Report 3

Sultan Aitzhan

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Contents

1 Time stepping and finite differences: the whole line

Recall the equation we obtained for the surface elevation on the whole line:

$$\eta_{tt} = \eta_{xx} + \mu^2 \left(\frac{1}{3}\eta_{xxxx} + \eta_{xt} \int_{-\infty}^{x} \eta_t \,\mathrm{d}x' + \eta_x^2 + \eta_t^2 + \eta\eta_{xx} + \partial_x^2 \left(\int_{-\infty}^{x} \eta_t \,\mathrm{d}x' \right)^2 \right). \tag{1}$$

To do time stepping, introduce

$$u = \eta_t. \tag{2}$$

Also, note that

$$\partial_x^2 \left(\int_{-\infty}^{x} \eta_t \,\mathrm{d}x' \right)^2 = 2(\eta_t^2 + \eta_{tx} \int_{-\infty}^{x} \eta_t \,\mathrm{d}x')$$

Then, combining (2) and (1), we obtain a two-dimensional system:

$$\partial_t \begin{bmatrix} u \\ \eta \end{bmatrix} = \begin{bmatrix} \mu^2 \left(2u_x \int_{-\infty}^x u \,\mathrm{d}x' + 2u^2\right) + \eta_{xx} + \mu^2 \left(\frac{1}{3}\eta_{xxxx} + \eta_x^2 + \eta\eta_{xx}\right) \\ u \end{bmatrix}. \tag{3}$$

Now, consider (1) on a finite interval $[a, b]$, and let partition the interval into $n+1$ points $\{x_k\}_{k=0}^n$, with $x_0 = a$ and $x_n = b$. This means that the integral terms becomes

$$\int_{-\infty}^x \eta_t \,\mathrm{d}x' = \left\{ \int_{-\infty}^a + \int_a^x \right\} \eta_t \,\mathrm{d}x' \approx \int_a^x \eta_t \,\mathrm{d}x',$$

while assuming that

$$\int_{-\infty}^a \eta_t \,\mathrm{d}x'$$

is small enough. Now, we employ forward Euler time stepping. First, observe that

$$\begin{aligned} u_t(x_k, t_j) &= \frac{u(x_k, t_{j+1}) - u(x_k, t_j)}{\Delta t} = f_1(\eta, u, t) && \implies u(x_k, t_{j+1}) = u(x_k, t_j) + \Delta t f_1(\eta, u, t) \\ \eta_t(x_k, t_j) &= \frac{\eta(x_k, t_{j+1}) - \eta(x_k, t_j)}{\Delta t} = f_2(\eta, u, t) && \implies \eta(x_k, t_{j+1}) = \eta(x_k, t_j) + \Delta t f_2(\eta, u, t), \end{aligned}$$

where

$$\begin{aligned} f_1(\eta, u, t) &= \eta_{xx} + \mu^2 \left(2u_x \int_{-\infty}^x u \,\mathrm{d}x' + 2u^2 + \frac{1}{3}\eta_{xxxx} + \eta_x^2 + \eta\eta_{xx}\right) \\ f_2(\eta, u, t) &= u(x_k, t_j). \end{aligned}$$

Observe that the highest order derivative is 4, so we need to use five-point stencils. In particular, we use five point midpoint stencil for $x_2, \ldots, x_{n-2}$, and five point, one-sided stencils at x_0, x_1, x_{n-1}, x_n. First, consider the system for $x_2, \ldots, x_{n-2}$:

$$\begin{aligned} f_1(\eta(x_k, t_j), u(x_k, t_j), t) = \eta(x_k, t_j)_{xx} &+ \frac{\mu^2}{3}\eta(x_k, t_j)_{xxxx} \\ &+ \mu^2 \left(2u(x_k, t_j)_x \int_{x_0}^{x_k} u \,\mathrm{d}x' + 2u(x_k, t_j)^2 + \eta(x_k, t_j)_x^2 + \eta(x_k, t_j)\eta(x_k, t_j)_{xx}\right), \end{aligned}$$

where we have separated the linear and nonlinear terms. Let $\Delta x = x_k - x_{k-1}$ and recall the finite difference formulas at x :

$$f'(x) = \frac{f(x - 2\Delta x) - 8f(x - \Delta x) + 8f(x + \Delta x) - f(x + 2\Delta x)}{12\Delta x} + \mathcal{O}((\Delta x)^4)$$

$$f''(x) = \frac{-f(x-2\Delta x) + 16f(x-\Delta x) - 30f + 16f(x+\Delta x) - f(x+2\Delta x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2)$$
$$f''''(x) = \frac{f(x-2\Delta x) - 4f(x-\Delta x) + 6f - 4f(x+\Delta x) + f(x+2\Delta x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2)$$

so that

$$(\eta_k)_x = \frac{\eta_{k-2} - 8\eta_{k-1} + 8\eta_{k+1} - \eta_{k+2}}{12\Delta x} + \mathcal{O}((\Delta x)^4)$$
$$(u_k)_x = \frac{u_{k-2} - 8u_{k-1} + 8u_{k+1} - u_{k+2}}{12\Delta x} + \mathcal{O}((\Delta x)^4)$$
$$(\eta_k)_{xx} = \frac{-\eta_{k-2} + 16\eta_{k-1} - 30\eta_k + 16\eta_{k+1} - \eta_{k+2}}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2)$$
$$(\eta_k)_{xxxx} = \frac{\eta_{k-2} - 4\eta_{k-1} + 6\eta_k - 4\eta_{k+1} + \eta_{k+2}}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2),$$

where it is assumed $t = t_j$. Also, by trapezoidal rule,

$$\int_{x_0}^{x_k} u \,\mathrm{d}x' = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} u \,\mathrm{d}x = \frac{\Delta x}{2} \sum_{i=0}^{n-1} u(x_i) + u(x_{i+1}).$$

At $x = x_0$, we have

$$f'(x) = \frac{-25f(x) + 48f(x+\Delta x) - 36f(x+2\Delta x) + 16f(x+3\Delta x) - 3f(x+4\Delta x)}{12\Delta x} + \mathcal{O}((\Delta x)^4)$$
$$f''(x) = \frac{35f(x) - 104f(x+\Delta x) + 114f(x+2\Delta x) - 56f(x+3\Delta x) + 11f(x+4\Delta x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2)$$
$$f''''(x) = \frac{f(x) - 4f(x+\Delta x) + 6f(x+2\Delta x) - 4f(x+3\Delta x) + f(x+4\Delta x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2),$$

so that

$$(\eta_0)_x = \frac{-25\eta_0 + 48\eta_1 - 36\eta_2 + 16\eta_3 - 3\eta_4}{12\Delta x}$$
$$(u_0)_x = \frac{-25u_0 + 48u_1 - 36u_2 + 16u_3 - 3u_4}{12\Delta x}$$
$$(\eta_0)_{xx} = \frac{35\eta_0 - 104\eta_1 + 114\eta_2 - 56\eta_3 + 11\eta_4}{12\Delta x^2}$$

$$(\eta_0)_{xxxx} = \frac{\eta_0 - 4\eta_1 + 6\eta_2 - 4\eta_3 + \eta_4}{\Delta x^4}$$

At $x = x_1$, we have

$$\begin{aligned}
f'(x) &= \frac{-3f(x-\Delta x) - 10f(x) + 18f(x+\Delta x) - 6f(x+2\Delta x) + f(x+3\Delta x)}{12\Delta x} + \mathcal{O}((\Delta x)^4) \\
f''(x) &= \frac{11f(x-\Delta x) - 20f(x) + 6f(x+\Delta x) + 4f(x+2\Delta x) - f(x+3\Delta x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2) \\
f''''(x) &= \frac{f(x-\Delta x) - 4f(x) + 6f(x+\Delta x) - 4f(x+2\Delta x) + f(x+3\Delta x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2),
\end{aligned}$$

so that

$$\begin{aligned}
(\eta_1)_x &= \frac{-3\eta_0 - 10\eta_1 + 18\eta_2 - 6\eta_3 + \eta_4}{12\Delta x} \\
(u_1)_x &= \frac{-3u_0 - 10u_1 + 18u_2 - 6u_3 + u_4}{12\Delta x} \\
(\eta_1)_{xx} &= \frac{11\eta_0 - 20\eta_1 + 6\eta_2 + 4\eta_3 - \eta_4}{12\Delta x^2} \\
(\eta_1)_{xxxx} &= \frac{\eta_0 - 4\eta_1 + 6\eta_2 - 4\eta_3 + \eta_4}{\Delta x^4}
\end{aligned}$$

