6

Nonlinear Schrödinger models and water waves

A different asymptotic regime will be investigated in this chapter. Since nonlinear, dispersive, energy-preserving systems generically give rise to the nonlinear Schrödinger (NLS) equation, it is important to study this situation. In this chapter first the NLS equation will be derived from a nonlinear Klein–Gordon and the KdV equation. Then a discussion of how the NLS equation arises in the limit of deep water will be undertaken. A number of results associated with NLS-type equations, including Benney–Roskes/Davey–Stewartson systems, will also be obtained and analyzed.

6.1 NLS from Klein–Gordon

We start with the so-called "u-4" model, which arises in theoretical physics, or the nonlinear Klein–Gordon (KG) equation – with two choices of sign for the cubic nonlinear term:

$$u_{tt} - u_{xx} + u \mp u^3 = 0. \tag{6.1}$$

Note, by multiplying the equation with u_t we obtain the conservation of energy law

$$\partial_t\left(u_t^2 + u_x^2 + u^2 \mp u^4/2\right) - \partial_x(2u_xu_t) = 0$$

and hence the equation has a conserved density $T_1 = u_t^2 + u_x^2 + u^2 \mp u^4/2$. Note the "potential" is proportional to $V(u) = u^2 \mp u^4/2$, hence the term "u-4" model is used. We assume a real solution to (6.1) with a "rapid" phase and slowly varying functions of x, t:

$$u(x,t) = u(\theta, X, T; \varepsilon); \quad X = \varepsilon x,\ T = \varepsilon t,\ \theta = kx - \omega t, |\varepsilon| \ll 1,$$

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where the dispersion relation for the linear problem in (6.1) is $\omega^2 = 1 + k^2$. Based on the form of assumed solution the following operators are introduced in the asymptotic analysis:

$$\partial_t = -\omega\partial_\theta + \varepsilon\partial_T$$
$$\partial_x = k\partial_\theta + \varepsilon\partial_X.$$

The weakly nonlinear asymptotic expansion

$$u = \varepsilon u_0 + \varepsilon^2 u_1 + \varepsilon^3 u_2 + \cdots,$$

and the differential operators are substituted into (6.1) leading to

$$\left[(-\omega\partial_\theta + \varepsilon\partial_T)^2 - (k\partial_\theta + \varepsilon\partial_X)^2\right] u + u \mp u^3 = 0$$

or

$$\begin{aligned}&\left[(\omega^2 - k^2)\partial_\theta^2 - \varepsilon(2\omega\partial_\theta\partial_T + 2k\partial_\theta\partial_X) + \varepsilon^2\left(\partial_T^2 - \partial_X^2\right)\right](\varepsilon u_0 + \varepsilon^2 u_1 + \cdots) + \\ &\quad + (\varepsilon u_0 + \varepsilon^2 u_1 + \cdots) \mp (\varepsilon u_0 + \varepsilon^2 u_1 + \cdots)^3 = 0.\end{aligned}$$

We will find the equations up to $O(\varepsilon^3)$; the first two are:

$$\begin{aligned}3O(\varepsilon): \qquad & (\omega^2 - k^2)\partial_\theta^2 u_0 + u_0 = 0 \\ & \Rightarrow u_0 = Ae^{i\theta} + \text{c.c.} \\ O(\varepsilon^2): \qquad & (\omega^2 - k^2)\partial_\theta^2 u_1 + u_1 = (2\omega i A_T + 2ki A_X)e^{i\theta} + \text{c.c.}\end{aligned}$$

where c.c. denotes the complex conjugate. Note: throughout the discussion we will use the dispersion relation $\omega^2 - k^2 = 1$. In order to remove secular terms, we require that the coefficient on the right-hand side of the $O(\varepsilon^2)$ equation be zero. This leads to an equation for $A(X, T)$ that, to go to higher order, we modify by using the asymptotic expansion

$$2i(\omega A_T + kA_X) = \varepsilon g_1 + \varepsilon^2 g_2 + \cdots . \tag{6.2}$$

The quantity $A(X, T)$ is often called the slowly varying envelope of the wave. As usual, we absorb the homogeneous solutions into u_0 so $u_1 = 0$. Now let us look at the next-order equation

$$\begin{aligned}O(\varepsilon^3): \qquad (\omega^2 - k^2)\partial_\theta^2 u_2 + u_2 &= -(A_{TT} - A_{XX})e^{i\theta} + \text{ c.c.} \\ &\quad \pm (Ae^{i\theta} + A^* e^{-i\theta})^3 + g_1 e^{i\theta} + \text{ c.c.}\end{aligned}$$

In order to remove secular terms, we eliminate the coefficient of $e^{i\theta}$. This implies

$$g_1 = (A_{TT} - A_{XX}) \mp 3A^2 A^*.$$

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Combining the above relation with the asymptotic expansion in (6.2) we have

$$2i\omega(A_T + \omega'(k)A_X) = \varepsilon(A_{TT} - A_{XX} \mp 3|A|^2 A), \tag{6.3}$$

where we used the dispersion relation to find $\omega'(k) = k/\omega$. After removing the secular terms, we can solve the $O(\varepsilon^3)$ equation for u_2 giving

$$u_2 = \mp\frac{1}{8}A^3 e^{3i\theta} + \text{ c.c.}$$

Note, in (6.3), we can transform the equation for A and remove the ε (group velocity) term. To do this, we change the coordinate system so that we are moving with the group velocity $\omega'(k)$ – sometimes this is called the group-velocity frame:

$$\begin{aligned} \xi &= X - \omega'(k)T & \qquad \partial_T &= -\omega'(k)\partial_\xi + \varepsilon\partial_\tau \\ \tau &= \varepsilon T = \varepsilon^2 t & \qquad \partial_X &= \partial_\xi. \end{aligned}$$

Now, we use (6.3) to find

$$2i\omega(-\omega' A_\xi + \varepsilon A_\tau + \omega' A_\xi) = \varepsilon((\omega')^2 A_{\xi\xi} - A_{\xi\xi} \mp 3|A|^2 A) + O(\varepsilon^2).$$

We can simplify this equation by using the dispersion relation as follows: $\omega' = k/\omega$ and

$$\omega'' = \frac{\omega - \omega' k}{\omega^2} = \frac{1}{\omega} - \frac{k\omega'}{\omega^2} = \frac{1}{\omega}\left[1 - (\omega')^2\right],$$

to arrive at the nonlinear Schrödinger equation in canonical form:

$$iA_\tau + \frac{\omega''}{2}A_{\xi\xi} \pm \frac{3}{2\omega}|A|^2 A = 0. \tag{6.4}$$

It is important to note that, like the derivation of the Korteweg–de Vries equation in the previous chapter, the small parameter, ε, has disappeared from the leading-order NLS equation. The NLS equation is a maximally balanced asymptotic system.

The behavior of (6.4) depends on the plus or minus sign. The NLS equation with a "+", here

$$\omega'' = \frac{\omega^2 - k^2}{\omega^3} = \frac{1}{\omega^3},$$

so $\omega''/\omega > 0$. The NLS with a "+" sign is said to be "focusing" and we will see gives rise to "bright" solitons. Bright solitons have a localized shape and decay at infinity. With a "−", NLS is said to be "defocusing" and we will see admits "dark" soliton solutions. Dark solitons have a constant amplitude at infinity. The terminology "bright" and "dark" solitons is typically used in

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nonlinear optics. We will discuss these solutions later, and nonlinear optics in a subsequent chapter.

In our derivation of NLS, we transformed our coordinate system to coincide with the group velocity of the wave. An alternative form, useful in nonlinear optics and called the retarded frame, is

$$t' = T - \frac{1}{\omega'(k)}X \qquad \partial_T = \partial_{t'}$$

$$\chi = \varepsilon X \qquad \partial_X = -\frac{1}{\omega'}\partial_{t'} + \varepsilon\partial_\chi.$$

Next, we substitute the above relations into (6.3) to find

$$2i\omega\left(A_{t'} + \omega'\left[\left(\frac{-1}{\omega'}\right)A_{t'} + \varepsilon A_\chi\right]\right) = \varepsilon\left(A_{t't'} - \frac{1}{(\omega')^2}A_{t't'} \mp 3|A|^2 A\right).$$

Again we simplify this equation into a nonlinear Schrödinger equation

$$iA_\chi + \frac{\omega''}{2(\omega')^3}A_{t't'} \pm \frac{3}{2\omega\omega'}|A|^2 A = 0. \tag{6.5}$$

Note that in the above equation, the "evolution" variable is χ and the "spatial" variable is t'.

6.2 NLS from KdV

We can also derive the NLS equation from the KdV equation asymptotically for small amplitudes. To do that, we expand the solution of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

as

$$u = \varepsilon u_0 + \varepsilon^2 u_1 + \varepsilon^3 u_2 + \cdots,$$

where for convenience the assumption of small amplitude is taken through the expansion of u rather than in the equation itself. Since the nonlinear term, uu_x, is quadratic rather than cubic, it will turn out that the reduction to the NLS equation requires more work than in the KG equation. As we will see, in this case one needs to remove secularity at one additional power of ε, which in this case is $O(\varepsilon^3)$. Another difficulty is manifested by the need to consider a *mean term* in the solution. Indeed, the equation to leading order is

$$u_{0,t} + u_{0,xxx} = 0,$$

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whose real solution is given by

$$u_0 = A(X,T)e^{i\theta} + \text{c.c.} + M(X,T), \tag{6.6}$$

where again $T = \varepsilon t$ and $X = \varepsilon x$ are the slow variables, $\theta = kx - \omega t$ is the fast variable, with the dispersion relation $\omega = -k^3$, and $M(T,X)$ is a real, slowly varying, mean term, i.e., it corresponds to the coefficient of $e^{in\theta}$ with $n = 0$. The quantity $A(X,T)$ is the slowly varying envelope of the rapidly varying wave contribution. As we will see, in this case the mean term turns out to be $O(\varepsilon)$, however, that is not always the case.[1] Note that the addition of the complex conjugate ("c.c.") in (6.6) corresponds to the first term and not to the mean term, which is real-valued.

Substituting $\partial_t = -\omega\partial_\theta + \varepsilon\partial_T$ and $\partial_x = k\partial_\theta + \varepsilon\partial_X$ in the KdV equation leads to

$$(-\omega\partial_\theta + \varepsilon\partial_T)u + 6u(k\partial_\theta + \varepsilon\partial_X)u + (k\partial_\theta + \varepsilon\partial_X)^3 u = 0.$$

Using the above expansion for the solution, $u = \varepsilon u_0 + \varepsilon^2 u_1 + \varepsilon^3 u_2 + \cdots$ gives

$$\begin{aligned}&(-\omega\partial_\theta + \varepsilon\partial_T)(\varepsilon u_0 + \varepsilon^2 u_1 + \varepsilon^3 u_2 + \cdots)\\ &+ 6(\varepsilon u_0 + \varepsilon^2 u_1 + \varepsilon^3 u_2 + \cdots)(k\partial_\theta + \varepsilon\partial_X)(\varepsilon u_0 + \varepsilon^2 u_1 + \varepsilon^3 u_2 + \cdots)\\ &+ (k\partial_\theta + \varepsilon\partial_X)^3(\varepsilon u_0 + \varepsilon^2 u_1 + \varepsilon^3 u_2 + \cdots) = 0.\end{aligned}$$

The leading-order solution of $\mathcal{L}u_0 = 0$, $\mathcal{L}$ defined below, is given by (6.6). We recall that the dispersion relation for the exponential term is given by $\omega(k) = -k^3$. The $O(\varepsilon^2)$ equation is

$$\begin{aligned}\mathcal{L}u_1 &= -u_{0,T} - 3k^2 u_{0,\theta\theta X} - 6u_0 k u_{0,\theta}\\ &= (-A_T e^{i\theta} + \text{c.c.}) - M_T + (3k^2 A_X e^{i\theta} + \text{c.c.})\\ &\quad - 6(Ae^{i\theta} + \text{c.c.}) + M)(Aike^{i\theta} + \text{c.c.}),\end{aligned}$$

where we have defined the linear operator (using $\omega = -k^3$),

$$\mathcal{L}u = k^3(u_\theta + u_{\theta\theta\theta}).$$

Thus, we rewrite the $O(\varepsilon)$ equation as

$$\mathcal{L}u_1 = (-A_T - \omega' A_X - 6iMkA)e^{i\theta} + \text{c.c.} - 6(ikA^2 e^{2i\theta} + \text{c.c.}) - M_T,$$

where we have used the dispersion relation, from which one gets that $\omega'(k) = -3k^2$. In order to remove secular terms we note that such terms arise

[1] For example, in finite-depth water waves the mean term is $O(1)$ (Benney and Roskes, 1969; Ablowitz and Segur, 1981).

