3

Weakly nonlinear dispersive waves

The old order changeth, yielding place to new
The Passing of Arthur

In Chapter 2 we presented some classical ideas in the theory of water waves. One particular concept that we introduced was the phenomenon of a balance between nonlinearity and dispersion, leading to the existence of the solitary wave, for example. Further, under suitable assumptions, this wave can be approximated by the sech^2 function, which is an exact solution of the Korteweg–de Vries (KdV) equation; see Section 2.9.1. We shall now use this result as the starting point for a discussion of the equations, and of the properties of corresponding solutions, that arise when we invoke the assumptions of small amplitude and long wavelength. In the modern theories of nonlinear wave propagation – and certainly not restricted only to water waves – this has proved to be an exceptionally fruitful area of study.

The results that have been obtained, and the mathematical techniques that have been developed, have led to altogether novel, important and deep concepts in the theory of wave propagation. Starting from the general method of solution for the initial value problem for the KdV equation, a vast arena of equations, solutions and mathematical ideas has evolved. At the heart of this panoply is the *soliton*, which has caused much excitement in the mathematical and physical communities over the last 30 years or so. It is our intention to describe some of these results, and their relevance to the theory of water waves, where, indeed, they first arose.

3.1 Introduction

The existence of a steadily propagating nonlinear wave of permanent shape, such as the solitary wave, probably seems altogether likely. On the other hand, that somewhat similar objects could exist in pairs (or

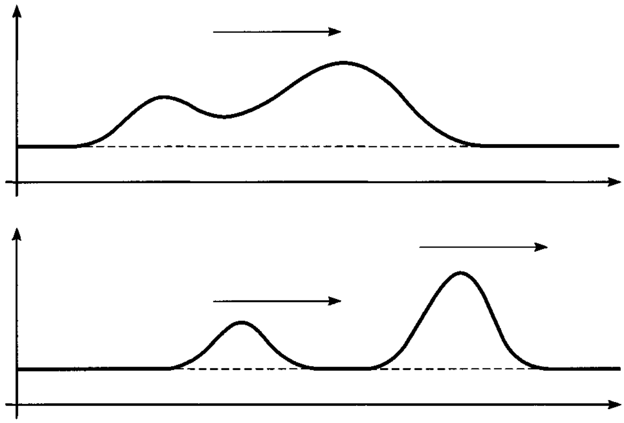


Figure 3.1. A sketch of J. Scott Russell's *compound* wave which 'represents the genesis by a large low column of fluid of a compound or double wave . . . the greater moving faster and altogether leaving the smaller'.

larger numbers) of different amplitude, interact nonlinearly and yet not destroy each other – indeed, retain their identities – would seem rather unlikely. However, precisely this phenomenon does occur for solutions of the KdV equation, and for the many other so-called *completely integrable* equations.

This very special type of interaction was first observed and described by Russell (1844); the essentials of his plate XLVII are shown in Figure 3.1. An obvious interpretation of the development represented in this figure is that an initial profile, which is not an exact solitary-wave solution, will evolve into two (or perhaps more) waves which move at different speeds and tend to individual solitary waves as time increases. Another observation, itself an extension of what Russell reported, is shown in Figure 3.2. This time we have an initial profile comprising two peaks, the taller to the left of the shorter, but both propagating to the right. The taller is moving faster (since it is locally similar to a solitary wave), and so catches up and then interacts with the shorter, and thereafter moves ahead of it. At first sight, the interaction appears to involve no interaction at all, as would be the case if the two waves satisfied the linear superposition principle; cf. equation (1.75) *et seq*. However, a nonlinear event does occur here, and that it is clearly *not* linear is confirmed by the fact that the waves are phase-shifted (by the interaction) from the

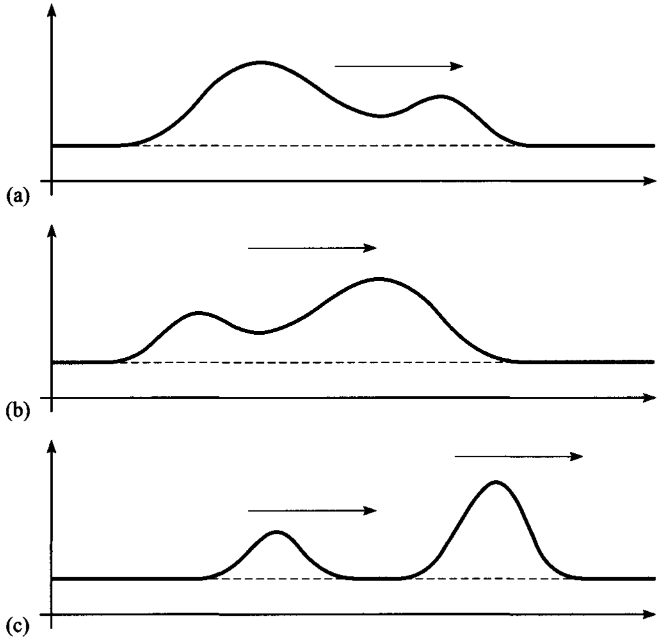


Figure 3.2. An extension of the situation depicted in Figure 3.1, where the larger wave is first to the left of the smaller; it catches up the smaller, interacts with it and then moves off to the right.

positions they would have taken had both waves travelled at constant speed througout. These, and many associated properties, will be briefly described in Section 3.3; our primary objective here is to show how this important class of completely integrable equations arise in water-wave theory. We shall then extend the ideas to more general problems, which usually do not give rise to completely integrable equations, but which do provide models for more realistic applications.

The various problems that we shall describe are based on the equation for an inviscid fluid, and for the propagation of gravity waves only (so $W_e = 0$, but see Q3.3). Although much of our early work will be for irrotational flow, some of the important applications presented later will allow an underlying rotational state; we therefore choose to develop all the work here from the Euler equations. (In contrast, a derivation of the KdV equation directly from Laplace's equation was given in Section 2.9.1.) The relevant governing equations will be found in Section 1.3.2, and they are reproduced here.

In rectangular Cartesian coordinates we have

$$\left.\begin{aligned} &\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x}, \quad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{\partial p}{\partial y}, \quad \delta^2 \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z} \\ &\text{where} \\ &\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \varepsilon\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right) \end{aligned}\right\} \tag{3.1}$$

with

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3.2}$$

and

$$p = \eta \quad \text{and} \quad w = \frac{\partial \eta}{\partial t} + \varepsilon\left(u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}\right) \quad \text{on} \quad z = 1 + \varepsilon\eta \tag{3.3}$$

$$w = u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y} \quad \text{on} \quad z = b(x, y). \tag{3.4}$$

Correspondingly, in cylindrical coordinates, we have

$$\left.\begin{aligned} &\frac{\mathrm{D}u}{\mathrm{D}t} - \frac{\varepsilon v^2}{r} = -\frac{\partial p}{\partial r}, \quad \frac{\mathrm{D}v}{\mathrm{D}t} + \frac{\varepsilon uv}{r} = -\frac{1}{r}\frac{\partial p}{\partial \theta}, \quad \delta^2 \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z} \\ &\text{where} \\ &\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \varepsilon\left(u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}\right) \end{aligned}\right\} \tag{3.5}$$

with

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \tag{3.6}$$

and

$$p = \eta \quad \text{and} \quad w = \frac{\partial \eta}{\partial t} + \varepsilon\left(u\frac{\partial \eta}{\partial r} + \frac{v}{r}\frac{\partial \eta}{\partial \theta}\right) \quad \text{on} \quad z = 1 + \varepsilon\eta \tag{3.7}$$

$$w = u\frac{\partial b}{\partial r} + \frac{v}{r}\frac{\partial b}{\partial \theta} \quad \text{on} \quad z = b(r, \theta). \tag{3.8}$$

In these equations, and for the following calculations, we consider only bottom topographies ($z = b$) which are independent of time.

3.2 The Korteweg–de Vries family of equations

We first present a derivation of the classical Korteweg–de Vries equation, from the Euler equations, being careful to describe the necessary (and minimal) assumptions that are required. We then show how this approach can be generalised to obtain corresponding equations valid in both different and higher-dimensional coordinate systems.

3.2.1 Korteweg–de Vries (KdV) equation

We consider surface gravity waves propagating in the positive x-direction over stationary water of constant depth (so $b = 0$). Thus, from equations (3.1)–(3.4), we have

$$\left.\begin{aligned} &u_t + \varepsilon(uu_x + wu_z) = -p_x; \quad \delta^2\{w_t + \varepsilon(uw_x + ww_z)\} = -p_z; \\ &\qquad\qquad u_x + w_z = 0, \\ &\text{with} \\ &\qquad p = \eta \quad \text{and} \quad w = \eta_t + \varepsilon u\eta_x \quad \text{on} \quad z = 1 + \varepsilon\eta \\ &\text{and} \\ &\qquad\qquad w = 0 \quad \text{on} \quad z = 0. \end{aligned}\right\} \tag{3.9}$$

This problem, when previously discussed via Laplace's equation in Section 2.9.1, led us to invoke a special choice of the parameters, namely $\delta^2 = O(\varepsilon)$ as $\varepsilon \to 0$. If this were to be a necessary condition in order to obtain the appropriate balance between nonlinearity and dispersion (and so to produce the KdV equation and hence to model solitary waves, for example), we might expect these waves to occur rather rarely in nature: solitary waves would be infrequently observed, and this is not the case. It is therefore no surprise that we can readily demonstrate that, for any δ as $\varepsilon \to 0$, there always exists a region of (x, t)-space where this balance comes about. Thus, for as long as no other physical effects intervene, we can expect to be able to generate KdV solitary waves (and solitons etc.) somewhere, provided only that the amplitude is small (in the sense of $\varepsilon \to 0$).

The region of interest is defined by a scaling of the independent variables. First we transform

$$x \to \frac{\delta}{\varepsilon^{1/2}}x, \quad t \to \frac{\delta}{\varepsilon^{1/2}}t, \tag{3.10}$$

for any ε and δ; this transformation then implies, for consistency from the equation of mass conservation, that we also transform

$$w \to \frac{\varepsilon^{1/2}}{\delta} w; \tag{3.11}$$

cf. Q1.34. The governing equations (3.9) then become

$$u_t + \varepsilon(uu_x + wu_z) = -p_x; \quad \varepsilon\{w_t + \varepsilon(uw_x + ww_z)\} = -p_z; \tag{3.12}$$

$$u_x + w_z = 0, \tag{3.13}$$

with

$$p = \eta \quad \text{and} \quad w = \eta_t + \varepsilon u \eta_x \quad \text{on} \quad z = 1 + \varepsilon\eta \tag{3.14}$$

and

$$w = 0 \quad \text{on} \quad z = 0, \tag{3.15}$$

so the net outcome of the transformation is to replace δ^2 by ε in equations (3.9). (The presence of δ in transformations (3.10) and (3.11) is merely equivalent to using h_0 alone as the relevant length scale; see Section 1.3.1.)

Now, for $\varepsilon \to 0$, we see that a first approximation to equations (3.12) and (3.14) is

$$p(x, t, z) = \eta(x, t), \quad 0 \le z \le 1$$

with

$$u_t + \eta_x = 0. \tag{3.16}$$

Then, from equation (3.13), we obtain

$$w = -zu_x$$

which satisfies (3.15), and from condition (3.14) we require

$$\eta_t = -u_x;$$

this combined with equation (3.16) yields

$$\eta_{tt} - \eta_{xx} = 0,$$

as we should expect (cf. equation (2.10)). We choose to follow right-going waves (but see Q3.2), and so we introduce

$$\xi = x - t. \tag{3.17}$$

However, an asymptotic expansion which is based on the classical wave equation (with higher-order nonlinear and dispersive terms) necessarily leads to a non-uniformity as t (or x) $\to \infty$; this is discussed in equation (1.95) *et seq*. Thus we define a suitable long-time variable

$$\tau = \varepsilon t; \tag{3.18}$$

cf. equation (1.99). Consequently $\xi = \mathrm{O}(1)$, $\tau = \mathrm{O}(1)$, together describe the *far-field* region for this problem, and therefore the region where we expect a KdV-type of balance to occur. (We observe that these scaling arguments have been generated by the existence of the surface wave propagating in the x-direction, and no scalings are required to describe different regions of the z-structure of the problem.)

With the choice of far-field variables, (3.17) and (3.18), the equations (3.12)–(3.15) become

$$-u_\xi + \varepsilon(u_\tau + uu_\xi + wu_z) = -p_\xi; \quad \varepsilon\{-w_\xi + \varepsilon(w_\tau + uw_\xi + ww_z)\} = -p_z; \tag{3.19}$$

$$u_\xi + w_z = 0, \tag{3.20}$$

with

$$p = \eta \quad \text{and} \quad w = -\eta_\xi + \varepsilon(\eta_\tau + u\eta_\xi) \quad \text{on} \quad z = 1 + \varepsilon\eta$$

and

$$w = 0 \quad \text{on} \quad z = 0. \tag{3.21}$$

We seek an asymptotic solution of this system of equations and boundary conditions in the form

$$q(\xi, \tau, z; \varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n q_n(\xi, \tau, z), \quad \eta(\xi, \tau; \varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n \eta_n(\xi, \tau), \tag{3.22}$$

where q (and correspondingly q_n) represents each of u, w and p. The leading-order then becomes

$$u_{0\xi} = p_{0\xi}; \quad p_{0z} = 0; \quad u_{0\xi} + w_{0z} = 0$$

with

$$p_0 = \eta_0 \quad \text{and} \quad w_0 = -\eta_{0\xi} \quad \text{on} \quad z = 1$$

and

$$w_0 = 0 \quad \text{on} \quad z = 0.$$

These equations directly lead to

$$p_0 = \eta_0, \quad u_0 = \eta_0, \quad w_0 = -z\eta_{0\xi}, \quad 0 \le z \le 1,$$

where we have imposed the condition that the perturbation in u is caused only by the passage of the wave; that is, $u_0 = 0$ whenever $\eta_0 = 0$. We see that the surface ($z = 1$) boundary condition on w_0 is automatically satisfied; thus, at this order, $\eta_0(\xi, \tau)$ is an arbitrary function. To proceed, and hence to determine η_0, we must first treat the surface boundary conditions with more care.

The two boundary conditions on $z = 1 + \varepsilon\eta$ are rewritten as evaluations on $z = 1$, by developing Taylor expansions of u, w and p about $z = 1$. (The usual convergence requirements need not be investigated since, strictly, these expansions are to exist only in the asymptotic sense as $\varepsilon \to 0$; see also Section 2.5.) These boundary conditions are therefore expressed in the form

$$p_0 + \varepsilon\eta_0 p_{0z} + \varepsilon p_1 = \eta_0 + \varepsilon\eta_1 + O(\varepsilon^2) \tag{3.23}$$

and

$$w_0 + \varepsilon\eta_0 w_{0z} + \varepsilon w_1 = -\eta_{0\xi} - \varepsilon\eta_{1\xi} + \varepsilon(\eta_{0\tau} + u_0\eta_{0\xi}) + O(\varepsilon^2) \tag{3.24}$$

on $z = 1$

which are to be used in conjunction with equations (3.19), (3.20) and (3.21).

The leading order has already been found, and the equations that define the next order are

$$-u_{1\xi} + u_{0\tau} + u_0 u_{0\xi} + w_0 u_{0z} = -p_{1\xi}; \quad p_{1z} = w_{0\xi}; \quad u_{1\xi} + w_{1z} = 0,$$

with

$$p_1 + \eta_0 p_{0z} = \eta_1 \quad \text{and} \quad w_1 + \eta_0 w_{0z} = -\eta_{1\xi} + \eta_{0\tau} + u_0\eta_{0\xi} \quad \text{on} \quad z = 1$$

and

$$w_1 = 0 \quad \text{on} \quad z = 0.$$

When we note that

$$u_{0z} = 0, \quad p_{oz} = 0 \quad \text{and} \quad w_{0z} = -\eta_{0\xi}; \tag{3.25}$$

then

$$p_1 = \frac{1}{2}(1 - z^2)\eta_{0\xi\xi} + \eta_1, \tag{3.26}$$

and hence

$$\begin{aligned} w_{1z} = -u_{1\xi} &= -\,p_{1\xi} - u_{0\tau} - u_0 u_{0\xi} \\ &= -\,\eta_{1\xi} - \frac{1}{2}(1 - z^2)\eta_{0\xi\xi\xi} - \eta_{0\tau} - \eta_0\eta_{0\xi}. \end{aligned}$$

Thus

$$w_1 = -(\eta_{1\xi} + \eta_{0\tau} + \eta_0\eta_{0\xi} + \frac{1}{2}\eta_{0\xi\xi\xi})z + \frac{1}{6}z^3\eta_{0\xi\xi\xi} \tag{3.27}$$

which satisfies the bottom boundary condition; finally the surface boundary condition (now on $z = 1$) yields

$$\begin{aligned} w_1|_{z=1} &= -(\eta_{1\xi} + \eta_{0\tau} + \eta_0\eta_{0\xi} + \frac{1}{2}\eta_{0\xi\xi\xi}) + \frac{1}{6}\eta_{0\xi\xi\xi} \\ &= -\eta_{1\xi} + \eta_{0\tau} + 2\eta_0\eta_{0\xi} \end{aligned}$$

resulting in

$$2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = 0; \tag{3.28}$$

at this order, η_1 is unknown. Equation (3.28) is the Korteweg–de Vries equation which describes the leading-order contribution to the surface wave; see also equation (1.102). This is the equation first derived (but not in our form) by Korteweg and de Vries (1895), which they did by seeking a solution of Laplace's equation as a power series in z. Furthermore, these authors also included the effects of surface tension, which here is left as an exercise (Q3.3). The significance of the KdV equation, together with some of its properties, will be discussed later.

Provided that bounded solutions of equation (3.28) exist, at least for $\xi = O(1)$, $\tau = O(1)$, and for all the higher-order terms, η_n, in the same region of space, then the function $\eta_0(\xi, \tau)$ gives the dominant behaviour that we seek. Clearly we may ask, in addition, if the asymptotic expansion for η (and hence for the other dependent variables) is uniformly valid as $|\xi| \to \infty$ and as $\tau \to \infty$. In the case of $\tau \to \infty$, this question is far from straightforward to answer completely, mainly because the equations for η_n, $n \geq 1$, are not readily solved. (The problem for η_1 is included as Q3.4; see also equation (1.100) *et seq.*) All the available evidence, some of which is numerical, suggests that our asymptotic representation of η is indeed uniformly valid as $\tau \to \infty$ (at least for solutions that satisfy $\eta \to 0$ as $|\xi| \to \infty$). The validity as $|\xi| \to \infty$ for $\tau < \infty$ does not normally raise any particular difficulties. These aspects are not pursued here since, although we believe that the theory just presented describes some important properties of real water waves, the relevance of $\tau \to \infty$ is questionable. Clearly, if the waves are allowed to propagate indefinitely, then other physical effects cannot be ignored; the most prominent of these is likely to be viscous damping (which we shall briefly discuss in Chapter 5). Usually, in practice, the damping is sufficiently weak to allow the nonlinear and dispersive effects to dominate before the waves eventually decay completely.

3.2.2 Two-dimensional Korteweg–de Vries (2D KdV) equation

The Korteweg–de Vries equation, (3.28), describes nonlinear plane waves that propagate in the x-direction. An obvious question to pose is: how is the wave propagation modified when the waves move on a two-dimensional surface (which, of course, is the physical situation)? Although a plane wave can propagate in any direction (at least on stationary water), and we may label this to be the x-direction, the waves that we wish to describe may not be plane. An important example arises when two (or more) waves, that are plane waves at infinity, cross; for the nonlinear interaction of these crossing waves, the y-dependence will not be trivial. We investigate the situation in which the wave configuration is propagating predominantly in the x-direction, with the appropriate balance of nonlinear and dispersive effects (also in the x-direction). However, in addition, we include the relevant dependence on the y-variable, this contribution appearing at the same order as the nonlinearity and dispersion.

The simplest way to see what this implies is to consider, first, the linear propagation of long waves on the surface; the leading-order problem is described by the classical wave equation

$$\eta_{tt} - (\eta_{xx} + \eta_{yy}) = 0,$$

written here in nondimensional variables (cf. equation (2.14)). This equation has a solution

$$\eta \propto \mathrm{e}^{\mathrm{i}(kx+ly-\omega t)} \quad \text{where} \quad \omega^2 = k^2 + l^2;$$

see Q2.7. Now, for waves that propagate predominantly in the x-direction, we require l to be small (since the wave propagates in the direction of the wave number vector $\mathbf{k} \equiv (k, l)$) and then the dispersion relation gives

$$\omega \sim k\left(1 + \frac{1}{2}\,\frac{l^2}{k^2}\right) \quad \text{as} \quad l \to 0.$$

This expression represents propagation at the (nondimensional) speed of unity (cf. equation (3.17)), together with a correction provided by the wave-number component in the y-direction. In order that this correction be the same size as the nonlinearity and dispersion, we require $l^2 = \mathrm{O}(\varepsilon)$ or $l = \mathrm{O}(\varepsilon^{1/2})$; equivalently, we may accommodate this by transforming $y \to \varepsilon^{1/2} y$ (and then we require $v \to \varepsilon^{1/2} v$ so that we have consistency with, for example, the representation in terms of a velocity potential, $\mathbf{u} = \nabla\phi$). Thus we introduce the variables

$$\xi = x - t, \quad \tau = \varepsilon t, \quad Y = \varepsilon^{1/2} y, \quad v = \varepsilon^{1/2} V \tag{3.29}$$

and then equations (3.1)–(3.4), with δ^2 replaced by ε (see Section 3.2.1) and with $b(x, y) = 0$, become

$$\begin{aligned} -u_\xi + \varepsilon(u_\tau + uu_\xi + \varepsilon V u_Y + wu_z) &= -p_\xi; \\ -V_\xi + \varepsilon(V_\tau + uV_\xi + \varepsilon V V_Y + wV_z) &= -p_Y; \\ \varepsilon\{-w_\xi + \varepsilon(w_\tau + uw_\xi + \varepsilon V w_Y + ww_z)\} &= -p_z; \\ u_\xi + \varepsilon V_Y + w_z &= 0, \end{aligned}$$

with

$$p = \eta \quad \text{and} \quad w = -\eta_\xi + \varepsilon(\eta_\tau + u\eta_\xi + \varepsilon V \eta_Y) \quad \text{on} \quad z = 1 + \varepsilon\eta$$

and

$$w = 0 \quad \text{on} \quad z = 0;$$

cf. equations (3.19)–(3.21). We seek an asymptotic solution, valid as $\varepsilon \to 0$, in the same form as before (see (3.22)); the leading order problem is then unchanged (except that the variables may now depend on Y), so that

$$p_0 = \eta_0, \quad u_0 = \eta_0, \quad w_0 = -z\eta_{0\xi}, \quad 0 \le z \le 1,$$

and with $V_{0\xi} = \eta_{0Y}$. At the next order, the only difference from the derivation of the KdV equation (Section 3.2.1) arises in the equation of mass conservation, which here reads

$$w_{1z} = -u_{1\xi} - V_{0Y}.$$

The result of this change is to give

$$w_1 = -(\eta_{1\xi} + \eta_{0\tau} + \eta_0\eta_{0\xi} + \frac{1}{2}\eta_{0\xi\xi\xi} + V_{0Y})z + \frac{1}{6}z^3\eta_{0\xi\xi\xi};$$

cf. equation (3.27). Thus we obtain the equation for the leading-order representation of the surface wave in the form

$$2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} + V_{0Y} = 0$$

where $V_{0\xi} = \eta_{0Y}$; upon the elimination of V_0 this yields

$$\left(2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi}\right)_\xi + \eta_{0YY} = 0, \tag{3.30}$$

the *two-dimensional Korteweg–de Vries* (*2D KdV*) equation. (The small amount of detail omitted from this derivation, which follows that leading to equations (3.23)–(3.28), is left as an exercise.) We note that this result recovers the KdV equation, (3.28), when there is no dependence on Y; that is, $V_{0Y} = 0$.

Equation (3.30), which in the literature is often called the *Kadomtsev–Petviashvili* (*KP*) equation (Kadomtsev & Petviashvili, 1970), turns out to be another of those very special completely integrable equations. This equation admits, as an exact analytical solution, any number of waves that cross obliquely and interact nonlinearly; this in turn, for the case of three such waves, leads to special solutions which correspond to a *resonance* condition. We shall comment in more detail about the solution of this equation later, and some of the exercises allow further exploration.

Before we leave this equation for the present, it is instructive to interpret the scalings, (3.29), that have led to the 2D KdV equation. This nonlinear dispersive wave appears at times $\tau = O(1)$, so t is large, and where $\xi = O(1)$; that is, the measure of the 'width' of the wave remains finite and non-zero as $\varepsilon \to 0$. However, the wave also depends on $Y = \varepsilon^{1/2}y$, and this is most conveniently thought of as weak dependence on the y-coordinate. That is, along any wavefront, $dy/d\xi = O(\varepsilon^{1/2})$ and so, in physical coordinates, the waves deviate only a little ($O(\varepsilon^{1/2})$) away from plane waves ($\xi =$ constant). Thus, for example, in the case of two obliquely crossing waves – an exact solution of the 2D KdV equation – the angle between them (in physical coordinates) is $O(\varepsilon^{1/2})$: they are nearly parallel waves in this approximation. We shall discuss more general aspects of obliquely crossing waves later (Section 3.4.5), as an example of a non-uniform environment.

3.2.3 Concentric Korteweg–de Vries (cKdV) equation

The two equations derived above, the KdV and 2D KdV equations, are the relevant weakly nonlinear dispersive wave equations that arise in Cartesian geometry. It is now reasonable to ask if a corresponding pair of equations exists in cylindrical geometry. In this section and the next we shall demonstrate that this is, indeed, the case, although the change of coordinates is not an altogether trivial exercise, since important differences arise. To see what the essential changes are we first consider, for purely concentric waves, the linearised problem for large radius.

The equation for linear concentric waves (in the long-wave approximation) is

$$\eta_{tt} - \left(\eta_{rr} + \frac{1}{r}\eta_r\right) = 0;$$

see equation (2.14), with the dependence on the angular coordinate, θ, absent. It is convenient to introduce the characteristic coordinate $\xi = r - t$ (for outward propagation) and $R = \alpha r$ (so that $\alpha \to 0$ will correspond to large radius; that is, $R = \mathrm{O}(1)$, $\alpha \to 0$, yields $r \to \infty$). The equation therefore becomes

$$2\eta_{\xi R} + \frac{1}{R}\eta_\xi + \alpha\left(\eta_{RR} + \frac{1}{R}\eta_R\right) = 0,$$

and as $\alpha \to 0$ we see that

$$\sqrt{R}\eta_\xi \sim g(\xi),$$

where $g(\xi)$ is an arbitrary function. Thus, for outwardly propagating waves, the relevant solution takes the form

$$\eta \sim \frac{1}{\sqrt{R}} f(\xi) \quad \text{as} \quad \alpha \to 0, \tag{3.31}$$

where $f = \int g \mathrm{d}\xi$ and we have chosen $\eta = 0$ when $f = 0$. This dominant behaviour, (3.31), for large radius, describes the expected geometrical decay of the wave: as the radius increases, so the length of the wavefront increases and the amplitude must correspondingly decrease. This presents a very different picture from that encountered in the derivation of the KdV equation. In that case the amplitude remained uniformly $\mathrm{O}(\varepsilon)$; here the amplitude decreases as the radius increases – and we expect the relevant region of balance to occur for some suitably large radius, yet this could imply that the amplitude is so small that the nonlinear terms play no rôle at leading order. We shall now establish that a scaling does exist which ensures that all the relevant conditions are met.

