saturated by the effect of the nonlinear term. Thus, an incoherent input signal $u(x,0)$ may give rise to a periodic pattern. This is an example of pattern formation in one spatial dimension. In multiple space dimensions, such behavior is more pronounced

and visually often very striking: out of a sea of white noise, a coherent pattern (squares, rectangles, hexagons, *etc*) may develop, seemingly spontaneous. As may be inferred from this example, pattern formation often requires the presence of dissipation and other non-dispersive processes. Because of this, beyond this chapter pattern formation and coherent structures are essentially ignored in the remainder of the text. The interested reader is referred to the seminal review paper [16] and the recent monograph [15].

2.6 Exercises

1. Consider the example on P.23 for the linear Schrödinger equation. Show that both approaches lead to the same answer for $u(x,t)$, whether a complex scalar equation or a real system is used.

2. We have seen the dispersion relationships for the linear KdV and BBM equations. Both equations can be used to model long waves in shallow water. Based on the dispersion relationships for these two equations, discuss which one would be advantageous for numerical purposes, from the point of view of suppressing unwanted oscillations, *etc*.

3. **The Benjamin-Ono equation**

$$u_t + uu_x + \mathcal{H}u_{xx} = 0$$

is used to describe internal waves in deep water [61]. Here $\mathcal{H}f(x)$ is the Hilbert transform of $f(x)$:

$$\mathcal{H}f(x) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{f(z)}{z-x} dz,$$

and $\fint$ denotes the Cauchy principal value integral. Write down the linear dispersion relationship for this equation.

4. Derive the linear dispersion relationship for the one-dimensional surface water wave problem by linearizing around the trivial solution $\zeta(x,t) = 0$, $\phi(x,z,t) = 0$:

$$\begin{aligned} \nabla^2\phi &= 0, && -h < z < \zeta(x,t) \\ \phi_z &= 0, && z = -h \\ \zeta_t + \phi_x\zeta_x &= \phi_z, && z = \zeta(x,t) \\ \phi_t + g\zeta + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) &= T\frac{\zeta_{xx}}{\left(1+\zeta_x^2\right)^{3/2}}, && z = \zeta(x,t) \end{aligned}$$

Here $z = \zeta(x,t)$ is the surface of the water, $\phi(x,z,t)$ is the velocity potential so that $v = \nabla\phi$ is the velocity of the water, g is the acceleration of gravity, and $T > 0$ is the coefficient of surface tension.

5. Having found that for the surface water wave problem without surface tension the linear dispersion relationship is $\omega^2 = gk\tanh(kh)$, find the group velocities for the case of long waves in shallow water (kh small), and for the case of deep water (kh big).

6. Consider the linear free Schrödinger ("free", because there's no potential) equation

$$i\psi_t + \psi_{xx} = 0, \qquad -\infty < x < \infty, \quad t > 0, \quad \psi \to 0 \ \text{ as } \ |x| \to \infty,$$

with $\psi(x,0) = \psi_0(x)$ such that $\int_{-\infty}^{\infty} |\psi_0|^2 dx < \infty$.

   (a) Using the Fourier transform, write down the solution of this problem.

   (b) Using the Method of Stationary Phase, find the dominant behavior as $t \to \infty$ of the solution, along lines of constant x/t.

   (c) With $\psi_0(x) = e^{-x^2}$, the integral can be worked out exactly. Compare (graphically or other) this exact answer with the answer you get from the Method of Stationary Phase. Use the lines $x/t = 1$ and $x/t = 2$ to compare.

   (d) Use your favorite numerical integrator (write your own, or use maple, mathematica or matlab) to compare (graphically or other) with the exact answer and the answer you get from the Method of Stationary Phase.

7. Everything that we have done for continuous space equations also works for equations with a discrete space variable. Consider the discrete linear Schrödinger equation:

$$i\frac{d\psi_n}{dt} + \frac{1}{h^2}(\psi_{n+1} - 2\psi_n + \psi_{n-1}) = 0,$$

where h is a real constant, n is any integer, $t > 0$, $\psi_n \to 0$ as $|n| \to \infty$, and $\psi_n(0) = \psi_{n,0}$ is given.

   (a) The discrete analogue of the Fourier transform is given by

$$\psi_n(t) = \frac{1}{2\pi i}\oint_{|z|=1} \hat{\psi}(z,t) z^{n-1} dz,$$

   and its inverse

$$\hat{\psi}(z,t) = \sum_{m=-\infty}^{\infty} \psi_m(t) z^{-m}.$$

   Show that these two transformations are indeed inverses of each other.

   (b) The dispersion relation of a semi-discrete problem is obtained by looking for solutions of the form $\psi_n = z^n e^{-i\omega t}$. Show that for the semi-discrete Schrödinger equation

$$\omega(z) = -\frac{(z-1)^2}{zh^2}.$$

   How does this compare to the dispersion relation of the continuous space problem? Specifically, demonstrate that you recover the dispersion relationship for the continuous problem as $h \to 0$.

8. **The FitzHugh-Nagumo system**

$$\begin{aligned} u_t &= u_{xx} + u(u-a)(1-u) + w \\ w_t &= \epsilon u \end{aligned}$$

(with ϵ and a positive) describes an excitable system. For instance it has been used to model the dynamics of a neuron [61]. Write down its linear dispersion relationship. Discuss how different "small" (so that your linearization is justified) initial conditions will evolve (which modes will grow, which will not, *etc*).

9. **The Swift-Hohenberg equation** is often used as the prototype equation to illustrate pattern formation, especially in two dimensions. The equation can be written as

$$\psi_t = \alpha\psi - (1 + \partial_x^2 + \partial_y^2)^2\psi + \beta\psi^2 - \psi^3.$$

Discuss the dynamics of this equation for different values of the parameters α and β. You may wish to visit Michael Cross' webpage `http://crossgroup.caltech.edu/Patterns/SwiftHohenberg.html`, illustrating the patterns that occur in this equation. These range from stripes over hexagons to quasicrystals.

Chapter 3

Quasi-linear partial differential equations. The method of characteristics

In the previous chapter we have examined the effects of dispersion on the evolution of initial profiles. If the initial condition of a given nonlinear PDE is small, we may expect that the dynamics of the linearized equation is not unlike that of the nonlinear equation with the same initial condition. In this chapter, we investigate the effect of quasi-linear first-order nonlinearities. These are nonlinear terms that contain the function and its first-order spatial derivative, and that are linear in this derivative. Although other nonlinearities are relevant in many equations, this class of nonlinearities is important: it is probably the most basic nonlinear effect one may encounter, while allowing us to build some intuition about the effect of nonlinearities in general. Further, many other nonlinearities may be converted to this form by various procedures: taking derivatives, rewriting equations as first-order systems, *etc.*

The study of such quasi-linear equations leads to the method of characteristics, which is the starting point for more general considerations such as Riemann's method and the classification of higher-order PDEs into equations of hyperbolic, parabolic and elliptic type, see for instance [28]. In this chapter we limit ourselves to solving scalar first-order equations using characteristics. We examine the formation of shocks, and what may happen after shocks form.

3.1 The method of characteristics

We consider partial differential equations of the form

$$u_t + c(u)u_x = F(x,t,u) \tag{3.1}$$

As stated, such an equation is called **quasi-linear**, because it is linear in the derivatives of $u(x,t)$. We could have written down a slightly more general equation, by allowing $c(u)$ to be an explicit function of x and t as well.

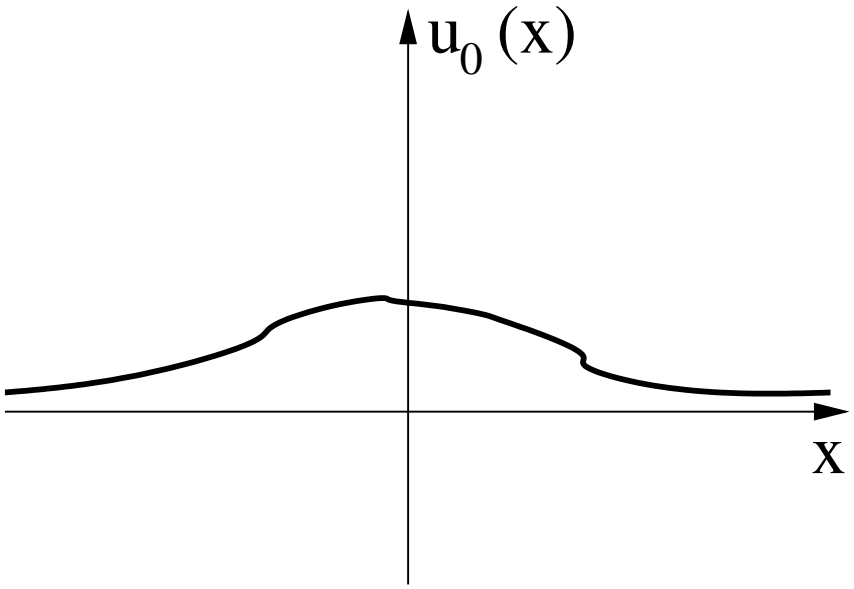


Figure 3.1: The initial condition $u = u_0(x)$ for equation (3.1)

We use the method of characteristics to solve equations like this. An excellent presentation of the method of characteristics and many of its applications (traffic flow, gas dynamics) is found in [69]. Let $u(x, 0) = u_0(x)$ be the initial condition for (3.1), given on the whole line $x \in \mathbb{R}$, see Fig. 3.1. We want to know what happens to $u(x, t)$ for $t > 0$. For our purposes, it suffices to consider the homogeneous case only, so that $F(x, t, u) = 0$.

Our equation reads

$$\frac{\partial u}{\partial t} + c(u)\frac{\partial u}{\partial x} = 0. \tag{3.2}$$

This can be rewritten as

$$\frac{du}{dt} = 0,$$

where

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x},$$

along curves for which

$$\frac{dx}{dt} = c(u).$$

Such curves are called **characteristic curves**. How do we use this? Along such a characteristic curve C

$$\begin{aligned} & \frac{du}{dt} = 0 \\ \Rightarrow \quad & u = \text{constant along C} \\ \Rightarrow \quad & u = u_0(\xi), \end{aligned}$$

along the characteristic curve that starts at $x = \xi$ in the (x, t)-plane. What is the equation of this curve? We have along this curve

$$\begin{aligned} & \frac{dx}{dt} = c(u(x,t)) \\ \Rightarrow \quad & \frac{dx}{dt} = c(u_0(\xi)) \\ \Rightarrow \quad & x = \xi + tc(u_0(\xi)). \end{aligned}$$

Thus, the solution of the equation (3.2) with initial condition $u(x, 0) = u_0(x)$ is given by

$$u = u_0(\xi),$$

where ξ is determined by the implicit equation

$$x = \xi + tc(u_0(\xi)).$$

This implicit equation should be solved for ξ, as a function of x and t.

Geometrically, it is clear what is happening, as is shown in Fig. 3.2. On every point of the x-axis, we can plot a vector with direction given by

$$\frac{dx}{dt} = c(u_0(\xi)),$$

which indicates how the initial condition is "transported" along the characteristic curves. For the equation we have considered, the characteristic curves are straight lines. This would not be the case if we allowed $c(u)$ to depend explicitly on x and/or t. For the straight-line case, there are different possibilities depending on $c(u)$ and $u_0(x)$:

- the characteristic lines could be parallel, or
- the characteristic lines could fan out, or
- the characteristic lines could cross.

Let us look at some examples.

Example. Consider the initial-value problem

$$\begin{cases} u_t + cu_x &= 0, \\ u(x,0) &= u_0(x), \end{cases}$$

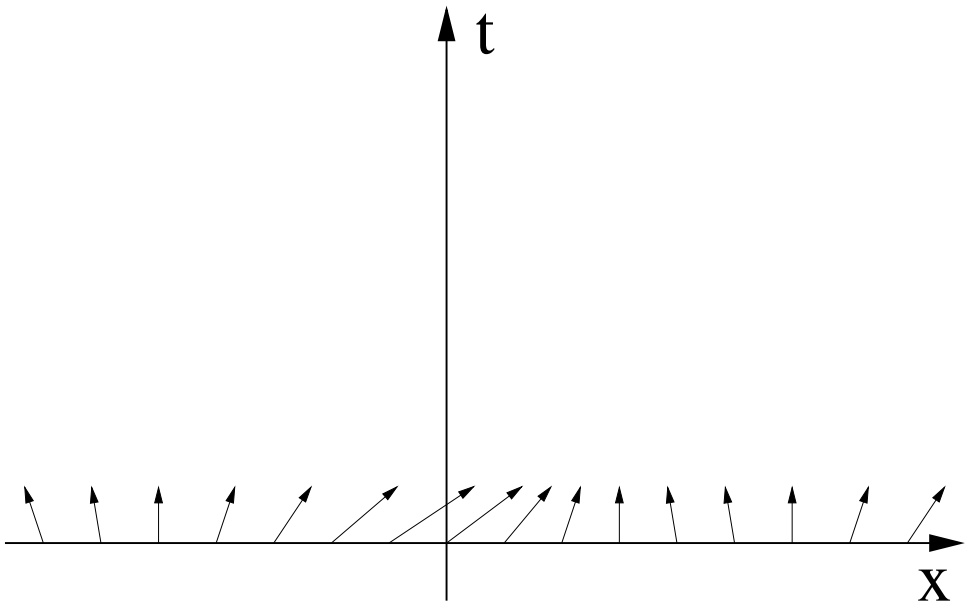


Figure 3.2: The initial condition $u(x,0) = u_0(x)$ being transported away from the x axis, in the directions of the arrows.

where c is a constant. The characteristic curves are determined by

$$\frac{dx}{dt} = c \quad \Rightarrow \quad x = \xi + ct \quad \Rightarrow \quad \xi = x - ct.$$

Thus the characteristics are all straight lines with slope $1/c$ in the (x,t)-plane. Furthermore, $u(x,t)$ is constant along these characteristics. Thus, along characteristics,

$$u(x,t) = u_0(\xi) = u_0(x - ct),$$

which provides the explicit solution to our initial-value problem. Since the characteristics are all parallel, no problems arise and the solution exists for all time. From Fig. 3.3 and our calculations we see that the initial condition is merely transported at constant speed along the characteristics.

Example. A more complicated example is given by

$$\begin{cases} u_t + uu_x &= 0, \\ u(x,0) &= u_0(x) = \arctan x. \end{cases}$$

This partial differential equation is known as the (dissipationless) **Burgers equation**. It has played and still plays an important role as the simplest nonlinear partial differential equation, and as such is a laboratory for new techniques, numerical methods, *etc.* Of course, we recognize it as the KdV equation minus the dispersion term.

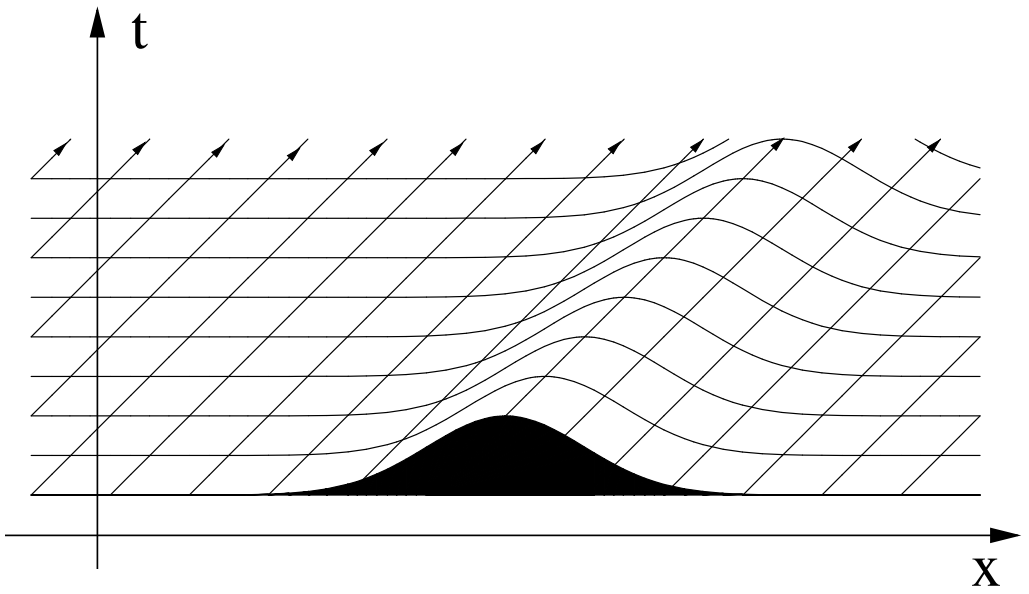


Figure 3.3: The characteristics for $u_t + cu_x = 0$, demonstrating the uniform translation of the initial condition over time. (3.1)

Noting that $u(x,t) = u_0(\xi)$ is constant along characteristics, and that we have $c(u) = u$ for this equation, it follows that the characteristics are determined by

$$\begin{aligned} & \frac{dx}{dt} = u \\ \Rightarrow \quad & x = \xi + u_0(\xi)t \\ \Rightarrow \quad & x = \xi + t\arctan\xi \\ \Rightarrow \quad & t = \frac{x-\xi}{\arctan\xi}. \end{aligned}$$

These characteristic lines are fanning out, thus the solution spreads out. It is defined for all $t \geq 0$, and is given by

$$u(x,t) = u_0(\xi) = \arctan\xi,$$

where $\xi = \xi(x,t)$ is determined by the implicit relationship

$$x = \xi + t\arctan\xi.$$

In this case, we cannot write down an explicit solution, as we cannot solve $x = \xi + t\,\text{arctan}\xi$ explicitly for ξ as a function of x and t. However, it is still straightforward to analyze the behavior of the solution. The solution at different times and the characteristics are shown in Fig. 3.4.

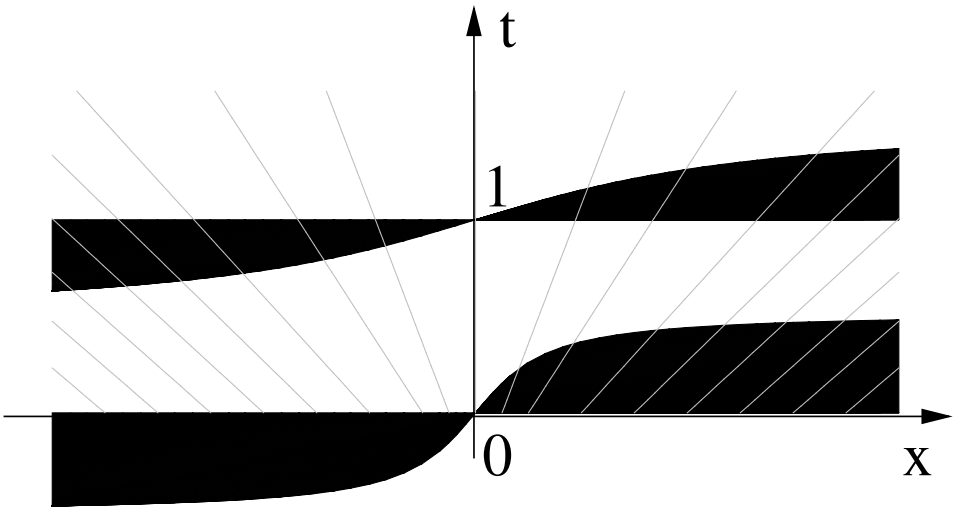


Figure 3.4: The characteristics for $u_t + uu_x = 0$, demonstrating the spreading of the initial condition $u_0 = \text{arctan}x$ over time.

Example. Consider the system

$$\begin{cases} u_t + uu_x &= 0, \\ u(x,0) &= -\text{arctan}x. \end{cases}$$

This is almost the same example, but now the sign of the initial condition is switched. Following what we did in the previous example, the solution is given by

$$u(x,t) = u_0(\xi) = -\text{arctan}\xi,$$

where $\xi = \xi(x,t)$ is determined by the implicit relationship

$$x = \xi - t\,\text{arctan}\xi.$$

As before, we cannot solve this explicitly for ξ, but we can analyze the behavior of the solution. The plot of the characteristics is shown in Fig. 3.5. We see there is a time t^* after which the solution obtained through the methods of characteristics is no longer valid:

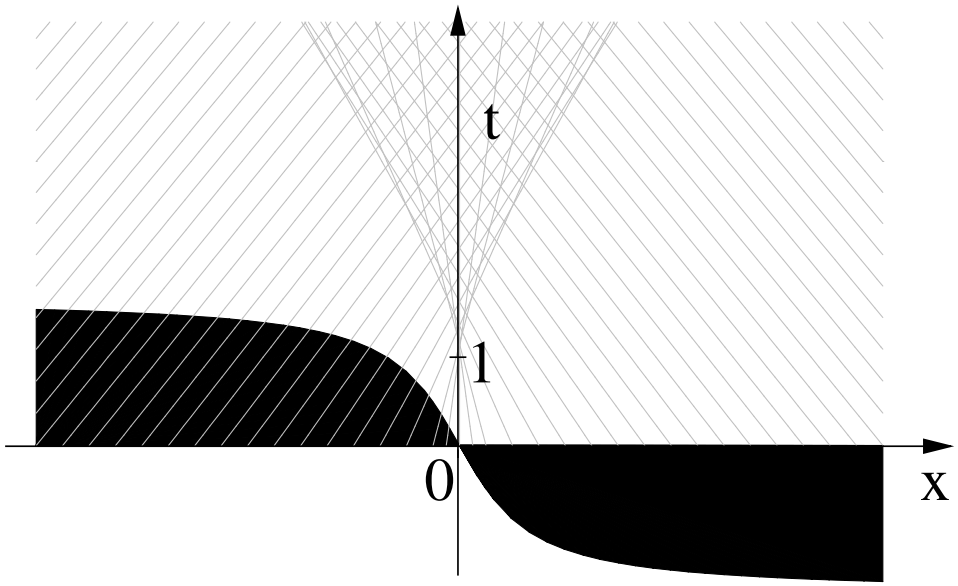


Figure 3.5: The crossing of the characteristics for $u_t + uu_x = 0$, demonstrating the steepening of the initial condition $u_0 = -\arctan x$ over time.

multiple characteristics cross, and in this wedge of overlapping characteristics, it is not possible to assign a unique value to $u(x,t)$.

We will return to this example soon, to resolve the issues we have just encountered. When characteristics cross, what happens for $t > t^*$, where t^* is the time at which they first cross? This is a question we will look into later. For now, let us find a condition to determine this time of first crossing t^*. This time is known as the breaking time, as it is the time at which the wave first "breaks": the solution develops an infinite derivative in space x or time t (if one happens, the other one happens as well, because of the partial differential equation). We have

$$u_t = u_{0\xi}\,\xi_t, \quad u_x = u_{0\xi}\,\xi_x,$$

using the chain rule. These become infinite when both of ξ_x and ξ_t become infinite, since u_0 is a well-behaved function. Taking x and t derivatives of the equation for the characteristics gives

$$\begin{aligned} & x = \xi + tc(u_0(\xi)) \\ \Rightarrow \qquad & 1 = \xi_x + tc'(u_0(\xi))u_{0\xi}\xi_x \qquad\qquad \Rightarrow \qquad \xi_x = \frac{1}{1 + tc'(u_0(\xi))u_{0\xi}} \end{aligned}$$

$$\Rightarrow \qquad 0 = \xi_t + tc'(u_0(\xi))u_{0\xi}\xi_t + c(u_0(\xi)) \qquad \Rightarrow \qquad \xi_t = \frac{-c(u_0(\xi))}{1 + tc'(u_0(\xi))u_{0\xi}}.$$

These expressions have the same denominators, so that the breaking time is determined by when these denominators are zero. Thus

$$t^* = -\frac{1}{c'(u_0(\xi^*))u_{0\xi}(\xi^*)},$$

where ξ^* corresponds to those characteristics for which

$$c'(u_0)u_{0\xi} < 0,$$

(to ensure a positive breaking time), and

$$|c'(u_0)u_{0\xi}| \text{ is maximal,}$$

(to ensure t^* is the smallest time at which crossing occurs).

Example. Let us return to our previous example to try to determine the breaking time t^*. We have

$$\begin{aligned} c(u) = u \qquad &\Rightarrow \qquad c' = 1, \\ u_0 = -\arctan\xi \qquad &\Rightarrow \qquad u_{0\xi} = \frac{-1}{1+\xi^2}. \end{aligned}$$

Thus ξ^* is determined by

$$c'u_{0\xi} < 0 \quad \Rightarrow \quad \frac{-1}{1+\xi^2} < 0,$$

which is always true, and

$$\frac{1}{1+\xi^2} \text{ is maximal,}$$

which implies $\xi^* = 0$. Thus

$$t^* = -\frac{1}{c'(u_0(\xi^*))u_{0\xi}(\xi^*)} = -\frac{1}{-\frac{1}{1+(\xi^*)^2}} = 1 + (\xi^*)^2 = 1,$$

which agrees with Fig. 3.5.

3.2 After shock formation: introducing dissipation or dispersion

Once a shock forms, it is clear that the equation

$$u_t + c(x,t,u)u_x = F(x,t,u)$$

is no longer valid. How do we modify it? This question has several answers, but none of them are determined by mathematics. At this point, the application that we are attempting to model needs to give us ideas to modify the model, so as to be able to come up with a unique solution. We will see two ways to do this. Both may give analytical results. More importantly they are used in numerical methods to either avoid shock formation, or else to dictate the motion and location of shocks, once they form.

The first method we discuss modifies the governing equation to prevent the formation of shocks. There are many ways of doing this, and typically one looks at the application being considered to figure out what the right way is to achieve this. Our equation is supposed to model some process associated with an application. When we get close to shock formation and derivatives get large, it is possible that other terms that we have previously omitted become important.

Example. We have found that the dissipationless Burgers equation

$$u_t + uu_x = 0$$

develops a shock if the initial condition has a profile as shown in Fig. 3.6. In general, any profile with negative derivative will lead to steepening. Now suppose that the problem we are modeling actually corresponds to the differential equation

$$u_t + uu_x = \epsilon u_{xx},$$

where $\epsilon > 0$ is very small. Because of the small coefficient ϵ we may have thought it was fine to ignore this second derivative term. This approximation will be valid, as long as derivatives in the solution are "small" (meaning "not large"). However, as we approach the shock, derivatives become very large and the term on the right does become significant.

Let us verify that this is indeed the case. Close to the shock time t^*, we have

$$u_x = u_0'(\xi)\xi_x = \frac{u_0'(\xi)}{1 + tu_0'(\xi)},$$

and the denominator approaches zero. Also,

$$u_{xx} = u_0''(\xi)\xi_x^2 + u_0'(\xi)\xi_{xx}.$$

To calculate ξ_{xx}, we return to the definition of the characteristic curve:

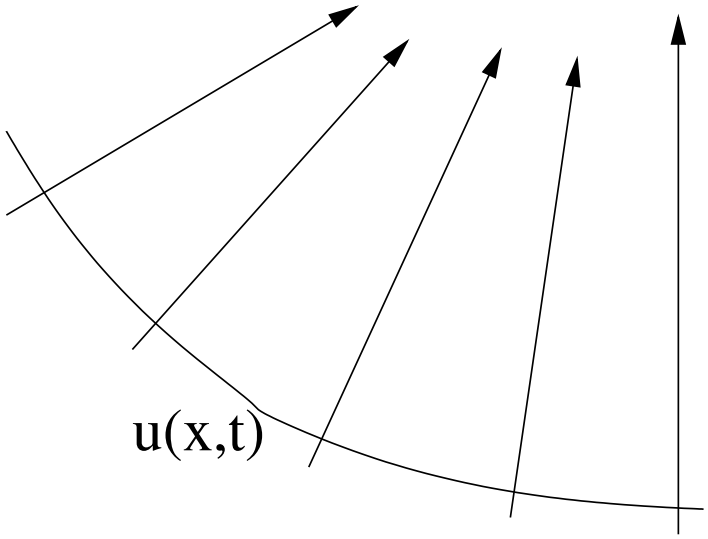


Figure 3.6: Part of an initial profile that will lead to shock formation in the dissipationless Burgers equation.

$$
\begin{aligned}
& & x &= \xi + t u_0(\xi) \\
\Rightarrow & & 1 &= \xi_x + t u_0'(\xi)\xi_x \\
\Rightarrow & & 0 &= \xi_{xx} + t u_0'(\xi)\xi_{xx} + t u_0''(\xi)\xi_x^2 \\
\Rightarrow & & \xi_{xx}(1 + t u_0'(\xi)) &= -t u_0''(\xi)\xi_x^2 \\
\Rightarrow & & \xi_{xx} &= \frac{-t u_0''(\xi)\xi_x^2}{1 + t u_0'(\xi)}.
\end{aligned}
$$

Using this, we obtain

$$
\begin{aligned}
u_{xx} &= u_0''(\xi)\xi_x^2 + u_0'(\xi)\xi_{xx} \\
&= u_0''(\xi)\xi_x^2 - u_0'(\xi)\frac{t u_0''(\xi)\xi_x^2}{1 + t u_0'(\xi)} \\
&= u_0''(\xi)\xi_x^2 \left(1 - \frac{t u_0'}{1 + t u_0'}\right) \\
&= \frac{u_0''(\xi)\xi_x^2}{1 + t u_0'} \\
&= \frac{u_0''(\xi)}{(1 + t u_0')^3},
\end{aligned}
$$

and we see that $u_{xx} \to \infty$ much faster than u_x or u_t. Thus, the right-hand side term cannot be ignored close to the shock formation.

Next, we'll show that taking this second-derivative term into account does indeed have the desired effect of arresting the shock formation. Near the shock we have

$$u_t = \epsilon u_{xx} - u u_x \sim \epsilon u_{xx},$$

Since u_{xx} is much larger than the product of uu_x just before the shock formation. Just before a shock is formed, the solution has a singular inflection point (*i.e.*, the second derivative changes sign, but by passing through infinity instead of zero). Above this inflection point $u_t \sim \epsilon u_{xx} < 0$, whereas below the inflection point $u_t \sim \epsilon u_{xx} > 0$. The combination of these is to counteract the steepening, as desired, see Fig. 3.7.

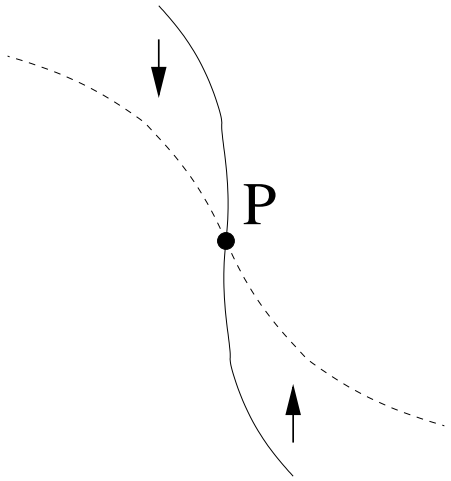


Figure 3.7: The effect of second-derivative dissipation on the near-formation of a shock near an inflection point P.

For other examples, and depending on the applications, different mechanisms may prevent shock formation. For instance for dispersive processes, the introduction of a third-derivative term may be more appropriate:

$$u_t + u u_x = \epsilon u_{xxx}.$$

The resulting partial differential equation is the KdV equation, of course. Although we will see in future chapters that the initial-value problem for this equation can be solved for initial data in a suitable function space, investigating the behavior of solutions as $\epsilon \to 0$ is nowhere near as straightforward as for the Burgers equation with dissipation. It is the topic of what is now known as Lax-Levermore theory [47], where one examines the zero-dispersion limit of wave equations with increasingly small dispersion. This is now well understood for the KdV equation and for the defocusing NLS equation

$$i\psi_t = -\epsilon \psi_{xx} + |\psi|^2 \psi.$$

For the focusing NLS equation

$$i\psi_t = -\epsilon\psi_{xx} - |\psi|^2\psi,$$

the small-dispersion problem is significantly more complicated, and serious progress has been made since the mid 1990s only. See [39] for an overview.

The main difference between dissipation and dispersion in the prevention of shock formation is that dissipation tends to smooth out the shock profile, and as the coefficient of the dissipative term approaches zero, the solution converges to the observed shock profile (in some norm). On the other hand, adding dispersion tends to result in high-frequency oscillations, and as the coefficient of the dispersive term tends to zero, the solution does not necessarily converge to the shock profile. Often there is only weak convergence[1], *i.e.*, the local average[2] of the solution converges to the shock profile. This is illustrated in Fig. 3.8

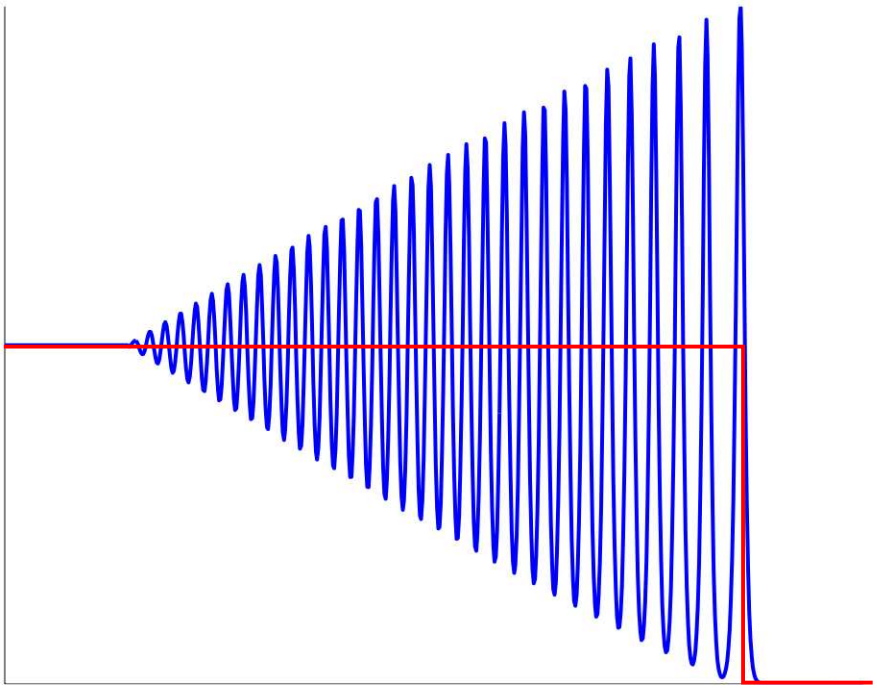


Figure 3.8: A solution profile (fast oscillations; blue) for small dispersion, and the shock (red) it is weakly converging to (Figure modified from [34]).

For yet different applications, one may have to resort to higher-order dissipation or dispersion, or different types of nonlinear terms.

[1] See any standard analysis text, *e.g.*, [25].

[2] The average of the solution over a small interval which contains a large number of oscillations of the solution.

3.3 After shock formation: Rankine-Hugoniot conditions

Another approach to the problem of shocks forming is to embrace them: instead of trying to prevent them, let us figure out what to do with them once they form. Let us determine where the shocks go, and how fast they move. The key to doing this is to work with an integral form of the partial differential equation. This integral form typically corresponds to the conservation law form of the application being modeled. Most partial differential equations arising in applications are originally derived in integral form, expressing the conservation of various quantities, such as mass, charge, momentum, energy, *etc.* This is true for Maxwell's equations in electro-magnetism, for the equations of continuum mechanics, including the Navier-Stokes equations, *etc.*.

The idea behind using an integral form of the partial differential equation is that integrals do not require continuous integrands to act on, whereas the arguments of derivatives should be smooth. Thus, using the integral form, we can deal with discontinuous solutions. Such solutions are also known as weak solutions of the differential equation.

Example. Again we consider the dissipationless Burgers equation

$$u_t + uu_x = 0.$$

We can rewrite this equation as

$$\frac{\partial}{\partial t}(u) + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0.$$

This form of the equation is called the conservation law form: integrating over the entire spatial domain, we find a conserved quantity for the partial differential equation if suitable boundary conditions are used. For instance

$$\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial t}(u) + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right)\right)dx = 0 \quad \Rightarrow \quad \frac{d}{dt}\int_{-\infty}^{\infty} u dx = 0,$$

where we have assumed that $u \to 0$ as $x \to \pm\infty$. Thus $\int_{-\infty}^{\infty} u dx$ is a constant of the motion.

Let us integrate the conservation law form over only part of space:

$$\begin{aligned} &\int_{x_1}^{x_2} \frac{\partial}{\partial t}(u)dx + \frac{1}{2}u^2\Big|_{x=x_1}^{x=x_2} = 0 \\ \Rightarrow \qquad &\frac{d}{dt}\int_{x_1}^{x_2} u dx = \frac{1}{2}\left(u_1^2 - u_2^2\right), \end{aligned}$$

where $u_1 = u(x_1, t)$ and $u_2 = u(x_2, t)$. As long as our solution has continuous derivatives, this integral equation, which is valid for all x_1 and x_2, is completely equivalent with the original partial differential equations form of the dissipationless Burgers equation.

Suppose that the solution exhibits a shock at $x = s(t)$. Also, assume that $x_2 > s(t)$ and $x_1 < s(t)$. Lastly, let $u^- = \lim_{x<s(t)} u(x,t)$, and $u^+ = \lim_{x>s(t)} u(x,t)$. Thus u^- and u^+ are the values of u just past and in front of the shock. Then

$$\frac{d}{dt}\left(\int_{x_1}^{s(t)} u dx + \int_{s(t)}^{x_2} u dx\right) = \frac{1}{2}\left(u_1^2 - u_2^2\right)$$

$$\Rightarrow \qquad u^- s' - u^+ s' + \int_{x_1}^{s(t)} u_t dx + \int_{s(t)}^{x_2} u_t dx = \frac{1}{2}\left(u_1^2 - u_2^2\right).$$

Now we let $x_1 \to s$ and $x_2 \to s$. Then both of

$$\int_{x_1}^{s(t)} u_t dx \quad \text{and} \quad \int_{s(t)}^{x_2} u_t dx$$

vanish, as u_t is smooth both in front of and past the shock. We obtain

$$(u^- - u^+)s' = \frac{1}{2}\left(u^{-2} - u^{+2}\right)$$

$$\Rightarrow \qquad s' = \frac{1}{2}\frac{(u^- - u^+)(u^- + u^+)}{u^- - u^+}$$

$$\Rightarrow \qquad s' = \frac{1}{2}(u^- + u^+).$$

We find that the velocity of the shock is equal to the average of the solution amplitude in front of and past the shock.

How can we use this? For typical initial conditions the values of u^- and u^+ are not constant, and the shock velocity condition represents an ordinary differential equation for the shock position. However, we know at which time t^* and on which characteristic ξ^* the shock first forms. We may use our condition to move the shock approximately, using an Euler-like numerical method, for instance.

Let us examine how this works: at time t^*, a shock forms on characteristic ξ^*, from which we find the initial position $s(t^*)$ of the shock. Then we know the values of $u^- = u(s(t^*), t^*) = u^+$. Note that $u^-(t^*) = u^+(t^*)$ for smooth initial conditions. This is not the case if the initial conditions are discontinuous. Using this, we may propagate the shock with a small time step Δt:

$$s(t^* + \Delta t) = s(t^*) + \frac{1}{2}(u^- + u^+)\Delta t + \mathcal{O}((\Delta t)^2).$$

Next, we update the values of u^+ and u^-, using the values transported to the shock line along the new characteristics that now contribute to the shock. The introduction of the shock, splits the region where characteristics used to cross in two regions, one to the left of the shock line, one to the right. The characteristics can now safely be followed all the way to the shock line, without risking any multivaluedness. A schematic of this idea is presented in Fig. 3.9.

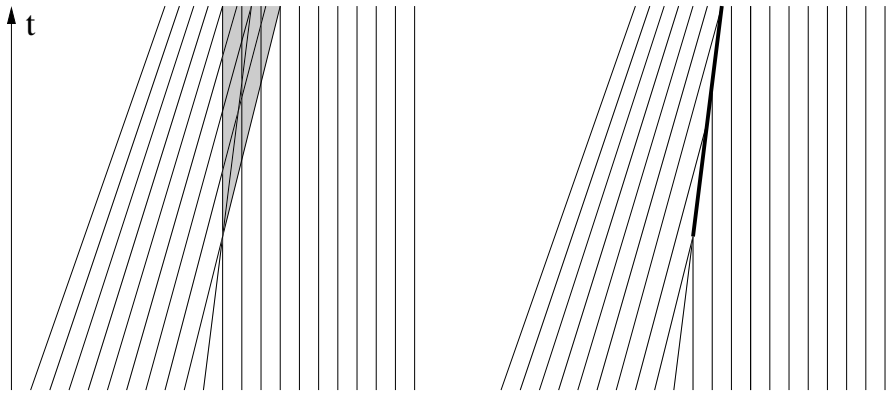


Figure 3.9: A shock region before and after the introduction of the shock. In the original shock region (left, filled in), the characteristics cross. After the shock is introduced (right, bold line), the characteristics are followed all the way to the shock.

In general, for an equation of the form

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}j = 0,$$

the equation for the shock gives

$$s' = \frac{\Delta j}{\Delta \rho},$$

where $\Delta j = j^- - j^+$, and $\Delta\rho = \rho^- - \rho^+$. This follows a derivation entirely similar to the one for the dissipationless Burgers equation. This condition is known as the **Rankine-Hugoniot condition**.

A word of warning... Suppose we wish to introduce a shock in the characteristic picture for the dissipationless Burgers equation

$$u_t + uu_x = 0.$$

We could also, perversely, rewrite this equation as

$$uu_t + u^2u_x = 0,$$

which has as its conservation law form the equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{3}u^3\right) = 0.$$

Using the Rankine-Hugoniot condition with $\rho = u^2/2$ and $j = u^3/3$, we find

$$s' = \frac{2}{3}\frac{u^{-3} - u^{+3}}{u^{-2} - u^{+2}} = \frac{2}{3}\frac{u^{-2} + u^{-}u^{+} + u^{+2}}{u^{-} + u^{+}}.$$

This is different from the result we obtained before! Thus, when going from the differential form of the partial differential equation to the integral form (which is of course what we are doing when we invoke the Rankine-Hugoniot condition), we might lose uniqueness. This is unsatisfying, but not unexpected. As previously announced, the application in question (physics, engineering, biology, *etc.*) will have to dictate which integral formulation to use. The above calculation merely illustrates that the mathematics alone cannot tell us the right answer.

Let us summarize: typically an integral formulation is obtained first for many applications. Then one often switches to a differential formulation, because differential equations are easier to solve than integral equation. Or maybe we just have more experience with them. In any case, the limit process of going from an integral equation to a differential equation is straightforward and it results in a unique differential formulation. Then, when shocks are found in the differential equation, it is important to know which integral formulation the differential equation originated from, so as to be able to resolve the shock in a consistent manner.

3.4 Exercises

1. Consider the initial-value problem

$$\begin{cases} u_t + tuu_x = 0, \\ u(x,0) = \sin(x), x \in \mathbb{R}. \end{cases}$$

   Solve this initial-value problem for $t \geq 0$. Draw the characteristics. For how long is the solution valid?

2. Consider the initial-value problem

$$\begin{cases} u_t + tuu_x = 0, \\ u(x,0) = \cos(x), x \in \mathbb{R}. \end{cases}$$

   Solve this initial-value problem for $t \geq 0$. Draw the characteristics. For how long is the solution valid?

3. Consider the initial-value problem

$$\begin{cases} u_t + 2xtu_x = 1, \\ u(x,0) = e^x, x \in \mathbb{R}. \end{cases}$$

   Solve this initial-value problem for $t \geq 0$. Draw the characteristics. For how long is the solution valid?

4. Consider the initial-value problem

$$\begin{cases} u_t + xu_x = 1, \\ u(x,0) = e^x, x \in \mathbb{R}. \end{cases}$$

Solve this initial-value problem for $t \geq 0$. Draw the characteristics. For how long is the solution valid?

5. Consider the initial-value problem

$$\begin{cases} u_t + uu_x = 0, \\ u(x,0) = u_0(x), x \in \mathbb{R}. \end{cases}$$

Solve this initial-value problem for $t \geq 0$ with (a) $u_0(x) = -x$, (b) $u_0(x) = 1 - x^2$, (c) $u_0(x) = \sin x$. For each case, draw the characteristics. For how long is the solution valid?

6. The method of characteristics is not restricted to initial-value problems. (a) Consider the problem

$$\begin{cases} xu_x + yu_y = u + 1, \\ u(x,x^2) = x^2, (x,y) \in \mathbb{R}^2. \end{cases}$$

This problem is due to Guenther and Lee, see [28]. Use our version of the method of characteristics to solve this problem. You can use either x or y as your "time-like" variable. Where is your solution defined? (b) Now consider the same problem, but with $u(x,x) = x^2$, instead of $u(x,x^2) = x^2$. Show that this problem cannot be solved using the method of characteristics.

Explanation: This is because the data from which you are to recover the solution is given on a characteristic curve (check this). Since the solution is transported along characteristic curves, giving data along a certain characteristic curve does not allow one to construct the solution along the other characteristics. In general, the method of characteristics will work when the data is given along a curve that crosses the characteristics.

7. We have seen an example where the breaking time t^* depends on the initial data. Now consider the initial-value problem

$$\begin{cases} (t-\alpha)u_t + xu_x = u, \\ u(x,0) = f(x), x \in \mathbb{R}, \alpha > 0. \end{cases}$$

Solve this initial-value problem for $t \geq 0$, *i.e.*, give an explicit solution for $u(x,t)$ in terms of the initial data $f(x)$. Draw the characteristics. For how long is the solution valid? As you see, in this problem the breaking time t^* is independent of the initial data $f(x)$.

8. The method of characteristics is easily extended to multidimensional problems. Consider the initial-value problem

$$\begin{cases} u_t + c_1(u)u_x + c_2(u)u_y = 0, \\ u(x,y,0) = u_0(x,y), (x,y) \in \mathbb{R}^2. \end{cases}$$

   - Generalize the method of characteristics to such two-dimensional equations: write down the equations for the characteristics and solve them if possible. Write down the exact solution for $u(x,y,t)$, in explicit form if possible, in implicit form otherwise.
   - Apply the above method to solve the two-dimensional transport equation $u_t + c_1 u_x + c_2 u_y = 0$, with initial condition $u(x,y,0) = u_0(x,y)$. Find the explicit solution for $u(x,y,t)$.
   - Use a slight variation of your method to solve the damped transport initial-value problem

$$\begin{cases} u_t + c_1 u_x + c_2 u_y + \alpha u = 0, \quad \alpha > 0 \\ u(x,y,0) = u_0(x,y), (x,y) \in \mathbb{R}^2. \end{cases}$$

9. Consider the IVP

$$\begin{cases} u_t + uu_x = 0, \\ u(x,0) = \phi(x), \end{cases}$$

   where $\phi(x)$ is determined by

$$\phi(x) = \begin{cases} 1, & x \le 0 \\ 1-x, & 0 < x < 1 \\ 0, & x \ge 1. \end{cases}$$

   (a) Show that a shock forms at $t_* = 1$. (b) You know that different integral representations of the differential equation can lead to different shock dynamics. By multiplying the differential equation by u^α, $\alpha \in (-1, +\infty)$, find different shock positions for any $t > 1$, depending on α. (c) Show that by a suitable choice of α, the shock can be continued along any of the characteristics starting in $x_0 \in (0,1)$. (d) For any choice of α, draw the solution $u(x,t)$ at $t = 0$, $t = 1/2$, $t = 1$, $t = 2$, and $t = 5$.

10. Consider the IVP

$$\begin{cases} u_t + uu_x = 0, \\ u(x,0) = \phi(x), \end{cases}$$

    where $\phi(x)$ is defined by

$$\phi(x) = \begin{cases} 1, & x < -1 \\ x^2, & -1 < x < 0 \\ 0, & 0 < x \end{cases}$$

Determine the solution of this problem in all regions of $x \in \mathbb{R}$ and $t > 0$, using the integral representation.

$$\frac{d}{dt}\int_a^b u dx + \frac{1}{2}u^2\Big|_a^b = 0.$$

Are there any shocks? If so, where and when do they occur? Also draw all relevant phases the solution goes through, and provide a picture of the characteristics. If there are shocks, include a picture of the characteristics without and with the shock lines.

11. **Rarefaction waves.** In class we considered several examples with characteristics. In many cases, each point of the (x,t)-plane had exactly one characteristic going through it. In other case, the characteristic crossed, and more than one characteristic could pass through a given point. There are other possibilities: it is possible that some regions of the plane have no characteristics in them.

    (a) Consider the following initial-value problem:

$$u_t + uu_x = 0, \quad x \in \mathbb{R}, \quad \text{and} \quad u(x,0) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1 & \text{for } x > 0. \end{cases}$$

    Graph the characteristics for this problem in the (x,t)-plane ($t \geq 0$) and show that there are no characteristics in a whole sector of the plane.

    (b) As with shocks, there are several ways to deal with this problem. We'll examine one here: consider the same differential equation as above, but with initial condition $u_\epsilon(x,0) = 1/2 + \tanh(x/\epsilon)/2$, with $\epsilon > 0$. Show that in this case, there is exactly one characteristic in every area of the (x,t)-plane ($t \geq 0$) (a graphical "proof" is fine). What happens to the initial condition $u_\epsilon(x,0)$ as $\epsilon \to 0$? What happens to the characteristics as $\epsilon \to 0$?

    (c) We now use the limit characteristics found above to solve the initial-value problem. It is clear what to do on the original characteristics (*i.e.*, the ones we had before we talked about this limit process), as we can propagate the initial condition along them. What do we do along the characteristics in the sector? The solution should be constant along them, but we don't know what initial condition to propate along them. In order to be constant along these characteristics, the solution $u(x,t)$ should be of the form $u(x,t) = g(x/t)$. Substitute this ansatz in the partial differential equation to find an equation for $g(z)$, $z = x/t$ (ignore the possibility $g' = 0$, which cannot result in a continuous solution. It can be shown that this choice can indeed be ignored).

    (d) Finally, write down the solution to the original initial-value problem obtained this way and plot it for various instances of time. This solution satisfies the original partial differential equation at all time, except at the corner points emanating from the original discontinuity.

12. **Rarefaction waves: another example.** Use the same process to solve the initial-value problem

$$u_t + u^2 u_x = 0, \quad x \in \mathbb{R}, \quad \text{and} \quad u(x,0) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1 & \text{for } x > 0. \end{cases}$$

13. **Shock waves.** It is not often that shock conditions like the ones we derived in class can be solved as a differential equation for the shock position, hence determining the shock position $s(t)$ at any time after its formation. But sometimes, this is possible.

    Consider the following initial-value problem:

$$u_t + u^2 u_x = 0, \quad x \in \mathbb{R}, \quad \text{and} \quad u(x,0) = \begin{cases} 1 & \text{for } x \leq 2, \\ 0 & \text{for } x > 2. \end{cases}$$

    Solve this initial-value problem in the region of the (x,t)-plane ($t \geq 0$) where no shock develops. Next, assuming that

$$\frac{d}{dt}\int_{x_1}^{x_2} u dx + \int_{x_1}^{x_2} u^2 u_x dx = 0$$

    is the correct integral formulation to use, derive the shock condition governing the motion of the shock position $x = s(t)$. In this case, this condition can be solved for the shock position. Proceed to solve it, and hence find the position of the shock for any time after its formation. Give the solution to the initial-value problem, for any time $t \geq 0$. Illustrate this with some appropriately chosen snapshots of the solution $u(x,t)$.

14. **Rarefaction waves and shock waves combined.** Consider the following initial-value problem:

$$u_t + u u_x = 0, \quad x \in \mathbb{R}, \quad \text{and} \quad u(x,0) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1 & \text{for } 0 < x < 1, \\ 0 & \text{for } x \geq 1. \end{cases}$$

    (a) Find the solution of this problem in all areas of the (x,t) ($t \geq 0$)-plane where there is exactly one characteristic through each point.

    (b) Using the same process as before, fill in the rarefaction region with characteristic lines, and write down the solution to the initial-value problem in this region.

    (c) Assuming that the integral formulation

$$\frac{d}{dt}\int_{x_1}^{x_2} u dx + \int_{x_1}^{x_2} \left(\frac{1}{2}u^2\right)_x dx = 0$$

    is the correct integral formulation corresponding to the partial differential equation, derive the shock speed condition for a shock moving on a path $x = s(t)$.

(d) Use this shock condition to find the location of the shock at any time $t > 0$. Note that there are two different regions for this.

(e) Knowing the path of the shock, give a complete plot of the characteristics in the (x, t)-plane ($t \geq 0$), and describe the complete solution to the initial-value problem. Also provide snapshots of the solution $u(x, t)$ at relevant times.

15. **Rarefaction waves and shock waves combined: good times were had by all.** Consider the following initial-value problem:

$$u_t + uu_x = 0, \quad x \in \mathbb{R}, \quad \text{and} \quad u(x,0) = \begin{cases} 0 & \text{for } x < 0, \text{ or } x > 3 \\ 3 & \text{for } 0 < x < 1, \\ 2 & \text{for } 1 < x < 2, \\ 1 & \text{for } 2 < x < 3. \end{cases}$$

Using rarefaction waves and the integral formulation you used in the previous problem, solve this problem for all $t > 0$.

16. Consider the nonlinear partial differential equation

$$\phi_{tt}\phi_x^2 - 2\phi_{xt}\phi_x\phi_t + \phi_t^2\phi_{xx} = 0.$$

This equation is a special case of the so-called Monge-Ampère equation. Show that in terms of the auxiliary function $u = \phi_t/\phi_x$ this equation may be reduced to a first-order equation. Assuming initial conditions $\phi(x, 0) = 1 + 2e^{3x}$ and $\phi_t(x, 0) = 4e^{3x}$, find $\phi(x, t)$ for $x \in \mathbb{R}$ and $t \geq 0$.

Chapter 4

Multiple scale perturbation theory

Typically, physical and other applications present us with systems of equations whose solutions are analytically untractable. Reductive perturbation theory is used to derive simpler equations that still incorporate many of the physical effects one wishes to investigate. For instance, starting from the Euler equations describing the dynamics of surface water waves under the influence of gravity, we may derive the KdV equation to investigate long waves in shallow water [43]. We will go through the details of this derivation below. Similarly, we can examine the evolution of modulated wave trains in nonlinear optics using the NLS equation [32], as opposed to having to undertake the study of Maxwell's equations with the appropriate polarization vector. Or we can examine Alfvén waves in space plasmas using the Derivative Nonlinear Schrödinger (DNLS) equation

$$u_t = iu_{xx} + \alpha(|u|^2 u)_x = 0, \quad \alpha = \pm 1,$$

as opposed to having to deal with Maxwell's equations coupled with the fluid equations [18].

The most general method for obtaining these reduced equations is the method of multiple scales. We will illustrate the use of the method on two examples in this chapter. Good discussions and examples of the multiple-scales method can be found in the classic text [40]. More examples, especially in the context of nonlinear wave equations, are found in [36], albeit often using more heuristic arguments than the method requires.

Since the equations obtained are often simpler than those from which they originated[1], we may anticipate that analytical methods may prove useful in their study. Sometimes, much more is true and the equations found are integrable. As we have already remarked, integrable equations are the exception rather than the rule. This makes it all the more amazing how often these equations show up in applications. The main reason is that they incorporate and balance effects that occur in numerous application settings, making them universal. For instance, the KdV equation arises in *all* applications where long waves propagate in a dispersive medium, just like the NLS equation is found to govern the dynamics of modulated wave trains in dispersive media. These statements are independent of the specific application

[1]This is not always the case. In this chapter, we will derive the NLS equation, starting from the KdV equation. It is hard to argue that one equation is simpler than the other.

studied, whether it is Bose-Einstein condensation, fluid mechanics, nonlinear optics, plasma physics, solid state physics, *etc.* Of course, the specific derivation of these equations depends heavily on the application equations from which we start, as is obvious from the examples below.

4.1 Waves in shallow water: the KdV equation

Our first case study for the method of multiple scales is a careful derivation of the KdV equation in the context of water waves. Historically, the works of Boussinesq [10] and Korteweg and de Vries [43] are the most relevant. The books by Ablowitz & Segur [4] and Infeld & Rowlands [36], for instance, contain derivations as well, with varying amounts of detail.

4.1.1 Problem set-up

Consider the problem of describing the surface dynamics of water waves in shallow water, under the influence of gravity. We assume that the dynamics inside the water is irrotational (no vorticity), and inviscid (no viscocity or other dissipative effects). Further, we ignore the effects of surface tension and assume that the water has constant density. We only consider the one-dimensional problem (one vertical direction z and one horizontal direction x). A sketch of this set-up is given in Fig. 4.1.

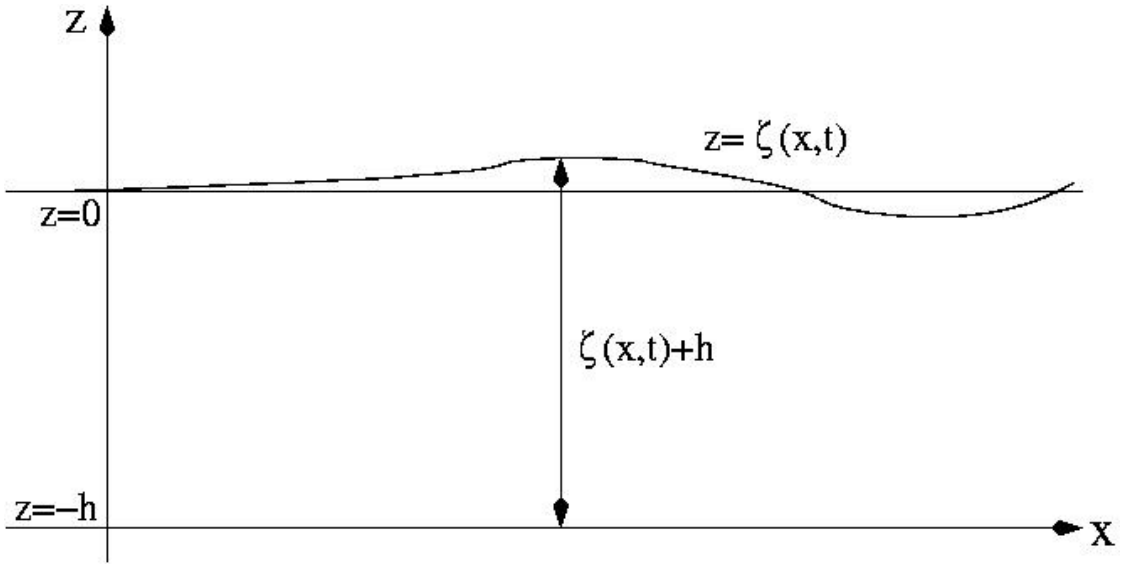


Figure 4.1: The set-up of the shallow water wave problem.

The equation for the surface of the water is $z = \zeta(x,t)$. We assume that $z = 0$ corresponds to the equilibrium situation (flat water surface). The bottom of the water is assumed to

have no topography and is at $z = -h$. Since the motion is irrotational, it may be derived from a velocity potential, thus the velocity (u, w) is given by $(u, w) = \nabla\phi = (\phi_x, \phi_z)$.