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not only from the coefficients of $e^{i\theta}$, but also from the coefficients of "$e^{i0\theta} = 1$" (which correspond to a mean term). Therefore, according to our methodology, we require that

$$A_T + \omega' A_X + 6iMkA = \varepsilon g_1 + \varepsilon^2 g_2 + \cdots, \tag{6.7}$$

$$M_T = \varepsilon f_1 + \varepsilon^2 f_2 + \cdots \tag{6.8}$$

and solve u_1 for what remains, i.e., solve,

$$\mathcal{L}u_1 = -6(ikA^2 e^{2i\theta} + \text{c.c.}).$$

The latter equation can be solved by letting $u = \alpha e^{2i\theta} + \text{c.c.}$, which gives that $\alpha = A^2/k^2$ and (omitting homogeneous terms as was done earlier for perturbations in ODE problems)

$$u_1 = \frac{A^2}{k^2} e^{2i\theta} + \text{c.c.} = \alpha e^{2i\theta} + \text{c.c.},$$

where $\alpha = A^2/k^2$. Inspecting equation (6.8) we see that $M = O(\varepsilon)$. Thus, in retrospect, we could (or should) have introduced the mean term at the $O(\varepsilon)$ solution, i.e., in u_1. However, that will not change the result of the analysis, provided we work consistently.

Next we inspect the $O(\varepsilon^3)$ equation:

$$\begin{aligned}&\mathcal{L}u_2 + 6(u_0 k u_{1,\theta} + u_0 u_{0,X} + u_1 k u_{0,\theta}) + 3k^2 u_{1,\theta\theta X} + 3k u_{0,\theta XX} + u_{1T} \\ &\quad = -f_1 - (g_1 e^{i\theta} + \text{c.c.}),\end{aligned}$$

where the f_1 and g_1 terms arise from equations (6.7) and (6.8). Using (6.6) and the fact that $M = O(\varepsilon)$ we get that to leading order

$$\begin{aligned}\mathcal{L}u_2 = &-6(Ae^{i\theta} + \text{c.c.})(2ik\alpha e^{2i\theta} + \text{c.c.}) - 6(Ae^{i\theta} + \text{c.c.})(A_X e^{i\theta} + \text{c.c.}) - \\ &- 6(\alpha e^{2i\theta} + \text{c.c.})(ikAe^{i\theta} + \text{c.c.}) - 3k^2[(2i)^2 \alpha_X e^{2i\theta} + \text{c.c.}] - \\ &- 3k(iA_{XX} e^{i\theta} + \text{c.c.}) - f_1 - (g_1 e^{i\theta} + \text{c.c.}) - (\alpha_T e^{2i\theta} + \text{c.c.}).\end{aligned}$$

The reader is cautioned about the "i" terms. For example, note that $(iA_{XX}e^{i\theta} + \text{c.c.}) = iA_{XX}e^{i\theta} - iA_{XX}e^{-i\theta}$. Here we need to remove secularity of the $e^{i\theta}$ and mean terms in order to find the f_1 and g_1 terms. Using $\alpha = A^2/k^2$ this leads to

$$\begin{aligned}g_1 &= -3ikA_{XX} - 6ik\alpha A^* = -3ikA_{XX} - \frac{6i}{k} A^2 A^*, \\ f_1 &= -6(AA_X^* + \text{c.c.}) = -6(|A|^2)_X,\end{aligned}$$

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where A^* is the complex conjugate of A. Thus, from equations (6.7) and (6.8) using f_1, g_1, we obtain to leading order that

$$A_T + \omega' A_X + 6iMkA = \varepsilon\left(-3ikA_{XX} - \frac{6i}{k}A^2A^*\right), \tag{6.9}$$

$$M_T = -6\varepsilon(|A|^2)_X. \tag{6.10}$$

As such, we have solved the problem of removing secularity at $O(\varepsilon^3)$. However, in order to obtain the NLS equation one additional step is required. We transform to a moving reference frame, i.e., $A(X, T) = A(\xi, \tau)$, where the new variables are given by

$$\xi = X - \omega'(k)T, \quad \tau = \varepsilon T. \tag{6.11}$$

Therefore, the derivatives transform according to

$$\partial_X = \partial_\xi, \quad \partial_T = \varepsilon\partial_\tau - \omega'\partial_\xi$$

and equations (6.9) and (6.10) become

$$\varepsilon A_\tau + 6iMkA = \varepsilon\left(-3ikA_{\xi\xi} - \frac{6i}{k}A^2A^*\right), \tag{6.12}$$

$$\varepsilon M_\tau - \omega' M_\xi = -6\varepsilon(|A|^2)_\xi. \tag{6.13}$$

The latter equation can be simplified using $M = O(\varepsilon)$, which leads to

$$-\omega' M_\xi = -6\varepsilon(|A|^2)_\xi + O(\varepsilon^2),$$

whose (leading-order) solution is given by (omitting the integration constant)

$$M \sim -\frac{2\varepsilon|A|^2}{k^2},$$

where we have used $\omega' = -3k^2$. Upon substituting the solution for M in (6.12) we have that

$$A_\tau + 3ikA_{\xi\xi} + \frac{6i}{k}(-2|A|^2)A + \frac{6i}{k}|A|^2A = 0.$$

Therefore, we arrive at the following *defocusing* NLS equation

$$iA_\tau - 3kA_{\xi\xi} + \frac{6}{k}|A|^2A = 0.$$

Using $\omega'' = -6k$ leads to the canonical (or generic) form of the NLS equation,

$$iA_\tau + \frac{\omega''(k)}{2}A_{\xi\xi} + \frac{6}{k}|A|^2A = 0.$$

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An alternative, more direct, and perhaps "faster", way of obtaining the NLS equation from the KdV equation is described next (Ablowitz et al., 1990, where it was used in the study of the KP equation and the associated boundary conditions). This method consists of making the ansatz (sometimes referred to as the "quasi-monochromatic" assumption. This method can also be used for the Klein–Gordon model in Section 6.1.):

$$u_0 = \varepsilon\Big[A(T,X)e^{i\theta} + \text{c.c.} + M(X,T)\Big] + \varepsilon^2(\alpha e^{2i\theta} + \text{c.c.}) + \cdots,$$

i.e., we add the $O(\varepsilon^2)$ second-harmonic term to the ansatz that has a fundamental and a mean term. Below we outline the derivation. The KdV terms are then respectively given by

$$u_t = \varepsilon\Big[(Aik^3 + \varepsilon A_T)e^{i\theta} + \text{c.c.} + \varepsilon M_T\Big] + \varepsilon^2\Big[(2ik^3\alpha + \varepsilon\alpha_T)e^{2i\theta} + \text{c.c.}\Big] + \cdots,$$

$$\begin{aligned} 6uu_x = {} & 6\varepsilon\Big[(Ae^{i\theta} + \text{c.c.} + M) + \varepsilon(\alpha e^{2i\theta} + \text{c.c.})\Big]\varepsilon\Big[\Big((ikA + \varepsilon A_X)e^{i\theta} + \\ & + \text{c.c.}\Big) + \varepsilon M_X + \varepsilon\Big((2ik\alpha + \varepsilon\alpha_X)e^{2i\theta} + \text{c.c.}\Big)\Big] + \cdots, \end{aligned}$$

and

$$\begin{aligned} u_{xxx} = {} & \varepsilon\Big\{\Big[(ik + \varepsilon\partial_X)^3 A\Big] e^{i\theta} + \text{c.c.} + \varepsilon^3 M_{XXX} + \\ & + \varepsilon^2\Big[(2ik + \varepsilon\partial_X)^3 \alpha e^{2i\theta} + \text{c.c.}\Big]\Big\} + \cdots. \end{aligned}$$

We then set the coefficients of the $e^{i\theta}$ and mean terms to zero. The remaining terms are the coefficients of $e^{2i\theta}$, which are also set to zero (similarly if we add terms $e^{in\theta}$ at higher order):

$$\varepsilon^2\left(2ik^3\alpha + 6ikA^2 - 8ik^3\alpha\right) = 0.$$

This equation implies that $\alpha = A^2/k^2$, i.e., the same as in the preceding derivation. Removing the secular mean term gives

$$M_T + 6\varepsilon(|A|^2)_X + 6\varepsilon M M_X = 0.$$

As before, one obtains $M = O(\varepsilon)$, from which it follows that to leading order the equation for M is

$$M_T + 6\varepsilon(|A|^2)_X \sim 0.$$

Then recalling the earlier change of variables, (6.11),

$$\partial_T = \varepsilon\partial_\tau - \omega'\partial_\xi, \partial_X = \partial_\xi$$

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we find, after integration (and omitting the integration constant)

$$M \sim -\frac{2\varepsilon|A|^2}{k^2},$$

which agrees with what we found before. Next, removing the $e^{i\theta}$ secular terms leads to

$$\varepsilon^2 \left(A_T - 3k^2 A_X + 6ikMA\right) + \varepsilon^3 \Big(3ikA_{XX} + 12ik\alpha A^2 A^* \\ - 6ik\alpha A^2 A^* + 6MA_X\Big) = 0.$$

Note that the last term in the $O(\varepsilon^3)$ parentheses is negligible, since it is smaller by $M = O(\varepsilon)$ than the other terms. Upon substituting the solution for M one arrives at the equation

$$A_T - 3k^2 A_X - \frac{12i\varepsilon|A|^2}{k}A + \varepsilon\left(3ikA_{XX} + \frac{6i}{k}|A|^2 A\right) = 0.$$

Finally, transforming the coordinates to the moving-frame (6.11) leads to

$$A_\tau + 3ikA_{\xi\xi} - \frac{6i}{k}|A|^2 A = 0,$$

and, using the dispersion relation $\omega'' = -3k$, results in the NLS equation in canonical form

$$iA_\tau + \frac{\omega''(k)}{2}A_{\xi\xi} + \frac{6}{k}|A|^2 A = 0.$$

Note again, given the signs (i.e. the product of nonlinear and dispersive coefficients), the above NLS equation is of defocusing type.

6.3 Simplified model for the linear problem and "universality"

It is noteworthy that in the derivations of the NLS equation from both the KG and KdV equations, the linear terms in the NLS equation turn out to be

$$iA_\tau + \frac{\omega''(k)}{2}A_{\xi\xi}.$$

It might seem that the $\omega''(k)/2$ coefficient is a coincidence from these two different derivations. However, as explained below, this is not the case. In fact, this coefficient will always arise in the derivation because it manifests an inherent property of the slowly varying amplitude approximation of a constant coefficient linearized dispersive equation, in the moving-frame system.

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To see that concretely, let us recall the linear KdV equation

$$A_t = -A_{xxx},$$

whose dispersion relation is $\omega(k) = -k^3$. We can solve this equation explicitly (e.g., using Fourier transforms), by looking for wave solutions

$$A(x,t) = \tilde{A}(X,T)e^{i(kx-\omega t)} + \text{c.c.}$$

The slowly varying amplitude assumption corresponds to a superposition of waves of the form:

$$\tilde{A}(X,T) = A_0 e^{i(\varepsilon Kx - \varepsilon\Omega t)} = A_0 e^{i(KX-\Omega T)}; \quad A_0 \text{ const.}$$

This gives us

$$\begin{aligned} A(x,t) &= A_0 e^{i(kx+\varepsilon Kx-\omega t-\varepsilon\Omega t)} + \text{c.c.} \\ &= A_0 e^{i(kx+KX-\omega t-\Omega T)} + \text{c.c.}, \end{aligned}$$

where $X = \varepsilon x$ and $T = \varepsilon t$. The quantities K and Ω are sometimes referred to as the sideband wavenumber and frequency, respectively, because they correspond to a small deviation from the central wavenumber k and central frequency ω. It is useful to look at these deviations from the point of view of operators, whereby $\Omega \to i\partial_T$ and $K \to -i\partial_X$. Thus,

$$\omega_{\text{tot}} \sim \omega + \varepsilon\Omega \to \omega + i\varepsilon\partial_T.$$

Similarly,

$$k_{\text{tot}} \sim k + \varepsilon K \to k - i\varepsilon\partial_X.$$

We can expand $\omega(k)$ in a Taylor series around the central wavenumber as

$$\omega_{\text{tot}}(k + \varepsilon K) \sim \omega(k) + \varepsilon K\omega' + \varepsilon^2 K^2 \frac{\omega''}{2}.$$

Then using the operator $K = -i\partial_X$ we have

$$\omega_{\text{tot}}(k - i\varepsilon\partial_X) \sim \omega(k) - i\varepsilon\omega'\partial_X - \varepsilon^2 \frac{\omega''}{2}\partial_{XX}.$$

Alternatively, if we use the operator $\Omega = i\partial_T$ this yields

$$\omega_{\text{tot}}(k)A = (\omega + \varepsilon\Omega)A \sim [\omega(k) + i\varepsilon\partial_T]A \sim \left(\omega(k) - i\varepsilon\omega'\partial_X - \varepsilon^2 \frac{\omega''}{2}\partial_{XX}^2\right)A.$$

Then to leading order

$$i\varepsilon(A_T + \omega' A_X) + \varepsilon^2 \frac{\omega''}{2} A_{XX} = 0 \tag{6.14}$$

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and in the moving frame ("water waves variant"), $\xi = X - \omega'(k)T$, $\tau = \varepsilon T$, this equation transforms to

$$\varepsilon^2\left(iA_\tau + \frac{\omega''}{2}A_{\xi\xi}\right) = 0,$$

which is the linear Schrödinger equation with the canonical $\omega''(k)/2$ coefficient.