The equations that describe concentric gravity waves are (from (3.5)–(3.8))

$$u_t + \varepsilon(uu_r + wu_z) = -p_r; \quad \delta^2\{w_t + \varepsilon(uw_r + ww_z)\} = -p_z;$$

$$u_r + \frac{1}{r}u + w_z = 0,$$

with

$$p = \eta \quad \text{and} \quad w = \eta_t + \varepsilon u \eta_r \quad \text{on} \quad z = 1 + \varepsilon \eta$$

and

$$w = 0 \quad \text{on} \quad z = 0,$$

where, as before, we have chosen $b = 0$. To proceed, we introduce

$$\xi = \frac{\varepsilon^2}{\delta^2}(r - t), \quad R = \frac{\varepsilon^6}{\delta^4} r, \tag{3.32}$$

where a large radius variable is used here in preference to large time (but see below), and we write

$$\eta = \frac{\varepsilon^3}{\delta^2} H, \quad p = \frac{\varepsilon^3}{\delta^2} P, \quad u = \frac{\varepsilon^3}{\delta^2} U, \quad w = \frac{\varepsilon^5}{\delta^4} W; \tag{3.33}$$

in this transformation, large distance/time is measured by the scale δ^4/ε^6, so $1/\sqrt{\delta^4/\varepsilon^6} = \varepsilon^3/\delta^2$, which is the scale of the wave amplitude, consistent with the decay at large radius. The original amplitude parameter, ε, is now to be interpreted as based on the amplitude of the wave for $r = \mathrm{O}(1)$ and $t = \mathrm{O}(1)$. The governing equations thus become

$$-U_\xi + \Delta(UU_\xi + WU_z + \Delta UU_R) = -(P_\xi + \Delta P_R)$$

$$\Delta\{-W_\xi + \Delta(UW_\xi + WW_z + \Delta UW_R)\} = -P_z;$$

$$U_\xi + W_z + \Delta\left(U_R + \frac{1}{R}U\right) = 0,$$

with

$$P = H \quad \text{and} \quad W = -H_\xi + \Delta(UH_\xi + \Delta UH_R) \quad \text{on} \quad z = 1 + \Delta H$$

and

$$W = 0 \quad \text{on} \quad z = 0,$$

where $\Delta = \varepsilon^4/\delta^2$. These equations are identical, in terms of their general structure, to those discussed in Sections 3.2.1 and 3.2.2, with ε replaced by Δ; here we therefore require only that $\Delta \to 0$. This condition is satisfied, for example, with δ fixed and $\varepsilon \to 0$, and the scalings (3.32) then describe the region where the required balance occurs; the amplitude of the wave in this region is $\mathrm{O}(\Delta)$.

We seek an asymptotic solution in the usual form

$$Q \sim \sum_{n=0}^{\infty} \Delta^n Q_n, \quad \Delta \to 0,$$

where Q represents each of H, P, U and W. Directly, we see that the leading order yields the familiar result

$$P_0 = H_0, \quad U_0 = H_0, \quad W_0 = -zH_{0\xi}, \quad 0 \le z \le 1,$$

and then (similar to the derivation of the 2D KdV equation) we observe that the new important ingredient comes from the equation of mass conservation:

$$W_{1z} = -U_{1\xi} - \left(U_{0R} + \frac{1}{R}U_0\right).$$

We leave the few small details in this calculation to the reader; it should be clear, however, that the equation for $H_0(\xi, \tau)$ will be

$$2H_{0R} + \frac{1}{R}H_0 + 3H_0H_{0\xi} + \frac{1}{3}H_{0\xi\xi\xi} = 0, \tag{3.34}$$

the *concentric Korteweg–de Vries* (*cKdV*) equation. (Equivalently, we could work throughout using a large time variable, $\tau = \varepsilon^6 t/\delta^4$, and write $R = \tau + \Delta\xi \sim \tau$; this option is left as an exercise in Q3.7.)

This equation is, in a significant way, different from the first two KdV-type equations that we have derived: equation (3.34) includes a term (H_0/R) which involves a variable coefficient. Nevertheless, the cKdV equation is also one of the set of completely integrable equations, as we shall describe later.

3.2.4 Nearly concentric Korteweg–de Vries (ncKdV) equation

In Section 3.2.2 we described how the classical KdV equation can be extended to include (weak) dependence on the y-coordinate; this addition leads to the 2D KdV equation. We now explore how the concentric KdV equation might have a counterpart which represents an appropriate (weak) dependence on the angular variable, θ. Indeed, if we follow the philosophy adopted for the 2D KdV equation (where the relevant scaling in the y-direction was $\varepsilon^{1/2}$), then we might expect that the scaling on θ is $\Delta^{1/2}$. (Remember that the small parameter for the cKdV equation, after rescaling all the variables, turned out to be Δ – which plays the rôle of ε

in that derivation). Thus we use the variables introduced for the cKdV, namely (3.32) and (3.33), and also define

$$\theta = \Delta^{1/2}\Theta = \frac{\varepsilon^2}{\delta}\Theta \tag{3.35}$$

so that the angular distortion away from purely concentric is small. This is precisely equivalent to the case of nearly plane waves which satisfy $\mathrm{d}y/\mathrm{d}\xi = \mathrm{O}(\varepsilon^{1/2})$, so that $\theta = \mathrm{O}(\varepsilon^{1/2})$ with $\mathrm{d}y/\mathrm{d}\xi = \tan\theta$. Further, as we found before (see (3.29)), a scaling is then implied for the θ-component of the velocity vector; here we write

$$v = \frac{\varepsilon^5}{\delta^3} V \tag{3.36}$$

in order to maintain consistency between the scalings on $u = \phi_r$ and on $v = \phi_\theta / r$.

The governing equations, which follow from equations (3.5)–(3.8) with (3.32), (3.33), (3.35) and (3.36), become

$$-U_\xi + \Delta\left\{UU_\xi + WU_z + \Delta\left(UU_R + \frac{V}{R}U_\Theta\right)\right\} - \frac{\Delta^3}{R}V^2 = -(P_\xi + \Delta P_R);$$

$$-V_\xi + \Delta\left\{UV_\xi + WV_z + \Delta\left(UV_R + \frac{1}{R}VV_\Theta\right)\right\} + \Delta^2\frac{UV}{R} = -\frac{1}{R}P_\Theta;$$

$$\Delta\left\{-W_\xi + \Delta\left(UW_\xi + WW_z + \Delta UW_R + \frac{\Delta}{R}VW_\Theta\right)\right\} = -P_z;$$

$$U_\xi + W_z + \Delta\left\{U_R + \frac{1}{R}(U + V_\Theta)\right\} = 0,$$

with

$$P = H \quad \text{and} \quad W = -H_\xi + \Delta\left\{UH_\xi + \Delta\left(UH_R + \frac{1}{R}VH_\Theta\right)\right\} \quad \text{on } z = 1 + \Delta H$$

and

$$W = 0 \quad \text{on} \quad z = 0.$$

We have written $\Delta = \varepsilon^4/\delta^2$ and used the large radius variable, $R = \varepsilon^6 r/\delta^4$, as we did in Section 3.2.3. The asymptotic solution follows the familiar pattern, with

$$P_0 = H_0, \quad U_0 = H_0, \quad W_0 = -zH_{0\xi}, \quad 0 \le z \le 1,$$

and $V_{0\xi} = H_{0\Theta}/R$. At the next order the important difference again arises from the equation of mass conservation, where

$$W_{1z} = -U_{1\xi} - U_{0R} - \frac{1}{R}(U_0 + V_{0\Theta})$$

which leads to a fourth KdV-type equation

$$2H_{0R} + \frac{1}{R}H_0 + 3H_0H_{0\xi} + \frac{1}{3}H_{0\xi\xi\xi} + \frac{1}{R}V_{0\Theta} = 0$$

where $V_{0\xi} = H_{0\Theta}/R$. When we eliminate $V_0(\xi, R, \Theta)$, we obtain the single equation

$$\left(2H_{0R} + \frac{1}{R}H_0 + 3H_0H_{0\xi} + \frac{1}{3}H_{0\xi\xi\xi}\right)_\xi + \frac{1}{R^2}H_{0\Theta\Theta} = 0, \qquad (3.37)$$

the *nearly concentric Korteweg–de Vries* (*ncKdV*) equation (although a few authors have named this *Johnson's* equation since it appeared first in Johnson (1980)). When there is no dependence on the angular coordinate, Θ, we have $V_{0\Theta} = 0$ and the equation becomes the cKdV equation.

3.2.5 Boussinesq equation

The four equations derived in the previous sections all relate to propagation in one direction. We now consider the problem of describing waves that propagate in both the positive and negative x-directions and which are also weakly nonlinear and weakly dispersive. To start, we recall the governing equations for one-dimensional propagation, incorporating the scaling that replaces δ^2 by ε; these are equation (3.12)–(3.15):

$$u_t + \varepsilon(uu_x + wu_z) = -p_x; \quad \varepsilon\{w_t + \varepsilon(uw_x + ww_z)\} = -p_z;$$
$$u_x + w_z = 0,$$

with

$$p = \eta \quad \text{and} \quad w = \eta_t + \varepsilon u\eta_x \quad \text{on} \quad z = 1 + \varepsilon\eta$$

and

$$w = 0 \quad \text{on} \quad z = 0.$$

We seek an asymptotic solution, as $\varepsilon \to 0$, in the usual form (that is, in integer powers of ε) and so obtain, at leading order,

$$p_0 = \eta_0, \quad u_{0t} = -\eta_{0x}, \quad w_0 = -zu_{0x}, \quad u_{0x} = -\eta_{0t}, \quad 0 \le z \le 1, \quad (3.38)$$

and hence

$$\eta_{0tt} - \eta_{0xx} = 0 \tag{3.39}$$

exactly as in Section 3.2.1. Note that, in this development, we are seeking a solution which is valid for $x = \mathrm{O}(1)$ and $t = \mathrm{O}(1)$; cf. the far-field scalings adopted in Sections 3.2.1–3.2.4.

At $\mathrm{O}(\varepsilon)$ we see that

$$p_1 = -\frac{1}{2}(1 - z^2)u_{0xt} + \eta_1,$$

so

$$u_{1t} + u_0 u_{0x} = \frac{1}{2}(1 - z^2)u_{0xxt} - \eta_{1x};$$

then

$$w_{1z} = -u_{1x}$$

gives

$$w_{1zt} = -u_{1xt} = \eta_{1xx} - \frac{1}{2}(1 - z^2)u_{0xxxt} + (u_0 u_{0x})_x.$$

Thus

$$w_{1t} = \left\{(u_0 u_{0x})_x + \eta_{1xx} - \frac{1}{2}u_{0xxxt}\right\}z + \frac{1}{6}z^3 u_{0xxxt},$$

which satisfies $w_{1t} = 0$ (equivalently $w_1 = 0$) on $z = 0$; the boundary condition on $z = 1$ then yields, after differentiating with respect to t,

$$(w_1 + \eta_0 w_{0z})_t = (\eta_{1t} + u_0 \eta_{0x})_t,$$

and so

$$(u_0 u_{0x})_x + \eta_{1xx} - \frac{1}{3}u_{0xxxt} - (\eta_0 u_{0x})_t = \eta_{1tt} + (u_0 \eta_{0x})_t.$$

This equation can be rewritten as

$$\eta_{1tt} - \eta_{1xx} - \left(\frac{1}{2}\eta_0^2 + u_0^2\right)_{xx} - \frac{1}{3}\eta_{oxxxx} = 0, \tag{3.40}$$

where we have used the equations (3.38), as necessary. Finally, we combine equations (3.39) and (3.40) to obtain a single equation for

$$\eta = \eta_0 + \varepsilon\eta_1 + \mathrm{O}(\varepsilon^2)$$

which is correct at O(ε); this is

$$\eta_{tt} - \eta_{xx} - \varepsilon\left\{\frac{1}{2}\eta^2 + \left(\int_{-\infty}^{x} \eta_t \mathrm{d}x\right)^2\right\}_{xx} - \frac{\varepsilon}{3}\eta_{xxxx} = \mathrm{O}(\varepsilon^2), \qquad (3.41)$$

where we have written

$$u_0 = -\int_{-\infty}^{x} \eta_{0t} \mathrm{d}x,$$

with the assumption that $u_0 \to 0$ as $x \to -\infty$. (We could equally have chosen

$$u_0 = \int_{x}^{\infty} \eta_{0t} \mathrm{d}x,$$

so that $u_0 \to 0$ as $x \to +\infty$, if that was appropriate.)

Equation (3.41) (or, more precisely, equation (3.41) with zero on the right-hand side) is one version of the *Boussinesq equation* (Boussinesq, 1871). This equation possesses solutions that describe propagation both to the left and to the right; furthermore, the waves also interact *weakly* and are *weakly dispersive*. Nevertheless, these O(ε) terms are exactly the ones associated with the KdV equation and, indeed, equation (3.41) recovers precisely the KdV equation of our earlier work; see Q3.9. So, although these terms are O(ε) here, they are the *relevant* and *dominant* contributions in the characterisation of our nonlinear dispersive waves.

We have mentioned that the equations which describe unidirectional propagation belong to the class of completely integrable equations. The Boussinesq equation, suitably approximated (Q3.9), gives rise to the KdV equation which is one of these remarkable equations. At first sight the Boussinesq equation, (3.41), appears rather more complicated (for example, second derivative in time) than our previous equations and is therefore, perhaps, not a member of this special class. However, if we set

$$H = \eta - \varepsilon\eta^2$$

and define

$$X = x + \varepsilon \int_{-\infty}^{x} \eta(x, t; \varepsilon)\, \mathrm{d}x$$

then the equation for $H(X, t; \varepsilon)$ becomes

$$H_{tt} - H_{XX} - \frac{3\varepsilon}{2}(H^2)_{XX} - \frac{\varepsilon}{3}H_{XXXX} = \mathrm{O}(\varepsilon^2); \tag{3.42}$$

see Q3.10. Equation (3.42) is the more conventional version of the Boussinesq equation; this equation, with zero on the right-hand side, turns out to be completely integrable for any $\varepsilon > 0$. That is, the equation

$$H_{tt} - H_{XX} + 3(H^2)_{XX} - H_{XXXX} = 0, \tag{3.43}$$

written now in its most usual form, is completely integrable. (The transformation from (3.42) to (3.43) is simply

$$H \to -\frac{2}{\varepsilon}H, \quad X \to \sqrt{\frac{\varepsilon}{3}}X, \quad t \to \sqrt{\frac{\varepsilon}{3}}\,t,$$

the confirmation of which is left to the reader.)

3.2.6 Transformations between these equations

We have already commented that, under suitable conditions, the Boussinesq equation recovers the KdV equation, a demonstration that has been left as an exercise (Q3.9). Here, we examine the nature of transformations between the KdV, cKdV, 2D KdV and ncKdV equations; that transformations should exist is easily established. These four equations are written in either Cartesian or cylindrical coordinates, so

$$r^2 = x^2 + y^2, \quad \tan\theta = y/x,$$

for the variables used in equations (3.1)–(3.8). Thus for a nearly plane wavefront, for which $y/x \to 0$, we may write

$$r - t \sim x\left(1 + \frac{1}{2}\frac{y^2}{x^2}\right) - t,$$

and because we are in the neighbourhood of the wavefront (that is, $\xi = \mathrm{O}(1)$, $t = \mathrm{O}(\varepsilon^{-1})$; see (3.29)) we obtain

$$\begin{aligned} r - t &\sim x - t + \frac{1}{2}y^2/t \\ &= \xi + \frac{1}{2}Y^2/\tau. \end{aligned}$$

Of course, this is only a rough-and-ready argument, but the suggestion is, for example, that we should seek a solution of the 2D KdV equation, (3.30), namely

$$(2\eta_\tau + 3\eta\eta_\xi + \frac{1}{3}\eta_{\xi\xi\xi})_\xi + \eta_{YY} = 0,$$

in the form

$$\eta = H(\zeta, \tau), \quad \zeta = \xi + \frac{1}{2}Y^2/\tau. \tag{3.44}$$

This yields

$$\frac{\partial}{\partial\zeta}(2H_\tau - \frac{Y^2}{\tau^2}H_\zeta + 3HH_\zeta + \frac{1}{3}H_{\zeta\zeta\zeta}) + \frac{1}{\tau}H_\zeta + \frac{Y^2}{\tau^2}H_{\zeta\zeta} = 0$$

which gives, after one integration in ζ (and upon assuming decay conditions for $\zeta \to \infty$, for example), the cKdV equation

$$2H_\tau + \frac{1}{\tau}H + 3HH_\zeta + \frac{1}{3}H_{\zeta\zeta\zeta} = 0. \tag{3.45}$$

This is the form of equation (3.34), when we read R for τ and ξ for ζ, and is even closer to the equation derived in Q3.7. In confirmation of our earlier derivation, Section 3.2.3, we also note that equation (3.45) is invariant under the scaling transformation

$$\zeta \to \frac{\delta}{\varepsilon^{3/2}}\zeta, \quad \tau \to \frac{\delta^3}{\varepsilon^{9/2}}\tau, \quad H \to \frac{\varepsilon^2}{\delta^2}H$$

which describes the choice of variables consistent with (3.10), (3.18) and (3.32) (with τ replacing R).

To take this idea further, we might now expect that a corresponding transformation exists which involves the angular variable (Θ) in the ncKdV equation, and which takes this equation into the KdV equation. Following the same philosophy as above, we write (with $x = r\cos\theta$)

$$\begin{aligned} x - t &\sim r - t - \frac{1}{2}r\theta^2 \quad \text{as} \quad \theta \to 0 \\ &= \frac{\delta^2}{\varepsilon^2}(\xi - \frac{1}{2}R\Theta^2); \end{aligned}$$

see (3.32) and (3.35). This suggests that we seek a solution of

$$(2H_R + \frac{1}{R}H + 3HH_\xi + \frac{1}{3}H_{\xi\xi\xi})_\xi + \frac{1}{R^2}H_{\Theta\Theta} = 0 \tag{3.46}$$

in the form

$$H = \eta(\zeta, R), \quad \zeta = \xi - \frac{1}{2}R\Theta^2 \tag{3.47}$$

which yields

$$2\eta_R + 3\eta\eta_\zeta + \frac{1}{3}\eta_{\zeta\zeta\zeta} = 0 \tag{3.48}$$

after one integration (as described above). This is the KdV equation, with R replacing τ; since $R = \tau + \Delta\xi \sim \tau$ as $\Delta \to 0$, we may interpret the R derivative as a τ derivative, to leading order, and hence recover equation (3.28).

These two results show, for example, that for suitable initial data our four KdV-type equations can be reduced to the solution of just two of them (the KdV and cKdV equations). Of course, in general, the initial profiles will not necessarily conform with the transformations (3.44) or (3.47), and then we must seek solutions of the original 2D KdV and ncKdV equations. We shall briefly describe the near-field problems, and their rôle in providing initial data for our various KdV-type equations, in the next section.

Finally, we remark that the transformations we have presented here are capable of a small extension which then enables the 2D KdV and ncKdV equations to be directly related; this is explored in Q3.12. (This idea turns out to be useful in obtaining certain classes of solution of the ncKdV equation; see Q3.13.)

3.2.7 Matching to the near-field

The equations that we have derived in this chapter, with the exception of the Boussinesq equation, describe waves that are characterised by the balance of nonlinear and dispersive effects in an appropriate far-field. The complete prescription for the solution of these equations requires boundary conditions (such as decay behaviour ahead and behind the wavefront) and initial data provided by the near-field problem; cf. equation (1.94) *et seq*. (The Boussinesq equation is written in near-field variables, and its far-field is represented by a KdV equation, as described in Q3.9.) We now briefly explore the relation between the near-field and far-field problems.

First, for the KdV equation for $\eta_0(\xi, \tau)$,

$$2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = 0,$$

(3.28), we require the initial profile $\eta_0(\xi, 0)$. From the derivation given in Section 3.2.1, and using the near-field variables (x, t) (defined in (3.10) with (3.11)), we showed that

$$\eta_{tt} - \eta_{xx} = 0,$$

to leading order. Thus, for right running waves, we have

$$\eta = f(x - t) = f(\xi),$$

were $f(\cdot)$ is determined by the initial conditions provided (on $t = 0$) for the wave equation. The matching of the near-field and far-field solutions is then stated as: the two functions

$$\begin{array}{lll} f(\xi) & \text{as } t \to \infty & \text{for } \xi = \mathrm{O}(1) \\ \eta_0(\xi, \tau) & \text{as } \tau \to 0 & \text{for } \xi = \mathrm{O}(1) \end{array}$$

are to be identical. Hence, to leading order, we must have the initial condition

$$\eta_0(\xi, 0) = f(\xi);$$

this shows that the solution of the KdV equation provides a uniformly valid solution for $\tau \in [0, T]$, certainly for any $T = \mathrm{O}(1)$, to leading order in ε.

The corresponding development for the 2D KdV equation is essentially the same, after the additional variable Y (see (3.29)) is introduced. Then the near-field yields

$$\eta = f(\xi, Y),$$

to leading order, which provides the matching condition for the 2D KdV, for the function $\eta_0(\xi, Y, \tau)$, as the initial condition

$$\eta_0(\xi, Y, 0) = f(\xi, Y).$$

Finally, we turn to the related problem for the concentric KdV equation, (3.34). In terms of the appropriate near-field variables, defined by the transformation

$$r \to \frac{\delta^2}{\varepsilon^2} r, \quad t =\to \frac{\delta^2}{\varepsilon^2} t, \quad \eta \to \frac{\varepsilon}{\delta} \eta,$$

we obtain, to leading order, the concentric wave equation

$$\eta_{tt} - \left(\eta_{rr} + \frac{1}{r}\eta_r\right) = 0,$$

already mentioned in Section 3.2.3. For large r this yields

$$\eta \sim \frac{1}{\sqrt{r}} f(r-t) \quad \text{as} \quad r \to \infty, \quad r - t = \xi = \mathrm{O}(1);$$

see equation (3.31). On the other hand, the cKdV equation has a solution of the form

$$H_0(\xi, R) \sim \frac{1}{\sqrt{R}} F(\xi) \quad \text{as} \quad R \to 0, \quad \xi = \mathrm{O}(1)$$

(obtained from the dominant balance $2H_{0R} \sim -H_0/R$); the matching condition therefore provides the initial condition (that is, as $R \to 0$) for the cKdV equation:

$$F(\xi) = f(\xi).$$

The function $f(\cdot)$ is available from the solution of the concentric wave equation which is valid in the near-field.

This leaves the nearly concentric KdV equation for consideration. Unhappily, this equation is not so easily analysed; either the dependence on Θ is absent from the leading-order near-field problem (in which case the calculation reduces, essentially, to that for the cKdV equation) or the terms involving Θ in the solution of the ncKdV equation are exponentially small as $R \to 0$. The structure of the near-field in this latter case is then quite involved. This description is beyond the scope of our text, but the ideas are touched on in Johnson (1980), where the problem of matching to *similarity solutions* of the various KdV equations is also discussed in some detail.

3.3 Completely integrable equations: some results from soliton theory

Wave after wave, each mightier than the last.

The Coming of Arthur

In the introduction to this chapter we mentioned the existence of special equations together with solitary waves, solitons and complete integrability. We have now met a number of these special equations, and our purpose here is to write a little about these equations, their properties and methods of solution. It is not the rôle of this text to provide a comprehensive discussion of these equations, nor to present a careful development of *inverse scattering transform theory* (to use the more accurate title for these studies). Certainly these ideas, usually grouped together under the simpler title of *soliton theory*, are relevant to our further exploration of water-wave theory, but only to the extent of having available solutions and, perhaps, some methods of solution. There are many good texts now available which provide the basis for further study; some of these offer excellent introductions to the theory, whereas others describe advanced and deep ideas. An extensive list of Further Reading is provided at the end of this chapter.

The last 25 years or so have seen the rise of this exciting and powerful approach to our understanding of wave propagation. In particular, the existence of families of solutions of nonlinear wave equations that describe nonlinear interactions without the expected destruction (and, perhaps, resulting chaos), was a considerable surprise. Apart from the diverse observations in nature of many of these phenomena, from our wave interactions on water to the red spot on Jupiter, this work has also led, for example, to the important and very practical application to signal propagation along fibre-optic cables of great length. Furthermore, it turns out that many fundamental concepts in various branches of physics, applied mathematics and pure mathematics also have an important place in this work. Thus both Hamiltonian mechanics and the geometry of surfaces – to mention but two – play a fundamental rôle in soliton theory. In addition, quite new mathematical techniques have been developed and, even more, some longstanding mathematical problems have been solved using soliton theory (for example, the solution of Painlevé equations).

Briefly, the story begins with the KdV equation and its numerical solution, first in a related problem by Fermi, Pasta and Ulam in 1955, and then by Zabusky and Kruskal in 1965. (It was Zabusky and Kruskal who coined the word 'soliton' to describe these new nonlinear waves, because they possess the properties of both solitary waves and particles such as the electron and the photon.) The results were so surprising – primarily the nonlinear interaction of waves that retain their identities – that a group at Princeton University (Gardner, Greene, Kruskal and Miura) set out to understand the processes involved. This led them to develop (in 1967) a method of solution which treats a function that satisfies the KdV equation – the required solution – as the (time-dependent) potential of a one-dimensional linear scattering problem.

The linear scattering problem, and the associated inverse scattering problem, coupled with the time evolution deduced by invoking the KdV equation, produce a solution method which ultimately transforms the *nonlinear* partial differential equation into a *linear* integral equation. Although this integral equation cannot be solved in closed form for arbitrary initial data of the KdV equation, it does possess simple exact solutions which correspond to the soliton solutions (and which enable these solutions of the KdV equation to be written down in a fairly simple and compact form).

From this small beginning – one equation and apparently a tailor-made method of solution – has sprung a whole range of methods which are applicable to many different equations; it has also led to many alternative approaches to the construction of some of the special solutions. We shall present the method of solution for the KdV equation (but without its rather lengthy derivation), and likewise for a few other equations that are important and relevant to water-wave theory. We shall also describe one of the simple direct methods of solution (Hirota's bilinear transformation) and the rôle of conservation laws both in the theory of these equations and, of course, in their application to water waves. In the space available, and in the context of water waves, we cannot explore the many other equally important aspects of soliton theory, such as the Bäcklund transform, Hamiltonian systems and prolongation structure.

