CHAPTER

2

Cartesian Tensors

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CHAPTER OBJECTIVES

- To define the notation used in this text for scalars, vectors, and tensors
- To review the basic algebraic manipulations of vectors and matrices
- To present how vector differentiation is applied to scalars, vectors, and tensors
- To review the fundamental theorems of vector field theory

2.1. SCALARS, VECTORS, TENSORS, NOTATION

The physical quantities in fluid mechanics vary in their complexity, and may involve multiple spatial directions. Their proper specification in terms of *scalars*, *vectors*, and (second order) *tensors* is the subject of this chapter. Here, three independent spatial dimensions are assumed to exist. The reader can readily simplify, or extend, the various results presented here for fewer, or more, independent spatial dimensions.

Scalars or zero-order tensors may be defined with a single magnitude and appropriate units, may vary with spatial location, but are independent of coordinate directions. Scalars are typically denoted herein by italicized symbols. For example, common scalars in fluid mechanics are pressure p, temperature T, and density ρ.

Vectors or first-order tensors have both a magnitude and a direction. A vector can be completely described by its components along three orthogonal coordinate directions. Thus, the components of a vector may change with a change in coordinate system. A vector is usually denoted herein by a boldface symbol. For example, common vectors in fluid mechanics are position $\mathbf{x}$, fluid velocity $\mathbf{u}$, and gravitational acceleration $\mathbf{g}$. In a Cartesian coordinate system with unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, in the three mutually perpendicular directions, the position vector $\mathbf{x}$, OP in Figure 2.1, may be written

$$\mathbf{x} = \mathbf{e}_1 x_1 + \mathbf{e}_2 x_2 + \mathbf{e}_3 x_3, \tag{2.1}$$

where x_1, x_2, and x_3 are the components of $\mathbf{x}$ along each Cartesian axis. Here, the subscripts of $\mathbf{e}$ do *not* denote vector components but rather reference the coordinate axes 1, 2, and 3; hence, the $\mathbf{e}$s are vectors themselves. Sometimes, to save writing, the components of a vector are denoted with an italic symbol having one index—such as i, j, or k—that implicitly is known to take on three possible values: 1, 2, or 3. For example, the components of $\mathbf{x}$ can be denoted by x_i or x_j (or x_k, etc.). For algebraic manipulation, a vector is written as a column matrix; thus, (2.1) is consistent with the following vector specifications:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{where} \quad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The *transpose* of the matrix (denoted by a superscript T) is obtained by interchanging rows and columns, so the transpose of the column matrix $\mathbf{x}$ is the row matrix:

$$\mathbf{x}^{\mathrm{T}} = [\,x_1 \quad x_2 \quad x_3\,].$$

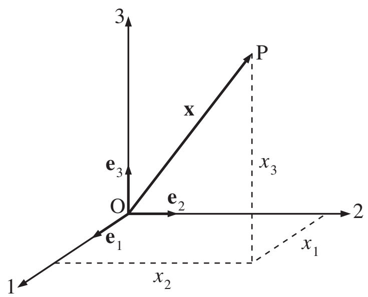

FIGURE 2.1 Position vector OP and its three Cartesian components (x_1, x_2, x_3). The three unit vectors for the coordinate directions are $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. Once the coordinate system is chosen, the vector $\mathbf{x}$ is completely defined by its components, x_i where $i = 1, 2,$ or 3.

However, to save space in the text, the square-bracket notation for vectors shown here is typically replaced by triplets (or doublets) of values separated by commas and placed inside ordinary parentheses, for example, $\mathbf{x} = (x_1, x_2, x_3)$.

Second-order tensors have a component for each *pair* of coordinate directions and therefore may have as many as $3 \times 3 = 9$ separate components. A second-order tensor is sometimes denoted by a boldface symbol. For example, a common second-order tensor in fluid mechanics is the stress $\boldsymbol{\tau}$. Like vector components, second-order tensor components change with a change in coordinate system. Once a coordinate system is chosen, the nine components of a second-order tensor can be represented by a 3×3 matrix, or by an italic symbol having two indices, such as τ_{ij} for the stress tensor. Here again the indices i and j are known implicitly to separately take on the values 1, 2, or 3. Second-order tensors are further discussed in Section 2.4.

A second implicit feature of *index-based* or *indicial* notation is the implied sum over a repeated index in terms involving multiple indices. This notational convention can be stated as follows: *Whenever an index is repeated in a term, a summation over this index is implied, even though no summation sign is explicitly written*. This notational convention saves writing and increases mathematical precision when dealing with products of first- and higher-order tensors. It was introduced by Albert Einstein and is sometimes referred to as the *Einstein summation convention*. It can be illustrated by a simple example involving the ordinary dot product of two vectors **a** and **b** having components a_i and b_j, respectively. Their dot product is the sum of component products,

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^{3} a_i b_i \equiv a_i b_i, \tag{2.2}$$

where the final three-line *definition* equality ($\equiv$) follows from the repeated-index implied-sum convention. Since this notational convention is unlikely to be comfortable to the reader after a single exposure, it is repeatedly illustrated via definition equalities in this chapter before being adopted in the remainder of this text wherever indicial notation is used.

Both boldface (aka, *vector* or *dyadic*) and indicial (aka, *tensor*) notations are used throughout this text. With boldface notation the physical meaning of terms is generally clearer, and there are no subscripts to consider. Unfortunately, algebraic manipulations may be difficult and not distinct in boldface notation since the product **ab** may not be well defined nor equal to **ba** when **a** and **b** are second-order tensors. Boldface notation has other problems too; for example, the order or rank of a tensor is not clear if one simply calls it **a**.

Indicial notation avoids these problems because it deals only with tensor *components*, which are *scalars*. Algebraic manipulations are simpler and better defined, and special attention to the ordering of terms is unnecessary (unless differentiation is involved). In addition, the number of indices or subscripts clearly specifies the order of a tensor. However, the physical structure and meaning of terms written with index notation only become apparent after an examination of the indices. Hence, indices must be clearly written to prevent mistakes and to promote proper understanding of the terms they help define. In addition, the cross product involves the possibly cumbersome alternating tensor ε_{ijk} as described in Sections 2.7 and 2.9.

2.2. ROTATION OF AXES: FORMAL DEFINITION OF A VECTOR

A vector can be formally defined as any quantity whose components change similarly to the components of the position vector under rotation of the coordinate system. Let O123 be the original coordinate system, and O1′2′3′ be the rotated system that shares the same origin O (see Figure 2.2). The position vector **x** can be written in either coordinate system,

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \quad \text{or} \quad \mathbf{x} = x'_1\mathbf{e}'_1 + x'_2\mathbf{e}'_2 + x'_3\mathbf{e}'_3, \tag{2.1, 2.3}$$

where the components of **x** in O123 and O1′2′3′ are x_i and x'_j, respectively, and the $\mathbf{e}'_j$ are the unit vectors in O1′2′3′. Forming a dot product of **x** with $\mathbf{e}'_1$, and using both (2.1) and (2.3) produces

$$\mathbf{x}\cdot\mathbf{e}'_1 = x_1\mathbf{e}_1\cdot\mathbf{e}'_1 + x_2\mathbf{e}_2\cdot\mathbf{e}'_1 + x_3\mathbf{e}_3\cdot\mathbf{e}'_1 = x'_1, \tag{2.4}$$

where we recognize the dot products between unit vectors as direction cosines; $\mathbf{e}_1\cdot\mathbf{e}'_1$ is the cosine of the angle between the 1 and 1′ axes, $\mathbf{e}_2\cdot\mathbf{e}'_1$ is the cosine of the angle between the 2 and 1′ axes, and $\mathbf{e}_3\cdot\mathbf{e}'_1$ is the cosine of the angle between the 3 and 1′ axes. Forming the dot products $\mathbf{x}\cdot\mathbf{e}'_2 = x'_2$ and $\mathbf{x}\cdot\mathbf{e}'_3 = x'_3$, and then combining these results with (2.3) produces

$$x'_j = x_1C_{1j} + x_2C_{2j} + x_3C_{3j} = \sum_{i=1}^{3} x_iC_{ij} \equiv x_iC_{ij}, \tag{2.5}$$

where $C_{ij} = \mathbf{e}_i\cdot\mathbf{e}'_j$ is a 3 × 3 matrix of direction cosines and the definition equality follows from the summation convention. In (2.5) the *free* or not-summed-over index is j, while the

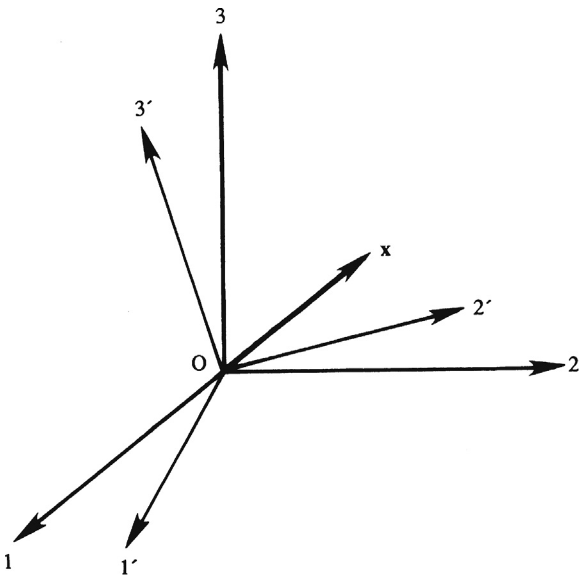

FIGURE 2.2 A rotation of the original Cartesian coordinate system O123 to a new system O1′2′3′. Here the **x** vector is unchanged, but its components in the original system x_i and in the rotated system x'_i will not be the same.

repeated or summed-over index can be any letter other than j. Thus, the rightmost term in (2.5) could equally well have been written $x_k C_{kj}$ or $x_m C_{mj}$. Similarly, any letter can also be used for the free index, as long as the same free index is used on *both* sides of the equation. For example, denoting the free index by i and the summed index by k allows (2.5) to be written with indicial notation as

$$x'_i = x_k C_{ki}. \tag{2.6}$$

This index-choice flexibility exists because the three algebraic equations represented by (2.5), corresponding to the three values of j, are the same as those represented by (2.6) for the three values of i.

