

Report 3

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February 10, 2020

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1 Time stepping and finite differences: the whole line

Recall the equation we obtained for the surface elevation on the whole line:

$$\eta_{tt} = \eta_{xx} + \mu^2 \left(\frac{1}{3} \eta_{xxxx} + \eta_{xt} \int_{-\infty}^x \eta_t \, dx' + \eta_x^2 + \eta_t^2 + \eta \eta_{xx} + \partial_x^2 \left(\int_{-\infty}^x \eta_t \, dx' \right)^2 \right). \quad (1)$$

To do time stepping, introduce

$$u = \eta_t. \quad (2)$$

Also, note that

$$\partial_x^2 \left(\int_{-\infty}^x \eta_t \, dx' \right)^2 = 2(\eta_t^2 + \eta_{tx} \int_{-\infty}^x \eta_t \, dx')$$

Then, combining (2) and (1), we obtain a two-dimensional system:

$$\partial_t \begin{bmatrix} u \\ \eta \end{bmatrix} = \begin{bmatrix} \mu^2 \left(2u_x \int_{-\infty}^x u \, dx' + 2u^2 \right) + \eta_{xx} + \mu^2 \left(\frac{1}{3} \eta_{xxxx} + \eta_x^2 + \eta \eta_{xx} \right) \\ u \end{bmatrix}. \quad (3)$$

Now, consider (1) on a finite interval $[a, b]$, and let partition the interval into $n + 1$ points $\{x_k\}_{k=0}^n$, with $x_0 = a$ and $x_n = b$. This means that the integral terms becomes

$$\int_{-\infty}^x \eta_t \, dx' = \left\{ \int_{-\infty}^a + \int_a^x \right\} \eta_t \, dx' \approx \int_a^x \eta_t \, dx',$$

while assuming that

$$\int_{-\infty}^a \eta_t \, dx'$$

is small enough. Now, we employ forward Euler time stepping. First, observe that

$$\begin{aligned} u_t(x_k, t_j) &= \frac{u(x_k, t_{j+1}) - u(x_k, t_j)}{\Delta t} = f_1(\eta, u, t) & \implies u(x_k, t_{j+1}) &= u(x_k, t_j) + \Delta t f_1(\eta, u, t) \\ \eta_t(x_k, t_j) &= \frac{\eta(x_k, t_{j+1}) - \eta(x_k, t_j)}{\Delta t} = f_2(\eta, u, t) & \implies \eta(x_k, t_{j+1}) &= \eta(x_k, t_j) + \Delta t f_2(\eta, u, t), \end{aligned}$$

where

$$\begin{aligned} f_1(\eta, u, t) &= \eta_{xx} + \mu^2 \left(2u_x \int_{-\infty}^x u \, dx' + 2u^2 + \frac{1}{3} \eta_{xxxx} + \eta_x^2 + \eta \eta_{xx} \right) \\ f_2(\eta, u, t) &= u(x_k, t_j). \end{aligned}$$

Observe that the highest order derivative is 4, so we need to use five-point stencils. In particular, we use five point midpoint stencil for x_2, \dots, x_{n-2} , and five point, one-sided stencils at x_0, x_1, x_{n-1}, x_n . First, consider the system for x_2, \dots, x_{n-2} :

$$\begin{aligned} f_1(\eta(x_k, t_j), u(x_k, t_j), t) &= \eta(x_k, t_j)_{xx} + \frac{\mu^2}{3} \eta(x_k, t_j)_{xxxx} \\ &\quad + \mu^2 \left(2u(x_k, t_j)_x \int_{x_0}^{x_k} u \, dx' + 2u(x_k, t_j)^2 + \eta(x_k, t_j)_x^2 + \eta(x_k, t_j) \eta(x_k, t_j)_{xx} \right), \end{aligned}$$

where we have separated the linear and nonlinear terms. Let $\Delta x = x_k - x_{k-1}$ and recall the finite difference formulas at x :

$$f'(x) = \frac{f(x - 2\Delta x) - 8f(x - \Delta x) + 8f(x + \Delta x) - f(x + 2\Delta x)}{12\Delta x} + \mathcal{O}((\Delta x)^4)$$

$$f''(x) = \frac{-f(x-2\Delta x) + 16f(x-\Delta x) - 30f(x) + 16f(x+\Delta x) - f(x+2\Delta x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2)$$

$$f'''(x) = \frac{f(x-2\Delta x) - 4f(x-\Delta x) + 6f(x) - 4f(x+\Delta x) + f(x+2\Delta x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2)$$

so that

$$(\eta_k)_x = \frac{\eta_{k-2} - 8\eta_{k-1} + 8\eta_{k+1} - \eta_{k+2}}{12\Delta x} + \mathcal{O}((\Delta x)^4)$$

$$(u_k)_x = \frac{u_{k-2} - 8u_{k-1} + 8u_{k+1} - u_{k+2}}{12\Delta x} + \mathcal{O}((\Delta x)^4)$$

$$(\eta_k)_{xx} = \frac{-\eta_{k-2} + 16\eta_{k-1} - 30\eta_k + 16\eta_{k+1} - \eta_{k+2}}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2)$$

$$(\eta_k)_{xxx} = \frac{\eta_{k-2} - 4\eta_{k-1} + 6\eta_k - 4\eta_{k+1} + \eta_{k+2}}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2),$$

where it is assumed $t = t_j$. Also, by trapezoidal rule,

$$\int_{x_0}^{x_k} u \, dx' = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} u \, dx = \frac{\Delta x}{2} \sum_{i=0}^{n-1} u(x_i) + u(x_{i+1}).$$

At $x = x_0$, we have

$$f'(x) = \frac{-25f(x) + 48f(x+\Delta x) - 36f(x+2\Delta x) + 16f(x+3\Delta x) - 3f(x+4\Delta x)}{12\Delta x} + \mathcal{O}((\Delta x)^4)$$

$$f''(x) = \frac{35f(x) - 104f(x+\Delta x) + 114f(x+2\Delta x) - 56f(x+3\Delta x) + 11f(x+4\Delta x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2)$$

$$f'''(x) = \frac{f(x) - 4f(x+\Delta x) + 6f(x+2\Delta x) - 4f(x+3\Delta x) + f(x+4\Delta x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2),$$

so that

$$(\eta_0)_x = \frac{-25\eta_0 + 48\eta_1 - 36\eta_2 + 16\eta_3 - 3\eta_4}{12\Delta x}$$

$$(u_0)_x = \frac{-25u_0 + 48u_1 - 36u_2 + 16u_3 - 3u_4}{12\Delta x}$$

$$(\eta_0)_{xx} = \frac{35\eta_0 - 104\eta_1 + 114\eta_2 - 56\eta_3 + 11\eta_4}{12\Delta x^2}$$

$$(\eta_0)_{xxxx} = \frac{\eta_0 - 4\eta_1 + 6\eta_2 - 4\eta_3 + \eta_4}{\Delta x^4}$$

At $x = x_1$, we have

$$\begin{aligned} f'(x) &= \frac{-3f(x - \Delta x) - 10f(x) + 18f(x + \Delta x) - 6f(x + 2\Delta x) + f(x + 3\Delta x)}{12\Delta x} + \mathcal{O}((\Delta x)^4) \\ f''(x) &= \frac{11f(x - \Delta x) - 20f(x) + 6f(x + \Delta x) + 4f(x + 2\Delta x) - f(x + 3\Delta x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2) \\ f'''(x) &= \frac{f(x - \Delta x) - 4f(x) + 6f(x + \Delta x) - 4f(x + 2\Delta x) + f(x + 3\Delta x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2), \end{aligned}$$

so that

$$\begin{aligned} (\eta_1)_x &= \frac{-3\eta_0 - 10\eta_1 + 18\eta_2 - 6\eta_3 + \eta_4}{12\Delta x} \\ (u_1)_x &= \frac{-3u_0 - 10u_1 + 18u_2 - 6u_3 + u_4}{12\Delta x} \\ (\eta_1)_{xx} &= \frac{11\eta_0 - 20\eta_1 + 6\eta_2 + 4\eta_3 - \eta_4}{12\Delta x^2} \\ (\eta_1)_{xxx} &= \frac{\eta_0 - 4\eta_1 + 6\eta_2 - 4\eta_3 + \eta_4}{\Delta x^4} \end{aligned}$$

