2

Linear and nonlinear wave equations

In Chapter 1 we saw how the KdV equation can be derived from the FPU problem. We also mentioned that the KdV equation was originally derived for weakly nonlinear water waves in the limit of long or shallow water waves. Researchers have subsequently found that the KdV equation is "universal" in the sense that it arises whenever we have a weakly dispersive and a weakly quadratic nonlinear system. Thus the KdV equation has also been derived from other physical models, such as internal waves, ocean waves, plasma physics, waves in elastic media, etc. In later chapters we will analyze water waves in depth, but first we will discuss some basic aspects of waves.

Broadly speaking, the study of wave propagation is the study of how signals or disturbances or, more generally, information is transmitted (cf. Bleistein, 1984). In this chapter we begin with a study of "dispersive waves" and we will introduce the notion of phase and group velocity. We will then briefly discuss: the linear wave equation, the concept of characteristics, shock waves in scalar first-order partial differential equations (PDEs), traveling waves of the viscous Burgers equation, classification of second-order quasilinear PDEs, and the concept of the well-posedness of PDEs.

2.1 Fourier transform method

Consider a PDE in evolution form, first order in time, and in one spatial dimension,

$$u_t = F[u, u_x, u_{xx}, \dots],$$

where F is, say, a polynomial function of its arguments. We will consider the initial value problem on $|x| < \infty$ and assume $u \to 0$ sufficiently rapidly as $|x| \to \infty$ with $u(x, 0) = u_0(x)$ given. To begin with, suppose we consider the linear homogeneous case, i.e.,

$$u_t = \sum_{j=0}^{N} a_j(x,t) u_{jx},$$

where $u_{jx} \equiv \partial^j u / \partial x^j$ and $a_j(x,t)$ are prescribed coefficients. When the a_j are constants,

$$u_t = \sum_{j=0}^{N} a_j u_{jx}, \tag{2.1}$$

and $u_0(x)$ decays fast enough, then we can use the method of Fourier transforms to solve this equation. Before we do that, however, let us recall some basic facts about Fourier transforms (cf. Ablowitz and Fokas, 2003).

The function $u(x,t)$ can be expressed using the (spatial) Fourier transform as

$$u(x,t) = \frac{1}{2\pi} \int b(k,t) e^{ikx}\, dk, \tag{2.2}$$

where it is assumed that u is smooth and $|u| \to 0$ as $|x| \to \infty$ sufficiently rapidly; also, unless otherwise specified, $\int$ represents an integral from $-\infty$ to $+\infty$ in this chapter. For our purposes it suffices to require that $u \in L_1 \bigcap L_2$, meaning that $\int |u|\, dx$ and $\int |u|^2\, dx$ are both finite. Substituting (2.2) into (2.1), assuming the interchange of derivatives and integral, leads to

$$\int e^{ikx} \left\{ b_t - b \sum_{j=0}^{N} (ik)^j a_j \right\} dk = 0.$$

It follows that

$$b_t = b \sum_{j=0}^{N} (ik)^j a_j,$$

or that

$$b_t = -i\omega(k) b,$$

with

$$-i\omega(k) = \sum_{j=0}^{N} (ik)^j a_j. \tag{2.3}$$

We call $\omega(k)$, which we assume is real, the *dispersion relation* corresponding to (2.1). For example, if $N = 3$, then $\omega(k) = ia_0 - ka_1 - ik^2 a_2 + k^3 a_3$ and $\omega(k)$ is real if a_1, a_3 are real and a_0, a_2 are pure imaginary. We can solve this ODE to get

$$b(k,t) = b_0(k) e^{-i\omega(k)t},$$

where $b_0(k) \equiv b(k, 0)$. From this one obtains

$$u(x,t) = \frac{1}{2\pi} \int b_0(k) e^{i[kx - \omega(k)t]}\, dk.$$

Hence $b_0(k)$ plays the role of a weight function, which depends on the initial conditions according to the inverse Fourier transform:

$$b_0(k) = \int u(x,0) e^{-ikx}\, dx.$$

Strictly speaking, we now have an "algorithm" in terms of integrals for solving our problem for $u(x,t)$.

Note that, in retrospect, we can also obtain this relation by substituting

$$u(x,t) = \frac{1}{2\pi} \int b_0(k) e^{i[kx - \omega(k)t]}\, dk$$

into the PDE.

There is, in fact, an alternative method for obtaining the dispersion relation. It is based on the observation that in this case one can substitute $u_s = e^{i[kx-\omega(k)t]}$ into the PDE and replace the time and spatial derivatives by

$$\partial_t \to -i\omega, \qquad \partial_x \to ik.$$

Then (2.3) follows from (2.1) directly.

2.2 Terminology: Dispersive and non-dispersive equations

Let us define and then explain the terminology that is frequently used in conjunction with these wave problems and Fourier transforms: k is usually called the *wavenumber*, ω is the *frequency*, k and ω real, and $\theta \equiv kx - \omega(k)t$ is the *phase* in the exponent or simply the phase.

The temporal *period* (or period for short) is denoted by

$$T \equiv \frac{2\pi}{\omega}.$$

The meaning of the period is that whenever $t \to t + nT$, where n is an integer, then the phase remains the same modulo 2π and therefore $e^{i\theta}$ remains unchanged. Similarly, we call $\lambda = 2\pi/k$ the *wavelength* and note that whenever $x \to x + n\lambda$ the phase remains the same modulo 2π and therefore $e^{i\theta}$ remains unchanged. Furthermore, we call

$$c(k) \equiv \frac{\omega(k)}{k}$$

the *phase velocity*, since $\theta = k(x - c(k)t)$. There is also the notion of *group velocity* $v_g(k) \equiv \omega'(k)$; i.e., the speed of a slowly varying group of waves. We will discuss its importance later.

An equation in one space and one time dimension is said to be dispersive when $\omega(k)$ is real-valued and $\omega''(k) \not\equiv 0$. The meaning of dispersion will be further elucidated when we discuss the long time asymptotics of these equations. Consider the first-order linear equation

$$u_t - a_1 u_x = 0, \tag{2.4}$$

where $a_1 \neq 0$, real and constant. We see that

$$-i\omega = ika_1,$$

which gives the linear dispersion relation

$$\omega(k) = -a_1 k.$$

In this case $\omega''(k) \equiv 0$, which means the PDE is non-dispersive.

Now let us look at the first-order constant coefficient equation (2.1). In this case the dispersion relation (2.3) shows that if all the a_j are real, then ω is real if and only if all the even powers of k (or even values of the index j) vanish; i.e., $a_j = 0$ for $j = 2, 4, 6, \ldots$. In that case it follows from (2.3) that $\omega(k)$ is an odd function of k, in which case the dispersion relation takes the general form

$$\omega(k) = -\sum_{j=0}^{N} (-1)^j a_{2j+1} k^{2j+1}.$$

A further example is the linearized KdV equation given by

$$u_t + u_{xxx} = 0, \tag{2.5}$$

i.e., (1.5) without the nonlinear term. By substituting $u_s = e^{i[kx-\omega(k)t]}$ one obtains its dispersion relation,

$$\omega(k) = -k^3.$$

Thus ω is real and $\omega'' = -6k \not\equiv 0$, which means that this is a dispersive equation.