3.3.1 Solution of the Korteweg–de Vries equation

The solution, $u(x, t)$, of the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{3.49}$$

(which is obtained by a simple scaling transforming from equation (3.28); see Q3.1) is related to a function $K(x, z; t)$ by the transformation

$$u(x, t) = -2\frac{\mathrm{d}}{\mathrm{d}x}K(x, x; t) \tag{3.50}$$

where K satisfies the integral equation

$$K(x, z; t) + F(x, z, t) + \int_x^\infty K(x, y; t)F(y, z, t)\mathrm{d}y = 0, \tag{3.51}$$

usually called the *Marchenko* (or sometimes *Gel'fand–Levitan*) *equation.* In this equation, $F(x, z, t)$ satisfies both the equations

$$F_{xx} - F_{zz} = 0; \quad F_t + 4(F_{xxx} + F_{zzz}) = 0, \tag{3.52}$$

but since the final evaluation which leads to u is on $z = x$, the relevant solution for F is a function only of $(x + z)$; thus it is convenient to write $F = F(x + z, t)$ so that we now have

$$F_t + 8F_{\xi\xi\xi} = 0 \quad (\xi = x + z) \tag{3.53}$$

and

$$K(x, z; t) + F(x + z, t) + \int_x^\infty K(x, y; t)F(y + z, t)\mathrm{d}y = 0. \tag{3.54}$$

The formulation of this method of solution, via the scattering and inverse scattering problems, enables the initial-value (Cauchy) problem for the KdV equation to be solved, at least provided that certain existence conditions are satisfied, for example

$$\int_{-\infty}^\infty |u(x, t)|\mathrm{d}x < \infty, \qquad \int_{-\infty}^\infty (1 + |x|)|u(x, t)|\mathrm{d}x \leq \infty, \quad \forall t.$$

In particular the initial profile, $u(x, 0)$, must satisfy these conditions. (The first of these says that u must be absolutely integrable and the second – the *Faddeev condition* – says that u must actually decay quite rapidly at infinity.) The solitary wave and soliton solutions certainly do satisfy these conditions (because they decay exponentially as $|x| \to \infty$ for all t), although periodic solutions clearly do not.

In order to gain some familiarity with these equations, and with the method of solution, we shall show how the solitary-wave solution can be recovered. We then extend the technique to obtain the two-soliton solution, and thereafter the generalisation to N–solitons is easily explained (although the details of the calculation are rather lengthy and are not reproduced here).

Example 1: solitary-wave solution

We start from the simplest exponential solution of equation (3.53), which we choose to write as

$$F = \mathrm{e}^{-k\xi+\omega t+\alpha}, \quad \xi = x + z, \tag{3.55}$$

where k (> 0) is a constant, α is an arbitrary constant (equivalent to writing $F = Ae^{-k\xi+\omega t}$) and $\omega(k)$ is to be determined; this solution ensures that $F \to 0$ as $x \to +\infty$. Substitution of (3.55) into (3.53) yields directly the dispersion relation for ω in terms of the wave number (k)

$$\omega = 8k^3,$$

and then the integral equation, (3.51), becomes

$$K(x, z; t) + \exp\{-k(x+z) + 8k^3 t + \alpha\} + \int_x^\infty K(x, y; t) \exp\{-k(y+z) + 8k^3 t + \alpha\} \, \mathrm{d}y = 0.$$

It is immediately clear that the solution takes the (separable) form

$$K(x, z; t) = \mathrm{e}^{-kz} L(x; t) \tag{3.56}$$

so that

$$L + \exp(-kx + 8k^3 t + \alpha) + L \exp(8k^3 t + \alpha) \int_x^\infty \mathrm{e}^{-2ky} \mathrm{d}y = 0$$

and hence

$$L\left\{1 + \frac{1}{2k}\exp(-2kx + 8k^3 t + \alpha)\right\} + \exp(-kx + 8k^3 t + \alpha) = 0,$$

which gives $L(x; t)$. Thus

$$K(x, x; t) = \mathrm{e}^{-kx} L(x; t) = \frac{-1}{\dfrac{1}{2k} + \exp(2kx - 8k^3 t - \alpha)}$$

and then

$$u(x, t) = -2\frac{\mathrm{d}}{\mathrm{d}x} K(x, x; t) = \frac{-4k \exp(2kx - 8k^3 t - \alpha)}{\left(\dfrac{1}{2k} + \exp(2kx - 8k^3 t - a)\right)^2}$$

$$= \frac{-8k^2}{(\sqrt{2k}\mathrm{e}^{\theta} + \mathrm{e}^{-\theta}/\sqrt{2k})^2}, \quad \theta = kx - 4k^3 t - \alpha/2,$$

or

$$u(x, t) = -2k^2 \operatorname{sech}^2\{k(x - x_0) - 4k^3 t\} \tag{3.57}$$

where we have written

$$\sqrt{2k}\mathrm{e}^{-\alpha/2} = \mathrm{e}^{-kx_0}$$

so that x_0 now describes an arbitrary shift in x. Solution (3.57) is the solitary-wave solution, of amplitude $-2k^2$, of the KdV equation; cf. equation (2.174) and Q1.55(a).

Example 2: two-soliton solution

The extension to two (and eventually N) solitons is surprisingly simple, even though the solution that we obtain describes the *nonlinear* interaction of two (or more) waves. Such a description cannot apply to the solitary wave, since it propagates at constant speed with unchanging form: it is merely an example of a travelling-wave solution. The method hinges on the property that both the equations for F and K are linear, and therefore we may choose to develop a more general solution by taking a linear combination of functions. Thus we write, in place of (3.55),

$$F = \exp(\theta_1) + \exp(\theta_2), \quad \theta_i = -k_i(x+z) + 8k_i^3 t + \alpha_i,$$

where the dispersion relation ($\omega_i = 8k_i^3$) has been incorporated; we are interested in solutions for which $k_1 \neq k_2$, α_1 and α_2 are arbitrary constants. The integral equation for $K(x, z; t)$ now becomes

$$\begin{aligned} K(x,z;t) + \exp\{-k_1(x+z) + 8k_1^3 t + \alpha_1\} + \exp\{-k_2(x+z) + 8k_2^3 t + \alpha_2\} \\ + \int_x^\infty K(x,y;t)\{\exp[-k_1(y+z) + 8k_1^3 t + \alpha_1] \\ + \exp[-k_2(y+z) + 8k_2^3 t + \alpha_2]\}\mathrm{d}y = 0 \end{aligned}$$

so that the solution must take the form

$$K(x,z;t) = \exp(-k_1 z)L_1(x;t) + \exp(-k_2 z)L_2(x;t);$$

our problem is an example of a *separable* integral equation, leading to this simple method of solution, which extends what we did in Example 1.

Since $k_1 \neq k_2$, the integral equation separates into two (algebraic) equations for L_1 and L_2; these are

$$L_1 + \exp(-k_1 x + 8k_1^3 t + \alpha_1)$$
$$+ \exp(8k_1^3 + \alpha_1)\left\{ L_1 \int_x^\infty \exp(-2k_1 y)\mathrm{d}y + L_2 \int_x^\infty \exp[-(k_1 + k_2)y]\mathrm{d}y \right\} = 0;$$
$$L_2 + \exp(-k_2 x + 8k_2^3 t + \alpha_2)$$
$$+ \exp(8k_2^3 + \alpha_2)\left\{ L_1 \int_x^\infty \exp[-(k_1 + k_2)y]\mathrm{d}y + L_2 \int_x^\infty \exp(-2k_2 y)\mathrm{d}y \right\} = 0.$$

The integrations, like that in Example 1, are very easily accomplished, yielding the pair of equations

$$L_1 + \exp(-k_1 x + \phi_1)$$
$$+ \frac{L_1}{2k_1}\exp(-2k_1 x + \phi_1) + \frac{L_2}{k_1 + k_2}\exp\{-(k_1 + k_2)x + \phi_1\} = 0;$$
$$L_2 + \exp(-k_2 x + \phi_2)$$
$$+ \frac{L_1}{k_1 + k_2}\exp\{-(k_1 + k_2)x + \phi_2\} + \frac{L_2}{2k_2}\exp(-2k_2 + \phi_2) = 0$$

where

$$\phi_i = 8k_i^3 t + \alpha_i.$$

These equations are solved for L_1 and L_2, and then we form

$$K(x, x; t) = \exp(-k_1 x)L_1(x; t) + \exp(-k_2 x)L_2(x; t)$$

from which we then calculate

$$u(x, t) = -2\frac{\mathrm{d}}{\mathrm{d}x}K(x, x; t).$$

The manipulative details, which are altogether straightforward, are left as an exercise; the resulting solution can be expressed in a number of ways, one of which is

$$u(x, t) = -8\frac{k_1^2 E_1 + k_2^2 E_2 + 2(k_1 - k_2)^2 E_1 E_2 + A(k_2^2 E_1 + k_1^2 E_2)E_1 E_2}{(1 + E_1 + E_2 + AE_1 E_2)^2} \tag{3.58}$$

where

$$E_i = \exp\{2k_i(x - x_{0i}) - 8k_i^3 t\}, \quad i = 1, 2,$$

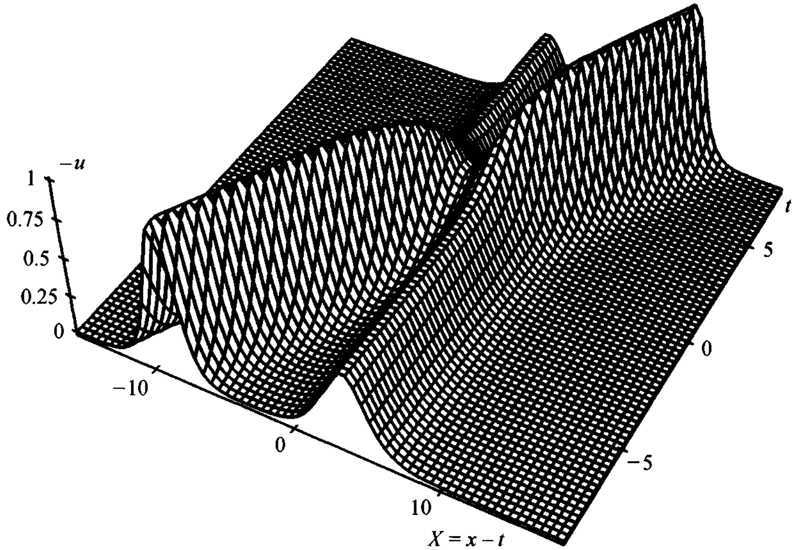


Figure 3.3. A perspective view of a two-soliton solution of the Korteweg–de Vries equation (for $k_1 = 1$ and $k_2 = \sqrt{2}$), drawn in the frame $X = x - t$. Note that $-u$ is plotted here.

and

$$A = (k_1 - k_2)^2/(k_1 + k_2)^2.$$

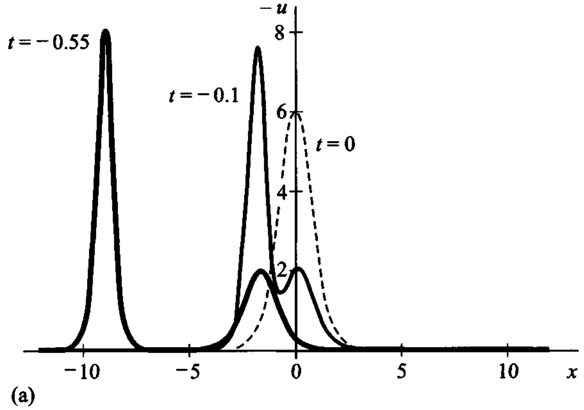
The two arbitrary phase shifts are x_{0i}, $i = 1, 2$. Solution (3.58) is the most general two-soliton solution of the KdV equation, an example of which is shown in Figure 3.3. A special case of this solution, in which $k_1 = 1$, $k_2 = 2$, $x_{01} = x_{02} = 0$, is

$$u(x, t) = -12\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{\{3\cosh(x - 28t) + \cosh(3x - 36t)\}^2} \tag{3.59}$$

(after some further manipulation); this particular solution is the first that was obtained (Gardner *et al.*, 1967) and corresponds to the initial profile

$$u(x, 0) = -6\,\mathrm{sech}^2 x.$$

The observed water wave (of positive amplitude) is recovered by transforming $u \to -u$ (cf. equations (3.28), (3.49)), and this solution (3.59) is depicted in Figure 3.4.



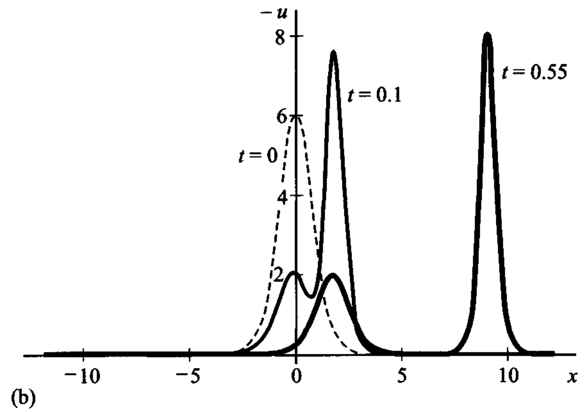


Figure 3.4. The two-soliton solution of the Korteweg–de Vries equation with $u(x,0) = -6\,\mathrm{sech}^2 x$, shown at times (a) $t = -0.55$, -0.1, 0 and (b) $t = 0$, 0.1, 0.55. Note that $-u$ is plotted against x.

The generalisation to N solitons is obtained in the obvious way by writing

$$F = \sum_{i=1}^{N} \exp(\theta_i), \quad \theta_i = -k_i(x+z) + 8k_i^3 t + \alpha_i; \tag{3.60}$$

the 3-soliton solution is explored in Q3.17 and Q3.25. For N solitons, the initial profile, which is simply a sech^2 function (which arises when the k_i are suitable integers), takes the form

$$u(x,0) = -N(N+1)\,\mathrm{sech}^2 x.$$

Both specific and general forms of the N-soliton solution are discussed in the literature, and many interesting and useful properties have been described. A particularly significant property of these nonlinear wave interactions is evident in Figure 3.4, which represents solution (3.59). The two waves, which are locally almost solitary waves for $t < 0$, combine to form a single wave (the $-6\,\mathrm{sech}^2 x$ profile) at the instant $t = 0$. Thereafter, the taller wave, which had caught up the shorter, moves ahead and away from the shorter as t increases. The two waves that move into $x > 0$ are (asymptotically) identical to the pair that moved in $x < 0$. It might seem, at first sight, that this process is purely linear: the faster (taller) wave catches up and then overtakes the slower (shorter) one, the full solution at any time being the sum of the two. However, a more careful examination of the sequence shown in Figure 3.4 makes it clear that the taller wave has moved *forward*, and the shorter one *backward*, relative to the positions that they would have reached if the two waves had moved at constant speeds throughout. The net result of the nonlinear interaction, therefore, is to produce a *phase shift* of the waves; this property is generally regarded as the hallmark of this type of nonlinear interaction; that is, of soliton solutions. The relevant calculation for solution (3.59) is left as an exercise (Q3.14), and the phase shifts are represented in Figure 3.5.

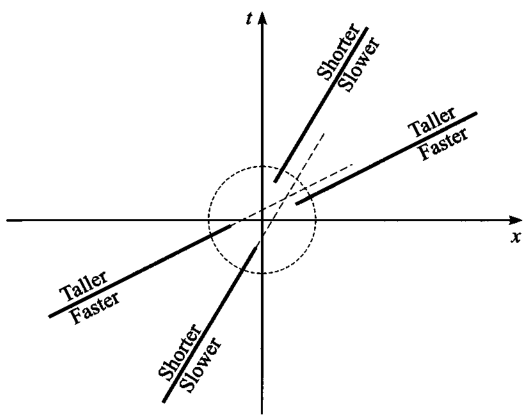


Figure 3.5. A representation of the paths of the two wave crests in a two-soliton solution of the KdV equation. The circle indicates the region inside which the dominant interaction occurs, and the dotted lines show the paths that would have been taken by the waves if no interaction had occurred.

3.3.2 Soliton theory for other equations

The development for the KdV equation is now extended to other equations that are relevant to water waves. We shall present the details for the methods of solution for the 2D KdV and cKdV equations. (Another important example, the Nonlinear Schrödinger (NLS) equation, will be described in the next chapter.)

The solution of the two-dimensional Korteweg–de Vries equation

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0 \tag{3.61}$$

follows that for the KdV equation very closely (Dryuma, 1974). The transformation between u and $K(x, x; t, y)$ is the same, namely

$$u(x, t, y) = -\frac{\mathrm{d}K}{\mathrm{d}x} K(x, x; t, y), \tag{3.62}$$

where K satisfies

$$K(x, z; t, y) + F(x, z, t, y) + \int_x^\infty K(x, Y; t, y) F(Y, z, t, y) \mathrm{d}Y = 0$$

(and we have written the integration variable here as Y, to avoid the obvious confusion). The function F satisfies the pair of equations

$$F_{xx} - F_{zz} - F_y = 0, \quad F_t + 4(F_{xxx} + F_{zzz}) = 0; \tag{3.63}$$

cf. equation (3.52). Then, for example, the solitary-wave solution is obtained by choosing

$$F = \exp\{-(kx + lz) + (k^2 - l^2)y + 4(k^3 + l^3)t + \alpha\};$$

see Q3.18 and Q3.19.

The concentric Korteweg–de Vries equation,

$$u_t + \frac{u}{2t} - 6uu_x + u_{xxx} = 0, \tag{3.64}$$

is solved by a similar method, although the details are not so straightforward. As before, $K(x, z; t)$ is a solution of the Marchenko equation

$$K(x, z; t) + F(x, z, t) + \int_x^\infty K(x, y; t) F(y, z, t) \mathrm{d}y = 0$$

where F is now a solution of the pair of equations

$$\left.\begin{aligned} F_{xx} - F_{zz} &= (x - z)F; \\ 3tF_t - F + F_{xxx} + F_{zzz} &= xF_x + zF_z. \end{aligned}\right\} \tag{3.65}$$

The solution of the cKdV equation is then obtained in the form

$$u(x, t) = -2(12t)^{-2/3}\frac{\mathrm{d}K}{\mathrm{d}\xi}, K = K(\xi, \xi; t), \tag{3.66}$$

where $\xi = x/(12t)^{1/3}$; it is the requirement to use the similarity variable that particularly complicates the procedure in this case. Another mild irritant is that equations (3.65) do not admit exponential solutions, the simplest solutions being based on the *Airy* functions. Further exploration of this method is provided in Q3.22.

3.3.3 Hirota's bilinear method

The solitary-wave solution of the KdV equation,

$$u(x, t) = -2k^2 \operatorname{sech}^2\{k(x - x_0) - 4k^3 t\},$$

as given in (3.57), can be written as

$$\begin{aligned} u(x, t) &= -2k\frac{\partial}{\partial x}\tanh\{k(x - x_0) - 4k^3 t\} \\ &= -2\frac{\partial^2}{\partial x^2}\log(\mathrm{e}^{\theta} + \mathrm{e}^{-\theta}), \quad \theta = k(x - x_0) - 4k^3 t, \\ &= -2\frac{\partial^2}{\partial x^2}\left\{-k(x - x_0) + 4k^3 t + \log(1 + \mathrm{e}^{2\theta})\right\} \\ &= -2\frac{\partial^2}{\partial x^2}\log f, \quad f = 1 + \exp\{2k(x - x_0) - 8k^3 t\}. \end{aligned} \tag{3.67}$$

Indeed, the N-soliton solution can be written in precisely the same form:

$$u = -2\frac{\partial^2}{\partial x^2}\log f, \tag{3.68}$$

where $f(x, t)$ turns out to be the determinant of an $N \times N$ matrix of coefficients that arise in the solution of the Marchenko equation, when F is a sum of N exponential terms (see (3.60)). Hirota's idea, first published in 1971, was to explore the possibility of solving the KdV equation (at least for the soliton solutions) by constructing $f(x, t)$ directly. At first sight it might appear that the problem for f is more difficult than that for u; however, Hirota showed that it eventually leads to a very neat method of solution. And the idea is not restricted to the KdV equation: all the soliton-type equations can be tackled in a similar way (although the transformation (3.68) is not always the relevant one). In the context of solitary wave and soliton solutions, which we have already suggested are of some interest in water-wave theory, this method often provides a convenient method for their construction. We regard this technique as a powerful addition to the more general method of solution that is based on the Marchenko integral equation. There are yet other approaches available, but we believe that Hirota's method is sufficiently simple and useful to warrant a place in this text.

We describe the elements of the method by developing the details for the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0,$$

with

$$u = -2\frac{\partial^2}{\partial x^2}\log f,$$

where $f_x, f_t, f_{xx}, f_{tt}, \ldots \to 0$ as $x \to +\infty$ or $x \to -\infty$ (see (3.67)). The process is made a little simpler if, first, we write $u = \phi_x$ and then integrate once in x to yield

$$\phi_t - 3\phi_x^2 + \phi_{xxx} = 0, \tag{3.69}$$

where the decay conditions on f imply corresponding conditions on ϕ $(= -2f_x/f)$ and these have been used to give equation (3.69). (This version of the KdV equation is often called the *potential KdV equation*.) Now we introduce f so that

$$\phi_t = -2(ff_{xt} - f_x f_t)/f^2, \quad \phi_x = -2(ff_{xx} - f_x^2)/f^2$$

and

$$\phi_{xxx} = -2(ff_{xxxx} - 4f_x f_{xxx} - 3f_{xx}^2)/f^2 - 24 f_{xx} f_x^2/f^3 + 12 f_x^4/f^4;$$

it is clear that when these are substituted into equation (3.69) we obtain (after multiplication by f^2)

$$ff_{xt} - f_x f_t + ff_{xxxx} - 4f_x f_{xxx} + 3f_{xx}^2 = 0. \tag{3.70}$$

This equation certainly appears more difficult to solve than the original KdV equation, although we do note one significant improvement: every term is now quadratic in f. So how do we tackle the solution of equation (3.70)?

The crucial step was provided by Hirota when he introduced the *bilinear operator*, $\mathrm{D}_t^m\mathrm{D}_x^n(a\cdot b)$, defined as

$$\mathrm{D}_t^m\mathrm{D}_x^n(a\cdot b)=\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^m\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^n a(x,t)b(x',t')\bigg|_{\substack{x'=x\\ t'=t}} \tag{3.71}$$

for non-negative integers m and n. As an example, let us consider the case of $m=n=1$ for which

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)a(x,t)b(x',t')=\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)(a_xb-ab_{x'})$$
$$=a_{xt}b-a_tb_{x'}-a_xb_{t'}+ab_{x't'}.$$

This is evaluated on $x'=x$, $t'=t$, to give

$$\mathrm{D}_t\mathrm{D}_x(a\cdot b)=a_{xt}b+ab_{xt}-a_tb_x-a_xb_t$$

and if we choose the special case of $a=b$, for all x, t, then

$$\mathrm{D}_t\mathrm{D}_x(a\cdot b)=2(aa_{xt}-a_xa_t). \tag{3.72}$$

Another useful example is to find $\mathrm{D}_x^4(a\cdot b)$, so that now $m=0$ and $n=4$; this yields

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^4 a(x,t)b(x',t')$$
$$=a_{xxxx}b-4a_{xxx}b_{x'}+6a_{xx}b_{x'x'}-4a_xb_{x'x'x'}+ab_{x'x'x'x'}.$$

We evaluate on $x'=x$, $t'=t$, and again make the special choice $a=b$, to give

$$\mathrm{D}_x^4(a\cdot a)=2(aa_{xxxx}-4a_xa_{xxx}+3a_{xx}^2). \tag{3.73}$$

It is immediately clear, if we compare equations (3.72) and (3.73) with (3.70), that our equation for f can be expressed as

$$(\mathrm{D}_x\mathrm{D}_t+\mathrm{D}_x^4)(f\cdot f)=0, \tag{3.74}$$

the *bilinear form* of the KdV equation.

Before we describe how the bilinear equation, (3.74), is solved, we offer two comments. First, examination of the differential operator

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial t'}\right)^m\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial x'}\right)^n a(x,t)b(x',t')\bigg|_{\substack{x'=x\\ t'=t}},$$

which is left as an exercise, shows that this is precisely the familiar derivative of a product:

$$\frac{\partial^{m+n}}{\partial t^m \partial x^n}(ab).$$

In other words, Hirota's novel differential operator simply uses the difference – rather than the sum – of the derivatives in t and t', and in x and x'. Second, and often used as a guide in the quick construction of a bilinear form, is the *interpretation* of D_x and D_t as the conventional derivatives $\partial/\partial x$ and $\partial/\partial t$, respectively. If we use this interpretation of $\mathrm{D}_t + \mathrm{D}_x^3$ then this operator becomes the linearised operator in the KdV equation, obtained by letting $u \to 0$. Thus the underlying structure of the bilinear form is that of the corresponding *linear* differential equation, at least here for KdV equation; we shall meet later some other equations which possess this same property.

In order to solve the bilinear equation we require some properties of the bilinear operator, and in particular the two results

$$\mathrm{D}_t^m \mathrm{D}_x^n(a \cdot 1) = \mathrm{D}_t^m \mathrm{D}_x^n(1 \cdot a) = \frac{\partial^{m+n} a}{\partial t^m \partial x^n}, \text{ for } m+n \text{ even}, \tag{3.75}$$

and

$$\mathrm{D}_t^m \mathrm{D}_x^n\{\exp(\theta_1) \cdot \exp(\theta_2)\} = (\omega_2 - \omega_1)^m (k_1 - k_2)^n \exp(\theta_1 + \theta_2), \tag{3.76}$$

where $\theta_i = k_i x - \omega_i t + \alpha_i$; these and other properties are explored in Q3.24. Now for B any bilinear operator and

$$f = 1 + \mathrm{e}^\theta, \quad \theta = 2k(x - x_0) - 8k^3 t, \tag{3.77}$$

(see (3.74)), then

$$\mathrm{B}(f \cdot f) = \mathrm{B}(1 \cdot 1) + \mathrm{B}(1 \cdot \mathrm{e}^\theta) + \mathrm{B}(\mathrm{e}^\theta \cdot 1) + \mathrm{B}(\mathrm{e}^\theta \cdot \mathrm{e}^\theta).$$

Consequently, with (3.75) and (3.76), the bilinear form of our KdV equation gives

$$(\mathrm{D}_x \mathrm{D}_t + \mathrm{D}_x^4)(f \cdot f) = 2(2k)(-8k^3) + (2k)^4\}\mathrm{e}^\theta = 0$$

which confirms that (3.77) is an exact solution (which, of course, generates the solitary-wave solution). The extension of this approach to the construction of the N-soliton solution is now addressed.