At $x = x_{n-1}$, we have

$$\begin{aligned} f'(x) &= \frac{-f(x - 3\Delta x) + 6f(x - 2\Delta x) - 18f(x - \Delta x) + 10f(x) + 3f(x + \Delta x)}{12\Delta x} + \mathcal{O}((\Delta x)^4) \\ f''(x) &= \frac{-f(x - 3\Delta x) + 4f(x - 2\Delta x) + 6f(x - \Delta x) - 20f(x) + 11f(x + \Delta x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2) \\ f'''(x) &= \frac{f(x - 3\Delta x) - 4f(x - 2\Delta x) + 6f(x - \Delta x) - 4f(x) + f(x + \Delta x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2), \end{aligned}$$

so that

$$\begin{aligned} (\eta_{n-1})_x &= \frac{-\eta_{n-4} + 6\eta_{n-3} - 18\eta_{n-2} + 10\eta_{n-1} + 3\eta_n}{12\Delta x} \\ (u_{n-1})_x &= \frac{-u_{n-4} + 6u_{n-3} - 18u_{n-2} + 10u_{n-1} + 3u_n}{12\Delta x} \end{aligned}$$

$$\begin{aligned}
(\eta_{n-1})_{xx} &= \frac{-\eta_{n-4} + 4\eta_{n-3} + 6\eta_{n-2} - 20\eta_{n-1} + 11\eta_n}{12\Delta x^2} \\
(\eta_{n-1})_{xxx} &= \frac{\eta_{n-4} - 4\eta_{n-3} + 6\eta_{n-2} - 4\eta_{n-1} + \eta_n}{\Delta x^4}
\end{aligned}$$

At $x = x_n$, we have

$$\begin{aligned}
f'(x) &= \frac{f(x-4\Delta x) - 4f(x-3\Delta x) + 6f(x-2\Delta x) - 4f(x-\Delta x) + f(x)}{12\Delta x} + \mathcal{O}((\Delta x)^4) \\
f''(x) &= \frac{11f(x-4\Delta x) - 56f(x-3\Delta x) + 114f(x-2\Delta x) - 104f(x-\Delta x) + 35f(x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2) \\
f'''(x) &= \frac{f(x-4\Delta x) - 4f(x-3\Delta x) + 6f(x-2\Delta x) - 4f(x-\Delta x) + f(x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2),
\end{aligned}$$

so that

$$\begin{aligned}
(\eta_n)_x &= \frac{3\eta_{n-4} - 16\eta_{n-3} + 36\eta_{n-2} - 48\eta_{n-1} + 25\eta_n}{12\Delta x} \\
(u_n)_x &= \frac{3u_{n-4} - 16u_{n-3} + 36u_{n-2} - 48u_{n-1} + 25u_n}{12\Delta x} \\
(\eta_n)_{xx} &= \frac{11\eta_{n-4} - 56\eta_{n-3} + 114\eta_{n-2} - 104\eta_{n-1} + 35\eta_n}{12\Delta x^2} \\
(\eta_n)_{xxx} &= \frac{\eta_{n-4} - 4\eta_{n-3} + 6\eta_{n-2} - 4\eta_{n-1} + \eta_n}{\Delta x^4}.
\end{aligned}$$

All in all, we obtain:

$$\begin{aligned}
f_1(\eta(x_0, t_j), u(x_0, t_j), t) &= \eta(x_0, t_j)_{xx} + \frac{\mu^2}{3}\eta(x_0, t_j)_{xxx} + \mu^2 \left(2u(x_0, t_j)^2 + \eta(x_0, t_j)_x^2 + \eta(x_0, t_j)\eta(x_0, t_j)_{xx} \right), \\
f_1(\eta(x_1, t_j), u(x_1, t_j), t) &= \eta(x_1, t_j)_{xx} + \frac{\mu^2}{3}\eta(x_1, t_j)_{xxx} \\
&\quad + \mu^2 \left(2u(x_1, t_j)_x \int_{x_0}^{x_1} u \, dx' + 2u(x_1, t_j)^2 + \eta(x_1, t_j)_x^2 + \eta(x_1, t_j)\eta(x_1, t_j)_{xx} \right), \\
f_1(\eta(x_2, t_j), u(x_2, t_j), t) &= \eta(x_2, t_j)_{xx} + \frac{\mu^2}{3}\eta(x_2, t_j)_{xxx} \\
&\quad + \mu^2 \left(2u(x_2, t_j)_x \int_{x_0}^{x_2} u \, dx' + 2u(x_2, t_j)^2 + \eta(x_2, t_j)_x^2 + \eta(x_2, t_j)\eta(x_2, t_j)_{xx} \right),
\end{aligned}$$

$$\begin{aligned}
& \dots \\
& f_1(\eta(x_k, t_j), u(x_k, t_j), t) = \eta(x_k, t_j)_{xx} + \frac{\mu^2}{3} \eta(x_k, t_j)_{xxxx} \\
& \quad + \mu^2 \left(2u(x_k, t_j)_x \int_{x_0}^{x_k} u \, dx' + 2u(x_k, t_j)^2 + \eta(x_k, t_j)_x^2 + \eta(x_k, t_j) \eta(x_k, t_j)_{xx} \right), \\
& \dots \\
& f_1(\eta(x_{n-1}, t_j), u(x_{n-1}, t_j), t) = \eta(x_{n-1}, t_j)_{xx} + \frac{\mu^2}{3} \eta(x_{n-1}, t_j)_{xxxx} \\
& \quad + \mu^2 \left(2u(x_{n-1}, t_j)_x \int_{x_0}^{x_{n-1}} u \, dx' + 2u(x_{n-1}, t_j)^2 + \eta(x_{n-1}, t_j)_x^2 + \eta(x_{n-1}, t_j) \eta(x_{n-1}, t_j)_{xx} \right), \\
& f_1(\eta(x_n, t_j), u(x_n, t_j), t) = \eta(x_n, t_j)_{xx} + \frac{\mu^2}{3} \eta(x_n, t_j)_{xxxx} \\
& \quad + \mu^2 \left(2u(x_n, t_j)_x \int_{x_0}^{x_n} u \, dx' + 2u(x_n, t_j)^2 + \eta(x_n, t_j)_x^2 + \eta(x_n, t_j) \eta(x_n, t_j)_{xx} \right),
\end{aligned}$$

Now, we obtain the discretised problem. First, consider the column of linear terms:

$$\begin{aligned}
& (\eta_0)_{xx} + \frac{\mu^2}{3} (\eta_0)_{xxxx} \\
& = \frac{35\eta_0 - 104\eta_1 + 114\eta_2 - 56\eta_3 + 11\eta_4}{12\Delta x^2} + \frac{\mu^2}{3} \frac{\eta_0 - 4\eta_1 + 6\eta_2 - 4\eta_3 + \eta_4}{\Delta x^4} \\
& = \frac{35\Delta x^2 + 4\mu^2}{12\Delta x^4} \eta_0 - \frac{104\Delta x^2 + 16\mu^2}{12\Delta x^4} \eta_1 + \frac{114\Delta x^2 + 24\mu^2}{12\Delta x^4} \eta_2 - \frac{56\Delta x^2 + 16\mu^2}{12\Delta x^4} \eta_3 + \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4} \eta_4 \\
& (\eta_1)_{xx} + \frac{\mu^2}{3} (\eta_1)_{xxxx} \\
& = \frac{11\eta_0 - 20\eta_1 + 6\eta_2 + 4\eta_3 - \eta_4}{12\Delta x^2} + \frac{\mu^2}{3} \frac{\eta_0 - 4\eta_1 + 6\eta_2 - 4\eta_3 + \eta_4}{\Delta x^4} \\
& = \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4} \eta_0 - \frac{20\Delta x^2 + 16\mu^2}{12\Delta x^4} \eta_1 + \frac{6\Delta x^2 + 24\mu^2}{12\Delta x^4} \eta_2 + \frac{4\Delta x^2 - 16\mu^2}{12\Delta x^4} \eta_3 + \frac{-\Delta x^2 + 4\mu^2}{12\Delta x^4} \eta_4 \\
& \dots \\
& (\eta_k)_{xx} + \frac{\mu^2}{3} (\eta_k)_{xxxx}
\end{aligned}$$