If, however, the coordinates transform to the "optics variant" using the "retarded" frame, i.e., $\chi = \varepsilon X$ and $t' = T - X/\omega'$, then equation (6.14) becomes

$$i\varepsilon\left(A_{t'} + \omega'\left(-\frac{1}{\omega'}A_{t'} + \varepsilon A_\chi\right)\right) + \frac{\omega''}{2}\varepsilon^2\frac{1}{(\omega')^2}A_{t't'} = 0,$$

which simplifies to

$$iA_\chi + \frac{\omega''}{2(\omega')^3}A_{t't'} = 0; \tag{6.15}$$

i.e., the "optical" variant of the linear Schrödinger equation has a canonical $\omega''/2(\omega')^3$ coefficient in front of the dispersive term [see (6.5) for a typical NLS equation]. The above interchange of coordinates from $\xi = x - \omega'(k)T$, $T = \epsilon t$ to $t' = T - x/\omega'$, $\chi = \epsilon x$ is also reflected in terms of the dispersion relation; namely, instead of considering $\omega = \omega(k)$, let us consider $k = k(\omega)$. Then

$$\frac{d^2k}{d\omega^2} = \frac{d}{d\omega}\frac{dk}{d\omega} = \frac{dk}{d\omega}\frac{d}{dk}\left(\frac{1}{d\omega/dk}\right) = -\frac{1}{\omega'^3}\frac{d^2\omega}{dk^2}$$

so that (6.15) can be written as

$$iA_\chi - \frac{k''}{2}A_{t't'} = 0.$$

On the other hand, suppose we consider rather general conservative nonlinear wave problems with leading quadratic or cubic nonlinearity. We have seen earlier that a multiple-scales analysis, or Stokes–Poincaré frequency-shift analysis (see also Chapter 4), shows, omitting dispersive terms, a wave solution of the form

$$u(x,t) = \varepsilon A(\tau)e^{i(kx-\omega t)} + \text{c.c.},$$

with $\tau = \varepsilon t$ has $A(\tau)$ satisfying

$$i\frac{\partial A}{\partial \tau} + n|A|^2A = 0.$$

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The constant coefficient n depends on the particular equation studied. Putting the linear and nonlinear effects together implies that an NLS equation of the form

$$i\frac{\partial A}{\partial \tau} + \frac{\omega''}{2}\frac{\partial^2 A}{\partial \xi^2} + n|A|^2 A = 0$$

is "natural". Indeed, the NLS equation can be viewed as a "universal" equation as it generically governs the slowly varying envelope of a monochromatic wave train (see also Benney and Newell, 1967).

6.4 NLS from deep-water waves

In this section we discuss the derivation of the NLS equation from the Euler–Bernoulli equations in the limit of infinitely deep (1+1)-dimensional water waves, i.e.,

$$\phi_{xx} + \phi_{zz} = 0, \quad -\infty < z < \varepsilon\eta(x,t) \tag{6.16}$$

$$\phi_z = 0, \quad z \to -\infty \tag{6.17}$$

$$\phi_t + \frac{\varepsilon}{2}\left(\phi_x^2 + \phi_z^2\right) + g\eta = 0, \quad z = \varepsilon\eta \tag{6.18}$$

$$\eta_t + \varepsilon\eta_x\phi_x = \phi_z, \quad z = \varepsilon\eta. \tag{6.19}$$

There are major differences between this model and the shallow-water model discussed in Chapter 5 that require our attention. To begin with, equations (6.16) and (6.17) are defined for $z \to -\infty$, as opposed to $z = -1$. In addition, the parameter $\mu = h/\lambda_x$, which was taken to be very small for shallow-water waves, is not small in this case. In fact, taking $h \to \infty$ in this case would imply $\mu \to \infty$, which is not a suitable limit, so we will not use the parameter μ (or the previous non-dimensional scaling). We will use (6.16)–(6.19) in dimensional form and begin by only assuming that the nonlinear terms are small.

The idea of the derivation is as follows. We have already seen in the previous section that the linear model always gives rise to the same linear Schrödinger equation. Since water waves have leading quadratic nonlinearity, general considerations mentioned earlier suggest that, for the nonlinear model, we expect to obtain the NLS equation in the form

$$iA_\tau + \frac{\omega''}{2}A_{\xi\xi} + n|A|^2 A = 0,$$

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where n is a coefficient (that may depend on $\omega(k)$ and its derivatives) that is yet to be found. Our goal is to obtain the dispersive and nonlinear coefficients. The dispersive term is canonical so we will first find the Stokes frequency shift; i.e., the coefficient n in the above equation.

6.4.1 Derivation of the frequency shift

Let us first notice that (6.18) and (6.19) are defined on the free surface, which creates difficulties in the analysis. Therefore, the first step is to approximate the boundary conditions using a Taylor expansion of $z = \varepsilon\eta$ around the stationary limit (i.e., the fixed free surface), which is $z = 0$. Thus, one has that

$$\phi_t(t, x, \varepsilon\eta) = \phi_t(t, x, 0) + \varepsilon\eta\phi_{tz}(t, x, 0) + \frac{1}{2}(\varepsilon\eta)^2\phi_{tzz}(t, x, 0) + \cdots ,$$
$$\phi_x(t, x, \varepsilon\eta) = \phi_x(t, x, 0) + \varepsilon\eta\phi_{xz}(t, x, 0) + \frac{1}{2}(\varepsilon\eta)^2\phi_{xzz}(t, x, 0) + \cdots ,$$
$$\phi_z(t, x, \varepsilon\eta) = \phi_z(t, x, 0) + \varepsilon\eta\phi_{zz}(t, x, 0) + \frac{1}{2}(\varepsilon\eta)^2\phi_{zzz}(t, x, 0) + \cdots .$$

Using multiple scales in the time variable (only!) and defining $T = \varepsilon t$, we can rewrite (6.18) and (6.19) to $O(\varepsilon^2)$ as

$$\left[\phi_t + \varepsilon\eta\phi_{tz} + \frac{1}{2}(\varepsilon\eta)^2\phi_{tzz} + \varepsilon(\phi_T + \varepsilon\eta\phi_{Tz}) + \cdots\right] + \frac{\varepsilon}{2}\left(\phi_x^2 + \phi_z^2 + 2\varepsilon\eta\phi_x\phi_{xz} + 2\varepsilon\eta\phi_z\phi_{zz} + \cdots\right) + g\eta = 0, \tag{6.20}$$

and

$$\eta_t + \varepsilon\eta_T + \varepsilon\eta_x(\phi_x + \varepsilon\eta\phi_{xz}) + \cdots = \phi_z + \varepsilon\eta\phi_{zz} + \frac{1}{2}(\varepsilon\eta)^2\phi_{zzz} + \cdots \tag{6.21}$$

where all the functions in (6.20) and (6.21) are understood to be evaluated at $z = 0$ for all x.

Next we expand ϕ and η as

$$\phi = \phi^{(0)} + \varepsilon\phi^{(1)} + \varepsilon^2\phi^{(2)} + \cdots ,$$
$$\eta = \eta^{(0)} + \varepsilon\eta^{(1)} + \varepsilon^2\eta^{(2)} + \cdots ,$$

substitute them into the equations, and study the corresponding equations at $O(\varepsilon^0)$, $O(\varepsilon^1)$, and $O(\varepsilon^2)$.

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Leading order, $O(\varepsilon^0)$

From (6.16) one gets to leading order that

$$\phi^{(j)}_{xx} + \phi^{(j)}_{zz} = 0,$$

i.e., this is Laplace's equation (at every order of ε). Together with the boundary condition

$$\lim_{z\to-\infty} \phi^{(j)}_z = 0$$

(also true at every order of ε). The solution is given by

$$\phi^{(j)} \sim \sum_m A^{(j)}_m(T)e^{im\theta+m|k|z} + \text{c.c.}, \tag{6.22}$$

where $\theta = kx - \omega t$ and the summation[2] is carried over $m = 0, 1, 2, 3, \ldots$ Note that the choice of $+m|k|z$ in the exponent assures that the solution is decaying as z approaches $-\infty$. In addition, the summation over all possible modes is necessary, since even if the initial conditions excite only a single mode, the nonlinearity will generate the other modes.

To describe the analysis, we will explicitly show (once) the perturbation terms in detail. Keeping up to $O(\varepsilon^2)$ terms in (6.20) and (6.21) on $z = 0$ lead to

$$\begin{aligned}&\left[\left(\phi^{(0)}_t + \varepsilon\phi^{(1)}_t + \varepsilon^2\phi^{(2)}_t\right) + \varepsilon(\eta^{(0)} + \varepsilon\eta^{(1)})\left(\phi^{(0)}_{tz} + \varepsilon\phi^{(1)}_{tz}\right) + \frac{1}{2}\varepsilon^2\eta^{(0)2}\phi^{(0)}_{tzz}\right]\\
&\quad + \varepsilon\left(\phi^{(0)}_T + \varepsilon\phi^{(1)}_T + \varepsilon\eta^{(0)}\phi^{(0)}_{Tz}\right) + \frac{1}{2}\varepsilon\left[\left(\phi^{(0)}_x + \varepsilon\phi^{(1)}_x\right)^2 + 2\varepsilon\eta^{(0)}\phi^{(0)}_x\phi^{(0)}_{xz}\right]\\
&\quad + \frac{1}{2}\varepsilon\left[\left(\phi^{(0)}_z + \varepsilon\phi^{(1)}_z\right)^2 + 2\varepsilon\eta^{(0)}\phi^{(0)}_z\phi^{(0)}_{zz}\right] + g(\eta^{(0)} + \varepsilon\eta^{(1)} + \varepsilon^2\eta^{(2)}) = 0\end{aligned}$$

and

$$\begin{aligned}&\left(\eta^{(0)}_t + \varepsilon\eta^{(1)}_t + \varepsilon^2\eta^{(2)}_t\right) + \varepsilon\left(\eta^{(0)}_T + \varepsilon\eta^{(1)}_T\right)\\
&\quad \varepsilon\left(\eta^{(0)}_x + \varepsilon\eta^{(1)}_x\right)\left(\phi^{(0)}_x + \varepsilon\phi^{(1)}_x\right) + \varepsilon^2\eta^{(0)}\eta^{(0)}_x\phi^{(0)}_{xz} = \phi^{(0)}_z + \varepsilon\phi^{(1)}_z + \varepsilon^2\phi^{(2)}_z\\
&\qquad + \varepsilon\left(\eta^{(0)} + \varepsilon\eta^{(1)}\right)\left(\phi^{(0)}_{zz} + \varepsilon\phi^{(1)}_{zz}\right) + \frac{1}{2}\varepsilon^2\eta^{(0)2}\phi^{(0)}_{zzz}.\end{aligned}$$

Below we study these equations up to $O(\varepsilon^2)$.

At $O(\varepsilon^0)$ we obtain

$$\phi^{(0)}_t + g\eta^{(0)} = 0, \tag{6.23}$$

$$\eta^{(0)}_t - \phi^{(0)}_z = 0. \tag{6.24}$$

[2] It will turn out that the mean term $m = 0$ is of low order.

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It follows from (6.22), assuming only one harmonic at leading order (recall $\theta = kx - \omega t$),

$$\phi^{(0)} \sim A_1(T)e^{i\theta+|k|z} + \text{c.c.}$$

and that

$$\eta^{(0)} = N_1(T)e^{i\theta} + \text{c.c.}$$

and from (6.23) and (6.24) we get the system

$$\begin{aligned} -i\omega A_1 + gN_1 &= 0, \\ -i\omega N_1 &= |k|A_1. \end{aligned} \tag{6.25}$$

This is a linear system in A_1 and N_1 that can also be written in matrix form as follows

$$\begin{pmatrix} -i\omega & g \\ -|k| & -i\omega \end{pmatrix} \begin{pmatrix} A_1 \\ N_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The solution of this linear homogeneous system is unique if and only if its determinant is zero, which leads to the dispersion relation

$$\omega^2(k) = g|k|. \tag{6.26}$$

Note that this dispersion relation can be viewed as the formal limit as $h \to \infty$ of the more general dispersion relation (at any water depth),

$$\omega^2(k) = gk\tanh(kh),$$

since $\tanh(x) \to \text{sgn}(x)$ as $x \to \infty$ and $|k| = k \cdot \text{sgn}(k)$.

