YaleNUSCollege

APPROXIMATE EQUATIONS IN THE SHALLOW WATER REGIME OF THE WATER WAVE PROBLEM ON UNBOUNDED DOMAINS

SULTAN AITZHAN

Capstone Final Report for BSc (Honours) in

Mathematical, Computational and Statistical Sciences

Supervised by: Prof. Katie Oliveras, Prof. Dave Smith

AY 2019/2020

Yale-NUS College Capstone Project

DECLARATION & CONSENT

1. I declare that the product of this Project, the Thesis, is the end result of my own work and that due acknowledgement has been given in the bibliography and references to ALL sources be they printed, electronic, or personal, in accordance with the academic regulations of Yale-NUS College.

2. I acknowledge that the Thesis is subject to the policies relating to Yale-NUS College Intellectual Property (Yale-NUS HR 039).

ACCESS LEVEL

3. I agree, in consultation with my supervisor(s), that the Thesis be given the access level specified below: [check one only]

   o Unrestricted access
   Make the Thesis immediately available for worldwide access.

   o Access restricted to Yale-NUS College for a limited period
   Make the Thesis immediately available for Yale-NUS College access only from _____________ (mm/yyyy) to _______________ (mm/yyyy), up to a maximum of 2 years for the following reason(s): (please specify; attach a separate sheet if necessary):
   ____.

   After this period, the Thesis will be made available for worldwide access.

   o Other restrictions: (please specify if any part of your thesis should be restricted)
   ____
   ____

____
Name & Residential College of Student

____________________________________ ___________________________
Signature of Student Date

____________________________________ ___________________________
Name & Signature of Supervisor Date

Contents

Frequently Used Notation

$\mathbb{R}$	Set of real numbers
$\mathbf{v}$	Vector in $\mathbb{R}^3$
$v(\mathbf{x})$	Scalar field, i.e. map from $\mathbb{R}^3$ to $\mathbb{R}$
$\mathbf{v}(\mathbf{x})$	Vector field, i.e. map from $\mathbb{R}^3$ to $\mathbb{R}^3$
$\dfrac{\partial f}{\partial x_i}$ or $\partial_{x_i} f$ or f_{x_i}	Partial derivative of f with respect to variable x_i
∇	Gradient operator given by transpose of $(\partial_x, \partial_y, \partial_z)$
Δ	Divergence operator $\Delta = \nabla \cdot \nabla$
ε	Parameter for nonlinearity (Chapter 3)
μ	Parameter for amplitude (Chapter 3)
$f(x) = \mathcal{O}(g(x))$	There is $M > 0$ such that for sufficiently large x, $$\lvert f(x) \geqslant M \lvert g(x) \rvert$$
$f(x) \ll g(x)$	$f(x)$ is much less than $g(x)$
$f(x) \gg g(x)$	$f(x)$ is much greater than $g(x)$
$\mathcal{F}_k\left\{f(x)\right\}$ or $\widehat{f}_k$:	Fourier transform of $f(x)$, given by $$\mathcal{F}_k\left\{f(x)\right\} = \int_{-\infty}^{\infty} e^{-ikx} f(x)\,\mathrm{d}x$$
$\mathcal{F}_s^k\left\{f(x)\right\}$ or $\widehat{f}_c^k$:	Fourier Sine transform of $f(x)$, given by $$\mathcal{F}_s^k\left\{f(x)\right\} = \int_0^{\infty} \sin(kx) f(x)\,\mathrm{d}x$$
$\mathcal{F}_c^k\left\{f(x)\right\}$ or $\widehat{f}_c^k$:	Fourier Cosine transform of $f(x)$, given by

$$\mathcal{F}_c^k\left\{f(x)\right\} = \int_0^\infty \cos(kx) f(x)\,\mathrm{d}x$$

$\mathcal{F}_k^{-1}\left\{F(k)\right\}$: Inverse Fourier transform of $f(k)$, given by

$$\mathcal{F}_k^{-1}\left\{F(k)\right\} = \frac{1}{2\pi}\int_{-\infty}^\infty e^{ikx} F(k)\,\mathrm{d}k$$

$\left\{\mathcal{F}_s^k\right\}^{-1}\left\{F(k)\right\}$: Inverse Sine Fourier transform of $f(k)$, given by

$$\left\{\mathcal{F}_s^k\right\}^{-1}\left\{F(k)\right\} = \frac{2}{\pi}\int_0^\infty \sin(kx) F(k)\,\mathrm{d}k$$

$\left\{\mathcal{F}_c^k\right\}^{-1}\left\{F(k)\right\}$: Inverse Fourier Cosine transform of $F(k)$, given by

$$\left\{\mathcal{F}_c^k\right\}^{-1}\left\{F(k)\right\} = \frac{2}{\pi}\int_0^\infty \cos(kx) F(k)\,\mathrm{d}k$$

1

Chapter 1

Introduction

Often, mathematical modelling of the real-world phenomena results in ordinary and partial differential equations. Depending on the equations, mathematicians may or may not have the tools to obtain solutions or understand the phenomenon very well. Fortunately, there are techniques that allow one to analyse differential equations without directly solving them. One such tool is asymptotic analysis which leads to simplified equations that are very similar to original equations. As such, solutions of the simplified equations model the real-world phenomenon subject to some error estimates.

In particular, one problem that is amenable to asymptotic methods is the *water wave problem*, which describes the behaviour of water and its surface under certain conditions. Assuming an irrotational, incompressible, and inviscid fluid, and a domain with flat bottom, the equations of fluid motion are given by

$$\phi_{xx} + \phi_{zz} = 0, \qquad -h < z < \eta(x,t), \tag{1.1a}$$

$$\phi_z = 0, \qquad z = -h, \tag{1.1b}$$

$$\phi_t + g\eta + \frac{1}{2}(\phi_x^2 + \phi_z^2) = 0, \qquad z = \eta(x,t), \tag{1.1c}$$

$$\eta_t + \phi_x\eta_x = \phi_z, \qquad z = \eta(x,t), \tag{1.1d}$$

where $\phi(x,z,t)$ is the fluid velocity and $\eta(x,t)$ is the surface elevation. In addition, z is the vertical coordinate, x is the horizontal direction, and g is acceleration due to gravity. We let $x \in \mathbb{R}$, so that (1.1) is the water wave problem *on the whole line*. Although nonlinear partial differential equations (PDEs) (1.1d) and (1.1c) are hard to solve on their own, what makes the problem (1.1) truly difficult is the need to solve the Laplace's equation (1.1a) on a domain with an unknown shape.

To make the equations of motion more tractable, one can reformulate the problem and apply the tools of asymptotics. Of particular interest is the work of Ablowitz, Fokas, and Musslimani, 2006, henceforth referred to as the AFM formulation. In this paper, authors rewrite (1.1) as a system of two equations, for the surface variable $q(x,t) = \phi(x,\eta(x,t))$, i.e. the velocity evaluated at the surface. Taking advantage of a new system, various asymptotic reductions are performed. In well-understood physical conditions, the new formulation corresponds to the original model (Ablowitz, Fokas, and Musslimani, 2006)

One interesting physical condition is the shallow water regime, which is defined by small-amplitude waves that have a small depth relative to the wavelength. In this regime, asymptotic tools reveal that the fluid behaviour is governed by the following approximate equations: the *wave* equation in the leading order,

$$q_{tt} - q_{xx} = 0, \tag{1.2}$$

and two *Korteweg de Vries (KdV)* equations in the higher order,

$$\begin{aligned} F_T + \frac{1}{3}F_{\xi\xi\xi} + 3FF_\xi &= 0, \\ G_T + \frac{1}{3}G_{\zeta\zeta\zeta} + 3GG_\zeta &= 0, \end{aligned} \tag{1.3}$$

where $q(x,t) = F(\xi, T) + G(\zeta, T)$. Physically, F and G are right-going and left going waves, respectively. We refer to the model given by (1.2) and (1.3) as the *KdV model on the whole line*. We observe that on *the half-line*, when waves are bounded from one side by a wall, the rigorous derivation of the corresponding model remains unknown.

In this capstone project, we consider an alternative formulation of the problem (1.1), as presented in Oliveras and Vasan, 2013. Although slightly different from AFM formulation, it is contended that this formulation is well-suited for performing asymptotics. We further advocate the efficacy of this formulation, by deriving the KdV model. As a brief outline, in Chapter 2, we introduce the reader to asymptotic and perturbative methods. In Chapter 3, we explain the physical assumptions of the problem and describe the shallow-water regime. In Chapter 4, we reformulate the problem and derive the approximate equations. In Chapter 5, we describe an application of the formulation to a water wave problem on *the half line*, while attempting to rigorously derive a half-line model.

Chapter 2

Asymptotic and perturbative methods

In this chapter, we present an introduction to perturbation theory, and the most relevant technique for this project, the multiple scale analysis. The focus is on the illustration of ideas through examples, rather than rigorous justification and proofs. Examples are from Chapters 7 and 11 of Bender and Orszag, 1999.

Perturbation theory is a collection of techniques used for obtaining approximate solutions to problems typically involving some small parameter ε. The main idea is to represent the unknown variable f as a *perturbation series*, which is a formal power series

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots . \tag{2.1}$$

Substituting (2.1) into the original problem decomposes what is a difficult problem into many simpler ones. Solving for the first n terms in the series yields an approximated solution $f \approx f_0 + \varepsilon f_1 + \dots + \varepsilon^n f_n$. It should be recognised that with this approach, perturbation theory is most useful when the first few problems reveal the important features of the solution, and the remaining problems give only small corrections. The perturbative techniques are applied in numerous settings, including finding roots of polynomials and solving initial value problems for differential equations.

Since we deal with differential equations, we illustrate the method with an ordinary differential equation (ODE).

Example 1. Consider the following boundary value problem (BVP):

$$y'' + y = \frac{\cos x}{1+y}, \qquad y(0) = y(\pi/2) = 1.$$

Introduce a small parameter $\varepsilon > 0$ into the problem

$$y'' + y = \frac{\cos x}{1+\varepsilon y} = \cos x(1 - \varepsilon y + \mathcal{O}(\varepsilon^2)),$$

where we use geometric series in the last equality. Expanding $y = y_0 + \varepsilon y_1 + \mathcal{O}(\varepsilon^2)$ and substituting into the above equation gives

$$(y_0 + \varepsilon y_1)'' + y_0 + \varepsilon y_1 = \cos x(1 - \varepsilon y_0) + \mathcal{O}(\varepsilon^2).$$

By ordering the above into powers of ε, we simplify the original problem into the following problems

$$\mathcal{O}(\varepsilon^0): \qquad y_0'' + y_0 = \cos x, \qquad y_0(0) = y_0(\pi/2) = 1, \tag{2.2}$$

$$\mathcal{O}(\varepsilon^1): \qquad y_1'' + y_1 = -y_0 \cos x, \qquad y_1(0) = y_1(\pi/2) = 0. \tag{2.3}$$

Solving equations (2.2) and (2.3) recursively yields y_0 and y_1. An approximated solution becomes $y \approx y_0 + \varepsilon y_1 \to y_0 + y_1$, as $\varepsilon \to 1$. Thus, using perturbation series yields an approximate solution of the given BVP.

Although the idea is simple, problems may contain subtleties that require a careful examination before using perturbation series.