It can be shown (see Exercise 2.2) that the components of $\mathbf{x}$ in O123 are related to those in O1′2′3′ by

$$x_j = \sum_{i=1}^{3} x'_i C_{ji} \equiv x'_i C_{ji}. \tag{2.7}$$

The indicial positions on the right side of this relation are different from those in (2.5), because the first index of C_{ij} is summed in (2.5), whereas the second index of C_{ij} is summed in (2.7).

We can now formally define a Cartesian vector as any quantity that transforms like the position vector under rotation of the coordinate system. Therefore, by analogy with (2.5), $\mathbf{u}$ is a vector if its components transform as

$$u'_j = \sum_{i=1}^{3} u_i C_{ij} \equiv u_i C_{ij}. \tag{2.8}$$

EXAMPLE 2.1

Convert the two-dimensional vector $\mathbf{u} = (u_1, u_2)$ from Cartesian (x_1, x_2) to polar (r, θ) coordinates (see Figure 3.3a).

Solution

Clearly $\mathbf{u}$ can be represented in either coordinate system: $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta$, where u_r and u_θ are the components in polar coordinates, and $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the unit vectors in polar coordinates. Here the polar coordinate system is rotated compared to the Cartesian system, as illustrated in Figure 2.3. Forming the dot product of the above equation with $\mathbf{e}_r$ and $\mathbf{e}_\theta$ produces two algebraic equations that are equivalent to (2.5)

$$u_r = u_1\mathbf{e}_1\cdot\mathbf{e}_r + u_2\mathbf{e}_2\cdot\mathbf{e}_r$$
$$u_\theta = u_1\mathbf{e}_1\cdot\mathbf{e}_\theta + u_2\mathbf{e}_2\cdot\mathbf{e}_\theta,$$

with subscripts r and θ replacing $j = 1$ and 2 in (2.5). Evaluation of the unit vector dot products leads to

$$u_r = u_1 \cos\theta + u_2 \cos\left(\frac{\pi}{2} - \theta\right) = u_1 \cos\theta + u_2 \sin\theta, \text{ and}$$
$$u_\theta = u_1 \cos\left(\theta + \frac{\pi}{2}\right) + u_2 \cos\theta = -u_1 \sin\theta + u_2 \cos\theta.$$

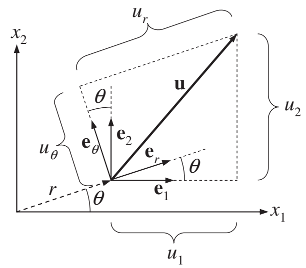

FIGURE 2.3 Resolution of a two-dimensional vector $\mathbf{u}$ in (x_1,x_2)-Cartesian and (r,θ)-polar coordinates. The angle between the $\mathbf{e}_1$ and $\mathbf{e}_r$ unit vectors, and the $\mathbf{e}_2$ and $\mathbf{e}_\theta$ unit vectors, is θ. The angle between the $\mathbf{e}_r$ and $\mathbf{e}_2$ unit vectors is $\pi/2-\theta$, and the angle between the $\mathbf{e}_1$ and $\mathbf{e}_\theta$ unit vectors is $\pi/2+\theta$. Here $\mathbf{u}$ does not emerge from the origin of coordinates (as in Figure 2.2) but it may be well defined in either coordinate system even though its components are not the same in the (x_1,x_2)- and (r,θ)-coordinates.

Thus, in this case:

$$C_{ij} = \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{e}_r & \mathbf{e}_1 \cdot \mathbf{e}_\theta \\ \mathbf{e}_2 \cdot \mathbf{e}_r & \mathbf{e}_2 \cdot \mathbf{e}_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

2.3. MULTIPLICATION OF MATRICES

Let $\mathbf{A}$ and $\mathbf{B}$ be two 3×3 matrices. The inner product of $\mathbf{A}$ and $\mathbf{B}$ is defined as the matrix $\mathbf{P}$ whose elements are related to those of $\mathbf{A}$ and $\mathbf{B}$ by

$$P_{ij} = \sum_{k=1}^{3} A_{ik}B_{kj} \equiv A_{ik}B_{kj}, \quad \text{or} \quad \mathbf{P} = \mathbf{A}\cdot\mathbf{B}, \tag{2.9, 2.10}$$

where the definition equality in (2.9) follows from the summation convention, and the single dot between $\mathbf{A}$ and $\mathbf{B}$ in (2.10) signifies that a single index is summed to find $\mathbf{P}$. An important feature of (2.9) is that the elements are summed over the inner or *adjacent* index k. It is sometimes useful to write (2.9) as

$$P_{ij} = A_{ik}B_{kj} = (\mathbf{A}\cdot\mathbf{B})_{ij},$$

where the last term is to be read as, "the ij-element of the inner product of matrices $\mathbf{A}$ and $\mathbf{B}$." In explicit form, (2.9) is written as

$$\begin{bmatrix} P_{11} & \boxed{P_{12}} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} \boxed{A_{11} \quad A_{12} \quad A_{13}} \\ A_{21} \quad A_{22} \quad A_{23} \\ A_{31} \quad A_{32} \quad A_{33} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}. \tag{2.11}$$

This equation signifies that the ij-element of $\mathbf{P}$ is determined by multiplying the elements in the i-row of $\mathbf{A}$ and the j-column of $\mathbf{B}$, and summing. For example,

$$P_{12} = A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32},$$

as indicated by the dashed-line boxes in (2.11). Naturally, the inner product $\mathbf{A} \cdot \mathbf{B}$ is only defined if the number of columns of $\mathbf{A}$ equals the number of rows of $\mathbf{B}$.

Equation (2.9) also applies to the inner product of a 3×3 matrix and a column vector. For example, (2.6) can be written as $x_i' = C_{ik}^{\mathrm{T}} x_k$, which is now of the form of (2.9) because the summed index k is adjacent. In matrix form, (2.6) can therefore be written as

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Symbolically, the preceding is $\mathbf{x}' = \mathbf{C}^{\mathrm{T}} \cdot \mathbf{x}$, whereas (2.7) is $\mathbf{x} = \mathbf{C} \cdot \mathbf{x}'$.

2.4. SECOND-ORDER TENSORS

A simple-to-complicated hierarchical description of physically meaningful quantities starts with scalars, proceeds to vectors, and then continues to second- and higher-order tensors. A scalar can be represented by a single value. A vector can be represented by three components, one for each of three orthogonal spatial directions denoted by a single free index. A second-order tensor can be represented by nine components, one for each pair of directions, and denoted by two free indices. Nearly all the tensors considered in Newtonian fluid mechanics are zero-, first-, or second-order tensors.

To better understand the structure of second-order tensors, consider the stress tensor $\boldsymbol{\tau}$ or τ_{ij}. Its two free indices specify two directions; the first indicates the orientation of the *surface* on which the stress is applied while the second indicates the component of the *force per unit area* on that surface. In particular, the first (i) index of τ_{ij} denotes the direction of the surface normal, and the second (j) index denotes the force component direction. This situation is illustrated in Figure 2.4, which shows the normal and shear stresses on an infinitesimal cube having surfaces parallel to the coordinate planes. The stresses are positive if they are directed as shown in this figure. The sign convention is that, on a surface whose outward normal points in the positive direction of a coordinate axis, the normal and shear stresses are positive if they point in the positive directions of the other axes. For example, on the surface ABCD, whose outward normal points in the positive x_2 direction, the positive stresses τ_{21}, τ_{22}, and τ_{23} point in the x_1, x_2, and x_3 directions, respectively. Normal stresses are positive if they are tensile and negative if they are compressive. On the opposite face EFGH the stress components have the same value as on ABCD, but their directions are reversed. This is because Figure 2.4 represents stresses *at a point*. The cube shown is intended to be vanishingly small, so that the faces ABCD and EFGH are just opposite sides of a plane perpendicular to the x_2-axis. Thus, stresses on the opposite faces are equal and opposite, and satisfy Newton's third law.