First order, $O(\varepsilon)$

The $O(\varepsilon)$ equation corresponding to the system (6.23), (6.24) reads

$$\phi_t^{(1)} + g\eta^{(1)} = -\left(\eta^{(0)}\phi_{tz}^{(0)} + \phi_T^{(0)}\right) - \frac{1}{2}\left(\phi_x^{(0)2} + \phi_z^{(0)2}\right), \tag{6.27}$$

$$\eta_t^{(1)} - \phi_z^{(1)} = -\eta_x^{(0)}\phi_x^{(0)} - \eta_T^{(0)} + \eta^{(0)}\phi_{zz}^{(0)}. \tag{6.28}$$

It follows from (6.25) that

$$A_1 = -\frac{ig}{\omega}N_1.$$

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Using this relation and substituting the solution we found for $\phi^{(0)}$ and $\eta^{(0)}$ into (6.27) and (6.28) leads to[3]

$$\phi_t^{(1)} + g\eta^{(1)} = g|k|N_1^2 e^{2i\theta} - A_{1,T}e^{i\theta} + \text{c.c.}, \tag{6.29}$$

$$\eta_t^{(1)} - \phi_z^{(1)} = -\frac{2ig}{\omega}k^2 N_1^2 e^{2i\theta} - N_{1,T}e^{i\theta} + \text{c.c.} \tag{6.30}$$

Removing the secular terms $e^{i\theta}$ requires taking $A_{1,T} = N_{1,T} = 0$ at this order. But, as usual we expand

$$A_{1,T} = \varepsilon f_1 + \varepsilon^2 f_2 + \cdots$$

and

$$N_{1,T} = \varepsilon g_1 + \varepsilon^2 g_2 + \cdots ,$$

in which case remaining terms in the equation lead us to a solution of the form

$$\phi^{(1)} = A_2 e^{2i\theta + 2|k|z} + \text{c.c.}$$

and

$$\eta^{(1)} = N_2 e^{2i\theta} + \text{c.c.}$$

Substituting this ansatz into (6.29) and (6.30) yields the system

$$-2i\omega A_2 + gN_2 = g|k|N_1^2,$$
$$-2i\omega N_2 - 2|k|A_2 = -\frac{2ig}{\omega}k^2 N_1^2.$$

This time we arrived at an inhomogeneous linear system for A_2 and N_2. Note that its solution is unique on account of the fact that its determinant is non-zero: this follows from the dispersion relation (6.26). Using the dispersion relation, the solution is found to be

$$A_2 = 0$$

and

$$N_2 = |k|N_1^2.$$

Therefore,

$$\phi^{(1)} = 0$$

and

$$\eta^{(1)} = |k|N_1^2 e^{2i\theta} + \text{c.c.}$$

[3] Note that the mean terms cancel in this case, which is the reason we need not have considered them in the expansion.

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Second order, $O(\varepsilon^2)$

Now for the $O(\varepsilon^2)$ equation corresponding to (6.20) and (6.21):

$$\begin{aligned}
\phi_t^{(2)} + g\eta^{(2)} &= -\eta^{(1)}\phi_{tz}^{(0)} - \eta^{(0)}\phi_{tz}^{(1)} - \frac{1}{2}\eta^{(0)2}\phi_{tzz}^{(0)} - (f_1 e^{i\theta} + \text{c.c.}) \\
&\quad - \left[\phi_x^{(0)}\phi_x^{(1)} + \eta^{(0)}\phi_x^{(0)}\phi_{xz}^{(0)} + \phi_z^{(0)}\phi_z^{(1)} + \eta^{(0)}\phi_z^{(0)}\phi_{zz}^{(0)}\right] \\
&\quad - \phi_T^{(1)} - \eta^{(0)}\phi_{Tz}^{(0)}, \\
\eta_t^{(2)} - \phi_z^{(2)} &= -\left(\eta_x^{(0)}\phi_x^{(1)} + \eta_x^{(1)}\phi_x^{(0)} + \eta^{(0)}\eta_x^{(0)}\phi_{xz}^{(0)}\right) + \eta^{(0)}\phi_{zz}^{(1)} \\
&\quad + \eta^{(1)}\phi_{zz}^{(1)} + \frac{1}{2}\eta^{(0)2}\phi_{zzz}^{(0)} - (g_1 e^{i\theta} + \text{c.c.}) - \eta_T^{(1)}.
\end{aligned}$$

Substituting $\phi^{(1)} = 0$ and taking into account the residual terms f_1 and g_1 after the removal of the previous secularities leads to the system

$$\begin{aligned}
\phi_t^{(2)} + g\eta^{(2)} &= -\eta^{(1)}\phi_{tz}^{(0)} - \left(f_1 e^{i\theta} + \text{c.c.}\right) - \frac{1}{2}\eta^{(0)2}\phi_{tzz}^{(0)} \\
&\quad - \left(\eta^{(0)}\phi_x^{(0)}\phi_{xz}^{(0)} + \eta^{(0)}\phi_z^{(0)}\phi_{zz}^{(0)}\right) - \eta^{(0)}\phi_{Tz}^{(0)} \\
\eta_t^{(2)} - \phi_z^{(2)} &= -\left(\eta_x^{(1)}\phi_x^{(0)} + \eta^{(0)}\eta_x^{(0)}\phi_{xz}^{(0)}\right) \\
&\quad - \left(g_1 e^{i\theta} + \text{c.c.}\right) + \eta^{(1)}\phi_{zz}^{(0)} + \frac{1}{2}\eta^{(0)2}\phi_{zzz}^{(0)} - \eta_T^{(1)}.
\end{aligned}$$

As always, we will remove secular terms and solve for the remaining equations. In doing so we will substitute the previous solutions

$$\begin{cases}
\phi^{(0)} = A_1 e^{i\theta+|k|z} + \text{c.c.} \\
\eta^{(0)} = N_1 e^{i\theta} + \text{c.c.} \\
\eta^{(1)} = |k|^2 N_1^2 e^{i2\theta} + \text{c.c.} \\
\phi^{(1)} = 0.
\end{cases}$$

Thus, by substituting the previous solutions we arrive at

$$\begin{aligned}
\phi_t^{(2)} + g\eta^{(2)} &= (C_1 e^{i\theta} + \text{c.c.}) + (C_2 e^{2i\theta} + \text{c.c.} + (C_3 e^{3i\theta} + \text{c.c.}) + C_0, \\
\eta_t^{(2)} - \phi_z^{(2)} &= (D_1 e^{i\theta} + \text{c.c.}) + (D_2 e^{2i\theta} + \text{c.c.}) + (D_3 e^{3i\theta} + \text{c.c.}),
\end{aligned}$$

where the coefficients $C_1, C_2, C_3, C_0, D_1, D_2, D_3$ depend on A_1, N_1, k and ω. Removal of secular terms requires that the coefficients of $e^{\pm i\theta}$ be zero. To do this, we will look for a solution of the form

$$\begin{aligned}
\phi^{(2)} &= \left(A_1^{(2)} e^{i\theta+|k|z} + A_2^{(2)} e^{2i\theta+|k|z} + A_3 e^{3i\theta+|k|z} + \text{c.c.}\right) + A_0, \\
\eta^{(2)} &= \left(N_1^{(2)} e^{i\theta} + N_2^{(2)} e^{2i\theta} + N_3 e^{3i\theta} + \text{c.c.}\right) + N_0
\end{aligned}$$

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and use the method of undetermined coefficients. In doing so we call $f_1 = A_{1,\tau}$ and $g_1 = N_{1,\tau}$, where $\tau = \varepsilon T = \varepsilon^2 t$. Therefore, using previous solutions and removing the coefficients of $e^{\pm i\theta}$, we find the system

$$-i\omega A_1^{(2)} + gN_1^{(2)} = -\frac{3}{2}gk^2|N_1|^2N_1 - A_{1,\tau}, \tag{6.31}$$

$$-i\omega N_1^{(2)} - |k|A_1^{(2)} = -\frac{5i}{2}\omega k^2|N_1|^2N_1 - N_{1,\tau}. \tag{6.32}$$

Using (6.31) in (6.32) one arrives at

$$A_1^{(2)} = \frac{g}{i\omega}N_1^{(2)} + \frac{3}{2i\omega}gk^2|N_1|^2N_1 + \frac{1}{i\omega}A_{1,\tau}$$

and

$$-i\omega N_1^{(2)} - \frac{g|k|}{i\omega}N_1^{(2)} = \frac{3}{2i\omega}|k|k^2|N_1|^2N_1 + \frac{|k|}{i\omega}A_{1,\tau} - \frac{5i}{2}k^2\omega|N_1|^2N_1 - N_{1,\tau}.$$

Using $A_1 = -\dfrac{ig}{\omega}N_1$ and $\omega^2 = g|k|$ leads to

$$-2N_{1,\tau} - 4ik^2\omega|N_1|^2N_1 = 0,$$

or

$$N_{1,\tau} = -2ik^2\omega|N_1|^2N_1.$$

We have studied this type of equation before in Chapter 5 and have shown that $|N_1|^2(\tau) = |N_1|^2(0)$ and, therefore, that

$$N_1(\tau) = N_1(0)e^{-2ik^2\omega|N_1(0)|^2\tau}.$$

Hence the original free-surface solution is given to leading order by

$$\eta = N_1(0)e^{ikx-2i\omega(1+2\varepsilon^2k^2|N_1(0)|^2)t} + \text{c.c.},$$

or

$$\eta = a\cos\left[kx - \omega\left(1 + \frac{\varepsilon^2a^2k^2}{2}\right)t\right],$$

where $a = 2|N_1(0)|$. The total frequency is therefore approximately given by

$$\begin{aligned}\omega_{\text{new}} &= \omega\left(1 + 2\varepsilon^2k^2|N_1(0)|^2\right)\\ &= \omega\left(1 + \frac{\varepsilon^2a^2k^2}{2}\right).\end{aligned}$$

The $O(\varepsilon^2)$ term, i.e., $2\varepsilon^2\omega k^2|N_1(0)|^2$, corresponds to the nonlinear frequency shift.

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Note that we can also derive an equation for A_1. Indeed, using $N_1 = \frac{i\omega}{g} A_1$ gives that

$$A_{1,\tau} = -2ik^2 \frac{\omega^3}{g^2} |A_1|^2 A_1.$$

Using $\omega^2 = g|k|$ we get that

$$iA_{1,\tau} - \frac{2k^4}{\omega} |A_1|^2 A_1 = 0. \tag{6.33}$$

Similar to the derivation of N_1, the solution of this equation is given by

$$A_1(\tau) = A_1(0) e^{-2i\frac{k^4}{\omega}|A_1(0)|^2\tau} = A_1(0) e^{-2ik^2\omega|N_1(0)|^2\tau},$$

where in the last equation we have used $A_1 = -\frac{ig}{\omega} N_1$. This shows that A_1 and N_1 have the same nonlinear frequency shift.

It is remarkable that Stokes obtained this nonlinear frequency shift in 1847 (Stokes, 1847)! While his derivation method (a variant of the "Stokes–Poincaré" frequency-shift method we have described) was different from the one we use here in terms of multiple scales, and he used different nomenclature (sines and cosines instead of exponentials), he nevertheless obtained the same result to leading order, i.e.,

$$\omega_{\text{new}} = \omega \left(1 + \frac{\varepsilon^2 k^2 a^2}{2} + \cdots \right),$$

where $a = 2|N_1(0)|$.

6.5 Deep-water theory: NLS equation

In the previous section, we were concerned with deriving the nonlinear term of the NLS equation and thus allowed the slowly varying envelope A to depend only on the slow time $T = \varepsilon t$. Here, however, we outline the calculation when slow temporal and spatial variations are included. Since the water wave equations have an additional depth variable, z, we need to take some additional care. Therefore we discuss the calculation in some detail.

