

# Report 3

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## 1 The half line problem

In this section, we deal with this term

$$\partial_x(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x\eta_t\,dx'\right)\}\}$$

More generally, we have the following result:

**Theorem 1.** *For nice enough  $f$  defined on  $x \geq 0$ , we have*

$$(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} = \frac{1}{\pi} \int_0^\infty f(y) \left( \frac{1}{x-y} + \frac{1}{x+y} \right) dy.$$

Before we begin, recall the Riemann-Lebesgue lemma:

**Lemma 2** (Theorem 11.6, [1]). *Assume that  $f \in L(I)$ . Then, for each real  $\beta$ , we have*

$$\lim_{\alpha \rightarrow \infty} \int_I f(t) \sin(\alpha t + \beta) dt = 0.$$

*Proof of Theorem 1.* Consider

$$(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\}.$$

For generality, we consider  $(\mathcal{F}_s^k)^{-1}\{G(k)\}$ , where  $G$  is a function of  $k$  defined on  $k \geq 0$ . Expanding the integral, we obtain:

$$\begin{aligned} (\mathcal{F}_s^k)^{-1}\{G(k)\} &= \int_0^\infty \sin(kx)G(k) dk \\ &= \frac{1}{2i} \int_0^\infty (e^{ikx} - e^{-ikx})G(k) dk \\ &= \frac{1}{2i} \left[ \int_0^\infty e^{ikx}G(k) dk - \int_0^\infty e^{-ikx}G(k) dk \right] \\ &= \frac{1}{2i} \left[ \int_0^\infty e^{ikx}G(k) dk + \int_0^{-\infty} e^{ikx}G(-k) dk \right] && \text{(apply } k \mapsto -k \text{ in the 2nd term)} \\ &= \frac{1}{2i} \left[ \int_0^\infty e^{ikx}G(k) dk + \int_{-\infty}^0 e^{ikx}(-G(-k)) dk \right], \end{aligned}$$

where  $-G(-k)$  is an odd extension to  $k < 0$ . Now, observe the following:

$$\begin{aligned} \frac{2}{\pi} \int_0^\infty \cos(kx)f(x) dx &= \frac{1}{\pi} \int_0^\infty (e^{ikx} + e^{-ikx})f(x) dx \\ &= \frac{1}{\pi} \left[ \int_0^\infty e^{ikx}f(x) dx + \int_0^\infty e^{-ikx}f(x) dx \right] \\ &= \frac{1}{\pi} \left[ - \int_0^{-\infty} e^{-ikx}f(-x) dx + \int_0^\infty e^{-ikx}f(x) dx \right] && \text{(apply } x \mapsto -x \text{ in the 1st term)} \\ &= \frac{1}{\pi} \left[ \int_{-\infty}^0 e^{-ikx}f(-x) dx + \int_0^\infty e^{-ikx}f(x) dx \right] \\ &= \frac{1}{\pi} \int_{-\infty}^\infty e^{-ikx}F(x) dx, \end{aligned}$$

where we used an even extension to  $x < 0$  and defined

$$F(x) = \begin{cases} f(x) & x > 0 \\ f(-x) & x < 0 \end{cases}.$$

For  $k > 0$ , we have

$$G(k) = \mathcal{F}_c^k\{f\} = \frac{2}{\pi} \int_0^\infty \cos(kx) f(x) dx = \frac{1}{\pi} \int_{-\infty}^\infty e^{-ikx} F(x) dx. \quad (1)$$

For  $k < 0$ , we have

$$-G(-k) = -\mathcal{F}_c^{-k}\{f\} = -\frac{2}{\pi} \int_0^\infty \cos(-kx) f(x) dx = -\frac{2}{\pi} \int_0^\infty \cos(kx) f(x) dx = -\frac{1}{\pi} \int_{-\infty}^\infty e^{-ikx} F(x) dx, \quad (2)$$

since cosine is an even function. Thus, using (1) and (2), we obtain

$$\begin{aligned} (\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} &= \frac{1}{2i} \left[ \int_0^\infty e^{ikx} \mathcal{F}_c^k\{f\} dk + \int_{-\infty}^0 e^{ikx} (-\mathcal{F}_c^{-k}\{f\}) dk \right] \\ &= \frac{1}{2\pi i} \left[ \int_0^\infty e^{ikx} \int_{-\infty}^\infty e^{-iky} F(y) dy dk - \int_{-\infty}^0 e^{ikx} \int_{-\infty}^\infty e^{-iky} F(y) dy dk \right] \\ &= \frac{1}{2\pi i} \left[ \int_0^\infty e^{ikx} \int_{-\infty}^\infty e^{ik(x-y)} F(y) dy dk - \int_{-\infty}^0 \int_{-\infty}^\infty e^{ik(x-y)} F(y) dy dk \right]. \end{aligned} \quad (3)$$

Let

$$\begin{aligned} V(k) &= \int_{-\infty}^\infty \sin(k(x-y)) F(y) dy = -V(-k), \\ U(k) &= \int_{-\infty}^\infty \cos(k(x-y)) F(y) dy = U(-k), \end{aligned}$$

so that  $V$  is odd and  $U$  is even. This allows to rewrite (3) as:

$$\begin{aligned} (\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} &= \frac{1}{2\pi i} \left[ \int_0^\infty e^{ikx} \int_{-\infty}^\infty e^{ik(x-y)} F(y) dy dk - \int_{-\infty}^0 \int_{-\infty}^\infty e^{ik(x-y)} F(y) dy dk \right] \\ &= \frac{1}{2\pi i} \left[ \int_0^\infty U(k) + iV(k) dk - \int_{-\infty}^0 U(k) + iV(k) dk \right] \\ &= \frac{1}{2\pi i} \left[ \int_0^\infty U(k) + iV(k) dk + \int_{-\infty}^0 U(-k) + iV(-k) dk \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi i} \left[ \int_0^\infty U(k) + iV(k) dk + \int_0^\infty -U(-k) + i(-V(-k)) dk \right] \\
&= \frac{1}{2\pi i} \left[ \int_0^\infty U(k) + iV(k) dk + \int_0^\infty -U(k) + iV(k) dk \right] \\
&= \frac{1}{\pi} \int_0^\infty V(k) dk,
\end{aligned}$$

where on the third last line, we flipped the bounds of integration and brought the minus sign inside the integral, and on the second last line, we used that  $U$  is even and  $V$  is odd. Thus, we obtain

$$(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} = \frac{1}{\pi} \int_0^\infty V(k) dk = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty \sin(k(x-y))F(y) dy dk.$$

Note that the integral in  $k$  is an improper integral, so

$$\int_0^\infty \int_{-\infty}^\infty \sin(k(x-y))F(y) dy dk = \lim_{\alpha \rightarrow \infty} \int_0^\alpha \int_{-\infty}^\infty \sin(k(x-y))F(y) dy dk.$$

Now, interchanging the order of integration, we have

$$\begin{aligned}
\int_0^\alpha \int_{-\infty}^\infty \sin(k(x-y))F(y) dy dk &= \int_{-\infty}^\infty F(y) \int_0^\alpha \sin(k(x-y)) dk dy \\
&= \int_{-\infty}^\infty F(y) \left[ -\frac{\cos(k(x-y))}{x-y} \Big|_0^\alpha \right] dy \\
&= \int_{-\infty}^\infty F(y) \left[ \frac{1}{x-y} - \frac{\cos(\alpha(x-y))}{x-y} \right] dy \\
&= \int_{-\infty}^\infty F(y) \frac{1 - \cos(\alpha(x-y))}{x-y} dy.
\end{aligned}$$

The interchange is justified, since sine is bounded and differentiable on  $\mathbb{R}$ . Finally, we use the Riemann-Lebesgue lemma to deal with the last term:

$$\begin{aligned}
\int_{-\infty}^\infty F(y) \frac{1 - \cos(\alpha(x-y))}{x-y} dy &= \int_0^\infty f(y) \frac{1 - \cos(\alpha(x-y))}{x-y} dy + \int_{-\infty}^0 f(-y) \frac{1 - \cos(\alpha(x-y))}{x-y} dy \\
&= \int_0^\infty f(y) \frac{1 - \cos(\alpha(x-y))}{x-y} dy - \int_\infty^0 f(y) \frac{1 - \cos(\alpha(x+y))}{x+y} dy \\
&= \int_0^\infty f(y) \frac{1 - \cos(\alpha(x-y))}{x-y} dy + \int_0^\infty f(y) \frac{1 - \cos(\alpha(x+y))}{x+y} dy
\end{aligned}$$

$$\begin{aligned}
&= \int_0^\infty f(y) \frac{1}{x-y} dy - \int_0^\infty f(y) \frac{\cos(\alpha(x-y))}{x-y} dy \\
&+ \int_0^\infty f(y) \frac{1}{x+y} dy - \int_0^\infty f(y) \frac{\cos(\alpha(x+y))}{x+y} dy.
\end{aligned}$$

As  $\alpha \rightarrow \infty$ , the terms

$$\int_0^\infty f(y) \frac{\cos(\alpha(x-y))}{x-y} dy, \quad \int_0^\infty f(y) \frac{\cos(\alpha(x+y))}{x+y} dy \rightarrow 0$$

by the Riemann-Lebesgue lemma with  $\beta = \pi/2$ , so that

$$\int_{-\infty}^\infty F(y) \frac{1 - \cos(\alpha(x-y))}{x-y} dy = \int_0^\infty f(y) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy.$$

Thus,

$$(\mathcal{F}_s^k)^{-1} \{ \mathcal{F}_c^k \{ f \} \} = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty \sin(k(x-y)) F(y) dy dk = \frac{1}{\pi} \int_0^\infty f(y) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy.$$

The proof is complete. □

*Remark 3.* Note that the integral

$$\frac{1}{\pi} \int_0^\infty f(y) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy =$$

looks like a convolution-type transform. In fact, the term with  $1/(x-y)$  is pretty much the Hilbert transform, but on a half-line.