$$\begin{aligned}
&= \frac{-\eta_{k-2} + 16\eta_{k-1} - 30\eta_k + 16\eta_{k+1} - \eta_{k+2}}{12(\Delta x)^2} + \frac{\mu^2}{3} \frac{\eta_{k-2} - 4\eta_{k-1} + 6\eta_k - 4\eta_{k+1} + \eta_{k+2}}{(\Delta x)^4} \\
&= \frac{-\Delta x^2 + 4\mu^2}{12\Delta x^4} \eta_{k-2} + \frac{16(\Delta x^2 - \mu^2)}{12\Delta x^4} \eta_{k-1} + \frac{-30\Delta x^2 + 24\mu^2}{12\Delta x^4} \eta_k - \frac{16(\Delta x^2 - \mu^2)}{12\Delta x^4} \eta_{k+1} + \frac{-\Delta x^2 + 4\mu^2}{12\Delta x^4} \eta_{k+2} \\
&\dots \\
&(\eta_{n-1})_{xx} + \frac{\mu^2}{3} (\eta_{n-1})_{xxxx} \\
&= \frac{-\eta_{n-4} + 4\eta_{n-3} + 6\eta_{n-2} - 20\eta_{n-1} + 11\eta_n}{12\Delta x^2} + \frac{\mu^2}{3} \frac{\eta_{n-4} - 4\eta_{n-3} + 6\eta_{n-2} - 4\eta_{n-1} + \eta_n}{\Delta x^4} \\
&= \frac{-\Delta x^2 + 4\mu^2}{12\Delta x^4} \eta_{n-4} + \frac{4\Delta x^2 - 16\mu^2}{12\Delta x^4} \eta_{n-3} + \frac{6\Delta x^2 + 24\mu^2}{12\Delta x^4} \eta_{n-2} - \frac{20\Delta x^2 + 16\mu^2}{12\Delta x^4} \eta_{n-1} + \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4} \eta_n \\
&(\eta_n)_{xx} + \frac{\mu^2}{3} (\eta_n)_{xxxx} \\
&= \frac{11\eta_{n-4} - 56\eta_{n-3} + 114\eta_{n-2} - 104\eta_{n-1} + 35\eta_n}{12\Delta x^2} + \frac{\mu^2}{3} \frac{\eta_{n-4} - 4\eta_{n-3} + 6\eta_{n-2} - 4\eta_{n-1} + \eta_n}{\Delta x^4} \\
&= \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4} \eta_{n-4} - \frac{56\Delta x^2 + 16\mu^2}{12\Delta x^4} \eta_{n-3} + \frac{114\Delta x^2 + 24\mu^2}{12\Delta x^4} \eta_{n-2} - \frac{104\Delta x^2 + 16\mu^2}{12\Delta x^4} \eta_{n-1} + \frac{35\Delta x^2 + 4\mu^2}{12\Delta x^4} \eta_n.
\end{aligned}$$

Then, the matrix becomes

$$\begin{bmatrix}
(\eta_0)_{xx} + \frac{\mu^2}{3} (\eta_0)_{xxxx} \\
(\eta_1)_{xx} + \frac{\mu^2}{3} (\eta_1)_{xxxx} \\
(\eta_2)_{xx} + \frac{\mu^2}{3} (\eta_2)_{xxxx} \\
\vdots \\
(\eta_k)_{xx} + \frac{\mu^2}{3} (\eta_k)_{xxxx} \\
\vdots \\
(\eta_{n-2})_{xx} + \frac{\mu^2}{3} (\eta_{n-2})_{xxxx} \\
(\eta_{n-1})_{xx} + \frac{\mu^2}{3} (\eta_{n-1})_{xxxx} \\
(\eta_n)_{xx} + \frac{\mu^2}{3} (\eta_n)_{xxxx}
\end{bmatrix}$$

$$= \begin{bmatrix} \frac{35\Delta x^2+4\mu^2}{12\Delta x^4} & -\frac{104\Delta x^2+16\mu^2}{12\Delta x^4} & \frac{114\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{56\Delta x^2+16\mu^2}{12\Delta x^4} & \frac{11\Delta x^2+4\mu^2}{12\Delta x^4} & 0 & 0 & 0 & \dots \\ \frac{11\Delta x^2+4\mu^2}{12\Delta x^4} & -\frac{20\Delta x^2+16\mu^2}{12\Delta x^4} & \frac{6\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{4\Delta x^2-16\mu^2}{12\Delta x^4} & -\frac{\Delta x^2+4\mu^2}{12\Delta x^4} & 0 & 0 & 0 & \dots \\ \frac{12\Delta x^4}{12\Delta x^4} & -\frac{12\Delta x^4}{12\Delta x^4} & \frac{12\Delta x^4}{12\Delta x^4} & -\frac{12\Delta x^4}{12\Delta x^4} & \frac{12\Delta x^4}{12\Delta x^4} & 0 & 0 & 0 & \dots \\ -\frac{\Delta x^2+4\mu^2}{12\Delta x^4} & \frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & -\frac{30\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & -\frac{\Delta x^2+4\mu^2}{12\Delta x^4} & \vdots & \vdots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & \dots & 0 & -\frac{\Delta x^2+4\mu^2}{12\Delta x^4} & \frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & -\frac{30\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & -\frac{\Delta x^2+4\mu^2}{12\Delta x^4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ \dots & 0 & 0 & 0 & -\frac{\Delta x^2+4\mu^2}{12\Delta x^4} & \frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & -\frac{30\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & -\frac{\Delta x^2+4\mu^2}{12\Delta x^4} \\ \dots & 0 & 0 & 0 & -\frac{\Delta x^2+4\mu^2}{12\Delta x^4} & \frac{4\Delta x^2-16\mu^2}{12\Delta x^4} & \frac{6\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{20\Delta x^2-16\mu^2}{12\Delta x^4} & \frac{11\Delta x^2+4\mu^2}{12\Delta x^4} \\ \dots & 0 & 0 & 0 & \frac{11\Delta x^2+4\mu^2}{12\Delta x^4} & -\frac{56\Delta x^2+16\mu^2}{12\Delta x^4} & \frac{114\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{104\Delta x^2+16\mu^2}{12\Delta x^4} & \frac{35\Delta x^2+4\mu^2}{12\Delta x^4} \end{bmatrix} \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_k \\ \vdots \\ \eta_{n-2} \\ \eta_{n-1} \\ \eta_n \end{bmatrix} \quad 0$$

For simplicity, let \mathcal{A} represent the above matrix. Now, recall the system:

$$\begin{aligned} u(x_k, t_{j+1}) &= u(x_k, t_j) + \Delta t f_1(\eta(x_k, t_j), u(x_k, t_j), t), \\ \eta(x_k, t_{j+1}) &= \eta(x_k, t_j) + \Delta t u(x_k, t_j), \end{aligned}$$

where

$$\begin{aligned} f_1(\eta(x_k, t_j), u(x_k, t_j), t) &= \eta(x_k, t_j)_{xx} + \frac{\mu^2}{3} \eta(x_k, t_j)_{xxx} \\ &+ \mu^2 \left(2u(x_k, t_j)_x \int_{x_0}^{x_k} u \, dx' + 2u(x_k, t_j)^2 + \eta(x_k, t_j)_x^2 + \eta(x_k, t_j) \eta(x_k, t_j)_{xx} \right). \end{aligned}$$

For convenience, let

$$B_k = \left(2u(x_k, t_j)_x \int_{x_0}^{x_k} u \, dx' + 2u(x_k, t_j)^2 + \eta(x_k, t_j)_x^2 + \eta(x_k, t_j) \eta(x_k, t_j)_{xx} \right),$$

Let \mathcal{B} represent the column vector of B_k 's. Then, we can write the system

$$\begin{aligned} u(x_0, t_{j+1}) &= u(x_0, t_j) + \Delta t f_1(\eta(x_0, t_j), u(x_0, t_j), t) \\ u(x_1, t_{j+1}) &= u(x_1, t_j) + \Delta t f_1(\eta(x_1, t_j), u(x_1, t_j), t) \\ u(x_2, t_{j+1}) &= u(x_2, t_j) + \Delta t f_1(\eta(x_2, t_j), u(x_2, t_j), t) \\ &\vdots \\ u(x_{n-2}, t_{j+1}) &= u(x_{n-2}, t_j) + \Delta t f_1(\eta(x_{n-2}, t_j), u(x_{n-2}, t_j), t) \end{aligned}$$

$$\begin{aligned}
u(x_{n-1}, t_{j+1}) &= u(x_{n-1}, t_j) + \Delta t f_1(\eta(x_{n-1}, t_j), u(x_{n-1}, t_j), t) \\
u(x_n, t_{j+1}) &= u(x_n, t_j) + \Delta t f_1(\eta(x_n, t_j), u(x_n, t_j), t)
\end{aligned}$$

as follows:

$$\begin{aligned}
\begin{bmatrix} u(x_0, t_{j+1}) \\ u(x_1, t_{j+1}) \\ u(x_2, t_{j+1}) \\ \vdots \\ u(x_{n-2}, t_{j+1}) \\ u(x_{n-1}, t_{j+1}) \\ u(x_n, t_{j+1}) \end{bmatrix} &= \begin{bmatrix} u(x_0, t_j) \\ u(x_1, t_j) \\ u(x_2, t_j) \\ \vdots \\ u(x_{n-2}, t_j) \\ u(x_{n-1}, t_j) \\ u(x_n, t_j) \end{bmatrix} + \Delta t \begin{bmatrix} f_1(\eta(x_0, t_j), u(x_0, t_j), t) \\ f_1(\eta(x_1, t_j), u(x_1, t_j), t) \\ f_1(\eta(x_2, t_j), u(x_2, t_j), t) \\ \vdots \\ f_1(\eta(x_{n-2}, t_j), u(x_{n-2}, t_j), t) \\ f_1(\eta(x_{n-1}, t_j), u(x_{n-1}, t_j), t) \\ f_1(\eta(x_n, t_j), u(x_n, t_j), t) \end{bmatrix} \\
&= \begin{bmatrix} u(x_0, t_j) \\ u(x_1, t_j) \\ u(x_2, t_j) \\ \vdots \\ u(x_{n-2}, t_j) \\ u(x_{n-1}, t_j) \\ u(x_n, t_j) \end{bmatrix} + \Delta t \mathcal{A} \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-2} \\ \eta_{n-1} \\ \eta_n \end{bmatrix} + \Delta t \mathcal{B}
\end{aligned}$$