The neatest way to set up this problem is to introduce an arbitrary parameter ε, with the assumption that f can be expanded in integral powers of ε. The aim is to show that the series that we obtain terminates after a finite number of terms; in this situation we may then arbitrarily assign ε: for example, $\varepsilon = 1$. Thus we write

$$f = 1 + \sum_{n=1}^{\infty} \varepsilon^n f_n(x, t)$$

and then the general bilinear form becomes

$$\begin{aligned} \mathrm{B}(f \cdot f) &= \mathrm{B}(1 \cdot 1) + \varepsilon \mathrm{B}(f_1 \cdot 1 + 1 \cdot f_1) + \varepsilon^2 \mathrm{B}(f_2 \cdot 1 + f_1 \cdot f_1 + 1 \cdot f_2) \\ &\quad + \cdots + \varepsilon^r \mathrm{B}\left(\sum_{m=0}^{\infty} f_{r-m} \cdot f_m \right) + \cdots = 0 \end{aligned}$$

where $f_0 = 1$. We know that $\mathrm{B}(1 \cdot 1) = 0$, and we ask that each coefficient of ε^r $(r = 1, 2, \ldots)$ be zero, so

$$\mathrm{B}(f_1 \cdot 1 + 1 \cdot f_1) = 0, \tag{3.78}$$

$$\mathrm{B}(f_2 \cdot 1 + f_1 \cdot f_1 + 1 \cdot f_2) = 0, \text{ etc.} \tag{3.79}$$

Because B is a linear differential operator, we have

$$\mathrm{B}(a \cdot b + c \cdot d) = \mathrm{B}(a \cdot b) + \mathrm{B}(c \cdot d)$$

and then equation (3.78), for our KdV equation, becomes

$$2 \frac{\partial}{\partial x} \left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3} \right) f_1 = 0$$

or, after one integration,

$$\mathrm{D} f_1 = 0, \quad \mathrm{D} \equiv \frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}, \tag{3.80}$$

where we have again used $f_{1t}, f_{1x}, \ldots \to 0$ as $x \to +\infty$ or $x \to -\infty$. The next two equations in this sequence are written as

$$2 \frac{\partial}{\partial x}(\mathrm{D} f_2) = -\mathrm{B}(f_1 \cdot f_1); \quad 2 \frac{\partial}{\partial x}(\mathrm{D} f_3) = -\mathrm{B}(f_1 \cdot f_2 + f_2 \cdot f_1), \tag{3.81}$$

where $\mathrm{B} = \mathrm{D}_x \mathrm{D}_t + \mathrm{D}_x^4$. It is immediately clear that a solution of this set is

$$f_1 = \mathrm{e}^{\theta}, \quad \theta = kx - k^3 t + \alpha; \quad f_n = 0, \ \forall n \geq 2,$$

where we have written k here for $2k$ and α for $-2kx_0$; see (3.67). Thus we have a solution which terminates after $n = 1$, and so we may set $\varepsilon = 1$; this recovers (3.67) for the solitary wave. (We note that the presence of ε is equivalent to a phase shift, but the term α already provides an arbitrary phase shift in the solution.)

The equation for f_1, (3.78), is linear and so we may construct more general solutions by taking a linear combination of exponential terms; let us choose

$$f_1 = \exp(\theta_1) + \exp(\theta_2), \quad \theta_i = k_i x - k_i^3 t + \alpha_i.$$

The equation for f_2, from (3.81), now becomes

$$2\frac{\partial}{\partial x}(\mathrm{D}f_2) = -\,\mathrm{B}\{\exp(\theta_1)\cdot\exp(\theta_1)\} - \mathrm{B}\{\exp(\theta_1)\cdot\exp(\theta_2)\}$$
$$- \mathrm{B}\{\exp(\theta_2)\cdot\exp(\theta_1)\} - \mathrm{B}\{\exp(\theta_2)\cdot\exp(\theta_2)\}$$

and the terms involving either only θ_1 or only θ_2 are zero. Otherwise, we see that

$$2\frac{\partial}{\partial x}(\mathrm{D}f_2) = -2\{(k_1 - k_2)(k_2^3 - k_1^3) + (k_1 - k_2)^4\}\exp(\theta_1 + \theta_2)$$

and this equation clearly has a particular integral of the form

$$f_2 = A\exp(\theta_1 + \theta_2). \tag{3.82}$$

This yields the equation

$$A\{-(k_1 + k_2)(k_1^3 + k_2^3) + (k_1 + k_2)^4\}$$
$$= (k_1 - k_2)^2\{k_1^2 + k_1 k_2 + k_2^2 - (k_1 - k_2)^2\}$$

for the constant A, which simplifies to give

$$A = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2. \tag{3.83}$$

We use only the particular integral for f_2; any additional contributions (as part of a complementary function) could be moved from f_2 to f_1 – at least when $\varepsilon = 1$ – and we have already made a choice for f_1.

The equation for f_3 then becomes

$$2\frac{\partial}{\partial x}(\mathrm{D}f_3) = -A\mathrm{B}\{\exp(\theta_1)\cdot\exp(\theta_1 + \theta_2)\} - A\mathrm{B}\{\exp(\theta_1 + \theta_2)\cdot\exp(\theta_1)\}$$
$$- A\mathrm{B}\{\exp(\theta_2)\cdot\exp(\theta_1 + \theta_2)\} - A\mathrm{B}\{\exp(\theta_1 + \theta_2)\cdot\exp(\theta_2)\}$$
$$= -2A\{-k_2(k_2^3) + k_2^4\}\exp(2\theta_1 + \theta_2)$$
$$- 2A\{-k_1(k_1^3) + k_1^4\}\exp(2\theta_2 + \theta_1)$$
$$= 0,$$

and so a solution for f_3 is $f_3 = 0$. The equation for f_4 is

$$2\frac{\partial}{\partial x}(\mathrm{D}f_4) = -\mathrm{B}(f_1 \cdot f_3 + f_2 \cdot f_2 + f_3 \cdot f_1)$$
$$= 0$$

since $f_3 = 0$ and f_2 is a single exponential (from (3.82)); it is clear, therefore, that we may choose $f_n = 0$, $\forall n \geq 3$. Thus we have another exact solution which, for $\varepsilon = 1$, is

$$f = 1 + \exp(\theta_1) + \exp(\theta_2) + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \exp(\theta_1 + \theta_2); \tag{3.84}$$

this generates the most general two-soliton solution of the KdV equation, previously written down in (3.58).

This method of solution can be extended to produce an exact solution which represents the N-soliton solution of the KdV equation. This is accomplished simply by writing

$$f_1 = \sum_{i=1}^{N} \exp(\theta_i)$$

and then it can be shown that the series for f terminates after the term f_N. The construction of this solution is routine but rather tedious and therefore will not be pursued here, although the case $N = 3$ is set as an exercise in Q3.25, and a 3-soliton solution is depicted in Figure 3.6. The form that f takes, for example as given in (3.84) for $N = 2$, represents a *nonlinear superposition principle* for the soliton solutions from which their explicit construction follows directly.

Finally, the other nonlinear equations that we have introduced in Section 3.2 can also be written in bilinear form. (The details are left for the reader to explore in the exercises.) Thus we find that the 2D KdV equation

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0$$

has the bilinear form

$$(\mathrm{D}_x\mathrm{D}_t + \mathrm{D}_x^4 + 3\mathrm{D}_y^2)(f \cdot f) = 0,$$

and the cKdV equation

$$u_t + \frac{u}{2t} - 6uu_x + u_{xxx} = 0$$

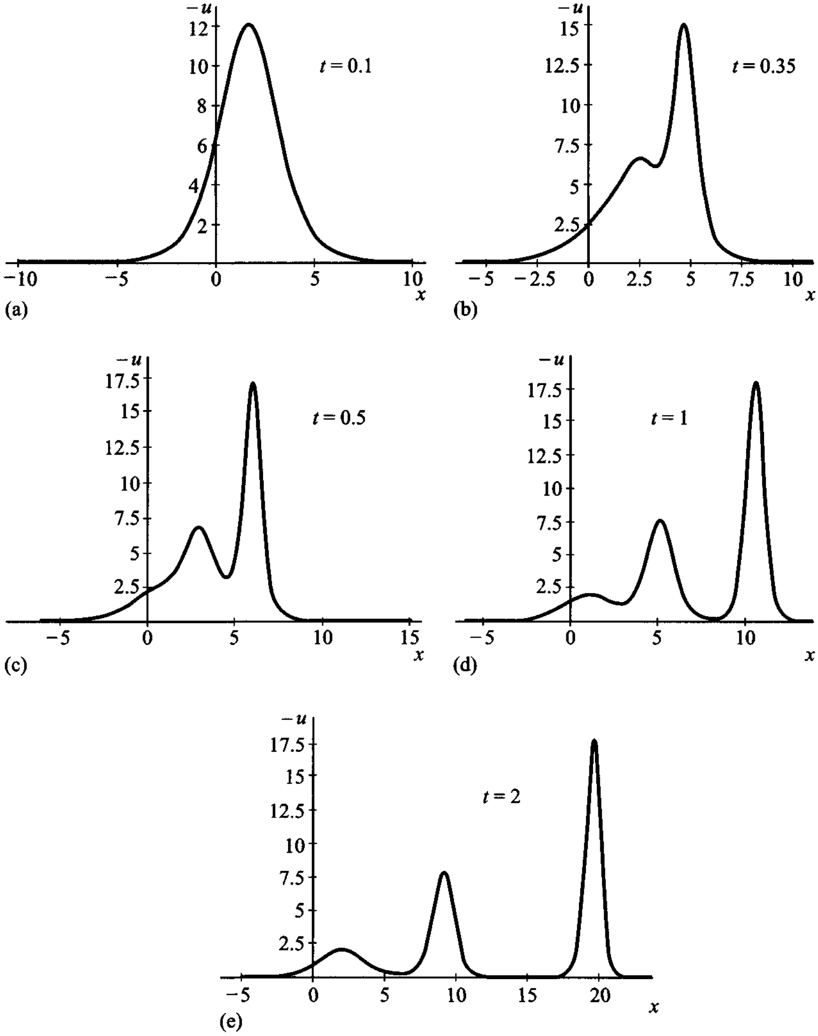


Figure 3.6. A 3-soliton solution of the Korteweg–de Vries equation, for $k_1 = 1$, $k_2 = 2$, and $k_3 = 3$, at times $t = 0.1$(a), 0.35(b), 0.5(c), 1(d) and 2(e). Note that $-u$ is plotted here.

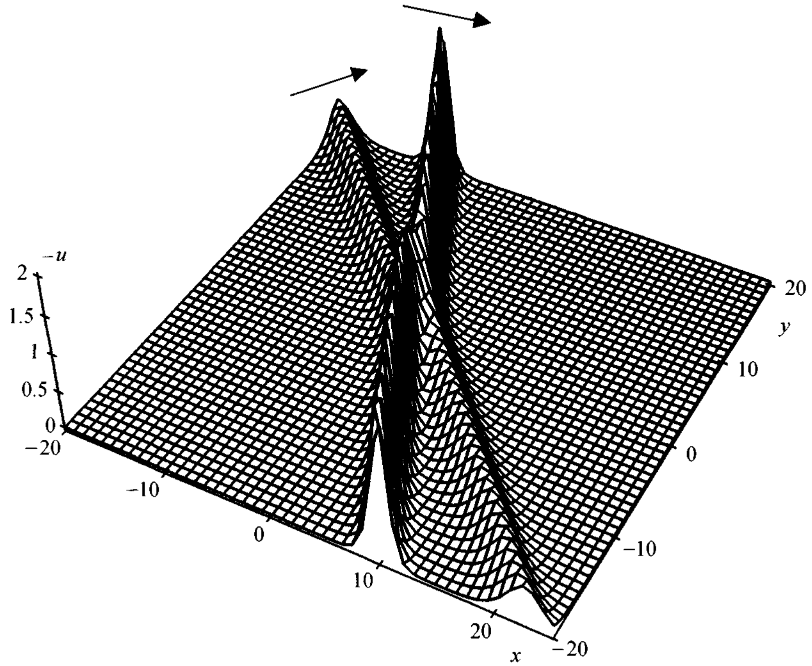


Figure 3.7. A 2-soliton solution of the 2D Korteweg–de Vries equation, for $l_1 = l_2 = 1$ and $k_1 = 1$, $k_2 = 2$. Note that $-u$ is plotted here.

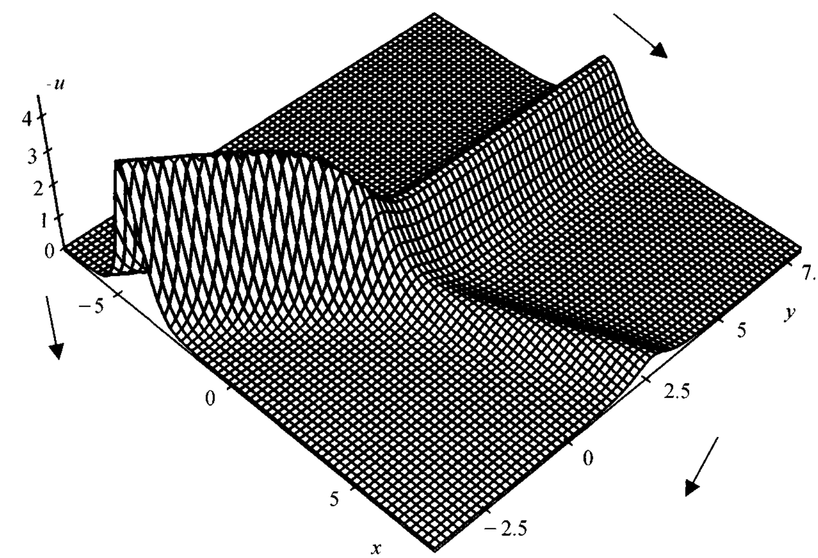


Figure 3.8. A *resonant* 2-soliton solution of the 2D KdV equation, for $l_1 = 0$, $l_2 = -3$, $k_1 = 2$, and $k_2 = 3$. Note that $-u$ is plotted here.

becomes

$$\left(\mathrm{D}_x\mathrm{D}_t + \mathrm{D}_x^4 + \frac{1}{2t}\frac{\partial}{\partial x}\right)(f \cdot f) = 0$$

where

$$\frac{\partial}{\partial x}(f \cdot f) = f\frac{\partial f}{\partial x};$$

in both these equations the transformation is

$$u = -2\frac{\partial^2}{\partial x^2}\log f.$$

With this same transformation, the Boussinesq equation

$$u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} = 0$$

has the bilinear form

$$(\mathrm{D}_t^2 - \mathrm{D}_x^2 - \mathrm{D}_x^4)(f \cdot f) = 0.$$

As the exercises should demonstrate, the construction of solitary wave and soliton solutions from the bilinear form is, in most cases, a fairly straightforward and routine operation. A 2-soliton solution of the 2D KdV equation is shown in Figure 3.7 (see Q3.30), and a *resonant* solution is shown in Figure 3.8 (see Q3.32). A solution of the Boussinesq equation, which describes both head-on and overtaking soliton collisions, is given in Figure 3.9.

3.3.4 Conservation laws

We are already familiar with the equation of mass conservation (Section 1.1.1) and how this equation can be integrated in z (Section 1.2.4) to produce the form

$$d_t + \nabla_\perp \cdot \bar{\mathbf{u}}_\perp = 0$$

(equation (1.38)). This is a general equation for water waves, where $d = h - b$ is the local depth and

$$\bar{\mathbf{u}}_\perp = \int_b^h \mathbf{u}_\perp \mathrm{d}z.$$

Furthermore, in the case of one-dimensional propagation with decay conditions at infinity, we found (equation (1.40)) that

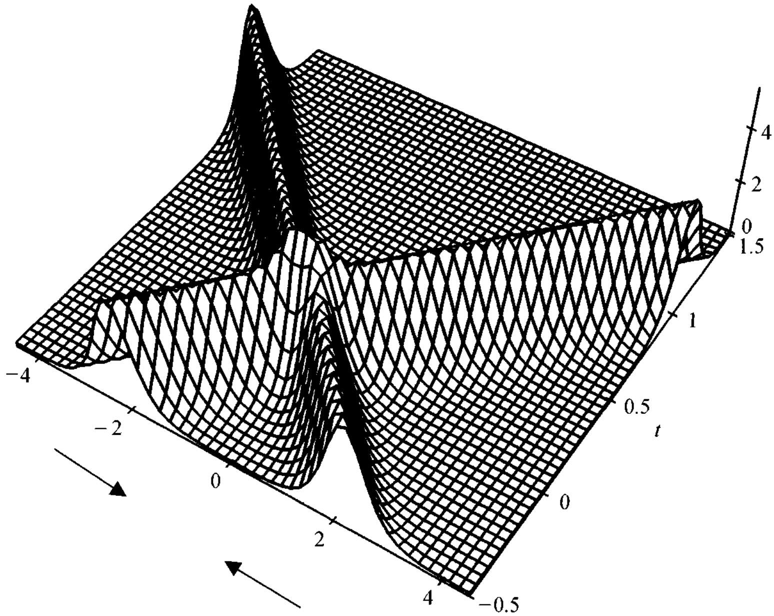


Figure 3.9. A solution of the Boussinesq equation depicting the head-on collision of two solitons, each of amplitude 2.

$$\int_{-\infty}^{\infty} H(x,t)\mathrm{d}x = \text{constant},$$

where $h(x,t) = h_0 + H(x,t)$ and $H \to 0$ as $|x| \to \infty$; this is a very convenient and transparent version of the statement of mass conservation in water waves. Of course, this result can be obtained – very simply – directly from the equations for one-dimensional gravity-wave propagation:

$$\left.\begin{aligned}
&u_t + \varepsilon(uu_x + wu_z) = -p_x; \quad \delta^2\{w_t + \varepsilon(uw_x + ww_z)\} = -p_z;\\
&\qquad\qquad u_x + w_z = 0,\\
&\text{with}\\
&\qquad p = \eta \quad \text{and} \quad w = \eta_t + \varepsilon u\eta_x \quad \text{on} \quad z = 1 + \varepsilon\eta\\
&\text{and}\\
&\qquad\qquad w = 0 \quad \text{on} \quad z = 0;
\end{aligned}\right\} \tag{3.85}$$

see equations (3.9). Thus, employing the technique of differentiation under the integral sign, we obtain

$$\frac{\partial}{\partial x}\left(\int_0^{1+\varepsilon\eta} u\,\mathrm{d}z\right) - \varepsilon u\eta_x\Big|_{1+\varepsilon\eta} + [w]_0^{1+\varepsilon\eta} = 0,$$

so

$$\eta_t + \frac{\partial}{\partial x}\left(\int_0^{1+\varepsilon\eta} u\,\mathrm{d}z\right) = 0 \tag{3.86}$$

from which we get

$$\int_{-\infty}^{\infty} \eta(x, t)\mathrm{d}x = \text{constant.} \tag{3.87}$$

Similarly, for energy (see Section 1.2.5), we obtain directly

$$\frac{\partial}{\partial t}\left\{\frac{1}{2}\eta^2 + \frac{1}{2}\int_0^{1+\varepsilon\eta} (u^2 + \delta^2 w^2)\mathrm{d}z\right\} + \frac{\partial}{\partial x}\left\{\int_0^{1+\varepsilon\eta}\left(\frac{\varepsilon}{2}u^3 + \frac{\varepsilon\delta^2}{2}uw^2 + up\right)\mathrm{d}z\right\} = 0 \tag{3.88}$$

from equations (3.85); cf. equation (1.47). (The derivation of this result is left as an exercise (Q3.33), although all the essential details are described in Section 1.2.5.) The resulting conserved energy in the motion is therefore

$$\int_{-\infty}^{\infty}\left\{\eta^2 + \int_0^{1+\varepsilon\eta} (u^2 + \delta^2 w^2)\mathrm{d}z\right\}\mathrm{d}x = \text{constant,} \tag{3.89}$$

where decay conditions at infinity have again been invoked. Generally, expressions of the form

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0, \tag{3.90}$$

where $T(x, t)$ (the *density*) and $X(x, t)$ (the *flux*) do not normally contain derivatives with respect to t, are called *conservation laws*. If both T and X_x are integrable over all x, so that

$$X \to X_0 \quad \text{as} \quad |x| \to \infty,$$

where X_0 is a constant, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\infty}^{\infty} T\mathrm{d}x\right) = 0 \quad \text{or} \quad \int_{-\infty}^{\infty} T(x, t)\,\mathrm{d}x = \text{constant}: \tag{3.91}$$

the integral of $T(x, t)$ over all x is a *conserved quantity* (often called a *constant of the motion*, especially when t is interpreted as a time-like variable). We have so far written down the conservation laws for mass (3.86) and for energy (3.88), as they apply to our one-dimensional water-wave problem. It is no surprise that there is a third conservation law, namely the one that describes the conservation of momentum. There has been no need to express this in the form (3.90) in our earlier work, but we will now show how it can be obtained very simply from equations (3.85). (Ideas that are closely related to all three conservation laws have been developed in the discussion of the jump conditions in Section 2.7.)

Referring to equations (3.85), we see that the first added to εu times the third yields

$$u_t + 2\varepsilon u u_x + \varepsilon(uw)_z + p_x = 0,$$

and so

$$\int_0^{1+\varepsilon\eta} (u_t + 2\varepsilon u u_x + p_x)\,\mathrm{d}z + \varepsilon[uw]_0^{1+\varepsilon\eta} = 0.$$

The boundary conditions that describe w then give

$$\int_0^{1+\varepsilon\eta} (u_t + 2\varepsilon u u_x + p_x)\,\mathrm{d}z + \varepsilon u_s(\eta_t + \varepsilon u_s \eta_x) = 0$$

where u_s is $u(x, t, z)$ evaluated on the surface, $z = 1 + \varepsilon\eta$. Again, application the method of differentiating under the integral sign produces

$$\frac{\partial}{\partial t}\left(\int_0^{1+\varepsilon\eta} u\,\mathrm{d}z\right) + \frac{\partial}{\partial x}\left\{\int_0^{1+\varepsilon\eta} (\varepsilon u^2 + p)\,\mathrm{d}z - \frac{1}{2}\varepsilon\eta^2\right\} = 0 \tag{3.92}$$

where we have used the surface boundary condition for p (where $p = \eta$). Equation (3.92) is the conservation law for momentum which, with an undisturbed background state in place, gives

$$\int_{-\infty}^{\infty}\left(\int_{0}^{1+\varepsilon\eta} u\,\mathrm{d}z\right)\mathrm{d}x = \text{constant}, \tag{3.93}$$

the conserved momentum in the motion. Thus, as we must expect, the passage of the gravity wave (within the confines of our model), conserves the three fundamental properties of motion: mass, momentum and energy. The question that we now address is how these conservation laws – and associated conserved quantities – manifest themselves in our various KdV-type equations.

We begin this discussion with the KdV equation itself; the leading-order contribution (η_0) to the representation of the surface wave satisfies equation (3.28):

$$2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = 0$$

or

$$\frac{\partial}{\partial\tau}(2\eta_0) + \frac{\partial}{\partial\xi}\left(\frac{3}{2}\eta_0^2 + \frac{1}{3}\eta_{0\xi\xi}\right) = 0,$$

when written in the form of a conservation law. With the assumption that the wave decays at infinity, so that $\eta_0 \to 0$ as $|\xi| \to \infty$ (which is equivalent to $|x| \to \infty$ for any t), we have

$$\int_{-\infty}^{\infty} \eta_0\,\mathrm{d}\xi = \text{constant}; \tag{3.94}$$

this is the conservation of mass, equation (3.87). Another conserved quantity is obtained by multiplying the KdV equation by η_0, to give

$$\frac{\partial}{\partial\tau}(\eta_0^2) + \frac{\partial}{\partial\xi}\left\{\eta_0^3 + \frac{1}{3}\left(\eta_0\eta_{0\xi\xi} - \frac{1}{2}\eta_{0\xi}^2\right)\right\} = 0,$$

so

$$\int_{-\infty}^{\infty} \eta_0^2\,\mathrm{d}\xi = \text{constant}. \tag{3.95}$$

It is clear that the two constants of the motion, (3.94) and (3.95), are general properties of all solutions of the KdV equations which decay rapidly enough at infinity. (This also means that, for example, periodic solutions do not satisfy these particular integral constraints, although an analogous set of results can be obtained if the integral is taken over just one period.) The second result, (3.95), should – we must surmise – correspond to the conservation of momentum, equation (3.93). Now from Section 3.2.1 we find that

$$u \sim u_0 + \varepsilon\left\{\eta_1 + \left(\frac{1}{3} - \frac{1}{2}z^2\right)\eta_{0\xi\xi} - \frac{1}{4}\eta_0^2\right\}; \quad u_0 = \eta_0$$

(where the KdV equation has been used to eliminate the term $\eta_{0\tau}$ in u_1), so

$$\int_{-\infty}^{\infty}\left(\int_0^{1+\varepsilon\eta} u\,\mathrm{d}z\right)\mathrm{d}x \sim \int_{-\infty}^{\infty}\left\{u_0 + \varepsilon(u_0\eta_0 + \eta_1 + \frac{1}{3}\eta_{0\xi\xi} - \frac{1}{4}\eta_0^2)\right\}\mathrm{d}\xi$$
$$- \frac{\varepsilon}{2}\int_{-\infty}^{\infty}\eta_{0\xi\xi}\left(\int_0^1 z^2\mathrm{d}z\right)\mathrm{d}\xi$$
$$= \int_{-\infty}^{\infty}\left(\eta_0 + \varepsilon\eta_1 + \frac{3\varepsilon}{4}\eta_0^2\right)\mathrm{d}\xi,$$

which is correct at $\mathrm{O}(\varepsilon)$. The first two terms in this integral appear in the conservation of mass, and consequently we must have

$$\int_{-\infty}^{\infty}\eta_0^2\,\mathrm{d}\xi = \text{constant},$$

which recovers (3.95). (The confirmation that the integral of η_1 alone is itself a constant follows directly from the equation for η_1 obtained in Q3.4.)

We have obtained two conserved densities for our KdV equation, η_0 and η_0^2, which correspond to the conservation of mass and momentum, respectively. We can anticipate that the equation possesses a third conserved density, which is associated with the energy of the motion. To see that this is indeed the case, we construct $3\eta_0^2 \times$ (KdV) minus $(2\eta_{0\xi}/3) \times (\partial/\partial\xi)$(KdV) to give

$$3\eta_0^2\left(2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi}\right)$$
$$- \frac{2}{3}\eta_{0\xi}(2\eta_{0\xi\tau} + 3\eta_0\eta_{0\xi\xi} + 3\eta_{0\xi}^2 + \frac{1}{3}\eta_{0\xi\xi\xi\xi}) = 0$$

which can be written as

$$2\frac{\partial}{\partial\tau}\left(\eta_0^3-\frac{1}{3}\eta_{0\xi}^2\right)+\frac{\partial}{\partial\xi}\left(\frac{9}{4}\eta_0^4+\eta_0^2\eta_{0\xi\xi}-2\eta_0\eta_{0\xi}^2-\frac{2}{9}\eta_{0\xi}\eta_{0\xi\xi\xi}+\frac{1}{9}\eta_{0\xi\xi}^2\right)=0.$$

This is in the form of a conservation law, so we obtain a third conserved quantity

$$\int_{-\infty}^{\infty}\left(\eta_0^3-\frac{1}{3}\eta_{0\xi}^2\right)\mathrm{d}\xi=\text{constant}, \tag{3.96}$$

which is indeed directly related to the total energy given in (3.89) (see Q3.34). (In the context of the KdV equation treated in isolation, it would seem reasonable to regard (3.95) as a statement of energy conservation, since the integrand is a square (that is, proportional to (amplitude)^2). However, as we have seen, when the appropriate physical interpretation is adopted, it is (3.96) which corresponds to the conservation of energy.)