We can use Fourier transforms to solve this equation. As indicated above, using the Fourier transform, $u(x, t)$ can be expressed as

$$u(x,t) = \frac{1}{2\pi} \int b(k,t) e^{ikx}\, dk,$$

and for the linearized KdV equation (2.5) one finds that $b_t = ik^3 b$ hence

$$u(x,t) = \frac{1}{2\pi}\int b_0(k)e^{i(kx+k^3t)}\,dk.$$

However, by itself this is not a particularly insightful solution, since one cannot evaluate this integral explicitly. Often having an integral formula alone does not give useful qualitative information. Similarly, we usually cannot explicitly evaluate the integrals corresponding to most linear dispersive equations. This is exactly the place where asymptotics of integrals plays an extremely important role: it will allow us to approximate the integrals with simple, understandable expressions. This will be a recurring theme in this book: namely, analytical methods can yield solutions that are inconvenient or uninformative and asymptotics can be used to obtain valuable information about these problems. Later we will briefly discuss the long-time asymptotic analysis of the linearized KdV equation.

In so far as we are dealing with dispersive equations, we have seen that requiring $\omega(k)$ to be real implies that $\omega(k)$ is odd. If in addition, $u(x,t)$ is real, this knowledge can be encoded into $b_0(k)$ as follows. We have that

$$u^*(x,t) = \frac{1}{2\pi}\int b_0^*(k)e^{-i[kx-\omega(k)t]}\,dk,$$

where u^* denotes the complex conjugate of u. Calling $k' = -k$ yields

$$\begin{aligned} u^*(x,t) &= -\frac{1}{2\pi}\int_{\infty}^{-\infty} b_0^*(-k')e^{-i[-k'x-\omega(-k')t]}\,dk' \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} b_0^*(-k')e^{-i[-k'x-\omega(-k')t]}\,dk'. \end{aligned}$$

Then if we require that u is real-valued then $u(x,t) = u^*(x,t)$. In addition, $\omega(-k') = -\omega(k)$. Combined, one obtains that

$$\frac{1}{2\pi}\int b_0(k)e^{i[kx-\omega(k)t]}\,dk = \frac{1}{2\pi}\int b_0^*(-k')e^{i[k'x-\omega(k')t]}\,dk'.$$

This identity is satisfied for all (x,t) if and only if

$$b_0^*(k) = b_0(-k).$$

Note that we cannot apply the Fourier transform method for the nonlinear problem, sometimes called the inviscid Burgers equation,

$$u_t + uu_x = 0,$$

because of the nonlinear product. We study the solution of this equation by using the method of characteristics; this is briefly discussed later in this chapter.

We also note that for some problems, $u_s = \exp(i(kx - \omega(k)t))$ yields an $\omega(k)$ that is not purely real. For example, for the so-called heat or diffusion equation

$$u_t = u_{xx},$$

we find $\omega(k) = -ik^2$. The Fourier transform method works as before giving

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} b_0(k)e^{ikx-k^2t}\,dk.$$

We see that the solution decays, i.e., the solution diffuses with increasing t.

2.3 Parseval's theorem

The L_2-norm of a function $f(x)$ is defined by

$$\|f\|_2^2 \equiv \int |f(x)|^2\,dx.$$

Let $\hat{f}(k)$ be the Fourier transform of $f(x)$. Parseval's theorem [see e.g., Ablowitz and Fokas (2003)] states that

$$\|f(x)\|_2^2 = \frac{1}{2\pi}\|\hat{f}(k)\|_2^2.$$

In many cases the L_2-norm of the solutions of PDEs has the meaning of energy (i.e., is proportional to the energy in physical units). The physical meaning associated with Parseval's relation is that the energy in physical space is equal to the energy in frequency (or sometimes called spectral or Fourier) space. Moreover, we know that when ω is real-valued then

$$\int |u(x,t)|^2\,dx = \frac{1}{2\pi}\int |b_0(k)|^2\,dk = \text{constant}.$$

Hence, it follows from Parseval's theorem that energy is conserved in linear dispersive equations. While it is possible to prove this result using direct integration methods, we get it "for free" in linear PDEs using Parseval's theorem.

2.4 Conservation laws

We saw above how Parseval's theorem is used to prove energy conservation. However, there may be other conserved quantities. These often play a very useful role in the analysis of problems, as we will see later.

A conservation law (or relation) has the general form

$$\frac{\partial}{\partial t}T(x,t) + \frac{\partial}{\partial x}F(x,t) = 0;$$

we call T the density of the conserved quantity, and F the flux. Let us integrate this relation from $x = -\infty$ to $x = \infty$:

$$\frac{\partial}{\partial t}\int T\,dx + F(x,t)\Big|_{x\to-\infty}^{x\to+\infty} = 0.$$

The second term is zero, since we assume that F decays at infinity, which leads to

$$\frac{\partial}{\partial t}\int T\,dx = 0,$$

i.e., $\int T\,dx =$ constant.

For example, let us study the conservation laws for the linearized KdV equation:

$$u_t + u_{xxx} = 0.$$

This equation is already in the form of a conservation law, with $T_1 = u$ and $F_1 = u_{xx}$; that is, $\int u\,dx$ is conserved. This, however, is only one of many conservation laws corresponding to (2.5). Another example is energy conservation. Using Parseval's theorem, we have already proven that linear dispersive equations with constant coefficients satisfy energy conservation. Since (2.5) is solvable by Fourier transforms we know from Parseval's theorem that energy is conserved. An alternative way of seeing this is by multiplying (2.5) by u and integrating with respect to x. It can be checked that this leads to

$$\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) + \frac{\partial}{\partial x}\left(uu_{xx} - \frac{1}{2}u_x^2\right) = 0,$$

from which it follows that $\int |u|^2\,dx =$ constant.

2.5 Multidimensional dispersive equations

So far we have focused on dispersive equations in one spatial dimension. The method of Fourier transforms can be generalized to solve constant coefficient multidimensional equations, where a typical solution takes the form:

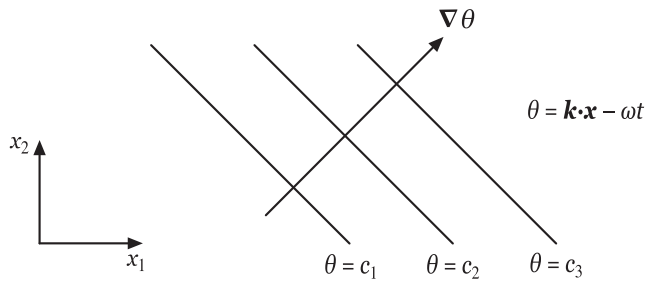

Figure 2.1 Phase contours in two dimensions ($n = 2$).

$$u(\boldsymbol{x}, t) = \frac{1}{2\pi} \int b_0(\boldsymbol{k}) e^{i[\boldsymbol{k}\cdot\boldsymbol{x} - \omega(\boldsymbol{k})t]}\, d\boldsymbol{k}$$

and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $\boldsymbol{k} = (k_1, k_2, \ldots, k_n)$, cf. Whitham (1974). As before, ω is the frequency (assumed real) and $T = 2\pi/\omega$ is the period. The wavenumber becomes the vector $\boldsymbol{k} = \nabla\theta$, where $\theta \equiv \boldsymbol{k} \cdot \boldsymbol{x} - \omega(\boldsymbol{k})t$, and the wavelength $\boldsymbol{\lambda} = 2\pi\hat{\boldsymbol{k}}/|\boldsymbol{k}|$, where $\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$ and $|\cdot|$ is the (Euclidean) modulus of the vector. Thus $\theta(\boldsymbol{x} + \boldsymbol{\lambda}) = \theta(\boldsymbol{x}) + 2\pi$. The condition that θ be constant describes the phase contours (see Figure 2.1). We can also define the phase speed as $\boldsymbol{c}(\boldsymbol{k}) = \omega(\boldsymbol{k})\hat{\boldsymbol{k}}/|\boldsymbol{k}|$, from which it follows that $\boldsymbol{c} \cdot \boldsymbol{k} = \omega$.