The water wave problem is governed by the following four equations [45, 64]:

1. $\nabla^2\phi = 0$, for $-h < z < \zeta(x,t)$. This follows from conservation of mass, having an incompressible fluid and potential flow. This equation governs the motion in the bulk of the water, *i.e.*, away from the boundaries. The other three equations are boundary conditions.

2. $\phi_z = 0$ at $z = -h$. This equation dictates that there is no flow through the bottom.

3. $\zeta_t + \phi_x\zeta_x = \phi_z$, at $z = \zeta(x,t)$ (at the surface). This equation is known as the kinematic condition. It states that fluid particles on the surface remains on the surface: $\frac{d}{dt}(\zeta - z) = 0$.

4. $\phi_t + g\zeta + \frac{1}{2}|\nabla\phi|^2 = 0$, ignoring surface tension. Like the kinematic condition, this equation is also specified at the surface $z = \zeta(x,t)$. It is known as the dynamic condition. It is essentially Bernoulli's equation.

Lastly, we impose that the fluid is in equilibrium as $x \to \pm\infty$. In summary, we are attempting to solve the following free-boundary value problem:

$$\begin{cases} \phi_{xx} + \phi_{zz} = 0, & \text{for} \quad -h < z < \zeta(x,t), \\ \phi_z = 0, & \text{at} \quad z = -h, \\ \zeta_t + \phi_x\zeta_x = \phi_z, & \text{at} \quad z = \zeta(x,t), \\ \phi_t + g\zeta + \frac{1}{2}(\phi_x^2 + \phi_z^2) = 0 & \text{at} \quad z = \zeta(x,t), \end{cases} \tag{4.1}$$

with $\zeta, \phi \to 0$ as $|x| \to \infty$. It is bad enough that these equations are nonlinear. What makes them truly hard to deal with is that we are solving a partial differential equation (Laplace's equation) inside a domain whose boundaries we do not know. Finding these boundaries is part of the problem and, in this case, the essence of the problem.

4.1.2 The linear dispersion relation

Linearizing the above equations around the equilibrium solution $\zeta(x,t) = 0$ and $\phi(x,z,t) = 0$, we find the linear dispersion relationship to be

$$\omega^2 = gk\tanh(kh), \tag{4.2}$$

see Problem 4 on Page 36. Note that the right-hand side is indeed positive for all values of k. A plot of $\omega(k)$ is shown in Fig. 4.2. We discuss a few interesting limits.

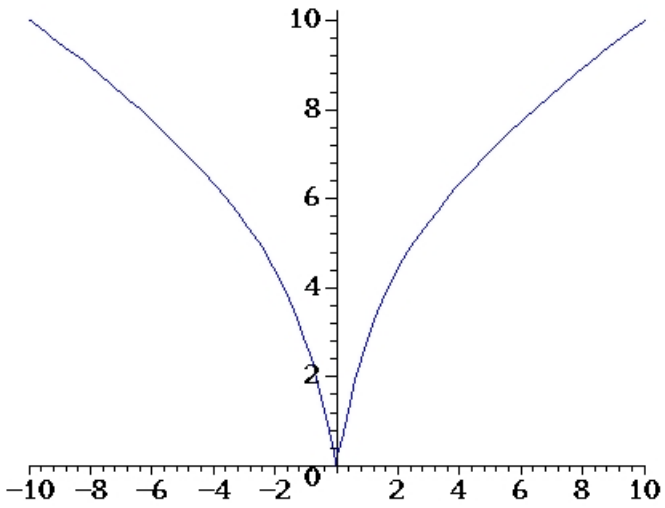


Figure 4.2: The linear dispersion relationship for the water wave problem, with $g = 10$ and $h = 1$.

Long waves in shallow water

If $|k|$ is small so that $|k|h \ll 1$, then we find $\omega^2 = ghk^2$, or $\omega_\pm = \pm\sqrt{gh}|k|$. In this case the waves are long (wavelength=$2\pi/k$) compared to the depth of the water. The phase and group velocities for the + and − branch are given by

$$c_p = \frac{\omega_\pm(k)}{k} = \pm\mathrm{sgn}(k)\sqrt{gh}, \quad c_g = \frac{d\omega_\pm(k)}{dk} = \pm\mathrm{sgn}(k)\sqrt{gh}.$$

Thus waves in shallow water are essentially non-dispersive: wave crests move with the same speed as wave packets.

Short waves in deep water

If $kh \gg 1$ then $\omega^2 = g|k|$ and $\omega_\pm = \pm\sqrt{g|k|}$. These waves are short compared to the depth of the water. The phase and group velocities for the + and − branch are given by

$$c_p = \frac{\omega_\pm(k)}{k} = \pm sgn(k)\sqrt{\frac{g}{|k|}}, \quad c_g = \frac{d\omega_\pm(k)}{dk} = \pm\frac{\mathrm{sgn}(k)}{2}\sqrt{\frac{g}{|k|}}.$$

In this case, wave crests move twice as fast as wave packets.

Both types of wave may exist in a given wave field. For instance, Fig. 4.3 displays both long and short waves on the Atlantic Ocean near Panama. The dominant waves (large wave crests moving towards the lower-left corner of the photograph) are long waves: their crest-to-crest wavelength is large compared to the local depth of the water. On the other hand,

the small ripples on the surface have a short wavelength compared to this depth. Note that there appears to be little or no interaction between these different waves. Of course, it is not clear from Fig. 4.3 that both types of waves are gravity waves. That requires detailed measurements and data analysis. In fact, the short waves in the picture are more likely the result of wind blowing across the water surface. Nevertheless, the photograph illustrates the simultaneous presence in time of different kinds of water waves at the same location.

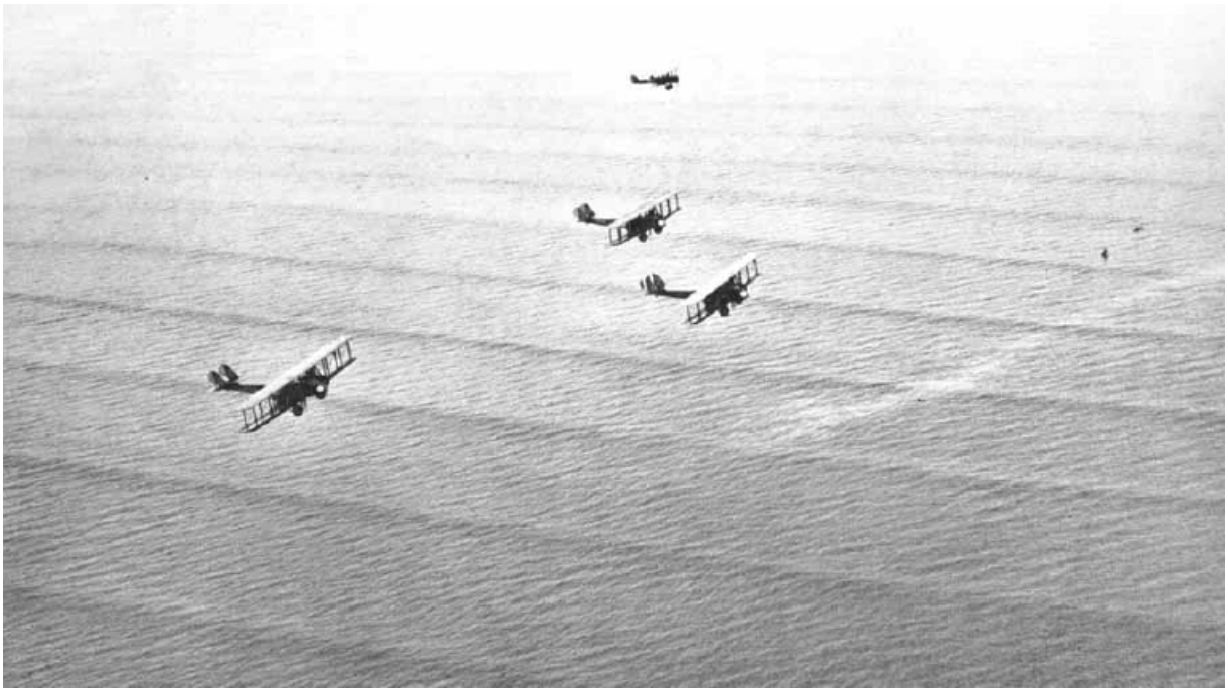


Figure 4.3: Surface water waves near the coast of Panama. The original caption read: "As they near shallow water close to the coast of Panama, huge deep-sea waves, relics of a recent storm, are transformed into waves that have crests, but little or no troughs. A light breeze is blowing diagonally across the larger waves to produce a cross-chop. Three Army bombers, escorted by a training ship, are proceeding from Albrook Field, Canal Zone, to David, Panama." (Taken from National Geographic 63 (1933).)

4.1.3 The principle of Maximal Balance

At this point we state which type of waves we are interested in. We introduce two small parameters ϵ_1 and ϵ_2 as follows:

- $\epsilon_1 = (kh)^2 \ll 1$, *i.e.*, long waves compared to the depth of the water.
- $\epsilon_2 = |\zeta|_{\max}/h \ll 1$, thus the waves have small amplitude.

The condition that these parameters are small represents different physical assumptions, as is obvious from their definition. **Kruskal's principle of maximal balance** [44] states that in order to obtain an equation that is interesting, we should balance these different physical assumptions, taking all of them into account. Thus we let $\epsilon = \epsilon_1 = \mathcal{O}(\epsilon_2)$, and we continue working with one small parameter ϵ only, representing both assumptions. Note that we could have incorporated both effects by setting $\sqrt{\epsilon_1} = \epsilon_2$, *etc.* Indeed, there are many possibilities. Some of these lead to different interesting equations from the one that we are about to obtain, while others do not lead to anything interesting at all. Kruskal's principle does not tell us how to balance different effects[2], and we cannot expect it to do so. As Fig. 4.3 illustrates, different wave phenomena may be present simultaneously in an application settings. Although a lot of hard work goes into using the method of multiple scales (as in what follows), it is the seemingly small choices that are made in this balancing step that determine the relevance of the final equation or equations obtained. This step is the one that requires the practitioner's creative input.

4.1.4 Nondimensionalization

When working with small parameters, one should always work in dimensionless variables. Indeed, the magnitude of variables with dimensions may be changed arbitrarily by changing its measuring units, thus it is impossible to decide whether a dimensional quantity is small or not[3].

We define new dimensionless variables as follows:

$$\begin{aligned} z^* &= z/h, \\ x^* &= \sqrt{\epsilon}\frac{x}{h}, \\ t^* &= \sqrt{\frac{\epsilon g}{h}}t, \\ \zeta &= \epsilon h \zeta^*, \\ \phi &= h\sqrt{\epsilon g h}\phi^*. \end{aligned}$$

It is convenient to define new independent variables in terms of old ones, and old dependent variables in terms of new ones, because of how the substitution rules work out, using the chain rule with the independent variables. The first two equations state that variations in the horizontal direction are slower than in the vertical direction. In other words, they

[2]Perhaps we should write: fortunately, Kruskal's principle does not tell us how to balance different effects. If it did asymptotics in general and the method of multiple scales would be reduced to a simple mechanical crank, without any creative input from the executioner.

[3]Another reason for working with dimensionless quantities is that we can introduce Taylor or Laurent series, as necessary. The arguments of functions whose series expansions contain more than one term has to be dimensionless. Indeed, different powers of dimensional quantities have different dimensions, and quantities of different dimensions cannot be added together. As a special case, the argument of transcendental functions (*e.g.*, exp, sin, ...) has to be dimensionless.

encode that we are working with long waves in shallow water. The third equation follows from the second one and the dispersion relation for shallow water waves by ensuring that the relevant velocity is $\sqrt{gh}$: the quantity t^* is defined so that the velocity-like quantity $\Delta x/\Delta t = \sqrt{gh}\,\Delta x^*/\Delta t^*$. This is why it was necessary to examine the linear dispersion relation first. The fourth equation encodes the small-amplitude assumption, while, as for the third equation, the last definition follows from $\Delta\phi/\Delta x$ being a velocity. That does not determine the accompanying power of ϵ, which is partially a choice.s Obviously, a different choice leads to different equations in what follows, and perturbation series assumptions have to be changed accordingly to ensure desirable results. The benefit of the present choice is that the equations below involve integer powers of ϵ only. No square roots of ϵ appear.

In these new variables, our first equation becomes:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=0 \Rightarrow \frac{\epsilon}{h}\frac{\partial^2\phi^*}{\partial x^{*2}}+\frac{1}{h}\frac{\partial^2\phi^*}{\partial z^{*2}}=0 \Rightarrow \epsilon\frac{\partial^2\phi^*}{\partial x^{*2}}+\frac{\partial^2\phi^*}{\partial z^{*2}}=0,$$

valid for $-1<z^*<\epsilon\zeta^*$. The second equation gives

$$\frac{\partial\phi}{\partial z}=0 \Rightarrow \frac{\partial\phi^*}{\partial z^*}=0,$$

valid on $z^*=-1$. From the third equation, we obtain

$$\begin{aligned}
& & \frac{\partial\phi}{\partial z} &= \frac{\partial\zeta}{\partial t}+\frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x}\\
\Rightarrow & & \sqrt{\epsilon gh}\frac{\partial\phi^*}{\partial z^*} &= \epsilon h\sqrt{\frac{\epsilon g}{h}}\frac{\partial\zeta^*}{\partial t^*}+\epsilon h\frac{\epsilon}{h^2}h\sqrt{\epsilon gh}\frac{\partial\phi^*}{\partial x^*}\frac{\partial\zeta^*}{\partial x^*}\\
\Rightarrow & & \frac{\partial\phi^*}{\partial z^*} &= \epsilon\frac{\partial\zeta^*}{\partial t^*}+\epsilon^2\frac{\partial\phi^*}{\partial x^*}\frac{\partial\zeta^*}{\partial x^*},
\end{aligned}$$

valid on $z^*=\epsilon\zeta^*$. The last equation becomes

$$\begin{aligned}
& & \frac{\partial\phi}{\partial t}+g\zeta+\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial\phi}{\partial z}\right)^2 &= 0\\
\Rightarrow & \quad h\sqrt{\epsilon gh}\sqrt{\frac{\epsilon g}{h}}\frac{\partial\phi^*}{\partial t^*}+\epsilon gh\zeta^*+\frac{1}{2}h^2(\epsilon gh)\frac{\epsilon}{h^2}\left(\frac{\partial\phi^*}{\partial x^*}\right)^2+\frac{1}{2}h^2(\epsilon gh)\frac{1}{h^2}\left(\frac{\partial\phi^*}{\partial z^*}\right)^2 &= 0\\
\Rightarrow & & \frac{\partial\phi^*}{\partial t^*}+\zeta+\frac{1}{2}\left(\frac{\partial\phi^*}{\partial z^*}\right)^2+\frac{\epsilon}{2}\left(\frac{\partial\phi^*}{\partial x^*}\right)^2 &= 0,
\end{aligned}$$

which is valid on $z^*=\epsilon\zeta^*$ as well.

So as not to burden the notation with all these stars, we omit them from here on out. Our final system of dimensionless equations is

$$\begin{aligned}
&\phi_{zz} + \epsilon\phi_{xx} = 0, && \text{for} && -1 < z < \epsilon\zeta,\\
&\phi_z = 0, && \text{at} && z = -1,\\
&\epsilon\zeta_t + \epsilon^2\phi_x\zeta_x = \phi_z, && \text{at} && z = \epsilon\zeta,\\
&\phi_t + \zeta + \frac{1}{2}\phi_z^2 + \frac{\epsilon}{2}\phi_x^2 = 0 && \text{at} && z = \epsilon\zeta.
\end{aligned}$$

4.1.5 Determining the dependence on z

Using the first two of these equations, it is easy to determine the z dependence of $\phi(x, z, t)$. Since $\phi_z = 0$ at $z = -1$, it is convenient to expand $\phi(x, z, t)$ as a series in $z + 1$:

$$\phi(x, z, t) = \sum_{n=0}^{\infty} (z+1)^n \phi_n(x, t).$$

Substitution in $\phi_{zz} + \epsilon\phi_{xx} = 0$ gives:

$$\begin{aligned}
&\epsilon \sum_{n=0}^{\infty} (z+1)^n \phi_{nxx}(x,t) + \sum_{n=2}^{\infty} (z+1)^{n-2} n(n-1)\phi_n(x,t) = 0\\
\Rightarrow\quad &\epsilon \sum_{n=0}^{\infty} (z+1)^n \phi_{nxx}(x,t) + \sum_{n=0}^{\infty} (z+1)^n (n+2)(n+1)\phi_{n+2}(x,t) = 0\\
\Rightarrow\quad &\epsilon\phi_{nxx} + (n+2)(n+1)\phi_{n+2}(x,t) = 0, \qquad n = 0, 1, \ldots,\\
\Rightarrow\quad &\phi_{n+2} = \frac{-\epsilon\phi_{nxx}}{(n+1)(n+2)}, \qquad n = 0, 1, \ldots.
\end{aligned}$$

We know that $\phi_z = 0$ at $z = -1$, thus $\phi_1(x, t) = 0$. Using the above recursion relation, we find

$$0 = \phi_1(x,t) = \phi_3(x,t) = \phi_5(x,t) = \ldots,$$

hence

$$\phi(x, z, t) = \phi_0(x,t) - \epsilon\frac{(z+1)^2}{2!}\phi_{0xx} + \epsilon^2\frac{(z+1)^4}{4!}\phi_{0xxxx} - \ldots.$$

Two equations remain to be solved, both at the surface $z = \epsilon\zeta(x, t)$:

$$\begin{cases} \phi_z = \epsilon\zeta_t + \epsilon^2\phi_x\zeta_x, \\ \phi_t + \zeta + \dfrac{1}{2}\phi_z^2 + \dfrac{\epsilon}{2}\phi_x^2 = 0. \end{cases} \tag{4.3}$$

The above result for $\phi(x, z, t)$ gives $\phi_z = -\epsilon(z+1)\phi_{0xx} + \ldots$. Evaluating this at $z = \epsilon\zeta$ gives:

$$\phi_z = -\epsilon(\epsilon\zeta + 1)\phi_{0xx} + \ldots = -\epsilon\phi_{0xx} + \ldots,$$

valid at $z = \epsilon\zeta(x,t)$. All the terms that were ignored are of order at least ϵ^2. The first of the above equations becomes

$$-\epsilon\phi_{0xx} + \ldots = \epsilon\zeta_t + \epsilon^2\phi_{0x}\zeta_x + \ldots,$$

so that

$$\zeta_t + \partial_x\phi_{0x} = \theta(\epsilon).$$

Similarly, from the second equation we obtain

$$\phi_{0t} + \zeta = \theta(\epsilon) \Rightarrow (\phi_{0x})_t + \partial_x\zeta = \theta(\epsilon).$$

4.1.6 The perturbation expansions and the slow time scales

At this point, two functions remain to be determined: $\phi_0(x,t)$ and $\zeta(x,t)$. We assume the following expansions:

$$\phi_{0x} = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots, \tag{4.4}$$
$$\zeta = \zeta_0 + \epsilon\zeta_1 + \epsilon^2\zeta_2 + \ldots. \tag{4.5}$$

It turns out that the relevant quantity for our calculations is the spatial derivative of $\phi_0(x,t)$, rather than the function itself. Next, we introduce the slow time scales τ_1, τ_2, *etc*:

$$\tau_0 = t, \tau_1 = \epsilon t, \tau_2 = \epsilon^2 t, \ldots,$$

which implies that

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau_0} + \epsilon\frac{\partial}{\partial \tau_1} + \epsilon^2\frac{\partial}{\partial \tau_2} + \ldots$$

It is clear at this stage why the perturbation expansions for $\phi_{0x}(x,t)$ and $\zeta(x,t)$ are introduced. Indeed, the remaining equations depend on ϵ, and it is natural to determine their solution order by order in ϵ. During the course of our calculations, we have used $\sqrt{\epsilon}$. However, the two surface equations (4.3) only depend on integer powers of ϵ, and it is not necessary to include half-integer powers of ϵ in the perturbation expansions (4.4-4.5). The introduction of the slow time scales τ_1, τ_2, *etc.* may appear puzzling. In fact, currently there is no reason for introducing it. If we were to proceed with t as the only temporal variable, soon we would find ourselves stuck, having too many conditions on too few variables. At that stage, the slow time scales would have to be introduced, resulting in a plethora of calculations that need to be redone. To avoid this backtracking, we introduce the slow time scales here. We comment below at which stage the reader can observe the necessity of their use.

Some clarification can be given as to why the first slow time scale τ_1 should be of first order in ϵ. Consider the linear dispersion relation (4.2). Since we are interested in long waves, we expand this near $k = 0$, obtaining

$$\omega_{1,2} = \pm\sqrt{gh}k\left(1 + \frac{1}{6}(kh)^2 + \mathcal{O}((kh)^4)\right),$$

where the + sign is to be used for $kh > 0$, the − sign for $kh < 0$. Using $kh = \sqrt{\epsilon}$ (see Section 4.1.3) and the nondimensionalizations for x and t, we obtain

$$e^{ikx - i\omega_{1,2}t} = e^{i(x^* \mp t^*) \mp \epsilon t^*/6 + \mathcal{O}(\epsilon^2)}.$$

Here the entries in the exponent on the left-hand side have dimensions, whereas those on the right-hand side do not. It follows that we should expect wave propagation in two directions, left and right, corresponding to the bottom and top signs in the above equation. This wave propagation phenomenon is dominant. In addition, we expect to observe secondary dynamics, at the time scale $\tau_1 = \epsilon t^*$.

It should be clear to the reader that the presentation of examples of the method of multiple scales one finds in textbooks or research articles is the result of much backtracking, fine tuning, and polishing. The use of the method of multiple scales is truly a situation where hindsight is twenty-twenty!

4.1.7 Lowest-order results

With all of this newly introduced notation, the above equations, at lowest order become

$$\begin{cases} \dfrac{\partial \zeta_0}{\partial \tau_0} + \dfrac{\partial u_0}{\partial x} = 0 \\ \dfrac{\partial u_0}{\partial \tau_0} + \dfrac{\partial \zeta_0}{\partial x} = 0 \end{cases}.$$

Combining these equations gives

$$\begin{cases} \dfrac{\partial^2 \zeta_0}{\partial \tau_0^2} - \dfrac{\partial^2 \zeta_0}{\partial x^2} = 0, \\ \dfrac{\partial^2 u_0}{\partial \tau_0^2} - \dfrac{\partial^2 u_0}{\partial x^2} = 0. \end{cases}$$

Both ζ_0 and u_0 satisfy wave equations with (dimensionless) velocity 1 as functions of x and τ_0. Thus

$$\begin{cases} \zeta_0 = f(x - \tau_0, \tau_1, \tau_2, \ldots) + g(x + \tau_0, \tau_1, \tau_2, \ldots) \\ u_0 = f(x - \tau_0, \tau_1, \tau_2, \ldots) - g(x + \tau_0, \tau_1, \tau_2, \ldots) \end{cases}.$$

Thus, on the fastest time scale τ_0, all initial data splits in two directions, with one part (f) propagating to the right, and the other part (g) propagating to the left. The only relevant dynamics on this time scale is simple wave propagation, according to the wave equation. This behavior is to be expected from the linear dispersion relation, as remarked above.

4.1.8 Higher order results

Let us proceed to higher order. First we determine $\phi_z(x, \epsilon\zeta(x,t), t)$ to second order in ϵ.

$$\begin{aligned}
\phi_z(x,\epsilon\zeta(x,t),t) &= -\epsilon(1+z)\phi_{0xx} + \epsilon^2\frac{(1+z)^3}{6}\phi_{0xxxx} + \theta(\epsilon^3) \quad \text{at } z = \epsilon\zeta(x,t) \\
&= -\epsilon(1+\epsilon\zeta)\phi_{0xx} + \epsilon^2\frac{(1+\epsilon\zeta)^3}{6}\phi_{0xxxx} + \theta(\epsilon^3) \\
&= -\epsilon(1+\epsilon\zeta_0)\phi_{0xx} + \epsilon^2\frac{1}{6}\phi_{0xxxx} + \theta(\epsilon^3) \\
&= -\epsilon(1+\epsilon\zeta_0)(u_{0x}+\epsilon u_{1x}) + \epsilon^2\frac{1}{6}u_{0xxx} + \theta(\epsilon^3) \\
&= -\epsilon u_{0x} + \epsilon^2\left(-\zeta_0 u_{0x} - u_{1x} + \frac{1}{6}u_{0xxx}\right) + \theta(\epsilon^3).
\end{aligned}$$

Next we determine $\phi_x(x, \epsilon\zeta(x,t), t)$ to first order in ϵ,

$$\begin{aligned}
\phi_x(x,\epsilon\zeta(x,t),t) &= \phi_{0x} - \epsilon\frac{(1+z)^2}{2}\phi_{0xxx} + \theta(\epsilon^2) \quad \text{at } z = \epsilon\zeta(x,t) \\
&= \phi_{0x} - \epsilon\frac{1}{2}\phi_{0xxx} + \theta(\epsilon^2) \\
&= u_0 + \epsilon u_1 - \epsilon\frac{1}{2}u_{0xx} + \theta(\epsilon^2) \\
&= u_0 + \epsilon\left(u_1 - \frac{1}{2}u_{0xx}\right) + \theta(\epsilon^2),
\end{aligned}$$

and $\phi_{xt}(x, \epsilon\zeta(x,t), t)$, also to first order in ϵ.

$$\begin{aligned}
\phi_{xt}(x,\epsilon\zeta(x,t),t) &= \left(\partial_{\tau_0} + \epsilon\partial_{\tau_1}\right)\left(u_0 + \epsilon(u_1 - u_{0xx}/2)\right) + \theta(\epsilon^2) \\
&= u_{0\tau_0} + \epsilon\left(u_{0\tau_1} + u_{1\tau_0} - u_{0xx\tau_0}/2\right) + \theta(\epsilon^2).
\end{aligned}$$

Last, we find an expression for ζ_t to first order in ϵ.

$$\begin{aligned}
\zeta_t &= \left(\frac{\partial}{\partial\tau_0} + \epsilon\frac{\partial}{\partial\tau_1}\right)\left(\zeta_0 + \epsilon\zeta_1\right) + \theta(\epsilon^2) \\
&= \zeta_{0\tau_0} + \epsilon\left(\zeta_{1\tau_0} + \zeta_{0\tau_1}\right) + \theta(\epsilon^2).
\end{aligned}$$

Using all of these expressions, the kinematic equation becomes, to second order in ϵ,

$$\begin{aligned}
\epsilon^1: \quad & -u_{0x} = \zeta_{0\tau_0} \\
\epsilon^2: \quad & -\zeta_0 u_{0x} - u_{1x} + \frac{1}{6}u_{0xxx} = \zeta_{1\tau_0} + \zeta_{0\tau_1} + u_0\zeta_{0x} \quad \text{or} \\
& \zeta_{1\tau_0} + u_{1x} = -\left(\zeta_{0\tau_1} + u_0\zeta_{0x} + \zeta_0 u_{0x} - \frac{1}{6}u_{0xxx}\right).
\end{aligned}$$

The first equation we had already found by looking at the leading order terms. We will hang on to the second equation for now. Bernoulli's equation $\phi_{xt} + \zeta_x + \frac{1}{2}\partial_x(\phi_z^2 + \epsilon\phi_x^2) = 0$, evaluated at $z = \epsilon\zeta(x,t)$ becomes, to second order in ϵ:

$$\begin{aligned}
\epsilon^0: \quad & u_{0\tau_0} + \zeta_{0x} = 0, \\
\epsilon^1: \quad & u_{0\tau_1} + u_{1\tau_0} - \frac{1}{2}u_{0xx\tau_0} + \zeta_{1x} + u_0 u_{0x} = 0 \quad \text{or} \\
& u_{1\tau_0} + \zeta_{1x} = -\left(u_{0\tau_1} - \frac{1}{2}u_{0xx\tau_0} + u_0 u_{0x}\right).
\end{aligned}$$

As for the kinematic condition, we had already found the first equation by considering the leading-order terms. Thus we are left with both of the higher-order equations

$$\begin{cases} \zeta_{1\tau_0} + u_{1x} &= -\left(\zeta_{0\tau_1} + u_0\zeta_{0x} + \zeta_0 u_{0x} - \frac{1}{6}u_{0xxx}\right), \\ u_{1\tau_0} + \zeta_{1x} &= -\left(u_{0\tau_1} - \frac{1}{2}u_{0xx\tau_0} + u_0 u_{0x}\right). \end{cases}$$

As expected, the left-hand side of these equations is identical to what was found at the leading order, but for the next set of unknown functions ζ_1 and u_1. The right-hand side of these equations consists of functions whose dependence on the fast time scale τ_0 has already been determined. We wish to determine both how u_1 and ζ_1 depend on τ_0, but even more how u_0 and ζ_0 depend on the slower time scale τ_1.

4.1.9 Characteristic variables

Let us change variables on these two equations, to express them in characteristic variables. We introduce the variables that "diagonalize" the system of equations on the left-hand side:

$$l = x + \tau_0, \quad r = x - \tau_0.$$

Here l (left) is the characteristic variable for a left-moving wave, and r (right) is the characteristic variable for a right-moving wave. Then

$$\partial_x = \partial_r + \partial_l, \quad \partial_{\tau_0} = \partial_l - \partial_r,$$

and $\zeta_0 = f(r) + g(l)$, $u_0 = f(r) - g(l)$. Note that $f(r)$ and $g(l)$ each depend on all stretched time variables τ_1, τ_2, *etc*, but not on the other characteristic variable. The transformed equations are

$$\begin{cases} \zeta_{1l} - \zeta_{1r} + u_{1r} + u_{1l} &= -\left(f_{\tau_1} + g_{\tau_1} + (f-g)(f_r + g_l) + (f+g)(f_r - g_l) - \frac{1}{6}(f_{rrr} - g_{lll})\right), \\ u_{1l} - u_{1r} + \zeta_{1r} + \zeta_{1l} &= -\left(f_{\tau_1} - g_{\tau_1} - \frac{1}{2}(f_{rrr} + g_{lll}) + (f-g)(f_r - g_l)\right). \end{cases}$$

Adding and subtracting these two results in

$$\begin{cases} 2(\zeta_{1l} + u_{1l}) &= -(2f_{\tau_1} + 3ff_r + \frac{1}{3}f_{rrr}) + f_r g + g g_l + f g_l - \frac{2}{3}g_{lll}, \\ 2(u_{1r} - \zeta_{1r}) &= -(2g_{\tau_1} - 3g g_l - \frac{1}{3}g_{lll}) - g_l f - f f_r - g f_r + \frac{2}{3}f_{rrr}. \end{cases}$$

The left-hand sides may be integrated:

$$\begin{cases} 2(\zeta_1 + u_1) &= -(2f_{\tau_1} + 3ff_r + \frac{1}{3}f_{rrr})l + f_r \int g dl + \frac{1}{2}g^2 + fg - \frac{2}{3}g_{ll} + C_1, \\ 2(u_1 - \zeta_1) &= -(2g_{\tau_1} - 3g g_l - \frac{1}{3}g_{lll})r - g_l \int f dr - \frac{1}{2}f^2 - gf + \frac{2}{3}f_{rr} + C_2, \end{cases}$$

where C_1 (C_2) may depend on all variables but l (r). All terms on the right-hand sides are bounded, except for the first terms (which are secular: they grow without bound as $r, l \to \pm\infty$) and the second terms. The second terms of both equations may be eliminated by imposing that we consider profiles for which the right- and left-translating frames of reference do not interact. Thus the equations we are about to derive are valid in frames moving along with the shallow-water velocity ($\sqrt{gh}$ in dimensional variables), and after a sufficient amount of time has elapsed so that the right- and left-moving waves no longer interact. This leaves only the growth of the first secular terms. This growth is unphysical, as the quantities on the left-hand side are clearly bounded. Thus we impose that the secular terms vanish. This gives

$$2\frac{\partial f}{\partial \tau_1} + 3f\frac{\partial f}{\partial r} + \frac{1}{3}\frac{\partial^3 f}{\partial r^3} = 0,$$

for the right-going wave, and

$$2\frac{\partial g}{\partial \tau_1} - 3g\frac{\partial g}{\partial l} - \frac{1}{3}\frac{\partial^3 g}{\partial l^3} = 0,$$

for the left-going wave. Note that if we had not introduced the slow time scale τ_1, we would have found two ordinary differential equations, one to be satisfied by f, the other by g. This would determine the profiles f and g up to some arbitrary constants, instead of allowing for general initial conditions. Having introduced the slow time scale τ_1, we have obtained two PDEs satisfied by f and g. Both are KdV equations. Thus, in their respective frames of reference, both parts of the wave profile are described by a KdV equation which describes how the right- or left-going part of the profile evolves in its moving frame of reference.

4.1.10 The one-soliton solution

The one-soliton solution wave profile for the right- and left-going KdV equations is given by

$$f = \frac{4}{3}a^2\mathrm{sech}^2 a\left(r - \frac{2}{3}a^2\tau_1 - x_0\right), \quad g = \frac{4}{3}a^2\mathrm{sech}^2 a\left(l + \frac{2}{3}a^2\tau_1 - x_0\right),$$

as the reader can easily check. In these formulae, a is a free parameter, determining the amplitude, width and speed of the soliton. The constant x_0 is also a free parameter, determining the initial position of the peak of the soliton. Note that the parameters for the right-going soliton (f) and for the left-going one (g) are unrelated. More compactly,

$$\begin{aligned} g, f &= \frac{4}{3}a^2\mathrm{sech}^2 a\left(x \pm \tau_0 \pm \frac{2}{3}a^2\tau_1 - x_0\right) \\ &= \frac{4}{3}a^2\mathrm{sech}^2 a\left(x - x_0 \pm \left(\tau_0 + \frac{2}{3}a^2\tau_1\right)\right), \end{aligned}$$

where the $+$ ($-$) sign corresponds to g (f). Returning to dimensional variables, the one-soliton wave profile is

$$\begin{aligned} &\frac{4}{3}a^2\epsilon h\,\mathrm{sech}^2 a\left(\frac{\sqrt{\epsilon}}{h}(x - x_0) \pm \left(\sqrt{\frac{\epsilon g}{h}}t + \frac{2}{3}a^2\epsilon\sqrt{\frac{\epsilon g}{h}}t\right)\right) = \\ &\quad \frac{4}{3}(a^2\epsilon)h\,\mathrm{sech}^2\left[\frac{a\sqrt{\epsilon}}{h}\left(x - x_0 \pm \sqrt{gh}\left(1 + \frac{2}{3}a^2\epsilon\right)t\right)\right]. \end{aligned}$$

Some comments are in order.

- In the above, ϵ is an arbitrary parameter, introduced as a measure of smallness for both amplitude and 1/wavelength. At this point, it is clear that the exact value of ϵ does not matter, as only the combination $a\sqrt{\epsilon}$ shows up, thus ϵ may conveniently be absorbed by a.

- A soliton raises the surface of the water: there is no dip leading or trailing the soliton.

- A soliton moves with a speed slightly higher than the shallow-water wave speed $\sqrt{gh}$.

- The above conclusions were obtained because we assumed surface tension was a small effect. If surface tension is a major effect (as for instance in thin films), the last two conclusions don't necessarily hold.

The soliton solution for the KdV equation has been compared extensively with water-wave experiments, see [4] for an overview, or [29, 30, 31, 62]. In all cases, good agreement was found with experimental observations.

4.2 Slowly modulated waves: the NLS equation

The derivation of the KdV equation from the water wave equations illustrates all aspects and intricacies of the method of multiple scales, including how complicated and messy it often gets. In this section, we present a derivation of the equally seminal NLS equation, starting from the KdV equation. We use the method of multiple scales, but its application is nowhere near as complicated as before, mainly due to the fact that our starting equation (the KdV equation) is significantly simpler than the Euler equations (4.1).

The NLS equation arises in the study of modulated wave trains in dispersive media. In this section, we derive the equation from the KdV equation: starting from a periodic wave train solution of the KdV equation, we examine the dynamics of wave trains as they experience modulation. Our derivation follows [12]. Similar derivations are found in [1, 37, 38] and elsewhere.

4.2.1 Problem set-up

Small-amplitude waves in dispersive media are governed by the KdV equation, which may be written as

$$u_t = uu_x + u_{xxx}. \tag{4.6}$$

We already know this equation has solitary wave solutions. We will see in Chapter 5 that it has periodic solutions as well. Let us investigate what the dynamics would be of a small-amplitude wave train solution that undergoes a slow modulation. In an application problem, such modulations can be caused by perturbations in material properties, the presence of other signals, *etc.*

4.2.2 The linear dispersion relation and modulated wave packets.

Since the wave trains we examine have small amplitude, we begin by investigating the linearized KdV equation

$$u_t = u_{xxx}.$$

Using what we have learned in Chapter 2, the dispersion relation is

$$\omega(k) = k^3,$$

so that a localized solution on the whole line is written as

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(k)e^{ikx-i\omega(k)t}\,dk, \tag{4.7}$$

where $U(y)$ is the Fourier transform of the initial condition. This solution can be rewritten using some simple manipulations. Suppose that in Fourier space, the solution is centered around wave number k_0. In other words, the localized solution (4.7) contains a pronounced

component with period $2\pi/k_0$. One way this may occur is if the solution consists of a localized envelope of an underlying "carrier" wave with wave number k_0. Such a solution is illustrated in Fig. 4.4.

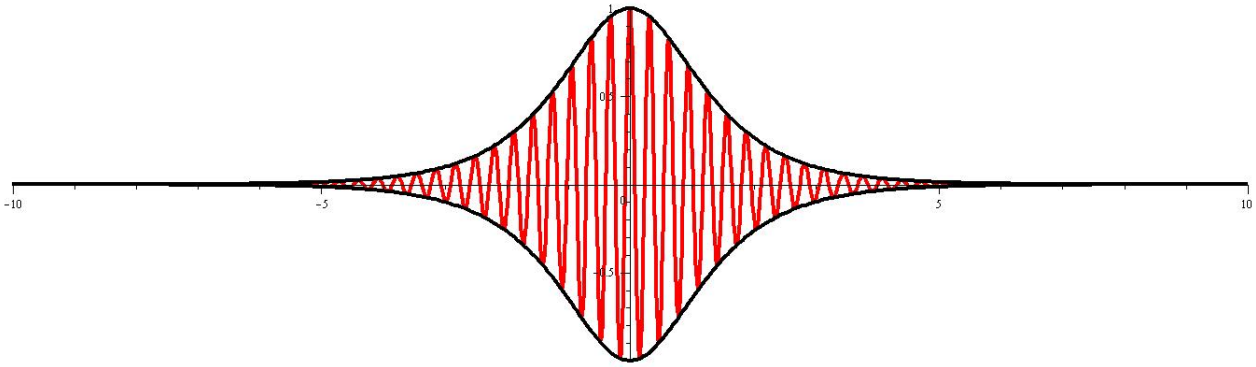


Figure 4.4: A wave packet given by $\text{sech}(x)\cos(20x) = e^{i20x}\text{sech}(x)/2 + e^{-i20x}\text{sech}(x)/2$. Also plotted is its envelope consisting of $\text{sech}(x)$ and $-\text{sech}(x)$. The Fourier spectrum of this solution has two pronounced peaks at $k_0 = 20$ and at $k_0 = -20$.

Assuming a wave-packet-like solution, we have

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} U(k)e^{ikx-it(\omega_0+(k-k_0)\omega_0'+\tilde{\Omega}(k))}dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} U(k)e^{ik_0x-i\omega_0t+i(k-k_0)(x-i\omega_0't)-i\tilde{\Omega}(k)t}dk \\ &= \frac{1}{2\pi}e^{ik_0x-i\omega_0t}\int_{-\infty}^{\infty} U(k)e^{i(k-k_0)\xi-i\tilde{\Omega}(k)t}dk \\ &= \frac{1}{2\pi}e^{ik_0x-i\omega_0t}\int_{-\infty}^{\infty} U(k+k_0)e^{ik\xi-i\Omega(k)t}dk \\ &= e^{ik_0x-i\omega_0t}A(\xi,t), \end{aligned} \tag{4.8}$$

where $\omega_0 = \omega(k_0)$, $\omega_0' = c_g = \omega'(k_0)$, $\tilde{\Omega}(k) = \omega(k) - \omega_0 - (k-k_0)\omega_0'$, and $\Omega(k) = \tilde{\Omega}(k+k_0)$. Finally, the envelope $A(\xi,x)$ is given by

$$A(\xi,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(k+k_0)e^{ik\xi-i\Omega(k)t}dk.$$

For a wave packet solution, we may assume that the dependence of the envelope function $A(\xi,t)$ on its variables is slow in comparison to the dependence of $\exp(ik_0x - i\omega_0t)$ on x and t. Thus, the wave number k_0 and the frequency ω_0 are much higher than any dominant wave number and frequency present in $A(\xi,t)$. We use this in what follows to set up the slow variables for the method of multiple scales. It should be mentioned that to ensure reality of

the solution $u(x,t)$, it is necessary to add the complex conjugate to any ansatz of the form (4.8). Indeed, the Fourier transform of a real solution $u(x,t)$ will be peaked at $-k_0$ if it is peaked at k_0, see also Fig. 4.4.

4.2.3 The perturbation expansion and the slow time scales

Since we are assuming the solutions have small amplitude, we let

$$u = \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \dots, \tag{4.9}$$

where ϵ is a small parameter. Substituting this in (4.6), we find that u_1 satisfies the linear KdV equation. Assuming a wave packet form with carrier wave number k_0 and carrier wave frequency ω_0, we let

$$u_1(x,t) = e^{ik_0x - i\omega_0 t} A(\xi, \tau_2) + c.c., \tag{4.10}$$

where *c.c.* denotes the complex conjugate of the first term. We will use this notation in what follows: we suppress all terms with negative wave numbers, accounting for them using the $+c.c.$ notation. Note that this does not apply to real terms, all of which are written explicitly. Also, We have introduced the slow envelope variables $\xi = \epsilon(x - c_g t)$ and $\tau_2 = \epsilon^2 t$. Similarly, we could introduce slower time scales $\tau_3 = \epsilon^3 t$, and so on. It turns out that for the derivation of the NLS equation, the introduction of τ_2 suffices. We have already motivated that the envelope functions should depend only on slow variables. This is now made explicit. The presence of the scaling factor ϵ^2 with τ_2 is motivated by our calculations in the previous section: the frequency function $\Omega(k)$ is related to a second-order term in the Taylor expansion of the $\omega(k)$. Allowing $A(\xi, \tau_2)$ to depend also on a time variable $\tau_1 = \epsilon t$ merely results in a redefining of c_g.

With the introduction of the slow variables, we need to replace derivatives using the rules

$$\begin{aligned}
\frac{\partial}{\partial x} &\to \frac{\partial}{\partial x} + \epsilon \frac{\partial}{\partial \xi}, \\
\frac{\partial}{\partial t} &\to \frac{\partial}{\partial t} - \epsilon c_g \frac{\partial}{\partial \xi} + \epsilon^2 \frac{\partial}{\partial \tau_2} + \dots .
\end{aligned}$$

4.2.4 Results at consecutive orders

Substituting the perturbation expansion (4.9) in (4.6) and using the derivative replacement rules, we find

$$\begin{aligned}
\epsilon u_{1t} + \epsilon^2 u_{2t} + \epsilon^3 u_{3t} - \epsilon^2 c_g u_{1\xi} - \epsilon^3 c_g u_{2\xi} + \epsilon^3 u_{1\tau_2} = \\
\epsilon^2 u_1 u_{1x} + \epsilon^3 u_1 u_{2x} + \epsilon^3 u_1 u_{1\xi} + \epsilon^3 u_2 u_{1x} + \\
\epsilon u_{1xxx} + \epsilon^2 u_{2xxx} + \epsilon^3 u_{3xxx} + 3\epsilon^2 u_{1\xi xx} + 3\epsilon^3 u_{2\xi xx} + 3\epsilon^3 u_{1\xi\xi x} + \mathcal{O}(\epsilon^4).
\end{aligned}$$

Here the different lines of the equation correspond to the different terms in (4.6).

Results at order ϵ

Collecting all coefficients of ϵ, we find $u_{1t} - u_{1xxx} = 0$, as expected. Substitution of (4.10) gives $\omega_0 = k_0^3$, confirming the dispersion relationship.

Results at order ϵ^2

At second order in ϵ, we find

$$u_{2t} - u_{2xxx} = c_g u_{1\xi} + u_1 u_{1x} + 3u_{1\xi xx}.$$

Substituting of (4.10) gives

$$\begin{aligned} u_{2t} - u_{2xxx} &= c_g(e^{ik_0x - i\omega_0 t}A_\xi + c.c.) + (e^{ik_0x-i\omega_0 t}A + c.c.)(ik_0 e^{ik_0x-i\omega_0 t}A + c.c.) + \\ &\quad \left(3(ik_0)^2 e^{ik_0x-i\omega_0 t} + c.c.\right) \\ &= (c_g - 3k_0^2)e^{ik_0x-i\omega_0 t}A_\xi + ik_0 e^{2ik_0x-2i\omega_0 t}A^2 + c.c.. \end{aligned}$$

Note that $e^{ik_0x-i\omega_0 t}$ is in the kernel of the left-hand side. Thus the first term on the right should be eliminated to avoid secular terms in u_2 that grow linearly in x and t. This results in the requirement $c_g = 3k_0^2$, another confirmation of the dispersion relation. We are left with

$$u_{2t} - u_{2xxx} = ik_0 e^{2ik_0x-2i\omega_0 t}A^2 + c.c.. \tag{4.11}$$

Using the method of undetermined coefficients, we let

$$u_2 = e^{2ik_0x-2i\omega_0 t}B(\xi,\tau_2) + c.c. + C(\xi,\tau_2).$$

The equation (4.11) for u_2 imposes no conditions on the slow function $C(\xi,\tau_2)$. Note that $C(\xi,\tau_2)$ may be chosen to be real, since u_2 is real and $C(\xi,\tau_2)$ does not multiply any factors of $e^{ik_0x-i\omega_0 t}$. We find for $B(\xi,\tau_2)$ that

$$B(\xi,\tau_2) = \frac{1}{6k_0^2}A^2,$$

so that

$$u_2 = \frac{1}{6k_0^2}e^{2ik_0x-2i\omega_0 t}A^2(\xi,\tau_2) + c.c. + C(\xi,\tau_2). \tag{4.12}$$

It is clear that our expansion method fails if $k_0 = 0$, but this is to be expected. Our expansion method assumes that k_0 is a high wave number, in comparison to any other wave numbers involved.

Results at order ϵ^3

At third order in ϵ, we find

$$u_{3t} - u_{3xxx} = c_g u_{2\xi} - u_{1\tau_2} + u_1 u_{2x} + u_1 u_{1\xi} + u_2 u_{1x} + 3u_{2\xi xx} + 3u_{1\xi\xi x}.$$

Next, we substitute the expressions for u_1 (4.10) and for u_2 (4.12). This results in a right-hand side which contains terms with $\exp(ik_0x - i\omega_0 t)$, $\exp(2ik_0x - 2i\omega_0 t)$, $\exp(3ik_0x - 3i\omega_0 t)$, and their complex conjugates. There are also a few terms which depend only on the slow variables. As before, all terms in the kernel of the left-hand side should be made to vanish, to avoid secular growth in u_3. Thus, the terms only depending on the slow variables, and the coefficients of $\exp(ik_0x - i\omega_0 t)$ and $\exp(-ik_0x + i\omega_0 t)$ must be equated to zero.

First, we investigate the slow-variable term. It is given by

$$\begin{aligned} & 3k_0^2 C_\xi + AA_\xi^* + A^* A_\xi = 0 \\ \Rightarrow \quad & 3k_0^2 C_\xi + \left(|A|^2\right)_\xi = 0 \\ \Rightarrow \quad & 3k_0^2 C + |A|^2 = \tilde{D}(\tau_2), \end{aligned}$$

so that

$$C(\xi, \tau_2) = -\frac{1}{3k_0^2}|A|^2 + D(\tau_2), \tag{4.13}$$

where $D(\tau_2) = \tilde{D}(\tau_2)/3k_0^2$.

The coefficient of $\exp(ik_0x - i\omega_0 t)$ is given by

$$-A_{\tau_2} + \frac{i}{6k_0}|A|^2 A + 3ik_0 A_{\xi\xi} + ik_0 AC = 0,$$

which results in

$$(i\partial_{\tau_2} + k_0 D)A = -3k_0 A_{\xi\xi} + \frac{1}{6k_0}|A|^2 A,$$

upon using (4.13). The coefficient of $\exp(-ik_0x + i\omega_0 t)$ gives the complex conjugate equation, of course. The equation we have obtained is almost the NLS equation. Equating

$$A(\xi, \tau_2) = a(\xi, \tau_2) e^{ik_0 \int D d\tau_2}$$

gives

$$ia_{\tau_2} = -3k_0 a_{\xi\xi} + \frac{1}{6k_0}|a|^2 a, \tag{4.14}$$

the NLS equation! Because the sign of the ratio of the coefficient of the dispersive term (the first term on the right-hand side) to the coefficient of the nonlinear term is always negative, the equation is known as the *defocusing* NLS equation. The reader should convince her/himself that this cannot be altered: no matter which coefficients are used in the KdV equation, the equation is always of defocusing type, see Problem 2 on Page 86.

4.2.5 Defocusing *vs.* focusing

In general, we may write the NLS equation as

$$ia_t = -a_{xx} + \sigma|a|^2 a, \tag{4.15}$$

where $\sigma = \pm 1$. By scaling both the independent and the dependent variables, the size of the coefficients of the different terms in the equation is arbitrary, but the sign of the ratio of the two terms on the right-hand side cannot be changed. This is encoded in σ. The second-derivative term is the dispersive term. If the dispersive term and the nonlinear term have the same sign, the equation is of defocusing type. Otherwise it is of focusing type. This nomenclature originates in nonlinear optics where the different equation gives rise to light bundles that are either spreading out (defocusing) or focusing, depending on the nature of the material [51]. In the context of Bose-Einstein condensates, the equations are referred to instead as repulsive or attractive, respectively [58].

Using ideas from quantum mechanics [49], a heuristic explanation can be given for the different behavior exhibited by the defocusing and the focusing equation. Let us rewrite (4.15) suggestively as

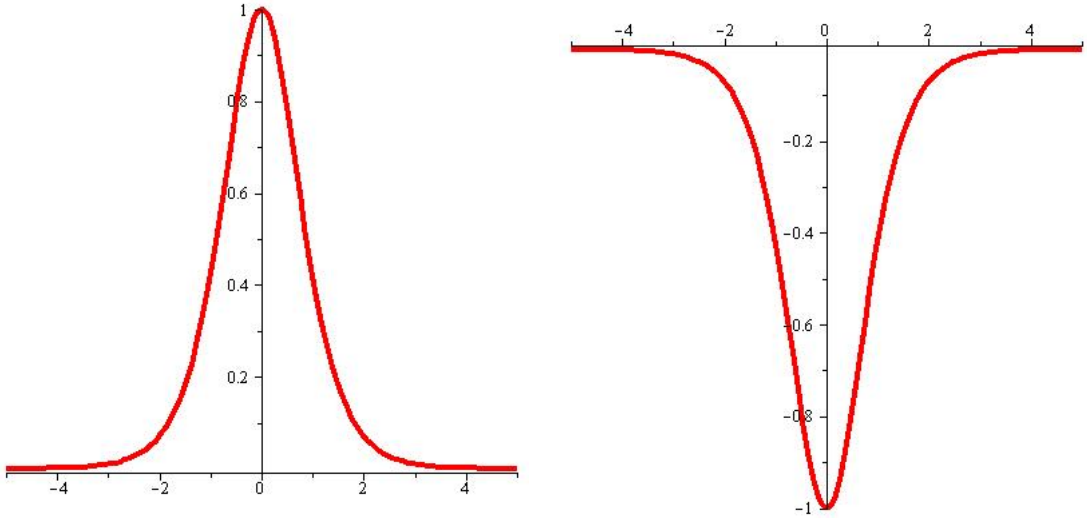
$$ia_t = -a_{xx} + V\, a.$$

This equation is similar to the linear Schrödinger equation with inter-particle potential given by $V = \sigma|a|^2$. Suppose that the solution of the NLS equation is a localized envelope. Then the potential V looks like the profile displayed in Fig. 4.5a for $\sigma = 1$ (defocusing) or Fig. 4.5b for $\sigma = -1$ (focusing). Thus, for the defocusing equation the "potential" the wave function a experiences is positive, leading to a repulsive interaction: particles with this type of interaction try to maximize the interparticle distance, leading to a phenonomen of defocusing. For the focusing equation, this "potential" is negative, and particles experience a potential well, leading to an attractive interaction among particles. All particle positions seek to line up with the center of the well, leading to a focusing phenomenon.

Remarks.

- The NLS equation is sometimes referred to as the Gross-Pitaevski equation, after Gross [27] and Pitaevski [59] who derived the equation in the context of Bose-Einstein condensation in 1961. This is 7 years prior to Zakharov's derivation in the context of deep water waves. It should be emphasized that the context of Bose-Einstein condensation is the one application where the equation does not describe the modulation of a wave train. Instead, it arises from a Hartree-Fock mean-field approximation (see, for instance, [49]) to the many-particle linear Schrödinger equation [27, 59].

- The NLS equation has been derived in a plethora of application fields. A few prominent examples are: (a) Waves in deep water. Such waves are susceptible to the so-called Benjamin-Feir instability [7, 9, 68]. The NLS equation describes the evolution of this instability, past its linear onset [73]. (b) In plasma physics, the equation is used to



(a) (b)

Figure 4.5: (a) The repulsive potential $|a|^2$ for a localized solution a. (b) The attractive potential $-|a|^2$ for a localized solution a.

describe so-called Langmuir waves, among many other phenomena [36]. (c) In nonlinear optics, it describes the propagation of localized wave packets in χ_3 materials [32, 51]. (d) As mentioned above, it describes the mean-field dynamics of the ground-state of a Bose-Einstein condensate [27, 59].

- The equation can also be derived in applications where multiple space dimensions enter the picture. In that case the dispersive term is replaced by either a Laplacian in the appropriate number of dimensions, or by a hyperbolic operator (*e.g.*, two-dimensional waves in deep water [19], depending on the magnitude of surface tension). Many interesting new issues require further study at this point. The investigation of so-called collapse and blow-up of a wave packet has been of especially great interest. An overview of the rich literature is presented by Sulem & Sulem [66].

4.2.6 The one-soliton solution

In this section we examine the qualitative aspects of the soliton solutions of both the defocusing and the focusing NLS equation. The derivation of some of these results is left as an exercise to the reader, see Problem 1 on Page 124. We begin with the soliton solution of the focusing NLS equation.

The bright soliton

The reader should verify that

$$a(x,t) = \sqrt{2}\, k e^{i\theta + irx + i(k^2 - r^2)t} \operatorname{sech} k(x - x_0 - 2rt)$$

satisfies (4.15) with $\sigma = -1$. It is referred to as the *bright soliton* solution of the focusing NLS equation. It consists of a complex oscillation modulated by a sech envelope. The different parameters correspond to different characteristic properties of the soliton. All of these parameters are assumed to be real.

- The parameter x_0 determines the location of the peak of the envelope at $t = 0$.
- The parameter r determines the velocity $2r$ of the peak of the envelope.
- The parameter k determines both the amplitude $\sqrt{2}k$ of the soliton and its width.
- Lastly, the parameter *theta* gives rise to a constant phase shift.

Often, the physically important quantity is the function $u_1 = a\exp(ik_0x - i\omega_0 t) + c.c. = \mathrm{Re}(a\exp(ik_0x - i\omega_0 t))$, as we have learned from the derivation leading to (4.15). It appears that the parameters r and, to a lesser extent, k, can be used to tune the carrier wave number k_0 and frequency ω_0. Sometimes, only the so-called intensity of the soliton solution is considered. This is

$$I(x,t) = |a(x,t)|^2 = 2k^2 \operatorname{sech}^2 k(x - x_0 - 2rt).$$

The intensity of the bright soliton is illustrated in Fig. 4.6. In the context of nonlinear optics, this appears as a bright spot on a dark background [33, 50].

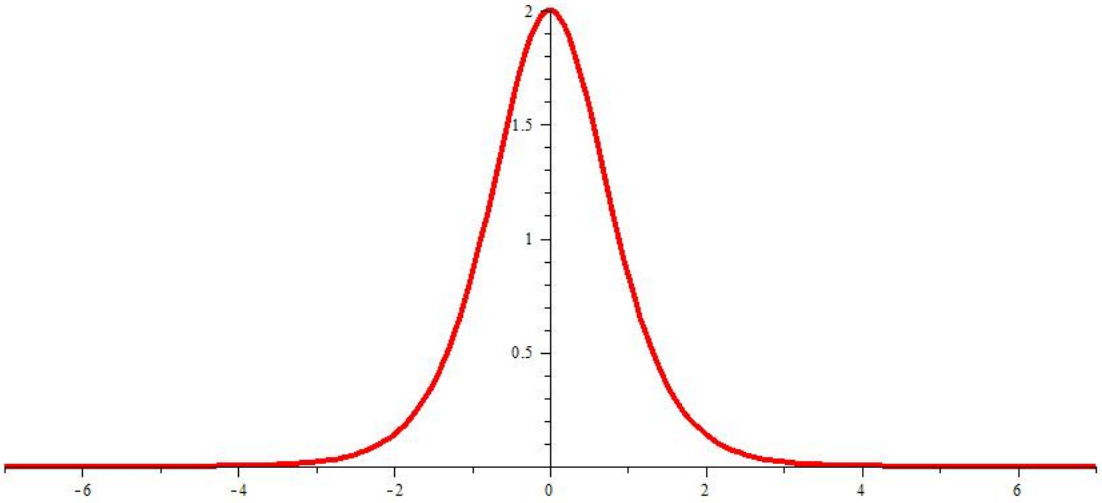


Figure 4.6: The intensity of a bright soliton with $x_0 = 0$, $k = 1$, $r = 0$, and $\theta = 0$, evaluated at $t = 0$.

Dark and grey solitons

If $\sigma = 1$, then

$$a(x,t) = \sqrt{2}\, k e^{i\theta + irx - i(k^2 - r^2)t} \tanh\, k(x - x_0 - 2rt)$$

is a solution of the NLS equation. This solution is referred to as a *dark soliton*. The reason for this is that intensity $I = |a|^2$ is given by

$$I(x,t) = 2k^2 \tanh^2 k(x - x_0 - 2rt) = 2k^2(1 - \mathrm{sech}^2 k(x - x_0 - 2rt)).$$

Thus, the intensity of the dark soliton is visible as a "dark" spot on a bright background. In the context of nonlinear optics, this requires fibers to be flooded with light, while the dark solitons travel back and forth. For this and other reasons dark solitons are not considered much in optical fiber communications, but they have been observed [21]. The intentity of the dark soliton is displayed in Fig. 4.7. The solution parameters have similar meanings as for the bright soliton.