At $x = x_{n-1}$, we have

$$\begin{aligned}
f'(x) &= \frac{-f(x-3\Delta x) + 6f(x-2\Delta x) - 18f(x-\Delta x) + 10f(x) + 3f(x+\Delta x)}{12\Delta x} + \mathcal{O}((\Delta x)^4) \\
f''(x) &= \frac{-f(x-3\Delta x) + 4f(x-2\Delta x) + 6(x-\Delta x) - 20f(x) + 11f(x+\Delta x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2) \\
f''''(x) &= \frac{f(x-3\Delta x) - 4f(x-2\Delta x) + 6f(x-\Delta x) - 4f(x) + f(x+\Delta x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2),
\end{aligned}$$

so that

$$\begin{aligned}
(\eta_{n-1})_x &= \frac{-\eta_{n-4} + 6\eta_{n-3} - 18\eta_{n-2} + 10\eta_{n-1} + 3\eta_n}{12\Delta x} \\
(u_{n-1})_x &= \frac{-u_{n-4} + 6u_{n-3} - 18u_{n-2} + 10u_{n-1} + 3u_n}{12\Delta x}
\end{aligned}$$

$$\begin{aligned}(\eta_{n-1})_{xx} &= \frac{-\eta_{n-4}+4\eta_{n-3}+6\eta_{n-2}-20\eta_{n-1}+11\eta_n}{12\Delta x^2}\\ (\eta_{n-1})_{xxxx} &= \frac{\eta_{n-4}-4\eta_{n-3}+6\eta_{n-2}-4\eta_{n-1}+\eta_n}{\Delta x^4}\end{aligned}$$

At $x = x_n$, we have

$$\begin{aligned}f'(x) &= \frac{f(x-4\Delta x)-4f(x-3\Delta x)+6f(x-2\Delta x)-4f(x-\Delta x)+f(x)}{12\Delta x}+\mathcal{O}((\Delta x)^4)\\ f''(x) &= \frac{11f(x-4\Delta x)-56f(x-3\Delta x)+114f(x-2\Delta x)-104f(x-\Delta x)+35f(x)}{12(\Delta x)^2}+\mathcal{O}((\Delta x)^2)\\ f''''(x) &= \frac{f(x-4\Delta x)-4f(x-3\Delta x)+6f(x-2\Delta x)-4f(x-\Delta x)+f(x)}{(\Delta x)^4}+\mathcal{O}((\Delta x)^2),\end{aligned}$$

so that

$$\begin{aligned}(\eta_n)_x &= \frac{3\eta_{n-4}-16\eta_{n-3}+36\eta_{n-2}-48\eta_{n-1}+25\eta_n}{12\Delta x}\\ (u_n)_x &= \frac{3u_{n-4}-16u_{n-3}+36u_{n-2}-48u_{n-1}+25u_n}{12\Delta x}\\ (\eta_n)_{xx} &= \frac{11\eta_{n-4}-56\eta_{n-3}+114\eta_{n-2}-104\eta_{n-1}+35\eta_n}{12\Delta x^2}\\ (\eta_n)_{xxxx} &= \frac{\eta_{n-4}-4\eta_{n-3}+6\eta_{n-2}-4\eta_{n-1}+\eta_n}{\Delta x^4}.\end{aligned}$$

All in all, we obtain:

$$\begin{aligned}f_1(\eta(x_0,t_j),u(x_0,t_j),t) &= \eta(x_0,t_j)_{xx}+\frac{\mu^2}{3}\eta(x_0,t_j)_{xxxx}+\mu^2\left(2u(x_0,t_j)^2+\eta(x_0,t_j)_x^2+\eta(x_0,t_j)\eta(x_0,t_j)_{xx}\right),\\ f_1(\eta(x_1,t_j),u(x_1,t_j),t) &= \eta(x_1,t_j)_{xx}+\frac{\mu^2}{3}\eta(x_1,t_j)_{xxxx}\\ &\quad+\mu^2\left(2u(x_1,t_j)_x\int_{x_0}^{x_1}u\,\mathrm{d}x'+2u(x_1,t_j)^2+\eta(x_1,t_j)_x^2+\eta(x_1,t_j)\eta(x_1,t_j)_{xx}\right),\\ f_1(\eta(x_2,t_j),u(x_2,t_j),t) &= \eta(x_2,t_j)_{xx}+\frac{\mu^2}{3}\eta(x_2,t_j)_{xxxx}\\ &\quad+\mu^2\left(2u(x_2,t_j)_x\int_{x_0}^{x_2}u\,\mathrm{d}x'+2u(x_2,t_j)^2+\eta(x_2,t_j)_x^2+\eta(x_2,t_j)\eta(x_2,t_j)_{xx}\right),\end{aligned}$$

$$\begin{aligned}
&\dots \\
f_1(\eta(x_k,t_j),u(x_k,t_j),t) &= \eta(x_k,t_j)_{xx} + \frac{\mu^2}{3}\eta(x_k,t_j)_{xxxx} \\
&+ \mu^2\left(2u(x_k,t_j)_x\int_{x_0}^{x_k} u\,\mathrm{d}x' + 2u(x_k,t_j)^2 + \eta(x_k,t_j)_x^2 + \eta(x_k,t_j)\eta(x_k,t_j)_{xx}\right), \\
&\dots \\
f_1(\eta(x_{n-1},t_j),u(x_{n-1},t_j),t) &= \eta(x_{n-1},t_j)_{xx} + \frac{\mu^2}{3}\eta(x_{n-1},t_j)_{xxxx} \\
&+ \mu^2\left(2u(x_{n-1},t_j)_x\int_{x_0}^{x_{n-1}} u\,\mathrm{d}x' + 2u(x_{n-1},t_j)^2 + \eta(x_{n-1},t_j)_x^2 + \eta(x_{n-1},t_j)\eta(x_{n-1},t_j)_{xx}\right), \\
f_1(\eta(x_n,t_j),u(x_n,t_j),t) &= \eta(x_n,t_j)_{xx} + \frac{\mu^2}{3}\eta(x_n,t_j)_{xxxx} \\
&+ \mu^2\left(2u(x_n,t_j)_x\int_{x_0}^{x_n} u\,\mathrm{d}x' + 2u(x_n,t_j)^2 + \eta(x_n,t_j)_x^2 + \eta(x_n,t_j)\eta(x_n,t_j)_{xx}\right),
\end{aligned}$$

Now, we obtain the discretised problem. First, consider the column of linear terms:

