

## CHAPTER

# 1

## Introduction

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### CHAPTER OBJECTIVES

- To properly introduce the subject of fluid mechanics and its importance
- To state the assumptions upon which the subject is based
- To review the basic background science of liquids and gases
- To present the relevant features of fluid statics
- To establish dimensional analysis as an intellectual tool for use in the remainder of the text

## 1.1. FLUID MECHANICS

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Fluid mechanics is the branch of science concerned with moving and stationary fluids. Given that the vast majority of the observable mass in the universe exists in a fluid state, that life as we know it is not possible without fluids, and that the atmosphere and oceans covering this planet are fluids, fluid mechanics has unquestioned scientific and practical importance. Its allure crosses disciplinary boundaries, in part because it is described by a nonlinear field theory and also because it is readily observed. Mathematicians, physicists, biologists, geologists, oceanographers, atmospheric scientists, engineers of many types, and even artists have been drawn to study, harness, and exploit fluid mechanics to develop and test formal and computational techniques, to better understand the natural world, and to attempt to improve the human condition. The importance of fluid mechanics cannot be overstated for applications involving transportation, power generation and conversion, materials processing and manufacturing, food production, and civil infrastructure. For example, in the twentieth century, life expectancy in the United States approximately doubled. About half of this increase can be traced to advances in medical practice, particularly antibiotic therapies. The other half largely resulted from a steep decline in childhood mortality from water-borne diseases, a decline that occurred because of widespread delivery of clean water to nearly the entire population—a fluids-engineering and public-works achievement. Yet, the pursuits of mathematicians, scientists, and engineers are interconnected: Engineers need to understand natural phenomena to be successful, scientists strive to provide this understanding, and mathematicians pursue the formal and computational tools that support these efforts.

Advances in fluid mechanics, like any other branch of physical science, may arise from mathematical analyses, computer simulations, or experiments. Analytical approaches are often successful for finding solutions to idealized and simplified problems and such solutions can be of immense value for developing insight and understanding, and for comparisons with numerical and experimental results. Thus, some fluency in mathematics, especially multivariable calculus, is helpful in the study of fluid mechanics. In practice, drastic simplifications are frequently necessary to find analytical solutions because of the complexity of real fluid flow phenomena. Furthermore, it is probably fair to say that some of the greatest theoretical contributions have come from people who depended rather strongly on their physical intuition. Ludwig Prandtl, one of the founders of modern fluid mechanics, first conceived the idea of a boundary layer based solely on physical intuition. His knowledge of mathematics was rather limited, as his famous student Theodore von Karman (1954, page 50) testifies. Interestingly, the boundary layer concept has since been expanded into a general method in applied mathematics.

As in other scientific fields, mankind's mathematical abilities are often too limited to tackle the full complexity of real fluid flows. Therefore, whether we are primarily interested in understanding flow physics or in developing fluid-flow applications, we often must depend on observations, computer simulations, or experimental measurements to test hypotheses and analyses, and develop insights into the phenomena under study. This book is an introduction to fluid mechanics that should appeal to anyone pursuing fluid mechanical inquiry. Its emphasis is on fully presenting fundamental concepts and illustrating them with examples drawn from various scientific and engineering fields. Given its finite size, this book provides—at best—an incomplete description of the subject. However, the purpose of this

book will be fulfilled if the reader becomes more curious and interested in fluid mechanics as a result of its perusal.

## 1.2. UNITS OF MEASUREMENT

For mechanical systems, the units of all physical variables can be expressed in terms of the units of four basic variables, namely, *length*, *mass*, *time*, and *temperature*. In this book, the international system of units (Système international d'unités) commonly referred to as SI (or MKS) units, is preferred. The basic units of this system are *meter* for length, *kilogram* for mass, *second* for time, and *Kelvin* for temperature. The units for other variables can be derived from these basic units. Some of the common variables used in fluid mechanics, and their SI units, are listed in Table 1.1. Some useful conversion factors between different systems of units are listed in Appendix A. To avoid very large or very small numerical values, prefixes are used to indicate multiples of the units given in Table 1.1. Some of the common prefixes are listed in Table 1.2.

Strict adherence to the SI system is sometimes cumbersome and will be abandoned occasionally for simplicity. For example, temperatures will be frequently quoted in degrees Celsius ( $^{\circ}\text{C}$ ), which is related to Kelvin (K) by the relation  $^{\circ}\text{C} = \text{K} - 273.15$ . However, the English system of units (foot, pound,  $^{\circ}\text{F}$ ) will not be used, even though this unit system remains in use in some places in the world.

## 1.3. SOLIDS, LIQUIDS, AND GASES

The various forms of matter may be broadly categorized as being fluid or solid. A fluid is a substance that deforms continuously under an applied shear stress or, equivalently, one that does not have a preferred shape. A solid is one that does not deform continuously under an applied shear stress, and does have a preferred shape to which it relaxes when external forces on it are withdrawn. Consider a rectangular element of a solid ABCD (Figure 1.1a).

TABLE 1.1 SI Units

Quantity	Name of unit	Symbol	Equivalent
Length	Meter	m	
Mass	Kilogram	kg	
Time	Second	s	
Temperature	Kelvin	K	
Frequency	Hertz	Hz	$\text{s}^{-1}$
Force	Newton	N	$\text{kg ms}^{-2}$
Pressure	Pascal	Pa	$\text{N m}^{-2}$
Energy	Joule	J	$\text{N m}$
Power	Watt	W	$\text{J s}^{-1}$

TABLE 1.2 Common Prefixes

Prefix	Symbol	Multiple
Mega	M	$10^6$
Kilo	k	$10^3$
Deci	d	$10^{-1}$
Centi	c	$10^{-2}$
Milli	m	$10^{-3}$
Micro	$\mu$	$10^{-6}$

Under the action of a shear force  $F$  the element assumes the shape  $ABC'D'$ . If the solid is perfectly elastic, it returns to its preferred shape  $ABCD$  when  $F$  is withdrawn. In contrast, a fluid deforms *continuously* under the action of a shear force, *however small*. Thus, the element of the fluid  $ABCD$  confined between parallel plates (Figure 1.1b) successively deforms to shapes such as  $ABC'D'$  and  $ABC''D''$ , and keeps deforming, as long as the force  $F$  is maintained on the upper plate. When  $F$  is withdrawn, the fluid element's final shape is retained; it does not return to a prior shape. Therefore, we say that a fluid flows.

The qualification “however small” in the description of a fluid is significant. This is because some solids also deform continuously if the shear stress exceeds a certain limiting value, corresponding to the *yield point* of the solid. A solid in such a state is known as *plastic*, and plastic deformation changes the solid object's unloaded shape. Interestingly, the distinction between solids and fluids may not be well defined. Substances like paints, jelly, pitch, putty, polymer solutions, and biological substances (for example, egg whites) may simultaneously display both solid and fluid properties. If we say that an elastic solid has a perfect memory of its preferred shape (because it always springs back to its preferred shape when unloaded) and that an ordinary viscous fluid has zero memory (because it never springs back when unloaded), then substances like egg whites can be called *viscoelastic* because they partially rebound when unloaded.

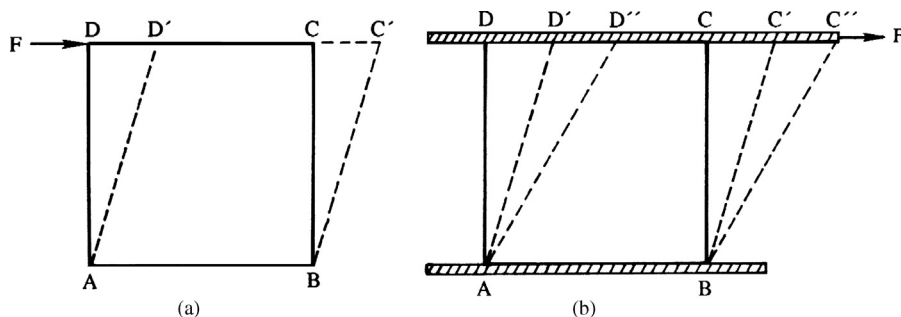


FIGURE 1.1 Deformation of solid and fluid elements under a constant externally applied shear force. (a) Solid; here the element deflects until its internal stress balances the externally applied force. (b) Fluid; here the element deforms continuously as long as the shear force is applied.

Although solids and fluids behave very differently when subjected to shear stresses, they behave similarly under the action of compressive normal stresses. However, tensile normal stresses again lead to differences in fluid and solid behavior. Solids can support both tensile and compressive normal stresses, while fluids typically expand or change phase (i.e., boil) when subjected to tensile stresses. Some liquids can support a small amount of tensile stress, the amount depending on the degree of molecular cohesion and the duration of the tensile stress.

Fluids generally fall into two classes, liquids and gases. A gas always expands to fill the entire volume of its container. In contrast, the volume of a liquid changes little, so that it cannot completely fill a large container; in a gravitational field, a free surface forms that separates a liquid from its vapor.

## 1.4. CONTINUUM HYPOTHESIS

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A fluid is composed of a large number of molecules in constant motion undergoing collisions with each other, and is therefore discontinuous or discrete at the most microscopic scales. In principle, it is possible to study the mechanics of a fluid by studying the motion of the molecules themselves, as is done in kinetic theory or statistical mechanics. However, we are generally interested in the *average manifestation* of the molecular motion. For example, forces are exerted on the boundaries of a fluid's container due to the constant bombardment of the fluid molecules; the statistical average of these collision forces per unit area is called *pressure*, a macroscopic property. So long as we are not interested in the molecular mechanics of the origin of pressure, we can ignore the molecular motion and think of pressure as simply the average force per unit area exerted by the fluid.

When the molecular density of the fluid and the size of the region of interest are large enough, such average properties are sufficient for the explanation of macroscopic phenomena and the discrete molecular structure of matter may be ignored and replaced with a continuous distribution, called a *continuum*. In a continuum, fluid properties like temperature, density, or velocity are defined at every point in space, and these properties are known to be appropriate averages of molecular characteristics in a small region surrounding the point of interest. The continuum approximation is valid when the Knudsen number,  $Kn = l/L$  where  $l$  is the mean free path of the molecules and  $L$  is the length scale of interest (a body length, a pore diameter, a turning radius, etc.), is much less than unity. For most terrestrial situations, this is not a great restriction since  $l \approx 50 \text{ nm}$  for air at room temperature and pressure, and  $l$  is more than two orders of magnitude smaller for water under the same conditions. However, a molecular-kinetic-theory approach may be necessary for analyzing flows over very small objects or in very narrow flow paths, or in the tenuous gases at the upper reaches of the atmosphere.

## 1.5. MOLECULAR TRANSPORT PHENOMENA

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Although the details of molecular motions may be locally averaged to compute temperature, density, or velocity, random molecular motions still lead to diffusive transport of

molecular species, temperature, or momentum that impact fluid properties at macroscopic scales.

Consider a surface area AB within a mixture of two gases, say, nitrogen and oxygen (Figure 1.2), and assume that the nitrogen mass fraction  $Y$  varies across AB. Here the mass of nitrogen per unit volume is  $\rho Y$  (sometimes known as the nitrogen *concentration* or *density*), where  $\rho$  is the overall density of the gas mixture. Random migration of molecules across AB in both directions will result in a *net* flux of nitrogen across AB, from the region of higher  $Y$  toward the region of lower  $Y$ . To a good approximation, the flux of one constituent in a mixture is proportional to its gradient:

$$\mathbf{J}_m = -\rho\kappa_m\nabla Y. \quad (1.1)$$

Here the vector  $\mathbf{J}_m$  is the mass flux ( $\text{kg m}^{-2} \text{s}^{-1}$ ) of the constituent,  $\nabla Y$  is the mass-fraction gradient of that constituent, and  $\kappa_m$  is a (positive) constant of proportionality that depends on the particular pair of constituents in the mixture and the local thermodynamic state. For example,  $\kappa_m$  for diffusion of nitrogen in a mixture with oxygen is different than  $\kappa_m$  for diffusion of nitrogen in a mixture with carbon dioxide. The linear relation (1.1) for mass diffusion is generally known as *Fick's law*, and the minus sign reflects the fact that species diffuse from higher to lower concentrations. Relations like this are based on empirical evidence, and

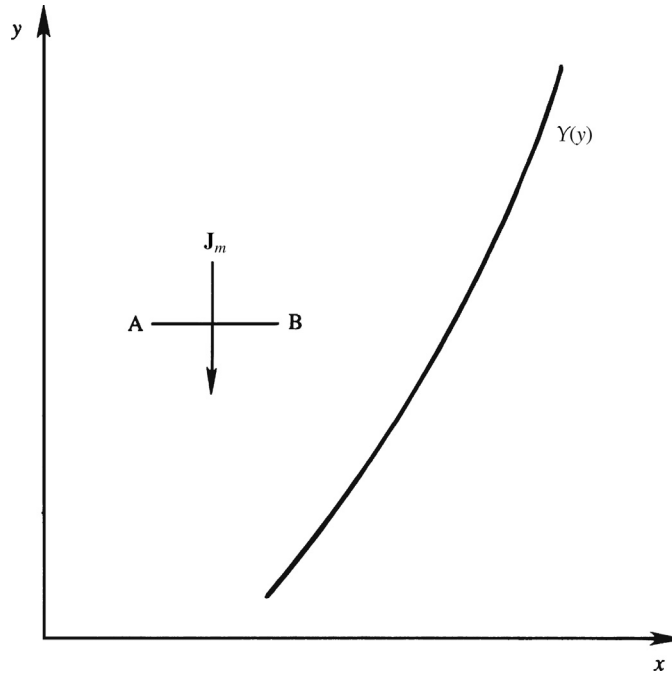


FIGURE 1.2 Mass flux  $\mathbf{J}_m$  due to variation in the mass fraction  $Y(y)$ . Here the mass fraction profile increases with increasing  $Y$ , so Fick's law of diffusion states that the diffusive mass flux that acts to smooth out mass-fraction differences is downward across AB.

are called *phenomenological laws*. Statistical mechanics can sometimes be used to derive such laws, but only for simple situations.