The theorem yields

$$\partial_x (\mathcal{F}_s^k)^{-1} \{ \mathcal{F}_c^k \{ \partial_t \left( \eta \int_0^x \eta_t dx' \right) \} \} = \partial_x \left( \frac{1}{\pi} \int_0^\infty \partial_t \left( \eta \int_0^y \eta_t dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \right).$$

For generality, let  $f(y) = \partial_t \left( \eta \int_0^y \eta_t dy' \right)$ . Note the following:

$$\begin{aligned}
\partial_x \left( \frac{1}{\pi} \int_0^\infty f(y) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \right) &= \frac{1}{\pi} \int_0^\infty f(y) \partial_x \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&= -\frac{1}{\pi} \int_0^\infty f(y) \left[ \frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] dy,
\end{aligned}$$

so that

$$\partial_x (\mathcal{F}_s^k)^{-1} \{ \mathcal{F}_c^k \{ \partial_t \left( \eta \int_0^x \eta_t dx' \right) \} \} = -\frac{1}{\pi} \int_0^\infty \partial_t \left( \eta \int_0^y \eta_t dy' \right) \left[ \frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] dy. \tag{4}$$

As can be seen, the integral (4) is singular whenever  $x = y$  or  $x = -y$ , over  $y$ . To deal with this issue, one may need to use a Residue theorem. To conclude, the surface expression on a half-line becomes:

$$\begin{aligned}\eta_{tt} - \eta_{xx} &= \mu^2 \left( \frac{1}{3} \eta_{xxxx} + \partial_x (\mathcal{F}_s^k)^{-1} \{ \mathcal{F}_c^k \{ \partial_t \left( \eta \int_0^x \eta_t dx' \right) \} \} + \frac{1}{2} \partial_x^2 \left( \int_0^x \eta_t dx' \right)^2 \right) \\ &= \mu^2 \left( \frac{1}{3} \eta_{xxxx} - \frac{1}{\pi} \int_0^\infty \partial_t \left( \eta \int_0^y \eta_t dy' \right) \left[ \frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] dy + \frac{1}{2} \partial_x^2 \left( \int_0^x \eta_t dx' \right)^2 \right).\end{aligned}$$

## 2 Approximate equations: half-line

In this section, we derive the approximate equations from

$$\eta_{tt} - \eta_{xx} = \varepsilon \left( \frac{1}{3} \eta_{xxxx} + \frac{d}{dx} \frac{1}{\pi} \int_0^\infty \partial_t \left( \eta \int_0^y \eta_t dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy + \frac{1}{2} \partial_x^2 \left( \int_0^x \eta_t dx' \right)^2 \right). \quad (5)$$

As we approximate, we assume an expansion of  $\eta$  in  $\varepsilon$ :

$$\eta = \eta_0 + \varepsilon \eta_1 + \mathcal{O}(\varepsilon^2). \quad (6)$$

### 2.1 First order approximation

Substitution of (6) into equation (5) yields

$$\eta_{0tt} - \eta_{0xx} + \varepsilon (\eta_{1tt} - \eta_{1xx}) = \varepsilon \left[ \frac{1}{3} \eta_{0xxxx} + \frac{d}{dx} \frac{1}{\pi} \int_0^\infty \partial_t \left( \eta \int_0^y \eta_t dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy + \frac{1}{2} \partial_x^2 \left( \int_{-\infty}^x (\eta_0 + \varepsilon \eta_1)_t dx' \right)^2 \right] + \mathcal{O}(\varepsilon^2). \quad (7)$$

In the leading order  $\mathcal{O}(\varepsilon^0)$ , equation (7) becomes

$$\eta_{0tt} - \eta_{0xx} = 0. \quad (8)$$

This is the wave equation with velocity 1, whose solution depends on the type of boundary conditions we prescribe for  $\eta$  at  $x = 0$ . For now, we prescribe

$$\eta_x(0, t) = 0.$$

The general solution is

$$\eta(x, t) = \begin{cases} F(x-t) + G(x+t) & x > t \\ F(t-x) + G(x+t) & x < t \end{cases},$$

where  $F, G$  are to be determined.

## 2.2 Second order approximation

As in the velocity potential case, we employ multiple scales. First, we find an expression for  $\eta_0$ . We introduce

$$\tau_0 = t, \quad \tau_1 = \varepsilon t, \quad \tau_2 = \varepsilon^2 t, \dots,$$

so that

$$\eta(x, t) = \eta(x, \tau_0, \tau_1, \dots).$$

With this in mind, the expansion (6) becomes

$$\eta(x, \tau_0, \tau_1, \dots) = \eta_0(x, \tau_0, \tau_1, \dots) + \mathcal{O}(\varepsilon^1). \quad (9)$$

Substituting (9) into (5), within  $\mathcal{O}(\varepsilon^0)$ , we obtain

$$\eta_{0\tau_0\tau_0} - \eta_{0xx} = 0, \quad (10)$$

so that the general solution is

$$\eta_0(x, \tau_0, \tau_1, \dots) = \begin{cases} F_2(x - \tau_0, \tau_1, \dots) + G_2(x + \tau_0, \tau_1, \dots) & x \geq \tau_0 \\ F_1(\tau_0 - x, \tau_1, \dots) + G_1(x + \tau_0, \tau_1, \dots) & x < \tau_0 \end{cases}.$$

Now, although we have found an expression for  $\eta_0$ , the functions  $F_i, G_i$  used are still general functions. To determine  $F_i, G_i$ , we proceed to the next order, i.e.  $\mathcal{O}(\varepsilon^1)$ . We introduce

$$\xi = x - \tau_0 \quad \zeta = x + \tau_0$$

so that

$$\eta_0(x, \tau_0, \tau_1, \dots) = \begin{cases} F_1(\xi, \tau_1, \dots) + G_1(\zeta, \tau_1, \dots) & x \geq t \\ F_2(-\xi, \tau_1, \dots) + G_2(\zeta, \tau_1, \dots) & x < t \end{cases},$$

and

$$\begin{aligned} \partial_x &= \partial_\xi \frac{d\xi}{dx} + \partial_\zeta \frac{d\zeta}{dx} = \partial_\xi + \partial_\zeta, \\ \partial_t &= \partial_\xi \frac{d\xi}{dt} + \partial_\zeta \frac{d\zeta}{dt} + \partial_{\tau_1} \frac{d\tau_1}{dt} = -\partial_\xi + \partial_\zeta + \varepsilon \partial_{\tau_1}. \end{aligned}$$

*Remark 4.* We emphasize the piecewise nature of solutions, which is why we write that  $F_1, F_2$  as different functions even though they share the same variable  $\xi$ . It is very important to be aware which  $F_i$  we need to use, as we will demonstrate when dealing with the non-local terms. In addition, we also need to impose some more conditions at  $x = \tau_0$ , to reinforce some sort of continuity between  $F_1$  and  $F_2$ .

### 2.2.1 The case $x < \tau_0$

We consider the case  $x < t$ . First, we use

$$\begin{aligned}\eta &= \eta_0 + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2) \\ &= F_1(t - x, \varepsilon t, \dots) + G_1(x + t, \varepsilon t, \dots) + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2) \\ &= F_1(-\xi, \tau_1, \dots) + G_1(\zeta, \tau_1, \dots) + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2) \\ &= F_1 + G_1 + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2).\end{aligned}$$

For ease of writing, we suppress explicit dependence on variables, though the reader should bear in mind that function  $F_1, (G_1)$  depend on  $-\xi, (\zeta), \tau_1, \tau_2$ , etc. In addition, observe that

$$(\partial_t^2 - \partial_x^2) = (-4\partial_\xi\partial_\zeta + 2\varepsilon(\partial_\zeta\partial_{\tau_1} - \partial_\xi\partial_{\tau_1}) + \varepsilon^2\partial_{\tau_1}^2),$$

so that the LHS of (5) becomes

$$(\partial_t^2 - \partial_x^2)\eta = \varepsilon(-4\eta_{1\xi\zeta} - 2\partial_\xi\partial_{\tau_1}(F_1)_{\tau_1} + 2\partial_\zeta\partial_{\tau_1}G_1) + \mathcal{O}(\varepsilon^2). \quad (11)$$

Now, we deal with the RHS of (5). By appropriate substitutions, the terms become:

$$\begin{aligned}\frac{1}{3}\eta_{xxxx} &= \frac{1}{3}(\partial_\xi^4 F_1 + \partial_\zeta^4 G_1 + \mathcal{O}(\varepsilon)); \\ \left(\int_0^x \eta_t dx'\right)^2 &= \left(\int_0^x \eta_{0t} dx'\right)^2 + \mathcal{O}(\varepsilon) \\ &= \left(\int_0^x (-\partial_{\xi'} + \partial_{\zeta'} + \varepsilon\partial_{\tau_1})(F_1 + G_1) dx'\right)^2 + \mathcal{O}(\varepsilon) \\ &= \left(\int_0^x -\partial_{\xi'} F_1 + \partial_{\zeta'} G_1 dx'\right)^2 + \mathcal{O}(\varepsilon) \\ &= \left(\int_0^x -\partial_{\xi'} F_1 dx'\right)^2 - 2\left(\int_0^x (\partial_{\xi'} F_1 dx')\right)\left(\int_0^x \partial_{\zeta'} G_1 dx'\right) + \left(\int_0^x \partial_{\zeta'} G_1 dx'\right)^2 + \mathcal{O}(\varepsilon) \\ &= (F_1 - F_1(\tau_0))^2 - 2(F_1 - F_1(\tau_0))(G_1 - G_1(\tau_0)) + (G_1 - G_1(\tau_0))^2 + \mathcal{O}(\varepsilon),\end{aligned}$$

where for the last line we translate  $\xi' = x' - t, \zeta' = x' + t$  to obtain

$$\int_0^x -\partial_{\xi'}(F_1(\tau_0 - \xi')) dx' = \int_{-t}^{x-t} (F_1)_{\xi'}(-\xi', \tau_1) d\xi' = \int_{-\tau_0}^{\xi} (F_1)_{\xi'}(-\xi', \tau_1) d\xi' = F_1 - F_1(\tau_0),$$



$$\int_0^x (G_1)_{\zeta'}(x' + \tau_0, \tau_1) dx' = \int_t^{x+t} (G_1)_{\zeta'}(\zeta', \tau_1) d\zeta' = \int_{\tau_0}^{\zeta} (G_1)_{\zeta'}(\zeta', \tau_1) d\zeta' = G_1 - G_1(\tau_0).$$

Note that previously we wrongly assumed that there is some strange term  $F(-t)$ . But  $F$  that we used was rather  $F_2$ , which is appropriate when  $x \geq \tau_0$ . In this case,  $x < \tau_0$ , so we need to use  $F_1$ , which provides the right viewpoint. Next, from Proposition 6, we also have

$$\begin{aligned} & (\partial_\xi + \partial_\zeta) \int_0^\infty \partial_t \left( \eta \int_0^y \eta_t dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\ &= \partial_\xi \left( \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_0^\infty 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} d\zeta' + \int_{2\tau_0}^\infty 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} d\zeta' \right) \\ &+ \partial_\zeta \left( \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\zeta + \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' + \int_0^\infty 2F_1 \partial_{\xi'} F_1 \frac{1}{\zeta + \xi'} d\xi' + \int_{2\tau_0}^\infty 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' \right) \\ &+ \partial_\xi \left( A \int_{-\tau_0}^0 -\partial_\xi F_1 \frac{1}{\xi - \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \frac{1}{\xi + \zeta'} d\zeta' + (A+B) \int_0^\infty -\partial_\xi F_2 \frac{1}{\xi - \xi'} d\xi' + \int_{2\tau_0}^\infty \partial_\zeta G_2 \frac{1}{\xi + \zeta'} d\zeta' \right) \\ &+ \partial_\zeta \left( A \int_{-\tau_0}^0 -\partial_\xi F_1 \frac{1}{\zeta + \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \frac{1}{\zeta - \zeta'} d\zeta' + (A+B) \int_0^\infty -\partial_\xi F_2 \frac{1}{\zeta + \xi'} d\xi' + \int_{2\tau_0}^\infty \partial_\zeta G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right). \end{aligned}$$