In the multidimensional linear case, an equation is said to be dispersive if ω is real and (instead of $\omega'' \not\equiv 0$) its Hessian is non-singular, that is

$$\det \left| \frac{\partial^2 \omega(\boldsymbol{k})}{\partial k_i \partial k_j} \right| \not\equiv 0.$$

If we have a nonlinear equation, we call it a *nonlinear dispersive wave equation* if its linear part is dispersive and the equation is energy-preserving.

2.6 Characteristics for first-order equations

First let us use the Fourier transform method to solve (2.4), $u_t - a_1 u_x = 0$. As before, we get $\omega(k) = -a_1$ and

$$u(x, t) = \frac{1}{2\pi} \int b_0(k) e^{ik(x + a_1 t)}\, dk.$$

Calling

$$f(x) \equiv u(x, 0) = \frac{1}{2\pi} \int b_0(k) e^{ikx}\, dk,$$

we observe that $u(x,t) = f(x + a_1 t)$. This solution can be checked directly by substitution into the PDE.

We can also solve (2.4) using the so-called method of characteristics. We can think of the left-hand side of (2.4) as a total derivative that is expanded according to the chain rule. That is,

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} = 0. \tag{2.6}$$

It follows from (2.6) that $du/dt = 0$ along the curves $dx/dt = -a_1$, or, said differently, that $u = c_2$, a constant, along the curves $x + a_1 t = c_1$. Thus for the initial value problem $u(x, t = 0) = f(x)$, at any point $x = \xi$ we can identify the constant c_2 with $f(\xi)$ and c_1 with ξ. Hence the solution to the initial value problem is given by

$$u = f(x + a_1 t).$$

Alternatively, from (2.4) and (2.6) (solving for du), we can write the above in the succinct form

$$\frac{dt}{1} = \frac{dx}{-a_1} = \frac{du}{0}.$$

These equations imply that

$$t = \frac{-1}{a_1}x + \frac{c_1}{a_1}, \qquad u = c_2,$$

where c_1 and c_2 are constants. Hence

$$x + a_1 t = c_1, \qquad u = c_2.$$

For first-order PDEs the solution has one arbitrary function degree of freedom and the constants are related by this function. In this case $c_2 = g(c_1)$, where g is an arbitrary function, leads to

$$u = g(x + a_1 t).$$

As above, suppose we specify the initial condition $u(x, 0) = f(x)$. Then if we take $c_1 = \xi$, so that $t = 0$ corresponds to the point $x = \xi$ on the initial data, this then implies that $u(\xi, t) = g(\xi) = f(\xi)$; now in general, along $\xi = x + a_1 t$, that agrees with the solution obtained by Fourier transforms. The meaning of ξ is that of a characteristic curve (a line in this case) in the (x, t)-plane, along which the solution is non-unique; in other words, along a characteristic ξ, the solution can be specified arbitrarily. Moreover as we move from one characteristic, say ξ_1, to another, say ξ_2, the solution can change abruptly.

The method of characteristics also applies to quasilinear first-order equations. For example, let us consider the inviscid Burgers equation mentioned earlier:

$$u_t + uu_x = 0. \tag{2.7}$$

From (2.7) we have that $du/dt = 0$ along the curves $dx/dt = u$, or, said differently, that $u = c_2$, a constant, along the curves $x - ut = c_1$. Hence an implicit solution is given by

$$u = f(x - ut).$$

Alternatively, if we specify the initial condition $u(x, 0) = f(x)$, with $f(x)$ a smooth function of x, and we take $c_1 = \xi$ so that corresponding to $t = 0$ is a point $x = \xi$ on the initial data, this then means that $u(x, t) = f(\xi)$ along $x = \xi + f(\xi)t$. The latter equation implies that ξ is a function of time, i.e., $\xi = \xi(x, t)$, which in turn leads to the solution $u = u(x, t)$. If we have a "hump-like" initial condition such as $u(x, 0) = \operatorname{sech}^2 x$ then either points at the top of the curve, e.g., the maximum, "move" faster than the points of lower amplitude and eventually break, or a multi-valued solution occurs. The breaking time follows from $x = \xi + f(\xi)t$. Taking the derivative of this equation yields $\partial\xi/\partial x = \xi_x$ and ξ_t:

$$\xi_x = \frac{1}{f'(\xi)t + 1}, \quad \xi_t = -\frac{f(\xi)}{f'(\xi)t + 1} \tag{2.8}$$

and the breaking time is given by $t = t_B = 1/\max(-f')$. This is the breaking time, depicted in Figure 1.2, that is associated with the KdV equation (dashed line) in Chapter 1.

Thus the solution to (2.7) can be written in the form

$$u = u(\xi(x, t)).$$

Prior to the breaking time $t = t_B$ the solution is single valued. So we can verify that the solution (2.8) satisfies the equation:

$$u_t = u'(\xi)\xi_t$$
$$u_x = u'(\xi)\xi_x$$

and using (2.7) and (2.8)

$$u_t + uu_x = -\frac{f(\xi)}{f'(\xi)t + 1} + \frac{f(\xi)}{f'(\xi)t + 1} = 0.$$

In the exercises, other first-order equations are studied by the method of characteristics.

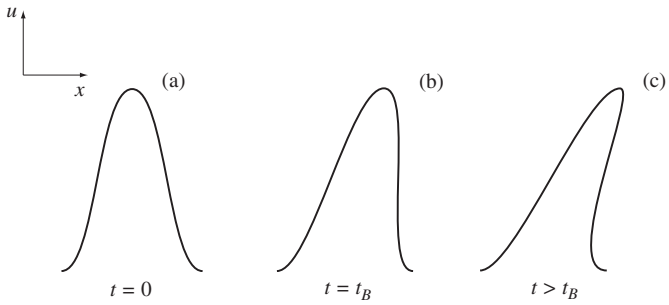

Figure 2.2 The solution of (2.7) at different times.

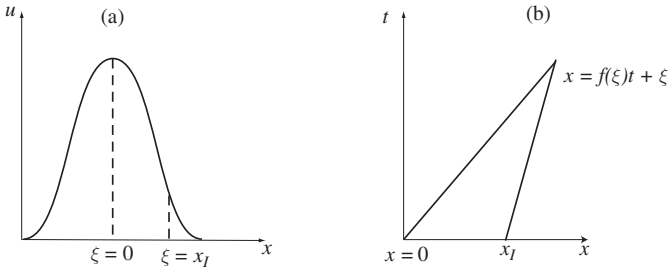

Figure 2.3 (a) A typical function $u(\xi, 0) = f(\xi)$. (b) Characteristics associated with points $\xi = 0$, $\xi = x_I$ that intersect at $t = t_B$. The point x_I is the inflection point of $f(\xi)$.

In Figure 2.2 is shown a typical case and we can formally find the solution for values $t > t_B$ ignoring the singularities and triple-valued solution. The value of time t, when characteristics first intersect, is denoted by t_B – see Figure 2.3.