The analogous relation for heat transport via a temperature gradient  $\nabla T$  is *Fourier's law*,

$$\mathbf{q} = -k\nabla T, \quad (1.2)$$

where  $\mathbf{q}$  is the heat flux ( $J m^{-2} s^{-1}$ ), and  $k$  is the material's thermal conductivity.

The analogous relationship for momentum transport via a velocity gradient is qualitatively similar to (1.1) and (1.2) but is more complicated because momentum and velocity are vectors. So as a first step, consider the effect of a vertical gradient,  $du/dy$ , in the horizontal velocity  $u$  (Figure 1.3). Molecular motion and collisions cause the faster fluid above AB to pull the fluid underneath AB forward, thereby speeding it up. Molecular motion and collisions also cause the slower fluid below AB to pull the upper faster fluid backward, thereby slowing it down. Thus, without an external influence to maintain  $du/dy$ , the flow profile shown by the solid curve will evolve toward a profile shown by the dashed curve. This is analogous to saying that  $u$ , the horizontal momentum per unit mass (a momentum *concentration*), *diffuses* downward. Here, the resulting momentum flux, from high to low  $u$ , is equivalent to a shear stress,  $\tau$ , existing in the fluid. Experiments show that the magnitude of  $\tau$  along a surface such as AB is, to a good approximation, proportional to the local velocity gradient,

$$\tau = \mu(du/dy), \quad (1.3)$$

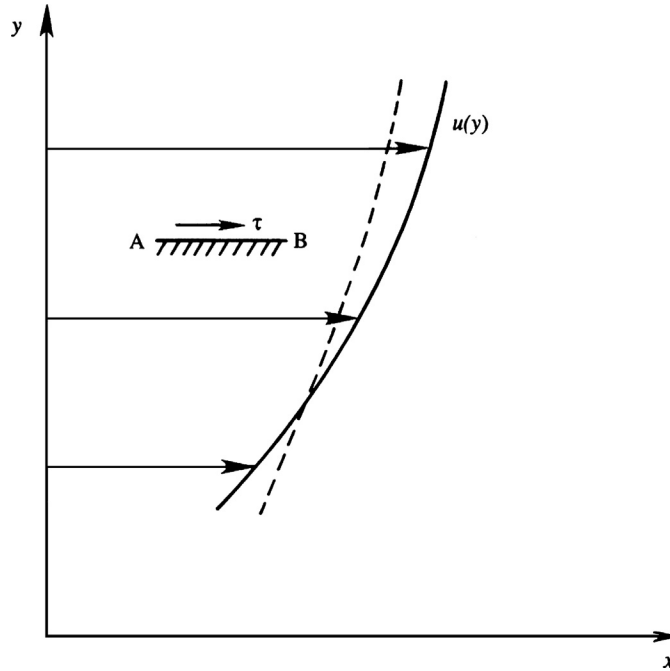


FIGURE 1.3 Shear stress  $\tau$  on surface AB. The diffusive action of fluid viscosity tends to decrease velocity gradients, so that the continuous line tends toward the dashed line.

where the constant of proportionality  $\mu$  (with units of  $\text{kg m}^{-1} \text{s}^{-1}$ ) is known as the *dynamic viscosity*. This is *Newton's law of friction*. It is analogous to (1.1) and (1.2) for the simple unidirectional shear flow depicted in Figure 1.3. However, it is an incomplete scalar statement of molecular momentum transport when compared to the more complete vector relationships (1.1) and (1.2) for species and thermal molecular transport. A more general tensor form of (1.3) that accounts for three velocity components and three possible orientations of the surface AB is presented in Chapter 4 after the mathematical and kinematical developments in Chapters 2 and 3. For gases and liquids,  $\mu$  depends on the local temperature  $T$ . In ideal gases, the random thermal speed is roughly proportional to  $T^{1/2}$ , so molecular momentum transport, and consequently  $\mu$ , also vary approximately as  $T^{1/2}$ . For liquids, shear stress is caused more by the intermolecular cohesive forces than by the thermal motion of the molecules. These cohesive forces decrease with increasing  $T$  so  $\mu$  for a liquid decreases with increasing  $T$ .

Although the shear stress is proportional to  $\mu$ , we will see in Chapter 4 that the tendency of a fluid to transport velocity gradients is determined by the quantity

$$\nu \equiv \mu/\rho, \quad (1.4)$$

where  $\rho$  is the density ( $\text{kg m}^{-3}$ ) of the fluid. The units of  $\nu$  ( $\text{m}^2 \text{s}^{-1}$ ) do not involve the mass, so  $\nu$  is frequently called the *kinematic viscosity*.

Two points should be noticed about the transport laws (1.1), (1.2), and (1.3). First, only *first* derivatives appear on the right side in each case. This is because molecular transport is carried out by a nearly uncountable number of molecular interactions at length scales that are too small to be influenced by higher derivatives of the species mass fractions, temperature, or velocity profiles. Second, nonlinear terms involving higher powers of the first derivatives, for example  $|\nabla u|^2$ , do not appear. Although this is only expected for small first-derivative magnitudes, experiments show that the linear relations are accurate enough for most practical situations involving mass fraction, temperature, or velocity gradients.

## 1.6. SURFACE TENSION

A density discontinuity may exist whenever two immiscible fluids are in contact, for example at the interface between water and air. Here unbalanced attractive intermolecular forces cause the interface to behave as if it were a stretched membrane under tension, like the surface of a balloon or soap bubble. This is why small drops of liquid in air or small gas bubbles in water tend to be spherical in shape. Imagine a liquid drop surrounded by an insoluble gas. Near the interface, all the liquid molecules are trying to pull the molecules on the interface inward toward the center of the drop. The net effect of these attractive forces is for the interface area to contract until equilibrium is reached with other surface forces. The magnitude of the tensile force that acts per unit length to open a line segment lying in the surface like a seam is called *surface tension*  $\sigma$ ; its units are  $\text{N m}^{-1}$ . Alternatively,  $\sigma$  can be thought of as the energy needed to create a unit of interfacial area. In general,  $\sigma$  depends on the pair of fluids in contact, the temperature, and the presence of surface-active chemicals (surfactants) or impurities, even at very low concentrations.

An important consequence of surface tension is that it causes a pressure difference across curved interfaces. Consider a spherical interface having a radius of curvature  $R$  (Figure 1.4a).



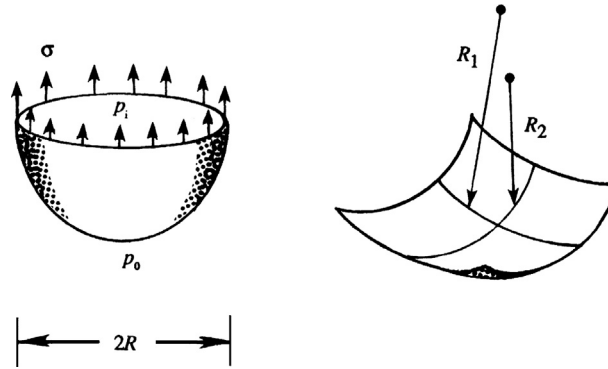


FIGURE 1.4 (a) Section of a spherical droplet, showing surface tension forces. (b) An interface with radii of curvatures  $R_1$  and  $R_2$  along two orthogonal directions.

If  $p_i$  and  $p_o$  are the pressures on the inner and outer sides of the interface, respectively, then a static force balance gives

$$\sigma(2\pi R) = (p_i - p_o)\pi R^2,$$

from which the pressure jump is found to be

$$p_i - p_o = 2\sigma/R,$$

showing that the pressure on the concave side (the inside) is higher.

The curvature of a general surface can be specified by the radii of curvature along two orthogonal directions, say,  $R_1$  and  $R_2$  (Figure 1.4b). A similar analysis shows that the pressure difference across the interface is given by

$$p_i - p_o = \sigma \left( \frac{1}{R_1} + \frac{1}{R_2} \right), \quad (1.5)$$

which agrees with the spherical interface result when  $R_1 = R_2$ . This pressure difference is called the Laplace pressure.

It is well known that the free surface of a liquid in a narrow tube rises above the surrounding level due to the influence of surface tension. This is demonstrated in Example 1.1. Narrow tubes are called *capillary tubes* (from Latin *capillus*, meaning hair). Because of this, the range of phenomena that arise from surface tension effects is called *capillarity*. A more complete discussion of surface tension is presented at the end of Chapter 4 as part of the section on boundary conditions.

## 1.7. FLUID STATICS

The magnitude of the force per unit area in a static fluid is called the *pressure*; pressure in a moving medium will be defined in Chapter 4. Sometimes the ordinary pressure is called the

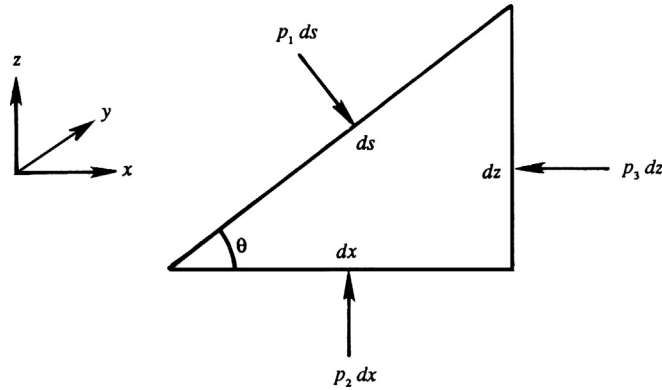


FIGURE 1.5 Demonstration that  $p_1 = p_2 = p_3$  in a static fluid. Here the vector sum of the three arrows is zero when the volume of the element shrinks to zero.

*absolute pressure*, in order to distinguish it from the *gauge pressure*, which is defined as the absolute pressure minus the atmospheric pressure:

$$p_{\text{gauge}} = p - p_{\text{atm}}.$$

The standard value for atmospheric pressure  $p_{\text{atm}}$  is  $101.3 \text{ kPa} = 1.013 \text{ bar}$  where  $1 \text{ bar} = 10^5 \text{ Pa}$ . An absolute pressure of zero implies vacuum while a gauge pressure of zero implies atmospheric pressure.

In a fluid at rest, tangential viscous stresses are absent and the only force between adjacent surfaces is normal to the surface. We shall now demonstrate that in such a case the surface force per unit area (or pressure) is equal in all directions. Consider a small volume of fluid with a triangular cross section (Figure 1.5) of unit thickness normal to the paper, and let  $p_1$ ,  $p_2$ , and  $p_3$  be the pressures on the three faces. The  $z$ -axis is taken vertically upward. The only forces acting on the element are the pressure forces normal to the faces and the weight of the element. Because there is no acceleration of the element in the  $x$  direction, a balance of forces in that direction gives

$$(p_1 ds) \sin \theta - p_3 dz = 0.$$

Because  $dz = \sin \theta ds$ , the foregoing gives  $p_1 = p_3$ . A balance of forces in the vertical direction gives

$$-(p_1 ds) \cos \theta + p_2 dx - (1/2) \rho g dx dz = 0.$$

As  $\cos \theta ds = dx$ , this gives

$$p_2 - p_1 - (1/2) \rho g dz = 0.$$

As the triangular element is shrunk to a point, that is,  $dz \rightarrow 0$  with  $\theta = \text{constant}$ , the gravity force term drops out, giving  $p_1 = p_2$ . Thus, at a point in a static fluid, we have

$$p_1 = p_2 = p_3, \quad (1.6)$$

so that the force per unit area is independent of the angular orientation of the surface. The pressure is therefore a scalar quantity.

We now proceed to determine the *spatial distribution* of pressure in a static fluid. Consider an infinitesimal cube of sides  $dx$ ,  $dy$ , and  $dz$ , with the  $z$ -axis vertically upward (Figure 1.6). A balance of forces in the  $x$  direction shows that the pressures on the two sides perpendicular to the  $x$ -axis are equal. A similar result holds in the  $y$  direction, so that

$$\partial p / \partial x = \partial p / \partial y = 0. \quad (1.7)$$

This fact is expressed by *Pascal's law*, which states that all points in a resting fluid medium (and connected by the *same* fluid) are at the same pressure if they are at the same depth. For example, the pressure at points F and G in Figure 1.7 are the same.