$$\begin{aligned}
&(\eta_0)_{xx} + \frac{\mu^2}{3}(\eta_0)_{xxxx} \\
&\quad = \frac{35\eta_0 - 104\eta_1 + 114\eta_2 - 56\eta_3 + 11\eta_4}{12\Delta x^2} + \frac{\mu^2}{3}\frac{\eta_0 - 4\eta_1 + 6\eta_2 - 4\eta_3 + \eta_4}{\Delta x^4} \\
&\quad = \frac{35\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_0 - \frac{104\Delta x^2 + 16\mu^2}{12\Delta x^4}\eta_1 + \frac{114\Delta x^2 + 24\mu^2}{12\Delta x^4}\eta_2 - \frac{56\Delta x^2 + 16\mu^2}{12\Delta x^4}\eta_3 + \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_4 \\
&(\eta_1)_{xx} + \frac{\mu^2}{3}(\eta_1)_{xxxx} \\
&\quad = \frac{11\eta_0 - 20\eta_1 + 6\eta_2 + 4\eta_3 - \eta_4}{12\Delta x^2} + \frac{\mu^2}{3}\frac{\eta_0 - 4\eta_1 + 6\eta_2 - 4\eta_3 + \eta_4}{\Delta x^4} \\
&\quad = \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_0 - \frac{20\Delta x^2 + 16\mu^2}{12\Delta x^4}\eta_1 + \frac{6\Delta x^2 + 24\mu^2}{12\Delta x^4}\eta_2 + \frac{4\Delta x^2 - 16\mu^2}{12\Delta x^4}\eta_3 + \frac{-\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_4 \\
&\quad \dots \\
&(\eta_k)_{xx} + \frac{\mu^2}{3}(\eta_k)_{xxxx}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-\eta_{k-2} + 16\eta_{k-1} - 30\eta_k + 16\eta_{k+1} - \eta_{k+2}}{12(\Delta x)^2} + \frac{\mu^2}{3}\frac{\eta_{k-2} - 4\eta_{k-1} + 6\eta_k - 4\eta_{k+1} + \eta_{k+2}}{(\Delta x)^4} \\
&= \frac{-\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_{k-2} + \frac{16(\Delta x^2 - \mu^2)}{12\Delta x^4}\eta_{k-1} + \frac{-30\Delta x^2 + 24\mu^2}{12\Delta x^4}\eta_k - \frac{16(\Delta x^2 - \mu^2)}{12\Delta x^4}\eta_{k+1} + \frac{-\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_{k+2} \\
&\cdots \\
(\eta_{n-1})_{xx} &+ \frac{\mu^2}{3}(\eta_{n-1})_{xxxx} \\
&= \frac{-\eta_{n-4} + 4\eta_{n-3} + 6\eta_{n-2} - 20\eta_{n-1} + 11\eta_n}{12\Delta x^2} + \frac{\mu^2}{3}\frac{\eta_{n-4} - 4\eta_{n-3} + 6\eta_{n-2} - 4\eta_{n-1} + \eta_n}{\Delta x^4} \\
&= \frac{-\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_{n-4} + \frac{4\Delta x^2 - 16\mu^2}{12\Delta x^4}\eta_{n-3} + \frac{6\Delta x^2 + 24\mu^2}{12\Delta x^4}\eta_{n-2} - \frac{20\Delta x^2 + 16\mu^2}{12\Delta x^4}\eta_{n-1} + \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_n \\
(\eta_n)_{xx} &+ \frac{\mu^2}{3}(\eta_n)_{xxxx} \\
&= \frac{11\eta_{n-4} - 56\eta_{n-3} + 114\eta_{n-2} - 104\eta_{n-1} + 35\eta_n}{12\Delta x^2} + \frac{\mu^2}{3}\frac{\eta_{n-4} - 4\eta_{n-3} + 6\eta_{n-2} - 4\eta_{n-1} + \eta_n}{\Delta x^4} \\
&= \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_{n-4} - \frac{56\Delta x^2 + 16\mu^2)}{12\Delta x^4}\eta_{n-3} + \frac{114\Delta x^2 + 24\mu^2}{12\Delta x^4}\eta_{n-2} - \frac{104\Delta x^2 + 16\mu^2}{12\Delta x^4}\eta_{n-1} + \frac{35\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_n.
\end{aligned}
$$

Then, the matrix becomes

$$
\begin{bmatrix}
(\eta_0)_{xx} + \frac{\mu^2}{3}(\eta_0)_{xxxx} \\
(\eta_1)_{xx} + \frac{\mu^2}{3}(\eta_1)_{xxxx} \\
(\eta_2)_{xx} + \frac{\mu^2}{3}(\eta_2)_{xxxx} \\
\vdots \\
(\eta_k)_{xx} + \frac{\mu^2}{3}(\eta_k)_{xxxx} \\
\vdots \\
(\eta_{n-2})_{xx} + \frac{\mu^2}{3}(\eta_{n-2})_{xxxx} \\
(\eta_{n-1})_{xx} + \frac{\mu^2}{3}(\eta_{n-1})_{xxxx} \\
(\eta_n)_{xx} + \frac{\mu^2}{3}(\eta_n)_{xxxx}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\frac{35\Delta x^2+4\mu^2}{12\Delta x^4} & -\frac{104\Delta x^2+16\mu^2}{12\Delta x^4} & \frac{114\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{56\Delta x^2+16\mu^2}{12\Delta x^4} & \frac{11\Delta x^2+4\mu^2}{12\Delta x^4} & 0 & 0 & 0 & \dots & \\
\frac{11\Delta x^2+4\mu^2}{12\Delta x^4} & -\frac{20\Delta x^2+16\mu^2}{12\Delta x^4} & \frac{6\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{4\Delta x^2-16\mu^2}{12\Delta x^4} & \frac{-\Delta x^2+4\mu^2}{12\Delta x^4} & 0 & 0 & 0 & \dots & \\
\frac{-\Delta x^2+4\mu^2}{12\Delta x^4} & \frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & \frac{-30\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & \frac{-\Delta x^2+4\mu^2}{12\Delta x^4} & 0 & 0 & 0 & \dots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \\
0 & \dots & 0 & \frac{-\Delta x^2+4\mu^2}{12\Delta x^4} & \frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & \frac{-30\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & \frac{-\Delta x^2+4\mu^2}{12\Delta x^4} & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \\
\dots & 0 & 0 & 0 & \frac{-\Delta x^2+4\mu^2}{12\Delta x^4} & \frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & \frac{-30\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & \frac{-\Delta x^2+4\mu^2}{12\Delta x^4} & \\
\dots & 0 & 0 & 0 & -\frac{\Delta x^2+4\mu^2}{12\Delta x^4} & \frac{4\Delta x^2-16\mu^2)}{12\Delta x^4} & \frac{6\Delta x^2+24\mu^2}{12\Delta x^4} & \frac{-20\Delta x^2-16\mu^2}{12\Delta x^4} & \frac{11\Delta x^2+4\mu^2}{12\Delta x^4} & \\
\dots & 0 & 0 & 0 & \frac{11\Delta x^2+4\mu^2}{12\Delta x^4} & -\frac{56\Delta x^2+16\mu^2)}{12\Delta x^4} & \frac{114\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{104\Delta x^2+16\mu^2}{12\Delta x^4} & \frac{35\Delta x^2+4\mu^2}{12\Delta x^4} &
\end{bmatrix}
\begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_k \\ \vdots \\ \eta_{n-2} \\ \eta_{n-1} \\ \eta_n \end{bmatrix}
$$

For simplicity, let $\mathcal{A}$ represent the above matrix. Now, recall the system:

$$
\begin{aligned}
u(x_k, t_{j+1}) &= u(x_k, t_j) + \Delta t f_1(\eta(x_k, t_j), u(x_k, t_j), t), \\
\eta(x_k, t_{j+1}) &= \eta(x_k, t_j) + \Delta t u(x_k, t_j),
\end{aligned}
$$

where

$$
\begin{aligned}
f_1(\eta(x_k, t_j), u(x_k, t_j), t) &= \eta(x_k, t_j)_{xx} + \frac{\mu^2}{3}\eta(x_k, t_j)_{xxxx} \\
&\quad + \mu^2 \left(2u(x_k, t_j)_x \int_{x_0}^{x_k} u \, \mathrm{d}x' + 2u(x_k, t_j)^2 + \eta(x_k, t_j)_x^2 + \eta(x_k, t_j)\eta(x_k, t_j)_{xx} \right).
\end{aligned}
$$

For convenience, let

$$
B_k = \left(2u(x_k, t_j)_x \int_{x_0}^{x_k} u \, \mathrm{d}x' + 2u(x_k, t_j)^2 + \eta(x_k, t_j)_x^2 + \eta(x_k, t_j)\eta(x_k, t_j)_{xx} \right),
$$

Let $\mathcal{B}$ represent the column vector of B_k's. Then, we can write the system

$$
\begin{aligned}
u(x_0, t_{j+1}) &= u(x_0, t_j) + \Delta t f_1(\eta(x_0, t_j), u(x_0, t_j), t) \\
u(x_1, t_{j+1}) &= u(x_1, t_j) + \Delta t f_1(\eta(x_1, t_j), u(x_1, t_j), t) \\
u(x_2, t_{j+1}) &= u(x_2, t_j) + \Delta t f_1(\eta(x_2, t_j), u(x_2, t_j), t) \\
&\vdots \\
u(x_{n-2}, t_{j+1}) &= u(x_{n-2}, t_j) + \Delta t f_1(\eta(x_{n-2}, t_j), u(x_{n-2}, t_j), t)
\end{aligned}
$$

$$
\begin{aligned}
u(x_{n-1},t_{j+1}) &= u(x_{n-1},t_j) + \Delta t f_1(\eta(x_{n-1},t_j), u(x_{n-1},t_j), t) \\
u(x_n,t_{j+1}) &= u(x_n,t_j) + \Delta t f_1(\eta(x_n,t_j), u(x_n,t_j), t)
\end{aligned}
$$

as follows:

$$
\begin{aligned}
\begin{bmatrix} u(x_0,t_{j+1}) \\ u(x_1,t_{j+1}) \\ u(x_2,t_{j+1}) \\ \vdots \\ u(x_{n-2},t_{j+1}) \\ u(x_{n-1},t_{j+1}) \\ u(x_n,t_{j+1}) \end{bmatrix}
&= \begin{bmatrix} u(x_0,t_j) \\ u(x_1,t_j) \\ u(x_2,t_j) \\ \vdots \\ u(x_{n-2},t_j) \\ u(x_{n-1},t_j) \\ u(x_n,t_j) \end{bmatrix}
+ \Delta t \begin{bmatrix} f_1(\eta(x_0,t_j), u(x_0,t_j), t) \\ f_1(\eta(x_1,t_j), u(x_1,t_j), t) \\ f_1(\eta(x_2,t_j), u(x_2,t_j), t) \\ \vdots \\ f_1(\eta(x_{n-2},t_j), u(x_{n-2},t_j), t) \\ f_1(\eta(x_{n-1},t_j), u(x_{n-1},t_j), t) \\ f_1(\eta(x_n,t_j), u(x_n,t_j), t) \end{bmatrix} \\
&= \begin{bmatrix} u(x_0,t_j) \\ u(x_1,t_j) \\ u(x_2,t_j) \\ \vdots \\ u(x_{n-2},t_j) \\ u(x_{n-1},t_j) \\ u(x_n,t_j) \end{bmatrix}
+ \Delta t \mathcal{A} \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-2} \\ \eta_{n-1} \\ \eta_n \end{bmatrix} + \Delta t \mathcal{B}
\end{aligned}
$$