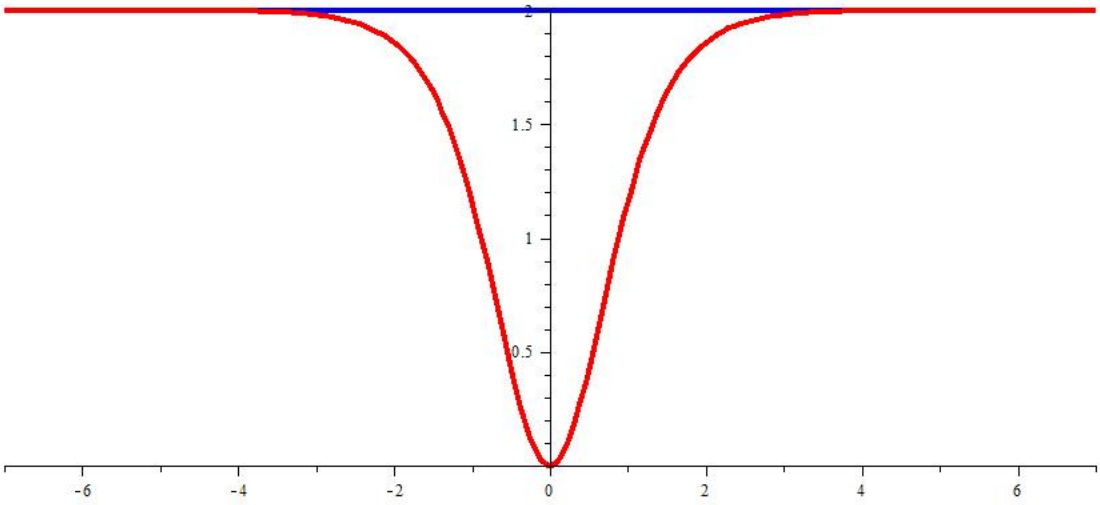


Figure 4.7: The intensity $I(x,t)$ of a dark soliton with $x_0 = 0$, $k = 1$, $r = 0$, and $\theta = 0$, evaluated at $t = 0$.

The dark soliton is often referred to as a *topological soliton*, because it cannot be interpreted as a perturbation of its background, unlike the bright soliton. Indeed, the bright soliton appears as a localized disturbance to its background state: far away from the peak of the bright soliton, the background state appears completely unperturbed. The envelope of the dark soliton has different limiting values as $x \to \pm\infty$, as illustrated in Fig. 4.8. Thus, the presence of a dark soliton imparts a large, non-localized effect on its surroundings. The use of the word "topological" refers to the fact that the dark soliton changes the space it is present in. In this sense, the dark soliton does not fit our original make-shift definition of

a soliton on Page 2. As stated there, we will change this definition as necessary. A more precise definition will be given in later chapters.

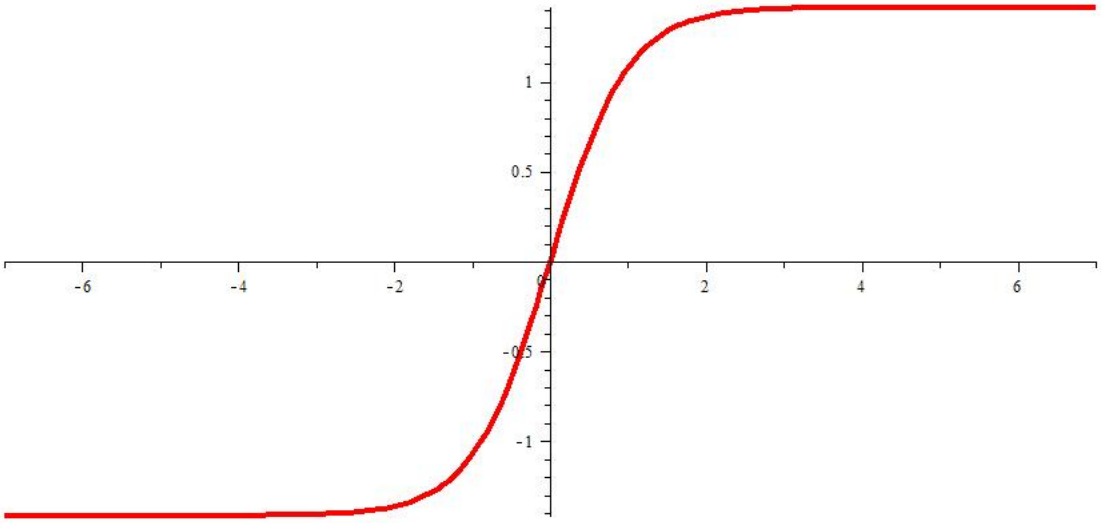


Figure 4.8: The envelope $\sqrt{2}\,k \tanh k(x - x_0 - 2rt)$ of a dark soliton with $x_0 = 0$, $k = 1$, and $r = 0$, evaluated at $t = 0$.

In addition to the dark soliton solution, the defocusing NLS has other topological soliton solutions, known as the *grey solitons*. Their intensity is given by

$$I(x,t) = |a(x,t)|^2 = 2k^2(\tanh^2 k(x - x_0 - 2rt) + B) = 2k^2(B + 1 - \operatorname{sech}^2 k(x - x_0 - 2rt)),$$

where B is a positive parameter. Because of the presence of this parameter, the intensity does not dip down to 0 at $t = 0$ and $x = x_0$. In other words, the dark soliton still appears as a darker spot, compared to its bright background, but the intensity at its darkest point does not vanish. This is illustrated in Fig. 4.9. Clearly, the dark soliton is a special case of the grey solitons, obtained for $B = 0$. The explicit solution formula for the grey soliton $a(x,t)$ is more complicated than any of the other solutions previously mentioned and we omit it here.

4.3 Exercises

1. **The KdV equation for ion-acoustic waves in plasmas.** Ion-acoustic waves are low-frequency electrostatic waves in a plasma consisting of electrons and ions. We consider the case with a single ion species.

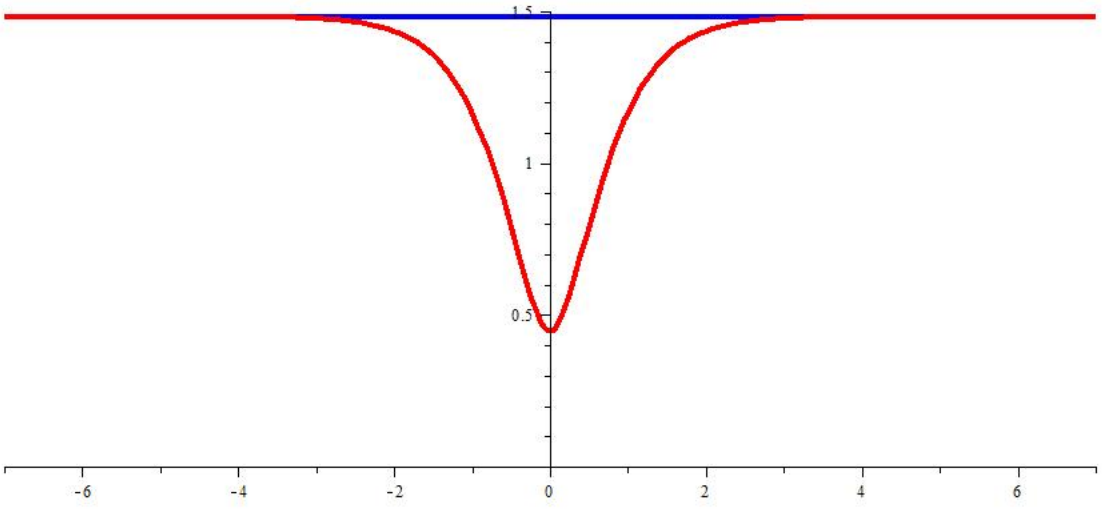


Figure 4.9: The intensity $I(x,t)$ of a grey soliton with $x_0 = 0$, $k = 1$, and $r = 0$, evaluated at $t = 0$.

Consider the following system of one-dimensional equations

$$\begin{cases} \dfrac{\partial n}{\partial t} + \dfrac{\partial}{\partial z}(nv) &= 0 \\ \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial z} &= -\dfrac{e}{m}\dfrac{\partial \phi}{\partial z} \\ \dfrac{\partial^2 \phi}{\partial z^2} &= \dfrac{e}{\varepsilon_0}\left[N_0 \exp\left(\dfrac{e\phi}{\kappa T_e}\right) - n\right] \end{cases}.$$

Here n denotes the ion density, v is the ion velocity, e is the electron charge, m is the mass of an ion, ϕ is the electrostatic potential, ε_0 is the vacuum permittivity, N_0 is the equilibrium density of the ions, κ is Boltzmann's constant, and T_e is the electron temperature.

(a) Verify that $c_s = \sqrt{\dfrac{\kappa T_e}{m}}$, $\lambda_{De} = \sqrt{\dfrac{\varepsilon_0 \kappa T_e}{N_0 e^2}}$, and $\omega_{pi} = \sqrt{\dfrac{N_0 e^2}{\varepsilon_0 m}}$ have dimensions of velocity, length and frequency, respectively. These quantities are known as the ion acoustic speed, the Debye wavelength for the electrons, and the ion plasma frequency.

(b) Nondimensionalize the above system, using

$$n = N_0 n^*, \quad v = c_s v^*, \quad z = \lambda_{De} z^*, \quad t = \frac{t^*}{\omega_{pi}}, \quad \phi = \frac{\kappa T_e}{e}\phi^*.$$

(c) You have obtained the system

$$\begin{cases} \dfrac{\partial n}{\partial t} + \dfrac{\partial}{\partial z}(nv) &= 0 \\ \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial z} &= -\dfrac{\partial \phi}{\partial z} \\ \dfrac{\partial^2 \phi}{\partial z^2} &= e^{\phi} - n \end{cases}$$

for the dimensionless variables. Note that we have dropped the $*$'s, to ease the notation. Find the linear dispersion relation for this system, linearized around the trivial solution $n = 1$, $v = 0$, and $\phi = 0$.

(d) Rewrite the system using the "stretched variables"

$$\xi = \epsilon^{1/2}(z - t), \quad \tau = \epsilon^{3/2}t.$$

Given that we are looking for low-frequency waves, explain how these variables are inspired by the dispersion relation.

(e) Expand the dependent variables as

$$\begin{cases} n &= 1 + \epsilon n_1 + \epsilon^2 n_2 + \dots, \\ v &= \epsilon v_1 + \epsilon^2 v_2 + \dots, \\ \phi &= \epsilon \phi_1 + \epsilon^2 \phi_2 + \dots. \end{cases}$$

Using that all disturbances return to their equilibrium values as $\xi \to \pm\infty$, $\tau \to \infty$, find a governing equation which determines how ϕ_1 depends on ξ and τ.

2. Consider the KdV equation:

$$u_t = auu_x + bu_{xxx}.$$

Here a and b are real, constant coefficients, and p is an integer greater than one. Using the same scalings as in Section 4.2, show that the NLS equation governing the dynamics of modulated wave trains of this equation is always of defocusing type.

3. **Obtaining the KdV equation from the NLS equation.** We have shown that the NLS equation may be used to describe the slow modulation of periodic wave trains of the KdV equation. In this problem we show that the KdV equation describes the dynamics of long-wave solutions of the NLS equation. We follow the derivation presented in [57].

Consider the defocusing NLS equation

$$ia_t = -a_{xx} + |a|^2 a.$$

(a) Let

$$a(x,t) = e^{i\int V\,dx}\rho^{1/2}.$$

Derive a system of equations for the phase function $V(x,t)$ and for the amplitude function $\rho(x,t)$, by substituting this form of $a(x,t)$ in the NLS equation, dividing out the exponential, and separating real and imaginary parts. Write your equations in the form $\rho_t = \ldots$, and $V_t = \ldots$. Due to their similarity with the equations of hydrodynamics, this new form of the NLS equation is referred to as its *hydrodynamic form*.

(b) Find the linear dispersion relation for the hydrodynamic form of the defocusing NLS equation, linearized around the trivial solution $V = 0$, $\rho = 1$. In other words, we are examining perturbations of the so-called Stokes wave solution of the NLS equation, which is given by a signal of constant amplitude.

(c) Rewrite the system using the "stretched variables"

$$\xi = \epsilon(z - \beta t), \quad \tau = \epsilon^3 t.$$

Given that we are looking for long waves, explain how these variables are inspired by the dispersion relation. What should the value of β be?

(d) Expand the dependent variables as

$$\begin{cases} V &= \epsilon^2 V_1 + \epsilon^4 V_2 + \ldots, \\ \rho &= 1 + \epsilon^2\rho_1 + \epsilon^4\rho_2 + \ldots. \end{cases}$$

Using that all disturbances return to their equilibrium values as $\xi \to \pm\infty$, $\tau \to \infty$, find a governing equation which determines how V_1 depends on ξ and τ. This equation should be equivalent to the KdV equation.

4. Consider the previous problem, but with the focusing NLS equation

$$ia_t = -a_{xx} - |a|^2 a.$$

The method presented in the previous problem does not allow one to describe the dynamics of long-wave solutions of the focusing NLS equation using the KdV equation. How does this show up in the calculations?

Chapter 5

Special solutions of one-dimensional nonlinear partial differential equations

Here's a secret: whenever we solve partial differential equations analytically, we always reduce the problem to solving ordinary differential equations. If the equation we are solving is linear with constant coefficients and has simple boundary conditions, we attempt to do this using separation of variables. This reduces the problem to that of solving an infinite set of decoupled ordinary differential equations. The outcome of this procedure is a set of so-called fundamental solutions of our partial differential equation. What are these good for? Since the equation is linear, we can construct more general solutions by superposition of the fundamental solutions. This is especially useful if the set of fundamental solutions is complete in some suitable function space, like the square-integrable functions. In that case the superposition constitutes the general solution of the partial differential equation within that function space.

If the partial differential equation at hand is nonlinear, then superposition will no longer give us more general solutions. So why should we care about getting some exact solutions?

- The analysis of exact solutions should result in a better understanding of the partial differential equation.
- Exact solutions are a great way to test if numerical methods are doing what they are supposed to do.
- In many cases, the long-time behavior of solutions of the partial differential equation may be described asymptotically by different exact solutions. We will have more to say about this later.

5.1 Traveling wave solutions

Our visual perception is very good at isolating features of patterns that do not change. Thus, if the solution of a partial differential equation contains features with a shape that

is invariant under the dynamics, it will jump out at us. This is the case with the solitons that we've already seen. Such invariant shapes might be described by special solutions of the partial differential equation. These special solutions are the traveling wave or stationary solutions. Traveling wave or stationary solutions of an equation are defined to be solutions that are stationary in some frame of reference that is moving with constant velocity v with respect to the original frame of reference. Thus

$$u(x,t) = U(x - vt) = U(z),$$

where $z = x - vt$, and v is constant. This ansatz does indeed state that we are looking for solutions whose shape does not change in time, if we put ourselves in a suitably moving coordinate frame.

Here's the general idea: let

$$u_t = N(u, u_x, u_{xx}, \ldots)$$

be some nonlinear partial differential equation. Thus $N(u, u_x, \ldots)$ is a nonlinear function of u and its x-derivatives. Substituting our traveling wave ansatz into the original partial differential equation gives

$$-vU' = N(U, U', U'', \ldots),$$

which is an ordinary differential equation for $U(z)$. All solutions of this ordinary differential equation give traveling wave solutions of the partial differential equations. But not all of these solutions will be of interest to us: they may not satisfy the conditions (boundary, initial or other) that we may want them to satisfy. For instance, we may impose that our solutions remain bounded for all x and t. In this section, we examine the ordinary differential equation from different points of view. We use techniques familiar from physics and differential equations. At the end, we will have achieved an understanding of how the soliton solutions fit into a larger class of stationary solutions. Few issues in this section will be dealt with in all generality. For most, we will restrict ourselves to a class of examples.

5.1.1 Energy integral approach

Let's consider the class of examples

$$4u_t = 6u^p u_x + u_{xxx},$$

for $p = 0, 1, 2, \ldots$. The case of $p = 0$ corresponds to a linear equation and is not very interesting. Stationary solutions satisfy

$$-4vU' = 6U^p U' + U'''.$$

This equation may be integrated once to obtain

$$-4vU = \frac{6}{p+1}U^{p+1} + U'' + \alpha, \tag{5.1}$$

where α is an integration constant. We will come back to this equation when we use phase plane analysis to examine the qualitative behavior of our solutions. Multiplying by U', we obtain an equation that may be integrated once more:

$$\begin{aligned} -4vUU' &= \frac{6}{p+1}U^{p+1}U' + U''U' + \alpha U' \\ \Rightarrow -2vU^2 &= \frac{6}{(p+1)(p+2)}U^{p+2} + \frac{1}{2}U'^2 + \alpha U - \beta, \end{aligned}$$

where β is a second integration constant. This equation is rewritten as

$$\frac{1}{2}U'^2 + V(U; v, \alpha) = \beta,$$

where

$$V(U; v, \alpha) = \frac{6}{(p+1)(p+2)}U^{p+2} + 2vU^2 + \alpha U.$$

The above equation is the equation of conservation of energy for the motion of a particle with unit mass under the influence of a conservative force, whose potential is given by $V(U; v, \alpha)$. The first term of the energy equation denotes the kinetic energy, the second one the potential energy. The right-hand side denotes the constant total energy. From this simple analogy, many conclusions may be drawn. For starters, if $p = 0$ then V is a quadratic function of U and in essence we are solving the harmonic oscillator, with its trigonometric solutions.

- p **even.**

  If p is even then $V \to \infty$ as $U \to \pm\infty$. The potential looks like indicated in Figs. 5.1 and 5.2 below. In order to have real-valued solutions, it is necessary that the kinetic energy is positive. Since the kinetic energy is the difference between the total energy and the potential energy, this requires a sufficiently high value of β. If β is chosen sufficiently high, it follows that real-valued solutions exist, and that these are bounded: $U_{\min} < U < U_{\max}$.

  More can be said. The "time" (in z, thus x, or up to a multiplicative constant t, in terms of the original variables) it takes U to travel from $U_{\max} = U(z_2)$ to $U_{\min} = U(z_1)$ is given by (doing a simple separation of variables):

$$z_2 - z_1 = \int_{U_{\min}}^{U_{\max}} \frac{dU}{\sqrt{2(\beta - V(U; v, \alpha))}}.$$

  The integral on the right-hand side is singular, as $V(U; v, \alpha) = \beta$ at both $U_{\min}$ and $U_{\max}$. If the equation $V(U; v, \alpha) = \beta$ has simple roots at both $U_{\min}$ and $U_{\max}$ (as is the case

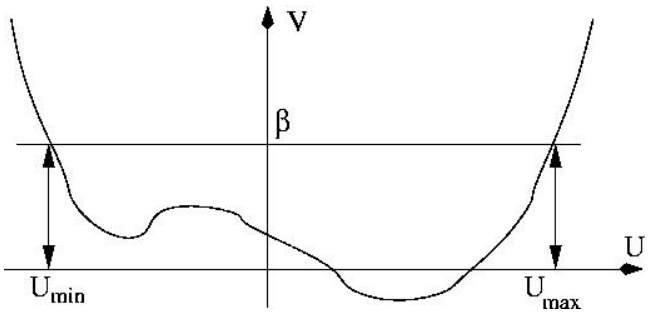


Figure 5.1: The potential $V(U; v, \alpha)$ with p even. Here the total energy was chosen to give rise to a periodic solution.

in Fig. 5.1), both singularities are square-root singularities. Square-root singularities are integrable and the resulting $z_2 - z_1$ is finite. This is the case of periodic solutions, and $z_2 - z_1$ is half the period of the solution. On the other hand, if $V(U; v, \alpha) = \beta$ has at least a double root at either $U_{\min}$ or $U_{\max}$ (as is the case in Fig. 5.2), the corresponding singularity is not integrable and the resulting $z_2 - z_1$ is infinite. This will correspond to the soliton case. This simple argument validates the often made statement that solitons are limits of periodic solutions of integrable systems where the period has limited to infinity.

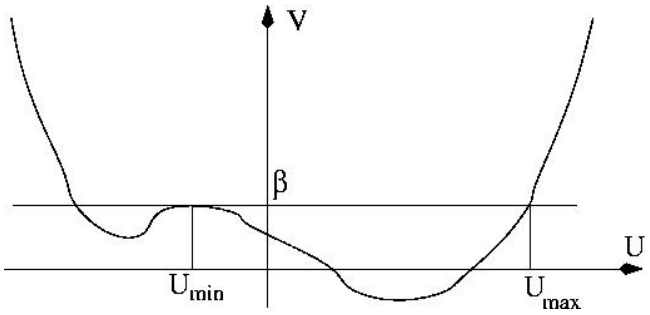


Figure 5.2: The potential $V(U; v, \alpha)$ with p even. Here the total energy was chosen so as to obtain a soliton solution.

- p **odd**

  If p is odd then $V \to \infty$ as $U \to \infty$ and $V \to -\infty$ as $U \to -\infty$. The potential looks like indicated in Figs. 5.3 and 5.4 below. It is clear that for all values of β real-valued solutions exist, but it is not immediately clear whether any bounded real-valued solutions exist. If the potential is monotone as in Fig. 5.3, there is no value of β for which bounded solutions exist.

  On the other hand, if the potential is not monotone and has one or more local minima, bounded solutions exist for values of β slightly above local minimum values of the

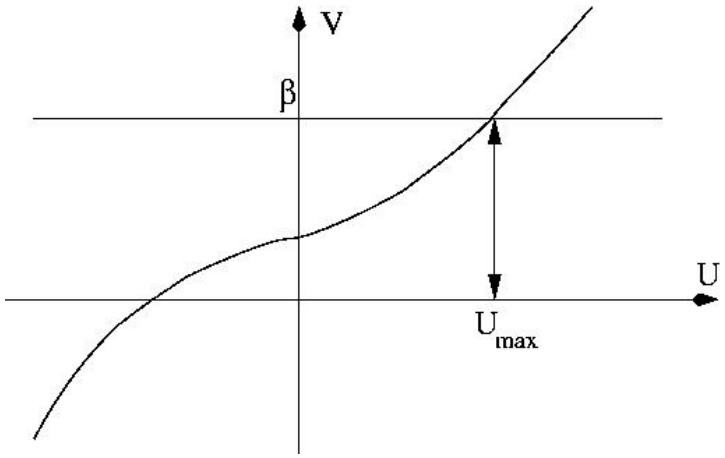


Figure 5.3: The potential $V(U; v, \alpha)$ with p odd. In this case, no bounded solutions exist.

potential, as indicated in Fig. 5.4. In that case, solitons may be found at the edge of the region of existence of such bounded solutions, namely at energy levels β which pass through the lowest of the local maxima adjacent to the local minimum.

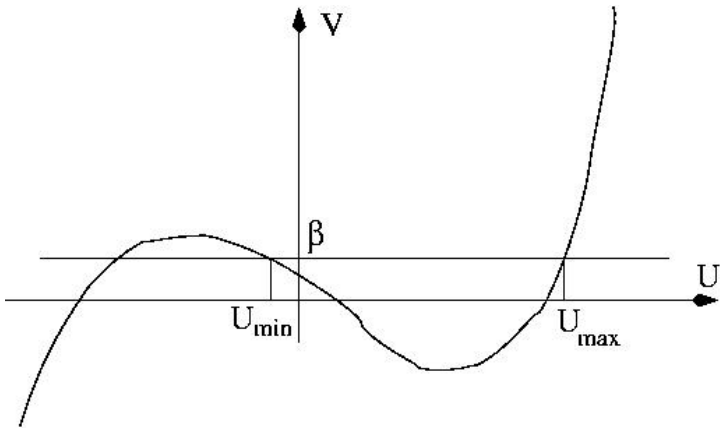


Figure 5.4: The potential $V(U; v, \alpha)$ with p odd. In this case bounded solutions exist.

From the potential energy picture, it also follows that any equilibrium solutions must satisfy $V'(U; v, \alpha) = 0$. Thus such equilibrium solutions are found at the local minima or maxima of the potential. The ones at the minima are stable within the class of stationary solutions, whereas the equilibrium solutions at the maxima are unstable. Looking ahead at the phase plane analysis, we conclude that since energy is conserved, the only type of equilibrium points that can occur are either centers (stable) or saddles (unstable).

5.1.2 Phase plane analysis

To use phase plane analysis, we return to Eqn. (5.1). This equation is rewritten as

$$U'' = -\frac{\partial V}{\partial U}(U; v, \alpha).$$

In order to write this as a first-order system, we introduce $u_1 = U$, $u_2 = U'$ and find

$$\left\{ \begin{array}{lcl} u_1' & = & u_2 \\ u_2' & = & -\frac{\partial V}{\partial u_1}(u_1; v, \alpha) \end{array} \right. .$$

From our previous results, it follows immediately that

$$E(u_1, u_2; v, \alpha) = \frac{1}{2}u_2^2 + V(u_1; v, \alpha) = \beta$$

is a conserved quantity for the first-order dynamical system. Thus in the (u_1, u_2)-plane the trajectories of the system are confined to the level curves of $E(u_1, u_2; v, \alpha)$. Further note that all equilibrium solutions (*i.e.*, solutions for which $(u_1', u_2') = (0, 0)$) are on the horizontal axis. Lastly, if $(u_1(t), u_2(t))$ is a solution, then so is $(u_1(-t), -u_2(-t))$. This is easily verified by direct substitution.

Let's look at some specific cases.

The KdV equation, $p = 1$.

With $p = 1$ we obtain

$$V = U^3 + 2vU^2 + \alpha U,$$

or

$$\frac{\partial V}{\partial U} = 3U^2 + 4vU + \alpha.$$

We know equilibrium solutions satisfy $\frac{\partial V}{\partial U} = 0$, thus the number of these will be determined by the discriminant of the above equation. This discriminant is

$$\Delta = 16v^2 - 12\alpha = 4(4v^2 - 3\alpha).$$

There are three possible scenarios: $v^2 < 3\alpha/4$, $v^2 = 3\alpha/4$, and $v^2 > 3\alpha/4$. Let's examine these cases in detail.

- $v^2 < 3\alpha/4$**:**

  In this case $V(U; v, \alpha)$ is a monotone increasing function of U. As discussed, there are real solutions for all values of the total energy β, but none of these are bounded. Such a potential and its corresponding phase plane are illustrated in Fig. 5.5. There are no equilibrium solutions.

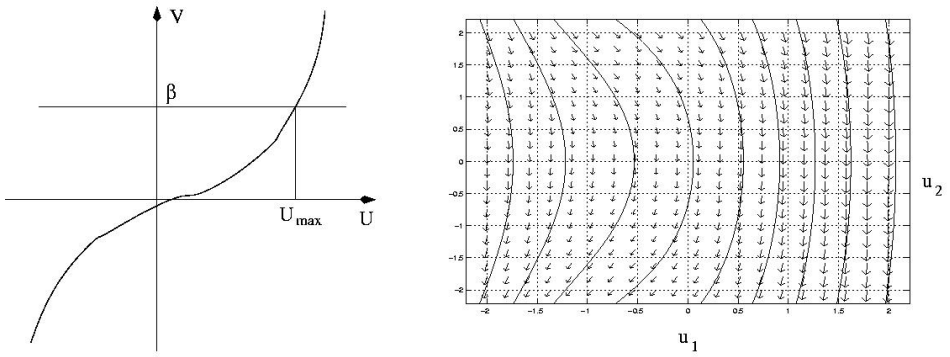


Figure 5.5: The potential $V(U; v, \alpha)$ for the KdV equation for $v^2 < 3\alpha/4$ (left). The corresponding phase portrait is on the right.

- $v^2 = 3\alpha/4$**:**

  Now $V(U; v, \alpha)$ is still a monotone increasing function of U, but now with a single inflection point at U_0, see Fig. 5.6. There are real solutions for all values of the total energy β, but none of these are bounded, as before. This potential and its corresponding phase plane are illustrated in Fig. 5.6. There is one equilibrium solution, which is quasi-stable.

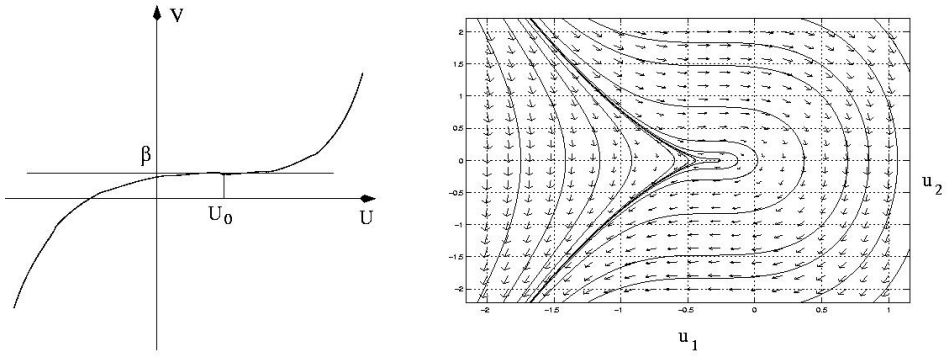


Figure 5.6: The potential $V(U; v, \alpha)$ for the KdV equation for $v^2 = 3\alpha/4$ (left). The corresponding phase portrait is on the right.

- $v^2 > 3\alpha/4$**:**

  Now $V(U; v, \alpha)$ is no longer monotone. It has one local minimum (at U_b, see Fig. 5.7),

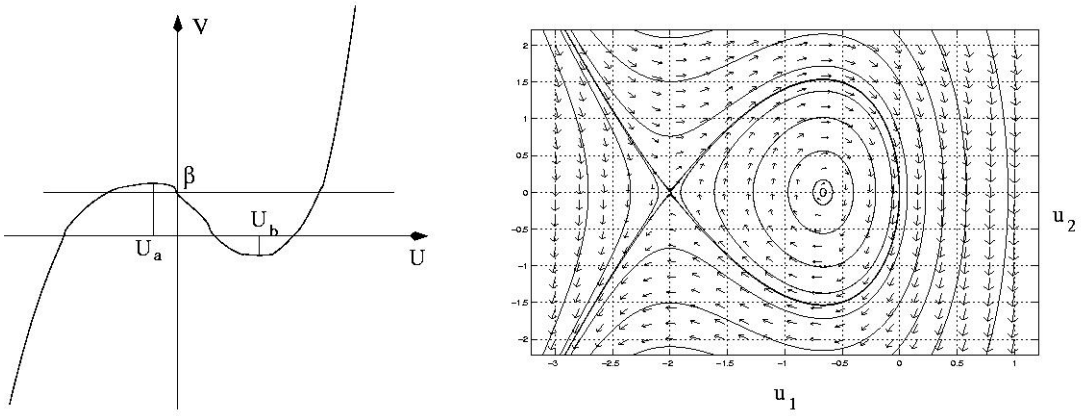


Figure 5.7: The potential $V(U; v, \alpha)$ for the KdV equation for $v^2 > 3\alpha/4$ (left). The corresponding phase portrait is on the right.

and one local maximum (at U_a, see Fig. 5.7). There are real solutions for all values of the total energy β, and some of these are bounded, namely if $V(U_b; v, \alpha) \leq \beta \leq V(U_a; v, \alpha)$. This potential and its corresponding phase plane are illustrated in Fig. 5.7. There are two equilibrium solutions. One if located at U_a and is unstable, giving rise to a saddle point in the phase plane. The other one is at U_b and is stable. It is a center in the phase plane. The center is surrounded by periodic solutions. The saddle gives rise to a homoclinic connection, enclosing all periodic solutions. This homoclinic connection is the separatrix between the periodic solutions and the unbounded solutions. As discussed before, this is where we find the soliton solution. Thus the soliton solution is obtained for $\beta = V(U_a; v, \alpha)$.

Lastly, we show that by the use of a Galilean transformation, the value of u as $x \to \pm\infty$ may be chosen arbitrarily. Define new variables by

$$\hat{x} = x - ct, \quad \hat{t} = t, \quad u = \hat{u} + d.$$

Then

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \hat{x}}, \quad \frac{\partial}{\partial t} = -c\frac{\partial}{\partial \hat{x}} + \frac{\partial}{\partial \hat{t}}.$$

The KdV equation in the new coordinates becomes

$$4\left(-c\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{u}}{\partial \hat{t}}\right) = 6(\hat{u} + d)\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial^3 \hat{u}}{\partial \hat{x}^3}.$$

Thus the KdV equation is invariant under this Galilean transformation if d is chosen to be $d = -2c/3$. Then the equation remains the same, but the transformation may be used to

change the overall value of u: by choosing an appropriately translating Galilean frame, the value of u may always be chosen to be zero. Note that we have assumed here that the values of $u(x,t)$ as $x \to \infty$ and $u(x,t)$ as $x \to -\infty$ are the same. For the KdV equation this is indeed the case, as according to the phase plane analysis KdV solitons arise from homoclinic connections.

The Modified KdV equation, $p = 2$.

For the case of the modified KdV (MKdV) equation ($p = 2$) we find

$$V(U; v, \alpha) = \frac{1}{2}U^4 + 2vU^2 + \alpha U.$$

As before, the equilibrium solutions are given by the critical points of the potential:

$$\frac{\partial V}{\partial U} = 2U^3 + 4vU + \alpha = 0.$$

This is a cubic equation with real coefficients, thus there is always at least one real root. The possible scenarios are illustrated in Fig. 5.8, and correspond to one inflection point (a), one degenerate inflection point (b) and two inflection points (c), respectively.

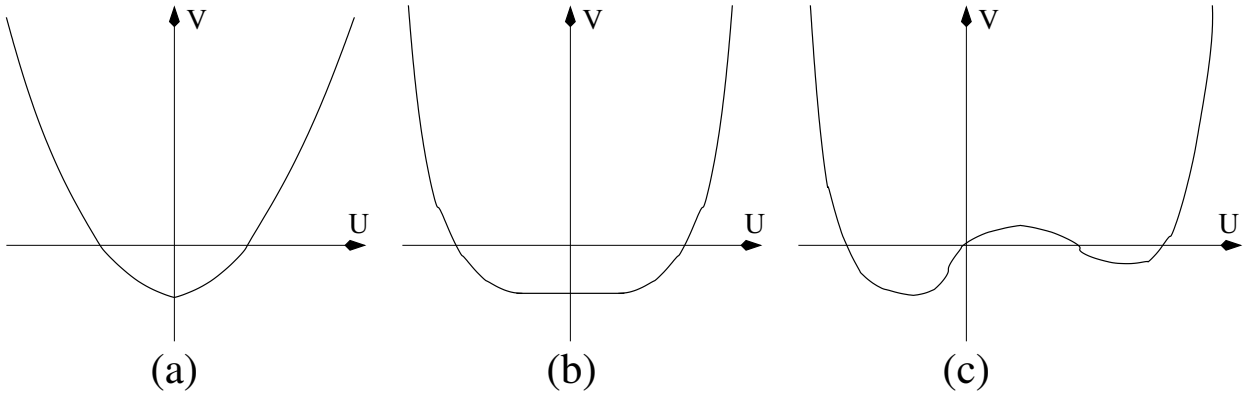


Figure 5.8: Possible potentials $V(U; v, \alpha)$ for the MKdV equation.

- **(a) One equilibrium point**

  This case is illustrated in Fig. 5.9. There is a single equilibrium point, at the global minimum of the potential well. There are no soliton solutions, as there are no local maxima. All real solutions are bounded and periodic. The equilibrium point in the phase plane is a center, surrounded by closed orbits.

- **(b) Two equilibrium points**

  This case is illustrated in Fig. 5.10. There is one equilibrium point, which is stable. It is at the global minimum of the potential well. There is a degenerate inflection point.

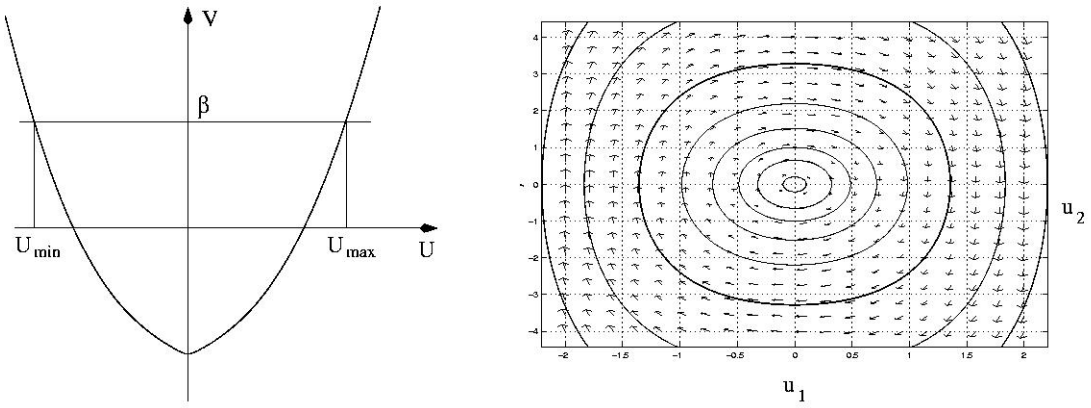


Figure 5.9: The potential $V(U; v, \alpha)$ (left) and phase plane (right) for stationary solutions of the MKdV equation, for case (a).

There are no soliton solutions. All real solutions are bounded and periodic. The stable equilibrium point in the phase plane is a center, surrounded by closed orbits.

- **(c) Three equilibrium points**

  This case is illustrated in Fig. 5.11. There are three equilibrium points. Two are at local minima of the potential well, the third one is at a local maximum, in between these two minima. Depending on the parameter values of α and V, this potential may be symmetric around the local maximum with equally deep wells on either side. Typically it is not symmetric, and the wells will have different depths. For case (c) there are two different soliton solutions. One (above the right well) is a soliton elevated above its background, whereas the other one (above the left well) is a soliton of depression in its background. If the wells are asymmetric, these solitons will have different amplitudes and widths. In the phase plane, the solitons arise as homoclinic connections, separating two distinct classes of periodic solutions.

5.1.3 Explicit integration: soliton solutions

To obtain the explicit functional form of the soliton solution we return to the energy equation:

$$\frac{1}{2}U'^2 + V(U; v, \alpha) = \beta,$$

which gives

$$U' = \pm\sqrt{2(\beta - V(U; v, \alpha))}.$$

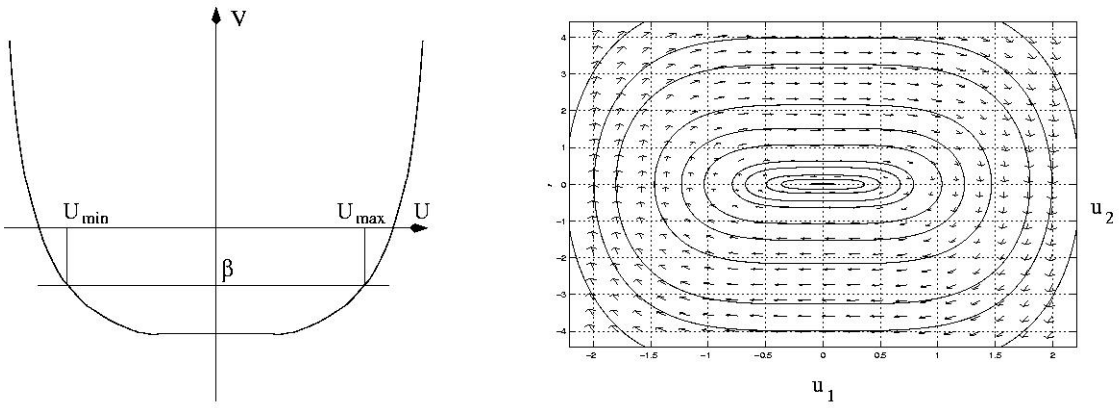


Figure 5.10: The potential $V(U; v, \alpha)$ (left) and phase plane (right) for stationary solutions of the MKdV equation, for case (b).

Separating variables gives an implicit solution

$$\int_{U_0}^{U} \frac{dU}{\sqrt{2(\beta - V(U; v, \alpha))}} = \pm \int_0^z dz = \pm z.$$

Written out, this implicit solution is

$$\pm z = \int_{U_0}^{U} \frac{dU}{\sqrt{2\left(\beta - \frac{6}{(p+1)(p+2)} U^{p+2} - 2vU^2 - \alpha U\right)}}.$$

If $p = 0$ (linear equation, quadratic potential) this results in trigonometric or hyperbolic solutions. This is expected for solutions of linear equations with constant coefficients. If $p = 1$ or $p = 2$ (KdV or MKdV), the integral on the right-hand side has a polynomial of degree 3 or 4 under the square root, resulting in elliptic functions. For any other value of $p \geq 3$, this integral results in hyperelliptic functions. Elliptic functions and their limits are meromorphic functions, whereas hyperelliptic functions are not typically meromorphic. This is a first hint as to why KdV and MKdV are special and the other ($p \geq 3$) equations are not.

Since we are specifically looking for soliton solutions that vanish at $x \to \pm\infty$, we restrict ourselves to putting $\alpha = 0$ and $\beta = 0$. This implies that the soliton solutions we will find have the same limits as $x \to \pm\infty$. The zero values of α and β are consequences of previous equations, expressing both α and β as differential polynomials of U. The implicit solution simplifies to

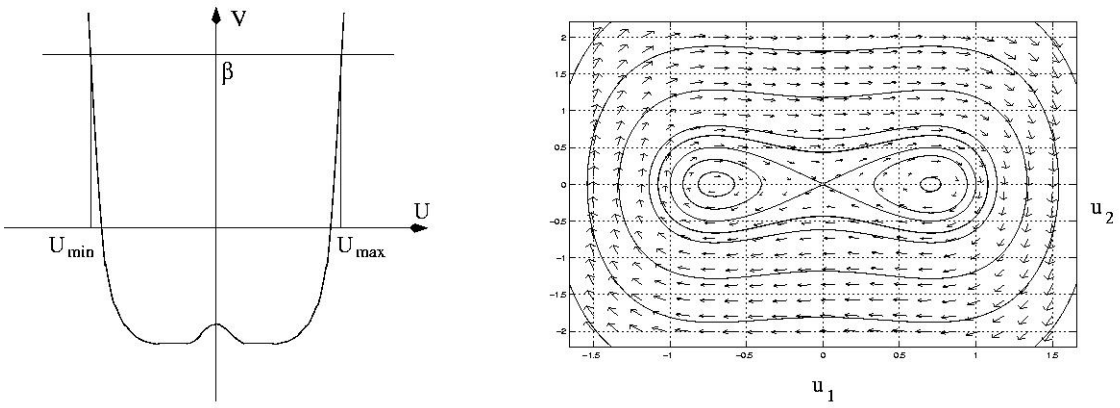


Figure 5.11: The potential $V(U; v, \alpha)$ (left) and phase plane (right) for stationary solutions of the MKdV equation, for case (c).

$$\pm z = \int_{U_0}^{U} \frac{dU}{\sqrt{2\left(-\frac{6}{(p+1)(p+2)}U^{p+2} - 2vU^2\right)}}.$$

As $z \to \pm\infty$, $U \to 0$. Thus, for a whole range of x values the second term under the square root dominates, requiring that v be negative. Thus we let

$$v = -a^2.$$

Thus

$$\pm z = \int \frac{dU}{U\sqrt{4a^2 - \frac{12}{(p+1)(p+2)}U^p}}.$$

This integral may be done by substituting

$$w = \sqrt{4a^2 - \frac{12}{(p+1)(p+2)}U^p} \quad \Rightarrow \quad w^2 = 4a^2 - \frac{12}{(p+1)(p+2)}U^p.$$

Then

$$2wdw = \frac{-12pU^{p-1}dU}{(p+1)(p+2)}.$$

The implicit solution becomes

$$\begin{aligned}
\pm z &= \int \frac{2wU^{1-p}\frac{(p+1)(p+2)}{-12p}dw}{Uw} \\
&= 2\int \frac{(p+1)(p+2)}{-12pU^p}dw \\
&= \frac{2}{p}\int \frac{1}{w^2-4a^2}dw \\
&= \frac{1}{2ap}\int\left(\frac{dw}{w-2a}-\frac{dw}{w+2a}\right) \\
&= \frac{1}{2ap}\ln\left(C\frac{w-2a}{w+2a}\right).
\end{aligned}$$

Here C is a constant of integration, which may be used to absorb the absolute value we would normally need inside the ln. Then

$$\begin{aligned}
C\frac{w-2a}{w+2a} &= e^{\pm 2apz} \\
\Rightarrow \frac{w-2a}{w+2a} &= \frac{1}{C}e^{\pm 2apz} \\
\Rightarrow w &= 2a\frac{1+\frac{1}{C}e^{\pm 2apz}}{1-\frac{1}{C}e^{\pm 2apz}}.
\end{aligned}$$

It follows that in order to have bounded solutions, we need $C<0$, otherwise the denominator will be zero for some real z. Set

$$C = -e^{\pm 2ap\alpha}.$$

Then

$$\begin{aligned}
w &= 2a\frac{1-e^{\pm(2apz-2ap\alpha)}}{1+e^{\pm(2apz-2ap\alpha)}} \\
&= 2a\frac{e^{\mp ap(z-\alpha)}-e^{\pm ap(z-\alpha)}}{e^{\mp ap(z-\alpha)}+e^{\pm ap(z-\alpha)}} \\
&= 2a\tanh \mp ap(z-\alpha) \\
&= \mp 2a\tanh ap(z-\alpha).
\end{aligned}$$

Solving for U, we obtain

$$
\begin{aligned}
w^2 &= 4a^2 \tanh^2 ap(z-\alpha) \\
\Rightarrow 4a^2 - \frac{12}{(p+1)(p+2)} U^p &= 4a^2 \tanh^2 ap(z-\alpha) \\
\Rightarrow \frac{12}{(p+1)(p+2)} U^p &= 4a^2 - 4a^2 \tanh^2 ap(z-\alpha) \\
\Rightarrow \frac{12}{(p+1)(p+2)} U^p &= 4a^2 \mathrm{sech}^2 ap(z-\alpha) \\
\Rightarrow U &= \left[\frac{(p+1)(p+2)}{3} a^2 \mathrm{sech}^2 ap(z-\alpha) \right]^{1/p}.
\end{aligned}
$$

In the original variables, we get

$$
\begin{aligned}
u(x,t) &= U(x-vt) \\
&= \left[\frac{(p+1)(p+2)}{3} a^2 \mathrm{sech}^2 ap(x-vt-\alpha) \right]^{1/p} \\
&= \left[\frac{(p+1)(p+2)}{3} a^2 \mathrm{sech}^2 ap(x+a^2t-\alpha) \right]^{1/p}.
\end{aligned}
$$

This is the solitary wave solution of the p-KdV equation. Some special cases are:

- $p = 1$ **(KdV)**: $u(x,t) = 2a^2 \,\mathrm{sech}^2 a(x + a^2 t + \alpha)$
- $p = 2$ **(MKdV)**: $u(x,t) = \pm 2a \,\mathrm{sech}\, 2a(x + a^2 t + \alpha)$

Both are meromorphic functions in the complex x-plane. The KdV soliton has double poles lined up on a vertical line in the complex x-plane. For the MKdV soliton, these poles are simple. For $p > 2$, there are branch points at each of the poles, thus the solitary wave solution is no longer meromorphic as a function of x. We will get back to this when we discuss the Painlevé property.

5.2 Similarity solutions

Most of the partial differential equations we are interested in have been derived from an application problem through the use of a nondimensionalization and a reductive perturbation analysis. The original variables in this set-up have physical dimensions. There is choice in the units used to measure these dimensions. For instance, length may be measured in meters, or in kilometers. Switching from kilometers to meters will introduce a scaling factor of 1000 in the equations. Clearly, choosing different units cannot affect the problem under considerations, which implies that the scaling factor has to dissapear from the equations

determining the solutions of the problem. This is one instance of a scaling symmetry that all of us are familiar with.

In general, an equation or a system of equations possesses a scaling symmetry if there is a scaling transformation on both the dependent and independent variables that leaves the equations invariant. It follows from the above observation that any system with physical units possesses a scaling symmetry. Going beyond this, it follows that every system derived from a system with dimensions through consistent nondimensionalization has a scaling symmetry. This scaling symmetry may be thought of as the remnant of the original dimensions.

In this section we will learn how to find solutions that are constructed from specific multiplicative combinations of independent and dependent variables that are scaling independent. This is somewhat reminiscent of the Buckingham Pi theorem, where physical laws are expressed in terms of a minimal number of dimensionless quantities. In terms of these scaling independent groups, we find a reduced differential equation that has to be satisfied. The equation is reduced in that it depends on fewer independent variables. The most important case is that of a partial differential equation with one spatial variable, which reduces to an ordinary differential equation.

5.2.1 Scaling symmetries

An equation

$$u_t = N(u, u_x, u_{xx}, \ldots, x, t)$$

possesses a scaling symmetry if it is invariant under the scaling transformation

$$x = \frac{\hat{x}}{a}, \quad t = \frac{\hat{t}}{b}, \quad u = c\hat{u},$$

where a, b and c are scalars, at least one of which can be chosen arbitrarily. The equation is invariant if the new variables $\hat{x}$, $\hat{t}$, and $\hat{u}$ satisfy the same equation as the original variables. Thus

$$\hat{u}_{\hat{t}} = N(\hat{u}, \hat{u}_{\hat{x}}, \hat{u}_{\hat{x}\hat{x}}, \ldots, \hat{x}, \hat{t}).$$

Example. Let us consider the nonlinear Schrödinger equation (NLS)

$$iq_t = -q_{xx} + \sigma|q|^2 q.$$

It has the scaling symmetry

$$x = \frac{\hat{x}}{a}, \quad t = \frac{\hat{t}}{a^2}, \quad q = a\hat{q}.$$

which may be verified by direct substitution. How does one find such a scaling symmetry? Returning to the definition of a scaling symmetry, we have that

$$x = \frac{\hat{x}}{a}, \quad t = \frac{\hat{t}}{b}, \quad q = c\hat{q},$$

which implies that

$$\frac{\partial}{\partial x} = a\frac{\partial}{\partial \hat{x}}, \quad \frac{\partial}{\partial t} = b\frac{\partial}{\partial \hat{t}}.$$

This implies that the transformed equation becomes

$$\begin{aligned} ibc\hat{q}_{\hat{t}} &= -ca^2\hat{q}_{\hat{x}\hat{x}} + \sigma c^3|\hat{q}|^2\hat{q} \\ \Rightarrow \qquad i\hat{q}_{\hat{t}} &= -\frac{a^2}{b}\hat{q}_{\hat{x}\hat{x}} + \sigma\frac{c^2}{b}|\hat{q}|^2\hat{q}, \end{aligned}$$

from which it follows by requiring invariance that

$$b = a^2,\ c^2 = b \quad \Rightarrow \quad b = a^2,\ c = a,$$

resulting in the scaling symmetry given before.

5.2.2 Scaling reductions

The scaling symmetries of differential equations may be used to construct special solutions, referred to as **similarity solutions**. These similarity solutions satisfy reduced differential equations, depending on fewer independent variables.

Example. Let us again consider the NLS equation, and see how a simpler equation, satisfied by the similarity solution is obtained. From the scaling symmetry, is is clear that

$$z = xt^{-1/2}$$

and

$$t^{1/2}q$$

are both scaling invariant. We impose that these two scaling invariant groups are functions of each other:

$$t^{1/2}q(x,t) = F(z) \quad \Rightarrow \quad q(x,t) = t^{-1/2}F(z).$$

This is an ansatz for a similarity solution. Substitution of this ansatz in the partial equation will result in an ordinary differential equation for the as yet unknown function $F(z)$. Let us proceed to determine this differential equation. We need:

$$\begin{aligned} q_t &= -\frac{1}{2}t^{-3/2}F(z) + t^{-1/2}F'(z)z_t = -\frac{1}{2}t^{-3/2}F(z) - \frac{1}{2}t^{-2}xF'(z), \\ q_x &= t^{-1/2}F'(z)z_x = t^{-1}F'(z), \\ q_{xx} &= t^{-3/2}F''(z). \end{aligned}$$

This gives

$$\begin{aligned} i\left(-\frac{1}{2}t^{-3/2}F(z) - \frac{1}{2}t^{-2}xF'(z)\right) &= -t^{-3/2}F''(z) + \sigma t^{-3/2}|F(z)|^2F(z) \\ \Rightarrow \qquad i\left(-\frac{1}{2}F(z) - \frac{1}{2}t^{-1/2}xF'(z)\right) &= -F''(z) + \sigma|F(z)|^2F(z) \\ \Rightarrow \qquad i\left(-\frac{1}{2}F(z) - \frac{1}{2}zF'(z)\right) &= -F''(z) + \sigma|F(z)|^2F(z), \end{aligned}$$

which is an ordinary differential equation, to be solved (in principle) for $F(z)$. It can be rewritten as

$$\left(F' - \frac{i}{2}zF\right)' = \sigma|F|^2F.$$

Note that it is possible to construct other scaling invariant combinations of the original independent and dependent variables. You should convince yourself that the choices we made are not restrictive in any way. All other choices result in the same ansatz we made, perhaps after a redefinition of the function $F(z)$.

The next step would be to solve the ordinary differential equation we found, if possible. As the case is here, this is not easily done. However, one should realize that progress has been made, as we have reduced the original partial differential equation to an ordinary differential equation. From this equation many properties of the corresponding solutions may be deduced.

Example. As a second example, consider the KdV equation

$$u_t + uu_x + u_{xxx} = 0.$$

Since the calculations proceed similarly to the previous example, we only give the results.

- First, we find the scaling symmetry to be

$$x = \frac{\hat{x}}{a}, \quad t = \frac{\hat{t}}{a^3}, \quad u = a^2\hat{u}.$$

- Next, we determine the similarity ansatz to be

$$u = t^{-2/3}F(z), \quad \text{with} \quad z = xt^{-1/3}.$$

- Lastly, substituting this ansatz in the KdV equation results in the ordinary differential equation

$$-\frac{2}{3}F - \frac{1}{3}zF' + FF' + F''' = 0.$$

As before, we have constructed an ordinary differential equation whose solution specifies the similarity solution.

A special note is in order about similarity solutions of linear equations, or for that matter, for equations that are homogeneous in any of their dependent or independent variables. For the sake of explicitness, let us look at linear equations. When trying to determine the scaling symmetry, one finds that the scaling of the dependent variable is undetermined, as it appears once in every term of the equation. Therefore its scaling factors out. As a consequence, a two-parameter family of scaling symmetries is found (with an undetermined scaling parameter,

and an undetermined exponent for the scaling of the dependent variable), instead of a one-parameter scaling symmetry (with merely an undetermined scaling parameter). Let's see this in an example.

Example. Consider the linear Schrödinger equation

$$iq_t = -q_{xx}.$$

- It has a scaling symmetry

$$x = \frac{\hat{x}}{a}, \quad t = \frac{\hat{t}}{a^2}, \quad q = a^c \hat{q},$$

  for any value of c.

- Next, we determine the similarity ansatz to be

$$q = t^{-c/2} F(z), \quad \text{with} \quad z = xt^{-1/2}.$$

- Lastly, substituting this ansatz in the linear Schrödinger equation results in the ordinary differential equation

$$F'' - \frac{i}{2}(zF' + cF) = 0.$$

- This equation is linear and can be solved for any c in terms of Whittaker functions. It suffices for us to note that

$$F = e^{iz^2/4}$$

  is a solution for $c = 1$ (check!), so that

$$q(x,t) = \frac{1}{t^{1/2}} e^{-ix^2/4t}$$

  is a solution of the linear Schrödinger equation.

This example illustrates an important fact about similarity solutions: they are often outside of the function space in which we might want to solve the problem. A natural function space for this linear would be $L^2(\mathbb{R})$, the space of square integrable functions on the whole line. Note that the explicit solution found in the last example is not square integrable, although it is bounded for all real $t > 0$. As a consequence, it is a solution we would not have found using the standard technique of separation of variables.

5.3 Symmetry solutions of differential equations

The traveling-wave or stationary solutions and the similarity solutions are two of the simplest examples of classes of symmetry solutions. Before we talk about symmetry solutions in general, we need to discuss symmetries of differential equations in some detail.

Definition 1 *Consider the equation $u_t = N(x, u, u_x, \ldots)$. The transformation*

$$\begin{cases} x &= x(\hat{x}, \hat{t}, \hat{u}) \\ t &= t(\hat{x}, \hat{t}, \hat{u}) \\ u &= u(\hat{x}, \hat{t}, \hat{u}) \end{cases}$$

is a symmetry of this equation if it leaves the equation invariant. In other words, the new dependent variable $\hat{u}$ satisfies the equation $\hat{u}_{\hat{t}} = N(\hat{x}, \hat{u}, \hat{u}_{\hat{x}}, \ldots)$.

This definition is given specifically for the case of a scalar partial differential equation. It generalizes in an obvious way to systems of equations, with more dependent or independent variables. Alternatively, we can also state it for ordinary differential equations and systems of such.

Example. Let

$$\frac{dx}{dt} = f(x),$$

an autonomous ordinary differential equation. The transformation

$$\begin{cases} x &= \hat{x} \\ t &= \hat{t} + \epsilon \end{cases},$$

where $\epsilon \in \mathbb{R}$ is a free parameter, is a symmetry of the equation. Showing this is an easy calculation:

$$\frac{dx}{dt} = \frac{d\hat{x}}{d\hat{t}} = f(\hat{x}).$$

Example. Consider the system of ordinary differential equations

$$\begin{cases} \dfrac{dx}{dt} &= x^2 + y^2 \\ \dfrac{dy}{dt} &= 2xy \end{cases}.$$

It has the symmetry

$$\begin{cases} x &= \hat{x} \\ y &= -\hat{y} \\ t &= \hat{t} \end{cases}.$$

This is also an almost trivial calculation.

There is a fundamental difference between the symmetries in these two examples. The symmetry for the first example depends on an arbitrary real parameter. By letting this parameter be zero, our symmetry becomes the identity transformation. Thus it is correct to say that the symmetry of the first example is a continuous deformation of the identity. Such a statement is not possible for the symmetry for the second example: it contains no free parameter, and it is fundamentally different from the identity transformation. Because

of this difference, the first symmetry is referred to as a *continuous* symmetry, whereas the second symmetry is called a *discrete* symmetry. Continuous symmetries are also called *Lie symmetries*. Lie symmetries are members of one-or more parameter families of symmetries, whereas discrete symmetries are isolated. Lie symmetries are the only symmetries we will consider in the remainder of this section.

Remark. The Lie symmetries we discuss in this chapter are so-called Lie point symmetries, because the transformation from (u, x, t) to $(\hat{u}, \hat{x}, \hat{t})$ does not depend on any of the the derivatives of u at the point (x, t), but only on the values of u at (x, t). The more general symmetries where the transformation does depend on derivatives are called nonclassical.

