2

Some classical problems in water-wave theory

Yet let us hence, and find or feel a way
Thro' this blind haze
The Passing of Arthur

The study of problems in water-wave theory, particularly under the umbrella of the linear approximation, goes back over 150 years. In the intervening time, many different problems – and extensions of standard problems – have been discussed by many authors. In a text such as ours, it is necessary to make a selection from this body of classical work; we cannot hope to describe all the various problems, nor all the subtle variants of standard problems. Our intention is, of course, to include the simplest and most fundamental results (such as, for example, the speed of waves over constant depth and the description of particle paths), but otherwise we choose those topics which contain some interesting and relevant mathematics. However, since we shall not present all that some readers might, perhaps, expect or prefer, we endeavour to remedy this by introducing additional examples through the exercises. The sufficiently dedicated reader is therefore directed to the exercises, particularly if a broader spectrum of water-wave theory is desired.

The material here is presented under two separate headings. The first is *linear problems*, where, apart from the elementary aspects mentioned above, we single out those topics that are attractive and which will prove relevant to some of our later discussions. Thus we describe *waves on sloping beaches*, as well as the phenomenon of *edge waves*. We shall also develop some rather general ideas associated with *ray theory*, and apply the results to *variable depth*, *ship waves*, and *waves on currents*. Under the second heading, *nonlinear problems*, we extend the application to waves on a sloping beach in order to include the effects of nonlinearity. We also describe the *Stokes expansion* (which produces higher approximations to the classical linear wave), and introduce the fully nonlinear *solitary wave* – a very famous wave. Other nonlinear waves that we shall describe include the *hydraulic jump* and *bore*, and we shall explain the

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analogy between nonlinear water waves and (nonlinear) *gas dynamics*; this leads us to introduce the notion of *simple waves* and the rôle of the *Riemann invariants*.

I Linear problems

Our hoard is little, but our hearts are great.

The Marriage of Geraint

The linear equations (defined by $\varepsilon \to 0$, keeping all other parameters fixed) have been described, for an inviscid fluid, in Section 1.3.1. These equations, expressed in Cartesian coordinates, are

$$\left.\begin{aligned} &u_t = -p_x; \quad v_t = -p_y; \quad \delta^2 w_t = -p_z; \quad u_x + v_y + w_z = 0, \\ &\text{with} \\ &w = \eta_t \quad \text{and} \quad p = \eta - \delta^2 W(\eta_{xx} + \eta_{yy}) \text{ on } z = 1 \\ &\text{and} \\ &w = ub_x + vb_y \text{ on } z = b. \end{aligned}\right\} \tag{2.1}$$

Correspondingly, written in cylindrical coordinates, these equations become

$$\left.\begin{aligned} &u_t = -p_r; \quad v_t = -\frac{1}{r}p_\theta; \quad \delta^2 w_t = -p_z; \quad \frac{1}{r}(ru)_r + \frac{1}{r}v_\theta + w_z = 0, \\ &\text{with} \\ &w = \eta_t \quad \text{and} \quad p = \eta - \delta^2 W\left(\eta_{rr} + \frac{1}{r}\eta_r + \frac{1}{r^2}\eta_{\theta\theta}\right) \text{ on } z = 1 \\ &\text{and} \\ &w = ub_r + \frac{v}{r}b_\theta \text{ on } z = b(r, \theta). \end{aligned}\right\} \tag{2.2}$$

Most of the problems that we present, however, will be based on rectangular Cartesian geometry.

2.1 Wave propagation for arbitrary depth and wavelength

We consider, first, the simplest problem of all: the propagation of a plane harmonic wave in the x-direction over constant depth. The depth, in nondimensional variables, is $1 - b$ (> 0), but we may choose $b = 0$

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(since the actual depth is subsumed into the length scale in the z-direction; see Section 1.3.1). The governing equations, (2.1), therefore reduce to

$$\left.\begin{aligned} &u_t = -p_x; \quad \delta^2 w_t = -p_z; \quad u_x + w_z = 0 \\ &\text{with} \\ &w = \eta_t, \quad p = \eta - \delta^2 W \eta_{xx} \text{ on } z = 1; \quad w = 0 \text{ on } z = 0. \end{aligned}\right\} \qquad (2.3)$$

The surface wave is described by

$$\eta = A\mathrm{e}^{\mathrm{i}(kx - \omega t)} + \text{c.c.}, \qquad (2.4)$$

where A is a complex constant; this represents a wave whose initial form (at $t = 0$) is

$$\eta = A\mathrm{e}^{\mathrm{i}kx} + \text{c.c.},$$

where k is the (nondimensional) wave number.

It is convenient to write

$$E = \exp\{\mathrm{i}(kx - \omega t)\},$$

and then to seek a solution (upon the suppression of the complex conjugate) in the form

$$u = U(z)E, \quad w = W(z)E, \quad p = P(z)E. \qquad (2.5)$$

To avoid the obvious confusion, the Weber number is rewritten here as W_{e}; the equations (2.3) now give, presented in the same order as above,

$$\frac{\omega}{k} U = P; \quad P' = \mathrm{i}\omega\delta^2 W; \quad W' + \mathrm{i}kU = 0 \qquad (2.6)$$

(where the prime denotes the derivative with respect to z), with

$$W(1) = -\mathrm{i}\omega A; \quad P(1) = (1 + \delta^2 k^2 W_{\mathrm{e}})A; \quad W(0) = 0. \qquad (2.7)$$

From equations (2.6) we see, directly, that

$$W'' = -\mathrm{i}kU' = -\frac{k^2}{\omega} P' = \delta^2 k^2 W,$$

so the general solution for $W(z)$ is

$$W = B\mathrm{e}^{\delta kz} + C\mathrm{e}^{-\delta kz},$$

where B and C are arbitrary constants. The two boundary conditions for $W(z)$ (given in (2.7)) then yield the solution

$$W = -\mathrm{i}\omega A\left(\frac{\sinh \delta kz}{\sinh \delta k}\right). \qquad (2.8)$$

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Also, equations (2.6) show that

$$P(1) = \frac{\omega}{k} U(1) = \frac{\mathrm{i}\omega}{k^2} W'(1)$$

and hence the boundary condition on P (in (2.7)) gives

$$1 + \delta^2 k^2 W_e = \frac{\delta\omega^2}{k} \frac{\cosh \delta k}{\sinh \delta k}$$

or

$$\left(\frac{\omega}{k}\right)^2 = c_p^2 = (1 + \delta^2 k^2 W_e) \frac{\tanh \delta k}{\delta k} \quad (> 0); \tag{2.9}$$

this is the *dispersion relation* for (plane) surface waves and so determines $\omega(k)$ (and hence the phase speed $c_p(k)$).

Thus for waves of any wave number, k, and with the surface tension contribution included, we can find the speed, c_p, of these waves. (We observe that (2.9) is an expression for c_p^2, so it is possible to have propagation both to the right ($c_p > 0$) and to the left ($c_p < 0$), as we would expect.) The dispersion relation is a function of $\delta k = h_0/\Lambda$, where $\Lambda = \lambda/k$ is the (physical) wavelength of the wave initiated at $t = 0$. We may now examine the special cases of $\delta k \to 0$ and $\delta k \to \infty$.

The first case, $\delta k \to 0$, which describes long waves (or shallow water), gives rise to the very simple result

$$c_p^2 \sim 1, \tag{2.10}$$

which, in original physical variables, produces the speeds of propagation

$$c_p \sim \pm\sqrt{gh_0}, \tag{2.11}$$

which is independent of the wave number, and so these waves are *non-dispersive*. (This speed of propagation, $\sqrt{gh_0}$, confirms the choice of scales adopted in Section 1.3.1.) The speeds given by (2.10) are also independent of the Weber number, but directly related to g, so waves that travel at these speeds are called *gravity waves* (see (2.11)). Indeed, the gravity wave describes an oscillatory balance between kinetic and potential energy, in the gravitational field.

On the other hand the limit $\delta k \to \infty$, which describes short waves (or deep water), yields

$$c_p^2 \sim \delta k W_e, \tag{2.12}$$

and waves moving at the speeds obtained from (2.12) are called *capillary waves* (or, sometimes, *ripples*). We comment that our preferred

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terminology is to emphasize the wavelength of the wave rather than the depth of the water (provided that this depth remains finite); we therefore discuss *long waves* or *short waves* as the limiting forms.

Now, if we consider an environment for which it is reasonable to ignore the effects of surface tension altogether (that is, W_e is always negligibly small), equation (2.9) becomes

$$c_p^2 = \frac{\tanh \delta k}{\delta k}, \tag{2.13}$$

for gravity waves of any wavelength. Then for short waves, where $\delta k \to \infty$, we obtain

$$c_p \sim \pm \frac{1}{\sqrt{\delta k}} \quad (\text{or } \pm\sqrt{g\Lambda} \text{ in dimensional variables});$$

this time the speed is not dependent on the depth. These various properties of the dispersion relation, expressed in terms of the phase speed c_p, are shown in Figure 2.1. It is evident that there is a minimum speed of propagation defined by equation (2.9); see Q2.1, Q2.2. Furthermore, at

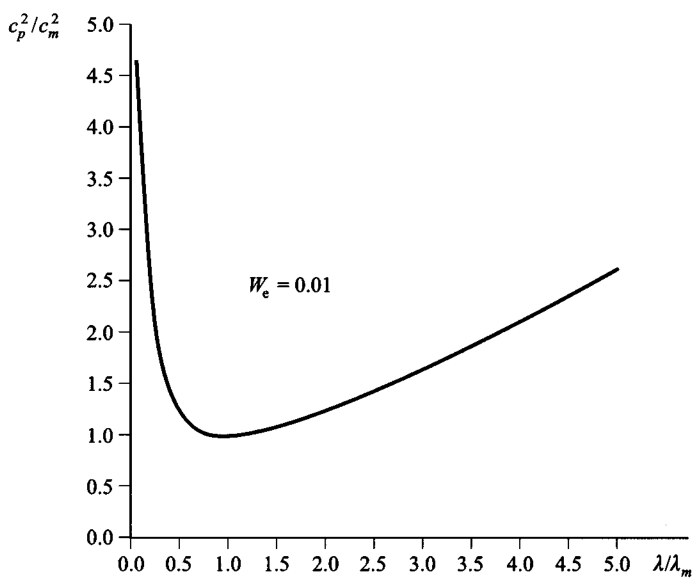

Figure 2.1. The wave speed obtained from equation (2.9), expressed as $(c_p/c_m)^2$ against λ/λ_m, where $\lambda = \delta k$, for W_e (or W) $= 0.01$; the subscript m denotes the value at the minimum point (see Q2.1 and Q2.2).