Now, let us see how one would perform time-stepping. As such, impose initial conditions

$$
\eta(x,t_0) = f(x), \qquad u(x,t_0) = \eta_t(x,t_0) = g(x).
$$

Let $j = 0$, and for simplicity, pick $k \in [0,n]$. The system is

$$
\begin{aligned}
u(x_k,t_1) &= u(x_k,t_0) + \Delta t f_1(\eta(x_k,t_0), u(x_k,t_0)), \\
\eta(x_k,t_1) &= \eta(x_k,t_0) + \Delta t u(x_k,t_0),
\end{aligned}
$$

where

$$
\begin{aligned}
f_1(\eta(x_k,t_0), u(x_k,t_0)) = \eta(x_k,t_0)_{xx} &+ \frac{\mu^2}{3}\eta(x_k,t_0)_{xxxx} \\
&+ \mu^2 \left(2u(x_k,t_0)_x \int_{x_0}^{x_k} u \, \mathrm{d}x' + 2u(x_k,t_0)^2 + \eta(x_k,t_0)_x^2 + \eta(x_k,t_0)\eta(x_k,t_0)_{xx} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-\eta(x_{k-2},t_0)+16\eta(x_{k-1},t_0)-30\eta(x_k,t_0)+16\eta(x_{k+1},t_0)-\eta(x_{k+2},t_0)}{12(\Delta x)^2} \\
&+ \frac{\mu^2}{3}\frac{\eta(x_{k-2},t_0)-4\eta(x_{k-1},t_0)+6\eta(x_k,t_0)-4\eta(x_{k+1},t_0)+\eta(x_{k+2},t_0)}{(\Delta x)^4} \\
&+ \mu^2\left(2u(x_k,t_0)_x\int_{x_0}^{x_k}u(x',t_0)\,\mathrm{d}x' + 2u(x_k,t_0)^2+\eta(x_k,t_0)_x^2+\eta(x_k,t_0)\eta(x_k,t_0)_{xx}\right)
\end{aligned}
$$

Note that all the terms on the last line can be computed via finite differences and both initial conditions. With this, we obtain the values of u, η at point x_k and time t_1. Performing this calculation for all k, we move on to compute u, η at time t_2, and so on.

2 The half line problem

In this section, we deal with this term

$$\partial_x(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\}\}$$

More generally, we have the following result:

Theorem 1. *For nice enough f defined on $x \geqslant 0$, we have*

$$(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} = \frac{1}{\pi}\int_0^\infty f(y)\left(\frac{1}{x-y}+\frac{1}{x+y}\right)\mathrm{d}y.$$

Before we begin, recall the Riemann-Lebesgue lemma:

Lemma 2 (Theorem 11.6, [1])**.** *Assume that $f \in L(I)$. Then, for each real β, we have*

$$\lim_{\alpha\to\infty}\int_I f(t)\sin(\alpha t+\beta)\,\mathrm{d}t = 0.$$

Proof of Theorem 1. Consider

$$(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\}.$$

For generality, we consider $(\mathcal{F}_s^k)^{-1}\{G(k)\}$, where G is a function of k defined on $k \geqslant 0$. Expanding the integral, we obtain:

$$(\mathcal{F}_s^k)^{-1}\{G(k)\} = \int_0^\infty \sin(kx)G(k)\,\mathrm{d}k$$

$$
\begin{aligned}
&= \frac{1}{2i}\int_0^\infty (e^{ikx} - e^{-ikx})G(k)\,\mathrm{d}k \\
&= \frac{1}{2i}\left[\int_0^\infty e^{ikx}G(k)\,\mathrm{d}k - \int_0^\infty e^{-ikx}G(k)\,\mathrm{d}k\right] \\
&= \frac{1}{2i}\left[\int_0^\infty e^{ikx}G(k)\,\mathrm{d}k + \int_0^{-\infty} e^{ikx}G(-k)\,\mathrm{d}k\right] && \text{(apply } k \mapsto -k \text{ in the 2nd term)} \\
&= \frac{1}{2i}\left[\int_0^\infty e^{ikx}G(k)\,\mathrm{d}k + \int_{-\infty}^{0} e^{ikx}(-G(-k))\,\mathrm{d}k\right],
\end{aligned}
$$

where $-G(-k)$ is an odd extension to $k < 0$. Now, observe the following:

$$
\begin{aligned}
\frac{2}{\pi}\int_0^\infty \cos(kx)f(x)\,\mathrm{d}x &= \frac{1}{\pi}\int_0^\infty (e^{ikx} + e^{-ikx})f(x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\int_0^\infty e^{ikx}f(x)\,\mathrm{d}x + \int_0^\infty e^{-ikx}f(x)\,\mathrm{d}x\right] \\
&= \frac{1}{\pi}\left[-\int_0^{-\infty} e^{-ikx}f(-x)\,\mathrm{d}x + \int_0^\infty e^{-ikx}f(x)\,\mathrm{d}x\right] && \text{(apply } x \mapsto -x \text{ in the 1st term)} \\
&= \frac{1}{\pi}\left[\int_{-\infty}^0 e^{-ikx}f(-x)\,\mathrm{d}x + \int_0^\infty e^{-ikx}f(x)\,\mathrm{d}x\right] \\
&= \frac{1}{\pi}\int_{-\infty}^\infty e^{-ikx}F(x)\,\mathrm{d}x,
\end{aligned}
$$

where we used an even extension to $x < 0$ and defined

$$
F(x) = \begin{cases} f(x) & x > 0 \\ f(-x) & x < 0 \end{cases}.
$$

For $k > 0$, we have

$$
G(k) = \mathcal{F}_c^k\{f\} = \frac{2}{\pi}\int_0^\infty \cos(kx)f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_{-\infty}^\infty e^{-ikx}F(x)\,\mathrm{d}x. \tag{4}
$$

For $k < 0$, we have

$$
-G(-k) = -\mathcal{F}_c^{-k}\{f\} = -\frac{2}{\pi}\int_0^\infty \cos(-kx)f(x)\,\mathrm{d}x = -\frac{2}{\pi}\int_0^\infty \cos(kx)f(x)\,\mathrm{d}x = -\frac{1}{\pi}\int_{-\infty}^\infty e^{-ikx}F(x)\,\mathrm{d}x, \tag{5}
$$

since cosine is an even function. Thus, using (4) and (5), we obtain

$$\begin{aligned}
(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} &= \frac{1}{2i}\left[\int_0^\infty e^{ikx}\mathcal{F}_c^k\{f\}\,\mathrm{d}k + \int_{-\infty}^0 e^{ikx}(-\mathcal{F}^(- k)_c\{f\})\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty e^{ikx}\int_{-\infty}^\infty e^{-iky}F(y)\,\mathrm{d}y\,\mathrm{d}k - \int_{-\infty}^0 e^{ikx}\int_{-\infty}^\infty e^{-iky}F(y)\,\mathrm{d}y\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty e^{ikx}\int_{-\infty}^\infty e^{ik(x-y)}F(y)\,\mathrm{d}y\,\mathrm{d}k - \int_{-\infty}^0 \int_{-\infty}^\infty e^{ik(x-y)}F(y)\,\mathrm{d}y\,\mathrm{d}k\right].
\end{aligned} \tag{6}$$