We will now use the structure of the water wave equations to suggest an ansatz for our perturbative calculation. The equations we will consider are

$$\phi_{xx} + \phi_{zz} = 0, \qquad -\infty < z < \varepsilon\eta \tag{6.34a}$$

$$\lim_{z\to-\infty} \phi_z = 0 \tag{6.34b}$$

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$$\phi_t + \frac{\varepsilon}{2}\left(\phi_x^2 + \phi_z^2\right) + g\eta = 0, \qquad z = \varepsilon\eta \tag{6.34c}$$

$$\eta_t + \varepsilon\eta_x\phi_x = \phi_z, \qquad z = \varepsilon\eta, \tag{6.34d}$$

i.e., the water wave equations in the deep-water limit. There are three distinct steps in the calculation. First, because of the free boundary, we expand $\phi = \phi(t, x, \varepsilon\eta)$ for $\varepsilon \ll 1$:

$$\phi = \phi(t, x, 0) + \varepsilon\eta\phi_z(t, x, 0) + \frac{(\varepsilon\eta)^2}{2}\phi_{zz}(t, x, 0) + \cdots \tag{6.35}$$

We similarly expand ϕ_t, ϕ_x, and ϕ_z. Then the free-surface equations (6.34c) and (6.34d) expanded around $z = 0$ take the form:

$$\left[\phi_t + \varepsilon\eta\phi_{tz} + \frac{1}{2}(\varepsilon\eta)^2\phi_{tzz}\right] + \frac{\varepsilon}{2}\left(\phi_x^2 + \phi_z^2 + 2\varepsilon\eta\phi_x\phi_{xz} + 2\varepsilon\eta\phi_z\phi_{zz}\right) + g\eta = 0,$$

and

$$\eta_t + \varepsilon\eta_x(\phi_x + \varepsilon\eta\phi_{xz}) = \phi_z + \varepsilon\eta\phi_{zz} + \frac{1}{2}(\varepsilon\eta)^2\phi_{zzz}.$$

Second, introduce slow temporal and spatial scales:

$$\begin{aligned}\phi(t, x, z) &= \phi(t, x, z, T, X, Z; \varepsilon)\\ \eta(t, x) &= \phi(t, x, \varepsilon\eta, T, X, \varepsilon),\end{aligned}$$

where $X = \varepsilon x$, $Z = \varepsilon z$, and $T = \varepsilon t$. Finally, because of the quadratic nonlinearity, we expect second harmonics and mean terms to be generated. This suggests the ansatz

$$\phi = \left(Ae^{i\theta+|k|z} + \text{c.c.}\right) + \varepsilon\left(A_2e^{2i\theta+2|k|z} + \text{c.c.} + \overline{\phi}\right) \tag{6.36a}$$

$$\eta = \left(Be^{i\theta} + \text{c.c.}\right) + \varepsilon\left(B_2e^{2i\theta} + \text{c.c.} + \overline{\eta}\right). \tag{6.36b}$$

The coefficients A, A_2 and $\overline{\phi}$ depend on X, Z, and T while B, B_2 and $\overline{\eta}$ depend on X, T. The rapid phase is given by $\theta = kx - \omega t$, with the dispersion relation $\omega^2 = g|k|$. Substituting the ansatz for ϕ into Laplace's equation (6.34a) we find

$$\begin{aligned} e^{i\theta}\left[2\varepsilon k\left(iA_X + \operatorname{sgn}(k)A_Z\right) + \varepsilon^2\left(A_{XX} + A_{ZZ}\right) + \cdots\right] &= 0,\\ e^0\left[\overline{\phi}_{XX} + \overline{\phi}_{ZZ}\right] &= 0.\end{aligned}$$

The first equation implies

$$\begin{aligned} A_Z &= -i\operatorname{sgn}(k)A_X - \frac{\varepsilon\operatorname{sgn}(k)}{2k}\left(A_{XX} + A_{ZZ}\right) + O(\epsilon^2)\\ &= -i\operatorname{sgn}(k)A_X + O(\varepsilon).\end{aligned} \tag{6.37}$$

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Taking the derivative of the above expression with respect to the slow variable Z gives

$$\begin{aligned} A_{ZZ} &= -i\,\mathrm{sgn}(k)A_{XZ} + O(\varepsilon) \\ &= -i\,\mathrm{sgn}(k)(-i\,\mathrm{sgn}(k))A_{XX} + O(\varepsilon) \\ &= -A_{XX} + O(\varepsilon), \end{aligned} \tag{6.38}$$

where we differentiated (6.37) with respect to X and substituted the resulting expression for A_{XZ} to get the second line. Since the $O(\varepsilon)$ term in (6.37) is proportional to $A_{XX} + A_{ZZ}$, we can use (6.38) to obtain

$$A_Z = -i\,\mathrm{sgn}(k)A_X + O(\varepsilon^2).$$

Substituting our ansatz (6.36) into the Bernoulli equation (6.34c) and kinematic equation (6.34d) with (6.35), we find, respectively,

$$\begin{aligned} e^{i\theta}\Big\{(-\,i\omega A + gB) + \varepsilon A_T + \varepsilon^2\Big[-\,i\omega k^2 A|B|^2 + 4k^2|k||A|^2 B + + 2k^2|k|A^2 B^* \\ + \frac{i}{2}\omega k^2 B^2 A^* + 4k^2 A_2 A^* - i\omega|k|A\overline{\eta} + i\omega|k|B_2 A^* \\ -4i\omega|k|A_2 B^*\Big] + \cdots\Big\} = 0, \end{aligned} \tag{6.39}$$

$$e^{2i\theta}\left\{A_2 - \varepsilon\frac{4k^2 A}{2|k|g}\left(A_T + \frac{\omega}{2k}A_X\right) + \cdots\right\} = 0, \tag{6.40}$$

$$e^0\left\{\overline{\phi}_Z - \overline{\eta}_T - \frac{2\omega k}{g}\frac{\partial}{\partial X}|A|^2 + \cdots\right\} = 0, \tag{6.41}$$

and

$$\begin{aligned} e^{i\theta}\Big\{(-i\omega B - |k|A) + \varepsilon\,[B_T + i\,\mathrm{sgn}(k)A_X] + \varepsilon^2\Big[\frac{k^2|k|}{2}\left(B^2A^* - 2|B|^2A\right) \\ +k^2\left(B_2A^* - 2B^*A_2\right) - k^2\overline{\eta}A\Big] + \cdots\Big\} = 0, \end{aligned} \tag{6.42}$$

$$e^{2i\theta}\left\{B_2 + \frac{k^2A^2}{g} - \varepsilon\frac{2ik}{g}AA_X + \cdots\right\} = 0, \tag{6.43}$$

$$e^0\left\{\overline{\eta} + O(\varepsilon)\right\} = 0. \tag{6.44}$$

Note that we used the result $A_Z = -i\,\mathrm{sgn}(k)A_X$ found earlier. Using (6.44), we find from (6.41) that

$$\overline{\phi}_Z = \frac{2\omega k}{g}\frac{\partial}{\partial X}|A|^2,$$

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i.e., up to $O(\varepsilon^2)$ the mean velocity potential depends explicitly on $|A|^2$. We also note that if A is independent of X these results agree with those from the previous section. Setting the coefficients of each power of ε to zero in (6.39) and (6.42) we get to leading order

$$-i\omega A + gB = 0$$
$$-|k|A - i\omega B = 0,$$

which, since the dispersion relationship $\omega^2 = g|k|$ is satisfied, has the non-trivial solution $B = \frac{i\omega}{g}A$. From (6.39) – (6.44), we now have

$$B = \frac{i\omega}{g}A - \frac{\varepsilon A_T}{g} + \varepsilon^2 \left[i\frac{\omega k^2}{g}A|B|^2 - 4\frac{k^2|k|}{g}|A|^2 B \right.$$
$$\left. - \frac{i\omega k^2}{2g}B^2 A^* + \frac{i\omega|k|k^2}{g^2}A^2 A^* \right] + O(\varepsilon^3).$$

Substituting this into (6.42), and with (6.39)–(6.44), yields

$$2i\omega\left(A_T + v_g A_X\right) - \varepsilon\left(A_{TT} + 4k^4|A|^2 A\right) + O(\varepsilon^2) = 0,$$

where we have defined the group velocity as $v_g = \omega'(k) = \omega/2k$. From this and (6.40), we see that $A_2 \sim O(\varepsilon^2)$. If we neglect the $O(\varepsilon^2)$ terms in the above equation and make the change of variables $\tau = \varepsilon T$, $\xi = X - v_g T$, we get the focusing NLS equation

$$iA_\tau + \frac{\omega''}{2}A_{\xi\xi} - \frac{2k^4}{\omega}|A|^2 A = 0. \tag{6.45a}$$

With $\omega'' = -v_g^2/\omega$, the above equation can be written as

$$iA_\tau - \left(\frac{v_g^2}{2\omega}A_{\xi\xi} + \frac{2k^4}{\omega}|A|^2 A\right) = 0, \tag{6.45b}$$

which is the typical formulation of the focusing NLS equation found in water wave theory. We also note that in terms of B, which is associated with the wave elevation η, using $A = \frac{g}{i\omega}B$, equation (6.45b) becomes

$$iB_\tau + \frac{\omega''}{2}B_{\xi\xi} - 2k^2\omega|B|^2 B = 0.$$

We note the important point that the coefficient of the nonlinear term in (6.45b) agrees with the Stokes frequency shift discussed in the previous section. As

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discussed earlier an alternative change of coordinates to a retarded time frame is to let $t' = T - X/v_g$ and $\chi = \varepsilon X$. We then get

$$iA_\chi + \frac{\omega''}{2(\omega')^3}A_{t't'} - \frac{2k^4}{\omega\omega'}|A|^2A = 0. \tag{6.46}$$

This formulation is more commonly found in the context of nonlinear optics.

This derivation of the NLS equation for deep-water waves was done in 1968 by Zakharov for deep water, including surface tension (Zakharov, 1968) and in the context of finite depth by Benney and Roskes (1969). It took more than a century from Stokes' (Stokes, 1847) initial discovery of the nonlinear frequency shift until these NLS equations were derived.

We remark that this NLS equation is called the "focusing" NLS because the signs of the dispersive and nonlinear terms are the same in (6.45b). To see that in (6.46), we recall that $\omega^2(k) = g|k|$ and therefore, for positive k, one gets that $\omega = \sqrt{gk}$, $v_g = \omega' = \sqrt{g/4k}$, and $\omega'' = -\sqrt{g}/4k^{3/2} = -v_g^2/\omega$; we also note $\frac{\omega''}{2(\omega')^2} = -\frac{1}{\omega}$. These results imply that the coefficient of the second derivative term and the nonlinear coefficient have the same sign. As we will see, the focusing NLS equation admits "bright" soliton solutions, i.e., solutions that are traveling localized "humps".

6.6 Some properties of the NLS equation

Note that the linear operator in (6.45a) is

$$\widehat{L} = i\partial_\tau + \frac{\omega''}{2}\partial_\xi^2,$$

and is what we expected to find from our earlier considerations (see Section 6.3). Similarly, the nonlinear part:

$$iA_\tau - \frac{2k^4}{\omega}|A|^2A = 0,$$

is what we expected from the frequency shift analysis in Section 6.4.1; see (6.33). We can rescale (6.45b) by $\xi = \frac{v_g}{\sqrt{2}}x$, $A = k^2u$, and $\tau = -2\omega^2 t$ to get the focusing NLS equation in standard form:

$$iu_t + u_{xx} + 2|u|^2u = 0. \tag{6.47}$$

Remarkably, this equation can be solved exactly using the so-called inverse scattering transform (Zakharov and Shabat, 1972); see also (Ablowitz et al., 2004b). One special solution is a "bright" soliton:

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$$u = \eta \operatorname{sech}\left[\eta\left(x + 2\xi t - x_0\right)\right] e^{-i\Theta},$$

where $\Theta = \xi x + (\xi^2 - \eta^2)t + \Theta_0$. The parameters ξ and η are related to an eigenvalue from the inverse scattering transform analysis via $\lambda = \xi/2 + i\eta/2$ where λ is the eigenvalue. If, instead of (6.47), we had the defocusing NLS equation,

$$iu_t + u_{xx} - 2|u|^2 u = 0, \tag{6.48}$$

then we can find "dark" – "black" or more generally "gray" – soliton solutions. Letting $t \to -t/2$, (6.48) goes to

$$iu_t - \frac{1}{2}u_{xx} + |u|^2 u = 0$$

and has a black soliton solution whose amplitude vanishes at the origin

$$u = \eta \tanh\left(\eta x\right) e^{i\eta^2 t}.$$

Note that $u \to \pm\eta$ as $x \to \pm\infty$. A gray soliton solution is given by

$$u(x,t) = \eta e^{2i\eta^2 t + i\psi_0}\left[\cos\alpha + i\sin\alpha\, \tanh\left[\sin\alpha\, \eta(x - 2\eta\cos\alpha\, t - x_0)\right]\right]$$

with $\eta, \alpha, x_0, \psi_0$ arbitrary real parameters. In Figure 6.1 a "bright" and the two dark (black and gray) are depicted.

These solutions satisfy the boundary conditions

$$u(x,t) \to u_\pm(t) = \eta e^{2i\eta^2 t + i\psi_0 \pm i\alpha} \qquad \text{as} \quad x \to \pm\infty$$

and appear as localized dips of intensity $\eta^2 \sin^2\alpha$ on the background field η. The gray soliton moves with velocity $2\eta\cos\alpha$ and reduces to the dark (black) soliton when $\alpha \to \pi/2$ with $\psi_0 = -\pi/2$ (Hasegawa and Tappert, 1973b; Zakharov and Shabat, 1973); see also Prinari et al. (2006) where the vector IST problem for non-decaying data is discussed in detail.

A property of the NLS equation we will investigate next is its Galilean invariance. That is, if $u_1(x,t)$ is a solution of (6.47) then so is

$$u_2(x,t) = u_1(x - vt, t)e^{i(kx - \omega t)},$$

with $k = v/2$ and $\omega = k^2$. Substituting u_2 into (6.47) we find u_1 satisfies:

$$iu_{1,t} + \omega u_1 - ivu_{1,x} + (u_{1,xx} + 2iku_{1,x} - k^2 u_1) + 2|u_1|^2 u_1 = 0.$$

Using the fact that u_1 is assumed to be a solution of (6.47) and using the values for k, ω, this implies u_2 also satisfies (6.47).