Example 2. Consider the following initial value problem (IVP) for the weakly nonlinear Duffing oscillator

$$y'' + y + \varepsilon y^3 = 0, \qquad y(0) = 1 \qquad y'(0) = 0. \tag{2.4}$$

Application of the regular perturbation series yields

$$y(t) \approx y_0 + \varepsilon y_1 = \cos t + \varepsilon \left(\frac{1}{32} \cos 3t - \frac{1}{32} \cos t + \frac{3}{8} t \sin t \right), \tag{2.5}$$

which converges as $\varepsilon \to 0$ for fixed x. Note that the convergence is only pointwise, not uniform. Indeed, for values $t \sim 1/\varepsilon$ or larger, the presence of $t \sin t$ implies that y_1 is unbounded in t; terms such as $t \sin t$ are called *secular*. However, solutions of the Duffing oscillator are known to be bounded. In particular, this suggests that the usual perturbation expansion of y is not sufficient, and that the secularity is an outcome of this misfortune.

Multiple scale analysis

When ordinary perturbative methods fail to give a uniformly accurate approximation, the method of *multiple scales* comes in handy. The idea is to introduce a new variable $\tau = \varepsilon t$. Physically, τ represents a longer time scale than t, since τ is not negligible when t is of order $1/\varepsilon$ or larger. Although $y(t)$ is a function of t alone, through introduction of τ, $y(t)$ becomes a function of t and τ, i.e $y(t, \tau)$. As such, the multiple scales looks for solutions as functions of both t and τ, treating these variables independently. It must be emphasised that such treatment in two variables is rather an artifice, introduced to eliminate secularities.

We illustrate the method on the Duffing oscillator, in Example (2). Formally, we write $y(t) = Y_0(t,\tau) + \varepsilon Y_1(t,\tau) + \mathcal{O}(\varepsilon^2)$. By chain rule, we have $\dfrac{\mathrm{d}}{\mathrm{d}t} = \dfrac{\partial}{\partial t} + \varepsilon \dfrac{\partial}{\partial \tau}$. Note

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial Y_0}{\partial t} + \varepsilon\left(\frac{\partial Y_0}{\partial \tau} + \frac{\partial Y_1}{\partial t}\right) + \mathcal{O}(\varepsilon^2), \tag{2.6}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{\partial^2 Y_0}{\partial t^2} + \varepsilon\left(2\frac{\partial^2 Y_0}{\partial \tau \partial t} + \frac{\partial^2 Y_1}{\partial t^2}\right) + \mathcal{O}(\varepsilon^2). \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.4) and collecting in powers of ε yields

$$\mathcal{O}(\varepsilon^0): \quad \frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0, \tag{2.8}$$

$$\mathcal{O}(\varepsilon^1): \quad \frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -Y_0^3 - 2\frac{\partial^2 Y_0}{\partial \tau \partial t}. \tag{2.9}$$

The general solution of (2.8) is $Y_0(t,\tau) = A(\tau)e^{it} + A^*(\tau)e^{-it}$, where $A(\tau)$ is an arbitrary complex function in τ. We determine $A(\tau)$ by requiring that secular terms do not appear in $Y_1(t,\tau)$. Substituting Y_0 into (2.9) gives

$$\begin{aligned}\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = &\left(-3A^2A^* - 2i\frac{\mathrm{d}A}{\mathrm{d}\tau}\right)e^{it} + \left(-3A(A^*)^2 + 2i\frac{\mathrm{d}A^*}{\mathrm{d}\tau}\right)e^{-it}\\ &- A^3e^{3it} - (A^*)^3e^{-3it}.\end{aligned} \tag{2.10}$$

Notice that e^{it} and e^{-it} appear in the solution of (2.8), which is the homogeneous version of (2.9). Therefore, unless the coefficients of e^{it} and e^{-it}

vanish, Y_1 will be secular in τ. To preclude secularity, we must have

$$\begin{aligned} -3A^2A^* - 2i\frac{\mathrm{d}A}{\mathrm{d}\tau} &= 0, \\ -3A(A^*)^2 + 2i\frac{\mathrm{d}A^*}{\mathrm{d}\tau} &= 0. \end{aligned}$$

Observe that the two equations are complex conjugate of each other, so $A(\tau)$ is not overdetermined. Solving for $A(\tau)$ along with initial conditions $y(0) = 1, y'(0) = 0$ yields $A(\tau) = \frac{1}{2}e^{i3\tau/8}$. Thus, we obtain $Y_0(t,\tau) = \cos(t + \frac{3}{8}\tau)$. Finally, using that $\tau = \varepsilon t$, we obtain the approximated solution

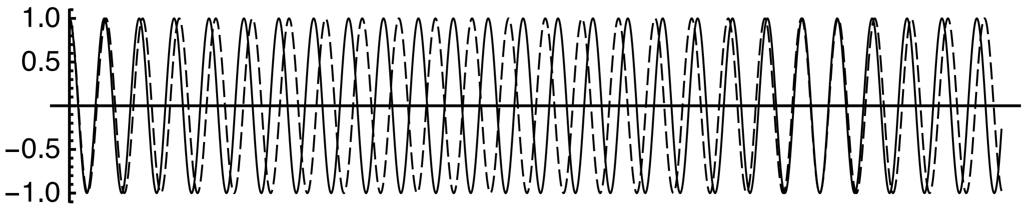
$$y(t) = \cos\left[t(1+\varepsilon\frac{3}{8})\right] + \mathcal{O}(\varepsilon), \qquad \varepsilon \to 0, \quad \varepsilon t = \mathcal{O}(1). \tag{2.11}$$

To conclude, we note that while (2.5) approximates y well for $0 \leqslant t \ll \mathcal{O}(1/\varepsilon)$, (2.11) approximates over a much larger range of t. Figure 2.1 illustrates this.

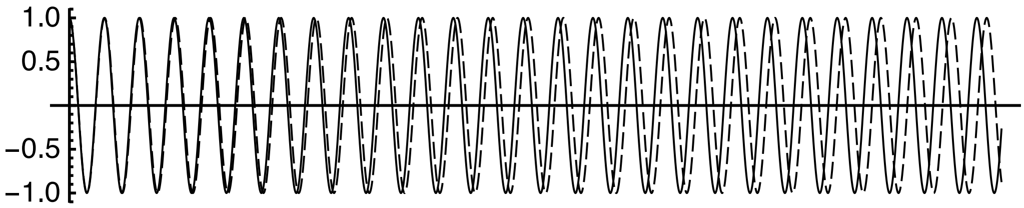
We remark that the choice of scales $\tau = \varepsilon t$ tends to be example-specific. More generally, one may choose $\tau = f(t)$, where f can be any function. For example, for an IVP $\{y'' + y - \varepsilon t y = 0, y(0) = 1, y'(0) = 0\}$, one uses $\tau = \sqrt{\varepsilon}t$, and for $y'' + \omega^2(\varepsilon t)y = 0$, one uses $\tau = \int^t \omega(\varepsilon s)\,\mathrm{d}s$.

Remark 3. It is important to understand the need for multiple scales. In the example of the Duffing oscillator, we could solve the problem numerically for the exact solution, or approximate the solution via multiple scales. A question arises: why use multiple scales when we can solve the differential equations numerically?

Note that many real-world phenomena are expressed in terms of PDEs, which are much harder to solve numerically than ODEs. Due to many



(A) Exact $y(t)$ (line) vs. $y(t) = \cos t$ (dashed)



(B) Exact $y(t)$ (line) vs. $y(t) = \cos\left[t(1+\varepsilon\frac{3}{8})\right]$ (dashed)

FIGURE 2.1: Superposed solutions of the Duffiing oscillator IVP for $\varepsilon = 0.1$, for $t \in [0, 160]$. Note that $\cos t$ is not valid for large values of t; when $t = 160$, $\cos t$ is one cycle out of phase with the exact solution $y(t)$ (Plot 2.1a). However, multiple scales solution approximates $y(t)$ closely, even for large values of t (Plot 2.1b).

differences in the physical phenomena, there is no unified analytic and numerical treatment. In addition, developing numerical schemes can be very tricky, since issues such as stability, error as well as the physical conditions need to be carefully addressed to obtain an effective software. Furthermore, numerically solving PDEs can be time-consuming and require large computational power, especially if the great precision is required. The latter is particularly important for real time predicting. This is another reason to prefer multiple scales. When appropriately applied, multiple scales and perturbation methods turn the original, difficult problem into many simpled problems. These problems are much easier to solve and provide further insight into the mathematics and physics of the problem.

Chapter 3

Water wave problem

In this chapter, we present the water wave problem in greater detail and discuss the shallow water limit.

3.1 The water wave problem

Conservation of mass (3.1) and conservation of momentum (3.2) are the two core principles that provide the relevant equations of fluid dynamics. The resulting equations are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{3.1}$$

$$\rho \left[v \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = \mathbf{F} - \nabla P + v_\star \Delta \mathbf{v}, \tag{3.2}$$

where, $\rho = \rho(\mathbf{x}, t)$ denotes the fluid mass density, $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ is the fluid velocity, $P(\mathbf{x}, t)$ refers to pressure, $\mathbf{F}(\mathbf{x})$ is an external force, and $v_\star$ is the kinematic viscosity due to frictional forces. Derivations of (3.1) and (3.2) can be found in Johnson, 1997, Chapter 3. Assuming that the fluid is inviscid, incompressible, and irrotational, one can follow Ablowitz, 2011, Section 5.1 to obtain the system,

$$\phi_{xx} + \phi_{zz} = 0 \qquad -h < z < \eta(x,t) \tag{3.3a}$$

$$\phi_z = 0 \qquad z = -h \tag{3.3b}$$

$$\phi_t + g\eta + \frac{1}{2}(\phi_x^2 + \phi_z^2) = 0 \qquad z = \eta(x,t) \tag{3.3c}$$

$$\eta_t + \phi_x \eta_x = \phi_z \qquad z = \eta(x,t) \tag{3.3d}$$

where $\phi(x,z,t)$ is a scalar field such that $\mathbf{v} = \nabla\phi$ and $\eta(x,t)$ is the surface elevation. In addition, z is the vertical coordinate, x is the horizontal direction, and g is acceleration due to gravity. See Figure 3.1 for a visual representation of domain, which we denote $S = \mathbb{R} \times (-h, \eta)$. In deriving the problem (3.3), the following main assumptions are made:

- The problem has 1 horizontal dimension $x \in \mathbb{R}$.
- The fluid velocity $\mathbf{v}$ tends to equilibrium as $|x| \to \infty$.
- The external force is the buoyancy driven by gravity, i.e. $\mathbf{F} = -\nabla(\rho_0 g z)$.
- The pressure vanishes on the surface, $P = 0$ at $z = \eta(x,t)$.
- The fluid density is constant, i.e. $\rho(\mathbf{x}, t) = \rho_0$.

Since the problem is expressed in terms of ϕ, the scalar potential of velocity $\mathbf{v}$, (3.3) is called the *velocity potential* formulation of the water wave problem.

We begin to describe the physical relevance of the problem (3.3). First, we elaborate on each equation:

(3.3a): The assumption that the fluid is irrotational means that the curl vanishes: $\nabla \times \mathbf{v} = 0$. The conservation of mass (3.1) then becomes

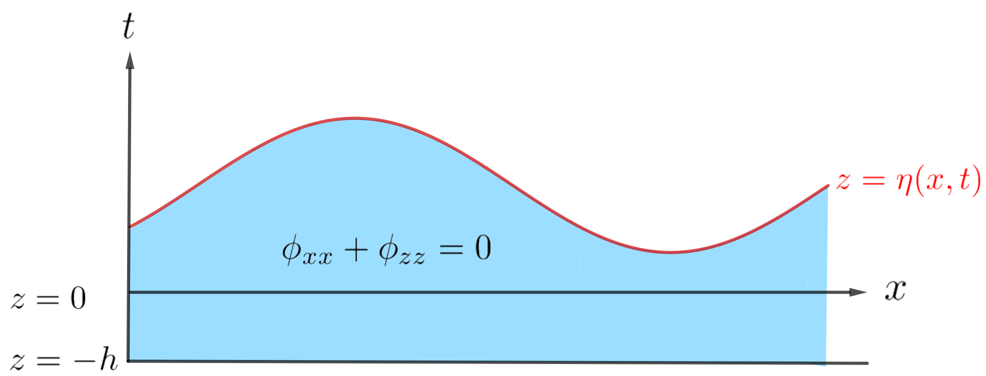


FIGURE 3.1: Schematic for the domain of the water wave problem (3.3).