Let

$$\begin{aligned}
V(k) &= \int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y = -V(-k), \\
U(k) &= \int_{-\infty}^\infty \cos(k(x-y))F(y)\,\mathrm{d}y = U(-k),
\end{aligned}$$

so that V is odd and U is even. This allows to rewrite (6) as:

$$\begin{aligned}
(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} &= \frac{1}{2\pi i}\left[\int_0^\infty e^{ikx}\int_{-\infty}^\infty e^{ik(x-y)}F(y)\,\mathrm{d}y\,\mathrm{d}k - \int_{-\infty}^0 \int_{-\infty}^\infty e^{ik(x-y)}F(y)\,\mathrm{d}y\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty U(k)+iV(k)\,\mathrm{d}k - \int_{-\infty}^0 U(k)+iV(k)\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty U(k)+iV(k)\,\mathrm{d}k + \int_{\infty}^0 U(-k)+iV(-k)\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty U(k)+iV(k)\,\mathrm{d}k + \int_0^\infty -U(-k)+i(-V(-k))\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty U(k)+iV(k)\,\mathrm{d}k + \int_0^\infty -U(k)+iV(k)\,\mathrm{d}k\right] \\
&= \frac{1}{\pi}\int_0^\infty V(k)\,\mathrm{d}k,
\end{aligned}$$

where on the third last line, we flipped the bounds of integration and brought the minus sign inside the integral, and on the second last line, we used that U is even and V is odd. Thus, we obtain

$$(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} = \frac{1}{\pi}\int_0^\infty V(k)\,\mathrm{d}k = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y\,\mathrm{d}k.$$

Note that the integral in k is an improper integral, so

$$\int_0^\infty \int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y\,\mathrm{d}k = \lim_{\alpha\to\infty}\int_0^\alpha \int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y\,\mathrm{d}k.$$

Now, interchanging the order of integration, we have

$$\begin{aligned}
\int_0^\alpha \int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y\,\mathrm{d}k &= \int_{-\infty}^\infty F(y)\int_0^\alpha \sin(k(x-y))\,\mathrm{d}k\,\mathrm{d}y \\
&= \int_{-\infty}^\infty F(y)\left[-\frac{\cos(k(x-y))}{x-y}\,\Big|_0^\alpha\right]\mathrm{d}y \\
&= \int_{-\infty}^\infty F(y)\left[\frac{1}{x-y}-\frac{\cos(\alpha(x-y))}{x-y}\right]\mathrm{d}y \\
&= \int_{-\infty}^\infty F(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y.
\end{aligned}$$

The interchange is justified, since sine is bounded and differentiable on $\mathbb{R}$. Finally, we use the Riemann-Lebesgue lemma to deal with the last term:

$$\begin{aligned}
\int_{-\infty}^\infty F(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y &= \int_0^\infty f(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y + \int_{-\infty}^0 f(-y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y \\
&= \int_0^\infty f(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y - \int_\infty^0 f(y)\frac{1-\cos(\alpha(x+y))}{x+y}\,\mathrm{d}y \\
&= \int_0^\infty f(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y + \int_0^\infty f(y)\frac{1-\cos(\alpha(x+y))}{x+y}\,\mathrm{d}y \\
&= \int_0^\infty f(y)\frac{1}{x-y}\,\mathrm{d}y - \int_0^\infty f(y)\frac{\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y \\
&\quad + \int_0^\infty f(y)\frac{1}{x+y}\,\mathrm{d}y - \int_0^\infty f(y)\frac{\cos(\alpha(x+y))}{x+y}\,\mathrm{d}y.
\end{aligned}$$

As $\alpha\to\infty$, the terms

$$\int_0^\infty f(y)\frac{\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y, \qquad \int_0^\infty f(y)\frac{\cos(\alpha(x+y))}{x+y}\,\mathrm{d}y \to 0$$

by the Riemann-Lebesgue lemma with $\beta=\pi/2$, so that

$$\int_{-\infty}^\infty F(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y = \int_0^\infty f(y)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y.$$

Thus,

$$(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y\,\mathrm{d}k = \frac{1}{\pi}\int_0^\infty f(y)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y.$$

The proof is complete. □

Remark 3. Note that the integral

$$\frac{1}{\pi}\int_0^\infty f(y)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y =$$

looks like a convolution-type transform. In fact, the term with $1/(x-y)$ is pretty much the Hilbert transform, but on a half-line.

The theorem yields

$$\partial_x(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\}\} = \partial_x\left(\frac{1}{\pi}\int_0^\infty \partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right).$$

For generality, let $f(y) = \partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)$. Note the following:

$$\begin{aligned}\partial_x\left(\frac{1}{\pi}\int_0^\infty f(y)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\right) &= \frac{1}{\pi}\int_0^\infty f(y)\partial_x\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\,\mathrm{d}y\\ &= -\frac{1}{\pi}\int_0^\infty f(y)\left[\frac{1}{(x-y)^2}+\frac{1}{(x+y)^2}\right]\,\mathrm{d}y,\end{aligned}$$

so that

$$\partial_x(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\}\} = -\frac{1}{\pi}\int_0^\infty \partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)\left[\frac{1}{(x-y)^2}+\frac{1}{(x+y)^2}\right]\,\mathrm{d}y. \tag{7}$$

As can be seen, the integral (7) is singular whenever $x=y$ or $x=-y$, over y. To deal with this issue, one may need to use a Residue theorem. To conclude, the surface expression on a half-line becomes:

$$\begin{aligned}\eta_{tt}-\eta_{xx} &= \mu^2\left(\frac{1}{3}\eta_{xxxx} + \partial_x(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\}\} + \frac{1}{2}\partial_x^2\left(\int_0^x \eta_t\,\mathrm{d}x'\right)^2\right)\\ &= \mu^2\left(\frac{1}{3}\eta_{xxxx} - \frac{1}{\pi}\int_0^\infty \partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)\left[\frac{1}{(x-y)^2}+\frac{1}{(x+y)^2}\right]\,\mathrm{d}y + \frac{1}{2}\partial_x^2\left(\int_0^x \eta_t\,\mathrm{d}x'\right)^2\right).\end{aligned}$$

3 Time stepping and finite differences: the half line

On the half line, the equation for the surface elevation is given by:

$$\eta_{tt} = \eta_{xx} + \mu^2 \left(\frac{1}{3}\eta_{xxxx} - \frac{1}{\pi} \int_0^\infty \partial_t \left(\eta \int_0^y \eta_t \, \mathrm{d}y' \right) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] \mathrm{d}y + \frac{1}{2} \partial_x^2 \left(\int_0^x \eta_t \, \mathrm{d}x' \right)^2 \right) \tag{8}$$

To do time stepping, introduce

$$u = \eta_t. \tag{9}$$

Also, note that

$$\frac{1}{2}\partial_x^2 \left(\int_0^x \eta_t \, \mathrm{d}x' \right)^2 = \eta_t^2 + \eta_{tx} \int_0^x \eta_t \, \mathrm{d}x' = u^2 + u_x \int_0^x u \, \mathrm{d}x',$$

and

$$\partial_t \left(\eta \int_0^y \eta_t \, \mathrm{d}y' \right) = \eta_t \int_0^x \eta_t \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)) = u \int_0^x u \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)).$$

Then, combining (9) and (8), we obtain a two-dimensional system:

$$\partial_t \begin{bmatrix} u \\ \eta \end{bmatrix} = \begin{bmatrix} \eta_{xx} + \mu^2 \left(\frac{1}{3}\eta_{xxxx} + u^2 + u_x \int_0^x u \, \mathrm{d}x' - \frac{1}{\pi} \int_0^\infty u \int_0^x u \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] \mathrm{d}y \right) \\ u \end{bmatrix}. \tag{10}$$

Now, consider (8) on a finite interval $[a, b]$, and let partition the interval into $n+1$ points $\{x_k\}_{k=0}^n$, with $x_0 = a$ and $x_n = b$. Note that we need to pick the partition such that

$$\int_0^\infty u \int_0^x u \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] \mathrm{d}y \approx \int_{x_0}^{x_n} u \int_0^x u \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] \mathrm{d}y.$$