Now, let us see how one would perform time-stepping. As such, impose initial conditions

$$\eta(x, t_0) = f(x), \quad u(x, t_0) = \eta_t(x, t_0) = g(x).$$

Let $j = 0$, and for simplicity, pick $k \in [0, n]$. The system is

$$\begin{aligned}
u(x_k, t_1) &= u(x_k, t_0) + \Delta t f_1(\eta(x_k, t_0), u(x_k, t_0)), \\
\eta(x_k, t_1) &= \eta(x_k, t_0) + \Delta t u(x_k, t_0),
\end{aligned}$$

where

$$\begin{aligned}
f_1(\eta(x_k, t_0), u(x_k, t_0)) &= \eta(x_k, t_0)_{xx} + \frac{\mu^2}{3} \eta(x_k, t_0)_{xxxx} \\
&\quad + \mu^2 \left(2u(x_k, t_0)_x \int_{x_0}^{x_k} u \, dx' + 2u(x_k, t_0)^2 + \eta(x_k, t_0)_x^2 + \eta(x_k, t_0) \eta(x_k, t_0)_{xx} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{-\eta(x_{k-2}, t_0) + 16\eta(x_{k-1}, t_0) - 30\eta(x_k, t_0) + 16\eta(x_{k+1}, t_0) - \eta(x_{k+2}, t_0)}{12(\Delta x)^2} \\
&+ \frac{\mu^2}{3} \frac{\eta(x_{k-2}, t_0) - 4\eta(x_{k-1}, t_0) + 6\eta(x_k, t_0) - 4\eta(x_{k+1}, t_0) + \eta(x_{k+2}, t_0)}{(\Delta x)^4} \\
&+ \mu^2 \left(2u(x_k, t_0)_x \int_{x_0}^{x_k} u(x', t_0) dx' + 2u(x_k, t_0)^2 + \eta(x_k, t_0)_x^2 + \eta(x_k, t_0)\eta(x_k, t_0)_{xx} \right)
\end{aligned}$$

Note that all the terms on the last line can be computed via finite differences and both initial conditions. With this, we obtain the values of u, η at point x_k and time t_1 . Performing this calculation for all k , we move on to compute u, η at time t_2 , and so on.

2 The half line problem

In this section, we deal with this term

$$\partial_x (\mathcal{F}_s^k)^{-1} \{ \mathcal{F}_c^k \{ \partial_t \left(\eta \int_0^x \eta_t dx' \right) \} \}$$

More generally, we have the following result:

Theorem 1. *For nice enough f defined on $x \geq 0$, we have*

$$(\mathcal{F}_s^k)^{-1} \{ \mathcal{F}_c^k \{ f \} \} = \frac{1}{\pi} \int_0^\infty f(y) \left(\frac{1}{x-y} + \frac{1}{x+y} \right) dy.$$

Before we begin, recall the Riemann-Lebesgue lemma:

Lemma 2 (Theorem 11.6, [1]). *Assume that $f \in L(I)$. Then, for each real β , we have*

$$\lim_{\alpha \rightarrow \infty} \int_I f(t) \sin(\alpha t + \beta) dt = 0.$$

Proof of Theorem 1. Consider

$$(\mathcal{F}_s^k)^{-1} \{ \mathcal{F}_c^k \{ f \} \}.$$

For generality, we consider $(\mathcal{F}_s^k)^{-1} \{ G(k) \}$, where G is a function of k defined on $k \geq 0$. Expanding the integral, we obtain:

$$(\mathcal{F}_s^k)^{-1} \{ G(k) \} = \int_0^\infty \sin(kx) G(k) dk$$

$$\begin{aligned}
&= \frac{1}{2i} \int_0^\infty (e^{ikx} - e^{-ikx}) G(k) dk \\
&= \frac{1}{2i} \left[\int_0^\infty e^{ikx} G(k) dk - \int_0^\infty e^{-ikx} G(k) dk \right] \\
&= \frac{1}{2i} \left[\int_0^\infty e^{ikx} G(k) dk + \int_0^{-\infty} e^{ikx} G(-k) dk \right] && \text{(apply } k \mapsto -k \text{ in the 2nd term)} \\
&= \frac{1}{2i} \left[\int_0^\infty e^{ikx} G(k) dk + \int_{-\infty}^0 e^{ikx} (-G(-k)) dk \right],
\end{aligned}$$

where $-G(-k)$ is an odd extension to $k < 0$. Now, observe the following:

$$\begin{aligned}
\frac{2}{\pi} \int_0^\infty \cos(kx) f(x) dx &= \frac{1}{\pi} \int_0^\infty (e^{ikx} + e^{-ikx}) f(x) dx \\
&= \frac{1}{\pi} \left[\int_0^\infty e^{ikx} f(x) dx + \int_0^\infty e^{-ikx} f(x) dx \right] \\
&= \frac{1}{\pi} \left[- \int_0^{-\infty} e^{-ikx} f(-x) dx + \int_0^\infty e^{-ikx} f(x) dx \right] && \text{(apply } x \mapsto -x \text{ in the 1st term)} \\
&= \frac{1}{\pi} \left[\int_{-\infty}^0 e^{-ikx} f(-x) dx + \int_0^\infty e^{-ikx} f(x) dx \right] \\
&= \frac{1}{\pi} \int_{-\infty}^\infty e^{-ikx} F(x) dx,
\end{aligned}$$

where we used an even extension to $x < 0$ and defined

$$F(x) = \begin{cases} f(x) & x > 0 \\ f(-x) & x < 0 \end{cases}.$$

For $k > 0$, we have

$$G(k) = \mathcal{F}_c^k \{f\} = \frac{2}{\pi} \int_0^\infty \cos(kx) f(x) dx = \frac{1}{\pi} \int_{-\infty}^\infty e^{-ikx} F(x) dx. \quad (4)$$

For $k < 0$, we have

$$-G(-k) = -\mathcal{F}_c^{-k} \{f\} = -\frac{2}{\pi} \int_0^\infty \cos(-kx) f(x) dx = -\frac{2}{\pi} \int_0^\infty \cos(kx) f(x) dx = -\frac{1}{\pi} \int_{-\infty}^\infty e^{-ikx} F(x) dx, \quad (5)$$

since cosine is an even function. Thus, using (4) and (5), we obtain

$$\begin{aligned}
(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} &= \frac{1}{2i} \left[\int_0^\infty e^{ikx} \mathcal{F}_c^k\{f\} dk + \int_{-\infty}^0 e^{ikx} (-\mathcal{F}_c^k\{f\}) dk \right] \\
&= \frac{1}{2\pi i} \left[\int_0^\infty e^{ikx} \int_{-\infty}^\infty e^{-iky} F(y) dy dk - \int_{-\infty}^0 e^{ikx} \int_{-\infty}^\infty e^{-iky} F(y) dy dk \right] \\
&= \frac{1}{2\pi i} \left[\int_0^\infty e^{ikx} \int_{-\infty}^\infty e^{ik(x-y)} F(y) dy dk - \int_{-\infty}^0 \int_{-\infty}^\infty e^{ik(x-y)} F(y) dy dk \right]. \tag{6}
\end{aligned}$$

Let

$$\begin{aligned}
V(k) &= \int_{-\infty}^\infty \sin(k(x-y)) F(y) dy = -V(-k), \\
U(k) &= \int_{-\infty}^\infty \cos(k(x-y)) F(y) dy = U(-k),
\end{aligned}$$

so that V is odd and U is even. This allows to rewrite (6) as:

$$\begin{aligned}
(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} &= \frac{1}{2\pi i} \left[\int_0^\infty e^{ikx} \int_{-\infty}^\infty e^{ik(x-y)} F(y) dy dk - \int_{-\infty}^0 \int_{-\infty}^\infty e^{ik(x-y)} F(y) dy dk \right] \\
&= \frac{1}{2\pi i} \left[\int_0^\infty U(k) + iV(k) dk - \int_{-\infty}^0 U(k) + iV(k) dk \right] \\
&= \frac{1}{2\pi i} \left[\int_0^\infty U(k) + iV(k) dk + \int_{-\infty}^0 U(-k) + iV(-k) dk \right] \\
&= \frac{1}{2\pi i} \left[\int_0^\infty U(k) + iV(k) dk + \int_0^\infty -U(-k) + i(-V(-k)) dk \right] \\
&= \frac{1}{2\pi i} \left[\int_0^\infty U(k) + iV(k) dk + \int_0^\infty -U(k) + iV(k) dk \right] \\
&= \frac{1}{\pi} \int_0^\infty V(k) dk,
\end{aligned}$$

where on the third last line, we flipped the bounds of integration and brought the minus sign inside the integral, and on the second last line, we used that U is even and V is odd. Thus, we obtain

$$(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} = \frac{1}{\pi} \int_0^\infty V(k) dk = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty \sin(k(x-y)) F(y) dy dk.$$