5.3.1 Transformation groups and their generators

Consider a transformation which depends on one continuous parameter ϵ

$$\left\{ \begin{array}{rcl} x & = & x(\hat{x}, \hat{t}, \hat{u}, \epsilon) \\ t & = & t(\hat{x}, \hat{t}, \hat{u}, \epsilon) \\ u & = & u(\hat{x}, \hat{t}, \hat{u}, \epsilon) \end{array} \right. .$$

We assume that this transformation is analytic with respect to ϵ. If we think of ϵ as being small, then

$$\left\{ \begin{array}{rcl} x & = & x(\hat{x}, \hat{t}, \hat{u}, 0) + \epsilon \dfrac{\partial x}{\partial \epsilon}(\hat{x}, \hat{t}, \hat{u}, 0) + \mathcal{O}(\epsilon^2) \\ t & = & t(\hat{x}, \hat{t}, \hat{u}, 0) + \epsilon \dfrac{\partial t}{\partial \epsilon}(\hat{x}, \hat{t}, \hat{u}, 0) + \mathcal{O}(\epsilon^2) \\ u & = & u(\hat{x}, \hat{t}, \hat{u}, 0) + \epsilon \dfrac{\partial u}{\partial \epsilon}(\hat{x}, \hat{t}, \hat{u}, 0) + \mathcal{O}(\epsilon^2) \end{array} \right. .$$

Possibly by redefining the parameter, we can always arrange it so that the transformation is equal to the identity transformation when $\epsilon = 0$, so that we have

$$\left\{ \begin{array}{rcl} x & = & \hat{x} + \epsilon X(\hat{x}, \hat{t}, \hat{u}) + \mathcal{O}(\epsilon^2) \\ t & = & \hat{t} + \epsilon T(\hat{x}, \hat{t}, \hat{u}) + \mathcal{O}(\epsilon^2) \\ u & = & \hat{u} + \epsilon U(\hat{x}, \hat{t}, \hat{u}) + \mathcal{O}(\epsilon^2) \end{array} \right. ,$$

where we have defined

$$\left\{ \begin{array}{rcl} X(\hat{x}, \hat{t}, \hat{u}) & = & \dfrac{\partial x}{\partial \epsilon}(\hat{x}, \hat{t}, \hat{u}, 0) \\ T(\hat{x}, \hat{t}, \hat{u}) & = & \dfrac{\partial t}{\partial \epsilon}(\hat{x}, \hat{t}, \hat{u}, 0) \\ U(\hat{x}, \hat{t}, \hat{u}) & = & \dfrac{\partial u}{\partial \epsilon}(\hat{x}, \hat{t}, \hat{u}, 0) \end{array} \right. .$$

The functions X, T and U are called the *generators* of the transformation group.

Claim. The ϵ-dependent transformation $(x,t,u) \to (\hat{x},\hat{t},\hat{u})$ may be recovered from its generators, for ϵ in some interval containing zero.

Proof. The components of the transformation satisfy the system of ordinary differential equations

$$\left\{ \begin{array}{rcl} \dfrac{\partial x}{\partial \epsilon} & = & X(x,t,u) \\ \dfrac{\partial t}{\partial \epsilon} & = & T(x,t,u) \\ \dfrac{\partial u}{\partial \epsilon} & = & U(x,t,u) \end{array} \right. ,$$

with "initial conditions" $x(\epsilon = 0) = \hat{x}$, $t(\epsilon = 0) = \hat{t}$, and $u(\epsilon = 0) = \hat{u}$. The standard existence theorem for ordinary differential equations (ϵ is the only independent variable in these equations) now guarantees the existence of a solution for some interval of ϵ values, around these initial conditions. ■

Before we move on to an example, it should be noted that the near-identity transformation $(x,t,u) \to (\hat{x},\hat{t},\hat{u})$ may be formally inverted to give

$$\left\{ \begin{array}{rcl} \hat{x} & = & x + \epsilon X(x,t,u) + \mathcal{O}(\epsilon^2) \\ \hat{t} & = & t + \epsilon T(x,t,u) + \mathcal{O}(\epsilon^2) \\ \hat{u} & = & u + \epsilon U(x,t,u) + \mathcal{O}(\epsilon^2) \end{array} \right. .$$

Example. Consider the one-parameter family of (linear) transformations

$$\left\{ \begin{array}{rcl} x & = & \hat{x}\cos\epsilon + \hat{t}\sin\epsilon \\ t & = & -\hat{x}\sin\epsilon + \hat{t}\cos\epsilon \\ u & = & \hat{u} \end{array} \right. .$$

Note that $\epsilon = 0$ does indeed give rise to the identity transformation. The generators of this transformation may be found by Taylor expanding around $\epsilon = 0$:

$$\left\{ \begin{array}{rcl} x & = & \hat{x} + \epsilon\hat{t} + \mathcal{O}(\epsilon^2) \\ t & = & \hat{t} - \epsilon\hat{x} + \mathcal{O}(\epsilon^2) \\ u & = & \hat{u} \end{array} \right. ,$$

from which it follows that the generators are

$$X = \hat{t}, \quad T = -\hat{x}, \quad U = 0.$$

We could have also found this by taking a derivative of the transformation with respect to ϵ, and equating ϵ to zero.

The transformation is recovered from its generators by solving the differential equations

$$\begin{cases} \dfrac{\partial x}{\partial \epsilon} &= t \\ \dfrac{\partial t}{\partial \epsilon} &= -x \\ \dfrac{\partial u}{\partial \epsilon} &= 0 \\ x(0) &= \hat{x},\ t(0) = \hat{t},\ u(0) = \hat{u}. \end{cases}$$

It is an easy check to see that the original transformation is indeed the only solution of this system.

In general, the transformation determines a Lie group of symmetries of the underlying differential equation, whereas the generators determine its associated Lie algebra. The process of recovering the group from its algebra is a standard formal exponentiation.

5.3.2 Finding symmetries

In this section we illustrate the algorithmic way of finding symmetries by considering two examples. From these two examples, it should be clear how to proceed in general. One aspect learned from these examples is that the calculations to determine the Lie symmetries are tedious, but algorithmic. Many software packages exist to assist with these calculations, and some have automated them completely.

Example. Our first example is that of the **Emden-Fowler equation**

$$x^2 \frac{d^2u}{dx^2} + 2x\frac{du}{dx} + x^2u^5 = 0.$$

This is a nonlinear ordinary differential equation. Because of this, our set-up from the previous section may be downscaled a little. We look for symmetries of the form

$$\begin{cases} \hat{x} &= x + \epsilon X(x,u) + \mathcal{O}(\epsilon^2), \\ \hat{u} &= u + \epsilon U(x,u) + \mathcal{O}(\epsilon^2). \end{cases}$$

Imposing that the hatted variables $\hat{x}$ and $\hat{u}$ satisfy the same Emden-Fowler equation, we have

$$\hat{x}^2 \frac{d^2\hat{u}}{d\hat{x}^2} + 2\hat{x}\frac{d\hat{u}}{d\hat{x}} + \hat{x}^2\hat{u}^5 = 0.$$

We now rewrite this equation using our near-identity transformation. To this end we need

to compute both $d\hat{u}/d\hat{x}$ and $d^2\hat{u}/d\hat{x}^2$. We have

$$\begin{aligned}
\frac{d\hat{u}}{d\hat{x}} &= \frac{d\hat{u}/dx}{d\hat{x}/dx} \\
&= \frac{du/dx + \epsilon dU/dx}{1 + \epsilon dX/dx} + \mathcal{O}(\epsilon^2) \\
&= \left(\frac{du}{dx} + \epsilon\frac{dU}{dx}\right)\left(1 - \epsilon\frac{dX}{dx}\right) + \mathcal{O}(\epsilon^2) \\
&= \frac{du}{dx} + \epsilon\left(\frac{dU}{dx} - \frac{du}{dx}\frac{dX}{dx}\right) + \mathcal{O}(\epsilon^2) \\
&= u' + \epsilon\left(\frac{\partial U}{\partial x} + u'\frac{\partial U}{\partial u} - u'\left(\frac{\partial X}{\partial x} + u'\frac{\partial X}{\partial u}\right)\right) + \mathcal{O}(\epsilon^2) \\
&= u' + \epsilon\left(U_x + u'U_u - u'X_x - u'^2X_u\right) + \mathcal{O}(\epsilon^2),
\end{aligned}$$

where we have used primes to denote derivatives of u with respect to x, and indices to denote partial derivatives of X and U with respect to x and u. Similarly,

$$\begin{aligned}
\frac{d^2\hat{u}}{d\hat{x}^2} &= \frac{d\frac{d\hat{u}}{d\hat{x}}/dx}{d\hat{x}/dx} \\
&= \frac{u'' + \epsilon\frac{d}{dx}\left(U_x + u'U_u - u'X_x - u'^2X_u\right)}{1 + \epsilon dX/dx} + \mathcal{O}(\epsilon^2) \\
&= \big(u'' + \epsilon\left(U_{xx} + 2u'U_{xu} + u''U_u + u'^2U_{uu} - u''X_x - u'X_{xx}\right. \\
&\qquad \left.-2u'^2X_{xu} - 2u'u''X_u - u'^3X_{uu}\right)\big)\left(1 - \epsilon\frac{dX}{dx}\right) + \mathcal{O}(\epsilon^2) \\
&= u'' + \epsilon\left(U_{xx} + 2u'U_{xu} + u''U_u + u'^2U_{uu} - 2u''X_x - u'X_{xx}\right. \\
&\qquad \left.-2u'^2X_{xu} - 3u'u''X_u - u'^3X_{uu}\right) + \mathcal{O}(\epsilon^2).
\end{aligned}$$

These two expressions and the near-identity transformation are substituted in

$$\hat{x}^2\frac{d^2\hat{u}}{d\hat{x}^2} + 2\hat{x}\frac{d\hat{u}}{d\hat{x}} + \hat{x}^2\hat{u}^5 = 0,$$

and the coefficients of different powers of ϵ are equated to zero. At order zero in ϵ we recover the Emden-Fowler equation expressed using the independent variable x and the dependent variable u. This is no surprise. To coefficient of the first-order term is

$$\begin{aligned}
u''\left(2xX + x^2U_u - 2x^2X_x - 3u'x^2X_u\right) \\
+u'^3\left(-x^2X_{uu}\right) \\
+u'^2\left(x^2U_{uu} - 2x^2X_{xu} - 2xX_u\right) \\
+u'\left(2x^2U_{xu} - x^2X_{xx} + 2xU_u - 2xX_x + 2X\right) \\
+x^2U_{xx} + 2xU_x + 2xu^5X + 5x^2u^4U = 0.
\end{aligned}$$

We have conveniently sorted the terms by their dependence on the derivatives of $u(x)$. First note that the quantity u'' appears in this equation. It may be eliminated by using the Emden-Fowler equation: $u'' = (-2u' - xu^5)/x$. This gives

$$\begin{aligned} u'^3\left(-x^2X_{uu}\right) \\ +u'^2\left(x^2U_{uu} - 2x^2X_{xu} + 4xX_u\right) \\ +u'\left(2x^2U_{xu} - x^2X_{xx} + 2xX_x - 2X + 3x^2u^5X_u\right) \\ +x^2U_{xx} + 2xU_x + 5x^2u^4U - x^2u^5U_u + 2x^2u^5X_x = 0. \end{aligned}$$

Since X and U are functions of x and u only, but not of u', we may interpret the above equation as a third-order polynomial in u', which has to vanish identically. As a consequence, it's coefficients have to vanish:

$$\begin{cases} -x^2X_{uu} &= 0 \\ x^2U_{uu} - 2x^2X_{xu} + 4xX_u &= 0 \\ 2x^2U_{xu} - x^2X_{xx} + 2xX_x - 2X + 3x^2u^5X_u &= 0 \\ x^2U_{xx} + 2xU_x + 5x^2u^4U - x^2u^5U_u + 2x^2u^5X_x &= 0 \end{cases}.$$

The main idea is to solve these equations one-by-one, breaking down the u- and x-dependence of X and U as we go. For instance, from the first equation

$$X = A(x) + uB(x).$$

This solves the first equation, leaving us with the last three. Substituting this result in the second equation results in

$$U = u^2B_x - \frac{2u^2}{x}B + C(x)u + D(x).$$

At this point we have completely determined the u-dependence of both X and U. This means that upon substitution in the remaining equations we may also equate powers of u, leaving us to determine the x-dependence. The third equation separates in the following equations:

$$\begin{aligned} u^5: &\quad B = 0, \\ u^1: &\quad 4x^2B_{xx} - 8x^2\left(\frac{B}{x}\right)_x - x^2B_{xx} + 2xB_x - 2B = 0, \\ u^0: &\quad 2x^2C_x - x^2A_{xx} + 2xA_x - 2A = 0. \end{aligned}$$

From the first of these equations,

$$B = 0,$$

and the second equation is satisfied. Turning to the fourth of the original equations, it separates in four equations:

$$\begin{aligned} u^5: &\quad 4x^2C + 2x^2A_x = 0, \\ u^4: &\quad D = 0, \\ u^1: &\quad x^2C_{xx} + 2xC_x = 0, \\ u^0: &\quad x^2D_{xx} + 2xD_x = 0. \end{aligned}$$

From the second equation, the fourth one is automatically satisfied. The first equation gives

$$C = -\frac{1}{2}A_x.$$

The remaining two equations may be written as equations for A only. They are both Euler equations. Their solution is

$$A = x,$$

from which it follows that

$$U = -\frac{u}{2}.$$

This last step should make it clear that not every equation has symmetries: the conditions to be satisfied for symmetries to exist are typically overdetermined and some cancelation has to occur for these overdetermined equations to have a solution, as happened here.

We are now in a position to state the infinitesimal symmetry transformation of the Emden-Fowler equation:

$$\hat{x} = x(1+\epsilon) + \mathcal{O}(\epsilon^2), \quad \hat{u} = u\left(1 - \frac{1}{2}\epsilon\right) + \mathcal{O}(\epsilon^2).$$

Its full form (obtained by solving the ϵ differential equations) is

$$\hat{x} = xe^{\epsilon}, \quad \hat{u} = ue^{-\epsilon/2}.$$

We see that this is nothing but a regular scaling symmetry. Using the results from the previous section, we could use it to obtain a similarity solution of the Emden-Fowler equation.

Example. As a second example, we find the Lie-point symmetries of the **KdV equation**

$$u_t = uu_x + u_{xxx}.$$

We look for a symmetry transformation of the form

$$\begin{cases} \hat{x} &= x + \epsilon X(x,t,u) + \mathcal{O}(\epsilon^2), \\ \hat{t} &= t + \epsilon T(x,t,u) + \mathcal{O}(\epsilon^2), \\ \hat{u} &= u + \epsilon U(x,t,u) + \mathcal{O}(\epsilon^2). \end{cases}$$

Imposing that the hatted variables $\hat{x}$, $\hat{t}$ and $\hat{u}$ satisfy the same KdV equation, we have

$$\hat{u}_{\hat{t}} = \hat{u}\hat{u}_{\hat{x}} + \hat{u}_{\hat{x}\hat{x}\hat{x}}.$$

We now rewrite this equation using our near-identity transformation. To this end we need to compute $\hat{u}_{\hat{x}}$, $\hat{u}_{\hat{t}}$ and $\hat{u}_{\hat{x}\hat{x}\hat{x}}$. To this end, we check how the derivatives of a function $\hat{f}(\hat{x},\hat{t})$ transform. Suppose that $\hat{f}(\hat{x},\hat{t}) = f(x,t) + \epsilon F(x,t) + \mathcal{O}(\epsilon^2)$. We have

$$\frac{\partial \hat{f}}{\partial x} = \frac{\partial \hat{f}}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial x} + \frac{\partial \hat{f}}{\partial \hat{t}}\frac{\partial \hat{t}}{\partial x},$$
$$\frac{\partial \hat{f}}{\partial t} = \frac{\partial \hat{f}}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial t} + \frac{\partial \hat{f}}{\partial \hat{t}}\frac{\partial \hat{t}}{\partial t},$$

from which it follows by Kramer's rule that

$$\frac{\partial \hat{f}}{\partial \hat{x}} = \frac{\dfrac{\partial \hat{f}}{\partial x}\dfrac{\partial \hat{t}}{\partial t} - \dfrac{\partial \hat{f}}{\partial t}\dfrac{\partial \hat{t}}{\partial x}}{\dfrac{\partial \hat{x}}{\partial x}\dfrac{\partial \hat{t}}{\partial t} - \dfrac{\partial \hat{x}}{\partial x}\dfrac{\partial \hat{t}}{\partial t}},$$

and a similar formula for $\partial \hat{u}/\partial \hat{t}$. Expanding this results in

$$\begin{aligned}
\frac{\partial \hat{f}}{\partial \hat{x}} &= \frac{(f_x + \epsilon F_x)(1 + \epsilon T_t) - (f_t + \epsilon F_t)(\epsilon T_x)}{(1 + \epsilon X_x)(1 + \epsilon T_t) - \epsilon T_x \epsilon X_t} \\
&= \frac{f_x + \epsilon F_x + \epsilon f_x T_t - \epsilon f_t T_x}{1 + \epsilon X_x + \epsilon T_t} + \mathcal{O}(\epsilon^2) \\
&= (f_x + \epsilon F_x + \epsilon f_x T_t - \epsilon f_t T_x)(1 - \epsilon X_x - \epsilon T_t) + \mathcal{O}(\epsilon^2) \\
&= \hat{f}_x - \epsilon(f_t T_x + f_x X_x) + \mathcal{O}(\epsilon^2).
\end{aligned}$$

During the above calculation we have assumed that f, X and T depend on x and t only explicitly, and not through any dependence on u and its derivatives. If such is the case, the above formula needs to be modified as

$$\hat{f}_{\hat{x}} = D_x \hat{f} - \epsilon(D_t f D_x T + D_x f D_x X) + \mathcal{O}(\epsilon^2),$$

where

$$\begin{aligned}
D_x &= \partial_x + u_x \partial_u + u_{xx} \partial_{u_x} + u_{xxx} \partial_{u_{xx}} + \dots, \\
D_t &= \partial_t + u_t \partial_u + u_{tt} \partial_{u_t} + u_{ttt} \partial_{u_{tt}} + \dots,
\end{aligned}$$

and u, u_x, u_t, *etc.* are all regarded as independent variables.

Applying this formula to $\hat{f} = \hat{u} = u + \epsilon U(x,t,u) + \mathcal{O}(\epsilon^2)$ gives

$$\begin{aligned}
\hat{u}_{\hat{x}} &= D_x \hat{u} - \epsilon(D_t u D_x T + D_x u D_x X) + \mathcal{O}(\epsilon^2) \\
&= D_x(u + \epsilon U(x,t,u)) - \epsilon(D_t u D_x T + D_x u D_x X) + \mathcal{O}(\epsilon^2) \\
&= u_x + \epsilon(U_x + u_x U_u) - \epsilon(u_t(T_x + u_x T_u) + u_x(X_x + u_x X_u)) + \mathcal{O}(\epsilon^2) \\
&= u_x + \epsilon\left(U_x + (U_u - X_x)u_x - T_x u_t - X_u u_x^2 - T_u u_x u_t\right) + \mathcal{O}(\epsilon^2).
\end{aligned}$$

We may now apply the same formula to find $\hat{u}_{\hat{x}\hat{x}}$ using $\hat{f} = \hat{u}_{\hat{x}}$, having determined $f = u_x$ and $F = U_x + (U_u - X_x)u_x - T_x u_t - X_u u_x^2 - T_u u_x u_t$ above. Note that F explicitly depends on u_x and u_t, making the use of the derivative rules a little more complicated. Once this is done, we may iterate again to find $\hat{u}_{\hat{x}\hat{x}\hat{x}}$. We could continue this, but no more x-derivatives are needed for the treatment of the KdV equation. As you can and should check, we find

$$\begin{aligned}
\hat{u}_{\hat{x}\hat{x}} =& u_{xx} + \epsilon(U_{xx} + (2U_{xu} - X_{xx})u_x - T_{xx}u_t + (U_{uu} - 2X_{xu})u_x^2 \\
&- 2T_{xu}u_x u_t - X_{uu}u_x^3 - T_{uu}u_x^2 u_t + (U_u - 2X_x)u_{xx} \\
&- 2T_x u_{xt} - 3X_u u_x u_{xx} - T_u u_{xx} u_t - 2T_u u_x u_{xt}) + \mathcal{O}(\epsilon^2),
\end{aligned}$$

and

$$\begin{aligned}\hat{u}_{\hat{x}\hat{x}\hat{x}} =& u_{xxx} + \epsilon(U_{xxx} + (3U_{xxu} - X_{xxx})u_x - T_{xxx}u_t + (3U_{xuu} - 3X_{xxu})u_x^2 \\ & - 3T_{xxu}u_xu_t + (-3X_{xuu} + U_{uuu})u_x^3 - 3T_{xuu}u_x^2u_t - X_{uuu}u_x^4 \\ & + (3U_{xu} - 3X_{xx})u_{xx} - T_{xxx}u_x^3u_t - 3T_{xx}u_{xt} + (2U_{uu} - 7X_{xu})u_xu_{xx} \\ & - 6X_{uu}u_x^2u_{xx} - 3T_{xu}u_{xx}u_t - 6T_{xu}u_xu_{xt} - 3T_{uu}u_x^2u_{xt} - 3T_{uu}u_{xx}u_xu_t \\ & - 3X_uu_{xx}^2 - 3T_uu_{xx}u_{xt} + (U_u - 3X_x)u_{xxx} - 4X_uu_xu_{xxx} - T_uu_{xxx}u_t \\ & - 3T_xu_{xxt} - 3T_uu_xu_{xxt}) + \mathcal{O}(\epsilon^2).\end{aligned}$$

Using the similar formula for $\hat{f}_{\hat{t}}$, we obtain

$$\hat{u}_{\hat{t}} = u_t + \epsilon(U_t - X_tu_x + (U_u - T_t)u_t - X_uu_xu_t - T_uu_t^2) + \mathcal{O}(\epsilon^2).$$

The collection of these results allows us to express the KdV equation in the hatted variables. Doing this results in the KdV equation in terms of the original variables at order ϵ^0, as expected. The first-order condition determines the infinitesimal symmetries. This first-order condition depends on x, t and u through its dependence on $X(x,t,u)$, $T(x,t,u)$ and $U(x,t,u)$. However, it also explicitly depends on u_x, u_t, u_{xx}, u_{xt}, u_{xxx}, u_{xxt} and u_{5x}. As a first step, the dependence on u_t, u_{xt} and u_{xxt} is eliminated by letting

$$\begin{aligned}u_t &= uu_x + u_{xxx},\\ u_{xt} &= u_x^2 + uu_{xx} + u_{xxxx},\\ u_{xxt} &= 3u_xu_{xx} + uu_{xxx} + u_{5x}.\end{aligned}$$

Having eliminated all t derivatives of u and its x derivatives, we obtain an equation that explicitly depends on the x derivatives of u, up to order 5. Moreover, this explicit dependence is polynomial. This implies that the coefficients of all different powers of these derivatives must vanish, resulting in many differential equations to be satisfied by $X(x,t,u)$, $T(x,t,u)$ and $U(x,t,u)$.

Equating the coefficients of u_{5x} to zero, we find two conditions:

$$\begin{aligned}T_x &= 0,\\ t_u &= 0,\end{aligned}$$

from which it follows that

$$T = T(t),$$

without any dependence on x or u. The remaining condition is

$$\begin{aligned}-X_tu_x + (U_u - T_t)(uu_x + u_{xxx}) - X_uu_x(uu_x + u_{xxx}) + U_t =\\ U_{ux} + uU_x + uu_xU_u - uu_xX_x - uu_x^2X_u + U_{xxx} +\\ (3U_{xxu} - X_{xxx})u_x + (3U_{xuu} - 3X_{uxx})u_x^2 + (-3X_{xuu} + U_{uuu})u_x^3 -\\ X_{uuu}u_x^4 + (3U_{ux} - 3X_{xx})u_{xx} + (2U_{uu} - 7X_{xu})u_xu_{xx} - 6X_{uu}u_x^2u_{xx} -\\ 3X_uu_{xx}^2 + (U_u - 3X_x)u_{xxx} - 4X_uu_xu_{xxx}.\end{aligned}$$

Now we equate coefficients of u_{xxx}, to find that

$$\begin{aligned} X_u &= 0, \\ T_t &= 3X_x, \end{aligned}$$

from which it follows that X only depends on x and t, and since T is a function of t only, we obtain that X is linear in x:

$$X = a(t) + xb(t),$$

and

$$T_t = 3b(t).$$

Next we peel off the terms with u_{xx} contributions. These result in

$$\begin{aligned} U_{uu} &= 0, \\ U_{ux} &= 0, \end{aligned}$$

giving rise to

$$U = c(x,t) + ud(t).$$

Proceeding by eliminating terms with u_x we finally get that

$$\begin{aligned} X &= -\alpha t + \beta - \frac{1}{2}\gamma x, \\ T &= -\frac{3}{2}\gamma t + \delta, \\ U &= \alpha + \gamma u, \end{aligned}$$

where α, β, γ and δ are arbitrary constants. We conclude that the Lie point symmetry group of the KdV equation is 4-dimensional. Thus, this group contains 4 fundamentally independent symmetries. Let us examine these in more detail. Each fundamental symmetry may be obtained by equating 3 of the 4 parameters to zero, and the other one to 1.

1. $\boldsymbol{(\alpha, \beta, \gamma, \delta) = (1, 0, 0, 0)}$: the infinitesimal symmetry is given by

$$\begin{aligned} \hat{u} &= u + \epsilon + \mathcal{O}(\epsilon^2), \\ \hat{x} &= x - \epsilon t + \mathcal{O}(\epsilon^2), \\ \hat{t} &= t + \mathcal{O}(\epsilon^2). \end{aligned}$$

   We may integrate our symmetry group equations to recover the symmetry for finite values of ϵ. We find

$$\begin{aligned} \hat{u} &= u + \epsilon, \\ \hat{x} &= x - \epsilon t, \\ \hat{t} &= t, \end{aligned}$$

   for $\epsilon \in \mathbb{R}$. This is a Galilean boost: putting the solution in a moving coordinate frame results in a change of the background level of the solution: if $u(x,t)$ is a solution, then so is $u(x + \epsilon t, t) + \epsilon$.

2. $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}) = (\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0})$: the infinitesimal symmetry is given by

$$\begin{aligned}\hat{u} &= u(1+\epsilon) + \mathcal{O}(\epsilon^2),\\ \hat{x} &= x(1-\epsilon/2) + \mathcal{O}(\epsilon^2),\\ \hat{t} &= t(1-3\epsilon/2) + \mathcal{O}(\epsilon^2).\end{aligned}$$

We may integrate our symmetry group equations to recover the symmetry for finite values of ϵ. We find

$$\begin{aligned}\hat{u} &= ue^{\epsilon},\\ \hat{x} &= xe^{-\epsilon/2},\\ \hat{t} &= te^{-3\epsilon/2},\end{aligned}$$

for $\epsilon \in \mathbb{R}$. This is a scaling transformation: if $u(x,t)$ is a solution, then so is $e^{\epsilon}u(xe^{\epsilon/2}, te^{3\epsilon/2})$.

3. $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}) = (\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0})$: the infinitesimal symmetry is given by

$$\begin{aligned}\hat{u} &= u + \mathcal{O}(\epsilon^2),\\ \hat{x} &= x + \epsilon + \mathcal{O}(\epsilon^2),\\ \hat{t} &= t + \mathcal{O}(\epsilon^2).\end{aligned}$$

We may integrate our symmetry group equations to recover the symmetry for finite values of ϵ. We find

$$\begin{aligned}\hat{u} &= u,\\ \hat{x} &= x + \epsilon,\\ \hat{t} &= t,\end{aligned}$$

for $\epsilon \in \mathbb{R}$. This is a spatial translational invariance: if $u(x,t)$ is a solution, then so is $u(x-\epsilon, t)$. Thus the choice of spatial origin does not matter.

4. $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}) = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1})$: the infinitesimal symmetry is given by

$$\begin{aligned}\hat{u} &= u + \mathcal{O}(\epsilon^2),\\ \hat{x} &= x + \mathcal{O}(\epsilon^2),\\ \hat{t} &= t + \epsilon + \mathcal{O}(\epsilon^2).\end{aligned}$$

We may integrate our symmetry group equations to recover the symmetry for finite values of ϵ. We find

$$\begin{aligned}\hat{u} &= u,\\ \hat{x} &= x,\\ \hat{t} &= t + \epsilon,\end{aligned}$$

for $\epsilon \in \mathbb{R}$. This is a temporal translational invariance: if $u(x,t)$ is a solution, then so is $u(x, t-\epsilon)$. In other words, it does not matter where we start our clock.

Example. For **linear homogeneous partial differential equations with constant coefficients**, we know that if $u(x,t)$ and $U(x,t)$ are both solutions, then so is

$$\hat{u}(\hat{x},\hat{t}) = u(x,t) + \epsilon U(x,t),$$

where $\hat{x} = x$ and $\hat{t} = t$. Thus, linear homogeneous partial differential equations with constant coefficients have infinite-dimensional families of point symmetries. In other words, these symmetries depend on free functions. For instance, for the heat equation

$$u_t = \sigma u_{xx},$$

one finds the infinitesimal symmetries

$$\begin{aligned}
\hat{u} &= u + \epsilon \left(\left(x\eta(t) + \chi(t)\right) u + \frac{x^3}{12\sigma} T''(t) + \frac{x^2}{2\sigma} A'(t) + x\gamma(t) + \delta(t) \right) + \mathcal{O}(\epsilon^2), \\
\hat{x} &= x + \epsilon \left(\frac{x}{2} T'(t) + A(t) \right) \mathcal{O}(\epsilon^2), \\
\hat{t} &= t + \epsilon T(t) + \mathcal{O}(\epsilon^2),
\end{aligned}$$

as the reader can (and should) check, using the formulas for $\hat{u}_{\hat{t}}$ and $\hat{u}_{\hat{x}\hat{x}}$ from the previous example. Thus, the point symmetries for the heat equation depend on six arbitrary functions of t.

Example. Consider Burgers' equation

$$u_t + uu_x = \sigma u_{xx}.$$

The reader may verify that looking for the Lie point symmetries of this equation results in a 5-dimensional set of infinitesimal symmetries. The usual culprits are encountered: translation in x, translation in t, a scaling transformation, a Galilean shift. The fifth symmetry is more complicated, but of no direct interest to us right now.

Let us rewrite the Burgers' equation in its potential form: introduce $v(x,t)$ by

$$u(x,t) = v_x(x,t).$$

Substitution in the Burgers' equation and integrating once results in

$$v_t + \frac{1}{2} v_x^2 = \sigma v_{xx}.$$

We now look for Lie-point symmetries of this equation. We may use the formulae we derived and used for the KdV example. To first order in ϵ we find

$$\begin{aligned}
&v_t + \epsilon \left(V_t - X_t v_x + (V_v - T_t) v_t - X_v v_x v_t - T_v v_t^2 \right) + \\
&\frac{1}{2} v_x^2 + \epsilon v_x \left(V_x - T_x v_t + (V_v - X_x) v_x - X_v v_x^2 - T_v v_x v_t \right) = \\
&\sigma v_{xx} + \sigma\epsilon \left(V_{xx} + (2V_{xv} - X_{xx}) v_x - T_{xx} v_t + (V_{vv} - 2X_{xv}) v_x^2 - \right. \\
&2T_{xv} v_x v_t - X_{vv} v_x^3 - T_{vv} v_x^2 v_t + (V_v - 2X_x) v_{xx} - 2T_x v_{xt} - \\
&\left. 3X_v v_x v_{xx} - T_v v_{xx} v_t - 2T_v v_x v_{xt} \right) + \mathcal{O}(\epsilon^2).
\end{aligned}$$

Next we replace

$$v_t = \sigma v_{xx} - \frac{1}{2} v_x^2,$$
$$v_{xt} = \sigma v_{xxx} - v_v v_{xx}.$$

This imports dependence on v_{xxx} in the equation determining the infinitesimal symmetries. Equating its coefficients to zero results in

$$T_u = 0, \quad T_x = 0,$$

so that T depends only on t. This simplifies the remaining expression significantly. Equating its coefficients of v_{xx} gives

$$X = \frac{x}{2} T'(t) + A(t).$$

Continuing this way, the coefficients of the different powers of v_x give rise to

$$V_v = 2\sigma V_{vv},$$
$$-\frac{x}{2} T''(t) - A'(t) + V_x = 2\sigma V_{xv},$$
$$V_t = \sigma V_{xx}.$$

The first of these equations may be solved:

$$V = B(x,t) + C(x,t) e^{v/2\sigma}.$$

Substituting this in the second equation determines $B(x,t)$,

$$B(x,t) = \frac{1}{4} x^2 T''(t) + x A'(t) + D(t),$$

whereas the third equation states that $C(x,t)$ solves the heat equation:

$$C_t = \sigma C_{xx}.$$

Rounding up the remaining equations, we finally find

$$V = \frac{\gamma}{2} x^2 + \delta x + \sigma\gamma t + \chi + C(x,t) e^{v/2\sigma},$$
$$X = \frac{\beta}{2} x + \gamma x t + \delta t + \eta,$$
$$T = \alpha + \beta t + \gamma t^2.$$

Here α, β, γ, δ, η, and χ are arbitrary real constants. The function $C(x,t)$ is any solution of the heat equation. We conclude that the Lie-point symmetry group of Burgers' equation in its potential form is infinite-dimensional: it is determined by six arbitrary constants, and any solution of the one-dimensional heat equation. We can examine the point-symmetry group in more detail.

1. $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\eta}, \boldsymbol{\chi}, \boldsymbol{C}(\boldsymbol{x}, \boldsymbol{t})) = (\mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$: the infinitesimal symmetry is given by

$$\begin{aligned}\hat{v} &= v + \mathcal{O}(\epsilon^2),\\ \hat{x} &= x + \mathcal{O}(\epsilon^2),\\ \hat{t} &= t + \epsilon + \mathcal{O}(\epsilon^2).\end{aligned}$$

This corresponds to translation in t, which is expected as the potential Burgers' equation is autonomous in t. If $v(x,t)$ is a solution, then so is $v(x, t+\epsilon)$, for any $\epsilon \in \mathbb{R}$.

2. $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\eta}, \boldsymbol{\chi}, \boldsymbol{C}(\boldsymbol{x}, \boldsymbol{t})) = (\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$: the infinitesimal symmetry is given by

$$\begin{aligned}\hat{v} &= v + \mathcal{O}(\epsilon^2),\\ \hat{x} &= x(1+\epsilon/2) + \mathcal{O}(\epsilon^2),\\ \hat{t} &= t(1+\epsilon) + \mathcal{O}(\epsilon^2).\end{aligned}$$

Integrating the group equations gives

$$\begin{aligned}\hat{v} &= v,\\ \hat{x} &= xe^{\epsilon/2},\\ \hat{t} &= te^{\epsilon},\end{aligned}$$

which is a scaling transformation. It is a little peculiar, since v does not scale, despite it solving a nonlinear problem.

3. $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\eta}, \boldsymbol{\chi}, \boldsymbol{C}(\boldsymbol{x}, \boldsymbol{t})) = (\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$: the infinitesimal symmetry is given by

$$\begin{aligned}\hat{v} &= v + \epsilon\left(\frac{x^2}{2} + \sigma t\right) + \mathcal{O}(\epsilon^2),\\ \hat{x} &= x + \epsilon x t + \mathcal{O}(\epsilon^2),\\ \hat{t} &= t + \epsilon t^2 + \mathcal{O}(\epsilon^2).\end{aligned}$$

This is the analog for the potential Burgers' equation of the previously mentioned fifth point symmetry of the Burgers' equation.

4. $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\eta}, \boldsymbol{\chi}, \boldsymbol{C}(\boldsymbol{x}, \boldsymbol{t})) = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0})$: the infinitesimal symmetry is given by

$$\begin{aligned}\hat{v} &= v + \epsilon x + \mathcal{O}(\epsilon^2),\\ \hat{x} &= x + \epsilon t + \mathcal{O}(\epsilon^2),\\ \hat{t} &= t + \mathcal{O}(\epsilon^2).\end{aligned}$$

This is the infinitesimal version of the Galilean boost.

5. $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\eta}, \boldsymbol{\chi}, \boldsymbol{C}(\boldsymbol{x}, \boldsymbol{t})) = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0})$: the infinitesimal symmetry is given by

$$\begin{aligned}\hat{v} &= v + \mathcal{O}(\epsilon^2),\\ \hat{x} &= x + \epsilon + \mathcal{O}(\epsilon^2),\\ \hat{t} &= t + \mathcal{O}(\epsilon^2),\end{aligned}$$

which corresponds to a translation of the x variable. This is expected since the potential Burgers' equation is autonomous in x.

6. $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\eta}, \boldsymbol{\chi}, \boldsymbol{C}(\boldsymbol{x}, \boldsymbol{t})) = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0})$: the infinitesimal symmetry is

$$\begin{aligned}\hat{v} &= v + \epsilon + \mathcal{O}(\epsilon^2),\\ \hat{x} &= x + \mathcal{O}(\epsilon^2),\\ \hat{t} &= t + \mathcal{O}(\epsilon^2),\end{aligned}$$

which corresponds to a translation of the v variable. This is expected since the potential Burgers' equation depends only on derivatives of v.

7. $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\eta}, \boldsymbol{\chi}, \boldsymbol{C}(\boldsymbol{x}, \boldsymbol{t})) = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \boldsymbol{C}(\boldsymbol{x}, \boldsymbol{t}))$: the infinitesimal symmetry is

$$\begin{aligned}\hat{v} &= v + \epsilon C(x,t) e^{v/s\sigma} + \mathcal{O}(\epsilon^2),\\ \hat{x} &= x + \mathcal{O}(\epsilon^2),\\ \hat{t} &= t + \mathcal{O}(\epsilon^2).\end{aligned}$$

Integrating the group equations results in

$$\hat{v} = -2\sigma \ln\left(e^{-v/s\sigma} - \frac{\epsilon}{2\sigma} C(x,t)\right).$$

This relationship may be viewed as our first example of a Bäcklund transformation: a transformation from solutions of one equation to solutions of another. In this case, we find a new solution ($\hat{v}$) of the potential Burgers' equation from a known one (v) given any solution of the heat equation. In particular, by letting $v \to -\infty$ and choosing $\epsilon = -2\sigma$, we obtain (dropping the hats)

$$v = -2\sigma \ln C(x,t),$$

where $C(x,t)$ solves the heat equation. Transforming back to the variables for the heat equation, we have

$$u = -2\sigma \frac{C_x}{C},$$

the **Cole-Hopf transformation**! The use of Lie-point symmetries has allowed us to *systematically construct* the Cole-Hopf transformation, hence establishing that the Burgers' equation may be linearized and related to the heat equation. It should be

noted that our detour via the potential Burgers' equation was required to do this, as in the original variables, the Bäcklund transformation leading to the Cole-Hopf transformation depends not only on u but also on its integrand and therefore cannot be found using point symmetries[1].

As all these examples illustrate, the explicit determination of the generators of symmetries is a tedious task, albeit a systematic one. Indeed, the steps taken above are easily automated and computer algorithms have been developed to facilitate these calculations, and the more complicated calculations one encounters in the case of higher-dimensional equations or systems of equations. Further, the determination of non-classical symmetries where the generators may explicitly depend on derivatives of the dependent variable is equaly easily automated.

5.3.3 Using symmetries

The examples from the previous section have already illustrated the main ways symmetries may be used to analyze nonlinear equations.

- **Generalizing solutions:** symmetries allow us to generalize entire classes of solutions, from knowing a single solution: if $u(x,t)$ is a solution of the differential equation, then the action of the symmetry group on this solution gives rise to a more general solution, depending on free parameters, or even free functions.

  Example. Consider the KdV equation $u_t = uu_x + u_{xxx}$. It is straightforward to check that
  $$u(x,t) = 12\mathrm{sech}^2(x-4t)$$
  is a solution. It contains no free parameters. Using the scaling symmetry $\hat{u} = ue^{\epsilon}$, $x = \hat{x}e^{\epsilon/2}$, $t = \hat{t}e^{3\epsilon/2}$, we obtain that
  $$\hat{u}(\hat{x},\hat{t}) = 12k^2\mathrm{sech}^2(k\hat{x} - 4k^3\hat{t})$$
  is a one-parameter family of solutions of the same equation, so the hats may safely be dropped. Note that we have set $k = e^{\epsilon/2}$.

  It should be noted that the Lie-point symmetries are statements about the differential equation only. They do not take nonlocal considerations such as boundary conditions into account. Specifically, it is quite common for Lie-point symmetries to transform solutions to new solutions of the differential equations that are not in the function space of interest, or that do not satisfy the imposed boundary conditions. For instance, applying the Galilean boost to the KdV solution given above results in a new solution that approaches a non-zero constant as $|x| \to \infty$.

[1] This is easily seen by taking an x-derivative of the transformation, which results in $\hat{u} = \hat{v}_{\hat{x}}$. Using the chain rule on the right-hand side brings in u and v, without derivatives. Eliminating v gives rise to integrals over x of u.

- **Group-invariant solutions:** having found the Lie-point symmetries, we may look for solutions of the differential equation that are invariant under the action of the symmetry group, or under the action of some of its subgroups. For instance, we have already considered similarity solutions of the differential equations, namely solutions that are invariant under the action of a scaling transformation. For partial differential equations with one spatial variable that are autonomous with respect to x and t both, we have looked for stationary or equilibrium solutions. These are solutions that depend only on the specific combination $z = x - Vt$, *i.e.*, they are invariant under the action $x \to x + (\alpha - 1)Vt$, $t \to \alpha t$. This is a particular combination of the actions of the two translation symmetries, with respect to x and t respectively.

  Since the restriction of invariance under the action of any subgroup imposes conditions among the independent and dependent variables, the group-invariant solutions satisfy equations that depend on fewer variables, and may be easier to analyze. In some cases they may even be solved, leading to exact solutions of the differential equation.

- **Conserved quantities:** as we will see in the next chapter, Noether's theorem states that there is an intimate connection between symmetries of Lagrangian differential equations and conserved quantities of these equations. We have seen here that the construction of the Lie-point symmetries proceeds in an algorithmic way, due to the determining equations of the infinitesimal symmetries being linear. Similarly, Noether's theorem allows for the algorithmic construction of their corresponding conserved quantities, given the infinitesimal symmetries.

- **Transformation of the differential equation:** as the Burgers' equation example has demonstrated, the construction of Lie-point symmetries may be used to systematically establish transformations between the equation under investigation and other, hopefully simpler equations. In the ideal case, we may establish a connection between the original equation and one that we know how to solve explicitly. Whenever a symmetry depends on functions, instead of mere parameters, these functions will satisfy linear differential equations, since the equations determining the infinitesimal symmetries are linear in the generators of the symmetries.

  In all fairness, it should be mentioned that most often the transformation to a simpler differential equation will depend not only on the values of the dependent variables, but also on one or more derivatives. Because of this, Lie-point symmetries will usually not accomplish this goal, if such a transformation exists at all. This also was illustrated by the construction of the Cole-Hopf transformation for the Burgers' equation. Only by choosing to write the equation in potential form were we able to achieve a form where Lie-point symmetries were adequate for the task at hand. This judicious choice of variables for the underlying equation is non-algorithmic. It could have been avoided by considering nonclassical Lie symmetries where the generators are allowed to explicitly depend on derivatives of the dependent variables as well.

5.4 Exercises

1. For the Nonlinear Schrödinger equation (NLS)

$$i\phi_t = -\phi_{xx} + \sigma|\phi|^2\phi,$$

examine the behavior of solutions of the form

$$\phi(x,t) = e^{-i\omega t}\psi(x),$$

where $\psi(x)$ is real.

   a) Do this using phase-plane analysis and the potential energy method; consider different cases: $\sigma = -1$ (focusing, or attractive case), and $\sigma = 1$ (defocusing, or repulsive case). Indicate for which values of the parameters you will find soliton solutions.

   b) Find the explicit expression for the soliton solutions, using the appropriate boundary conditions as $|x| \to \infty$.

2. Consider a Modified KdV equation

$$u_t - 6u^2u_x + u_{xxx} = 0.$$

   (a) Find its scaling symmetry.

   (b) Using the scaling symmetry, write down an ansatz for any similarity solutions of the equation.

   (c) Show that your ansatz is compatible with $u = (3t)^{-1/3}w(z)$, with $z = x/(3t)^{1/3}$.

   (d) Use the above form of u to find an ordinary differential equation for $w(z)$. This equation will be of third order. It can be integrated once (do this) to obtain a second-order equation. The second-order equation you obtain this way is known as the second of the Painlevé equations. We'll see more about these later.

3. Consider the Emden-Fowler equation

$$x^2\frac{d^2u}{dx^2} + 2x\frac{du}{dx} + x^2u^5 = 0.$$

Use the scaling symmetry found in this chapter to construct a similarity solution for this equation.

4. Find the Lie-point symmetries of the nonlinear Schrödinger equation

$$iq_t = -\frac{1}{2}q_{xx} + \alpha|q|^2q,$$

where $\alpha = \pm 1$.

5. Construct the Lie-point symmetries for the heat equation

$$u_t = \sigma u_{xx}, \qquad \text{with} \;\; \sigma > 0.$$

6. Find all five Lie-point symmetries of Burgers' equation

$$u_t + uu_x = \sigma u_{xx},$$

and interpret them, when possible.

Chapter 6

An overview of Hamiltonian systems

One of the main questions we wish to answer in this course is "What is an integrable system?" In the finite-dimensional case, the answer to this question is well understood in the context of Hamiltonian systems. In this chapter, we review aspects of finite-dimensional Hamiltonian systems that are important for our purpose of generalizing the theory to infinite-dimensional systems. We end this chapter by demonstrating that the KdV equation is an infinite-dimensional Hamiltonian system, with non-canonical Poisson structure.

6.1 Finite-dimensional Hamiltonian systems

6.1.1 Definition of a finite-dimensional Hamiltonian system

Let $z : D \subset \mathbb{R} \to \mathbb{R}^n$. Further, let $H(z)$ be a function defined on $\mathbb{R}^n$. We refer to this function as the Hamiltonian. Next, define a bilinear operation $\{\cdot, \cdot\}$ on the space of functions of $\mathbb{R}^n$. This bilinear operation is called the Poisson bracket. The Poisson bracket of two functions is another function, such that

- $\{z_i, az_j + bz_k\} = a\{z_i, z_j\} + b\{z_i, z_k\}$, for all $1 \leq i, j, k \leq n$, and $a, b \in \mathbb{R}$, constant. Thus the Poisson bracket is linear in its first component.

- $\{z_i, z_j\} = -\{z_j, z_i\}$, for all $1 \leq i, j \leq n$. Thus the Poisson bracket is antisymmetric. Together with the first property this shows that the Poisson bracket is also linear in its second component. Thus the Poisson bracket is indeed bilinear.

- $\{z_i, z_j z_k\} = z_j\{z_i, z_k\} + z_k\{z_i, z_j\}$, for all $1 \leq i, j, k \leq n$. This is known as the Leibniz property, because of its similarity to the product rule. Indeed, this property makes the Poisson bracket a formal derivative in an algebraic sense.

- $\{z_i, \{z_j, z_k\}\} + \{z_j, \{z_k, z_i\}\} + \{z_k, \{z_i, z_j\}\} = 0$, for all $1 \leq i, j, k \leq n$. This is the Jacobi identity.

Using the Poisson bracket and the Hamiltonian $H(z)$, one may define dynamics on any function $f(z)$ on $\mathbb{R}^n$ by

$$f'(z) = \{f(z), H(z)\}.$$

For convenience, we assume that $f(z)$ is analytic in the components of z. The above equation then holds for any analytic function on $\mathbb{R}^n$. In particular, it holds for the coordinate functions z_i, $i = 1, \ldots, n$. Thus

$$z_i' = \{z_i, H(z)\}.$$

Thus given a Poisson bracket, any function $H(z)$ on $\mathbb{R}^n$ defines a Hamiltonian system, using the above differential equations.

6.1.2 Elementary properties of finite-dimensional Hamiltonian systems

- By the Leibniz property, the above two definitions of Hamiltonian dynamics are equivalent.
  - Clearly the first set of differential equations implies the second set, as the first equation holds for any function defined on $\mathbb{R}^n$.
  - From the Leibniz property, it follows easily using induction that $\{z_i^l, g(z)\} = lz_i^{l-1}\{z_i, g(z)\}$, for any $1 \leq i, k \leq n$, l a natural number, and any function $g(z)$ defined on $\mathbb{R}^n$. (Check this!) For analytic functions, it then follows that $\{f(z), g(z)\} = \sum_{k=1}^n \frac{\partial f}{\partial z_k}\{z_k, g(z)\}$, using the power series representation. Note that this can be interpreted as a chain rule.

    In general $f'(z) = \sum_{i=1}^n \frac{\partial f}{\partial z_i} z_i' = \sum_{i=1}^n \frac{\partial f}{\partial z_i}\{z_i, H(z)\} = \{f(z), H(z)\}$, where we have used the above chain rule. Thus indeed, one may use either set of differential equations to define Hamiltonian dynamics.

- Using the Leibniz property, we have

$$z_i' = \sum_{j=1}^n \{z_i, z_j\}\frac{\partial H}{\partial z_j} = \sum_{j=1}^n \omega_{ij}(z)\frac{\partial H}{\partial z_j},$$

  for all i, or equivalently

$$f'(z) = \sum_{i,j=1}^n \omega_{ij}(z)\frac{\partial f}{\partial z_i}\frac{\partial H}{\partial z_j} = (\nabla f)^T \cdot \omega \cdot (\nabla H).$$

  Thus the matrix $\omega = (\omega_{ij})_{i,j=1}^n$, with $\omega_{ij} = \{z_i, z_j\}$ determines the coordinate representation of the Poisson bracket.

- $H'(z) = \{H(z), H(z)\} = 0$, using the antisymmetry of the Poisson bracket. Thus the Hamiltonian is a constant of the motion, *i.e.*, a conserved quantity of the dynamics.

- If a function $G(z)$ Poisson commutes with the Hamiltonian, *i.e.*, $\{G(z), H(z)\} = 0$, for all $z \in \mathbb{R}^n$, then $G(z)$ is a constant of the motion: $G'(z) = \{G(z), H(z)\} = 0$.

- A function $C(z)$ such that

$$\{f(z), C(z)\} = 0,$$

for any function $f(z)$ defined on $\mathbb{R}^n$ is called a Casimir function of the Poisson bracket. Thus a Casimir of the Poisson bracket Poisson commutes with any other function. It is clear that the existence of any Casimirs is indeed a property of the Poisson bracket (the geometry of the system), and has nothing to do with the Hamiltonian (the dynamics of the system). It follows from the previous property that a Casimir is trivially a conserved quantity of the Hamiltonian system. As a matter of fact, it is a conserved quantity for any Hamiltonian system with the given Poisson structure, independent of what the Hamiltonian is. The defining equation for the Casimir function may be written as

$$(\nabla f)^T \omega (\nabla C) = 0.$$

Since this holds for any function $f(z)$, this is equivalent to

$$\omega \nabla C = 0,$$

implying that the Poisson matrix $\omega = (\omega_{ij})_{i,j=1}^n$ is singular. The number of independent Casimirs is equal to the co-rank of ω.

6.1.3 Darboux's theorem

Next, we present Darboux's theorem, without proof. We will not use the results of this theorem, but it is too important to ignore. Further, it allows us to connect our rather unconventional definition of what a Hamiltonian system is to any other definitions you may have seen in physics, optimization, *etc.*

Theorem (Darboux) Given a Hamiltonian structure defined by a Poisson structure ω and a Hamiltonian $H(z)$, it is always possible to find a coordinate transformation $z \to (q, p, C_1, \ldots, C_r)$, where r denotes the co-rank of the Poisson matrix ω, and q and p are N-dimensional vectors with $2N + r = n$, such that in these new variables the Hamiltonian system has Poisson matrix

$$\omega = \begin{pmatrix} 0 & 1_N & 0 \\ -1_N & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where 1_N is the identity matrix of size $N \times N$ and 0 denotes the zero matrix of suitable size. The $C_i(z)$'s are the Casimirs of the original Poisson structure.

Note that ω is antisymmetric, thus its co-rank r will be odd if n is odd, and even if n is so. Thus $n - r$ is indeed even. This preferred set of coordinates is usually referred to as canonical variables. The variables q and p are referred to as mutually conjugate. In canonical variables the dynamical equations are

$$\begin{cases} q_i' = \dfrac{\partial \hat{H}}{\partial p_i}, & i = 1, \ldots, N, \\ p_i' = -\dfrac{\partial \hat{H}}{\partial q_i}, & i = 1, \ldots, N, \\ C_i' = 0, & i = 1, \ldots, r, \end{cases}$$

where $\hat{H} = H(z((q, p, C_1, \ldots, C_r)))$ is the transformed Hamiltonian.

In this new coordinate system

$$\{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij},$$

for all $i, j = 1, \ldots, N$. In canonical variables, the Casimirs may be ignored and the dimension of the phase space is reduced from n to $2N = n - r$ dimensions. Then

$$\omega = \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix},$$

and

$$\{f, g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right),$$

for any two functions f and g of the canonical variables q and p.

6.1.4 Some examples

Mechanical systems

Let

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + V(q_1, \ldots, q_N),$$

and

$$\omega = \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix}.$$

Then we may use this choice of ω and $H(q,p)$ to define a Hamiltonian system with canonical variables. Its equations of motion are:

$$\begin{cases} q_i' = \dfrac{\partial H}{\partial p_i} = \dfrac{p_i}{m_i}, \\ p_i' = -\dfrac{\partial H}{\partial q_i} = -\dfrac{\partial V}{\partial q_i}, \end{cases}$$

for $i = 1, \ldots, N$. Eliminating the momentum variables p_i from these equations results in

$$m_i q_i'' = -\frac{\partial V}{\partial q_i}, \qquad i = 1, \ldots, N.$$

These are Newton's equations of motion for N (one-dimensional) particles of different mass in a conservative force field. The Hamiltonian is the total energy, being the sum of the kinetic and the potential energy. Thus we see that Newton's equations are a special case of a Hamiltonian system. It was indeed in this context (and that of geometric optics) that Hamilton first proposed his equations. The setting of Hamilton's equations is to be preferred over that of the Newtonian formulation for many reasons: the equations may all be derived from a scalar quantity, namely the Hamiltonian; the equations can be derived from a variational principle (see below), offering the advantage that geometrical constraints on the dynamics may be taken into consideration without changing the form of the equations of motion. This is in sharp contrast to the Newtonian form, where such situations require the use of d'Alembert's principle to eliminate unknown reaction forces, which is cumbersome, to put it mildly.

The Heisenberg magnet

Consider the equations of motion of the Heisenberg magnet

$$S' = J \times S,$$

where the vector $S = (S_x, S_y, S_z)^T$, and the magnetic momentum J is constant. Written out explicitly, we obtain

$$\frac{d}{dt}\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \begin{pmatrix} 0 & -J_z & J_y \\ J_z & 0 & -J_x \\ -J_y & J_x & 0 \end{pmatrix}\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}.$$

This system is Hamiltonian with (check this!)

$$\omega = \begin{pmatrix} 0 & -J_z & J_y \\ J_z & 0 & -J_x \\ -J_y & J_x & 0 \end{pmatrix}$$

and

$$H(S) = \frac{1}{2}S^2 = \frac{1}{2}\left(S_x^2 + S_y^2 + S_z^2\right).$$

With this ω the Poisson bracket is given by

$$\begin{aligned}
\{f,g\} &= (\nabla f)^T \omega (\nabla g) \\
&= \begin{pmatrix} \frac{\partial f}{\partial S_x} & \frac{\partial f}{\partial S_y} & \frac{\partial f}{\partial S_z} \end{pmatrix} \begin{pmatrix} 0 & -J_z & J_y \\ J_z & 0 & -J_x \\ -J_y & J_x & 0 \end{pmatrix} \begin{pmatrix} \partial g/\partial S_x \\ \partial g/\partial S_y \\ \partial g/\partial S_z \end{pmatrix} \\
&\overset{\text{Check!}}{=} [\nabla f, J, \nabla g],
\end{aligned}$$

where $[A, B, C]$ denotes the triple product of the vectors A, B and C: $[A, B, C] = A \cdot (B \times C)$. As this system is odd dimensional, there has to be at least one Casimir (why?). One easily sees that there is indeed only one Casimir, as ω has rank 2. Since the gradient of the Casimir with respect to the components of S is in the kernel of ω, we find

$$C = J \cdot S = J_x S_x + J_y S_y + J_z S_z.$$

This is easily verified:

$$\{f, C\} = [\nabla f, J, J] = (\nabla f)^T \cdot (J \times J) = 0.$$

According to Darboux's theorem there exist canonical coordinates q and p (scalar in this case) such that the system is Hamiltonian with the canonical Poisson structure. However, finding these coordinates in practice is not easy, even for a simple case as the one considered here. For the Heisenberg magnet the canonical variables are the coordinates parametrizing the plane $J \cdot S =$ constant. The formulation of the problem in these new variables loses a lot of symmetry. For this reason, one often prefers to continue working in non-canonical variables, even if the transformation to canonical variables may be worked out.

6.1.5 More properties of finite-dimensional Hamiltonian systems

For cultural reasons, we finish this fast overview of finite-dimensional Hamiltonian system by mentioning a few more properties of Hamiltonian systems, mostly without proof.

- A **canonical transformation** is a transformation on a Hamiltonian system specified by the canonical Poisson structure ω and $H(z)$ that leaves the canonical Poisson structure invariant. In other words, the Poisson brackets of the new variables are equal to those of the corresponding old variables:

$$\{Z_i, Z_j\} = \{z_i, z_j\},$$

with Z=Z(z). For instance, for a mechanical system this means that any transformation on the coordinates q is allowed, but this then implies a transformation on the momentum variables p:

$$q \to Q = Q(q) \Rightarrow p \to P = P(q,p).$$

Such a special canonical transformation where the new coordinates Q only depend on the old coordinates and not on the old momenta is called a contact transformation. A general canonical transformation may mix old coordinates and old momenta in the new variables.

Given a Hamiltonian system with canonical coordinates, one usually does not allow any transformations that are not canonical. Note that the transformation mentioned in Darboux's theorem is obviously not canonical, as it changes the Poisson structure which is canonical in the new variables, but not in the old ones.

- The flow of a Hamiltonian system preserves volume. Phrased differently, the flow is divergence free. Thus, if a set of initial conditions is chosen in phase space, the volume of this set does not change under time evolution.