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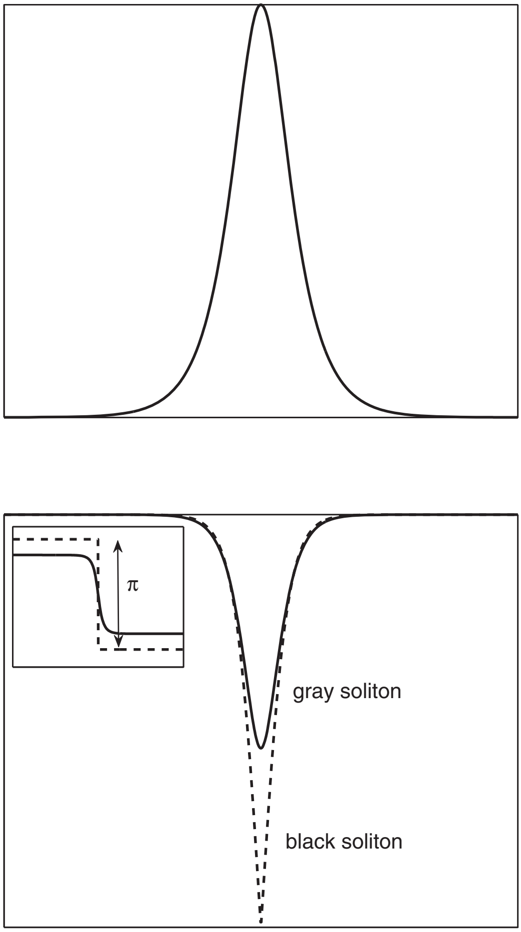

Figure 6.1 Bright (top) and dark (bottom) solitons of the NLS. In the inset we plot the relative phases.

Another important result involves the linear stability of a special periodic solution of (6.47). In (6.45a), i.e., the standard water wave formulation, take A independent of ξ to get

$$iA_\tau = \frac{2k^4}{\omega}|A|^2 A.$$

Note that this agrees with the results of the Stoke's frequency shift calculation, (6.33). With the change of variables mentioned above in terms of the standard NLS, (6.47), this means:

$$iu_t = -2|u|^2 u,$$

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which has the plane wave solution $u = ae^{2ia^2t}$, $a = u(0)$; for convenience take a real. We now perturb this solution: $u = ae^{2ia^2t}(1 + \varepsilon(x,t))$, where $|\varepsilon| \ll 1$. Substituting this into (6.47) and linearizing (i.e., assuming $|\varepsilon| \ll 1$) we find,

$$i\varepsilon_t + \varepsilon_{xx} + 2a^2 (\varepsilon + \varepsilon^*) = 0.$$

We will consider the linearized problem on the periodic spatial domain $0 < x < L$. Thus, $\varepsilon(x,t)$ has the Fourier expansion

$$\varepsilon(x,t) = \sum_{-\infty}^{\infty} \widehat{\varepsilon}_n(t) e^{i\mu_n x},$$

where $\mu_n = 2\pi n/L$. Note that $\widehat{\varepsilon}_{-n}(t)$ is not the complex conjugate of $\widehat{\varepsilon}_n(t)$, since ε is not necessarily real. Since the PDE is linear, it is sufficient to consider $\varepsilon = \widehat{\varepsilon}_n(t)e^{i\mu_n x} + \widehat{\varepsilon}_{-n}(t)e^{-i\mu_n x}$. Thus, with $\varepsilon'_n \equiv \partial\varepsilon_n/\partial t$,

$$\begin{aligned} i\left(\widehat{\varepsilon}'_n e^{i\mu_n x} + \widehat{\varepsilon}'_{-n} e^{-i\mu_n x}\right) - \mu_n^2 \left(\widehat{\varepsilon}_n e^{i\mu_n x} + \widehat{\varepsilon}_{-n} e^{-i\mu_n x}\right) \\ + 2a^2 \left(\widehat{\varepsilon}_n e^{i\mu_n x} + \widehat{\varepsilon}_{-n} e^{-i\mu_n x} + \widehat{\varepsilon}^*_n e^{-i\mu_n x} + \widehat{\varepsilon}^*_{-n} e^{i\mu_n x}\right) = 0. \end{aligned}$$

Setting to zero the coefficients of $e^{i\mu_n x}$ and $e^{-i\mu_n x}$, we find, respectively,

$$\begin{aligned} i\widehat{\varepsilon}'_n - \mu_n^2 \widehat{\varepsilon}_n + 2a^2 (\widehat{\varepsilon}_n + \widehat{\varepsilon}^*_{-n}) = 0 \\ i\widehat{\varepsilon}'_{-n} - \mu_n^2 \widehat{\varepsilon}_{-n} + 2a^2 (\widehat{\varepsilon}_{-n} + \widehat{\varepsilon}^*_n) = 0. \end{aligned}$$

Taking the conjugate of the last equation and multiplying it by -1, we have the system

$$i\frac{\partial}{\partial t}\begin{pmatrix} \widehat{\varepsilon}_n \\ \widehat{\varepsilon}^*_{-n} \end{pmatrix} + \begin{pmatrix} 2a^2 - \mu_n^2 & 2a^2 \\ -2a^2 & -2a^2 + \mu_n^2 \end{pmatrix}\begin{pmatrix} \widehat{\varepsilon}_n \\ \widehat{\varepsilon}^*_{-n} \end{pmatrix} = 0,$$

to solve. Assuming a solution of the form

$$\begin{pmatrix} \widehat{\varepsilon}_n \\ \widehat{\varepsilon}^*_{-n} \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} e^{i\sigma_n t},$$

we find

$$\det\begin{pmatrix} 2a^2 - \mu_n^2 - \sigma_n & 2a^2 \\ -2a^2 & -2a^2 + \mu_n^2 - \sigma_n \end{pmatrix} = 0$$

must be true, hence with $\mu_n^2 = (2n\pi/L)^2$

$$\sigma_n^2 = \left(\frac{2\pi n}{L}\right)^2 \left[\left(\frac{2\pi n}{L}\right)^2 - 4a^2\right].$$

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Thus, when

$$\frac{aL}{\pi} > n$$

the system is unstable, since $\sigma_n^2 < 0$ leads to exponential growth. Note that there are only a finite number of unstable modes. In the context of water waves, we have deduced the famous result by Benjamin and Feir (1967) that the Stokes water wave is unstable. Later, Benney and Roskes (1969) (BR) showed that all periodic wave solutions of the slowly varying envelope water wave equations in 2 + 1 dimensions are unstable. The BR equations are also discussed below. We also remark that Zakharov and Rubenchik (1974) showed that solitons are unstable to weak transverse modulations, i.e., one-dimensional soliton solutions of

$$iu_t + u_{xx} + \varepsilon^2 u_{yy} + 2|u|^2 u = 0$$

are unstable. (See Ablowitz and Segur, 1979 for further discussion.)

6.7 Higher-order corrections to the NLS equation

As mentioned earlier, the nonlinear Schrödinger equation was first derived in 1968 in deep water with surface tension by Zakharov (1968) and its finite depth analog in 1969 by Benney and Roskes (1969). It took some ten years until in 1979 Dysthe (1979) derived, in the context of deep-water waves, the next-order correction to the NLS equation. We simply quote the $(1+1)$-dimensional result:

$$\begin{aligned} &2i\omega\left(A_T + v_g A_X\right) - \varepsilon\left(v_g^2 A_{XX} + 4k^2|A|^2 A\right) \\ &= \varepsilon^2\left(\frac{i\omega^2}{8k^3}A_{XXX} + 2ik^3A^2A_X^* - 12ik^3|A|^2A_X + 2\omega k\overline{\phi}_X A\right). \end{aligned} \tag{6.49}$$

The mean term $\overline{\phi}_X$ is unique to deep-water waves and arises from the quadratic nonlinearity in the water wave equations. The mean field satisfies

$$\overline{\phi}_{XX} + \overline{\phi}_{ZZ} = 0, \qquad -\infty < z < 0,$$

with $\overline{\phi}_Z \to 0$ as $z \to -\infty$ and

$$\overline{\phi}_Z = \frac{2\omega k}{g}\frac{\partial}{\partial X}|A|^2, \qquad z = 0.$$

Non-dimensionalizing (6.49) by setting $T' = \omega T$, $X' = kX$, $Z' = kZ$, $\eta' = k\eta$, $\overline{\phi}' = (2k^2/\omega)\overline{\phi}$, $u = \left(2\sqrt{2}k^2/\omega\right)A$, $\tau = -\varepsilon T'/8$, and $\xi = X' - T'/2$, we get

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$$iu_\tau + u_{\xi\xi} + 2|u|^2 u + \varepsilon\left(\frac{i}{2}u_{\xi\xi\xi} - 6i|u|^2 u_\xi + iu^2 u_\xi^* + 2u\mathcal{H}[|u|^2]_\xi\right) = 0, \qquad (6.50)$$

where

$$\mathcal{H}[f] = \frac{1}{\pi}\int \frac{f(u)}{u-x}\,du$$

is the so-called Hilbert transform. (The integral is understood in the principle-value sense.) Ablowitz, Hammack, Henderson, and Schober (Ablowitz et al., 2000a, 2001) showed that periodic solutions of (6.50) can actually be chaotic. Experimentally and analytically, a class of periodic solutions is found not to be "repeatable", while soliton solutions are repeatable.

To go from the term including $\overline{\phi}_x$ in (6.49) to the Hilbert transform requires some attention. The velocity potential satisfies Laplace's equation:

$$\overline{\phi}_{xx} + \overline{\phi}_{zz} = 0.$$

In the Fourier domain, with $\widehat{\overline{\phi}} = \displaystyle\int \overline{\phi} e^{-i\xi x}\,dx$, we find

$$\widehat{\overline{\phi}}_{zz} - \xi^2\widehat{\overline{\phi}} = 0.$$

Using the boundary condition $\widehat{\overline{\phi}}_z \to 0$ as $z \to -\infty$, we get

$$\widehat{\overline{\phi}}(\xi, z) = C(\xi)e^{|\xi|z},$$

where C is to be determined. The factor C is fixed by the boundary condition

$$\overline{\phi}_z = \frac{2\omega k}{g}\frac{\partial}{\partial x}|A|^2,$$

on $z = 0$. Hence

$$C(\xi) = \frac{2i\omega k}{g}\frac{\xi}{|\xi|}\mathcal{F}(|A|^2) = \frac{2i\omega k}{g}\mathrm{sgn}(\xi)|\mathcal{F}(|A|^2)$$

where $\mathcal{F}$ represents the Fourier transform. Using a well-known result from Fourier transforms, $\mathcal{F}^{-1}[\mathrm{sgn}\xi\hat{F}(\xi)] = \mathcal{H}[f(x)]$, where f is the inverse Fourier transform of $\hat{F}$ and $\mathcal{H}[\cdot]$ is the Hilbert transform (Ablowitz and Fokas, 2003) and noting that $\mathcal{H}\left[e^{i\xi x}\right] = i\,\mathrm{sgn}(\xi)e^{i\xi x}$, we have

$$\overline{\phi} = \frac{2\omega k}{g}\mathcal{H}\left[\ |A|^2\right]$$

$$\overline{\phi}_x = \frac{2\omega k}{g}\frac{\partial}{\partial x}\mathcal{H}\left[\ |A|^2\right].$$

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6.8 Multidimensional water waves

For multidimensional water waves in finite depth with surface tension, the velocity potential has the form

$$\phi = \varepsilon(\tilde{\phi}(X, Y, T) + \frac{\cosh(k(z+h))}{\cosh(kh)}(\tilde{A}(X, Y, T)e^{i(kx-\omega t)} + \text{c.c.})) + O(\varepsilon^2),$$

where $X = \varepsilon x$, $Y = \varepsilon y$, $T = \varepsilon t$ and $\tilde{\phi}$, $\tilde{A}$ satisfy coupled nonlinear wave equations. Benney and Roskes (1969) derived this system without surface tension. It was subsequently rederived by Davey and Stewartson (1974) who put the system in a simpler form. Later, Djordjevic and Redekopp (1977) included surface tension.

After non-dimensionalization and rescaling, the equations can be put into the following form; we call it a Benney–Roskes (BR) system:

$$\begin{aligned} iA_t + \sigma_1 A_{xx} + A_{yy} &= \sigma_2 |A|^2 A + A\Phi_x \\ a\Phi_{xx} + \Phi_{yy} &= -b(|A|^2)_x \\ \sigma_1 = \pm 1, \qquad & \sigma_2 = \pm 1. \end{aligned} \tag{6.51}$$

The parameters σ_1, σ_2, a, and b are dimensionless and depend on the dimensionless fluid depth and surface tension. Here we have presented it in a rescaled, normalized form that helps in analyzing its behavior for different choices of σ_1 and σ_2. The quantity A is related to the slowly varying envelope of the first harmonic of the potential velocity field and Φ is related to the slowly varying mean potential velocity field.