$\nabla \cdot \mathbf{v} = \nabla \cdot \nabla\phi = 0$. In other words, (3.3a) represents that the fluid inside the domain S is incompressible and irrotational.

(3.3b): This equation is an assumption that the bottom is a flat and impermeable surface, so that the fluid cannot escape through the bottom. Since ϕ_z is the vertical velocity, (3.3b) means that there is no flow through the bottom. We call (3.3b) *the bottom condition*.

(3.3c): This equation is the conservation of momentum applied at the surface $z = \eta(x,t)$. Since the conservation of momentum is a statement about the balance of external forces and fluid at the fluid's surface, this equation describes the dynamics of the velocity potential on the boundary. As such, we term (3.3c) as *the dynamic boundary condition*.

(3.3d): This equation represents the condition that the surface η is a surface of the fluid. In other words, the surface $z = \eta(x,t)$ is always composed of fluid particles that remain on the surface. We call (3.3d) *the kinematic boundary condition*, since it is about the geometry and shape of the surface. The condition should be contrasted with the dynamic condition, which is about the interaction of forces acting at the surface.

Second, we explain the physical assumptions and considerations leading to the mathematical formulation of the problem. While mostly following Lannes, 2013, Chapter 1, we also expound on some points and provide additional references. Roughly speaking, the model is the most useful in applications to coastal oceanography, specifically in modelling tsunamis.

- The fluid is assumed to be continuous. We mainly deal with behaviour on scales that are large compared to the distance between molecules that comprise the fluid. Physical quantities such as mass and velocity are spread continuously throughout the region; this is *the continuity assumption*. This is important to note, since water may have discontinuities in the form of air bubbles. Whenever this happens to a significant degree, e.g. when waves break, the problem (3.3) is no longer valid. A microscopic description of fluids is given by the Newton's laws and Boltzmann equation (see Gallagher, 2019 for a survey).

- The fluid is assumed to be incompressible and inviscid. Since the fluid under interest is water: its density does not change both in space and time, and it is mostly inviscid (see Cohen and Kundu, 2008, Appendix A2). This justifies the assumption. One can extend the problem to work with compressible fluids. However, to account for viscosity, one needs to work with Navier-Stokes equations (see Chorin and Marsden, 1993, Section 1.3.

- The fluid is assumed to be irrotational. In applications considered here (e.g. tsunamis), rotational effects contribute little. In reality, when the scales are much larger, rotational flows give rise to tornadoes, vortices and eddies. The situation is delicate when rotating effects are considered: the assumption $\mathbf{v} = \nabla\phi$ no longer holds, and the velocity formulation is replaced with the stream-function formulation. See Lannes, 2013, p.32-35 for a detailed overview.

- The bottom is assumed to be flat. In reality the bottom may differ drastically. Since the degree of bed rigidity, the porosity, and the roughness all influence the fluid to varying degrees, it is also more challenging to work with such bottoms. In addition, when working in the shallow water regime (as explained in the next section), it is necessary to make strong assumptions on the bottom to obtain correct asymptotics. As such, we choose to deal with the simplest case of a flat bottom. For a discussion of waves over more realistic seabeds, see Dean and Dalrymple, 1991, Chapter 9.

- We assume that the surface and the bottom of the domain can be parametrised as graphs. Neither the surface nor the bottom need to be (classical) functions; indeed, modelling overhanging waves is one such example. While this is true, assuming that the surface and the bottom are functions provides an easier setting to work in. The model can be extended by considering parametric representation of the interface (see Lannes, 2013, Section 1.2.2).

- The fluid is contained in its domain. We assume that the fluid does not leak through its bottom, nor do the fluid particles leave the fluid surface. These two conditions are given by (3.3b) and (3.3d), respectively.

- The fluid tends to equilibrium. This is a natural condition so long as one considers infinite domains such as the whole line. Note that any discussion of the rate of convergence relates to the *stability* theory of water waves, and is not touched upon in this project.

- The water depth is assumed to be bounded by some nonnegative constant. In other words, we always have that $\eta - h > 0$ (see Figure 3.2). This is a major limitation, as it excludes vanishes shorelines. However, removing this assumption is an open problem. As the depth is vanishing, the size of computational domain becomes part of the solution ϕ. This issue leads to theoretical and numerical difficulties in describing the shoreline. There are some models that are extended to this case, though they are limited to one dimension and are not rigorously justified (see Bonneton et al., 2011).

- The external force is conservative. This assumption follows naturally from conservation laws for water waves. For our problem, relevant external forces are a *body force*, which is the force due to an outside source and is identical for all fluid particles, and a *local force*, which is the force exerted on a fluid particle by other particles. For our problem, gravity and pressure are pertinent body forces, and friction is the local force. Since the fluid is inviscid, no local force is present. See Cohen and Kundu, 2008, Chapter 4 for a detailed discussion.

- The surface tension is negligible and the surface pressure is constant. In applications to coastal oceanography, the surface tension tends to be very small, which justifies the assumption (see Lannes, 2013, Example 9.1, Chapter 9). Since we expect no large weather variations, we can also assume that locally pressure is constant. Of course, surface tension is relevant when describing some smaller scale phenomena such as ripples. If one is to incorporate non-constant pressure and/or surface tension, the dynamic condition (3.3c) needs to be changed accordingly.

- The system is two dimensional: there is one spatial dimension x and one vertical dimension z. By ignoring the remaining dimension y, we assume that the fluid is moving in x direction only. In other words, we consider the special case of *no transverse waves*. Incorporating weak transverse variation leads to a generalisation of the KdV equation, the Kadomtsev-Petviashvili equation:

$$\partial_x(u_t + 6uu_x + u_{xxx}) + 3\rho u_{yy} = 0. \tag{3.4}$$

  In (3.4), if $\rho = 1$, then water waves with small surface tension are modelled, and if $\rho = -1$, then water waves with large surface tension are modelled.

Remark 4. We briefly mention the role of initial conditions in the water waves problem. While the wave motion is expected to be initiated in some fashion, we are mainly interested in the evolution of the wave motion. As such, initial conditions are not mentioned explicitly.

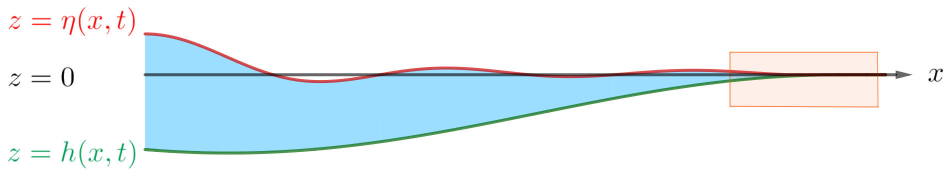


FIGURE 3.2: The region in orange is the zone of vanishing water depth. In this region, the domain becomes part of the solution, which complicates the problem. Therefore, the problem (3.3) is valid in the zone to the left of the orange region.

3.2 Shallow water regime

For now, the problem (3.3) admits numerous types of water waves: short waves, long waves, intermediate waves. In particular, we have yet to specify how wavelength relates to the water depth. To understand the relation between wave velocity and wavelength, we first examine the *dispersion relation*. With this in mind, we focus on the shallow water regime, characterised by small-amplitude waves that have long wavelength, relative to the water depth. In nature, tsunamis and tidal waves are examples of this regime. In coastal engineering, this regime has implications in the design of harbours and in studying estuaries and lagoons.

First, we consider small amplitude waves, or equivalently, we assume that $|\eta| \ll 1$ and $\|\nabla\phi\| \ll 1$. The dispersion relation is obtained by linearising the problem (3.3) around $z = 0$. More concretely, we begin by assuming the special form of the solutions $\phi_s(x, z, t) = A(k, z, t)\exp(ikx)$, and $\eta_s(x, t) = \tilde{\eta}(k, t)\exp(ikx)$. We then may follow Section 5.2 of Ablowitz, 2011, to obtain the following ODE:

$$\frac{\partial^2 \tilde{\eta}}{\partial t^2} + gk\tanh(kh)\tilde{\eta} = 0.$$

Assuming that $\tilde{\eta}(k,t) = \tilde{\eta}(k,0)\exp(-i\omega t)$, the above equation yields the dispersion relation

$$\omega^2 = gk\tanh(kh).$$

Here, ω is a frequency, k is a wave number, and g is gravity. For shallow water, the wavelength $\frac{2\pi}{k}$ is much bigger than the depth h, so $kh \ll 1$. Expansion of $\tanh(kh)$ in kh leads to

$$\omega^2 = gk(kh - \frac{(kh)^3}{3} + \ldots) \simeq ghk^2.$$

Thus, small amplitude water waves in shallow water have frequency $\omega = \pm\sqrt{gh}|k|$, or equivalently, velocity $c_0 = \sqrt{gh}$. Understanding the dispersion relation allows us to *rescale* the problem (3.3), so that the rescaled problem models shallow water waves with velocity $c_0 = \sqrt{gh}$.

In addition, the problem (3.3) does not specify the physical dimensions. Since dimensions of the problem are directly related to the units of variables (wavelength, time, height), it can be difficult to decide which terms are negligible when performing an approximation procedure. The process of *non-dimensionalisation* removes the dimensions of the problem, allowing us to work with "pure" numbers. Letting primes denote the dimensionless variables, we introduce

$$z = hz' \qquad x = \lambda x' \qquad t = \frac{\lambda}{c_0}t' \qquad \eta = a\eta' \qquad \phi = \frac{\lambda g a}{c_0}\phi', \tag{3.5}$$

where $c_0 = \sqrt{gh}$ is the shallow water speed, λ is the wavelength of the initial data, and a is the maximum amplitude of initial data. See Lannes, 2013, Sections 1.3.2-1.3.3 for a detailed discussion of (3.5).

We further define the following parameters $\varepsilon = \frac{a}{h}, \mu = \frac{h}{\lambda}$. Physically, ε is an amplitude of the water wave relative to depth, while μ is a ratio of depth to a typical wavelength. Alternatively, we regard that ε measures nonlinearity and μ measures dispersion. Transforming (3.3) via chain rule and dropping the primed notation yields

$$\mu^2\phi_{xx} + \phi_{zz} = 0, \qquad -1 < z < \varepsilon\eta, \tag{3.6a}$$

$$\phi_z = 0, \qquad z = -1, \tag{3.6b}$$

$$\phi_t + \frac{\varepsilon}{2}\left(\phi_x^2 + \frac{1}{\mu^2}\phi_z^2\right) + \eta = 0, \qquad z = \varepsilon\eta(x,t), \tag{3.6c}$$

$$\mu^2\left[\eta_t + \varepsilon\phi_x\eta_x\right] = \phi_z, \qquad z = \varepsilon\eta(x,t). \tag{3.6d}$$

The problem (3.6) is a "normalised" problem that models shallow water waves, when ε and μ are considered small parameters.