  We demonstrate this here for Hamiltonian systems that are given with canonical Poisson structure. If one regards the solutions of the Hamiltonian system $z(0) \to z(t)$ as a transformation on the variables (it is the best kind of transformation, as it solves the system), then one can show that this transformation is canonical. The divergence of the flow is given by

$$\begin{aligned} \nabla \cdot z' &= \sum_{i=1}^{N} \frac{\partial q_i'}{\partial q_i} + \sum_{i=1}^{N} \frac{\partial p_i'}{\partial p_i} \\ &= \sum_{i=1}^{N} \frac{\partial^2 H}{\partial q_i \partial p_i} - \sum_{i=1}^{N} \frac{\partial^2 H}{\partial p_i \partial q_i} \\ &= 0, \end{aligned}$$

  thus the flow is divergence free from which it follows that the Jacobian of the transformation from initial conditions to solutions is

$$J = \det\left(\frac{\partial(q(t),p(t))}{\partial(q(0),p(0))}\right) = 1,$$

  and phase space volume is conserved.

- A canonical Hamiltonian system is derivable from Hamilton's variational principle:

$$\delta \int \left(H - \sum_{i=1}^{N} p_i q_i' \right) dt = 0.$$

We will review variational methods later.

- One can show that equilibrium solutions of Hamiltonian systems are at best neutrally stable. Thus, in the case of a neutrally stable equilibrium point, all eigenvalues of the linearized problem are pure imaginary. If not all eigenvalues are pure imaginary, the spectrum is symmetric accross the imaginary line. If the system is also real-valued on the real line, one ends up with a spectrum that has four-fold symmetry, both accross the real and the imaginary line. Thus, for every growing mode, there is a decaying mode. This implies that a Hamiltonian system cannot have sources or sinks.

  Specifically, in two dimensions, this implies that the equilibrium points are either centers (neutrally stable) or saddles (unstable).

- **Poincaré recurrence**

  We have seen that a Hamiltonian system preserves the Hamiltonian. We refer to the surface $H(z) = E = H(z(0))$ (constant) as the energy surface. It is an $(n-1)$-dimensional surface in phase space. Poincaré showed that if this surface is compact, then the motion of the Hamiltonian system is recurrent, for arbitrary initial conditions. In mathematical ϵ, δ language:

  $$\forall\, \epsilon, \delta > 0, \exists\, T > delta : |z(T) - z(0)| < \epsilon.$$

  Returning to the FPU problem, it was hypothesized at first that the recurrence of Hamiltonian systems was responsible for the recurrence observed in the FPU numerics. This was quickly dismissed, as the estimated recurrence times for a system with that many degrees of freedom were several orders of magnitude larger than those observed.

6.2 Integrability of finite-dimensional Hamiltonian systems

Let us talk about integrability of any arbitrary system of ordinary differential equations. To solve an N-th order system

$$x' = f(x)$$

requires $N-1$ integration constants. The last one is then provided by the arbitrariness of the initial t_0: if $x(t)$ is a solution, so is $x(t-t_0)$, as the differential system is autonomous. At an existence-uniqueness level, these integration constants are just the initial conditions for the initial-value problem. A differential equation is called **solvable** if its solution can be

written down in terms of quadratures. As stated, this requires the existence of N constants of the motion.

Hamiltonian systems are very special. Let us assume that we have used Darboux's theorem to obtain a set of canonical variables. This leaves us with two sets of mutually canonically conjugate variables $q_1, \ldots, q_N$ and $p_1, \ldots, p_N$. By using canonical transformations, these may be transformed to new canonical variables $Q_1, \ldots, Q_N$ and $P_1, \ldots, P_N$. The main idea is to find a canonical transformation so that in these new variables the equation for one of the P variables read

$$P_k' = 0,$$

for some $1 \leq k \leq N$. This will happen if the new transformed Hamiltonian no longer depends on the new coordinate variable Q_k. In that case we say that Q_k is a cyclic coordinate. Every time we manage to transform to a system of canonical coordinates so that one of the new coordinates is cyclic (*i.e.*, the new Hamiltonian does not depend on it), the corresponding equation for the new conjugate momentum variable may be trivially integrated. Hence we effectively reduce the number of degrees of freedom on which our dynamical system depends. As stated, for a general dynamical system with $2N$ variables , we would expect to have to do this $2N$ times.

One can show (but we won't do this here) that for every variable J_k (the new momentum variable) for which $J_k' = 0$, we may write for the canonically conjugate variable θ_k:

$$\theta_k' = \Omega_k = \text{constant} \quad \Rightarrow \quad \theta_k = \Omega_k t + \theta_k(0),$$

thus the equation for the conjugate variable is also easily integrated. In summary, we find that in order to completely solve a Hamiltonian system, we are required to find N variables J_k, $k = 1, \ldots, N$ such that

$$\begin{cases} J_k' = -\dfrac{\partial H}{\partial \theta_k} \\ \theta_k' = \dfrac{\partial H}{\partial J_k} = \Omega_k \end{cases},$$

and thus $H = \sum_{k=1}^{N} \Omega_k J_k$. Lastly, we need

$$\{J_i, J_k\} = 0,$$

for $i, k = 1, \ldots, N$. This condition is necessary, since we are requiring the variables J_k, $k = 1, \ldots, N$ to act as new momentum variables. With the canonical Poisson structure, the momentum variables have mutually vanishing Poisson brackets. Two variables whose Poisson bracket vanishes are said to be in involution.

We find that a Hamiltonian system with canonical Poisson structure is solvable if we can find N conserved quantities that are mutually in involution. Such a system is called integrable.

The new variables $J_1, \ldots, J_N$ are called action variables, and their conjugate variables

$\theta_1, \ldots, \theta_N$ are angle variables. Finding action-angle variables for a given Hamiltonian system is the topic of Hamilton-Jacobi theory. Action-angle variables have added importance, as they are the preferred starting point of any perturbation approach of a near-integrable system, *i.e.*, a system with an ϵ-perturbation that is integrable in the $\epsilon = 0$ limit.

6.2.1 The Liouville-Arnol'd theorem

Consider the $2N$-dimensional Hamiltonian system specified by the Hamiltonian H and the canonical Poisson structure. Let $F_1, \ldots, F_N$ be a set of functionally independent conserved quantities for this system, so that all elements are mutually in involution:

$$\{F_j, F_k\} = 0,$$

for $j, k = 1, \ldots, N$. Note that these conserved quantities are not necessarily action variables. We are not imposing that the Hamiltonian is a linear function of the $F_1, \ldots, F_N$. Next, consider the level set of these functions:

$$M = \{(q,p) : F_j(q,p) = F_{j0}(\text{constant}), \text{ for } \ j = 1, \ldots, N\}.$$

Then the Liouville-Arnol'd theorem states that for allmost all[1] values of $F_{10}, \ldots, F_{N0}$:

- M is a manifold.
- If M is compact and connected, it is topologically an N-dimensional torus $T^{(N)}$

$$T^{(N)} = \{(\varphi_1, \ldots, \varphi_N) \mod 2\pi\}.$$

- The flow of the Hamiltonian system is quasi-periodic on this torus. In angle variables:

$$\theta_j' = \Omega_j(F_1, \ldots, F_N),$$

  for $j = 1, \ldots, N$ with quasi-periods $2\pi/\Omega_j(F_1, \ldots, F_N)$ for $j = 1, \ldots, N$.

- The Hamiltonian system can be solved in terms of quadratures, and is thus solvable in the ordinary sense of solvability of a dynamical system.

6.2.2 Example: the harmonic oscillator

Consider the Hamiltonian (total energy)

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2$$

[1] This "almost all" statement is to be taken in a measure theory sense.

of a particle under the influence of a Hooke's law type force, resulting in a Harmonic oscillator. In this case $N = 1$, thus we need only one conserved quantity. Since the Hamiltonian is conserved, we may use it as the only required conserved quantity. The set M is given by

$$M = \left\{ (q,p) : \frac{p^2}{2m} + \frac{1}{2}kq^2 = E \right\},$$

for some constant E, determined by the initial conditions. Thus M is the set of all points on an ellipsoid in the (q,p)-plane, as shown in Fig. 6.1. It is immediately obvious that M is indeed a manifold, for all possible initial conditions. Further, indeed M is topologically a 1-dimensional torus $T^{(1)}$, *i.e.*, a circle. We now proceed to prove the last two statements of the Liouville-Arnol'd theorem for this example.

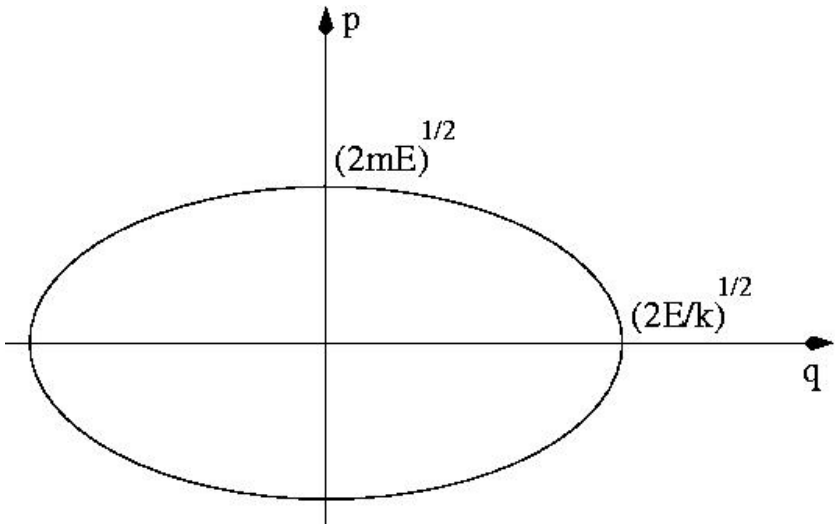


Figure 6.1: The energy surface for the harmonic oscillator

The equations of motion are

$$\begin{cases} q' = \dfrac{\partial H}{\partial p} = p/m \\ p' = -\dfrac{\partial H}{\partial q} = -kq \end{cases},$$

which gives

$$q'' + \omega^2 q = 0,$$

with $\omega = \sqrt{k/m}$ (not to be confused with the Poisson structure). This equation is easily solved, resulting in

$$q = A\sin(\omega t + \phi_0), \quad p = A\sqrt{k\,m}\cos(\omega t + \phi_0),$$

where A and ϕ_0 are arbitrary constants. This demonstrates that the equations may indeed be integrated, and in this case the integrals (quadratures) may be worked out explicitly. It is also clear that the motion is quasi-periodic, as it is periodic.

Lastly, we find the action-angle variables (θ, J) for this problem, and check that (1) the transformation from $(q,p) \to (\theta, J)$ is canonical. Since the Hamiltonian is a linear combination of the action variables in general, we find in this case

$$H = \Omega J.$$

For this example there is an obvious candidate for the angle variable, namely $\theta = \omega t + \phi_0$, or $\theta' = \omega$, from which $\Omega = \omega$. Thus

$$J = \frac{H}{\omega} = \frac{A^2}{2}\sqrt{k\,m}.$$

Thus the transformation from (q,p) to action-angle variables is given by

$$\begin{cases} q = \sqrt{\dfrac{2J}{\sqrt{k\,m}}}\sin\theta \\ p = \sqrt{2J\sqrt{k\,m}}\cos\theta \end{cases},$$

with inverse

$$\begin{cases} J = \dfrac{1}{\omega}\left(\dfrac{p^2}{2m} + \dfrac{1}{2}kq^2\right) \\ \theta = \arctan\left(\dfrac{q\sqrt{k\,m}}{p}\right) \end{cases}.$$

This allows us to calculate the Poisson bracket $\{\theta, J\}$:

$$\{\theta, J\} = \frac{\partial\theta}{\partial q}\frac{\partial J}{\partial p} - \frac{\partial\theta}{\partial p}\frac{\partial J}{\partial q} \overset{\text{Check!}}{=} 1.$$

6.3 The calculus of variations

In order to generalize the concept of a Hamiltonian system to partial differential equations, we need the essentials of the calculus of variations. Consider two points $P = (x_1, x_2, \ldots, x_n)$ and $Q = (y_1, y_2, \ldots, y_n)$, both in $\mathbb{R}^n$. Next, consider the set of all smooth curves joining these two points:

$$\gamma : t \in [a,b] \to x(t) = (x_1(t), \ldots, x_n(t)),$$

such that $x(a) = P$ and $x(b) = Q$, as illustrated in Fig. 6.2. In what follows we assume that γ is sufficiently smooth so that all derivatives make sense.

Define the functional

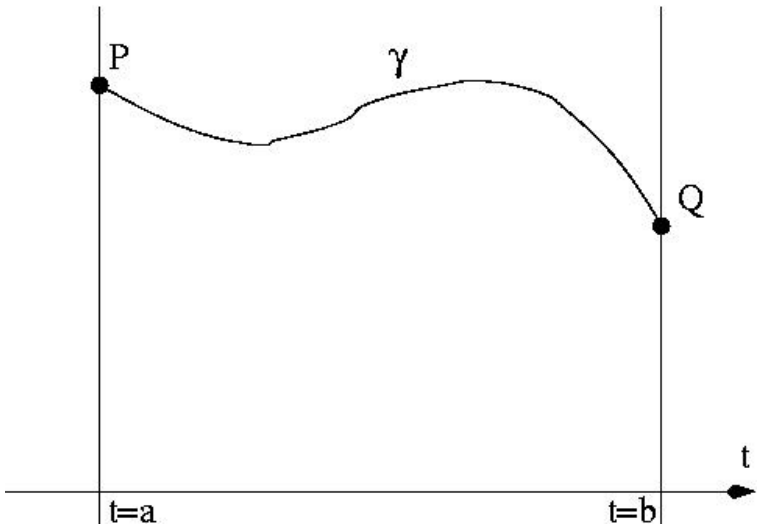


Figure 6.2: The set-up of the classical variational problem

$$S[\gamma] = \int_a^b L\left(x, x', x'', \ldots, x^{(N)}\right) dt,$$

where $L(x, x', x'', \ldots, x^{(N)})$ is a function of $x(t)$ and its first N derivatives, referred to as the Lagrangian. Sometimes the notation $S[x]$ is used, instead of $S[\gamma]$. It is referred to as the action. Thus γ (or $x(t)$) is the input, resulting in a scalar output S.

6.3.1 The fundamental theorem of the calculus of variations

In the calculus of variations the goal is to characterize for which γ the action $S[\gamma]$ has a critical value. With some additional conditions that we will ignore, a critical point of $S[\gamma]$ is a local minimum or maximum value of $S[\gamma]$, attained at some extremizer $x(t)$. The following is the fundamental theorem of the calculus of variations.

Theorem [Fundamental theorem of the calculus of variations]: If $\gamma_0 : t \in [a, b] \to x(t)$ is a critical point of the action $S[\gamma]$ then along γ_0

$$0 = \frac{\delta S}{\delta x_i} = \frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial x_i'} + \frac{d^2}{dt^2}\frac{\partial L}{\partial x_i''} - \ldots + (-1)^N \frac{d^N}{dt^N}\frac{\partial L}{\partial x_i^{(N)}},$$

for all $i = 1, \ldots, n$. Conversely, if these equations are satisfied, then γ_0 is a critical point of the action $S[\gamma]$.

These equations are known as the Euler-Lagrange equations. In the case where the Lagrangian depends on more than first derivatives of the unknown function, the equations

may also be referred to as the Ostrogradskii equations. The expression on the right is referred to as the functional or variational derivative of S. Sometimes, we will loosely refer to it as the variational derivative of L. The Euler-Lagrange equations are differential equations for the components of $x(t)$. They give necessary and sufficient conditions to characterize a critical point of the action.

Proof. First, we prove that the Euler-Lagrange equations give sufficient conditions to characterize a critical point of the action. To this end, we consider a small variation of the path γ, as indicated in Fig. 6.3.

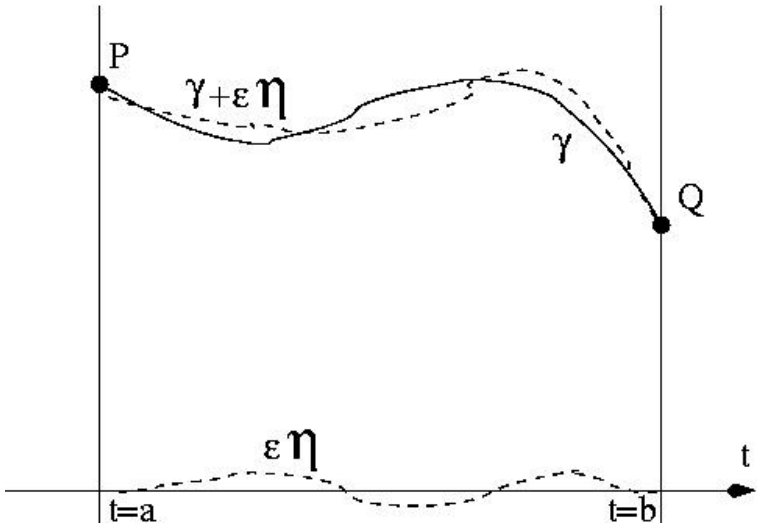


Figure 6.3: A small variation $\epsilon\eta(t)$ of the path γ specified by $x(t)$

Here ϵ is a real number, thought of as small. Further, $\eta(t)$ is a curve $t \in [a, b] \to \eta(t)$, such that $\eta(a) = 0 = \eta(b)$ so that the end points of the path from a to b are fixed. Then we calculate

$$\frac{dS}{d\epsilon}[x(t) + \epsilon\eta(t)]\bigg|_{\epsilon=0} = \lim_{\epsilon\to 0} \frac{S[x + \epsilon\eta] - S[x]}{\epsilon}.$$

First, consider

$$S[x(t) + \epsilon\eta(t)] = \int_a^b L(x + \epsilon\eta, x' + \epsilon\eta', x'' + \epsilon\eta'', \ldots, x^{(N)} + \epsilon\eta^{(N)})dt.$$

In order to calculate the above derivative, we expand this is a Taylor series in ϵ:

$$\begin{aligned} S[x(t)+\epsilon\eta(t)] &= \int_a^b L\left(x+\epsilon\eta, x'+\epsilon\eta', x''+\epsilon\eta'', \ldots, x^{(N)}+\epsilon\eta^{(N)}\right) dt \\ &= \int_a^b L\left(x, x', x'', \ldots, x^{(N)}\right) dt + \\ &\quad \epsilon\sum_{i=1}^{n}\int_a^b \left(\frac{\partial L}{\partial x_i}\eta_i + \frac{\partial L}{\partial x_i'}\eta_i' + \ldots \frac{\partial L}{\partial x_i^{(N)}}\eta_i^{(N)}\right) dt + \theta(\epsilon^2). \end{aligned}$$

Here all partial derivative terms in the $\theta(\epsilon)$ term are evaluated at $\epsilon = 0$. Now consider the second one of these terms. For each $i = 1, \ldots, n$

$$\int_a^b \frac{\partial L}{\partial x_i'}\eta_i' dt = \eta_i \frac{\partial L}{\partial x_i'}\bigg|_a^b - \int_a^b \eta_i \frac{d}{dt}\frac{\partial L}{\partial x_i'} dt,$$

using integration by parts. The first term vanishes due to the fixed boundaries ($\eta_i(a) = 0 = \eta_i(b)$, for all i), thus

$$\int_a^b \frac{\partial L}{\partial x_i'}\eta_i' dt = -\int_a^b \eta_i \frac{d}{dt}\frac{\partial L}{\partial x_i'} dt,$$

Similarly for the second term, after two integration by parts steps,

$$\int_a^b \frac{\partial L}{\partial x_i''}\eta_i'' dt = (-1)^2 \int_a^b \eta_i \frac{d^2}{dt^2}\frac{\partial L}{\partial x_i''} dt,$$

and so on for the other terms. Note that these calculations require that higher-order derivatives of $\eta(t)$ vanish at $t = a$ and $t = b$ as well. Thus

$$S[x+\epsilon\eta] = S[x] + \epsilon\sum_{i=1}^{n}\int_a^b \eta_i \left(\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial x_i'} + \ldots + (-1)^N \frac{d^N}{dt^N}\frac{\partial L}{\partial x_i^{(N)}}\right) dt + \theta(\epsilon^2)$$

Now we may calculate our derivative:

$$\frac{dS}{d\epsilon}[x+\epsilon\eta]\bigg|_{\epsilon=0} = \sum_{i=1}^{n}\int_a^b \eta_i \left(\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial x_i'} + \ldots + (-1)^N \frac{d^N}{dt^N}\frac{\partial L}{\partial x_i^{(N)}}\right) dt.$$

It follows immediately that if the Euler-Lagrange equations are satisfied, then the action $S[x]$ has a critical point.

Next we show that these conditions are also necessary for the action $S[x]$ to have a critical point. Assume that the Euler-Lagrange equations are not satisfied, but the action has a critical point nonetheless so that

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial x_i'} + \ldots + (-1)^N \frac{d^N}{dt^N}\frac{\partial L}{\partial x_i^{(N)}} \neq 0,$$

at least not identically. Assume that the left-hand side of this equation has a finite number of zeros at a finite number of t values in $[a, b]$. Then away from these zeros, the left-hand side is either positive or negative. The equation

$$0 = \frac{dS}{d\epsilon}[x + \epsilon\eta]\Big|_{\epsilon=0} = \sum_{i=1}^{n} \int_a^b \eta_i \left(\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial x_i'} + \ldots + (-1)^N \frac{d^N}{dt^N}\frac{\partial L}{\partial x_i^{(N)}} \right) dt$$

holds for arbitrary η, as long as $N-1$ derivatives of each component of $\eta(t)$ vanish at the boundary points. In particular, we may choose all components of $\eta(t)$ to vanish identically, except the first one. The non-zero component may be chosen to be positive (negative) whenever the variational derivative with respect to the first component is positive (negative), resulting in an integrand that is strickly positive, except at a few isolated zeros. But the integral needs to vanish, as the left-hand side is zero. This is only possible if the first Euler-Lagrange equation holds. The argument may be repeated with any other component, leading to the conclusion that all Euler-Lagrange equations are satisfied.

In summary we find that the Euler-Lagrange equations are necessary and sufficient conditions for the action functional to have a critical point. This concludes the proof of the fundamental theorem.

6.3.2 Examples

1. **Lagrangian mechanics**

   Let the Lagrangian be the difference between the kinetic energy and the potential energy for an n particle system moving in one-dimension, for convenience:

   $$L = \sum_{i=1}^{n} \frac{1}{2} m_i x_i'^2 - V(x_1, \ldots, x_n).$$

   The Euler-Lagrange equations for each $i = 1, \ldots, n$ become

   $$\begin{aligned} \frac{\partial L}{\partial x_i} = -\frac{\partial V}{\partial x_i}, \quad \frac{\partial L}{\partial x_i'} &= m_i x_i' \\ \Rightarrow 0 = -\frac{\partial V}{\partial x_i} - \frac{d}{dt} m_i x_i' & \\ \Rightarrow m_i x_i'' = -\frac{\partial V}{\partial x_i}, & \end{aligned}$$

   Newton's equations in a conservative force field! Thus Newton's equations may be derived from a variational principle. This specific principle is due to Lagrange. One of the many advantages of Lagrange's approach is the ease in which coordinate changes may be performed: the variational principle is independent of the choice of coordinates of the curve γ. A different parametrization $x(t)$ gives the same curve, but results in

different Euler-Lagrange equations, namely those in the new coordinates. Thus, in order to transform coordinates on equations derivable from a variational principle it suffices to perform the coordinate transformations on the Lagrangian, followed by writing down the Euler-Lagrange equations for the new Lagrangian in the new variables. This allows us to take constraints on the dynamics into account easily, without having to resort to d'Alembert's principle to eliminate unknown reaction forces.

2. **Hamilton's principle**

   Consider the action

$$S = \int_a^b \left(H(q_1, \ldots, q_n, p_1, \ldots, p_n) - \sum_{k=1}^{n} p_k q_k' \right) dt,$$

   with Lagrangian

$$H(q_1, \ldots, q_n, p_1, \ldots, p_n) - \sum_{k=1}^{n} p_k q_k'.$$

   The Euler-Lagrange equations become:

$$\begin{aligned}
\frac{\delta S}{\delta q_i} &= \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial q_i'} = 0 \\
&\Rightarrow \frac{\partial H}{\partial q_i} - \frac{d}{dt}(-p_i) = 0 \\
&\Rightarrow p_i' = -\frac{\partial H}{\partial q_i}, \\
\frac{\delta S}{\delta p_i} &= \frac{\partial L}{\partial p_i} = 0 \\
&\Rightarrow \frac{\partial H}{\partial p_i} - q_i' = 0 \\
&\Rightarrow q_i' = \frac{\partial H}{\partial p_i}.
\end{aligned}$$

   Thus Hamilton's equations in canonical variables may be derived from a variational principle. At this point we can rephrase the definition of a canonical transformation: a canonical transformation is a transformation that leaves the form of Hamilton's variational principle invariant. In practice, this is the most convenient way to generate canonical transformations.

3. **Generalization to functions of more than one variable**

How should we generalize the Euler-Lagrange equations to deal with surfaces (maybe more than two-dimensional) instead of curves? The derivation of the fundamental theorem may be repeated, by using Green's theorem instead of integration by parts. Assume the Lagrangian depends only on one function of $t_1, \ldots, t_N$ and its first derivatives with respect to these variables

$$L = L(q, \nabla q),$$

then the Euler-Lagrange equation is given by

$$\frac{\partial L}{\partial q} - \sum_{i=1}^{N} \frac{\partial}{\partial t_i} \frac{\partial L}{\partial q_{t_i}} = 0.$$

If multiple functions appear in the Lagrangian, then this prescription is repeated for every such function. As an example, consider the Lagrangian

$$L = \frac{1}{2}u_t^2 - \frac{1}{2}c^2(\nabla u)^2 = \frac{1}{2}u_t^2 - \frac{1}{2}c^2\left(u_x^2 + u_y^2 + u_z^2\right).$$

We obtain

$$\begin{aligned} 0 &= \frac{\partial L}{\partial u} - \frac{\partial}{\partial t}\frac{\partial L}{\partial u_t} - \frac{\partial}{\partial x}\frac{\partial L}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial L}{\partial u_y} - \frac{\partial}{\partial z}\frac{\partial L}{\partial u_z} \\ &= 0 - \frac{\partial}{\partial t}u_t - \frac{\partial}{\partial x}(-c^2 u_x) - \frac{\partial}{\partial y}(-c^2 u_y) - \frac{\partial}{\partial z}(-c^2 u_z) \\ &= -u_{tt} + c^2 \Delta u, \end{aligned}$$

the wave equation! Thus the wave equation may be derived from this variational principle. Thus, instead of solving the wave equation, we could prefer to look for critical points of its action. From a mathematical point of view this could be beneficial, as the Lagrangian involves only first derivatives whereas the equation involves second derivatives. Thus this allows us to find a "solution" of the wave equation that may not have enough regularity to be considered a solution of the equation, but can be considered a weak solution. In general, many equations arising in pure and applied mathematics and their applications are derived from variational principles. These may be used to find numerical solutions, as it may be more convenient to extremize one scalar quantity *vs.* solving a large number of coupled equations. As stated, this allows us to find weak solutions to the equations, as the variational principle involves fewer derivatives than the partial differential equations that are its Euler-Lagrange equations. One important application of this is the study of shock formation and propagation.

6.4 Infinite-dimensional Hamiltonian systems

6.4.1 Defining the Hamiltonian

In the framework of partial differential equations with one spatial variable x, a Hamiltonian is a functional

$$H(u) = \int \mathcal{H}(u)dx,$$

where the Hamiltonian density $\mathcal{H}(u)$ is a function of u and a finite number of its spatial derivatives u_x, u_{xx}, ... The bounds on the integral depend on the boundary conditions given for the partial differential equation: if the equation is defined on a finite domain $x \in [a, b]$ then a and b are the lower and upper bounds for the integral defining the Hamiltonian. If the equation is defined over all of $\mathbb{R}$, then the integration is over $\mathbb{R}$. Note that for both periodic boundary conditions (finite domain) and vanishing boundary conditions the integral vanishes if $\mathcal{H}(u)$ is a total derivative, *i.e.*, $\mathcal{H}(u) = \partial_x F(u)$. For this reason, Hamiltonian densities differing by a total derivative are considered to be equivalent.

6.4.2 Defining the Poisson bracket

In order to define a Poisson bracket, we use an antisymmetric differential operator B, instead of an antisymmetric matrix:

$$(f, Bg) = \int fBgdx = -\int (Bf)gdx = -(Bf, g).$$

Then the Poisson bracket of any two functionals is defined by

$$\{F, G\} = \int \frac{\delta F}{\delta u} B \frac{\delta G}{\delta u} dx.$$

Bilinearity and antisymmetry follow immediately. The Leibniz rule is no longer required, as it does not make sense to talk about the product of two functionals. The Jacobi identity is still imposed and seriously restricts the set of possible antisymmetric differential operators B. Three commonly used B's are

$$\begin{aligned} B &= \partial_x \\ B &= u\partial_x + \partial_x u = 2u\partial_x + u_x \\ B &= \partial_x^3 + 2(u\partial_x + \partial_x u). \end{aligned}$$

It is easy to verify that these are antisymmetric, but checking the Jacobi identity is much harder. We skip this here.

6.4.3 Hamiltonian dynamics for partial differential equations

Having defined a Poisson bracket and a Hamiltonian, Hamiltonian dynamics on any functional on the phase space may be defined as before:

$$\frac{\partial K}{\partial t} = \{K(u), H(u)\}.$$

The following result handily allows us to define dynamics on the coordinate functions themselves:

$$u(x,t) = \int \delta(x-y)u(y,t)dy \quad \Rightarrow \quad \frac{\delta u(x)}{\delta u(y)} = \delta(x-y),$$

where $\delta(x-y)$ denotes the Dirac delta function (whose name is unfortunately identical to the symbol used for the variational derivative). Then

$$\begin{aligned} u_t &= \frac{\partial}{\partial t}u(x,t) \\ &= \{u(x,t), H(u(x,t))\} \\ &= \int \frac{\delta u(x,t)}{\delta u(y,t)} B \frac{\delta H(u(x,t))}{\delta u(y,t)} dy \\ &= \int \delta(x-y) B \frac{\delta H(u(x,t))}{\delta u(y,t)} dy \\ &= B\frac{\delta H(u(x,t))}{\delta u(x,t)}, \end{aligned}$$

or in short:

$$u_t = B\frac{\delta H}{\delta u}.$$

In the above, we tacitly assumed that u is a scalar function. If the Hamiltonian density depends on more than one function, all the above concepts may be generalized for vector functions. For instance, if one can identify coordinate functions q and p such that the equations of motion are given by

$$\frac{\partial q}{\partial t} = \frac{\delta H}{\delta p}, \quad \frac{\partial p}{\partial t} = -\frac{\delta H}{\delta q},$$

then the system is Hamiltonian with canonical Poisson bracket

$$\{F,G\} = \int \begin{pmatrix} \frac{\delta F}{\delta q} & \frac{\delta F}{\delta p} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta G/\delta q \\ \delta G/\delta p \end{pmatrix} dx = \int \left(\frac{\delta F}{\delta q}\frac{\delta G}{\delta p} - \frac{\delta F}{\delta p}\frac{\delta G}{\delta q} \right) dx,$$

which is easily verified. In this case

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

just like for a finite-dimensional system. In this case B is an antisymmetric differential operator of order 0.

6.5 The KdV equation as a Hamiltonian system

The KdV equation

$$u_t + uu_x + u_{xxx} = 0$$

is Hamiltonian with Poisson structure given by

$$B = \partial_x,$$

and Hamiltonian

$$H = \int \left(\frac{1}{2} u_x^2 - \frac{1}{6} u^3 \right) dx.$$

This is easily shown:

$$\begin{aligned} \frac{\delta H}{\delta u} &= \frac{\partial \mathcal{H}}{\partial u} - \frac{\partial}{\partial x} \frac{\partial \mathcal{H}}{\partial u_x} \\ &= -\frac{1}{2} u^2 - \frac{\partial}{\partial x} u_x \\ &= -\frac{1}{2} u^2 - u_{xx}, \end{aligned}$$

hence the dynamics is given by

$$\begin{aligned} u_t &= \partial_x \frac{\delta H}{\delta u} \\ &= \partial_x \left(-\frac{1}{2} u^2 - u_{xx} \right) \\ &= -uu_x - u_{xxx}, \end{aligned}$$

the KdV equation!

Remarks

- Through all the machinery we have set up, we immediately obtain that the Hamiltonian $H = \int \left(\frac{1}{2} u_x^2 - \frac{1}{6} u^3 \right) dx$ is a conserved quantity for the KdV equation.

- The Poisson operator B has Casimirs if $B(\delta C/\delta u) = 0$, in analogy with the finite-dimensional case. If such $C(u)$ may be found, it follows that every other functional, including the Hamiltonian, Poisson commutes with $C(u)$. With $B = \partial_x$, this requires C to be a linear functional of u and its derivatives. Since functional densities are considered equivalent if they are the same up to a total derivative, only

$$C = \int u dx$$

is important. Thus, $C = \int u dx$ is a Casimir for the KdV equation with Poisson structure ∂_x. It is also a conserved quantity, as it trivially Poisson commutes with the Hamiltonian.

This is easily verified directly: the KdV equation is rewritten as

$$\partial_t(u) + \partial_x \left(\frac{1}{2}u^2 + u_{xx} \right) = 0.$$

Integrating over all values of x ($\mathbb{R}$ or a finite interval for the periodic boundary condition case) gives

$$\begin{aligned} & \int \left(\partial_t(u) + \partial_x \left(\frac{1}{2}u^2 + u_{xx} \right) \right) dx = 0 \\ \Rightarrow \quad & \int (\partial_t u) dx = 0 \\ \Rightarrow \quad & \frac{d}{dt} \int u dx = 0 \\ \Rightarrow \quad & \frac{dC}{dt} = 0, \end{aligned}$$

and $C(u)$ is conserved. The integral of the x-derivative term vanishes because of the boundary conditions.

- Similarly, by multiplying the equation by u and integrating over all x-values, we obtain

$$\begin{aligned} & u u_t + u^2 u_x + u u_{xxx} = 0 \\ \Rightarrow \quad & \partial_t \left(\frac{1}{2}u^2 \right) + \partial_x \left(\frac{1}{3}u^3 + u u_{xx} - \frac{1}{2}u_x^2 \right) = 0 \\ \Rightarrow \quad & \int \left(\partial_t \left(\frac{1}{2}u^2 \right) + \partial_x \left(\frac{1}{3}u^3 + u u_{xx} - \frac{1}{2}u_x^2 \right) \right) dx = 0 \\ \Rightarrow \quad & \frac{d}{dt} \int \frac{1}{2}u^2 dx = 0, \end{aligned}$$

so $\frac{1}{2}\int u^2 dx$ is a conserved quantity for the KdV equation as well.

Thus we have constructed three conserved quantities for the KdV equation:

$$\begin{aligned} F_{-1} &= \int u dx, \\ F_0 &= \int \frac{1}{2} u^2 dx, \\ F_1 &= \int \left(\frac{1}{2} u_x^2 - \frac{1}{6} u^3 \right) dx = H(u). \end{aligned}$$

These are not unexpected. In many physical applications where the KdV equation arises, these correspond to the conservation of mass, momentum and energy in the system that is being modeled. The real question is whether there are any others. If there are, how many? In the next chapter we will demonstrate that the KdV equation has an infinite number of conserved quantities, that are all functionally independent. By itself, this is not enough to prove integrability of the KdV equation, but it is a strong hint. In order to prove integrability, we also need to prove that the constructed set of conserved quantities is complete, *i.e.*, spans half the space of the canonical variables for the KdV equation. This is a far more technical matter, which is often ignored.

6.6 The NLS equation as a Hamiltonian system

The Nonlinear Schrödinger (NLS) equation describes the slow modulation of a wave train due to nonlinear effects and dispersion. The equation is given by

$$iq_t = -q_{xx} + \sigma |q|^2 q.$$

If $\sigma = 1$ the equation is known as the defocusing or repulsive NLS equation, whereas if $\sigma = -1$, the equation is said to be focusing or attractive.

The NLS equation may be embedded in the system of NLS-like equations

$$\begin{cases} iq_t &= -q_{xx} + \sigma q^2 p, \\ -ip_t &= -p_{xx} + \sigma p^2 q, \end{cases},$$

which contains the NLS equation using the consistent reduction $p = q^*$, the complex conjugate of q. Then the first equation is the NLS equation and the second equation is its complex conjugate equation.

This system of equations is Hamiltonian with canonical Poisson structure and Hamiltonian

$$H = \int \mathcal{H}(q, p, q_x, p_x) dx = -i \int \left(q_x p_x + \frac{\sigma}{2} q^2 p^2 \right) dx.$$

This is an easy calculation. We need to verify that the equations

$$\begin{pmatrix} q_t \\ p_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta H/\delta q \\ \delta H/\delta p \end{pmatrix} \quad \Rightarrow \quad \begin{cases} q_t &= \dfrac{\delta H}{\delta p} \\ p_t &= -\dfrac{\delta H}{\delta q} \end{cases}$$

are equivalent to the given NLS-like system. We have

$$\begin{aligned} \frac{\delta H}{\delta q} &= \frac{\partial \mathcal{H}}{\partial q} - \partial_x \frac{\partial \mathcal{H}}{\partial q_x} = -i\left(\sigma q p^2 - \partial_x p_x\right) = -i\left(\sigma q p^2 - p_{xx}\right), \\ \frac{\delta H}{\delta p} &= \frac{\partial \mathcal{H}}{\partial p} - \partial_x \frac{\partial \mathcal{H}}{\partial p_x} = -i\left(\sigma p q^2 - \partial_x q_x\right) = -i\left(\sigma p q^2 - q_{xx}\right), \end{aligned}$$

and, indeed, taking into account the $-$ sign in the definition of the dynamical equation for p, the NLS-like system is recovered, containing the NLS equation as a special case.

Since the Poisson structure for this Hamiltonian system is canonical, there are no Casimirs. However, one easily verifies that the following are conserved quantities:

$$\begin{aligned} F_0 &= \int qp\, dx, \\ F_1 &= i\int \left(qp_x - q_x p\right) dx, \\ F_2 &= -i\int \left(q_x p_x + \frac{\sigma}{2} q^2 p^2\right) dx, \end{aligned}$$

where $F_2 = H$, the Hamiltonian. These imply conserved quantities for the NLS equation by letting $p = q^*$:

$$\begin{aligned} F_0 &= \int |q|^2 dx, \\ F_1 &= i\int \left(qq_x^* - q^* q_x\right) dx, \\ F_2 &= -i\int \left(|q_x|^2 + \frac{\sigma}{2}|q|^4\right) dx. \end{aligned}$$

As for the KdV equation, the existence of these three conserved quantities is not a surprise. Depending on the application, they often correspond to the conservation of mass, momentum and energy. Again, the real question is whether any other conserved quantities exist. Like for the KdV equation, one may construct an infinite sequence of independent conserved quantities, of which F_0, F_1 and F_2 are the first members.

6.7 Exercises

1. Consider the Modified Vector Derivative NLS equation

$$\boldsymbol{B}_t + (\|\boldsymbol{B}\|^2\boldsymbol{B})_x + \gamma(\boldsymbol{e}_1 \times \boldsymbol{B}_0)\,(\boldsymbol{e}_1 \cdot (\boldsymbol{B}_x \times \boldsymbol{B}_0)) + \boldsymbol{e}_1 \times \boldsymbol{B}_{xx} = 0.$$

This equation describes the transverse propagation of nonlinear Alfvén waves in magnetized plasmas. Here $\boldsymbol{B} = (0, u, v)$, $\boldsymbol{e}_1 = (1, 0, 0)$, $\boldsymbol{B}_0 = (0, B_0, 0)$, and γ is a constant. The boundary conditions are $\boldsymbol{B} \to \boldsymbol{B}_0$, $\boldsymbol{B}_x \to 0$ as $|x| \to \infty$. By looking for stationary solutions $\boldsymbol{B} = \boldsymbol{B}(x - Wt)$, one obtains a system of ordinary differential equations. Integrating once, one obtains a first-order system of differential equations for u and v.

   a) Show that this system is Hamiltonian with canonical Poisson structure, by constructing its Hamiltonian $H(u, v)$.

   b) Find the value of the Hamiltonian such that the boundary conditions are satisfied. Then $H(u, v)$ equated to this constant value defines a curve in the (u, v)-plane on which the solution lives. In the equation of this curve, let $U = u/B_0$, $V = v/B_0$, and $W_0 = W/B_0^2$. Now there are only two parameters in the equation of the curve: W_0 and γ.

   c) With $\gamma = 1/10$, plot the curve for $W_0 = 3$, $W_0 = 2$, $W_0 = 1.1$, $W_0 = 1$, $W_0 = 0.95$, $W_0 = 0.9$. All of these curves have a singular point at $(1, 0)$. This point is an equilibrium point for the Hamiltonian system, corresponding to the constant solution which satisfies the boundary condition. The curves beginning and ending at this equilibrium point correspond to soliton solutions of the Modified Vector Derivative NLS equation. How many soliton solutions are there for the different velocity values you considered? Draw a qualitatively correct picture of the solitons for all these cases.

2. Show that the canonical Poisson bracket

$$\{f, g\} = \sum_{j=1}^{N} \left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} \right)$$

satisfies the Jacobi identity

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0.$$

3. Show that the Sine-Gordon equation

$$u_{tt} - u_{xx} + \sin(u) = 0$$

is Hamiltonian with canonical Poisson structure and Hamiltonian

$$H = \int \left(\frac{1}{2}p^2 + \frac{1}{2}q_x^2 + 1 - \cos(q) \right) dx,$$

where $q = u$, and $p = u_t$.

4. Check explicitly that the conserved quantities $F_{-1} = \int u dx$, $F_0 = \int \frac{1}{2}u^2 dx$, $F_1 = \int \left(\frac{1}{6}u^3 - \frac{1}{2}u_x^2\right) dx$, $F_2 = \int \left(\frac{1}{24}u^4 - \frac{1}{2}uu_x^2 + \frac{3}{10}u_{xx}^2\right) dx$ are mutually in involution with respect to the Poisson bracket defined by the Poisson structure given by ∂_x.

5. **Bi-Hamiltonian systems** We know that the KdV equation $u_t = uu_x + u_{xxx}$ is Hamiltonian with Poisson structure defined by ∂_x and Hamiltonian $F_1 = \int \left(\frac{1}{6}u^3 - \frac{1}{2}u_x^2\right) dx$. Show that the KdV equation is also Hamiltonian with Poisson structure $\partial_{xxx} + \frac{1}{3}(u\partial_x + \partial_x u)$ and Hamiltonian $F_0 = \frac{1}{2}\int u^2 dx$. An equation which is Hamiltonian with two different Poisson structures is called Bi-Hamiltonian. One can show that the presence of a Bi-Hamiltonian structure implies the existence of an infinite number of conserved quantities, which are in involution under the Poisson Brackets defined by both Poisson structures.

Chapter 7

Conserved quantities and hierarchies of integrable equations

Throughout these notes the importance of conserved quantities has been emphasized. In this chapter we examine conserved quantities in detail. First, we examine direct methods to calculate them if they exist. Next, we prove the existence of an infinite sequence of conserved quantities for the KdV equation. Using this sequence of conserved quantities, we define the KdV hierarchy as an infinite sequence of equations whose flows commute with the flow of the KdV equation.

7.1 Calculating conserved quantities using the scaling symmetry

7.1.1 Generalities

Repeating the argument from the previous chapter,

$$F(u, u_x, \ldots) = \int f(u, u_x, \ldots) dx$$

is a conserved quantity for the equation

$$u_t = N(u, u_x, \ldots, u_{nx})$$

if

$$\frac{dF}{dt} = 0 \Leftrightarrow \int \frac{\partial}{\partial t} f(u, u_x, \ldots) dx = 0,$$

which occurs when

$$\frac{\partial}{\partial t} f(u, u_x, \ldots) = -\frac{\partial}{\partial x} g(u, u_x, \ldots),$$

(the $-$ sign is for convenience) for some function $g(u, u_x, \ldots)$ of u and its x-derivatives. Conversely, if

$$\frac{\partial f}{\partial t} + \frac{\partial g}{\partial x} = 0,$$

for some functions f and g depending on u and its x-derivatives then $F = \int f dx$ is a conserved quantity for the equation we are considering. The quantity f is referred to as the conserved density; j is called its corresponding flux. At this point, we should also note that if $f(u, u_x, \ldots) = \partial_x h(u, u_x, \ldots)$, then F is trivial and such conserved densities should be ignored. Thus, as for Hamiltonians and other functional densities, conserved densities are determined up to total x-derivatives.

7.1.2 Using the scaling symmetry

Following the scheme of the previous chapter, we could attempt to contruct more conserved quantities by multiplying the KdV equation by increasingly large powers of u, but this fails quickly. How then do we proceed to construct further conserved quantities, if there are any?

At this point, we turn to the scaling symmetry of the KdV equation. The KdV equation is invariant under the scaling transformation (check!)

$$\begin{cases} x' &= \epsilon\, x \\ t' &= \epsilon^3\, t \\ u' &= u/\epsilon^2 \end{cases},$$

implying that the KdV equation expressed in terms of the new function $u'(x', t')$ of the new variables x' and t' is identical to the KdV equation expressed in terms of its original variables x, t and $u(x, t)$. Thus, we see that u scales with a factor of ϵ^2, ∂_x scales with a factor of ϵ and ∂_t scales with a factor of ϵ^3. With this knowledge, a scaling factor may be associated with every term that is a combination of u, and its x- and t-derivatives. Since any conserved quantity statement of an equation should inherit the scaling symmetry of that equation, it is only sensible to add or subtract terms in a conserved density flux that scale in the same way. This statement is completely analogous to the physics statement that only quantities with the same units may be added or subtracted.

It is clear from the above argument that any combination of u and its x- and t-derivatives scales as a certain power of ϵ. At this point, we may forget about ϵ and focus on its exponent. We refer to the exponent of the scaling factor of a given term as the **weight** of that terms. Denoting the weight by square parentheses $[\cdot]$, we have

$$\begin{aligned} [\partial_x] &= 1, \\ [\partial_t] &= 3, \\ [u] &= 2. \end{aligned}$$

With this information, it is easy to see that all terms of the KdV equation do indeed have the same weight, *i.e.*, scale the same way:

$$\begin{aligned}[u_t] &= [u] + [\partial_t] = 2 + 3 = 5,\\ [uu_x] &= 2[u] + [\partial_x] = 4 + 1 = 5,\\ [u_{xxx}] &= [u] + 3[\partial_x] = 2 + 3 = 5.\end{aligned}$$

Weights are also an easy way to determine scaling symmetries of an equation. Suppose we don't know the scaling symmetry. Imposing that all terms of the KdV equation have the same weight, we obtain

$$[u] + [\partial_t] = 2[u] + [\partial_x] = [u] + 3[\partial_x].$$

These equations have the above weights as solutions. Since these equations are homogeneous, *i.e.*, $(0, 0, 0)$ is a solution, any multiple of $([\partial_x], [\partial_t], [u]) = \lambda(1, 3, 2)$ is a solution as well. This corresponds to using ϵ^λ as a scaling parameter, as opposed to ϵ.

7.1.3 Examples

Let us use the principle of equal weight to construct some conserved quantities for the KdV equation. Let us start with some results that we already know. Choosing a weight for f, $[f]$, in

$$f_t + g_x = 0$$

implies that $[g] = [f] + 2$. In what follows we construct conserved densities of weight 2, 3, 4, 5, 6 and 8.

- $\boldsymbol{[f] = 2}$**.** What terms can appear in f, if $[f] = 2$? Since $[u] = 2$, the only possibility is $f = u$. Next we require g such that $[g] = 4$. Possible terms for such g are u^2, u_{xx}. Thus

  $$g = c_1 u^2 + c_2 u_{xx}.$$

  Expressing the equality $f_t + g_x = 0$ gives an expression involving u and its x- and t-derivatives. All t-derivatives are replaced using the KdV equation, resulting in an expression involving u and some of its x-derivatives. The coefficients of all the remaining terms necessarily cancel, resulting in linear algebraic equations for the unknown coefficients c_1 and c_2:

$$\begin{aligned} u_t + 2c_1 u u_x + c_2 u_{xxx} &= 0 \\ \Rightarrow -u u_x - u_{xxx} + 2c_1 u u_x + c_2 u_{xxx} &= 0 \\ \Rightarrow (-1 + 2c_1) u u_x + (-1 + c_2) u_{xxx} &= 0 \\ \Rightarrow -1 + 2c1 = 0, \quad -1 + c_2 &= 0 \\ \Rightarrow c_1 = 1/2, \quad c_2 &= 1, \end{aligned}$$

resulting in the conservation law

$$\partial_t u + \partial_x \left(\frac{1}{2} u^2 + u_{xx} \right) = 0,$$

and the conserved quantity $F_{-1} = \int u dx$.

- $[\boldsymbol{f}] = \mathbf{3}$. With $[f] = 3$, only $f = u_x$ is possible. Since $u_x = \partial_x(u)$ is an x-derivative, this results in a trivial conservation law:

$$\partial_t(u_x) + \partial_x(-u_t) = 0,$$

and no nontrivial conservation law of weight 3 is found for the KdV equation.

- $[\boldsymbol{f}] = \mathbf{4}$. With $[f] = 4$, only the terms u^2 and u_{xx} are possible. This second term is an x-derivative, and only has a trivial contribution. We may restrict ourselves to $f = u^2$. Then $[g] = 6$, with possible terms for g being u^3, u_x^2, $u u_{xx}$, u_{xxxx}. Thus

$$g = c_1 u^3 + c_2 u_x^2 + c_3 u u_{xx} + c_4 u_{xxxx},$$

resulting in

$$\begin{aligned} \partial_t(u^2) + \partial_x(c_1 u^3 + c_2 u_x^2 + c_3 u u_{xx} + c_4 u_{xxxx}) &= 0 \\ \Rightarrow 2 u u_t + 3c_1 u^2 u_x + 2c_2 u_x u_{xx} + c_3 u_x u_{xx} + c_3 u u_{xxx} + c_4 u_{5x} &= 0 \\ \Rightarrow 2u(-u u_x - u_{xxx}) + 3c_1 u^2 u_x + 2c_2 u_x u_{xx} + c_3 u_x u_{xx} + c_3 u u_{xxx} + c_4 u_{5x} &= 0 \\ \Rightarrow (-2 + 3c_1) u^2 u_x + (-2 + c_3) u u_{xxx} + (2c_2 + c_3) u_x u_{xx} + c_4 u_{5x} &= 0 \\ \Rightarrow -2 + 3c_1 = 0, \quad -2 + c_3 = 0, \quad 2c_2 + c_3 = 0, \quad c_4 &= 0 \\ \Rightarrow c_1 = 2/3, \quad c_2 = -1, \quad c_3 = 2, \quad c_4 &= 0, \end{aligned}$$

resulting in the conservation law

$$\partial_t \left(u^2 \right) + \partial_x \left(\frac{2}{3} u^3 + -u_x^2 + 2 u u_{xx} \right) = 0,$$

and the conserved quantity

$$F_0 = \frac{1}{2}\int u^2 dx.$$

The factor 1/2 is for convenience. Clearly, any constant multiple of a conserved quantity is again a conserved quantity.

Note that the scaling symmetry approach reduces the problem of finding conserved quantities to one of solving a system of linear algebraic equations for the unknown coefficients c_i. This approach is highly algorithmic and is ideally suited for automatization using a symbolic computation package such as Maple$^{(c)}$ or Mathematica$^{(c)}$.

- $[\boldsymbol{f}] = \mathbf{5}$. With $[f] = 5$, only the terms uu_x and u_{xxx} need to be considered. But both terms are x-derivatives: $uu_x = \partial_x(u^2/2)$ and $u_{xxx} = \partial_x(u_{xx})$. Thus $[f] = 5$ only results in a trivial conservation law. It is generally true that odd values of $[f]$ result in trivial conservation laws for the KdV equation.

- $[\boldsymbol{f}] = \mathbf{6}$. With $[f] = 6$, the possible terms are u^3, u_x^2, uu_{xx} and u_{xxxx}. The second and third term are equivalent, up to an x-derivative: $uu_{xx} = \partial_x(uu_x) - u_x^2$, and the last term is an x-derivative. Therefore we have

$$f = a_1 u^3 + a_2 u_x^2,$$

where a_1 and a_2 are coefficients, to be determined. With $[f] = 6$, we need $[g] = 8$, with possible terms u^4, u^2u_{xx}, uu_x^2, u_{xx}^2, u_xu_{xxx}, uu_{xxxx} and u_{6x}:

$$g = c_1 u^4 + c_2 u^2 u_{xx} + c_3 u u_x^2 + c_4 u_{xx}^2 + c_5 u_x u_{xxx} + c_6 u u_{xxxx} + c_7 u_{6x}.$$

In order to eliminate t-derivatives from $f_t = 3a_1u^2u_t + 2a_2u_xu_{xt}$, we need

$$u_t = -uu_x - u_{xxx} \quad \Rightarrow \quad u_{xt} = -u_x^2 - uu_{xx} - u_{xxxx}.$$

Then

$$\begin{gathered} f_t + g_x = 0 \\ \Rightarrow 3a_1u^2u_t + 2a_2u_xu_{xt} + \partial_x\left(c_1u^4 + c_2u^2u_{xx} + c_3uu_x^2 + c_4u_{xx}^2 + c_5u_xu_{xxx} + c_6uu_{xxxx} + c_7u_{6x}\right) = 0 \\ \Rightarrow 3a_1u^2\left(-uu_x - u_{xxx}\right) + 2a_2u_x\left(-u_x^2 - uu_{xx} - u_{xxxx}\right) + 4c_1u^3u_x + 2c_2uu_xu_{xx} + \\ c_2u^2u_{xxx} + c_3u_x^3 + 2c_3uu_xu_{xx} + 2c_4u_{xx}u_{xxx} + c_5u_{xx}u_{xxx} + c_5u_xu_{xxxx} + \\ c_6u_xu_{xxxx} + c_6uu_{5x} + c_7u_{7x} = 0, \end{gathered}$$

resulting in the equations:

$$\begin{aligned}
u^3u_x: &\quad -3a_1 + 4c_1 = 0,\\
u^2u_{xxx}: &\quad -3a_1 + c_2 = 0,\\
u_x^3: &\quad -2a_2 + c_3 = 0,\\
uu_xu_{xx}: &\quad -2a_2 + 2c_2 + 2c_3 = 0,\\
u_xu_{xxxx}: &\quad -2a_2 + c_5 + c_6 = 0,\\
u_{xx}u_{xxx}: &\quad 2c_4 + c_5 = 0,\\
uu_{5x}: &\quad c_6 = 0,\\
u_{7x}: &\quad c_7 = 0.
\end{aligned}$$

Choosing $a_2 = 1/2$, we obtain the solution $a_1 = -1/6$, $c_1 = -1/8$, $c_2 = -1/2$, $c_3 = 1$, $c_4 = -1/2$, $c_5 = 1$, $c_6 = 0$ and $c_7 = 0$. Thus, the conservation law is

$$\partial_t\left(\frac{1}{2}u_x^2 - \frac{1}{6}u^3\right) + \partial_x\left(-\frac{1}{8}u^4 - \frac{1}{2}u^2u_{xx} + uu_x^2 - \frac{1}{2}u_{xx}^2 + u_xu_{xxx}\right) = 0,$$

resulting in the converved quantity

$$F_1 = \int\left(\frac{1}{2}u_x^2 - \frac{1}{6}u^3\right)dx,$$

the Hamiltonian for the KdV equation.

- $[\boldsymbol{f}] = \mathbf{8}$**.** With $[f] = 8$, the possible terms are u^4, u^2u_{xx}, uu_x^2, u_{xx}^2, u_xu_{xxx}, uu_{xxxx} and u_{6x}. The second and third term are equivalent, up to an x-derivative, as are the fourth, fifth and sixth terms. The last term is an x-derivative. Therefore we have

$$f = a_1u^4 + a_2uu_x^2 + a_3u_{xx}^2,$$

where a_1, a_2 and a_3 are coefficients, to be determined. With $[f] = 8$, we need $[g] = 10$, with possible terms u^5, u^3u_{xx}, $u^2u_x^2$, u^2u_{xxxx}, uu_xu_{xxx}, uu_{xx}^2, $u_x^2u_{xx}$, uu_{6x}, u_xu_{5x}, $u_{xx}u_{xxxx}$, u_{xxx}^2 and u_{8x}. Proceeding in a similar way to the above calculations, we find the next conserved quantity to be

$$F_2 = \int\left(\frac{1}{24}u^4 - \frac{1}{2}uu_x^2 + \frac{3}{10}u_{xx}^2\right)dx.$$

It is clear that we could proceed this way. It is equally clear that the calculations are quickly becoming very tedious. Given that we have constructed one more conserved quantity than one could be led to believe exists from physical considerations, and that it seems even more such quantities could be constructed, is it possible there are an infinite number?

7.2 The KdV equation has an infinite number of conserved quantities

In this section, we prove that the KdV equation possesses an infinite number of conserved quantities. The main tool for this is the Miura transformation:

$$u = -6(v^2 + v_x),$$

which transforms a function $v(x,t)$ to a function $u(x,t)$. Notice that this transformation is not invertible: both $v(x,t) = 0$ and $v(x,t) = 1/x$ give rise to $u(x,t) = 0$. We have the following

Claim: If $v(x,t)$ satisfies the Modified KdV equation

$$v_t - 6v^2 v_x + v_{xxx} = 0,$$

then its Miura tranform $u(x,t)$ satisfies the KdV equation $u_t + uu_x + u_{xxx} = 0$.

Before proceeding to prove the claim, we should pause and wonder at the marvel that is the Miura transformation: given one solution to a certain nonlinear partial differential equation, we may produce a solution to a different nonlinear partial differential equation. The Miura transformation is our first encounter with a Bäcklund transformation: a Bäcklund transformation is a transformation that transforms solutions of one equation to solutions of another one. If the source and target differential equations are the same, the transformation is called an auto-bäcklund transformation.

Proof of the claim:

$$\begin{aligned}
u_t + uu_x + u_{xxx} &= -6(2vv_t + v_{xt}) + 36(v^2 + v_x)(2vv_x + v_{xx}) - 6\partial_x^2(2vv_x + v_{xx}) \\
&= -12v(6v^2 v_x - v_{xxx}) - 6\partial_x(6v^2 v_x - v_{xxx}) + 36(v^2 + v_x)(2vv_x + v_{xx}) \\
&\quad -6\partial_x(2v_x^2 + 2vv_{xx} + v_{xxx}) \\
&= -12v(6v^2 v_x - v_{xxx}) - 6(12vv_x^2 + 6v^2 v_{xx} - v_{xxxx}) + \\
&\quad 36(v^2 + v_x)(2vv_x + v_{xx}) - 6(4v_x v_{xx} + 2v_x v_{xx} + 2vv_{xxx} + v_{xxxx}) \\
&= 0.
\end{aligned}$$

Using the Miura transformation to show that the KdV equation has an infinite number of conserved quantities proceeds in four steps.