The existence of these three conservation laws is to be expected since our underlying water-wave equations exhibit this same property (where only conservative forces are involved). However, there is now a real surprise: the KdV equation possesses an *infinite* number of conservation laws. In the early stages of the study of the KdV equation (Miura, Gardner & Kruskal, 1968), eight further conservation laws were written down explicitly (having been obtained by extraordinary perseverance); for example, the next two conserved densities are

$$\frac{45}{4}\eta_0^4-15\eta_0\eta_{0\xi}^2+\eta_{0\xi\xi}^2$$

and

$$63\eta_0^5-210\eta_0^2\eta_{0\xi}^2+28\eta_0\eta_{0\xi\xi}^2-\frac{8}{9}\eta_{0\xi\xi\xi}^2;$$

see Q3.35. The existence of an infinite set of conservation laws (which will not be proved here) relates directly to the important idea that the KdV equation, and other 'soliton' equations, each constitute a *completely integrable Hamiltonian system*; equivalently, this is to say that the KdV equation can be written as a *Hamiltonian flow*. This aspect of soliton theory is quite beyond the scope of a text that is centred on water-wave theory, but much has been written on these matters; see the section on Further Reading at the end of this chapter.

Finally, we briefly indicate the form of some of the conservation laws that are associated with the standard KdV-type equations. We shall use here the simplest – we might say normalised – versions of these equations that were introduced in Section 3.3. First we consider the concentric KdV equation, (3.64):

$$u_t + \frac{u}{2t} - 6uu_x + u_{xxx} = 0.$$

It is clear that, for the first two terms, $t^{1/2}$ is an integrating factor; thus we multiply by $t^{1/2}$ to give

$$\frac{\partial}{\partial t}(t^{1/2}u) + \frac{\partial}{\partial x}\left\{t^{1/2}(u_{xx} - 3u^2)\right\} = 0,$$

so $t^{1/2}u$ is a conserved density. This describes the geometrical decay that is required to maintain the conservation of mass (cf. equation (3.31)). Similarly, if we multiply by $2tu$, then we obtain

$$\frac{\partial}{\partial t}(tu^2) + \frac{\partial}{\partial x}\left\{t(2uu_{xx} - u_x^2 - 4u^3)\right\} = 0$$

so that another conserved density is tu^2; further conserved densities are discussed in Q3.39.

The Boussinesq equation, (3.43), is

$$H_{tt} - H_{XX} + 3(H^2)_{XX} - H_{XXXX} = 0$$

which, for our current purpose, is most conveniently written as the pair of equations

$$H_t = -U_X, \quad U_t + H_X - 3(H^2)_X + H_{XXX} = 0;$$

cf. equation (3.38). The second equation here is obtained after one integration in X, coupled with the assumption of decay conditions as $|X| \to \infty$. We now obtain directly

$$\int_{-\infty}^{\infty} H_t \,\mathrm{d}X = -[U]_{-\infty}^{\infty}; \qquad \int_{-\infty}^{\infty} U_t \,\mathrm{d}X = -[H - 3H^2 + H_{XX}]_{-\infty}^{\infty}$$

and so

$$\int_{-\infty}^{\infty} H \,\mathrm{d}X = \text{constant} \quad \text{and} \quad \int_{-\infty}^{\infty} U_t \,\mathrm{d}X = \text{constant}. \tag{3.97}$$

The first of these is the conservation of mass and the second is the conservation of momentum, an identification which becomes clearer if we revert to the original x, where

$$X = x + \varepsilon \int_{-\infty}^{x} \eta \, dx$$

(see Section 3.2.5), so that

$$\int_{-\infty}^{\infty} U(1 + \varepsilon\eta) \, dx = \text{constant};$$

cf. equation (3.93). A few other conserved densities are given in Q3.40.

Our third and final example is the two-dimensional KdV equation which is written here as

$$u_t - 6uu_x + u_{xxx} + 3v_y = 0, \quad u_y = v_x;$$

see equation (3.61) and Section 3.2.2. No longer do we have the classical form of a (two-dimensional) conservation law: u is a function of three variables here. This complication produces a development that is less straightforward. When we integrate the second equation with respect to x, and impose decay conditions at infinity, we obtain

$$\frac{\partial}{\partial y}\left(\int_{-\infty}^{\infty} u \, dx\right) = 0 \quad \text{so} \quad \int_{-\infty}^{\infty} u \, dx = f(t).$$

However, this is true for all y; let us evaluate the integral for any y that is far-removed from any wave interaction in, say, the N-soliton solution. (The N-soliton solution of the 2D KdV equation describes the interaction of waves that asymptote to plane oblique solitary waves at infinity; see Section 3.3.2 and Q3.19.) In this situation, the function $f(t)$ is a constant; consequently we obtain

$$\int_{-\infty}^{\infty} u \, dx = \text{constant}, \tag{3.98}$$

at least for this class of solutions. A similar argument yields the result

$$\int_{-\infty}^{\infty} v \, dy = \text{constant}; \tag{3.99}$$

these two conserved quantities are analogous to the pair (3.97) that we derived for the Boussinesq equation. To proceed, the integral in x of the first equation of this pair yields

$$\frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty} u\,\mathrm{d}x\right)+[-3u^2+u_{xx}]_{-\infty}^{\infty}+3\frac{\partial}{\partial y}\left(\int_{-\infty}^{\infty} v\,\mathrm{d}x\right)=0$$

and so, making use of (3.98) and the same argument as above, we also have

$$\int_{-\infty}^{\infty} v\,\mathrm{d}x=\text{constant}. \tag{3.100}$$

The obvious interpretation of equations (3.99) and (3.100) is that momentum is conserved in both the y- and x- directions; other conservation laws are even less straightforward to obtain and to interpret.

As an intriguing postscript, we mention the equations for shallow water (obtained in Section 2.6). Following the choices we made there (of setting $\varepsilon = 1$ and writing $1+\varepsilon\eta(x, t) = h(x, t)$), these equations are

$$\left.\begin{aligned} &u_t+uu_x+wu_z+h_x=0; \quad u_x+w_z=0,\\ \text{with}\\ &w=h_t+uh_x \text{ on } z=h \text{ and } w=0 \text{ on } z=0.\end{aligned}\right\} \tag{3.101}$$

We have seen that our water-wave equations, (3.85), admit just the three physical conservation laws (of mass, momentum and energy). On the other hand, all our KdV-type equations – that is, completely integrable equations – possess an infinity of conservation laws. The question we pose is: how many conservation laws does the set (3.101) possess? The obvious answer, surely, is just three; let us investigate further.

First, the now very familiar procedure of forming

$$\int_0^h u_x\,\mathrm{d}z+[w]_0^h=\frac{\partial}{\partial x}\left(\int_0^h u\,\mathrm{d}z\right)+\frac{\partial h}{\partial t}=0$$

yields the conservation of mass

$$\int_{-\infty}^{\infty} h(x,t)\,\mathrm{d}x=\text{constant}, \tag{3.102}$$

provided decay conditions obtain. Next, the second equation in (3.101) is multiplied by u and added to the first to give

$$u_t+2uu_x+(uw)_z+h_x=0,$$

so (cf. equation (3.92)) we obtain

$$\frac{\partial}{\partial t}\left(\int_0^h u\,\mathrm{d}z\right)+\frac{\partial}{\partial x}\left(\frac{1}{2}h^2+\int_0^h u^2\,\mathrm{d}z\right)=0,$$

from which we obtain the conservation of momentum

$$\int_{-\infty}^{\infty}\left(\int_0^h u\,\mathrm{d}z\right)\mathrm{d}x=\text{constant}. \tag{3.103}$$

Finally, multiply the first equation by u to produce

$$\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right)+\frac{\partial}{\partial x}\left(\frac{1}{3}u^3\right)+\frac{\partial}{\partial z}\left(\frac{1}{2}u^2 w\right)-\frac{1}{2}u^2 w_z+\frac{\partial}{\partial x}(uh)-hu_x=0$$

and then substitute from the second equation for w_z and for u_x:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right)+\frac{\partial}{\partial x}\left(\frac{1}{2}u^3+uh\right)+\frac{\partial}{\partial z}\left(\frac{1}{2}u^2 w+hw\right)=0$$

since $h=h(x,t)$ only. Consequently the integration in z, coupled with differentiation under the integral sign, yields

$$\frac{\partial}{\partial t}\left\{\frac{1}{2}h^2+\frac{1}{2}\int_0^h u^2\,\mathrm{d}z\right\}+\frac{\partial}{\partial x}\left\{\int_0^h\left(\frac{1}{2}u^3+uh\right)\mathrm{d}z\right\}=0,$$

so

$$\int_{-\infty}^{\infty}\left(h^2+\int_0^h u^2\,\mathrm{d}z\right)\mathrm{d}x=\text{constant}, \tag{3.104}$$

the conserved energy. These three conserved quantities, (3.102), (3.103) and (3.104), are to be compared with those derived earlier ((3.87), (3.93) and (3.89)). No surprises here: we have derived the expected conservation laws for mass, momentum and energy.

We now explore an extension of this process by multiplying the first equation of (3.101) by u^2 (and follow the development described by Benney (1974) and Miura (1974)), to give

$$\frac{\partial}{\partial t}\left(\frac{1}{3}u^3\right)+\frac{\partial}{\partial x}\left(\frac{1}{4}u^4\right)+wu^2u_z+u^2h_x=0;$$

this is rewritten as

$$\frac{\partial}{\partial t}\left(\frac{1}{3}u^3\right)+\frac{\partial}{\partial x}\left(\frac{1}{4}u^4+u^2h\right)+\frac{\partial}{\partial z}\left(\frac{1}{3}u^3w\right)-\frac{1}{3}u^3w_z-2huu_x=0, \quad (3.105)$$

The same equation is multiplied by h to produce

$$(hu)_t-uh_t+huu_x+whu_z+\left(\frac{1}{2}h^2\right)_x=0$$

which is added to equation (3.105) to yield

$$\frac{\partial}{\partial t}\left(\frac{1}{3}u^3+hu\right)+\frac{\partial}{\partial x}\left(\frac{1}{3}u^4+u^2h+\frac{1}{2}h^2\right)$$
$$+\frac{\partial}{\partial z}\left(\frac{1}{3}u^3w\right)-huu_x-uh_t+hwu_z=0. \qquad (3.106)$$

Here we write

$$hwu_z=(huw)_z-huw_z$$

and then introduce

$$[w]_0^h+\int_0^h u_x\,\mathrm{d}z=0; \quad \text{that is,} \quad h_t+m_x=0, \quad m=\int_0^h u\,\mathrm{d}z,$$

to give

$$\begin{aligned} hwu_z-uh_t-huu_x &= (huw)_z-huw_z+um_x-huu_x \\ &= (huw)_z+(um)_x-u_xm=(huw)_z+(um)_x+(mw)_z. \end{aligned}$$

Thus equation (3.106) becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{3}u^3+hu\right)+\frac{\partial}{\partial x}\left(\frac{1}{3}u^4+hu^2+\frac{1}{2}h^2+um\right)$$
$$+\frac{\partial}{\partial z}\left\{w\left(\frac{1}{3}u^3+hu+m\right)\right\}=0,$$

which provides a *fourth* conservation law which, after an integration in z over $(0, h)$, yields

$$\left(\frac{1}{3}m_3+hm_1\right)_t+\left(\frac{1}{3}m_4+hm_2+\frac{1}{2}m_1^2+\frac{1}{3}h^3\right)_x=0,$$

where we have written

$$m_n = \int_0^h u^n \, dz.$$

Indeed, as Benney and Miura demonstrate, the set of equations (3.101) – like our special evolution equations – possesses an *infinite* set of conservation laws; see Q3.42 and Q3.43 for more about these laws.

We have now introduced some of the important equations of soliton theory that arise in the study of water waves, together with a description of some of their properties. We now extend our studies to show how other physically relevant properties can be introduced into our nonlinear evolution equations (although the resulting equations that govern the wave propagation are unlikely to be completely integrable).

3.4 Waves in a non-uniform environment

The equations that we have derived so far – the KdV family of equations – appear to arise in very special circumstances. In particular, we have assumed that the water is stationary and that the bottom is both flat and horizontal. It is clear that any application of these methods to situations that model physical reality more closely must encompass variable depth and an underlying (non-uniform) flow, at the very least. Certainly it would be a disappointment to find that all the interesting phenomena of nonlinear wave propagation (that we have described earlier) occur only under ideal conditions that hardly ever obtain in the physical world. One of our objectives in this section will be to demonstrate that the derivation and existence of KdV equations (in water-wave theory) are fairly robust to changes in the underlying physical properties. Specifically, we shall see how the derivation of some of the family of KdV equations is affected by the inclusion of (a) an underlying shear flow and (b) variable depth. In addition we shall briefly look at some properties of obliquely interacting waves.

3.4.1 Waves over a shear flow

The purpose here is to derive the classical Korteweg–de Vries equation, for long gravity waves, propagating in the x-direction, over water which is moving only in the x-direction, with a velocity profile which depends only on z: $u = U(z)$. This is the prescribed underlying shear flow, although the terminology that we adopt is not to imply that the profile is generated by viscous stresses. This description is used in order to indicate what type of profile could be chosen; in the undisturbed state – no waves – the governing equations (for inviscid flow) admit a solution for arbitrary $U(z)$, provided that the depth is constant; see Q1.13. Thus we set $b = 0$ for all x in the equations for one-dimensional flow; we use equations (3.12)–(3.15), where the shear flow is introduced by writing

$$U(z) + \varepsilon u \quad \text{for} \quad \varepsilon u \tag{3.107}$$

since we want $U = \mathrm{O}(1)$ as $\varepsilon \to 0$. Hence the equations that we shall now examine are

$$\left.\begin{aligned}
&u_t + Uu_x + U'w + \varepsilon(uu_x + wu_z) = -p_x\\
&\varepsilon\{w_t + Uw_x + \varepsilon(uw_x + ww_z)\} = -p_z;\\
&u_x + w_z = 0,\\
&\text{with}\\
&p = \eta \quad \text{and} \quad w = \eta_t + U\eta_x + \varepsilon u\eta_x \quad \text{on} \quad z = 1 + \varepsilon\eta\\
&\text{and}\\
&w = 0 \quad \text{on} \quad z = 0,
\end{aligned}\right\} \tag{3.108}$$

where $U' \equiv \mathrm{d}U/\mathrm{d}z$. (We note in passing that, indeed, these equations are satisfied with $u = w = p = \eta = 0$ – no disturbances – for arbitrary $U(z)$.)

The first task here is to determine the nature of the linear problem, that is, the leading order problem in the asymptotic expansion for $\varepsilon \to 0$. This is described by the equations

$$\left.\begin{aligned}
&u_t + Uu_x + U'w = -p_x; \quad p_z = 0; \quad u_x + w_z = 0,\\
&\text{with}\\
&p = \eta \quad \text{and} \quad w = \eta_t + U\eta_x \quad \text{on} \quad z = 1\\
&\text{and}\\
&w = 0 \quad \text{on} \quad z = 0.
\end{aligned}\right\} \tag{3.109}$$

We are interested (at this order) in waves that propagate at constant speed with unchanging form. Do any such solutions of equations (3.109) exist? Let us suppose that they do, and so introduce a coordinate that is moving with the waves at a constant speed c; we therefore transform from (x, t, z) to $(x - ct, z)$. Our equations (3.109) become

$$(U - c)u_\xi + U'w = -p_\xi; \quad u_\xi + w_z = 0; \quad p = \eta \quad (0 \le z \le 1), \tag{3.110}$$

with

$$w = (U - c)\eta_\xi \quad \text{on} \quad z = 1; \quad w = 0 \quad \text{on} \quad z = 0. \tag{3.111}$$

Here we have obtained $p(= \eta)$ in the familiar way, and have written $\xi = x - ct$.

To proceed, we eliminate u_ξ between the equations in (3.110) to give

$$U'w - (U - c)w_z = \eta_\xi \quad \text{or} \quad (U - c)^2 \frac{\partial}{\partial z}\left(\frac{w}{U - c}\right) = \eta_\xi,$$

so

$$w = (U - c)\eta_\xi \int_0^z \frac{\mathrm{d}z}{(U - c)^2}$$

which satisfies the bottom boundary condition (in (3.111)). The surface boundary condition (on $z = 1$; see (3.111) again) requires that

$$\int_0^1 \frac{\mathrm{d}z}{(U - c)^2} = 1, \tag{3.112}$$

and then $\eta(\xi)$ is arbitrary: the waves propagate at a constant speed (c) determined by equation (3.112), given $U(z)$, and – at this order – they move with an unchanging shape which is arbitrary. The equation for c, (3.112), is very different from that which has appeared in any of our other work that has led to an expression for the speed of propagation for gravity waves. This is an important equation in water-wave theory (and its counterparts appear in other problems which incorporate an underlying flow); it is known as the *Burns condition* (Burns (1953), although it seems to have appeared first in Thompson (1949)). But it turns out that its real interest is evident in the cases where solutions of (3.112) do not exist!

Solutions of (3.112) for c exist only provided $U(z) \neq c$ for $0 \leq z \leq 1$. If $U(z_c) = c$ for some $z_c \in (0, 1)$ – and $U(0) = c$ or $U(1) = c$ can never happen – then the left-hand side of (3.112) is not defined; $z = z_c$ is called a *critical level* or *layer*. We shall make a few comments about the nature of the Burns condition later (Section 3.4.2), but it is sufficient for our present purposes to assume that solutions of (3.112) *do* exist. For example, the simple choice

$$U(z) = U_0 + (U_1 - U_0)z, \tag{3.113}$$

where U_0 and U_1 are constants, yields

$$-\frac{1}{(U_1-U_0)}\left[\frac{1}{U_1-c}-\frac{1}{U_0-c}\right]=1$$

and so

$$c=\frac{1}{2}\left[U_0+U_1\pm\sqrt{4+(U_1-U_0)^2}\right]. \qquad (3.114)$$

This solution describes two possible speeds of propagation, one of which satisfies $c > U_1$ and the other $c < U_0$. That is, for the linear shear (3.113), the two speeds of propagation are: greater than the surface speed of the flow and less than the bottom speed. We note that, for $U(z) = 0$, $0 \leq z \leq 1$, (that is, $U_0 = U_1 = 0$) we recover $c = \pm 1$ (see equation (2.10)). (The case of a uniform stream corresponds to the choice $U_0 = U_1$; cf. Q2.11.)

We now proceed to the derivation of the KdV equation as relevant to this problem, and to accomplish this we follow the method described in Section 3.2.1. Thus we introduce a local characteristic variable (ξ) and a far-field variable (τ) defined by

$$\xi = x - ct, \quad \tau = \varepsilon t, \qquad (3.115)$$

where c is a solution of the Burns condition. Equations (3.108) become

$$\left.\begin{aligned}
&(U-c)u_\xi + U'w + \varepsilon(u_\tau + uu_\xi + wu_z) = -p_\xi;\\
&\varepsilon\{(U-c)w_\xi + \varepsilon(w_\tau + uw_\xi + ww_z)\} = -p_z;\\
&u_\xi + w_z = 0,\\
&\text{with}\\
&p = \eta \quad \text{and} \quad w = (U-c)\eta_\xi + \varepsilon(\eta_\tau + u\eta_\xi)\\
&\qquad \text{on} \quad z = 1 + \varepsilon\eta\\
&\text{and}\\
&w = 0 \quad \text{on} \quad z = 0.
\end{aligned}\right\} \qquad (3.116)$$

We seek a solution of this set, as usual, in the form of an asymptotic expansion

$$q \sim \sum_{n=0}^{\infty} \varepsilon^n q_n, \quad \varepsilon \to 0,$$

where q represents each of u, w, p and η. Thus, at leading order, we have

$$(U-c)u_{0\xi} + U'w_0 = -p_{0\xi}; \quad p_{0z} = 0; \quad u_{0\xi} + w_{0z} = 0$$

with

$$p_0 = \eta_0 \quad \text{and} \quad w_0 = (U-c)\eta_{0\xi} \quad \text{on} \quad z = 1$$

and

$$w_0 = 0 \quad \text{on} \quad z = 0.$$

This is, as expected, the linear problem that we have just described: see equations (3.109)–(3.111). Thus c is a solution of

$$\int_0^1 \frac{\mathrm{d}z}{(U-c)^2} = 1, \tag{3.117}$$

the Burns condition, and

$$\left.\begin{aligned} w_0 &= (U-c)\eta_{0\xi}\int_0^z \frac{\mathrm{d}z}{(U-c)^2}; \\ u_0 &= -\eta_0\left\{\frac{1}{U-c} + U'\int_0^z \frac{\mathrm{d}z}{(U-c)^2}\right\}; \quad p_0 = \eta_0, \end{aligned}\right. \tag{3.118}$$

where we have assumed that $u_0 = 0$ wherever $\eta_0 = 0$. The solution (3.118), with (3.117), is valid for arbitrary $\eta_0(\xi, \tau)$, at this order; to find η_0 we must construct the problem at $\mathrm{O}(\varepsilon)$.

The $\mathrm{O}(\varepsilon)$ terms from equations (3.116) give rise to the set of equations

$$(U-c)u_{1\xi} + U'w_1 + u_{0\tau} + u_0u_{0\xi} + w_0u_{0z} = -p_{1\xi}; \tag{3.119}$$

$$(U-c)w_{0\xi} = -p_{1z}; \quad u_{1\xi} + w_{1z} = 0 \tag{3.120}$$

with

$$\left.\begin{aligned} p_1 + \eta_0 p_{0z} &= \eta_1 \qquad (3.121) \\ w_1 + \eta_0 w_{0z} &= (U-c)\eta_{1\xi} + U'\eta_0\eta_{0\xi} + \eta_{0\tau} + u_0\eta_{0\xi} \qquad (3.122) \end{aligned}\right\} \text{ on } z = 1$$

and

$$w_1 = 0 \quad \text{on} \quad z = 0. \tag{3.123}$$

At this stage it is convenient to introduce a compact notation to cope with the integrals that arise, namely

$$I_n(z) = \int_0^z \frac{\mathrm{d}z}{(U-c)^n}; \tag{3.125}$$

then, for example, the Burns condition (3.117) becomes simply

$$I_2(1)\ (= I_{21}) = 1. \tag{3.126}$$

Similarly, equations (3.118) are written as

$$w_0 = (U-c)I_2\eta_{0\xi}; \quad u_0 = -\{(U-c)^{-1} + U'I_2\}\eta_0; \quad p_0 = \eta_0 \tag{3.127}$$

and then from equations (3.120) and (3.121) we obtain

$$p_1 = \eta_1 + \eta_{0\xi\xi}\int_z^1 (U-c)^2 I_2 \mathrm{d}z.$$

Now we eliminate $u_{1\xi}$ between equations (3.119) and (3.120) to give

$$(U-c)^2\left\{\frac{w_1}{U-c}\right\}_z + \{(U-c)^{-1} + U'I_2\}\eta_{0\tau} - \{(U-c)^{-1} + U'I_2\}^2\eta_0\eta_{0\xi}$$
$$+ (U-c)U''I_2^2\eta_0\eta_{0\xi} = \eta_{1\xi} + \eta_{0\xi\xi\xi}\int_z^1 (U-c)^2 I_2 \mathrm{d}z,$$

and then the solution which satisfies the bottom boundary condition, (3.123), can be written as

$$w_1 = (U-c)\left\{\left(\frac{I_2}{U-c} - 2I_3\right)\eta_{0\tau} + \left(I_4 + 4\int_0^z \frac{U'I_2}{(U-c)^3}\mathrm{d}z - \frac{U'I_2}{U-c}\right)\eta_0\eta_{0\xi}\right.$$
$$\left. + I_2\eta_{1\xi} + \eta_{0\xi\xi\xi}\int_0^z (U-c)^{-2}\left[\int_z^1 (U-c)^2 I_2 \mathrm{d}z\right]\mathrm{d}z\right\}.$$

Finally, the surface boundary condition for w_1, (3.121), requires that

$$(U_1 - c)\eta_{1\xi} + U_1'\eta_0\eta_{0\xi} + \eta_{0\tau} - 2\left\{\frac{1}{U_1 - c} + U_1'I_{21}\right\}\eta_0\eta_{0\xi}$$
$$= (U_1 - c)\left\{\left(\frac{I_{21}}{U_1 - c} - 2I_{31}\right)\eta_{0\tau}\right.$$
$$\left. + \left(I_{41} + 4\int_0^1 \frac{U'I_2}{(U-c)^3}\,\mathrm{d}z - \frac{U_1'I_{21}}{U_1 - c}\right)\eta_0\eta_{0\xi} + I_{21}\eta_{1\xi} + J_1\eta_{0\xi\xi\xi}\right\}$$

where the additional subscript '1' denotes evaluation on $z = 1$, and

$$J_1 = \int_0^1 \int_z^1 \int_0^{z_1} \frac{[U(z_1) - c]^2}{[U(z) - c]^2[U(z_2) - c]^2}\,\mathrm{d}z_2\mathrm{d}z_1\mathrm{d}z.$$

After we use $I_{21} = 1$ (see (3.126)) and simplify, the equation for η_0 becomes

$$-2I_{31}\eta_{0\tau} + 3I_{41}\eta_0\eta_{0\xi} + J_1\eta_{0\xi\xi\xi} = 0, \tag{3.128}$$

since η_1 cancels identically from the problem at this order.

Equation (3.128) is an altogether satisfying result: it is a (classical) Korteweg–de Vries equation, since it has constant coefficients (and so may be transformed into any suitable variant of the KdV equation; see Q3.1). The presence of an ambient arbitrary velocity profile (and hence an arbitrary vorticity distribution in the flow) is evident only through the three constants I_{31}, I_{41} and J_1. Thus a problem that, we might have supposed, is significantly more involved than for the case of propagation on a stationary flow, reduces essentially to the same result. Hence nonlinear dispersive waves (for example the solitary wave) can exist on arbitrary flows. It is now a simple exercise, first, to check that we recover our previous KdV equation for stationary flow and, second, to obtain the form of the KdV equation for any given flow (at least in the absence of a critical level); see Q3.44 & Q3.45. (More details of this derivation will be found in Freeman & Johnson (1970); the corresponding calculation for the sech^2 solitary wave is described by Benjamin (1962).)

3.4.2 The Burns condition

The derivation of the Korteweg–de Vries equation for flow over an arbitrary shear, as we have presented it, is valid only if a critical level does not arise. If $U(z)$ and c are such that $U(z_c) = c$ for some $z_c\,(0 < z_c < 1)$, it is clear that we must examine the nature of the problem in the neighbourhood of $z = z_c$ (since, for example, the integrals over z no longer exist). It turns out that, in the context of inviscid fluid dynamics, a region exists in the neighbourhood of $z = z_c$ (in fact, where $z - z_c = O(\varepsilon^{1/2})$) where nonlinear effects are important. The inclusion of the appropriate contribution from the nonlinearity enables the singularity at $z = z_c$ to be removed. This calculation is, however, altogether beyond the scope of this text; those interested in this aspect of the problem should consult some of the references given in the Further Reading at the end of this chapter. Suffice it to record here that the Burns condition and, indeed, the KdV equation, are both recovered even when a critical level is present. The only change from the results that we have described is that all the integrals are now defined by their *finite parts*, that is, their *Cauchy principal values*.