2.7 Shock waves and the Rankine–Hugoniot conditions

Often we wish to take the solution further in time, beyond $t = t_B$, but do not want the multi-valued solution for physical or mathematical reasons. In many important cases the solution has a nearly discontinuous structure. This is shown schematically in Figure 2.4. Such a situation occurs in the case of the so-called viscous Burgers equation

$$u_t + uu_x = \nu u_{xx} \tag{2.9}$$

where ν is a constant; in this case there is a rapidly changing solution that can be viewed as an approximation to a discontinuous solution for small

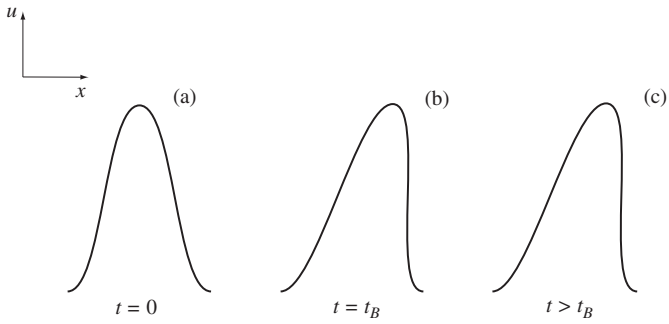

Figure 2.4 Evolution of the solution to (2.7) with a "shock wave".

ν ($0 < \nu \ll 1$). One way of introducing discontinuities, i.e., shocks, without resorting to adding a "viscous" term (i.e., we keep $\nu = 0$) is to consider (2.7) as coming from a conservation law and its corresponding integral form. In other words, (2.7) can be written in conservation law form as

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$$

which in turn can be derived from the *integral* form

$$\frac{d}{dt}\int_x^{x+\Delta x} u(x,t)\,dx + \frac{1}{2}\left(u^2(x+\Delta x,t) - u^2(x,t)\right) = 0, \tag{2.10}$$

in the limit $\Delta x \to 0$. Equation (2.10) can support a shock wave since it is an integral relation. Suppose between two points x_1 and x_2 we have a discontinuity that can change in time $x = x_s(t)$ – see Figure 2.5. Then (2.10) reads

$$\frac{d}{dt}\left(\int_{x_1}^{x_s(t)} u(x,t)\,dx + \int_{x_s(t)}^{x_2} u(x,t)\,dx\right) + \frac{1}{2}\left(u^2(x_2,t) - u^2(x_1,t)\right) = 0.$$

Letting $x_2 = x_s(t) + \epsilon$, $x_1 = x_s(t) - \epsilon$, and $\epsilon > 0$, then as $\epsilon \to 0$, we have $x_2 \to x_+$ and $x_1 \to x_-$ and so

$$(u(x_-,t) - u(x_+,t))\frac{dx_s}{dt} = -\frac{1}{2}\left(u^2(x_+,t) - u^2(x_-,t)\right)$$

and find

$$v_s = \frac{dx_s}{dt} = \frac{1}{2}\left(u(x_+,t) - u(x_-,t)\right) = \frac{u_+ + u_-}{2}, \tag{2.11}$$

where $u_\pm = u(x_\pm, t)$. The last equation, (2.11), describes the speed of the shock, $v_s = dx_s/dt$, in terms of the jump discontinuities.

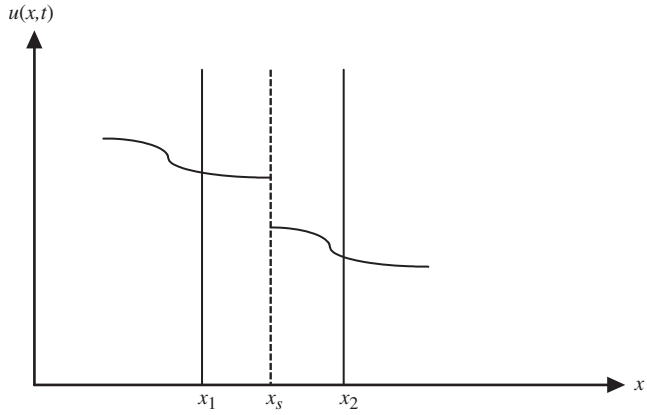

Figure 2.5 A discontinuity in $u(x,t)$ at $x = x_s(t)$.

We note that if (2.7) is generalized to

$$u_t + c(u)u_x = 0,$$

the corresponding conservation law is

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}(q(u)) = 0, \tag{2.12}$$

where $q'(u) = c(u)$, and its integral form is

$$\frac{d}{dt}\int_x^{x+\Delta x} u(x,t)\,dx + q(u(x+\Delta x)) - q(u(x,t)) = 0.$$

Then following the same procedure as above, we find that the shock speed is

$$v_s = \frac{dx_s}{dt} = \frac{q(x_+,t) - q(x_-,t)}{u_+ - u_-}. \tag{2.13}$$

We note that in the limit $x_+ \to x_-$ we have $v_s \to q'(x_+)$.

Equations (2.11) and (2.13) are sometimes referred to as the *Rankine–Hugoniot* (RH) relations that were originally derived for the Euler equations of fluid dynamics, cf. Lax (1987) and LeVeque (2002).

The RH relations, sometimes referred to as shock conditions, are used to avoid multi-valued behavior in the solution, which would otherwise occur after characteristics cross – see Figure 2.6. In order to make the problem well-posed one needs to add admissibility conditions, sometimes called entropy-satisfying conditions, to the relation. In their simplest form these conditions indicate characteristics should be going into a shock as time increases, rather than emanating, as that would be unstable, see Lax (1987) and LeVeque (2002).

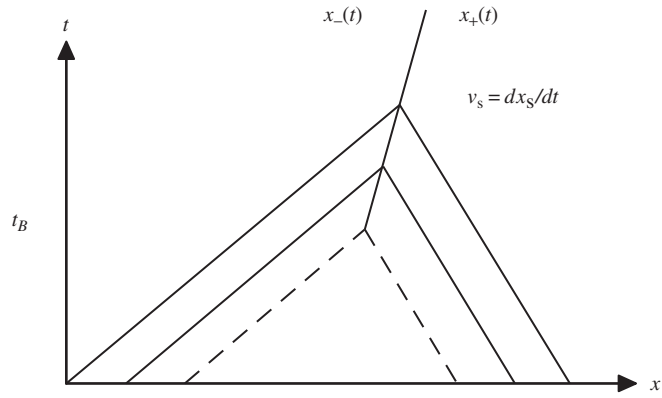

Figure 2.6 Typical characteristic diagram.

Another viewpoint is that an admissibility relation should be the limit of, for example, a viscous solution to Burgers' equation, see (2.9), where the admissible solution $u_A(x,t) = \lim_{\nu\to 0} u(x,t;\nu)$. We discuss Burgers' equation in detail later.

Let us consider the problem of fitting the discontinuities satisfying the shock conditions into the smooth part of the solution, e.g., for $x \to x_+(t)$, $x \to x_-(t)$.

For example, consider (2.7) with the following initial condition:

$$u(x,0) = \begin{cases} u_+, & x > 0 \\ u_-, & x < 0. \end{cases}$$

where $u_\pm$ are constants. In the case where $u_+ < u_-$, we see that the characteristics cross immediately; i.e., they satisfy $dx/dt = u$, so

$$x = u_+ t + \xi, \quad \xi > 0,$$
$$x = u_- t + \xi, \quad \xi < 0,$$

as indicated in Figure 2.7. The speed of the shock is given by

$$v_s = \frac{dx}{dt} = \frac{u_+ + u_-}{2}.$$

If $u_+ = -u_-$ the shock is stationary.

