4

Slow modulation of dispersive waves

'But let me tell thee now another tale'
The Coming of Arthur

In ever climbing up the climbing wave
The Lotos-Eaters: Choric song IV

The Korteweg–de Vries equation, members of its family and the applications to more realistic situations, cover only one general area of interest in the modern theory of nonlinear water waves. In particular, all our discussions in Chapter 3 have been based on the requirement that the waves are long; this was accomplished by the condition $\delta \to 0$ or, rather, by the rescaling

$$x \to \frac{\delta}{\varepsilon^{\frac{1}{2}}} x, \quad t \to \frac{\delta}{\varepsilon^{\frac{1}{2}}} t,$$

with $\varepsilon \to 0$; see equation (3.10). In this discussion we shall now allow the wave to be of any wavelength, so that the wave number (k) plays the rôle of a parameter in our calculations. The amplitude parameter, ε, is then used to describe the slow evolution of an harmonic wave of wave number k; the wave is thus slowly modulated as described by $\varepsilon \to 0$. The approach that we adopt is to be found in Section 1.4.2 (equation (1.103) *et seq.*) where the appropriate multiple-scale technique is used there to obtain the asymptotic solution of a partial differential equation.

We shall follow a similar route to that developed in Chapter 3, namely, a presentation of the derivation of the basic evolution equation together with the application of these ideas to more realistic situations. It turns out that the fundamental equation (the *Nonlinear Schrödinger equation*) – and some of its relations – are again special equations of the completely integrable (soliton) type. We shall describe a few properties of these equations, and how solutions can be readily obtained. Not surprisingly, the long-wave limit of these various problems that we present here recovers the essential features of the KdV description; we shall show how this comes about.

4.1 The evolution of wave packets

We shall present two derivations that lead to a description of the evolution of wave packets (for gravity waves) on the surface of water of finite depth. First we examine the problem of the propagation of a plane wave and then, just as in Chapter 3, we construct a two-dimensional version of this problem that incorporates a suitable (weak) dependence on the coordinate that is transverse to the predominant direction of propagation; cf. the 2D KdV equation. This two-dimensional surface wave is described by a pair of equations: the *Davey–Stewartson* (DS) equations.

4.1.1 Nonlinear Schrödinger (NLS) equation

In keeping with much that has gone before, we shall start with an examination of gravity waves (moving in one direction) on stationary water of constant depth ($b = 0$). The most direct approach is to formulate the problem in terms of the equations for irrotational flow (although we shall not always be able to follow this route). Thus, from equations (2.132) with $\partial/\partial y \equiv 0$, we have

$$\left.\begin{aligned} &\phi_{zz} + \delta^2\phi_{xx} = 0; \\ &\left.\begin{aligned} &\phi_z = \delta^2(\eta_t + \varepsilon\phi_x\eta_x) \\ &\phi_t + \eta + \frac{1}{2}\varepsilon\left(\frac{1}{\delta^2}\phi_z^2 + \phi_x^2\right) = 0 \end{aligned}\right\} \text{ on } z = 1 + \varepsilon\eta \end{aligned}\right\} \tag{4.1}$$

and

$$\phi_z = 0 \quad \text{on} \quad z = 0.$$

In these equations we have retained the shallowness parameter, δ, and we shall consider $\varepsilon \to 0$ for δ fixed. The solution that we seek is a harmonic wave with wave number k – a solution of the linear equations ($\varepsilon = 0$) – which is allowed to evolve slowly on scales determined by ε. We have already seen (equation (1.103) *et seq.*) that the relevant scales would seem to be associated with both ε and ε^2. Here, therefore, we introduce

$$\xi = x - c_p t, \quad \zeta = \varepsilon(x - c_g t), \quad \tau = \varepsilon^2 t, \tag{4.2}$$

where $c_p(k)$ and $c_g(k)$ are to be determined (but the notation should be suggestive!). The justification for this choice is, ultimately, that it produces a consistent solution of the equations; a simple argument based on the Fourier integral representation of a general plane wave also confirms this choice (Q4.1).

The governing equations, (4.1), under the transformation (4.2), become

$$\left.\begin{aligned}
&\phi_{zz} + \delta^2\left(\phi_{\xi\xi} + 2\varepsilon\phi_{\xi\zeta} + \varepsilon^2\phi_{\zeta\zeta}\right) = 0;\\
&\left.\begin{aligned}
\phi_z &= \delta^2\{\varepsilon^2\eta_\tau - \varepsilon c_g\eta_\zeta - c_p\eta_\xi\\
&\qquad + \varepsilon\left(\phi_\xi + \varepsilon\phi_\zeta\right)\left(\eta_\xi + \varepsilon\eta_\zeta\right)\}\\
\varepsilon^2\phi_\tau &- \varepsilon c_g\phi_\zeta - c_p\phi_\xi + \eta\\
&+ \frac{1}{2}\varepsilon\left\{\frac{1}{\delta^2}\phi_z^2 + \left(\phi_\xi + \varepsilon\phi_\zeta\right)^2\right\} = 0
\end{aligned}\right\} \text{ on } z = 1 + \varepsilon\eta\\
&\text{and}\\
&\qquad \phi_z = 0 \quad \text{on} \quad z = 0.
\end{aligned}\right\} \tag{4.3}$$

We seek an asymptotic solution of these equations in the form

$$\phi \sim \sum_{n=0}^{\infty} \varepsilon^n \phi_n(\xi, \zeta, \tau, z); \quad \eta \sim \sum_{n=0}^{\infty} \varepsilon^n \eta_n(\xi, \zeta, \tau) \quad \text{as} \quad \varepsilon \to 0,$$

which is to be periodic in ξ. The leading-order problem is clearly

$$\phi_{0zz} + \delta^2\phi_{0\xi\xi} = 0 \tag{4.4}$$

with

$$\phi_{0z} = -\delta^2 c_p \eta_{0\xi} \quad \text{and} \quad -c_p\phi_{0\xi} + \eta_0 = 0 \quad \text{on} \quad z = 1 \tag{4.5}$$

and

$$\phi_{0z} = 0 \quad \text{on} \quad z = 0. \tag{4.6}$$

The solution of interest to us takes the form

$$\eta_0 = A_0 E + \text{c.c.}; \quad \phi_0 = f_0 + F_0 E + \text{c.c.}, \tag{4.7}$$

where $E = \exp(\mathrm{i}k\xi)$, $A_0 = A_0(\zeta, \tau)$, $F_0 = F_0(z, \zeta, \tau)$, $f_0 = f_0(\zeta, \tau)$ and c.c. denotes the complex conjugate of the terms in E. The real term $f_0(\zeta, \tau)$ is required in order to accommodate the mean drift component; see Section 2.5. This solution describes a single harmonic wave, of wave number k, which is propagating at speed c_p. We see that Laplace's equation, (4.4), with (4.7), becomes

$$F_{0zz} - \delta^2 k^2 F_0 = 0,$$

so the solution which satisfies the bottom boundary condition, (4.6), is

$$F_0 = G_0(\zeta, \tau)\cosh(\delta k z),$$

where $G_0(\zeta, \tau)$ is yet to be determined. The two surface boundary conditions, (4.5), yield

$$\delta k G_0 \sinh \delta k = -\mathrm{i}\delta^2 k c_p A_0 \quad \text{and} \quad \mathrm{i}k c_p G_0 \cosh \delta k = A_0,$$

from which we obtain

$$c_p^2 = \frac{\tanh \delta k}{\delta k} \quad \text{and} \quad G_0 = -\frac{\mathrm{i}A_0}{k c_p}\operatorname{sech}\delta k; \tag{4.8}$$

thus we may write

$$F_0 = -\mathrm{i}\delta c_p A_0 \frac{\cosh \delta k z}{\sinh \delta k}, \tag{4.9}$$

all of which is familiar; see equation (2.4) *et seq.*, equation (2.13) and Q2.5. At this order, the amplitude function $A_0(\zeta, \tau)$ is unknown; we now proceed to the problem given by the $\mathrm{O}(\varepsilon)$ terms.

Equations (4.3), upon collecting the terms of $\mathrm{O}(\varepsilon)$ and expanding about $z = 1$ in the surface boundary conditions, yield

$$\phi_{1zz} + \delta^2 \phi_{1\xi\xi} + 2\delta^2 \phi_{0\xi\zeta} = 0; \tag{4.10}$$

with

$$\left.\begin{aligned} &\phi_{1z} + \eta_0 \phi_{0zz} = \delta^2(-c_g \eta_{0\zeta} - c_p \eta_{1\xi} + \phi_{0\xi}\eta_{0\xi}) && (4.11)\\ &-c_g \phi_{0\zeta} - c_p(\phi_{1\xi} + \eta_0 \phi_{0\xi z}) + \eta_1 + \frac{1}{2}\left(\frac{1}{\delta^2}\phi_{0z}^2 + \phi_{0\xi}^2\right) = 0 && (4.12)\end{aligned}\right\} \text{ on } z = 1$$

and

$$\phi_{1z} = 0 \quad \text{and} \quad z = 0. \tag{4.13}$$

It is clear that these equations will produce terms E^2, E^{-2} and E^0 (from $E^1 E^{-1}$) by virtue of the nonlinearity of the surface boundary conditions. The contributions E^2 (with E^{-2}, the complex conjugate) are the first of the higher harmonics that are generated by the nonlinear interaction; the fundamental is E^1 (with E^{-1}), introduced in (4.7). Since we are seeking a solution which is periodic in ξ, we choose to build in this requirement at this stage. We do this by imposing a periodic structure on the form of solution that we seek hereafter. An alternative approach is to solve at each $\mathrm{O}(\varepsilon^n)$ and determine the various functions that are available, in such a fashion as to remove all those terms that contribute to the non-periodic (or *secular*) terms; this is how we tackled the problem in Section 1.4.2 (equations (1.103) *et seq.*). The two methods produce, eventually, exactly the same result, but the former presents us with a more straightforward calculation, as we shall now demonstrate.

In order to implement this idea, we write

$$\phi_n = \sum_{m=0}^{n+1} F_{nm} E^m + \text{c.c.}; \quad \eta_n = \sum_{m=0}^{n+1} A_{nm} E^m + \text{c.c.} \tag{4.14}$$

for $n = 1, 2, \ldots$, where $F_{nm}(z, \zeta, \tau)$ and $A_{nm}(\zeta, \tau)$ are to be determined; the complex conjugate relates only to terms in E^m, $m = 1, 2, \ldots$. We note that terms $m = 0$, although not harmonic (oscillatory) functions, do not destroy the periodicity in ξ. The solution described by (4.14) incorporates the phenomenon that, at each higher order in ε, higher harmonics progressively appear, so E^2 appears first at $O(\varepsilon)$, E^3 first at $O(\varepsilon^2)$, and so on.

Laplace's equation for ϕ_1, (4.10), therefore gives

$$F_{10zz} = 0; \quad F_{12zz} - 4\delta^2 k^2 F_{12} = 0$$

and

$$F_{11zz} - \delta^2 k^2 F_{11} + 2\mathrm{i}\delta^2 k F_{0\zeta} = 0,$$

which have solutions (see Q4.2) satisfying the bottom boundary condition, (4.13),

$$\left.\begin{aligned} &F_{10} = G_{10}(\zeta, \tau); \quad F_{12} = G_{12}(\zeta, \tau)\cosh(2\delta k z) \\ \text{and} \quad & \\ &F_{11} = G_{11}(\zeta, \tau)\cosh(\delta k z) - \mathrm{i}\delta G_{0\zeta} z \sinh(\delta k z), \end{aligned}\right\} \tag{4.15}$$

where the $G_{1m}(\zeta, \tau)$ are arbitrary functions at this stage. These results are then used in the two surface boundary conditions, (4.11) and (4.12), to give six equations (arising from the coefficients of E^0, E^1 and E^2 in each). Equation (4.11) yields (with the asterisk denoting the complex conjugate)

$$E^0\!: (A_0 G_0^* + A_0^* G_0)\delta^2 k^2 \cosh \delta k = \delta^2 k^2 (A_0 G_0^* + A_0^* G_0) \cosh \delta k; \tag{4.16}$$

$$E^1\!: \delta k G_{11} \sinh \delta k - \mathrm{i}\delta G_{0\zeta}(\sinh \delta k + \delta k \cosh \delta k) = -\delta^2 (c_g A_{0\zeta} + \mathrm{i}k c_p A_{11}) \tag{4.17}$$

$$E^2\!: 2\delta k G_{12} \sinh 2\delta k + \delta^2 k^2 A_0 G_0 \cosh \delta k = -\delta^2 (2\mathrm{i}k c_p A_{12} + k^2 A_0 G_0 \cosh \delta k), \tag{4.18}$$

and, correspondingly, equation (4.12) gives

$$E^0: \; -c_g f_{0\zeta} + \mathrm{i}\delta k^2 c_p(A_0 G_0^* + A_0^* G_0)\sinh\delta k + A_{10} \\ + k^2 G_0 G_0^*(\sinh^2\delta k + \cosh^2\delta k) = 0; \quad (4.19)$$

$$E^1: \; -c_g G_{0\zeta}\cosh\delta k - \mathrm{i}kc_p(G_{11}\cosh\delta k - \mathrm{i}\delta G_{0\zeta}\sinh\delta k) + A_{11} = 0; \quad (4.20)$$

$$E^2: \; -\mathrm{i}kc_p(2G_{12}\cosh 2\delta k + \delta k A_0 G_0 \sinh\delta k) + A_{12} - \frac{1}{2}k^2 G_0^2 = 0. \quad (4.21)$$

We see that equation (4.16) is identically satisfied, and that with equation (4.8) (for G_0) used in (4.19) we obtain

$$A_{10} = -\frac{2\delta k}{\sinh 2\delta k} A_0 A_0^* + c_g f_{0\zeta}. \quad (4.22)$$

Equation (4.20) gives us directly that

$$A_{11} = c_g G_{0\zeta}\cosh\delta k + \mathrm{i}kc_p(G_{11}\cosh\delta k - \mathrm{i}\delta G_{0\zeta}\sinh\delta k),$$

and when this is used in equation (4.17) we find, first, that G_{11} cancels identically when we invoke the expression for c_p^2 (in (4.8)); then, with G_0 from (4.8), we see that $A_{0\zeta}$ ($\neq 0$) also cancels, leaving

$$c_g = \frac{1}{2}c_p(1 + 2\delta k \operatorname{cosech} 2\delta k), \quad (4.23)$$

which is the group speed for gravity waves; see equation (2.29) *et seq.*, and Q2.26. Finally, equations (4.18) and (4.21) are solved for G_{12} and A_{12} (using (4.8) as necessary) to give

$$G_{12} = -\frac{3\mathrm{i}}{4}\frac{\delta^2 k c_p A_0^2}{\sinh^4\delta k}; \quad A_{12} = \frac{\delta k\cosh\delta k}{2\sinh^3\delta k}(2\cosh^2\delta k + 1)A_0^2, \quad (4.24)$$

and $A_0(\zeta, \tau)$ is still undetermined. (It is clear that the solution of this problem requires some fairly extensive manipulation – and it is considerably worse at the next order – the details of which are left to the sufficiently enthusiastic reader.) We now examine the next order, $O(\varepsilon^2)$, where we expect the equation for A_0 to emerge.