Vertical equilibrium of the element in Figure 1.6 requires that

$$p \, dx \, dy - (p + dp) \, dx \, dy - \rho g \, dx \, dy \, dz = 0,$$

which simplifies to

$$dp/dz = -\rho g. \quad (1.8)$$

This shows that the pressure in a static fluid subject to a constant gravitational field decreases with height. For a fluid of uniform density, (1.8) can be integrated to give

$$p = p_0 - \rho g z, \quad (1.9)$$

where  $p_0$  is the pressure at  $z = 0$ . Equation (1.9) is the well-known result of *hydrostatics*, and shows that the pressure in a liquid decreases *linearly* with increasing height. It implies that

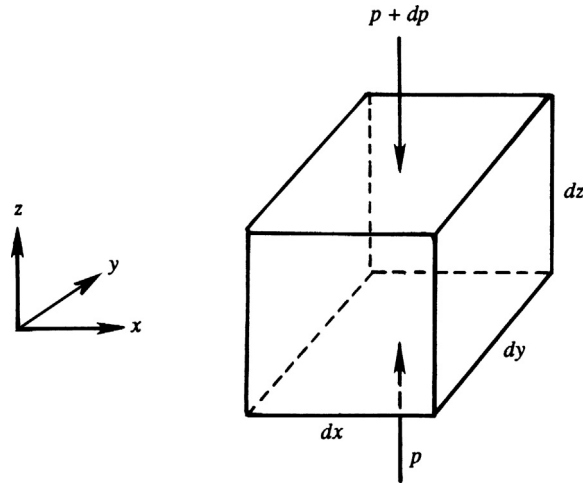


FIGURE 1.6 Fluid element at rest. Here the pressure difference between the top and bottom of the element balances the element's weight.

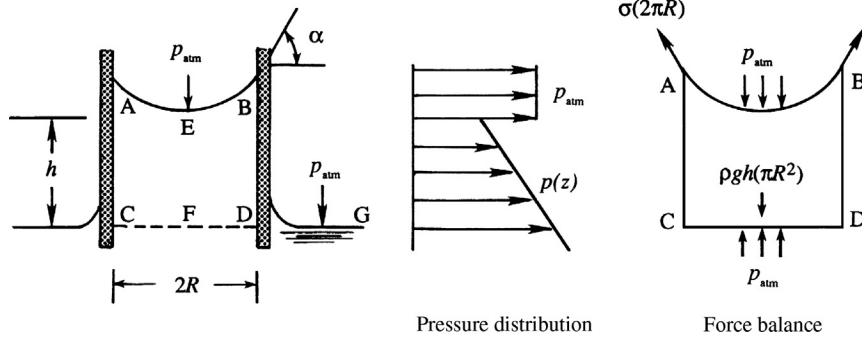


FIGURE 1.7 Rise of a liquid in a narrow tube (Example 1.1) because of the action of surface tension. The curvature of the surface and the surface tension cause a pressure difference to occur across the surface.

the pressure rise at a depth  $h$  below the free surface of a liquid is equal to  $\rho gh$ , which is the weight of a column of liquid of height  $h$  and unit cross section.

### EXAMPLE 1.1

Using Figure 1.7, show that the rise of a liquid in a narrow tube of radius  $R$  is given by

$$h = \frac{2\sigma \sin \alpha}{\rho g R},$$

where  $\sigma$  is the surface tension and  $\alpha$  is the *contact angle* between the fluid and the tube's inner surface.

### Solution

Since the free surface is concave upward and exposed to the atmosphere, the pressure just below the interface at point E is below atmospheric. The pressure then increases linearly along EF. At F the pressure again equals the atmospheric pressure, since F is at the same level as G where the pressure is atmospheric. The pressure forces on faces AB and CD therefore balance each other. Vertical equilibrium of the element ABCD then requires that the weight of the element balances the vertical component of the surface tension force, so that

$$\sigma(2\pi R)\sin \alpha = \rho gh(\pi R^2),$$

which gives the required result.

## 1.8. CLASSICAL THERMODYNAMICS

Classical thermodynamics is the study of equilibrium states of matter, in which the properties are assumed uniform in space and time. Here, the reader is assumed to be familiar with

the basic thermodynamic concepts, so this section merely reviews the main ideas and the most commonly used relations in this book.

A thermodynamic *system* is a quantity of matter that exchanges heat and work, but no mass, with its surroundings. A system in equilibrium is free of fluctuations, such as those generated during heat or work input from, or output to, its surroundings. After any such thermodynamic change, fluctuations die out or *relax*, a new equilibrium is reached, and once again the system's properties, such as pressure and temperature, are well defined. Here, the system's *relaxation time* is defined as the time taken by the system to adjust to a new thermodynamic state.

This thermodynamic system concept is obviously not directly applicable to a macroscopic volume of a moving fluid in which pressure and temperature may vary considerably. However, experiments show that classical thermodynamics does apply to small fluid volumes commonly called *fluid particles*. A fluid particle is a small deforming volume carried by the flow that: 1) always contains the same fluid molecules, 2) is large enough so that its thermodynamic properties are well defined when it is at equilibrium, but 3) is small enough so that its *relaxation time* is short compared to the time scales of fluid-motion-induced thermodynamic changes. Under ordinary conditions (the emphasis in this text), molecular densities, speeds, and collision rates are high enough so that the conditions for the existence of fluid particles are met, and classical thermodynamics can be directly applied to flowing fluids. However, there are circumstances involving rarified gases, shock waves, and high-frequency acoustic waves where one or more of the fluid particle requirements are not met and molecular-kinetic and quantum theories are needed.

The basic laws of classical thermodynamics are empirical, and cannot be derived from anything *more* fundamental. These laws essentially establish *definitions*, upon which the subject is built. The first law of thermodynamics can be regarded as a principle that defines the *internal energy* of a system, and the second law can be regarded as the principle that defines the *entropy* of a system.

## First Law of Thermodynamics

The first law of thermodynamics states that the energy of a system is conserved;

$$\delta q + \delta w = \Delta e, \quad (1.10)$$

where  $\delta q$  is the heat added to the system,  $\delta w$  is the work done on the system, and  $\Delta e$  is the increase of the system's *internal energy*. All quantities in (1.10) are normalized by the mass of the system and therefore have units of  $\text{J kg}^{-1}$  and appear as lowercase letters. When (1.10) is written with capital letters,  $\delta Q + \delta W = \Delta E$ , it portrays the same thermodynamic law without normalization by the system mass. The internal energy (aka, thermal energy) is a manifestation of the random molecular motion of the system's constituents. In fluid flows, the kinetic energy of the fluid particles' macroscopic motion has to be included in the  $e$ -term in (1.10) in order that the principle of conservation of energy is satisfied. For developing the relations of classical thermodynamics, however, we shall only include the thermal energy in the term  $e$ .

It is important to realize the difference between heat and internal energy. Heat and work are forms of *energy in transition*, which appear at the *boundary* of the system and are *not contained* within the matter. In contrast, the internal energy resides within the matter. If two equilibrium states 1 and 2 of a system are known, then  $Q$  and  $W$  depend on the *process* or *path* followed by the system in going from state 1 to state 2. The change  $\Delta e = e_2 - e_1$ , in contrast, does not depend on the path. In short,  $e$  is a thermodynamic property and is a function of the thermodynamic state of the system. Thermodynamic properties are called *state functions*, in contrast to heat and work, which are *path functions*.

Frictionless quasi-static processes, carried out at an extremely slow rate so that the system is at all times in equilibrium with the surroundings, are called *reversible processes*. For a compressible fluid, the most common type of reversible work is by the expansion or contraction of the boundaries of the fluid particle. Let  $v = 1/\rho$  be the *specific volume*, that is, the volume per unit mass. The work done per unit mass by a fluid particle in an infinitesimal reversible process is  $-p dv$ , where  $dv$  is the increase of  $v$ . The first law (1.10) for a reversible process then becomes

$$de = dq - p dv, \quad (1.11)$$

provided that  $q$  is also reversible. Note that irreversible forms of work, such as those done against frictional stresses, are excluded from (1.11).

## Equations of State

A relation defining one state function in terms of two or more others is called an *equation of state*. For a simple compressible substance composed of a single component (the applicable model for nearly all pure fluids), the specification of two independent thermodynamic properties completely determines the state of the system. We can write relations such as the *thermal* and *caloric equations of state*:

$$p = p(v, T) \text{ or } e = e(p, T), \quad (1.12)$$

respectively. For more complicated systems composed of more than one component, the specification of additional properties is needed to completely determine the state. For example, seawater contains dissolved salt so its density is a function of temperature, pressure, and salinity.

## Specific Heats

Before we define the specific heats of a substance, we define the thermodynamic property *enthalpy* as

$$h \equiv e + pv. \quad (1.13)$$

It is the sum of the thermal energy and the pressure-volume potential energy, and arises naturally in the study of compressible fluid flows.

For single-component systems, the specific heat capacities at constant pressure and constant volume are defined as

$$C_p \equiv (\partial h / \partial T)_p \text{ and } C_v \equiv (\partial e / \partial T)_v, \quad (1.14, 1.15)$$

respectively. Here, (1.14) means that we regard  $h$  as a function of  $p$  and  $T$ , and find the partial derivative of  $h$  with respect to  $T$ , keeping  $p$  constant. Equation (1.15) has an analogous interpretation. The specific heats as defined are thermodynamic properties because they are defined in terms of other properties of the system. That is,  $C_p$  and  $C_v$  can be determined when two other system properties (say,  $p$  and  $T$ ) are known.

For certain processes common in fluid flows, the heat exchange can be related to the specific heats. Consider a reversible process in which the work done is given by  $p dv$ , so that the first law of thermodynamics has the form of (1.11). Dividing by the change of temperature, it follows that the heat transferred per unit mass per unit temperature change in a constant volume process is

$$(\partial Q / \partial T)_v = (\partial e / \partial T)_v = C_v.$$

This shows that  $C_v dT$  represents the heat transfer per unit mass in a reversible constant-volume process, in which the only type of work done is of the  $p dv$  type. It is misleading to define  $C_v = (dQ/dT)_v$  without any restrictions imposed, as the temperature of a constant-volume system can increase without heat transfer, such as by vigorous stirring.

Similarly, the heat transferred at constant pressure during a reversible process is given by

$$(\partial Q / \partial T)_p = (\partial h / \partial T)_p = C_p.$$

## Second Law of Thermodynamics

The second law of thermodynamics restricts the direction in which real processes can proceed as time increases. Its implications are discussed in Chapter 4. Some consequences of this law are the following:

- (i) There must exist a thermodynamic property  $s$ , known as *entropy*, whose change between states 1 and 2 is given by

$$s_2 - s_1 = \int_1^2 \frac{dq_{\text{rev}}}{T}, \quad (1.16)$$

where the integral is taken along any reversible process between the two states.

- (ii) For an *arbitrary* process between states 1 and 2, the entropy change is

$$s_2 - s_1 \geq \int_1^2 \frac{dq}{T} \text{ (Clausius-Duhem),}$$

which states that the entropy of an isolated system ( $dq = 0$ ) can only increase. Such increases are caused by friction, mixing, and other irreversible phenomena.

- (iii) Molecular transport coefficients such as viscosity  $\mu$  and thermal conductivity  $k$  must be positive. Otherwise, spontaneous unmixing or momentum separation would occur and lead to a decrease of entropy of an isolated system.

## Property Relations

Two common relations are useful in calculating entropy changes during a process. For a reversible process, the entropy change is given by

$$Tds = dq. \quad (1.17)$$

On substituting into (1.11) and using (1.13), we obtain

$$Tds = de + pdv, \text{ or } Tds = dh - vdp. \text{ (Gibbs)} \quad (1.18)$$

It is interesting that these relations (1.18) are also valid for irreversible (frictional) processes, although the relations (1.11) and (1.17), from which equations (1.18) are derived, are true for reversible processes only. This is because (1.18) are relations between thermodynamic *state functions* alone and are therefore true for *any* process. The association of  $Tds$  with heat and  $-pdv$  with work does not hold for irreversible processes. Consider stirring work done at constant volume that raises a fluid element's temperature; here  $de = Tds$  is the increment of stirring work done.

## Speed of Sound

In a compressible fluid, infinitesimal isentropic changes in density and pressure propagate through the medium at a finite speed,  $c$ . In Chapter 15, we shall prove that the square of this speed is given by

$$c^2 = (\partial p / \partial \rho)_s, \quad (1.19)$$

where the subscript  $s$  signifies that the derivative is taken at constant entropy. This is the speed of sound. For incompressible fluids,  $\partial \rho / \partial p \rightarrow 0$  under all conditions so  $c \rightarrow \infty$ .