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any given speed above this minimum, two waves – a gravity wave and a capillary wave – can coexist at the same speed. This is sometimes observed when capillary waves are seen 'riding on' gravity waves, both moving at essentially the same speed. However, a more dramatic phenomenon occurs if a disturbance is generated in a *moving* stream. Provided that the stream is moving faster than the minimum propagation speed, two sets of standing (stationary) waves can often be observed: one of rather long waves (gravity waves) behind the disturbance, the other of rather short waves (capillary waves) *ahead* of the disturbance; see Figure 2.2. (That some waves can propagate *forward* of the disturbance is, perhaps, rather surprising; this will be explained in due course.) The inclusion of a stream moving at a constant speed (for all x and z) is described in Q2.11.

Corresponding calculations are also possible in cylindrical geometry (and based, therefore, on equations (2.2)). One of the simplest cases arises for long waves ($\delta \to 0$) with $b = 0$; see Q2.17. The surface wave is then described by the classical wave equation, written in cylindrical coordinates

$$\eta_{tt} - \left(\eta_{rr} + \frac{1}{r}\eta_r + \frac{1}{r^2}\eta_{\theta\theta}\right) = 0. \tag{2.14}$$

This equation can be solved by using the conventional method of separation of variables, perhaps coupled with the use of an integral transform; see Q2.18 and Q2.19. Indeed, if we seek a solution for purely concentric waves, $\eta(r, t)$, and make use of the *Hankel transform*

$$\hat{y}(p) = \int_0^\infty r y(r) J_0(pr)\,\mathrm{d}r \quad (p > 0),$$

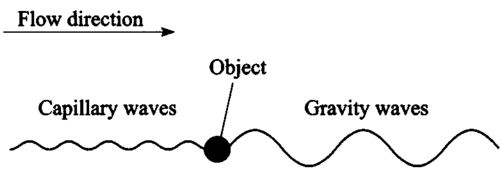

Figure 2.2. Schematic representation of the generation of capillary waves and gravity waves by a fixed object in the surface of a moving stream.

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then the Hankel transform of $\eta(r, t)$ (written as $\hat{\eta}(t; p)$) satisfies

$$\hat{\eta}'' + p^2\hat{\eta} = 0;$$

see Q2.18. (This result does require the introduction of appropriate boundedness and decay conditions.) Then given, at $t = 0$, that

$$\eta = f(r) \quad \text{and} \quad \eta_t = 0,$$

we obtain

$$\hat{\eta} = \hat{f}(p)\cos pt,$$

where $\hat{f}(p)$ is the transform of $f(r)$; thus, using the inverse transform, we obtain

$$\eta(r, t) = \int_0^\infty p\hat{f}(p)\cos(tp)\, J_0(rp)\mathrm{d}p.$$

This type of solution, suitably adjusted for deep water (see Q2.19), will provide the basis for a brief description of the propagation of concentric waves in Section 2.1.3. (We comment that some authors prefer to use the symmetric version of the Hankel transform:

$$\hat{y}(p) = \int_0^\infty (pr)^{1/2} y(r) J_0(pr)\mathrm{d}r.)$$

2.1.1 Particle paths

An important consideration in any wave motion is to find what, if anything, is actually moved (presumably in the direction of propagation) as the wave progresses. This might involve, for example, mass or momentum or energy. In water waves, a first calculation of this type is to find the particle paths that describe the motion of the fluid particles on and below the surface. Then, for example, any motion that occurs near the bottom of the flow will provide the necessary source for the displacement of the sediment (if the bed of the flow is so comprised).

In our simple linear calculation, we have so far determined the vertical velocity component, from (2.8), and the horizontal velocity component (in Q2.3); these are

$$w = -\mathrm{i}\omega A \frac{\sinh \delta kz}{\sinh \delta k} E + \text{c.c.}; \quad u = \delta\omega A \frac{\cosh \delta kz}{\sinh \delta k} E + \text{c.c.},$$

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respectively, where $E = \exp\{i(kx - \omega t)\}$. The particle paths are then defined by

$$\frac{dx}{dt} = \varepsilon u, \quad \frac{dz}{dt} = \varepsilon w, \tag{2.15}$$

and note the inclusion of the parameter ε, required since the particle paths are, in general,

$$\frac{d\mathbf{x}}{dt} = \mathbf{u} \quad \text{and} \quad \mathbf{u} \equiv \varepsilon(u, w) \text{ here.}$$

Thus equations (2.15) describe paths whose amplitude is $O(\varepsilon)$; it is therefore convenient to introduce

$$x = x_0 + \varepsilon X, \quad z = z_0 + \varepsilon Z,$$

where x_0 and z_0 are treated as fixed (and $O(1)$). The particle paths as $\varepsilon \to 0$ – the approximation used throughout this work on linear waves – are now described by

$$\frac{dX}{dt} \sim \delta\omega A \frac{\cosh \delta k z_0}{\sinh \delta k} E_0 + \text{c.c.}; \quad \frac{dZ}{dt} \sim -i\omega A \frac{\sinh \delta k z_0}{\sinh \delta k} E_0 + \text{c.c.},$$

where $E_0 = \exp\{i(kx_0 - \omega t)\}$. These may be integrated directly to give

$$X \sim i\delta A \frac{\cosh \delta k z_0}{\sinh \delta k} E_0 + \text{c.c.}, \quad Z \sim A \frac{\sinh \delta k z_0}{\sinh \delta k} E_0 + \text{c.c.},$$

where the arbitrary constant is set to zero in each case (so that $X = 0$ and $Z = 0$ when $A = 0$). This representation of the particle paths is usefully recast as

$$\left(\frac{X}{\delta \cosh \delta k z_0}\right)^2 + \left(\frac{Z}{\sinh \delta k z_0}\right)^2 = \frac{4|A|^2}{(\sinh \delta k)^2}, \quad 0 < z_0 \le 1, \tag{2.16}$$

to leading order as $\varepsilon \to 0$.

The fluid particles, in the neighbourhood of the point (x_0, z_0), move on ellipses for which

$$\frac{\text{major axis}}{\text{minor axis}} = \delta \coth \delta k z_0,$$

(and which collapse to a point when $A = 0$, as one would expect). For long waves, $\delta \to 0$, the major and minor axes become, respectively,

$$4|A|/k \quad \text{and} \quad 4|A|z_0,$$

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which describe different ellipses at different depths, and which approach the (degenerate) horizontal path as $z_0 \to 0$. On the other hand, for short waves ($\delta \to \infty$), the corresponding results are

$$4\delta|A|\mathrm{e}^{-\delta k(1-z_0)} \quad \text{and} \quad 4|A|\mathrm{e}^{-\delta k(1-z_0)},$$

whose ratio does not vary as z_0 varies. In this case the ellipses are all of the same eccentricity, but of decreasing size as z_0 decreases. (Indeed, in original physical variables, these trajectories become *circles* of decreasing radius as z_0 decreases; see Q2.4.)

We have found, therefore, that (in this first approximation) as the small-amplitude wave propagates on the surface, the fluid particles follow closed paths. Consequently there is no net transfer of material particles due to the passage of the wave (at least, at this order of approximation). In particular, near the bottom of the flow there is, predominantly, a horizontal *oscillatory* motion of the fluid as a long wave propagates overhead. Clearly, there is (at this order) no net flow of matter, but what of energy, for example?

2.1.2 Group velocity and the propagation of energy

We return to our first analysis in which we examined the solution initiated by a pure harmonic wave of fixed amplitude. This time, however, we construct the solution to equations (2.3) with the initial surface profile now given by

$$\eta = A(\alpha x)\mathrm{e}^{\mathrm{i}kx} + \mathrm{c.c.},$$

where A is a complex-valued function. For $\alpha \to 0$, this describes (with k fixed) another pure harmonic wave, but here with a slowly varying amplitude; this is obviously an improvement on our simplest case. (Another generalisation is to allow for many – perhaps all – wave numbers, k; this choice is discussed in Q2.22.) The purpose is to obtain the appropriate solution of equations (2.3) which is uniformly valid as $\alpha \to 0$; see Section 1.4.2. The parameter, δ, is held fixed and, for simplicity, we consider only gravity waves (so the Weber number, W_e, is set to zero); the corresponding calculation for $W_\mathrm{e} \neq 0$ is described in Q2.26.