Finally, note that

$$\begin{aligned} & \frac{1}{2} (\partial_\xi^2 + 2\partial_\xi \partial_\zeta + \partial_\zeta^2) ((F_1 - F_1(\tau_0))^2 - 2(F_1 - F_1(\tau_0))(G_1 - G_1(\tau_0)) + (G_1 - G_1(\tau_0))^2) \\ &= \frac{1}{2} \partial_\xi^2 ((F_1 - F_1(\tau_0))^2 - 2(F_1 - F_1(\tau_0))(G_1 - G_1(\tau_0))) + \frac{1}{2} \partial_\zeta^2 (-2(F_1 - F_1(\tau_0))(G_1 - G_1(\tau_0)) + (G_1 - G_1(\tau_0))^2) \\ &- 2\partial_\xi \partial_\zeta ((F_1 - F_1(\tau_0))(G_1 - G_1(\tau_0))) \\ &= \partial_\xi ((F_1 - F_1(\tau_0))\partial_\xi F_1 - \partial_\xi F_1(G_1 - G_1(\tau_0))) + \partial_\zeta (-(F_1 - F_1(\tau_0))\partial_\zeta G_1 + (G_1 - G_1(\tau_0))\partial_\zeta G_1) - 2\partial_\xi F_1 \partial_\zeta G_1 \\ &= \partial_\xi ((F_1 - F_1(\tau_0) - G_1 + G_1(\tau_0))\partial_\xi F_1) + \partial_\zeta ((G_1 - G_1(\tau_0) - F_1 + F_1(\tau_0))\partial_\zeta G_1) - 2\partial_\xi F_1 \partial_\zeta G_1 \\ &= \partial_\xi ((F_1 - G_1 - A)\partial_\xi F_1) + \partial_\zeta ((G_1 - F_1 + A)\partial_\zeta G_1) - 2\partial_\xi F_1 \partial_\zeta G_1, \end{aligned}$$

where we set  $A = F_1(\tau_0) - G_1(\tau_0)$ .

Substitution of terms into the RHS of (5) leads to:

$$\begin{aligned} & \frac{1}{3} \eta_{xxxx} + \frac{d}{dx} \frac{1}{\pi} \int_0^\infty \partial_t \left( \eta \int_0^y \eta_t dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy + \frac{1}{2} \partial_x^2 \left( \int_0^x \eta_t dx' \right)^2 \\ &= \frac{1}{3} (\partial_\xi^4 F_1 + \partial_\zeta^4 G_1) + \partial_\xi ((F_1 - G_1 - A)\partial_\xi F_1) + \partial_\zeta ((G_1 - F_1 + A)\partial_\zeta G_1) - 2\partial_\xi F_1 \partial_\zeta G_1 \end{aligned}$$

$$\begin{aligned}
& + \partial_\xi \frac{1}{\pi} \left( \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_0^\infty 2F_2 \partial_{\xi'} F_2 \frac{1}{\xi - \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} d\zeta' + \int_{2\tau_0}^\infty 2G_2 \partial_{\zeta'} G_2 \frac{1}{\xi + \zeta'} d\zeta' \right) \\
& + \partial_\zeta \frac{1}{\pi} \left( \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi + \zeta'} d\xi' + \int_0^\infty 2F_2 \partial_{\xi'} F_2 \frac{1}{\xi + \zeta'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' + \int_{2\tau_0}^\infty 2G_2 \partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right) \\
& + \partial_\xi \left( A \int_{-\tau_0}^0 -\partial_\xi F_1 \frac{1}{\xi - \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \frac{1}{\xi + \zeta'} d\zeta' + (A+B) \int_0^\infty -\partial_\xi F_2 \frac{1}{\xi - \xi'} d\xi' + \int_{2\tau_0}^\infty \partial_\zeta G_2 \frac{1}{\xi + \zeta'} d\zeta' \right) \\
& + \partial_\zeta \left( A \int_{-\tau_0}^0 -\partial_\xi F_1 \frac{1}{\zeta + \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \frac{1}{\zeta - \zeta'} d\zeta' + (A+B) \int_0^\infty -\partial_\xi F_2 \frac{1}{\zeta + \xi'} d\xi' + \int_{2\tau_0}^\infty \partial_\zeta G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right) + \mathcal{O}(\varepsilon). \tag{12}
\end{aligned}$$

Combining (11) and (12), in  $\mathcal{O}(\varepsilon^1)$  we have

$$\begin{aligned}
-4\eta_{1\xi\zeta} & = 2\partial_\xi \partial_{\tau_1} F_1 - 2\partial_\zeta \partial_{\tau_1} G_1 + \frac{1}{3}(\partial_\xi^4 F_1 + \partial_\zeta^4 G_1) + \partial_\xi ((F_1 - G_1 - A)\partial_\xi F_1) + \partial_\zeta ((G_1 - F_1 + A)\partial_\zeta G_1) - 2\partial_\xi F_1 \partial_\zeta G_1 \\
& + \partial_\xi \frac{1}{\pi} \left( \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_0^\infty 2F_2 \partial_{\xi'} F_2 \frac{1}{\xi - \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} d\zeta' + \int_{2\tau_0}^\infty 2G_2 \partial_{\zeta'} G_2 \frac{1}{\xi + \zeta'} d\zeta' \right) \\
& + \partial_\zeta \frac{1}{\pi} \left( \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi + \zeta'} d\xi' + \int_0^\infty 2F_2 \partial_{\xi'} F_2 \frac{1}{\xi + \zeta'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' + \int_{2\tau_0}^\infty 2G_2 \partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right) \\
& + \partial_\xi \frac{1}{\pi} \left( A \int_{-\tau_0}^0 -\partial'_\xi F_1 \frac{1}{\xi - \xi'} d\xi' + A \int_{\tau_0}^{2\tau_0} \partial'_\zeta G_1 \frac{1}{\xi + \zeta'} d\zeta' + (A+B) \int_0^\infty -\partial'_\xi F_2 \frac{1}{\xi - \xi'} d\xi' + (A+B) \int_{2\tau_0}^\infty \partial'_\zeta G_2 \frac{1}{\xi + \zeta'} d\zeta' \right) \\
& + \partial_\zeta \frac{1}{\pi} \left( A \int_{-\tau_0}^0 -\partial'_\xi F_1 \frac{1}{\zeta + \xi'} d\xi' + A \int_{\tau_0}^{2\tau_0} \partial'_\zeta G_1 \frac{1}{\zeta - \zeta'} d\zeta' + (A+B) \int_0^\infty -\partial'_\xi F_2 \frac{1}{\zeta + \xi'} d\xi' + (A+B) \int_{2\tau_0}^\infty \partial'_\zeta G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right) \tag{13}
\end{aligned}$$

By rearranging appropriately, (13) becomes

$$\begin{aligned}
-4\eta_{1\xi\zeta} = & \partial_\xi(2\partial_{\tau_1}F_1 + \frac{1}{3}\partial_\xi^3F_1 + (F_1 - A)\partial_\xi F_1 + \frac{1}{\pi} \left( \int_{-\tau_0}^0 2F_1\partial_{\xi'}F_1 \frac{1}{\xi - \xi'} d\xi' + \int_0^\infty 2F_2\partial_{\xi'}F_2 \frac{1}{\xi - \xi'} d\xi' + A \int_{-\tau_0}^0 -\partial_{\xi'}F_1 \frac{1}{\xi - \xi'} d\xi' + (A + B) \int_0^\infty -\partial_{\xi'}F_2 \frac{1}{\xi - \xi'} d\xi' \right)) \\
& + \partial_\zeta(-2\partial_{\tau_1}G_1 + \frac{1}{3}\partial_\zeta^3G_1 + (G_1 + A)\partial_\zeta G_1 + \frac{1}{\pi} \left( \int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'}G_1 \frac{1}{\zeta - \zeta'} d\zeta' + \int_{2\tau_0}^\infty 2G_2\partial_{\zeta'}G_2 \frac{1}{\zeta - \zeta'} d\zeta' + A \int_{\tau_0}^{2\tau_0} \partial_{\zeta'}G_1 \frac{1}{\zeta - \zeta'} d\zeta' + (A + B) \int_{2\tau_0}^\infty \partial_{\zeta'}G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right)) \\
& + \partial_\xi(-G_1\partial_\xi F_1 + \frac{1}{\pi} \left( \int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'}G_1 \frac{1}{\xi + \zeta'} d\zeta' + \int_{2\tau_0}^\infty 2G_2\partial_{\zeta'}G_2 \frac{1}{\xi + \zeta'} d\zeta' + A \int_{\tau_0}^{2\tau_0} \partial_{\zeta'}G_1 \frac{1}{\xi + \zeta'} d\zeta' + (A + B) \int_{2\tau_0}^\infty \partial_{\zeta'}G_2 \frac{1}{\xi + \zeta'} d\zeta' \right)) \\
& + \partial_\zeta(-F_1\partial_\zeta G_1 + \frac{1}{\pi} \left( \int_{-\tau_0}^0 2F_1\partial_{\xi'}F_1 \frac{1}{\xi + \zeta'} d\xi' + \int_0^\infty 2F_2\partial_{\xi'}F_2 \frac{1}{\xi + \zeta'} d\xi' + A \int_{-\tau_0}^0 -\partial_{\xi'}F_1 \frac{1}{\zeta + \xi'} d\xi' + (A + B) \int_0^\infty -\partial_{\xi'}F_2 \frac{1}{\zeta + \xi'} d\xi' \right)) \\
& - 2\partial_\xi F_1 \partial_\zeta G_1.
\end{aligned} \tag{14}$$

Integration of (14) with respect to  $\zeta$  yields

$$\begin{aligned}
-4\eta_{1\xi} = & \zeta \partial_\xi(2\partial_{\tau_1}F_1 + \frac{1}{3}\partial_\xi^3F_1 + (F_1 - A)\partial_\xi F_1 + \frac{1}{\pi} \left( \int_{-\tau_0}^0 (2F_1 - A)\partial_{\xi'}F_1 \frac{1}{\xi - \xi'} d\xi' + \int_0^\infty (2F_2 - (A + B))\partial_{\xi'}F_2 \frac{1}{\xi - \xi'} d\xi' \right)) \\
& + (-2\partial_{\tau_1}G_1 + \frac{1}{3}\partial_\zeta^3G_1 + (G_1 + A)\partial_\zeta G_1 + \frac{1}{\pi} \left( \int_{\tau_0}^{2\tau_0} (2G_1 + A)\partial_{\zeta'}G_1 \frac{1}{\zeta - \zeta'} d\zeta' + \int_{2\tau_0}^\infty (2G_2 + (A + B))\partial_{\zeta'}G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right)) \\
& + \partial_\xi \int (G_1\partial_\xi F_1 + \frac{1}{\pi} \left( \int_{\tau_0}^{2\tau_0} (2G_1 + A)\partial_{\zeta'}G_1 \frac{1}{\xi + \zeta'} d\zeta' + \int_{2\tau_0}^\infty (2G_2 + (A + B))\partial_{\zeta'}G_2 \frac{1}{\xi + \zeta'} d\zeta' \right)) d\zeta \\
& + (-F_1\partial_\zeta G_1 + \frac{1}{\pi} \left( \int_{-\tau_0}^0 (2F_1 - A)\partial_{\xi'}F_1 \frac{1}{\xi + \zeta'} d\xi' + \int_0^\infty (2F_2 - (A + B))\partial_{\xi'}F_2 \frac{1}{\xi + \zeta'} d\xi' \right)) - 2\partial_\xi F_1 G_1.
\end{aligned} \tag{15}$$