## Thermal Expansion Coefficient

When fluid density is a function of temperature, we define the thermal expansion coefficient

$$\alpha \equiv -\frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_p, \quad (1.20)$$

where the subscript  $p$  signifies that the partial derivative is taken at constant pressure. This expansion coefficient appears frequently in the study of nonisothermal systems.

## 1.9. PERFECT GAS

A basic result from kinetic theory and statistical mechanics for the thermal equation of state for  $n$  identical noninteracting gas molecules confined within a container having volume  $V$  is

$$pV = nk_B T, \quad (1.21)$$

where  $p$  is the average pressure on the inside surfaces of the container,  $k_B = 1.381 \times 10^{-23} \text{ J K}^{-1}$  is Boltzmann's constant, and  $T$  is the absolute temperature. Equation (1.21) is



the molecule-based version of the perfect gas law. It is valid when attractive forces between the molecules are negligible and when  $V/n$  is much larger than the (average) volume of an individual molecule. When used with the continuum approximation, (1.21) is commonly rearranged by noting that  $\rho = mn/V$ , where  $m$  is the (average) mass of one gas molecule. Here  $m$  is calculated (in SI units) as  $M_w/A_o$  where  $M_w$  is the (average) molecular weight in kg (kg-mole)<sup>-1</sup> of the gas molecules, and  $A_o$  is the kilogram-based version of Avogadro's number,  $6.023 \times 10^{26}$  (kg-mole)<sup>-1</sup>. With these replacements, (1.21) becomes

$$p = \frac{n}{V} k_B T = \frac{nm}{V} \left( \frac{k_B}{m} \right) T = \rho \left( \frac{k_B A_o}{M_w} \right) T = \rho \left( \frac{R_u}{M_w} \right) T = \rho R T, \quad (1.22)$$

where the product  $k_B A_o = R_u = 8314 \text{ J kmol}^{-1} \text{ K}^{-1}$  is the *universal gas constant*, and  $R = R_u/M_w$  is the *gas constant* for the gas under consideration. A perfect gas is one that obeys (1.22), even if it is a mixture of several different molecular species. For example, the average molecular weight of dry air is  $28.966 \text{ kg kmol}^{-1}$ , for which (1.22) gives  $R = 287 \text{ J kg}^{-1} \text{ K}^{-1}$ . At ordinary temperatures and pressures most gases can be treated as perfect gases.

The gas constant for a particular gas is related to the specific heats of the gas through the relation

$$R = C_p - C_v, \quad (1.23)$$

where  $C_p$  and  $C_v$  are the specific heat capacities at constant pressure and volume, respectively. In general,  $C_p$  and  $C_v$  increase with temperature. The ratio of specific heats

$$\gamma \equiv C_p/C_v \quad (1.24)$$

is important in compressible fluid dynamics. For air at ordinary temperatures,  $\gamma = 1.40$  and  $C_p = 1004 \text{ J kg}^{-1} \text{ K}^{-1}$ . It can be shown that (1.21) or (1.22) is equivalent to  $e = e(T)$  and  $h = h(T)$ , and conversely, so that the internal energy and enthalpy of a perfect gas are only functions of temperature (Exercise 1.10).

A process is called *adiabatic* if it takes place without the addition of heat. A process is called *isentropic* if it is adiabatic and frictionless, for then the entropy of the fluid does not change. From (1.18) it can be shown (Exercise 1.11) that isentropic flow of a perfect gas with constant specific heats obeys

$$p/\rho^\gamma = \text{const}. \quad (1.25)$$

Using (1.22) and (1.25), the temperature and density changes during an isentropic process from a reference state (subscript 0) to a current state (no subscript) are

$$T/T_0 = (p/p_0)^{(\gamma-1)/\gamma} \text{ and } \rho/\rho_0 = (p/p_0)^{1/\gamma} \quad (1.26)$$

(see Exercise 1.8). In addition, simple expressions can be found for the speed of sound  $c$  and the thermal expansion coefficient  $\alpha$  for a perfect gas:

$$c = \sqrt{\gamma R T} \text{ and } \alpha = 1/T. \quad (1.27, 1.28)$$

### 1.10. STABILITY OF STRATIFIED FLUID MEDIA

In a static fluid environment subject to a gravitational field,  $p$ ,  $\rho$ , and  $T$  may vary with height  $z$ , but (1.8) and (1.12) provide two constraints so the  $p$ ,  $\rho$ , and  $T$  variations cannot be arbitrary. Furthermore, these constraints imply that the specification of the vertical profile of any one thermodynamic variable allows the profiles of the others to be determined. In addition, our experience suggests that the fluid medium will be stable if  $\rho(z)$  decreases with increasing  $z$ . Interestingly, the rate at which the density decreases also plays a role in the stability of the fluid medium when the fluid is compressible, as in a planetary atmosphere.

To assess the stability of a static fluid medium, consider a fluid particle with density  $\rho(z_0)$  in an atmosphere (or ocean) at equilibrium at height  $z_0$  that is displaced upward a small distance  $\zeta$  via a frictionless adiabatic process and then released from rest. At its new height,  $z_0 + \zeta$ , the fluid particle will have a different density,  $\rho(z_0) + (d\rho_a/dz)\zeta + \dots$ , where  $d\rho_a/dz$  is the isentropic density gradient for the displaced particle at height  $z_0$  (see Figure 1.8). The density of the fluid particles already at height  $z_0 + \zeta$  is  $\rho(z_0 + \zeta) = \rho(z_0) + (d\rho/dz)\zeta + \dots$ , where  $d\rho/dz$  is the equilibrium density gradient at height  $z_0$  in the fluid medium. A vertical-direction application of Newton's second law including weight and buoyancy for the displaced element leads to

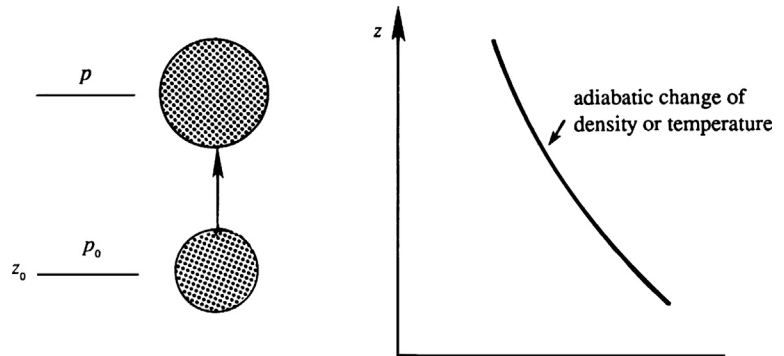
$$\frac{d^2\zeta}{dt^2} - \frac{g}{\rho(z_0)} \left( \frac{d\rho}{dz} - \frac{d\rho_a}{dz} \right) \zeta = 0$$

when first-order terms in  $\zeta$  are retained (see Exercise 1.13). The coefficient of  $\zeta$  in the second term is the square of the *Brunt-Väisälä frequency*,  $N$ ,

$$N^2 = -\frac{g}{\rho(z_0)} \left( \frac{d\rho}{dz} - \frac{d\rho_a}{dz} \right). \quad (1.29)$$

When  $N^2$  is positive, the fluid medium is *stable*; the displaced fluid particle will accelerate back toward  $z_0$  after release and the action of viscous forces and thermal conduction will arrest any oscillatory motion. Thus, a stable atmosphere (or ocean) is one in which the density decreases with height *faster* than in an isentropic atmosphere (or ocean). When  $N^2$  is negative, the fluid medium is *unstable*; the displaced fluid element will accelerate away from  $z_0$  after

**FIGURE 1.8** Adiabatic expansion of a fluid particle displaced upward in a compressible medium. In a static pressure field, if the fluid particle rises it encounters a lower pressure and may expand adiabatically.



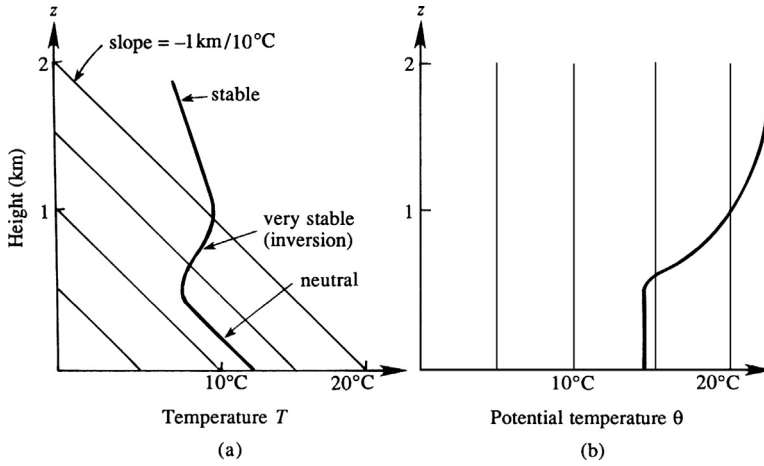


FIGURE 1.9 Vertical variation of the (a) actual and (b) potential temperature in the atmosphere. Thin straight lines represent temperatures for a neutral atmosphere. Slopes less than the neutral atmosphere lines lead to atmospheric instability. Slopes greater than the neutral atmosphere lines indicate a stable atmosphere.

release and further increase its displacement  $\zeta$ . When  $N^2$  is zero, the fluid medium is *neutrally stable* and the element will not move if released from rest; it will have zero vertical acceleration. There are two ways to achieve neutral stability: 1) the fluid density may be independent of the vertical coordinate so that  $d\rho/dz = d\rho_a/dz = 0$ , or 2) the equilibrium density gradient in the fluid medium may equal the isentropic density gradient,  $d\rho/dz = d\rho_a/dz$ . The former case implies that constant-density fluid media are neutrally stable. The latter case requires a *neutrally stable atmosphere* to be one where  $p$ ,  $\rho$ , and  $T$  decrease with increasing height in such a way that the entropy is constant.

In atmospheric science,  $\Gamma \equiv dT/dz$  is the atmospheric temperature gradient or *lapse rate*. The rate of temperature decrease in an isentropic atmosphere  $\Gamma_a$  is

$$dT_a/dz \equiv \Gamma_a = -g\alpha T/C_p, \quad (1.30)$$

(see Exercise 1.14) and is called the *adiabatic temperature gradient* or *adiabatic lapse rate*. It is the steepest rate at which the temperature can decrease with increasing height without causing instability. In the earth's atmosphere, the adiabatic lapse rate is approximately  $-10^\circ\text{C km}^{-1}$ .

Figure 1.9a shows a typical distribution of temperature in the earth's atmosphere. The lower part has been drawn with a slope nearly equal to the adiabatic temperature gradient because mixing processes near the ground tend to form a neutral (constant entropy) atmosphere. Observations show that the neutral atmosphere ends at a layer where the temperature increases with height, a very stable situation. Meteorologists call this an *inversion*, because the temperature gradient changes signs here. Atmospheric turbulence and mixing processes below such an inversion typically cannot penetrate above it. Above this inversion layer the temperature decreases again, but less rapidly than near the ground, which again corresponds to stability. An isothermal atmosphere (a vertical line in Figure 1.9a) is quite stable.

## Potential Temperature and Density

The foregoing discussion of static stability of a compressible atmosphere can be expressed in terms of the concept of *potential temperature*, which is generally denoted by  $\theta$ . Suppose the

pressure and temperature of a fluid particle at a height  $z$  are  $p(z)$  and  $T(z)$ . Now if we take the particle *adiabatically* to a standard pressure  $p_o = p(0)$  (say, the sea level pressure, nearly equal to 100 kPa), then the temperature  $\theta$  attained by the particle is called its *potential temperature*. Using (1.26) for a perfect gas, it follows that the actual temperature  $T$  and the potential temperature  $\theta$  are related by

$$T(z) = \theta(z)(p(z)/p_o)^{(\gamma-1)/\gamma}. \quad (1.31)$$

Taking the logarithm and differentiating, we obtain

$$\frac{1}{T} \frac{dT}{dz} = \frac{1}{\theta} \frac{d\theta}{dz} + \frac{(\gamma-1)}{\gamma p} \frac{dp}{dz}.$$

Substituting  $dp/dz = -\rho g$ ,  $p = \rho RT$ , and  $\alpha = 1/T$  produces

$$\frac{T}{\theta} \frac{d\theta}{dz} = \frac{dT}{dz} + \frac{g}{C_p} = \frac{d}{dz}(T - T_a) = \Gamma - \Gamma_a. \quad (1.32)$$

If the temperature decreases at a rate  $\Gamma = \Gamma_a$ , then the potential temperature  $\theta$  (and therefore the entropy) is uniform with height. It follows that an atmosphere is stable, neutral, or unstable depending upon whether  $d\theta/dz$  is positive, zero, or negative, respectively. This is illustrated in [Figure 1.9b](#). It is the gradient of *potential* temperature that determines the stability of a column of gas, not the gradient of the actual temperature. However, this difference is negligible for laboratory-scale phenomena. For example, a 1.0 m vertical change may result in an air temperature decrease of only  $1.0 \text{ m} \times (10^\circ\text{C km}^{-1}) = 10^{-2}^\circ\text{C}$ .