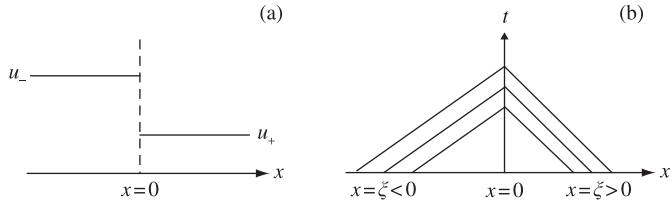

Figure 2.7 (a) Shock wave $u_+ > u_-$. (b) Characteristics cross.

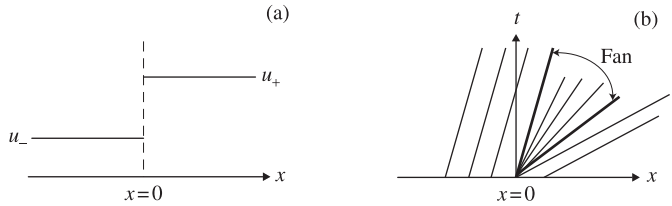

Figure 2.8 (a) "Rarefaction wave" $u_+ < u_-$. (b) Characteristics do not cross. There is a "fan" at $x = 0$.

On the other hand, if $u_+ < u_-$, as in Figure 2.8, then the discontinuities do not cross. In this case

$$\begin{aligned} x &= u_+ t + \xi, \quad \xi > 0, \\ x &= u_- t + \xi, \quad \xi < 0, \\ x &= ut, \quad \xi = 0, \end{aligned}$$

and the solution is termed a *rarefaction wave* with a *fan* at $\xi = 0$, $u = x/t$ where $u_- < x/t < u_+$.

Next we return briefly to discuss the viscous Burgers equation (2.9). If we look for a traveling solution $u = u(\zeta)$, $\zeta = x - Vt - x_0$, where V, x_0 are constants, and V the velocity of the (viscous shock) traveling wave, then

$$-Vu_\zeta + uu_\zeta = \nu u_{\zeta\zeta}.$$

Integrating yields

$$-Vu + \frac{1}{2}u^2 + A = \nu u_\zeta, \tag{2.14}$$

where A is the constant of integration. Further, if $u \to u_\pm$ as $x \to \pm\infty$, where $u_\pm$ are constants, then

$$A = -\left(\frac{1}{2}u_\pm^2 - Vu_\pm\right)$$

and by eliminating the constant A, we find

$$V = \frac{1}{2}\frac{u_+^2 - u_-^2}{u_+ - u_-} = \frac{1}{2}(u_+ + u_-),$$

which is the same as the shock wave speed found earlier for the inviscid Burgers equation. Indeed, if we replace Burgers' equation by the generalization

$$u_t + (q(u))_x = \nu u_{xx},$$

then the corresponding traveling wave $u = u(\zeta)$ would have its speed satisfy

$$V = \frac{q(u_+) - q(u_-)}{u_+ - u_-}$$

[by the same method as for (2.7)], which agrees with the shock wave velocity (2.13) associated with (2.12). For the Burgers equation we can give an explicit expression for the solution u by carrying out the integration of (2.14)

$$d\zeta = \frac{\nu du}{u^2/2 - Vu + A}$$

with $A = -((1/2)u_+^2 - (1/2)(u_+ + u_-)u_+) = (1/2)u_+u_-$. This results in

$$\begin{aligned} u &= \frac{u_+ + u_- e^{(u_+ - u_-)\zeta/(2\nu)}}{1 + e^{(u_+ - u_-)\zeta/(2\nu)}} \\ &= \frac{u_+ + u_-}{2} + \frac{u_- - u_+}{2}\tanh\left(\frac{u_+ - u_-}{4\nu}\zeta\right), \end{aligned}$$

like in Figure 2.9. The *width* of the viscous shock wave is proportional to $w = 4\nu/(u_+ - u_-)$.

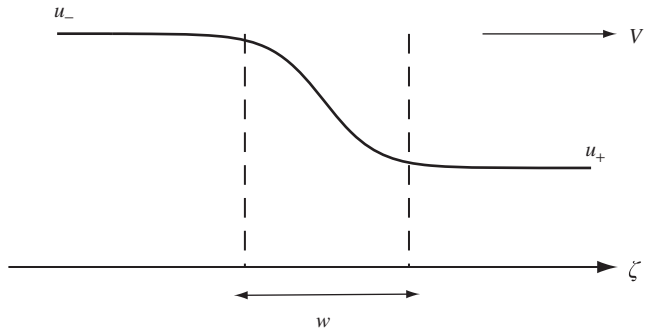

Figure 2.9 Traveling viscous shock wave with velocity V and width $w = 4\nu/(u_+ - u_-)$.

2.8 Second-order equations: Vibrating string equation

Let us turn our attention to deriving an approximate equation governing the transverse vibration of a long string that is initially plucked and left to vibrate (see Figure 2.10).

Let ρ be the mass density of the string: we assume it to be constant with $\rho = m/L$, where m is the total mass of the string and L is its length. The string is in equilibrium (i.e., at rest) when it is undisturbed along a straight line, which we will choose to be the x-axis. When the string is plucked we can approximately describe its vertical displacement from equilibrium at each point x as a function $y(x, t)$. Consider an infinitesimally small segment of the string, i.e., a segment of length $|\Delta s| \ll 1$, where $\Delta s^2 = \Delta x^2 + \Delta y^2$ (see Figure 2.11). We will assume no external forces and that horizontal acceleration is negligible. It follows from Newton's second law that

- difference in vertical tensions:

$$(T \sin\theta)|_{x+\Delta x} - (T \sin\theta)|_x = \Delta m \, y_{tt},$$

- difference in horizontal tensions:

$$(T \cos\theta)|_{x+\Delta x} - (T \cos\theta)|_x = 0.$$

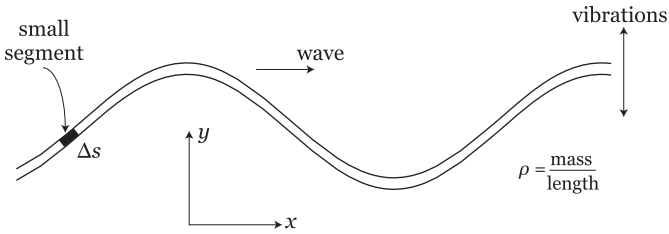

Figure 2.10 Vibrations of a long string.

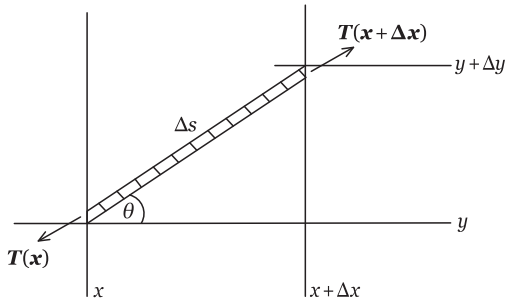

Figure 2.11 Tension forces acting on a small segment of the string.