1. **The KdV equation has a Galilean invariance.** Indeed, the KdV equation is invariant under the transformation of $(x,t,u) \to (x',t',u')$ given by

$$\begin{cases} x' &= x + t/\epsilon^2 \\ t' &= t \\ u &= u' - 1/\epsilon^2. \end{cases}$$

This is verified directly:

$$\begin{aligned} u_t + uu_x + u_{xxx} &= \left(\partial_{t'} + \frac{1}{\epsilon^2}\partial_{x'}\right)\left(u' - \frac{1}{\epsilon^2}\right) + \left(u' - \frac{1}{\epsilon^2}\right)\frac{\partial u'}{\partial x'} + \frac{\partial^3 u'}{\partial x'^3} \\ &= u'_{t'} + u'u'_x + u'_{x'x'x'}. \end{aligned}$$

2. **The Gardner equation.** The transformation

$$v = -\epsilon w(x', t') + \frac{\beta}{\epsilon},$$

with $\beta = 1/\sqrt{6}$, and x' and t' defined by the above Galilean transformation, when applied to the Modified KdV equation, results in the Gardner equation

$$w_{t'} + \partial_{x'}\left(w_{x'x'} + \sqrt{6}w^2 - 2\epsilon^2 w^3\right) = 0.$$

As before, the proof is by direct verification. The Gardner equation is written as a conservation law, therefore it is clear that $\int w dx'$ is a conserved quantity for the Gardner equation, or

$$\frac{d}{dt'}\int w dx' = 0.$$

3. **The Miura transformation for the Gardner equation.** In these new variables, the Miura transformation $w(x', t') \to u'(x', t')$ becomes

$$\begin{aligned} u' - \frac{1}{\epsilon^2} &= -6\left(\left(-\epsilon w + \frac{\beta}{\epsilon}\right)^2 - \epsilon w_{x'}\right) \\ &= -6\left(\epsilon^2 w^2 - 2w\beta + \frac{1}{6\epsilon^2}\right) + 6\epsilon w_{x'} \\ \Rightarrow \quad u' &= 12\beta w + 6\epsilon w_{x'} - 6\epsilon^2 w^2. \end{aligned}$$

4. **Inverting the Miura transform.** Thinking of ϵ as small, this last expression for the Miura transform may be inverted to express $w(x', t')$ as a function of $u'(x', t')$, for small ϵ. In what follows, all primes are dropped for the sake of simplicity. Let

$$w = \sum_{n=0}^{\infty} \epsilon^n w_n = w_0 + \epsilon w_1 + \epsilon^2 w_2 + \ldots.$$

Before proceeding to invert the Miura transformation, it is important to consider the bigger picture: since $\int w dx$ is conserved, and ϵ is independent of x and t, so is everyone of its coefficients of any power of ϵ^n, $n = 0, 1, \ldots$. But all these coefficients are functions of u and its derivatives, through the inverted Miura transformation. Thus we construct an infinite sequence of conserved quantities for the KdV equation. The main principle behind this is that we have constructed one conserved quantity $\int w dx$, depending on a parameter ϵ. Since ϵ is arbitrary (as long as it is small), the Taylor coefficients of this conserved quantity, expanded as a series in ϵ are all conserved individually.

Substituting the expansion for w in the Miura transformation gives

$$\begin{aligned} u &= 12\beta \sum_{n=0}^{\infty} \epsilon^n w_n + 6\epsilon \sum_{n=0}^{\infty} \epsilon^n w_{n,x} - 6\epsilon^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \epsilon^{i+j} w_i w_j \\ &= 12\beta w_0 + 12\beta\epsilon w_1 + 6\epsilon w_{0,x} + 12\beta \sum_{n=2}^{\infty} \epsilon^n w_n + 6 \sum_{n=2}^{\infty} \epsilon^n w_{n-1,x} - 6 \sum_{n=2}^{\infty} \epsilon^n \sum_{k=0}^{n-2} w_k w_{n-2-k}, \end{aligned}$$

resulting in

$$\begin{aligned} \epsilon^0 : &\quad u = 12\beta w_0 \quad \Rightarrow \quad w_0 = \frac{1}{2\sqrt{6}} u \\ \epsilon^1 : &\quad 0 = 12\beta w_1 + 6 w_{0,x} \quad \Rightarrow \quad w_1 = -\frac{1}{4} u_x \\ \epsilon^n : &\quad 0 = 12\beta w_n + 6 w_{n-1,x} - 6 \sum_{k=0}^{n-2} w_k w_{n-2-k}, \quad \text{for} \quad n \geq 2. \end{aligned}$$

This last equation gives

$$w_n = -\frac{1}{2\beta} w_{n-1,x} + \frac{1}{2\beta} \sum_{k=0}^{n-2} w_k w_{n-2-k},$$

valid for $n = 2, 3, \ldots$. Together with the initiations for w_0 and w_1, this is a recursion relationship allowing for the determination of all w_n's, $n = 0, 1, \ldots$. We obtain

$$\begin{aligned} w_2 &= \frac{\sqrt{6}}{8} u_{xx} + \frac{1}{8\sqrt{6}} u^2, \\ w_3 &= -\frac{1}{8} \partial_x \left(3u_{xx} + u^2 \right), \end{aligned}$$

and so on. One can show that all w_n's with odd n are x-derivatives. They give rise to trivial conserved quantities. For n even, the entire KdV sequence of conserved quantities is obtained, each conserved density containing a term with a higher power of u, allowing us to conclude functional independence of the conserved quantities. Note that the conserved densities with even n may contain terms that are x-derivatives. These terms may be ignored when writing down the corresponding conserved quantity, but they must be retained when using the recursion relationship to find high-order terms. It is possible, but significantly harder than the proof for functional independence, to prove that these conserved quantities are mutually in involution with respect to the KdV Poisson structure ∂_x. This is beyond the scope of these notes.

In summary, we have shown in a constructive way that the KdV equation has an infinite set of conserved quantities

$$\int w_0 dx, \quad \int w_1 dx, \quad \int w_2 dx, \quad \int w_3 dx, \ldots.$$

Our method of construction may seem like it relied heavily on various tricks. After we introduce Lax pairs, we will see more systematic ways of demonstrating the existence of an infinite number of conserved quantities.

7.3 The KdV hierarchy

Previously we found that the KdV equation may be written as a Hamiltonian system with Poisson structure $B = \partial_x$ and Hamiltonian given by the conserved quantity F_1:

$$F_1 = \int \left(\frac{1}{2} u_x^2 - \frac{1}{6} u^3 \right) dx.$$

Any functional F may be used to define a Hamiltonian system by $u_t = \partial_x \frac{dF}{du}$, thus specifically we may define an infinite hierarchy of equations by using the infinite sequence of conserved quantities F_n, $n = -1, 0, 1, \ldots$ of the KdV equation. This hierarchy of equations is referred to as the KdV hierarchy, as it contains the KdV equation as its $n = 1$ member. Thus, the n-th member of the KdV hierarchy is defined by

$$\frac{\partial u}{\partial t_n} = \partial_x \frac{dF_n}{du},$$

valid for $n = -1, 0, 1, \ldots$. In order to set the KdV time variable apart from the time variable for the other members of the hierarchy, we have introduced new time-like variables t_n, $n = -1, 0, 1, \ldots$. Let's consider the first few members of this infinite hierarchy.

- $\boldsymbol{n = -1}$**.** Then $F_{-1} = \int u dx$, and the $n = -1$ member of the KdV hierarchy is

$$u_{t_{-1}} = \partial_x 1 \quad \Rightarrow u_{t_{-1}} = 0.$$

This is the reason we started counting the sequence of conserved quantities from $n = -1$. The above calculation is nothing but a verification of the fact that F_{-1} is the Casimir functional for the KdV Poisson structure. In what follows, the members of the KdV hierarchy will be counted starting from $n = 0$.

- $\boldsymbol{n = 0}$. Then $F_0 = \frac{1}{2}\int u^2 dx$, and

$$u_{t_0} = \partial_x u \quad \Rightarrow u_{t_0} = u_x.$$

This is the first-order wave equation, or the transport equation with unit velocity. For this reason, the x variable of the KdV hierarchy is often identified with t_1. This is the last linear equation of the hierarchy.

- $\boldsymbol{n = 1}$. Then $F_1 = \int \left(\frac{1}{2}u_x^2 - \frac{1}{6}u^3\right) dx$, and

$$u_{t_1} = \partial_x \left(-\frac{1}{2}u^2 - \partial_x u_x\right) \quad \Rightarrow u_{t_1} = -uu_x - u_{xxx}.$$

Thus the KdV equation is the "first" member of the KdV hierarchy.

- $\boldsymbol{n = 2}$. Then $F_2 = \int \left(\frac{1}{24}u^4 - \frac{1}{2}uu_x^2 + \frac{3}{10}u_{xx}^2\right) dx$, and

$$u_{t_2} = \partial_x \left(\frac{1}{6}u^3 - \frac{1}{2}u_x^2 - \partial_x(-uu_x) + \partial_x^2\left(\frac{3}{5}u_{xx}\right)\right) \;\Rightarrow\; u_{t_2} = \frac{1}{2}u^2u_x + 2u_xu_{xx} + uu_{xxx} + \frac{3}{5}u_{5x}.$$

This is the second KdV equation. As n increases, the equations become more complicated, more nonlinear, but also more dispersive.

Claim: F_{-1}, F_0, F_1, ... are a common set of conserved quantities for all members of the KdV hierarchy.

Proof: Consider the n-th KdV equation, $n \in \mathbb{N}$,

$$u_{t_n} = \partial_x \frac{dF_n}{du}.$$

We need to show that $\partial F_k/\partial t_n = 0$, $(n \geq -1)$ which is equivalent to

$$\{F_k, F_n\} = 0,$$

since F_n is the Hamiltonian for the n-th KdV equation. But this is true for any k and n, as all conserved quantities F_k, $k = -1, 0, 1, \ldots$ are mutually in involution under the KdV Poisson structure, and this Poisson structure is identical for all members of the KdV hierarchy.

Next we prove

Claim: All members of the KdV hierarchy commute. In other words, if $S_k(t_{k1}, t_{k0})$ is the solution operator for the k-th KdV equation (thus $S_k(t_{k1}, t_{k0})$ evolves a solution of the k-th KdV equation from $t_k = t_{k0}$ to $t_k = t_{k1}$), then $S_j S_k = S_k S_j$, for any $j, k = 0, 1, \ldots$.

Proof: We prove an infinitesimal version of the claim. A small change in t_j is given by ∂_{t_j}, while a small change in t_k is given by ∂_{t_k}. Thus, we verify that

$$\frac{\partial}{\partial t_j}\frac{\partial u}{\partial t_k} = \frac{\partial}{\partial t_k}\frac{\partial u}{\partial t_j}.$$

Starting from the left-hand side

$$\begin{aligned}
\frac{\partial}{\partial t_j}\frac{\partial u}{\partial t_k} &= \left\{\frac{\partial u}{\partial t_k}, F_j\right\} \\
&= \{\{u, F_k\}, F_j\} \\
&= -\{\{F_k, F_j\}, u\} - \{\{F_j, u\}, F_k\} \\
&= -\{\{F_j, u\}, F_k\} \\
&= \{\{u, F_j\}, F_k\} \\
&= \left\{\frac{\partial u}{\partial t_j}, F_k\right\} \\
&= \frac{\partial}{\partial t_k}\frac{\partial u}{\partial t_j},
\end{aligned}$$

where we have used the Jacobi identity and the fact that $\{F_k, F_j\} = 0$. This proves the claim.

Example Let's use an example to illustrate that the above property is not trivial. Consider the two ordinary differential equations

$$\frac{dz_1}{dx} = -z_1^2,$$

and

$$\frac{dz_2}{dy} = z_2 - 1.$$

Then

$$z_1(x) = \frac{z_1(0)}{1 + x z_1(0)},$$

and

$$z_2(y) = (z_2(0) - 1)e^y - 1.$$

Using $z_1(x)$ with $z_1(0) = \alpha$ as an initial condition for $z_2(y)$ gives

$$z_{12}(x, y, \alpha) = \frac{(\alpha(1 - x) - 1)\, e^y - 1 - \alpha x}{1 + \alpha x},$$

whereas using $z_2(y)$ with $z_2(0) = \alpha$ as an initial condition for $z_1(x)$ gives

$$z_{21}(x, y, \alpha) = \frac{(\alpha - 1)e^y - 1}{1 + x\,((\alpha - 1)e^y - 1)}.$$

Clearly $z_{12}(x, y, \alpha) \neq z_{21}(x, y, \alpha)$, and the flows of these two differential equations do not commute. In this example we could explicitly calculate the solution operators of the equations. Typically, this is not possible. In that case, the commutativity or the absence thereof may be tested by computing

$$\begin{aligned} \frac{d}{dx}\frac{d}{dy}z - \frac{d}{dy}\frac{d}{dx}z &= \frac{d}{dx}(z - 1) - \frac{d}{dy}(-z^2) \\ &= \frac{dz}{dx} + 2z\frac{dz}{dy} \\ &= -z^2 + 2z(z - 1) \\ &= z^2 - 2z \neq 0, \end{aligned}$$

which is a far more straightforward calculation than actually solving both equations.

Because the different members of the KdV hierarchy commute, they have a common set of solutions. Thus, it is possible to construct solutions $u(x, t_j, t_k)$ that are simultaneous solutions to the j-th and k-th KdV equations. We have already seen and used this: when we derived the KdV equation in the water wave problem, we first encountered the first-order one-directional wave equation. Then we constructed soliton solutions that depended on both τ_0 and τ_1, the time variable for the wave equation and the KdV equation respectively. Because of this it may be tempting to guess that the higher members of the KdV hierarchy correspond to higher-order evolution of the wave form in (for our case) the water wave problem. Unfortunately, this is not true.

7.4 The stationary KdV hierarchy

Previously, we studied the one-soliton solution of the KdV equation by considering a stationary reduction of the KdV equation. In this section, we generalize this to study multi-soliton solutions which may be found as stationary solutions of higher-order members of the KdV hierarchy.

The stationary KdV hierarchy is defined as the collection of ordinary differential equations given by

$$\partial_x \frac{dT_n}{du} = 0,$$

for $n = 0, 1, \ldots$, where

$$T_n = \sum_{k=-1}^{n} c_k F_k = F_n + c_{n-1}F_{n-1} + \ldots + c_1 F_1 + c_0 F_0 + c_{-1}F_{-1},$$

and $c_n = 1$. Thus, the n-th stationary KdV equation is defined by the stationarity of the n-th KdV equation, where all lower-order KdV flows have been included, as in our example of the one-soliton where the stationarity assumption led to the inclusion of a wave equation term. The stationary KdV equation is easily integrated once:

$$\frac{dT_n}{du} = \alpha \quad \Rightarrow \frac{d}{du}\left(T_n - \alpha F_{-1}\right) = 0.$$

Note that the inclusion of α merely alters the value of the constant c_{-1}, thus it may be ommited. This leads to the definition of the stationary KdV hierarchy as the collection of ordinary differential equations

$$\frac{dT_n}{du} = 0.$$

It follows immediately that this equation is Lagrangian. Through the use of a generalized Legendre transformation (the Ostrogradski transformation), it may be transformed into a Hamiltonian system with canonical Poisson structure. Bogoyavlenski and Novikov showed that this family of Hamiltonian systems is integrable and constructed a complete set of conserved quantities.

Remarks:

- The approach of Bogoyavlenski and Novikov allows for the construction of large classes of finite-dimensional solutions of the KdV equation (as well as other members of the KdV hierarchy) by solving ordinary differential equations. Analytically, this is beneficial, as these ordinary differential equations are Hamiltonian with canonical Poisson structure and a complete set of conserved quantities for this system is known. Numerically this is beneficial as well. The numerical solution of ordinary differential equations is significantly easier than that of partial differential equations. Furthermore, the knowledge of conserved quantities imposes many restrictions on the numerical results. If these are satisfied we are guaranteed accurate numerical results that mimic the qualitative properties of the solutions we are looking for.

- Since any equation of the KdV hierarchy commutes with any of the stationary KdV equations, the set of solutions of the n-th stationary KdV equation is invariant under the KdV hierarchy. The set of all solutions of the n-th stationary KdV hierarchy

depends on $3n+1$ arbitrary constants (the $n+1$ constants c_{-1}, c_0, ..., c_{n-1}, and the constants of the integration for the $2n$-th order differential equation). Thus this set is $(3n+1)$-dimensional. If any element of this set is used as an initial condition for the k-th KdV equation, then the resulting solution of the k-th KdV equation remains a solution of the n-th stationary KdV equation.

Let $u(x,c,d)$ be a solution of the n-th stationary KdV equation. Here $c = (c_{-1}, c_0, c_1, \ldots, c_{n-1})$ and $d = (d_1, \ldots, d_{2n})$, the constants of integration for the n-th stationary KdV equation. The above statement implies that using $u(x,c,d)$ as an initial condition in the k-th KdV equation results in a different solution of the n-th stationary KdV equation. In other words, the constants c and d then become functions of t_k.

In general, finite-dimensional classes of solutions of the entire KdV hierarchy may be generated by solving one of the members of the stationary KdV hierarchy, and allowing the constant parameters to be dependent on all variables t_k, $k = 0, 1, \ldots$.

- What is the relevance of these finite-dimensional classes of solutions? Suppose one wants to solve the following partial differential equation initial-value problem:

$$\begin{cases} u_t + uu_x + u_{xxx} = 0, \\ u(x,0) = U(x) \in L_2(a,b), \end{cases}$$

where (a,b) may be $(-\infty, \infty)$ for the case of the whole line, and $(a,b) = (a, a+L)$ for the periodic boundary condition problem with period L. McKean and others showed that any $U(x) \in L_2(a,b)$ may be approximated arbitrarily close by solutions of the stationary KdV hierarchy. Thus,

$$\forall \epsilon > 0, \quad \exists N,\, c,\, d : n > N \Rightarrow \int_a^b |U(x) - U_n(x,c,d)|^2 dx < \epsilon,$$

where $U_n(x,c,d)$ is a solution of the n-th KdV equation. Then, if such a $U_n(x,c,d)$ is used as an initial condition for the KdV equation, the solution corresponding to this initial condition remains close to the solution with initial condition $U(x)$ in an orbital sense: amplitudes will match well over long times, but there may be a phase drift. In this sense, the KdV initial-value problem in $L_2(a,b)$ may be solved with any degree of accuracy by considering solutions of the stationary KdV hierarchy, for sufficiently high n.

7.5 Exercises

1. Find the fourth conserved quantity for the KdV equation $u_t = uu_x + u_{xxx}$, *i.e.*, the conserved quantity which contains $\frac{1}{24}\int u^4 dx$.

2. **Recursion operator** For a Bi-Hamiltonian system with two Poisson structures given by B_0, B_1, one defines a recursion operator $R = B_1 B_0^{-1}$, which takes one element of the hierarchy of equations to the next element. For the KdV equation with $B_0 = \partial_x$ and $B_1 = \partial_{xxx} + \frac{1}{3}(u\partial_x + \partial_x u)$, we get $B_0^{-1} = \partial_x^{-1}$, integration with respect to x. Write down the recursion operator. Apply it to u_x (the zero-th KdV flow) to obtain the first KdV flow. Now apply it to $uu_x + u_{xxx}$ to get (up to rescaling of t_2) the second KdV equation. What is the third KdV equation?

3. Consider the function $U(x) = 2\partial_x^2 \ln\left(1 + e^{kx+\alpha}\right)$. Show that for a suitable k, $U(x)$ is a solution of the first member of the stationary KdV hierarchy (as you've already seen, it's the one-soliton solution):

$$6uu_x + u_{xxx} + c_0 u_x = 0.$$

(Note: it may be easier to define c_0 in terms of k, instead of the other way around)

Having accomplished this, let $u(x, t_1, t_2, t_3, \ldots) = U(x)|_{\alpha=\alpha(t_1,t_2,t_3,\ldots)}$. Determine the dependence of α on t_1, t_2 and t_3 such that $u(x, t_1, t_2, t_3, \ldots)$ is simultaneously a solution of the first, second and third KdV equations:

$$\begin{aligned}
u_{t_1} &= 6uu_x + u_{xxx}, \\
u_{t_2} &= 30u^2u_x + 20u_x u_{xx} + 10uu_{xxx} + u_{5x}, \\
u_{t_3} &= 140u^3u_x + 70u_x^3 + 280uu_x u_{xx} + 70u_{xx}u_{xxx} + 70u^2u_{xxx} + 42u_x u_{xxxx} + 14uu_{5x} + u_{7x}.
\end{aligned}$$

Based on this, write down a guess for the one-soliton solution that solves the entire KdV hierarchy.

4. **Warning: maple/mathematica-intensive.** Consider the function

$$U(x) = 2\partial_x^2 \ln\left(1 + e^{k_1 x+\alpha} + e^{k2_x+\beta} + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 e^{k_1 x + k_2 x + \alpha + \beta}\right).$$

Show that for a suitable k_1, k_2, $U(x)$ is a solution of the second member of the stationary KdV hierarchy:

$$30u^2u_x + 20u_x u_{xx} + 10uu_{xxx} + u_{5x} + c_1(6uu_x + u_{xxx}) + c_0 u_x = 0.$$

(Note: it may be easier to define c_1, c_0 in terms of k_1 and k_2 instead of the other way around)

Having accomplished this, let $u(x, t_1, t_2, t_3, \ldots) = U(x)|_{\alpha=\alpha(t_1,t_2,t_3,\ldots),\beta=\beta(t_1,t_2,t_3,\ldots)}$. Determine the dependence of α and β on t_1, t_2 and t_3 such that $u(x, t_1, t_2, t_3, \ldots)$ is simultaneously a solution of the first, second and third KdV equations, given above.

Based on this, write down a guess for the two-soliton solution of the entire KdV hierarchy.

Part II

Local and algebraic aspects of integrable equations

Chapter 8

Lax pairs

The Lax pair is the most fundamental object in the theory of integrable systems. The first Lax pair was proposed for the KdV equation in 1968 by Peter Lax. Although Lax pairs were originally introduced for partial differential equation, they are now an essential object in the theory of integrable ordinary differential equations as well. The importance of a Lax pair is not only due to the fact that most of the other miracles of integrables systems theory may be derived from it, but mostly because the Lax pair is the starting point for using the inverse scattering method on a given integrable equation, as we will see in the next chapter.

8.1 Definition

Consider a partial differential equation in evolution form

$$u_t = F(u, u_x, u_{xx}, \ldots).$$

A Lax pair for this equation is a pair of linear operators X and T which depend on u and its derivatives with respect to x, as well as on a parameter λ in an essential way[1]:

$$X = X(u, \lambda), \quad T = T(u, \lambda),$$

such that the partial differential equation $u_t = F(u, u_x, \ldots)$ is the **compatibility condition** of the two linear ordinary differential equations

$$\left\{ \begin{array}{rcl} \psi_x & = & X(u, \lambda)\psi \\ \psi_t & = & T(u, \lambda)\psi. \end{array} \right.$$

Remarks:

- In most cases, the parameter λ does not depend on x or t.

[1]Here "in an essential way" means that this dependence on λ may not be eliminated by any kind of transformation.

- The operators $X(u,\lambda)$ and $T(u,\lambda)$ are usually square matrices and simply act on their right-hand side by multiplication, as any system of differential equations may be written in first-order form, as in the above.
- The two ordinary differential equations for $\psi(x,t)$ do not have constant coefficients, due to the dependence of X and T on u and its derivatives.

Imposing the compatibility of the two differential equations for $\psi(x,t)$ is the same as imposing that their flows commute. As we've seen this is equivalent to imposing that mixed derivatives are equal:

$$\begin{aligned}
& & \psi_{xt} &= \psi_{tx} \\
\Rightarrow & & \partial_t \psi_x &= \partial_x \psi_t \\
\Rightarrow & \partial_t \left(X(u,\lambda)\psi\right) &= & \partial_x \left(T(u,\lambda)\psi\right) \\
\Rightarrow & X_t\psi + X\psi_t &= & T_x\psi + T\psi_x \\
\Rightarrow & X_t\psi + XT\psi &= & T_x\psi + TX\psi \\
\Rightarrow & X_t + XT &= & T_x + TX,
\end{aligned}$$

or, introducing the matrix commutator $[T,X]$,

$$X_t - T_x = [T,X].$$

In the above we have assumed that the solutions $\psi(x,t)$ form a complete set in a suitable space, since we concluded that if an operator acting on all ψ gives a zero results, then that operator is zero. Summarizing, we have the following

Definition: The pair of matrices $X(u,\lambda)$, $T(u,\lambda)$ is a Lax pair for the partial differential equation $u_t = F(u,u_x,\ldots)$ if (i) X and T depend on the parameter λ is an essential way, (ii) the given partial differential equation is equivalent to the compatibility condition

$$X_t - T_x = [T,X].$$

8.1.1 A Lax pair for the KdV equation

As a first example, let us verify that

$$X = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}, \quad T = \begin{pmatrix} -u_x & 2u + 4\lambda \\ -u_{xx} - 2u\lambda + 4\lambda^2 - 2u^2 & u_x \end{pmatrix},$$

is a Lax pair for the KdV equation $u_t = 6uu_x + u_{xxx}$. The different terms of the compatibility condition $X_t + XT = T_x + TX$ are:

$$X_t = \begin{pmatrix} 0 & 0 \\ -u_t & 0 \end{pmatrix},$$

(where we used that λ is independent of t),

$$T_x = \begin{pmatrix} -u_{xx} & 2u_x \\ -u_{xxx} - 2u_x\lambda - 4uu_x & u_{xx} \end{pmatrix},$$

(where we used that λ is independent of x), and

$$\begin{aligned} XT &= \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix} \begin{pmatrix} -u_x & 2u + 4\lambda \\ -u_{xx} - 2u\lambda + 4\lambda^2 - 2u^2 & u_x \end{pmatrix} \\ &= \begin{pmatrix} -u_{xx} - 2u\lambda + 4\lambda^2 - 2u^2 & u_x \\ -u_x(\lambda - u) & (2u + 4\lambda)(\lambda - u) \end{pmatrix} \\ &= \begin{pmatrix} -u_{xx} - 2u\lambda + 4\lambda^2 - 2u^2 & u_x \\ -u_x\lambda + uu_x & -2u\lambda + 4\lambda^2 - 2u^2 \end{pmatrix}, \end{aligned}$$

and lastly

$$\begin{aligned} TX &= \begin{pmatrix} -u_x & 2u + 4\lambda \\ -u_{xx} - 2u\lambda + 4\lambda^2 - 2u^2 & u_x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix} \\ &= \begin{pmatrix} (2u + 4\lambda)(\lambda - u) & -u_x \\ u_x(\lambda - u) & -u_{xx} - 2u\lambda + 4\lambda^2 - 2u^2 \end{pmatrix} \\ &= \begin{pmatrix} -2u\lambda + 4\lambda^2 - 2u^2 & -u_x \\ u_x\lambda - uu_x & -u_{xx} - 2u\lambda + 4\lambda^2 - 2u^2 \end{pmatrix}. \end{aligned}$$

The four components of the compatibility condition become:

$$\begin{aligned} (1,1): \quad & -u_{xx} - 2u\lambda + 4\lambda^2 - 2u^2 = -u_{xx} - 2u\lambda + 4\lambda^2 - 2u^2 \\ & \Rightarrow 0 = 0, \\ (1,2): \quad & u_x = 2u_x - u_x \\ & \Rightarrow 0 = 0, \\ (2,1): \quad & -u_t - u_x\lambda + uu_x = -u_{xxx} - 2u_x\lambda - 4uu_x + u_x\lambda - uu_x \\ & \Rightarrow u_t = 6uu_x + u_{xxx}, \\ (2,2): \quad & -2u\lambda + 4\lambda^2 - 2u^2 = u_{xx} - u_{xx} - 2u\lambda + 4\lambda^2 - 2u^2 \\ & \Rightarrow 0 = 0. \end{aligned}$$

Thus three of these conditions are identically satisfied. However, in order for the compatibility condition to hold, the condition for the $(2,1)$ component requires that $u(x,t)$ satisfies the KdV equation. Thus the pair X, T is a Lax pair for the KdV equation, as was claimed.

The ordinary differential equations for ψ are referred to as the Lax equations. Note that for the KdV equation, the first one of the Lax equations may be rewritten as

$$\left\{ \begin{array}{lcl} \psi_{1x} &=& \psi_2 \\ \psi_{2x} &=& (\lambda - u)\psi_1 \end{array} \right. \quad \Rightarrow \psi_{1xx} = (\lambda - u)\psi_1 \quad \Rightarrow \psi_{1xx} + u\psi_1 = \lambda\psi_1,$$

which is the time-independent linear Schrödinger equation, up to scaling of the coefficients. We will return to this when we discuss the inverse scattering transform.

8.1.2 A Lax pair for the NLS equation

8.2 Consequences of the existence of a Lax pair

In this section we illustrate how to deduce many of the miracles that exist for integrable equations in a systematic way from the existence of a Lax pair.

8.2.1 An infinite number of conserved quantities

Previously we proved the existence of an infinite number of conserved quantities for the KdV equation from the Miura transform connection with the modified KdV equation and the Gardner equation. None of the steps of that approach were systematic. Here we illustrate how the same sequence of conserved quantities may be deduced systematically from the Lax pair for the KdV equation.

Consider the Lax pair for the KdV equation as $x \to \pm\infty$. If we are considering the KdV equation with vanishing conditions as $x \to \pm\infty$, then, as $x \to \pm\infty$,

$$\psi_x = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \psi,$$

and

$$\psi_t = \begin{pmatrix} 0 & 4\lambda \\ 4\lambda^2 & 0 \end{pmatrix} \psi.$$

Thus, eliminating the second component of ψ from these equations, we obtain

$$\psi_{1xx} = \lambda\psi_1, \quad \psi_{1tt} = 16\lambda^3\psi_1.$$

Thus, as $x \to \pm\infty$, ψ_1 is a linear combination of modes of the form

$$\psi_1 \sim e^{\sqrt{\lambda}x + 4\lambda\sqrt{\lambda}t}.$$

At this point, we are not concerned with which values of λ are allowed, and which are not (since $\psi(x,t)$ may be thought of as an eigenfunction, we wish it to be bounded, thus its behavior at ∞ should not be exponential growth, *etc.*), we merely consider it an accounting parameter, to be used in this faculty somewhat later. For this reason, the appearance of $\sqrt{\lambda}$ is of no concern. Based on this asymptotic behavior, consider the substitution

$$\psi_1 = e^{\sqrt{\lambda}x+4\lambda\sqrt{\lambda}t+\varphi(x,t)}.$$

It follows from the above analysis that

$$\lim_{x\to\infty} \varphi(x,t) = \text{constant},$$

independent of x or t. Then

$$\begin{aligned} \psi_{1x} &= (\sqrt{\lambda}+\varphi_x)\psi_1 \\ \psi_{1xx} &= (\sqrt{\lambda}+\varphi_x)^2\phi + \varphi_{xx}\psi_1, \end{aligned}$$

and the differential equation for $\varphi(x,t)$ is

$$\begin{aligned} & \left(\lambda^2+\varphi_x^2+2\sqrt{\lambda}\varphi_x\right)\phi + \varphi_{xx}\phi = (\lambda-u)\phi \\ \Rightarrow \quad & \varphi_{xx}+2\sqrt{\lambda}\varphi_x+\varphi_x^2+u=0 \\ \Rightarrow \quad & \chi_x+2\sqrt{\lambda}\chi+\chi^2+u=0, \end{aligned}$$

where $\chi(x,t)=\varphi_x(x,t)$. This is a Ricatti equation for $\chi(x,t)$. At the price of transforming the linear problem into a nonlinear problem, we obtained a first-order problem, as opposed to a second order problem. As for the Miura-transformation problem, for large absolute values of $\sqrt{\lambda}$, the Ricatti equation may be solved by assuming a series ansatz for $\chi(x,t)$:

$$\chi = \frac{1}{2\sqrt{\lambda}}\sum_{n=0}^{\infty}\frac{\chi_n(x,t)}{(2\sqrt{\lambda})^n}.$$

Then:

$$\begin{aligned} \chi_x &= \frac{1}{2\sqrt{\lambda}}\sum_{n=0}^{\infty}\frac{\chi_{n,x}}{(2\sqrt{\lambda})^n} \\ &= \sum_{n=0}^{\infty}\frac{\chi_{n,x}}{(2\sqrt{\lambda})^{n+1}} \\ &= \sum_{n=1}^{\infty}\frac{\chi_{n-1,x}}{(2\sqrt{\lambda})^n}, \\ 2\sqrt{\lambda}\,\chi &= \sum_{n=0}^{\infty}\frac{\chi_n}{(2\sqrt{\lambda})^n}, \\ \chi^2 &= \frac{1}{(2\sqrt{\lambda})^2}\sum_{n=0}^{\infty}\frac{\chi_n}{(2\sqrt{\lambda})^n}\sum_{k=0}^{\infty}\frac{\chi_k}{(2\sqrt{\lambda})^k} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(2\sqrt{\lambda})^2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\chi_n\chi_k}{(2\sqrt{\lambda})^{n+k}} \\
&= \frac{1}{(2\sqrt{\lambda})^2}\sum_{N=0}^{\infty}\sum_{k=0}^{N}\frac{\chi_{N-k}\chi_k}{(2\sqrt{\lambda})^{N}} \\
&= \sum_{N=0}^{\infty}\frac{1}{(2\sqrt{\lambda})^{N+2}}\sum_{k=0}^{N}\chi_{N-k}\chi_k \\
&= \sum_{n=2}^{\infty}\frac{1}{(2\sqrt{\lambda})^{n}}\sum_{k=0}^{n-2}\chi_{n-k-2}\chi_k.
\end{aligned}
$$

Substituting these in the Ricatti equation and equating equal powers of $2\sqrt{\lambda}$ gives:

$$
\begin{aligned}
n=0: &\quad \chi_0 + u = 0 \quad\Rightarrow\quad \chi_0 = -u, \\
n=1: &\quad \chi_{0,x} + \chi_1 = 0 \quad\Rightarrow\quad \chi_1 = -\chi_{0,x} \quad\Rightarrow\quad \chi_1 = u_x, \\
n\geq 2: &\quad \chi_{n-1,x} + \chi_n + \sum_{k=0}^{n-2} = 0\chi_{n-k-2}\chi_k \\
&\quad\Rightarrow\quad \chi_n = -\chi_{n-1,x} - \sum_{k=0}^{n-2}\chi_{n-k-2}\chi_k.
\end{aligned}
$$

Thus, all the coefficients $\chi_n(x,t)$ are found recursively this way. How do we obtain conserved quantities from them? Recall that

$$
\chi(x,t) = \varphi_x(x,t) \quad\Rightarrow\quad \varphi(x,t) = \int_{-\infty}^{x}\chi(x,t)dx.
$$

Integrating the ansatz for $\chi(x,t)$ term-by-term, we find

$$
\varphi(x,t) = \frac{1}{2\sqrt{\lambda}}\sum_{n=0}^{\infty}\frac{\varphi_n(x,t)}{(2\sqrt{\lambda})^n},
$$

where

$$
\varphi_n(x,t) = \int_{-\infty}^{x}\chi_n(x,t), \quad \text{for} \quad n=0,1,\ldots.
$$

Lastly, recall that $\lim_{x\to\infty}\varphi(x,t)$ =constant, independent of x and t. Thus

$$
\lim_{x\to\infty}\varphi(x,t) = \text{constant} = \frac{1}{2\sqrt{\lambda}}\sum_{n=0}^{\infty}\frac{\lim_{x\to\infty}\int_{-\infty}^{x}\chi_n(x,t)dx}{(2\sqrt{\lambda})^n} = \frac{1}{2\sqrt{\lambda}}\sum_{n=0}^{\infty}\frac{\int_{-\infty}^{\infty}\chi_n(x,t)dx}{(2\sqrt{\lambda})^n}.
$$

Since this constant does not depend on t, neither do any of its expansion coefficients, thus all

$$\int_{-\infty}^{\infty} \chi_n dx, \quad n = 0, 1, \ldots$$

are conserved quantities for the KdV equation. Just like when we used the Miura transformation, all coefficients with odd index result in trivial conserved quantities (their densities are x-derivatives), but the coefficients with even index give the same sequence of conserved quantities that we know from the previous chapter.

This is a general method: Given a Lax pair for an integrable partial differential equation, an infinite sequence of conserved quantities is constructed this way: (i) use the behavior as $x \to \infty$ where the Lax pair has constant coefficients; (ii) solve the resulting constant coefficient equations; (iii) use an exponential transformation to transform the problem into one for a new function which has constant behavior as $x \to \infty$. This new function is often the logarithmic derivative for one of the eigenfunction components; (iv) the resulting differential equation will be nonlinear, but it may be solved recursively. All coefficients give rise to conserved densities.

8.3 Exercises

1. Show that
$$X = \begin{pmatrix} -i\zeta & q \\ \pm q^* & i\zeta \end{pmatrix}, T = \begin{pmatrix} -i\zeta^2 \mp \frac{i}{2}|q|^2 & q\zeta + \frac{i}{2}q_x \\ \pm q^*\zeta \mp \frac{i}{2}q_x^* & i\zeta^2 \pm \frac{i}{2}|q|^2 \end{pmatrix}$$
are two Lax Pairs for the Nonlinear Schrödinger equations
$$iq_t = -\frac{1}{2}q_{xx} \pm |q|^2 q.$$
Here the top (bottom) signs of one matrix correspond to the top (bottom) signs of the other.

2. Let $\psi_n = \psi_n(t)$, $n \in \mathbb{Z}$. Consider the difference equation
$$\psi_{n+1} = X_n \psi_n,$$
and the differential equation
$$\frac{\partial \psi_n}{\partial t} = T_n \psi_n.$$
What is the compatibility condition of these two equations? Using this result, show that
$$X_n = \begin{pmatrix} z & q_n \\ q_n^* & 1/z \end{pmatrix}, T_n = \begin{pmatrix} iq_n q_{n-1}^* - \frac{i}{2}(1/z - z)^2 & \frac{i}{z}q_{n-1} - izq_n \\ -izq_{n-1}^* + \frac{i}{z}q_n^* & -iq_n^* q_{n-1} + \frac{i}{2}(1/z - z)^2 \end{pmatrix}$$

is a Lax Pair for the semi-discrete equation

$$i\frac{\partial q_n}{\partial t} = q_{n+1} - 2q_n + q_{n-1} - |q_n|^2(q_{n+1} + q_{n-1})$$

Note that this is a discretization of the NLS equation. It is known as the Ablowitz-Ladik lattice. It is an integrable discretization of NLS. For numerical purposes, it is far superior in many ways to the "standard" discretization of NLS:

$$i\frac{\partial q_n}{\partial t} = q_{n+1} - 2q_n + q_{n-1} - 2|q_n|^2 q_n.$$

Chapter 9

Darboux Transformations

Having found a Lax pair for an integrable equation, it may be used to set up a Darboux transformation for this equation. Darboux transformations will allow us to construct special solutions of the equation using elementary methods. Specifically, we will construct general soliton solutions starting from the most elementary soliton solutions. Darboux transformations are closely related to Bäcklund transformations. The main difference is that Bäcklund transformations are direct methods in the sense that they do not rely on the existence of a Lax pair. On the other hand, using the Lax pair allows for a more systematic set-up of the transformation, due to the linearity of the Lax pair.

Again, consider the KdV equation in the form

$$u_t = 6uu_x + u_{xxx}.$$

It has a Lax pair as obtained in the previous chapter. This Lax pair may be rewritten as a scalar second-order equation:

$$\left\{ \begin{array}{rcl} \psi_{xx} & = & (\lambda - u)\psi, \\ \psi_t & = & -u_x\psi + (2u + 4\lambda)\psi_x. \end{array} \right.$$

We will use the Lax pair to construct soliton solutions of the KdV equation from simpler solutions of the same equation. The simplest solution we can start from is the $u = 0$ solution.

Consider the first equation of the Lax pair:

$$\psi_{xx} = (\lambda - u)\psi.$$

Take $\lambda = \lambda_1$, fixed. Corresponding to this value of λ, the equation has two linearly independent solutions, as it is a linear ordinary equation of second order. Denote one of these solutions by

$$\psi_1 = \psi(x, \lambda_1),$$

where the dependence on λ is indicated explicitly. Define

$$\sigma_1 = \frac{\psi_{1x}}{\psi_1} = \partial_x \ln \psi.$$

The following theorem holds:

Theorem: (Darboux) The function $\psi[1] = (\partial_x - \sigma_1)\psi$ satisfies the differential equation

$$\psi[1]_{xx} = (\lambda - u[1])\psi[1],$$

with $u[1] = u + 2\sigma_{1x}$.

Remark: Given a potential u for the linear Schrödinger equation, and one solution of the equation corresponding to this potential, the Darboux theorem provides a way to construct the general solution of a related linear Schrödinger equation, from the general solution of the original equation.

Proof: The proof is by direct verification. Consider

$$\begin{aligned}
&\psi[1]_{xx} - \lambda\psi[1] + u[1]\psi[1] = \\
&\quad (\psi_x - \sigma_1\psi)_{xx} - \lambda\,(\psi_x - \sigma_1\psi) + u[1]\,(\psi_x - \sigma_1\psi) = \\
&\quad \psi_{xxx} - (\sigma_{1x}\psi + \sigma_1\psi_x)_x - \lambda\psi_x + \lambda\sigma_1\psi + u[1]\psi_x - u[1]\sigma_1\psi = \\
&\quad ((\lambda - u)\psi)_x - \sigma_{1xx}\psi - 2\sigma_{1x}\psi_x - \sigma_1\psi_{xx} - \lambda\psi_x + \lambda\sigma_1\psi + u[1]\psi_x - u[1]\sigma_1\psi = \\
&\quad - u_x\psi + (\lambda - u)\psi_x - \sigma_{1xx}\psi - 2\sigma_{1x}\psi_x - \sigma_1(\lambda - u)\psi - \lambda\psi_x + \lambda\sigma_1\psi + u[1]\psi_x - u[1]\sigma_1\psi.
\end{aligned}$$

Now we look at the coefficients of ψ and ψ_x separately. The coefficient of ψ_x is:

$$(\lambda - u) - 2\sigma_{1x} - \lambda + u[1] \;=\; -u - 2\sigma_{1x} + u + 2\sigma_{1x} = 0.$$

It remains to check the coefficient of ψ:

$$\begin{aligned}
-u_x - \sigma_{1xx} - \sigma_1(\lambda - u) + \lambda\sigma_1 - u[1]\sigma_1 &= \\
-u_x - \sigma_{1xx} + \sigma_1 u - u[1]\sigma_1 &= \\
-u_x - \sigma_{1xx} + \sigma_1 u - (u + 2\sigma_{1x})\sigma_1 &= \\
-u_x - \sigma_{1xx} - 2\sigma_{1x}\sigma_1 &= \\
-\partial_x\left(u + \sigma_{1x} + \sigma_1^2\right) &= \\
-\partial_x\left(u + \left(\frac{\psi_{1x}}{\psi_1}\right)_x + \frac{\psi_{1x}^2}{\psi_1^2}\right) &= \\
-\partial_x\left(u + \frac{\psi_1\psi_{1xx} - \psi_{1x}^2}{\psi_1^2} + \frac{\psi_{1x}^2}{\psi_1^2}\right) &= \\
-\partial_x\left(u + \frac{\psi_1\psi_{1xx}}{\psi_1^2}\right) &= \\
-\partial_x\left(\lambda_1\right) &= 0.
\end{aligned}$$

This finishes the proof of Darboux's theorem.

Let us summarize how the Darboux theorem may be used:

1. Specify a potential for the Schrödinger equation; find the general solution of this Schrödinger equation, for all λ. Single out a solution corresponding to a specific $\lambda = \lambda_1$.

2. From this knowledge, Darboux's theorem allows us to construct a second Schrödinger equation defined by a new potential $u[1]$, and its general solution $\psi[1]$ for all λ.

For future use, it is convenient to notice that

$$\left\{ \begin{array}{rcl} \psi[1] & = & \dfrac{W(\psi_1, \psi)}{\psi_1}, \\ u[1] & = & u + 2\partial_x^2 \ln \psi_1, \end{array} \right.$$

where $W(\psi_1, \psi) = \psi_1 \psi_x - \psi_{1x} \psi$ is the Wronskian of the two functions ψ_1 and ψ.

Example: Let $u = 0$, then the Schrödinger equation of the first step is

$$\psi_{xx} = \lambda \psi,$$

with general solution e^{ikx}, where $\lambda = -k^2$. Choose $\psi_1 = \cosh \kappa_1 (x - x_1)$, with $\lambda_1 = \kappa_1^2$. It is typical to choose ψ_1 such that λ_1 is not in the spectrum, *i.e.*, ψ_1 is unbounded. Then $\sigma_1 = \kappa_1 \tanh \kappa_1 (x - x_1)$ and

$$\begin{array}{rcl} \psi[1] & = & \left(\dfrac{d}{dx} - \sigma_1 \right) e^{ikx} \\ & = & (ik - \sigma_1) e^{ikx} \\ & = & (ik - \kappa_1 \tanh \kappa_1 (x - x_1)) e^{ikx} \end{array}$$

This is the general solution for a Schrödinger equation with $\lambda = -k^2$ and

$$\begin{array}{rcl} u[1] & = & 2\partial_x^2 \ln \cosh \kappa_1 (x - x_1) \\ & = & 2\kappa_1 \partial_x \tanh \kappa_1 (x - x_1) \\ & = & 2\kappa_1^2 \mathrm{sech}^2 \kappa_1 (x - x_1), \end{array}$$

and the reader will recognize the one-soliton solution of the KdV equation.

It is clear that this procedure may be repeated. Let $\psi_2[1] = W(\psi_1, \psi_2)/\psi_1$, where ψ_2 is a solution of the original Schrödinger equation, but with $\lambda = \lambda_2$. Then

$$\sigma_2 = \frac{\psi_2[1]_x}{\psi_2[1]},$$

and

$$\left\{ \begin{array}{rcl} \psi[2] & = & \left(\dfrac{d}{dx} - \sigma_2 \right) \psi[1] = \dfrac{W(\psi_2[1], \psi[1])}{\psi_2[1]}, \\ u[2] & = & u[1] + 2\sigma_{2x} = u[1] + 2\partial_x^2 \ln \psi_2[1]. \end{array} \right.$$

The formula for $u[2]$ can be rewritten as:

$$\begin{aligned} u[2] &= u + 2\sigma_{1x} + 2\sigma_2 x \\ &= u + 2\partial_x(\sigma_1 + \sigma_2) \\ &= u + 2\partial_x^2 \ln \psi_1 \psi_2[1] \\ &= u + 2\partial_x^2 \ln W(\psi_1, \psi_2). \end{aligned}$$

This formula expresses the potential $u[2]$ in terms of two different eigenfunctions of the original Schrödinger equation. With significantly more work, one can rewrite the formula for $\psi[2]$ as well:

$$\psi[2] = \frac{W(\psi_1, \psi_2, \psi)}{W(\psi_1, \psi_2)},$$

where $W(\psi_1, \psi_2, \psi)$ denotes the Wronskian of the three functions ψ_1, ψ_2 and ψ. Instead of working through the algebra to prove this result, we prove the general **Crum theorem:**

Theorem (Crum): After N steps of the Darboux transformation, we obtain

$$\begin{aligned} \psi[N] &= \frac{W(\psi_1, \psi_2, \ldots, \psi_N, \psi)}{W(\psi_1, \psi_2, \ldots, \psi_N)}, \\ u[N] &= u + 2\partial_x^2 \ln W(\psi_1, \psi_2, \ldots, \psi_N). \end{aligned}$$

Chapter 10

The Painlevé property

In this chapter we start by looking at a seemingly unrelated property of functions of a complex variable. Consider $f(z)$, $z \in \mathbb{C}$. Such a function is called analytic in $D \subset \mathbb{C}$, if for any $z_0 \in D$, $f(z)$ has a Taylor expansion around z_0. A possibly less familiar definition is the following:

Definition 2 (Meromorphic function) *A function $f(z)$ is called meromorphic in $D \subset \mathbb{C}$ if for every $z_0 \in D$, $f(z)$ has a finite-order Laurent expansion around z_0,* i.e., *for any $z_0 \in D$, there is a finite $n \in \mathbb{Z}$ such that*

$$f(z) = \alpha_0(z - z_0)^n + \alpha_1(z - z_0)^{n+1} + \dots .$$

Those points $z_0 \in D$ for which $n < 0$ are called poles of the function $f(z)$ of order $|n|$. The other points are referred to as regular points of $f(z)$.

Complex functions may have more severe singularities than poles, such as branch points or essential singularities. The "nice" thing about poles is that the behavior of the function near the pole remains single-valued, despite the presence of a singularity.

In what follows, we discuss the behavior of solutions of ordinary differential equations in the complex plane: we regard the independent variable of the differential equation as a complex variable, thus the solution of the differential equation becomes a function of a complex variable. We will be especially interested in the possible singularities in the complex plane of the solutions of the differential equations at hand.

10.1 Solutions of linear ordinary differential equations

Consider a general homogeneous linear ordinary differential equation of order N:

$$\frac{d^N y}{dx^N} + p_{N-1}(x)\frac{d^{N-1} y}{dx^{N-1}} + \ldots + p_1(x)\frac{dy}{dx} + p_0(x)y = 0,$$

where we now think of x as a complex variable. If x_0 is a regular point of the differential equation, *i.e.*, the coefficients $p_k(x)$, $k = 0, 1, \ldots, N-1$ are analytic at x_0, then the equation

has a unique solution with initial conditions specified at x_0. This solution is analytic at x_0, which is an immediate consequence of Frobenius' method for finding series solutions of linear differential equations.

It follows from these conclusions that the singularities of a solution of a linear ordinary differential equation can only occur at the singularities, if any, of the coefficient functions. Such singularities are called *fixed*, since their location is determined by the differential equations and is independent of the initial conditions.

Example 1 (Hypergeometric equation) *Consider the hypergeometric equation*

$$x(1-x)y'' + (c - (a+b+c+1)x)\,y' - aby = 0.$$

Here a, b and c are constant parameters. The possible singularities of the hypergeometric function, which is the solution of the hypergeometric equation, occur at $x = 0$ and $x = 1$, which are the singularities of the coefficients

$$p_1(x) = \frac{c - (a+b+c+1)x}{x(1-x)}, \quad p_0(x) = -\frac{ab}{x(1-x)}.$$

Thus for a linear ordinary differential equation, it is easy to find out where its solutions may have singularities. The equation gives all the warnings necessary.

10.2 Solutions of nonlinear ordinary differential equations

For nonlinear differential equations, the scenario is entirely different. We still should be aware of the singularities of whatever functions are present in the differential equation, but other singularities may arise as well.

Example 2 *Consider the nonlinear first-order differential equation*

$$y' + y^2 = 0,$$

with initial conditions specified by $y(x_0) = y_0$. Note that we could put $x_0 = 0$, since the equation is autonomous, so x may always be shifted by a constant amount. Then

$$y = \frac{y_0}{y_0(x - x_0) + 1},$$

which is singular at

$$x = x_0 - \frac{1}{y_0},$$

which is a pole of order one, whose location depends on the initial condition.

This example illustrates that even for seemingly simple nonlinear ordinary differential equations, the solution may have singularities whose existence cannot be conjured from direct inspection of the differential equation. Singularities whose location depends on the initial conditions are called *moveable*. Solutions of nonlinear differential equations can have all kinds of singularities:

Example 3 *Consider*

$$y' + y^3 = 0.$$

Separation of variables gives

$$y = \frac{1}{\sqrt{2(x - x_0)}},$$

where x_0 is a constant of integration. In this case, the singularity is a moveable branch point.

Example 4 *Consider the nonlinear second-order differential equation*

$$yy'' - y' + 1 = 0.$$

Then it is easy to check that

$$y = (x - x_0)\ln(x - x_0) + \alpha(x - x_0)$$

is a solution. Here α and x_0 are integration constants. There is a singularity at $x = x_0$, which is an infinite-order moveable branch point.

Example 5 *Consider the nonlinear second-order differential equation with parameter α:*

$$\alpha yy'' + (1 - \alpha)y'^2 = 0.$$

It is easy to check that

$$y = a(x - x_0)^\alpha$$

is a solution. Here a and x_0 are integration constants. There is a singularity at $x = x_0$, which is a moveable branch point whose precise nature depends on α.

Example 6 *Consider the nonlinear second-order differential equation*

$$\left(yy'' - y'^2\right)^2 + 4xy'^3 = 0.$$

Again, it is easy to check that

$$y = \alpha e^{1/(x - x_0)}$$

is a solution. Here α and x_0 are integration constants. There is a singularity at $x = x_0$, which is an essential singularity.

The above examples should make it clear that nonlinear ordinary differential equations that appear simple may have a very complicated singularity structure in the complex plane. In the following sections, we will examine the singularities of solutions of ordinary differential equations in more detail. We will proceed in more or less historical order, starting with first-order equations, following with a discussion of the work of Kovaleskaya, and finishing with the work of Painlevé and his collaborators on second-order nonlinear differential equations.

10.2.1 First-order equations

Painlevé showed in 1888 that the first-order differential equation

$$G(y'(x), y(x), x) = 0,$$

where G is polynomial in its entries, has solutions with moveable singularities which are either poles or algebraic (finite-order) branch points.

Fuchs proceeded to show that the only equation of the form

$$y' = \frac{P(y,x)}{Q(y,x)},$$

where P and Q are polynomial in their entries, whose solutions have moveable singularities that are at worst poles (thus excluding branch points) is the Ricatti equation

$$y' = p_0(x) + p_1(x)y + p_2(x)y^2,$$

where $p_0(x)$, $p_1(x)$ and $p_2(x)$ are polynomials in x. We are familiar with the Ricatti equation, so the study of first-order nonlinear differential equations from the point of view of the singularities of their solutions does not give any surprising results.

10.2.2 The work of Kovaleskaya

In 1889, Sofia Kovaleskaya was awarded the Bordin prize for her work on the motion of a top (formally defined as a rigid body with a fixed point under the influence of gravity). The prize money was trippled to acknowledge the exceptional quality of the work.

The equations describing the motion of the top are a system of sixth order. For this system to be integrable, six conserved quantities need to exist. Some are easy to find, such as t_0 (arbitrary time shift), E (energy), L_z (angular momentum around the z-axis, *i.e.*, the axis of rotation), *etc.* Two cases were known where a total of six conserved quantities could be found:

- **The Euler top:** In this case, there is additional symmetry in the problem, by imposing that the center of mass of the top is its fixed point. This results in gravity not having any net effect.
- **The Lagrange top:** In this case, the fixed point of the top and its center of gravity lie on a line which is a symmetry axis for the top.

The solutions of both cases were obtained in their final, explicit form by Jacobi and Lagrange, respectively. Both involve elliptic functions, which are meromorphic. Furthermore, the poles of the solutions were all moveable. Looking at the solutions of Jacobi and Lagrange in this light was already a major leap: the independent variable for these differential equations and their solutions is t (time), which is real, of course, in the physical problem.

Kovaleskaya, motivated by the work of Fuchs, decided to look for all values of the parameters for which the solution as a function of complex time has as its only singularities moveable poles. To this end, she considered an ansatz for the solutions of the form

$$y = \frac{1}{(t-t_0)^n} \sum_{j=0}^{\infty} y_j (t-t_0)^j,$$

for all unknown functions. Note that the value of n may be different for different unknown functions. Kovaleskaya was able to find all cases for which the solution in this series form depends on five undetermined constants. With the addition of t_0, these five undetermined constants are a full set of constants of the motion for the spinning top, implying that in these cases the motion of the top is completely integrable. These cases included the Euler and Lagrange top, but there was one new case, now known as the Kovaleskaya top. For the Kovaleskaya top two of the principal moments of inertia of the rigid body are equal, and the third one is half of what the others are. There is no clear physical interpretation as to what makes this top stand out, in the sense that it is more symmetrical than other tops. The extra symmetry leading to the extra required conserved quantity is hidden.

Kovaleskaya was able to perform the explicit integration of the equations of the motion in this third case. Elliptic functions were not sufficient anymore, and hyperelliptic functions came into play. The only type of singularities of hyperelliptic functions are poles, but typically hyperelliptic functions are not globally meromorphic as there may be accumulation points of poles, where the functions may lose their single-valued character. Nevertheless, the specific combinations Kovaleskaya required to solve the equations of motion were single-valued. In later years (after 1974), hyperelliptic functions have played an increasingly important role in the theory of integrable functions.

10.2.3 Second-order equations

After their work on first-order equations, Painlevé and coworkers set out to classify second-order differential equations, written in normal form as

$$y'' = F(x, y, y'),$$

where F is a rational function of x, y and y'. In 1887, Picard first put forth the question of the classification of all such equations by the singularities of their solution. Specifically, Picard asked for the characterization of all such equations whose solutions have only moveable poles, and no other types of singularities, apart from those already present in the differential equation. A differential equation whose solutions have this property now is said to be of

Painlevé type, and the property itself is referred to as the Painlevé property. Picard's problem was solved in a series of papers by Painlevé, Gambier and others from 1887 to 1909.

In these papers, Painlevé and his collaborators found that (up to transformations of a certain class), there are 50 equations with the Painlevé property. The solutions of 44 of these equations were already known in terms of elementary functions (sin, cos, *etc.*) or known transcendental special functions (Bessel, elliptic, *etc.*). The solutions of the other six equations define new functions, now referred to as the *Painlevé transcendents*. These six equations are:

$$
\begin{aligned}
P_I: &\quad y'' = 6y^2 + x,\\
P_{II}: &\quad y'' = 2y^3 + xy + \alpha,\\
P_{III}: &\quad y'' = \frac{1}{y}y'^2 - \frac{1}{x}y' + \frac{1}{x}\left(\alpha y^2 + \beta\right) + \gamma y^3 + \frac{\delta}{y},\\
P_{IV}: &\quad y'' = \frac{1}{2y^2}y'^2 + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \alpha)y + \frac{\beta}{y},\\
P_V: &\quad y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)y'^2 - \frac{1}{x}y' + \frac{(y-1)^2}{x^2}\left(\alpha + \frac{\beta}{y}\right) + \frac{\gamma}{x}y + \delta\frac{y(y+1)}{y-1},\\
P_{VI}: &\quad y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)y'^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y' +\\
&\qquad \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha + \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \delta\frac{x(x-1)}{(y-x)^2}\right),
\end{aligned}
$$

where α, β, γ and δ are constant parameters. Although the P_I and P_{II} definitely don't look bad, the others have a distinct aura of invincibility about them!