Similarly, *potential density*  $\rho_\theta$  is the density attained by a fluid particle if taken via an isentropic process to a standard pressure  $p_o$ . Using (1.26), the actual density  $\rho(z)$  and potential density are related by

$$\rho(z) = \rho_\theta(z)(p(z)/p_o)^{1/\gamma}. \quad (1.33)$$

Multiplying (1.31) and (1.33), and using  $p = \rho RT$ , we obtain  $\theta\rho_\theta = p_o/R = \text{const}$ . Taking the logarithm and differentiating, we obtain

$$-\frac{1}{\rho_\theta} \frac{d\rho_\theta}{dz} = \frac{1}{\theta} \frac{d\theta}{dz}. \quad (1.34)$$

Thus, an atmosphere is stable, neutral, or unstable depending upon whether  $d\rho_\theta/dz$  is negative, zero, or positive, respectively.

Interestingly, compressibility effects are also important in the deep ocean where saltwater density depends not only on the temperature and pressure, but also on the *salinity* ( $S$ ) defined as kilograms of salt per kilogram of water. The average salinity of seawater is approximately 3.5%. Here, the potential density is defined as the density attained if a fluid particle is taken to a reference pressure via an isentropic process *and* at constant salinity. The potential density thus defined must decrease with height for stable water column conditions. Oceanographers automatically account for the compressibility of seawater by converting their density measurements at any depth to the sea level pressure, which serves as the reference pressure.

Because depth change–induced density changes are relatively small in percentage terms ( $\sim 0.5\%$  for a 1.0 km change in depth) for seawater, the static stability of the *ocean* is readily determined from (1.29). In particular, the vertical isentropic density gradient in (1.29) may be rewritten using  $dp_a/dz = -\rho_a g$  and the definition of the sound speed  $c$  (1.19) to find

$$\frac{d\rho_a}{dz} = \left(\frac{\partial\rho_a}{\partial p}\right)_{s,S} \frac{dp_a}{dz} = -\left(\frac{\partial\rho_a}{\partial p}\right)_{s,S} \rho_a g = -\frac{\rho_a g}{c^2} \cong -\frac{\rho g}{c^2},$$

where the approximation  $\rho_a \cong \rho$  produces the final result. Thus, (1.29) and its ensuing discussion imply that the ocean is stable, neutral, or unstable depending upon whether

$$\frac{d\rho_\theta}{dz} = \frac{d\rho}{dz} - \frac{d\rho_a}{dz} \cong \frac{d\rho}{dz} + \frac{\rho g}{c^2} \quad (1.35)$$

is negative, zero, or positive, respectively.

## Scale Height of the Atmosphere

Approximate expressions for the pressure distribution and the thickness or *scale height* of the atmosphere can be obtained by assuming isothermal conditions. This is a reasonable assumption in the lower 70 km of the atmosphere, where the absolute temperature generally remains within 15% of 250 K. The hydrostatic distribution (1.8) and perfect gas law (1.22) require

$$dp/dz = -\rho g = -pg/RT.$$

When  $g$ ,  $R$ , and  $T$  are constants, integration gives

$$p(z) = p_0 e^{-gz/RT},$$

where  $p_0$  is the pressure at  $z = 0$ . The pressure therefore falls to  $e^{-1}$  of its surface value in a height  $H = RT/g$ . Thus, the quantity  $RT/g$  is called the *scale height* of the atmosphere, and it provides a reasonable quantitative measure of the thickness of the atmosphere. For an average atmospheric temperature of  $T = 250$  K, the scale height is  $RT/g = 7.3$  km.

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## 1.11. DIMENSIONAL ANALYSIS

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Interestingly, a physical quantity's units may be exploited to learn about its relationship to other physical quantities. This possibility exists because the natural realm does not need mankind's units of measurement to function. Natural laws are independent of any unit system imposed on them by human beings. Consider Newton's second law, generically stated as *force* = (*mass*)  $\times$  (*acceleration*); it is true whether a scientist or engineer uses cgs (centimeter, gram, second), MKS (meter, kilogram, second), or even English (inch or foot, pound, second) units in its application. Because nature is independent of our systems of units, we can draw two important conclusions: 1) all correct physical relationships can be stated in dimensionless form, and 2) in any comparison, the units of the items being compared must be the same for the

comparison to be valid. The first conclusion leads to the problem-simplification or scaling-law-development technique known as *dimensional analysis*. The second conclusion is known as the principle of *dimensional homogeneity*. It requires all terms in an equation to have the same dimension(s) and thereby provides an effective means for error catching within derivations and in derived answers. If terms in an equation do not have the same dimension(s) then the equation is not correct and a mistake has been made.

Dimensional analysis is a broadly applicable technique for developing scaling laws, interpreting experimental data, and simplifying problems. Occasionally it can even be used to solve problems. Dimensional analysis has utility throughout the physical sciences and it is routinely taught to students of fluid mechanics. Thus, it is presented here for subsequent use in this chapter's exercises and in the remaining chapters of this text.

Of the various formal methods of dimensional analysis, the description here is based on Buckingham's method from 1914. Let  $q_1, q_2, \dots, q_n$  be  $n$  variables and parameters involved in a particular problem or situation, so that there must exist a functional relationship of the form

$$f(q_1, q_2, \dots, q_n) = 0. \quad (1.36)$$

Buckingham's theorem states that the  $n$  variables can always be combined to form exactly  $(n - r)$  independent dimensionless parameter groups, where  $r$  is the number of independent dimensions. Each dimensionless parameter group is commonly called a  $\Pi$ -group or a *dimensionless group*. Thus, (1.36) can be written as a functional relationship

$$\phi(\Pi_1, \Pi_2, \dots, \Pi_{n-r}) = 0 \text{ or } \Pi_1 = \phi(\Pi_2, \Pi_3, \dots, \Pi_{n-r}). \quad (1.37)$$

The dimensionless groups are not unique, but  $(n - r)$  of them are *independent* and form a *complete set* that spans the parametric solution space of (1.37). The power of dimensional analysis is most apparent when  $n$  and  $r$  are single-digit numbers of comparable size so (1.37), which involves  $n - r$  dimensionless groups, represents a significant simplification of (1.36), which has  $n$  parameters. The process of dimensional analysis is presented here as a series of six steps that should be followed by a seventh whenever possible. Each step is described in the following paragraphs and illustrated via the example of determining the functional dependence of the pressure difference  $\Delta p$  between two locations in a round pipe carrying a flowing viscous fluid.

## Step 1. Select Variables and Parameters

Creating the list of variables and parameters to include in a dimensional analysis effort is the most important step. The parameter list should usually contain only one unknown variable, the *solution variable*. The rest of the variables and parameters should come from the problem's geometry, boundary conditions, initial conditions, and material parameters. Physical constants and other fundamental limits may also be included. However, shorter parameter lists tend to produce the most powerful dimensional analysis results; expansive lists commonly produce less useful results.

For the round-pipe pressure drop example, select  $\Delta p$  as the solution variable, and then choose as additional parameters: the distance  $\Delta x$  between the pressure measurement

locations, the inside diameter  $d$  of the pipe, the average height  $\varepsilon$  of the pipe's wall roughness, the average flow velocity  $U$ , the fluid density  $\rho$ , and the fluid viscosity  $\mu$ . The resulting functional dependence between these seven parameters can be stated as

$$f(\Delta p, \Delta x, d, \varepsilon, U, \rho, \mu) = 0. \quad (1.38)$$

Note, (1.38) does not include the fluid's thermal conductivity, heat capacities, thermal expansion coefficient, or speed of sound, so this dimensional analysis example will not account for the thermal or compressible flow effects embodied by these missing parameters.

## Step 2. Create the Dimensional Matrix

Fluid flow problems without electromagnetic forces and chemical reactions involve only mechanical variables (such as velocity and density) and thermal variables (such as temperature and specific heat). The dimensions of all these variables can be expressed in terms of four basic dimensions—mass  $M$ , length  $L$ , time  $T$ , and temperature  $\theta$ . We shall denote the dimension of a variable  $q$  by  $[q]$ . For example, the dimension of the velocity  $u$  is  $[u] = L/T$ , that of pressure is  $[p] = [\text{force}]/[\text{area}] = MLT^{-2}/L^2 = M/LT^2$ , and that of specific heat is  $[C_p] = [\text{energy}]/[\text{mass}][\text{temperature}] = MLT^{-2}L/M\theta = L^2/\theta T^2$ . When thermal effects are not considered, all variables can be expressed in terms of three fundamental dimensions, namely,  $M$ ,  $L$ , and  $T$ . If temperature is considered only in combination with Boltzmann's constant ( $k_B\theta$ ), a gas constant ( $R\theta$ ), or a specific heat ( $C_p\theta$ ), then the units of the combination are simply  $L^2/T^2$ , and only the three dimensions  $M$ ,  $L$ , and  $T$  are required.

The dimensional matrix is created by listing the powers of  $M$ ,  $L$ ,  $T$ , and  $\theta$  in a column for each parameter selected. For the pipe-flow pressure difference example, the selected variables and their dimensions produce the following dimensional matrix:

$$\begin{array}{c|ccccccc} & \Delta p & \Delta x & d & \varepsilon & U & \rho & \mu \\ \hline M & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ L & -1 & 1 & 1 & 1 & 1 & -3 & -1 \\ T & -2 & 0 & 0 & 0 & -1 & 0 & -1 \end{array} \quad (1.39)$$

where the seven variables have been written above the matrix entries and the three units have been written in a column to the left of the matrix. The matrix in (1.39) portrays  $[\Delta p] = ML^{-1}T^{-2}$  via the first column of numeric entries.

## Step 3. Determine the Rank of the Dimensional Matrix

The *rank*  $r$  of any matrix is defined to be the size of the largest square submatrix that has a nonzero determinant. Testing the determinant of the first three rows and columns of (1.39), we obtain

$$\begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -2 & 0 & 0 \end{vmatrix} = 0.$$

However, (1.39) does include a nonzero third-order determinant, for example, the one formed by the last three columns:

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & -3 & -1 \\ -1 & 0 & -1 \end{vmatrix} = -1.$$

Thus, the rank of the dimensional matrix (1.39) is  $r = 3$ . If *all* possible third-order determinants were zero, we would have concluded that  $r < 3$  and proceeded to testing second-order determinants.

For dimensional matrices, the rank is less than the number of rows only when one of the rows can be obtained by a linear combination of the other rows. For example, the matrix (not from 1.39)

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ -1 & 2 & 1 & -2 \\ -1 & 4 & 1 & 0 \end{vmatrix}$$

has  $r = 2$ , as the last row can be obtained by adding the second row to twice the first row. A rank of less than 3 commonly occurs in statics problems, in which mass or density is really not relevant but the dimensions of the variables (such as force) involve M. In most fluid mechanics problems without thermal effects,  $r = 3$ .

#### Step 4. Determine the Number of Dimensionless Groups

The number of dimensionless groups is  $n - r$  where  $n$  is the number of variables and parameters, and  $r$  is the rank of the dimensional matrix. In the pipe-flow pressure difference example, the number of dimensionless groups is  $4 = 7 - 3$ .

#### Step 5. Construct the Dimensionless Groups

This can be done by exponent algebra or by inspection. The latter is preferred because it commonly produces dimensionless groups that are easier to interpret, but the former is sometimes required. Examples of both techniques follow here. Whatever the method, the best approach is usually to create the first dimensionless group with the solution variable appearing to the first power.

When using exponent algebra, select  $r$  parameters from the dimensional matrix as *repeating parameters* that will be found in all the subsequently constructed dimensionless groups. These repeating parameters must span the appropriate  $r$ -dimensional dimension space of M, L, and/or T, that is, the determinant of the dimensional matrix formed from these  $r$  parameters must be nonzero. For many fluid-flow problems, a characteristic velocity, a characteristic length, and a fluid property involving mass are ideal repeating parameters.