A vector $\mathbf{u}$ is completely specified by the three components u_i (where $i = 1, 2, 3$) because the components of $\mathbf{u}$ in any direction other than the original axes can be found from (2.8). Similarly, the state of stress at a point can be completely specified by the nine components τ_{ij} (where $i, j = 1, 2, 3$) that can be written as the matrix

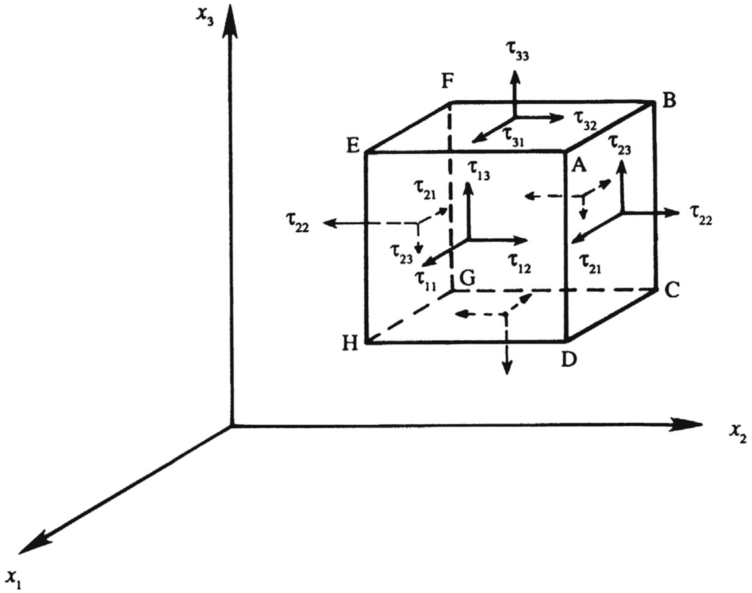

FIGURE 2.4 Illustration of the stress field at a point via stress components on a cubic volume element. Here each surface may experience one normal and two shear components of stress. The directions of positive normal and shear stresses are shown. For clarity, the stresses on faces FBCG and CDHG are not labeled.

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}.$$

The specification of these nine stress components on surfaces perpendicular to the coordinate axes completely determines the state of stress at a point because the stresses on any arbitrary plane can be determined from them. To find the stresses on any arbitrary surface, we can consider a rotated coordinate system $O1'2'3'$ having one axis perpendicular to the given surface. It can be shown by a force balance on a tetrahedron element (see, e.g., Sommerfeld, 1964, page 59) that the components of $\boldsymbol{\tau}$ in the rotated coordinate system are

$$\tau'_{mn} = \sum_{i=1}^{3} \sum_{j=1}^{3} C_{im}C_{jn}\tau_{ij} \equiv C_{im}C_{jn}\tau_{ij}, \tag{2.12}$$

where the definition equality follows from the summation convention. This equation may also be written as: $\tau'_{mn} = C^{T}_{mi}\tau_{ij}C_{jn}$ or $\boldsymbol{\tau}' = \mathbf{C}^{T}\boldsymbol{\cdot}\boldsymbol{\tau}\boldsymbol{\cdot}\mathrm{C}$. Note the similarity between the vector transformation rule (2.8) and (2.12). In (2.8) the first index of $\mathbf{C}$ is summed, while its second index is free. Equation (2.12) is identical, except that $\mathbf{C}$ is used twice. A quantity that obeys (2.12) is called a *second-order tensor*.

Tensor and matrix concepts are not quite the same. A matrix is any *arrangement* of elements, written as an array. The elements of a matrix represent the components of a second-order tensor only if they obey (2.12). In general, tensors can be of any order and the number of free indices corresponds to the order of the tensor. For example, $\mathbf{A}$ is a fourth-order tensor if it has four free indices, and the associated $3^4 = 81$ components change under a rotation of the coordinate system according to

$$A'_{mnpq} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} C_{im}C_{jn}C_{kp}C_{lq}A_{ijkl} \equiv C_{im}C_{jn}C_{kp}C_{lq}A_{ijkl}, \tag{2.13}$$

where again the definition equality follows from the summation convention. Tensors of various orders arise in fluid mechanics. Common second-order tensors are the stress tensor τ_{ij} and the velocity-gradient tensor $\partial u_i/\partial x_j$. The nine products $u_i v_j$ formed from the components of the two vectors **u** and **v** also transform according to (2.12), and therefore form a second-order tensor. In addition, the *Kronecker-delta* and *alternating tensors* are also frequently used; these are defined and discussed in Section 2.7.

2.5. CONTRACTION AND MULTIPLICATION

When the two indices of a tensor are equated, and a summation is performed over this repeated index, the process is called *contraction*. An example is

$$\sum_{j=1}^{3} A_{jj} \equiv A_{jj} = A_{11} + A_{22} + A_{33},$$

which is the sum of the diagonal terms of A_{ij}. Clearly, A_{jj} is a scalar and therefore independent of the coordinate system. In other words, A_{jj} is an *invariant*. (There are three independent invariants of a second-order tensor, and A_{jj} is one of them; see Exercise 2.9.)

Higher-order tensors can be formed by multiplying lower-order tensors. If **A** and **B** are two second-order tensors, then the 81 numbers defined by $P_{ijkl} \equiv A_{ij}B_{kl}$ transform according to (2.13), and therefore form a fourth-order tensor.

Lower-order tensors can be obtained by performing a contraction within a multiplied form. The four contractions of $A_{ij}B_{kl}$ are

$$\begin{aligned}
\sum_{i=1}^{3} A_{ij}B_{ki} &\equiv A_{ij}B_{ki} = B_{ki}A_{ij} = (\mathbf{B}\cdot\mathbf{A})_{kj},\\
\sum_{i=1}^{3} A_{ij}B_{ik} &\equiv A_{ij}B_{ik} = A^{T}_{ji}B_{ik} = \left(\mathbf{A}^{\mathrm{T}}\cdot\mathbf{B}\right)_{jk},\\
\sum_{j=1}^{3} A_{ij}B_{kj} &\equiv A_{ij}B_{kj} = A_{ij}B^{T}_{jk} = \left(\mathbf{A}\cdot\mathbf{B}^{\mathrm{T}}\right)_{ik},\\
\sum_{j=1}^{3} A_{ij}B_{jk} &\equiv A_{ij}B_{jk} = (\mathbf{A}\cdot\mathbf{B})_{ik},
\end{aligned} \tag{2.14}$$

where all the definition equalities follow from the summation convention. All four products in (2.14) are second-order tensors. Note also in (2.14) how the terms have been rearranged until the summed index is adjacent; at this point they can be written as a product of matrices.

The contracted product of a second-order tensor **A** and a vector **u** is a vector. The two possibilities are

$$\sum_{j=1}^{3} A_{ij}u_j \equiv A_{ij}u_j = (\mathbf{A}\cdot\mathbf{u})_i, \text{ and}$$

$$\sum_{i=1}^{3} A_{ij}u_i \equiv A_{ij}u_i = A^{\mathrm{T}}_{ji}u_i = \left(\mathbf{A}^{\mathrm{T}}\cdot\mathbf{u}\right)_j,$$

where again the definition equalities follow from the summation convention. The doubly contracted product of two second-order tensors **A** and **B** is a scalar. Using all three notations, the two possibilities are

$$\sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}B_{ji} \equiv A_{ij}B_{ji} (= \mathbf{A} : \mathbf{B}) \text{ and } \sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}B_{ij} \equiv A_{ij}B_{ij} (= \mathbf{A} : \mathbf{B}^{\mathrm{T}}),$$

where the bold colon (**:**) implies a *double* contraction or double dot product.

2.6. FORCE ON A SURFACE

A surface area element has a size (or magnitude) and an orientation, so it can be treated as a vector $d\mathbf{A}$. If dA is the surface element's size, and **n** is its normal unit vector, then $d\mathbf{A} = \mathbf{n}dA$.

Suppose the nine components, τ_{ij}, of the stress tensor with respect to a given set of Cartesian coordinates O123 are given, and we want to find the force per unit area, $\mathbf{f}(\mathbf{n})$ with components f_i, on an arbitrarily oriented surface element with normal **n** (see Figure 2.5). One way of completing this task is to switch to a rotated coordinate system, and use (2.12) to find the normal and shear stresses on the surface element. An alternative method is described here. Consider the tetrahedral element shown in Figure 2.5. The net force f_1 on the element in the first direction produced by the stresses τ_{ij} is

$$f_1 dA = \tau_{11}dA_1 + \tau_{21}dA_2 + \tau_{31}dA_3.$$

The geometry of the tetrahedron requires: $dA_i = n_i dA$, where n_i are the components of the surface normal vector **n**. Thus, the net force equation can be rewritten as

$$f_1 dA = \tau_{11}n_1 dA + \tau_{21}n_2 dA + \tau_{31}n_3 dA.$$

Dividing by dA then produces $f_1 = \tau_{j1}n_j$ (with summation implied), or for any component of **f**,

$$f_i = \sum_{j=1}^{3} \tau_{ji}n_j \equiv \tau_{ji}n_j \quad \text{or} \quad \mathbf{f} = \mathbf{n}\cdot\boldsymbol{\tau}, \tag{2.15}$$

where the boldface-only version of (2.15) follows when $\tau_{ij} = \tau_{ji}$, a claim that is proved in Chapter 4. Therefore, the contracted or inner product of the stress tensor $\boldsymbol{\tau}$ and the unit

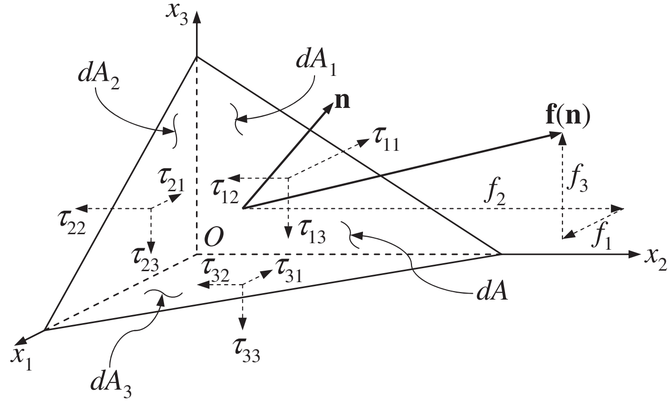

FIGURE 2.5 Force **f** per unit area on a surface element whose outward normal is **n**. The areas of the tetrahedron's faces that are perpendicular to the ith coordinate axis are dA_i. The area of the largest tetrahedron face is dA. As in Figure 2.4, the directions of positive normal and shear stresses are shown.

normal vector **n** gives the force per unit area on a surface perpendicular to **n**. This result is analogous to $u_n = \mathbf{u}\cdot\mathbf{n}$, where u_n is the component of the vector **u** along **n**; however, whereas u_n is a scalar, **f** in (2.15) is a vector.