One way to define the finite part of our integrals is as the

$$\text{finite part as } \varepsilon \to 0^+ \left\{ \int_0^{z_c-\varepsilon} f(z)\,\mathrm{d}z + \int_{z_c+\varepsilon}^{1} f(z)\,\mathrm{d}z \right\}, \tag{3.129}$$

from which it is clear that the finite part recovers the classical value of any integral which is defined for all $z \in [0, 1]$. The usual shorthand for the finite part is to write $⨍$ for $\int$, or $\mathrm{I}\!\!\!-$ for I; in this notation, the Burns condition (3.117) becomes

$$\mathrm{I}\!\!\!-_{21} = ⨍_0^1 \frac{\mathrm{d}z}{[U(z) - c]^2} = 1. \tag{3.130}$$

It can be shown (Burns, 1953) that, for a monotonic profile which satisfies $U(0) \le U(z) \le U(1)$, there are always at least two solutions of equation (3.130):

$$c > U(1) \quad \text{and} \quad c < U(0),$$

exactly as we mentioned earlier. Depending on the form of the function $U(z)$ there may, or may not, be one or more *critical-layer solutions* for which $c = U(z_c)$, $0 < z_c < 1$. We conclude by noting that both the linear-shear profile given in (3.113) and the parabolic (Poiseuille) profile

$$U(z) = U_1(2z - z^2), \quad U_1 = \text{constant},$$

do not admit any critical-layer solutions; see Q3.46. Two 'model' profiles that do give rise to critical-layer solutions – one each – are discussed in Q3.47, Q3.48.

3.4.3 Ring waves over a shear flow

In the two preceding sections we have presented some linear and nonlinear aspects of the problem of *unidirectional* propagation over an arbitrary shear flow. We now address the corresponding problem of a *ring wave* moving over a (unidirectional) shear flow. We refer to this wave as a ring wave, rather than a concentric wave (cf. Sections 2.1.3 and 3.2.3) because it turns out that the wave is concentric only in the case of uniform flow (U = constant everywhere). The presentation here will include some details of the linear problem, and we will mention only briefly the related nonlinear problem.

The underlying shear flow, exactly as in Section 3.4.1, is written as $u = U(z)$, and this is given; the wave, however, propagates outwards from some initial central disturbance. This coordinate mix – rectangular Cartesian for the shear flow and plane polar for the wave – leads to a rather involved formulation of this problem. First we recall the governing equations expressed in rectangular Cartesian coordinates, suitably written with our choice of parameters. Thus we use equations (3.1)–(3.4), but with δ^2 replaced by ε; see equations (3.12)–(3.15) *et seq*. In addition we introduce our standard representation for a horizontal flat bottom ($b = 0$), and then replace εu by $U(z) + \varepsilon u$; see equation (3.107). These manoeuvres yield the set of equations

$$\left.\begin{aligned} u_t + (U + \varepsilon u)u_x + \varepsilon v u_y + U' w + \varepsilon w u_z &= -p_x;\\ v_t + (U + \varepsilon u)v_x + \varepsilon v v_y + \varepsilon w v_z &= -p_y;\\ \varepsilon\{w_t + (U + \varepsilon u)w_x + \varepsilon v w_y + \varepsilon w w_z\} &= -p_z;\\ u_x + v_y + w_z &= 0,\\ \text{with}\qquad\qquad & \\ p = \eta \quad \text{and} \quad w = \eta_t + (U + \varepsilon u)\eta_x &+ \varepsilon v \eta_y\\ \text{on} \quad z = 1 + \varepsilon\eta & \\ \text{and}\qquad\qquad & \\ w = 0 \quad \text{on} \quad z = 0. & \end{aligned}\right\} \tag{3.31}$$

Now we introduce a plane polar coordinate system which is moving at a constant speed c in the x-direction; we shall make a suitable choice of c later. Thus we transform $(x, y, t) \to (r, \theta, t)$, where

$$x = ct + r\cos\theta, \quad y = r\sin\theta \tag{3.132}$$

and, correspondingly, we define the velocity perturbation in the (x, y)-plane by the transformation

$$u \to u\cos\theta - v\sin\theta, \quad v \to u\sin\theta + v\cos\theta, \tag{3.133}$$

so that (u, v) now represents the perturbation velocity vector for the horizontal components of the motion, written in the polar coordinate frame. Further, we choose to describe a wave whose wavefront has reached an appropriate far-field (so that we may, eventually, construct the relevant KdV equation); thus we define

$$\xi = rk(\theta) - t, \quad R = \varepsilon rk(\theta), \tag{3.134}$$

where the wavefront is represented by $\xi =$ constant and $k(\theta)$ is to be determined. (A concentric wave corresponds to the case $k(\theta) =$ constant, for all θ.) This choice of far-field variables, (3.134), is to be compared with those used in Sections 3.2.1 and 3.2.3; in particular, in this latter case, we see that (3.134) is equivalent to setting $\delta = \varepsilon$ there and then writing ε for ε^2.

The set of equations (3.131), under the transformations (3.132)–(3.134), becomes

$$\left.\begin{aligned}
&(D_1 + \varepsilon D_2 + \varepsilon D_3 + \varepsilon^2 D_4)u + \varepsilon(U - c)\frac{k}{R}v\sin\theta + U'w\cos\theta \\
&\qquad\qquad\qquad\qquad - \varepsilon^2\frac{kv^2}{R} = -k(p_\xi + \varepsilon p_R); \\
&(D_1 + \varepsilon D_2 + \varepsilon D_3 + \varepsilon^2 D_4)v - \varepsilon(U - c)\frac{k}{R}u\sin\theta - U'u\sin\theta \\
&\qquad\qquad\qquad\qquad + \varepsilon^2\frac{kuv}{R} = -k'(p_\xi + \varepsilon p_R) - \varepsilon\frac{k}{R}p_\theta; \\
&\varepsilon\left\{(D_1 + \varepsilon D_2 + \varepsilon D_3 + \varepsilon^2 D_4)w\right\} = -p_z \\
&ku_\xi + k'v_\xi + w_z + \varepsilon\left(ku_R + \frac{ku}{R} + k'v_R + \frac{k}{R}v_\theta\right) = 0, \\
&\text{with} \\
&\quad p = \eta \quad \text{and} \quad \left.\begin{aligned} w = (D_1 + \varepsilon D_2 + \varepsilon^2 D_4)\eta + \varepsilon(ku + k'v)\eta_\xi \\ \text{on } z = 1 + \varepsilon\eta \end{aligned}\right\}
\end{aligned}\right\} \tag{3.135}$$

and

$$w = 0 \quad \text{on} \quad z = 0.$$

The differential operators (D_n) are defined by

$$D_1 \equiv \{-1 + [U(z) - c](k\cos\theta - k'\sin\theta)\}\frac{\partial}{\partial\xi};$$

$$D_2 \equiv [U(z) - c]\left\{(k\cos\theta - k'\sin\theta)\frac{\partial}{\partial R} - \frac{k}{R}\sin\theta\frac{\partial}{\partial\theta}\right\};$$

$$D_3 \equiv (ku + k'v)\frac{\partial}{\partial\xi} + w\frac{\partial}{\partial z}; \quad D_4 \equiv (ku + k'v)\frac{\partial}{\partial R} + \frac{kv}{R}\frac{\partial}{\partial\theta},$$

where $k' = \mathrm{d}k/\mathrm{d}\theta$ and, as before, $U' = \mathrm{d}U/\mathrm{d}z$. (The routine but rather tedious calculation that leads to equations (3.135) is left as an exercise.)

We seek an asymptotic solution of the set (3.135) in the usual fashion:

$$q \sim \sum_{n=0}^{\infty} \varepsilon^n q_n, \quad \varepsilon \to 0,$$

where q represents each of u, v, w, p and η. The leading-order, linear problem is therefore

$$\begin{gathered}
-u_{0\xi} + (U - c)(k\cos\theta - k'\sin\theta)u_{0\xi} + U'w_0\cos\theta = -k'p_{0\xi};\\
-v_{0\xi} + (U - c)(k\cos\theta - k'\sin\theta)v_{0\xi} - U'w_0\sin\theta = -k'p_{0\xi};\\
p_{0z} = 0; \quad ku_{0\xi} + k'v_{0\xi} + w_{0z} = 0,
\end{gathered}$$

with

$$p_0 = \eta_0 \text{ and } w_0 = -\eta_{0\xi} + (U - c)(k\cos\theta - k'\sin\theta)\eta_{0\xi} \text{ on } z = 1$$

and

$$w_0 = 0 \quad \text{on} \quad z = 0.$$

In common with our previous calculation of this type, we see that $p_0 = \eta_0$ for $z \in [0, 1]$, and then adding the first equation $\times k$ to $k' \times$ the second and eliminating $(ku_{0\xi} + k'v_{0\xi})$ with the equation of mass conservation yields

$$-Fw_{0z} + F_z w_0 = -(k^2 + k'^2)\eta_{0\xi}$$

where we have written

$$F(z, \theta) = -1 + \{U(z) - c\}(k\cos\theta - k'\sin\theta).$$

Thus

$$w_0 = (k^2 + k'^2)F\eta_{0\xi}\int_0^z \frac{\mathrm{d}z}{F^2}$$

satisfies the bottom boundary condition, and then the surface boundary condition for w_0 requires that

$$(k^2 + k'^2)\int_0^1 \frac{\mathrm{d}z}{[1 - \{U(z) - c\}(k\cos\theta - k'\sin\theta)]^2} = 1, \tag{3.136}$$

for arbitrary η_0.

Equation (3.136) is a *generalised Burns condition*, which reduces to our previous Burns condition, (3.117), when we introduce the choice for one-dimensional plane waves: $k(\theta) = 1$, $\theta = 0$ and write c for $1 + c$ (since the characteristic, ξ, contributes a wave speed of 1). In this case, equation (3.136) is used to determine c for a given $U(z)$. However, in the context of a ring wave, this equation is used to define $k(\theta)$ given both $U(z)$ and the speed (c) of the frame of reference. (We note that, in this frame, the speed of the outward propagating wave is $1/k(\theta)$ at any θ, provided $k(\theta) > 0$.) The derivation that has been described assumes that a critical level, $z = z_c$ ($z_c \in (0, 1)$), is not present; if a critical level does occur, so that $F(z_c, \theta) = 0$, then the generalised Burns condition is still (3.136) but now interpreted as the finite part of the integral.

A simple example of the use of the generalised Burns condition is afforded by the choice (see Q3.47)

$$U(z) = \begin{cases} U_1, & d \le z \le 1 \\ U_1 z/d, & 0 \le z < d, \end{cases} \tag{3.137}$$

where U_1 and $d \in [0, 1]$ are constants; this model shear flow was used in Johnson (1990), where more properties of the ring wave are described. The generalised Burns condition (3.136), with (3.137), becomes

$$(k^2 + k'^2)\left\{\frac{1-d}{[1 - (U_1 - c)(k\cos\theta - k'\sin\theta)]^2} + \left[\frac{d/\{U_1(k\cos\theta - k'\sin\theta)\}}{\{1 - (U_1 z/d - c)(k\cos\theta - k'\sin\theta)\}}\right]_0^d\right\} = 1,$$

and we now make a choice for c (the speed of the polar coordinate frame). The form of this expression for $k(\theta)$ suggests that we set $c = U_1$, an obvious selection on physical grounds since this ensures that the frame is moving at the surface speed of the shear flow. Our equation for $k(\theta)$ then reduces to

$$(k^2 + k'^2)[1 - d + d/\{1 + U_1(k\cos\theta - k'\sin\theta)\}] = 1, \tag{3.138}$$

a nonlinear first-order ordinary differential equation for $k(\theta)$. This equation possesses, quite clearly, the general solution

$$\left.\begin{aligned} &k(\theta) = a\cos\theta + b(a)\sin\theta \\ &\text{where} \\ &(a^2 + b^2)\{1 - d + d/(1 + aU_1)\} = 1, \end{aligned}\right\} \tag{3.139}$$

an approach that can also be adopted for general $U(z)$; see Q3.49. Unfortunately, solutions of the form (3.139) for any $a > 0$ (provided that b is real) do not admit $k(\theta) > 0$ for all θ: at some $\theta \in (0, \pi)$ (and also again for $\theta \in (0, -\pi)$), $k(\theta) = 0$ and thereafter $k(\theta) < 0$. Thus at two (symmetric) points the wavefront has moved to infinity (that is, $r = \{t + \text{constant}\}/k(\theta) \to \infty$) and, where $k(\theta) < 0$, it is moving *inwards*. But we are seeking an outward propagating wave and this, it turns out, is represented by the *singular solution* of equation (3.138). This solution (see Q3.52) can be written in the form

$$\left.\begin{aligned} &k(\theta) = a\cos\theta + b(a)\sin\theta \\ &\text{with} \\ &\tan\theta = -1/b'(a) \\ &\text{where} \\ &(a^2 + b^2)\{1 - d + d/(1 + aU_1)\} = 1. \end{aligned}\right\} \tag{3.140}$$

Three examples ($d = 0.5$; $U_1 = 0.5, 1, 2$) are presented in Figure 3.10, which shows clearly how the shear flow distorts the wavefront from the circular; these results have been obtained directly from equations (3.140) that define the singular solution.

Finally, we briefly state what happens when we construct the problem that arises at $O(\varepsilon)$. As we know, at this order, we shall find that η_1 is arbitrary, but we expect to obtain a KdV-type equation that describes the evolution of η_0. The calculation follows the lines of that already presented in Section 3.4.1, although it involves more complicated integrals that define the coefficients of the equation for η_0. This equation takes the form

$$A\eta_{0R} + \frac{B}{R}\eta_0 + \frac{C}{R}\eta_{0\theta} + D\eta_0\eta_{0\xi} + E\eta_{0\xi\xi\xi} = 0, \tag{3.141}$$

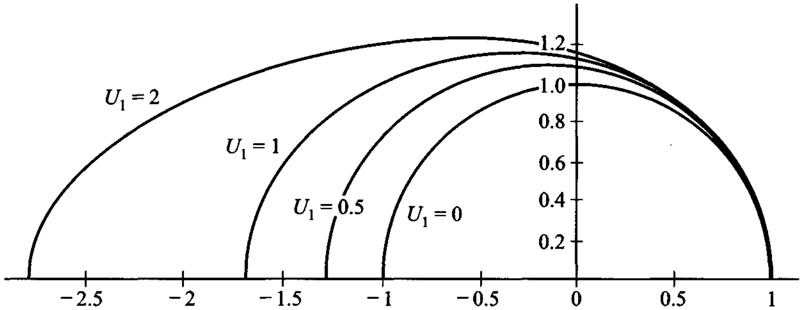


Figure 3.10. The shape of the wavefronts for the ring wave over a shear flow (equation (3.137)) for $0 \leq \theta \leq \pi$, with $d = 0.5$ and $U_1 = 0.5$, 1, 2. The corresponding circular ring wave ($U_1 = 0$) is included for comparison.

where A–E are the coefficients (which here depend on θ); more details can be found in Johnson (1990). This equation is clearly of KdV-type but, for arbitrary $U(z)$ (and therefore general coefficients), it is not one of the family of completely integrable equations. It does, however, recover the concentric KdV (cKdV) equation when $U(z) = $ constant (and we set $k(\theta) = 1$) for then $F(z, \theta) = -1$ and $A = 2$, $B = 1$, $C = 0$, $D = 3$, $E = 1/3$. Equation (3.141) can be discussed, in the general case, only via a numerical approach (which we do not pursue here).

3.4.4 The Korteweg–de Vries equation for variable depth

Another problem of some practical interest is the propagation of nonlinear dispersive waves (such as a solitary wave) over variable depth. We now address this situation in the case of one-dimensional propagation. Here, as we shall see, the important decision that we must make concerns the scale on which the depth variation occurs. In order to explain what is involved, we consider the classical situation that gives rise to the KdV equation (described in Section 3.2.1). We have shown that the relevant scales are

$$\xi = x - t, \quad \tau = \varepsilon t,$$

for right-going waves; thus $x - t = O(1)$ and $t = O(\varepsilon^{-1})$. This is equivalent to the choice $x - t = O(1)$ and $x = O(\varepsilon^{-1})$ (cf. Figure 1.7), which is the convenient interpretation to adopt here, for, if the depth varies on a scale which is either faster or slower than $O(\varepsilon^{-1})$, we shall obtain appropriately simplified KdV problems. In the former case, we have a situation where, to leading order as $\varepsilon \to 0$, the depth changes rapidly relative to any changes due to the natural evolution of the nonlinear wave. On the other hand, in the latter case, the wave will evolve at essentially the (local) constant depth. Of course, both these problems are of considerable interest in their own right and they have received some attention – particularly the latter choice; more details can found from the Further Reading at the end of this chapter. However, for the development that we present here, the most interesting case arises when the scale of the depth variation is the same as the scale on which the wave will naturally evolve even over constant depth. In the context of the KdV derivations that we have presented so far, this can be thought of as the 'worst case' scenario. Of course, we may then use this case to gain some insight into the problems for faster and for slower depth variations; we shall touch on these two extremes later.

The governing equations for one-dimensional propagation (cf. equations (3.12)–(3.15) with (3.4)) are

$$\left.\begin{aligned} u_t + \varepsilon(uu_x + wu_z) &= -p_x; \\ \varepsilon\left\{w_t + \varepsilon(uw_x + ww_z)\right\} &= -p_z; \\ u_x + w_z &= 0, \\ \text{with} \qquad & \\ p = \eta \text{ and } w = \eta_t + \varepsilon u\eta_x &\text{ on } z = 1 + \varepsilon\eta \\ \text{and} \qquad & \\ w = 0 \quad \text{on} \quad z &= 0. \end{aligned}\right\} \qquad (3.142)$$

The important choice, described above, is to set

$$b(x) = B(\varepsilon x),$$

and we shall usually define $B(\varepsilon x) = 0$ in $x < 0$, so that the wave propagates (rightwards) from a region of constant depth. The appropriate variables to use for the far-field (cf. $\xi = x - t$, $\tau = \varepsilon t$) must accommodate the variation of wave speed with depth (see equation (2.47) and Q2.33), and the slow spatial scale (εx). Thus we introduce

$$\xi = \frac{1}{\varepsilon}\chi(X) - t, \qquad X = \varepsilon x,$$

where $\chi(X)$ is to be determined, so we transform according to

$$\frac{\partial}{\partial x} \equiv \chi' \frac{\partial}{\partial \xi} + \varepsilon \frac{\partial}{\partial X}, \qquad \frac{\partial}{\partial t} \equiv -\frac{\partial}{\partial \xi}.$$

This makes clear why the factor ε^{-1} is required in the definition of ξ: $\chi' = \mathrm{O}(1)$ plays the rôle of the speed of propagation, $c = \mathrm{O}(1)$ (actually χ' is equivalent to $1/c$). Further, we anticipate that, for constant depth ($B = 0$), we shall have $\chi(X) = X = \varepsilon x$ and then ξ recovers our former expression $(x - t)$. The set of governing equations, (3.142), therefore becomes

$$\left.\begin{aligned}
&-u_\xi + \varepsilon\{u(\chi' u_\xi + \varepsilon u_X) + w u_z\} = -(\chi' p_\xi + \varepsilon p_X);\\
&\varepsilon[-w_\xi + \varepsilon\{u(\chi' w_\xi + \varepsilon w_X) + w w_z\}] = -p_z;\\
&\chi' u_\xi + \varepsilon u_X + w_z = 0,\\
&\text{with}\\
&p = \eta \quad \text{and} \quad w = -\eta_\xi + \varepsilon u(\chi' \eta_\xi + \varepsilon \eta_X)\\
&\qquad \text{on} \quad z = 1 + \varepsilon\eta\\
&\text{and}\\
&w = \varepsilon u B'(X) \quad \text{on} \quad z = B(X).
\end{aligned}\right\} \tag{3.143}$$

We adopt the standard form of solution:

$$q \sim \sum_{n=0}^{\infty} \varepsilon^n q_n, \quad \varepsilon \to 0,$$

where q represents each of u, w, p and η; the leading-order problem from (3.143) is then described by the equations

$$u_{0\xi} = \chi' p_{0\xi}; \quad p_{0z} = 0; \quad \chi' u_{0\xi} + w_{0z} = 0,$$

with

$$p_0 = \eta_0 \quad \text{and} \quad w_0 = -\eta_{0\xi} \quad \text{on} \quad z = 1$$

and

$$w_0 = 0 \quad \text{on} \quad z = B(X).$$

These equations are in a form that we recognise as typical of these problems; the solution is immediately

$$p_0 = \eta_0, \quad 0 \le z \le 1; \quad u_0 = \chi' \eta_0; \quad w_0 = (B - z)\chi'^2 \eta_{0\xi}, \tag{3.144}$$

where we have chosen $u_0 = 0$ wherever $\eta_0 = 0$. The function w_0 satisfies the bottom boundary condition (on $z = B$), and in order to satisfy the corresponding surface condition we require

$$\chi'^2 = \frac{1}{D(X)}, \tag{3.145}$$

where $D(X) = 1 - B(X)$ (> 0) is the local depth. Thus we write

$$\chi(X) = \int_0^X \frac{\mathrm{d}X'}{\sqrt{D(X')}} \tag{3.146}$$

for right-going waves, which agrees precisely with the form given in equation (2.47). Here, for convenience, we have chosen $\chi(0) = 0$. At this order $\eta_0(\xi, X)$ is arbitrary, so we move to the O(ε) terms, which will provide the equation for η_0.

From equations (3.143) we see that

$$\left.\begin{aligned} &-u_{1\xi} + \chi' u_0 u_{0\xi} + w_0 u_{0z} = -(\chi' p_{1\xi} + p_{0X});\\ &w_{0\xi} = p_{1z}; \quad \chi' u_{1\xi} + u_{0X} + w_{1z} = 0,\\ &\text{with}\\ &p_1 + \eta_0 p_{0z} = \eta_1 \text{ and } w_1 + \eta_0 w_{0z} = -\eta_{1\xi} + \chi' u_0 \eta_{0\xi}\\ &\qquad \text{on } z = 1\\ &\text{and}\\ &w_1 = u_0 B'(X) \quad \text{on} \quad z = B(X). \end{aligned}\right\} \tag{3.147}$$

(We note in passing that, at least in this problem, there is no need to expand χ as

$$\chi \sim \sum_{n=0}^{\infty} \varepsilon^n \chi_n(X),$$

although in other problems this might be necessary in order to obtain a uniformly valid representation.) Thus we obtain (with $\chi' = 1\sqrt{D}$)

$$p_1 = \frac{1}{D}\left\{B(z-1) + \frac{1}{2}(1-z^2)\right\}\eta_{0\xi\xi} + \eta_1,$$

since $p_{0z} = 0$ and we have used (3.144). From the first and third equations in (3.147), upon the elimination of $u_{1\xi}$, we obtain

$$\begin{aligned} &u_{0X} + w_{1z} + \frac{1}{D}u_0 u_{0\xi} + \frac{1}{\sqrt{D}} w_0 u_{0z}\\ &\qquad = -\frac{1}{D}\left[\frac{1}{D}\left\{B(z-1) + \frac{1}{2}(1-z^2)\right\}\eta_{0\xi\xi\xi} + \eta_{1\xi}\right] - \frac{1}{\sqrt{D}}\eta_{0X} \end{aligned}$$

which yields

$$w_1 = (B - z)\left\{ \left(\frac{1}{\sqrt{D}} \eta_0 \right)_X + \frac{1}{D^2} \eta_0 \eta_{0\xi} + \frac{1}{\sqrt{D}} \eta_{0X} + \frac{1}{D} \eta_{1\xi} \right\}$$
$$- \frac{1}{D^2} \left\{ B\left(\frac{1}{2} z^2 - z \right) + \frac{1}{2}\left(z - \frac{z^3}{3} \right) - \frac{1}{3} B^3 + B^2 - \frac{1}{2} B \right\} \eta_{0\xi\xi\xi} + \frac{B'}{\sqrt{D}} \eta_0,$$

where the bottom boundary condition (on $z = B$) is satisfied. Finally, the surface boundary condition requires

$$-\eta_{1\xi} + \frac{2}{D} \eta_0 \eta_{0\xi} = -D\left\{ \left(\frac{1}{\sqrt{D}} \eta_0 \right)_X + \frac{1}{D^2} \eta_0 \eta_{0\xi} + \frac{1}{\sqrt{D}} \eta_{0X} + \frac{1}{D} \eta_{1\xi} \right\}$$
$$- \frac{1}{3} D \eta_{0\xi\xi\xi} - \frac{D'}{\sqrt{D}} \eta_0$$

in which, as expected, $\eta_{1\xi}$ cancels identically, leaving

$$2\sqrt{D}\eta_{0X} + \frac{1}{2}\frac{D'}{\sqrt{D}}\eta_0 + \frac{3}{D}\eta_0\eta_{0\xi} + \frac{1}{3}D\eta_{0\xi\xi\xi} = 0, \tag{3.148}$$

where $D = D(X)$. This is a variable-coefficient KdV equation, which clearly reduces to our classical KdV equation, (3.28), when we introduce the constant depth, $D = 1$:

$$2\eta_{0X} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = 0,$$

although we must now interpret X as τ (which is legitimate at this order).

Our new KdV-type equation, (3.148), is not one of the special completely integrable equations (for arbitrary $D(X)$), but special reductions are possible (as for $D = 1$; see also Q3.53). However, the general equation can be usefully written by first multiplying by $D^{-1/4}$, to give

$$2(D^{1/4}\eta_0)_X + \frac{3}{D^{5/4}}\eta_0\eta_{0\xi} + \frac{1}{3}D^{3/4}\eta_{0\xi\xi\xi} = 0,$$

where the first term embodies Green's law (as described in equation (2.47) *et seq.* and in Q2.34). Indeed, it is convenient to introduce

$$H_0(\xi, X) = D^{1/4}\eta_0$$

so that we obtain

$$2H_{0X} + \frac{3}{D^{7/4}}H_0H_{0\xi} + \frac{1}{3}D^{1/2}H_{0\xi\xi\xi} = 0. \tag{3.149}$$

Although this equation can be solved in any complete sense only numerically, we can make some important observations about the nature of its solutions.

For a wave profile that tends to zero both ahead of and behind the wavefront, we see that the integral in ξ of equation (3.149) yields

$$2\frac{\mathrm{d}}{\mathrm{d}X}\int_{-\infty}^{\infty} H_0 \mathrm{d}\xi + \left[\frac{3}{2}D^{-7/4}H_0^2 + \frac{1}{3}D^{1/2}H_{0\xi\xi}\right]_{-\infty}^{\infty} = 0,$$

so

$$\int_{-\infty}^{\infty} H_0(\xi, X)\mathrm{d}\xi = \text{constant},$$

which is equivalent to the conservation of mass. However, this does not describe the correct mass conservation for the water-wave problem. To see this, consider

$$\int_{-\infty}^{\infty} H_0 \mathrm{d}\xi = D^{1/4}(X) \int_{-\infty}^{\infty} \eta_0(\xi, X)\,\mathrm{d}\xi = \text{constant};$$

let us suppose that a wave is moving in a region of constant depth ($D = 1$) and is carrying a total mass of m_0; then

$$D^{1/4} \int_{-\infty}^{\infty} \eta_0(\xi, X)\,\mathrm{d}\xi = m_0.$$

But the mass carried by the wave is always

$$\int_{-\infty}^{\infty} \eta_0(\xi, X)\,\mathrm{d}\xi$$

and this is clearly *not* conserved as D varies, since

$$\int_{-\infty}^{\infty} \eta_0(\xi, X)\,\mathrm{d}\xi = m_0 D^{-1/4}(X).$$

The difficulty has arisen because the mass conservation applies to the complete water-wave problem, and not necessarily to a single element of the solution taken in isolation – here the solution of our KdV equation. Indeed, it is this inconsistency which has led to much detailed study of this problem (particularly in the cases of faster and slower depth variations, where considerable headway can be made). The important observation (see Miles, 1979; Knickerbocker and Newell 1980, 1985) is that other wave components of smaller amplitude, but which carry O(1) mass, are required to complete the description. In particular it has been found that a *left-going* wave (that is, a *reflected* wave) is necessary, and that this supplies the major correction to the overall mass conservation. (Other conservation laws for equation (3.149) are discussed in Q3.54.)