After Painlevé's work, the six Painlevé transcendents were mostly forgotten, until they came to the forefront again through their connection with soliton equations, to be described in Section 10.4. Since then, the Painlevé equations have arisen in many subject areas, and their solutions have found many uses. They are now in the pantheon of the classical special functions.

There have been efforts to extend the methods of Painlevé and Gambier to third-order differential equations by Bureau, Cosgrove, Hitchin and others. Needless to say, the number of equations with the Painlevé property for third-order equations increases drastically. Apparently the classification remains unfinished as of today.

10.3 How does the Painlevé method work

To illustrate how the Painlevé method works, consider the first Painlevé equation P_I:

$$y'' = 6y^2 + x.$$

We are looking for solutions of the form

$$\begin{aligned} y &= \frac{1}{(x-x_0)^p}\sum_{k=0}^{\infty}\alpha_k(x-x_0)^k \\ &= \sum_{k=0}^{\infty}\alpha_k(x-x_0)^{k-p}, \end{aligned}$$

where p is an integer, to be determined. We expect it to be positive, so that y has a pole of order $p>0$ at $x=x_0$. Further, α_k, for $k=0,1,\ldots$ are complex constants.

10.3.1 Singularity analysis

First, we determine the possible orders p of the pole. To this end, we consider the most singular terms of the differential equations. Close to x_0, we have

$$\begin{aligned} y &\sim \alpha_0(x-x_0)^{-p}, \\ y' &\sim -p\alpha_0(x-x_0)^{-p-1}, \\ y'' &\sim p(p+1)\alpha_0(x-x_0)^{-p-2}. \end{aligned}$$

Substitution in the differential equation gives

$$\begin{aligned} p(p+1)\alpha_0(x-x_0)^{-p-2} &= 6\alpha_0^2(x-x_0)^{-2p}+\ldots \\ p(p+1)(x-x_0)^{-p-2} &= 6\alpha_0(x-x_0)^{-2p}+\ldots. \end{aligned}$$

where the x-term has been ignored, as it is not singular. There are several possibilities:

- **$-p-2$ is dominant:** In order to have a solution, the coefficient of terms with this power has to vanish, leading to

  $$p(p+1)=0,$$

  and thus $p=0$ or $p=-1$. Our assumption also requires $-p-2<-2p$, which is satisfied for either $p=0$ or $p=-1$. But both of these lead to solutions that are not singular at x_0, so they are of no interest to us now.

- **$-2p$ is dominant:** In order to have a solution, the coefficient of terms with this power has to vanish, leading to

  $$6=0,$$

  which is not possible.

- **$-2p$ and $-p-2$ are equal and are both dominant:** Then

$$-2p = -p - 2 \quad \Rightarrow \quad p = 2,$$

which is a positive integer indeed. The vanishing of its coefficient leads to

$$p(p+1) = 6\alpha_0 \quad \Rightarrow \quad \alpha_0 = 1.$$

Only the third of these three possibilities leads to a solution which is meromorphic at $x = x_0$, with a pole of order 2. In general, it is possible that several of these possibilities lead to solutions. If this happens, usually one of the solution branches will depend on a full set of undetermined constants, giving rise to the general solution. The other branches may give rise to special solutions, depending on less than a full set of arbitrary constants. The above calculation is another example of the principle of maximal balance, stating that the maximal number of terms have to be taken into account to obtain the most interesting solution.

10.3.2 Construction of the solution

We have determined a meromorphic solution should be of the form

$$\begin{aligned} y &= \frac{1}{(x-x_0)^2}\sum_{k=0}^{\infty}\alpha_k(x-x_0)^k \\ &= \sum_{k=0}^{\infty}\alpha_k(x-x_0)^{k-2}, \end{aligned}$$

with $\alpha_0 = 1$. Then

$$\begin{aligned} y' &= \sum_{k=0}^{\infty}\alpha_k(k-2)(x-x_0)^{k-3}, \\ y'' &= \sum_{k=0}^{\infty}\alpha_k(k-2)(k-3)(x-x_0)^{k-4}. \end{aligned}$$

Substitution of these results in the differential equation gives (with $x - x_0 = X$)

$$\begin{aligned} \sum_{k=0}^{\infty}\alpha_k(k-2)(k-3)X^{k-4} &= 6\sum_{k=0}^{\infty}\alpha_k X^{k-2}\sum_{m=0}^{\infty}\alpha_m X^{m-2} + X + x_0 \\ \sum_{n=-4}^{\infty}\alpha_{n+4}(n+2)(n+1)X^n &= 6\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\alpha_k\alpha_m X^{k+m-4} + X + x_0 \\ \sum_{n=-4}^{\infty}\alpha_{n+4}(n+2)(n+1)X^n &= 6\sum_{n=-4}^{\infty}X^n\sum_{m=0}^{n+4}\alpha_{n+4-m}\alpha_m + X + x_0. \end{aligned}$$

Equating coefficients of equal powers results in a recursion relationship for the coefficients α_k, $k = 0, 1, \ldots$:

$$\begin{aligned} \alpha_{n+4}(n+2)(n+1) &= 6\sum_{m=0}^{n+4} \alpha_{n+4-m}\alpha_m + \delta_{n1} + x_0\delta_{n0} \\ &= 6\left(2\alpha_0\alpha_{n+4} + \sum_{m=1}^{n+3} \alpha_{n+4-m}\alpha_m\right) + \delta_{n1} + x_0\delta_{n0} \\ \Rightarrow \quad \left((n+2)(n+1) - 12\alpha_0\right)\alpha_{n+4} &= 6\sum_{m=1}^{n+3} \alpha_{n+4-m}\alpha_m + \delta_{n1} + x_0\delta_{n0} \end{aligned}$$

where we have assumed that $n > -4$, so that these results are valid for $n = -3, -2, \ldots$, so that finally

$$\alpha_{n+4} = \frac{6\sum_{m=1}^{n+3} \alpha_{n+4-m}\alpha_m + \delta_{n1} + x_0\delta_{n0}}{(n+1)(n+2) - 12\alpha_0},$$

valid for $n = -3, -2, \ldots$, unless the denominator vanishes.

Let's see how this recursion relationship works:

$$\begin{aligned} \text{from before} &: \alpha_0 = 1, \\ n = -3 &: \alpha_1 = 0, \\ n = -2 &: \alpha_2 = 0, \\ n = -1 &: \alpha_3 = 0, \\ n = 0 &: \alpha_4 = -\frac{x_0}{10}, \\ n = 1 &: \alpha_5 = -\frac{1}{6}, \end{aligned}$$

At $n = 2$, the denominator vanishes. Whenever the denominator vanishes, we were not allowed to divide by it, to obtain an expression for the corresponding α_k. Two conclusions are drawn from this: (i) there is no equation that determines the coefficients α_k for which the denominator of the recursion relationship vanishes, and (ii) if this happens, the numerator better vanish as well, otherwise the equation "0 = something that is not 0" is obtained. For our example, the recursion relationship fails when

$$(n+1)(n+2) - 12 = 0 \quad \Rightarrow \quad n = -5 \ \text{ or } \ n = 2.$$

The first of these possibilities should be rejected as $n = -5$ is not an allowed value for n. The second value is an allowed value for n. Thus $\alpha_{n+4} = \alpha_6$ is undetermined provided $\sum_{m=1}^{5} \alpha_m\alpha_{6-m} = 0$, but

$$\sum_{m=1}^{5} \alpha_m \alpha_{6-m} = 2\alpha_1\alpha_5 + 2\alpha_2\alpha_4 + \alpha_3^2 = 0,$$

so that indeed α_6 is undetermined. All other α_k, $k = 7, 8, \ldots$ are determined from the recursion relationship, so that the Laurent series solution for $y(x)$ does depend on two arbitrary constants, x_0 (the location of the pole), and α_6. The coefficients (in general there could be more than one) that are not determined by the recursion relationship are called **resonances**.

The solution we have obtained is

$$y = \frac{1}{(x-x_0)^2}\left(1 - \frac{x_0}{10}(x-x_0)^4 - \frac{1}{6}(x-x_0)^5 + \alpha_6(x-x_0)^6 + \frac{x_0^2}{300}(x-x_0)^7 + \ldots\right).$$

The last remaining task is to determine the radius of convergence of this expression. This is far more difficult than for linear differential equations, where the recursion relationship is linear. In that case the radius of convergence follows from a simple application of the ratio test. For nonlinear differential equations, with an accompanying nonlinear recursion relation the situation is more complicated and it will not be considered here.

Let us summarize the main steps of the Painlevé algorithm:

1. **Singularity analysis:** determine the dominant powers of a meromorphic expansion. This power achieves a balance between at least two terms of the differential equation. It is possible that there are several possibilities. Each of these possibilities gives rise to a different solution branch.

2. **Leading-order coefficient:** from the singularity analysis, determine the coefficient of the first term of the meromorphic expansion. As before, there may be several possibilities.

3. **Recursion relationship:** using the form of meromorphic function, find a recursion relationship for the remaining coefficients from the differential equation.

4. **Resonances:** find the resonances, *i.e.*, those coefficients for which the recursion relationship fails.

5. **Radius of convergence:** determine the radius of convergence of all of the resulting meromorphic series solutions.

This procedure typically results in one solution branch which depends on a full set of arbitrary constants. This solution branch is referred to as the principal branch. The other branches are called special branches.

10.4 The relationship of the Painlevé property to the integrability of partial differential equations

In 1978, Ablowitz, Ramani and Segur formulated the following conjecture:

Conjecture 1 *Every ordinary differential equation which arises as a symmetry reduction of a completely integrable partial differential equation is of Painlevé type (*i.e.*, has the Painlevé property), perhaps after a transformation of variables.*

This conjecture gives a necessary condition for integrability: when faced with a partial differential equation, find all its symmetry reductions, and check to see that these reductions possess the Painlevé property. Finding symmetry reductions is an algorithmic problem, the solution of which has been implemented in a variety of software packages. It should be noted that the last statement of the conjecture may make it hard to use: if the ordinary differential equation found is not of Painlevé type, it is not always obvious whether or not a transformation exists that may transform the ordinary equation to one that does have the Painlevé property.

Example 7 *The KdV equation $u_t + 6uu_x + u_{xxx} = 0$ has the symmetry reduction*

$$u = U(x - Vt),$$

leading to the ordinary differential equation

$$-VU' + 6UU' + U''' = 0,$$

which leads to the class of stationary solutions of the KdV equation. It is solvable in terms of elliptic functions, which are meromorphic. The familiar soliton solution, expressed in terms of hyperbolic functions, may be obtained as a limit of these.

Example 8 *The Modified KdV equation $u_t - 6u^2u_x + u_{xxx} = 0$ has the similarity reduction*

$$u = (3t)^{-1/3}w(z), \quad z = x(3t)^{-1/3}.$$

Then

$$\begin{aligned}
u_t &= -\frac{1}{3}(3t)^{-4/3}3w(z) + (3t)^{-1/3}w'(z)\left(\frac{-1}{3}\right)(3t)^{-4/3}3 \\
&= -(3t)^{-4/3}w - zw'(z)(3t)^{-4/3} \\
u_x &= (3t)^{-1/3}w'(z)(3t)^{-1/3} \\
&= (3t)^{-2/3}w'(z) \\
u_{xx} &= (3t)^{-3/3}w''(z) \\
u_{xxx} &= (3t)^{-4/3}w'''(z),
\end{aligned}$$

leading to the ordinary differential equation

$$\begin{aligned}
& -(3t)^{-4/3}w - zw'(z)(3t)^{-4/3} - 6(3t)^{-2/3}w^2(z)(3t)^{-2/3}w'(z) + (3t)^{-4/3}w'''(z) = 0 \\
\Rightarrow \quad & -w - zw'(z) - 6w^2(z)w'(z) + w'''(z) = 0 \\
\Rightarrow \quad & -(zw)' - 6w^2w' + w''' = 0 \\
\Rightarrow \quad & -zw - 2w^3 + w'' = \alpha \\
\Rightarrow \quad & w'' = 2w^3 + zw + \alpha,
\end{aligned}$$

which is the P_{II} equation.

Can something be said about the partial differential equation directly? In other words, is it possible to come up with some type of integrability test that bypasses the symmetry reductions? If so, we are now dealing with functions of more than one variable, x and t. One of these variables is complex, or maybe they both are. Such a function typically will not have isolated singularities. In the case of complex x but real t, these singularities will fill out a smooth curve, say

$$\varphi(x,t) = 0.$$

For a fixed t, we expect this equation to have isolated roots. As t varies these roots will vary smoothly, except perhaps at some times where the roots may collide. Thus, for a given partial differential equation we could look for solutions of the form

$$u(x,t) = \frac{1}{\varphi(x,t)^p}\sum_{n=0}^{\infty}\alpha_n(t)\varphi(x,t)^n,$$

where the α_n, $n = 0, 1, \ldots$ are analytic functions and $\alpha_0 \neq 0$. Using the implicit function theorem, the determining equation for the roots may be solved for one of them, giving

$$\varphi(x,t) = 0 \quad \Rightarrow \quad x = x_0(t),$$

allowing us to look for expansions of the form

$$\begin{aligned}
u(x,t) &= \frac{1}{(x-x_0)^p}\sum_{n=0}^{\infty}\alpha_n(t)(x-x_0)^n \\
&= \sum_{n=0}^{\infty}\alpha_n(t)(x-x_0)^{n-p},
\end{aligned}$$

where x_0 is the location of the pole, which will typically vary in t.

Example 9 *As an example, we consider the equation*

$$u_t + 6uu_x + u_{xxx} + \alpha(t)u = 0.$$

This equation arises in various water-wave scenarios where the KdV equation shows up, but with an additional slowly varying bottom topography. This example equation contains at least one integrable equation as a special case, for $\alpha(t) \equiv 0$. Are there any other choices for $\alpha(t)$ for which the equation is integrable?

We look for solutions of the form

$$\begin{aligned} u(x,t) &= \frac{1}{(x-x_0)^p}\sum_{n=0}^{\infty}\alpha_n(t)(x-x_0)^n \\ &= \sum_{n=0}^{\infty}\alpha_n(t)(x-x_0)^{n-p}. \end{aligned}$$

First, we consider the most singular terms:

$$u \sim \alpha_0(x-x_0)^{-p}.$$

Then

$$\begin{aligned} u_x &\sim -p\alpha_0(x-x_0)^{-p-1}, \\ u_{xx} &\sim p(p+1)\alpha_0(x-x_0)^{-p-2}, \\ u_{xxx} &\sim -p(p+1)(p+2)\alpha_0(x-x_0)^{-p-3}, \\ u_t &\sim \alpha_0'(x-x_0)^{-p} + p\alpha_0 x_0'(x-x_0)^{-p-1}. \end{aligned}$$

Here we have used $'$ to denote time differentiation. Substitution in the partial differential equation gives

$$\begin{aligned} &\alpha_0'(x-x_0)^{-p} + p\alpha_0 x_0'(x-x_0)^{-p-1} - 6p\alpha_0^2(x-x_0)^{-2p-1} + \\ &\quad -p(p+1)(p+2)\alpha_0(x-x_0)^{-p-3} + \alpha(t)\alpha_0(x-x_0)^{-p} = 0. \end{aligned}$$

The third and fourth terms are the only candidates to be the most singular. As before, no meromorphic solutions are found by considering one of these to be singular (check!), but balancing both gives

$$-2p-1 = -p-3 \quad \Rightarrow \quad p = 2.$$

Equating the coefficients of the most singular terms gives

$$-6p\alpha_0^2 - p(p+1)(p+2)\alpha = 0 \quad \Rightarrow \quad \alpha_0 = -2.$$

Thus we are looking for solutions of the form

$$\begin{aligned} u(x,t) &= \frac{1}{(x-x_0)^2}\sum_{n=0}^{\infty}\alpha_n(t)(x-x_0)^n \\ &= \sum_{n=0}^{\infty}\alpha_n(t)(x-x_0)^{n-2}. \end{aligned}$$

Next this ansatz is substituted in the partial differential equation, in order to find a recursion relationship for the coefficients $\alpha_n(t)$, $n = 0, 1, 2, \ldots$. We have

$$\begin{aligned} u_t &= \sum_{n=0}^{\infty}\alpha_n'(x-x_0)^{n-2} - x_0'\sum_{n=0}^{\infty}\alpha_n(n-2)(x-x_0)^{n-3}, \\ u_x &= \sum_{n=0}^{\infty}\alpha_n(t)(n-2)(x-x_0)^{n-3}, \\ u_{xx} &= \sum_{n=0}^{\infty}\alpha_n(t)(n-2)(n-3)(x-x_0)^{n-4}, \\ u_{xxx} &= \sum_{n=0}^{\infty}\alpha_n(t)(n-2)(n-3)(n-4)(x-x_0)^{n-5}. \end{aligned}$$

We obtain

$$\begin{aligned} &\sum_{n=0}^{\infty}\alpha_n'(x-x_0)^{n-2} - x_0'\sum_{n=0}^{\infty}\alpha_n(n-2)(x-x_0)^{n-3} + \\ &\qquad 6\sum_{n=0}^{\infty}\alpha_n(x-x_0)^{n-2}\sum_{k=0}^{\infty}\alpha_k(k-2)(x-x_0)^{k-3} + \\ &\sum_{n=0}^{\infty}\alpha_n(n-2)(n-3)(n-4)(x-x_0)^{n-5} + \alpha(t)\sum_{n=0}^{\infty}\alpha_n(x-x_0)^{n-2} = 0 \\ \Rightarrow\quad &\sum_{n=0}^{\infty}\alpha_n'(x-x_0)^{n-2} - x_0'\sum_{n=-1}^{\infty}\alpha_{n+1}(n-1)(x-x_0)^{n-2} + \\ &\qquad 6\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(k-2)\alpha_n\alpha_k(x-x_0)^{k+n-5} + \\ &\sum_{n=-3}^{\infty}\alpha_{n+3}(n+1)n(n-1)(x-x_0)^{n-2} + \alpha(t)\sum_{n=0}^{\infty}\alpha_n(x-x_0)^{n-2} = 0 \\ \Rightarrow\quad &\sum_{n=0}^{\infty}\alpha_n'(x-x_0)^{n-2} + 2x_0'\alpha_0(x-x_0)^{-3} - x_0'\sum_{n=0}^{\infty}\alpha_{n+1}(n-1)(x-x_0)^{n-2} + \end{aligned}$$

$$
\begin{aligned}
6\sum_{N=-3}^{\infty}(x-x_0)^{N-2}\sum_{k=0}^{N+3}(k-2)\alpha_{N+3-k}\alpha_k - 24\alpha_0(x-x_0)^{-5} - 6\alpha_1(x-x_0)^{-4} + \\
\sum_{n=0}^{\infty}\alpha_{n+3}(n+1)n(n-1)(x-x_0)^{n-2} + \alpha(t)\sum_{n=0}^{\infty}\alpha_n(x-x_0)^{n-2} &= 0 \\
\Rightarrow \quad \sum_{n=0}^{\infty}\alpha_n'(x-x_0)^{n-2} - 4x_0'(x-x_0)^{-3} - x_0'\sum_{n=0}^{\infty}\alpha_{n+1}(n-1)(x-x_0)^{n-2} + \\
-48(x-x_0)^{-5} + 36\alpha_1(x-x_0)^{-4} + 6(4\alpha_2-\alpha_1^2)(x-x_0)^{-3} + \\
6\sum_{n=0}^{\infty}(x-x_0)^{n-2}\sum_{k=0}^{n+3}(k-2)\alpha_{n+3-k}\alpha_k + 48(x-x_0)^{-5} - 6\alpha_1(x-x_0)^{-4} + \\
\sum_{n=0}^{\infty}\alpha_{n+3}(n+1)n(n-1)(x-x_0)^{n-2} + \alpha(t)\sum_{n=0}^{\infty}\alpha_n(x-x_0)^{n-2} &= 0.
\end{aligned}
$$

The most singular term cancels, as it should. At the next order (terms of order -4):

$$36\alpha_1 - 6\alpha_1 = 0 \quad \Rightarrow \quad \alpha_1 \equiv 0.$$

The terms of order -3 result in the condition

$$-4x_0' + 6(4\alpha_2 - \alpha_1^2) = 0 \quad \Rightarrow \quad \alpha_2 = \frac{1}{6}x_0'.$$

The remaining terms can all be grouped in a general recursion relationship: for $n = 0, 1, 2, \ldots$

$$
\begin{aligned}
\alpha_n' - x_0'\alpha_{n+1}(n-1) + 6\sum_{k=0}^{n+3}(k-2)\alpha_k\alpha_{n+3-k} + \alpha_{n+3}(n+1)n(n-1) + \alpha(t)\alpha_n &= 0 \\
\Rightarrow \quad \alpha_n' - x_0'\alpha_{n+1}(n-1) - 12\alpha_0\alpha_{n+3} + 6\sum_{k=1}^{n+2}(k-2)\alpha_k\alpha_{n+3-k} + \\
6(n+1)\alpha_{n+3}\alpha_0 + \alpha_{n+3}n(n^2-1) + \alpha(t)\alpha_n &= 0 \\
\Rightarrow \quad \alpha_{n+3}(24 - 12(n+1) + n(n^2-1)) = \\
x_0'\alpha_{n+1}(n-1) - \alpha_n' - 6\sum_{k=1}^{n+2}(k-2)\alpha_k\alpha_{n+3-k} - \alpha(t)\alpha_n \\
\Rightarrow \quad \alpha_{n+3}(n-1)(n-3)(n+4) = \\
x_0'\alpha_{n+1}(n-1) - \alpha_n' - 6\sum_{k=1}^{n+2}(k-2)\alpha_k\alpha_{n+3-k} - \alpha(t)\alpha_n.
\end{aligned}
$$

Thus, if $n = 0, 2, 4, 5, 6, \ldots$

$$\alpha_{n+3} = \frac{x_0'\alpha_{n+1}(n-1) - \alpha_n' - 6\sum_{k=1}^{n+2}(k-2)\alpha_k\alpha_{n+3-k} - \alpha(t)\alpha_n}{(n-1)(n-3)(n+4)}.$$

If $n = 1$ or $n = 3$, the denominator of this expression is zero. Thus α_4 and α_6 are potential resonances. Note that the third root of the denominator is not an allowed value of n. For $n = 0$, we find

$$\alpha_3 = \frac{2\alpha(t)}{12} = \frac{1}{6}\alpha(t).$$

Next up is $n = 1$. We know that this corresponds to a potential resonance. The numerator of the recursion relationship is

$$-6\sum_{k=1}^{2}(k-2)\alpha_k\alpha_{4-k} = -6\alpha_1\alpha_3,$$

which is zero. Thus α_4 is indeed a resonance. For $n = 2$, we obtain

$$\alpha_5 = \frac{1}{36}(x_0'' + x_0'\alpha).$$

The next term again gives rise to a potential resonance. The numerator of the recursion relationship is

$$2x_0'\alpha_4 - \alpha_3' - 6\sum_{k=1}^{5}(k-2)\alpha_k\alpha_{6-k} - \alpha(t)\alpha_3 = -\frac{1}{3}(\alpha^2 + \alpha'/2).$$

If this numerator does not vanish, α_6 is infinite, and the partial differential equations has no solutions that are meromorphic in x. Thus, the vanishing of the above numerator is a solvability condition for the partial differential equation. If it is satisfied then α_6 is undetermined, and it is a second resonance for the equation, resulting in a general solution branch, as there are three arbitrary functions of t in the solution: $x_0(t)$, $\alpha_4(t)$ and $\alpha_6(t)$. The solvability condition gives

$$\alpha' = -2\alpha^2,$$

which is an ordinary differential equation of Painlevé type. It has two solution branches

- *$\alpha = 0$: in this case, the original equation is the KdV equation. We know the KdV equation is integrable, which is confirmed here by the Painlevé test.*
- *$\alpha \neq 0$: then*

$$\alpha = \frac{1}{2(t - t_0)}.$$

Often one chooses $t_0 = 0$ and the partial differential equation becomes

$$u_t + 6uu_x + u_{xxx} + \frac{1}{2t}u = 0,$$

which is known as the Cylindrical KdV equation. It is known to be integrable. This is nicely confirmed by the Painlevé test.

This example illustrates that the partial differential equation Painlevé test offers a straighforward algorithmic way to test the integrability of a given partial differential equation, without having to consider all of its symmetry reductions.

10.5 Exercises

1. Using the Painlevé test, discuss the integrability of

$$u_t = uu_x^p + u_{xxx},$$

as a function of p.

2.

Chapter 11

The prolongation method

Throughout this course we have derived many benefits from a nonlinear partial differential equation being written as the compatibility condition of two linear differential equations. Thus, given a Lax pair for a nonlinear partial differential equation, one may very reasonably call it integrable. Is there a way to see if a given partial differential equation has a Lax pair or not? Answering this question is the goal of this chapter.

11.1 Outline

Consider a nonlinear partial differential equation in evolution form

$$u_t = F(u, u_x, u_{xx}, \dots,),$$

where $F(u, u_x, \dots)$ depends on u and a finite number of its derivatives with respect to x. In order for this equation to have a Lax pair, we need there to exist two matrices X and T such that the nonlinear partial differential equation is the compatibility condition of the two linear equations

$$\begin{cases} \psi_x = X\psi, \\ \psi_t = T\psi. \end{cases}$$

Here X and T depend on u and a finite number of its x-derivatives. There is no need to have X and T depend on t-derivatives, as these can always be eliminated using the evolution equation. The compatibility condition is, as before, given by

$$X_t - T_x + [X, T] = 0.$$

At this point we don't know the dimensions of the matrices X and T. We don't know what their functional dependence on u and its derivatives is. At this point we don't even know on how many of these derivatives it depends! It appears we don't have much to work with.

Here's an outline of how the prolongation method works:

1. Choose a positive integer N. Assume that X depends on u, u_x, ..., u_{Nx} (N x-derivatives). This is the only assumption we'll make. We won't even guess the dimension of the matrices!

2. The compatibility condition will impose the number of x-derivatives that T depends on.

3. At this point, the dependence of the compatibility condition on the highest x-derivative of u will be explicit. Its coefficient is equated to zero. The resulting equation is solved. We now know the dependence of the compatibility condition on the next highest x-derivative of u, at which point the process repeats until done.

4. When done, a number of matrix integration constants has been introduced. Because of the presence of the matrix commutator in the compatibility condition, a number of commutation relations between these matrix integration constants exists. Thus these matrices are elements of a Lie algebra.

5. Once it is determined which Lie algebra has been found, the lowest-dimensional matrix representation of this algebra is used to find a matrix representation for the matrix integration constants. From these, the forms of X and T are reconstructed.

If all of the above is successful, we have constructed a Lax pair for the equation and it may reasonably be said to be integrable. If one of the above steps failed, this does not imply that no Lax pair exists. This is a possibility, but it is also possible that our assumption in the first step for the number of x-derivatives of u appearing in X was wrong. At this point, we may decide to repeat all of the steps with a larger choice of N.

A bit of history: Estabrook and Wahlquist were the first ones to demonstrate the prolongation method for the KdV and NLS equations. They used the method in a differential geometric setting, using differential forms. This is where the method got its name. The functions $u(x,t)$, *etc* are prolonged by considering the jet space consisting of a function and its derivatives u_x, u_{xx}, ... Then the evolution equation is an algebraic relationship among the variables in this jet space. None of the differential-geometric language was required for the method to work, and the method became more popular after Fordy and others emphasized its algorithmic approach, in a setting that was more compatible with the typical background of an applied mathematician.

11.2 Example: the KdV equation

Consider the KdV equation

$$u_t = 6uu_x + u_{xxx}.$$

11.2.1 Step 1. Choosing the dependence of X

For step 1, we choose $N = 0$, which is the easiest choice. This first choice is typical. Thus we assume

$$X = X(u),$$

and X depends on $u(x,t)$, but not on any of its derivatives.

11.2.2 Step 2. Determine the dependence of T

Next we determine the dependence of T: how many derivatives of $u(x,t)$ does it depend on? The compatibility condition requires the calculation of X_t, which is

$$X_t = X_u u_t = X_u(6uu_x + u_{xxx}),$$

by the chain rule. Hence there is a term in the compatibility condition with u_{xxx}. In order for this term to balance, another such term needs to be present. It may be obtained by imposing that T depends on u_{xx}. In that case, the chain rule will indeed give rise to a u_{xxx} term. It is also easy to see that T should not contain u_{xxx}. If it did, the chain rule would give rise to a term with u_{xxxx}, which cannot be balanced by any other term. Thus we have

$$T = T(u, u_x, u_{xx}),$$

and

$$T_x = T_u u_x + T_{u_x} u_{xx} + T_{u_{xx}} u_{xxx}.$$

The compatibility condition becomes

$$X_u(6uu_x + u_{xxx}) - T_u u_x - T_{u_x} u_{xx} - T_{u_{xx}} u_{xxx} + [X, T] = 0.$$

This is the equation that has to be solved for $X(u)$ and $T(u, u_x, u_{xx})$. If no solution can be found, we may start over, assuming $X = X(u, u_x)$, whence $T = T(u, u_x, u_{xx}, u_{xxx}, u_{xxxx})$, and so on.

11.2.3 Step 3. Determine the explicit dependence of X and T

This last equation is an algebraic relationship in which u, u_x, u_{xx} and u_{xxx} may all be regarded as independent variables. Indeed, the only relationship that holds between the derivatives of u is the KdV equation. But this was already used to eliminate any time derivatives. In the above equation, X_u does not depend on u_{xxx}, nor do any of the partial derivatives of T. Thus, all dependence on u_{xxx} is explicit. This dependence is polynomial (linear, actually). Thus all coefficients of different powers of u_{xxx} need to vanish. We obtain two equations:

$$\begin{aligned} u^1_{xxx}: &\quad X_u - T_{u_{xx}} = 0, \\ u^0_{xxx}: &\quad 6uu_xX_u - T_u u_x - T_{u_x}u_{xx} + [X,T] = 0. \end{aligned}$$

The first of these equations can be integrated with respect to u_{xx}, since X_u does not depend on it:

$$T = u_{xx}X_u + A(u,u_x),$$

with some integration "constant" $A(u,u_x)$, which may depend on u and u_x, but not u_{xx}. We have already successfully determined the u_{xx}-dependence of T! This result is now substituted in the remaining equation, resulting in an equation for A and X:

$$\begin{aligned} & 6uu_xX_u - (u_{xx}X_{uu} + A_u)u_x - A_{u_x}u_{xx} + [X, u_{xx}X_u + A] &=& \ 0 \\ \Rightarrow \quad & 6uu_xX_u - u_xu_{xx}X_{uu} - u_xA_u - A_{u_x}u_{xx} + u_{xx}[X,X_u] + [X,A] &=& \ 0. \end{aligned}$$

All the unknowns in this equation depend on u and possibly u_x, but not on u_{xx}. As before, this implies that all of its coefficients vanish:

$$\begin{aligned} u^1_{xx}: &\quad -u_xX_{uu} - A_{u_x} + [X,X_u] = 0, \\ u^0_{xx}: &\quad 6uu_xX_u - u_xA_u + [X,A] = 0. \end{aligned}$$

Again, the first of these equations is easily integrated, as X and X_u do not depend on u_x:

$$\begin{aligned} & -\frac{1}{2}u_x^2X_{uu} - A + u_x[X,X_u] = -B(u) \\ \Rightarrow \quad & A = -\frac{1}{2}u_x^2X_{uu} + u_x[X,X_u] + B(u), \end{aligned}$$

for some new constant of integration $B(u)$, which only depends on u. At this point, we have determined the u_x dependence of A, and thus of T. It remains to determine the u-dependence of both X and T. Substituting the newly found form for A in the remaining equation gives

$$\begin{aligned} & 6uu_xX_u - u_x\left(-\frac{1}{2}u_x^2X_{uuu} + u_x[X,X_{uu}] + B_u\right) + \left[X, -\frac{1}{2}u_x^2X_{uu} + u_x[X,X_u] + B\right] = 0 \\ \Rightarrow \quad & 6uu_xX_u + \frac{1}{2}u_x^3X_{uuu} - u_x^2[X,X_{uu}] - u_xB_u - \frac{1}{2}u_x^2[X,X_{uu}] + u_x[X,[X,X_u]] + [X,B] = 0. \end{aligned}$$

We proceed as before: B and X and their derivatives only depend on u, not on u_x. Thus the coefficients of all powers of u_x identically vanish:

$$
\begin{aligned}
u_x^3: &\quad \frac{1}{2}X_{uuu} = 0,\\
u_x^2: &\quad -[X, X_{uu}] - \frac{1}{2}[X, X_{uu}] = 0,\\
u_x^1: &\quad 6uX_u - B_u + [X, [X, X_u]] = 0,\\
u_x^0: &\quad [X, B] = 0,
\end{aligned}
$$

or, after simplification,

$$
\begin{aligned}
u_x^3: &\quad X_{uuu} = 0,\\
u_x^2: &\quad [X, X_{uu}] = 0,\\
u_x^1: &\quad 6uX_u - B_u + [X, [X, X_u]] = 0,\\
u_x^0: &\quad [X, B] = 0.
\end{aligned}
$$

The first of these equations gives

$$X = G_1 + uG_2 + u^2G_3,$$

where G_1, G_2 and G_3 are matrix constants of integration. Their dimension is undetermined, for now. The three remaining equations become

$$
\begin{aligned}
\left[G_1 + uG_2 + u^2G_3, 2G_3\right] &= 0,\\
6u(G_2 + 2uG_3) - B_u + \left[G_1 + uG_2 + u^2G_3, \left[G_1 + uG_2 + u^2G_3, G_2 + 2uG_3\right]\right] &= 0,\\
\left[G_1 + uG_2 + u^2G_3, B\right] &= 0.
\end{aligned}
$$

The first one of these equations is a polynomial in u, with constant coefficients. It is identically zero, thus all its coefficients vanish, resulting in

$$
\begin{aligned}
[G_1, G_3] &= 0,\\
[G_2, G_3] &= 0.
\end{aligned}
$$

Using these conditions, the equation containing B_u is rewritten after simplification as

$$B_u = 6uG_2 + 12u^2G_3 + [G_1, [G_1, G_2]] + u\,[G_2, [G_1, G_2]]\,.$$

To obtain this result, we have used the Jacobi identity for the commutator of two matrices:

$$[a, [b, c]] + [a, [b, c]] + [a, [b, c]] = 0,$$

for any three square matrices a, b and c of equal dimension. This is used to show that $[G_3, [G_1, G_2]] = 0$, using the above conditions on the commutators of G_3 with G_1 and G_2.

Integrating the expression for B_u results in

$$B = 3u^2 G_2 + 4u^3 G_3 + u\,[G_1, G_4] + \frac{1}{2}u^2\,[G_2, G_4] + G_0,$$

for a new matrix constant G_0. We have introduced the constant matrix G_4,

$$G_4 = [G_1, G_2],$$

to avoid writing nested commutators.

The remaining equation is

$$[G_1 + uG_2 + u^2 G_3, B] = 0.$$

Using the newly-found functional form of B, this becomes

$$\left[G_1 + uG_2 + u^2 G_3, 3u^2 G_2 + 4u^3 G_3 + u\,[G_1, G_4] + \frac{1}{2}u^2\,[G_2, G_4] + G_0\right] = 0.$$

This is a polynomial in u. The coefficients of all powers of u vanish identically:

$$\begin{aligned}
u^5: &\quad [G_3, G_3] = 0,\\
u^4: &\quad 4[G_2, G_3] + 3[G_3, G_2] + \frac{1}{2}\,[G_3, [G_2, G_4]] = 0,\\
u^3: &\quad 4[G_1, G_3] + 3[G_2, G_2] + \frac{1}{2}[G_2, [G_2, G_4]] + [G_3, [G_1, G_4]] = 0,\\
u^2: &\quad 3[G_1, G_2] + \frac{1}{2}[G_1, [G_2, G_4]] + [G_2, [G_1, G_4]] + [G_3, G_0] = 0,\\
u^1: &\quad [G_1, [G_1, G_4]] + [G_2, G_0] = 0,\\
u^0: &\quad [G_1, G_0] = 0.
\end{aligned}$$

In order to avoid further use of nested commutators, we introduce the constant matrices G_5 and G_6:

$$\begin{aligned}
G_5 &= [G_1, G_4],\\
G_6 &= [G_2, G_4].
\end{aligned}$$

Simplifying the above conditions on the constant matrices gives

$$\begin{aligned}
u^4: &\quad [G_3, G_6] = 0,\\
u^3: &\quad \frac{1}{2}[G_2, G_6] + [G_3, G_5] = 0,\\
u^2: &\quad 3G_4 + \frac{1}{2}[G_1, G_6] + [G_2, G_5] + [G_3, G_0] = 0,\\
u^1: &\quad [G_1, G_5] + [G_2, G_0] = 0,\\
u^0: &\quad [G_1, G_0] = 0.
\end{aligned}$$

Let's summarize: we have completely determined the functional dependence of X and T to be

$$\begin{aligned} X &= G_1 + uG_2 + u^2G_3, \\ T &= u_{xx}X_u - \frac{1}{2}u_x^2X_{uu} + u_x[X, X_u] + 3u^2G_2 + 4u^3G_3 + u[G_1, G_4] + \frac{1}{2}u^2[G_2, G_4] + G_0 \\ &= u_{xx}(G_2 + 2uG_3) - u_x^2G_3 + u_xG_4 + 3u^2G_2 + 4u^3G_3 + uG_5 + \frac{1}{2}u^2G_6 + G_0 \\ &= u_{xx}G_2 + 2uu_{xx}G_3 - u_x^2G_3 + u_xG_4 + 3u^2G_2 + 4u^3G_3 + uG_5 + \frac{1}{2}u^2G_6 + G_0. \end{aligned}$$

11.2.4 Step 4. The commutation table

Here G_0, G_1, G_2, G_3, G_4, G_5 and G_6 are constant matrices of unknown dimension. They satisfy the following equations:

$$\begin{gathered} [G_1, G_3] = 0, \quad [G_2, G_3] = 0, \quad G_4 = [G_1, G_2], \\ [G_3, G_4] = 0, \quad [G_3, G_6] = 0, \quad \tfrac{1}{2}[G_2, G_6] + [G_3, G_5] = 0, \\ 3G_4 + \tfrac{1}{2}[G_1, G_6] + [G_2, G_5] + [G_3, G_0] = 0, \\ [G_1, G_5] + [G_2, G_0] = 0, \quad [G_1, G_0] = 0 \end{gathered} .$$

More of such conditions may be obtained from the Jacobi identity. For instance,

$$\begin{aligned} & [G_1, [G_3, G_4]] + [G_3, [G_4, G_1]] + [G_4, [G_1, G_3]] = 0 \\ \Rightarrow \quad & [G_1, 0] + [G_3, -G_5] + [G_4, 0] = 0 \\ \Rightarrow \quad & [G_5, G_3] = 0, \end{aligned}$$

from which it follows that $[G_2, G_6] = 0$, using the conditions we already found. Also,

$$\begin{aligned} & [G_1, [G_2, G_4]] + [G_2, [G_4, G_1]] + [G_4, [G_1, G_2]] = 0 \\ \Rightarrow \quad & [G_1, G_6] + [G_2, -G_5] + [G_4, G_4] = 0 \\ \Rightarrow \quad & [G_1, G_6] = [G_2, G_5]. \end{aligned}$$

These conditions are most conveniently collected in a commutation table, where the entry at the position determined by G_i and $G_k$is their commutator. The commutation table for the matrix constants of the KdV equation is given in Table 11.1. The "new" matrices G_7 and G_8 were introduced so that all information contained in the relationships for the matrix constants could be encoded in the commutation table.

$[\ ,\ ]$	G_0	G_1	G_2	G_3	G_4	G_5	G_6
G_0	0	0	G_8	$3(G_4+G_7/2)$			
G_1	0	0	G_4	0	G_5	G_8	G_7
G_2	$-G_8$	$-G_4$	0	0	G_6	G_7	0
G_3	$-3(G_4+G_7/2)$	0	0	0	0	0	0
G_4		$-G_5$	$-G_6$	0	0		
G_5		$-G_8$	$-G_7$	0		0	
G_6		$-G_7$	0	0			0

Table 11.1: The commutation table for the matrix constants determining the Lax pair of the KdV equation.

11.2.5 Step 5. Wanted: a Lie algebra

If we were able to complete the commutation table, *i.e.*, express all commutators $[G_i, G_j]$ $(i, j = 0, \ldots, 6)$ as a linear combination of the G_i $(i = 0, \ldots, 6)$, then these G_i would be part of a Lie algebra, determined by this commutation table. At this point, it is not essential for us to know much about Lie algebras. It suffices to know that any Lie algebra may be decomposed into semi-simple Lie algebras and these have been completely classified. In other words, once the commutation table is completed, we may consult any mathematician who knows about Lie algebras[1] and she will be able to tell us (1) which Lie algebra we have encountered, and (2) what its lowest-dimensional matrix representation is. Thus, the final result of this is an explicit matrix form of the constants G_i $(i = 0, \ldots, 6)$, usually depending on one or more undetermined scalar constants. Together with the previously determined functional form of X and T, this gives the complete form of the Lax pair of the evolution equation we started with.

So, what is the catch? The catch is that we have not filled in the commutation table. Perhaps we could find a few more entries using the Jacobi identity. However, typically this does not allow for a full determination of the commutation table. Nor does it here, for the KdV equation. We may attempt to add a few entries to the list of generators G_i, $(i = 0, 1, \ldots, 6)$, such as G_7 and G_8. Now the table has an extra two rows and columns. Again, we might use the Jacobi identity to fill in more empty spaces. Once more we will fail and we will have to add more generators to the list. Proceeding this way, we are lead to an infinite-dimensional Lie algebra or Kac-Moody algebra. Such algebras do not have finite-dimensional representations and it appears that no matrix representation for the Lax pair will be found.

11.2.6 Step 6. The closure conditions

An alternative approach is to try to impose additional conditions on the algebra so as to complete the commutation table. Such conditions are refered to as closure conditions. If

[1] As a matter of fact, this is how I first used the prolongation method.

this approach fails, it may be because the imposed closure are too strict, or it may be that the evolution equation is not integrable.

Let us use the commutation table as a guide to impose some closure conditions. We see that G_3 commutes with any other generator, except for possibly G_0. It seems natural to impose that G_3 does indeed commute with all generators. This results in two closure conditions:

$$G_7 = -2G_4,$$

and

$$G_3 = \omega I,$$

where I denotes the identity matrix and ω is a scalar constant. This is the most general choice for a matrix commuting with any other matrix of the same dimensions. Note that this choice assumes that the generators do indeed span the Lie algebra we are hoping to find. Assume that $\omega \neq 0$. The only effect G_3 has on the Lax pair is in the $X_t - T_x$ terms, as it commutes with all matrices. But the coefficient of G_3 in X is u^2, and we see that the only consequence of non-zero ω is the inclusion of the second conservation law for the KdV equation in the Lax pair. We are out to obtain the simplest Lax pair for the KdV equation, thus we may choose $\omega = 0$, or

$$G_3 = 0.$$

Next, we note from the commutation table that if G_5 and G_6 can be written as linear combinations of G_1, G_2 and G_4, then these last three generators will form a subalgebra: their commutation table will not involve any generators outside this list of three. We set

$$\left\{ \begin{array}{rcl} G_5 &=& \alpha G_1 + \beta G_2 + \gamma G_4 \\ G_6 &=& \delta G_1 + \kappa G_2 + \epsilon G_4 \end{array} \right. ,$$

where α, β, γ, δ, κ and γ are scalar constants, to be determined.

From $[G_1, G_6] = G_7$, we obtain

$$\begin{array}{rrcl} & [G_1, \delta G_1 + \kappa G_2 + \epsilon G_4] &=& -2G_4 \\ \Rightarrow & \kappa[G_1, G_2] + \epsilon[G_1, G_4] &=& -2G_4 \\ \Rightarrow & \kappa G_4 + \epsilon G_5 &=& -2G_4 \\ \Rightarrow & \kappa G_4 + \epsilon(\alpha G_1 + \beta G_2 + \gamma G_4) &=& -2G_4 \\ \Rightarrow & (\kappa + \epsilon\gamma) G_4 + \epsilon\alpha G_1 + \epsilon\beta G_2 &=& -2G_4 \\ \Rightarrow & \kappa + \epsilon\gamma = -2, \quad \epsilon\alpha = 0, \quad \epsilon\beta &=& 0. \end{array}$$

Similarly, the commutator relationships $[G_1, G_5] = G_8$, $[G_2, G_5] =$ and $[G_2, G_6] = 0$ result in the conditions

$$\alpha = 2, \quad \gamma = 0, \quad \delta = 0, \quad \kappa = -2, \quad \epsilon = 0,$$

and thus

$$\begin{cases} G_5 &= 2G_1 + \beta G_2 \\ G_6 &= -2G_2 \end{cases},$$

where β remains undetermined.

$[\ ,\ ]$	G_1	G_2	G_4
G_1	0	G_4	$2G_1 + \beta G_2$
G_2	$-G_4$	0	$-2G_2$
G_4	$-2G_1 - \beta G_2$	$2G_2$	0

Table 11.2: The commutation table of the subalgebra $\{G_1, G_2, G_4\}$.

11.2.7 Step 7. A representation for the algebra

The commutation table for the subalgebra generated by $\{G_1, G_2, G_4\}$ is given in Table 11.2. From it, anyone who knows about Lie algebras will tell you that it is isomorfic with $sl(2, \mathbb{C})$, which is a Lie algebra with a representation in terms of 2×2 traceless matrices. This algebra is generated by three basis elements. One choice for this basis is

$$\begin{aligned} e_+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ e_- &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ h &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

The commutation table for $sl(2, \mathbb{C})$ is easily computed. It is given in Table 11.3.

$[\ ,\ ]$	e_+	e_-	h
e_+	0	h	$-2e_+$
e_-	$-h$	0	$2e_-$
h	$2e_+$	$-2e_-$	0

Table 11.3: The commutation table of the Lie algebra $sl(2, \mathbb{C})$.

Next, we determine the relationships between our generators and the basis elements of $sl(2, \mathbb{C})$:

$$\left\{\begin{array}{lcl} G_1 &=& \alpha_1 e_+ + \alpha_2 e_- \\ G_2 &=& \beta_1 e_+ + \beta_2 e_- \\ G_4 &=& \gamma_3 h \end{array}\right.$$

In general, the generators G_1, G_2 and G_4 should be expanded as linear combinations with constant coefficients of all basis elements. The similarity between the two commutation tables 11.2 and 11.3 allows for the reduced ansatz used here. Now all the existing commutation relations for $\{G_1, G_2, G_4\}$ are imposed in order to determine as many of the α_1, α_2, β_1, β_2 and γ_3 as possible. For instance

$$\begin{array}{rrcl} & [G_1, G_2] &=& G_4 \\ \Rightarrow & [\alpha_1 e_+ + \alpha_2 e_-, \beta_1 e_+ + \beta_2 e_-] &=& \gamma_3 h \\ \Rightarrow & \alpha_1\beta_2 h - \alpha_2\beta_1 h &=& \gamma_3 h \\ \Rightarrow & \alpha_1\beta_2 - \alpha_2\beta_1 &=& \gamma_3. \end{array}$$

Similarly, we obtain the conditions

$$\beta_1\gamma_3 = \beta_1, \quad \beta_2\gamma_3 = -\beta_2,$$

and

$$\left\{\begin{array}{rcl} 2\gamma_3\alpha_1 &=& -2\alpha_1 - \beta\beta_1 \\ -2\gamma_3\alpha_2 &=& -2\alpha_2 - \beta\beta_2 \end{array}\right. .$$

From these conditions it follows that either $\beta_1 = 0$ or $\beta_2 = 0$. In what follows we work with $\beta_1 = 0$ and $\beta_2 \neq 0$. Equivalent results are obtained with $\beta_2 = 0$ and $\beta_1 \neq 0$. Indeed, many different representations of the Lax pair for an integrable equation may be obtained from a given commutation table. All these representations are equivalent in the sense that their compatibility condition gives rise to the same integrable partial differential equation. Often these different representations are related by similarity or other simple transformations.

With our choice

$$\gamma_3 = -1,$$

so that

$$G_4 = -h = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The remaining equations are

$$\alpha_1 = -\frac{1}{\beta_2}, \quad \alpha_2 = -\frac{\beta\beta_2}{4},$$

from which

$$\begin{cases} G_1 &= -\dfrac{1}{\beta_2}e_+ - \dfrac{\beta\beta_2}{4}e_- = \begin{pmatrix} 0 & -1/\beta_2 \\ -\beta\beta_2/4 & 0 \end{pmatrix} \\ G_2 &= \beta_2 e_- = \begin{pmatrix} 0 & 0 \\ \beta_2 & 0 \end{pmatrix} \end{cases}.$$

11.2.8 Step 8. The Lax pair

We are now in a position to reconstruct the Lax pair. First,

$$\begin{aligned} X &= G_1 + G_2 u \\ &= \begin{pmatrix} 0 & -\frac{1}{\beta_2} \\ -\frac{\beta\beta_2}{4} + \beta_2 u & 0 \end{pmatrix}. \end{aligned}$$

The first part of the scattering problem is

$$\begin{aligned} &\begin{cases} \psi_{1x} &= -\dfrac{1}{\beta_2}\psi_2, \\ \psi_{2x} &= \left(-\dfrac{\beta\beta_2}{4} + \beta_2 u\right)\psi_1 \end{cases} \\ \Rightarrow \qquad &\psi_{1xx} = -\frac{1}{\beta_2}\left(-\frac{\beta\beta_2}{4} + \beta_2 u\right)\psi_1 \\ &= \left(\frac{\beta}{4} - u\right)\psi_1, \end{aligned}$$

which is the familias Schrödinger scattering problem, with $\lambda = \beta/4$. We see that it was important not to choose a value for β when it was left undetermined by the algebra, as it results in the scattering parameter for the forward scattering problem. From the above calculation, it is however clear that the value of β_2 is irrelevant, so we let $\beta_2 = -1$ from now on. Thus, setting $\beta = 4\lambda$

$$X = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}.$$

Next, we reassemble T:

$$\begin{aligned} T &= u_{xx}G_2 + 2uu_{xx}G_3 - u_x^2 G_3 + u_x G_4 + 3u^2 G_2 + 4u^3 G_3 + uG_5 + \frac{1}{2}u^2 G_6 + G_0 \\ &= \left(u_{xx} + 3u^2\right)G_2 + u_x G_4 + uG_5 + \frac{1}{2}u^2 G_6 + G_0 \\ &= \left(u_{xx} + 3u^2\right)\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} + u_x \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + u\begin{pmatrix} 0 & 2 \\ -2\lambda & 0 \end{pmatrix} + \frac{1}{2}u^2\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} + G_0 \\ &= \begin{pmatrix} -u_x + G_{011} & 2u + G_{012} \\ -u_{xx} - 2u^2 - 2\lambda u + G_{021} & u_x - G_{011} \end{pmatrix}, \end{aligned}$$

where we have introduced the components of G_0:

$$G_0 = \begin{pmatrix} G_{011} & G_{012} \\ G_{021} & -G_{011} \end{pmatrix},$$

which we have chosen so that G_0 is traceless. There are different ways to determine G_0, but the most direct one is to use the newly found representations of X and T and plug them in the compatibility condition $X_t + XT = T_x + TX$. Its different components give:

$$\begin{aligned} 11: &\quad G_{012} = 4\lambda, \quad G_{021} = 4\lambda^2, \\ 12: &\quad G_{011} = 0. \end{aligned}$$

The other components are identically satisfied, provided $u(x,t)$ satisfies the KdV equation. Finally,

$$T = \begin{pmatrix} -u_x & 2u + 4\lambda \\ -u_{xx} - 2u^2 - 2\lambda u + 4\lambda^2 & u_x \end{pmatrix},$$

and we have indeed reconstructed the Lax pair for the KdV equation, using only very elementary assumptions.

Part III

Global aspects of integrable equations

Chapter 12

The inverse scattering method

The inverse scattering method to solve the initial-value problem for a given integrable equation is undoubtably *the* pinnacle of the theory of integrable systems, and one of the major achievements in nonlinear science. Many authors list the method of inverse scattering for integrable systems and the understanding of chaos for almost all other systems as the two major pillars that were added to nonlinear science in the twentieth century.

12.1 The big picture

Given a partial differential equation, the main problem of interest is the initial-value problem with given boundary conditions. For instance, for the KdV equation, we could set out to solve

$$\begin{cases} u_t = 6uu_x + u_{xxx}, \\ u(x,0) = U(x) \in H, \\ \lim_{|x|\to\infty} u(x,t) = 0, \end{cases}$$

where H is the function space of choice. Thus H could be $L_2(-\infty,\infty)$, *etc.* The method that does this is the inverse scattering method. When Gardner, Greene, Kruskal and Miura first used it to solve the KdV equation, their knowledge of both scattering and inverse scattering in the context of quantum mechanics was crucial.

12.1.1 A little bit of quantum mechanics

In non-relativistic quantum mechanics, a particle or wave interacting with a potential is described by the one-particle time-independent linear Schrödinger equation

$$\psi_{1xx} + u\psi_1 = \lambda\psi_1.$$

We have rescaled the dependent and independent variables for our convenience. The wave function ψ_1 is denoted with the index to remind ourselves it is the first component of the

Lax pair vector ψ for the KdV equation. In quantum mechanics ψ_1 has a probabilistic interpretation:

$$P(a < \text{ particle } < b) = \int_a^b |\psi_1|^2 dx,$$

where $P(a <$ particle $< b)$ denotes the probability to find the particle (one-dimensional for our purposes) between $x = a$ and $x = b$. The function $u(x, t)$ is referred to as the potential. Note that its t-dependence is purely parametric, as the time-independent Schrödinger equation is an ordinary differential equation in x, and does not involve t. Specifically, t, which is the time-variable for the KdV equation, should not be confused with the time variable for the time-dependent Schrödinger equation, which does not arise in the method of inverse scattering for the KdV equation. Lastly, λ is the spectral parameter for the above equation. It represents the measurable values of the energy of the particle or wave described by ψ_1. These values are real, as they should be physically. Mathematically, this is due to the self-adjointness of the linear Schrödinger equation.

Here's how scattering theory works in the context of quantum mechanics. There are two main branches:

1. **Forward scattering:**

   Given a potential $u(x)$ (we'll surpress the t-dependence when it might cause confusion), determine the spectrum of the linear Schrödinger equation. In other words, determine the set of all possible energy values that may be observed: find the λ's for which there are bounded ψ_1. For these λ's, characterize the nature of the corresponding eigenfunctions ψ_1. It is clear how to do this: solve the linear Schrödinger equation for all values of λ. It is a second-order ordinary differential equation for ψ_1, for any given value of λ, thus for any λ we find a set of two fundamental solutions, spanning the general solution. Next we retain all λ for which at least one of the two fundamental solutions is bounded. The collection of all these λ's make up the spectrum of the problem. This problem is referred to as the forward scattering problem, or as the direct problem: given a potential, determine how waves scatter of it.

2. **Inverse scattering:**

   The inverse scattering problem asks for the determination of $u(x)$ from the knowledge of the spectrum and some characteristics of the corresponding eigenfunctions. Figure 12.1 presents a cartoon of the experimental set-up.

   First we send in a wave with a certain value of λ, say from $-\infty$. As this wave approaches the region where $u(x)$ has an important contribution, it starts to interact with $u(x)$. Some of the wave gets reflected, and returns to $-\infty$. Some other part of the wave gets transmitted and proceeds to $+\infty$. In the experimental set-up, one controls the wave that is sent in, and one measures the reflected and transmitted wave. Is it possible to determine which potential $u(x)$ is causing this reflection and transmission? This is the question posed by inverse scattering: if we know how waves scatter of a potential,

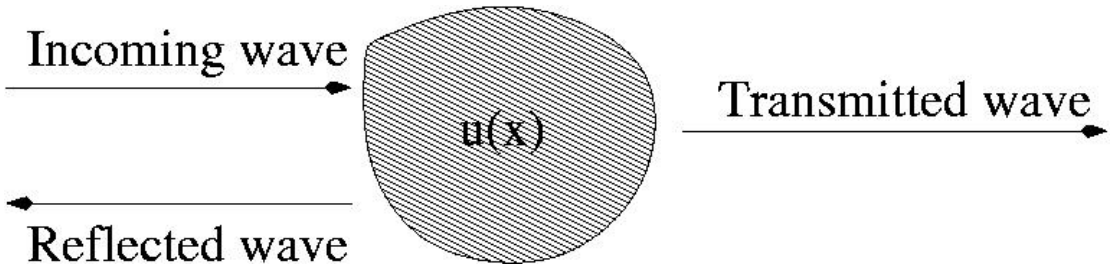


Figure 12.1: The cartoon of inverse scattering in quantum mechanics

is it possible to determine the potential? Mathematically the problem is formulated as determining the coefficients of a linear differential equation, if some properties of its solutions are known. It may be surprising that the answer to these questions is affirmative as first shown by Gel'fand and Levitan. Marchenko contributed to the solution of the inverse problem for the Schrödinger equation as well.

Since $\lim_{x\to\pm\infty} u = 0$, everything becomes easy as $x \to \pm\infty$. There is no potential, and the problem becomes easy to solve, as the Schrödinger equation has constant coefficients. This simple behavior at infinity is essential to the success of the solution method of the inverse problem. Mathematically, the solution of the inverse problem traditionally reduces to the problem of solving a linear integral equation (called the Gel'fand-Levitan-Marchenko equation in the context of the Schrödinger problem). In more modern treatments, the problem is reduced to that of solving a Riemann-Hilbert problem, as we will see later.

12.1.2 A preview of coming attractions

Let us examine how the above will help us solve the initial-value problem for the KdV equation. First, let us recall how the Fourier transform is used to solve a one-dimensional (one spatial variable) partial differential equation. Consider the scheme outlined in Fig. 12.2. Thus in order to solve a one-dimensional partial differential equation, we first take a Fourier transform of its initial condition, then evolve this Fourier transform in time. This time-evolution is obtained from solving an ordinary differential equation, instead of a partial differential equation. Lastly, we take an inverse Fourier transform of the time-evolved Fourier transform of the initial data. The scheme of the inverse scattering method is completely parallel to this, as illustrated in Fig. 12.3.