EXAMPLE 2.2

In two spatial dimensions, x_1 and x_2, consider parallel flow through a channel (see Figure 2.6). Choose x_1 parallel to the flow direction. The viscous stress tensor at a point in the flow has the form

$$\tau = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix},$$

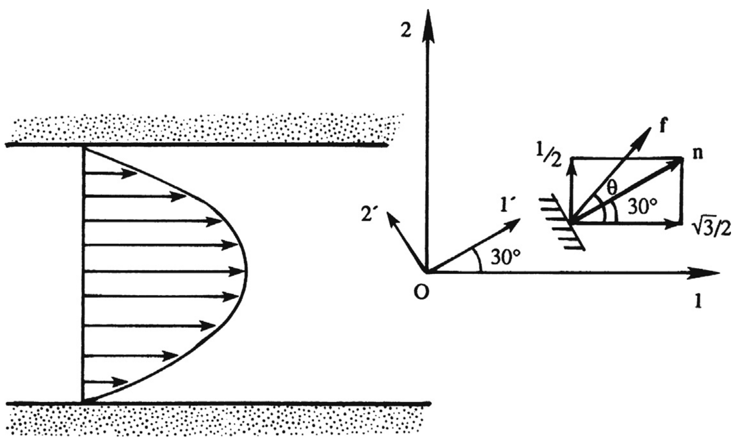

FIGURE 2.6 Determination of the force per unit area on a small area element with a normal vector rotated 30° from the flow direction in a simple unidirectional shear flow parallel to the x_1-axis.

where a is positive in one half of the channel, and negative in the other half. Find the magnitude and direction of the force per unit area **f** on an element whose outward normal points $\phi = 30°$ from the flow direction.

Solution by Using (2.15)

Start with the definition of **n** in the given coordinates:

$$\mathbf{n} = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}.$$

The force per unit area is therefore

$$\mathbf{f} = \tau_{ji}n_j = \tau_{ij}n_j = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}\begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix} = \begin{bmatrix} a\sin\phi \\ a\cos\phi \end{bmatrix} = \begin{bmatrix} a/2 \\ a\sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

The magnitude of **f** is

$$f = |\mathbf{f}| = \left(f_1^2 + f_2^2\right)^{1/2} = |a|.$$

If θ is the angle of **f** with respect to the x_1 axis, then

$$\sin\theta = f_2/f = \left(\sqrt{3}/2\right)(a/|a|) \quad \text{and} \quad \cos\theta = f_1/f = (1/2)(a/|a|).$$

Thus $\theta = 60°$ if a is positive (in which case both $\sin\theta$ and $\cos\theta$ are positive), and $\theta = 240°$ if a is negative (in which case both $\sin\theta$ and $\cos\theta$ are negative).

Solution by Using (2.12)

Consider a rotated coordinate system $O1'2'$ with the x_1'-axis coinciding with **n** as shown in Figure 2.6. Using (2.12), the components of the stress tensor in the rotated frame are

$$\tau'_{11} = C_{11}C_{21}\tau_{12} + C_{21}C_{11}\tau_{21} = (\cos\phi\sin\phi)a + (\sin\phi\cos\phi)a = \frac{\sqrt{3}}{2}\frac{1}{2}a + \frac{1}{2}\frac{\sqrt{3}}{2}a = \frac{\sqrt{3}}{2}a \text{ and}$$

$$\tau'_{12} = C_{11}C_{22}\tau_{12} + C_{21}C_{12}\tau_{21} = (\cos\phi)^2 a - (\sin\phi)^2 a = \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2}a - \frac{1}{2}\frac{1}{2}a = \frac{1}{2}a,$$

where $C_{ij} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$. The normal stress is therefore $\sqrt{3}a/2$, and the shear stress is $a/2$. These results again provide the magnitude of a and a direction of 60° or 240° depending on the sign of a.

2.7. KRONECKER DELTA AND ALTERNATING TENSOR

The *Kronecker delta* is defined as

$$\delta_{ij} = \left\{ \begin{array}{l} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{array} \right\}. \tag{2.16}$$

In three spatial dimensions it is the 3×3 identity matrix:

$$\boldsymbol{\delta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In matrix multiplication operations involving the Kronecker delta, it simply replaces its summed-over index by its other index. Consider

$$\sum_{j=1}^{3} \delta_{ij} u_j \equiv \delta_{ij} u_j = \delta_{i1} u_1 + \delta_{i2} u_2 + \delta_{i3} u_3;$$

the right-hand side is u_1 when $i = 1$, u_2 when $i = 2$, and u_3 when $i = 3$; thus

$$\delta_{ij} u_j = u_i. \tag{2.17}$$

From its definition it is clear that δ_{ij} is an *isotropic tensor* in the sense that its components are unchanged by a rotation of the frame of reference, that is, $\delta'_{ij} = \delta_{ij}$. Isotropic tensors can be of various orders. There is no isotropic tensor of first order, and δ_{ij} is the only isotropic tensor of second order. There is also only one isotropic tensor of third order. It is called the *alternating tensor* or *permutation symbol*, and is defined as

$$\varepsilon_{ijk} = \left\{ \begin{array}{rl} 1 & \text{if } ijk = 123, 231, \text{ or } 312 \text{ (cyclic order)}, \\ 0 & \text{if any two indices are equal}, \\ -1 & \text{if } ijk = 321, 213, \text{ or } 132 \text{ (anti-cyclic order)} \end{array} \right\}. \tag{2.18}$$

From this definition, it is clear that *an index on ε_{ijk} can be moved two places* (*either to the right or to the left*) *without changing its value*. For example, $\varepsilon_{ijk} = \varepsilon_{jki}$ where i has been moved two places to the right, and $\varepsilon_{ijk} = \varepsilon_{kij}$ where k has been moved two places to the left. For a movement of one place, however, the sign is reversed. For example, $\varepsilon_{ijk} = -\varepsilon_{ikj}$ where j has been moved one place to the right.

A very frequently used relation is the *epsilon delta relation:*

$$\sum_{k=1}^{3} \varepsilon_{ijk} \varepsilon_{klm} \equiv \varepsilon_{ijk} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}. \tag{2.19}$$

The reader can verify the validity of this relationship by choosing some values for the indices $ijlm$. This relationship can be remembered by noting the following two points: 1) The adjacent index k is summed; and 2) the first two indices on the right side, namely, i and l, are the first index of ε_{ijk} and the first *free* index of ε_{klm}. The remaining indices on the right side then follow immediately.

2.8. VECTOR, DOT, AND CROSS PRODUCTS

The dot product of two vectors **u** and **v** is defined as the scalar

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{i=1}^{3} u_i v_i \equiv u_i v_i.$$

It is easy to show that $\mathbf{u} \cdot \mathbf{v} = uv \cos\theta$, where u and v are the vectors' magnitudes and θ is the angle between the vectors (see Exercises 2.12 and 2.13). The dot product is therefore the magnitude of one vector times the component of the other in the direction of the first. The dot product $\mathbf{u} \cdot \mathbf{v}$ is equal to the sum of the diagonal terms of the tensor $u_i v_j$.

The cross product between two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as the vector $\mathbf{w}$ whose magnitude is $uv \sin\theta$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, and whose direction is perpendicular to the plane of $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ form a right-handed system. Clearly, $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$. Furthermore, unit vectors in right-handed coordinate systems obey the cyclic rule $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$. From these requirements it can be shown that

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{e}_1 + (u_3 v_1 - u_1 v_3)\mathbf{e}_2 + (u_1 v_2 - u_2 v_1)\mathbf{e}_3 \tag{2.20}$$

(see Exercise 2.14). Equation (2.20) can be written as the determinant of a matrix

$$\mathbf{u} \times \mathbf{v} = \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}.$$

In indicial notation, the k-component of $\mathbf{u} \times \mathbf{v}$ can be written as

$$(\mathbf{u} \times \mathbf{v})_k = \sum_{i=1}^{3} \sum_{j=1}^{3} \varepsilon_{ijk} u_i v_j \equiv \varepsilon_{ijk} u_i v_j = \varepsilon_{kij} u_i v_j. \tag{2.21}$$

As a check, for $k = 1$ the nonzero terms in the double sum in (2.21) result from $i = 2, j = 3$, and from $i = 3, j = 2$. This follows from the definition (2.18) that the permutation symbol is zero if any two indices are equal. Thus, (2.21) gives

$$(\mathbf{u} \times \mathbf{v})_1 = \varepsilon_{ij1} u_i v_j = \varepsilon_{231} u_2 v_3 + \varepsilon_{321} u_3 v_2 = u_2 v_3 - u_3 v_2,$$

which agrees with (2.20). Note that the third form of (2.21) is obtained from the second by moving the index k two places to the left; see the remark following (2.18).

2.9. GRADIENT, DIVERGENCE, AND CURL

The vector-differentiation operator "del"[i] is defined symbolically by

$$\nabla = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3} = \sum_{i=1}^{3} \mathbf{e}_i \frac{\partial}{\partial x_i} \equiv \mathbf{e}_i \frac{\partial}{\partial x_i}. \tag{2.22}$$

When operating on a scalar function of position ϕ, it generates the vector

$$\nabla \phi = \sum_{i=1}^{3} \mathbf{e}_i \frac{\partial \phi}{\partial x_i} \equiv \mathbf{e}_i \frac{\partial \phi}{\partial x_i},$$

[i]The inverted Greek delta is called a "nabla" ($\nu\alpha\beta\lambda\alpha$). The word originates from the Hebrew word for lyre, an ancient harp-like stringed instrument. It was on his instrument that the boy, David, entertained King Saul (Samuel II) and it is mentioned repeatedly in Psalms as a musical instrument to use in the praise of God.