From equations (4.3), with the usual expansion of the surface boundary conditions about $z = 1$, we obtain the problem at $O(\varepsilon^2)$ as

$$\phi_{2zz} + \delta^2\phi_{2\xi\xi} + 2\delta^2\phi_{1\xi\zeta} + \delta^2\phi_{0\zeta\zeta} = 0; \quad (4.25)$$

with

$$\phi_{2z} + \eta_0\phi_{1zz} + \frac{1}{2}\eta_0^2\phi_{0zzz} + \eta_1\phi_{0zz}$$
$$= \delta^2\{\eta_{0\tau} - c_g\eta_{1\zeta} - c_p\eta_{2\xi} + \phi_{0\xi}(\eta_{1\xi} + \eta_{0\zeta}) + \eta_{0\xi}(\eta_0\phi_{0\xi z} + \phi_{1\xi} + \phi_{0\zeta})\}; \quad (4.26)$$

$$\phi_{0\tau} - c_g\eta_0\phi_{0\zeta z} - c_p\phi_{1\zeta} - c_p(\phi_{2\xi} + \eta_0\phi_{1\xi z} + \eta_1\phi_{0\xi z} + \frac{1}{2}\eta_0^2\phi_{0\xi zz})$$
$$+ \eta_2 + \frac{1}{\delta^2}\phi_{0z}(\eta_0\phi_{0zz} + \phi_{1z}) + \phi_{0\xi}(\eta_0\phi_{0\xi z} + \phi_{1\xi} + \phi_{0\zeta}) = 0, \quad (4.27)$$

both on $z = 1$, and

$$\phi_{2z} = 0 \quad \text{on} \quad z = 0. \quad (4.28)$$

The periodic solution described in equations (4.14) is now used for $n = 2$ (so the higher harmonic E^3 now appears for the first time); equation (4.25) then gives

$$F_{20zz} + \delta^2 f_{0\zeta\zeta} = 0; \quad F_{21zz} - \delta^2k^2F_{21} + 2\mathrm{i}k\delta^2F_{11\zeta} + \delta^2F_{0\zeta\zeta} = 0,$$

and so on. It soon becomes evident that the equation for A_0 appears from the terms that arise at E^1 (because this is equivalent to the removal of secular terms at E^1; cf. equation (1.108) *et seq.*), so we examine only this problem in any detail. The solution for $F_{21}(z, \zeta, \tau)$ which satisfies the bottom boundary condition, (4.28), becomes

$$F_{21} = G_{21}\cosh\delta kz - (\mathrm{i}\delta G_{11\zeta} + \frac{\delta}{2k}G_{0\zeta\zeta})z\sinh\delta kz$$
$$+ \frac{1}{2}\delta^2G_{0\zeta\zeta}\left(\frac{1}{\delta k}z\sinh\delta kz - z^2\cosh\delta kz\right); \quad (4.29)$$

see Q4.2. The boundary condition (4.26) gives, for terms E^1,

$$F_{21z} + A_0^*F_{12zz} + \frac{1}{2}(A_0^2F_{0zzz}^* + 2A_0A_0^*F_{0zzz}) + A_{10}F_{0zz} + A_{12}F_{0zz}^*$$
$$= \delta^2\{A_{0\tau} - c_gA_{11\zeta} - \mathrm{i}kc_pA_{21} + 2k^2A_{12}F_0^*$$
$$-k^2A_0(A_0^*F_{0z} - A_0F_{0z}^*) + k^2A_0^*(A_0F_{0z} + 2F_{12})\} \quad (4.30)$$

on $z=1$. The second boundary condition on $z=1$, (4.27), similarly gives

$$\begin{aligned} F_{0\tau} - c_g F_{11\zeta} - \mathrm{i}kc_p F_{21} - 2\mathrm{i}kc_p A_0^* F_{12z} - \mathrm{i}kc_p(A_{10}F_{0z} - A_{12}F_{0z}^*) \\ - \frac{1}{2}\mathrm{i}kc_p(2A_0A_0^*F_{0zz} - A_0^2F_{0zz}^*) + A_{21} \\ + \frac{1}{\delta^2}\{(A_0F_{0zz}^* + A_0^*F_{0zz})F_{0z} + (A_0F_{0zz} + F_{12z})F_{0z}^*\} \\ - k^2(A_0^*F_{0z} - A_0F_{0z}^*)F_0 + k^2(A_0F_{0z} + 2F_{12})F_0^* = 0. \qquad (4.31) \end{aligned}$$

The procedure that we follow is easy to describe, but rather tiresome to perform: eliminate A_{21} between equations (4.30) and (4.31), introduce the functions obtained at earlier stages (including F_{21} from (4.29)) and simplify. We find that G_{21} cancels identically by virtue of the definition of c_p^2, and that A_{11} (or G_{11}) also cancels when the expression for c_g, (4.23), is used. This leaves an equation for $A_0(\zeta, \tau)$, incorporating the terms $A_{0\tau}$, $A_{0\zeta\zeta}$ and $A_0|A_0|^2$:

$$-2\mathrm{i}kc_pA_{0\tau} + \alpha A_{0\zeta\zeta} + \beta A_0|A_0|^2 = 0, \qquad (4.32)$$

which is one form of the *Nonlinear Schrödinger* (*NLS*) *equation*, where here

$$\alpha = c_g^2 - (1 - \delta k \tanh \delta k)\mathrm{sech}^2 \delta k \qquad (4.33)$$

and

$$\beta = \frac{k^2}{c_p^2}\left\{\frac{1}{2}(1 + 9\coth^2\delta k - 13\mathrm{sech}^2\delta k - 2\tanh^4\delta k) - (2c_p + c_g\mathrm{sech}^2\delta k)^2(1 - c_g^2)^{-1}\right\}, \qquad (4.34)$$

although a more instructive form for α is

$$\alpha = -kc_p\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}, \quad \omega(k) = kc_p;$$

cf. Q4.1 and see Q4.3. (The NLS equation is sometimes called the *cubic Schrödinger equation*.) It is a straightforward calculation – which is left as an exercise – to show that $\alpha > 0$ for all δk, but that β changes in sign from positive to negative as δk decreases, across $\delta k \approx 1.363$; see Section 4.3.1 and Figure 4.6 (on p. 336). We comment here that the condition $\alpha\beta > 0$ turns out to be significant for the existence of certain important solutions of the NLS equation; see Section 4.2. The consideration of terms that arise at $\varepsilon^2 E^0$ is left as an exercise; see Q4.4.

Some relevant properties of the NLS equation, and the interpretation of its solutions in the understanding of water-wave phenomena, will be presented later.

4.1.2 Davey–Stewartson (DS) equations

The classical NLS equation applies to the situation where the wave properties only in one direction, and for which the profile evolves only in this same direction. Such a wave would be generated by an initial profile which takes the form

$$A(\varepsilon x)\mathrm{e}^{\mathrm{i}kx} + \text{c.c.};$$

we now consider (following Davey & Stewartson, 1974) the development of a wave which, at $t = 0$, is described by

$$A(\varepsilon x, \varepsilon y)\mathrm{e}^{\mathrm{i}kx} + \text{c.c.}$$

We see that the slow (or weak) dependence occurs equally in both the x- and y-directions, but that the fast oscillation is only in the x-direction: the wave packet will propagate in the x-direction with a slowly evolving structure in both x- and y-directions. The group speed is, of course, still associated with the propagation in the x-direction. The appropriate form of solution will be sought from the governing equations (see equations (2.132)); these are

$$\left.\begin{array}{l} \phi_{zz} + \delta^2(\phi_{xx} + \phi_{yy}) = 0; \\ \text{with} \\ \left.\begin{array}{l} \phi_z = \delta^2\{\eta_t + \varepsilon(\phi_x\eta_x + \phi_y\eta_y)\} \\ \phi_t + \eta + \frac{1}{2}\varepsilon\left(\frac{1}{\delta^2}\phi_z^2 + \phi_x^2 + \phi_y^2\right) = 0 \end{array}\right\} \text{ on } z = 1 + \varepsilon\eta \\ \text{and} \\ \phi_z = 0 \quad \text{on} \quad z = 0. \end{array}\right\} \qquad (4.35)$$

We introduce the variables (cf. equation (4.2))

$$\xi = x - c_p t, \quad \zeta = \varepsilon(x - c_g t), \quad Y = \varepsilon y, \quad \tau = \varepsilon^2 t$$

and so equations (4.35) become (cf. equations (4.3))

$$\phi_{zz} + \delta^2(\phi_{\xi\xi} + 2\varepsilon\phi_{\xi\zeta} + \varepsilon^2\phi_{\zeta\zeta} + \varepsilon^2\phi_{YY}) = 0;$$

$$\left.\begin{aligned} &\phi_z = \delta^2\{\varepsilon^2\eta_\tau - \varepsilon c_g\eta_\zeta - c_p\eta_\xi \\ &\qquad + \varepsilon(\phi_\xi + \varepsilon\phi_\zeta)(\eta_\xi + \varepsilon\eta_\zeta) + \varepsilon^3\phi_Y\eta_Y\} \\ &\varepsilon^2\phi_\tau - \varepsilon c_g\phi_\zeta - c_p\phi_\xi + \eta \\ &\qquad + \frac{1}{2}\varepsilon\left\{\frac{1}{\delta^2}\phi_z^2 + (\phi_\xi + \varepsilon\phi_\zeta)^2 + \varepsilon\phi_Y^2\right\} = 0 \end{aligned}\right\} \text{ on } z = 1 + \varepsilon\eta$$

and

$$\phi_z = 0 \quad \text{on} \quad z = 0.$$

It is immediately clear that, if we proceed no further than $O(\varepsilon^2)$, the only contribution from the dependence in Y will arise from the term ϕ_{YY} in Laplace's equation. The other terms involving derivatives in Y produce new nonlinear interactions that appear first at $O(\varepsilon^3)$. The calculation therefore follows very closely that already presented for the NLS equation, so we shall not give the details here.

We seek a solution in the form

$$\left.\begin{aligned} &\phi \sim f_0(\zeta, Y, \tau) + \sum_{n=0}^{\infty}\varepsilon^n\left\{\sum_{m=0}^{n+1}F_{nm}(z, \zeta, Y, \tau)E^m + \text{c.c.}\right\}; \\ &\eta \sim \sum_{n=0}^{\infty}\varepsilon^n\left\{\sum_{m=0}^{n+1}A_{nm}(\zeta, Y, \tau)E^m + \text{c.c.}\right\}; \end{aligned}\right\} \tag{4.36}$$

where $E = \exp(\mathrm{i}k\xi)$ and $A_{00} = 0$ (so that the first approximation to the surface wave is purely harmonic). The results mirror those already obtained, for all the terms at $O(1)$ and $O(\varepsilon)$; the differences first appear at $O(\varepsilon^2)$. It turns out that the problem at ε^2E^0 gives

$$(1 - c_g^2)f_{0\zeta\zeta} + f_{0YY} = -\frac{1}{c_p^2}(2c_p + c_g\operatorname{sech}^2\delta k)(|A_0|^2)_\zeta, \tag{4.37}$$

the equation for f_0, given $A_0\,(\equiv A_{01})$; the surface boundary conditions for the terms ε^2E^1 produce

$$\left.\begin{aligned} &-2\mathrm{i}kc_pA_{0\tau} + \alpha A_{0\zeta\zeta} - c_pc_gA_{0YY} \\ &\qquad + \frac{k^2}{2c_p^2}(1 + 9\coth^2\delta k - 13\operatorname{sech}^2\delta k - 2\tanh^4\delta k)A_0|A_0|^2 \\ &\qquad\qquad + k^2(2c_p + c_g\operatorname{sech}^2\delta k)A_0f_{0\zeta} = 0. \end{aligned}\right\} \tag{4.38}$$

These two equations, (4.37) and (4.38), are the *Davey–Stewartson* (*DS*) *equations* for the modulation of harmonic waves. It is clear that for no dependence on Y, so that (4.37) gives

$$(1 - c_g^2)f_{0\zeta} = -\frac{1}{c_p^2}(2c_p + c_g \operatorname{sech}^2 \delta k)|A_0|^2 \tag{4.39}$$

(on the assumption that $f_{0\zeta} = 0$ where $A_0 = 0$), equation (4.38) then recovers the NLS equation

$$-2\mathrm{i}kc_p A_{0\tau} + \alpha A_{0\zeta\zeta} + \beta A_0 |A_0|^2 = 0$$

as given in (4.32); also see Q4.24. Equation (4.39) provides the leading contribution to the mean drift generated by the nonlinear interaction of the wave; see Q4.4 and Q2.32.

The DS equations are more compactly written in the form

$$(1 - c_g^2)f_{0\zeta\zeta} + f_{0YY} = -\frac{\gamma}{c_p^2}(|A_0|^2)_\zeta; \tag{4.40}$$

$$-2\mathrm{i}kc_p A_{0\tau} + \alpha A_{0\zeta\zeta} - c_p c_g A_{0YY} + \left\{\beta + \frac{\gamma^2 k^2}{c_p^2(1 - c_g^2)}\right\} A_0 |A_0|^2 + k^2 \gamma A_0 f_{0\zeta} = 0, \tag{4.41}$$

where α and β were given earlier ((4.33), (4.34)) and

$$\gamma = 2c_p + c_g \operatorname{sech}^2 \delta k; \tag{4.42}$$

we observe that $\gamma > 0$ and note that $c_p c_g > 0$. These equations (and, of course, the NLS equation) may be further approximated for long or short waves ($\delta \to 0$, $\delta \to \infty$, respectively), although their validity must remain in doubt: that is, for sufficiently small/large δ, other terms will presumably become important. However, as model equations for the evolution of wave packets in these two limits, they do provide useful insights; these limiting cases are considered in Q4.6. Furthermore, as a mathematical exercise to confirm the overall consistency of our equations, the result of letting $\delta \to 0$ (so that we have long waves) is important. We know, for the one-dimensional propagation of long waves, that the relevant equation is the Korteweg–de Vries equation. The existence of a close relationship between the NLS and KdV equations is now explored.

4.1.3 Matching between the NLS and KdV equations

The two fundamental equations for weakly nonlinear waves that we have introduced are the KdV and NLS equations. The former equation describes long waves, which can be obtained by letting $\delta \to 0$ and $\varepsilon \to 0$ with $\delta^2 = \mathrm{O}(\varepsilon)$; see Section 2.9.1. Alternatively, and more generally, we use a suitable rescaling of the variables which allows us to obtain the KdV equation for arbitrary δ. However, this transformation results in the replacement of δ^2 by ε in the governing equations (see equations (3.10)–(3.15)) with $\varepsilon \to 0$; thus the transformation, coupled with $\varepsilon \to 0$, is equivalent to $\delta \to 0$: long waves. On the other hand, the NLS equation uses scaled variables which are defined with respect to ε only, with δ $(= \mathrm{O}(1))$ retained as a parameter throughout. Thus, at least for a class of waves, we have two representations:

$$\eta(x, t; \varepsilon, \delta) \text{ with } \varepsilon \to 0, \ \ \delta \to 0 - \text{KdV};$$
$$\eta(x, t; \varepsilon, \delta) \text{ with } \varepsilon \to 0, \ \ \delta \text{ fixed } \ - \text{NLS}.$$

We might, therefore, suppose that the two descriptions satisfy some appropriate matching condition in δ. That is, the KdV representation with $\delta \to \infty$ might match with the NLS representation with $\delta \to 0$. So we take the short-wave limit of the KdV equation (but, as we shall see, written in an appropriate form) and the long-wave limit of the NLS equation.

Let us first construct the limiting form of the NLS equation as $\delta \to 0$; this requires that we determine the dominant behaviours of the coefficients of the equation

$$-2\mathrm{i}kc_p A_{0\tau} + \alpha A_{0\zeta\zeta} + \beta A_0 |A_0|^2 = 0,$$

where α and β are given in equations (4.33) and (4.34). (The details of this calculation, for the DS equations and then for the NLS equation, are rehearsed in Q4.6 but we shall record the salient features here.) From Q4.5 we have that

$$c_p \sim 1 - \frac{1}{6}\delta^2 k^2; \quad c_g \sim 1 - \frac{1}{2}\delta^2 k^2 \quad \text{as} \quad \delta \to 0 \tag{4.43}$$

(cf. equation (2.137) *et seq.*; the behaviours of c_p and c_g, as functions of δk, are also shown in Figure 4.1), and so

$$-2\mathrm{i}kc_p \sim -2\mathrm{i}k$$

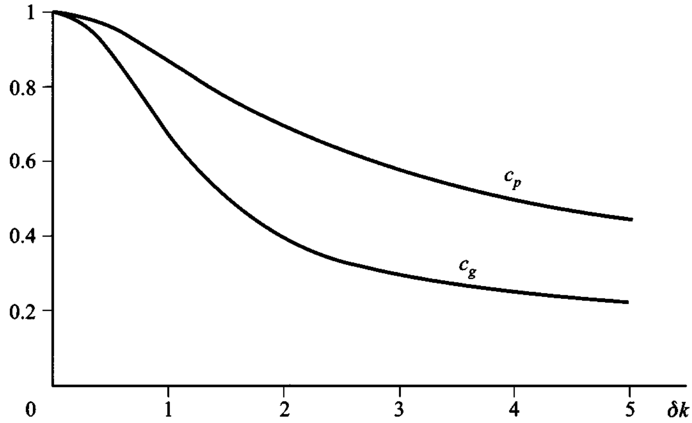

Figure 4.1. Plots of c_p and c_g as functions of $\delta k\,(\geq 0)$.

and

$$\begin{aligned}\alpha &\sim (1-\tfrac{1}{2}\delta^2k^2)^2-(1-\delta^2k^2)(1-\tfrac{1}{2}\delta^2k^2)^2\\ &\sim \delta^2k^2,\end{aligned}$$

both as $\delta \to 0$. Similarly we obtain

$$\begin{aligned}\beta \sim k^2\Bigg\{&\frac{1}{2}\left[1+9\left(\frac{1}{\delta k}+\frac{\delta k}{3}\right)^2-13(1-\frac{1}{2}\delta^2k^2)^2-2(\delta k)^4\right]\\ &-\left[2(1-\frac{1}{6}\delta^2k^2)+(1-\frac{1}{2}\delta^2k^2)(1-\frac{1}{2}\delta^2k^2)^2\right]^2\\ &\times\left[1-\left(1-\frac{1}{2}\delta^2k^2\right)^2\right]^{-1}\Bigg\}\\ \sim k^2&\left(\frac{9}{2\delta^2k^2}-\frac{9}{\delta^2k^2}\right)=-\frac{9}{2\delta^2},\end{aligned}$$

and hence our NLS equation, approximated for long waves, becomes

$$-2\mathrm{i}kA_{0\tau}+\delta^2k^2A_{0\zeta\zeta}-\frac{9}{2\delta^2}A_0|A_0|^2=0;$$

in the light of what we describe below, it is convenient to multiply by δ^2 to give

$$-2\mathrm{i}k\delta^2A_{0\tau}+\delta^4k^2A_{0\zeta\zeta}-\frac{9}{2}A_0|A_0|^2=0. \tag{4.44}$$

Now we turn to the examination of the KdV equation, for $\delta \to \infty$, which proves to be rather less straightforward.