Note that the integral in k is an improper integral, so

$$\int_0^\alpha \int_{-\infty}^\infty \sin(k(x-y))F(y) \, dy \, dk = \lim_{\alpha \rightarrow \infty} \int_0^\alpha \int_{-\infty}^\infty \sin(k(x-y))F(y) \, dy \, dk.$$

Now, interchanging the order of integration, we have

$$\begin{aligned} \int_0^\alpha \int_{-\infty}^\infty \sin(k(x-y))F(y) \, dy \, dk &= \int_{-\infty}^\infty F(y) \int_0^\alpha \sin(k(x-y)) \, dk \, dy \\ &= \int_{-\infty}^\infty F(y) \left[-\frac{\cos(k(x-y))}{x-y} \right]_0^\alpha \, dy \\ &= \int_{-\infty}^\infty F(y) \left[\frac{1}{x-y} - \frac{\cos(\alpha(x-y))}{x-y} \right] \, dy \\ &= \int_{-\infty}^\infty F(y) \frac{1 - \cos(\alpha(x-y))}{x-y} \, dy. \end{aligned}$$

The interchange is justified, since sine is bounded and differentiable on \mathbb{R} . Finally, we use the Riemann-Lebesgue lemma to deal with the last term:

$$\begin{aligned} \int_{-\infty}^\infty F(y) \frac{1 - \cos(\alpha(x-y))}{x-y} \, dy &= \int_0^\infty f(y) \frac{1 - \cos(\alpha(x-y))}{x-y} \, dy + \int_{-\infty}^0 f(-y) \frac{1 - \cos(\alpha(x-y))}{x-y} \, dy \\ &= \int_0^\infty f(y) \frac{1 - \cos(\alpha(x-y))}{x-y} \, dy - \int_\infty^0 f(y) \frac{1 - \cos(\alpha(x+y))}{x+y} \, dy \\ &= \int_0^\infty f(y) \frac{1 - \cos(\alpha(x-y))}{x-y} \, dy + \int_0^\infty f(y) \frac{1 - \cos(\alpha(x+y))}{x+y} \, dy \\ &= \int_0^\infty f(y) \frac{1}{x-y} \, dy - \int_0^\infty f(y) \frac{\cos(\alpha(x-y))}{x-y} \, dy \\ &\quad + \int_0^\infty f(y) \frac{1}{x+y} \, dy - \int_0^\infty f(y) \frac{\cos(\alpha(x+y))}{x+y} \, dy. \end{aligned}$$

As $\alpha \rightarrow \infty$, the terms

$$\int_0^\infty f(y) \frac{\cos(\alpha(x-y))}{x-y} \, dy, \quad \int_0^\infty f(y) \frac{\cos(\alpha(x+y))}{x+y} \, dy \rightarrow 0$$

by the Riemann-Lebesgue lemma with $\beta = \pi/2$, so that

$$\int_{-\infty}^\infty F(y) \frac{1 - \cos(\alpha(x-y))}{x-y} \, dy = \int_0^\infty f(y) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] \, dy.$$

Thus,

$$(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty \sin(k(x-y))F(y) dy dk = \frac{1}{\pi} \int_0^\infty f(y) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy.$$

The proof is complete. \square

Remark 3. Note that the integral

$$\frac{1}{\pi} \int_0^\infty f(y) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy =$$

looks like a convolution-type transform. In fact, the term with $1/(x-y)$ is pretty much the Hilbert transform, but on a half-line.

The theorem yields

$$\partial_x (\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t \left(\eta \int_0^x \eta_t dx' \right)\}\} = \partial_x \left(\frac{1}{\pi} \int_0^\infty \partial_t \left(\eta \int_0^y \eta_t dy' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy \right).$$

For generality, let $f(y) = \partial_t \left(\eta \int_0^y \eta_t dy' \right)$. Note the following:

$$\begin{aligned} \partial_x \left(\frac{1}{\pi} \int_0^\infty f(y) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy \right) &= \frac{1}{\pi} \int_0^\infty f(y) \partial_x \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy \\ &= -\frac{1}{\pi} \int_0^\infty f(y) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] dy, \end{aligned}$$

so that

$$\partial_x (\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t \left(\eta \int_0^x \eta_t dx' \right)\}\} = -\frac{1}{\pi} \int_0^\infty \partial_t \left(\eta \int_0^y \eta_t dy' \right) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] dy. \quad (7)$$

As can be seen, the integral (7) is singular whenever $x = y$ or $x = -y$, over y . To deal with this issue, one may need to use a Residue theorem. To conclude, the surface expression on a half-line becomes:

$$\begin{aligned} \eta_{tt} - \eta_{xx} &= \mu^2 \left(\frac{1}{3} \eta_{xxxx} + \partial_x (\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t \left(\eta \int_0^x \eta_t dx' \right)\}\} + \frac{1}{2} \partial_x^2 \left(\int_0^x \eta_t dx' \right)^2 \right) \\ &= \mu^2 \left(\frac{1}{3} \eta_{xxxx} - \frac{1}{\pi} \int_0^\infty \partial_t \left(\eta \int_0^y \eta_t dy' \right) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] dy + \frac{1}{2} \partial_x^2 \left(\int_0^x \eta_t dx' \right)^2 \right). \end{aligned}$$

3 Time stepping and finite differences: the half line

On the half line, the equation for the surface elevation is given by:

$$\eta_{tt} = \eta_{xx} + \mu^2 \left(\frac{1}{3} \eta_{xxx} - \frac{1}{\pi} \int_0^\infty \partial_t \left(\eta \int_0^y \eta_t dy' \right) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] dy + \frac{1}{2} \partial_x^2 \left(\int_0^x \eta_t dx' \right)^2 \right) \quad (8)$$

To do time stepping, introduce

$$u = \eta_t. \quad (9)$$

Also, note that

$$\frac{1}{2} \partial_x^2 \left(\int_0^x \eta_t dx' \right)^2 = \eta_t^2 + \eta_{tx} \int_0^x \eta_t dx' = u^2 + u_x \int_0^x u dx',$$

and

$$\partial_t \left(\eta \int_0^y \eta_t dy' \right) = \eta_t \int_0^x \eta_t dx' + \eta(\eta_x - \eta_x(0)) = u \int_0^x u dx' + \eta(\eta_x - \eta_x(0)).$$

Then, combining (9) and (8), we obtain a two-dimensional system:

$$\partial_t \begin{bmatrix} u \\ \eta \end{bmatrix} = \begin{bmatrix} \eta_{xx} + \mu^2 \left(\frac{1}{3} \eta_{xxx} + u^2 + u_x \int_0^x u dx' - \frac{1}{\pi} \int_0^\infty u \int_0^x u dx' + \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] dy \right) \\ u \end{bmatrix}. \quad (10)$$

Now, consider (8) on a finite interval $[a, b]$, and let partition the interval into $n + 1$ points $\{x_k\}_{k=0}^n$, with $x_0 = a$ and $x_n = b$. Note that we need to pick the partition such that

$$\int_0^\infty u \int_0^x u dx' + \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] dy \approx \int_{x_0}^{x_n} u \int_0^x u dx' + \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] dy.$$

We proceed to forward Euler time stepping. First, observe that

$$\begin{aligned} u_t(x_k, t_j) &= \frac{u(x_k, t_{j+1}) - u(x_k, t_j)}{\Delta t} = f_1(\eta, u, t) & \implies u(x_k, t_{j+1}) &= u(x_k, t_j) + \Delta t f_1(\eta, u, t) \\ \eta_t(x_k, t_j) &= \frac{\eta(x_k, t_{j+1}) - \eta(x_k, t_j)}{\Delta t} = f_2(\eta, u, t) & \implies \eta(x_k, t_{j+1}) &= \eta(x_k, t_j) + \Delta t f_2(\eta, u, t), \end{aligned}$$

where

$$f_1(\eta, u, t) = \eta_{xx} + \mu^2 \left(\frac{1}{3} \eta_{xxx} + u^2 + u_x \int_0^x u dx' - \frac{1}{\pi} \int_0^\infty u \int_0^x u dx' + \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] dy \right)$$

$$f_2(\eta, u, t) = u(x_k, t_j).$$

Observe that the highest order derivative is 4, so we need to use five-point stencils. In particular, we use five point midpoint stencil for x_2, \dots, x_{n-2} , and five point, one-sided stencils at x_0, x_1, x_{n-1}, x_n . First, consider the system for x_2, \dots, x_{n-2} :

$$\begin{aligned} f_1(\eta(x_k, t_j), u(x_k, t_j), t) &= \eta(x_k, t_j)_{xx} + \frac{\mu^2}{3} \eta(x_k, t_j)_{xxxx} \\ &+ \mu^2 \left(u^2(x_k) + u_x(x_k) \int_0^{x_k} u \, dx' - \frac{1}{\pi} \int_0^{x_n} u \int_0^y u \, dx' + \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] dy \right), \end{aligned}$$