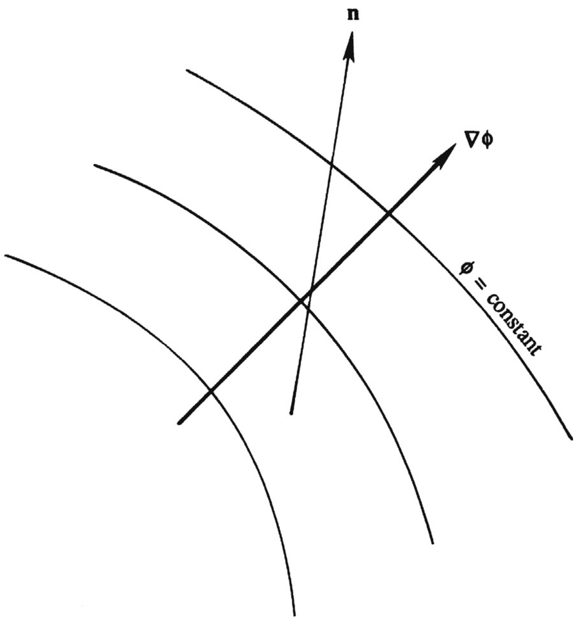

FIGURE 2.7 An illustration of the gradient, $\nabla\phi$, of a scalar function ϕ. The curves of constant ϕ and $\nabla\phi$ are perpendicular, and the spatial derivative of ϕ in the direction $\mathbf{n}$ is given by $\mathbf{n}\cdot\nabla\phi$. The most rapid change in ϕ is found when $\mathbf{n}$ and $\nabla\phi$ are parallel.

whose i-component is $(\nabla\phi)_i = \partial\phi/\partial x_i$. The vector $\nabla\phi$ is called the *gradient* of ϕ, and $\nabla\phi$ is perpendicular to surfaces defined by $\phi = \text{constant}$. In addition, it specifies the magnitude and direction of the *maximum* spatial rate of change of ϕ (Figure 2.7). The spatial rate of change of ϕ in any other direction $\mathbf{n}$ is given by

$$\partial\phi/\partial n = \nabla\phi\cdot\mathbf{n}.$$

In Cartesian coordinates, the *divergence* of a vector field $\mathbf{u}$ is defined as the scalar

$$\nabla\cdot\mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i} \equiv \frac{\partial u_i}{\partial x_i}. \tag{2.23}$$

So far, we have defined the operations of the gradient of a scalar and the divergence of a vector. We can, however, generalize these operations. For example, the divergence of a second-order tensor $\boldsymbol{\tau}$ can be defined as the vector whose i-component is

$$(\nabla\cdot\boldsymbol{\tau})_i = \sum_{j=1}^{3}\frac{\partial\tau_{ij}}{\partial x_j} \equiv \frac{\partial\tau_{ij}}{\partial x_j}.$$

It is evident that the divergence operation *decreases* the order of the tensor by one. In contrast, the gradient operation *increases* the order of a tensor by one, changing a zero-order tensor to a first-order tensor, and a first-order tensor to a second-order tensor, i.e., $\partial u_i/\partial x_j$.

The *curl* of a vector field $\mathbf{u}$ is defined as the vector $\nabla \times \mathbf{u}$, whose i-component can be written as

$$(\nabla \times \mathbf{u})_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \equiv \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \tag{2.24}$$

using (2.21) and (2.22). The three components of the vector $\nabla \times \mathbf{u}$ can easily be found from the right-hand side of (2.24). For the $i = 1$ component, the nonzero terms in the double sum result from $j = 2$, $k = 3$, and from $j = 3$, $k = 2$. The three components of $\nabla \times \mathbf{u}$ are finally found as

$$(\nabla \times \mathbf{u})_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \quad (\nabla \times \mathbf{u})_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \quad \text{and} \quad (\nabla \times \mathbf{u})_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}. \tag{2.25}$$

A vector field $\mathbf{u}$ is called *solenoidal* or *divergence free* if $\nabla \cdot \mathbf{u} = 0$, and *irrotational* or *curl free* if $\nabla \times \mathbf{u} = 0$. The word solenoidal refers to the fact that the divergence of the magnetic induction is always zero because of the absence of magnetic monopoles. The reason for the word irrotational is made clear in Chapter 3.

EXAMPLE 2.3

If a is a positive constant and $\mathbf{b}$ is a constant vector, determine the divergence and the curl of a vector field that diverges from the origin of coordinates, $\mathbf{u} = a\mathbf{x}$, and a vector field indicative of solid body rotation about a fixed axis, $\mathbf{u} = \mathbf{b} \times \mathbf{x}$.

Solution

Using $\mathbf{u} = a\mathbf{x} = ax_1\mathbf{e}_1 + ax_2\mathbf{e}_2 + ax_3\mathbf{e}_3$ in (2.23) and (2.25) produces:

$$\nabla \cdot \mathbf{u} = \frac{\partial ax_1}{\partial x_1} + \frac{\partial ax_2}{\partial x_2} + \frac{\partial ax_3}{\partial x_3} = a + a + a = 3a,$$

$$(\nabla \times \mathbf{u})_1 = \frac{\partial ax_3}{\partial x_2} - \frac{\partial ax_2}{\partial x_3} = 0, \quad (\nabla \times \mathbf{u})_2 = \frac{\partial ax_1}{\partial x_3} - \frac{\partial ax_3}{\partial x_1} = 0, \quad \text{and}$$

$$(\nabla \times \mathbf{u})_3 = \frac{\partial ax_2}{\partial x_1} - \frac{\partial ax_1}{\partial x_2} = 0.$$

Thus, $\mathbf{u} = \mathrm{a}\mathbf{x}$ has a constant nonzero divergence and is irrotational. Using $\mathbf{u} = (b_2x_3 - b_3x_2)\mathbf{e}_1 + (b_3x_1 - b_1x_3)\mathbf{e}_2 + (b_1x_2 - b_2x_1)\mathbf{e}_3$ in (2.23) and (2.25) produces:

$$\nabla \cdot \mathbf{u} = \frac{\partial (b_2x_3 - b_3x_2)}{\partial x_1} + \frac{\partial (b_3x_1 - b_1x_3)}{\partial x_2} + \frac{\partial (b_1x_2 - b_2x_1)}{\partial x_3} = 0,$$

$$(\nabla \times \mathbf{u})_1 = \frac{\partial (b_1x_2 - b_2x_1)}{\partial x_2} - \frac{\partial (b_3x_1 - b_1x_3)}{\partial x_3} = 2b_1,$$

$$(\nabla \times \mathbf{u})_2 = \frac{\partial (b_2x_3 - b_3x_2)}{\partial x_3} - \frac{\partial (b_1x_2 - b_2x_1)}{\partial x_1} = 2b_2, \quad \text{and}$$

$$(\nabla \times \mathbf{u})_3 = \frac{\partial (b_3x_1 - b_1x_3)}{\partial x_1} - \frac{\partial (b_2x_3 - b_3x_2)}{\partial x_2} = 2b_3.$$

Thus, $\mathbf{u} = \mathbf{b} \times \mathbf{x}$ is divergence free and rotational.

2.10. SYMMETRIC AND ANTISYMMETRIC TENSORS

A tensor **B** is called *symmetric* in the indices i and j if the components do not change when i and j are interchanged, that is, if $B_{ij} = B_{ji}$. Thus, the matrix of a symmetric second-order tensor is made up of only six distinct components (the three on the diagonal where $i = j$, and the three above or below the diagonal where $i \neq j$). On the other hand, a tensor is called *antisymmetric* if $B_{ij} = -B_{ji}$. An antisymmetric tensor must have zero diagonal components, and is made up of only three distinct components (the three above or below the diagonal). Any tensor can be represented as the sum of a symmetric part and an antisymmetric part. For if we write

$$B_{ij} = \frac{1}{2}(B_{ij} + B_{ji}) + \frac{1}{2}(B_{ij} - B_{ji}) = S_{ij} + A_{ij},$$

then the operation of interchanging i and j does not change the first term, but changes the sign of the second term. Therefore, $(B_{ij} + B_{ji})/2 \equiv S_{ij}$ is called the symmetric part of B_{ij}, and $(B_{ij} - B_{ji})/2 \equiv A_{ij}$ is called the antisymmetric part of B_{ij}.

Every vector can be associated with an antisymmetric tensor, and vice versa. For example, we can associate the vector $\boldsymbol{\omega}$ having components ω_i, with an antisymmetric tensor:

$$\mathbf{R} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \tag{2.26}$$

The two are related via

$$R_{ij} = \sum_{k=1}^{3} -\varepsilon_{ijk}\omega_k \equiv -\varepsilon_{ijk}\omega_k, \quad \text{and} \quad \omega_k = \sum_{i-1}^{3}\sum_{j=1}^{3} -\frac{1}{2}\varepsilon_{ijk}R_{ij} \equiv -\frac{1}{2}\varepsilon_{ijk}R_{ij}. \tag{2.27}$$

As a check, (2.27) gives $R_{11} = 0$ and $R_{12} = -\varepsilon_{123}\omega_3 = -\omega_3$, in agreement with (2.26). (In Chapter 3, **R** is recognized as the *rotation* tensor corresponding to the *vorticity* vector $\boldsymbol{\omega}$.)