In the limit $|\Delta x| \to 0$ we assume that $y = y(x,t)$; then one has (in that limit)

$$\sin\theta = \frac{\Delta y}{\Delta s} = \frac{\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \to \frac{y_x}{\sqrt{1+y_x^2}},$$
$$\cos\theta = \frac{\Delta x}{\Delta s} \to \frac{1}{\sqrt{1+y_x^2}}.$$

Hence with these assumptions, in this limit, and using $dm = \rho ds$, Newton's equations become

$$\left(\rho\frac{ds}{dx}\right)y_{tt} - \frac{\partial}{\partial x}\left(T\frac{y_x}{\sqrt{1+y_x^2}}\right) = 0,$$
$$\frac{\partial}{\partial x}\left(T\frac{1}{\sqrt{1+y_x^2}}\right) = 0.$$

From the second equation we get that $T = T_0\sqrt{1+y_x^2}$, T_0 constant and, using $ds/dx = \sqrt{1+y_x^2}$, the first equation yields

$$y_{tt} = \frac{T_0}{\rho\sqrt{1+y_x^2}}y_{xx}. \tag{2.15}$$

This is a nonlinear equation governing the vibrations of the string. If we consider small-amplitude vibrations (i.e., $|y_x| \ll 1$) and so neglect y_x^2 in the denominator of the right-hand side we obtain, as an approximation, the linear wave equation,

$$y_{tt} - c^2 y_{xx} = 0,$$

where $c^2 = T_0/\rho$ is the wave speed (speed of sound). However, we can approximate (2.15) by assuming that $|y_x|$ is small but without neglecting it completely. That can be done by taking the first correction term from a Taylor series of the denominator:

$$\frac{1}{\sqrt{1+y_x^2}} \approx \frac{1}{1+y_x^2/2} \approx 1 - \frac{1}{2}y_x^2.$$

Doing so leads to the nonlinear equation

$$y_{tt} = c^2 y_{xx}\left(1 - \frac{1}{2}y_x^2\right),$$

which is more amenable to analysis than (2.15), but still retains some of its nonlinearity. It is interesting that this equation is a cubic nonlinear version of (1.2) when $\varepsilon \gg h^2/12$. Note: if y_x becomes large (e.g., a "shock" is formed), then we need to go back to (2.15) and reconsider our assumptions.

2.9 Linear wave equation

Consider the linear wave equation

$$u_{tt} - c^2 u_{xx} = 0,$$

where c = constant with the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. Below we derive the solution of this equation given by D'Alembert.

It is helpful to change to the coordinate system

$$\xi = x - ct, \qquad \eta = x + ct.$$

Thus, from the chain rule we have that

$$\partial_x = \frac{1}{2}(\partial_\xi + \partial_\eta), \qquad \partial_t = \frac{1}{2c}(-\partial_\xi + \partial_\eta)$$

and the equation transforms into

$$\left[(\partial_\xi + \partial_\eta)^2 - (-\partial_\xi + \partial_\eta)^2\right]u = 0.$$

Simplifying leads to

$$4u_{\xi\eta} = 0.$$

The latter equation can be integrated with respect to ξ to give

$$u_\eta = \tilde{F}(\eta),$$

where $\tilde{F}$ is an arbitrary function. Integrating once more gives

$$u(\xi,\eta) = F(\eta) + G(\xi) = F(x+ct) + G(x-ct).$$

This is the general solution of the wave equation. It remains to incorporate the initial data. We have that

$$\begin{aligned} u(x,0) &= F(x) + G(x) = f(x), \\ u_t(x,0) &= cF'(x) - cG'(x) = g(x). \end{aligned}$$

Differentiating the first equation leads to

$$\begin{aligned} f'(x) &= F'(x) + G'(x), \\ g(x) &= cF'(x) - cG'(x). \end{aligned}$$

By addition and subtraction of these equations one gets that

$$F' = \frac{1}{2}\left(f' + \frac{g}{c}\right), \qquad G' = \frac{1}{2}\left(f' - \frac{g}{c}\right).$$

It follows that

$$F(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(\zeta)\,d\zeta + c_1,$$
$$G(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\zeta)\,d\zeta + c_2,$$

where c_1, c_2 are constants. Hence,

$$u(x,t) = \frac{1}{2}\Big[f(x+ct) + f(x-ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\zeta)\,d\zeta + c_3, c_3 = c_1 + c_2.$$

Since $u(x,0) = f(x)$ it follows that $c_3 = 0$ and that

$$u(x,t) = \frac{1}{2}\Big[f(x+ct) + f(x-ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\zeta)\,d\zeta.$$

This is the well-known solution of D'Alembert to the wave equation. From this solution we can see that the wave (or disturbance) or a discontinuity propagates along the lines $\xi = x + ct =$ constant and $\eta = x - ct =$ constant, called the characteristics. More precisely, when looking forward in time we see that a disturbance at $t = 0$ located in the region $a < x < b$ propagates within the "range of influence" that is bounded by the lines $x - ct = \xi_R = b$ and $x + ct = \eta_L = a$ (see Figure 2.12). Similarly, looking back in time we see that the solution at time t was generated from a region in space that is bounded by the "domain of dependence" bounded by the characteristics $x - ct = a$ and $x + ct = b$. Thus no "information" originating inside $a < x < b$ can affect the region outside the range of influence; c can be thought of as the "speed of light".

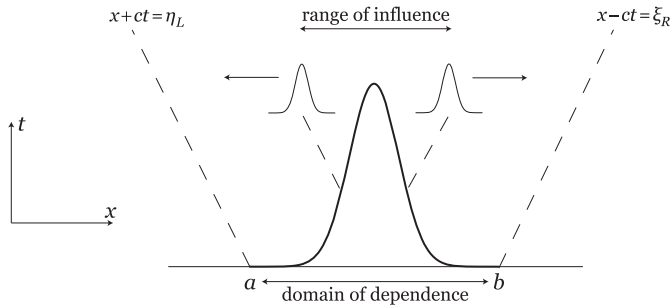

Figure 2.12 The characteristics of the wave equation; domain of dependence and range of influence.

2.10 Characteristics of second-order equations

We already encountered the method of characteristics in Section 2.6 for the case of first-order equations (in the evolution variable t). The notion of characteristics, however, can be extended to higher-order equations, as we just saw in D'Alembert's solution to the wave equation. Loosely speaking, characteristics are those curves along which discontinuities can propagate. A more formal definition for characteristics is that they are those curves along which the Cauchy problem does not have a unique solution. For second-order quasilinear PDEs the Cauchy problem is given by

$$Au_{xx} + Bu_{xy} + Cu_{yy} = D, \tag{2.16}$$

where we assume that A, B, C, and D are real functions of u, u_x, u_y, x, and y and the boundary values are given on a curve C in the (x, y)-plane in terms of u or $\partial u/\partial n$ (the latter being the normal derivative of u with respect to the curve C).

We will not go into the formal derivation of the equations for the characteristics, which can be found in many PDE textbooks (see, e.g., Garabedian, 1984). A quick method for deriving these equations can be accomplished by keeping in mind the basic property of a characteristic – a curve along which discontinuities can propagate, cf. Whitham (1974). One assumes that u has a small discontinuous perturbation of the form

$$u = U_s + \varepsilon\Theta(\nu(x, y))V_s, \tag{2.17}$$

where U_s, V_s are smooth functions, ε is small, and

$$\Theta(\nu) = \begin{cases} 0, & \nu < 0 \\ 1, & \nu > 0 \end{cases}$$

is the Heaviside function (sometimes denoted by $H(\nu)$). Here $\nu(x, y) =$ constant are the characteristic curves to be found. We recall that $\Theta'(\nu) = \delta(\nu)$, i.e., the derivative of a Heaviside function is the Dirac delta function (cf. Lighthill 1958). Roughly speaking, δ is less smooth than Θ and near $\nu = 0$ is taken to be "much larger" than Θ, which is less smooth than U_s, etc. In turn, δ' is more singular (much larger) than δ near $\nu = 0$, etc. When we substitute (2.17) into (2.16) we keep the highest-order terms, which are those terms that multiply δ', and neglect the less singular terms that multiply δ, Θ, and U_s. This gives us

$$\left(A\nu_x^2 + B\nu_x\nu_y + C\nu_y^2\right)\varepsilon\delta' V_s + \text{l.o.t.} = 0,$$

(here l.o.t., lower-order terms, means terms that are less singular) or

$$Av_x^2 + Bv_xv_y + Cv_y^2 = 0. \tag{2.18}$$

Since $v(x, y) =$ constant are the characteristics, we have that $dv = v_x dx + v_y dy = 0$ and therefore

$$\frac{dy}{dx} = -\frac{v_x}{v_y}.$$

Combined with (2.18), we obtain

$$A\frac{dy}{dx}^2 - B\frac{dy}{dx} + C = 0 \tag{2.19}$$

as the equation for the characteristic $y = y(x)$.