Observe that we have yet to make any assumptions about ε and μ, nor is any relationship between the two parameters prescribed. To obtain interesting limiting equations, we make the following assumptions:

- Assume $\mu \ll 1$. Recall that μ is a ratio of depth to wavelength, and in shallow water regime, we expect depth is much smaller compared to wavelength. This justifies the assumption.

- To obtain equations that are interesting, we should balance the parameters by connecting them to each other. This is known as the principle of maximal balance (see Kruskal, 1963). We choose $\varepsilon = \mu^2$, which reflects the balance of weak nonlinearity and weak dispersion.

- From the maximal balance, it follows that $\varepsilon \ll 1$. Physically, water waves have small amplitude relative to depth.

Thus, the nondimensional problem (3.6) becomes:

$$\varepsilon\phi_{xx} + \phi_{zz} = 0, \qquad -1 < z < \varepsilon\eta, \tag{3.7a}$$

$$\phi_z = 0, \qquad z = -1, \tag{3.7b}$$

$$\phi_t + \frac{1}{2}\left(\varepsilon\phi_x^2 + \phi_z^2\right) + \eta = 0, \qquad z = \varepsilon\eta(x,t), \tag{3.7c}$$

$$\varepsilon\left[\eta_t + \varepsilon\phi_x\eta_x\right] = \phi_z, \qquad z = \varepsilon\eta(x,t). \tag{3.7d}$$

Remark 5. Note that there is no reason not to balance in other ways, say $\varepsilon = \sqrt{\mu}$. There are many options: some give interesting equations, while others do not lead to anything interesting. Indeed, it is this assumption in the procedure that determines the relevance of to-be-derived equations.

3.3 Deriving the KdV model

Finally, we describe the chief result that we aim to obtain in this project. Let us assume an expansion $\phi(x,z,t) = \phi_0(x,z,t) + \varepsilon\phi_1(x,z,t) + \mathcal{O}(\varepsilon^2)$. Substituting the perturbation series into (3.7a) and using (3.7b) yields an approximation

$$\phi = A - \frac{\varepsilon}{2}A_{xx}(z+1)^2 + \frac{\varepsilon^2}{4!}A_{xxxx}(z+1)^4 + \mathcal{O}(\varepsilon^3), \tag{3.8}$$

where $\phi_0 = A(x,t)$. This approximation is valid in $-1 < z < \varepsilon\eta$. Substitution of the series (3.8) into (3.7c) and (3.7d), along with appropriate manipulations yields

$$A_{tt} - A_{xx} = \varepsilon \left(\frac{A_{xxxx}}{3} - 2A_x A_{xt} - A_{xx} A_t \right), \tag{3.9}$$

valid up to $\mathcal{O}(\varepsilon)$. Note that obtaining (3.9) is an especially lengthy calculation, since both the dynamic and kinematic conditions must be expanded carefully. Assume an expansion $A := A_0 + \varepsilon A_1 + \mathcal{O}(\varepsilon^2)$; substituting this expansion into (3.9) yields

$$A_{0tt} - A_{0xx} = 0. \tag{3.10}$$

This is the wave equation, valid within $\mathcal{O}(\varepsilon^0)$. The general solution is $A_0 = F(x-t) + G(x+t)$, for some general functions F, G.

We would like to determine the functions F, G. First, we observe that in parallel to the Duffing oscillar, (3.9) contains secularities when attempting to solve in the higher order. This can be shown directly via dispersion relation, or one could solve (3.9) numerically and see that the solution is unbounded in time. The presence of secular terms warrants an introduction of time scales: $\tau_0 = t, \tau_1 = \varepsilon t$. We also introduce $\xi = x - \tau_0$ and $\zeta = x + \tau_0$, so that $A_0 = F(\xi, \tau_1) + G(\zeta, \tau_1)$. Via appropriate calculations, within $\mathcal{O}(\varepsilon)$, one obtains

$$2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + 3FF_{\xi} = 0 \tag{3.11}$$

$$2G_{\tau_1} - \frac{1}{3}G_{\zeta\zeta\zeta} - 3GG_{\zeta} = 0. \tag{3.12}$$

In other words, we have obtained two KdV equations, (3.11) and (3.12), which determine the dependence of F, G on τ_1. leading order solution A_0

of the problem. Solving the KdV equations via initial conditions, we then determine the leading order solution A_0.

The wave and KdV equations are special PDEs. The wave equation arises as a model in numerous fields of classical physics, such as electrodynamics, plasma physics, and general relativity. The KdV equation appears whenever long waves propagate over dispersive medium, be it fluid mechanics, nonlinear optics, or Bose-Einstein condensation. Because of how they occur independently of the applications, the wave and KdV equations have been studied extensively. Furthermore, the KdV equation is special: despite being nonlinear, this PDE can be solved exactly.

In conclusion, the leading order solution of the water wave problem for small-amplitude, shallow water waves is described by the wave equation (3.10) and two KdV equations (3.11), (3.12). This is the result we seek to obtain for the whole line problem.

3.4 Are asymptotic methods reliable?

Given the emphasis on asymptotic and perturbative methods, one is interested whether these methods provide “reliable" solutions. Formally, how can we justify that solutions obtained from asymptotic equations converge to the solutions of the original problem? In the context of the water waves problem, we wonder if the KdV model, provided by the shallow water approximation, is "reasonable". Of course, in asking these questions, one needs to specify the meaning of “reliable" and “reasonable". Following Lannes, 2013, the validity of our asymptotic model can be understood from the following questions:

1. Do the solutions of the water waves problem exist on the required time scale?

2. Do the solutions of the asymptotic, KdV model exist on the same time scale?

3. Are the asymptotic solutions close to the actual solutions with the corresponding initial data? If so, how close?

If the answer to all three questions is positive, then the asymptotic model is *fully justified*. Indeed, the KdV model is fully justified (see Lannes, 2013, p. 297-298). However, actual proofs of the answers are involved and require advanced mathematics (see Chapter 7 and Appendix C in Lannes, 2013). Since the mathematical justification of the KdV model is beyond the scope of the capstone project, we choose not to discuss this topic.

Chapter 4

Non-local derivation on the whole line

As briefly mentioned in Chapter 3, a direct derivations of the KdV model is subject to lengthy calculations and careful bookkeeping. If we are to change domains (say, the half-line), then additional care must be taken to ensure that the model is correct. As such, we look for a more efficient way of deriving asymptotic models on various domains.

For the water wave problem, recall that the equations of motion are challenging to work with directly, due to the nonlinear boundary conditions and the domain with an unknown shape. To address these issues, reformulations of the problem are introduced: they result in equivalent problems that are more tractable. Below, we give a short overview of these formulations, along with explaining the pros and cons of each.

For one-dimensional surfaces (no y dimension), conformal mappings can be used to eliminate these problems (for an overview, see Dyachenko et al., 1996). However, this approach is limited to one-dimensional surfaces. For both one- and two-dimensional surfaces, other formulations (such as the Hamiltonian formulation given in Zakharov, 1968 or the

Zakharov–Craig–Sulem formulation, Craig and Sulem, 1993) reduce the problem to a system of two equations, in terms of surface variables $q(x,t) = \phi(x,\eta(x,t),t)$ and $\eta(x,t)$, by introducing a Dirichlet-to-Neumann operator (DNO). Another non-local formulation is introduced in Ablowitz, Fokas, and Musslimani, 2006 (the AFM formulation), which results in a system of two equations for the same variables as in the DNO formulation. Both the DNO and AFM formulations reduce the problem from the full fluid domain to a system of equations in $\eta(x,t)$ and $q(x,t)$. However, these formulations involve solving for $q(x,t)$, which may be of little relevance in applications and hard to measure in experiments. We are typically interested in the wave height $\eta(x,t)$.

A new formulation is introduced in Oliveras and Vasan, 2013. In the work, the authors formally eliminate $q(x,t)$, which reduces the water waves problem to a system of two equations, in one variable $\eta(x,t)$. This formulation allows to rigorously investigate one- and two-dimensional water waves. The computation of Stokes-wave asymptotic expansions for periodic waves justifies the use of the formulation; indeed, following Oliveras and Vasan, 2013, the computations can be performed with arguably less effort, especially for two-dimensional waves. Our goal is to further justify the use of this formulation, which we call the $\mathcal{H}$ formulation.

In this chapter, we first rewrite the water wave problem by introducing a normal-to-tangential operator. We then perform a perturbation expansion for the operator, and proceed to obtain an expression for the surface elevation. Finally, performing asymptotics and applying time scales yields the desired approximate equations. We emphasise that it is not our intention here to further the study of the water wave problem per se, but rather to demonstrate the efficacy of the $\mathcal{H}$ formulation for doing asymptotics. We caution the reader that due to the page limit, many steps in calculations are omitted.

4.1 Water-wave problem on the whole line: non-local formulation

Recall the water wave problem (3.3) and consider the velocity potential evaluated at the surface $q(x,t) = \phi(x, \eta(x,t), t)$. We seek to reformulate the problem (3.3). Combining (3.3d) and (3.3c), evaluated at $z = \eta$, we obtain

$$q_t + \frac{1}{2}q_x^2 + g\eta - \frac{1}{2}\frac{(\eta_t + q_x\eta_x)^2}{1+\eta_x^2} = 0, \tag{4.1}$$

which is an equation for two unknowns $q(x,t), \eta(x,t)$. We aim to find an equation in $\eta(x,t)$ only.

Let $\mathbf{N} = [-\eta_x, 1]^T$ and $\mathbf{T} = [1, \eta_x]^T$ be vectors normal and tangent to the surface $z = \eta(x,t)$, respectively. We introduce an operator that maps the normal derivative at a surface η to the tangential derivative at the surface:

$$\mathcal{H}(\eta, D)\{\nabla\phi \cdot \mathbf{N}\} = \nabla\phi \cdot \mathbf{T}, \tag{4.2}$$

where $D = -i\partial_x$. Note that by (3.3d), $\nabla\phi \cdot \mathbf{N} = \phi_z - \phi_x\eta_x = \eta_t$, and by chain rule, $\nabla\phi \cdot \mathbf{T} = \phi_x + \eta_x\phi_z = q_x$. This lets us rewrite (4.2) as

$$\mathcal{H}(\eta, D)\{\eta_t\} = q_x. \tag{4.3}$$

Together, (4.1) and (4.3) form a system of two equations for two unknowns $q(x,t)$, and $\eta(x,t)$. Differentiating (4.1) with respect to x and (4.3) with respect to t allows to further reduce the system to a single equation for $\eta(x,t)$:

$$\begin{aligned} &\partial_t \left(\mathcal{H}(\eta, D)\{\eta_t\}\right) \\ &\quad + \partial_x \left(\frac{1}{2}\left(\mathcal{H}(\eta, D)\{\eta_t\}\right)^2 + \varepsilon\eta - \frac{1}{2}\frac{(\eta_t + \eta_x \mathcal{H}(\eta, D)\{\eta_t\})^2}{1+\eta_x^2}\right) = 0. \end{aligned} \tag{4.4}$$

Equation (4.4) represents a scalar equation for the water wave surface η, incorporating the dynamic and kinematic boundary conditions. The utility of (4.4) depends on whether we can find a useful representation for the operator $\mathcal{H}(\eta, D)$.

4.2 Behaviour of the $\mathcal{H}$ operator

In the previous section, we obtain a scalar equation (4.4), which has an unknown variable $\eta(x,t)$ and an operator $\mathcal{H}$. In this section, we derive another, nonlocal equation for η and $\mathcal{H}$, thereby completing the system.