A commonly occurring operation is the doubly contracted product, P, of a *symmetric* tensor $\boldsymbol{\tau}$ and another tensor **B**:

$$P = \sum_{k=1}^{3}\sum_{l=1}^{3} \tau_{kl}B_{kl} \equiv \tau_{kl}B_{kl} = \tau_{kl}(S_{kl} + A_{kl}) = \tau_{kl}S_{kl} + \tau_{kl}A_{kl} = \tau_{ij}S_{ij} + \tau_{ij}A_{ij}, \tag{2.28}$$

where **S** and **A** are the symmetric and antisymmetric parts of **B** (see above). The final equality follows from the index-summation convention; sums are completed over both k and l, so these indices can be replaced by any two distinct indices. Exchanging the indices of **A** in the final term of (2.28) produces $P = \tau_{ij}S_{ij} - \tau_{ij}A_{ji}$, but this can also be written $P = \tau_{ji}S_{ji} - \tau_{ji}A_{ji}$ because S_{ij} and τ_{ij} are symmetric. Now, replace the index j by k and the index i by l to find:

$$P = \tau_{kl}S_{kl} - \tau_{kl}A_{kl}. \tag{2.29}$$

This relationship and the fourth part of the extended equality in (2.28) require that $\tau_{ij}A_{ij} = \tau_{kl}A_{kl} = 0$, and

$$\tau_{ij}B_{ij} = \tau_{ij}S_{ij} = \frac{1}{2}\tau_{ij}\left(B_{ij} + B_{ji}\right).$$

Thus, the doubly contracted product of a symmetric tensor $\boldsymbol{\tau}$ with any tensor $\mathbf{B}$ equals $\boldsymbol{\tau}$ doubly contracted with the symmetric part of $\mathbf{B}$, and the doubly contracted product of a symmetric tensor and an antisymmetric tensor is zero. The latter result is analogous to the fact that the definite integral over an even (symmetric) interval of the product of a symmetric and an antisymmetric function is zero.

2.11. EIGENVALUES AND EIGENVECTORS OF A SYMMETRIC TENSOR

The reader is assumed to be familiar with the concepts of eigenvalues and eigenvectors of a matrix, so only a brief review of the main results is provided here. Suppose $\boldsymbol{\tau}$ is a symmetric tensor with real elements, for example, the stress tensor. Then the following facts can be proved:

(1) There are three real eigenvalues λ^k ($k = 1, 2, 3$), which may or may not all be distinct. (Here, the superscript k is not an exponent, and λ^k does not denote the k-component of a vector.) These eigenvalues (λ^1, λ^2, and λ^3) are the roots or solutions of the third-degree polynomial

$$\det\left|\tau_{ij} - \lambda\delta_{ij}\right| = 0.$$

(2) The three eigenvectors $\mathbf{b}^k$ corresponding to distinct eigenvalues λ^k are mutually orthogonal. These eigenvectors define the directions of the *principal axes* of $\boldsymbol{\tau}$. Each $\mathbf{b}$ is found by solving three algebraic equations,

$$\left(\tau_{ij} - \lambda\delta_{ij}\right)b_j = 0$$

($i = 1, 2,$ or 3), where the superscript k on λ and $\mathbf{b}$ has been omitted for clarity because there is no sum over k.

(3) If the coordinate system is rotated so that its unit vectors coincide with the eigenvectors, then $\boldsymbol{\tau}$ is diagonal with elements λ^k in this rotated coordinate system,

$$\tau' = \begin{bmatrix} \lambda^1 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^3 \end{bmatrix}.$$

(4) Although the elements τ_{ij} change as the coordinate system is rotated, they cannot be larger than the largest λ or smaller than the smallest λ; the λ^k represent the extreme values of τ_{ij}.

EXAMPLE 2.4

The strain rate tensor $\mathbf{S}$ is related to the velocity vector $\mathbf{u}$ by

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

For a two-dimensional flow parallel to the 1-direction,

$$\mathbf{u} = \begin{bmatrix} u_1(x_2) \\ 0 \end{bmatrix},$$

show how S is diagonalized in a frame of reference rotated to coincide with the principal axes.

Solution

For the given velocity profile $u_1(x_2)$, it is evident that $S_{11} = S_{22} = 0$, and $2S_{12} = 2S_{21} = du_1/dx_2 = 2\Gamma$. The strain rate tensor in the original coordinate system is therefore

$$\mathbf{S} = \begin{bmatrix} 0 & \Gamma \\ \Gamma & 0 \end{bmatrix}.$$

The eigenvalues are determined from

$$\det|S_{ij} - \lambda\delta_{ij}| = \det\begin{vmatrix} -\lambda & \Gamma \\ \Gamma & -\lambda \end{vmatrix} = \lambda^2 - \Gamma^2 = 0,$$

which has solutions $\lambda^1 = \Gamma$ and $\lambda^2 = -\Gamma$. The first eigenvector b^1 is given by

$$\begin{bmatrix} 0 & \Gamma \\ \Gamma & 0 \end{bmatrix} \begin{bmatrix} b_1^1 \\ b_2^1 \end{bmatrix} = \lambda^1 \begin{bmatrix} b_1^1 \\ b_2^1 \end{bmatrix},$$

which has solution $b_1^1 = b_2^1 = 1/\sqrt{2}$, when b^1 is normalized to have magnitude unity. The second eigenvector is similarly found so that

$$\mathbf{b}^1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \text{and} \quad \mathbf{b}^2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

These eigenvectors are shown in Figure 2.8. The direction cosine matrix of the original and the rotated coordinate system is therefore

$$\mathbf{C} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

which represents rotation of the coordinate system by 45°. Using the transformation rule (2.12), the components of **S** in the rotated system are found as follows:

$$\begin{aligned} S'_{12} &= C_{i1}C_{j2}S_{ij} = C_{11}C_{22}S_{12} + C_{21}C_{12}S_{21} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\Gamma - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\Gamma = 0, \\ S'_{11} &= C_{i1}C_{j1}S_{ij} = C_{11}C_{21}S_{12} + C_{21}C_{11}S_{21} = \Gamma, \quad \text{and} \\ S'_{22} &= C_{i2}C_{j2}S_{ij} = C_{12}C_{22}S_{12} + C_{22}C_{12}S_{21} = -\Gamma. \end{aligned}$$

(Instead of using (2.12), all the components of **S** in the rotated system can be found by carrying out the matrix product $\mathrm{C}^{\mathrm{T}} \cdot \mathbf{S} \cdot \mathrm{C}$.) The matrix of S in the rotated frame is therefore:

$$\mathbf{S}' = \begin{bmatrix} \Gamma & 0 \\ 0 & -\Gamma \end{bmatrix}.$$

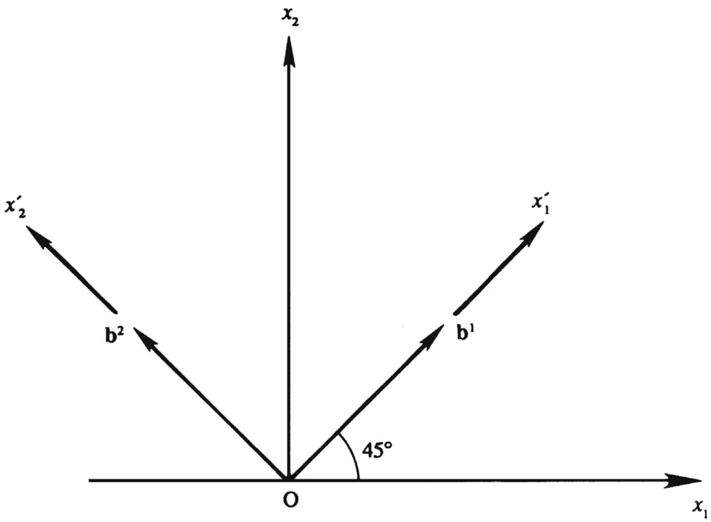

FIGURE 2.8 Original coordinate system Ox_1x_2 and the rotated coordinate system $Ox'_1x'_2$ having unit vectors that coincide with the eigenvectors of the strain-rate tensor in Example 2.4. Here the strain rate is determined from a unidirectional flow having only cross-stream variation, and the angle of rotation is determined to be 45°.

The foregoing matrix contains only diagonal terms. For positive Γ, it will be shown in the next chapter that it represents a linear stretching at a rate Γ along one principal axis, and a linear compression at a rate $-\Gamma$ along the other; the shear strains are zero in the principal-axis coordinate system of the strain rate tensor.

2.12. GAUSS' THEOREM

This very useful theorem relates volume and surface integrals. Let V be a volume bounded by a closed surface A. Consider an infinitesimal surface element dA having outward unit normal $\mathbf{n}$ with components n_i (Figure 2.9), and let $Q(\mathbf{x})$ be a scalar, vector, or tensor field of any order. Gauss' theorem states that

$$\iiint_V \frac{\partial Q}{\partial x_i} dV = \iint_A n_i Q dA. \tag{2.30}$$

The most common form of Gauss' theorem is when $\mathbf{Q}$ is a vector, in which case the theorem is

$$\iiint_V \sum_{i=1}^{3} \frac{\partial Q_i}{\partial x_i} dV \equiv \iiint_V \frac{\partial Q_i}{\partial x_i} dV = \iint_A \sum_{i=1}^{3} n_i Q_i \, dA \equiv \iint_A n_i Q_i \, dA, \quad \text{or} \quad \iiint_V \nabla\cdot\mathbf{Q} dV$$
$$= \iint_A \mathbf{n}\cdot\mathbf{Q} dA,$$

which is commonly called the *divergence theorem*. In words, the theorem states that the volume integral of the divergence of $\mathbf{Q}$ is equal to the surface integral of the outflux of $\mathbf{Q}$.