Our KdV equation is

$$2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = 0 \tag{4.45}$$

(equation (3.28)), where

$$\xi = \frac{\varepsilon^{1/2}}{\delta}(x - t), \quad \tau = \frac{\varepsilon^{3/2}}{\delta}t,$$

and x, t are the original nondimensional variables. In order to produce the explicit dependence on δ in the equation, we write

$$\xi = \frac{\varepsilon^{1/2}}{\delta}\hat{\xi}, \quad \tau = \frac{\varepsilon^{1/2}}{\delta}\hat{\tau} \quad (\text{so } \hat{\xi} = x - t, \hat{\tau} = \varepsilon t)$$

to give

$$2\eta_{0\hat{\tau}} + 3\eta_0\eta_{0\hat{\xi}} + \frac{\lambda}{3}\eta_{0\hat{\xi}\hat{\xi}\hat{\xi}} = 0, \tag{4.46}$$

where $\lambda = \delta^2/\varepsilon$ (and we see here the relevance of the special choice $\delta^2 = \mathrm{O}(\varepsilon)$ alluded to above, and used in Section 2.9.1). In this form, the limiting process that allows us to describe short waves is $\lambda \to \infty$. However, we also require a solution which produces a direct correspondence with the form of solution used in the derivation of the NLS equation. Thus we seek a modulated harmonic wave solution of equation (4.45) with $\lambda \to \infty$; to this end we introduce

$$\left.\begin{aligned} X &= \hat{\xi} + c_{p_1}\lambda\hat{\tau} && [= x - (1 - \delta^2 c_{p_1})t]; \\ Z &= \lambda^{-1}(\hat{\xi} + c_{g_1}\lambda\hat{\tau}) && \left[= \frac{\varepsilon}{\delta^2}\{x - (1 - \delta^2 c_{g_1})t\}\right]; \\ T &= \lambda^{-1}\hat{\tau} && \left[= \frac{\varepsilon^2}{\delta^2}t\right], \end{aligned}\right\} \tag{4.47}$$

where the notation c_{p_1}, c_{g_1} indicates the correction – to be found – to the phase and group speeds, respectively. The KdV equation, (4.46), therefore becomes

$$\begin{aligned} 2(\lambda c_{p_1}\eta_{0X} + c_{g_1}\eta_{0Z} + \lambda^{-1}\eta_{0T}) + 3\eta_0(\eta_{0X} + \lambda^{-1}\eta_{0Z}) \\ + \frac{\lambda}{3}(\eta_{0XXX} + 3\lambda^{-1}\eta_{0ZXX} + 3\lambda^{-2}\eta_{0ZZX} + \lambda^{-3}\eta_{0ZZZ}) = 0 \end{aligned}$$

and we write the solution (cf. equation (4.36)) as

$$\eta_0 \sim \sum_{n=0}^{\infty} \lambda^{-n} \left\{ \sum_{m=0}^{n+1} A_{nm}(Z, T)E^m + \text{c.c.} \right\}, \tag{4.48}$$

where $E = \exp(\mathrm{i}kX)$, c.c. denotes the complex conjugate of the terms $m \geq 1$ and $A_{00} = 0$. The wave number of the fundamental is k, and this is a fixed parameter in the solution. We collect the various terms from the equation, and these are listed to the left; we obtain

$$\lambda E^1: \qquad c_{p_1} = k^2/6; \tag{4.49}$$

$$E^1: \qquad c_{g_1} = k^2/2 \text{ (provided } A_{01Z} \neq 0); \tag{4.50}$$

$$E^2: \qquad A_{12} = \tfrac{3}{2k^2} A_{01}^2;$$

$$\lambda^{-1}E^0: \quad A_{10} = -\tfrac{3}{k^2}|A_{01}|^2;$$

$$\lambda^{-1}E^1: \quad -2\mathrm{i}kA_{01T} + k^2 A_{01ZZ} - \tfrac{9}{2}A_{01}|A_{01}|^2 = 0 \tag{4.51}$$

where each earlier result is used, as necessary, to produce later results. We note that the corrections (to c_p and c_g) given in (4.49) and (4.50) agree precisely with the approximations used in the NLS equation; see equations (4.43). Equation (4.51) is the required NLS equation which describes the evolution of the amplitude of the fundamental; this is to be compared with the NLS equation valid for long waves, (4.44):

$$-2\mathrm{i}k\delta^2 A_{0\tau} + \delta^4 k^2 A_{0\zeta\zeta} - \frac{9}{2}A_0|A_0|^2 = 0. \tag{4.52}$$

When we introduce the variables (4.47) into (4.52), that is

$$\tau = \varepsilon^2 t = \delta^2 T, \quad \zeta = \varepsilon\{x - (1 - \frac{1}{2}\delta^2 k^2)t\} = \delta^2 Z,$$

we obtain

$$-2\mathrm{i}kA_{0T} + k^2 A_{0ZZ} - \frac{9}{2}A_0|A_0|^2 = 0,$$

which is precisely equation (4.51) (since $A_0 \equiv A_{01}$): the short-wave limit of the KdV equation (for harmonic waves) matches the long-wave limit of the NLS equation. The same match also occurs between the Davey–Stewartson equations, (4.37) and (4.38), and the 2D KdV equation, (3.30); this is discussed in Q4.7 and by Freeman & Davey (1975).

4.2 NLS and DS equations: some results from soliton theory

The Nonlinear Schrödinger (NLS) equation, which in one version is often written in the simple (normalised) form as

$$\mathrm{i}u_t + u_{xx} + u|u|^2 = 0 \tag{4.53}$$

(see Q4.8 and below), is one of the *completely integrable* equations; we will call (4.53) the NLS + equation (see below). The method of solution involves an important extension of that used for the solution of the KdV equation (described in Section 3.3.1). The central idea is to replace the scalar functions F and K (as used, for example, in equations (3.51) and (3.52)) by 2×2 matrix functions, in an approach developed first by Zakharov & Shabat (1972); see also Shabat (1973), Zakharov & Shabat (1974). On this basis we shall present the general method of solution for the NLS equation, written both in the form (4.53) and also in the (normalised) form (called NLS–)

$$\mathrm{i}u_t + u_{xx} - u|u|^2 = 0. \tag{4.54}$$

It turns out that the relative sign of the terms u_{xx} and $u|u|^2$ is important in determining the essential character of the solution of the NLS equation (hence: NLS+, NLS–); for some simple solutions see Q4.9–Q4.12. To change the sign of the term $\mathrm{i}u_t$ is simply equivalent to taking the complex conjugate of the equation. We shall later mention the Davey–Stewartson equations, and how the bilinear method (see Section 3.3.3) and conservation laws (Section 3.3.4) are relevant to this NLS family of equations.

4.2.1 Solution of the Nonlinear Schrödinger equation

We follow the notation used in Section 3.3.1; but here, $F(x, z, t)$ is a 2×2 matrix function which satisfies the pair of (matrix) equations

$$\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} F_x + F_z \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} = 0; \quad F_{xx} - F_{zz} - \mathrm{i}\alpha F_t = 0, \tag{4.55}$$

where l, m and α are arbitrary real constants. The 2×2 function $K(x, z; t)$ is then a solution of the *matrix* Marchenko equation

$$K(x, z; t) + F(x, z, t) + \int_x^\infty K(x, y; t) F(y, z, t)\,\mathrm{d}y = 0. \tag{4.56}$$

We write this solution as

$$K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad \text{then} \quad u(x, t) = b(x, x; t) \tag{4.57}$$

is a solution of the general NLS± equation

$$\mathrm{i}\alpha(l-m)u_t + (l+m)u_{xx} \pm \frac{2}{lm}(l-m)(l^2-m^2)u|u|^2 = 0; \tag{4.58}$$

see Q4.17. The choice of signs in equation (4.58) is governed by the two possibilities for $c(x, x; t)$, namely $\pm u^*$ (where the asterisk denotes the complex conjugate). Armed with this information, we will describe how the equations are solved in order to generate the solitary-wave solution of the NLS+ equation; we shall then indicate how this approach is extended to embrace the N-soliton solution.

Example: solitary-wave solution

The required solutions are obtained when we set

$$F = \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix};$$

then the first of equations (4.55) yields

$$f = f(mx - lz, t) \quad g = g(lx - mz, t),$$

but f and g are otherwise arbitrary functions. The second of equations (4.55) is now satisfied by the choice of the (single) exponential solutions

$$\left. \begin{aligned} f &= f_0 \exp\{\lambda(mx - lz) + \mathrm{i}\lambda^2(l^2 - m^2)t/\alpha\}; \\ g &= g_0 \exp\{\mu(lx - mz) + \mathrm{i}\mu^2(m^2 - l^2)t/\alpha\}, \end{aligned} \right\} \tag{4.59}$$

where f_0, g_0, λ and μ are arbitrary constants. This introduction of an exponential solution is to be compared with equation (3.55), for the KdV equation.

The matrix Marchenko equation, (4.56), with K given by the expression in (4.57), becomes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} + \int_x^\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} \mathrm{d}y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.60}$$

where we have suppressed the arguments of the functions; the functions f and g are given by (4.59). Thus, from equation (4.60), we obtain four scalar equations:

$$a(x, z; t) + \int_x^\infty b(x, y; t) g_0 \exp\{\mu(ly - mz) + \mathrm{i}\mu^2(m^2 - l^2)t/\alpha\} \mathrm{d}y = 0; \quad (4.61)$$

$$b(x, z; t) + f_0 \exp\{\lambda(mx - lz) + \mathrm{i}\lambda^2(l^2 - m^2)t/\alpha\}$$
$$+ \int_x^\infty a(x, y; t) f_0 \exp\{\lambda(my - lz) + \mathrm{i}\lambda^2(l^2 - m^2)t/\alpha\} \mathrm{d}y = 0, \quad (4.62)$$

and two similar equations for $c(x, z; t)$ and $d(x, z; t)$ (whose identification is left as an exercise in Q4.18). The integral equations, (4.61) and (4.62), clearly possess solutions

$$a(x, z; t) = \mathrm{e}^{-\mu m z} M(x; t), \quad b(x, z; t) = \mathrm{e}^{-\lambda l z} L(x; t),$$

respectively, and so (4.61) and (4.62) yield

$$M + g_0 L \int_x^\infty \exp\{l(\mu - \lambda)y + \mathrm{i}\mu^2(m^2 - l^2)t/\alpha\} \mathrm{d}y = 0;$$

$$L + f_0 \exp\{\lambda m x + \mathrm{i}\lambda^2(l^2 - m^2)t/\alpha\}$$
$$+ f_0 M \int_x^\infty \exp\{m(\lambda - \mu)y + \mathrm{i}\lambda^2(l^2 - m^2)t/\alpha\} \mathrm{d}y = 0.$$

These two equations exist only if

$$\mathscr{R}\{l(\mu - \lambda)\} < 0 \quad \text{and} \quad \mathscr{R}\{m(\lambda - \mu)\} < 0, \quad (4.63)$$

for otherwise the integrals would not be finite. These two conditions imply that l and m must be of opposite sign; let us, for ease of further calculation, choose

$$l = 2, \quad m = -1 \quad \text{and} \quad \alpha = \frac{1}{3}. \quad (4.64)$$

Equation (4.58), with these choices, then becomes

$$\mathrm{i}u_t + u_{xx} \mp 9u|u|^2 = 0 \quad (4.65)$$

(and consequently equations (4.53) and (4.54) are recovered if we transform $u \to u/3$). The solution for $L(x; t)$, with $\mathscr{R}(\mu - \lambda) < 0$, is now obtained directly as

$$L(x; t) = \frac{f_0 \exp(-\lambda x + 9\mathrm{i}\lambda^2 t)}{\frac{1}{2} f_0 g_0 (\lambda - \mu)^{-2} \exp\{3(\mu - \lambda)[x - 3\mathrm{i}(\lambda + \mu)t]\} - 1}$$

and a convenient choice affording further simplification is

$$\frac{1}{2} f_0 g_0 (\lambda - \mu)^{-2} = -1; \tag{4.66}$$

the solution that we seek is therefore

$$\begin{aligned} u(x, t) = b(x, x; t) &= \mathrm{e}^{-\lambda l x} L(x; t) \\ &= \frac{-f_0 \exp\{-3\lambda(x - 3\mathrm{i}\lambda t)\}}{1 + \exp\{3(\mu - \lambda)[x - 3\mathrm{i}(\lambda + \mu)t]\}}. \end{aligned} \tag{4.67}$$

Finally, we introduce real parameters k and p such that

$$\lambda = k + \mathrm{i}p, \quad \mu = -k + \mathrm{i}p \quad (k > 0)$$

and then $f_0 g_0 = -8k^2$.

The corresponding calculation for $d(x, z; t)$ and $c(x, z; t)$ can be followed through (see Q4.18, Q4.19); when we impose the condition $c(x, x; t) = -u^*$, we find that $g_0 = -f_0^*$ and hence $f_0 = \pm 2\sqrt{2}k$ (choosing a real f_0). The solution (4.67) now becomes

$$u(x, t) = \pm\sqrt{2}k \exp\{-3\mathrm{i}px + 9\mathrm{i}(k^2 - p^2)t\} \operatorname{sech}(3kx + 18kpt)$$

and if we identify

$$a = 3\sqrt{2}k, \quad c = -6p \quad \text{and} \quad n = \frac{1}{2}(a^2 + \frac{1}{2}c^2)$$

we obtain the solitary-wave solution

$$u(x, t) = \pm a \exp\{\mathrm{i}[\frac{1}{2}c(x - ct) + nt]\} \operatorname{sech}\{a(x - ct)/\sqrt{2}\} \tag{4.68}$$

of the NLS+ equation

$$\mathrm{i}u_t + u_{xx} + u|u|^2 = 0,$$

which in the form (4.68) is discussed in Q4.9. The NLS solitary wave is an oscillatory wave packet which propagates at a speed c, the underlying oscillation being governed by the frequency n (which is a function of the wave amplitude and speed). An example of this wave is given in Figure 4.2 (for the choice $a = 1$, $c = 10$ and at two different times) where we have elected to show only the real part of u. The imaginary part is, of course, very similar, and the modulus of u is simply

$$|u| = a \operatorname{sech}\{a(x - ct)/\sqrt{2}\},$$

a sech profile. Other simple solutions of the NLS+ equation are possible; see Q4.11 and Q4.12. Examples of these two solutions (the *Ma* and

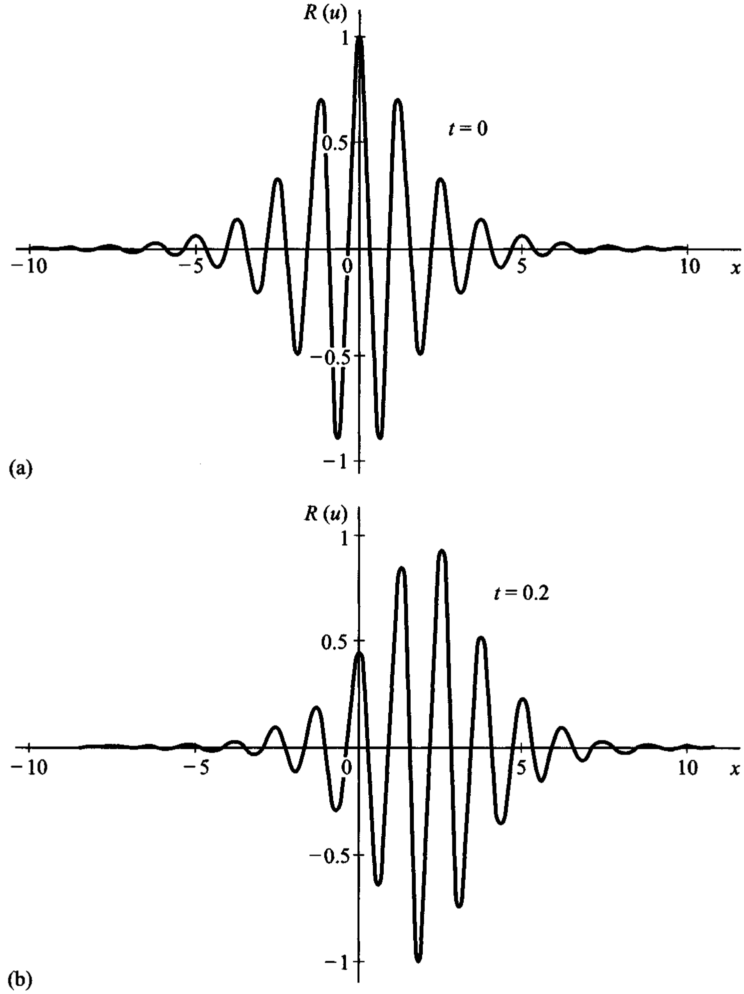

Figure 4.2. Real part of the solitary-wave solution, (4.68), of the NLS+ equation, for $a = 1$ and $c = 10$, at times $t = 0$(a), 0.2(b).

rational-cum-oscillatory waves) are depicted in Figure 4.3. As the figures clearly demonstrate, these solutions take the form of standing waves; not surprisingly, the solutions of most interest to use are those that represent the propagation – and interaction – of the nonlinear waves. Finally, we comment that the N-soliton solution of the NLS equation is obtained in the obvious way. That is, for the function f, for example (see equation (4.59)), we write

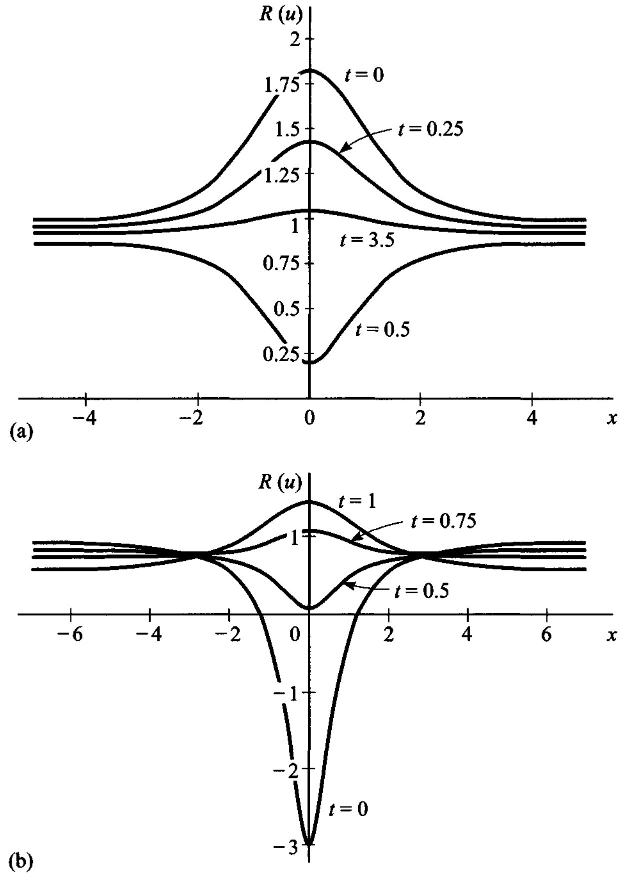

Figure 4.3. (a) Real part of the Ma solitary wave, given in Q4.11, for $a = m = 1$, at times $t = 0$, 0.25, 0.35, 0.5. (b) Real part of the rational-cum-oscillatory solution, given in Q4.12, at times $t = 0$, 0.5, 0.75, 1.

$$f(x, z; t) = \sum_{n=1}^{N} f_n \exp\{\lambda_n(mx - lz) + \mathrm{i}\lambda_n^2(l^2 - m^2)t/\alpha\}, \tag{4.69}$$

where f_n and λ_n are arbitrary constants. We shall write more about these solutions, and their interpretation in the context of water-wave theory, later. Here, we comment that the N-soliton solution is more readily obtained by direct methods, such as Hirota's bilinear method, which we now present.