Consider the following boundary value problem:

$$\phi_{xx} + \phi_{zz} = 0, \qquad -h < z < \eta(x,t), \tag{4.5a}$$

$$\phi_z = 0, \qquad z = -h, \tag{4.5b}$$

$$\nabla\phi \cdot \mathbf{N} = f(x), \qquad z = \eta(x,t). \tag{4.5c}$$

Let φ be harmonic on $S = \mathbb{R} \times (-h, \eta)$. Using (4.5a) and that φ_z is also harmonic on S, we have

$$\varphi_z(\phi_{xx} + \phi_{zz}) - \phi((\varphi_z)_{zz} + (\varphi_z)_{xx}) = 0.$$

Taking the integral over the domain yields

$$\int_{-\infty}^{\infty} \int_{-h}^{\eta(x)} \varphi_z(\phi_{xx} + \phi_{zz}) - \phi((\varphi_z)_{zz} + (\varphi_z)_{xx}) \, \mathrm{d}z \, \mathrm{d}x = 0.$$

An application of Green's theorem gives

$$\int_{\partial S} \varphi_z(\nabla \phi \cdot \mathbf{n}) - \phi(\nabla \varphi_z \cdot \mathbf{n}) \, \mathrm{d}s = 0, \tag{4.6}$$

where ∂S is the boundary of the domain, $\mathrm{d}s$ is the area element, and $\mathbf{n}$ is the normal vector. Now, observe that $-\nabla \varphi_z \cdot \mathbf{n} = \nabla \varphi_x \cdot \mathbf{t}$, where $\mathbf{t}$ is the tangential vector. We use this to rewrite (4.6) and obtain the following contour integral:

$$\begin{aligned} 0 &= \int_{\partial S} \varphi_z(\nabla \phi \cdot \mathbf{n}) + \phi(\nabla \varphi_z \cdot \mathbf{t}) \, \mathrm{d}s \\ &= \int_{\partial S} \varphi_z(\phi_z \, \mathrm{d}x - \phi_x \, \mathrm{d}z) + \phi(\varphi_{xx} \, \mathrm{d}x + \varphi_{xz} \, \mathrm{d}z). \end{aligned} \tag{4.7}$$

Splitting the contour and rewriting the improper integral as a limit yields

$$\begin{aligned} &\int_{\partial S} (\cdot) \, \mathrm{d}s \\ &\quad = \left\{ \int_{-\infty}^{\infty} (\cdot) \Big|_{z=-h} + \lim_{R\to\infty} \int_{-h}^{\eta(x)} (\cdot) \Big|_{x=R} + \int_{\infty}^{-\infty} (\cdot) \Big|_{z=\eta(x)} + \lim_{R\to\infty} \int_{\eta(x)}^{-h} (\cdot) \Big|_{x=-R} \right\} \mathrm{d}s \end{aligned}$$

We consider each segment. As $R \to \infty$, we require that ϕ and its gradient vanish, so these integrals vanish. At $z = -h$, we can pick φ such that $\varphi_x(x, -h) = 0$. The bottom condition and integration by parts then reveal that the contribution at $z = -h$ vanishes as well. Finally, at $z = \eta(x, t)$, integration parts and recognising normal and tangential derivatives yield

$$\int_{-\infty}^{\infty} \varphi_z(\phi_z - \phi_x\eta_x) + \phi(\varphi_{xx} + \varphi_{xz}\eta_x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \varphi_z \nabla\phi \cdot \mathbf{N} - \varphi_x \nabla\phi \cdot \mathbf{T}\,\mathrm{d}x.$$

Finally, recalling (4.5c) and (4.2), we reduce the contour integral (4.7) to

$$\int_{-\infty}^{\infty} \varphi_z f(x) - \varphi_x(x, \eta)\mathcal{H}(\eta, D)\{f(x)\}\,\mathrm{d}x = 0. \tag{4.8}$$

Note that $\varphi(x, z) = e^{-ikx}\sinh(k(z + h)), k \in \mathbb{R}$ is one solution of the problem $\Delta\varphi = 0$, $\varphi_z(-h, z) = 0$. Then, (4.8) becomes

$$\int_{-\infty}^{\infty} e^{-ikx}(k\cosh(k(\eta + h))f(x) + ik\sinh(k(\eta + h))\mathcal{H}(\eta, D)\{f(x)\})\,\mathrm{d}x = 0. \tag{4.9}$$

Taking out k in the integral gives

$$\int_{-\infty}^{\infty} e^{-ikx}(i\cosh(k(\eta + h))f(x) - \sinh(k(\eta + h))\mathcal{H}(\eta, D)\{f(x)\})\,\mathrm{d}x = 0, \tag{4.10}$$

for all $k \in \mathbb{R}$. The equation (4.10) gives a description for the operator $\mathcal{H}(\eta, D)$ in dimensional coordinates.

As mentioned in Chapter 3, to derive asymptotic models we need to work in non-dimensional variables. Using the same rescaling and nondimensionalisation as in (3.5), one may continue the same procedure to obtain the nondimensional version of (4.10):

$$\int_{-\infty}^{\infty} e^{-ikx}(i\cosh(\mu k(\eta+1))g(x) - \sinh(\mu k(\eta+1))\mathcal{H}(\varepsilon\eta, D)\{g(x)\})\,\mathrm{d}x = 0, \tag{4.11}$$

where $k \in \mathbb{R}$ and the primed notation is dropped for convenience. In addition, note that g and f are related by $g(x') = \frac{\sqrt{gh}}{ga} f(x)$.

In summary, introduction of the normal-to-tangential operator $\mathcal{H}(\eta, D)$ allows to reduce the water waves problem (3.3) to a scalar equation (4.4) for η, where the operator $\mathcal{H}$ is described via (4.10). This is the nonlocal formulation described in Oliveras and Vasan, 2013. We emphasise that (4.10) only provides an implicit relationship that can be solved for $\mathcal{H}(\eta, D)\{f(x)\}$. In the next section, we apply perturbation methods to approximate the operator $\mathcal{H}$.

Remark 6. We observe that (4.10) actually holds only for $k \neq 0$. However, for the water wave problem, $f(x) = \eta_t$. As $k \to 0$, (4.9) then reduces to

$$\int_{-\infty}^{\infty} \eta_t\,\mathrm{d}x = 0.$$

This equation is known to be true and represents the conservation of mass. See Benjamin and Olver, 1982 for details.

4.3 Perturbation expansion of the $\mathcal{H}$ operator

As the relation in (4.11) is implicit, it is difficult to determine the operator $\mathcal{H}$ directly. Therefore, following Craig and Sulem, 1993, we find a formal series expansion for the operator via perturbative methods. Since $\varepsilon \ll 1$, we expand the hyperbolic functions as a Taylor series in ε :

$$\cosh(\mu k(\eta+1)) = \cosh(\mu k) + \mu k \varepsilon \eta \sinh(\mu k) + \mathcal{O}(\varepsilon^2),$$
$$\sinh(\mu k(\eta+1)) = \sinh(\mu k) + \mu k \varepsilon \eta \cosh(\mu k) + \mathcal{O}(\varepsilon^2).$$

Now, we note that the idea of formally expanding as a perturbation series applies not only to classical functions but also to operators. Therefore, we formally expand

$$\mathcal{H}(\varepsilon\eta, D)\{g(x)\} = \sum_{j=0}^{\infty} \mathcal{H}_j(\varepsilon\eta, D)\{g(x)\},$$

where $\mathcal{H}_j$ is homogeneous of degree j, i.e. $\mathcal{H}_j(\varepsilon\eta, D) = \varepsilon^j \mathcal{H}_j(\eta, D)$. Equation (4.11) becomes:

$$\int_{-\infty}^{\infty} e^{-ikx} \Big(i \left[\cosh(\mu k) + \mu k \varepsilon \eta \sinh(\mu k) + \mathcal{O}(\varepsilon^2)\right] g(x) \\ - \left[\sinh(\mu k) + \mu k \varepsilon \eta \cosh(\mu k) + \mathcal{O}(\varepsilon^2)\right] \left[\mathcal{H}_0 + \varepsilon \mathcal{H}_1 + \mathcal{O}(\varepsilon^2)\right] (\varepsilon\eta, D)\{g(x)\} \Big)\, \mathrm{d}x = 0. \tag{4.12}$$

At leading order $\mathcal{O}(\varepsilon^0)$: Using (4.12), we obtain

$$\int_{-\infty}^{\infty} e^{-ikx} (i \cosh(\mu k) g(x) - \sinh(\mu k) \mathcal{H}_0(\varepsilon\eta, D)\{g(x)\})\, \mathrm{d}x = 0.$$

For $k \neq 0$, dividing by $\sinh(\mu k)$ yields

$$\int_{-\infty}^{\infty} e^{-ikx} (i \coth(\mu k) g(x) - \mathcal{H}_0(\varepsilon\eta, D)\{g(x)\})\, \mathrm{d}x = 0.$$

Splitting the integrand and recognizing the Fourier transform yields:

$$\begin{aligned}\mathcal{F}_k\left\{\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\}\right\} &= \int_{-\infty}^{\infty} e^{-ikx} i\coth(\mu k) g(x)\,\mathrm{d}x \\ &= i\coth(\mu k)\mathcal{F}_k\left\{g(x)\right\}.\end{aligned}$$

Finally, we invert Fourier transform to obtain

$$\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\} = \mathcal{F}_k^{-1}\left\{i\coth(\mu k)\mathcal{F}_k\left\{g(x)\right\}\right\}, \tag{4.13}$$

where we write out k's explicitly to keep track of transforms. Provided that $\mathcal{F}_k\left\{g(x)\right\} \to 0$ faster than $\mathcal{O}(\mu k)$ as $k \to 0$, (4.13) is defined for all $k \in \mathbb{R}$ (see Remark 7).

In the next order $\mathcal{O}(\varepsilon^1)$: From (4.12), we obtain

$$\int_{-\infty}^{\infty} e^{-ikx}(i\mu k\eta\sinh(\mu k)g - \left[\sinh(\mu k)\mathcal{H}_1 + \mu k\eta\cosh(\mu k)\mathcal{H}_0\right]\{g\})\,\mathrm{d}x = 0,$$

where we drop $(\varepsilon\eta, D)$ and write $g = g(x)$for notational convenience. For $k \neq 0$, dividing by $\sinh(\mu k)$ yields

$$\int_{-\infty}^{\infty} e^{-ikx}(i\mu k\eta g(x) - \left[\mathcal{H}_1 + \mu k\eta\coth(\mu k)\mathcal{H}_0\right]\{g(x)\})\,\mathrm{d}x = 0.$$

Splitting the integral and recognising the Fourier transform yields:

$$\begin{aligned}\mathcal{F}_k\left\{\mathcal{H}_1\{g\})\right\} &= \int_{-\infty}^{\infty} e^{-ikx}(i\mu k\eta g - \mu k\eta\coth(\mu k)\mathcal{H}_0\{g\})\,\mathrm{d}x \\ &= \mu ik\mathcal{F}_k\left\{\eta g\right\} - \mu k\coth(\mu k)\mathcal{F}_k\left\{\eta\mathcal{H}_0\{g\})\right\}.\end{aligned}$$

Inverting Fourier transform and using (4.13), we obtain an expression for $\mathcal{H}_1$:

$$\begin{aligned}\mathcal{H}_1\{g(x)\} &= \mathcal{F}_k^{-1}\left\{ik\mathcal{F}_k\left\{\mu\eta g\right\}\right\} - \mathcal{F}_k^{-1}\left\{\mu k\coth(\mu k)\mathcal{F}_k\left\{\eta\mathcal{H}_0\{g\}\right)\right\}\\ &= \mu\partial_x(\eta g) - \mathcal{F}_k^{-1}\left\{\mu k\coth(\mu k)\mathcal{F}_k\left\{\eta\mathcal{F}_l^{-1}\left\{i\coth(\mu l)\mathcal{F}_l\left\{g\right\}\right\}\right\}\right\}.\end{aligned}$$

In sum, we approximate the $\mathcal{H}$ operator within two orders:

$$\mathcal{H}(\varepsilon\eta, D)\{g(x)\} = [\mathcal{H}_0 + \varepsilon\mathcal{H}_1](\varepsilon\eta, D)\{g(x)\} + \mathcal{O}(\varepsilon^2),$$

where

$$\mathcal{H}_0(\varepsilon\eta, D)\{g\} = \mathcal{F}_k^{-1}\left\{i\coth(\mu k)\hat{g}_k\right\} \tag{4.14a}$$

$$\mathcal{H}_1(\varepsilon\eta, D)\{g\} = \mu\partial_x(\eta g) - \mathcal{F}_k^{-1}\left\{\mu k\coth(\mu k)\mathcal{F}_k\left\{\eta\mathcal{F}_l^{-1}\left\{i\coth(\mu l)\hat{g}_l\right\}\right\}\right\}. \tag{4.14b}$$

Following this procedure, a formal recursion formula can be obtained, so that each $\mathcal{H}_j$ can be written in terms of $\mathcal{H}_i, i = 0, 1, \ldots, j-1$. Since our interest is in approximating, the first two terms are sufficient.