and further integration with respect to  $\xi$  leads to

$$\begin{aligned}
-4\eta_1 = & \zeta(2\partial_{\tau_1}F_1 + \frac{1}{3}\partial_\xi^3F_1 + (F_1 - A)\partial_\xi F_1 + \frac{1}{\pi} \left( \int_{-\tau_0}^0 (2F_1 - A)\partial_{\xi'}F_1 \frac{1}{\xi - \xi'} d\xi' + \int_0^\infty (2F_2 - (A + B))\partial_{\xi'}F_2 \frac{1}{\xi - \xi'} d\xi' \right)) \\
& + \xi(-2\partial_{\tau_1}G_1 + \frac{1}{3}\partial_\zeta^3G_1 + (G_1 + A)\partial_\zeta G_1 + \frac{1}{\pi} \left( \int_{\tau_0}^{2\tau_0} (2G_1 + A)\partial_{\zeta'}G_1 \frac{1}{\zeta - \zeta'} d\zeta' + \int_{2\tau_0}^\infty (2G_2 + (A + B))\partial_{\zeta'}G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right)) \\
& + \int (G_1\partial_\xi F_1 + \frac{1}{\pi} \left( \int_{\tau_0}^{2\tau_0} (2G_1 + A)\partial_{\zeta'}G_1 \frac{1}{\xi + \zeta'} d\zeta' + \int_{2\tau_0}^\infty (2G_2 + (A + B))\partial_{\zeta'}G_2 \frac{1}{\xi + \zeta'} d\zeta' \right)) d\zeta \\
& + \int (-F_1\partial_\zeta G_1 + \frac{1}{\pi} \left( \int_{-\tau_0}^0 (2F_1 - A)\partial_{\xi'}F_1 \frac{1}{\xi + \zeta'} d\xi' + \int_0^\infty (2F_2 - (A + B))\partial_{\xi'}F_2 \frac{1}{\xi + \zeta'} d\xi' \right)) d\xi - 2F_1 G_1.
\end{aligned}$$

Since  $\eta_1$  must be bounded, we must have

$$2\partial_{\tau_1} F_1 + \frac{1}{3}\partial_{\xi}^3 F_1 + (F_1 - A)\partial_{\xi} F_1 + \frac{1}{\pi} \left( \int_{-\tau_0}^0 (2F_1 - A)\partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_0^{\infty} (2F_2 - (A + B))\partial_{\xi'} F_2 \frac{1}{\xi - \xi'} d\xi' \right) = 0, \quad (16)$$

$$-2\partial_{\tau_1} G_1 + \frac{1}{3}\partial_{\zeta}^3 G_1 + (G_1 + A)\partial_{\zeta} G_1 + \frac{1}{\pi} \left( \int_{\tau_0}^{2\tau_0} (2G_1 + A)\partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' + \int_{2\tau_0}^{\infty} (2G_2 + (A + B))\partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right) = 0. \quad (17)$$

In other words, we have obtained two KdV-like equations, (16) and (17), whose solutions  $F_1, G_1$  describe behaviour of the surface elevation in the leading order, when  $x < \tau_0$ .

### 2.2.2 The case $x \geq \tau_0$

On the domain  $x \geq \tau_0$ , we use

$$\begin{aligned} \eta &= \eta_0 + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2) \\ &= F_2(x - t, \varepsilon t, \dots) + G_2(x + t, \varepsilon t, \dots) + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2) \\ &= F_2(\xi, \tau_1, \dots) + G_2(\zeta, \tau_1, \dots) + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2) \\ &= F_2 + G_2 + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2). \end{aligned}$$

For ease of writing, we suppress explicit dependence on variables, though the reader should bear in mind that function  $F_2, (G_2)$  depend on  $\xi, (\zeta), \tau_1, \tau_2$ , etc. In addition, D'Alembert operator becomes

$$(\partial_t^2 - \partial_x^2) = (-4\partial_{\xi}\partial_{\zeta} + 2\varepsilon(\partial_{\zeta}\partial_{\tau_1} - \partial_{\xi}\partial_{\tau_1}) + \varepsilon^2\partial_{\tau_1}^2),$$

so that the LHS of (5) becomes

$$(\partial_t^2 - \partial_x^2)\eta = \varepsilon(-4\eta_{1\xi\zeta} - 2\partial_{\xi}\partial_{\tau_1}F_1 + 2\partial_{\zeta}\partial_{\tau_1}G_1) + \mathcal{O}(\varepsilon^2). \quad (18)$$

Now, we deal with the RHS of (5). By appropriate substitutions, the terms become:

$$\begin{aligned} \frac{1}{3}\eta_{xxxx} &= \frac{1}{3}(\partial_{\xi}^4 F_2 + \partial_{\zeta}^4 (G_2) + \mathcal{O}(\varepsilon)); \\ \left( \int_0^x \eta_t dx' \right)^2 &= \left( \int_0^{\tau_0} \eta_t dx' + \int_{\tau_0}^x \eta_t dx' \right)^2 \\ &= \left( \int_0^{\tau_0} \eta_{0t} dx' + \int_{\tau_0}^x \eta_{0t} dx' \right)^2 + \mathcal{O}(\varepsilon) \end{aligned}$$

$$\begin{aligned}
&= \left( \int_0^{\tau_0} (-\partial_{\xi'} + \partial_{\zeta'} + \varepsilon \partial_{\tau_1})(F_1 + G_1) dx' + \int_{\tau_0}^x (-\partial_{\xi'} + \partial_{\zeta'} + \varepsilon \partial_{\tau_1})(F_2 + G_2) dx' \right)^2 + \mathcal{O}(\varepsilon) \\
&= \left( \int_0^{\tau_0} (-\partial_{\xi'} + \partial_{\zeta'})(F_1 + G_1) dx' + \int_{\tau_0}^x (-\partial_{\xi'} + \partial_{\zeta'})(F_2 + G_2) dx' \right)^2 + \mathcal{O}(\varepsilon), \\
&= (-F_2 + G_2 + A + B)^2 + \mathcal{O}(\varepsilon),
\end{aligned}$$

where for the last line we have simplified as follows:

$$\begin{aligned}
\int_0^{\tau_0} -\partial_{\xi'}(F_1(\tau_0 - \xi')) dx' &= - \int_{-\tau_0}^0 \partial_{\xi'} F_1(-\xi', \tau_1) d\xi' = -F_1(0) + F_1(\tau_0), \\
\int_0^{\tau_0} \partial_{\zeta'} G_1(x' + \tau_0, \tau_1) dx' &= \int_{\tau_0}^{2\tau_0} \partial_{\zeta'} G_1(\zeta', \tau_1) d\zeta' = G_1(2\tau_0) - G_1(\tau_0) \\
\int_{\tau_0}^x -\partial_{\xi'}(F_2(\tau_0 - \xi')) dx' &= - \int_0^{x-\tau_0} \partial_{\xi'} F_2(\xi', \tau_1) d\xi' = -F_2 + F_2(0), \\
\int_{\tau_0}^x \partial_{\zeta'} G_2(x' + \tau_0, \tau_1) dx' &= \int_{2\tau_0}^{x+\tau_0} \partial_{\zeta'} G_2(\zeta', \tau_1) d\zeta' = G_2 - G_2(2\tau_0).
\end{aligned}$$

Addition of terms yields

$$\begin{aligned}
\int_0^{\tau_0} (-\partial_{\xi'} + \partial_{\zeta'})(F_1 + G_1) dx' + \int_{\tau_0}^x (-\partial_{\xi'} + \partial_{\zeta'})(F_2 + G_2) dx' &= -F_1(0) + F_1(\tau_0) + G_1(2\tau_0) - G_1(\tau_0) - F_2 + F_2(0) + G_2 - G_2(2\tau_0) \\
&= -F_2 + G_2 - F_1(0) + F_2(0) + G_1(2\tau_0) - G_2(2\tau_0) + F_1(\tau_0) - G_1(\tau_0) \\
&= -F_2 + G_2 + A + B,
\end{aligned}$$

where we write

$$F_1(\tau_0) - G_1(\tau_0) = A \quad -F_1(0) + F_2(0) + G_1(2\tau_0) - G_2(2\tau_0) = B.$$

Next, by Proposition 6, we also have

$$\begin{aligned}
&(\partial_{\xi} + \partial_{\zeta}) \int_0^{\infty} \partial_t \left( \eta \int_0^y \eta_t dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&= \partial_{\xi} \left( \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_0^{\infty} 2F_2 \partial_{\xi'} F_2 \frac{1}{\xi - \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} d\zeta' + \int_{2\tau_0}^{\infty} 2G_2 \partial_{\zeta'} G_2 \frac{1}{\xi + \zeta'} d\zeta' \right) \\
&+ \partial_{\zeta} \left( \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi + \zeta'} d\xi' + \int_0^{\infty} 2F_2 \partial_{\xi'} F_2 \frac{1}{\xi + \zeta'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' + \int_{2\tau_0}^{\infty} 2G_2 \partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right)
\end{aligned}$$

$$\begin{aligned}
& + \partial_\xi \left( A \int_{-\tau_0}^0 -\partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} d\zeta' + (A+B) \int_0^\infty -\partial_{\xi'} F_2 \frac{1}{\xi - \xi'} d\xi' + \int_{2\tau_0}^\infty \partial_{\zeta'} G_2 \frac{1}{\xi + \zeta'} d\zeta' \right) \\
& + \partial_\zeta \left( A \int_{-\tau_0}^0 -\partial_{\xi'} F_1 \frac{1}{\zeta + \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' + (A+B) \int_0^\infty -\partial_{\xi'} F_2 \frac{1}{\zeta + \xi'} d\xi' + \int_{2\tau_0}^\infty \partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right) + \mathcal{O}(\varepsilon).
\end{aligned}$$

Finally, note that

$$\begin{aligned}
\frac{1}{2}(\partial_\xi^2 + 2\partial_\xi \partial_\zeta + \partial_\zeta^2)(-F_2 + G_2 + A + B)^2 &= \frac{1}{2}\partial_\xi^2(-F_2 + G_2 + A + B)^2 + \frac{1}{2}\partial_\zeta^2(-F_2 + G_2 + A + B)^2 + \partial_\xi \partial_\zeta(-F_2 + G_2 + A + B)^2 \\
&= \partial_\xi((-F_2 + G_2 + A + B)\partial_\xi(-F_2)) + \partial_\zeta((-F_2 + G_2 + A + B)\partial_\zeta G_2) - 2\partial_\xi F_2 \partial_\zeta G_2.
\end{aligned}$$