4.2.2 Bilinear method for the NLS equation

In Section 3.3.3 we introduced Hirota's bilinear method for the KdV equation, which led to the bilinear form of that equation:

$$(\mathrm{D}_x\mathrm{D}_t + \mathrm{D}_x^4)(f \cdot f) = 0.$$

This equation and its solution will be found in (3.74) *et seq*. Of some importance for us was that this approach provided a rather direct route to the construction of N-soliton solutions. Furthermore, we also gave the bilinear form of a number of other equations that belong to the KdV family of completely integrable equations.

Now, the NLS equation is a completely integrable equation (as we mentioned in Section 4.2.1), so it is no surprise to learn that this equation can be expressed in a bilinear form. It can be shown (see Q4.20) that the NLS equation

$$\mathrm{i}u_t + u_{xx} + \varepsilon u|u|^2 = 0 \quad (\varepsilon = \pm 1) \tag{4.70}$$

with $u = g/f$, where f is a real function, can be written as a *pair* of bilinear equations:

$$(\mathrm{i}\mathrm{D}_t + \mathrm{D}_x^2)(g \cdot f) = 0; \quad \mathrm{D}_x^2(f \cdot f) = \varepsilon|g|^2. \tag{4.71}$$

We observe, just as we found with the KdV family, that the linearised operator which appears in the NLS equation (that is, $\mathrm{i}\partial/\partial t + \partial^2/\partial x^2$) has a direct counterpart in equations (4.71), namely $(\mathrm{i}\mathrm{D}_t + \mathrm{D}_x^2)$. As an example of the method of solution here, we seek the solitary-wave solution of (4.70) via (4.71).

From equations (4.67) or (4.68), we see that an obvious way to proceed is, first, to introduce

$$\theta = kx + \omega t + \alpha, \tag{4.72}$$

where k, ω and α are complex constants, and then to write

$$g = \mathrm{e}^{\theta}, \quad f = 1 + A\exp(\theta + \theta^*) \tag{4.73}$$

where A is a real constant (and the asterisk denotes the complex conjugate). On recalling the properties of the bilinear operator (described in Section 3.3.3 and in Q3.24), we find that the second equation in (4.71) gives (the non-zero contributions)

$$\mathrm{D}_x^2\{1 \cdot A\exp(\theta + \theta^*) + A\exp(\theta + \theta^*) \cdot 1\} = \varepsilon\exp(\theta + \theta^*),$$

so

$$A = \frac{1}{2}\varepsilon(k + k^*)^{-2}. \qquad (4.74)$$

The first equation of (4.71) becomes

$$(\mathrm{iD}_t + \mathrm{D}_x)\{\mathrm{e}^\theta \cdot 1 + \mathrm{e}^\theta \cdot A \exp(\theta + \theta^*)\} = 0$$

which yields

$$(\mathrm{i}\omega + k^2)\mathrm{e}^\theta + A\left[\mathrm{i}(\omega - \omega - \omega^*) + (k - k - k^*)^2\right]\exp(2\theta + \theta^*) = 0,$$

which requires

$$\mathrm{i}\omega + k^2 = 0 \quad \text{and} \quad -\mathrm{i}\omega^* + k^{*2} = 0.$$

These are clearly consistent with

$$\omega = \mathrm{i}k^2; \qquad (4.75)$$

thus we have a solution

$$u = g/f = \mathrm{e}^\theta \Big/ \left\{1 + \frac{1}{2}\varepsilon(k + k^*)^{-2}\exp(\theta + \theta^*)\right\} \qquad (4.76)$$

with

$$\theta = kx + \mathrm{i}k^2 t + \alpha,$$

where k and α are arbitrary (complex) parameters. It is clear that (4.76) provides a bounded solution only in the case $\varepsilon = +1$; that is, for the NLS+ equation; cf. Q4.9 and Q4.10. Then, for this case, solution (4.76) can be recast precisely in the form of (4.68) if we make the identification

$$k = \frac{a}{\sqrt{2}} + \mathrm{i}\frac{c}{2} \qquad (4.77)$$

and choose α to be real, and such that

$$\sqrt{A}\,\mathrm{e}^\alpha = 1. \qquad (4.78)$$

(The rôle of α is simply to provide a constant phase-shift in the solution.) The use of (4.77) and (4.78) in (4.76) gives, after a little manipulation,

$$u = \pm a \exp\left\{\mathrm{i}\left[\frac{1}{2}c(x - ct) + nt\right]\right\}\operatorname{sech}\left\{a(x - ct)/\sqrt{2}\right\},$$

with $n = \frac{1}{2}(a^2 + \frac{1}{2}c^2)$, all exactly as in (4.68). In summary, therefore, the solitary-wave solution of the NLS+ equation can be expressed as

$$g = \mathrm{e}^{\theta}, \quad f = 1 + A\exp(\theta + \theta^*), \quad \theta = kx + \mathrm{i}k^2 t + \alpha.$$

The method that we have described can be extended to obtain the N-soliton solution, although the calculation – even for the case $N = 2$ – is considerably more involved than for the KdV family of equations. We shall present the results that produce the 2-soliton solution, but the details are left as an exercise (Q4.30). First, we write

$$g = E_1(1 + b_2 E_2 E_2^*) + E_2(1 + b_1 E_1 E_1^*) \tag{4.79}$$

with

$$E_m = \exp(k_m x + \mathrm{i}k_m^2 t + \alpha_m), \quad m = 1, 2,$$

where b_m are constants. Correspondingly, we have

$$f = 1 + f_1 E_1 E_1^* + f_2 E_2 E_2^* + cE_1 E_2^* + c^* E_1^* E_2 + dE_1 E_1^* E_2 E_2^*, \tag{4.80}$$

where f_m, c and d are constants. These two expressions are substituted into

$$(\mathrm{iD}_t + \mathrm{D}_x^2)(g \cdot f) = 0, \quad \mathrm{D}_x^2(f \cdot f) = |g|^2;$$

we find that the given f and g satisfy both equations provided

$$f_m = \frac{1}{2}(k_m + k_m^*)^{-2}; \quad c = \frac{1}{2}(k_1 + k_2^*)^{-2}; \quad d = \left(\frac{k_1 + k_2^*}{k_1^* + k_2}\right)^2 b_1^* b_2$$

with

$$b_m = \frac{(k_1 - k_2)^2}{2(k_m + k_m^*)^2(k_m^* + k_n)^2} \quad (n = 1, 2; n \neq m).$$

The solution, (4.79) with (4.80), represents the interaction of two solitons which asymptotically take the form

$$u_m = a_m \exp\left\{\mathrm{i}\left[\frac{1}{2}c_m(x - c_m t) + n_m t\right]\right\}\operatorname{sech}\left\{a_m(x - c_m t)/\sqrt{2}\right\} \tag{4.81}$$

where

$$k_m = \frac{a_m}{\sqrt{2}} + \mathrm{i}\frac{c_m}{2} \quad \text{with} \quad n_m = \frac{1}{2}(a_m^2 + \frac{1}{2}c_m^2), \tag{4.82}$$

for $m = 1, 2$. An example of this solution is depicted in Figure 4.4 (where we have chosen $a_1 = \sqrt{2}$, $a_2 = 2\sqrt{2}$, $c_1 = -2$ and $c_2 = 2$ and we have

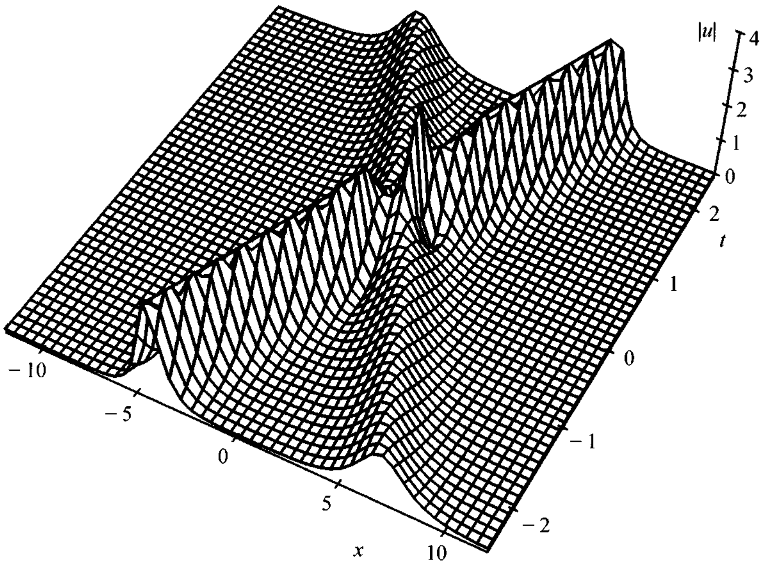

Figure 4.4. Two-soliton solution of the NLS+ equation (based on equations (4.79) and (4.80)); $|u|$ is plotted here for the case $a_1 = \sqrt{2}$, $a_2 = 2\sqrt{2}$, $c_1 = -2$, $c_2 = 2$.

plotted $|u|$); the interaction, together with one of the resulting phase shifts, are clearly shown in this figure.

Finally, before we leave this 2-soliton example altogether, we make an important observation that distinguishes this type of interaction from the KdV-type. In the case of KdV solitons (and others of this family), each separate soliton must have its own distinct speed at infinity. (The general 2-soliton solution of the KdV equation, (3.58), with $k_1 = k_2$ merely recovers the solitary-wave solution with parameter k_1.) However, the 2-soliton solution here, (4.79) and (4.80), contains essentially the four real parameters a_m, c_m $(m = 1, 2)$ given by (4.82) (since the α_m simply provide arbitrary phase shifts at a prescribed instant in time). The speed of the NLS+ soliton – the envelope function is the relevant part – is given by c_m; see (4.81). If we set $c_1 = c_2$, and retain $a_1 \neq a_2$, the two solitons remain distinct but *do not move apart*: they stay bound together and forever interact. The object so produced is itself a new type of solitary wave (with three parameters: a_1, a_2 and $c_1 = c_2$); it is called a *bound soliton* or *bi-soliton*, and it can interact with other similar or different solitons. These different solitons might be the classical ones for the NSL+

equation (such as given in (4.81)) or higher-order bound solitons formed by producing the N-soliton solution (following (4.79) and (4.80)) and then choosing $c_m = c$, for $m = 1, 2, \ldots, N$. A description of the bi-soliton solution that is obtained from (4.79) and (4.80) is left as an exercise (Q4.31); an example of a bi-soliton solution is given in Figure 4.5 (where we have used $a_1 = \sqrt{2}$, $a_2 = 2\sqrt{2}$, $c_1 = c_2 = -2$). The bound, but varying, nature of this solution is quite evident from the figure.

It is clear from our examination of some of the solutions of the NLS+ equation that it possesses a very rich set of solutions – far more than for the KdV equation and other members of that family. Of course, which solution is the relevant one in a given situation is controlled by the precise details of the initial profile. As a significant addition to this brief description of the solutions of the NLS+ equation, we shall later present an important application of the equation (to study the stability of the Stokes wave; Section 4.3.1).

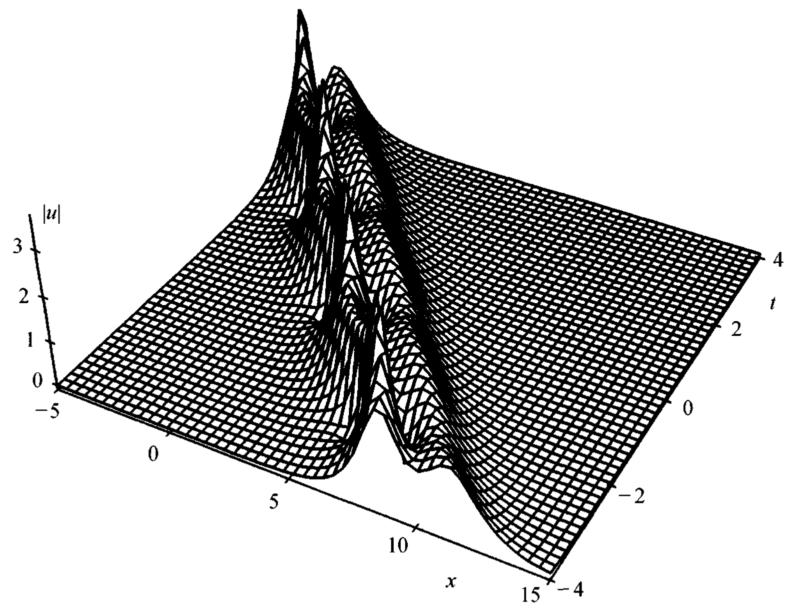

Figure 4.5. Bi-soliton (or *bound* soliton) of the NLS+ equation (given in Q4.31); $|u|$ is plotted here for the case $a_1 = \sqrt{2}$, $a_2 = 2\sqrt{2}$, $c_1 = c_2 = -2$.

4.2.3 Bilinear form of the DS equations for long waves

The long wave ($\delta \to 0$) approximation of the Davey–Stewartson equations (see Section 4.1.2), which is discussed in Q4.6, can be written as

$$-2\mathrm{i}kA_{0\tau} + \delta^2k^2A_{0\zeta\zeta} - A_{0YY} + \frac{9}{2\delta^2}A_0|A_0|^2 + 3k^2A_0f_{0\zeta} = 0 \qquad (4.83)$$

with

$$\delta^2k^2f_{0\zeta\zeta} + f_{0YY} = -3(|A_0|^2)_\zeta, \qquad (4.84)$$

and these equations possess a compact bilinear representation. However, it is necessary first to change the variables (both dependent and independent) by introducing a more convenient set; we follow the ideas described by Anker & Freeman (1978) and Freeman (1984).

First we define

$$f_{0\zeta} = \phi_{YY} + \lambda|A_0|^2$$

where $\phi = \phi(\zeta, Y, \tau)$ and λ is a (complex) constant to be determined; then we see that equation (4.84) can be written, after one differentiation with respect to ζ, as

$$\delta^2k^2f_{0\zeta\zeta\zeta} + f_{0YY\zeta} = -3(|A_0|^2)_{\zeta\zeta},$$

so

$$\delta^2k^2(\phi_{YY} + \lambda|A_0|^2)_{\zeta\zeta} + (\phi_{YY} + \lambda|A_0|^2)_{YY} = -3(|A_0^2|)_{\zeta\zeta}.$$

We choose $\delta^2k^2\lambda = -3$ (so λ turns out to be real), to leave

$$\delta^2k^2\phi_{YY\zeta\zeta} + \phi_{YYYY} - \frac{3}{\delta^2k^2}|A_0|^2_{YY} = 0$$

or

$$\delta^2k^2\phi_{\zeta\zeta} + \phi_{YY} = \frac{3}{\delta^2k^2}|A_0|^2 \qquad (4.85)$$

when we integrate and then invoke decay conditions at infinity. In equation (4.83) we substitute for $f_{0\zeta}$ to give

$$-2\mathrm{i}kA_{0\tau} + \delta^2k^2A_{0\zeta\zeta} - A_{0YY} + \frac{3}{2}k^2A_0\left(2\phi_{YY} - \frac{3}{\delta^2k^2}|A_0|^2\right) = 0$$

and then upon substituting for $|A_0|^2$ from (4.85) this yields

$$-2\mathrm{i}kA_{0\tau} + \delta^2k^2A_{0\zeta\zeta} - A_{0YY} + \frac{3}{2}k^2A_0(\phi_{YY} - \delta^2k^2\phi_{\zeta\zeta}) = 0. \qquad (4.86)$$

At this stage we define

$$x = \frac{\zeta}{\delta k} - Y, \quad y = \frac{\zeta}{\delta k} + Y, \tag{4.87}$$

and hence equations (4.85) and (4.86) become

$$2(\phi_{xx} + \phi_{yy}) = \frac{3}{\delta^2 k^2}|A_0|^2;$$

$$-\mathrm{i}kA_{0\tau} + 2A_{0xy} + 3k^2 A_0 \phi_{xy} = 0.$$

Let us write

$$\phi_x = u, \quad \phi_y = -v$$

(where u and v are real which, with the definition of λ from above, implies then f_0 is real), then we obtain the pair of equations

$$2\delta^2 k^2 (u_x - v_y) = 3|A_0|^2;$$

$$-\mathrm{i}kA_{0\tau} + 2A_{0xy} + 3k^2 A_0 u_y = 0.$$

Finally, we define the complex function

$$Z = u + \mathrm{i}v$$

so that

$$u = \frac{1}{2}(Z + Z^*) \quad \text{and} \quad u_x - v_y = Z_x + \mathrm{i}Z_y$$

(since $u_y = -v_x$); our pair of equations therefore becomes

$$Z_x + \mathrm{i}Z_y = \frac{3}{2\delta^2 k^2}|A_0|^2;$$

$$-\mathrm{i}\frac{k}{2}A_{0\tau} + A_{0xy} + \frac{3k^2}{4}A_0(Z + Z^*)_y = 0$$

which, with the scaling transformations

$$\tau \to -\frac{2\tau}{k}, \quad Z \to \frac{8}{3k^2}Z, \quad A_0 \to \frac{4}{3\delta}A_0$$

yields

$$\mathrm{i}A_{0\tau} + A_{0xy} + 2A_0(Z + Z^*)_y = 0; \quad Z_x + \mathrm{i}Z_y = |A_0|^2; \tag{4.88}$$

this is essentially the form of the equations discussed by Anker & Freeman (1978) and Freeman (1984). (Our scaling transformation also involves a change in sign of τ; this is avoided if the second equation is expressed in terms of the conjugate, A_0^*.) It can be shown (Q4.27) that equations (4.88) possess a simple bilinear representation:

$$(\mathrm{iD}_\tau + \mathrm{D}_x\mathrm{D}_y)(g\cdot f) = 0; \quad (\mathrm{D}_x^2 + \mathrm{D}_y^2)(f\cdot f) = 2|g|^2 \tag{4.89}$$

where

$$A_0 = \frac{g}{f}, \quad Z = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right)\ln f \quad (f \text{ real});$$

cf. equations (4.71) for the NLS equation.