To form dimensionless groups for the pipe-flow problem, choose  $U$ ,  $d$ , and  $\rho$  as the repeating parameters. The determinant of the dimensional matrix formed by these three parameters is nonzero. Other repeating parameter choices will result in a different set of dimensionless groups, but any such alternative set will still span the solution space of the



problem. Thus, any satisfactory choice of the repeating parameters is equivalent to any other, so choices that simplify the work are most appropriate. Each dimensionless group is formed by combining the three repeating parameters, raised to unknown powers, with one of the nonrepeating variables or parameters from the list constructed for the first step. Here we ensure that the first dimensionless group involves the solution variable raised to the first power:

$$\Pi_1 = \Delta p U^a d^b \rho^c.$$

The exponents  $a$ ,  $b$ , and  $c$  are obtained from the requirement that  $\Pi_1$  is dimensionless. Replicating this equation in terms of dimensions produces

$$M^0 L^0 T^0 = [\Pi_1] = [\Delta p U^a d^b \rho^c] = (ML^{-1}T^{-2})(LT^{-1})^a (L)^b (ML^{-3})^c = M^{c+1} L^{a+b-3c-1} T^{-a-2}.$$

Equating exponents between the two extreme ends of this extended equality produces three algebraic equations that are readily solved to find  $a = -2$ ,  $b = 0$ ,  $c = -1$ , so

$$\Pi_1 = \Delta p / \rho U^2.$$

A similar procedure with  $\Delta p$  replaced by the other unused variables ( $\Delta x$ ,  $\varepsilon$ ,  $\mu$ ) produces

$$\Pi_2 = \Delta x / d, \quad \Pi_3 = \varepsilon / d, \quad \text{and} \quad \Pi_4 = \mu / \rho U d.$$

The inspection method proceeds directly from the dimensional matrix, and may be less tedious than exponent algebra. It involves selecting individual parameters from the dimensional matrix and sequentially eliminating their M, L, T, and  $\theta$  units by forming ratios with other parameters. For the pipe-flow pressure difference example we again start with the solution variable  $[\Delta p] = ML^{-1}T^{-2}$  and notice that the next entry in (1.39) that includes units of mass is  $[\rho] = ML^{-3}$ . To eliminate M from a combination of  $\Delta p$  and  $\rho$ , we form the ratio  $[\Delta p / \rho] = L^2 T^{-2} = [\text{velocity}^2]$ . An examination of (1.39) shows that  $U$  has units of velocity,  $LT^{-1}$ . Thus,  $\Delta p / \rho$  can be made dimensionless if it is divided by  $U^2$  to find:  $[\Delta p / \rho U^2] =$  dimensionless. Here we have the good fortune to eliminate L and T in the same step. Therefore, the first dimensionless group is  $\Pi_1 = \Delta p / \rho U^2$ . To find the second dimensionless group  $\Pi_2$ , start with  $\Delta x$ , the left most unused parameter in (1.39), and note  $[\Delta x] = L$ . The first unused parameter to the right of  $\Delta x$  involving only length is  $d$ . Thus,  $[\Delta x / d] =$  dimensionless so  $\Pi_2 = \Delta x / d$ . The third dimensionless group is obtained by starting with the next unused parameter,  $\varepsilon$ , to find  $\Pi_3 = \varepsilon / d$ . The final dimensionless group must include the last unused parameter  $[\mu] = ML^{-1}T^{-1}$ . Here it is better to eliminate the mass dimension with the density since reusing  $\Delta p$  would place the solution variable in two places in the final scaling law, an unnecessary complication. Therefore, form the ratio  $\mu / \rho$  which has units  $[\mu / \rho] = L^2 T^{-1}$ . These can be eliminated with  $d$  and  $U$ ,  $[\mu / \rho U d] =$  dimensionless, so  $\Pi_4 = \mu / \rho U d$ .

Forming the dimensionless groups by inspection becomes easier with experience. For example, since there are three length scales  $\Delta x$ ,  $d$ , and  $\varepsilon$  in (1.39), the dimensionless groups  $\Delta x / d$  and  $\varepsilon / d$  can be formed immediately. Furthermore, Bernoulli equations (see Section 4.9, "Bernoulli Equations") tell us that  $\rho U^2$  has the same units as  $p$  so  $\Delta p / \rho U^2$  is readily identified as a dimensionless group. Similarly, the dimensionless group that describes viscous

effects in the fluid mechanical equations of motion is found to be  $\mu/\rho U d$  when these equations are cast in dimensionless form (see Section 4.11).

Other dimensionless groups can be obtained by combining established groups. For the pipe-flow example, the group  $\Delta p d^2 \rho / \mu^2$  can be formed from  $\Pi_1 / \Pi_4^2$ , and the group  $\varepsilon / \Delta x$  can be formed as  $\Pi_3 / \Pi_2$ . However, only four dimensionless groups will be independent in the pipe-flow example.

### Step 6. State the Dimensionless Relationship

This step merely involves placing the  $(n - r)$   $\Pi$ -groups in one of the forms in (1.37). For the pipe-flow example, this dimensionless relationship is

$$\frac{\Delta p}{\rho U^2} = \varphi \left( \frac{\Delta x}{d}, \frac{\varepsilon}{d}, \frac{\mu}{\rho U d} \right), \quad (1.40)$$

where  $\varphi$  is an undetermined function. This relationship involves only four dimensionless groups, and is therefore a clear simplification of (1.36) which lists seven independent parameters. The four dimensionless groups in (1.40) have familiar physical interpretations and have even been given special names. For example,  $\Delta x / d$  is the pipe's aspect ratio, and  $\varepsilon / d$  is the pipe's roughness ratio. Common dimensionless groups in fluid mechanics are presented and discussed in Section 4.11.

### Step 7. Use Physical Reasoning or Additional Knowledge to Simplify the Dimensionless Relationship

Sometimes there are only two extensive thermodynamic variables involved and these must be proportional in the final scaling law. An overall conservation law can be applied that restricts one or more parametric dependencies, or a phenomena may be known to be linear, quadratic, etc., in one of the parameters and this dependence must be reflected in the final scaling law. This seventh step may not always be possible, but when it is, significant and powerful results may be achieved from dimensional analysis.

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#### EXAMPLE 1.2

Use dimensional analysis to find the parametric dependence of the scale height  $H$  in a static isothermal atmosphere at temperature  $T_o$  composed of a perfect gas with average molecular weight  $M_w$  when the gravitational acceleration is  $g$ .

#### Solution

Follow the six steps just described.

1. The parameter list must include  $H$ ,  $T_o$ ,  $M_w$  and  $g$ . Here there is no velocity parameter, and there is no need for a second specification of a thermodynamic variable since a static pressure gradient prevails. However, the universal gas constant  $R_u$  must be included to help relate the thermal variable  $T_o$  to the mechanical ones.

2. The dimensional matrix is:

	$H$	$T_o$	$M_w$	$g$	$R_u$
M	0	0	1	0	1
L	1	0	0	1	2
T	0	0	0	-2	-2
$\theta$	0	1	0	0	-1

Note that the  $kmole^{-1}$  specification of  $M_w$  and  $R_u$  is lost in the matrix above since a  $kmole$  is a pure number.

3. The rank of this matrix is four, so  $r = 4$ .
4. The number of dimensionless groups is:  $n - r = 5 - 4 = 1$ .
5. Use  $H$  as the solution parameter, and the others as the repeating parameters. Proceed with exponent algebra to find the dimensionless group:

$$\begin{aligned} M^0 L^0 T^0 \theta^0 &= [\Pi_1] = [HT_o^a M_w^b g^c R_u^d] = (L)(\theta)^a (M)^b (LT^{-2})^c (ML^2 T^{-2} \theta^{-1})^d \\ &= M^{b+d} L^{1+c+2d} T^{-2c-2d} \theta^{a-d}. \end{aligned}$$

Equating exponents yields four linear algebraic equations:

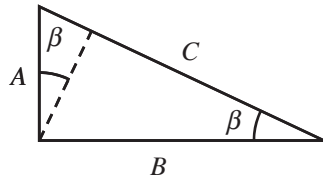
$$b + d = 0, \quad 1 + c + 2d = 0, \quad -2c - 2d = 0, \quad \text{and} \quad a - d = 0,$$

which are solved by:  $a = -1$ ,  $b = 1$ ,  $c = 1$ , and  $d = -1$ . Thus, the lone dimensionless group is:  $\Pi_1 = HgM_w/R_u T_o$ .

6. Because there is only a single dimensionless group, its most general behavior is to equal a constant, so  $HgM_w/R_u T_o = \varphi(\dots) = \text{const.}$ , or  $H = \text{const.} (R_u T_o / g M_w)$ . Based on the finding at the end of the previous section and  $R = R_u / M_w$  from (1.22), this parametric dependence is correct and the constant is unity in this case.

### EXAMPLE 1.3

Use dimensional analysis and [Figure 1.10](#) to prove the Pythagorean theorem based on a right triangle's area  $a$ , the radian measure  $\beta$  of its most acute angle, and the length  $C$  of its longest side ([Barenblatt, 1979](#)).



**FIGURE 1.10** A right triangle with area  $a$ , smallest acute angle  $\beta$ , and hypotenuse  $C$ . The dashed line is perpendicular to side  $C$ .

### Solution

Follow the six steps given earlier and then consider similarity between the main triangle and two sub-triangles.

1. The parameter list ( $a, \beta, C$ ) is given in the problem statement so  $n = 3$ .
2. The dimensional matrix is:

	$a$	$\beta$	$C$
M	0	0	0
L	2	0	1
T	0	0	0

3. With no M or T units, the rank of this matrix is one, so  $r = 1$ .
4. The number of dimensionless groups is:  $n - r = 3 - 1 = 2$ .
5. Let the triangle's area  $a$  be the solution parameter. By inspection,  $\Pi_1 = a/C^2$ , and  $\Pi_2 = \beta$ .
6. Therefore, the dimensionless relationship is:  $a/C^2 = \varphi(\beta)$  or  $a = C^2\varphi(\beta)$ .
7. When the dashed line is perpendicular to side  $C$ , then the large triangle is divided into two smaller ones that are similar to the larger one. These sub-triangles have  $A$  and  $B$  as their longest sides and both have the same acute angle as the large triangle. Therefore, the sub-triangle areas can be written as  $A^2\varphi(\beta)$  and  $B^2\varphi(\beta)$ . Summing the sub-triangle areas produces:  $A^2\varphi(\beta) + B^2\varphi(\beta) = C^2\varphi(\beta)$  or  $A^2 + B^2 = C^2$  when  $\varphi(\beta) \neq 0$ .

### EXAMPLE 1.4

Use dimensional analysis to determine the energy  $E$  released in an intense point blast if the blast-wave propagation distance  $D$  into an undisturbed atmosphere of density  $\rho$  is known as a function of time  $t$  following the energy release (Taylor, 1950; see Figure 1.11).



FIGURE 1.11 In an atmosphere with undisturbed density  $\rho$ , a point release of energy  $E$  produces a hemispherical blast wave that travels a distance  $D$  in time  $t$ .

### Solution

Again follow the six steps given earlier.

1. The parameter list ( $E, D, \rho, t$ ) is given in the problem statement so  $n = 4$ .
2. The dimensional matrix is:

	$E$	$D$	$\rho$	$t$
M	1	0	1	0
L	2	1	-3	0
T	-2	0	0	1

3. The rank of this matrix is three, so  $r = 3$ .
4. The number of dimensionless groups is:  $n - r = 4 - 3 = 1$ .
5. Let the point-blast energy be the solution parameter and construct the lone dimensionless group by inspection. First use  $E$  and  $\rho$  to eliminate M:  $[E/\rho] = L^5T^{-2}$ . Next use  $D$  to eliminate L:  $[E/\rho D^5] = T^{-2}$ . Then use  $t$  to eliminate T:  $[Et^2/\rho D^5] = \text{dimensionless}$ , so  $\Pi_1 = Et^2/\rho D^5$ .
6. Here there is only a single dimensionless group, so it must be a constant ( $K$ ). This produces:  $Et^2/\rho D^5 = \varphi(\dots) = K$  which implies:  $E = K\rho D^5/t^2$ , where  $K$  is not determined by dimensional analysis.
7. The famous fluid mechanician G. I. Taylor was able to estimate the yield of the first atomic-bomb test conducted on the White Sands Proving Grounds in New Mexico in July 1945 using: 1) the dimensional analysis shown above, 2) a declassified movie made by J. E. Mack, and 3) timed photographs supplied by the Los Alamos National Laboratory and the Ministry of Supply. He determined the fireball radius as a function of time and then estimated  $E$  using a nominal atmospheric value for  $\rho$ . His estimate of  $E = 17$  kilotons of TNT was very close to the actual yield (20 kilotons of TNT) in part because the undetermined constant  $K$  is close to unity in this case. At the time, the movie and the photographs were not classified but the yield of the bomb was entirely secret.

## EXAMPLE 1.5

Use dimensional analysis to determine how the average light intensity  $S$  (Watts/m<sup>2</sup>) scattered from an isolated particle depends on the incident light intensity  $I$  (Watts/m<sup>2</sup>), the wavelength of the light  $\lambda$  (m), the volume of the particle  $V$  (m<sup>3</sup>), the index of refraction of the particle  $n_s$  (dimensionless), and the distance  $d$  (m) from the particle to the observation point. Can the resulting dimensionless relationship be simplified to better determine parametric effects when  $\lambda \gg V^{1/3}$ ?

### Solution

Again follow the six steps given earlier, knowing that the seventh step will likely be necessary to produce a useful final relationship.