Substitution of terms into the RHS of (5) leads to:

$$\begin{aligned}
& \frac{1}{3}\eta_{xxxx} + \frac{d}{dx} \frac{1}{\pi} \int_0^\infty \partial_t \left( \eta \int_0^y \eta_t dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy + \frac{1}{2}\partial_x^2 \left( \int_0^x \eta_t dx' \right)^2 \\
&= \frac{1}{3}(\partial_\xi^4 F_2 + \partial_\zeta^4 G_2) + \partial_\xi((-F_2 + G_2 + A + B)\partial_\xi(-F_2)) + \partial_\zeta((-F_2 + G_2 + A + B)\partial_\zeta G_2) - 2\partial_\xi F_2 \partial_\zeta G_2 \\
&+ \partial_\xi \frac{1}{\pi} \left( \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_0^\infty 2F_2 \partial_{\xi'} F_2 \frac{1}{\xi - \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} d\zeta' + \int_{2\tau_0}^\infty 2G_2 \partial_{\zeta'} G_2 \frac{1}{\xi + \zeta'} d\zeta' \right) \\
&+ \partial_\zeta \frac{1}{\pi} \left( \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\zeta + \xi'} d\xi' + \int_0^\infty 2F_2 \partial_{\xi'} F_2 \frac{1}{\zeta + \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' + \int_{2\tau_0}^\infty 2G_2 \partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right) \\
&+ \partial_\xi \frac{1}{\pi} \left( A \int_{-\tau_0}^0 -\partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} d\zeta' + (A+B) \int_0^\infty -\partial_{\xi'} F_2 \frac{1}{\xi - \xi'} d\xi' + \int_{2\tau_0}^\infty \partial_{\zeta'} G_2 \frac{1}{\xi + \zeta'} d\zeta' \right) \\
&+ \partial_\zeta \frac{1}{\pi} \left( A \int_{-\tau_0}^0 -\partial_{\xi'} F_1 \frac{1}{\zeta + \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' + (A+B) \int_0^\infty -\partial_{\xi'} F_2 \frac{1}{\zeta + \xi'} d\xi' + \int_{2\tau_0}^\infty \partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right) + \mathcal{O}(\varepsilon) \tag{19}
\end{aligned}$$

Combining (18) and (19), in  $\mathcal{O}(\varepsilon^1)$  we have

$$\begin{aligned}
-4\eta_{1\xi\zeta} &= 2\partial_\xi\partial_{\tau_1}F_2 - 2\partial_\zeta\partial_{\tau_1}G_2 + \frac{1}{3}(\partial_\xi^4F_2 + \partial_\zeta^4G_2) + \partial_\xi((-F_2 + G_2 + A + B)\partial_\xi(-F_2)) + \partial_\zeta((-F_2 + G_2 + A + B)\partial_\zeta G_2) - 2\partial_\xi F_2\partial_\zeta G_2 \\
&+ \partial_\xi \frac{1}{\pi} \left( \int_{-\tau_0}^0 2F_1\partial_{\xi'}F_1 \frac{1}{\xi - \xi'} d\xi' + \int_0^\infty 2F_2\partial_{\xi'}F_2 \frac{1}{\xi - \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'}G_1 \frac{1}{\xi + \zeta'} d\zeta' + \int_{2\tau_0}^\infty 2G_2\partial_{\zeta'}G_2 \frac{1}{\xi + \zeta'} d\zeta' \right) \\
&+ \partial_\zeta \frac{1}{\pi} \left( \int_{-\tau_0}^0 2F_1\partial_{\xi'}F_1 \frac{1}{\xi + \zeta'} d\xi' + \int_0^\infty 2F_2\partial_{\xi'}F_2 \frac{1}{\xi + \zeta'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'}G_1 \frac{1}{\zeta - \zeta'} d\zeta' + \int_{2\tau_0}^\infty 2G_2\partial_{\zeta'}G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right) \\
&+ \partial_\xi \frac{1}{\pi} \left( A \int_{-\tau_0}^0 -\partial_{\xi'}F_1 \frac{1}{\xi - \xi'} d\xi' + A \int_{\tau_0}^{2\tau_0} \partial_{\zeta'}G_1 \frac{1}{\xi + \zeta'} d\zeta' + (A + B) \int_0^\infty -\partial_{\xi'}F_2 \frac{1}{\xi - \xi'} d\xi' + (A + B) \int_{2\tau_0}^\infty \partial_{\zeta'}G_2 \frac{1}{\xi + \zeta'} d\zeta' \right) \\
&+ \partial_\zeta \frac{1}{\pi} \left( A \int_{-\tau_0}^0 -\partial_{\xi'}F_1 \frac{1}{\zeta + \xi'} d\xi' + A \int_{\tau_0}^{2\tau_0} \partial_{\zeta'}G_1 \frac{1}{\zeta - \zeta'} d\zeta' + (A + B) \int_0^\infty -\partial_{\xi'}F_2 \frac{1}{\zeta + \xi'} d\xi' + (A + B) \int_{2\tau_0}^\infty \partial_{\zeta'}G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right)
\end{aligned}$$

By rearranging appropriately, (20) becomes

$$\begin{aligned}
-4\eta_{1\xi\zeta} &= \partial_\xi(2\partial_{\tau_1}F_2 + \frac{1}{3}\partial_\xi^3F_2 + F_2\partial_\xi F_2 + \frac{1}{\pi} \left( \int_{-\tau_0}^0 (2F_1 - A)\partial_{\xi'}F_1 \frac{1}{\xi - \xi'} d\xi' + \int_0^\infty (2F_2 - (A + B))\partial_{\xi'}F_2 \frac{1}{\xi - \xi'} d\xi' \right)) \\
&+ \partial_\zeta(-2\partial_{\tau_1}G_2 + \frac{1}{3}\partial_\zeta^3G_2 + G_2\partial_\zeta G_2 + \frac{1}{\pi} \left( \int_{\tau_0}^{2\tau_0} (2G_1 + A)\partial_{\zeta'}G_1 \frac{1}{\zeta - \zeta'} d\zeta' + \int_{2\tau_0}^\infty (2G_2 + A + B)\partial_{\zeta'}G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right)) \\
&+ \partial_\xi(-G_2\partial_\xi F_2 + \frac{1}{\pi} \left( \int_{\tau_0}^{2\tau_0} (2G_1 + A)\partial_{\zeta'}G_1 \frac{1}{\xi + \zeta'} d\zeta' + \int_{2\tau_0}^\infty (2G_2 + (A + B))\partial_{\zeta'}G_2 \frac{1}{\xi + \zeta'} d\zeta' \right)) \\
&+ \partial_\zeta(-F_2\partial_\zeta G_2 + \frac{1}{\pi} \left( \int_{-\tau_0}^0 (2F_1 - A)\partial_{\xi'}F_1 \frac{1}{\xi + \zeta'} d\xi' + \int_0^\infty (2F_2 - (A + B))\partial_{\xi'}F_2 \frac{1}{\xi + \zeta'} d\xi' \right)) - 2\partial_\xi F_2\partial_\zeta G_2.
\end{aligned} \tag{20}$$

Integration of (14) with respect to  $\zeta$  yields

$$\begin{aligned}
-4\eta_{1\xi} &= \zeta\partial_\xi(2\partial_{\tau_1}F_2 + \frac{1}{3}\partial_\xi^3F_2 + (F_2 - A - B)\partial_\xi F_2 + \frac{1}{\pi} \left( \int_{-\tau_0}^0 (2F_1 - A)\partial_{\xi'}F_1 \frac{1}{\xi - \xi'} d\xi' + \int_0^\infty (2F_2 - (A + B))\partial_{\xi'}F_2 \frac{1}{\xi - \xi'} d\xi' \right)) \\
&+ (-2\partial_{\tau_1}G_2 + \frac{1}{3}\partial_\zeta^3G_2 + (G_2 + A + B)\partial_\zeta G_2 + \frac{1}{\pi} \left( \int_{\tau_0}^{2\tau_0} (2G_1 + A)\partial_{\zeta'}G_1 \frac{1}{\zeta - \zeta'} d\zeta' + \int_{2\tau_0}^\infty (2G_2 + A + B)\partial_{\zeta'}G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right)) \\
&+ \partial_\xi \int (-G_2\partial_\xi F_2 + \frac{1}{\pi} \left( \int_{\tau_0}^{2\tau_0} (2G_1 + A)\partial_{\zeta'}G_1 \frac{1}{\xi + \zeta'} d\zeta' + \int_{2\tau_0}^\infty (2G_2 + (A + B))\partial_{\zeta'}G_2 \frac{1}{\xi + \zeta'} d\zeta' \right)) d\zeta \\
&+ (-F_2\partial_\zeta G_2 + \frac{1}{\pi} \left( \int_{-\tau_0}^0 (2F_1 - A)\partial_{\xi'}F_1 \frac{1}{\xi + \zeta'} d\xi' + \int_0^\infty (2F_2 - (A + B))\partial_{\xi'}F_2 \frac{1}{\xi + \zeta'} d\xi' \right)) - 2\partial_\xi F_2 G_2.
\end{aligned} \tag{21}$$

and further integration with respect to  $\xi$  leads to

$$\begin{aligned}
-4\eta_1 = & \zeta(2\partial_{\tau_1} F_2 + \frac{1}{3}\partial_\xi^3 F_2 + (F_2 - A - B)\partial_\xi F_2 + \frac{1}{\pi} \left( \int_{-\tau_0}^0 (2F_1 - A)\partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_0^\infty (2F_2 - (A + B))\partial_{\xi'} F_2 \frac{1}{\xi - \xi'} d\xi' \right)) \\
& + \xi(-2\partial_{\tau_1} G_2 + \frac{1}{3}\partial_\zeta^3 G_2 + (G_2 + A + B)\partial_\zeta G_2 + \frac{1}{\pi} \left( \int_{\tau_0}^{2\tau_0} (2G_1 + A)\partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' + \int_{2\tau_0}^\infty (2G_2 + A + B)\partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right)) \\
& + \int (-G_2\partial_\xi F_2 + \frac{1}{\pi} \left( \int_{\tau_0}^{2\tau_0} (2G_1 + A)\partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} d\zeta' + \int_{2\tau_0}^\infty (2G_2 + (A + B))\partial_{\zeta'} G_2 \frac{1}{\xi + \zeta'} d\zeta' \right)) d\zeta \\
& + \int (-F_2\partial_\zeta G_2 + \frac{1}{\pi} \left( \int_{-\tau_0}^0 (2F_1 - A)\partial_{\xi'} F_1 \frac{1}{\xi + \zeta'} d\xi' + \int_0^\infty (2F_2 - (A + B))\partial_{\xi'} F_2 \frac{1}{\xi + \zeta'} d\xi' \right)) d\xi - 2F_2 G_2.
\end{aligned}$$

Since  $\eta_1$  must be bounded, we must have

$$2\partial_{\tau_1} F_2 + \frac{1}{3}\partial_\xi^3 F_2 + (F_2 - A - B)\partial_\xi F_2 + \frac{1}{\pi} \left( \int_{-\tau_0}^0 (2F_1 - A)\partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_0^\infty (2F_2 - (A + B))\partial_{\xi'} F_2 \frac{1}{\xi - \xi'} d\xi' \right) = 0, \quad (22)$$

$$-2\partial_{\tau_1} G_2 + \frac{1}{3}\partial_\zeta^3 G_2 + (G_2 + A + B)\partial_\zeta G_2 + \frac{1}{\pi} \left( \int_{\tau_0}^{2\tau_0} (2G_1 + A)\partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' + \int_{2\tau_0}^\infty (2G_2 + A + B)\partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right) = 0. \quad (23)$$

In other words, we have obtained two KdV-like equations, (22) and (23), whose solutions  $F_2, G_2$  describe behaviour of the surface elevation in the leading order, when  $x \geq \tau_0$ .