where we have separated the linear and nonlinear terms. The only difference between this system and the whole-line system is the non-linear term; in other words, we can reuse our prior work on the linear term, and only deal with the non-linear term. Let

$$C_k = u^2(x_k) + u_x(x_k) \int_0^{x_k} u \, dx' - \frac{1}{\pi} \int_0^{x_n} \left(u \int_0^y u \, dx' + \eta(\eta_x - \eta_x(0)) \right) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] dy,$$

and let \mathcal{C} be the column vector of C_k ; thus, \mathcal{C} represents the non-linear part of the system. To discretise C_k , note that

$$\begin{aligned} \int_0^{x_k} u \, dx' &= \frac{\Delta x}{2} \sum_{i=0}^{k-1} u(x_{i+1}) + u(x_i) \\ \int_0^{x_n} u \int_0^y u \, dx' \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] dy &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \left(u \int_0^y u \, dx' \right) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] dy \\ &= \frac{\Delta x}{2} \sum_{i=0}^{n-1} u(x_{i+1}) \int_0^{x_{i+1}} u \, dx' \left[\frac{1}{(x_k - x_{i+1})^2} + \frac{1}{(x_k + x_{i+1})^2} \right] + u(x_i) \int_0^{x_i} u \, dx' \left[\frac{1}{(x_k - x_i)^2} + \frac{1}{(x_k + x_i)^2} \right] \\ &= \frac{\Delta x}{2} \sum_{i=0}^{n-1} \frac{\Delta x}{2} u(x_{i+1}) \left(\sum_{j=0}^i u(x_j) + u(x_{j+1}) \right) \left[\frac{1}{(x_k - x_{i+1})^2} + \frac{1}{(x_k + x_{i+1})^2} \right] \\ &\quad + \frac{\Delta x}{2} u(x_i) \left(\sum_{j=0}^{i-1} u(x_j) + u(x_{j+1}) \right) \left[\frac{1}{(x_k - x_i)^2} + \frac{1}{(x_k + x_i)^2} \right] \\ \int_0^{x_n} \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] dy &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] dy \\ &= \frac{\Delta x}{2} \sum_{i=0}^{n-1} \eta(x_{i+1})(\eta_x(x_{i+1}) - \eta_x(0)) \left[\frac{1}{(x_k - x_{i+1})^2} + \frac{1}{(x_k + x_{i+1})^2} \right] + \eta(x_i)(\eta_x(x_i) - \eta_x(0)) \left[\frac{1}{(x_k - x_i)^2} + \frac{1}{(x_k + x_i)^2} \right]. \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}
C_k &= u^2(x_k) + u_x(x_k) \int_0^{x_k} u \, dx' - \frac{1}{\pi} \int_0^{x_n} \left(u \int_0^y u \, dx' + \eta(\eta_x - \eta_x(0)) \right) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] dy \\
&= u^2(x_k) + \frac{\Delta x}{2} u_x(x_k) \sum_{i=0}^{k-1} u(x_{i+1}) + u(x_i) \\
&\quad - \frac{1}{\pi} \frac{\Delta x}{2} \sum_{i=0}^{n-1} \left(\frac{\Delta x}{2} u(x_{i+1}) \left(\sum_{j=0}^i u(x_j) + u(x_{j+1}) \right) + \eta(x_{i+1})(\eta_x(x_{i+1}) - \eta_x(0)) \right) \left[\frac{1}{(x_k - x_{i+1})^2} + \frac{1}{(x_k + x_{i+1})^2} \right] \\
&\quad + \left(\frac{\Delta x}{2} u(x_i) \left(\sum_{j=0}^{i-1} u(x_j) + u(x_{j+1}) \right) + \eta(x_i)(\eta_x(x_i) - \eta_x(0)) \right) \left[\frac{1}{(x_k - x_i)^2} + \frac{1}{(x_k + x_i)^2} \right],
\end{aligned}$$

where the expressions for derivatives depend on the point x_k . Let

$$F_i = \frac{\Delta x}{2} u(x_i) \left(\sum_{j=0}^{i-1} u(x_j) + u(x_{j+1}) \right) + \eta(x_i)(\eta_x(x_i) - \eta_x(0)), \quad i = 0, \dots, n,$$

which we will store as an array. This simplification, we obtain

$$C_k = u^2(x_k) + \frac{\Delta x}{2} u_x(x_k) \sum_{i=0}^{k-1} u(x_{i+1}) + u(x_i) - \frac{\Delta x}{2\pi} \sum_{i=0}^{n-1} F_{i+1} \left[\frac{1}{(x_k - x_{i+1})^2} + \frac{1}{(x_k + x_{i+1})^2} \right] + F_i \left[\frac{1}{(x_k - x_i)^2} + \frac{1}{(x_k + x_i)^2} \right],$$

With this in mind, we obtain the finite differences system:

$$\begin{bmatrix} u(x_0, t_{j+1}) \\ u(x_1, t_{j+1}) \\ u(x_2, t_{j+1}) \\ \vdots \\ u(x_{n-2}, t_{j+1}) \\ u(x_{n-1}, t_{j+1}) \\ u(x_n, t_{j+1}) \end{bmatrix} = \begin{bmatrix} u(x_0, t_j) \\ u(x_1, t_j) \\ u(x_2, t_j) \\ \vdots \\ u(x_{n-2}, t_j) \\ u(x_{n-1}, t_j) \\ u(x_n, t_j) \end{bmatrix} + \Delta t \begin{bmatrix} f_1(\eta(x_0, t_j), u(x_0, t_j), t) \\ f_1(\eta(x_1, t_j), u(x_1, t_j), t) \\ f_1(\eta(x_2, t_j), u(x_2, t_j), t) \\ \vdots \\ f_1(\eta(x_{n-2}, t_j), u(x_{n-2}, t_j), t) \\ f_1(\eta(x_{n-1}, t_j), u(x_{n-1}, t_j), t) \\ f_1(\eta(x_n, t_j), u(x_n, t_j), t) \end{bmatrix}$$

$$= \begin{bmatrix} u(x_0, t_j) \\ u(x_1, t_j) \\ u(x_2, t_j) \\ \vdots \\ u(x_{n-2}, t_j) \\ u(x_{n-1}, t_j) \\ u(x_n, t_j) \end{bmatrix} + \Delta t \mathcal{A} \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-2} \\ \eta_{n-1} \\ \eta_n \end{bmatrix} + \Delta t \mathcal{C}$$

4 Approximate equations: half-line

In this section, we derive the approximate equations from

$$\eta_{tt} - \eta_{xx} = \varepsilon \left(\frac{1}{3} \eta_{xxxx} + \frac{d}{dx} \frac{1}{\pi} \int_0^\infty \partial_t \left(\eta \int_0^y \eta_t dy' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy + \frac{1}{2} \partial_x^2 \left(\int_0^x \eta_t dx' \right)^2 \right). \quad (11)$$

As we approximate, we assume an expansion of η in ε :

$$\eta = \eta_0 + \varepsilon \eta_1 + \mathcal{O}(\varepsilon^2). \quad (12)$$

First order approximation

Substitution of (12) into equation (11) yields

$$\eta_{0tt} - \eta_{0xx} + \varepsilon(\eta_{1tt} - \eta_{1xx}) = \varepsilon \left[\frac{1}{3} \eta_{0xxxx} + \frac{d}{dx} \frac{1}{\pi} \int_0^\infty \partial_t \left(\eta \int_0^y \eta_t dy' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy + \frac{1}{2} \partial_x^2 \left(\int_{-\infty}^x (\eta_0 + \varepsilon \eta_1)_t dx' \right)^2 \right] + \mathcal{O}(\varepsilon^2). \quad (13)$$

In the leading order $\mathcal{O}(\varepsilon^0)$, equation (13) becomes

$$\eta_{0tt} - \eta_{0xx} = 0. \quad (14)$$

This is the wave equation with velocity 1, whose solution depends on the type of boundary conditions we prescribe for η at $x = 0$. For now, we prescribe

$$\eta_x(0, t) = 0.$$

The general solution is

$$\eta(x, t) = \begin{cases} F(x-t) + G(x+t) & x > t \\ F(t-x) + G(x+t) & x < t \end{cases},$$

where F, G are to be determined.

