Preface

The field of nonlinear dispersive waves has developed rapidly over the past 50 years. Its roots go back to the work of Stokes in 1847, Boussinesq in the 1870s and Korteweg and de Vries (KdV) in 1895, all of whom studied water wave problems. In the early 1960s researchers developed effective asymptotic methods, such as the method of multiple scales, that allow one to obtain nonlinear wave equations such as the KdV equation and the nonlinear Schrödinger (NLS) equation, as leading-order asymptotic equations governing a broad class of physical phenomena. Indeed, we now know that both the KdV and NLS equations are "universal" models. It can be shown that KdV-type equations arise whenever we have weakly dispersive and weakly nonlinear systems as the governing system. On the other hand, NLS equations arise from quasi-monochromatic and weakly nonlinear systems.

The discovery of solitons associated with the KdV equation in 1965 by Zabusky and Kruskal was a major development. They employed a synergistic approach: computational methods and analytical insight. This was soon followed by a remarkable publication in 1967 by Gardner, Greene, Kruskal and Miura that described the analytical method of solution to the KdV equation, with rapidly decaying initial data. They employed concepts of direct and inverse scattering in the solution of the KdV equation that was perceived by researchers then as nothing short of astonishing. It was the first time such a higher-order nonlinear dispersive wave equation (the KdV equation is third order in space and first order in time) was "solved" or linearized; moreover it was shown how solitons were related to discrete eigenvalues of the time-independent Schrödinger scattering problem. The question of whether this was a single event, i.e., special only to the KdV equation, was answered just a few years later. In 1971 Zakharov and Shabat, using ideas developed by Lax in 1968, obtained the method of solution to the NLS equation with rapidly decaying data. Their solution method also used direct and inverse scattering.

In 1973–1974 Ablowitz, Kaup, Newell and Segur showed that the methods used to solve the KdV and NLS equations applied to a class of nonlinear wave equations including physically important equations such as the modified KdV and sine–Gordon equations. They also showed that the technique was a natural generalization of the linear method of Fourier transforms. They termed the procedure the inverse scattering transform or IST. Subsequently researchers have found wide classes of equations, including numerous physically interesting nonlinear wave equations, solvable by IST, including higher-order PDEs in one space and one time dimension, multidimensional systems, discrete systems – i.e., differential–difference and partial difference equations and even singular integral equations. Solutions to the periodic initial value problem, direct methods to obtain soliton solutions, conservation laws, Hamiltonian structures associated with these equations, and much more, have been obtained. The development of IST has also motivated researchers to study many of these and related equations by functional analytic methods in order to establish local, and whenever feasible, global existence of solutions to the relevant initial value problems.

On the other hand, whenever physicists and engineers need to study a specific class of nonlinear wave equations, they invariably consider and frequently employ direct numerical simulation. This has the advantage of being applicable to a wide class of systems and is often readily carried out. But for complex multidimensional physical problems it can be extremely difficult or essentially impossible to carry out direct simulations. For example, researchers in optical communication rely on asymptotic reductions of Maxwell's equations (with nonlinear polarization terms) to fundamental NLS models because the scales of the dynamics differ enormously: indeed by many orders of magnitude (10^{15}). Once an asymptotic model is developed, direct numerical methods are usually feasible. However, to obtain general information related to specific classes of solutions, such as solitons or solitary waves, one often finds that an analytically based approach is highly desirable. Otherwise covering a range of interesting parameter values becomes a long and arduous chore.

This book aims to put into perspective concepts and asymptotic methods that researchers have found useful both for deriving important reduced asymptotic equations from physically significant models as well as for analyzing the asymptotic equations and solutions under perturbations.

Part I contains Chapters 1–7; here the fundamental aspects and basic applications of nonlinear waves and asymptotic analysis are discussed. Also included is some discussion of linear waves in order to help set ideas and concepts regarding nonlinear waves. Part II consists of Chapters 8 and 9. Here, the notion of exact solvability or integrability via associated linear compatible systems and the method of the inverse scattering transform (IST) is described. Each of the Chapters 1–9 has exercises that can be used for homework problems or may be considered by the reader as encouraging additional practice and thought. Part III contains applications of nonlinear waves. The material is by and large more recent in nature than Parts I and II. However, the mathematical methods and asymptotic analysis are similar to what has been developed earlier. In most respects the reader will not find the work technically difficult. Indeed the concepts often follow naturally and expand the scope and breadth of our understanding of nonlinear wave phenomena.