### 2.2.3 Conclusion

In summary, we have started out with a general solution of the wave equation on the right half-line:

$$\eta_0(x, \tau_0, \tau_1, \dots) = \begin{cases} F_2(x - \tau_0, \tau_1, \dots) + G_2(x + \tau_0, \tau_1, \dots) & x \geq \tau_0 \\ F_1(\tau_0 - x, \tau_1, \dots) + G_1(x + \tau_0, \tau_1, \dots) & x < \tau_0 \end{cases},$$



and obtained a system of 4 equations in four unknowns  $F_1, F_2, G_1, G_2$  :

$$\begin{aligned}
2\partial_{\tau_1} F_1 + \frac{1}{3}\partial_{\xi}^3 F_1 + (F_1 - A)\partial_{\xi} F_1 + \frac{1}{\pi} \left( \int_{-\tau_0}^0 (2F_1 - A)\partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_0^{\infty} (2F_2 - (A + B))\partial_{\xi'} F_2 \frac{1}{\xi - \xi'} d\xi' \right) &= 0, & x < \tau_0; \\
2\partial_{\tau_1} F_2 + \frac{1}{3}\partial_{\xi}^3 F_2 + (F_2 - A - B)\partial_{\xi} F_2 + \frac{1}{\pi} \left( \int_{-\tau_0}^0 (2F_1 - A)\partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_0^{\infty} (2F_2 - (A + B))\partial_{\xi'} F_2 \frac{1}{\xi - \xi'} d\xi' \right) &= 0, & x \geq \tau_0; \\
-2\partial_{\tau_1} G_1 + \frac{1}{3}\partial_{\zeta}^3 G_1 + (G_1 + A)\partial_{\zeta} G_1 + \frac{1}{\pi} \left( \int_{\tau_0}^{2\tau_0} (2G_1 + A)\partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' + \int_{2\tau_0}^{\infty} (2G_2 + (A + B))\partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right) &= 0, & x < \tau_0; \\
-2\partial_{\tau_1} G_2 + \frac{1}{3}\partial_{\zeta}^3 G_2 + (G_2 + A + B)\partial_{\zeta} G_2 + \frac{1}{\pi} \left( \int_{\tau_0}^{2\tau_0} (2G_1 + A)\partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' + \int_{2\tau_0}^{\infty} (2G_2 + A + B)\partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right) &= 0, & x \geq \tau_0,
\end{aligned} \tag{24}$$

where

$$A = F_1(\tau_0) - G_1(\tau_0), \quad B = F_2(0) - F_1(0) + G_1(2\tau_0) - G_2(2\tau_0).$$

One can decouple this system by further forcing

$$F_1(0) = F_2(0), \quad G_1(2\tau_0) = G_2(2\tau_0), \quad F_1(\tau_0) = G_1(\tau_0),$$

in which case,  $A, B = 0$ , and we obtain 2 systems of 2 equations: one in  $F_1, F_2$  :

$$\begin{aligned}
2\partial_{\tau_1} F_1 + \frac{1}{3}\partial_{\xi}^3 F_1 + F_1\partial_{\xi} F_1 + \frac{1}{\pi} \left( \int_{-\tau_0}^0 2F_1\partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_0^{\infty} 2F_2\partial_{\xi'} F_2 \frac{1}{\xi - \xi'} d\xi' \right) &= 0, & x < \tau_0; \\
2\partial_{\tau_1} F_2 + \frac{1}{3}\partial_{\xi}^3 F_2 + F_2\partial_{\xi} F_2 + \frac{1}{\pi} \left( \int_{-\tau_0}^0 2F_1\partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_0^{\infty} 2F_2\partial_{\xi'} F_2 \frac{1}{\xi - \xi'} d\xi' \right) &= 0, & x \geq \tau_0;
\end{aligned} \tag{25}$$

and another one in  $G_1, G_2$  :

$$\begin{aligned}
-2\partial_{\tau_1} G_1 + \frac{1}{3}\partial_{\zeta}^3 G_1 + G_1\partial_{\zeta} G_1 + \frac{1}{\pi} \left( \int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' + \int_{2\tau_0}^{\infty} 2G_2\partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right) &= 0, & x < \tau_0; \\
-2\partial_{\tau_1} G_2 + \frac{1}{3}\partial_{\zeta}^3 G_2 + G_2\partial_{\zeta} G_2 + \frac{1}{\pi} \left( \int_{\tau_0}^{2\tau_0} 2G_1\partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' + \int_{2\tau_0}^{\infty} 2G_2\partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right) &= 0, & x \geq \tau_0.
\end{aligned} \tag{26}$$

## 2.2.4 The Hilbert Transform term

*Proposition 5.* We have

$$\int_0^{\infty} \partial_t \left( \eta \int_0^y \eta_t dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy$$

$$\begin{aligned}
&= \int_0^{\tau_0} (2F_1 \partial_\xi F_1 + 2G_1 \partial_\zeta G_1) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy + \int_{\tau_0}^\infty (2F_2 \partial_\xi F_2 + 2G_2 \partial_\zeta G_2) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&+ A \int_0^{\tau_0} (-\partial_\xi F_1 + \partial_\zeta G_1) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy + (A+B) \int_{\tau_0}^\infty (-\partial_\xi F_2 + \partial_\zeta G_2) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy,
\end{aligned}$$

where

$$\begin{aligned}
A &= F_1(\tau_0) - G_1(\tau_0), \\
B &= F_2(0) - F_1(0) + G_1(2\tau_0) - G_2(2\tau_0).
\end{aligned}$$

*Proof.* Note:

$$\begin{aligned}
\int_0^\infty \partial_t \left( \eta \int_0^y \eta_t dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy &= \int_0^\infty (-\partial_\xi + \partial_\zeta + \varepsilon \partial_{\tau_1}) \left( (\eta_0 + \varepsilon \eta_1) \int_0^y (\eta_0 + \varepsilon \eta_1)_t dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy + \mathcal{O}(\varepsilon^2) \\
&= \int_0^\infty (-\partial_\xi + \partial_\zeta) \left( (\eta_0 + \varepsilon \eta_1) \int_0^y (\eta_0 + \varepsilon \eta_1)_t dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy + \mathcal{O}(\varepsilon) \\
&= \int_0^\infty (-\partial_\xi + \partial_\zeta) \left( \eta_0 \int_0^y (\eta_0)_t dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy + \mathcal{O}(\varepsilon) \\
&= \int_0^\infty (-\partial_\xi + \partial_\zeta) \left( \eta_0 \int_0^y (-\partial_\xi + \partial_\zeta + \varepsilon \partial_{\tau_1}) \eta_0 dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy + \mathcal{O}(\varepsilon) \\
&= \int_0^\infty (-\partial_\xi + \partial_\zeta) \left( \eta_0 \int_0^y (-\partial_\xi + \partial_\zeta) \eta_0 dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy + \mathcal{O}(\varepsilon).
\end{aligned}$$

Now, recalling that  $\eta_0$  is piecewise, we split the integral:

$$\int_0^{\tau_0} (-\partial_\xi + \partial_\zeta) \left( \eta_0 \int_0^y (-\partial_\xi + \partial_\zeta) \eta_0 dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy, \tag{27}$$

and

$$\int_{\tau_0}^\infty (-\partial_\xi + \partial_\zeta) \left( \eta_0 \int_0^y (-\partial_\xi + \partial_\zeta) \eta_0 dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy. \tag{28}$$

We deal with (27):

$$\int_0^{\tau_0} (-\partial_\xi + \partial_\zeta) \left( \eta_0 \int_0^y (-\partial_\xi + \partial_\zeta) \eta_0 dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy = \int_0^{\tau_0} (-\partial_\xi + \partial_\zeta) \left( (F_1 + G_1) \int_0^y (-\partial_\xi + \partial_\zeta) (F_1 + G_1) dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy$$

$$\begin{aligned}
&= \int_0^{\tau_0} (-\partial_\xi + \partial_\zeta) \left( (F_1 + G_1) \int_0^y (-\partial_\xi F_1 + \partial_\zeta G_1) dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&= \int_0^{\tau_0} (-\partial_\xi + \partial_\zeta) ((F_1 + G_1)(-F_1 - F_1(\tau_0)) + G_1 - G_1(\tau_0)) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&= \int_0^{\tau_0} (-\partial_\xi + \partial_\zeta) ((F_1 + G_1)(-F_1 + G_1 + F_1(\tau_0)) - G_1(\tau_0)) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&= \int_0^{\tau_0} (-\partial_\xi + \partial_\zeta)(-F_1^2 + G_1^2) + (-\partial_\xi + \partial_\zeta)(F_1 + G_1)(F_1(\tau_0)) - G_1(\tau_0)) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&= \int_0^{\tau_0} (2F_1 \partial_\xi F_1 + 2G_1 \partial_\zeta G_1 + A(-\partial_\xi F_1 + \partial_\zeta G_1)) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy,
\end{aligned}$$

where we can set  $F_1(\tau_0) - G_1(\tau_0) = A$  by imposing a free end condition  $\eta_x(0, t) = 0$ . Now, we deal with (28):

$$\begin{aligned}
&\int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta) \left( \eta_0 \int_0^y (-\partial_\xi + \partial_\zeta) \eta_0 dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&= \int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta) \left( (F_2 + G_2) \left( \int_0^{\tau_0} (-\partial_\xi + \partial_\zeta)(F_1 + G_1) dy' + \int_{\tau_0}^y (-\partial_\xi + \partial_\zeta)(F_2 + G_2) dy' \right) \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&= \int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta) ((F_2 + G_2)((-F_1(0) + F_1(\tau_0) + G_1(2\tau_0) - G_1(\tau_0) - F_2 + F_2(0) + G_2 - G_2(2\tau_0))) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&= \int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta) ((F_2 + G_2)(-F_2 + G_2 + F_2(0) - F_1(0) + F_1(\tau_0) + G_1(2\tau_0) - G_2(2\tau_0) - G_1(\tau_0))) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&= \int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta) ((F_2 + G_2)(-F_2 + G_2)) + (-\partial_\xi + \partial_\zeta) ((F_2 + G_2)(F_2(0) - F_1(0) + F_1(\tau_0) + G_1(2\tau_0) - G_2(2\tau_0) - G_1(\tau_0))) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy,
\end{aligned}$$

where for the second and third lines we translate  $\xi' = x' - t, \zeta' = x' + t$  to obtain

$$\begin{aligned}
\int_t^y -\partial_{\xi'}(F_2(\xi' - \tau_0)) dx' &= \int_0^{y-t} (F_2)_{\xi'}(\xi', \tau_1) d\xi' = \int_0^\xi (F_2)_{\xi'}(\xi', \tau_1) d\xi' = F_2(\xi, \tau_1) - F_2(0, \tau_1), \\
\int_t^y \partial_{\zeta'}(G_2)(\xi' + \tau_0, \tau_1) dx' &= \int_{2t}^{y+t} (G_2)_{\zeta'}(\zeta', \tau_1) d\zeta' = \int_{2\tau_0}^\zeta (G_2)_{\zeta'}(\zeta', \tau_1) d\zeta' = G_2(\zeta, \tau_1) - G_2(2\tau_0, \tau_1).
\end{aligned}$$