In conclusion, we briefly describe some properties of the wave component that is represented by the solution of the KdV equation, in the two extreme cases where

$$D(X) = \hat{D}(\sigma X), \quad \sigma \to 0 \quad \text{or} \quad \sigma \to \infty.$$

Of course, a more complete discussion of these problems – and indeed for the case of $\sigma = O(1)$ – requires a study of the other wave components, as we have just outlined, but this is beyond the scope of the presentation here. Nevertheless, the resulting wave evolution does give the correct picture to leading order in amplitude (even though the mass carried by the waves is incorrect to leading order).

First, in the case of $\sigma \to 0$, where the depth variation occurs on a scale that is slower than the evolution scale (X) of the wave, the variable coefficients in equation (3.149) are treated as independent of X. (This approach can be formalised by introducing an appropriate multiple-scale representation:

$$H_0 = H_0(\xi, X, \hat{X}), \quad \hat{X} = \sigma X, \quad \sigma \to 0.)$$

For example, the solitary-wave solution of this equation can be expressed, for η_0, as

$$\eta_0 = D^{-1/4} H_0 = \frac{A_0}{D} \operatorname{sech}^2 \left\{ \sqrt{\frac{3A_0}{4D^3}} \left(\xi - \frac{1}{2} D^{-5/2} A_0 X \right) \right\}, \tag{3.150}$$

where A_0 is the amplitude of the wave on the constant depth $D = 1$. We have chosen to write the solution in this form in order to ensure that the conservation law in H_0^2, for equation (3.149), is satisfied; see Q3.54 and Q3.55. An example of the evolution of the solitary wave, according to (3.150), is shown in Figure 3.11.

The second case that we describe is where the depth variation is fast ($\sigma \to \infty$) compared with the evolution of the wave. In this situation, the depth varies rapidly – instantaneously in the limit $\sigma \to \infty$ – so a wave moving on one depth must instantaneously begin to evolve as it adjusts to a new depth. As before, let us consider the example of a solitary wave, of

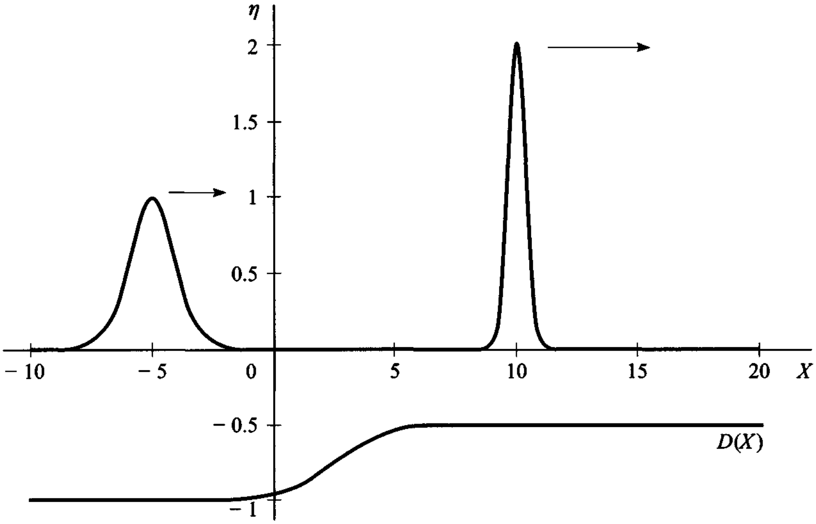


Figure 3.11. A representation of the distortion of a solitary wave (of amplitude 1) as it moves over a *slow* depth variation, from depth 1 to depth 0.5.

amplitude A_0, which is propagating in a region of constant depth, $D = 1$. Then, directly from equation (3.150), we have that

$$H_0 = A_0 \operatorname{sech}^2\left\{\frac{1}{2}\sqrt{3A_0}(\xi - \frac{1}{2}A_0 X)\right\}. \tag{3.151}$$

Now suppose that the depth changes suddenly from $D = 1$ (in $X < 0$, say) to $D = D_0$ (in $X > 0$); the profile (3.151) will move into $X > 0$ but cannot immediately adjust to the new depth. Thus this profile becomes an initial condition for the KdV equation, (3.149), evaluated for $D = D_0$:

$$2H_{0X} + 3D_0^{-7/4}H_0 H_{0\xi} + \frac{1}{3}D_0^{1/2}H_{0\xi\xi\xi} = 0.$$

We compare this version of the (constant coefficient) KdV equation with the standard form (see (3.49)):

$$u_t - 6uu_x + u_{xxx} = 0,$$

which possesses an N-soliton solution if

$$u(x, 0) = -N(N+1)\operatorname{sech}^2 x;$$

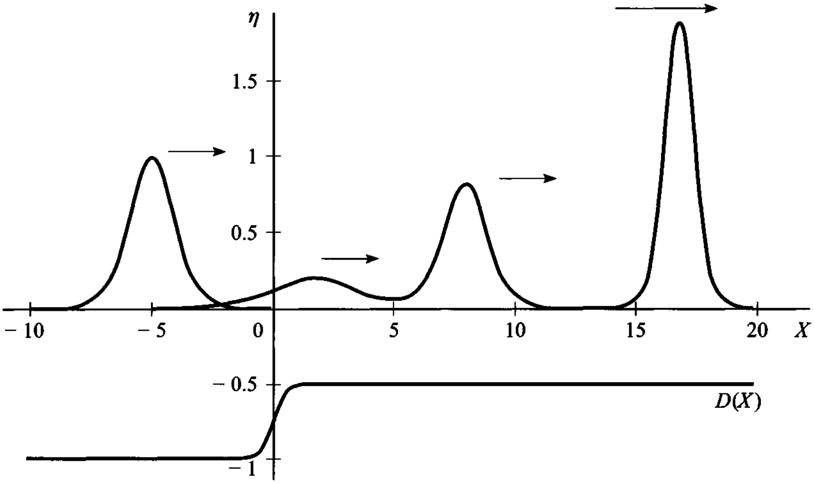


Figure 3.12. A representation of the distortion of a solitary wave (of amplitude 1) as it moves over a *fast* depth variation, from depth 1 to 0.451 (which corresponds to $N = 3$, the 3-soliton solution).

see equation (3.60) *et seq*. Thus we transform according to

$$\hat{\xi} = \frac{1}{2}\sqrt{3A_0}\xi, \quad \hat{X} = \frac{1}{6}D_0^{1/2}\left(\frac{1}{2}\sqrt{3A_0}\right)^3 X, \quad \hat{H}_0 = -\frac{2}{A_0}D_0^{-9/4}H_0$$

which gives

$$\hat{H}_{0\hat{X}} - 6\hat{H}_0\hat{H}_{0\hat{\xi}} + \hat{H}_{0\hat{\xi}\hat{\xi}\hat{\xi}} = 0;$$

an N-soliton solution is possible if

$$\hat{H}_0(\hat{\xi}, 0) = -N(N+1)\mathrm{sech}^2\hat{\xi}.$$

But

$$\hat{H}_0(\hat{\xi}, 0) = -\frac{2}{A_0}D_0^{-9/4}\hat{H}_0(\hat{\xi}, 0) = -2D_0^{-9/4}\mathrm{sech}^2\hat{\xi},$$

and so the solitary wave on $D = 1$ will evolve into N solitons on $D = D_0$ if

$$D_0 = \left\{\frac{1}{2}N(N+1)\right\}^{-4/9},$$

a result obtained and described in Tappert & Zabusky (1971) and Johnson (1973). We see immediately that solitons can appear only if the depth *decreases*, because $N = 2, 3, \ldots$ for two or more solitons. (If the depth increases, then the wave collapses into a nonlinear oscillatory wave; see Johnson (1973).) An example of 3-soliton production ($N = 3$, $D_0 \approx 0.451$) is shown in Figure 3.12, where $\eta_0 = D^{-1/4} H_0$ is reproduced.

3.4.5 Oblique interaction of waves

We have already met the two-dimensional KdV equation (Section 3.2.2), which admits solutions that represent obliquely crossing waves. In that analysis we were guided by the requirement to find the scaling that led to a KdV-type equation. Here, we address the problem of obliquely crossing waves (of small amplitude) directly from the governing equations, without the restriction to producing a KdV-type of balance. This approach will provide deeper insight into how such waves interact and, indeed, also provide a different interpretation of the rôle of the 2D KdV equation.

We shall consider the propagation of a plane wave – perhaps a solitary wave – moving in an arbitrary direction across the surface of stationary water of constant depth. However, the surface contains another plane wave which is also propagating in an arbitrary direction. Thus, as far as the first wave is concerned, the environment is no longer uniform. (This section will therefore complete a discussion of various non-uniform environments, namely: (a) an underlying shear flow; (b) variable depth; (c) a disturbed surface.)

The discussion of this problem that we shall present will follow closely the seminal work of Miles (1977a), although we shall cast it in a form that is consistent with much of our earlier work. This, it turns out, is an occasion when the most convenient approach is to take full advantage of the irrotationality of the flow, and so we shall formulate the problem in terms of Laplace's equation and the pressure equation; see Q1.38 and equations (2.132). We replace δ^2 by ε (as before) and set $b = 0$, so we have

$$\phi_{zz} + \varepsilon(\phi_{xx} + \phi_{yy}) = 0 \tag{3.152}$$

$$\left.\begin{aligned} &\phi_z = \varepsilon\{\eta_t + \varepsilon(\phi_x \eta_x + \phi_y \eta_y)\} \\ &\eta + \phi_t + \frac{1}{2}\phi_z^2 + \frac{1}{2}\varepsilon(\phi_x^2 + \phi_y^2) = 0 \end{aligned}\right\} \text{ on } z = 1 + \varepsilon\eta \tag{3.153}$$

and

$$\phi_z = 0 \quad \text{on} \quad z = 0. \tag{3.154}$$

(The way in which ϕ_z appears in these equations, and particularly the term ϕ_z^2, need cause no alarm since we shall find that, although $\phi = \mathrm{O}(1)$, $\phi_z = \mathrm{O}(\varepsilon)$.)

The Laplace equation, (3.152), can be solved in the form of an asymptotic expansion in ε which satisfies the bottom boundary condition, (3.154). To see how this proceeds, let us first write

$$\phi \sim \sum_{n=0}^{\infty} \varepsilon^n \phi_n$$

and then

$$\phi_{0zz} = 0; \quad \phi_{1zz} = -(\phi_{0xx} + \phi_{0yy}), \quad \text{etc.,}$$

and so

$$\phi_0 = f_0(x, y, t); \quad \phi_1 = -\frac{1}{2}z^2(f_{0xx} + f_{0yy}) + f_1(x, y, t), \quad \text{etc.,}$$

each satisfying $\phi_{nz} = 0$ on $z = 0$; the structure of this expansion is also evident from our work in Section 2.9.1. However, in this analysis we do not wish to be specific, at this early stage, about how f (that is, $f_0, f_1, \ldots$) is related to ε (and so how η relates to ε). It is clear that we may write the solution of Laplace's equation, and satisfy the bottom boundary condition, by writing

$$\phi \sim \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}(-\varepsilon\nabla_\perp^2)^n f, \quad \nabla_\perp^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

where $f = f(x, y, t; \varepsilon)$ is arbitrary. Of course, the complete asymptotic structure of ϕ, for $\varepsilon \to 0$, will be determined once we have settled on the form of f. The two equations that are required in order to define f and η are obtained by substituting for ϕ into the two surface boundary conditions, (3.153), with

$$\phi_z \sim \sum_{n=1}^{\infty} \frac{z^{2n-1}}{(2n-1)!}(-\varepsilon\nabla_\perp^2)^n f \quad (= \mathrm{O}(\varepsilon))$$

since the first term in ϕ is absent in ϕ_z. We must evaluate on $z = 1 + \varepsilon\eta$ and so, for example, ϕ_z becomes

$$\phi_z \sim \sum_{n=1}^{\infty} \frac{(1+\varepsilon\eta)^{2n-1}}{(2n-1)!}(-\varepsilon\nabla_\perp^2)^n f;$$

we shall retain terms in both boundary conditions that will allow us to find both the leading order and $O(\varepsilon)$ contributions. Thus equations (3.153) yield

$$-(1+\varepsilon\eta)\nabla_\perp^2 f + \frac{\varepsilon}{3!}\nabla_\perp^4 f \sim \eta_t + \varepsilon(f_x\eta_x + f_y\eta_y)$$

and

$$\eta + f_t - \frac{\varepsilon}{2!}\nabla_\perp^2 f_t + \frac{\varepsilon}{2}(f_x^2 + f_y^2) = O(\varepsilon^2) \tag{3.155}$$

from which a single equation for f may be obtained:

$$-(1-\varepsilon f_t)\nabla_\perp^2 f + \frac{\varepsilon}{6}\nabla_\perp^4 f + f_{tt} - \frac{\varepsilon}{2}\nabla_\perp^2 f_{tt} + 2\varepsilon(f_x f_{xt} + f_y f_{yt}) = O(\varepsilon^2). \tag{3.156}$$

Equation (3.156) enables $f(x, y, t; \varepsilon)$ to be determined, and then equation (3.155) gives $\eta(x, y, t; \varepsilon)$ directly. It is left as an exercise (Q3.56) to show that a single plane wave of the form

$$f = F(\xi, \tau; \varepsilon), \quad \eta = H(\xi, \tau; \varepsilon), \quad \xi = kx + ly - t, \quad \tau = \varepsilon t, \tag{3.157}$$

where $k^2 + l^2 = 1$ is the dispersion relation, recovers the KdV equation (cf. (3.28))

$$2H_\tau + 3HH_\xi + \frac{1}{3}H_{\xi\xi\xi} = 0, \tag{3.158}$$

to leading order. This wave propagates in the direction of the wave-number vector (k, l); indeed, it is convenient to write

$$k = \cos\theta, \;\; l = \sin\theta, \tag{3.159}$$

and then the wavefront moves in the direction that makes an angle θ with the positive x-axis.

The problem that we wish to address is the situation where a (nonlinear) plane wave, a solution of the KdV equation (3.158), moves on a surface (in an arbitrary direction) which contains another plane wave. Our first approach is to seek a solution which comprises two waves, each satisfying a KdV equation, together with an interaction between them that is *weak* (that is, $O(\varepsilon)$); see Q1.49. To this end we introduce

$$\zeta = px + qy - t; \quad p = \cos\psi, \quad q = \sin\psi, \tag{3.160}$$

and then write

$$f = F(\xi, \tau) + G(\zeta, \tau) + \varepsilon I(\xi, \zeta, \tau) + \mathrm{O}(\varepsilon^2), \tag{3.161}$$

where I represents the interaction of the two waves. Direct substitution into equation (3.156) then yields

$$\begin{aligned}
&-\big\{1 + \varepsilon(F_\xi + G_\zeta)\big\}\big\{F_{\xi\xi} + G_{\zeta\zeta} + \varepsilon(I_{\xi\xi} + I_{\zeta\zeta}) + 2\varepsilon(kp + lq)I_{\xi\zeta}\big\} \\
&\quad + \frac{\varepsilon}{6}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + F_{\xi\xi} + G_{\zeta\zeta} - 2\varepsilon(F_{\xi\tau} + G_{\zeta\tau}) \\
&\quad\quad + \varepsilon(I_{\xi\xi} + 2I_{\xi\zeta} + I_{\zeta\zeta}) - \frac{\varepsilon}{2}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) \\
&\quad\quad\quad - 2\varepsilon\big\{(kF_\xi + pG_\zeta)(kF_{\xi\xi} + pG_{\zeta\zeta}) + (lF_\xi + qG_\zeta)(lF_{\xi\xi} + qG_{\zeta\zeta})\big\} = \mathrm{O}(\varepsilon^2),
\end{aligned}$$

where we have used $k^2 + l^2 = 1$ and $p^2 + q^2 = 1$. But F_ξ and G_ζ satisfy appropriate KdV equations (see Q3.56); that is

$$2F_{\xi\tau} + 3F_\xi F_{\xi\xi} + \frac{1}{3}F_{\xi\xi\xi\xi} = 0; \quad 2G_{\zeta\tau} + 3G_\zeta G_{\zeta\zeta} + \frac{1}{3}G_{\zeta\zeta\zeta\zeta} = 0,$$

so we obtain

$$2\big\{1 - (kp + lq)\big\}I_{\xi\zeta} - \big\{1 + 2(kp + lq)\big\}(F_\xi G_{\zeta\zeta} + F_{\xi\xi}G_\zeta) = \mathrm{O}(\varepsilon) \tag{3.162}$$

which is the equation for I, at this order of approximation.

The coefficients of equation (3.162) are conveniently written in terms of

$$\begin{aligned}
kp + lq &= \cos\theta\cos\psi + \sin\theta\sin\psi \\
&= \cos(\theta - \psi) = 1 - 2\sin^2\{(\theta - \psi)/2\}
\end{aligned}$$

and we set $\lambda = \sin^2\{(\theta - \psi)/2\}$ so that

$$kp + lq = 1 - 2\lambda.$$

Further, on noting that $F_\zeta = G_\xi = 0$, we see that equation (3.162) reduces to

$$4\lambda I_{\xi\zeta} - (3 - 4\lambda)\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\zeta}\right)F_\xi G_\zeta = \mathrm{O}(\varepsilon) \tag{3.163}$$

which may be integrated directly and so, to leading order, we obtain

$$I = \left(\frac{3}{4}\lambda^{-1} - 1\right)(F_\xi G + FG_\zeta) \tag{3.164}$$

where we assume that $I = 0$ if either $F = 0$ or $G = 0$. The solution for f is therefore

$$f = F(\xi, \tau) + G(\zeta, \tau) + \varepsilon\left(\frac{3}{4}\lambda^{-1} - 1\right)(F_\xi G + FG_\zeta) + O(\varepsilon^2)$$

which can be written in the more compact form

$$f = F(\xi + \varepsilon\mu G, \tau) + G(\zeta + \varepsilon\mu F, \tau) + O(\varepsilon^2), \tag{3.165}$$

where $\mu = (3/4\lambda - 1)$; cf. Q1.53.

The surface wave, η, is obtained from equation (3.155) as

$$\eta = F_\xi + \varepsilon\left(\frac{1}{4}F_\xi^2 - \frac{1}{3}F_{\xi\xi\xi}\right) + G_\zeta + \varepsilon\left(\frac{1}{4}G_\zeta^2 - \frac{1}{3}G_{\zeta\zeta\zeta}\right) + \varepsilon\left(\frac{3}{2\lambda} - 3 + 2\lambda\right)F_\xi G_\zeta + O(\varepsilon^2), \tag{3.166}$$

where F_τ and G_τ have been eliminated by using the KdV equations for F_ξ and G_ζ (and by invoking decay conditions at $+\infty$); the derivation of (3.166) is left as an exercise (Q3.57). An example of the surface profile described by (3.166), for two solitary waves, is shown in Figure 3.13 (where we have taken $\varepsilon = 0.2$ to make clear the nature of the interaction). This figure also includes, for comparison, the solution for the same pair of plane waves when the interaction is absent; that is, $\varepsilon = 0$.

The solution that we have described so far assumes that the interaction between the waves is weak as $\varepsilon \to 0$ or, equivalently, $I = O(1)$. However, it is clear from equation (3.163), or from the solution (3.164), that I grows without bound as $\lambda \to 0$ (that is, as $\theta \to \psi$), so that the two waves approach the parallel orientation. This important observation was made by Miles (1977a), who proceeded to examine what happens as λ decreases; we shall follow a similar path here. Now, since the interaction term is εI, and $I = O(\lambda^{-1})$ as $\lambda \to$ O, our weak interaction theory is not uniformly valid when $\lambda = O(\varepsilon)$; this has become a *strong* interaction. Let us write $\psi = \theta + \alpha$, with $\alpha \to 0$; then

$$\lambda = \sin^2\{(\theta - \psi)/2\} = \sin^2(\alpha/2) = O(\alpha^2),$$

so the wave number (p, q) becomes

$$\begin{aligned}(p, q) = (\cos\psi, \sin\psi) &= (\cos\theta, \sin\theta) + \alpha(-\sin\theta, \cos\theta) + O(\alpha^2)\\ &= (k, l) + \alpha(-l, k) + O(\alpha^2).\end{aligned}$$

Thus, as we must expect, the wave number (p, q) is nearly parallel to the wave number (k, l); consequently we must use coordinates based on the

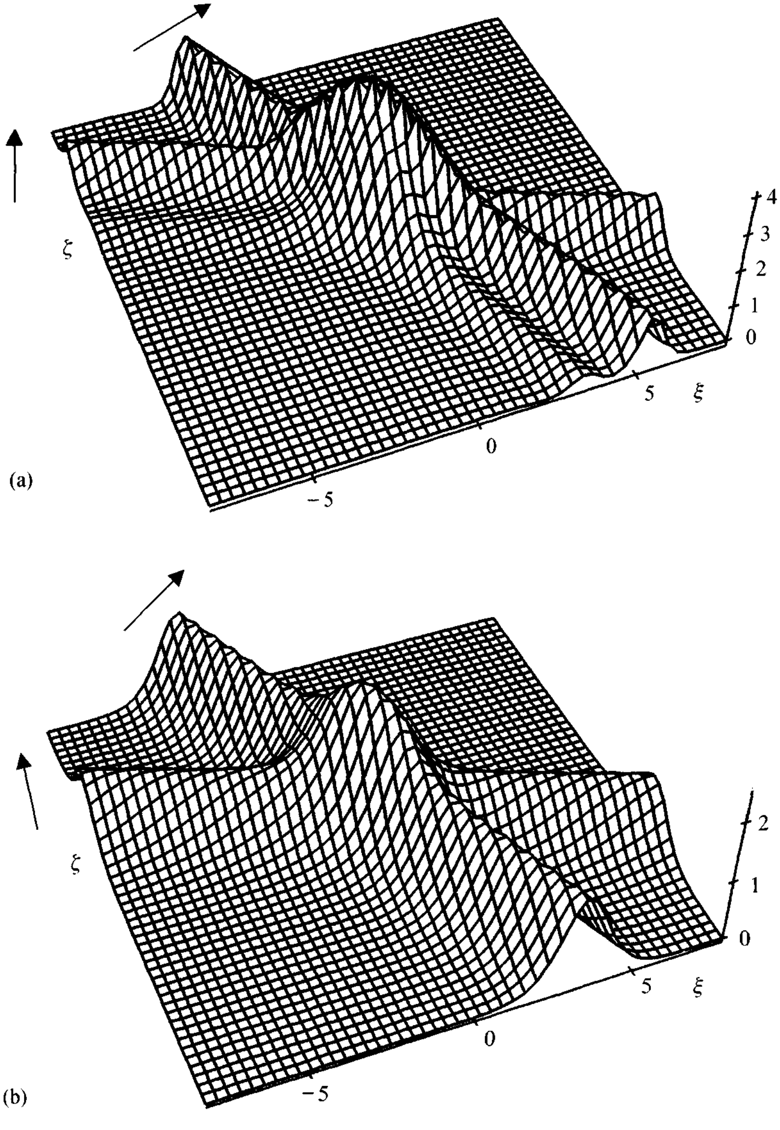


Figure 3.13. The oblique interaction of two solitary waves, for the case of a *weak* interaction, for amplitudes 1 and 1.5, with $\theta = 3\pi/8$ and $\psi = \pi/8$; (a) $\varepsilon = 0.2$, (b) $\varepsilon = 0$, no interaction.

wave number (k, l), and on $(-l, k)$ – this latter suitably scaled. Indeed, since $\lambda = O(\alpha^2)$ and the non-uniformity arises for $\lambda = O(\varepsilon)$, the relevant scaling is $\sqrt{\varepsilon}$; further, since $(-l, k)$ is perpendicular to (k, l), the configuration that we are led to is equivalent to that employed in the derivation of the 2D KdV equation (Section 3.2.2).

We introduce

$$\xi = kx + ly - t, \quad \zeta = \sqrt{\varepsilon}(-lk + ky), \quad \tau = \varepsilon t;$$

cf. equations (3.157) and (3.160), where ζ is now a (scaled) coordinate perpendicular to the characteristic coordinate, ξ. The equation for $f = f(\xi, \zeta, \tau; \varepsilon)$, obtained from (3.156), is therefore

$$-(1 + \varepsilon f_\xi)(f_{\xi\xi} + \varepsilon f_{\zeta\zeta}) + \frac{\varepsilon}{6} f_{\xi\xi\xi\xi} + f_{\xi\xi} - 2\varepsilon f_{\xi\tau} - \frac{\varepsilon}{2} f_{\xi\xi\xi\xi} - 2\varepsilon f_\xi f_{\xi\xi} = O(\varepsilon^2),$$

which simplifies to give

$$2f_{\xi\tau} + 3f_\xi f_{\xi\xi} + \frac{1}{3} f_{\xi\xi\xi\xi} + f_{\zeta\zeta} = O(\varepsilon).$$

And so, finally, with $\eta \sim -f_t \sim f_\xi$ (see equation (3.155)), we obtain for the surface wave

$$(2\eta_\tau + 3\eta\eta_\xi + \frac{1}{3}\eta_{\xi\xi\xi})_\xi + \eta_{\zeta\zeta} = 0,$$

to leading order: the 2D Korteweg–de Vries equation (Section 3.2.2, equation (3.30)). All that we have written about this equation is now applicable here.

In our earlier discussion of the 2D KdV equation (Section 3.2.2), we were guided by the requirement to obtain a KdV-type equation which incorporated some (weak) dependence on a transverse coordinate. This was a purely 'theoretical' exercise, whose success rested on a very special and precise choice for the way in which y (here, ζ) appears in the equation. What we have now demonstrated is that the 2D KdV equation arises quite naturally as the appropriate equation for the strong interaction of obliquely crossing waves. The interaction becomes more pronounced (eventually leading to a strongly nonlinear interaction) as the wave configuration is more nearly that of parallel waves. We can interpret this situation as one in which the waves interact over a much larger distance, thereby producing a greater effect – distortion – one upon the other.

This concludes all that we shall present in relation to the Korteweg–de Vries equation and other members of the family. We have demonstrated how the equations can arise in many different situations and, in particular, how they are relevant in configurations that model physical phenomena more closely (such as variable depth and shear flows). There is much that we have not included, not least the extensive breadth and depth of ideas that now constitute soliton theory. The exercises to some extent, and the Further reading more so, enable the interested reader to take many of these ideas much further.

Further reading

There are many texts now available that describe either general or very specific aspects of soliton theory; some of these texts are listed below. The derivation and properties of the various members of the KdV family of equations that arise in water-wave theory are described mainly in research papers; we provide a small selection for the interested reader.