As before, it is convenient to introduce

$$E = \exp\{\mathrm{i}(kx - \omega t)\},$$

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and then we seek a solution which also depends on the slow scales

$$X = \alpha x, \quad T = \alpha t. \tag{2.17}$$

The inclusion of T is a reasonable manoeuvre, since the solution is to be a wave that propagates in x and t, and the slow space scale ($X = \alpha x$, given in the initial datum) is therefore likely to have an associated slow time scale; of course, we lose nothing by including it (see also Q1.54). We seek a solution in the form

$$u = U(z, X, T; \alpha)E, \quad w = W(z, X, T; \alpha)E, \quad p = P(z, X, T; \alpha)E,$$

with

$$\eta = A(X, T; \alpha)E,$$

plus the complex conjugate in each case. The equations (2.3) yield

$$\left.\begin{aligned} \mathrm{i}\omega U - \alpha U_T = \mathrm{i}kP + \alpha P_X; \quad \delta^2(\mathrm{i}\omega W - \alpha W_T) = P_z; \\ \mathrm{i}kU + \alpha U_X + W_z = 0, \end{aligned}\right\} \tag{2.18}$$

with

$$W(1, X, T; \alpha) = -\mathrm{i}\omega A + \alpha A_T; \quad P(1, X, T; \alpha) = A; \quad W(0, X, T, \alpha) = 0. \tag{2.19}$$

If an appropriate solution of these equations exists (at least, as $\alpha \to 0$), then uniform validity as $|kx - \omega t| \to \infty$ is guaranteed since the complete solution has been constructed with only E^1 (but then E^{-1} as well) included: no higher harmonics and secular terms can be generated (cf. equation (1.106) *et seq.*).

Directly from equations (2.18) we see that

$$\left(\mathrm{i}\omega - \alpha\frac{\partial}{\partial T}\right)U_z = \left(\mathrm{i}k + \alpha\frac{\partial}{\partial X}\right)P_z = \delta^2\left(\mathrm{i}k + \alpha\frac{\partial}{\partial X}\right)\left(\mathrm{i}\omega - \alpha\frac{\partial}{\partial T}\right)W,$$

and the relevant solution here satisfies

$$U_z = \delta^2\left(\mathrm{i}k + \alpha\frac{\partial}{\partial X}\right)W;$$

thus we obtain

$$W_{zz} + \delta^2\left(\mathrm{i}k + \alpha\frac{\partial}{\partial X}\right)^2 W = 0. \tag{2.20}$$

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An asymptotic solution of the system (2.18) and (2.19) is sought in the form

$$Q \sim \sum_{n=0}^{\infty} \alpha^n Q_n, \quad \alpha \to 0. \tag{2.21}$$

where $Q \equiv U$, W, P, or A (and correspondingly for Q_n). Hence, with (2.21) used in (2.20), we obtain the equations

$$W_{0zz} - \delta^2 k^2 W_0 = 0; \quad W_{1zz} - \delta^2 k^2 W_1 = -2\mathrm{i}k\delta^2 W_{0X}, \tag{2.22}$$

and so on. From our previous calculation (Section 2.1), we have immediately that

$$W_0 = -\mathrm{i}\omega A_0 \left(\frac{\sinh \delta kz}{\sinh \delta k} \right), \tag{2.23}$$

where $c_p^2 = (\omega/k)^2 = (\tanh \delta k)/(\delta k)$; see equations (2.8) and (2.13). Now, for W_1, we obtain

$$W_{1zz} - \delta^2 k^2 W_1 = -2k\omega\delta^2 A_{0X} \left(\frac{\sinh \delta kz}{\sinh \delta k} \right) \tag{2.24}$$

which has the solution, for arbitrary $B_1(X, T)$,

$$W_1 = B_1 \sinh \delta kz - \delta\omega A_{0X} \frac{z \cosh \delta kz}{\sinh \delta k}, \tag{2.25}$$

which satisfies $W_1(0, X, T) = 0$. The other two boundary conditions at this order (see (2.19)) are

$$W_1 = -\mathrm{i}\omega A_1 + A_{0T} \quad \text{and} \quad P_1 = A_1 \text{ on } z = 1. \tag{2.26}$$

The first of these yields

$$\mathrm{i}\omega A_1 + A_{0T} = B_1 \sinh \delta k - \delta\omega A_{0X} \coth \delta k, \tag{2.27}$$

and the second uses (from equations (2.18))

$$\mathrm{i}kP_1 + P_{0X} = \mathrm{i}\omega U_1 - U_{0T} \quad \text{and} \quad \mathrm{i}kU_1 + U_{0X} + W_{1z} = 0 \text{ on } z = 1.$$

In Q2.3 we are led to the results

$$P_0 = \frac{\delta\omega^2}{k} A_0 \left(\frac{\cosh \delta kz}{\sinh \delta k} \right) \quad \text{and} \quad U_0 = \delta\omega A_0 \left(\frac{\cosh \delta kz}{\sinh \delta k} \right),$$

and so

$$\mathrm{i}kP_1 + \frac{\delta\omega^2}{k} A_{0X} \coth \delta k = -\delta\omega \left(\frac{\omega}{k} A_{0X} + A_{0T} \right) \coth \delta k - \frac{\omega}{k} W_{1z}$$

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on $z = 1$. Hence from (2.25) and (2.26) we obtain

$$\mathrm{i}kA_1 + \delta\omega\left(2\frac{\omega}{k}A_{0X} + A_{0T}\right)\coth\delta k = -\,\delta\omega B_1\cosh\delta k + \frac{\delta\omega^2}{k}A_{0X}(\delta k + \coth\delta k) \qquad (2.28)$$

and upon the elimination of B_1 between equations (2.27) and (2.28) we finally have

$$A_{0T} + \frac{\omega}{2k}\{1 + \delta k(\coth\delta k - \tanh\delta k)\}A_{0X} = 0. \qquad (2.29)$$

We have derived the equation which describes the variation of the leading-order approximation to the amplitude, A_0; this equation does not involve A_1, because this is eliminated with B_1 when $\omega(k)$ is used. The general solution of (2.29) is

$$A_0 = F(X - c_g T),$$

where F is determined by the initial datum (on $T = 0$) and

$$c_g = \frac{\omega}{2k}\{1 + \delta k(\coth\delta k - \tanh\delta k)\}.$$

It is left as an exercise to confirm that this speed of propagation is indeed the group speed:

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} \quad \text{where} \quad \omega^2 = \frac{k}{\delta}\tanh\delta k.$$

In another context, we provide curves of c_p and c_g (for gravity waves) as functions of δk; see Figure 4.1.

Thus, although the individual waves move forward at the phase speed ($c_p = \omega/k$), the *envelope* or *group* moves at the group speed, c_g. Indeed, this general property of a wave is easily explained by a simple (but heuristic) argument involving two waves of the same amplitude but differing slightly in wave number; see Q2.28. (The reason for our rather lengthier approach, apart from presenting a more careful treatment, is to introduce the techniques that we shall require later. A neater approach, which avoids finding A_1, is described in Q2.30.) The inclusion of the surface tension leads to the corresponding result, but with c_g now the appropriate group speed deduced from the dispersion relation (2.9); see Q2.26.

The connection between the propagation of the group and the propagation of energy is now easily stated. It is a familiar result that the energy in a wave motion is proportional to the square of the amplitude of the wave; here, this implies that the energy is proportional to $|A_0|^2$. But we

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have just demonstrated that A_0 is a function of $(X - c_g T)$, and so the energy propagates at the group speed. There are many ways of presenting this argument in a more precise form, some of which are rehearsed in the exercises; here we describe one such method that uses the general notion of energy, as developed in Section 1.2.5. From equation (1.48), we have that the total energy (per unit horizontal area) in the flow is

$$\mathscr{E} = \int_b^h \left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u} + \rho g z\right) \mathrm{d}z,$$

written in physical variables. This is re-expressed using our nondimensional and scaled variables (described in Section 1.3) as

$$\mathscr{E} = \int_b^{1+\varepsilon\eta} \left\{\frac{1}{2}\varepsilon^2(\mathbf{u}_\perp\cdot\mathbf{u}_\perp + \delta^2 w^2) + z\right\} \mathrm{d}z,$$

where $\mathscr{E} \to \rho g h_0^2 \mathscr{E}$ describes the nondimensionalisation of $\mathscr{E}$. For our simple problem of one-dimensional wave propagation (with $b = 0$), this becomes

$$\mathscr{E} = \int_0^{1+\varepsilon\eta} \left\{\frac{1}{2}\varepsilon^2(u^2 + \delta^2 w^2) + z\right\} \mathrm{d}z,$$

where u and w are given by U_0 and W_0, respectively, to leading order as $\alpha \to 0$.

Our primary concern here is with the energy carried by, let us say, one period of the wave. Thus we first introduce

$$u \sim U_0 E + \bar{U}_0 E^{-1}, \quad w \sim W_0 E + \bar{W}_0 E^{-1},$$

where $E = \exp(\mathrm{i}k\xi)$, $\xi = x - c_p t$ and the overbar denotes the complex conjugate, and then we define the energy carried by just one period of the wave: this is

$$\int_0^{2\pi/k} \mathscr{E}\, \mathrm{d}\xi.$$

Consistent with the linearisation ($\varepsilon \to 0$) that we have so far adopted, we therefore obtain

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$$\int_0^{2\pi/k} \mathscr{E}\,\mathrm{d}\xi$$

$$\sim \int_0^{2\pi/k}\int_0^1 \left\{\frac{1}{2}\varepsilon^2(U_0E+\bar{U}_0E^{-1})^2+\frac{1}{2}\varepsilon^2\delta^2(W_0E+\bar{W}_0E^{-1})^2+z\right\}\mathrm{d}z\,\mathrm{d}\xi,$$

where we have retained both the kinetic and potential contributions to the energy (although these are of different orders of magnitude as $\varepsilon \to 0$). It follows directly that we have

$$\int_0^{2\pi/k} \mathscr{E}\,\mathrm{d}\xi \sim \frac{2\pi}{k}\left\{\frac{1}{2}+\varepsilon^2\delta^2\omega^2|A_0|^2\int_0^1\left[\left(\frac{\cosh\delta kz}{\sinh\delta k}\right)^2+\left(\frac{\sinh\delta kz}{\sinh\delta k}\right)^2\right]\mathrm{d}z\right\},$$

where the term (π/k) represents the potential energy of the undisturbed fluid. (Because of our choice here of computing the energy in one period, this potential energy depends on the wavelength through k; it is quite usual, therefore, to define an *average energy* over one period:

$$\frac{k}{2\pi}\int_0^{2\pi/k} \mathscr{E}\,\mathrm{d}\xi.)$$

The second term is associated with the wave motion alone; because of our scaling, it is proportional to ε^2 (as $\varepsilon \to 0$) – which is to be expected – and it is also proportional to $|A_0|^2$, the required result.