We have that

$$\int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta) ((F_2 + G_2)(-F_2 + G_2)) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy = \int_{\tau_0}^{\infty} (-\partial_\xi + \partial_\zeta)(-F_2^2 + G_2^2) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy$$

$$\begin{aligned}
&= \int_{\tau_0}^{\infty} (\partial_{\xi}(F_2^2) + \partial_{\zeta}(G_2^2)) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&= \int_{\tau_0}^{\infty} (2F_2\partial_{\xi}F_2 + 2G_2\partial_{\zeta}G_2) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy.
\end{aligned}$$

Let

$$F_2(0) - F_1(0) + G_1(2\tau_0) - G_2(2\tau_0) = B,$$

so that

$$F_2(0) - F_1(0) + G_1(2\tau_0) - G_2(2\tau_0) + F_1(\tau_0) - G_1(\tau_0) = A + B.$$

We then see that:

$$\begin{aligned}
&\int_{\tau_0}^{\infty} (-\partial_{\xi} + \partial_{\zeta}) ((F_2 + G_2)(F_2(0) - F_1(0) + F_1(\tau_0) + G_1(2\tau_0) - G_2(2\tau_0) - G_1(\tau_0))) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&= (A + B) \int_{\tau_0}^{\infty} (-\partial_{\xi} + \partial_{\zeta})(F_2 + G_2) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&= (A + B) \int_{\tau_0}^{\infty} (-\partial_{\xi}F_2 + \partial_{\zeta}G_2) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy. \tag{29}
\end{aligned}$$

we observe interaction between  $F_i, G_i$  at the interface  $x = \tau_0$ . Note that if we impose continuity, then  $F_2(0) = F_1(0), G_2(2\tau_0) = G_1(2\tau_0)$ , which leaves us with  $F_1(\tau_0) - G_1(\tau_0)$ . As before, we can eliminate this term by imposing a free end condition  $\eta_x(0, t) = 0$ , and restricting the scalar of integration to be 0. In this case,  $A + B = 0$  and so the term (29) vanishes due to boundary conditions. More generally, we obtain

$$\begin{aligned}
&\int_{\tau_0}^{\infty} (-\partial_{\xi} + \partial_{\zeta}) \left( \eta_0 \int_0^y (-\partial_{\xi} + \partial_{\zeta}) \eta_0 dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&= \int_{\tau_0}^{\infty} (-\partial_{\xi} + \partial_{\zeta}) ((F_2 + G_2)(-F_2 + G_2)) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy + (A + B) \int_{\tau_0}^{\infty} (-\partial_{\xi}F_2 + \partial_{\zeta}G_2) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&= \int_{\tau_0}^{\infty} (2F_2\partial_{\xi}F_2 + 2G_2\partial_{\zeta}G_2) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy + (A + B) \int_{\tau_0}^{\infty} (-\partial_{\xi}F_2 + \partial_{\zeta}G_2) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy,
\end{aligned}$$

so that

$$\begin{aligned}
&\int_0^{\infty} \partial_t \left( \eta \int_0^y \eta_t dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&= \int_0^{\tau_0} (-\partial_{\xi} + \partial_{\zeta}) \left( \eta_0 \int_0^y (-\partial_{\xi} + \partial_{\zeta}) \eta_0 dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy + \int_{\tau_0}^{\infty} (-\partial_{\xi} + \partial_{\zeta}) \left( \eta_0 \int_0^y (-\partial_{\xi} + \partial_{\zeta}) \eta_0 dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy
\end{aligned}$$

$$\begin{aligned}
&= \int_0^{\tau_0} (2F_1 \partial_\xi F_1 + 2G_1 \partial_\zeta G_1) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy + \int_{\tau_0}^\infty (2F_2 \partial_\xi F_2 + 2G_2 \partial_\zeta G_2) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&+ A \int_0^{\tau_0} (-\partial_\xi F_1 + \partial_\zeta G_1) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy + (A+B) \int_{\tau_0}^\infty (-\partial_\xi F_2 + \partial_\zeta G_2) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy.
\end{aligned}$$

The proof is complete. □

*Proposition 6.* Let

$$\begin{aligned}
A &= F_1(\tau_0) - G_1(\tau_0), \\
B &= F_2(0) - F_1(0) + G_1(2\tau_0) - G_2(2\tau_0).
\end{aligned}$$

Then, we have

$$\begin{aligned}
&(\partial_\xi + \partial_\zeta) \int_0^\infty \partial_t \left( \eta \int_0^y \eta_t dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&= \partial_\xi \left( \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_0^\infty 2F_2 \partial_{\xi'} F_2 \frac{1}{\xi - \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} d\zeta' + \int_{2\tau_0}^\infty 2G_2 \partial_{\zeta'} G_2 \frac{1}{\xi + \zeta'} d\zeta' \right) \\
&+ \partial_\zeta \left( \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi + \zeta'} d\xi' + \int_0^\infty 2F_2 \partial_{\xi'} F_2 \frac{1}{\xi + \zeta'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' + \int_{2\tau_0}^\infty 2G_2 \partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right) \\
&+ \partial_\xi \left( A \int_{-\tau_0}^0 -\partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} d\zeta' + (A+B) \int_0^\infty -\partial_{\xi'} F_2 \frac{1}{\xi - \xi'} d\xi' + \int_{2\tau_0}^\infty \partial_{\zeta'} G_2 \frac{1}{\xi + \zeta'} d\zeta' \right) \\
&+ \partial_\zeta \left( A \int_{-\tau_0}^0 -\partial_{\xi'} F_1 \frac{1}{\zeta + \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' + (A+B) \int_0^\infty -\partial_{\xi'} F_2 \frac{1}{\zeta + \xi'} d\xi' + \int_{2\tau_0}^\infty \partial_{\zeta'} G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right)
\end{aligned}$$

*Proof.* By Proposition 5, we have

$$\begin{aligned}
&(\partial_\xi + \partial_\zeta) \int_0^\infty \partial_t \left( \eta \int_0^y \eta_t dy' \right) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&= (\partial_\xi + \partial_\zeta) \int_0^{\tau_0} (2F_1 \partial_\xi F_1 + 2G_1 \partial_\zeta G_1) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy + (\partial_\xi + \partial_\zeta) \int_{\tau_0}^\infty (2F_2 \partial_\xi F_2 + 2G_2 \partial_\zeta G_2) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&+ A(\partial_\xi + \partial_\zeta) \int_0^{\tau_0} (-\partial_\xi F_1 + \partial_\zeta G_1) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy + (A+B)(\partial_\xi + \partial_\zeta) \int_{\tau_0}^\infty (-\partial_\xi F_2 + \partial_\zeta G_2) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy,
\end{aligned}$$

Before we start, recall the following changes of variables; they will be used throughout the proof:

$$x - \tau_0 = \xi, \quad (30)$$

$$x + \tau_0 = \zeta, \quad (31)$$

$$y - \tau_0 = \xi', \quad (32)$$

$$y + \tau_0 = \zeta'. \quad (33)$$

Consider

$$(\partial_\xi + \partial_\zeta) \int_0^{\tau_0} (2F_1 \partial_\xi F_1 + 2G_1 \partial_\zeta G_1) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy.$$

First, observe that

$$\begin{aligned} \int_0^{\tau_0} 2F_1(\tau_0 - y) \partial_{\xi'} F_1(\tau_0 - y) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy &= \int_{-\tau_0}^0 2F_1(-\xi') \partial_{\xi'} F_1(-\xi') \left[ \frac{1}{x-\xi'-t} + \frac{1}{x+\xi'+t} \right] d\xi' && \text{(use (32))} \\ &= \int_{-\tau_0}^0 2F_1(-\xi') \partial_{\xi'} F_1(-\xi') \left[ \frac{1}{\xi + \tau_0 - \xi' - \tau_0} + \frac{1}{\zeta - \tau_0 + \xi' + \tau_0} \right] d\xi' && \text{(use (30) and (31))} \\ &= \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \left[ \frac{1}{\xi - \xi'} + \frac{1}{\zeta + \xi'} \right] d\xi', \end{aligned}$$

and

$$\begin{aligned} \int_0^{\tau_0} 2G_1(y + \tau_0) \partial_{\zeta'} G_1(y + \tau_0) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy &= \int_{\tau_0}^{2\tau_0} 2G_1(\zeta') \partial_{\zeta'} G_1(\zeta') \left[ \frac{1}{x-\zeta'-\tau_0} + \frac{1}{x+\zeta'+\tau_0} \right] d\zeta' && \text{(use (33))} \\ &= \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \left[ \frac{1}{\zeta - \tau_0 - \zeta' + \tau_0} + \frac{1}{\xi + \tau_0 + \zeta' - \tau_0} \right] d\zeta' && \text{(use (31)) and (30)} \\ &= \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \left[ \frac{1}{\zeta - \zeta'} + \frac{1}{\xi + \zeta'} \right] d\zeta'. \end{aligned}$$

This yields:

$$\begin{aligned} (\partial_\xi + \partial_\zeta) \int_0^{\tau_0} (2F_1 \partial_\xi F_1 + 2G_1 \partial_\zeta G_1) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy &= (\partial_\xi + \partial_\zeta) \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \left[ \frac{1}{\xi - \xi'} + \frac{1}{\zeta + \xi'} \right] d\xi' + (\partial_\xi + \partial_\zeta) \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \left[ \frac{1}{\zeta - \zeta'} + \frac{1}{\xi + \zeta'} \right] d\zeta' \\ &= \partial_\xi \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \partial_\zeta \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\zeta + \xi'} d\xi' \end{aligned}$$

$$\begin{aligned}
& + \partial_\xi \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} d\zeta' + \partial_\zeta \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' \\
& = \partial_\xi \left( \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} d\zeta' \right) \\
& + \partial_\zeta \left( \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\zeta + \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' \right). \tag{34}
\end{aligned}$$