A simple solution of equations (4.89) is obtained by following the development that has been described for the NLS equation; see equations (4.73) *et seq*. Thus, if we set

$$g = \mathrm{e}^\theta, \quad f = 1 + \mu\exp(\theta + \theta^*),$$

where $\theta = kx + ly + \omega\tau + \alpha$ and μ is a real constant, we find (Q4.28) that, for example,

$$A_0 = e^\theta/\left\{1 + [(k+k^*)^2 + (l+l^*)^2]^{-1}\exp(\theta+\theta^*)\right\} \tag{4.90}$$

with

$$\theta = kx + ly + \mathrm{i}kl\tau + \alpha,$$

where k, l and α are arbitrary (complex) constants. Solution (4.90) is the solitary wave solution of the long-wave Davey-Stewartson equations (although we should remember that x and y are not the physical coordinates used to describe the horizontal plane in which the wave propagates; see (4.87)). This solution, (4.90), should be compared (see Q4.29) with that discussed in Q4.26; the generalisation to N solitons follows the method adopted for the NLS equation, and presented in equations (4.79) and (4.80).

4.2.4 Conservation laws for the NLS and DS equations

All completely integrable equations possess an infinite number of conservation laws, the first few of which – certainly the first three – have simple and direct physical interpretations. These ideas were introduced and explored in the context of the KdV equation (and its associated family of equations) in Section 3.3.4. We now describe how the corresponding picture is developed for the NLS equation

$$\mathrm{i}u_t + u_{xx} + \varepsilon u|u|^2 = 0 \quad (\varepsilon = \pm 1). \tag{4.91}$$

The construction of the conservation laws for this equation is fairly straightforward, although the procedure becomes progressively more tedious for the 'higher' laws.

First, we write down the equation which is satisfied by u^*, the conjugate of u, namely

$$-\mathrm{i}u_t^* + u_{xx}^* + \varepsilon u^*|u|^2 = 0, \tag{4.92}$$

and then we form $u^* \times (4.91) - u \times (4.92)$ to give

$$\mathrm{i}(u^*u_t + uu_t^*) + u^*u_{xx} - uu_{xx}^* = 0.$$

This equation is immediately

$$\mathrm{i}\frac{\partial}{\partial t}(uu^*) + \frac{\partial}{\partial x}(u^*u_x - uu_x^*) = 0,$$

which we now integrate over all x; provided that ambient conditions exist at infinity (so that conditions at $\pm\infty$ are identical), we obtain

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} |u|^2\mathrm{d}x = 0,$$

so

$$\int_{-\infty}^{\infty} |u|^2\mathrm{d}x = \text{constant}. \tag{4.93}$$

Equation (4.93) is the first conservation law (for both versions of the NLS equation). Based on our experience with the KdV equation (see Section 3.3.4), we would expect this law to be associated with the conservation of mass; we shall confirm this interpretation shortly.

A second conservation law is derived by forming $u_x^* \times (4.91) + u_x \times (4.92)$, to produce

$$\mathrm{i}(u_x^*u_t - u_xu_t^*) + u_x^*u_{xx} + u_xu_{xx}^* + \varepsilon(u_x^*u + u_xu^*)|u|^2 = 0. \tag{4.94}$$

Further, we also construct $u^* \times (4.91)_x + u \times (4.92)_x$ to give

$$\mathrm{i}(u^*u_{xt} - uu_{xt}^*) + u^*u_{xxx} + uu_{xxx}^* + \varepsilon\left\{u^*(u|u|^2)_x + u(u^*|u|^2)_x\right\} = 0 \tag{4.95}$$

and then we form (4.94) – (4.95):

$$\mathrm{i}\frac{\partial}{\partial t}(uu_x^* - u^*u_x) + \frac{\partial}{\partial x}(u_xu_x^*) - (u^*u_{xxx} + uu_{xxx}^*) \\ + \varepsilon|u|^2(uu^*)_x - \varepsilon\left\{(uu^*)_x|u|^2 + 2uu^*(|u|^2)_x\right\} = 0.$$

This equation is re-expressed as

$$\mathrm{i}\frac{\partial}{\partial t}(uu_x^* - u^*u_x) + \frac{\partial}{\partial x}(u_xu_x^*) - \frac{\partial}{\partial x}(u^*u_{xx} + uu_{xx}^*) + \frac{\partial}{\partial x}(u_xu_x^*) - 2\varepsilon|u|^2(|u|^2)_x = 0$$

which is

$$\mathrm{i}\frac{\partial}{\partial t}(uu_x^* - u^*u_x) + \frac{\partial}{\partial x}\{2u_xu_x^* - (u^*u_{xx} + uu_{xx}^*) - \varepsilon|u|^4\} = 0.$$

Hence, with ambient conditions existing at infinity (as invoked above), this yields

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}(uu_x^* - u^*u_x)\,\mathrm{d}x = 0,$$

so

$$\int_{-\infty}^{\infty}(uu_x^* - u^*u_x)\,\mathrm{d}x = \text{constant} \tag{4.96}$$

is the second conservation law.

A third conservation law, whose derivation is left as an exercise (see Q4.32), is

$$\int_{-\infty}^{\infty}\left\{|u_x|^2 - \frac{1}{2}\varepsilon|u|^4\right\}\mathrm{d}x = \text{constant}; \tag{4.97}$$

this is the simplest law that includes $\varepsilon\,(=\pm1)$ and therefore takes different forms for the two NLS equations (NLS+, NLS–). We have produced the first three conservation laws; that an infinity exists is proved by Zakharov & Shabat (1972), and a little further exploration is provided in Q4.32 and Q4.33. The relation between these conservation laws, and the conserved quantities that arise directly from the governing equations, will now be briefly investigated.

The conservation of mass, equation (3.86), is

$$\eta_t + \frac{\partial}{\partial x}\left(\int_0^{1+\varepsilon\eta} u\,\mathrm{d}z\right) = 0; \tag{4.98}$$

this must be written in terms of the variables used in the derivation of the NLS equation (see (4.2)). Equation (4.98) therefore becomes

$$\varepsilon^2\eta_\tau - \varepsilon c_g \eta_\zeta - c_p \eta_\xi + \left(\frac{\partial}{\partial \xi} + \varepsilon \frac{\partial}{\partial \zeta}\right)\left\{\int_0^{1+\varepsilon\eta} (\phi_\xi + \varepsilon\phi_\zeta)\mathrm{d}z\right\} = 0 \tag{4.99}$$

where we have introduced $u = \phi_x = \phi_\xi + \varepsilon\phi_\zeta$. The conservation law we require is expressed in terms of (τ, ζ); see our NLS equation for water waves, (4.32). First, therefore, equation (4.99) is integrated in ξ over one period (for example, from 0 to $2\pi/k$) to give

$$\varepsilon\bar{\eta}_\tau - c_g\bar{\eta}_\zeta + \frac{\partial}{\partial\zeta}\left\{\overline{\int_0^{1+\varepsilon\eta} (\phi_\xi + \varepsilon\phi_\zeta)\,\mathrm{d}z}\right\} = 0 \tag{4.100}$$

where the overbar denotes the integral in ξ, and we have used the property that our solution is strictly periodic in ξ, at fixed τ, ζ. Now we integrate equation (4.100) over all ζ, and again use ambient conditions as $\zeta \to \pm\infty$, to obtain

$$\int_{-\infty}^{\infty} \bar{\eta}\,\mathrm{d}\zeta = \text{constant}, \tag{4.101}$$

which is the appropriate form of the conservation of mass that we need here. However, from equation (4.14) (see also equations (4.36)), we have

$$\eta \sim \sum_{n=0}^{\infty} \varepsilon^n \left\{\sum_{m=0}^{n+1} A_{nm}(\zeta, \tau)E^m + \text{c.c.}\right\}, \quad \varepsilon \to 0,$$

with $A_{00} = 0$; thus

$$\bar{\eta} = \int_0^{2\pi/k} \eta\,\mathrm{d}\xi \sim \frac{2\pi}{k}\sum_{n=1}^{\infty} \varepsilon^n A_{n0}. \tag{4.102}$$

The dominant behaviour (as $\varepsilon \to 0$) with (4.102) used in (4.101) then yields

$$\int_{-\infty}^{\infty} A_{10}\,\mathrm{d}\zeta = \text{constant},$$

and from equation (4.22) we have

$$A_{10} = -\frac{2\delta k}{\sinh \delta k}|A_0|^2 + c_g f_{0\zeta}$$

(where A_0 is written for A_{01}), and so

$$\int_{-\infty}^{\infty} |A_0|^2 \mathrm{d}\zeta = \text{constant}$$

which is therefore the conservation of mass for our NLS equation, (4.32), for $A_0(\zeta, \tau)$; this recovers our first conservation law (4.93), where for u read A_0.

The equivalent calculation for the conservation of momentum, starting from equation (3.92):

$$\frac{\partial}{\partial t}\left(\int_0^{1+\varepsilon\eta} u\,\mathrm{d}z\right) + \frac{\partial}{\partial x}\left\{\int_0^{1+\varepsilon\eta} (\varepsilon u^2 + p)\mathrm{d}z - \frac{1}{2}\varepsilon\eta^2\right\} = 0,$$

and leading to a conservation law of the type given in (4.96), is left as an exercise (Q4.34). The correspondence between the conservation of energy for the wave motion, and the third conservation law (4.97), is obtained in a similar way (although in this case the connection is less easily confirmed).

Before we leave the discussion of the conservation laws altogether, we briefly mention the situation with regard to the Davey–Stewartson equations. These are by no means straightforward to analyse, and this is because the waves depend on two variables (ζ and Y) in the horizontal plane. (Similar difficulties were encountered with the 2D KdV equation; see Section 3.3.4, equation (3.98) *et seq.*) However, we provide the first stage in the discussion of these equations; let us write them in the form (cf. equations (4.37), (4.38))

$$f_{0YY} + \lambda f_{0\zeta\zeta} = \mu(|A_0|^2)_\zeta$$

$$-\mathrm{i}\alpha A_{0\tau} + \beta A_{0\zeta\zeta} - \gamma A_{0YY} + \delta A_0|A_0|^2 + A_0 f_{0\zeta} = 0$$

where λ, μ, α, β, γ and δ are real constants. We take f_0 to be a real function (as we found for the solutions described in Section 4.2.3), so the conjugate of the second equation becomes

$$\mathrm{i}\alpha A^*_{0\tau} + \beta A^*_{0\zeta\zeta} - \gamma A^*_{0YY} + \delta A^*_0|A_0|^2 + A^*_0 f_{0\zeta} = 0.$$

The procedure adopted for the NLS equation (see equation (4.92) *et seq.*) then gives

$$\mathrm{i}\alpha\frac{\partial}{\partial\tau}(A_0A_0^*)+\beta\frac{\partial}{\partial\zeta}(A_0A_{0\zeta}^*-A_0^*A_{0\zeta})+\gamma\frac{\partial}{\partial Y}(A_0^*A_{0Y}-A_0A_{0Y}^*)=0,$$

which is in conservation form. Thus we obtain

$$\mathrm{i}\alpha\frac{\partial}{\partial\tau}\left(\int_{-\infty}^{\infty}|A_0|^2\,\mathrm{d}\zeta\right)+\gamma\frac{\partial}{\partial Y}\left(\int_{-\infty}^{\infty}(A_0^*A_{0Y}-A_0A_{0Y}^*)\,\mathrm{d}\zeta\right)=0$$

and

$$\mathrm{i}\alpha\frac{\partial}{\partial\tau}\left(\int_{-\infty}^{\infty}|A_0|^2\,\mathrm{d}Y\right)+\beta\frac{\partial}{\partial\zeta}\left(\int_{-\infty}^{\infty}(A_0A_{0\zeta}^*-A_0^*A_{0\zeta})\mathrm{d}Y\right)=0,$$

provided decay conditions exist as $|\zeta|\to\infty$, at fixed Y, and as $|Y|\to\infty$, at fixed ζ. This requires that the waves at infinity are not parallel to either the ζ or the Y coordinates; it is this type of additional assumption or restriction that complicates the issue. Furthermore, if decay conditions exist as $Y\to\pm\infty$, and as $\zeta\to\pm\infty$, that is, the solution vanishes (sufficiently rapidly) as $Y^2+\zeta^2\to\infty$, we see that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|A_0|^2\,\mathrm{d}\zeta\,\mathrm{d}Y=\text{constant},$$

a conserved quantity that applies only for a limited class of solutions. Nevertheless, albeit with some important restrictions, we have derived a rather conventional type of conservation law – clearly the conservation of mass.

Finally, the other equation in the DS pair is already in conservation form, namely

$$\frac{\partial}{\partial Y}(f_{0Y})+\frac{\partial}{\partial\zeta}(\lambda f_{0\zeta}-\mu|A_0|^2)=0$$

and so, for example, we obtain

$$\frac{\partial}{\partial\zeta}\left(\int_{-\infty}^{\infty}(\lambda f_{0\zeta}-\mu|A_0|^2)\,\mathrm{d}Y\right)+[f_{0Y}]_{-\infty}^{\infty}=0;$$

if conditions are the same as $\zeta \to \pm\infty$, then

$$\lambda \frac{\partial}{\partial \zeta}\left(\int_{-\infty}^{\infty} f_0 \, \mathrm{d}Y \right) - \mu \int_{-\infty}^{\infty} |A_0|^2 \, \mathrm{d}Y = g(\tau),$$

where $g(\tau)$ is an arbitrary function. If, for some ζ, the left-hand side of this equation is zero (because, for example, the solution decays in this region), we must have $g(\tau) = 0$ for all τ. Hence

$$\lambda \int_{-\infty}^{\infty} f_0 \, \mathrm{d}Y = \mu \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |A_0|^2 \mathrm{d}Y \, \mathrm{d}\zeta = \text{constant}$$

if we follow our previous discussion; thus

$$\int_{-\infty}^{\infty} f_0 \, \mathrm{d}Y = \text{constant}$$

is another conservation law. A further small exploration of some special conservation laws of the DS equations will be found in Q4.35; other conservation laws are to be found in Q4.36.

4.3 Applications of the NLS and DS equations

We have presented a theory which describes how modulated harmonic waves arise in the study of water waves. So far, for both the one-dimensional and two-dimensional problems, we have restricted the scenario to the simplest possible: constant depth and stationary water, in the undisturbed state. As we explained for the various KdV problems (Section 3.4), an important question to pose is whether the simple ideas and constructions carry over to more realistic situations. Thus we shall now – without spelling-out all the details, because of the complexity of much of the work – show how the effects of an underlying shear, and of variable depth, manifest themselves in the modulation problems. In addition, and as our first application, we shall use the NLS and DS equations (precisely as already derived) to examine the stability of wave trains; these procedures can also be employed, with appropriate adjustments, when a shear or variable depth is included. Other ingredients, such as the inclusion of surface tension, will not be entertained here (since our interest, in this introductory text, still remains principally the study of gravity waves). However, these and other aspects are left to the interested reader, who may follow the various avenues through the references that appear later.