Remark 7. As we proceed to use the operator $\mathcal{H}$, we must exercise caution. Recall the expression for $\mathcal{H}_0$:

$$\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\} = \mathcal{F}_k^{-1}\left\{i\coth(\mu k)\mathcal{F}_k\left\{g(x)\right\}\right\}.$$

Expanding $\coth(\mu k)$ via its Laurent series in μk gives

$$\coth(\mu k) = \frac{1}{\mu k} + \frac{\mu k}{3} + \mathcal{O}(\mu^3).$$

It is readily seen that since $\coth(\mu k)$ has a simple pole as $k \to 0$, so do $\mathcal{H}_0, \mathcal{H}_1$ and $\mathcal{H}$. As such, so long as $\mathcal{F}_k\{g(x)\} \to 0$ as $k \to 0$ at a rate faster than $\mathcal{O}(\mu k)$, then $\mathcal{F}_k\{\mathcal{H}_0\} \to 0$, as $k \to 0$. With this condition, $\mathcal{H}_0$ and $\mathcal{H}_1$ are defined for all $k \in \mathbb{R}$.

4.4 Deriving an expression for surface elevation

We proceed to derive approximate equation for the surface η, using (4.4) and the expansion for $\mathcal{H}$. The non-dimensional version of (4.4) is given by

$$\partial_t \left(\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right) + \partial_x \left(\frac{1}{2}\left(\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 + \varepsilon\eta - \frac{1}{2}\varepsilon^2\mu^2\frac{(\eta_t + \eta_x\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\})^2}{1+\varepsilon^2\mu^2\eta_x^2}\right) = 0. \tag{4.15}$$

Also, recall our maximal balance assumption $\varepsilon = \mu^2$. In the leading order, the equation (4.15) becomes

$$\mathcal{F}_k^{-1}\{i\coth(\mu k)\mathcal{F}_k\{\varepsilon\mu\eta_{tt}\}\} + \varepsilon\partial_x\eta + \mathcal{O}(\varepsilon^2) = 0.$$

Inverting the Fourier transform and multiplying by $k/(i\varepsilon)$ yields

$$\mu k \coth(\mu k)\widehat{\eta_{tt}}_k + k^2\widehat{\eta}_k + \mathcal{O}(\varepsilon) = 0.$$

Expanding $\coth(\mu k)$ in the leading order and noting that $\varepsilon = \mu^2$ gives

$$\widehat{\eta_{tt}}_k + k^2\widehat{\eta}_k + \mathcal{O}(\varepsilon) = 0.$$

Inverting the Fourier transform, we have

$$0 = \eta_{tt} + (-i\partial_x)^2\eta + \mathcal{O}(\varepsilon) = \eta_{tt} - \eta_{xx} + \mathcal{O}(\varepsilon),$$

which is the wave equation, as desired.

If we consider the next order terms, the non-dimensional equation (4.15) becomes

$$\partial_t \left(\mathcal{H}_0\{\varepsilon\mu\eta_t\} + \varepsilon\mathcal{H}_1\{\varepsilon\mu\eta_t\}\right) + \partial_x \left(\frac{1}{2}(\mathcal{H}_0\{\varepsilon\mu\eta_t\})^2 + \varepsilon\eta\right) = 0, \tag{4.16}$$

where we keep the terms up to $\mathcal{O}(\mu^2)$. Using (4.14a) and (4.14b), single equation (4.16) becomes

$$\begin{aligned}\varepsilon\mu\mathcal{F}_k^{-1}\left\{i\coth(\mu k)\widehat{\eta_{tt}}_k\right\} - \varepsilon^2\mathcal{F}_k^{-1}\left\{\mu k\coth(\mu k)\mathcal{F}_k\left\{\partial_t\left[\eta\mathcal{F}_l^{-1}\left\{i\mu\coth(\mu l)\widehat{\eta_t}_l\right\}\right]\right\}\right\}\\ + \varepsilon^2\mu^2(\eta\eta_t)_{tx} + \frac{\varepsilon^2}{2}\partial_x\left(\mathcal{F}_j^{-1}\left\{i\mu\coth(\mu j)\widehat{\eta_t}_j\right\}\right)^2 + \varepsilon\partial_x\eta = 0.\end{aligned} \tag{4.17}$$

Applying the Fourier transform and expanding $\coth(\mu k)$-like terms, the equation (4.17) becomes

$$\left(\frac{1}{k} + \frac{\mu^2 k}{3}\right)\widehat{\eta_{tt}}_k - \mu^2\mathcal{F}_k\left\{\partial_t\left[\eta\mathcal{F}_l^{-1}\left\{\frac{1}{l}\widehat{\eta_t}_l\right\}\right]\right\} - \frac{\mu^2}{2}k\mathcal{F}_k\left\{\left(\mathcal{F}_j^{-1}\left\{\frac{1}{j}\widehat{\eta_t}_j\right\}\right)^2\right\} + k\widehat{\eta}_k = 0,$$

where the terms up to $\mathcal{O}(\mu^2)$ remain. Finally, inverting the Fourier transform yields:

$$\eta_{tt} - \eta_{xx} + \mu^2\left(-\frac{\partial_x^2}{3}\eta_{tt} + i\partial_x\left(\partial_t\left[\eta\mathcal{F}_l^{-1}\left\{\frac{1}{l}\widehat{\eta_t}_l\right\}\right]\right) + \frac{1}{2}\partial_x^2\left(\mathcal{F}_j^{-1}\left\{\frac{1}{j}\widehat{\eta_t}_j\right\}\right)^2\right) = 0,$$

or more conveniently,

$$\eta_{tt}-\eta_{xx}=\mu^2\left(\frac{\partial_x^2}{3}\eta_{tt}-i\partial_x\left(\partial_t\left[\eta\mathcal{F}_l^{-1}\left\{\frac{1}{l}\widehat{\eta_t}_l\right\}\right]\right)-\frac{1}{2}\partial_x^2\left(\mathcal{F}_j^{-1}\left\{\frac{1}{j}\widehat{\eta_t}_j\right\}\right)^2\right). \tag{4.18}$$

We seek to simplify (4.18). First, integration by parts allows us to write

$$\mathcal{F}_l^{-1}\left\{\frac{1}{l}\widehat{\eta_t}_l\right\}=i\int_{-\infty}^{x}\eta_t(x',t)\,\mathrm{d}x', \tag{4.19}$$

Using (4.19) and that $\eta_{tt}=\eta_{xx}+\mathcal{O}(\mu^2)$, the equation (4.18) becomes

$$\eta_{tt}-\eta_{xx}=\varepsilon\left[\frac{1}{3}\eta_{xxxx}+\partial_x^2\left(\frac{\eta^2}{2}+\left(\int_{-\infty}^{x}\eta_t\,\mathrm{d}x'\right)^2\right)\right]. \tag{4.20}$$

In summary, the second order approximation of a scalar equation for η resulted into equation (4.20).

Remark 8. At the end of Section 2.4, we mention secularity in the next order. We now show this directly by examining the dispersion relation of (4.20). Assume a solution of the form $\tilde{\eta}(x,t)=\exp(i(kx-\omega t))$. Substituting $\tilde{\eta}$ into the linearised equation $\eta_{tt}-\eta_{xx}=\varepsilon\frac{1}{3}\eta_{xxxx}$ leads to the relation $-\omega^2+k^2=\varepsilon k^4/3$. For large k, $\omega\sim\pm i\sqrt{\varepsilon}k^2/\sqrt{3}$. Substituting the negative root of ω into the plane wave solution gives

$$\eta(x,t)\approx\exp(ikx)\exp\left(\sqrt{\frac{\varepsilon}{3}}k^2t\right).$$

As $k\to\infty$, this solution is unbounded in time. This is unphysical, since in reality the wave height is always bounded. This suggests that (4.20) contains secularities and hence warrants an application of time scales.

Remark 9. At the moment, it is important to explain why the derivation of (4.20) is included whereas that of (3.9) is omitted. First, the derivation of (4.20) is quicker than that of (3.9). In the velocity potential case, the expansion (3.8) is substituted into (3.7c), evaluated at $z = \varepsilon\eta$, to obtain an expression for η in terms of A. Then, the expansion (3.8) and the expression for η are substituted into (3.7d) obtain an equation in (3.9) Along the way, one needs to correctly estimate the contributions of every term. Thus, this derivation is rather long and requires the reader to be careful with computations.

In our derivation, we begin with a scalar equation (4.4) and a nonlocal equation (4.11). Substituting (4.14a) and (4.14b) into the equation, along with asymptotic reductions, we arrive at the expression (4.20). Clearly, there is much less algebra involved in the derivation. Asymptotic reductions are fairly immediate, since it is easy to estimate the order of the terms with $\mathcal{H}_0, \mathcal{H}_1$, due to presence of $\coth \mu k$ in each expression.

Second, the velocity potential derivation is well-known and can be found in textbooks on nonlinear waves. On the other hand, our derivation is a new result, illustrating for the first time how the nonlocal formulation of Oliveras and Vasan, 2013 can be used to derive a well-known asymptotic model.

4.5 Derivation of wave and KdV equations

We derive the approximate equations from (4.20). Anticipating secular terms, we introduce time scales $\tau_0 = t, \tau_1 = \varepsilon t$, so that $\eta(x,t) = \eta(x,\tau_0,\tau_1)$. We now expand

$$\eta(x,\tau_0,\tau_1) = \eta_0(x,\tau_0,\tau_1) + \varepsilon\eta_1(x,\tau_0,\tau_1) + \mathcal{O}(\varepsilon^2). \tag{4.21}$$

Substituting (4.21) into (4.20), within $\mathcal{O}(\varepsilon^0)$, we obtain

$$\eta_{0\tau_0\tau_0} - \eta_{0xx} = 0, \tag{4.22}$$

so that the general solution is $\eta_0(x, \tau_0, \tau_1) = F(x - \tau_0, \tau_1) + G(x + \tau_0, \tau_1)$. While the initial conditions allow to determine the dependence of F and G on τ_0, the dependence on τ_1 remains unknown. Thus, to find out η_1, we proceed to the next order: this will allow us to determine the dependence of F, G on the slow time scale τ_1. In addition, we introduce left-going and right-going variables $\xi = x - \tau_0, \zeta = x + \tau_0$. These new variables imply

$$\begin{aligned}\partial_x &= \partial_\xi \frac{\mathrm{d}\xi}{\mathrm{d}x} + \partial_\zeta \frac{\mathrm{d}\zeta}{\mathrm{d}x} = \partial_\xi + \partial_\zeta,\\ \partial_t &= \partial_\xi \frac{\mathrm{d}\xi}{\mathrm{d}t} + \partial_\zeta \frac{\mathrm{d}\zeta}{\mathrm{d}t} + \partial_{\tau_1} \frac{\mathrm{d}\tau_1}{\mathrm{d}t} = -\partial_\xi + \partial_\zeta + \varepsilon \partial_{\tau_1}.\end{aligned}$$

For ease of writing, we suppress explicit dependence on variables, though the reader should bear in mind that function F depends on ξ, τ_1, and G depends on ζ, τ_1.