A more detailed outline of this book is as follows.

Chapter 1 introduces the Korteweg–de Vries equation and the soliton concept from a historical perspective via the system of anharmonic oscillators originally studied by Fermi, Pasta and Ulam (FPU) in 1955. Kruskal and Zabusky (1965) showed how the KdV equation resulted from the FPU problem and they discussed why the soliton concept of "elastic interaction" explains the recurrence of initial states observed by FPU. In recent years many researchers have adopted the term soliton when they refer to a localized wave, and not necessarily one that maintains its speed/amplitude upon interaction. We will often use the more general notion when discussing physical problems. This chapter also gives additional historical background and examples.

Chapter 2 briefly discusses linear waves, the notion of dispersive and nondispersive wave systems, the technique of Fourier transforms, the method of characteristics and well-posedness.

Chapter 3 employs asymptotic methods of integrals to analyze the long-time asymptotic solution of linear dispersive wave systems. For the linear KdV equation it is shown that the long-time solution has three regions: exponential decay that matches to an Airy function connection region that in turn matches to a region with decaying oscillations. It is also shown how to extend Fourier analysis to linear differential–difference evolution systems.

Chapter 4 introduces perturbation methods, in particular the method of multiple scales and variants such as the Stokes–Poincaré frequency shift, in the context of ordinary differential equations. Linear and nonlinear equations are investigated including the nonlinear pendulum with slowly varying driving frequency.

In Chapter 5 the equations of water waves are introduced. In the limit of weak nonlinearity and long waves, i.e., shallow water, the KdV equation is derived. The extension to multidimensions of KdV, called the Kadomtsev–Petviashvili (KP) equation, is also discussed.

Nonlinear Schrödinger models are described in Chapter 6. The NLS equation is first derived from a model nonlinear Klein equation. Derivations of NLS equations from water waves in deep water with weak nonlinearity are outlined and some of the properties of NLS equations are described.

Chapter 7 introduces Maxwell's equations with nonlinear polarization terms such as those that arise in the context of nonlinear optics. The derivation of the NLS equation in bulk media is outlined. A brief discussion of how the NLS equation arises in the context of ferromagnetics is also included.

Although the primary focus of this book is directed towards physical problems and methods, the notion of integrable equations and solitons is still extremely useful, especially as a guide. In Chapters 8 and 9 some background information is given about these interesting systems. Chapter 8 shows how the Korteweg–de Vries (KdV), nonlinear Schrödinger (NLS), mKdV, sine–Gordon and other equations can be viewed as a compatibility condition of two linear equations: a linear scattering problem and associated linear time evolution equation under "isospectrality" (constancy of eigenvalues). In Chapter 9 the description of how one can obtain a linearization of these equations is given. It is shown how the solitons are related to eigenvalues of the linear scattering problem. The method is referred to as the inverse scattering transform (IST).

In Chapters 10 and 11 two applications of nonlinear optics are discussed: optical communications and mode-locked lasers. These areas are closely related and NLS equations play a central role.

In communications, NLS equations supplemented with rapidly varying coefficients that take into account damping, gain and dispersion variation is the relevant physically interesting asymptotic system. The latter is associated with the technology of dispersion-management (DM), i.e., the fusing together of optical fibers of substantially different, opposite in sign, dispersion coefficients. Dispersion-management, which is now used in commercial systems, significantly reduces penalties due to noise and multi-pulse interactions in wavelength division multiplexed (WDM) systems. WDM is the technology of the simultaneous transmission of pulses centered in widely separated frequency "windows". The analysis of these NLS systems centrally involves asymptotic analysis, in particular the technique of multiple scales. A key equation associated with DM systems is derived by the multiple-scales method. It is a non-local NLS-type equation that is referred to as the DMNLS equation. For these DM systems special solutions such as dispersion-managed solitons can be obtained and interaction phenomena are discussed.

The study of mode-locked lasers involves the study of NLS equations with saturable gain, filtering and loss terms. In many cases use of dispersion-management is useful. A well-known model, called the master equation, Taylor-expands the saturable power terms in the loss. It is found that keeping the full saturable loss model leads to mode-locking over wide parameter regimes for constant as well as dispersion-managed models. This equation provides insight to the phenomena that can occur. Localized modes and strings of solitons are found in the anomalous and normal dispersive regimes.