We proceed to forward Euler time stepping. First, observe that

$$\begin{aligned} u_t(x_k, t_j) &= \frac{u(x_k, t_{j+1}) - u(x_k, t_j)}{\Delta t} = f_1(\eta, u, t) & &\implies u(x_k, t_{j+1}) = u(x_k, t_j) + \Delta t f_1(\eta, u, t) \\ \eta_t(x_k, t_j) &= \frac{\eta(x_k, t_{j+1}) - \eta(x_k, t_j)}{\Delta t} = f_2(\eta, u, t) & &\implies \eta(x_k, t_{j+1}) = \eta(x_k, t_j) + \Delta t f_2(\eta, u, t), \end{aligned}$$

where

$$f_1(\eta, u, t) = \eta_{xx} + \mu^2 \left(\frac{1}{3}\eta_{xxxx} + u^2 + u_x \int_0^x u \, \mathrm{d}x' - \frac{1}{\pi} \int_0^\infty u \int_0^x u \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] \mathrm{d}y \right)$$

$$f_2(\eta, u, t) = u(x_k, t_j).$$

Observe that the highest order derivative is 4, so we need to use five-point stencils. In particular, we use five point midpoint stencil for $x_2, \ldots, x_{n-2}$, and five point, one-sided stencils at x_0, x_1, x_{n-1}, x_n. First, consider the system for $x_2, \ldots, x_{n-2}$:

$$\begin{aligned} f_1(\eta(x_k, t_j), u(x_k, t_j), t) &= \eta(x_k, t_j)_{xx} + \frac{\mu^2}{3}\eta(x_k, t_j)_{xxxx} \\ &+ \mu^2 \left(u^2(x_k) + u_x(x_k) \int_0^{x_k} u \, \mathrm{d}x' - \frac{1}{\pi} \int_0^{x_n} u \int_0^y u \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] \mathrm{d}y \right), \end{aligned}$$

where we have separated the linear and nonlinear terms. The only difference between this system and the whole-line system is the non-linear term; in other words, we can reuse our prior work on the linear term, and only deal with the non-linear term. Let

$$C_k = u^2(x_k) + u_x(x_k) \int_0^{x_k} u \, \mathrm{d}x' - \frac{1}{\pi} \int_0^{x_n} \left(u \int_0^y u \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)) \right) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] \mathrm{d}y,$$

and let $\mathcal{C}$ be the column vector of C_k; thus, $\mathcal{C}$ represents the non-linear part of the system. To discretise C_k, note that

$$\int_0^{x_k} u \, \mathrm{d}x' = \frac{\Delta x}{2} \sum_{i=0}^{k-1} u(x_{i+1}) + u(x_i)$$

$$\begin{aligned} \int_0^{x_n} u \int_0^y u \, \mathrm{d}x' \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] \mathrm{d}y &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \left(u \int_0^y u \, \mathrm{d}x' \right) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] \mathrm{d}y \\ &= \frac{\Delta x}{2} \sum_{i=0}^{n-1} u(x_{i+1}) \int_0^{x_{i+1}} u \, \mathrm{d}x' \left[\frac{1}{(x_k - x_{i+1})^2} + \frac{1}{(x_k + x_{i+1})^2} \right] + u(x_i) \int_0^{x_i} u \, \mathrm{d}x' \left[\frac{1}{(x_k - x_i)^2} + \frac{1}{(x_k + x_i)^2} \right] \\ &= \frac{\Delta x}{2} \sum_{i=0}^{n-1} \frac{\Delta x}{2} u(x_{i+1}) \left(\sum_{j=0}^{i} u(x_j) + u(x_{j+1}) \right) \left[\frac{1}{(x_k - x_{i+1})^2} + \frac{1}{(x_k + x_{i+1})^2} \right] \\ &+ \frac{\Delta x}{2} u(x_i) \left(\sum_{j=0}^{i-1} u(x_j) + u(x_{j+1}) \right) \left[\frac{1}{(x_k - x_i)^2} + \frac{1}{(x_k + x_i)^2} \right] \end{aligned}$$

$$\begin{aligned} \int_0^{x_n} \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] \mathrm{d}y &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] \mathrm{d}y \\ &= \frac{\Delta x}{2} \sum_{i=0}^{n-1} \eta(x_{i+1})(\eta_x(x_{i+1}) - \eta_x(0)) \left[\frac{1}{(x_k - x_{i+1})^2} + \frac{1}{(x_k + x_{i+1})^2} \right] + \eta(x_i)(\eta_x(x_i) - \eta_x(0)) \left[\frac{1}{(x_k - x_i)^2} + \frac{1}{(x_k + x_i)^2} \right]. \end{aligned}$$

Therefore, we obtain that

$$
\begin{aligned}
C_k &= u^2(x_k) + u_x(x_k)\int_0^{x_k} u\,\mathrm{d}x' - \frac{1}{\pi}\int_0^{x_n}\left(u\int_0^{y} u\,\mathrm{d}x' + \eta(\eta_x - \eta_x(0))\right)\left[\frac{1}{(x_k-y)^2} + \frac{1}{(x_k+y)^2}\right]\mathrm{d}y \\
&= u^2(x_k) + \frac{\Delta x}{2}u_x(x_k)\sum_{i=0}^{k-1} u(x_{i+1}) + u(x_i) \\
&\quad - \frac{1}{\pi}\frac{\Delta x}{2}\sum_{i=0}^{n-1}\left(\frac{\Delta x}{2}u(x_{i+1})\left(\sum_{j=0}^{i} u(x_j) + u(x_{j+1})\right) + \eta(x_{i+1})(\eta_x(x_{i+1}) - \eta_x(0))\right)\left[\frac{1}{(x_k-x_{i+1})^2} + \frac{1}{(x_k+x_{i+1})^2}\right] \\
&\quad + \left(\frac{\Delta x}{2}u(x_i)\left(\sum_{j=0}^{i-1} u(x_j) + u(x_{j+1})\right) + \eta(x_i)(\eta_x(x_i) - \eta_x(0))\right)\left[\frac{1}{(x_k-x_i)^2} + \frac{1}{(x_k+x_i)^2}\right],
\end{aligned}
$$

where the expressions for derivatives depend on the point x_k. Let

$$
F_i = \frac{\Delta x}{2}u(x_i)\left(\sum_{j=0}^{i-1} u(x_j) + u(x_{j+1})\right) + \eta(x_i)(\eta_x(x_i) - \eta_x(0)), \qquad i = 0, \dots n,
$$

which we will store as an array. This simplification, we obtain

$$
C_k = u^2(x_k) + \frac{\Delta x}{2}u_x(x_k)\sum_{i=0}^{k-1} u(x_{i+1}) + u(x_i) - \frac{\Delta x}{2\pi}\sum_{i=0}^{n-1} F_{i+1}\left[\frac{1}{(x_k-x_{i+1})^2} + \frac{1}{(x_k+x_{i+1})^2}\right] + F_i\left[\frac{1}{(x_k-x_i)^2} + \frac{1}{(x_k+x_i)^2}\right],
$$

With this in mind, we obtain the finite differences system:

$$
\begin{bmatrix} u(x_0, t_{j+1}) \\ u(x_1, t_{j+1}) \\ u(x_2, t_{j+1}) \\ \vdots \\ u(x_{n-2}, t_{j+1}) \\ u(x_{n-1}, t_{j+1}) \\ u(x_n, t_{j+1}) \end{bmatrix} = \begin{bmatrix} u(x_0, t_j) \\ u(x_1, t_j) \\ u(x_2, t_j) \\ \vdots \\ u(x_{n-2}, t_j) \\ u(x_{n-1}, t_j) \\ u(x_n, t_j) \end{bmatrix} + \Delta t \begin{bmatrix} f_1(\eta(x_0, t_j), u(x_0, t_j), t) \\ f_1(\eta(x_1, t_j), u(x_1, t_j), t) \\ f_1(\eta(x_2, t_j), u(x_2, t_j), t) \\ \vdots \\ f_1(\eta(x_{n-2}, t_j), u(x_{n-2}, t_j), t) \\ f_1(\eta(x_{n-1}, t_j), u(x_{n-1}, t_j), t) \\ f_1(\eta(x_n, t_j), u(x_n, t_j), t) \end{bmatrix}
$$

$$= \begin{bmatrix} u(x_0,t_j) \\ u(x_1,t_j) \\ u(x_2,t_j) \\ \vdots \\ u(x_{n-2},t_j) \\ u(x_{n-1},t_j) \\ u(x_n,t_j) \end{bmatrix} + \Delta t \mathcal{A} \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-2} \\ \eta_{n-1} \\ \eta_n \end{bmatrix} + \Delta t \mathcal{C}$$

4 Approximate equations: half-line

In this section, we derive the approximate equations from

$$\eta_{tt} - \eta_{xx} = \varepsilon \left(\frac{1}{3}\eta_{xxxx} + \frac{d}{dx}\frac{1}{\pi}\int_0^\infty \partial_t \left(\eta \int_0^y \eta_t \, \mathrm{d}y' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y + \frac{1}{2}\partial_x^2 \left(\int_0^x \eta_t \, \mathrm{d}x' \right)^2 \right). \tag{11}$$

As we approximate, we assume an expansion of η in ε :

$$\eta = \eta_0 + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2). \tag{12}$$