Finally, we briefly describe the particular form that c_g takes for our water-wave problem, and what this implies for the propagation of waves. We already have (from equation (2.9)) the dispersion relation

$$\omega^2 = \left(\frac{k}{\delta}+\delta k^3 W_\mathrm{e}\right)\tanh\delta k.$$

It is left as an exercise (Q2.26) to show that the group speed may be written as

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{1}{2}c_p\left\{\frac{1+3\delta^2k^2W_\mathrm{e}}{1+\delta^2k^2W_\mathrm{e}}+\frac{2\delta k}{\sinh 2\delta k}\right\},$$

where c_p is the phase speed. Then for long waves ($\delta k \to 0$) we see that $c_g \sim c_p$: the phase and group speeds are the same. On the other hand, for short waves ($\delta k \to \infty$), we see immediately that $c_g \sim 3c_p/2$: the group speed is *greater* than the phase speed. For the case of gravity waves only (so that $W_\mathrm{e} = 0$) we have

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$$c_g = \frac{1}{2} c_p (1 + 2\delta k \operatorname{cosech} 2\delta k),$$

and hence $\frac{1}{2} < c_g/c_p < 1$; on the other hand, for infinitely deep water with surface tension (see Q2.27) we obtain

$$c_g = \frac{1}{2} c_p \left\{ \frac{1 + 3\delta^2 k^2 W_e}{1 + \delta^2 k^2 W_e} \right\};$$

that is, $\frac{1}{2} \leq c_g/c_p < \frac{3}{2}$ (where equality occurs when $W_e = 0$). These few observations are sufficient to explain, for example, the phenomenon represented earlier in Figure 2.2. Waves produced by a fixed disturbance in a moving stream can be stationary (provided that the speed of the stream is greater than the minimum speed of propagation of waves). The energy in the gravity component (the left-hand branch in Figure 2.1) is always propagated at a speed *less* than c_p, so these gravity waves appear *behind* the disturbance. The capillary waves, however, always have a group speed which is *greater* than c_p, and consequently the forward propagation of energy for this mode generates these waves *ahead* of the disturbance. (It turns out that the attenuation of gravity waves is much less than that for capillary waves – mainly because of their significantly different wavelengths; see Chapter 5 – so gravity waves are seen to extend much further behind the disturbance than capillary waves are seen ahead.)

2.1.3 Concentric waves on deep water

In Section 2.1 we mentioned some results that can be obtained for wave propagation, which is governed by the classical wave equation written in cylindrical coordinates. It is now our intention to describe the character of purely concentric gravity waves (initiated by a central disturbance) as they propagate over deep water. Of course, corresponding calculations are possible for any depth and with surface tension included, but it is sufficient, both to give a flavour of the results and also for our future work, to examine this one example. We start with the representation of the solution obtained from Q2.19:

$$\eta(r, t) = \int_0^\infty p\hat{f}(p) \cos\left(t\sqrt{\frac{p}{\delta}} \right) J_0(rp)\,\mathrm{d}p, \tag{2.30}$$

which satisfies $\eta(r, 0) = f(r)$ (with transform $\hat{f}(p)$) and $\eta_t(r, 0) = 0$. It is immediately evident that any useful description of the wave profile, η, based on the solution (2.30), requires some approach that will produce a

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simplification. To this end we choose to analyse the solution in the regions where $t^2/r \to \infty$, a choice that will look reasonable when we recast the problem so that we may invoke the method of stationary phase.

First, we express the Bessel function, $J_0(pr)$, in the familiar integral form

$$J_0(pr) = \frac{2}{\pi}\int_0^{\pi/2} \cos(pr\cos\theta)\mathrm{d}\theta, \tag{2.31}$$

and then write (2.30) as

$$\eta = \mathscr{R}\frac{1}{\pi}\int_0^{\infty}\int_0^{\pi/2} p\hat{f}(p)\left[\exp\left\{\mathrm{i}\left(t\sqrt{\frac{p}{\delta}}+pr\cos\theta\right)\right\}\right.$$
$$\left. + \exp\left\{\mathrm{i}\left(t\sqrt{\frac{p}{\delta}}-pr\cos\theta\right)\right\}\right]\mathrm{d}\theta\,\mathrm{d}p, \tag{2.32}$$

where $\mathscr{R}$ denotes the real part of the double integral. To proceed, we introduce a new integration variable (q) and a parameter (σ) defined by

$$q^2 = \frac{\delta r^2}{t^2}p \quad \text{and} \quad \sigma = \frac{t^2}{\delta r}, \tag{2.33}$$

respectively. The expression for the surface wave given in equation (2.32) may therefore be rewritten as

$$\eta = \mathscr{R}\frac{2t^4}{\pi\delta^2 r^4}\int_0^{\infty}\int_0^{\pi/2} q^3\hat{f}\left(\frac{q^2t^2}{\delta r^2}\right)[\exp\{\mathrm{i}\sigma(q+q^2\cos\theta)\}$$
$$+ \exp\{\mathrm{i}\sigma(q-q^2\cos\theta)\}]\mathrm{d}\theta\,\mathrm{d}q, \tag{2.34}$$

and we examine this by employing *Kelvin's method of stationary phase* to give the asymptotic behaviour of $\eta(r,t)$ as $\sigma \to \infty$ (that is, as $t^2/\delta r \to \infty$). This very powerful and widely used result states (see Q2.16) that

$$\phi(\sigma) = \int_{-\infty}^{\infty} f(q)\exp\{\mathrm{i}\sigma\alpha(q)\}\mathrm{d}q$$
$$\sim f(Q)\sqrt{\frac{2\pi}{\sigma|\alpha''(Q)|}}\exp\left\{\mathrm{i}\sigma\alpha(Q)+\mathrm{i}\frac{\pi}{4}\operatorname{sgn}\alpha''(Q)\right\} \tag{2.35}$$

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as $\sigma \to +\infty$, where the point of *stationary phase* is defined by

$$\alpha'(Q) = 0;$$

the primes here denote derivatives with respect to Q, and sgn is the *signum* function (taking the values $+1$ or -1). Essentially the idea is that, for large σ, the integrand oscillates least rapidly near the point (or, perhaps, points) where $\alpha'(q) = 0$, and so this is where the dominant contribution will arise; elsewhere, rapid oscillations approximately cancel, although we might obtain a contribution from the end-points of the range since symmetry about these points is lost. The error in the behaviour given in (2.35) is $O(\sigma^{-1})$, in general, as $\sigma \to \infty$. This result is closely related to the *method of steepest descent*, and some standard references to these types of asymptotic evaluation are given in the section on Further Reading at the end of this chapter. We now apply (2.35) to the double integral (2.34), once in q and once in θ, and to both exponential terms.

First, in q, the points of stationary phase occur where

$$1 + 2q\cos\theta = 0; \quad 1 - 2q\cos\theta = 0,$$

which correspond, respectively, to the two exponental terms. However, since

$$0 \le q < \infty \quad \text{and} \quad 0 \le \theta \le \pi/2,$$

the dominant contribution will come only from the second term in the integral; that is, at

$$q = \frac{1}{2\cos\theta}.$$

The second derivative of the exponent, with respect to q, is then $-2\cos\theta$. Thus we have

$$\eta(r, t) \sim \mathscr{R}\frac{2t^4}{\pi\delta^2 r^4}\int_0^{\pi/2}(2\cos\theta)^{-3}\hat{f}\left(\frac{t^2}{4\delta r^2\cos^2\theta}\right)$$
$$\times\sqrt{\frac{\pi}{\sigma\cos\theta}}\exp\{i(-\pi + \sigma/\cos\theta)/4\}\,d\theta$$

which itself possesses a point of stationary phase where

$$\sin\theta = 0 \quad \text{or} \quad \theta = 0,$$

and so $q = \frac{1}{2}$. Since $\theta = 0$ occurs at the end of the range of integration, the method of stationary phase produces half the contribution represented in (2.35) (which there uses equal contributions from either side of $q = Q$).

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The second derivative of the exponent, evaluated at $\theta = 0$, takes the value $\frac{1}{4}$ and so

$$\eta(r,t) \sim \mathscr{R}\frac{t^4}{8\pi\delta^2 r^4}\hat{f}\left(\frac{t^2}{4\delta r^2}\right)\sqrt{\frac{\pi}{\sigma}}\sqrt{\frac{8\pi}{\sigma}}\mathrm{e}^{\mathrm{i}\sigma/4} \quad (\text{as } \sigma \to \infty)$$

$$= \frac{t^2}{2\sqrt{2}\delta r^3}\hat{f}\left(\frac{t^2}{4\delta r^2}\right)\cos\left(\frac{t^2}{4\delta r}\right), \tag{2.36}$$

which is the asymptotic behaviour as $t^2/\delta r \to \infty$. How can we make use of this result?