4.3.1 Stability of the Stokes wave

Our first and most direct application of the results obtained in Sections 4.1 and 4.2 is to the Stokes wave, which we introduced in Section 2.5. In order to see the relevance of the Nonlinear Schrödinger equation (and, indeed, the DS equations), we make use of a very special and simple solution of the NLS equation; for another, see Q4.40. From equation (4.32), our NLS equation is written as

$$-2ikc_pA_{0\tau} + \alpha A_{0\zeta\zeta} + \beta A_0|A_0|^2 = 0, \tag{4.103}$$

where α and β are given in (4.33) and (4.34). The nonlinear plane wave solution, with constant amplitude, of this equation is

$$A_0 = A\exp\{i(K\zeta - \Omega\tau)\}, \tag{4.104}$$

where A is a complex constant and K is a real constant; (4.104) is then a solution of (4.103) provided that Ω satisfies the dispersion relation

$$2kc_p\Omega = \beta|A|^2 - \alpha K^2. \tag{4.105}$$

Here, α, β and c_p are all functions of k (> 0), the wave number of the carrier wave (as described in equations (4.7)); K (> 0) is the wave number of the modulation. The primary wave (from (4.7)) therefore becomes

$$\eta_0 = A\exp\{i(kx + K\zeta - \omega t - \Omega\tau)\} + \text{c.c.},$$

and if we choose the wave number of this solution to be precisely k, then we set $K = 0$ to yield

$$\eta_0 = A\exp\{i[kx - (\omega + \varepsilon^2\Omega)t]\} + \text{c.c.}, \tag{4.106}$$

with $\tau = \varepsilon^2 t$ and Ω given by (4.105) with $K = 0$.

Solution (4.106) is the Stokes wave (cf. equation (2.133)) of amplitude A and wave number k, with the frequency (dispersion function) taken as far as terms of $O(\varepsilon^2)$ (cf. equation (2.137)). However, we note that our description via the NLS equation also incorporates an additional component to the *set-down*, which is associated with the *mean drift* (see Section 2.5 and Q4.4), although this is not needed here. Further details of this connection will be found in Q4.42. Now, with this identification as the Stokes-wave solution, we can use the NLS equation to provide an estimate for the stability of the Stokes wave.

The NLS equation describes the modulation of the amplitude of the harmonic wave, represented by $E\,(+\text{c.c.})$, for initial data which depends on x (and the parameter ε) in a way consistent with the NLS equation. The Stokes wave is recovered by introducing the plane wave solution of constant amplitude; thus we seek a solution which takes, as its initial form, a small perturbation about the nonlinear plane wave solution, (4.106). Thus we set

$$A_0 = A(1+\Delta a)\exp\{\mathrm{i}(-\Omega\tau + \Delta\theta)\} \tag{4.107}$$

where $a = a(\zeta, \tau)$, $\theta = \theta(\zeta, \tau)$ (both taken to be real functions) and we have chosen $K = 0$ (as used above), so

$$2kc_p\Omega = \beta|A|^2; \tag{4.108}$$

Δ is a parameter that we shall regard as small in what follows. Direct substitution of (4.107) into (4.103) yields

$$\begin{aligned} -\,2\mathrm{i}kc_p\{\Delta a_\tau + \mathrm{i}(1+\Delta a)(\Delta\theta_\tau - \Omega)\}A\mathscr{E} \\ +\,\alpha\{\Delta a_{\zeta\zeta} + 2\mathrm{i}\Delta^2 a_\zeta\theta_\zeta + \Delta(1+\Delta a)(\mathrm{i}\theta_{\zeta\zeta} - \Delta\theta_\zeta^2)\}A\mathscr{E} \\ +\,\beta(1+\Delta a)^3 A|A|^2\mathscr{E} = 0 \end{aligned}$$

where $\mathscr{E}$ is the exponential term in (4.107). The leading terms (that is, the O(1) terms as $\Delta \to 0$) cancel by virtue of (4.108), and then the leading perturbation terms (of O(Δ)) give

$$-2\mathrm{i}kc_p(a_\tau + \mathrm{i}\theta_\tau - \mathrm{i}\Omega) + \alpha(a_{\zeta\zeta} + \mathrm{i}\theta_{\zeta\zeta}) + 3\beta|A|^2 a = 0.$$

Since a and θ are real functions, and again invoking (4.108) to eliminate Ω, we obtain

$$\begin{aligned} 2kc_p\theta_\tau + \alpha a_{\zeta\zeta} + 2\beta|A|^2 a &= 0; \\ -2kc_p a_\tau + \alpha\theta_{\zeta\zeta} &= 0. \end{aligned}$$

This pair of equations is linear, with constant coefficients, so we have a solution

$$\begin{pmatrix} a \\ \theta \end{pmatrix} = \begin{pmatrix} a_0 \\ \theta_0 \end{pmatrix} \exp\{\mathrm{i}(\kappa\zeta - \lambda\tau)\} + \text{c.c.},$$

where a_0, θ_0, κ (> 0) and λ are constants; this solution exists provided (see Q4.43)

$$(2kc_p\lambda)^2 = \alpha^2\kappa^2(\kappa^2 - 2\beta|A|^2/\alpha), \tag{4.109}$$

the dispersion relation for λ.

Thus, from equation (4.109), it is immediately evident that for $\beta/\alpha < 0$ (or we could write $\alpha\beta < 0$), λ is real for all values of κ; the Stokes wave is stable or, more precisely, it is *neutrally* stable, since the wave perturbation persists but does not grow. However, if $\beta/\alpha > 0$, then λ will be imaginary for some κ, namely for

$$0 < \kappa < 2|A|\sqrt{\beta/\alpha}; \tag{4.110}$$

in this case a solution exists which grows exponentially as $\tau \to +\infty$: the Stokes wave is now unstable. We have already seen, in Section 4.2, that the sign of β/α is critical to the existence of certain types of solution of the NLS equations. In particular, for $\beta/\alpha > 0$ which corresponds to the NLS+ equation, we have a modulation which approaches zero at infinity (see equation (4.68)); no such solution exists for $\beta/\alpha < 0$. We might surmise that, for the unstable wave, there is a growth which continues until a balance is reached between the nonlinear and dispersive effects represented in the NLS+ equation. Once this has occurred, the amplitude modulation will evolve in line with the structure of a soliton solution: the solution will therefore not grow indefinitely. There is some observational and numerical evidence to support this sequence of events.

It is clear that, in order to see the relevance of the condition that heralds instability (that is, $\beta/\alpha > 0$ with equation (4.110)) – which we shall interpret more fully shortly – we need to know more about the coefficients of the NLS equation, (4.103). The expressions for the coefficients α and β are given in equations (4.33) and (4.34); these are rather complicated functions of δk. So that we can readily see their character, they are presented in Figure 4.6 where α and $\delta^2\beta$ are plotted against $\delta k > 0$. (It is usual practice to treat the given wave number, k, as positive.) The behaviours of α and β should be compared with the results obtained for $\delta k \to 0$ and $\delta k \to \infty$ in Q4.6. We see that the coefficient α is positive for all $\delta k\,(> 0)$, but that β changes sign from positive to negative as δk decreases across $\delta k = \delta k_0 \approx 1.363$. Thus the Stokes wave, based on the analysis above, is stable to small disturbances if $\delta k < \delta k_0$ (that is, for sufficiently long Stokes waves); on the other hand, if $\delta k > \delta k_0$ so that $\beta/\alpha > 0$, there exists a range of wave numbers κ for which the Stokes wave is unstable. (The problem in which δk is close to δk_0 must be treated separately; see Johnson (1977).) How should we interpret these κ?

The most straightforward approach is to construct the leading order term (the *fundamental*), η_0; further, its initial ($t = 0$) form is quite sufficient for our purposes, so we have

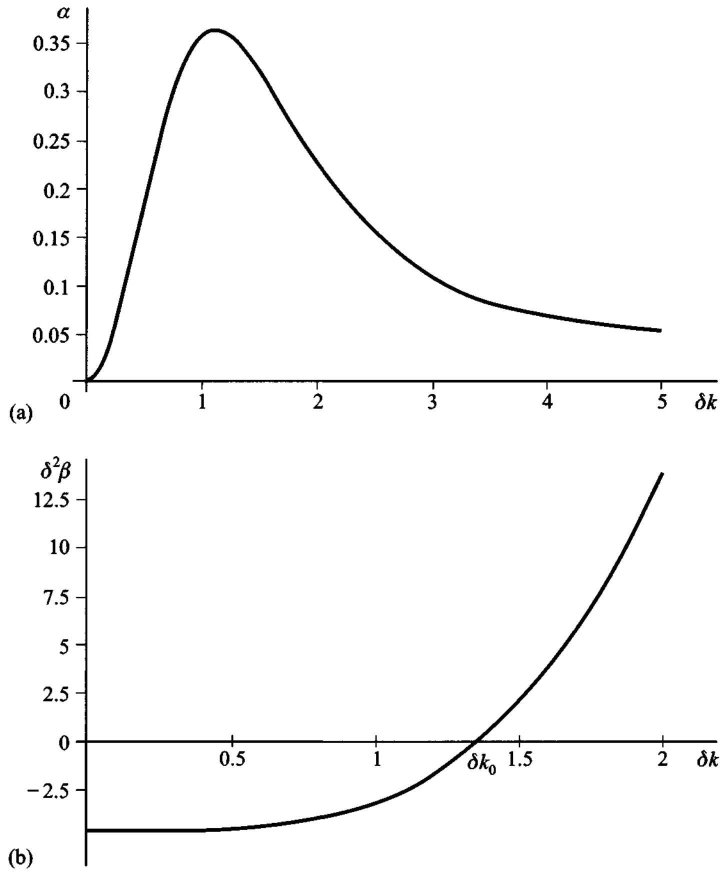

Figure 4.6. Plots of α (in (a)) and $\delta^2\beta$ (in (b)) as functions of δk, as required for the analysis of the stability of the Stokes wave.

$$\begin{aligned}\eta_0 &= A_0 E + \text{c.c.} = A(1+\Delta a)\mathscr{E}E + \text{c.c.}\\ &\sim A\exp\{\mathrm{i}(kx+\Delta\theta)\} + \Delta A a_0 \exp\{\mathrm{i}(k+\varepsilon\kappa)x + \mathrm{i}\Delta\theta\} + \text{c.c.}\end{aligned}$$

for $\Delta \to 0$ (at fixed x and ε). Here we have used $\zeta = \varepsilon(x - c_g t)$, with $t = 0$, and retained the term $\Delta\theta$ in the exponent (although it plays no rôle in this interpretation). The perturbation to the fundamental, the term in Δa_0, has a wave number $k + \varepsilon\kappa$; that is, close to k. Hence a perturbation which has a wave number close to that of the fundamental, will generate an instability whenever $\beta/\alpha > 0$. Since, both in nature and in the laboratory, it is impossible to produce waves with a precisely fixed wave number, waves with a small deviation in k will occur and give rise to this phenomenon. This is often observed: what starts out as a set of plane waves gradually breaks down (*along* the wavefronts) into a number of wave groups. This type of stability, because it is associated with a small change in the fundamental wave number, is called a *side-band* instability; it was first described in a seminal paper, in 1967, by Benjamin and Feir (and it is therefore often referred to as *Benjamin–Feir* instability).

We conclude this discussion of the rôle of the NLS equation, in the study of the Stokes wave, by extending the analysis to encompass the DS equations. In Q4.24, solutions of the DS equations which depend only on τ and $X = l\zeta + mY$ were obtained; with this choice of variables, coupled with appropriate decay conditions, this pair of equations then recovers the NLS equation in the form

$$- 2\mathrm{i}kc_p A_{0\tau} + (\alpha l^2 - c_p c_g m^2)A_{0XX} + \left\{\beta + \frac{\gamma^2 k^2 m^2}{c_p^2(1 - c_g^2)[m^2 + (1 - c_g^2)l^2]}\right\} A_0|A_0|^2 = 0. \quad (4.111)$$

Here, α and β are exactly as used above (and given in (4.33) and (4.34)), and γ is given in (4.42). The solution of this NLS equation, (4.111), describes a modulation that is oblique to the carrier wave, which itself propagates with its wavefronts normal to the x-direction. If we now use equation (4.111) as the basis for investigating the stability of the Stokes wave, then we are considering the perturbation to be at any angle relative to the carrier wave; this is clearly a more general perturbation. What effect does this have on the stability of the Stokes wave?

Following the analysis that we gave for the NLS equation, (4.103), which produced the stability condition $\beta/\alpha > 0$, we see that the corresponding condition for equation (4.111) is

$$(\alpha l^2 - c_p c_g m^2)\left\{\beta + \frac{\gamma^2 k^2 m^2}{c_p^2(1 - c_g^2)[m^2 + (1 - c_g^2)l^2]}\right\} < 0. \quad (4.112)$$

(Here we have chosen, for convenience, to express the condition as the product rather than the ratio of the coefficients; for the case $m = 0$ this yields $\alpha\beta < 0$, which is equivalent to $\beta/\alpha < 0$.) A slightly more transparent version of (4.112) is

$$(\alpha l^2 - c_p c_g m^2)\{(\beta + k^2\hat{\gamma}^2)m^2 + \beta(1 - c_g^2)l^2\} < 0 \quad (4.113)$$

where

$$\hat{\gamma}^2 = \gamma^2/\{c_p^2(1-c_g^2)\}$$

and we note that $\alpha > 0$, $c_p c_g > 0$, $c_g^2 < 1$ (cf. Figure 4.6), but that we can have $\beta < 0$. It is clear that it is always possible to find a pair (l, m) which leads to a violation of (4.113), except in one special case (and we consider only $\delta k > 0$). This case arises when the value of δk is such that

$$\frac{\beta + k^2\hat{\gamma}^2}{\beta(1-c_g^2)} = -\frac{c_p c_g}{\alpha}$$

for then (4.113) becomes

$$-(\alpha l^2 - c_p c_g m^2)^2 < 0$$

which is always true. (The very special case for which $l^2/m^2 = c_p c_g/\alpha$ is of no practical interest.) The value of δk where this occurs is $\delta k \approx 0.38$; thus for all other values of δk, the Stokes wave is always unstable to some oblique perturbation. Our conclusion, therefore, is that we cannot expect the Stokes wave to propagate without, eventually, suffering significant distortion.

4.3.2 Modulation of waves over a shear flow

The problem of nonlinear wave propagation, described by some type of KdV equation, in the presence of an underlying arbitrary shear has been described (Section 3.4.1). Where we might expect a quite dramatic disruption of the propagation process, we found that the effect of the shear was only to change the (constant) coefficients of the classical KdV equation. We now investigate how a shear flow manifests itself in the problem of the modulation of a wave. (We remember that there is no suggestion that the term 'shear flow' is to imply that our model accommodates any viscous contribution.) Again, if the presence of an arbitrary shear flow merely adjusts the constant coefficients of the NLS equation, then we should have much greater confidence in the predictions offered by that equation.

The starting point for this description is, in all essentials, the same as that adopted for the derivation of the KdV equation with shear. However, here we retain the parameter δ^2 because the waves are of arbitrary wavelength (and, therefore, we do not use the transformation which takes $\delta^2 \to \varepsilon$, with $\varepsilon \to 0$, in the equations). Thus from equations (3.108) (but see also (3.9)) we have

$$u_t + Uu_x + U'w + \varepsilon(uu_x + wu_z) = -p_x;$$
$$\delta^2\{w_t + Uw_x + \varepsilon(uw_x + ww_z)\} = -p_z;$$
$$u_x + w_z = 0,$$

with

$$p = \eta \quad \text{and} \quad w = \eta_t + U\eta_x + \varepsilon u\eta_x \quad \text{on} \quad z = 1 + \varepsilon\eta$$

and

$$w = 0 \quad \text{on} \quad z = 0.$$

The underlying shear flow is represented by $U(z)$ and $U' = dU/dz$. The solution that describes a modulated harmonic wave requires the choice of variables (see (4.2))

$$\xi = x - c_p t, \quad \zeta = \varepsilon(x - c_g t), \quad \tau = \varepsilon^2 t$$

and then we write, for $\varepsilon \to 0$

$$\eta \sim \sum_{n=0}^{\infty} \varepsilon^n \left\{ \sum_{m=0}^{n+1} A_{nm}(\zeta, \tau)E^m + \text{c.c.} \right\}$$

with $A_{00} = 0$, and

$$q \sim \sum_{n=0}^{\infty} \varepsilon^n \left\{ \sum_{m=0}^{n+1} Q_{nm}(\zeta, \tau, z)E^m + \text{c.c.} \right\}$$

where q (and Q_{nm}) stands for each of u, w and p; cf. equations (4.14) and (4.36).

The construction of the solution follows closely that described for both the NLS and DS equations (Section 4.1), but here the details are even more intricate. Thus we choose to present only the main features and results of the calculation, which, with the inclusion of a little more detail, can be found in Johnson (1976).

The terms $\varepsilon^0 E$, with $P_{01} = P(z)A_{01}$, yield the equation for $P(z)$:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{ \frac{1}{(U - c_p)^2} \frac{\mathrm{d}P}{\mathrm{d}z} \right\} - \frac{\delta^2 k^2}{(U - c_p)^2} P = 0, \tag{4.114}$$

which corresponds to the earlier equation for F_0; see equation (4.7) *et seq.* The boundary conditions for $P(z)$ are

$$P(1) = 1; \quad P'(0) = 0, \tag{4.115}$$

together with a third condition which leads to the determination of c_p, namely

$$P'(1) = (\delta k)^2 W_1$$

where we have written

$$W(z) = U(z) - c_p, \quad W_1 = W(1);$$

this gives

$$\int_0^1 \frac{P(z)}{\{W(z)\}^2}\,\mathrm{d}z = 1. \tag{4.116}$$

It is evident that equation (4.116) is a generalisation of the Burns condition given previously in (3.112); here it defines the phase speed, $c_p(k)$, for the given shear. A simple check on this result is afforded by the choice $U = 0$ (or, indeed, $U =$ constant), leading to the solution of (4.114) and then the determination of c_p from (4.116); see Q4.44. We now proceed on the assumption that there is no critical layer for the given $U(z)$, so that $W(z) \neq 0$, $z \in [0, 1]$.