First order approximation

Substitution of (12) into equation (11) yields

$$\eta_{0tt} - \eta_{0xx} + \varepsilon(\eta_{1tt} - \eta_{1xx}) = \varepsilon \left[\frac{1}{3}\eta_{0xxxx} + \frac{d}{dx}\frac{1}{\pi}\int_0^\infty \partial_t \left(\eta \int_0^y \eta_t \, \mathrm{d}y' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \mathrm{d}y + \frac{1}{2}\partial_x^2 \left(\int_{-\infty}^x (\eta_0 + \varepsilon\eta_1)_t \, \mathrm{d}x' \right)^2 \right) + \mathcal{O}(\varepsilon^2). \tag{13}$$

In the leading order $\mathcal{O}(\varepsilon^0)$, equation (13) becomes

$$\eta_{0tt} - \eta_{0xx} = 0. \tag{14}$$

This is the wave equation with velocity 1, whose solution depends on the type of boundary conditions we prescribe for η at $x = 0$. For now, we prescribe

$$\eta_x(0,t) = 0.$$

The general solution is

$$\eta(x,t) = \begin{cases} F(x-t) + G(x+t) & x > t \\ F(t-x) + G(x+t) & x < t \end{cases},$$

where F, G are to be determined.

Second order approximation

As in the velocity potential case, we employ multiple scales. First, we find an expression for η_0. We introduce

$$\tau_0 = t, \qquad \tau_1 = \varepsilon t, \qquad \tau_2 = \varepsilon^2 t, \ldots,$$

so that

$$\eta(x,t) = \eta(x, \tau_0, \tau_1, \ldots).$$

With this in mind, the expansion (12) becomes

$$\eta(x, \tau_0, \tau_1, \ldots) = \eta_0(x, \tau_0, \tau_1, \ldots) + \mathcal{O}(\varepsilon^1). \tag{15}$$

Substituting (15) into (11), within $\mathcal{O}(\varepsilon^0)$, we obtain

$$\eta_{0\tau_0\tau_0} - \eta_{0xx} = 0, \tag{16}$$

so that the general solution is

$$\eta_0(x, \tau_0, \tau_1, \ldots) = \begin{cases} F(x - \tau_0, \tau_1, \ldots) + G(x + \tau_0, \tau_1, \ldots) & x > t \\ F(\tau_0 - x, \tau_1, \ldots) + G(x + \tau_0, \tau_1, \ldots) & x < t \end{cases},$$

where we recalled the boundary conditions $\eta_x(0,t) = 0$. Now, although we have found an expression for η_0, the functions F, G used are still general functions. To determine F, G, we proceed to the next order, i.e. $\mathcal{O}(\varepsilon^1)$. We introduce

$$\xi = x - \tau_0 \qquad \zeta = x + \tau_0$$

so that

$$\eta_0(x, \tau_0, \tau_1, \ldots) = \begin{cases} F(\xi, \tau_1, \ldots) + G(\zeta, \tau_1, \ldots) & x > t \\ F(-\xi, \tau_1, \ldots) + G(\zeta, \tau_1, \ldots) & x < t \end{cases},$$

and

$$\begin{aligned} \partial_x &= \partial_\xi \frac{\mathrm{d}\xi}{\mathrm{d}x} + \partial_\zeta \frac{\mathrm{d}\zeta}{\mathrm{d}x} = \partial_\xi + \partial_\zeta, \\ \partial_t &= \partial_\xi \frac{\mathrm{d}\xi}{\mathrm{d}t} + \partial_\zeta \frac{\mathrm{d}\zeta}{\mathrm{d}t} + \partial_{\tau_1} \frac{\mathrm{d}\tau_1}{\mathrm{d}t} = -\partial_\xi + \partial_\zeta + \varepsilon \partial_{\tau_1}. \end{aligned}$$

We consider the case $x > t$. We can rewrite (15) as follows

$$\begin{aligned} \eta &= \eta_0 + \varepsilon \eta_1 + \mathcal{O}(\varepsilon^2) \\ &= F(x - t, \varepsilon t, \ldots) + G(x + t, \varepsilon t, \ldots) + \varepsilon \eta_1 + \mathcal{O}(\varepsilon^2) \end{aligned}$$

$$
\begin{aligned}
&= F(\xi,\tau_1,\ldots) + G(\zeta,\tau_1,\ldots) + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2)\\
&= F + G + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2).
\end{aligned}
$$

For ease of writing, we suppressed explicit dependence on variables, though the reader should bear in mind that function F (G) depend on ξ $(\zeta), \tau_1, \tau_2$, etc. In addition, observe that

$$(\partial_t^2 - \partial_x^2) = \left(-4\partial_\xi\partial_\zeta + 2\varepsilon(\partial_\zeta\partial_{\tau_1} - \partial_\xi\partial_{\tau_1}) + \varepsilon^2\partial_{\tau_1}^2\right),$$

so that the LHS of (**??**) becomes

$$(\partial_t^2 - \partial_x^2)\eta = \varepsilon\left(-4\eta_{1\xi\zeta} - 2F_{\tau_1\xi} + 2G_{\tau_1\zeta}\right) + \mathcal{O}(\varepsilon^2). \tag{17}$$

Now, we deal with the RHS of (**??**). By appropriate substitutions, the terms become:

$$
\begin{aligned}
\frac{1}{3}\eta_{xxxx} &= \frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta} + \mathcal{O}(\varepsilon));\\
\left(\int_0^x \eta_t \,\mathrm{d}x'\right)^2 &= \left(\int_0^x \eta_{0t}\,\mathrm{d}x'\right)^2 + \mathcal{O}(\varepsilon)\\
&= \left(\int_0^x (-\partial_{\xi'} + \partial_{\zeta'} + \varepsilon\partial_{\tau_1})(F+G)\,\mathrm{d}x'\right)^2 + \mathcal{O}(\varepsilon)\\
&= \left(\int_0^x -F_{\xi'} + G_{\zeta'}\,\mathrm{d}x'\right)^2 + \mathcal{O}(\varepsilon)\\
&= \left(\int_0^x F_{\xi'}\,\mathrm{d}x'\right)^2 - 2\left(\int_0^x F_{\xi'}\,\mathrm{d}x'\right)\left(\int_0^x G_{\zeta'}\,\mathrm{d}x'\right) + \left(\int_0^x G_{\zeta'}\,\mathrm{d}x'\right)^2 + \mathcal{O}(\varepsilon)\\
&= F^2 - 2FG + G^2 + \mathcal{O}(\varepsilon),
\end{aligned}
$$

where for the last line we translate $\xi' = x' - t, \zeta' = x' + t$ to obtain

$$
\begin{aligned}
\int_0^x F_{\xi'}(x' - \tau_0, \tau_1)\,\mathrm{d}x' &= \int_{-t}^{x-t} F_{\xi'}(\xi',\tau_1)\,\mathrm{d}\xi' = \int_{-\tau_0}^{\xi} F_{\xi'}(\xi',\tau_1)\,\mathrm{d}\xi' = F(\xi,\tau_1) - F(-\tau_0,\tau_1),\\
\int_0^x G_{\zeta'}(x' + \tau_0, \tau_1)\,\mathrm{d}x' &= \int_{t}^{x+t} G_{\zeta'}(\zeta',\tau_1)\,\mathrm{d}\zeta' = \int_{\tau_0}^{\zeta} G_{\zeta'}(\zeta',\tau_1)\,\mathrm{d}\zeta' = G(\zeta,\tau_1) - G(\tau_0,\tau_1).
\end{aligned}
$$