Second order approximation

As in the velocity potential case, we employ multiple scales. First, we find an expression for η_0 . We introduce

$$\tau_0 = t, \quad \tau_1 = \varepsilon t, \quad \tau_2 = \varepsilon^2 t, \dots,$$

so that

$$\eta(x, t) = \eta(x, \tau_0, \tau_1, \dots).$$

With this in mind, the expansion (12) becomes

$$\eta(x, \tau_0, \tau_1, \dots) = \eta_0(x, \tau_0, \tau_1, \dots) + \mathcal{O}(\varepsilon^1). \quad (15)$$

Substituting (15) into (11), within $\mathcal{O}(\varepsilon^0)$, we obtain

$$\eta_{0\tau_0\tau_0} - \eta_{0xx} = 0, \quad (16)$$

so that the general solution is

$$\eta_0(x, \tau_0, \tau_1, \dots) = \begin{cases} F(x - \tau_0, \tau_1, \dots) + G(x + \tau_0, \tau_1, \dots) & x > t \\ F(\tau_0 - x, \tau_1, \dots) + G(x + \tau_0, \tau_1, \dots) & x < t \end{cases},$$

where we recalled the boundary conditions $\eta_x(0, t) = 0$. Now, although we have found an expression for η_0 , the functions F, G used are still general functions. To determine F, G , we proceed to the next order, i.e. $\mathcal{O}(\varepsilon^1)$. We introduce

$$\xi = x - \tau_0 \quad \zeta = x + \tau_0$$

so that

$$\eta_0(x, \tau_0, \tau_1, \dots) = \begin{cases} F(\xi, \tau_1, \dots) + G(\zeta, \tau_1, \dots) & x > t \\ F(-\xi, \tau_1, \dots) + G(\zeta, \tau_1, \dots) & x < t \end{cases},$$

and

$$\begin{aligned} \partial_x &= \partial_\xi \frac{d\xi}{dx} + \partial_\zeta \frac{d\zeta}{dx} = \partial_\xi + \partial_\zeta, \\ \partial_t &= \partial_\xi \frac{d\xi}{dt} + \partial_\zeta \frac{d\zeta}{dt} + \partial_{\tau_1} \frac{d\tau_1}{dt} = -\partial_\xi + \partial_\zeta + \varepsilon \partial_{\tau_1}. \end{aligned}$$

We consider the case $x > t$. We can rewrite (15) as follows

$$\begin{aligned} \eta &= \eta_0 + \varepsilon \eta_1 + \mathcal{O}(\varepsilon^2) \\ &= F(x - t, \varepsilon t, \dots) + G(x + t, \varepsilon t, \dots) + \varepsilon \eta_1 + \mathcal{O}(\varepsilon^2) \end{aligned}$$

$$\begin{aligned}
&= F(\xi, \tau_1, \dots) + G(\zeta, \tau_1, \dots) + \varepsilon \eta_1 + \mathcal{O}(\varepsilon^2) \\
&= F + G + \varepsilon \eta_1 + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

For ease of writing, we suppressed explicit dependence on variables, though the reader should bear in mind that function F (G) depend on ξ (ζ), τ_1, τ_2 , etc. In addition, observe that

$$(\partial_t^2 - \partial_x^2) = (-4\partial_\xi \partial_\zeta + 2\varepsilon(\partial_\zeta \partial_{\tau_1} - \partial_\xi \partial_{\tau_1}) + \varepsilon^2 \partial_{\tau_1}^2),$$

so that the LHS of (??) becomes

$$(\partial_t^2 - \partial_x^2)\eta = \varepsilon(-4\eta_{1\xi\zeta} - 2F_{\tau_1\xi} + 2G_{\tau_1\zeta}) + \mathcal{O}(\varepsilon^2). \quad (17)$$

Now, we deal with the RHS of (??). By appropriate substitutions, the terms become:

$$\begin{aligned}
\frac{1}{3}\eta_{xxxx} &= \frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta} + \mathcal{O}(\varepsilon)); \\
\left(\int_0^x \eta_t dx'\right)^2 &= \left(\int_0^x \eta_{0t} dx'\right)^2 + \mathcal{O}(\varepsilon) \\
&= \left(\int_0^x (-\partial_{\xi'} + \partial_{\zeta'} + \varepsilon \partial_{\tau_1})(F + G) dx'\right)^2 + \mathcal{O}(\varepsilon) \\
&= \left(\int_0^x -F_{\xi'} + G_{\zeta'} dx'\right)^2 + \mathcal{O}(\varepsilon) \\
&= \left(\int_0^x F_{\xi'} dx'\right)^2 - 2\left(\int_0^x F_{\xi'} dx'\right)\left(\int_0^x G_{\zeta'} dx'\right) + \left(\int_0^x G_{\zeta'} dx'\right)^2 + \mathcal{O}(\varepsilon) \\
&= F^2 - 2FG + G^2 + \mathcal{O}(\varepsilon),
\end{aligned}$$

where for the last line we translate $\xi' = x' - t, \zeta' = x' + t$ to obtain

$$\begin{aligned}
\int_0^x F_{\xi'}(x' - \tau_0, \tau_1) dx' &= \int_{-t}^{x-t} F_{\xi'}(\xi', \tau_1) d\xi' = \int_{-\tau_0}^{\xi} F_{\xi'}(\xi', \tau_1) d\xi' = F(\xi, \tau_1) - F(-\tau_0, \tau_1), \\
\int_0^x G_{\zeta'}(x' + \tau_0, \tau_1) dx' &= \int_t^{x+t} G_{\zeta'}(\zeta', \tau_1) d\zeta' = \int_{\tau_0}^{\zeta} G_{\zeta'}(\zeta', \tau_1) d\zeta' = G(\zeta, \tau_1) - G(\tau_0, \tau_1).
\end{aligned}$$

For now, we assume that $F(x - \tau_0, \tau_1), G(x + \tau_0, \tau_1)$ vanish at $x = 0$. A brief discussion will be given at the end. Finally, we deal with the Hilbert transform term:

$$\int_0^\infty \partial_t \left(\eta \int_0^y \eta_t dy' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy = \int_0^\infty (-\partial_\xi + \partial_\zeta + \varepsilon \partial_{\tau_1}) \left((\eta_0 + \varepsilon \eta_1) \int_0^y (\eta_0 + \varepsilon \eta_1)_t dy' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy + \mathcal{O}(\varepsilon^2)$$

$$\begin{aligned}
&= \int_0^\infty (-\partial_\xi + \partial_\zeta) \left((\eta_0 + \varepsilon\eta_1) \int_0^y (\eta_0 + \varepsilon\eta_1)_t dy' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy + \mathcal{O}(\varepsilon) \\
&= \int_0^\infty (-\partial_\xi + \partial_\zeta) \left(\eta_0 \int_0^y (\eta_0)_t dy' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy + \mathcal{O}(\varepsilon) \\
&= \int_0^\infty (-\partial_\xi + \partial_\zeta) \left(\eta_0 \int_0^y (-\partial_\xi + \partial_\zeta + \varepsilon\partial_{\tau_1}) \eta_0 dy' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy + \mathcal{O}(\varepsilon) \\
&= \int_0^\infty (-\partial_\xi + \partial_\zeta) \left(\eta_0 \int_0^y (-\partial_\xi + \partial_\zeta) \eta_0 dy' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy + \mathcal{O}(\varepsilon) \\
&= \int_0^\infty (-\partial_\xi + \partial_\zeta) \left((F+G) \int_0^y (-F_\xi + G_\zeta) dy' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy + \mathcal{O}(\varepsilon) \\
&= \int_0^\infty (-\partial_\xi + \partial_\zeta) \left((F+G) \int_0^y (-F_{\xi'} + G_{\zeta'}) dy' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy + \mathcal{O}(\varepsilon) \\
&= \int_0^\infty (-\partial_\xi + \partial_\zeta) ((F+G)(-F+G)) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy + \mathcal{O}(\varepsilon) \\
&= \int_0^\infty (-\partial_\xi + \partial_\zeta) ((-F^2 + G^2)) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy + \mathcal{O}(\varepsilon) \\
&= \int_0^\infty (2FF_\xi + 2GG_\zeta) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy + \mathcal{O}(\varepsilon).
\end{aligned}$$

Substitution of terms into the RHS of (11) leads to:

$$\begin{aligned}
&\frac{1}{3}\eta_{xxxx} + \frac{d}{dx} \frac{1}{\pi} \int_0^\infty \partial_t \left(\eta \int_0^y \eta_t dy' \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy + \frac{1}{2} \partial_x^2 \left(\int_0^x \eta_t dx' \right)^2 \\
&= \frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + \frac{1}{\pi}(\partial_\xi + \partial_\zeta) \int_0^\infty (2FF_\xi + 2GG_\zeta) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy + \frac{1}{2}(\partial_\xi^2 + 2\partial_\xi\partial_\zeta + \partial_\zeta^2)(F^2 - 2FG + G^2) + \mathcal{O}(\varepsilon) \\
&= \frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + \frac{1}{\pi}(\partial_\xi + \partial_\zeta) \int_0^\infty (2FF_\xi + 2GG_\zeta) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy + \partial_\xi(FF_\xi - F_\xi G) + \partial_\zeta(GG_\zeta - FG_\zeta) - 2F_\xi G_\zeta + \mathcal{O}(\varepsilon). \tag{18}
\end{aligned}$$

Combining (17) and (18), in $\mathcal{O}(\varepsilon^1)$ we have

$$-4\eta_{1\xi\xi} = 2F_{\tau_1\xi} - 2G_{\tau_1\xi} + \frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + \frac{1}{\pi}(\partial_\xi + \partial_\zeta) \int_0^\infty (2FF_\xi + 2GG_\zeta) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy + \partial_\xi(FF_\xi - F_\xi G) + \partial_\zeta(GG_\zeta - FG_\zeta) - 2F_\xi G_\zeta \tag{19}$$