The terms that arise at εE generate an expression for the group speed, c_g, in the form

$$c_g = c_p - \left\{ \frac{\int_0^1 (WI')^2 \mathrm{d}z - 1}{W_1^{-1} + \int_0^1 II'W'\mathrm{d}z} \right\} \tag{4.117}$$

where we have written

$$I(z) = \int_0^z \frac{P(z)}{\{W(z)\}^2}\,\mathrm{d}z$$

(so the Burns condition, (4.116), now becomes $I_1 = I(1) = 1$). It is far from clear that the group speed given by (4.117) satisfies the classical relation (see, for example, Q2.26)

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}, \quad \text{where} \quad c_p = \frac{\omega}{k};$$

that this is indeed the case is left as an exercise (Q4.45).

Finally, terms $\varepsilon^2 E$ produce the Nonlinear Schrödinger equation for A_{01}:

$$2\mathrm{i}kW_1\left(1+W_1\int_0^1 \Pi' W'\,\mathrm{d}z\right)A_{01\tau}+\hat{\alpha}A_{01\zeta\zeta}+\hat{\beta}A_{01}|A_{01}|^2=0 \qquad (4.118)$$

where the coefficients $\hat{\alpha}$ and $\hat{\beta}$ are extremely complicated functions of k and $U(z)$. The expressions for $\hat{\alpha}$ and $\hat{\beta}$ are given in Johnson (1976). The important observation is that, for arbitrary $U(z)$ and given wave number k, all the coefficients of the NLS equation, (4.118), are constant. Thus the description of a modulated wave, its various properties via solutions of the NLS equation and, for example, its relevance to the stability of the Stokes wave, all follow the various analyses already given. The only requirement is, for a given $U(z)$, to compute the coefficients (as functions of k) and then to use this information in the desired solutions. This computation, however, is very lengthy except for the very simplest choices of $U(z)$.

We complete this section by applying our new NLS equation, (4.118), to the problem of the stability of Stokes waves that are moving over an arbitrary shear; stability is governed by the condition

$$\hat{\alpha}\hat{\beta}<0;$$

cf. equation (4.112). The details of where this condition is violated, for a given $U(z)$, require (as just mentioned) a lengthy computation that is quite beyond the scope of this text. Suffice it here to describe the situation that obtains for long waves; that is, $\delta k\to 0$. (We already know that the Stokes wave on stationary water is stable for $\delta k<\delta k_0\approx 1.363$.) For $\delta k\to 0$, but allowing $U(z)$ to be arbitrary, the NLS equation reduces (after much tiresome calculation) to

$$2\mathrm{i}k\delta^2 I_{31}A_{01\tau}+3k^2\delta^4 J_1 A_{01\zeta\zeta}-\frac{3}{2}\frac{(I_{41})^2}{J_1}A_{01}|A_{01}|^2=0; \qquad (4.119)$$

cf. equation (4.44). Here we have used the notation that was employed for the problem of the KdV equation associated with arbitrary shear (given in Section 3.4.1), namely

$$I_{n1}=\int_0^1\frac{\mathrm{d}z}{(U-c_p)^n};\quad J_1=\int_0^1\int_z^1\int_0^{z_1}\frac{[U(z_1)-c_p]^2}{[U(z)-c_p]^2[U(z_2)-c_p]^2}\,\mathrm{d}z_2\mathrm{d}z_1\mathrm{d}z.$$

The condition for stability of the Stokes wave, from equation (4.119) (and cf. (4.118)) is

$$\hat{\alpha}\hat{\beta} = -\frac{9}{2}k^2\delta^2(I_{41})^2 < 0,$$

which is clearly satisfied for *all* shear flows. Thus, for sufficiently long waves, the Stokes wave is stable *no matter* the form of the underlying shear (at least, in the absence of a critical layer). This result has important implications for Stokes waves that are observed in nature (or in the laboratory): for long waves, the underlying flow is essentially irrelevant. Of course, the value of δk at which the Stokes wave becomes *unstable* for a given shear – a far more significant result – cannot be obtained in any direct manner. Indeed, to be practically useful, an observed shear profile would have to be the basis for the choice of $U(z)$, followed by a computation of the coefficients for each δk.

A final mathematical comment: the NLS equation for long waves, (4.119), matches directly with the KdV equation for arbitrary shear, (3.128). This calculation is easily reproduced by following the method described in Section 4.1.3; indeed, merely noting the appropriate changes to the coefficients in the two pairs of equations confirms the matching. (A small additional calculation relevant to the derivation of equation (4.119) is discussed in Q4.46.)

4.3.3 Modulation of waves over variable depth

We have seen (Section 3.4.4) that the propagation of long waves, as they move over variable depth, produces a distortion of the waves; it is therefore no surprise to find that the same occurs for modulated harmonic waves. Since the derivation of the standard NLS equation (Section 4.1.1) is itself rather lengthy, we shall present here only briefest outline of the corresponding calculation for variable depth. Far more details, with a much fuller discussion, can be found in Djordjevic & Redekopp (1978), and in Turpin, Benmoussa & Mei (1983). This latter paper describes the result of combining both a slowly varying depth and a slowly varying current.

The problem is formulated in the same vein as we approached the derivation of the variable coefficient KdV equation (Section 3.4.4); that is, we first seek the appropriate scale on which the depth should vary. (Of course, other scales are possible – faster or slower – but these will generate simpler fundamental equations, in some sense.) The original

Nonlinear Schrödinger equation was obtained by introducing the variables

$$\xi = x - c_p t, \quad \zeta = \varepsilon(x - c_g t), \quad \tau = \varepsilon^2 t;$$

see equations (4.2). On the basis of this, we anticipate that the most general NLS equation will arise when the depth varies on the scale ε^2; cf. the argument used for the KdV equation with variable depth, given in Section 3.4.4. This assumption then requires some adjustments to our choice of variables here.

Let us write $X = \varepsilon^2 x$, so that the bottom is now defined by

$$z = b(x; \varepsilon) = B(X),$$

and we shall use X rather than $\tau = \varepsilon^2 t$ to represent the longest scale in the problem. The variable that is associated with the propagation of the group is written as

$$\zeta = \varepsilon\left(\frac{1}{\varepsilon^2}\int_0^X \gamma_g(X')\mathrm{d}X' - t\right),$$

where it is consistent to write $\gamma_g(X) = c_g^{-1}(X)$ with $c_g(X)$ the (local) group speed. The most convenient representation of the variable that provides the harmonic component is obtained by writing

$$E = \mathrm{e}^{\mathrm{i}\phi} \quad \text{with} \quad \frac{\partial \phi}{\partial x} = k \quad \text{and} \quad \frac{\partial \phi}{\partial t} = -kc_p(X).$$

The derivation follows precisely the route described in Section 4.1.1, and results in the Nonlinear Schrödinger equation with variable coefficients:

$$-2\mathrm{i}kc_p c_g A_X - \mathrm{i}k^2 c_p^2\left\{\frac{\partial}{\partial X}\left(\frac{c_g}{\omega}\right)\right\}A + \frac{\hat{\alpha}}{c_g^2}A_{\zeta\zeta} + \hat{\beta}A|A|^2 = 0; \tag{4.120}$$

cf. equation (4.32). The coefficients depend on X, through the local depth $D = 1 - B(X)$, with

$$c_p^2 = \frac{\tanh \delta kD}{\delta k}, \quad c_g = \frac{1}{2}c_p(1 + 2\delta kD \operatorname{cosech} 2\delta kD), \quad \omega = kc_p,$$

and $\hat{\alpha}$, $\hat{\beta}$ are precisely α, β (see equations (4.33) and (4.34)) with δ replaced by δD. In equation (4.120) we have the new term that arises by virtue of the dependence on X: a term proportional to A (which corresponds to the term in η_0 that appeared in the variable-depth KdV equation, (3.148)). It is clear that equation (4.120) recovers the standard NLS equation when we have constant coefficients, for then we set $D = 1$ and transform

$c_g\zeta \to \zeta$, $X \to c_g T$ (which is the appropriate leading-order equivalence for the propagation of the group); obviously $\hat{\alpha} \to \alpha$ and $\hat{\beta} \to \beta$.

The first two terms in equation (4.120) can be written as

$$-2ik^2c_p^2\left\{\frac{c_g}{\omega}A_X + \frac{1}{2}A\left(\frac{c_g}{\omega}\right)_X\right\} = -2ik^2c_p^2\sqrt{\frac{c_g}{\omega}}\left(\sqrt{\frac{c_g}{\omega}}A\right)_X$$

from which we see that we can write the equation as

$$-2ikc_pc_gB_X + \frac{\hat{\alpha}}{c_g^2}B_{\zeta\zeta} + \frac{\omega\hat{\beta}}{c_g}B|B|^2 = 0, \tag{4.121}$$

where $B = A\sqrt{c_g/\omega}$; see Q4.47. This equation, (4.121), can now be discussed in much the same way that we adopted for the variable coefficient KdV equation (in Section 3.4.4). That is, we may use the equation to give some insight into the development of, for example, a solitary wave as it enters a region of very rapid or very slow depth change; that is, on a scale shorter than ε^{-2}, or longer than ε^{-2}, respectively. Of course, as we mentioned in the case of the KdV equation, a complete study of these problems requires an analysis of the full equations, with the inclusion of the appropriate depth scales. The particular case of very slow depth change, which results in a distortion of the solitary wave only (in this representation), is left as an exercise (Q4.48); we shall, however, briefly describe the case of a rapid depth change.

Equation (4.121), with constant coefficients, has a solitary-wave solution (of amplitude b) if

$$|B| = b\,\mathrm{sech}\left(b\zeta\sqrt{\frac{\omega c_g\hat{\beta}}{2\hat{\alpha}}}\right)$$

on $X = 0$ (cf. equation (4.68) and Q4.9); we choose, in order to make the results more transparent, to work with the envelope $|B|$ rather than B itself. Equivalently, when we write $b = a\sqrt{c_g/\omega}$, we have

$$|A| = a\,\mathrm{sech}\left(ac_g\zeta\sqrt{\frac{\hat{\beta}}{2\hat{\alpha}}}\right)$$

as the corresponding initial ($X = 0$) profile for equation (4.120). Similarly, it turns out that if the initial profile is

$$|A| = a \operatorname{sech}\left(\frac{ac_g\zeta}{N}\sqrt{\frac{\hat{\beta}}{2\hat{\alpha}}}\right),$$

then N solitons will eventually appear as the solution evolves in X; see, for example, Satsuma & Yajima (1974). Thus, if there is a rapid change in depth so that

$$c_g\sqrt{\frac{\hat{\beta}}{2\hat{\alpha}}} = \mu_1 \text{ on } D = 1$$

changes to

$$\mu_0 = c_g\sqrt{\frac{\hat{\beta}}{2\hat{\alpha}}} = \mu_1/N \quad \text{on} \quad D = D_0,$$

an initial profile which is a solitary wave on $D = 1$ will evolve into N solitons on $D = D_0$; cf. the result for the KdV equation, given in equation (3.151) *et seq*. Figure 4.7 shows the result of plotting μ_1/μ_0 for various δk; we see that two solitons appear for the cases $\delta k = 20$, 30, but not for $\delta k = 10$. (Note that we are interested only in the solution for which D decreases monotonically to its final value of D_0, and therefore at the point on these curves where this is first attained.) When μ_1/μ_0 is not precisely

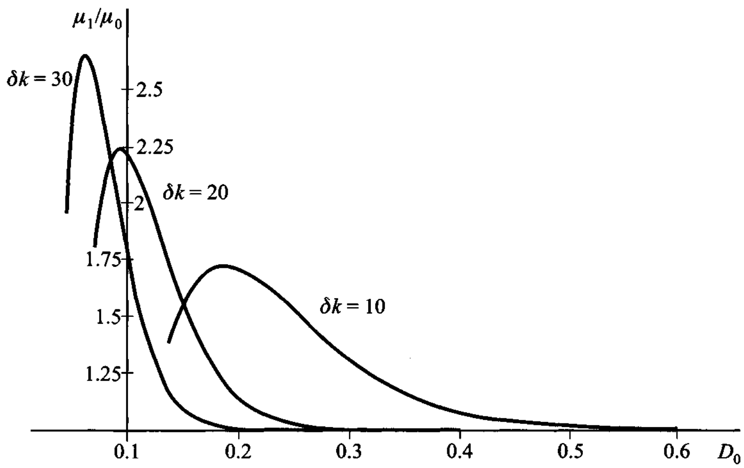

Figure 4.7. Plots of μ_1/μ_0 against D_0, for $\delta_k = 10$, 20, and 30, as used in the discussion of the solutions of the NLS equation with a rapid depth change.

an integer (that is, $\mu_1/\mu_0 = N + \Delta, 0 < \Delta < 1$), then the solution evolves into N solitons plus an oscillatory (dispersive) tail. These results mirror precisely those for the KdV equation, although here the relation between depth change and the number of solitons is considerably more involved.

Further reading

All the references to various aspects of soliton theory that were given at the end of the preceding chapter are relevant here (and will not be repeated). These texts describe the applications to both the NLS equations and the KdV family of equations. Below, we add a few further references that may prove of some interest to the reader who wishes to explore more deeply.

4.1 The initial work was done by Hasimoto & Ono (1972), Davey & Stewartson (1974) and Freeman & Davey (1975). Many other aspects of this work, which includes some mention of applications in other flow problems, can be found in Mei (1989), Infeld & Rowlands (1990) and Debnath (1994). An excellent text which touches on many more ideas in wave propagation, and which goes well beyond surface waves, is Craik (1988). All these texts and papers contain numerous references for still further reading.

4.3 A discussion of how these results apply to the stability of the Stokes wave is expanded in some of the references given above, and also in Whitham (1974). The particular applications that incorporate a shear or variable depth are mentioned in the texts by Mei and by Debnath. More information can be obtained from the papers by Johnson (1976), Djordjevic & Redekopp (1978) and Turpin *et al.* (1983).

Exercises

Q4.1 *Modulated wave from a Fourier representation.* Suppose that a wave is described by

$$\phi(x, t) = \int_{-\infty}^{\infty} F(k)\mathrm{e}^{\mathrm{i}(kx-\omega t)}\mathrm{d}k$$

for some given $F(k)$, and a given dispersion function $\omega = \omega(k)$. Consider the situation where the profile obtains its main contribution near the wave number $k = k_0$; define $k = k_0 + \varepsilon\kappa$, and assume that $\omega(k)$ may be expanded in a Taylor series about $k = k_0$ (as far as the term in ε^2). Write $F(k_0 + \varepsilon\kappa) = f(\kappa; \varepsilon)/\varepsilon$ and hence show that

$$\phi(x, t) \sim A(\zeta, \tau)\exp\{\mathrm{i}(k_0 x - \omega_0 t)\} \quad \text{as } \varepsilon \to 0,$$

where $\omega_0 = \omega(k_0)$, $\zeta = \varepsilon\{x - \omega'(k_0)t\}$ and $\tau = \varepsilon^2 t$, for some function $A(\zeta, \tau)$ which should be determined.

Q4.2 *Inhomogeneous differential equations.* Obtain the general solutions of the ordinary differential equations

$$\text{(a)}\ \frac{\mathrm{d}^2 F}{\mathrm{d}z^2} - \omega^2 F = \cosh(\omega z); \qquad \text{(b)}\ \frac{\mathrm{d}^2 F}{\mathrm{d}z^2} - \omega^2 F = z\sinh(\omega z),$$

where $\omega\,(> 0)$ is a constant.

Q4.3 *Second derivative of $\omega(k)$.* Given that

$$c_p^2 = \frac{\tanh \delta k}{\delta k} \quad \text{and} \quad c_g = \frac{\mathrm{d}}{\mathrm{d}k}(kc_p) = \frac{1}{2}c_p(1 + 2\delta k \operatorname{cosech} 2\delta k)$$

show that

$$kc_p\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2} = -\{c_g^2 - (1 - \delta k \tanh \delta k)\operatorname{sech}^2 \delta k\},$$

where $\omega = kc_p$.

[Observe how $\omega''(k_0)$ appears in the solution to Q4.1.]

Q4.4 *Modulated wave: mean drift component.* Use the terms that arise at $\varepsilon^2 E^0$ in the derivation of the NLS equation (and see equation (4.7)) to show that

$$f_{0\zeta} = -c_p^{-2}(1 - c_g^2)^{-1}(2c_p + c_g \operatorname{sech}^2 \delta k)|A_0|^2.$$

Hence show how this term is relevant to the particle velocity in the direction of propagation.

[This, you will find, provides the leading term to the non-periodic part of the velocity; it is a mean drift generated by the nonlinear interaction of the wave motion, usually called the *Stokes drift*.]

Q4.5 *Phase and group speeds for long waves.* Find the first two terms in the asymptotic expansions of c_p and c_g, as $\delta \to 0$; see equations (4.8) and (4.23).

Q4.6 *NLS and DS equations: long and short wave limits.* Obtain the long ($\delta \to 0$) and short ($\delta \to \infty$) wave limits of the Davey–Stewartson equations, retaining only the dominant contributions to each coefficient of the equations. Write down the corresponding Nonlinear Schrödinger equations that arise when there is no dependence on Y.

[The coefficients α and β are shown in Figure 4.6.]

Q4.7 *Matching of the DS and 2D KdV equations.* Follow the technique used in Section 4.1.3 to show that the DS equations in the long-wave limit ($\delta \to 0$; see Q4.6) match with the 2D KdV equation, (3.30),

$$(2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi})_\xi + \eta_{0YY} = 0$$

in the short-wave limit ($\delta \to \infty$). Construct the solution to this 2D KdV equation exactly as before, but now seek a solution which also depends on Y (the variable used in the 2D KdV equation). You will find that the correspondence requires that $A_{10} = f_{0z}$ (and you will need terms $\lambda^{-2}E^0$).