The three steps of this process are:

1. **Forward scattering**

   In this first step, the initial condition $U(x)$ is used as a potential in the linear Schrödinger equation and the scattering data S for this potential (to be defined later) is determined. Since in this step the "parameter" $t = 0$, we denote this set of scattering

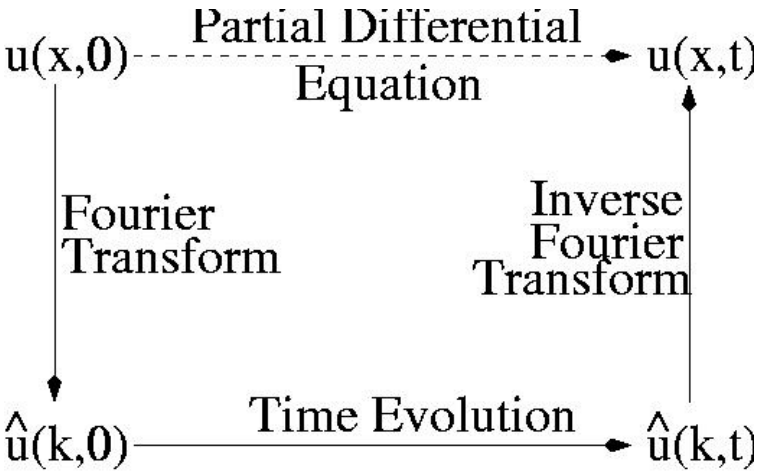


Figure 12.2: The solution scheme for the initial-value problem for a partial differential equation in one dimension, using the Fourier transform

data by $S(0)$. In general, this step requires solving the forward problem for the first equation of the Lax pair.

2. **Time evolution of the scattering data**

   In the second step, the scattering data $S(0)$ is evolved in time, using the second equation of the Lax pair. Since the scattering data does not depend on x, we may choose any value of x in this second Lax equation to determine the time evolution of $S(0)$. As $x \to \pm\infty$, the Lax equations have constant coefficients, and they become trivial to solve. Thus, we use the second Lax equation, evaluated at either $x = \infty$ or $x = -\infty$ to find $S(t)$, the scattering data as a function of the now arbitrary "parameter" t. As a matter of fact, this second step proves to be significantly easier to solve then either the first or third step.

3. **Inverse scattering**

   In this third and last step, the potential $u(x, t)$, for arbitrary values of the "parameter" t is reconstructed using Gel'fand-Levitan-Marchenko theory.

Many **remarks** are in order:

- Steps 1, 2, and 3 are all linear, as they all involve solving different types of linear problems (spectral problems for an linear ordinary differential equation with non-constant coefficients, a constant-coefficient linear ordinary different equation, and a linear integral equation). Thus we are solving an initial-value problem for a nonlinear partial differential equation by composing three linear steps!

- The space of initial data for which we can solve the initial-value problem is determined by step 1. During step 1, we encounter several restrictions on the initial data for the

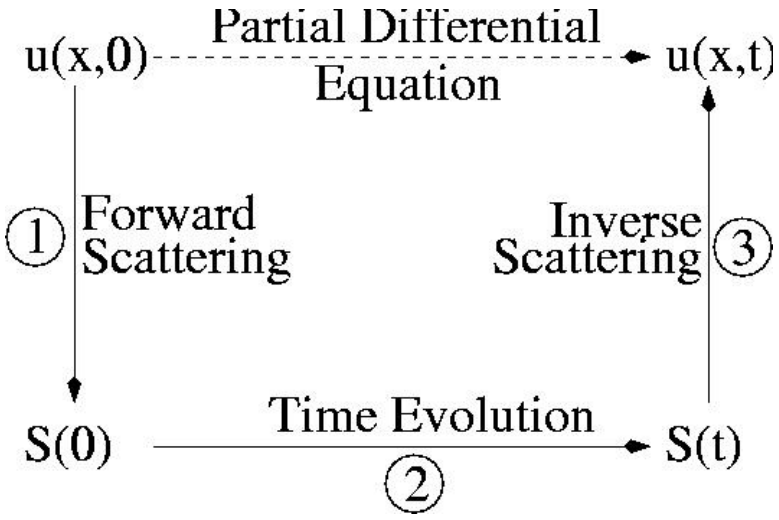


Figure 12.3: The solution scheme for the initial-value problem for an integrable nonlinear partial differential equation in one dimension, using the inverse scattering method

theory to work. The collection of these restrictions determines the function space in which $U(x)$ and $u(x,t)$ reside.

- As stated before, u depends on both x and t, where t is the time variable for the KdV equation, which has nothing to do with the time variable for the time-dependent Schrödinger equation. Rather, the t-dependence of $u(x,t)$ shows up as a parameter in the time-independent linear Schrödinger equation. Thus varying t gives rise to a one-parameter family of potentials for the linear Schrödinger equation, each having the same spectrum. Since the spectral parameter λ does not depend on t, all solutions $u(x,t)$ give rise to the same spectrum for the linear Schrödinger equation as their initial condition $u(x,0)$. This is a remarkable statement: imagine the two-soliton solution of the KdV equation. This solution looks significantly different from one time to the next. For each time, a different potential of the linear Schrödigner equation is obtained. All these potentials have the same spectrum!

- The hardest of the three steps is usually the third step. With typical initial data, this step can be reduced to the solution of a linear integral equation. However, if the initial data $U(x)$ is soliton initial data (*i.e.,* it consists of a finite number of solitons and nothing else), this integral equation reduces to a discrete integral equation, *i.e.,* a linear algebraic equation and the entire solution of the third step reduces to linear algebra.

- For general initial conditions $U(x)$, the spectrum of the linear Schrödinger equations contains a discrete part (eigenvalues) and a non-discrete part (the continuous spectrum). We'll show that the discrete spectrum, *i.e.,* the eigenvalues, correspond to

solitons in the initial data. In particular, the number of eigenvalues for given initial data is equal to the number of solitons that will evolve from this initial data. Furthermore, the location of these eigenvalues in the complex plane (on the real line for the linear Schrödinger operator, as it is self-adjoint) determines the characteristics (amplitude, velocity, width) of the solitons.

- For general initial data $U(x)$, its time evolution breaks up in a sum of solitons (whose number and characteristics are determined by the discrete part of the spectrum of $U(x)$), and a dispersive tail. This dispersive tail spreads and decreases in amplitude as $t \to \infty$, whereas the soliton amplitudes are constant in time. Thus one could very well argue that eventually (*i.e.,* after enough time has elapsed), we only need to consider the solitons and the dispersive tail may be ignored.

- The spectral data $S(t)$ essentially consists of the transmission and reflection coefficients from our quantum mechanical ramblings. For solitons, it turns out that the reflection coefficient is zero. Thus solitons of the KdV equation represent reflectionless potentials of the linear Schrödinger equation!

- The inverse scattering method may be considered an analogue for integrable partial differential equations of the Fourier transform for linear partial differential equations. This point may be pursued further. Ablowitz, Kaup, Newell and Segur demonstrated that the inverse scattering method is truly an extension of the Fourier transform, in the sense that it reduces to it, in the limit of small amplitude of the initial condition. In that case, all products may be linearized and they showed that the forward scattering step is in essence a Fourier transform and the inverse scattering step is an inverse Fourier transform.

12.2 A scalar Lax pair for the KdV equation

For the sake of simplicity, we will use a scalar form of the Lax pair for the KdV equation. A scalar Lax pair for a partial differential equation is a pair L and P, both scalar linear finite-order differential operators in x, such that the given equation is equivalent to the Lax equation

$$L_t = [P, L],$$

where $[P, L]$ is the operator commutator: $[P, L] = PL - LP$. One way to obtain the Lax equation is as the compatibility condition of the two linear equations

$$\begin{cases} L\psi = \lambda\psi, \\ \psi_t = P\psi. \end{cases}$$

Here λ is a spectral parameter, independent of x and t. The compatibility of these equations is obtained by taking the t derivative of the first equation:

$$\begin{aligned} \frac{\partial}{\partial t}L\psi &= \frac{\partial}{\partial t}\lambda\psi \\ \Rightarrow L_t\psi + L\psi_t &= \lambda\psi_t \\ \Rightarrow L_t\psi + LP\psi &= \lambda P\psi \\ \Rightarrow L_t\psi + LP\psi &= P\lambda\psi \\ \Rightarrow L_t\psi + LP\psi &= PL\psi, \end{aligned}$$

from which the Lax equation follows. The advantage of this approach is that the parameter λ is clearly the spectral parameter for the operator L, an interpretation which is not always clear from the systems Lax pair approach.

Theorem The KdV equation $u_t = uu_x + u_{xxx}$ has a scalar Lax pair given by

$$\begin{cases} L = \partial_x^2 + u, \\ P = 4\partial_x^3 + 3u\partial_x + 3\partial_x u \end{cases}$$

Note that two x-derivatives may be exchanged for multiplication by $\lambda - u$, using the first linear Lax equation. This allows us to rewrite the second operator as $P = -u_x + (2u + 4\lambda)\partial_x$.

Proof The different pieces for the Lax equation for these two operators are:

$$L_t = u_t,$$

and

$$\begin{aligned} [P, L] &= [4\partial_x^3 + 6u\partial_x + 3u_x, \partial_x^2 + u] \\ &= (4\partial_x^3 + 6u\partial_x + 3u_x)(\partial_x^2 + u) - (\partial_x^2 + u)(4\partial_x^3 + 6u\partial_x + 3u_x) \\ &= 4\partial_x^5 + 4u_{xxx} + 12u_{xx}\partial_x + 12u_x\partial_x^2 + 4u\partial_x^3 + 6u\partial_x^3 + 6uu_x + 6u^2\partial_x + 3u_x\partial_x^2 + 3uu_x \\ &\quad -(4\partial_x^5 + 6u_{xx}\partial_x + 12u_x\partial_x^2 + 6u\partial_x^3 + 3u_{xxx} + 6u_{xx}\partial_x + 3u_x\partial_x^2 + 4u\partial_x^3 + 6u^2\partial_x + 3uu_x) \\ &= u_{xxx} + uu_x, \end{aligned}$$

and the Lax equation is indeed identical to the KdV equation. This finishes the proof.

12.3 The forward problem

The goal of the forward scattering problem is to take initial data for the KdV equation in some function space, and determine from it scattering data. We will find out what is meant by "scattering data" and what the appropriate "function space" is, as we proceed.

Consider the forward scattering problem

$$\psi_{xx} + (u - \lambda)\psi = 0,$$

where $u = u(x,t)$ denotes the solution of the KdV equation. For the above ordinary differential equation, t shows up parametrically. More specific, we could let $t = 0$ and use initial data in the forward scattering problem. But we don't have to. First, we set

$$\lambda = -k^2.$$

This is motivated by the fact that λ plays the role of a spectral parameter for the operator $\partial_x^2 + u$. Since u is bounded, and $-\partial_x^2$ is a negative operator, the spectrum of the operator $\partial_x^2 + u$ is bounded from above. Further, since the operator is self-adjoint, the spectrum is contained on the real axis. It is possible for part of the spectrum $\sigma(\lambda) = \{\lambda \in \mathbb{C} : \psi \text{ is bounded}\}$ to be on the positive real axis. One can show that the continuous spectrum of $\partial_x^2 + u$ is contained on the negative real axis (including zero), because $u \to 0$ as $|x| \to \infty$. There may be isolated eigenvalues on the positive real axis. These correspond to solitons. For small enough (in a norm to be decided on later) initial data, no such eigenvalues exist, and no solitons evolve from the initial data. For sufficiently large initial data, one or more discrete positive eigenvalues may exist, giving rise to an equal number of solitons. For now, we have

$$\psi_{xx} + (u + k^2)\psi = 0.$$

This is a second-order ordinary differential equation with non-constant coefficients.

12.3.1 Two bases of eigenfunctions

Let us investigate the asymptotic behavior of this equation. Since $u \to 0$ as $|x| \to \infty$, as $x \to \pm\infty$, we have

$$\psi_{xx} + k^2\psi = 0.$$

From this, it is easy to deduce the asymptotic behavior of the solutions. Since the equation is of second order, it has two linearly independent solutions. Let us define two such solutions $\phi(x,k)$ and $\bar{\phi}(x,k)$ by their asymptotic behavior as $x \to -\infty$:

$$\left\{ \begin{array}{rcl} \phi(x,k) & \sim & e^{-ikx} \\ \bar{\phi}(x,k) & \sim & e^{ikx} \end{array} \right. , \quad \text{as } x \to -\infty.$$

Thus, $\phi(x,k)$ is the unique solution of the equation $\psi_{xx} + (u + k^2)\psi = 0$ that has the asymptotic behavior $\psi \sim e^{-ikx}$, with a similar statement for $\bar{\phi}$. It is equally easy to solve the equation as $x \to +\infty$. Thus we define a second set of linearly independent solutions $\varphi(x,k)$ and $\bar{\varphi}(x,k)$ so that

$$\left\{ \begin{array}{rcl} \varphi(x,k) & \sim & e^{ikx} \\ \bar{\varphi}(x,k) & \sim & e^{-ikx} \end{array} \right. , \quad \text{as } x \to +\infty.$$

Since we are solving a second-order differential equation, there are only two linearly independent solutions. We have defined two distinct sets of linearly independent solutions, thus the members of one set can be written as a linear combination of the members of the other set. Specifically,

$$\begin{cases} \phi(x,k) &= a(k)\bar{\varphi}(x,k) + b(k)\varphi(x,k) \\ \bar{\phi}(x,k) &= -\bar{a}(k)\varphi(x,k) + \bar{b}(k)\bar{\varphi}(x,k) \end{cases},$$

for four functions $a(k)$, $\bar{a}(k)$, $b(k)$, $\bar{b}(k)$ which may depend on the spectral parameter k (and possibly t, if we choose to write out the t-parameter explicitly), but not on x.

Remark. Since the equation is invariant under the change $k \to -k$, we have that

$$\begin{cases} \phi(x,-k) &= \bar{\phi}(x,k) \\ \varphi(x,-k) &= \bar{\varphi}(x,k) \end{cases},$$

since these relations hold at respectively $-\infty$ and ∞ and they are invariant under the flow of the differential equation. Hence they hold for all x. Then

$$\begin{array}{ll} & \begin{cases} \phi(x,-k) &= a(-k)\bar{\varphi}(x,-k) + b(-k)\varphi(x,-k) \\ \bar{\phi}(x,-k) &= -\bar{a}(-k)\varphi(x,-k) + \bar{b}(-k)\bar{\varphi}(x,-k) \end{cases} \\ \Rightarrow & \begin{cases} \bar{\phi}(x,k) &= a(-k)\varphi(x,k) + b(-k)\bar{\varphi}(x,k) \\ \phi(x,k) &= -\bar{a}(-k)\bar{\varphi}(x,k) + \bar{b}(-k)\varphi(x,k) \end{cases} \\ \Rightarrow & \begin{cases} -\bar{a}(k)\varphi(x,k) + \bar{b}(k)\bar{\varphi}(x,k) &= a(-k)\varphi(x,k) + b(-k)\bar{\varphi}(x,k) \\ a(k)\bar{\varphi}(x,k) + b(k)\varphi(x,k) &= -\bar{a}(-k)\bar{\varphi}(x,k) + \bar{b}(-k)\varphi(x,k) \end{cases} \\ \Rightarrow & -\bar{a}(k) = a(-k), \quad \bar{b}(k) = b(-k). \end{array}$$

Since the linear Schrödinger equation does not contain a first derivative term, it follows from Abel's formula that the Wronskian $W(f_1, f_2) = f_1 f_{2x} - f_{1x} f_2$ of any two solutions is constant, *i.e.*, independent of x. This implies that the value of such a Wronskian may be determined by using any convenient x-value. For instance,

$$\begin{aligned} W(\phi, \bar{\phi}) &= e^{-ikx} ike^{ikx} - (-ik)e^{-ikx}e^{ikx} \\ &= 2ik, \end{aligned}$$

using $x \to \infty$ to justify the first equality. Similarly, using $x \to \infty$, we get

$$\begin{aligned} W(\varphi, \bar{\varphi}) &= e^{ikx}(-ik)e^{-ikx} - ike^{ikx}e^{-ikx} \\ &= -2ik. \end{aligned}$$

Using these results with the above definitions of $a(k)$, $\bar{a}(k)$, $b(k)$ and $\bar{b}(k)$, we obtain

$$\begin{aligned} W(\phi,\bar{\phi}) &= W(a\bar{\varphi}+b\varphi,-\bar{a}\varphi+\bar{b}\bar{\varphi}) \\ &= W(a\bar{\varphi},-\bar{a}\varphi)+W(b\varphi,\bar{b}\bar{\varphi}) \\ &= a\bar{a}W(\bar{\varphi},\varphi)+b\bar{b}W(\varphi,\bar{\varphi}) \\ &= (a\bar{a}+b\bar{b})W(\varphi,\bar{\varphi}) \\ \Rightarrow \quad a\bar{a}+b\bar{b} &= -1. \end{aligned}$$

12.3.2 Scattering data

Definition. (Scattering data) *The collection*

$$\{a(k),\bar{a}(k),b(k),\bar{b}(k)\}$$

is a set of **scattering data**. *It is the goal of the forward problem to characterize for a given "potential" $u(x,t)$ this data and, if possible, obtain it.*

Remarks

- The scattering data $\{a(k),\bar{a}(k),b(k),\bar{b}(k)\}$ fully characterizes how eigenfunction information at $x=-\infty$ is related to eigenfunction information at $x=+\infty$.
- It is possible to use different collections of scattering data, and such is frequently done in the literature. Another popular choice is to use $\{T(k)=1/a(k),\rho(k)=b(k)/a(k),\bar{T}(k)=1/\bar{a}(k),\bar{\rho}(k)=\bar{b}(k)/\bar{a}(k)\}$, which may be interpreted as reflection and transmission coefficients in the quantum mechanical setting of the Schrödinger scattering problem. In that context, the relationship $a\bar{a}+b\bar{b}=-1$ may be interpreted as a statement of conservation of energy.

The expansion coefficients $a(k)$, $b(k)$ *etc.*may be expressed in terms of the eigenfunctions:

$$\begin{aligned} a(k) &= \frac{W(\phi,\varphi)}{2ik}, \\ b(k) &= \frac{W(\bar{\varphi},\phi)}{2ik}, \\ \bar{a}(k) &= \frac{W(\bar{\phi},\bar{\varphi})}{2ik}, \\ \bar{b}(k) &= \frac{W(\bar{\phi},\varphi)}{2ik}. \end{aligned}$$

All of these follow immediately from the expressions for ϕ and $\bar{\phi}$ in terms of φ and $\bar{\varphi}$.

12.3.3 Jost functions

Next, we define **Jost functions**. Jost functions are functions constructed from the eigenfunctions so that the asymptotics of some of them is particularly simple, as $x \to \pm\infty$. Specifically, half of the Jost functions have constant asymptotic behavior at either ∞ or $-\infty$. Let

$$\begin{aligned} M &= \phi e^{ikx}, \\ \bar{M} &= \bar{\phi} e^{ikx}, \\ N &= \varphi e^{ikx}, \\ \bar{N} &= \bar{\varphi} e^{ikx}, \end{aligned}$$

so that

$$\begin{aligned} \text{as } x \to -\infty \quad &, \quad M \to 1, \\ \text{as } x \to -\infty \quad &, \quad \bar{M} \to e^{2ikx}, \\ \text{as } x \to \infty \quad &, \quad N \to e^{2ikx}, \\ \text{as } x \to \infty \quad &, \quad \bar{N} \to 1. \end{aligned}$$

Only half of the oscillatory behavior at $\pm\infty$ is transformed away. The reason for this is that now the Jost function all satisfy the same differential equation

$$y_{xx} - 2iky_x + uy = 0.$$

The transformation of this differential equation to an integral equation allows us to determine the analyticity (or lack thereof) of the Jost functions in the complex k-plane, *i.e.*, as functions of k. We will not do this here, but the results will be given shortly. For now, let us relate the functions M and $\bar{M}$ to N and $\bar{N}$:

$$\left\{ \begin{array}{rcl} M &=& a\bar{N} + bN \\ \bar{M} &=& -\bar{a}N + \bar{b}\bar{N} \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{rcl} \dfrac{M}{a} &=& \bar{N} + \dfrac{b}{a}N \\ \dfrac{\bar{M}}{\bar{a}} &=& -N + \dfrac{\bar{b}}{\bar{a}}\bar{N} \end{array} \right. ,$$

where the last set of equations is valid for all k-values for which the first set is defined, with the exception of the roots of $a(k)$. Since

$$\varphi(x,k) = \bar{\varphi}(x,-k),$$

we have

$$N(x,k)e^{-ikx} = \bar{N}(x,-k)e^{ikx} \quad \Rightarrow \quad N(x,k) = \bar{N}(x,-k)e^{2ikx},$$

from which

$$\frac{M(x,k)}{a(k)} = \bar{N}(x,k) + \frac{b(k)}{a(k)} e^{2ikx} \bar{N}(x,-k).$$

This equation is known as the **fundamental equation**. Its fundamental role in the inverse scattering method will become clear in the section on inverse scattering.

12.3.4 Example: the delta function potential

Consider as a potential of the linear Schrödinger equation

$$u(x,0) = \alpha\delta(x),$$

where α is a real constant, and $\delta(x)$ denotes the Dirac delta function. Thus the forward scattering problem requires the construction of scattering data for the ordinary differential equation

$$\psi_{xx} + (\alpha\delta(x) + k^2)\psi = 0.$$

Determining ϕ and $\bar{\phi}$

As $x \to -\infty$ we have by definition

$$\phi \to e^{-ikx}, \quad \bar{\phi} \to e^{ikx}.$$

On the other hand, for all $x < 0$, the differential equation is $\psi_{xx} + k^2\psi = 0$, so that

$$\phi = e^{-ikx}, \quad \bar{\phi} = e^{ikx}, \quad \text{for all} \quad x < 0.$$

Similarly, for $x > 0$

$$\begin{cases} \phi &= c_1 e^{ikx} + c_2 e^{-ikx} \\ \bar{\phi} &= c_3 e^{ikx} + c_4 e^{-ikx} \end{cases}.$$

In order to completely determine the eigenfunctions $\phi(x,k)$ and $\bar{\phi}(x,k)$ for all x, the constants c_1, c_2, c_3 and c_4 need to be found. It is clear from the differential equation that the second derivative of the solutions will not be continuous. Actually, given that the equation shows that the second derivative blows up at $x = 0$, it appears the first derivative of the solutions will also be discontinous. However, we may impose that the solutions themselves are continuous. Taking left- and right-side limits, we obtain

$$c_1 + c_2 = 1, \quad c_3 + c_4 = 1.$$

The condition on the first derivative is obtained by integrating the differential equation over the singularity of the delta function. Let $\epsilon > 0$ be small. Then

$$
\begin{aligned}
& & \psi_{xx} + (\alpha\delta(x) + k^2)\psi = 0 \\
\Rightarrow & & \int_{-\epsilon}^{\epsilon} \psi_{xx} dx + \alpha \int_{-\epsilon}^{\epsilon} \delta(x)\psi dx + k^2 \int_{-\epsilon}^{\epsilon} \psi dx = 0 \\
\Rightarrow & & \int_{-\epsilon}^{\epsilon} \psi_{xx} dx + \alpha \int_{-\epsilon}^{\epsilon} \delta(x)\psi dx = 0 \\
\Rightarrow & & \psi_x(\epsilon) - \psi_x(-\epsilon) + \alpha\psi(0) = 0.
\end{aligned}
$$

Note that the contribution of the $k^2\psi$ term vanishes as $\epsilon \to 0$, since ψ is continuous. Applying this equation to ϕ, we have (as $\epsilon \to 0$)

$$
\left\{ \begin{array}{rcl} \phi_x(\epsilon) - \phi_x(-\epsilon) + \alpha\phi(0) & = & 0 \\ c_1 + c_2 & = & 1 \end{array} \right. \Rightarrow \left\{ \begin{array}{rcl} ikc_1 - ikc_2 - (-ik) + \alpha & = & 0 \\ c_1 + c_2 & = & 1 \end{array} \right. \Rightarrow \left\{ \begin{array}{rcl} c_1 & = & -\dfrac{\alpha}{2ik} \\ c_2 & = & 1 + \dfrac{\alpha}{2ik} \end{array} \right. .
$$

At this point, we have completely determined $\phi(x, k)$:

$$
\phi(x,k) = \begin{cases} e^{-ikx} & \text{if } x < 0, \\ \left(1 + \dfrac{\alpha}{2ik}\right) e^{-ikx} - \dfrac{\alpha}{2ik} e^{ikx} & \text{if } x > 0. \end{cases}
$$

In a similar way, the constants c_3 and c_4 may be found, at which point the function $\bar{\phi}$ has been completely determined. Alternatively, we may use that $\bar{\phi}(x, k) = \phi(x, -k)$, which gives the same result (check!):

$$
\bar{\phi}(x,k) = \begin{cases} e^{ikx} & \text{if } x < 0, \\ \left(1 - \dfrac{\alpha}{2ik}\right) e^{ikx} + \dfrac{\alpha}{2ik} e^{-ikx} & \text{if } x > 0. \end{cases}
$$

Determining φ and $\bar{\varphi}$

Using the asymptotics of φ and $\bar{\varphi}$ at $+\infty$, these functions may be determined in an analogous way. The final results are

$$
\varphi(x,k) = \begin{cases} \left(1 + \dfrac{\alpha}{2ik}\right) e^{ikx} - \dfrac{\alpha}{2ik} e^{-ikx} & \text{if } x < 0, \\ e^{ikx} & \text{if } x > 0. \end{cases}
$$

and

$$
\bar{\varphi}(x,k) = \begin{cases} \left(1 - \dfrac{\alpha}{2ik}\right) e^{-ikx} + \dfrac{\alpha}{2ik} e^{ikx} & \text{if } x < 0, \\ e^{-ikx} & \text{if } x > 0. \end{cases}
$$

Determining the scattering data

Having determined the two bases of eigenfunctions explicitly, the most convenient way to compute the scattering data at this point, is through the Wronskian relations. These may be evaluated at any x-value, since the final result is independent of x. In what follows, we use $x < 0$. The reader should check that the same results are obtained using $x > 0$.

$$\begin{aligned} a(k) &= \frac{W(\phi, \varphi)}{2ik} \\ &= \frac{1}{2ik} W\left(e^{-ikx}, \left(1 + \frac{\alpha}{2ik}\right) e^{ikx} - \frac{\alpha}{2ik} e^{-ikx}\right) \\ &= \frac{1}{2ik} W\left(e^{-ikx}, \left(1 + \frac{\alpha}{2ik}\right) e^{ikx}\right) \\ &= \frac{1}{2ik}\left(1 + \frac{\alpha}{2ik}\right) W(e^{-ikx}, e^{ikx}) \\ &= \frac{1}{2ik}\left(1 + \frac{\alpha}{2ik}\right)(2ik) \\ &= \frac{\alpha + 2ik}{2ik}, \end{aligned}$$

which is indeed independent of x. We may calculate $\bar{a}(k)$ in a similar way, or we may use

$$\begin{aligned} \bar{a}(k) &= -a(-k) \\ &= \frac{\alpha - 2ik}{2ik}. \end{aligned}$$

Next,

$$\begin{aligned} b(k) &= \frac{W(\bar{\varphi}, \phi)}{2ik} \\ &= \frac{1}{2ik} W\left(\left(1 - \frac{\alpha}{2ik}\right) e^{-ikx} + \frac{\alpha}{2ik} e^{ikx}, e^{-ikx}\right) \\ &= \frac{1}{2ik}\frac{\alpha}{2ik} W(e^{ikx}, e^{-ikx}) \\ &= -\frac{\alpha}{2ik}, \end{aligned}$$

from which

$$\begin{aligned} \bar{b}(k) &= b(-k) \\ &= \frac{\alpha}{2ik}. \end{aligned}$$

12.4 Time dependence of the scattering data

12.4.1 General strategy

Thus far we have only used the first half of the Lax pair. In order to determine the time dependence of the scattering data, the second half is used. We have

$$\begin{aligned} \psi_t &= 4\psi_{xxx} + 3u\psi_x + 3(u\psi)_x \\ &= 4\psi_{xxx} + 6u\psi_x + 3u_x\psi. \end{aligned}$$

Since the scattering data is independent of x, its time dependence may be obtained from the behavior at any favorably chosen x. The two "values" $x \to -\infty$ and $x \to +\infty$ are especially nice, as the two last terms of the ψ_t equation vanish there, and the equation becomes

$$\psi_t \sim 4\psi_{xxx}.$$

Let $\psi = c_1\phi$. Here c_1 is independent of x, so that ψ satisfies the first half of the Lax pair. Both c_1 and ϕ may depend on t and c_1 will be chosen to ensure that ψ is indeed a solution of the second equation of the Lax pair. Then, as $x \to -\infty$

$$\begin{aligned} \psi_t &= c_{1t}\phi + c_1\phi_t \\ &= c_{1t}e^{-ikx} + c_1(e^{-ikx})_t \\ &= c_{1t}e^{-ikx}, \end{aligned}$$

and, again as $x \to -\infty$

$$\begin{aligned} \psi_{xxx} &= c_1(-ik)^3 e^{-ikx} \\ &= c_1 ik^3 e^{-ikx}. \end{aligned}$$

Evaluating the second of the Lax pair equations as $x \to -\infty$ gives

$$\begin{aligned} c_{1t}e^{-ikx} &= 4ik^3 e^{-ikx} c_1 \\ \Rightarrow \quad c_{1t} &= 4ik^3 c_1 \\ \Rightarrow \quad c_1 &= e^{4ik^3 t}. \end{aligned}$$

Note that the overall constant of integration is irrelevant, as it may be factored out since we are using linear equations.

Having determined c_1, we now turn to the behavior as $x \to \infty$ of the second Lax equation. Then

$$\begin{aligned}\psi &= c_1\phi \\ &= c_1(a\bar{\varphi} + b\varphi) \\ &\sim c_1(ae^{-ikx} + be^{ikx}),\end{aligned}$$

as $x \to \infty$. Then

$$\begin{aligned}\psi_t &\sim c_{1t}(ae^{-ikx} + be^{ikx}) + c_1(a_t e^{-ikx} + b_t e^{ikx}) \\ &\sim 4ik^3 c_1(ae^{-ikx} + be^{ikx}) + c_1(a_t e^{-ikx} + b_t e^{ikx}),\end{aligned}$$

and

$$\begin{aligned}\psi_{xxx} &\sim c_1(a(-ik)^3 e^{-ikx} + b(ik)^3 e^{ikx}) \\ &\sim c_1(iak^3 e^{-ikx} - ibk^3 e^{ikx}).\end{aligned}$$

Equating the coefficients of e^{ikx} and e^{-ikx} on both sides of the Lax equation and canceling c_1 gives

$$\begin{aligned}e^{-ikx}: &\quad 4ik^3 a + a_t = 4iak^3 \quad \Rightarrow \quad a_t = 0 \quad \Rightarrow \quad a(k,t) = a(k,0), \\ e^{ikx}: &\quad 4ik^3 b + b_t = -4ik^3 \quad \Rightarrow \quad b_t = -8ik^3 b \quad \Rightarrow \quad b(k,t) = b(k,0)e^{-8ik^3 t}.\end{aligned}$$

The scattering coefficients $\bar{a}(k)$ and $\bar{b}(k)$ may be determined in a similar way, using the eigenfunction

$$\psi = c_2\bar{\phi}.$$

(check this!) However, it is even easier to use the relationships

$$\begin{aligned}\bar{a}(k,t) &= -a(-k,t) = -a(-k,0) = \bar{a}(k,0), \\ \bar{b}(k,t) &= b(-k,t) = b(-k,0)e^{-8i(-k)^3 t} = \bar{b}(k,0)e^{8ik^3 t}.\end{aligned}$$

In summary, we have found that the time-dependence of the scattering data for the KdV equation is governed by

$$\begin{aligned}a(k,t) &= a(k,0), \qquad b(k,t) = b(k,0)e^{-8ik^3 t}, \\ \bar{a}(k,t) &= \bar{a}(k,0), \qquad \bar{b}(k,t) = \bar{b}(k,0)e^{8ik^3 t}.\end{aligned}$$

12.4.2 Example: the delta function potential

As an example, consider again the delta function potential

$$u = \alpha\delta(x).$$

Using the above, the scattering data for all time is given by

$$a(k,t) = \frac{\alpha + 2ik}{2ik}, \qquad b(k,t) = -\frac{\alpha}{2ik}e^{-8ik^3t},$$
$$\bar{a}(k,t) = \frac{\alpha - 2ik}{2ik}, \qquad \bar{b}(k,t) = \frac{\alpha}{2ik}e^{8ik^3t}.$$

Summarizing again: suppose we wish to solve the KdV equation with delta-function initial condition

$$u(x, t = 0) = \alpha\delta(x).$$

As a first step, we solve the forward scattering problem and determine the scattering data $a(k,0)$, $b(k,0)$, $\bar{a}(k,0)$ and $\bar{b}(k,0)$. Next, the evolution of the scattering data is determined, as above. These are the scattering data if one were to solve the forward scattering problem with the KdV solution $u(x,t)$ evolving from the given initial condition.

12.5 The inverse problem

12.5.1 Scalar Riemann-Hilbert problems

Before we proceed with the description of the inverse problem in the method of inverse scattering, we need to define Riemann-Hilbert problems. For the purpose of solving the inverse problem associated with the KdV equation, we are only concerned with scalar Riemann-Hilbert problems, *i.e.*, the function to be determined is scalar. In the more general setting of matrix Riemann-Hilbert problems one attempts to solve for an unknown matrix.

Consider a contour C in the complex plane, and two functions $f(z)$ and $g(z)$ specified on it. Then the problem

$$\phi_+ - g(z)\phi_- = f(z)$$

asks one to find two functions $\phi_+(z)$ and $\phi_-(z)$ with the additional requirements that $\phi_+(z)$ is analytic on one side of C and $\phi_-(z)$ is analytic on the other side of C.

Historically, problems like this were first posed by Riemann, to be formalized more by Hilbert, who made the question of analyzing them one of his 23 famous problems. It may seem counterintuitive that two functions may be obtained from one equation, but such is the power of requiring analyticity. Below we will see how a Riemann-Hilbert problem specific to the KdV scattering problem may be solved.

12.5.2 The fundamental equation

The inverse scattering problem starts from the fundamental equation

$$\frac{M(x,k)}{a(k)} = \bar{N}(x,k) + \frac{b(k)}{a(k)} e^{2ikx} \bar{N}(x,-k).$$

The following properties show that the fundamental equation is a Riemann-Hilbert problem with the real line playing the role of the curve C. Actually, it is not quite in the form given above, but the slight discrepancy in form will not hurt us, as shown below. These properties are given here (for now) without proof.

- $M(x,k)$ and $a(k)$ are analytic functions of k in the upper-half k-plane $\Im(k) > 0$. Both $M(x,k)$ and $a(k) \to 1$ as $|k| \to \infty$ in the upper-half plane.
- $\bar{N}(x,k)$ is analytic in the lower-half k-plane, $\Im(k) < 0$, and $\bar{N}(x,k) \to 1$ as $|k| \to \infty$ in the upper-half plane.

The proof of these very important results follows from the integral equations satisfied by the Jost functions $M(x,k)$ and $\bar{N}(x,k)$. These integral equations are derived from the differential equations these functions satisfy. A Picard iteration may be used to determine that the functions are bounded as functions of k in their respective domains of claimed analyticity. Then a derivative with respect to k is taken to obtain the integral equation satisfied by the functions differentiated with respect to k. Again, the integral equation is used to establish these derivatives are bounded in the claimed domains. Since both the functions and their derivatives are bounded, the functions are analytic in their stated domains. These properties are summarized in Fig. 12.4.

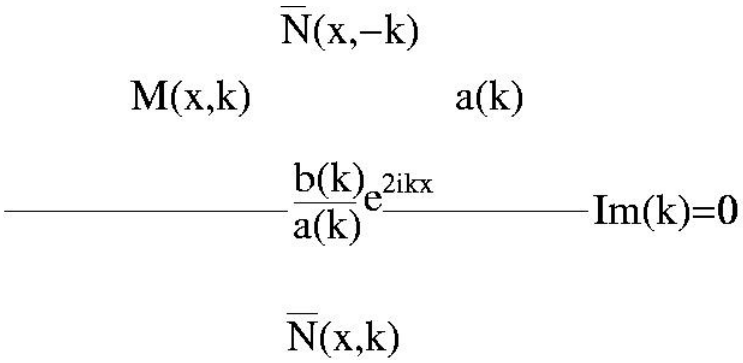


Figure 12.4: The regions of analyticity of the functions appearing in the fundamental equation. The expression $b(k)e^{2ikx}/a(k)$ is known on the real axis.

It is clear from the fundamental equation that the zeros of $a(k)$ play an important role. We do not prove this here, but it is known that all such zeros are simple, and that they are all on the imaginary axis.

12.5.3 The projection operators

Our main tools for solving the above Riemann-Hilbert-like problem are the **projection operators** P^+ and P^-: for any integrable function $f(k) \to 0$ as $|k| \to \infty$, $k \in \mathbb{C}$, we define

$$\begin{aligned}\left(P^{\pm} f\right)(k) &= \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - (k \pm i\epsilon)} d\xi \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - (k \pm i0)} d\xi,\end{aligned}$$

with $\epsilon > 0$. The last equality is purely notational: the $\pm i0$ simply denotes a limit process is implied.

First, let us show why these operators are refered to as projection operators. Consider a function $f(z)$, analytic in the upper half plane. Then

$$\left(P^{+} f\right)(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - (k + i0)} d\xi.$$

for $k \in \mathbb{R}$. We wish to use the residue theorem to calculate this integral. To do so, we need to close the integral using a large semi-circle, whose radius $\to \infty$. Since $f(\xi)$ is only defined in the upper-half k-plane, this semi-circle is chosen to lie there. The contribution from this semi-circle vanishes, as its radius approaches infinity, since $f(k) \to 0$ as $|k| \to \infty$ in the upper-half plane. Then

$$\left(P^{+} f\right)(k) = \frac{1}{2\pi i} \int_{C_{UHP}} \frac{f(\xi)}{\xi - (k + i0)} d\xi.$$

Here C_{UHP} denotes the closed contour around the upper-half plane, in a counter-clockwise sense. By the residue theorem, this integral divided by $2\pi i$ equals the sum of the residues of the poles enclosed by this contour. Since $f(\xi)$ is analytic inside the contour, all such poles are obtained from the zeros of the denominator of the integral. There is only one such zero, at $\xi = k + i0$, which is indeed in the upper-half plane (since k is on the real line). Thus

$$\left(P^{+} f\right)(k) = f(k + i0) = f(k).$$

(If the reader is uncomfortable with the $\pm i0$ notation he/she would be wise to repeat the exercise with ϵ in place, followed by an explicit limit process.) In other words, the action of the operator P^+ on a function analytic in the upper-half plane is the identity operation. Similarly, using f_+ and f_- to denote functions analytic in the upper- and lower-half planes, one finds

$$\begin{aligned}\left(P^{\pm} f_{\pm}\right)(k) &= f_{\pm}(k), \\ \left(P^{\pm} f_{\mp}\right)(k) &= 0,\end{aligned}$$

justifying the name projection operator.

12.5.4 An integral equation for $N(x,k)$

Let us rewrite the fundamental equation:

$$\left(\frac{M(x,k)}{a(k)} - 1\right) - \left(\bar{N}(x,k) - 1\right) = \frac{b(k)}{a(k)} e^{2ikx} \bar{N}(x,-k).$$

Both terms in the parentheses $\to 0$ as $|k| \to \infty$, in regions where the functions are defined. Now we apply the operator P^- to this equation:

$$\begin{aligned} P^-\left(\frac{M(x,k)}{a(k)} - 1\right) - P^-\left(\bar{N}(x,k) - 1\right) &= P^-\left(\frac{b(k)}{a(k)} e^{2ikx} \bar{N}(x,-k)\right) \\ \Rightarrow \qquad - \left(\bar{N}(x,k) - 1\right) &= P^-\left(\frac{b(k)}{a(k)} e^{2ikx} \bar{N}(x,-k)\right). \end{aligned}$$

The first term vanished because it is analytic in the upper-half plane, while the second-term is recovered due to its analyticity in the lower-half plane. We obtain

$$\bar{N}(x,k) = 1 - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{b(\xi)}{a(\xi)} \frac{e^{2i\xi x} \bar{N}(x,-\xi)}{\xi - (k - i0)} d\xi,$$

but $e^{2i\xi x}\bar{N}(x,-\xi) = N(x,\xi)$ and $\bar{N}(x,k) = N(x,-k)e^{2ikx}$, thus

$$N(x,-k)e^{2ikx} = 1 - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{b(\xi)}{a(\xi)} \frac{N(x,\xi)}{\xi - k + i0)} d\xi.$$

Evaluating this equation at $-k$ instead of k gives

$$N(x,k) = e^{2ikx}\left(1 - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{b(\xi)}{a(\xi)} \frac{N(x,\xi)}{(\xi + k + i0)} d\xi\right).$$

This is the desired linear integral equation for $N(x,k)$. Solving it results in $N(x,k,t)$ (provided $a(k,t)$ and $b(k,t)$ are used as inputs), from which $M(x,k,t)$ and the other Jost functions may be recovered.

12.5.5 Reconstructing $u(x,t)$

A second integral equation exists for the Jost functions. This is the integral equation mentioned earlier, used to establish the regions of analyticity of these different Jost functions. The coefficients of the kernel of that equation are given in terms of $u(x,t)$. Both integral equations may be used to obtain the large $|k|$ asymptotics of $N(x,k,t)$. Equating the first correction terms to the limiting behavior $\lim_{|k|\to\infty} e^{-2ikx} N(x,k,t) = 1$ gives

$$u(x,t) = -\frac{1}{\pi} \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \frac{b(k,t)}{a(k,t)} N(x,k,t) dk.$$

Thus, once $a(k,t)$ and $b(k,t)$ are known, and $N(x,k,t)$ is known from solving the integral equation given in the previous section, the solution of the KdV equation for all time t may be found from the above reconstruction formula.

As before, it is clear that the possible zeros of $a(k,t)$ play an important role. Since the time dependence of $a(k,t) = a(k,0)$ is trivial, these zeros are fixed in the complex k-plane. One can show that they are all simple and always contained on the positive imaginary axis. If $a(k,t)$ has any zeros there, the reconstruction formula needs to be modified to read

$$u(x,t) = -\frac{1}{\pi}\frac{\partial}{\partial x}\int_C \frac{b(k,t)}{a(k,t)}N(x,k,t)dk,$$

where C is a curve going from $-\infty$ to $+\infty$, passing above all zeros of $a(k,t)$, as indicated in Fig. 12.5. Such a contour may be deformed to C', as shown. In other words, the reconstruction formula needs to be modified as follows:

$$u(x,t) = -\frac{1}{\pi}\frac{\partial}{\partial x}\int_{-\infty}^{\infty} \frac{b(k,t)}{a(k,t)}N(x,k,t)dk + 2i\frac{\partial}{\partial x}\sum_{k=1}^{n}\frac{b(i\alpha_j)}{a'(i\alpha_j)}N(x,i\alpha_j,t),$$

where n is the number of zeros of $a(k)$, and $i\alpha_k$ is the k-th zero.

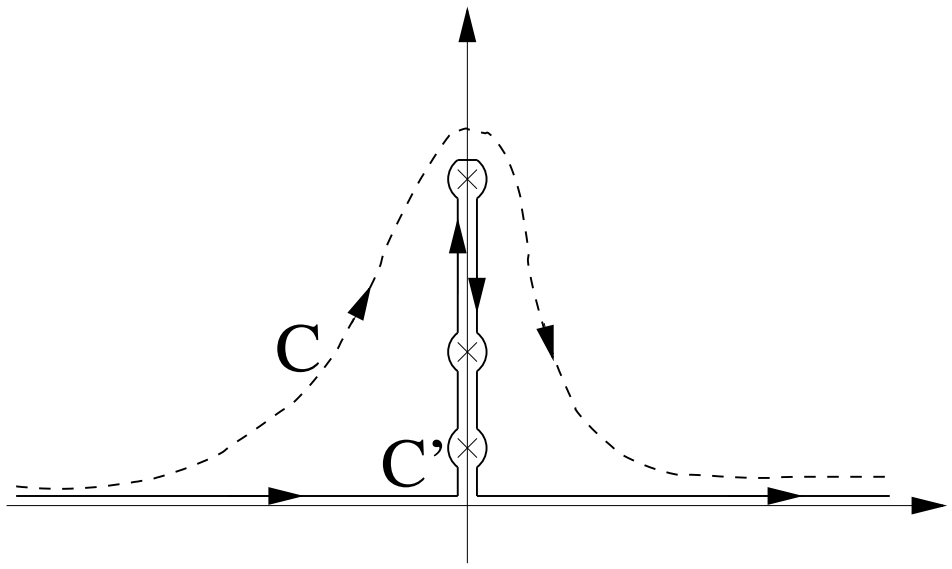


Figure 12.5: The contour of integration for the u-reconstruction formula if a has any zeros.

12.5.6 The delta function potential

As an example, consider the delta-function potential problem again. Here we have

$$a(k,t) = \frac{\alpha + 2ik}{2ik}, \quad b(k,t) = -\frac{\alpha}{2ik}e^{-8ik^3t}.$$

As stated, $a(k,t)$ has zeros only on the positive imaginary axis: $n=1$ and $i\alpha_1 = i\alpha/2$. This conclusion holds for $\alpha > 0$. If $\alpha < 0$, $a(k,t)$ has no zeros on the positive imaginary axis, and the summation term in the reconstruction formula vanishes. With $\alpha > 0$ the above reconstruction formula becomes

$$u(x,t) = \frac{\alpha}{\pi}\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\frac{e^{-8ik^3t}}{\alpha+2ik}N(x,k,t)dk - \alpha\frac{\partial}{\partial x}e^{-\alpha^3 t}N(x,i\alpha/2,t),$$

where $N(x,k,t)$ is determined by the integral equation

$$N(x,k) = e^{2ikx}\left(1+\frac{\alpha}{2\pi i}\int_{C'}\frac{e^{-8i\xi^3 t}}{(\alpha+2i\xi)}\frac{N(x,\xi)}{(\xi+k+i0)}d\xi\right).$$

It is not immediately clear that the second term in the reconstruction formula gives rise to a soliton, but it does appear that this term will contain exponential terms, whereas the integral has an oscillatory kernel, which gives rise to dispersive decay (as in the method of stationary phase) as $t\to\infty$.

Let us summarize our example again: suppose we wish to solve the KdV equation with delta-function initial condition

$$u(x,t=0) = \alpha\delta(x).$$

As a first step, we solve the forward scattering problem and determine the scattering data $a(k,0)$, $b(k,0)$, $\bar{a}(k,0)$ and $\bar{b}(k,0)$. Next, the evolution of the scattering data is determined, in a rather trivial way, as shown before. These are the scattering data if one were to solve the forward scattering problem with the KdV solution $u(x,t)$ evolving from the given initial condition. Using these time-dependent scattering data, the integral equation for the Jost function $N(x,k,t)$ may be solved, allowing for the reconstruction of the solution of the KdV equation for all time t.

12.6 The analyticity properties of the scattering data

In order to use the fundamental equation as the starting point for the inverse scattering, we relied heavily on the analyticity properties of the scattering data and the Jost functions in different parts of the complex k-plane. We show how to establish these properties in this section.

12.6.1 The analyticity properties of $M(x,k)$

We have defined the first Jost function as

$$M(x,k) = e^{ikx}\phi(x),$$

so that

$$\lim_{x\to-\infty} M(x,k) = 1.$$

We will assume that this limit is uniform, so that the derivatives with respect to x of $M(x,k)$ all approach 0 as $|x| \to \infty$. Also, we know that $M(x,k)$ satisfies the ordinary differential equation

$$M_{xx} - 2ikM_x + uM = 0.$$

The integral equation for $M(x,k)$

We start by rewriting the above differential equation for $M(x,k)$ as an integral equation. We will incorporate the boundary condition for $M(x,k)$ at the same time.

A lot of the philosophy of forward scattering is to regard the influence of $u(x)$ (again surpressing the t dependence, which is irrelevant for the forward and inverse scattering steps) as a perturbation to the constant coefficient behavior as $|x| \to \infty$. We continue that here. For starters, we move the term with $u(x)$ to the right-hand side, allowing us to think of it as a forcing.

$$\begin{aligned}
& & M_x x - 2ikM_x &= -uM \\
\Rightarrow & & e^{-2ikx}M_{xx} - 2ike^{-2ikx}M_x &= -e^{-2ikx}uM \\
\Rightarrow & & \left(e^{-2ikx}M_x\right)_x &= -e^{-2ikx}uM \\
\Rightarrow & & e^{-2ikx}M_x &= -\int_{-\infty}^{x} e^{-2ikz}u(z)M(z,k)dz \\
\Rightarrow & & M_x &= -\int_{-\infty}^{x} e^{2ikx-2ikz}u(z)M(z,k)dz \\
\Rightarrow & & M(x,k) &= 1 - \int_{-\infty}^{x} dy \int_{-\infty}^{y} dz e^{2ik(y-z)}u(z)M(z,k).
\end{aligned}$$

Note that we have already incorporated the boundary conditions satisfied by $M(x,k)$ and its derivative at $-\infty$. This is an integral equation satisfied by $M(x,k)$. We can simplify it quite significantly by switching the order of the integrations: since $u(z)$ and $M(z,k)$ do not depend on y, they will move out of the inside integral once we switch the order. The remaining inside integral will only depend on known functions, and may be explicitly integrated, leaving us with only the outside integral. Switching the order of the integrations is easily done using Fig. 12.6. The order given above corresponds to (i) for a fixed y, integrate over the infinite range of z values, followed by (ii) integrate over the infinite range of y values. This results in the integral bounds given above. Switching the order corresponds to (i) for a fixed z, integrate over the finite range of y values, followed by (ii) integrate over the infinite range of z values.

As shown in the figure, switching the order of integration results in

$$\begin{aligned}
& & M(x,k) &= 1 - \int_{-\infty}^{x} dz \int_{z}^{x} dy e^{2ik(y-z)}u(z)M(z,k) \\
\Rightarrow & & M(x,k) &= 1 - \int_{-\infty}^{x} u(z)M(z,k)dz \int_{z}^{x} e^{2ik(y-z)}dy
\end{aligned}$$

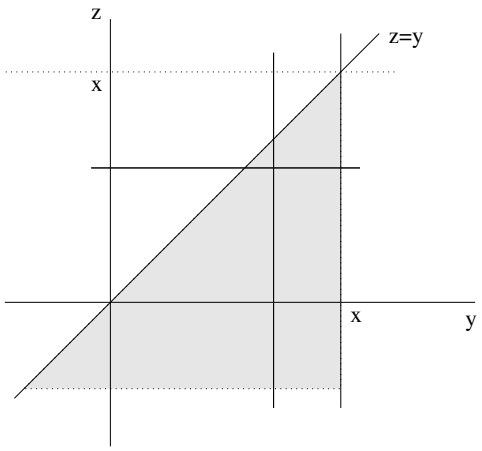


Figure 12.6: The domain of integration for the double integral in the integral equation for $M(x,k)$.

$$\Rightarrow \qquad M(x,k) = 1 - \int_{-\infty}^{x} u(z)M(z,k)\,\frac{e^{2ik(y-z)}}{2ik}\Big|_{y=z}^{y=x} dz$$

$$\Rightarrow \qquad M(x,k) = 1 - \int_{-\infty}^{x} u(z)M(z,k)\frac{e^{2ik(x-z)}-1}{2ik}dz$$

$$\Rightarrow \qquad M(x,k) = 1 - \int_{-\infty}^{x} \frac{e^{2ik(x-z)}-1}{2ik}u(z)M(z,k)dz.$$

This is the second, much improved, form of the integral equation satisfied by $M(x,k)$. It will be the starting point for establishing any analyticity properties of $M(x,k)$.

Bounds on $M(x,k)$ from the integral equation

In order to establish bounds on $M(x,k)$, we first solve the integral equation by iteration, using

$$M(x,k) = 1 + \sum_{j=1}^{\infty} M_j(x,k),$$

where $M_j(x,k)$ satisfies

$$M_j(x,k) = -\int_{-\infty}^{x} \frac{e^{2ik(x-z)}-1}{2ik}u(z)M_{j-1}(z,k)dz,$$

for $j = 1, 2, \ldots$. This is a recursion formula for $M_j(x,k)$, started by expressing $M_1(x,k)$ in terms of $M_0(x,k) \equiv 1$.

In what follows, we will use the following two bounds:

Lemma. (i) If $\Im(k) > 0$ then

$$\left|\frac{e^{2ikx}-1}{2ik}\right| \leq \frac{1}{|k|},$$

for all $x \geq 0$. Also, (ii) if $\Im(k) = 0$, then

$$\left|\frac{e^{2ikx}-1}{2ik}\right| \leq |x|,$$

for $x\mathbb{R}$.

Proof. (i) The proof is straightforward: let $k = k_R + ik_I$, with $k_I > 0$. Then

$$\begin{aligned}
\left|\frac{e^{2ikx}-1}{2ik}\right| &= \left|\frac{e^{2ik_Rx}e^{-2k_Ix}-1}{2ik}\right| \\
&\leq \frac{\left|e^{2ik_Rx}\right|\left|e^{-2k_Ix}\right|+1}{2|k|} \\
&= \frac{\left|e^{-2k_Ix}\right|+1}{2|k|} \\
&\leq \frac{1+1}{2|k|} \\
&= \frac{1}{|k|}.
\end{aligned}$$

Similarly, for (ii) we have

$$\begin{aligned}
\left|\frac{e^{2ikx}-1}{2ik}\right| &= \left|e^{ikx}\frac{e^{ikx}-e^{-ikx}}{2ik}\right| \\
&= \left|\frac{e^{ikx}-e^{-ikx}}{2ik}\right| \\
&= \left|\frac{\sin kx}{k}\right| \\
&= |x|\left|\frac{\sin kx}{kx}\right| \\
&\leq |x|,
\end{aligned}$$

since $|\sin\theta| \leq |\theta|$, for $\theta \in \mathbb{R}$. ■

With these bounds, we can establish bounds on the $M_j(x,k)$, for $j = 1, 2, \ldots$. First we investigate what happens when $\Im(k) > 0$, so that we may use the first bound we established

above. We have

$$\begin{aligned}|M_1(x,k)| &= \left|-\int_{-\infty}^{x} \frac{e^{2ik(x-z)}-1}{2ik}u(z)M_0(z,k)dz\right| \\ &\leq \int_{-\infty}^{x}\left|\frac{e^{2ik(x-z)}-1}{2ik}\right||u(z)|dz \\ &\leq \frac{1}{|k|}\int_{-\infty}^{x}|u(z)|dz = \frac{U(x)}{|k|},\end{aligned}$$

where we have defined

$$U(x) = \int_{-\infty}^{x}|u(z)|dz,$$

assuming that this integral is defined. Next,

$$\begin{aligned}|M_2(x,k)| &= \left|-\int_{-\infty}^{x} \frac{e^{2ik(x-z)}-1}{2ik}u(z)M_1(z,k)dz\right| \\ &\leq \int_{-\infty}^{x}\left|\frac{e^{2ik(x-z)}-1}{2ik}\right||u(z)|\frac{U(z)}{|k|}dz \\ &\leq \frac{1}{|k|^2}\int_{-\infty}^{x}U'(z)U(z)dz = \frac{U^2(x)}{2|k|^2}.\end{aligned}$$

We have used that $U'(x) = |u(x)|$. Proceeding this way,

$$\begin{aligned}|M_3(x,k)| &= \left|-\int_{-\infty}^{x} \frac{e^{2ik(x-z)}-1}{2ik}u(z)M_2(z,k)dz\right| \\ &\leq \int_{-\infty}^{x}\left|\frac{e^{2ik(x-z)}-1}{2ik}\right||u(z)|\frac{U^2(z)}{2|k|^2}dz \\ &\leq \frac{1}{2|k|^3}\int_{-\infty}^{x}U'(z)U^2(z)dz = \frac{U^3(x)}{3!|k|^2}.\end{aligned}$$

At this point, it is easy to see, and equally easy to establish using induction, that

$$|M_j(x,k)| \leq \frac{U^j(x)}{j!|k|^j},$$

for $j = 1, 2, \ldots$. Putting all these bounds together, we obtain

$$|M(x,k)| \leq \sum_{j=0}^{\infty}|M_j(x,k)| \leq \sum_{j=0}^{\infty}\frac{U^j(x)}{j!|k|^j} = e^{U(x)/|k|}.$$

This allows us to immediately conclude that if

$$\int_{-\infty}^{\infty}|u(x)|dx = U(\infty) < \infty,$$

then

$$|M(x,k)| \leq e^{U(\infty)/|k|},$$

and $M(x,k)$ is uniformly bounded as a function of x. As a function of k, $M(x,k)$ is bounded in the upper-half plane, but the bound is not uniform. In fact, the bound blows up as the real line (in particular the origin) is approached. In the above, we have assumed from the start that we were operating in the upper-half plane. The reader should appreciate that the decision to consider the upper-half plane would have naturally come out of the exponential kernel and its dependence on k.

Next we consider what happens for $k \in \mathbb{R}$. We have

$$\begin{aligned} |M_1(x,k)| &= \left| -\int_{-\infty}^{x} \frac{e^{2ik(x-z)}-1}{2ik} u(z) M_0(z,k) dz \right| \\ &\leq \int_{-\infty}^{x} \left| \frac{e^{2ik(x-z)}-1}{2ik} \right| |u(z)| dz \\ &\leq \int_{-\infty}^{x} (x-z)|u(z)| dz = U(x), \end{aligned}$$

where we have defined

$$U(x) = \int_{-\infty}^{x} (x-z)|u(z)| dz,$$

assuming that this integral is defined.

12.7 Summary

The inverse scattering method to solve the initial-value problem on the whole line ($x \in (-\infty, \infty)$) is a powerful machine as you may have concluded from this chapter. On the other hand, you have also seen that using this machine is not easy. Let us summarize what we gain by using this machine:

- It is possible to solve the initial-value problem for any initial data in a suitable space, like the square integrable functions on the whole line. The other methods we have seen allow for the construction of (sometimes large) classes of special solutions.
- Although the forward and inverse scattering step of the inverse scattering method are quite complicated, they only require the solution of linear problems of one variable. This should be contrasted with the equation we are trying to solve, which is a nonlinear partial differential equation.

12.8 Exercises

1. For the KdV equation $u_t = 6uu_x + u_{xxx}$ with initial condition $u(x,0) = \alpha\delta(x)$, consider the forward scattering problem. Show that the same function $a(k)$ is obtained by using the eigenfunctions for $x < 0$ and those for $x > 0$.

2. For the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ with initial condition $u(x,0) = 0$ for $x \in]-\infty, -L[\cup]L, \infty[$, and $u(x,0) = d$ for $x \in]-L, L[$, with L and d both positive, consider the forward scattering problem.

   - Find $a(k)$, for all time t.
   - Knowing that the number of solitons emanating from the initial condition is the number of zeros of $a(k)$ on the positive imaginary axis (*i.e.*, $k = i\kappa$, with $\kappa > 0$), discuss how many solitons correspond to the given initial condition, depending on the value of $2Ld$. You might want to use Maple or Matlab for this.
   - What happens for $d < 0$?
   - In the limit $L \to 0$, but $2dL = \alpha$, $u(x,0) \to \alpha\delta(x)$. What happens to $a(k)$ when you take this limit? Discuss.

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