A special solution to (6.51) is the following self-similar solution

$$\begin{aligned} A &= \frac{\Lambda}{t} \exp i\left(\frac{\sigma_1 x^2 + y^2}{4t} + \sigma_2 \frac{\Lambda^2}{t} + B(t) + \phi_0\right) \\ \Phi &= -B'(t)x + C(t)y + D(t). \end{aligned}$$

This is an analog of the similarity solution of the one-dimensional nonlinear Schrödinger (NLS) equation

$$\begin{gathered} iA_t + A_{xx} + \sigma |A|^2 A = 0 \\ A = \frac{\Lambda}{t^{1/2}} \exp i\left(\frac{x^2}{4t} + \sigma\Lambda^2 \log(t) + \phi_0\right). \end{gathered}$$

Since the above similarity solution of NLS approximates the long-time solution of NLS without solitons in the region $\left|\frac{x}{\sqrt{t}}\right| \le O(1)$, it can be expected that the similarity solution of the BR system is a candidate to approximate long-time "radiative" solutions.

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From the water wave equations in the limit $kh \to 0$ (the shallow-water limit) with suitable rescaling, cf. Ablowitz and Segur (1979), the BR system (6.51) reduces to

$$\begin{aligned} iA_t - \gamma A_{xx} + A_{yy} &= A(\gamma |A|^2 + \Phi_x) \\ \gamma \Phi_{xx} + \Phi_{yy} &= -2(|A|^2)_x, \qquad \gamma = \operatorname{sgn}\left(\tfrac{1}{3} - \hat{T}\right) = \pm 1. \end{aligned} \tag{6.52}$$

This is the so-called Davey–Stewartson (DS) equation. As we have mentioned, it describes multidimensional water waves in the slowly varying envelope approximation, with surface tension included, in the shallow-water limit. Here the normalized surface tension is defined as $\hat{T} = \dfrac{T_0}{\rho g h^2}$, T_0 being the surface tension coefficient. This system, (6.52), is integrable (Ablowitz and Clarkson, 1991) whereas the multidimensional deep-water limit

$$iA_t + \nabla^2 A + |A|^2 A = 0$$

is apparently not integrable. The concept of integrability is discussed in more detail in Chapters 8 and 9 (cf. also Ablowitz and Clarkson 1991).

The DS equation (6.52) can be generalized by making the following substitutions

$$\phi = \Phi_x, \qquad r = -\sigma q^* = -\sigma A^*, \qquad \sigma = \pm 1$$

and we write

$$\begin{aligned} iq_t - \gamma q_{xx} + q_{yy} &= q(\phi - qr) \\ \phi_{xx} + \gamma \phi_{yy} &= 2(qr)_{xx}. \end{aligned} \tag{6.53}$$

With $q = A$, $\gamma = -1$, and $\sigma = 1$, we get shallow-water waves with "large" surface tension, equation (6.52). It turns out that the generalized DS equation (GDS), (6.53), admits "localized" boundary induced pulse solutions when $\gamma = 1$ (Ablowitz and Clarkson, 1991) and weakly decaying "lump"-type solutions when $\gamma = -1$ (Villarroel and Ablowitz, 2003).

6.8.1 Special solutions of the Davey–Stewartson equations

Consider the case where $\gamma = -1$ with $\sigma = 1$. Then the DS system (6.53) becomes

$$\begin{aligned} 2iq_t + q_{xx} + q_{yy} &= 2(\phi - qr)q \\ \phi_{xx} - \phi_{yy} &= 2(qr)_{xx}. \end{aligned} \tag{6.54}$$

We call this the DSI system. Note that the DSI system admits an interesting class of solutions, cf. Fokas and Santini (1989, 1990) and Ablowitz et al.

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(2001a). In order to write down the pulse solution (also called a "dromion"), we will make a convenient rescaling and change of variable as follows:

$$Q = \phi - qr.$$

Then the second equation in GDS system (6.54) becomes

$$Q_{xx} - Q_{yy} = (qr)_{xx} + (qr)_{yy}. \tag{6.55}$$

Now we make the change of variable

$$\xi = \frac{x+y}{\sqrt{2}}, \quad \eta = \frac{x-y}{\sqrt{2}}$$

$$\partial_x = \frac{1}{\sqrt{2}}(\partial_\xi + \partial_\eta), \quad \partial_y = \frac{1}{\sqrt{2}}(\partial_\xi - \partial_\eta)$$

to transform (6.55) to

$$2Q_{\xi\eta} = (qr)_{\xi\xi} + (qr)_{\eta\eta}. \tag{6.56}$$

Finally, making the substitution

$$Q = -(U_1 + U_2),$$

and integrating with respect to ξ and η separates (6.56) into

$$U_1 = u_1(\eta) - \frac{1}{2}\int_{-\infty}^{\xi}(qr)_\eta \, d\xi', \quad U_2 = u_2(\xi) - \frac{1}{2}\int_{-\infty}^{\eta}(qr)_\xi \, d\eta.$$

These two equations along with the transformed second equation in (6.54),

$$2iq_t + q_{\xi\xi} + q_{\eta\eta} + 2(U_1 + U_2)q = 0,$$

give rise to the following "dromion" solution

$$u_1(\eta) = 2\lambda_R^2 \operatorname{sech}^2(\lambda_R(\hat{\eta} - \eta_0)), \quad \hat{\eta} = \eta - 2\lambda_I t$$

$$u_2(\xi) = 2\mu_R^2 \operatorname{sech}^2(\mu_R(\hat{\xi} - \xi_0)), \quad \hat{\xi} = \xi - 2\mu_I t$$

$$q = \frac{\rho\sqrt{\lambda_R\mu_R}e^{i\theta}}{\cosh(\mu_R(\hat{\xi} - \xi_0))\cosh(\lambda_R(\hat{\eta} - \eta_0)) + (|\rho|/2)^2 e^{(\lambda_R(\hat{\eta}-\eta_0))}e^{\mu_R(\hat{\xi}-\xi_0)}}$$

$$\theta = -(\mu_I\hat{\xi} + \lambda_I\hat{\eta}) + (|\mu|^2 + |\lambda|^2)\frac{t}{2} - \theta_0$$

$$\lambda = \lambda_R + i\lambda_I, \quad \mu = \mu_R + i\mu_I \qquad \text{constants.}$$

where $\lambda_R > 0, \mu_R > 0$, λ, μ, ρ are complex constants and ξ_0, η_0 are real constants. See Figure 6.2 where a typical dromion is depicted (with $\lambda_R = 1, \mu_R = 1, \lambda_I = 0, \mu_I = 0, \rho = 1$).

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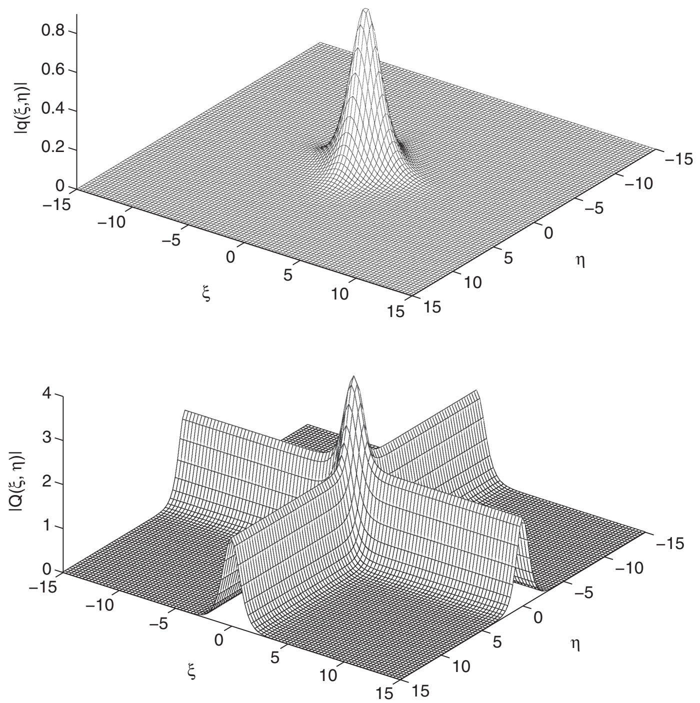

Figure 6.2 The dromion solution for the Benney–Roskes/Davey–Stewartson systems. The top figure represents the field q, while the bottom figure represents the mean field $Q = u_1 + u_2$.

6.8.2 Lump solution for small surface tension

It turns out that there are lump-type solutions to (6.52) when $\gamma = +1$; this is the so-called DSII system (Villarroel and Ablowitz, 2002). We now consider this case ($\gamma = +1$) corresponding to zero or "small" surface tension $\left(\hat{T} < \dfrac{1}{3}\right)$. Then the GDS system (6.53) is written, after substituting in $r = -\sigma q^*$, $\sigma = \pm 1$,

$$\begin{aligned} iq_t - q_{xx} + q_{yy} &= q(\phi + \sigma|q|^2) \\ \phi_{xx} + \phi_{yy} &= -2\sigma(|q|^2)_{xx}. \end{aligned} \tag{6.57}$$

The following is a lump solution

$$q = 2\rho\sigma \frac{e^{i\theta}}{\hat{x}^2 + \hat{y}^2 + \sigma|\rho|^2}$$

$$\phi + \sigma|q|^2 = R_{xx} - R_{yy}$$

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$$
\begin{aligned}
R &= \log(\hat{x}^2 + \hat{y}^2 + \sigma|\rho|^2) \\
\theta &= a\hat{x} - b\hat{y} + (b^2 - a^2)t - \theta_0 \\
\hat{x} &= x + at - x_0, \quad \hat{y} = y + bt - y_0 \\
k &= \tfrac{1}{2}(a + ib) \quad \text{constant.}
\end{aligned} \tag{6.58}
$$

This solution is non-singular when $\sigma = 1$, but for $\sigma = -1$, the above solution is singular. Alternatively we have the interesting case of the dark lump-type envelope hole solution (Satsuma and Ablowitz, 1979) when $\sigma = -1$. The case $\sigma = 1$ occurs in water waves; i.e., in this case, (6.57) can be transformed to the reduced water wave equation (6.52). But it has been shown that the lump solution (6.58) is unstable (Pelinovsky and Sulem, 2000).

We also note that Ablowitz et al. (1990) discussed the derivation of the integrable equations (6.54) and (6.57) from the KPI and KPII equations, respectively.

6.8.3 Multidimensional problems and wave collapse

As mentioned earlier, in 1969 Benny and Roskes derived a (2+1)-dimensional NLS-type equation for water waves (Benney and Roskes, 1969). In 1977 Djordjevic and Redekopp extended their results by including surface tension effects (Djordjevic and Redekopp, 1977). To get these equations the velocity potential expansion takes the form:

$$
\begin{gathered}
\phi = \varepsilon(\tilde{\phi}(X, Y, T) + \frac{\cosh\,[k(h + z)]}{\cosh kh}\left(\tilde{A}(X, Y, T)e^{i\theta} + \text{c.c.}\right)) + O(\varepsilon^2), \\
\theta = kx - \omega t, X = \varepsilon x, Y = \varepsilon y, T = \varepsilon t \\
\omega^2 = g\kappa \tanh(\kappa h)(1 + \widetilde{T}), \\
\kappa^2 = k^2 + l^2, \\
\widetilde{T} = \frac{k^2 T}{\rho g}.
\end{gathered}
$$

In terms of the redefined functions: $A = \tilde{A}\dfrac{k^2}{\sqrt{gk}}$, $\Phi = \dfrac{k^2}{\sqrt{gk}}\tilde{\phi}$, and variables: $\xi = k(X - \omega' T)$, $\tau = \sqrt{gk}\varepsilon^2 t$, $\eta = \varepsilon k Y$, we have that A, Φ satisfy the following coupled system

$$
iA_\tau + \lambda A_{\xi\xi} + \mu A_{\eta\eta} = \chi|A|^2 A + \chi_1 A\Phi_\xi \tag{6.59}
$$

$$
\alpha\Phi_{\xi\xi} + \Phi_{\eta\eta} = -\beta\left(|A|^2\right)_\xi. \tag{6.60}
$$

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The parameters λ, μ, χ, χ_1, α, and β depend on ω, k, $\widetilde{T}$, and gh (see Ablowitz and Segur, 1979, 1981). As $h \to \infty$, (6.59) reduces to an NLS equation

$$iA_\tau + \lambda_\infty A_{\xi\xi} + \mu_\infty A_{\eta\eta} - \chi_\infty |A|^2 A = 0,$$

where

$$\begin{aligned}
\lambda_\infty &= -\frac{\omega_0}{8\omega}\left(\frac{1-6\widetilde{T}-3\widetilde{T}^2}{1+\widetilde{T}}\right)\\
\mu_\infty &= \frac{\omega_0}{4\omega}\left(1+3\widetilde{T}\right)\\
\chi_\infty &= \frac{\omega_0}{4\omega}\left[\frac{8+\widetilde{T}+2\widetilde{T}^2}{\left(1-2\widetilde{T}\right)\left(1+\widetilde{T}\right)}\right]\\
\omega_0^2 &= g\kappa,
\end{aligned}$$

with $\lambda_\infty > 0$, $\mu_\infty > 0$ and $-\xi_\infty > 0$ for $\tilde{T}$ large enough. This equation has solutions that blow-up in finite time (Ablowitz and Segur, 1979). When $\widetilde{T} = 1/2$, the expansion breaks down due to second harmonic resonance: $\omega^2(2k) = [2\omega(k)]^2$. However, when $\widetilde{T} = 0$, i.e., when there is no surface tension, some of the coefficients in the above equation change sign and we have $\left(\frac{\omega_0}{\omega} > 0\right)$: $\lambda_\infty < 0$, $\mu_\infty > 0$ and $\chi_\infty > 0$. To date no blow-up has been found for the above NLS equation with the latter choices of signs.