Clearly, since the argument of the function $\hat{f}$ involves t^2/r^2, and $t^2/\delta r \to \infty$, we require to know the behaviour of $\hat{f}$ in order to describe $\eta(r,t)$. We consider the simple – and idealised – choice of initial disturbance given by

$$\eta(r,0) = f(r) = \begin{cases} A, & 0 \le r < a \\ 0, & r \ge a, \end{cases}$$

where A is a constant; this describes a 'top-hat' profile which is used here to generate the outward propagating concentric wave. The transform of this function is

$$\hat{f}(p) = A\int_0^a rJ_0(pr)\mathrm{d}r = \frac{A}{p^2}\int_0^{pa} yJ_0(y)\mathrm{d}y$$

$$= \frac{A}{p^2}[yJ_1(y)]_0^{pa} = \frac{Aa}{p}J_1(pa),$$

where a standard identity between the J_0 and J_1 Bessel functions:

$$\frac{\mathrm{d}}{\mathrm{d}y}\{yJ_1(y)\} = yJ_0(y),$$

has been employed. Thus equation (2.36) becomes

$$\eta(r,t) \sim \sqrt{2}\frac{Aa}{r}J_1\left(\frac{at^2}{4\delta r^2}\right)\cos\left(\frac{t^2}{4\delta r}\right),$$

and we choose to interpret the limiting process $t^2/\delta r \to \infty$ as $t \to \infty$ at fixed r (and fixed δ).

Then, upon the use of the asymptotic behaviour

$$J_1(y) \sim \sqrt{\frac{2}{\pi y}}\cos\left(y - \frac{3}{4}\pi\right) \quad \text{as} \quad y \to +\infty,$$

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we find that

$$\eta(r, t) \sim 4\frac{A\delta}{t}\sqrt{\frac{a}{\pi}}\cos\left(\frac{at^2}{4\delta r^2} - \frac{3}{4}\pi\right)\cos\left(\frac{t^2}{4\delta r}\right), \quad t \to \infty.$$

This describes, for example at fixed r, a wave whose amplitude decays like $1/t$ and for which the frequency increases; the general character of the wave is evident in Figure 2.3. This figure clearly demonstrates what is usually observed for cylindrical gravity waves: the wavelength decreases at any fixed radius or, equivalently, the wavelength increases outwards from the centre of the disturbance.

We have, thus far, presented only a rather brief introduction to the many elementary calculations that are possible in simple water-wave theory. As we mentioned earlier, other examples (such as waves on moving streams, standing waves and crossing waves) are explored in the exercises at the end of this chapter. We also take the opportunity there to expand on some of the results already discussed, and to describe alternative approaches to some of the standard problems. We now devote the rest of this section of the chapter to the study of a few slightly less routine calculations, but ones that begin to demonstrate the breadth and depth of what can be done even in the linear approximation. Furthermore, much of this will provide an excellent preparation for our work on the various nonlinear problems that we shall describe later.

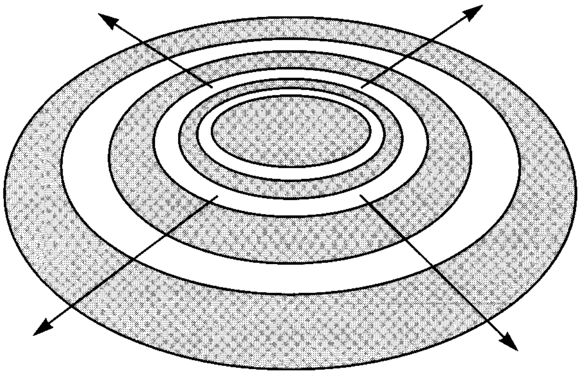

Figure 2.3. A representation of an outward propagating cylindrical (or concentric) gravity wave.

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2.2 Wave propagation over variable depth

It is a matter of everyday experience that most water waves do not propagate over constant depth, whether they be in a man-made reservoir, a river, or the ocean. Therefore a useful extension to our studies is to examine the effects of incorporating variable depth. We start with the most straightforward problem of this type: a plane wave propagating in the x-direction, with a depth which also varies only in x (so that $b = b(x)$). From equations (2.1) we therefore obtain

$$u_t = -p_x, \quad \delta^2 w_t = -p_z, \quad u_x + w_z = 0,$$

with

$$w = \eta_t \quad \text{and} \quad p = \eta - \delta^2 W \eta_{xx} \quad \text{on } z = 1,$$

and

$$w = ub'(x) \quad \text{on } z = b(x).$$

In order to make the problem even more manageable, we simplify further by considering only long waves, so $\delta \to 0$, and then we are left with

$$\left.\begin{aligned} &u_t = -p_x, \quad p_z = 0, \quad u_x + w_z = 0, \\ &\text{with} \\ &w = \eta_t, \quad \text{and} \quad p = \eta \quad \text{on } z = 1, \\ &\text{and} \\ &w = ub'(x) \quad \text{on } z = b(x). \end{aligned}\right\} \tag{2.37}$$

These equations immediately yield

$$p = \eta, \quad b \leq z \leq 1,$$

and so

$$u_t + \eta_x = 0 \quad \text{with} \quad w = (1 - z)u_x + \eta_t,$$

since u is not a function of z (in this approximation). Finally, evaluating w on $z = b$, we obtain the pair of equations

$$u_t + \eta_x = 0; \quad \eta_t + (du)_x = 0, \tag{2.38}$$

where $d(x) = 1 - b(x)$ is the local depth. These equations, (2.38), are usually called the *linearised shallow water* equations and, upon the elimination of u, they give

$$\eta_{tt} - (d\eta_x)_x = 0, \tag{2.39}$$

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the appropriate wave equation for the surface profile, $\eta(x, t)$. In some of our later work we shall examine the full governing equations, but incorporating a *slow* variation of the depth; for constant depth $d = 1$, we note that we recover the classical wave equation with propagation speeds ± 1 (cf. equation (2.10)). For our calculations with variable depth here we choose the example of propagation over a bed of constant slope. Thus we introduce

$$d(x) = \alpha(x_0 - x), \quad \alpha > 0, \quad x \leq x_0, \tag{2.41}$$

where the shoreline is to the right, at $x = x_0$ (in the absence of any surface disturbance); see Figure 2.4.

Before we proceed, however, we must add a word of caution: we cannot expect the results obtained in this calculation to be valid (or even meaningful) either as $d \to 0$ or as $d \to \infty$. Our original equations – the linearised shallow water equations, in particular – have been obtained under the assumption that d $(= 1 - b)$ is $O(1)$ (as $\varepsilon \to 0$ and $\delta \to 0$), and hence $d \to \infty$ is likely to be inadmissible, since this limit corresponds to short waves. Also, in this chapter, we are restricting the discussion to the linear approximation ($\varepsilon \to 0$), which is defined in terms of the ratio of a typical wave amplitude to a typical depth. At the shoreline, the depth decreases to zero, and so the (local) value of (amplitude/depth) will become large; this suggests that nonlinear terms cannot be

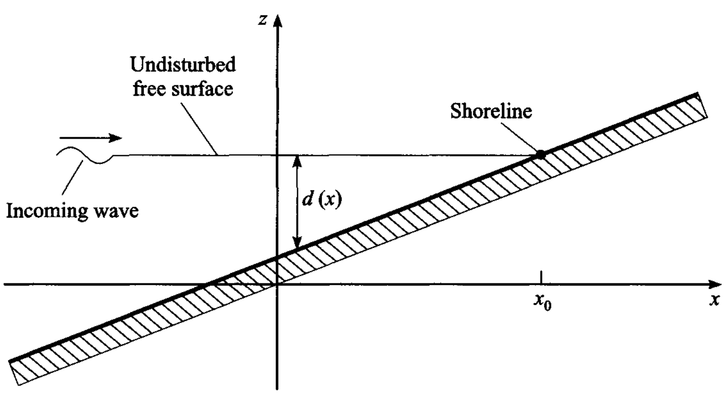

Figure 2.4. Defining sketch for a bed of constant slope; the shoreline is at $x = x_0$, and $d(x)$ is the depth below the undisturbed surface in $x < x_0$. The incoming wave from infinity moves from left to right.

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ignored as $d \to 0$. With these provisos in mind, let us proceed with the analysis.

The wave equation for $\eta(x, t)$, (2.40), with $d(x)$ given by (2.41), becomes

$$\eta_{tt} - \alpha(x_0 - x)\eta_{xx} + \alpha\eta_x = 0, \tag{2.42}$$

and we seek a solution which is harmonic in t:

$$\eta = A(x)\mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.}, \tag{2.43}$$

where ω is a real constant (the frequency) and $A(x)$ is an amplitude function (which, in general, is complex). (As we have mentioned before, this type of solution can be used as the basis for more general solutions by introducing, for example, the Fourier transform.) Equation (2.42) therefore yields the differential equation for $A(x)$,

$$\alpha(x_0 - x)A'' - \alpha A' + \omega^2 A = 0,$$

which shows that we may take $A(x)$ to be real (but see below). It is convenient to treat A as a function of $x_0 - x = X$, say, so that

$$\alpha X A'' + \alpha A' + \omega^2 A = 0,$$

for $A(X)$, which we recognise is related to the Bessel equation. To confirm this, we now regard A as a function of

$$2\omega\sqrt{\frac{X}{\alpha}} = \chi,$$

and so we obtain

$$\chi A'' + A' + \chi A = 0, \tag{2.44}$$

where $A = A(\chi) = A(2\omega\sqrt{(x_0 - x)/\alpha})$. We observe that the shoreline is at $X = 0$ (so $\chi = 0$ there) and that the undisturbed water exists in $X > 0$. Equation (2.44) is the Bessel equation of zero order, and we now require the appropriate solution in $\chi > 0$.