1. The parameter list ( $S, I, \lambda, V, n_s, d$ ) is given in the problem statement so  $n = 6$ .
2. The dimensional matrix is:

	$S$	$I$	$\lambda$	$V$	$n_s$	$d$
M	1	1	0	0	0	0
L	0	0	1	3	0	1
T	-3	-3	0	0	0	0

3. The rank of this matrix is 2 because all the dimensions are either intensity or length, so  $r = 2$ .
4. The number of dimensionless groups is:  $n - r = 6 - 2 = 4$ .
5. Let scattered light intensity  $S$  be the solution parameter. By inspection the four dimensionless groups are:

$$\Pi_1 = S/I, \Pi_2 = d/\lambda, \Pi_3 = V/\lambda^3, \text{ and } \Pi_4 = n_s.$$

6. Therefore, the dimensionless relationship is:  $S/I = \varphi_1(d/\lambda, V/\lambda^3, n_s)$ .
7. There are two physical features of this problem that allow refinement of this dimensional analysis result. First, light scattering from the particle must conserve energy and this implies:  $4\pi d^2 S = \text{const.}$  so  $S \propto 1/d^2$ . Therefore, the result in step 6 must simplify to:  $S/I = (\lambda/d)^2 \varphi_2(V/\lambda^3, n_s)$ . Second, when  $\lambda$  is large compared to the size of the scatterer, the scattered field amplitude will be produced from the dipole moment induced in the scatterer by the incident field, and this scattered field amplitude will be proportional to  $V$ . Thus,  $S$ , which is proportional to field amplitude squared, will be proportional to  $V^2$ . These deductions allow a further simplification of the dimensional analysis result to:

$$\frac{S}{I} = \left(\frac{\lambda}{d}\right)^2 \left(\frac{V}{\lambda^3}\right)^2 \varphi_3(n_s) = \frac{V^2}{d^2 \lambda^4} \varphi_3(n_s).$$

This is Lord Rayleigh's celebrated small-particle scattering law. He derived it in the 1870s while investigating light scattering from small scatterers to understand why the cloudless daytime sky was blue while the sun appeared orange or red at dawn and sunset. At the time, he imagined that the scatterers were smoke, dust, mist, aerosols, etc. However, the atmospheric abundance of these are insufficient to entirely explain the color change phenomena but the molecules that compose the atmosphere can accomplish enough scattering to explain the observations.

## EXERCISES

- 1.1. <sup>1</sup>Many centuries ago, a mariner poured 100 cm<sup>3</sup> of water into the ocean. As time passed, the action of currents, tides, and weather mixed the liquid uniformly throughout the earth's oceans, lakes, and rivers. Ignoring salinity, estimate the probability that the next cup of water you drink will contain at least one water molecule that was dumped by the mariner. Assess your chances of ever drinking truly pristine water. (*Consider the following facts:*  $M_w$  for water is 18.0 kg per kg-mole, the radius of the earth is 6370 km, and the mean depth of the oceans is approximately 3.8 km, and they cover 71% of the surface of the earth. One cup is  $\sim 240$  ml.)
- 1.2. <sup>1</sup>An adult human expels approximately 500 mL of air with each breath during ordinary breathing. Imagining that two people exchanged greetings (one breath each) many centuries ago and that their breath subsequently has been mixed uniformly throughout the atmosphere, estimate the probability that the next breath you take will contain at least one air molecule from that age-old verbal exchange. Assess your chances of ever getting a truly fresh breath of air. For this problem, assume that air is composed of identical molecules having  $M_w = 29.0$  kg per kg-mole and that the average atmospheric pressure on the surface of the earth is 100 kPa. Use 6370 km for the radius of the earth and 1.20 kg/m<sup>3</sup> for the density of air at room temperature and pressure.

<sup>1</sup>Based on a homework problem posed by Professor P. E. Dimotakis

- 1.3. In Cartesian coordinates, the Maxwell probability distribution,  $f(\mathbf{u}) = f(u_1, u_2, u_3)$ , of molecular velocities in a gas flow with average velocity  $\mathbf{U} = (U_1, U_2, U_3)$  is

$$f(\mathbf{u}) = \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp \left\{ -\frac{m}{2k_B T} |\mathbf{u} - \mathbf{U}|^2 \right\},$$

where  $n$  is the number of gas molecules in volume  $V$ ,  $m$  is the molecular mass,  $k_B$  is Boltzmann's constant and  $T$  is the absolute temperature.

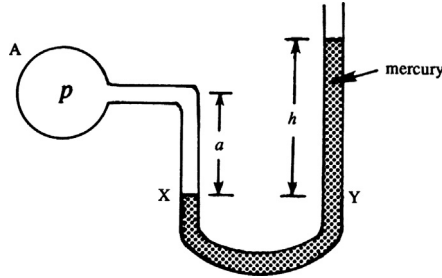
- a) Verify that  $\mathbf{U}$  is the average molecular velocity, and determine the standard deviations ( $\sigma_1, \sigma_2, \sigma_3$ ) of each component of  $\mathbf{U}$  using  $\sigma_i = [\iint_{all \mathbf{u}} (u_i - U_i)^2 f(\mathbf{u}) d^3 u]^{1/2}$  for  $i = 1, 2$ , and  $3$ .
  - b) Using (1.21), the molecular version of the perfect gas law, determine  $n/V$  at room temperature  $T = 295$  K and atmospheric pressure  $p = 101.3$  kPa.
  - c) Determine  $n$  for volumes  $V = (10 \mu\text{m})^3$ ,  $1 \mu\text{m}^3$ , and  $(0.1 \mu\text{m})^3$ .
  - d) For the  $i$ th velocity component, the standard deviation of the average,  $\sigma_{a,i}$ , over  $n$  molecules is  $\sigma_{a,i} = \sigma_i / \sqrt{n}$  when  $n \gg 1$ . For an airflow at  $\mathbf{U} = (1.0 \text{ ms}^{-1}, 0, 0)$ , compute the relative uncertainty,  $2\sigma_{a,1}/U_1$ , at the 95% confidence level for the average velocity for the three volumes listed in part c).
  - e) For the conditions specified in parts b) and d), what is the smallest volume of gas that ensures a relative uncertainty in  $U$  of one percent or less?
- 1.4. Using the Maxwell molecular velocity distribution given in Exercise 1.3 with  $\mathbf{U} = 0$ , determine the average molecular speed  $= \bar{v} = [\iint_{all \mathbf{u}} |\mathbf{u}|^2 f(\mathbf{u}) d^3 u]^{1/2}$  and compare it with  $c$  = speed of sound in a perfect gas under the same conditions.
- 1.5. By considering the volume swept out by a moving molecule, estimate how the mean-free path,  $l$ , depends on the average molecular cross section dimension  $\bar{d}$  and the molecular number density  $\tilde{n}$  for nominally spherical molecules. Find a formula for  $l\tilde{n}^{1/3}$  (the ratio of the mean-free path to the mean intermolecular spacing) in terms of the *molecular volume* ( $\bar{d}^3$ ) and the *available volume per molecule* ( $1/\tilde{n}$ ). Is this ratio typically bigger or smaller than one?
- 1.6. In a gas, the molecular momentum flux ( $MF_{ij}$ ) in the  $j$ -coordinate direction that crosses a flat surface of unit area with coordinate normal direction  $i$  is:

$$MF_{ij} = \frac{n}{V} \iiint_{all \mathbf{u}} m u_i u_j f(\mathbf{u}) d^3 u \text{ where } f(\mathbf{u}) \text{ is the Maxwell distribution given in Exercise}$$

1.3, and  $n$  is the number of molecules in volume  $V$ . For a perfect gas that is not moving on average (i.e.,  $\mathbf{U} = 0$ ), show that  $MF_{ij} = p$  (the pressure), when  $i = j$ , and that  $MF_{ij} = 0$ , when  $i \neq j$ .

- 1.7. Consider the viscous flow in a channel of width  $2b$ . The channel is aligned in the  $x$ -direction, and the velocity  $u$  in the  $x$ -direction at a distance  $y$  from the channel centerline is given by the parabolic distribution  $u(y) = U_0[1 - (y/b)^2]$ . Calculate the shear stress  $\tau$  as a function  $y, \mu, b$ , and  $U_0$ . What is the shear stress at  $y = 0$ ?
- 1.8. Estimate the height to which water at  $20^\circ\text{C}$  will rise in a capillary glass tube 3 mm in diameter that is exposed to the atmosphere. For water in contact with glass the wetting angle is nearly  $90^\circ$ . At  $20^\circ\text{C}$ , the surface tension of a water-air interface is  $\sigma = 0.073$  N/m.

- 1.9. A *manometer* is a U-shaped tube containing mercury of density  $\rho_m$ . Manometers are used as pressure-measuring devices. If the fluid in tank A has a pressure  $p$  and density  $\rho$ , then show that the gauge pressure in the tank is:  $p - p_{\text{atm}} = \rho_m g h - \rho g a$ . Note that the last term on the right side is negligible if  $\rho \ll \rho_m$ . (Hint: Equate the pressures at X and Y.)



- 1.10. Prove that if  $e(T, v) = e(T)$  only and if  $h(T, p) = h(T)$  only, then the (thermal) equation of state is (1.22) or  $p v = k T$ , where  $k$  is a constant.
- 1.11. Starting from the property relationships (1.18) prove (1.25) and (1.26) for a reversible adiabatic process when the specific heats  $C_p$  and  $C_v$  are constant.
- 1.12. A cylinder contains 2 kg of air at  $50^\circ\text{C}$  and a pressure of 3 bars. The air is compressed until its pressure rises to 8 bars. What is the initial volume? Find the final volume for both isothermal compression and isentropic compression.
- 1.13. Derive (1.29) starting from the arguments provided at the beginning of [Section 1.10](#) and [Figure 1.8](#).
- 1.14. Starting with the hydrostatic pressure law (1.8), prove (1.30) without using perfect gas relationships.
- 1.15. Assume that the temperature of the atmosphere varies with height  $z$  as  $T = T_0 + K z$  where  $K$  is a constant. Show that the pressure varies with height as  $p = p_0 \left[ \frac{T_0}{T_0 + K z} \right]^{g/KR}$ , where  $g$  is the acceleration of gravity and  $R$  is the gas constant for the atmospheric gas.
- 1.16. Suppose the atmospheric temperature varies according to:  $T = 15 - 0.001 z$ , where  $T$  is in degrees Celsius and height  $z$  is in meters. Is this atmosphere stable?
- 1.17. Consider the case of a pure gas planet where the hydrostatic law is:  $dp/dz = -\rho(z) G m(z)/z^2$ . Here  $G$  is the gravitational constant, and  $m(z) = 4\pi \int_0^z \rho(\zeta) \zeta^2 d\zeta$  is the planetary mass up to distance  $z$  from the center of the planet. If the planetary gas is perfect with gas constant  $R$ , determine  $\rho(z)$  and  $p(z)$  if this atmosphere is isothermal at temperature  $T$ . Are these vertical profiles of  $\rho$  and  $p$  valid as  $z$  increases without bound?
- 1.18. Consider a heat-insulated enclosure that is separated into two compartments of volumes  $V_1$  and  $V_2$ , containing perfect gases with pressures of  $p_1$  and  $p_2$ , and temperatures of  $T_1$  and  $T_2$ , respectively. The compartments are separated by an impermeable membrane that conducts heat (but not mass). Calculate the final steady-state temperature assuming each gas has constant specific heats.



- 1.19. Consider the initial state of an enclosure with two compartments as described in Exercise 1.18. At  $t = 0$ , the membrane is broken and the gases are mixed. Calculate the final temperature.
- 1.20. A heavy piston of weight  $W$  is dropped onto a thermally insulated cylinder of cross-sectional area  $A$  containing a perfect gas of constant specific heats, and initially having the external pressure  $p_1$ , temperature  $T_1$ , and volume  $V_1$ . After some oscillations, the piston reaches an equilibrium position  $L$  meters below the equilibrium position of a weightless piston. Find  $L$ . Is there an entropy increase?
- 1.21. <sup>2</sup>A gas of noninteracting particles of mass  $m$  at temperature  $T$  has density  $\rho$ , and internal energy per unit volume  $\varepsilon$ .
- Using dimensional analysis, determine how  $\varepsilon$  must depend on  $\rho$ ,  $T$ , and  $m$ . In your formulation use  $k_B$  = Boltzmann's constant,  $h$  = Planck's constant, and  $c$  = speed of light to include possible quantum and relativistic effects.
  - Consider the limit of slow-moving particles without quantum effects by requiring  $c$  and  $h$  to drop out of your dimensionless formulation. How does  $\varepsilon$  depend on  $\rho$  and  $T$ ? What type of gas follows this thermodynamic law?
  - Consider the limit of massless particles (i.e., photons) by requiring  $m$  and  $\rho$  to drop out of your dimensionless formulation of part a). How does  $\varepsilon$  depend on  $T$  in this case? What is the name of this radiation law?
- 1.22. Many flying and swimming animals—as well as human-engineered vehicles—rely on some type of repetitive motion for propulsion through air or water. For this problem, assume the average travel speed  $U$  depends on the repetition frequency  $f$ , the characteristic length scale of the animal or vehicle  $L$ , the acceleration of gravity  $g$ , the density of the animal or vehicle  $\rho_o$ , the density of the fluid  $\rho$ , and the viscosity of the fluid  $\mu$ .
- Formulate a dimensionless scaling law for  $U$  involving all the other parameters.
  - Simplify your answer for part a) for turbulent flow where  $\mu$  is no longer a parameter.
  - Fish and animals that swim at or near a water surface generate waves that move and propagate because of gravity, so  $g$  clearly plays a role in determining  $U$ . However, if fluctuations in the propulsive thrust are small, then  $f$  may not be important. Thus, eliminate  $f$  from your answer for part b) while retaining  $L$ , and determine how  $U$  depends on  $L$ . Are successful competitive human swimmers likely to be shorter or taller than the average person?
  - When the propulsive fluctuations of a surface swimmer are large, the characteristic length scale may be  $U/f$  instead of  $L$ . Therefore, drop  $L$  from your answer for part b). In this case, will higher speeds be achieved at lower or higher frequencies?
  - While traveling submerged, fish, marine mammals, and submarines are usually neutrally buoyant ( $\rho_o \approx \rho$ ) or very nearly so. Thus, simplify your answer for part b) so that  $g$  drops out. For this situation, how does the speed  $U$  depend on the repetition frequency  $f$ ?
  - Although fully submerged, aircraft and birds are far from neutrally buoyant in air, so their travel speed is predominately set by balancing lift and weight. Ignoring