2.11 Classification and well-posedness of PDEs

Based on the types of solutions to (2.19), it is possible to classify quasilinear PDEs in two independent variables. Letting $\lambda \equiv dy/dx$, we have the following classification (recall A, B, C are assumed real):

- Hyperbolic: Two, real roots λ_1 and λ_2;
- Parabolic: Two, real repeated roots $\lambda_1 = \lambda_2$; and
- Elliptic: Two, complex conjugate roots λ_1 and $\lambda_2 = \lambda_1^*$.

The terminology comes from an analogy between the quadratic form (2.18) and the quadratic form for a plane conic section (Courant and Hilbert, 1989).

Some prototypical examples are:

(a) The wave equation: $u_{tt} - c^2 u_{xx} = 0$ (note the interchange of variables $x \to t$, $y \to x$). Here $A = 1,\ B = 0,\ C = -c^2$. Thus (2.19) gives

$$\left(\frac{dx}{dt}\right)^2 - c^2 = 0,$$

which implies the characteristics are given by $x \pm ct =$ constant. There are two, *real* solutions. Therefore, the wave equation is hyperbolic.

(b) The heat equation: $u_t - u_{xx} = 0$ (in this example, $y \to t$). Here $A = -1,\ B = 0,\ C = 0$. Thus (2.19) gives

$$\left(\frac{dt}{dx}\right)^2 = 0,$$

which implies the characteristics are given by $t =$ constant. There are two, *repeated* real solutions. Therefore, the heat equation is parabolic.

(c) The potential equation (often called Laplace's equation): $u_{xx} + u_{yy} = 0$. Here $A = 1,\ B = 0,\ C = 1$. Thus (2.19) gives

$$\left(\frac{dy}{dx}\right)^2 + 1 = 0,$$

which implies the characteristics are given by $y = \pm ix +$ constant. There are two, *complex* solutions. Therefore, the potential equation is elliptic.

That the potential equation is elliptic suggests that we can specify "initial" data on any real curve and obtain a unique solution. However, we will see in doing this that something goes seriously wrong. This leads us to the concept of *well-posedness*. To this end, consider the problem

$$\begin{aligned} u_{yy} + u_{xx} &= 0, \\ u(x,0) &= f(x), \\ u_y(x,0) &= g(x), \end{aligned}$$

and look for solutions of the form $u_s = \exp\left[i(kx - \omega(k)y)\right]$. This leads to the dispersion relation $\omega = \pm ik$. We then have the formal solution

$$\begin{aligned} u(x,y) = &\frac{1}{2\pi}\int b_+(k,0)\exp(ikx)\exp(ky)\,dk \\ &+ \frac{1}{2\pi}\int b_-(k,0)\exp(ikx)\exp(-ky)\,dk, \end{aligned}$$

where b_+ and b_- are determined from the initial data. In general, these integrals do not converge. To make them converge we need to require that $b_\pm(k)$ decays faster than any exponential, which generally speaking is not physically reasonable. The "problem" is that the dispersion relation growth rate, $\omega(k) = \pm ik$, is unbounded and imaginary.

Now suppose we consider the following initial data on $y = 0$: $f(x) = 0$ and $g(x) = g_k(x) = \sin(kx)/k$, where k is a positive integer. Using separation of variables, we find, for each value of k, the solution is

$$u_k(x,y) = \left[A_0\cosh(ky) + B_0\sinh(ky)\right]\sin(kx),$$

where A_0 and B_0 are constants to be determined. Using $u_k(x,0) = 0$, we find that $A_0 = 0$. And

$$\frac{\partial u_k}{\partial y}(x,0) = \frac{\sin(kx)}{k} = B_0 k\sin(kx)$$

implies $B_0 = 1/k^2$. Thus, the solution is

$$u_k(x, y) = \frac{\sinh(ky)\sin(kx)}{k^2}.$$

Now from a physical point of view, we expect that if the initial data becomes "small", so should the solution. However, for any non-zero fixed value of y, we see that for each x, $u_k(x, y) \to \infty$ as $k \to \infty$, even though $g_k \to 0$. The solution does not depend continuously on the data. We say for a given initial/boundary value problem:

Definition 2.1 The problem is well-posed if there exists a unique solution and given a sequence of data g_k converging to g as $k \to \infty$, there exists a solution u_k and $u_k \to u_*$ as $k \to \infty$, where u_* is the solution corresponding to the data g.

In other words, for the equation to be well-posed it must have a unique solution that depends continuously on the data. Hence the potential equation (i.e., Laplace's equation) is not well-posed. See also Figure 2.13.

As a final example, we will look at the Klein–Gordon equation:

$$\begin{aligned} u_{tt} - c^2 u_{xx} + m^2 u(x, t) &= 0, \\ u(x, 0) = f(x), \qquad u_t(x, 0) &= g(x). \end{aligned} \tag{2.20}$$

This equation describes elastic vibrations with a restoring force proportional to the displacement and also occurs in the description of the so-called weak interaction, which is somewhat analogous to electromagnetic interactions. In the latter case, the constant m is proportional to the boson (analogous to the photon) mass and c is the speed of light. Remembering the formal correspondence $\partial_t \leftrightarrow -i\omega$ and $\partial_x \leftrightarrow ik$, we find the dispersion relationship $\omega_\pm = \pm\sqrt{c^2k^2 + m^2} = \pm\omega(k)$. Then

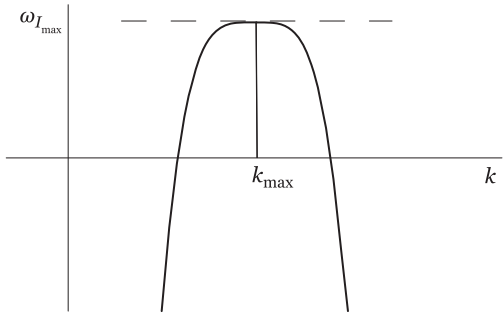

Figure 2.13 $\omega = \omega(k) = (\omega_R + i\omega_I)(k)$: ill-posed when $\omega_I \to \infty$ as $k \to \infty$; well-posed when ω is bounded as in the above figure.

$$u(x,t) = \frac{1}{2\pi}\int b_+(k,0)\exp\left[i(kx + ck\sqrt{1+\left(\frac{m}{ck}\right)^2}t\right]dk + \frac{1}{2\pi}\int b_-(k,0)\exp\left[i(kx - ck\sqrt{1+\left(\frac{m}{ck}\right)^2}t\right]dk, \quad (2.21)$$

where b_+ and b_- will be determined from the initial data. We have

$$f(x) = \frac{1}{2\pi}\left[\int b_+(k)\exp(ikx)dk + \int b_-(k)\exp(ikx)\,dk\right] \quad (2.22a)$$

$$g(x) = \frac{i}{2\pi}\left[\int b_+(k)\omega(k)\exp(ikx)\,dk - \int b_-(k)\omega(k)\exp(ikx)\,dk\right]. \quad (2.22b)$$