Through the change of variables, the LHS of (4.20) becomes

$$(\partial_t^2 - \partial_x^2)\eta = \varepsilon \left(-4\eta_{1\xi\zeta} - 2F_{\tau_1\xi} + 2G_{\tau_1\zeta}\right) + \mathcal{O}(\varepsilon^2), \tag{4.23}$$

while the RHS of (4.20) becomes

$$\begin{aligned}&\frac{1}{3}\eta_{xxxx} + \partial_x^2 \left(\frac{\eta^2}{2} + \left(\int_{-\infty}^{x} \eta_t \,\mathrm{d}x'\right)^2\right)\\ &= \varepsilon \frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + (\partial_\xi + \partial_\zeta)^2 \left(\frac{3}{2}F^2 + \frac{3}{2}G^2 - FG\right) + \mathcal{O}(\varepsilon),\end{aligned} \tag{4.24}$$

where we assume that F, G vanish as $\xi, \zeta \to -\infty$. Combining (4.23) and (4.24), in $\mathcal{O}(\varepsilon^1)$ we have

$$-4\eta_{1\xi\zeta} = (2F_{\tau_1\xi} - G_{\tau_1\zeta}) + \frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + (\partial_\xi + \partial_\zeta)^2\left(\frac{3}{2}F^2 + \frac{3}{2}G^2 - FG\right). \tag{4.25}$$

In the last term of (4.25), differentiation yields:

$$(\partial_\xi + \partial_\zeta)^2\left(\frac{3}{2}F^2 + \frac{3}{2}G^2 - FG\right) = \partial_\xi(3FF_\xi - GF_\xi) + \partial_\zeta(3GG_\zeta - FG_\zeta) - 2F_\xi G_\zeta,$$

so that (4.25) becomes

$$\begin{aligned}-4\eta_{1\xi\zeta} = {} & \partial_\xi(2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + 3FF_\xi) + \partial_\zeta(-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + 3GG_\zeta) \\ & - (GF_\xi + FG_\zeta) - 2F_\xi G_\zeta.\end{aligned} \tag{4.26}$$

Integration of (4.26) with respect to ζ yields

$$\begin{aligned}-4\eta_{1\xi} = {} & \partial_\xi(2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + 3FF_\xi)\zeta + (-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + 3GG_\zeta) \\ & - \left(F_\xi \int G\,\mathrm{d}\zeta + GF\right) - 2F_\xi G,\end{aligned}$$

and further integration with respect to ξ leads to

$$\begin{aligned}-4\eta_1 = {} & \underbrace{(2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + 3FF_\xi)\zeta}_{\to\infty \text{ as } \zeta\to\infty} + \underbrace{(-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + 3GG_\zeta)\xi}_{\to\infty \text{ as } \xi\to\infty} \\ & - \left(F\int G\,\mathrm{d}\zeta + G\int F\,\mathrm{d}\xi\right) - 2FG.\end{aligned} \tag{4.27}$$

From (4.27), we see that the under-braced terms contain ξ, ζ. This means that unless these terms are zero, η_1 grows without bound as $\xi, \zeta \to \infty$. In other words, these are the secular terms we wish to remove. Therefore, we must have

$$2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + 3FF_{\xi} = 0 \tag{4.28}$$

$$2G_{\tau_1} - \frac{1}{3}G_{\zeta\zeta\zeta} - 3GG_{\zeta} = 0. \tag{4.29}$$

The equations (4.28) and (4.29) are KdV equations, which allow us to determine the behaviour of F, G on a slow time scale τ_1.

In addition, requiring that η_1 is bounded for all ξ, ζ also implies that the terms

$$F, \qquad G, \qquad \int_{-\infty}^{\infty} G \, \mathrm{d}\zeta, \qquad \int_{-\infty}^{\infty} F \, \mathrm{d}\xi$$

are also bounded. This means that the left- and right-going waves do not affect each other long enough to interact strongly on the time scale τ_1. However, each of the two wave trains does interact with itself for a long time, and the two KdV equations govern the evolution of each wave train on the longer time scale τ_1 . The KdV equation governs the evolution on a slow time-scale τ_1 of a small amplitude wave that satisfies the linear wave equation on a fast time scale τ_0.

In conclusion, asymptotic analysis of the non-local formulation in shallow water limit gives rise to two KdV equations, (4.28) and (4.29), for the right-going and left-going waves. Given the right decay initial conditions, we can solve these PDEs by means of the inverse scattering transform (see Ablowitz, 2011, Chapter 9). Keeping the leading-order terms, we have an approximate solution for the wave height

$$\eta \approx \eta_0 = F(x - t, \varepsilon t) + G(x + t, \varepsilon t),$$

where F and G's dependence on $x-t$ and $x+t$ is determined via the initial conditions, and dependence on εt is determined via (4.28) and (4.29). The derivation is complete.

4.6 On KdV derivation

Although our goal is to approximate solutions of the water wave problem, the derivation of the KdV equations deserves a special consideration. Here, we obtain the KdV as equations needed to remove secular terms. In literature, another approach is given in Benjamin, Bona, and Mahony, 1972. The authors begin by considering an advection equation,

$$u_t + c_0 u_x = 0, \tag{4.30}$$

which is a model for small-amplitude, long waves, propagating in $+x$ direction, with speed c_0. The model (4.30) has limited utility, since nonlinear and dispersive effects accumulate and cause the model to lose its validity over large times. One can correct for these effects by considering each effect separately, through introduction of a small parameter ε. Nonlinearity is accounted by

$$u_t + c_0(1+\varepsilon u)u_x = u_t + c_0 u_x + c_0 \varepsilon u u_x = 0, \qquad \varepsilon \ll 1, \tag{4.31}$$

and dispersion is accounted by

$$u_t + c_0(Lu)_x = u_t + c_0 u_x + c_0 \varepsilon \alpha^2 u_{xxx} = 0, \qquad \varepsilon \ll 1. \tag{4.32}$$

We obtain (4.31) and (4.32) as the respective first-order approximations allowing for weak nonlinearity and dispersive effects, up to $\mathcal{O}(\varepsilon)$. Authors then argue that an approximation accounting for both effects can be anticipated by simply combining the ε terms:

$$u_t + u_x + \varepsilon(uu_x + \alpha^2 u_{xxx}) = 0, \tag{4.33}$$

where we set $c_0 = 1$. In nondimensional variables, we transform (4.33) to obtain $u_t + u_x + uu_x + u_{xxx} = 0$. Galilean transformations yield the usual form of the KdV: $u_t + uu_x + u_{xxx} = 0$.

The derivation is elegant, and certainly much shorter than the one presented in the previous section. Note that the use of (4.30) as the starting point is not problematic: one obtains the equation from the dynamic boundary condition, by imposing the shallow water limit. The issue is addition of the ε terms in (4.31) and (4.32): by doing so, the authors already presuppose a certain balance between nonlinearity and dispersion. However, there is no reason to assume this choice of balance; indeed, for a self-consistent theory we must account for the nonlinear and dispersive effects simultaneously.

Chapter 5

Water waves on the half-line

In the previous chapter, we use the $\mathcal{H}$ formulation to obtain expected, well-known results. In this chapter, we use the formulation to study a slightly different problem: the water waves problem, but on the half-line. We note that this is the first time this problem is studied, and none of the known formulations have been applied to this problem. Physically, we put up a tall, impenetrable barrier at $x = 0$. This requires imposing several conditions on both η, ϕ at $x = 0$. As such, we consider the problem (3.3) on the horizontal domain $x \in [0, \infty)$, along with the boundary conditions listed below:

$$\phi_x = 0, \qquad\qquad x = 0, \tag{5.1}$$

$$\phi_z(0, \eta, t) = \eta_t(0, t), \qquad\qquad (x, z) = (0, \eta), \tag{5.2}$$

In particular, (5.1) implies that the fluid does not leak through the barrier at $x = 0$, and (5.2) governs an interaction between the fluid and the surface at $x = 0$. Approximate equations are conjectured to be the wave and KdV-like equations. While the wave equation can be justified, there is no reason to expect the KdV equations. Indeed, a literature review reveals that a KdV-like equation has not been derived on the half-line, in the way that we derive the KdV equation on the whole line.

Using the $\mathcal{H}$ formulation, we derive an asymptotic model on the half-line, note the main differences, and discuss the difficulties that arise. To begin, we observe that the scalar equation (4.4) for η and $\mathcal{H}$ remains the same, while the non-local equation (4.10) changes to

$$\int_0^\infty \cos(kx)\cosh(k(\eta+h))f(x)+\sin(kx)\sinh(k(\eta+h))\mathcal{H}(\eta,D)\{f(x)\}\,\mathrm{d}x=0,$$

and the nondimensional version (4.11) becomes

$$\int_0^\infty \cos(kx)\cosh(\mu k(\eta+1))f(x)+\sin(kx)\sinh(\mu k(\eta+1))\mathcal{H}(\varepsilon\eta,D)\{f(x)\}\,\mathrm{d}x=0.$$

It is worth noting that taking the real part of the whole-line equations (4.10), (4.11) and restricting integrals to $[0,\infty)$ yields the half-line, non-local equations respectively.

By the same procedure, the first two terms in the expansion of $\mathcal{H}$ operator, (4.14a) and (4.14b), are given by cosine and sine transforms:

$$\mathcal{H}_0(\varepsilon\eta,D)\{f(x)\}=-\left\{\mathcal{F}_s^k\right\}^{-1}\left\{\coth(\mu k)\widehat{f_c^k}\right\},$$
$$\mathcal{H}_1(\varepsilon\eta,D)\{f(x)\}=-\left\{\mathcal{F}_s^k\right\}^{-1}\left\{\mu k\widehat{(\eta f(x))}_c^k+\mu k\coth(\mu k)\widehat{(\eta\mathcal{H}_0\{f(x)\})}_c^k\right\}.$$

The notable difference from the whole-line is the presence of Fourier cosine and sine transforms, in place of Fourier transform.

We apply the expansion to the scalar equation. In the leading order, this yields

$$-\mathcal{F}_s^k\left\{\int_0^x \eta_{tt}\,\mathrm{d}x'\right\}+\widehat{\eta_x}_s^k=0.$$

Inverting Fourier sine transform and differentiating with respect to x yields the wave equation on the half-line.

The next order approximation yields the equivalent of (4.20):

$$\eta_{tt}-\eta_{xx}=\mu^2\left(\frac{1}{3}\eta_{xxxx}+\partial_x\left\{\mathcal{F}_s^k\right\}^{-1}\left\{\mathcal{F}_c^k\left\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\right\}\right\}+\frac{1}{2}\partial_x^2\left(\int_0^x \eta_t\,\mathrm{d}x'\right)^2\right). \tag{5.3}$$

The notable difference between the equations on two domains is the presence of the inverse sine transform of the cosine transform. In particular, one can show that this term is the Hilbert transform (see Theorem 1 in Aitzhan, 2020 for the details).