The general solution for $A(\chi)$ is

$$A(\chi) = CJ_0(\chi) + DY_0(\chi),$$

where C and D are arbitrary (complex) constants. This solution contains a contribution (J_0) which is regular at the shoreline ($x = x_0$; that is, $\chi = 0$) since $J_0(\chi)$, as a power series, contains only even powers of χ (so only integer powers of $(x_0 - x)$ appear). On the other hand, the second part of the solution (Y_0) gives rise to a logarithmic singularity at $\chi = 0$, so we might expect that we should assign $D = 0$; we shall,

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however, retain this term for the present. The solution of our original equation, (2.42), therefore becomes

$$\eta(x,t)=\left\{CJ_0\left(2\omega\sqrt{\frac{x_0-x}{\alpha}}\right)+DY_0\left(2\omega\sqrt{\frac{x_0-x}{\alpha}}\right)\right\}\mathrm{e}^{-\mathrm{i}\omega t}+\text{c.c.}, \qquad (2.45)$$

and a natural next move is to examine the detailed character of this solution far away from the shore ($x_0 - x \to \infty$) and close to the shoreline at $x = x_0$. (We observe that, in equation (2.45), it is quite acceptable to choose both C and D to be real, as we mentioned earlier.)

First, for large values of the argument χ, we use the standard results

$$(J_0, Y_0)\sim\sqrt{\frac{2}{\pi\chi}}(\cos(\chi-\pi/4), \sin(\chi-\pi/4)) \quad \text{as } \chi\to+\infty,$$

and thus equation (2.45) yields

$$\eta(x,t)\sim\frac{1}{\sqrt{\pi\omega}}\left(\frac{\alpha}{x_0-x}\right)^{1/4}\left\{C\cos\left(2\omega\sqrt{\frac{x_0-x}{\alpha}}-\frac{\pi}{4}\right)\right.$$
$$\left.+D\sin\left(2\omega\sqrt{\frac{x_0-x}{\alpha}}-\frac{\pi}{4}\right)\right\}\mathrm{e}^{-\mathrm{i}\omega t}+\text{c.c.}$$

as $x_0 - x \to +\infty$. This is usefully rewritten in the form

$$\eta(x,t)\sim\frac{1}{2\sqrt{\pi\omega}}\left(\frac{\alpha}{x_0-x}\right)^{1/4}\left[(C+D)\exp\left\{\mathrm{i}\left(2\omega\sqrt{\frac{x_0-x}{\alpha}}-\omega t-\frac{\pi}{4}\right)\right\}\right.$$
$$\left.+(C-D)\exp\left\{-\mathrm{i}\left(2\omega\sqrt{\frac{x_0-x}{\alpha}}+\omega t-\frac{\pi}{4}\right)\right\}\right]+\text{c.c.} \qquad (2.46)$$

which describes two wave components, one moving to the right and one to the left. The first exponential term (with the coefficient $C + D$) is a right-going wave; it is therefore the plane wave which is approaching the shoreline (see Figure 2.4). The second exponential term represents the left-going component, and this is therefore a wave which is *reflected from* the shoreline. In both components we observe that the speed of propagation is not constant. The speed can be determined by considering the lines of constant phase, defined by

$$2\omega\sqrt{\frac{x_0-x}{\alpha}}\pm\omega t=\text{constant}.$$

Along these lines we have

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\pm\sqrt{\alpha(x_0-x)} \quad (=\pm\sqrt{d(x)}), \qquad (2.47)$$

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which shows that the characteristics for this propagation (drawn in (x, t)-space) are not straight lines; see Q2.33. We note, in passing, that the speed of propagation is the square root of the local depth (see equation (2.47)), which Q2.33 demonstrates is a general result. Furthermore, the wave decays as $x_0 - x \to \infty$; indeed, we see that the amplitude behaves like $(x_0 - x)^{-1/4}$ or $d^{-1/4}$ – a very famous result that we shall meet again later. (This is usually called *Green's law*; see Q2.34.) Finally, if we write the phase in the form

$$\frac{2\omega}{\sqrt{\alpha(x_0 - x)}}(x_0 - x) \pm \omega t,$$

we see that the wave number increases as $x \to x_0$, so the waves shorten as they approach the shore.

Near the shoreline $(x \to x_0)$ we make use of the familiar results

$$(J_0, Y_0) \sim (1, \frac{2}{\pi}\ln\chi) \quad \text{as } \chi \to 0^+,$$

and then equation (2.45) gives

$$\eta(x, t) \sim \left\{C + \frac{2D}{\pi}\ln\left(2\omega\sqrt{\frac{x_0 - x}{\alpha}}\right)\right\}\mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.}$$

as $x_0 - x \to 0^+$. As already mentioned, this exhibits the logarithmic behaviour at the shoreline; this singularity is removable only if $D = 0$ (and then the solution depends only on J_0). Apart from the presence of the singularity, the solution describes a wave which oscillates in time (t) at the shoreline $(x = x_0)$.

We may now collect together these various observations and hence describe the general nature of the solution (2.45) and, in particular, adumbrate its shortcomings. A reasonable problem in this context, we might suppose, is to prescribe an incoming wave at infinity which then moves towards the shoreline. To do this we must know the frequency of the wave and its amplitude; the frequency is no problem (it is ω), but at infinity its amplitude is zero – not what we want. This difficulty is associated with the inability of our shallow-water equations to describe accurately the effects of deep water – which is no surprise in view of the usual name for these equations! Furthermore, even if we are prepared to gloss over this problem, a reflected wave will also exist for all $x_0 - x$ unless we set $C = D\ (\neq 0)$; see equation (2.46). But now the coefficient of Y_0 is non-zero, so we have a singularity at the shoreline. Of course, the presence of this singularity is presumably indicative of the failure of the linear

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equations to cope with the increase in amplitude near the shoreline. We would expect, based on everyday experience, that the incoming wave will (almost always) break at the shoreline; a linear wave theory cannot accommodate this phenomenon. If we do set $D = 0$ then the singularity is removed, but both incoming and reflected waves exist everywhere in $\chi > 0$; the shoreline has become a perfect reflector: no singularity is necessary in this solution to account for the difference in energy between the incoming and outgoing waves.

2.2.1 *Linearised gravity waves of any wave number moving over a constant slope*

In the previous calculation we simplified the problem by considering only long waves, so $\delta \to 0$; this led us to a form of the so-called shallow water equations. As we have seen, the solution in this case is not wholly satisfactory. We now consider the problem of plane gravity waves (as above) *without* invoking the long-wave assumption. Of course, we are still operating within the confines of the linear theory, so again we cannot expect to be able to cope with the singularity at the shoreline (unless, perhaps, we happen upon a special pure-reflecting solution, similar to that described earlier).

We take equations (2.1), with $W_e = 0$, $\eta = \eta(x, t)$ and $b = b(x)$; these are

$$\left.\begin{aligned} &u_t = -p_x; \quad \delta^2 w_t = -p_z; \quad u_x + w_z = 0, \\ &\text{with} \\ &w = \eta_t \quad \text{and} \quad p = \eta \quad \text{on } z = 1, \\ &\text{and} \\ &w = ub'(x) \quad \text{on } z = b(x). \end{aligned}\right\} \qquad (2.48)$$

Again, for this particular calculation (with a constant slope), we require a depth variation which is linear in x but, for convenience, we translate the coordinates so that the shoreline is now along $x = 1/\alpha$, and so we write

$$d(x) = 1 - b(x) = 1 - \alpha x, \quad \alpha > 0, \quad x \leq 1/\alpha;$$

that is, $\alpha x_0 = 1$ in (2.41). We seek solutions (cf. equations (2.4), (2.5)) in the form

$$u = U(x, z)e^{-i\omega t}, \quad p = P(x, z)e^{-i\omega t}, \quad w = W(x, z)e^{-i\omega t}$$

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with

$$\eta(x, t) = A(x)e^{-i\omega t},$$

plus the complex conjugate in each case. Equations (2.48) then become

$$i\omega U = P_x; \quad i\omega\delta^2 W = P_z; \quad U_x + W_z = 0, \tag{2.49}$$

with

$$W(x, 1) = -i\omega A; \quad P(x, 1) = A, \tag{2.50}$$

and

$$W(x, b(x)) = \alpha U(x, b(x)) \text{ where } b(x) = \alpha x. \tag{2.51}$$

We see immediately that

$$W_{zz} + U_{xz} = 0 \quad \text{and} \quad i\omega U_{xz} = P_{xxz} = i\omega\delta^2 W_{xx},$$

so $W(x, z)$ satisfies Laplace's equation

$$W_{zz} + \delta^2 W_{xx} = 0; \tag{2.52}$$

this corresponds directly to the alternative formulation of this problem in terms of the velocity potential ϕ (which then satisfies the same Laplace equation; see Q2.5).

On the basis of our previous experience (again see Q2.5), we seek a solution by the method of separation of variables:

$$W(x, z) = \sum_n X_n(x)Z_n(z). \tag{2.53}$$

Then, for each n, equation (2.52) is replaced by the pair of equations

$$Z_n'' - \lambda_n\delta^2 Z_n = 0; \quad X_n'' + \lambda_n X_n = 0, \tag{2.54}$$

where λ_n is a parameter (an *eigenvalue*) yet to be determined. Further, also based on our earlier work (see, for example, Section 2.1), a reasonable choice for λ_n is

$$\lambda_n = k_n^2 \ (> 0)$$

and then the solution for Z_n becomes

$$Z_n = C_n \exp(\delta k_n z) + D_n \exp(-\delta k_n z),$$

where C_n and D_n are arbitrary constants. However, the depth increases indefinitely (as $x \to -\infty$ where $z \to -\infty$), so a bounded solution is possible only if $D_n = 0$. The complete solution for the nth component of W, W_n say, is therefore

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$$W_n(x, z) = \{A_n \exp(\mathrm{i}k_n x) + B_n \exp(-\mathrm{i}k_n x)\} \exp(\delta k_n z),$$

where C_n has been subsumed into the new arbitrary (complex) constants A_n and B_n.