By rearranging appropriately, (19) becomes

$$\begin{aligned}
-4\eta_{1\xi\zeta} = & \partial_\xi(2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + FF_\xi + \frac{1}{\pi} \int_0^\infty 2FF_\xi \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy) \\
& + \partial_\zeta(-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + GG_\zeta + \frac{1}{\pi} \int_0^\infty 2GG_\zeta \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy) \\
& + \partial_\xi(-F_\xi G + \frac{1}{\pi} \int_0^\infty 2GG_\zeta \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy) \\
& + \partial_\zeta(-FG_\zeta + \frac{1}{\pi} \int_0^\infty 2FF_\xi \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy) - 2F_\xi G_\zeta
\end{aligned} \tag{20}$$

Integration of (20) with respect to ζ yields

$$\begin{aligned}
-4\eta_{1\xi} = & \partial_\xi(2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + FF_\xi + \frac{1}{\pi} \int_0^\infty 2FF_\xi \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy)\zeta \\
& + (-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + GG_\zeta + \frac{1}{\pi} \int_0^\infty 2GG_\zeta \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy) \\
& + \partial_\xi(-F_\xi G + \frac{1}{\pi} \int_0^\infty 2GG_\zeta \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy) \\
& + \int (-FG_\zeta + \frac{1}{\pi} \int_0^\infty 2FF_\xi \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy) d\zeta - 2F_\xi G
\end{aligned}$$

and further integration with respect to ξ leads to

$$\begin{aligned}
-4\eta_{11} = & (2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + FF_\xi + \frac{1}{\pi} \int_0^\infty 2FF_\xi \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy)\zeta \\
& + (-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + GG_\zeta + \frac{1}{\pi} \int_0^\infty 2GG_\zeta \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy)\xi \\
& + \int \frac{1}{\pi} \int_0^\infty 2GG_\zeta \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy) d\xi + \int \frac{1}{\pi} \int_0^\infty 2FF_\xi \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy) d\zeta - 4FG.
\end{aligned}$$

Since η_{11} must be bounded, we must have

$$2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + FF_\xi + \frac{1}{\pi} \int_0^\infty 2FF_\xi \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy = 0, \tag{21}$$

$$-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + GG_{\zeta} + \frac{1}{\pi} \int_0^\infty 2GG_{\zeta} \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy = 0. \quad (22)$$

In other words, we have obtained two KdV-like equations, (21) and (22), whose solutions describe behaviour of the surface elevation in the leading order. Now, for $x < t$, by the same procedure, the equation for G remains the same but (21) becomes

$$2F(-\xi)_{\tau_1} + \frac{1}{3}F(-\xi)_{\xi\xi\xi} + F(-\xi)F(-\xi)_{\xi} + \frac{1}{\pi} \int_0^\infty 2F(-\xi)F(-\xi)_{\xi} \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy = 0,$$

though one has to assume that $F(\tau_0, \tau_1) = 0$, instead of $F(-\tau_0, \tau_1) = 0$.

4.1 Discussion of F, G at $x = 0$.

As is seen, in order to obtain

$$\begin{aligned} \left(\int_0^x \eta_t dx' \right)^2 &= F(\xi)^2 - 2F(\xi)G + G^2 + \mathcal{O}(\varepsilon), & x > t, \\ \left(\int_0^x \eta_t dx' \right)^2 &= F(-\xi)^2 + 2F(-\xi)G + G^2 + \mathcal{O}(\varepsilon), & x < t, \end{aligned}$$

we had to assume that at $x = 0$

$$\begin{aligned} F(-\tau_0, \tau_1), \quad G(\tau_0, \tau_1) &= 0, & x > t, \\ F(\tau_0, \tau_1), \quad G(\tau_0, \tau_1) &= 0, & x < t. \end{aligned} \quad (23)$$

Conditions in (23) therefore imply that at $x = 0$,

$$\begin{aligned} \eta(x, t, \varepsilon t) &= F(x - t, \varepsilon t) + G(x + t, \varepsilon t) = 0, & x > t, \\ \eta(x, t, \varepsilon t) &= F(t - x, \varepsilon t) + G(x + t, \varepsilon t) = 0, & x < t. \end{aligned}$$

In other words,

$$\eta(0, t, \varepsilon t) = F(-t, \varepsilon t) + G(t, \varepsilon t) = 0, \quad 0 > t, \quad (24)$$

$$\eta(0, t, \varepsilon t) = F(t, \varepsilon t) + G(t, \varepsilon t) = 0, \quad 0 < t. \quad (25)$$

Thus, it seems like $\eta(0, t) = 0$ is the right condition to obtain KdV-like equations. Namely, if we do not assume that $F(-t, \varepsilon t), G(t, \varepsilon t)$ vanish, then for $x > t$, we obtain the following general expression for the η_1 :

$$\begin{aligned}
-4\eta_1 = & \zeta(2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + (F - F(-\tau_0))F_{\xi} + \frac{1}{\pi} \int_0^{\infty} \left(2FF_{\xi} - F_{\xi}(F(-\tau_0) + G(\tau_0)) \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy) \\
& + \xi(-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + (G - G(\tau_0))G_{\zeta} + \frac{1}{\pi} \int_0^{\infty} \left(2GG_{\zeta} + G_{\zeta}(F(-\tau_0) + G(\tau_0)) \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy) \\
& + (-F_{\xi} \int G - G(t) d\zeta) + \int \frac{1}{\pi} \int_0^{\infty} \left(2GG_{\zeta} + G_{\zeta}(F(-\tau_0) + G(\tau_0)) \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy d\zeta \\
& + (-G_{\zeta} \int F - F(-t) d\xi) + \int \frac{1}{\pi} \int_0^{\infty} \left(2FF_{\xi} - F_{\xi}(F(-\tau_0) + G(\tau_0)) \right) \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy d\xi - 2FG.
\end{aligned} \tag{26}$$

If we impose that (24), then we obtain

$$\begin{aligned}
-4\eta_1 = & \zeta(2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + (F - F(-\tau_0))F_{\xi} + \frac{1}{\pi} \int_0^{\infty} 2FF_{\xi} \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy) \\
& + \xi(-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + (G - G(\tau_0))G_{\zeta} + \frac{1}{\pi} \int_0^{\infty} 2GG_{\zeta} \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy) \\
& + (-F_{\xi} \int G - G(t) d\zeta) + \int \frac{1}{\pi} \int_0^{\infty} 2GG_{\zeta} \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy d\zeta \\
& + (-G_{\zeta} \int F - F(-t) d\xi) + \int \frac{1}{\pi} \int_0^{\infty} 2FF_{\xi} \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy d\xi - 2FG,
\end{aligned}$$

which we can rewrite by recalling

$$F(\xi, \tau_1) - F(-\tau_0, \tau_1) = \int_0^x F_{\xi'}(\xi', \tau_1) dx', \quad G(\zeta, \tau_1) - G(\tau_0, \tau_1) = \int_0^x G_{\zeta'}(\zeta', \tau_1) dx'.$$

Thus,

$$\begin{aligned}
-4\eta_1 = & \zeta(2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + F_{\xi} \int_0^x F_{\xi'}(\xi', \tau_1) dx' + \frac{1}{\pi} \int_0^{\infty} 2FF_{\xi} \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy) \\
& + \xi(-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + G_{\zeta} \int_0^x G_{\zeta'}(\zeta', \tau_1) dx' + \frac{1}{\pi} \int_0^{\infty} 2GG_{\zeta} \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy) \\
& + (-F_{\xi} \int_0^x \int_0^x G_{\zeta'}(\zeta', \tau_1) dx' d\zeta) + \int \frac{1}{\pi} \int_0^{\infty} 2GG_{\zeta} \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy d\zeta \\
& + (-G_{\zeta} \int_0^x \int_0^x F_{\xi'}(\xi', \tau_1) dx' d\xi) + \int \frac{1}{\pi} \int_0^{\infty} 2FF_{\xi} \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy d\xi - 2FG.
\end{aligned}$$

Finally, the boundedness condition gives rise to two KdV-like, integro-differential equations:

$$\begin{aligned}
2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + F_{\xi} \int_0^x F_{\xi'}(\xi', \tau_1) dx' + \frac{1}{\pi} \int_0^{\infty} 2FF_{\xi} \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy &= 0, \\
-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + G_{\zeta} \int_0^x G_{\zeta'}(\zeta', \tau_1) dx' + \frac{1}{\pi} \int_0^{\infty} 2GG_{\zeta} \left[\frac{1}{x-y} + \frac{1}{x+y} \right] dy &= 0.
\end{aligned}$$

References

- [1] Tom M. Apostol, *Mathematical analysis*, Pearson, 1974.