Now, we seek an approximate solution of the half-line problem. Anticipating secularities, we introduce the time scales $\tau_0=t,\tau_1=\varepsilon t$. With an expansion $\eta=\eta_0+\varepsilon\eta_1$, within $\mathcal{O}(\varepsilon^0)$, we obtain $\eta_{0\tau_0\tau_0}-\eta_{0xx}=0$, which is the wave equation. The general solution on the half-line is

$$\eta_0(x,\tau_0,\tau_1)=\begin{cases}F_2(x-\tau_0,\tau_1)+G_2(x+\tau_0,\tau_1) & x\geqslant\tau_0\\ F_1(\tau_0-x,\tau_1)+G_1(x+\tau_0,\tau_1) & x\leqslant\tau_0\end{cases}, \tag{5.4}$$

where F_i and $G_i, i=1,2$ are some general functions. We emphasise that even though F_1 and F_2 are both left-going waves, they have different domains, and hence are different functions. Similarly, G_1 and G_2 are different functions even though they are both right-going waves.

As before, we note that the initial conditions only reveal that dependence of $F_i, G_i, i = 1,2$ on the time scale τ_0 but the dependence on τ_1 is unknown. Working with (5.3) and (5.4), following the same procedure as in Section 4.5, we obtain the two half-line expressions of (4.27), one valid on the domain $x \geqslant \tau_0$ and other valid on $x \leqslant \tau_0$. It is worth pointing out that derivations of approximate equations become complicated since the Hilbert-like transform requires additional care. The calculations on the half-line become much longer and therefore are omitted for brevity.

Removal of secular terms on the two domains yields the following system of 4 equations in four unknowns F_1, F_2, G_1, G_2 : for $\xi < 0$

$$\begin{aligned}
&2\partial_{\tau_1} F_1 + \frac{1}{3}\partial_\xi^3 F_1 + (F_1 - A)\partial_\xi F_1 \\
&\quad + \frac{1}{\pi}\left(\int_{-\tau_0}^{0} (2F_1 - A)\partial_{\xi'} F_1 \frac{1}{\xi - \xi'}\, \mathrm{d}\xi' + \int_0^\infty (2F_2 - (A+B))\partial_{\xi'} F_2 \frac{1}{\xi - \xi'}\, \mathrm{d}\xi' \right) = 0, \\
&-2\partial_{\tau_1} G_1 + \frac{1}{3}\partial_\zeta^3 G_1 + (G_1 + A)\partial_\zeta G_1 \\
&\quad + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0} (2G_1 + A)\partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'}\, \mathrm{d}\zeta' + \int_{2\tau_0}^\infty (2G_2 + (A+B))\partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'}\, \mathrm{d}\zeta' \right) = 0,
\end{aligned} \tag{5.5a}$$

and for $\xi \geqslant 0$

$$\begin{aligned}
&2\partial_{\tau_1} F_2 + \frac{1}{3}\partial_\xi^3 F_2 + (F_2 - A - B)\partial_\xi F_2 \\
&\quad + \frac{1}{\pi}\left(\int_{-\tau_0}^{0} (2F_1 - A)\partial_{\xi'} F_1 \frac{1}{\xi - \xi'}\, \mathrm{d}\xi' + \int_0^\infty (2F_2 - (A+B))\partial_{\xi'} F_2 \frac{1}{\xi - \xi'}\, \mathrm{d}\xi' \right) = 0, \\
&-2\partial_{\tau_1} G_2 + \frac{1}{3}\partial_\zeta^3 G_2 + (G_2 + A + B)\partial_\zeta G_2 \\
&\quad + \frac{1}{\pi}\left(\int_{\tau_0}^{2\tau_0} (2G_1 + A)\partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'}\, \mathrm{d}\zeta' + \int_{2\tau_0}^\infty (2G_2 + A + B)\partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'}\, \mathrm{d}\zeta' \right) = 0,
\end{aligned} \tag{5.5b}$$

where $A = F_1(\tau_0) - G_1(\tau_0)$, and $B = F_2(0) - F_1(0) + G_1(2\tau_0) - G_2(2\tau_0)$.

Roughly speaking, the term A comes from that nonlocal terms are evaluated at $x = 0$ and the term B arises since nonlocal terms are evaluated at $x = \tau_0$.

Each equation in the system (5.5) is only slightly similar to KdV: the time derivative is preserved but dispersive and nonlinear terms are very different. Interestingly, if $A = B = 0$, then dispersion is preserved, while nonlinearity is the same, except that Hilbert transform is applied to a fraction of it. That the approximate equations are more complicated is expected. Note that F_2 is defined on $x - t \geqslant 0$ and F_1 is defined on $x - t \leqslant 0$. As such, in an xt-diagram, we have two regions divided by a line $x = t$ (see Figure 5.1). Physically, the left-going wave gets reflected upon hitting the barrier at $x = 0$, which induces certain dynamics on the surface η in the *interaction* region $x - t < 0$. A priori, these dynamics are unknown, and understanding dynamics in this region becomes part of the problem.

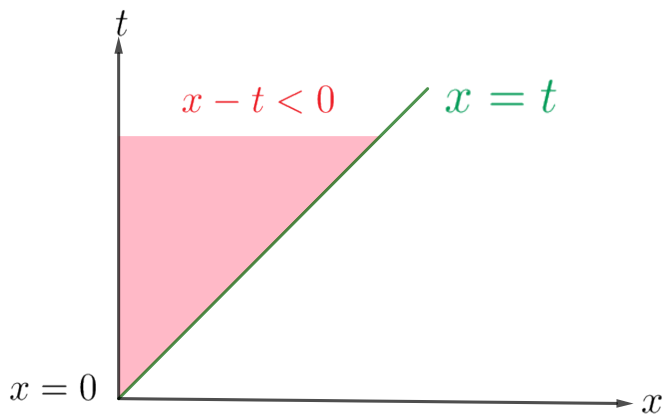


FIGURE 5.1: The interaction region in red, for the half-line water wave problem.

Further, a careful look at the system (5.5) reveals that unlike on the whole line case, the approximate equations for F_i and G_i are still dependent on the time scale τ_0. This is an issue, as the reason behind the time scales is to separate the dependence on different time scales. As of now, it is not clear why this issue appears. One possible reason is that the linear time scales $\tau_0 = t, \tau_1 = \varepsilon t$ should be replaced with different time scales.

In summary, although we do not obtain the approximate equations on the half-line, we see the utility of the $\mathcal{H}$ formulation in aiding to understand the physical and mathematical difficulties associated with the half-line problem. This should not be taken for granted: for example, if one conducts asymptotic expansions via the velocity potential formulation, one obtains 4 KdV equations for $F_i, G_i, i = 1, 2$, which clearly does not agree with the results of this section. As such, the $\mathcal{H}$ formulation shows that the half-line problem has several subtleties, which may not be readily seen via other formulations.

Concluding remarks

In this project, we examine the shallow water limit of the one dimensional water wave problem, using the $\mathcal{H}$ formulation as developed in Oliveras and Vasan, 2013. This capstone project presents two contributions to the field of nonlinear waves and raises two future directions.

First, using the $\mathcal{H}$ formulation, we show that water wave problem reduces to the well-known KdV model. The derivation is especially straightforward, since we can easily estimate the order of the approximated, normal-to-tangential operator $\mathcal{H}$. One subtle issue that we sidestep is the *equivalence* of the $\mathcal{H}$ formulation to the velocity potential formulation of the problem (3.3). While it is clear that solutions of (3.3) solve the $\mathcal{H}$ formulation, the opposite is not so obvious. Without such equivalence, we cannot investigate another important aspect of the problem, namely the stability of travelling wave solutions. Therefore, we would like to prove the equivalence, to establish the relevance of the $\mathcal{H}$ formulation with regards to other aspects of the water wave problem.

Second, we demonstrate the further utility of the $\mathcal{H}$ formulation by analysing the water wave problem on the half-line. Although we do not obtain the appropriate asymptotic model, equation (4.20) presents the distinct features and the associated difficulties of the problem. To the best of our knowledge, the asymptotic model on the half-line has yet to be derived, and achieving this task may provide additional insights into the physics of the problem and raise a number of mathematical questions. In particular, the asymptotic model might describe the surface dynamics in the interaction region $x - t < 0$. This direction is currently the focus of our research.

Bibliography

Ablowitz, M. J. (2011). *Nonlinear dispersive waves: asymptotic analysis and solitons*. Cambridge University Press.

Ablowitz, M. J., A.S. Fokas, and Z.H. Musslimani (2006). "On a new non-local formulation of water waves". In: *Journal of Fluid Mechanics*. DOI: 10.1017/S0022112006001091.

Aitzhan, S (2020). *Report 3*. github.com/sultanaitzhan/Capstone.

Bender, C. M. and S. A Orszag (1999). *Advanced mathematical methods for scientists and engineers: asymptotic methods and perturbation theory*. Springer.

Benjamin, T. B. and P. J. Olver (1982). "Hamiltonian structure, symmetries and conservation laws for water waves". In: *Journal of Fluid Mechanics*. DOI: 10.1017/S0022112082003292.

Benjamin, T.B., J. L. Bona, and J.J. Mahony (1972). "Model equations for long waves in nonlinear dispersive systems". In: *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*. DOI: 10.1098/rsta.1972.0032.

Bonneton, P. et al. (2011). "Model equations for long waves in nonlinear dispersive systems". In: *Journal of Computational Physics*. DOI: 10.1016/j.jcp.2010.11.015.

Chorin, A. J. and J. E. Marsden (1993). *A mathematical introduction to fluid mechanics*. Springer-Verlag.

Cohen, I. M. and P. K. Kundu (2008). *Fluid mechanics*. Academic Press.

Craig, W. and C. Sulem (1993). “Numerical simulation of gravity waves”. In: *Journal of Computational Physics*. DOI: 10.1006/jcph.1993.1164.

Dean, R. G. and R. A. Dalrymple (1991). *Water wave mechanics for scientists and engineers*. World Scientific.

Dyachenko, A. I. et al. (1996). “Analytical description of the free surface dynamics of an ideal fluid (canonical formalism and conformal mapping)”. In: *Physics Letters A*. DOI: 10.1016/0375-9601(96)00417-3.

Gallagher, I (2019). “From Newton to Navier-Stokes, or how to connect fluid mechanics equations from microscopic to macroscopic scales”. In: *Bulletin of the American Mathematical Society*. DOI: 10.1090/bull/1650.

Johnson, R. S. (1997). *A modern introduction to the mathematical theory of water waves*. Cambridge University Press.

Kruskal, M.D. (1963). “Asymptotology”. In: *Proceedings of the Conference on Mathematical Models in the Physical Sciences*. Ed. by S. Drobot. Englewood Cliffs, New Jersey.: Prentice-Hall, pp. 17–48.

Lannes, D. (2013). *The water waves problem: mathematical analysis and asymptotics*. American Mathematical Society.

Oliveras, K. and V. Vasan (2013). “A new equation describing travelling water waves”. In: *Journal of Fluid Mechanics*. DOI: 10.1017/jfm.2012.591.

Zakharov, V. E. (1968). “Stability of periodic waves of finite amplitude on the surface of a deep fluid”. In: *Journal of Applied Mechanics and Technical Physics*. DOI: 10.1007/BF00913182.