3.2 Two texts that cover some of the ground that we have described here are Infeld & Rowlands (1990) and Debnath (1994). Various derivations and discussions of these equations will be found in Korteweg & de Vries (1895), Kadomtsev & Petviashvili (1970), Miles (1978, 1981) and Johnson (1980).

3.3 A few of the texts that provide most of the essential features of soliton theory are Lamb (1980), Ablowitz & Segur (1981), Dodd *et al.* (1982), Drazin & Johnson (1994) and Ablowitz & Clarkson (1991). The texts by Lamb and Drazin & Johnson, in particular, present an elementary introduction to many of the ideas. More advanced texts, generally touching on deeper issues, are Calogero & Degasperis (1982) and Newell (1985). An excellent introduction to the ideas, coupled with a description of some simple experiments, is provided by Remoissenet (1994). In addition there are publications that describe specific topics in soliton theory: Rogers & Shadwick (1982) for Bäcklund transformations; Matsuno (1984) for the bilinear transform; and Schuur (1986) for the asymptotic structure of soliton solutions. Finally, the broader and deeper concepts that relate soliton theory to Hamiltonian methods are described by Faddeev & Takhtajan (1987) and Dickey (1991).

3.4 Nonlinear waves propagating over shear flows are described by Benjamin (1962) and Freeman & Johnson (1970); the Burns condition is discussed by Thompson (1949), Burns (1953), Velthuizen & van Wijngaarden (1969), Yih (1972), Brotherton-Ratcliffe & Smith (1989) and Johnson (1991). The nature of the critical layer, and particularly the rôle of nonlinearity, is described in Benney & Bergeron (1969), Davis (1969), Haberman (1972) and Varley & Blythe (1983); and the connection between nonlinear wave propagation and the critical layer is examined by Redekopp (1977), Maslowe & Redekopp (1980) and Johnson (1986). A theory for linear and nonlinear ring waves over a shear flow is presented in Johnson (1990).

The problems of waves moving over a variable depth have a long history, starting with Green (1837) and Boussinesq (1871). Some of the more recent papers, with the emphasis on nonlinear wave propagation, are Peregrine (1967), Grimshaw (1970, 1971), Kakutani (1971), Tappert & Zabusky (1971), Johnson (1973, 1994), Leibovich & Randall (1973), Miles (1979) and, most importantly, Knickerbocker & Newell (1980, 1985). The oblique interaction of nonlinear plane waves is described in Miles (1977a,b); the case of a large solitary wave interacting obliquely with a sech^2 wave is discussed in Johnson (1982); see also Tanaka (1993).

Exercises

Q3.1 *Standard KdV equation.* Show that a scaling transformation, $u \to \alpha u$, $x \to \beta x$, $t \to \gamma t$, for non-zero real constants α, β, γ, enables the general KdV equation

$$Au_t + Buu_x + Cu_{xxx} = 0$$

(for real, non-zero, constants A, B, C) to be transformed into

$$u_t - 6uu_x + u_{xxx} = 0.$$

Q3.2 *KdV for left-going waves.* Repeat the calculation given in Section 3.2.1, leading to the KdV equation (3.28), for waves that propagate to the left (cf. Q1.47 and Q1.48).

Q3.3 *KdV with surface tension.* Repeat the calculation given in Section 3.2.1, but retain the surface tension contribution (characterised by the parameter W (or W_e); see equation (1.64)) and derive the corresponding KdV equation. Show that the inclusion of surface tension alters only the coefficient of the third derivative term, that is, the dispersive contribution.

Q3.4 *Higher-order correction to the KdV equation.* Continue the calculation described in Section 3.2.1 to find the equation that defines $\eta_1(\xi, \tau)$. In the case of the travelling-wave solution, where both η_0

and η_1 are functions only of $\xi - c\tau$ (cf. Q1.55), obtain an expression for η_1 in terms of η_0 by seeking a solution $\eta_1 = \eta_0' F(\xi - c\tau)$ (where the prime denotes the derivative with respect to $\xi - c\tau$).

Q3.5 *KdV similarity solution.* Show that the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0$$

possesses a similarity solution of the form $u(x, t) = -(3t)^m F(\eta)$, $\eta = x(3t)^n$, for suitable values of the constants m and n. (The inclusion of the factor 3, and the use of the negative sign, are merely for convenience.) Hence obtain the equation for F:

$$F''' + (6F - \eta)F' - 2F = 0$$

and, by writing $F = \lambda V' - V^2$ with $V = V(\eta)$ and where λ is a constant to be determined, show that

$$V'' - \eta V - 2V^3 = 0$$

after two integrations, provided that V decays sufficiently rapidly as either $\eta \to +\infty$ or $\eta \to -\infty$.

[The equation for $V(\eta)$ is a *Painlevé equation* of the second kind; see Ince (1927). The use of soliton methods enables the Painlevé equations to be solved; see Ablowitz & Clarkson (1991), Drazin & Johnson (1993) and Airault (1979) as an introduction to these ideas.]

Q3.6 *KdV rational solution.* Show that

$$u(x, t) = 6x(x^3 - 24t)/(x^3 + 12t)^2$$

is a solution of the KdV equation given in Q3.5.

[This solution is not particularly useful since it is *singular* on $x^3 + 12t = 0$, although some other soliton equations do have rational solutions that exist everywhere.]

Q3.7 *A cKdV equation.* Follow the calculation described in Section 3.2.3 for the concentric KdV equation, but now use a large time variable $\tau = \varepsilon^6 t/\delta^4$; cf. equation (3.32). Hence obtain the appropriate cKdV equation.

Q3.8 *Solitary-wave solution of the Boussinesq equation.* Obtain the solitary-wave solution of the Boussinesq equation

$$u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} = 0$$

in the form $u(x,t) = a\,\mathrm{sech}^2\{b(x-ct)+\alpha\}$ for suitable relations between the constants a, b, c and α. Show that the wave may propagate in either direction.

Q3.9 *Boussinesq → KdV.* By means of a suitable choice of far-field variables, recover the KdV equation for right-going waves from the Boussinesq equation, (3.41). Repeat this calculation for left-going waves; cf. equation (3.28) and Q3.2.

Q3.10 *Boussinesq → standard Boussinesq.* Use the transformation

$$H = \eta - \varepsilon\eta^2, \quad X = x + \varepsilon \int_{-\infty}^{x} \eta(x', t; \varepsilon)\mathrm{d}x'$$

and thereby obtain equation (3.42) from equation (3.41).

[This transformation is equivalent to writing the equation in a Lagrangian rather than an Eulerian frame.]

Q3.11 *Solitary-wave solution of the 2D KdV equation I.* Show that the 2D KdV equation

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0$$

has the solitary-wave solution $u(x,t) = a\,\mathrm{sech}^2(kx + ly - \omega t + \alpha)$ for suitable relations between the constants a, k, l, ω and α.

Q3.12 *Transformations between ncKdV and 2D KdV equations.* Show that the nearly concentric KdV equation, (3.46), transforms into the 2D KdV equation if we write

$$H = \eta(\zeta, R, Y), \quad \zeta = \xi - \frac{1}{2}R\Theta^2, \quad Y = R\Theta.$$

Conversely, show that the choice

$$\eta = H(\zeta, \tau, \Theta), \quad \zeta = \xi + \frac{1}{2}Y^2/\tau, \quad \Theta = Y/\tau$$

transforms the 2D KdV equation, (3.20), into the ncKdV equation.

Q3.13 *Solution of the ncKdV equation.* Given that $v(x,t,y)$ is a solution of the 2D KdV equation

$$(v_t - 6vv_x + v_{xxx})_x + 3v_{yy} = 0,$$

show that $u(x,t,y) = v(x - y^2t/12, t, yt)$ is a soluton of the ncKdV equation

$$\left(u_t + \frac{u}{2t} - 6uu_x + u_{xxx}\right)_x + \frac{3}{t^2}u_{yy} = 0.$$

[This enables solutions for u, which decay sufficiently rapidly as $(x^2+y^2)^{-1} \to 0$, to be obtained from the solutions for v which satisfy this same condition. However, solutions for u which have a different behaviour at infinity cannot be obtained via this transformation; see Dryuma (1983), Matveev & Salle (1991).]

Q3.14 *Phase shifts for a 2-soliton KdV solution.* The phase shifts exhibited by the soliton behaviour in solution (3.59) can be examined in this fashion: we consider the asymptotic form of the solitary waves that appear as $t \to \pm\infty$. First, for $\xi = x - 16t = \mathrm{O}(1)$ as $t \to \pm\infty$, show that

$$u \sim -8\,\mathrm{sech}^2(2\xi \mp \frac{1}{2}\log 3),$$

and then for $\eta = x - 4t = \mathrm{O}(1)$ as $t \to \pm\infty$, show that

$$u \sim -2\,\mathrm{sech}^2(\eta \pm \frac{1}{2}\log 3);$$

all signs are vertically ordered. Hence deduce that the taller wave moves forward by an amount $x = \frac{1}{2}\log 3$, and the shorter back by $x = \log 3$, relative to where they would have been if moving throughout at constant speed.

Q3.15 *Phase-shifts: general.* Recast the calculation of Q3.14 in order to find the phase shifts for the general 2-soliton solution, (3.58).

Q3.16 *Character of the 2-soliton KdV solution.* Show that a special case of the 2-soliton solution, (3.58), takes the form of a sech^2 pulse at $t = 0$. Further, show that the pulse at $t = 0$ may have either one or two local maxima.

[In this calculation you should first define x so that a symmetric profile occurs at $t = 0$; for further details see Lax (1968).]

Q3.17 *Three-soliton solution of the KdV equation I.* Extend the calculation in Section 3.3.1 (Example 2) to obtain the general 3-soliton solution of the KdV equation. Show that this can be written in the form $u(x, t) = -2(\partial^2/\partial x^2)\log A$, where

$$A = 1 + \sum_{i=1}^{3} E_i + \sum_{<i=1>}^{3} A_{ij}E_iE_j + \prod_{<i=1>}^{3} A_{ij}E_i$$

with $E_i = \exp\{2k_i(x - x_{0i}) - 8k_i^3 t\}$, $A_{ij} = (k_i - k_j)^2/(k_i + k_j)^2$, and $< >$ denotes that j is to be chosen cyclically with respect to i.

Q3.18 *Solitary-wave solution of the 2D KdV equation II.* Use the choice

$$F = \exp\{-(kx + lz) + (k^2 - l^2)y + 4(k^3 + l^3)t + \alpha\}$$

in the Marchenko equation, and hence recover the solitary-wave solution obtained in Q3.11; see Section 3.3.2. (Direct correspondence with Q3.11 requires here that $k + l \to -2k$, $k^2 - l^2 \to 2l$, $k^3 + l^3 = -\omega/2$.)

Q3.19 *Two-soliton solution of the 2D KdV equation I.* See Q3.18; now write F as the sum of two appropriate exponentials and hence derive the two-soliton solution in the form

$$u(x, t) = -2\frac{\partial^2}{\partial x^2}\log(1 + E_1 + E_2 + AE_1E_2)$$

where $E_i = \exp\{-(k_i + l_i)x + (k_i^2 - l_i^2)y + 4(k_i^3 + l_i^3)t + \alpha_i\}$ and

$$A = \frac{(k_1 - k_2)(l_1 - l_2)}{(k_1 + k_2)(l_1 + l_2)}.$$

[This solution describes various configurations of two plane waves that intersect obliquely and suffer a nonlinear interaction; an excellent discussion of these solutions is to be found in Freeman (1980).]

Q3.20 *A 2D Boussinesq equation.* Follow the derivation of the Boussinesq equation (Section 3.2.5), but include the weak y-dependence as required for the two-dimensional KdV equation (Section 3.2.2). Hence show that, correct at $\mathrm{O}(\varepsilon)$, the surface wave satisfies the equation

$$\eta_{tt} - \eta_{xx} - \varepsilon\left\{\frac{1}{2}\eta^2 + \left(\int_{-\infty}^{x} \eta_t \mathrm{d}x\right)^2\right\}_{xx} - \frac{\varepsilon}{3}\eta_{xxxx} + \varepsilon V_{Yt} = \mathrm{O}(\varepsilon^2),$$

where $V_t = -\eta_Y$. Finally, transform and rescale (exactly as in Section 3.2.5) to obtain the 2D Boussinesq equation

$$H_{tt} - H_{XX} + 3(H^2)_{XX} - H_{XXXX} - H_{YY} = 0.$$

Q3.21 *Solitary-wave solution of the 2D Boussinesq equation.* Seek a solution of the equation for $H(X, t, Y)$ given in Q3.20 in the form $H = a\,\mathrm{sech}^2\{kX + lY - \omega t + \alpha\}$ for suitable relations between the constants a, k, l, ω and α. Confirm that your solution is an oblique wave that may propagate in one of two directions.

[This 2D Boussinesq equation is not a completely integrable equation, although it still provides a description of the *head-on* collision of *oblique* waves (cf. Q3.19) and it does possess some interesting properties; see Johnson (1996).]

Q3.22 *Solitary-wave solution of the cKdV equation.* Show that a solution for F of the pair of equations (3.65) is

$$F(x, z; t) = \int_{-\infty}^{\infty} f(st^{1/3}) Ai(x+s) Ai(s+z) \mathrm{d}s,$$

where f is an arbitrary function and Ai is the Airy function. The solitary-wave solution is usually regarded as that solution obtained from the choice $f(\cdot) = k\delta(\cdot)$, where δ is the Dirac delta function and k is a positive constant; construct the solitary-wave solution of the cKdV equation, (3.64).

Q3.23 *A similarity solution of the cKdV equation.* Show that

$$u(x, t) = -\frac{x}{12t} - \frac{2\lambda^2}{t}\operatorname{sech}^2\{\lambda(x + 8\lambda^2)/t^{1/2}\}, \quad t > 0,$$

is a solution of the concentric KdV equation, (3.64), for any real constant λ.

[This solution is undefined on $t = 0$, is not real for $t < 0$ and grows without bound as $|x| \to \infty$ at any fixed t.]

Q3.24 *Bilinear operator.* Prove these identities, where $\mathrm{D}_t^m \mathrm{D}_x^n (a \cdot b)$ is the bilinear operator defined in equation (3.71):

(a) $\mathrm{D}_t^m \mathrm{D}_x^n (a \cdot b) = \mathrm{D}_x^n \mathrm{D}_t^m (a \cdot b)$;

(b) $\mathrm{D}_x^n (a \cdot b) = (-1)^n \mathrm{D}_x^n (b \cdot a)$ and hence that $\mathrm{D}_x^n (a \cdot a) = 0$ for n odd;

(c) $\mathrm{D}_t^m \mathrm{D}_x^n (a \cdot 1) = \mathrm{D}_t^m \mathrm{D}_x^n (1 \cdot a) = \partial^{m+n} a / \partial x^n \partial t^m$ for $m + n$ even;

(d) $\mathrm{D}_t^m \mathrm{D}_x^n \{\exp(\theta_1) \cdot \exp(\theta_2)\} = (\omega_2 - \omega_1)^m (k_1 - k_2)^n \exp(\theta_1 + \theta_2)$ where $\theta_i = k_i x - \omega_i t + \alpha_i$, $i = 1, 2$.

Q3.25 *Three-soliton solution of the KdV equation II.* Use Hirota's bilinear method to find the expression for $f(x, t)$ which generates the 3-soliton solution of the KdV equation.

Q3.26 *Two-dimensional KdV equation.* Show that the bilinear form of the equation

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0$$

is

$$(\mathrm{D}_x\mathrm{D}_t + \mathrm{D}_x^4 + 3\mathrm{D}_y^2)(f \cdot f) = 0,$$

where $u(x, t) = -2(\partial^2/\partial x^2)\log f$.

Q3.27 *Boussinesq equation.* Show that the bilinear form of the equation

$$u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} = 0$$

is

$$(\mathrm{D}_t^2 - \mathrm{D}_x^2 - \mathrm{D}_x^4)(f \cdot f) = 0,$$

where $u(x, t) = -2(\partial^2/\partial x^2)\log f$.

Q3.28 *Concentric KdV equation.* Show that the bilinear form of the equation

$$u_t + \frac{u}{2t} - 6uu_x + u_{xxx} = 0$$

is

$$\left(\mathrm{D}_x\mathrm{D}_t + \mathrm{D}_x^4 + \frac{1}{2t}\frac{\partial}{\partial x}\right)(f \cdot f) = 0,$$

where $u(x, t) = -2(\partial^2/\partial x^2)\log f$ and $(\partial/\partial x)(f \cdot f) = ff_x$.

Q3.29 *Solitary-wave solutions.* Obtain the solitary-wave solutions of the equations given in Q3.26–Q3.28 by seeking appropriate simple solutions of the corresponding bilinear forms.

[Check your answers with those obtained in Q3.11, Q3.18, Q3.8 and Q3.22, and compare the various methods employed.]

Q3.30 *Two-soliton solution of the 2D KdV equation II.* See Q3.26 and Q3.29; obtain the expression for $f(x, t, y)$ from which the two-soliton solution of the 2D KdV can be constructed (cf. Q3.19).

Q3.31 *Two-soliton solution of the Boussinesq equation.* See Q3.27 and Q3.29; obtain the expression for $f(x, t)$ from which the two-soliton solution of the Boussinesq equation can be constructed. Show that your solution admits solitons which travel in either the same or opposite directions.

Q3.32 *A resonant solution of the 2D KdV equation.* The solutions obtained in Q3.30 can be written as

$$f = 1 + E_1 + E_2 + AE_1E_2$$

where $E_i = \exp(\theta_i)$, $\theta_i = k_i x + l_i y - \omega_i t + \alpha_i$ with $\omega_i = k_i^3 + 3l_i^2/k_i$; A is a function of the k_i and $l_i\,(\mathrm{i} = 1, 2)$. Show that this f is a solution even if $A = 0$, and describe this solution by examining $\theta_1 \to -\infty$ with θ_2 fixed; $\theta_2 \to -\infty$ with θ_1 fixed; $\theta_1 \to +\infty$ with $\theta_3 = \theta_1 - \theta_2$ fixed.

Introduce a parameterisation of the dispersion relation $\omega_i = k_i^3 + 3l_i^2/k_i$ in the form

$$k_i = m_i + n_i, \quad l_i = m_i^2 - n_i^2, \quad \omega_i = 4(m_i^3 + n_i^3), \quad i = 1, 2,$$

(cf. Q3.19). Hence show that $A = 0$ if, for example, $m_1 = m_2$. Write $\theta_3 = k_3 x + l_3 y - \omega_3 t + \alpha_3$ and show that, if $m_1 = m_2$, $n_3 = n_2$ and $m_3 = -n_1$, then $\omega_3 = k_3^2 + 3l_3^2/k_3$.

[These definitions of ω_3, k_3 and l_3 (that is, $\omega_3 = \omega_1 - \omega_2$, etc., and $\omega_i, k_i, l_i, i = 1, 2, 3$, satisfying the dispersion relation) are the conditions for a *resonant wave interaction* or *phase-locked waves*; see Miles (1977b), Freeman (1980).]

Q3.33 *Energy conservation law for water waves.* See equations (3.85); multiply the first by u, use the third twice (once for w_z and once for u_x) and then the second (for p_z), and hence derive equation (3.88). Also confirm that $\mathscr{E}$ (given in Section 2.1.2) can be used to obtain (3.89).

Q3.34 *Energy conservation for the KdV equation.* Show that the third conserved quantity for the KdV equation, (3.96), can be deduced from the statement of energy conservation for water waves, (3.89).

Q3.35 *KdV conserved density.* Show that

$$\frac{45}{4}u^4 - 15uu_x^2 + u_{xx}^2$$

is a conserved density of the KdV equation

$$2u_t + 3uu_x + \frac{1}{3}u_{xxx} = 0.$$

Q3.36 *KdV equation: another conserved density.* Show that $xu + 3tu^2$ is a conserved density for the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0.$$

Q3.37 *KdV equation: a 'centre of mass' property.* Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\infty}^{\infty} xu\,\mathrm{d}x\right) = \text{constant},$$

where u satisfies the KdV equation in Q3.36 (provided $u \to 0$ sufficiently rapidly as $|x| \to \infty$). Interpret this result as the conservation of linear momentum of a linear mass distribution with density $u(x, t)$. Further, confirm that this result is consistent with the phase shifts associated with the two-soliton solution (discussed in Q3.14 and Q3.15).

Q3.38 *N-soliton solution and the conserved quantities.* Given that $u(x, t)$ evolves, according to the KdV equation (Q3.36), into an N-soliton solution from a given initial profile $u(x, 0)$, consider the profile at $t = 0$ and the solution as $t \to \infty$; describe how the conserved quantities can be used to determine the amplitudes of the resulting solitons. Use the first two conservation laws, and then the first three, to verify your method for the 2-soliton and 3-soliton solutions, respectively.

[This idea is developed in Berezin & Karpman (1967).]

Q3.39 *cKdV: conserved densities.* Show that

$$xt^{1/2}u + 6t^{3/2}u^2$$

and

$$x^2t^{1/2}u + 12xt^{3/2}u^2 + 48t^{5/2}u^3 + 24t^{5/2}u_x^2$$

are conserved densities of the concentric KdV equation

$$u_t + \frac{u}{2t} - 6uu_x + u_{xxx} = 0.$$

[An interesting observation is that this cKdV equation is, approximately, the KdV equation (Q3.36) for large t with u_t dominating u/t. You may wish to confirm that the coefficients of the dominant terms in the first three conserved densities for the cKdV equation are the conserved densities of the KdV equation.]

Q3.40 *Boussinesq equation: conserved quantities.* Show that

$$\int_{-\infty}^{\infty} HU\mathrm{d}X, \quad \int_{-\infty}^{\infty} U\mathrm{d}t$$

and

$$\int_{-\infty}^{\infty} (H^2 + U^2 - 4H^3 + 2HH_{XX} - H_X^2)\mathrm{d}t$$

are conserved quantities for the Boussinesq equation written in the form

$$U_t + H_X - 3(H^2)_X + H_{XXX} = 0; \quad H_t = -U_X.$$

[See Hirota (1973).]

Q3.41 *Conserved quantities and the N-soliton solution.* Use the results obtained in Q3.40, and described in Section 3.3.4, to show in principle how the amplitudes of the solitons of the Boussinesq equation can be determined from given initial data; see Q3.38. Give an example of the method for the 2-soliton solution. (The solitary-wave solution of the Boussinesq equation is discussed in Q3.8.)

Q3.42 *Shallow water equations: conservation laws I.* Show that

$$\left(\frac{1}{4}u^4 + hu^2 + um_1 + \frac{1}{2}h^2\right)_t + \left(\frac{1}{4}u^5 + hu^3 + u^2m_1 + um_2 + \frac{3}{2}h^2u + hm_1\right)_x + \left\{\left(\frac{1}{4}u^4 + hu^2 + um_1 + m_2 + \frac{3}{2}h^2\right)w\right\}_z = 0$$

is a conservation law for the shallow water equations, (3.101). Hence obtain the corresponding conserved quantity.

Q3.43 *Shallow water equations: conservation laws II.* See Q3.42; show that

$$(tu)_t + \{t(u^2 + h) - xu\}_x + \{(tu - x)w\}_z = 0$$

and

$$\{t(u^2 + h) - xu\}_t + \{t(u^3 + 2h + m_{1x}) - x(u^2 + h)\}_x + \{[t(u^2 + 2h) - xu]w\}_z = 0$$

are also conservation laws.

Q3.44 *Reduction to the classical KdV equations.* Show that the KdV equation for shear flow, (3.128), together with the Burns condition, (3.117), lead to the classical KdV equations ((3.28), Q3.2) for right- and left-going waves, respectively, when $U(z) = 0$.

Q3.45 *KdV equation for linear shear.* Obtain the form of the KdV equation, (3.128), when the shear flow is

$$U(z) = U_0 + (U_1 - U_0)z, \quad 0 \le z \le 1;$$

see (3.113).

Q3.46 *Burns condition.* For the two shear profiles

(a) $U(z) = U_0 + (U_1 - U_0)z$;

(b) $U(z) = U_1(2z - z^2), \quad 0 \leq z \leq 1,$
show that no critical level exists.

[Hint: assume that a critical level does exist, use the definition (3.129) and then show that the *only* solutions are *not* critical.]

Q3.47 *Burns condition with critical level I.* Show that, for $0 < d < 1$, the Burns condition for the model profile

$$U(z) = \begin{cases} U_1, & d \leq z \leq 1 \\ U_1 z/d, & 0 \leq z < d \end{cases}$$

where U_1 is a constant, gives rise to three solutions for c, one of which corresponds to a critical level.

Q3.48 *Burns condition with critical level II.* Show that the conditions described in Q3.47 obtain also for the model profile

$$U(z) = \begin{cases} U_1, & d \leq z \leq 1 \\ U_1(2dz - z^2)/d^2, & 0 \leq z < d. \end{cases}$$

Q3.49 *Generalised Burns condition.* Show that the generalised Burns condition, (3.136), has a solution

$$k(\theta) = a\cos\theta + b(a)\sin\theta,$$

where a is a parameter, and $b(a)$ is to be determined.

Q3.50 *Generalised Burns condition for oblique waves.* Determine the generalised Burns condition, (3.136), for plane oblique waves; that is, $k(\theta) = 1$ and $\theta = \theta_0 =$ constant. In the case $U = U_0$ = constant, find the speed of the wave.

Q3.51 *Generalised Burns condition for linear shear.* Determine $k(\theta)$, using the method of Q3.49, for the case of a linear shear

$$U(z) = U_0 + (U_1 - U_0)z, \quad 0 \leq z \leq 1.$$

[Note: You are advised to make a convenient choice for c; see how we obtained (3.138).]

Q3.52 *Singular solution.* Derive the solution (3.140) from the general solution (3.139), using standard methods.

[Note: These ideas are described in any good text on (ordinary) differential equations, for example Forsyth (1921) or Piaggio (1933).]

Q3.53 *Variable coefficients → cKdV.* Show that the variable coefficient KdV equation, (3.148), transforms to the concentric KdV equation, (3.34), for H, where

$$\eta_0 = D^2 H\left(\int \sqrt{D}\,\mathrm{d}X, \xi\right),$$

provided that a special choice of $D(X)$ is made. What is this $D(X)$?

Q3.54 *Conservation laws.* Show that the variable coefficient KdV equation, with $D = D(X)$,

$$2H_{0X} + 3D^{-7/4}H_0H_{0\xi} + \frac{1}{3}D^{1/2}H_{0\xi\xi\xi} = 0$$

has a conserved density H_0^2. Also investigate the form of the next equation in this sequence (which involves $(H_0^3)_X$; cf. equation (3.96)).

Q3.55 *Variable depth solitary wave.* Obtain the most general solitary-wave solution of the equation given in Q3.54, where D is treated as a variable parameter. Now impose the conservation law associated with H_0^2 (also in Q3.54) and hence obtain the form of $\eta_0 = D^{-1/4}H_0$; see equation (3.150).

Q3.56 *Oblique plane wave.* Obtain, at leading order as $\varepsilon \to 0$, the KdV equation, (3.158), from equations (3.155) and (3.156), by seeking a solution which is a function of $\xi = kx + ly - t,\ \tau = \varepsilon t$.

Q3.57 *Oblique waves: weak interaction.* Obtain the expression for the surface wave, (3.166), correct at $\mathrm{O}(\varepsilon)$, from the solution for f, (3.165).