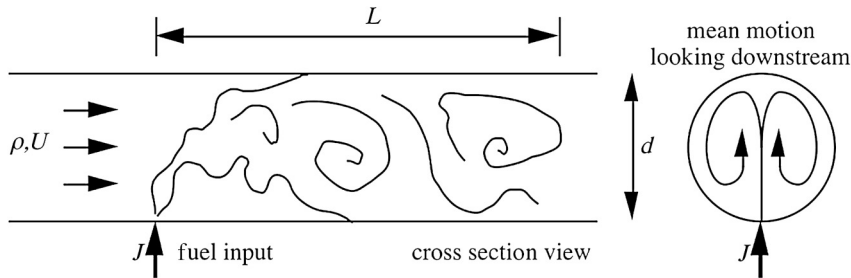
<sup>2</sup>Drawn from thermodynamics lectures of Prof. H. W. Liepmann

frequency and viscosity, use the remaining parameters to construct dimensionally accurate surrogates for lift and weight to determine how  $U$  depends on  $\rho_o/\rho$ ,  $L$ , and  $g$ .

- 1.23. The acoustic power  $W$  generated by a large industrial blower depends on its volume flow rate  $Q$ , the pressure rise  $\Delta P$  it works against, the air density  $\rho$ , and the speed of sound  $c$ . If hired as an acoustic consultant to quiet this blower by changing its operating conditions, what is your first suggestion?
- 1.24. A machine that fills peanut-butter jars must be reset to accommodate larger jars. The new jars are twice as large as the old ones but they must be filled in the same amount of time by the same machine. Fortunately, the viscosity of peanut butter decreases with increasing temperature, and this property of peanut butter can be exploited to achieve the desired results since the existing machine allows for temperature control.
  - a) Write a dimensionless law for the jar-filling time  $t_f$  based on: the density of peanut butter  $\rho$ , the jar volume  $V$ , the viscosity of peanut butter  $\mu$ , the driving pressure that forces peanut butter out of the machine  $P$ , and the diameter of the peanut butter-delivery tube  $d$ .
  - b) Assuming that the peanut butter flow is dominated by viscous forces, modify the relationship you have written for part a) to eliminate the effects of fluid inertia.
  - c) Make a reasonable assumption concerning the relationship between  $t_f$  and  $V$  when the other variables are fixed so that you can determine the viscosity ratio  $\mu_{new}/\mu_{old}$  necessary for proper operation of the old machine with the new jars.
- 1.25. As an idealization of fuel injection in a diesel engine, consider a stream of high-speed fluid (called a *jet*) that emerges into a quiescent air reservoir at  $t = 0$  from a small hole in an infinite plate to form a *plume* where the fuel and air mix.
  - a) Develop a scaling law via dimensional analysis for the penetration distance  $D$  of the plume as a function of:  $\Delta p$  the pressure difference across the orifice that drives the jet,  $d_o$  the diameter of the jet orifice,  $\rho_o$  the density of the fuel,  $\mu_\infty$  and  $\rho_\infty$  the viscosity and density of the air, and  $t$  the time since the jet was turned on.
  - b) Simplify this scaling law for turbulent flow where air viscosity is no longer a parameter.
  - c) For turbulent flow and  $D \ll d_o$ ,  $d_o$  and  $\rho_\infty$  are not parameters. Recreate the dimensionless law for  $D$ .
  - d) For turbulent flow and  $D \gg d_o$ , only the momentum flux of the jet matters, so  $\Delta p$  and  $d_o$  are replaced by the single parameter  $J_o = \text{jet momentum flux}$  ( $J_o$  has the units of force and is approximately equal to  $\Delta p d_o^2$ ). Recreate the dimensionless law for  $D$  using the new parameter  $J_o$ .
- 1.26. <sup>3</sup>One of the simplest types of gasoline carburetors is a tube with a small port for transverse injection of fuel. It is desirable to have the fuel uniformly mixed in the passing airstream as quickly as possible. A prediction of the mixing length  $L$  is sought. The parameters of this problem are:  $\rho$  = density of the flowing air,  $d$  = diameter of the tube,  $\mu$  = viscosity of the flowing air,  $U$  = mean axial velocity of the flowing air, and  $J$  = momentum flux of the fuel stream.

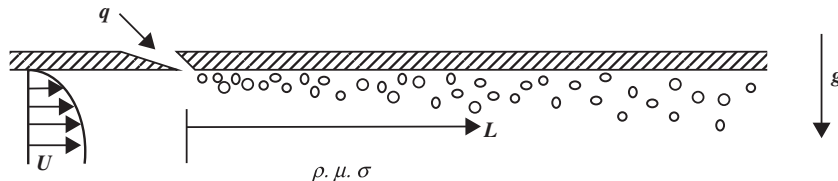
<sup>3</sup>Developed from research discussions with Professor R. Breidenthal

- a) Write a dimensionless law for  $L$ .
- b) Simplify your result from part a) for turbulent flow where  $\mu$  must drop out of your dimensional analysis.
- c) When this flow is turbulent, it is observed that mixing is essentially complete after one rotation of the counter-rotating vortices driven by the injected-fuel momentum (see the downstream view of the drawing for this problem), and that the vortex rotation rate is directly proportional to  $J$ . Based on this information, assume that  $L \propto (\text{rotation time})(U)$  to eliminate the arbitrary function in the result of part b). The final formula for  $L$  should contain an undetermined dimensionless constant.



- 1.27. Consider dune formation in a large horizontal desert of deep sand.
  - a) Develop a scaling relationship that describes how the height  $h$  of the dunes depends on the average wind speed  $U$ , the length of time the wind has been blowing  $\Delta t$ , the average weight and diameter of a sand grain  $w$  and  $d$ , and the air's density  $\rho$  and kinematic viscosity  $\nu$ .
  - b) Simplify the result of part a) when the sand-air interface is fully rough and  $\nu$  is no longer a parameter.
  - c) If the sand dune height is determined to be proportional to the density of the air, how do you expect it to depend on the weight of a sand grain?
- 1.28. An isolated nominally spherical bubble with radius  $R$  undergoes shape oscillations at frequency  $f$ . It is filled with air having density  $\rho_a$  and resides in water with density  $\rho_w$  and surface tension  $\sigma$ . What frequency ratio should be expected between two isolated bubbles with 2 cm and 4 cm diameters undergoing geometrically similar shape oscillations? If a soluble surfactant is added to the water that lowers  $\sigma$  by a factor of two, by what factor should air bubble oscillation frequencies increase or decrease?
- 1.29. In general, boundary layer skin friction,  $\tau_w$ , depends on the fluid velocity  $U$  above the boundary layer, the fluid density  $\rho$ , the fluid viscosity  $\mu$ , the nominal boundary layer thickness  $\delta$ , and the surface roughness length scale  $\varepsilon$ .
  - a) Generate a dimensionless scaling law for boundary layer skin friction.
  - b) For laminar boundary layers, the skin friction is proportional to  $\mu$ . When this is true, how must  $\tau_w$  depend on  $U$  and  $\rho$ ?
  - c) For turbulent boundary layers, the dominant mechanisms for momentum exchange within the flow do not directly involve the viscosity  $\mu$ . Reformulate your dimensional analysis without it. How must  $\tau_w$  depend on  $U$  and  $\rho$  when  $\mu$  is not a parameter?

- d) For turbulent boundary layers on smooth surfaces, the skin friction on a solid wall occurs in a viscous sublayer that is very thin compared to  $\delta$ . In fact, because the boundary layer provides a buffer between the outer flow and this viscous sub-layer, the viscous sublayer thickness  $l_v$  does not depend directly on  $U$  or  $\delta$ . Determine how  $l_v$  depends on the remaining parameters.
- e) Now consider nontrivial roughness. When  $\varepsilon$  is larger than  $l_v$  a surface can no longer be considered fluid-dynamically smooth. Thus, based on the results from parts a) through d) and anything you may know about the relative friction levels in laminar and turbulent boundary layers, are high- or low-speed boundary layer flows more likely to be influenced by surface roughness?
- 1.30. Turbulent boundary layer skin friction is one of the fluid phenomena that limit the travel speed of aircraft and ships. One means for reducing the skin friction of liquid boundary layers is to inject a gas (typically air) from the surface on which the boundary layer forms. The shear stress,  $\tau_w$  that is felt a distance  $L$  downstream of such an air injector depends on: the volumetric gas flux per unit span  $q$  (in  $m^2/s$ ), the free stream flow speed  $U$ , the liquid density  $\rho$ , the liquid viscosity  $\mu$ , the surface tension  $\sigma$ , and the gravitational acceleration  $g$ .
- a) Formulate a dimensionless law for  $\tau_w$  in terms of the other parameters.
- b) Experimental studies of air injection into liquid turbulent boundary layers on flat plates has found that the bubbles may coalesce to form an air film that provides near perfect lubrication,  $\tau_w \rightarrow 0$  for  $L > 0$ , when  $q$  is high enough and gravity tends to push the injected gas toward the plate surface. Reformulate your answer to part a) by dropping  $\tau_w$  and  $L$  to determine a dimensionless law for the minimum air injection rate,  $q_c$ , necessary to form an air layer.
- c) Simplify the result of part b) when surface tension can be neglected.
- d) Experimental studies (Elbing et al., 2008) find that  $q_c$  is proportional to  $U^2$ . Using this information, determine a scaling law for  $q_c$  involving the other parameters. Would an increase in  $g$  cause  $q_c$  to increase or decrease?



## Literature Cited

- Barenblatt, G. I. (1979). *"Similarity, Self-Similarity, and Intermediate Asymptotics"* (translated from Russian by Norman Stein). New York: Consultants Bureau.
- Elbing, B. R., Winkel, E. S., Lay, K. A., Ceccio, S. L., Dowling, D. R., and Perlin, M. (2008). "Bubble-induced skin friction drag reduction and the abrupt transition to air-layer drag reduction." *Journal of Fluid Mechanics*, 612, 201–236.

- Taylor, G. I. (1950). The formation of blast wave by a very intense explosion. Parts I and II. *Proceedings of the Royal Society of London, Series A*, 201, 159–174, and 175–186.
- von Karman, T. (1954). *Aerodynamics*. New York: McGraw-Hill.

## Supplemental Reading

- Batchelor, G. K. (1967). *An Introduction to Fluid Dynamics*. London: Cambridge University Press. (A detailed discussion of classical thermodynamics, kinetic theory of gases, surface tension effects, and transport phenomena is given.)
- Bridgeman, P. W. (1963). *Dimensional Analysis*. New Haven: Yale University Press.
- Çengel, Y. A., and Cimbala, J. M. (2010). *Fluid Mechanics: Fundamentals and Applications* (2nd ed.). New York: McGraw-Hill. (A fine undergraduate fluid mechanics textbook. Chapter 7 covers dimensional analysis.)
- Fox, R. W., Pritchard, P. J., and MacDonald, A. T. (2009). *Introduction to Fluid Mechanics* (7th ed.). New York: John Wiley. (A fine undergraduate fluid mechanics textbook. Chapter 7 covers dimensional analysis.)
- Hatsopoulos, G. N., and Keenan, J. H. (1981). *Principles of General Thermodynamics*. Melbourne, FL: Krieger Publishing Co. (This is a good text on thermodynamics.)
- Prandtl, L., and Tietjens, O. G. (1934). *Fundamentals of Hydro- and Aeromechanics*. New York: Dover Publications. (A clear and simple discussion of potential and adiabatic temperature gradients is given.)
- Taylor, G. I. (1974). "The interaction between experiment and theory in fluid mechanics." *Annual Review of Fluid Mechanics*, 6, 1–16.
- Tritton, D. J. (1988). *Physical Fluid Dynamics* (2nd ed.). Oxford: Clarendon Press. (A well-written book with broad coverage of fluid mechanics.)
- Vincenti, W. G., and Kruger, C. H. (1965). *Introduction of Physical Gas Dynamics*. Malabar, Florida: Krieger. (A fine reference for kinetic theory and statistical mechanics.)