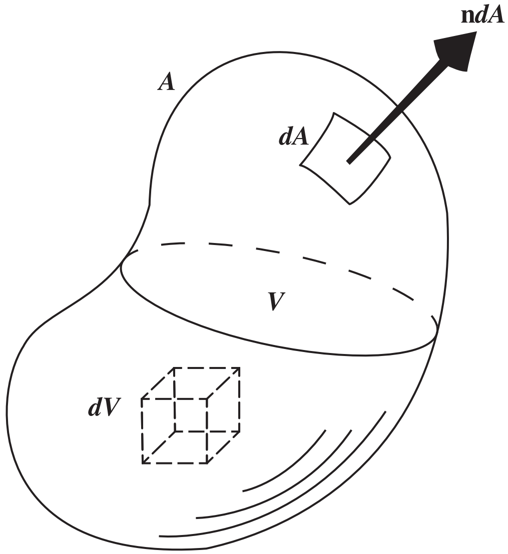

FIGURE 2.9 Illustration of Gauss' theorem for a volume V enclosed by surface area A. A small volume element, dV, and a small area element, dA, with outward normal $\mathbf{n}$ are shown.

Alternatively, (2.30) defines a generalized field derivative, denoted by $\mathcal{D}$, of Q when considered in its limiting form for a vanishingly small volume,

$$\mathcal{D}\mathrm{Q} = \lim_{V\to 0}\frac{1}{V}\iint\limits_{A} n_i Q dA. \tag{2.31}$$

Interestingly, this form is readily specialized to the gradient, divergence, and curl of any scalar, vector, or tensor Q. Moreover, by regarding (2.31) as a definition, the recipes for the computation of vector field derivatives may be obtained in any coordinate system. As stated, (2.31) defines the gradient of a tensor Q of any order. For a tensor of order one or higher, the divergence and curl are defined by including a dot (scalar) product or a cross (vector) product, respectively, under the integral:

$$\nabla\cdot\mathbf{Q} = \lim_{V\to 0}\frac{1}{V}\iint\limits_{A}\mathbf{n}\cdot\mathbf{Q}dA, \quad \text{and} \quad \nabla\times\mathbf{Q} = \lim_{V\to 0}\frac{1}{V}\iint\limits_{A}\mathbf{n}\times\mathbf{Q}dA. \tag{2.32, 2.33}$$

EXAMPLE 2.5

Obtain the recipe for the divergence of a vector $\mathbf{Q}(\mathbf{x})$ in Cartesian coordinates from the integral definition (2.32).

Solution

Consider an elemental rectangular volume centered on $\mathbf{x}$ with faces perpendicular to the coordinate axes (see Figure 2.4). Denote the lengths of the sides parallel to each coordinate axis by Δx_1, Δx_2, and Δx_3, respectively. There are six faces to this rectangular volume. First consider the two that are perpendicular to the x_1 axis, EADH with $\mathbf{n} = \mathbf{e}_1$ and FBCG with $\mathbf{n} = -\mathbf{e}_1$. A Taylor expansion of $\mathbf{Q}(\mathbf{x})$ from the center of the volume to the center of each of these sides produces

$$[\mathbf{Q}]_{EADH} = \mathbf{Q}(\mathbf{x}) + \frac{\Delta x_1}{2}\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial x_1} + \ldots \quad \text{and} \quad [\mathbf{Q}]_{FBCG} = \mathbf{Q}(\mathbf{x}) - \frac{\Delta x_1}{2}\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial x_1} + \ldots,$$

so that the x-direction contribution to the surface integral in (2.32) is

$$\begin{aligned}
([\mathbf{n}\cdot\mathbf{Q}]_{EADH} + [\mathbf{n}\cdot\mathbf{Q}]_{FBCG})dA &= \left(\left[\mathbf{e}_1\cdot\mathbf{Q}(\mathbf{x}) + \mathbf{e}_1\cdot\frac{\Delta x_1}{2}\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial x_1} + \ldots\right]\right. \\
&\quad \left. + \left[-\mathbf{e}_1\cdot\mathbf{Q}(\mathbf{x}) + \mathbf{e}_1\cdot\frac{\Delta x_1}{2}\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial x_1} + \ldots\right]\right)\Delta x_2\Delta x_3 \\
&= \left(\mathbf{e}_1\cdot\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial x_1} + \ldots\right)\Delta x_1\Delta x_2\Delta x_3.
\end{aligned}$$

Similarly for the other two directions:

$$([\mathbf{n}\cdot\mathbf{Q}]_{ABCD} + [\mathbf{n}\cdot\mathbf{Q}]_{EFGH})dA = \left(\mathbf{e}_2\cdot\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial x_2} + \ldots\right)\Delta x_1\Delta x_2\Delta x_3$$

$$([\mathbf{n}\cdot\mathbf{Q}]_{ABFE} + [\mathbf{n}\cdot\mathbf{Q}]_{DCGH})dA = \left(\mathbf{e}_3\cdot\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial x_3} + \ldots\right)\Delta x_1\Delta x_2\Delta x_3$$

Assembling the contributions from all six faces (or all three directions) to evaluate (2.32) produces

$$\begin{aligned}
\nabla\cdot\mathbf{Q} &= \lim_{V\to 0}\frac{1}{V}\iint_A \mathbf{n}\cdot\mathbf{Q}dA \\
&= \lim_{\substack{\Delta x_1\to 0\\ \Delta x_2\to 0\\ \Delta x_3\to 0}} \frac{1}{\Delta x_1\Delta x_2\Delta x_3}\left(\mathbf{e}_1\cdot\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial x_1} + \mathbf{e}_2\cdot\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial x_2} + \mathbf{e}_3\cdot\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial x_3} + \ldots\right)\Delta x_1\Delta x_2\Delta x_3,
\end{aligned}$$

and when the limit is taken, the expected Cartesian-coordinate form of the divergence emerges:

$$\nabla\cdot\mathbf{Q} = \mathbf{e}_1\cdot\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial x_1} + \mathbf{e}_2\cdot\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial x_2} + \mathbf{e}_3\cdot\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial x_3}.$$

2.13. STOKES' THEOREM

Stokes' theorem relates the integral over an open surface A to the line integral around the surface's bounding curve C. Here, unlike Gauss' theorem, the inside and outside of A are not well defined so an arbitrary choice must be made for the direction of the outward normal $\mathbf{n}$. Once this choice is made, the unit tangent vector to C, $\mathbf{t}$, points in the counter-clockwise direction when looking at the outside of A; it is defined as $\mathbf{t} = \mathbf{n_c} \times \mathbf{n}$, where

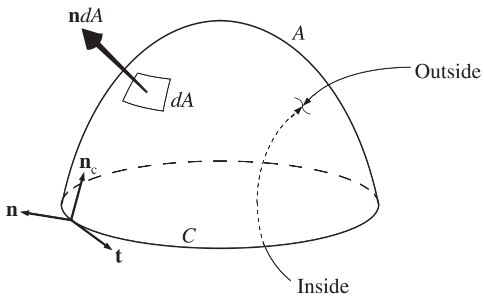

FIGURE 2.10 Illustration of Stokes' theorem for surface A bounded by the closed curve C. For the purposes of defining unit vectors, the *inside* and *outside* of A must be chosen, and one such choice is illustrated here. The unit vector $\mathbf{n}_c$ is perpendicular to C but is locally tangent to the surface A. The unit vector $\mathbf{n}$ is perpendicular to A and originates from the outside of A. The unit vector $\mathbf{t}$ is locally tangent to the curve C. The unit vectors $\mathbf{n}_c$, $\mathbf{n}$, and $\mathbf{t}$ define a right-handed triad of directions, $\mathbf{n}_c \times \mathbf{n} = \mathbf{t}$.

$\mathbf{n}_c$ is the unit normal to C that is locally tangent to A (Figure 2.10). For this geometry, Stokes' theorem states

$$\iint_A (\nabla \times \mathbf{u}) \cdot \mathbf{n}\, dA = \int_C \mathbf{u} \cdot \mathbf{t}\, ds, \tag{2.34}$$

where s is the arc length of the closed curve C. This theorem signifies that the surface integral of the curl of a vector field $\mathbf{u}$ is equal to the line integral of $\mathbf{u}$ along the bounding curve of the surface. In fluid mechanics, the right side of (2.34) is called the *circulation* of $\mathbf{u}$ about C. In addition, (2.34) can be used to define the curl of a vector through the limit of the circulation about an infinitesimal surface as

$$\mathbf{n} \cdot (\nabla \times \mathbf{u}) = \lim_{A \to 0} \frac{1}{A} \int_C \mathbf{u} \cdot \mathbf{t}\, ds. \tag{2.35}$$

The advantage of integral definitions of field derivatives is that such definitions do not depend on the coordinate system.

EXAMPLE 2.6

Obtain the recipe for the curl of a vector $\mathbf{u}(\mathbf{x})$ in Cartesian coordinates from the integral definition given by (2.35).