If, for a moment, we retain only one term in the expansion (2.53) then we find (by integrating P_z in equations (2.49) with respect to z) that

$$P(x, z) = \frac{\mathrm{i}\omega\delta}{k_n} \{A_n \exp(\mathrm{i}k_n x) + B_n(-\mathrm{i}k_n x)\}\{\exp(\delta k_n z) - \exp(\delta k_n)\} + A(x). \tag{2.55}$$

The two boundary conditions on W require

$$\left.\begin{aligned} &\{A_n \exp(\mathrm{i}k_n x) + B_n \exp(-\mathrm{i}k_n x)\} \exp(\delta k_n) = -\mathrm{i}\omega A \\ &\text{and} \\ &W(x, \alpha x) = \alpha U(x, \alpha x) = -\frac{\mathrm{i}\alpha}{\omega} P_x(x, \alpha x), \end{aligned}\right\} \tag{2.56}$$

where the partial derivative with respect to the first argument in $P(x, z)$ is implied, and P is given by (2.55). It should be clear that, upon eliminating $A(x)$ between the two equations in (2.56), we shall derive an identity involving terms

$$\exp(\mathrm{i}k_n x), \quad \exp(-\mathrm{i}k_n x) \quad \text{and} \quad \exp(\alpha\delta k_n x)$$

which cannot possibly be satisfied unless $A_n = B_n = 0$; we have apparently reached an impasse with this approach. However Hanson (1926), and others after him, made use of an important observation which allows some progress, at least for certain α.

At the heart of this discovery is the realisation that we might hope to satisfy all the boundary conditions if there can be arranged some appropriate symmetry between the x and z dependencies. In particular we seek a symmetry that will allow terms in x and z, when evaluated on the bottom ($z = b(x) = \alpha x$), to become essentially the same (but only for certain α). To demonstrate this idea, let us consider the simplest example of this type by using just two terms in the expansion (2.53). We write (cf. (2.54)), for $k > 0$,

$$Z_1'' - \delta^2 k^2 Z_1 = 0; \quad X_1'' + k^2 X_1 = 0,$$

so that we obtain the solution

$$W_1 = (A_1 \mathrm{e}^{\mathrm{i}kx} + B_1 \mathrm{e}^{-\mathrm{i}kx}) \mathrm{e}^{\delta kz},$$

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which is bounded as $z \to -\infty$. For the second component we write

$$Z_2'' + \delta^2 k^2 Z_2 = 0; \quad X_2'' - k^2 X_2 = 0,$$

and now we obtain

$$W_2 = (A_2 e^{i\delta k z} + B_2 e^{-i\delta k z}) e^{kx},$$

which is bounded as $x \to -\infty$. (Remember that the solution we seek is to be in $x \leq 1/\alpha$.) The solution

$$W = W_1 + W_2$$

therefore incorporates oscillatory structures in both x and z, which both decay (as $z \to -\infty$ and $x \to -\infty$, respectively).

As before, we first determine $P(x, z)$; this gives directly (as above, from equations (2.49))

$$P(x, z) = \frac{i\omega\delta}{k}(A_1 e^{ikx} + B_1 e^{-ikx})(e^{\delta k z} - e^{\delta k}) \\ + \frac{\omega\delta}{k}\{A_2(e^{i\delta k z} - e^{i\delta k}) - B_2(e^{-i\delta k z} - e^{-i\delta k})\} e^{kx} + A(x).$$

The boundary conditions on W yield (from equations (2.50))

$$-i\omega A(x) = (A_1 e^{ikx} + B_1 e^{-ikx}) e^{\delta k} + (A_2 e^{i\delta k} + B_2 e^{-i\delta k}) e^{kx}, \tag{2.57}$$

and from equations (2.49) and (2.51):

$$W(x, \alpha x) = \alpha U(x, \alpha x) = -\frac{i\alpha}{\omega} P_x(x, \alpha x)$$

that is,

$$(A_1 e^{ikx} + B_1 e^{-ikx}) e^{\alpha\delta k x} + (A_2 e^{i\alpha\delta k x} + B_2 e^{-i\alpha\delta k x}) e^{kx} \\ = i\alpha\delta(A_1 e^{ikx} - B_1 e^{-ikx})(e^{\alpha\delta k x} - e^{\delta k}) \\ - i\alpha\delta\{A_2(e^{i\alpha\delta k x} - e^{i\delta k}) - B_2(e^{-i\alpha\delta k x} - e^{-i\delta k})\} e^{kx} - \frac{i\alpha}{\omega} A'(x). \tag{2.58}$$

Finally, $A'(x)$ is obtained from equation (2.57). However, it is already clear that a consistent identity (for other than $A_1 = B_1 = A_2 = B_2 = 0$) is possible only if equation (2.58) involves terms $\exp(\pm ikx)$ and $\exp(kx)$, at most. This condition is satisfied if the slope of the bottom is such that $\alpha\delta = 1$. With this choice, then equation (2.58) (with A' from (2.57)) becomes

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$$
\begin{aligned}
(A_1E + B_1E^{-1} + A_2E + B_2E^{-1})\mathrm{e}^{kx} \\
&= \mathrm{i}(A_1E - B_1E^{-1})(\mathrm{e}^{kx} - \mathrm{e}^{\delta k}) - \mathrm{i}\{A_2(E - \mathrm{e}^{\mathrm{i}\delta k}) - B_2(E^{-1} - \mathrm{e}^{-\mathrm{i}\delta k})\}\mathrm{e}^{kx} \\
&\quad + \frac{\mathrm{i}k}{\delta\omega^2}(A_1E - B_1E^{-1})\mathrm{e}^{\delta k} + \frac{k}{\delta\omega^2}(A_2\mathrm{e}^{\mathrm{i}\delta k} + B_2\mathrm{e}^{-\mathrm{i}\delta k})\mathrm{e}^{kx}, \quad (2.59)
\end{aligned}
$$

where we have written $E = \exp(\mathrm{i}kx)$.

In order that equation (2.59) is an identity for arbitrary x, we require the following five equations to hold (each arising from the coefficient of the term given to the left); these are

$$
\begin{aligned}
&E\mathrm{e}^{kx}: && A_1 + A_2 = \mathrm{i}(A_1 - A_2); \\
&E^{-1}\mathrm{e}^{kx}: && B_1 + B_2 = \mathrm{i}(B_2 - B_1); \\
&E: && iA_1\left(\frac{k}{\delta\omega^2} - 1\right)\mathrm{e}^{\delta k} = 0; \\
&E^{-1}: && \mathrm{i}B_1\left(\frac{k}{\delta\omega^2} - 1\right)\mathrm{e}^{\delta k} = 0; \\
&\mathrm{e}^{kx}: && (1+\mathrm{i})A_2\mathrm{e}^{\mathrm{i}\delta k} = -(1-\mathrm{i})B_2\mathrm{e}^{-\mathrm{i}\delta k},
\end{aligned}
$$

for the five unknowns A_1, A_2, B_1, B_2, and $\omega(k)$. It is evident that the solution of this system is

$$\omega^2 = k/\delta \tag{2.60}$$

together with

$$A_2 = \mathrm{i}A_1, \quad B_2 = -\mathrm{i}B_1, \quad \text{and} \quad B_1 = \mathrm{i}A_1\mathrm{e}^{2\mathrm{i}\delta k},$$

where we recognise (2.60) as the dispersion relation for short gravity waves (or deep water) (see equation (2.13) *et seq.*), giving the speeds of propagation $c_p = \pm 1/\sqrt{\delta k}$. The surface wave, from equation (2.57) with the factor $\exp(-\mathrm{i}\omega t)$ reintroduced, becomes

$$\eta(x, t) = A_0\{\mathrm{e}^{\mathrm{i}(kx-\omega t-\delta k)} + \mathrm{e}^{-\mathrm{i}(kx+\omega t-\delta k)} + (1+\mathrm{i})\mathrm{e}^{k(x-\delta)-\mathrm{i}\omega t}\} + \text{c.c.}, \tag{2.61}$$

where A_0 is an arbitrary (complex) constant, which plays the rôle of the A_1 used earlier. We see that this solution, (2.61), possesses a number of important and special properties. First, the solution is everywhere regular in $x \le 1/\alpha$ ($= \delta$ since $\alpha\delta = 1$): there is no singularity at the shoreline, so there is some sort of perfect reflection here (cf. solution (2.45) with $D = 0$); indeed, at the shoreline ($x = \delta = 1/\alpha$) we have

$$\eta = (3 + \mathrm{i})A_0\mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.}$$

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The incoming and outgoing wave components travel at a fixed speed (for given k), which is certainly at variance with our previous result (equation (2.47)). Finally, the wave at infinity ($x \to -\infty$) exhibits a nonvanishing amplitude in both components. Clearly the two ingredients in this solution which make it particularly distinctive are (a) that it contains a contribution which does *not* represent a travelling wave (the term $\exp\{k(x-\delta) - \mathrm{i}\omega t\}$) and (b) that the amplitudes of the two waves at infinity are nonzero (but proportional; cf. equation (2.46)). Nevertheless, even though we have described a very special – and intriguing – solution of the governing (linear) equations, this does provide the basis for constructing more general and useful solutions (which may be investigated through the references in the Further Reading at the end of this chapter).