Let

$$\widehat{f}(k) \equiv \int f(x)\exp(-ikx)\,dx,$$
$$\widehat{g}(k) \equiv \int g(x)\exp(-ikx)\,dx;$$

then (2.22) implies that

$$\widehat{f} = b_+ + b_-,$$
$$\frac{-i\widehat{g}}{\omega} = b_+ - b_-.$$

Solving this system and substituting back into (2.21), we find that

$$u(x,t) = \frac{1}{4\pi}\int\left(\widehat{f}(k) - \frac{i\widehat{g}(k)}{\omega(k)}\right)\exp\left[i(kx+\omega(k)t\right]\,dk + \frac{1}{4\pi}\int\left(\widehat{f}(k) + \frac{i\widehat{g}(k)}{\omega(k)}\right)\exp\left[i(kx-\omega(k)t\right]\,dk.$$

As an example, for the specific initial data $u(x,0) = \delta(x)$, where $\delta(x)$ is the Dirac delta function, we have $u_t(x,0) = 0$, and so $\widehat{f}(k) = 1$ and $\widehat{g}(k) = 0$. Thus,

$$u(x,t) = \frac{1}{2\pi}\int e^{ikx}\cos\left[\omega(k)t\right]\,dk,$$

which due to $\sin(kx)$ being an odd function simplifies to

$$u(x,t) = \frac{1}{2\pi}\int \cos(kx)\cos\left[\omega(k)t\right]\,dk.$$

Finally, note that the solution method based on Fourier transforms may be summarized schematically:

$$\begin{array}{ccc} u(x,0) & \xrightarrow{\text{Direct transformation}} & \hat{u}(k,0) \\ & & \Big\downarrow \text{Evolution} \\ u(x,t) & \xleftarrow{\text{Inverse transformation}} & \hat{u}(k,t) \end{array}$$

where $\hat{u}(k,t)$ is the Fourier transform of $u(x,t)$. In Chapters 8 and 9, this will be generalized to certain types of nonlinear PDEs.

Exercises

2.1 Solve the following equations using Fourier transforms (all functions are assumed to be Fourier transformable):
- (a) $u_t + u_{5x} = 0, u(x,0) = f(x)$.
- (b) $u_t + \int K(x-\xi)u(\xi,t)d\xi = 0, u(x,0) = f(x), \hat{K(k)} = e^{-k^2}$. Recall the convolution theorem for Fourier transforms (cf. Ablowitz and Fokas, 2003: $\mathcal{F}\int f(x-\xi)g(\xi,t)d\xi = \hat{F}(k)\hat{G}(k)$ where $\mathcal{F}$ represents the Fourier transform).
- (c) $u_{tt} + u_{4x} = 0, u(x,0) = f(x),\ u_t(x,0) = g(x)$.
- (d) $u_{tt} - c^2u_{xx} - m^2u(x,t) = 0, u(x,0) = f(x), u_t(x,0) = g(x)$. Contrast this solution with that of the standard Klien–Gordon equation, see (2.20).

2.2 Analyze the following equations using the method of characteristics:
- (a) $u_t + c(u)u_x = 0$, with $u(x,0) = f(x)$ where $c(u)$ and $f(x)$ are smooth functions of u and x, respectively.
- (b) $u_t + uu_x = -\alpha u$, where α is a constant and $u(x,0) = f(x)$, with $f(x)$ a smooth function of x.

2.3 Find two conservation laws associated with
- (a) $u_{tt} - u_{xx} + m^2u = 0$, where $m \in \mathbb{R}$.
- (b) $u_{tt} + u_{xxxx} = 0$.

2.4 Show that the center of mass $\bar{x} = \int xu(x,t)\,dx$ satisfies the following relations:
- (a) For $u_t + u_{xxx} = 0$, $\dfrac{d\bar{x}}{dt} = 0$.
- (b) For $u_t + uu_x + u_{xxx} = 0$, $\dfrac{d\bar{x}}{dt} = c$, where c is proportional to $\int u^2\,dx$.

2.5 Use D'Alembert's solution to solve the initial value problem of the wave equation on the semi-infinite line

$$u_{tt} - c^2 u_{xx} = 0,$$

c constant, where $u(x,0) = f(x), u_t(x,0) = g(x), x > 0$ and either
(a) $u(0,t) = h(t)$ or
(b) $u_t(0,t) = h(t)$.

2.6 Consider the PDE

$$u_{tt} - u_{xx} + \sigma u_{xxxx} = 0,$$

where $\sigma = \pm 1$. Determine for which values of σ that the equation is well-posed and solve the Cauchy problem for the case where the equation is well-posed.

2.7 Consider the equation

$$u_t + uu_x = 0,$$

with initial condition $u(0,x) = \cos(\pi x)$.
(a) Find the (implicit) solution of the equation.
(b) Show that $u(x,t)$ has a point $t = t_B$ where u_x is infinite. Find t_B.
(c) Describe the solution beyond $t = 1/\pi$.

2.8 (a) Solve the Cauchy problem

$$u_x + uu_y = 1, \qquad u(0,y) = ay,$$

where a is a non-zero constant.
(b) Find the solution of the equation in part (a) with the data

$$x = 2t, \qquad y = t^2, \qquad u(0,t^2) = t.$$

Hint: The following forms are equivalent:

$$\frac{dx}{1} = \frac{dy}{u} = \frac{du}{1}$$

and

$$\frac{dx}{ds} = 1, \qquad \frac{dy}{ds} = u, \qquad \frac{du}{ds} = 1.$$

2.9 Consider the equation

$$u_t + uu_x = 1, \qquad -\infty < x < \infty, \qquad t > 0.$$

(a) Find the general solution.
(b) Discuss the solution corresponding to: $u = \frac{1}{2}t$ when $t^2 = 4x$.
(c) Discuss the solution corresponding to: $u = t$ when $t^2 = 2x$.

2.10 Solve the equation

$$u_t + uu_x = 0, \qquad -\infty < x < \infty, t > 0,$$

subject to the initial data

$$u(x,0) = \begin{cases} 0, & x \le 0 \\ x/a, & 0 < x < a \\ 1, & x \ge a. \end{cases}$$

Examine the solution as $a \to 0$.

2.11 Find the the solution of the equation

$$yu_x - xu_y = 0,$$

corresponding to the data $u(x,0) = f(x)$. Explain what happens if we give $u(x(s), y(s)) = f(s)$ along the curve defined by $\{s : x^2(s) + y^2(s) = a^2\}$.

2.12 Find the solution to the initial value problem

$$u_t + u^3 u_x = 0,$$
$$u(x,0) = f(x).$$

Find the solution when $f(x) = x^{1/3}$ and discuss its behavior as $t \to \infty$.

2.13 Use the traveling wave coordinate $\xi = x - ct$ to reduce the PDE

$$u_t - u_{xx} = \begin{cases} u, & 0 \le u \le 1/2 \\ 0, & 1/2 \le u \le 1. \end{cases}$$

to an ODE. Solve the ODE subject to the boundary conditions $u(-\infty) = 1$ and $u(+\infty) = 0$. Discuss the solution for $c = 2$ and $c \ne 2$.