Solution

This is obtained by considering rectangular contours in three perpendicular planes intersecting at the point (x,y,z). First, consider the elemental rectangle in the $x = const.$ plane. The central point in this plane is (x,y,z) and the element's area is $\Delta y \Delta z$. It may be shown by careful integration of a Taylor expansion of the integrand that the integral along each line segment may be represented by the product of the integrand at the center of the segment and the length of the segment with attention paid to the direction of integration ds. Thus we obtain

$$(\nabla \times \mathbf{u})_x = \lim_{\Delta y \to 0} \lim_{\Delta z \to 0} \left\{ \begin{array}{l} \dfrac{1}{\Delta y \Delta z}\left[u_z\left(x, y+\dfrac{\Delta y}{2}, z\right) - u_z\left(x, y-\dfrac{\Delta y}{2}, z\right)\right]\Delta z \\ +\dfrac{1}{\Delta y \Delta z}\left[u_y\left(x, y, z-\dfrac{\Delta z}{2}\right) - u_y\left(x, y, z+\dfrac{\Delta z}{2}\right)\right]\Delta y \end{array} \right\}.$$

Taking the limits produces

$$(\nabla \times \mathbf{u})_x = \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}.$$

Similarly, integrating around the elemental rectangles in the other two planes leads to

$$(\nabla \times \mathbf{u})_y = \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}, \quad \text{and} \quad (\nabla \times \mathbf{u})_z = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}.$$

2.14. COMMA NOTATION

Sometimes it is convenient to use an even more compact notation for partial derivatives

$$A_{,\,i} \equiv \partial A/\partial x_i, \tag{2.36}$$

where A is a tensor of any order. Here, the comma after the A indicates a spatial derivative in the direction of the following index or indices. Thus, as last illustrations of the implied-sum-over-repeated-index notation and additional examples of the comma notation, consider the divergence and curl of a vector $\mathbf{u}$ written in vector, ordinary, indicial, and comma notations:

$$\nabla \cdot \mathbf{u} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} \equiv \frac{\partial u_i}{\partial x_i} \equiv u_{i,i} \quad \text{and} \quad (\nabla \times \mathbf{u})_i = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j} \equiv \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j} \equiv \varepsilon_{ijk}\, u_{k,j}.$$

The comma notation has two advantages compared to the others. It is compact and allows all subscripts to be written on one line so that both indices of second-order tensors like $u_{i,j}$ are easily identified. Its disadvantages arise from its compactness. An imperfectly attentive reader may overlook a comma in a subscript listing. Plus, the comma must be written clearly in order to avoid confusion with other indices. The comma notation is adopted in Section 5.6 where the extent of the expressions is otherwise too cumbersome.

EXERCISES

2.1. For three spatial dimensions, rewrite the following expressions in index notation and evaluate or simplify them using the values or parameters given, and the definitions of δ_{ij} and ε_{ijk} wherever possible. In parts b) through e), $\mathbf{x}$ is the position vector, with components x_i.

a) $\mathbf{b} \cdot \mathbf{c}$, where $\mathbf{b} = (1, 4, 17)$ and $\mathbf{c} = (-4, -3, 1)$.

b) $(\mathbf{u} \cdot \nabla)\mathbf{x}$, where $\mathbf{u}$ = a vector with components u_i.

c) $\nabla\phi$, where $\phi = \mathbf{h}\cdot\mathbf{x}$ and $\mathbf{h}$ is a constant vector with components h_i.

d) $\nabla\times\mathbf{u}$, where $\mathbf{u} = \boldsymbol{\Omega}\times\mathbf{x}$ and $\boldsymbol{\Omega}$ is a constant vector with components Ω_i.

e) $\mathrm{C}\cdot\mathbf{x}$, where

$$\mathbf{C} = \begin{Bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{Bmatrix}.$$

2.2. Starting from (2.1) and (2.3), prove (2.7).

2.3. Using Cartesian coordinates where the position vector is $\mathbf{x} = (x_1, x_2, x_3)$ and the fluid velocity is $\mathbf{u} = (u_1, u_2, u_3)$, write out the three components of the vector: $(\mathbf{u}\cdot\nabla)\mathbf{u} = u_i(\partial u_j/\partial x_i)$.

2.4. Convert $\nabla\times\nabla\rho$ to indicial notation and show that it is zero in Cartesian coordinates for any twice-differentiable scalar function ρ.

2.5. Using indicial notation, show that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$. [*Hint*: Call $\mathbf{d}\equiv\mathbf{b}\times\mathbf{c}$. Then $(\mathbf{a}\times\mathbf{d})_m = \varepsilon_{pqm}a_p d_q = \varepsilon_{pqm}a_p\varepsilon_{ijq}b_i c_j$. Using (2.19), show that $(\mathbf{a}\times\mathbf{d})_m = (\mathbf{a}\cdot\mathbf{c})b_m - (\mathbf{a}\cdot\mathbf{b})c_m$.]

2.6. Show that the condition for the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ to be coplanar is $\varepsilon_{ijk}a_i b_j c_k = 0$.

2.7. Prove the following relationships: $\delta_{ij}\delta_{ij} = 3$, $\varepsilon_{pqr}\varepsilon_{pqr} = 6$, and $\varepsilon_{pqi}\varepsilon_{pqj} = 2\delta_{ij}$.

2.8. Show that $\mathbf{C}\cdot\mathbf{C}^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}}\cdot\mathbf{C} = \boldsymbol{\delta}$, where $\mathbf{C}$ is the direction cosine matrix and $\boldsymbol{\delta}$ is the matrix of the Kronecker delta. Any matrix obeying such a relationship is called an *orthogonal matrix* because it represents transformation of one set of orthogonal axes into another.

2.9. Show that for a second-order tensor $\mathbf{A}$, the following quantities are invariant under the rotation of axes:

$$I_1 = A_{ii}$$
$$I_2 = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix}$$
$$I_3 = \det(A_{ij}).$$

[*Hint*: Use the result of Exercise 2.8 and the transformation rule (2.12) to show that $I'_1 = A'_{ii} = A_{ii} = I_1$. Then show that $A_{ij}A_{ji}$ and $A_{ij}A_{jk}A_{ki}$ are also invariants. In fact, *all* contracted scalars of the form $A_{ij}A_{jk}\cdots A_{mi}$ are invariants. Finally, verify that

$$I_2 = \frac{1}{2}\left[I_1^2 - A_{ij}A_{ji}\right]$$
$$I_3 = \frac{1}{3}\left[A_{ij}A_{jk}A_{ki} - I_1 A_{ij}A_{ji} + I_2 A_{ii}\right].$$

Because the right-hand sides are invariant, so are I_2 and I_3.]

2.10. If $\mathbf{u}$ and $\mathbf{v}$ are vectors, show that the products $u_i v_j$ obey the transformation rule (2.12), and therefore represent a second-order tensor.

2.11. Show that δ_{ij} is an isotropic tensor. That is, show that $\delta'_{ij} = \delta_{ij}$ under rotation of the coordinate system. [*Hint*: Use the transformation rule (2.12) and the results of Exercise 2.8.]

2.12. If **u** and **v** are arbitrary vectors resolved in three-dimensional Cartesian coordinates, show that $\mathbf{u} \cdot \mathbf{v} = 0$ when **u** and **v** are perpendicular.

2.13. If **u** and **v** are vectors with magnitudes u and v, use the finding of Exercise 2.12 to show that $\mathbf{u} \cdot \mathbf{v} = uv\cos\theta$ where θ is the angle between **u** and **v**.

2.14. Determine the components of the vector **w** in three-dimensional Cartesian coordinates when **w** is defined by: $\mathbf{u} \cdot \mathbf{w} = 0$, $\mathbf{v} \cdot \mathbf{w} = 0$, and $\mathbf{w} \cdot \mathbf{w} = u^2v^2\sin^2\theta$, where **u** and **v** are known vectors with components u_i and v_i and magnitudes u and v, respectively, and θ is the angle between **u** and **v**. Choose the sign(s) of the components of **w** so that $\mathbf{w} = \mathbf{e}_3$ when $\mathbf{u} = \mathbf{e}_1$ and $\mathbf{v} = \mathbf{e}_2$.

2.15. If a is a positive constant and **b** is a constant vector, determine the divergence and the curl of $\mathbf{u} = a\mathbf{x}/x^3$ and $\mathbf{u} = \mathbf{b} \times (\mathbf{x}/x^2)$ where $x = \sqrt{x_1^2 + x_2^2 + x_3^2} \equiv \sqrt{x_i x_i}$ is the length of **x**.

2.16. Obtain the recipe for the gradient of a scalar function in cylindrical polar coordinates from the integral definition (2.32).

2.17. Obtain the recipe for the divergence of a vector function in cylindrical polar coordinates from the integral definition (2.32).

2.18. Obtain the recipe for the divergence of a vector function in spherical polar coordinates from the integral definition (2.32).

2.19. Use the vector integral theorems to prove that $\nabla \cdot (\nabla \times \mathbf{u}) = 0$ for any twice-differentiable vector function **u** regardless of the coordinate system.

2.20. Use Stokes' theorem to prove that $\nabla \times (\nabla\phi) = 0$ for any single-valued twice-differentiable scalar ϕ regardless of the coordinate system.

Literature Cited

Sommerfeld, A. (1964). *Mechanics of Deformable Bodies*. New York: Academic Press. (Chapter 1 contains brief but useful coverage of Cartesian tensors.)

Supplemental Reading

Aris, R. (1962). *Vectors, Tensors, and the Basic Equations of Fluid Mechanics*. Englewood Cliffs, NJ: Prentice-Hall. (This book gives a clear and easy treatment of tensors in Cartesian and non-Cartesian coordinates, with applications to fluid mechanics.)

Prager, W. (1961). *Introduction to Mechanics of Continua*. New York: Dover Publications. (Chapters 1 and 2 contain brief but useful coverage of Cartesian tensors.)