By a similar argument, one may show that

$$\int_{\tau_0}^{\infty} 2F_2 \partial_\xi F_2 \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy = \int_0^{\infty} 2F_2 \partial_\xi F_2 \left[ \frac{1}{\xi - \xi'} + \frac{1}{\zeta + \xi'} \right] d\xi',$$

and

$$\int_{\tau_0}^{\infty} 2G_2 \partial_\zeta G_2 \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy = \int_{2\tau_0}^{\infty} 2G_2 \partial_\zeta G_2 \left[ \frac{1}{\zeta - \zeta'} + \frac{1}{\xi + \zeta'} \right] d\zeta',$$

so that

$$\begin{aligned}
(\partial_\xi + \partial_\zeta) \int_{\tau_0}^{\infty} (2F_2 \partial_\xi F_2 + 2G_2 \partial_\zeta G_2) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy &= (\partial_\xi + \partial_\zeta) \int_0^{\infty} 2F_2 \partial_\xi F_2 \left[ \frac{1}{\xi - \xi'} + \frac{1}{\zeta + \xi'} \right] d\xi' + (\partial_\xi + \partial_\zeta) \int_{2\tau_0}^{\infty} 2G_2 \partial_\zeta G_2 \left[ \frac{1}{\zeta - \zeta'} + \frac{1}{\xi + \zeta'} \right] d\zeta' \\
&= \partial_\xi \int_0^{\infty} 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \partial_\zeta \int_0^{\infty} 2F_1 \partial_{\xi'} F_1 \frac{1}{\zeta + \xi'} d\xi' \\
&+ \partial_\xi \int_{2\tau_0}^{\infty} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} d\zeta' + \partial_\zeta \int_{2\tau_0}^{\infty} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' \\
&= \partial_\xi \left( \int_0^{\infty} 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_{2\tau_0}^{\infty} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} d\zeta' \right) \\
&+ \partial_\zeta \left( \int_0^{\infty} 2F_1 \partial_{\xi'} F_1 \frac{1}{\zeta + \xi'} d\xi' + \int_{2\tau_0}^{\infty} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' \right). \tag{35}
\end{aligned}$$

Finally, bringing (34) and (35) yields

$$\begin{aligned}
& (\partial_\xi + \partial_\zeta) \int_0^{\tau_0} (2F_1 \partial_{\xi'} F_1 + 2G_1 \partial_{\zeta'} G_1) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy + (\partial_\xi + \partial_\zeta) \int_{\tau_0}^{\infty} (2F_2 \partial_\xi F_2 + 2G_2 \partial_\zeta G_2) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&= \partial_\xi \left( \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} d\zeta' \right) + \partial_\zeta \left( \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\zeta + \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' \right)
\end{aligned}$$

$$\begin{aligned}
& + \partial_\xi \left( \int_0^\infty 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_{2\tau_0}^\infty 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} d\zeta' \right) + \partial_\zeta \left( \int_0^\infty 2F_1 \partial_{\xi'} F_1 \frac{1}{\zeta + \xi'} d\xi' + \int_{2\tau_0}^\infty 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' \right) \\
& = \partial_\xi \left( \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_0^\infty 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} d\zeta' + \int_{2\tau_0}^\infty 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} d\zeta' \right) \\
& + \partial_\zeta \left( \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\zeta + \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' + \int_0^\infty 2F_1 \partial_{\xi'} F_1 \frac{1}{\zeta + \xi'} d\xi' + \int_{2\tau_0}^\infty 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' \right),
\end{aligned}$$

which gives the first part of the identity. For the second part, it is straightforward that

$$\int_0^{\tau_0} (-\partial_\xi F_1 + \partial_\zeta G_1) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy = \int_{-\tau_0}^0 -\partial_\xi F_1 \left[ \frac{1}{\xi - \xi'} + \frac{1}{\zeta + \xi'} \right] d\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \left[ \frac{1}{\zeta - \zeta'} + \frac{1}{\xi + \zeta'} \right] d\zeta'$$

so that

$$\begin{aligned}
(\partial_\xi + \partial_\zeta) \int_0^{\tau_0} (-\partial_\xi F_1 + \partial_\zeta G_1) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy &= (\partial_\xi + \partial_\zeta) \int_{-\tau_0}^0 -\partial_\xi F_1 \left[ \frac{1}{\xi - \xi'} + \frac{1}{\zeta + \xi'} \right] d\xi' + (\partial_\xi + \partial_\zeta) \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \left[ \frac{1}{\zeta - \zeta'} + \frac{1}{\xi + \zeta'} \right] d\zeta' \\
&= \partial_\xi \left( \int_{-\tau_0}^0 -\partial_\xi F_1 \frac{1}{\xi - \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \frac{1}{\xi + \zeta'} d\zeta' \right) + \partial_\zeta \left( \int_{-\tau_0}^0 -\partial_\xi F_1 \frac{1}{\zeta + \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \frac{1}{\zeta - \zeta'} d\zeta' \right).
\end{aligned}$$

Similarly,

$$\int_{\tau_0}^\infty (-\partial_\xi F_2 + \partial_\zeta G_2) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy = \int_0^\infty -\partial_\xi F_2 \left[ \frac{1}{\xi - \xi'} + \frac{1}{\zeta + \xi'} \right] d\xi' + \int_{2\tau_0}^\infty \partial_\zeta G_2 \left[ \frac{1}{\zeta - \zeta'} + \frac{1}{\xi + \zeta'} \right] d\zeta'$$

so that

$$\begin{aligned}
(\partial_\xi + \partial_\zeta) \int_{\tau_0}^\infty (-\partial_\xi F_2 + \partial_\zeta G_2) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy &= (\partial_\xi + \partial_\zeta) \left( \int_0^\infty -\partial_\xi F_2 \left[ \frac{1}{\xi - \xi'} + \frac{1}{\zeta + \xi'} \right] d\xi' \right) + (\partial_\xi + \partial_\zeta) \left( \int_{2\tau_0}^\infty \partial_\zeta G_2 \left[ \frac{1}{\zeta - \zeta'} + \frac{1}{\xi + \zeta'} \right] d\zeta' \right) \\
&= \partial_\xi \left( \int_0^\infty -\partial_\xi F_2 \frac{1}{\xi - \xi'} d\xi' + \int_{2\tau_0}^\infty \partial_\zeta G_2 \frac{1}{\xi + \zeta'} d\zeta' \right) + \partial_\zeta \left( \int_0^\infty -\partial_\xi F_2 \frac{1}{\zeta + \xi'} d\xi' + \int_{2\tau_0}^\infty \partial_\zeta G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
A(\partial_\xi + \partial_\zeta) \int_0^{\tau_0} (-\partial_\xi F_1 + \partial_\zeta G_1) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy &+ (A+B)(\partial_\xi + \partial_\zeta) \int_{\tau_0}^\infty (-\partial_\xi F_2 + \partial_\zeta G_2) \left[ \frac{1}{x-y} + \frac{1}{x+y} \right] dy \\
&= A\partial_\xi \left( \int_{-\tau_0}^0 -\partial_\xi F_1 \frac{1}{\xi - \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \frac{1}{\xi + \zeta'} d\zeta' \right) + A\partial_\zeta \left( \int_{-\tau_0}^0 -\partial_\xi F_1 \frac{1}{\zeta + \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \frac{1}{\zeta - \zeta'} d\zeta' \right)
\end{aligned}$$



$$\begin{aligned}
& + (A + B)\partial_\xi \left( \int_0^\infty -\partial_\xi F_2 \frac{1}{\xi - \xi'} d\xi' + \int_{2\tau_0}^\infty \partial_\zeta G_2 \frac{1}{\xi + \zeta'} d\zeta' \right) + (A + B)\partial_\zeta \left( \int_0^\infty -\partial_\xi F_2 \frac{1}{\zeta + \xi'} d\xi' + \int_{2\tau_0}^\infty \partial_\zeta G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right) \\
& = \partial_\xi \left( A \int_{-\tau_0}^0 -\partial_\xi F_1 \frac{1}{\xi - \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \frac{1}{\xi + \zeta'} d\zeta' + (A + B) \int_0^\infty -\partial_\xi F_2 \frac{1}{\xi - \xi'} d\xi' + \int_{2\tau_0}^\infty \partial_\zeta G_2 \frac{1}{\xi + \zeta'} d\zeta' \right) \\
& + \partial_\zeta \left( A \int_{-\tau_0}^0 -\partial_\xi F_1 \frac{1}{\zeta + \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \frac{1}{\zeta - \zeta'} d\zeta' + (A + B) \int_0^\infty -\partial_\xi F_2 \frac{1}{\zeta + \xi'} d\xi' + \int_{2\tau_0}^\infty \partial_\zeta G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right),
\end{aligned}$$

which yields the second part of identity. Therefore,

$$\begin{aligned}
& \partial_\xi + \partial_\zeta \int_0^\infty \partial_t \left( \eta \int_0^y \eta_t dy' \right) \left[ \frac{1}{x - y} + \frac{1}{x + y} \right] dy \\
& = (\partial_\xi + \partial_\zeta) \int_0^{\tau_0} (2F_1 \partial_\xi F_1 + 2G_1 \partial_\zeta G_1) \left[ \frac{1}{x - y} + \frac{1}{x + y} \right] dy + (\partial_\xi + \partial_\zeta) \int_{\tau_0}^\infty (2F_2 \partial_\xi F_2 + 2G_2 \partial_\zeta G_2) \left[ \frac{1}{x - y} + \frac{1}{x + y} \right] dy \\
& + A(\partial_\xi + \partial_\zeta) \int_0^{\tau_0} (-\partial_\xi F_1 + \partial_\zeta G_1) \left[ \frac{1}{x - y} + \frac{1}{x + y} \right] dy + (A + B)(\partial_\xi + \partial_\zeta) \int_{\tau_0}^\infty (-\partial_\xi F_2 + \partial_\zeta G_2) \left[ \frac{1}{x - y} + \frac{1}{x + y} \right] dy \\
& = \partial_\xi \left( \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_0^\infty 2F_1 \partial_{\xi'} F_1 \frac{1}{\xi - \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} d\zeta' + \int_{2\tau_0}^\infty 2G_1 \partial_{\zeta'} G_1 \frac{1}{\xi + \zeta'} d\zeta' \right) \\
& + \partial_\zeta \left( \int_{-\tau_0}^0 2F_1 \partial_{\xi'} F_1 \frac{1}{\zeta + \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' + \int_0^\infty 2F_1 \partial_{\xi'} F_1 \frac{1}{\zeta + \xi'} d\xi' + \int_{2\tau_0}^\infty 2G_1 \partial_{\zeta'} G_1 \frac{1}{\zeta - \zeta'} d\zeta' \right) \\
& + \partial_\xi \left( A \int_{-\tau_0}^0 -\partial_\xi F_1 \frac{1}{\xi - \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \frac{1}{\xi + \zeta'} d\zeta' + (A + B) \int_0^\infty -\partial_\xi F_2 \frac{1}{\xi - \xi'} d\xi' + \int_{2\tau_0}^\infty \partial_\zeta G_2 \frac{1}{\xi + \zeta'} d\zeta' \right) \\
& + \partial_\zeta \left( A \int_{-\tau_0}^0 -\partial_\xi F_1 \frac{1}{\zeta + \xi'} d\xi' + \int_{\tau_0}^{2\tau_0} \partial_\zeta G_1 \frac{1}{\zeta - \zeta'} d\zeta' + (A + B) \int_0^\infty -\partial_\xi F_2 \frac{1}{\zeta + \xi'} d\xi' + \int_{2\tau_0}^\infty \partial_\zeta G_2 \frac{1}{\zeta - \zeta'} d\zeta' \right).
\end{aligned}$$

This gives the desired identity. □

## References

- [1] Tom M. Apostol, *Mathematical analysis*, Pearson, 1974.