For now, we assume that $F(x-\tau_0,\tau_1), G(x+\tau_0,\tau_1)$ vanish at $x = 0$. A brief discussion will be given at the end. Finally, we deal with the Hilbert transform term:

$$\int_0^\infty \partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)\left[\frac{1}{x-y} + \frac{1}{x+y}\right]\mathrm{d}y = \int_0^\infty (-\partial_\xi + \partial_\zeta + \varepsilon\partial_{\tau_1})\left((\eta_0 + \varepsilon\eta_1)\int_0^y (\eta_0 + \varepsilon\eta_1)_t\,\mathrm{d}y'\right)\left[\frac{1}{x-y} + \frac{1}{x+y}\right]\mathrm{d}y + \mathcal{O}(\varepsilon^2)$$

$$
\begin{aligned}
&= \int_0^\infty (-\partial_\xi + \partial_\zeta)\left((\eta_0 + \varepsilon\eta_1)\int_0^y (\eta_0 + \varepsilon\eta_1)_t \, \mathrm{d}y'\right)\left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \mathcal{O}(\varepsilon) \\
&= \int_0^\infty (-\partial_\xi + \partial_\zeta)\left(\eta_0 \int_0^y (\eta_0)_t \, \mathrm{d}y'\right)\left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \mathcal{O}(\varepsilon) \\
&= \int_0^\infty (-\partial_\xi + \partial_\zeta)\left(\eta_0 \int_0^y (-\partial_\xi + \partial_\zeta + \varepsilon\partial_{\tau_1})\eta_0 \, \mathrm{d}y'\right)\left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \mathcal{O}(\varepsilon) \\
&= \int_0^\infty (-\partial_\xi + \partial_\zeta)\left(\eta_0 \int_0^y (-\partial_\xi + \partial_\zeta)\eta_0 \, \mathrm{d}y'\right)\left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \mathcal{O}(\varepsilon) \\
&= \int_0^\infty (-\partial_\xi + \partial_\zeta)\left((F+G) \int_0^y (-F_\xi + G_\zeta) \, \mathrm{d}y'\right)\left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \mathcal{O}(\varepsilon) \\
&= \int_0^\infty (-\partial_\xi + \partial_\zeta)\left((F+G) \int_0^y (-F_{\xi'} + G_{\zeta'}) \, \mathrm{d}y'\right)\left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \mathcal{O}(\varepsilon) \\
&= \int_0^\infty (-\partial_\xi + \partial_\zeta)\left((F+G)(-F+G)\right)\left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \mathcal{O}(\varepsilon) \\
&= \int_0^\infty (-\partial_\xi + \partial_\zeta)\left((-F^2+G^2)\right)\left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \mathcal{O}(\varepsilon) \\
&= \int_0^\infty (2FF_\xi + 2GG_\zeta)\left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \mathcal{O}(\varepsilon).
\end{aligned}
$$

Substitution of terms into the RHS of (11) leads to:

$$
\begin{aligned}
&\frac{1}{3}\eta_{xxxx} + \frac{d}{dx}\frac{1}{\pi}\int_0^\infty \partial_t\left(\eta\int_0^y \eta_t \, \mathrm{d}y'\right)\left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \frac{1}{2}\partial_x^2\left(\int_0^x \eta_t \, \mathrm{d}x'\right)^2 \\
&\quad = \frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + \frac{1}{\pi}(\partial_\xi + \partial_\zeta)\int_0^\infty (2FF_\xi + 2GG_\zeta)\left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \frac{1}{2}(\partial_\xi^2 + 2\partial_\xi\partial_\zeta + \partial_\zeta^2)\left(F^2 - 2FG + G^2\right) + \mathcal{O}(\varepsilon) \\
&\quad = \frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + \frac{1}{\pi}(\partial_\xi + \partial_\zeta)\int_0^\infty (2FF_\xi + 2GG_\zeta)\left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \partial_\xi(FF_\xi - F_\xi G) + \partial_\zeta(GG_\zeta - FG_\zeta) - 2F_\xi G_\zeta + \mathcal{O}(\varepsilon).
\end{aligned}
\tag{18}
$$

Combining (17) and (18), in $\mathcal{O}(\varepsilon^1)$ we have

$$
-4\eta_{1\xi\zeta} = 2F_{\tau_1\xi} - 2G_{\tau_1\zeta} + \frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + \frac{1}{\pi}(\partial_\xi + \partial_\zeta)\int_0^\infty (2FF_\xi + 2GG_\zeta)\left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y + \partial_\xi(FF_\xi - F_\xi G) + \partial_\zeta(GG_\zeta - FG_\zeta) - 2F_\xi G_\zeta \tag{19}
$$

By rearranging appropriately, (19) becomes

$$
\begin{aligned}
-4\eta_{1\xi\zeta} &= \partial_\xi(2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + FF_\xi + \frac{1}{\pi}\int_0^\infty 2FF_\xi \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y) \\
&+ \partial_\zeta(-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + GG_\zeta + \frac{1}{\pi}\int_0^\infty 2GG_\zeta \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y) \\
&+ \partial_\xi(-F_\xi G + \frac{1}{\pi}\int_0^\infty 2GG_\zeta \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y) \\
&+ \partial_\zeta(-FG_\zeta + \frac{1}{\pi}\int_0^\infty 2FF_\xi \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y) - 2F_\xi G_\zeta
\end{aligned}
\tag{20}
$$

Integration of (20) with respect to ζ yields

$$
\begin{aligned}
-4\eta_{1\xi} &= \partial_\xi(2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + FF_\xi + \frac{1}{\pi}\int_0^\infty 2FF_\xi \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y)\zeta \\
&+ (-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + GG_\zeta + \frac{1}{\pi}\int_0^\infty 2GG_\zeta \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y) \\
&+ \partial_\xi(-F_\xi G + \frac{1}{\pi}\int_0^\infty 2GG_\zeta \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y) \\
&+ \int(-FG_\zeta + \frac{1}{\pi}\int_0^\infty 2FF_\xi \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y))\,\mathrm{d}\zeta - 2F_\xi G
\end{aligned}
$$

and further integration with respect to ξ leads to

$$
\begin{aligned}
-4\eta_1 &= (2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + FF_\xi + \frac{1}{\pi}\int_0^\infty 2FF_\xi \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y)\zeta \\
&+ (-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + GG_\zeta + \frac{1}{\pi}\int_0^\infty 2GG_\zeta \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y)\xi \\
&+ \int \frac{1}{\pi}\int_0^\infty 2GG_\zeta \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y)\,\mathrm{d}\xi + \int \frac{1}{\pi}\int_0^\infty 2FF_\xi \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y\,\mathrm{d}\zeta - 4FG.
\end{aligned}
$$

Since η_1 must be bounded, we must have

$$
2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + FF_\xi + \frac{1}{\pi}\int_0^\infty 2FF_\xi \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y) = 0, \tag{21}
$$

$$-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + GG_\zeta + \frac{1}{\pi}\int_0^\infty 2GG_\zeta \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y = 0. \tag{22}$$

In other words, we have obtained two KdV-like equations, (21) and (22), whose solutions describe behaviour of the surface elevation in the leading order. Now, for $x < t$, by the same procedure, the equation for G remains the same but (21) becomes

$$2F(-\xi)_{\tau_1} + \frac{1}{3}F(-\xi)_{\xi\xi\xi} + F(-\xi)F(-\xi)_\xi + \frac{1}{\pi}\int_0^\infty 2F(-\xi)F(-\xi)_\xi \left[\frac{1}{x-y} + \frac{1}{x+y}\right] \mathrm{d}y) = 0,$$

though one has to assume that $F(\tau_0, \tau_1) = 0$, instead of $F(-\tau_0, \tau_1) = 0$.

4.1 Discussion of F, G at $x = 0$.

As is seen, in order to obtain

$$\left(\int_0^x \eta_t \,\mathrm{d}x'\right)^2 = F(\xi)^2 - 2F(\xi)G + G^2 + \mathcal{O}(\varepsilon), \qquad x > t,$$
$$\left(\int_0^x \eta_t \,\mathrm{d}x'\right)^2 = F(-\xi)^2 + 2F(-\xi)G + G^2 + \mathcal{O}(\varepsilon), \qquad x < t,$$

we had to assume that at $x = 0$

$$\begin{aligned} F(-\tau_0, \tau_1),\ G(\tau_0, \tau_1) &= 0, \qquad x > t, \\ F(\tau_0, \tau_1),\ G(\tau_0, \tau_1) &= 0, \qquad x < t. \end{aligned} \tag{23}$$

Conditions in (23) therefore imply that at $x = 0$,

$$\begin{aligned} \eta(x, t, \varepsilon t) &= F(x - t, \varepsilon t) + G(x + t, \varepsilon t) = 0, \qquad x > t, \\ \eta(x, t, \varepsilon t) &= F(t - x, \varepsilon t) + G(x + t, \varepsilon t) = 0, \qquad x < t. \end{aligned}$$

In other words,

$$\begin{aligned} \eta(0, t, \varepsilon t) &= F(-t, \varepsilon t) + G(t, \varepsilon t) = 0, \qquad 0 > t, \\ \eta(0, t, \varepsilon t) &= F(t, \varepsilon t) + G(t, \varepsilon t) = 0, \qquad 0 < t. \end{aligned}$$

Thus, it seems like $\eta(0, t) = 0$ is the right condition to obtain KdV-like equations.

References

[1] Tom M. Apostol, *Mathematical analysis*, Pearson, 1974.