Blow-up

One can verify that

$$iA_t + \Delta A + |A|^2 A = 0, \qquad x \in \mathbb{R}^2$$

has the conserved quantities

$$\begin{aligned}
P(t) &= \int |A|^2 \, d\boldsymbol{x},\\
\underline{M}(t) &= \int A\nabla A \, d\boldsymbol{x}\\
H(t) &= \int |\nabla A|^2 - \frac{1}{2}|A|^4 \, d\boldsymbol{x},
\end{aligned}$$

i.e., mass (power), momentum, and energy (Hamiltonian) are conserved. Define

$$V(t) = \int |\mathbf{r}|^2 |A|^2 \, d\boldsymbol{x},$$

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where $|\mathbf{r}|^2 = x^2 + y^2$. Then one can show that in two dimensions (Vlasov et al., 1970),

$$\frac{d^2V}{dt^2} = 8H.$$

This is often called the virial theorem. Integrating this equation, we get

$$V(t) = 4Ht^2 + c_1 t + c_0.$$

If the initial conditions are such that $H < 0$, then there exists a time t^* when

$$\lim_{t\to t^*} V(t) = 0$$

and for $t > t^*$, $V(t) < 0$. However, V is a positive quantity. Thus a singularity in the solution has occurred in finite time $t = t^*$. Combining this result with the conservation of mass and a bit more analysis, one can show that in fact

$$\int |\nabla A|^2 \, d\boldsymbol{x}$$

becomes infinite as $t \to t^*$, which in turn implies that A also becomes infinite as $t \to t^*$. For an extensive discussion of the NLS equation in one and multidimensions, see Sulem and Sulem (1999).

The more general equation

$$i\psi_t + \Delta_d\psi + |\psi|^{2\sigma}\psi = 0, \qquad x \in \mathbb{R}^d, \tag{6.61}$$

where Δ_d is the d-dimensional Laplacian, has also been studied. There are three cases:

- "supercritical": $\sigma d > 2$ blow-up occurs;
- "critical": $\sigma d = 2$ blow-up can occur; collapse can be arrested with small perturbations;
- "subcritical": $\sigma d < 2$ global solutions exist.

A special solution to (6.61) can be found by assuming a solution of the form $\psi = f(r)e^{i\lambda t}$. In two dimensions, we find the nonlinear eigenvalue problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + f^3 - \lambda f = 0,$$

certain solutions to which are the so-called Townes modes. Asymptotically, $f \sim e^{-r}/\sqrt{r}$, $1 \ll r$ (Fibich and Papanicolaou, 1999). Weinstein (1983) showed that there is a critical energy (found from the Townes mode with the smallest energy),

$$E_c = \int |u|^2 dxdy = 2\pi \int_0^\infty rf^2(r)\, dr \simeq 2\pi(1.86) \simeq 11.68.$$

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Pulses with energy below E_c exist globally (Weinstein, 1983). In many cases when $E > E_c$, even by a small amount, blow-up occurs. In the supercritical case, the blow-up solution has a similarity form

$$\psi \sim \frac{1}{t'^{\alpha}} f\left(\frac{r}{t'^{\beta}}\right) e^{i\lambda \ln t'},$$

with suitable constants α and β. But for the critical case the structure is

$$\psi \sim \frac{\sqrt{L(t')}}{\sqrt{t'}} f\left(\frac{r\sqrt{L(t')}}{\sqrt{t'}}\right) e^{i\phi(t')},$$

where $t' = t_c - t$, $L(t) \sim \ln(\ln(1/t'))$ as $t \to t_c$ is the blow-up or collapse time.

The stationary states, collapse and properties of the BR equations (6.59) (sometimes referred to as Davey–Stewartson-type equations) as well as similar ones that arise in nonlinear optics were studied in detail by Papanicolaou et al. (1994) and Ablowitz et al. (2005). More specifically, the equations studied were

$$iU_z + \frac{1}{2}\Delta U + |U|^2 U - \rho U V_x = 0, \quad \text{and}$$
$$V_{xx} + \nu V_{yy} = (|U|^2)_x.$$

The case $\rho < 0$ corresponds to water waves, cf. Ablowitz and Segur (1979, 1981), and the case $\rho > 0$ corresponds to $\chi^{(2)}$ nonlinear optics (Ablowitz et al., 1997, 2001a; Crasovan et al., 2003).

When $\nu > 0$ collapse is possible. That there is a singularity in finite time can be shown by the virial theorem. In analogy with the Townes mode for NLS, there are stationary states (ground states) that satisfy $U = F(x, y)e^{i\mu z}$, $V = G(x, y)$

$$-\mu F + \frac{1}{2}\Delta F + |F|^2 F - \rho F G_x = 0$$
$$G_{xx} + \nu G_{yy} = (|F|^2)_x.$$

The stationary states F, G can be obtained numerically (Ablowitz et al., 2005). In Figure 6.3 slices of the modes along the $y = 0, x = 0$ axes, respectively, are given for different values of ρ. In Figure 6.3(c) and (d) contour plots of some typical modes F are shown. One can see that the modes are elliptical in nature when $\rho \neq 0$.

Quasi-self-similar collapse

Papanicolaou et al. (1994) showed that as collapse occurs, i.e., as $z \to z_c$, then U, V have a quasi-self-similar structure:

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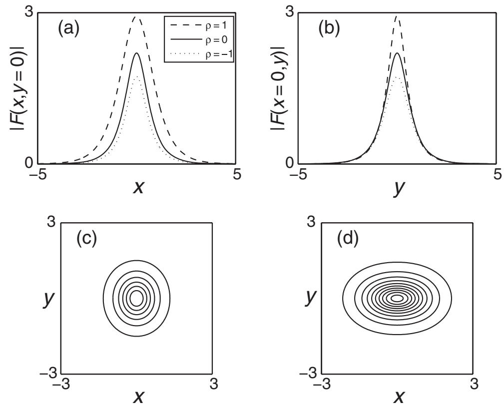

Figure 6.3 $F(x, y)$; $\nu = 0.5$; top (a), (b): "slices"; bottom: contour plots; (c) $\rho = -1$, (d)$\rho = 1$.

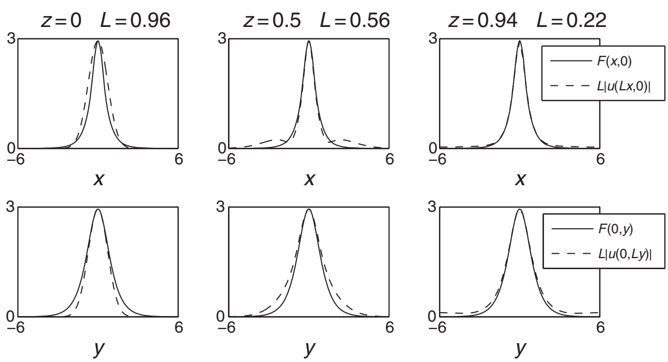

Figure 6.4 Wave collapse to the stationary state as $z \to z_c$.

$$U \sim \frac{1}{L(z)} F\left(\frac{x}{L(z)}, \frac{y}{L(z)}\right), \quad V \sim \frac{1}{L(z)} G\left(\frac{x}{L(z)}, \frac{y}{L(z)}\right)$$

where $L \to 0$. Further, one finds from direct simulation (Ablowitz et al., 2005) that the stationary modes are good approximations of collapse profiles. By taking typical initial conditions (of Gaussian type) that collapse, we find that the pulse structure is well approximated by the stationary modes in the neighborhood of the collapse point, where $L(z) = F(0,0)/U(0,0,z) \to 0$. In Figure 6.4 we compare $L(z)|U(Lx, Ly)|$ with $F(x, y)$ where $L(z) = F(0,0)/U(0,0,z)$. As collapse occurs, $L(z) \to 0$ for typical values $(\nu, \rho) = (0.5, 1)$. In Figure 6.4 we provide snapshots of the solution compared to the stationary state F as the wave

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collapse begins to occur. Along the $y = 0$ axis (top) and the $x = 0$ axis (bottom) at the edges of the solution, radiation is seen. But the behavior at the center of the wave indicates that collapse is well approximated by the stationary state.

Finally, we mention that a similar, but more complicated blow-up or wave collapse phenomenon occurs for the generalized KdV equation

$$u_t + u^p u_x + u_{xxx} = 0$$

when $p \geq 4$ (Ablowitz and Segur, 1981; Merle, 2001; Angulo et al., 2002).

Exercises

6.1 Derive the NLS equation from the following nonlinear equations.
(a) $u_{tt} - u_{xx} + sinu = 0$ (sine–Gordon).
(b) $u_{tt} - u_{xx} + u_{xxxx} + (uu_x)_x = 0$ (Boussinesq-type).
(c) $u_t + u^2 u_x + u_{xxx} = 0$ (mKdV) .

6.2 Given the equation

$$iu_t - u_{xxxx} + |u|^{2(n+1)} u = 0$$

with $n \geq 1$ an integer, substitute the mutli-scale and quasi-monochromatic assumption:

$$u \sim \mu e^{(ikx - \omega t)} A(\epsilon_1 x, \epsilon_2 t),$$

where μ, ϵ_1, and ϵ_2 are all asymptotically small parameters, into the equation. Choose a maximal balance between the small parameters to find a nonlinear Schrödinger-type wave equation for the slowly varying envelope A.

6.3 From the water wave equations with surface tension included, derive the dispersion relation

$$\omega^2 = g\kappa \tanh(\kappa h)(1 + \widetilde{T}),\ \kappa^2 = k^2 + l^2,\ \widetilde{T} = \frac{k^2 T_0}{\rho g}$$

in finite depth water waves with surface tension, where T_0 is the surface tension coefficient.

6.4 Beginning with the KP equation

$$\partial_x(u_t + 6uu_x + u_{xxx}) + 3\sigma u_{yy} = 0, \sigma = \pm 1$$

derive the integrable Davey–Stewartson equation by employing the slowly varying quasi-monochromatic wave expansion. Hint: see Ablowitz et al. (1990).

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6.5 Suppose we are given a "modified KP" equation

$$\partial_x\left(u_t+\frac{3}{n+1}(u^{2(n+1)})_x+u_{xxx}\right)+3\sigma u_{yy}=0, \sigma=\pm 1$$

where $n \geq 1$, an integer. Following a similar analysis as in the previous exercise, derive a "modified" Davey–Stewartson equation by employing the slowly varying quasi-monochromatic wave expansion.

6.6 Suppose we are given the "damped NLS" equation

$$iu_t+u_{xx}+2|u|^2u=-i\gamma u$$

where $0 < \gamma \ll 1$. Use the integral relation involving mass (power) $\int |u|^2 dx$ to derive an approximate evolution equation for the slowly varying soliton amplitude η where the unperturbed soliton is given by

$$u=\eta \operatorname{sech}\left[\eta\left(x-x_0\right)\right] e^{i\eta^2 t+i\theta_0}.$$

Hint: see Ablowitz and Segur (1981). Various perturbation problems involving related NLS-type equations are also discussed in Chapter 10.

6.7 Discuss the stability associated with the special solution

$$u=ae^{i(kx-(k^2-2\sigma|a|^2)t)}$$

where k, σ are constant, associated with the NLS equation

$$iu_t+u_{xx}+2\sigma|u|^2u=0$$

on the infinite interval.

6.8 Given the two-dimensional NLS equation

$$iA_t+\Delta A+|A|^2A=0, \qquad x\in\mathbb{R}^2$$

and associated "Townes" mode $A=f(r)e^{i\lambda t}$, where $r^2=x^2+y^2$, show that the "virial" equation reduces to

$$\frac{d^2V}{dt^2}=0$$

where $V(t)=\int(x^2+y^2)|A|^2\, d\boldsymbol{x}$. Hint: see the appendix in Ablowitz et al. (2005).

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