Q4.8 *Transformation of NLS equations.* Use scale transformations of u, x and t (as necessary) to transform

$$\mathrm{i}\alpha u_t + \beta u_{xx} \pm \gamma u|u|^2 = 0,$$

where α, β and γ are positive real constants, into

$$\mathrm{i}u_t + u_{xx} \pm u|u|^2 = 0.$$

Q4.9 *NLS+ equation: solitary wave.* Consider the NLS+ equation

$$\mathrm{i}u_t + u_{xx} + u|u|^2 = 0,$$

and seek a travelling-wave solution in the form

$$u = r\mathrm{e}^{\mathrm{i}(\theta + nt)}, \quad r = r(x - ct), \quad \theta = \theta(x - ct),$$

where r, θ, c and n are real (c, n being constants). Show that there is a solution for which

$$\theta' = \frac{1}{2}c; \quad 2(r')^2 = 2(n - \frac{1}{4}c^2)r^2 - r^4$$

and hence obtain the solitary-wave solution

$$u(x, t) = a\exp\left\{\mathrm{i}\left[\frac{1}{2}c(x - ct) + nt\right]\right\}\mathrm{sech}\{a(x - ct)/\sqrt{2}\}$$

for all $a^2 = 2(n - \frac{1}{4}c^2) > 0$.

[This solution represents an oscillatory wavepacket for which the amplitude approaches zero as $|x - ct| \to \infty$].

Q4.10 *NLS– equation: solitary wave.* Follow the procedure described in Q4.9, but now for the NLS– equation

$$\mathrm{i}u_t + u_{xx} - u|u|^2 = 0.$$

Show that there exists a solution for which

$$r^2 = -n - 2a^2\operatorname{sech}^2(a\xi), \quad \theta = -\arctan\left\{\frac{2a}{c}\tanh(a\xi)\right\},$$

where $\xi = x - ct$, for all c and $a = \frac{1}{2}\sqrt{-2n - c^2}$, provided $n < -\frac{1}{2}c^2$. What is the behaviour of this solution as $|\xi| \to \infty$?

[This solution is sometimes called a *dark* solitary wave because it describes a *depression* in a non-zero background state; it is not relevant in water-wave problems when there is no disturbance at infinity.]

Q4.11 *NLS+ equation: the Ma solitary wave.* Show that the NLS+ equation in Q4.9 has a solution

$$u(x, t) = a\exp(\mathrm{i}a^2t)\left\{1 + \left(\frac{2m(m\cos\theta + \mathrm{i}n\sin\theta)}{n\cosh(ma\sqrt{2}x) + \cos\theta}\right)\right\},$$

for all real a and m, where $n^2 = 1 + m^2$ and $\theta = 2mna^2t$. What is the behaviour of this solution as $|x| \to \infty$?

[Note that this solution does not represent a travelling wave; see Ma(1979), Peregrine (1983) and Figure 4.3.]

Q4.12 *A rational-cum-oscillatory solution.* Show that the NLS+ equation in Q4.9 has the solution

$$u(x, t) = \mathrm{e}^{\mathrm{i}t}\{1 - 4(1 + 2\mathrm{i}t)/(1 + 2x^2 + 4t^2)\}.$$

[This solution contains no free parameters, but see Q4.14 and Q4.15; this is not a travelling wave, as Figure 4.3 makes clear.]

Q4.13 *Behaviour of the Ma solitary wave.* Obtain the asymptotic behaviour of the Ma solitary wave (Q4.11) as $m \to \infty$ at fixed a. Retain terms of O(1) and O(m), and regard $mx = \mathrm{O}(1)$.

Q4.14 *A normalised Ma solution.* Show that the solution in Q4.11 can be 'normalised' by the removal of the amplitude a, under the transformation $x \to x/a$, $t \to t/a^2$, $u \to au$. Further, confirm that the NLS equation is invariant under this same transformation; see Q4.16.

Q4.15 *Ma → rational-cum-oscillatory.* For the solution given in Q4.11, set $a = 1$ and choose $n = -\sqrt{1 + m^2}$. Now let $m \to 0$ (for x and t fixed) and hence recover the solution in Q4.12. Repeat the calculation for arbitrary a, and compare your result with the general property described in Q4.14.

Q4.16 *Similarity solution of the NLS equation*. Show that the equation

$$\mathrm{i}u_t + u_{xx} + \varepsilon u|u|^2 = 0, \quad \varepsilon = \pm 1,$$

is invariant under each of the group transformation (a) $t \to t + \lambda$, $x \to x$, $u \to u$; (b) $t \to t$, $x \to x + \lambda$, $u \to u$; (c) $t \to \lambda^2 t$, $x \to \lambda x$, $u \to \lambda^{-1} u\,(\lambda \neq 0)$. Now use the property in (c) to obtain a similarity solution in the form $u(x, t) = t^m f(xt^n)$, for suitable m and n, and write down the equation for f.

Q4.17 *Normalised NLS± equations*. Use the results of Q4.8 to write equation (4.58):

$$\mathrm{i}\alpha(l - m)u_t + (l + m)u_{xx} \pm \frac{2}{lm}(l - m)(l^2 - m^2)u|u|^2 = 0,$$

in normalised form.

Q4.18 *Solution of the matrix Marchenko equation I*. Obtain the equations for c and d from equation (4.60), corresponding to equations (4.61) and (4.62) for a and b. Follow the same route as for a and b, and hence find the solutions for c and d.

Q4.19 *Solution of the matrix Marchenko equation II*. See Q4.18; impose the condition $c = -u^*$ and hence deduce that $g_0 = -f_0$ (for real f_0). Show, for the choice $c = u^*$ (which corresponds to the NLS– equation), that a solution of the form used in Q4.18 does not exist.

Q4.20 *NLS equation: bilinear form*. Show that the NLS equation

$$\mathrm{i}u_t + u_{xx} + \varepsilon u|u|^2 = 0 \quad (\varepsilon \text{ real constant})$$

can be written in the bilinear form

$$(\mathrm{iD}_t + \mathrm{D}_x^2)(g \cdot f) = 0; \quad \mathrm{D}_x^2(f \cdot f) = \varepsilon |g|^2$$

where $u = g/f$ and f is a real function.

Q4.21 *Generalised NLS equation*. Show that the equation

$$\mathrm{i}u_t + \beta u_{xx} + \mathrm{i}\gamma u_{xxx} + 3\mathrm{i}\delta|u|^2 u_x + \varepsilon u|u|^2 = 0,$$

where β, γ, δ and ε are real constants such that $\beta\delta = \gamma\varepsilon$, can be written in the bilinear form

$$(\mathrm{iD}_t + \beta \mathrm{D}_x^2 + \mathrm{i}\gamma \mathrm{D}_x^3)(g \cdot f) = 0; \quad \gamma \mathrm{D}_x^2(f \cdot f) = \delta |g|^2$$

where $u = g/f$ and f is a real function. Show how the equations used in Q4.20 can be recovered from these equations.

Q4.22 *Solitary-wave solution.* Obtain the solitary-wave solution of the generalised NLS equation (Q4.21) by seeking an appropriate solution of the bilinear form. (Follow the method described in Section 4.2.2.)

Q4.23 *NLS+ equation: a bi-soliton solution.* Seek a solution of the bilinear equations given in Q4.20 (for $\varepsilon = +1$), in the form of power series in the parameter δ, with

$$f = 1 + \sum_{n=1}^{\infty} \delta^{2n} f_{2n}; \quad g = \sum_{n=1}^{\infty} \delta^{2n-1} g_{2n-1},$$

which terminate (cf. Section 3.3.3). In particular obtain the solution

$$g_1 = 4\sqrt{2}(e^{it+x} + 3e^{9it+3x})$$

and hence determine corresponding expressions for g_3, f_2, and f_4; show that this solution terminates, so that $f_6 = f_8 = \ldots = 0$ and $g_5 = g_7 = \ldots = 0$. Finally, set $\delta = 1$ and write down a solution of the NLS+ equation.

[The confirmation that this is a bi-soliton solution is obtained by comparing it with the result of Q4.31, which provides a more general solution; this special bi-soliton solution is a standing wave.]

Q4.24 *DS equations $\rightarrow$ NLS equation I.* Seek a solution of the Davey–Stewartson equations, (4.40) and (4.41), which depend on ζ and Y only through the combination $(l\zeta + mY)$, for arbitrary constants l and m. Show that the resulting plane oblique waves satisfy a Nonlinear Schrödinger equation.

Q4.25 *DS equations $\rightarrow$ NLS equation II.* Repeat the calculation of Q4.24 (or start with the results of that calculation) to give the corresponding results for long waves; see Q4.6 and equations (4.83), (4.84).

Q4.26 *DS equations: solitary wave.* Use the results of Q4.25 and Q4.9 to find the solitary-wave solution of the Davey–Stewartson equations for long waves.

Q4.27 *Long-wave DS equations: bilinear form.* Show that the equations

$$iA_t + A_{xy} + 2A(Z + Z^*)_y = 0; \quad Z_x + iZ_y = |A|^2$$

(see (4.88)) can be written in the bilinear form

$$(\mathrm{iD}_t + \mathrm{D}_x\mathrm{D}_y)(g \cdot f) = 0; \quad (\mathrm{D}_x^2 + \mathrm{D}_y^2)(f \cdot f) = 2|g|^2$$

where $A = g/f$ and

$$Z = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right)\ln f,$$

for f real; see equations (4.89).

Q4.28 *DS bilinear form: solution*. Obtain the solitary-wave solution of the pair of bilinear equations given in Q4.27; see equation (4.90).

Q4.29 *Long-wave DS equations: solitary wave*. Show that your solutions obtained in Q4.28 and Q4.26 are equivalent.

Q4.30 *NLS+ equation: 2-soliton solution*. Use the bilinear form of the NLS+ equation (given in Q4.20) to obtain the 2-soliton solution of that equation; see equations (4.79), (4.80), *et seq*.

Q4.31 *NLS+ equation: bi-soliton solution*. From the 2-soliton solution obtained in Q4.30, construct the bi-soliton (or bound soliton) solution by choosing the two speeds to be equal (that is, $c_1 = c_2$); see equation (4.81) *et seq*., and Figure 4.5.

Q4.32 *NLS equation: two conservation laws*. Show that the NLS equation

$$\mathrm{i}u_t + u_{xx} + \varepsilon u|u|^2 = 0 \quad (\varepsilon = \pm 1),$$

possesses conserved quantities

$$\int_{-\infty}^{\infty} (|u_x|^2 - \frac{1}{2}\varepsilon|u|^4)\mathrm{d}x, \qquad \int_{-\infty}^{\infty} (uu_{xxx}^* + \frac{3}{2}\varepsilon|u|^2uu_x^*)\,\mathrm{d}x.$$

Q4.33 *NLS equation: a special conservation law*. Show that the NLS equation in Q4.32 has the conservation law

$$\int_{-\infty}^{\infty} \left\{\mathrm{i}x|u|^2 - t(u^*u_x - uu_x^*)\right\}\mathrm{d}x = \text{constant}.$$

Q4.34 *NLS equation: conservation of momentum*. Show that the conservation law

$$\int_{-\infty}^{\infty} (uu_x^* - u^*u_x)\,\mathrm{d}x = \text{constant}$$

corresponds to the leading term in the expression for the conservation of momentum in the wave motion; see equations (4.96) and (3.92).

Q4.35 *DS equations: special conservation laws*. Given the DS equation written as

$$f_{yy} + \lambda f_{xx} = \mu(|A|^2)_x;$$
$$-\mathrm{i}\alpha A_t + \beta A_{xx} - \gamma A_{yy} + \delta A|A|^2 + Af_x = 0,$$

where λ, μ, α, β, γ and δ are real constants, and given that

$$\int_{-\infty}^{\infty} |A|^2\,\mathrm{d}x \quad \text{and} \quad \int_{-\infty}^{\infty} |A|^2\,\mathrm{d}y$$

are constant, consider the following:

(a) What are the forms of

$$\int_{-\infty}^{\infty} (A^* A_y - AA_y^*)\,\mathrm{d}x \quad \text{and} \quad \int_{-\infty}^{\infty} (A^* A_x - AA_x^*)\,\mathrm{d}y\,?$$

(b) Are there conditions under which the two expressions in (a) are constants?

(c) What is the form of $\int_{-\infty}^{\infty} f\,\mathrm{d}y$?

(d) Are there conditions under which the expression in (c) is constant? If so, what is this constant?

Q4.36 *DS equations: conservation laws*. Show, provided solutions of the equations given in Q4.35 decay sufficiently rapidly as $x^2 + y^2 \to \infty$, that two constants of the motion are

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (AA_x^* - A^* A_x)\,\mathrm{d}x\,\mathrm{d}y; \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (AA_y^* - A^* A_y)\,\mathrm{d}x\,\mathrm{d}y.$$

Q4.37 *A 2D NLS equation*. A two-dimensional NLS+ equation is

$$\mathrm{i}u_t + u_{xx} + u_{yy} + u|u|^2 = 0;$$

obtain the plane solitary-wave solution of this equation; see Q4.9.

[This equation is a natural two-dimensional variant of the NLS equation; for more details in this direction, see Hui & Hamilton (1979) and Yuen & Lake (1982).]

Q4.38 *2D NLS equation: conservation laws.* Show that, with suitable decay conditions at infinity, solutions of the two-dimensional NLS+ equation given in Q4.37 possess the following conserved quantities:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u|^2\,\mathrm{d}x\,\mathrm{d}y; \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{|u_x|^2 + |u_y|^2 - \frac{1}{2}|u|^4\right\}\mathrm{d}x\,\mathrm{d}y.$$

Q4.39 *NLS equation: moment of inertia.* For the NLS equation given in Q4.32, define the moment of inertia

$$I = \int_{-\infty}^{\infty} x^2|u|^2\,\mathrm{d}x$$

and hence show that

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 8\int_{-\infty}^{\infty}\left[|u_x|^2 - \frac{\varepsilon}{4}|u|^4\right]\mathrm{d}x,$$

which is *not* a constant of the motion (see Q4.32).

Q4.40 *Another solution of the NLS equation.* Obtain a solution of the equation

$$\mathrm{i}u_t + u_{xx} + \varepsilon u|u|^2 = 0 \quad (\varepsilon = \pm 1)$$

in the form

$$u(x, t) = A(x)\exp(\mathrm{i}\omega t),$$

where ω is a real constant. Write down the equation for $A(x)$ and hence obtain, under suitable conditions that should be stated, the solution for which

(a) A is a *sech* function if $\varepsilon = +1$;

(b) A is a *tanh* function if $\varepsilon = -1$.

[See, for example, Hasimoto & Ono (1972).]

Q4.41 *Another representation of the NLS equation.* Seek a solution of the NLS equation given in Q4.40 in the form

$$u(x, t) = A(x, t)\exp\left(\mathrm{i}\int^{x} k(x', t)\,\mathrm{d}x'\right),$$

where both A and k are real functions. Obtain the two (real) equations that together describe A and k, and show that each can be written in conservation form. Explain how one of these equations gives a result consistent with the first conservation law, equation (4.93).

Q4.42 *Set-down and mean drift*. Show that the harmonic-wave solution,

$$\eta \sim \eta_0 + \varepsilon\eta_1,$$

of Section 4.1.1, generates a non-oscillatory component in η_1. Find this component, and confirm that one contribution corresponds to the set-down of the Stokes wave (given in Section 2.5) and the other to the mean drift (given in Q4.4).

Q4.43 *Derivation of a dispersion relation*. Find the solution of the pair of equations

$$2Kc_p\theta_t + \alpha a_{xx} + 2\beta|A|^2 a = 0; \quad -2Kc_p a_t + \alpha\theta_{xx} = 0,$$

where K, c_p, α, β and $|A|^2$ are real constants, for which both θ and a are proportional to

$$\exp\{\mathrm{i}(kx - \omega t)\} \quad (+\,\text{c.c.}).$$

Show that this solution exists provided ω and k satisfy a certain dispersion relation; what is it?

Q4.44 *Phase speed in the absence of shear*. For the choice $U(z) = 0$, obtain $P(z)$ from equations (4.114, 4.115), and hence determine c_p from the generalised Burns condition, (4.116). Confirm that your expression for c_p is the anticipated result.

Q4.45 *Classical result for c_g*. Show that the group speed, c_g, and the phase speed, c_p, are related by the classical identity

$$c_g = \frac{\mathrm{d}}{\mathrm{d}k}(kc_p),$$

where c_p and c_g are the expressions given for the NLS equation with shear; see equations (4.116, 4.117).

[Hint: formulate the problem for $\partial P/\partial k$, on the assumption that $P(z; k)$ may be differentiated with respect to k, where P satisfies equation (4.114).]

Q4.46 *Modulated waves over a shear: long wave limit*. Obtain the solution for $P(z)$ (defined by equations (4.114), 4.115)) as $\delta \to 0$, retaining terms as far as $\mathrm{O}(\delta^2)$. Hence show that

$$I_{21} \sim 1 + (\delta k)^2 J, \quad \delta \to 0;$$

see equation (4.119) *et seq*. for the notation adopted here.

Q4.47 *NLS equation for variable depth*. Obtain equation (4.121) from equation (4.120), where $B = A\sqrt{c_g/\omega}$.

Q4.48 *NLS equation for slow depth variation*. Seek a solution of equation (4.121) for which $B = B(\zeta, X, \sigma X)$, as $\sigma \to 0$, where $D = D(\sigma X)$; cf. equation (3.150) *et seq*. for the corresponding KdV problem. Write down the solitary-wave solution of the leading order NLS equation obtained as $\sigma \to 0$.