As a final comment on this solution, we briefly return to the assumption that made all this possible: the choice of slope with $\alpha\delta = 1$. It is reasonable to ask whether other choices of α lead to similar – or at least analogous – results. In order to describe what can be done, it is convenient first to write the bottom boundary condition

$$w = ub'(x) \quad \text{on} \quad z = b(x) = \alpha x$$

in the form

$$w\delta \cos\beta - u \sin\beta = 0 \quad \text{on} \quad z\delta\cos\beta - x\sin\beta = 0$$

where $\alpha\delta = \tan\beta$. The case that we have presented then corresponds to $\beta = \pi/4$. The generalisation is to angles $\beta = \pi/2n$ ($n = 2, 3, \dots$) where, for increasing n, there is a progressively increasing number of terms in the series (2.53), which are required to ensure that all the boundary conditions are satisfied; see Q2.36.

2.2.2 Edge waves over a constant slope

We now turn to a brief consideration of an altogether new phenomenon: the *edge wave*. It turns out that the linear equations (with or without the long-wave assumption) admit a solution which describes a wave which propagates *parallel* to the shoreline. In our notation, these waves propagate in the y-direction (sometimes called the *longshore* coordinate) and, as we shall demonstrate, their amplitude decays exponentially away from the shoreline (that is, as $x \to -\infty$); they are therefore usually called *trapped waves*.

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We start with equation (2.1) but, as before, our interest is in gravity waves only, and so we set W (that is, W_e) $= 0$; then, with the long-wave assumption $\delta \to 0$ (used here for simplicity), we have

$$\left.\begin{aligned} &u_t = -p_x; \quad v_t = -p_y; \quad p_z = 0; \quad u_x + v_y + w_z = 0, \\ &\text{with} \\ &w = \eta_t \quad \text{and} \quad p = \eta \quad \text{on} \quad z = 1, \\ &\text{and} \\ &w = ub'(x) \quad \text{on} \quad z = b(x). \end{aligned}\right\} \tag{2.62}$$

We choose the same depth variation as used in Section 2.2.1, so $b(x) = \alpha x$ with $x \leq 1/\alpha$, where the undisturbed shoreline is along $x = 1/\alpha$. We seek harmonic waves that are propagating in the y-direction, and thus we set

$$\left.\begin{aligned} &u = U(x,z)E, \quad v = V(x,z)E, \quad w = W(x,z)E \\ &\text{with} \\ &p = P(x,z)E \quad \text{and} \quad \eta = A(x)E, \end{aligned}\right\} \tag{2.63}$$

where $E = \exp\{\mathrm{i}(ly - \omega t)\}$, plus the complex conjugate in each case. Equations (2.62) therefore become

$$\mathrm{i}\omega U = P_x; \quad \omega V = lP; \quad P_z = 0; \quad U_x + \mathrm{i}lV + W_z = 0,$$

with

$$W = -\mathrm{i}\omega A \quad \text{and} \quad P = A \quad \text{on} \quad z = 1,$$

and

$$W = \alpha U \quad \text{on} \quad z = \alpha x.$$

Consequently we have that

$$P(x,z) = A(x), \quad 1 \geq z \geq x, \quad x \leq 1/\alpha$$

and hence

$$U = -\frac{\mathrm{i}}{\omega}A', \quad V = \frac{l}{\omega}A;$$

thus

$$W = \frac{\mathrm{i}}{\omega}(l^2 A - A'')(1 - z) - \mathrm{i}\omega A.$$

The final boundary condition on W then yields

$$(1 - \alpha x)(A'' - l^2 A) - \alpha A' + \omega^2 A = 0$$

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for $A(x)$. It is clearly convenient to regard $A = A(1-\alpha x)$ and then, with $X = 1 - \alpha x$, we have

$$XA'' + A' + \left(\frac{\omega^2}{\alpha^2} - l^2 X\right)A = 0$$

which can be put into a standard form if we now write

$$A(X) = \mathrm{e}^{-lX} L(2lX). \tag{2.64}$$

The equation for $L(Y)$, with $Y = 2lX$, is therefore

$$YL'' + (1-Y)L' + \gamma L = 0, \tag{2.65}$$

where

$$\gamma = \frac{1}{2}\left(\frac{\omega^2}{\alpha^2 l} - 1\right).$$

Now we recognise equation (2.65) as the equation that has as its solutions the *Laguerre polynomials*, $L_n(Y)$, whenever $\gamma = n$ $(n = 0, 1 \ldots)$. These are the only solutions of (2.65) which lead to a bounded solution for $A(X)$ in $x \leq 1/\alpha$; that is, for $X \geq 0$ (with $l > 0$). (In general, $A(X)$ is a linear combination of e^{-lX} and e^{lX} as $X \to \infty$; the Laguerre polynomials are those solutions for which the term e^{lX} is absent.) The dispersion relation for these waves is

$$\omega^2 = \alpha^2 l(2n+1),$$

and we write the solution $L_n(Y)$ in the usual form

$$L_n(Y) = \mathrm{e}^{Y} \frac{\mathrm{d}^n}{\mathrm{d}Y^n}(Y^n \mathrm{e}^{-Y}), \quad n = 0, 1, 2 \ldots.$$

The problem of finding the edge waves has therefore been reduced to a familiar exercise in the theory of eigenmodes and orthogonal polynomials. The first three modes (for $\omega > 0$) are

$$\begin{aligned}
n = 0&: \quad \omega = \alpha\sqrt{l}, \quad L_0 = 1;\\
n = 1&: \quad \omega = \alpha\sqrt{3l}, \quad L_1 = 1 - Y = 1 - 2lX;\\
n = 2&: \quad \omega = \alpha\sqrt{5l}, \quad L_2 = 2 - 4Y + Y^2 = 2 - 8lX + 4l^2X^2,
\end{aligned}$$

and these then lead to surface profiles such as

$$\eta(x,t) = A_0 \mathrm{e}^{-l(1-\alpha x)} \mathrm{e}^{\mathrm{i}(ly - \alpha\sqrt{l}t)} + \text{c.c.} \quad (n = 0),$$

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where A_0 is an arbitrary complex constant. By virtue of the general form exhibited in equation (2.64), where L is a Laguerre polynomial, all these modes decay exponentially as $x \to -\infty$.

To conclude, we make two observations. First, the dispersion relation is quite different from the others that we have encountered so far for gravity waves. We see that the frequency, ω, increases as the wave number (l) increases and, crucially, it also depends on the slope of the bottom (which, remember, is a slope in x – not y). Indeed, this dependence of ω on the slope (α) leads to the second point: if the bottom is flat, so that $\alpha = 0$, then $\omega = 0$ and no edge wave exists at all.

These waves are often generated by wind stresses (due to the passage of a storm, for example) if this disturbance moves parallel to the shoreline. They are of some significance because their largest amplitude occurs at the shoreline, and therefore they will contribute to the total *run-up* (the highest point reached by a wave on a beach).

2.3 Ray theory for a slowly varying environment

Many of the more general properties of water waves, some of which we have mentioned already, can be explored more fully if we examine propagation over a slowly varying depth or current. The restriction to a slowly varying environment – depth or current or both – enables us to exploit an asymptotic formulation without recourse to other assumptions (other than under the present umbrella of linearisation). Not surprisingly, water waves behave in a manner similar to light: the (slowly) varying conditions give rise to changes in wave number and phase speed, and so the waves, as they propagate, generally suffer refraction. It is possible to describe these and other phenomena in some detail; the results are usually collected together as *ray theory* (which is another name for the familiar *theory of geometrical optics*). In our presentation here we shall first describe the effects of a slowly varying depth, and then turn to a new area of study: slowly varying currents.

In contrast to much of our earlier work, we shall develop the theory of linear irrotational motion over a slowly varying depth from the point of view of Laplace's equation. We shall consider here only gravity waves (so we set the Weber number, W, to zero); then from equations (2.1), Q1.38 and Q2.5 we obtain

$$\phi_{zz} + \delta^2(\phi_{xx} + \phi_{yy}) = 0, \tag{2.66}$$

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with

$$\phi_z = \delta^2 \eta_t \quad \text{and} \quad \phi_t + \eta = 0 \quad \text{on} \quad z = 1, \tag{2.67}$$

and

$$\phi_z = \delta^2(\phi_x b_x + \phi_y b_y) \quad \text{on} \quad z = b(x, y). \tag{2.68}$$

(Q2.5 is the most useful guide to these equations, requiring only the addition of the second horizontal coordinate (y) and the variable depth.) It is convenient, first, to reduce the two boundary conditions on $z = 1$ to a single condition. Since these are evaluated on $z = 1$, we may take derivatives (as appropriate) in x, y, or t; in particular we can eliminate η altogether to give

$$\phi_z + \delta^2 \phi_{tt} = 0 \quad \text{on} \quad z = 1. \tag{2.69}$$

We then determine η at the end of the calculation as $(-\phi_t)$ on $z = 1$. Finally, the bottom topography is chosen to be

$$b(x, y) = B(\alpha x, \alpha y)$$

so that equation (2.68) now becomes

$$\phi_z = \alpha\delta^2(\phi_x B_X + \phi_y B_Y) \quad \text{on} \quad z = B(X, Y), \tag{2.70}$$

where $X = \alpha x$, $Y = \alpha y$. The compact form of this problem (equations (2.66), (2.69), and (2.70)), coupled with the asymptotic approach that we introduce below, will confirm the usefulness of the Laplace formulation here.

The analysis that we now present is driven by the choice of depth variation for which $\alpha \to 0$. It is clear that we must use the variables

$$X = \alpha x, \quad Y = \alpha y, \quad T = \alpha t, \tag{2.71}$$

the scaled time (T) being required as we have seen before; cf. Q1.54 and equations (2.17). In addition, we shall need a suitable way of describing the harmonic wave which propagates – albeit with slowly varying parameters – on the O(1) time and space scales. The neatest device in this type of problem is to introduce a (real) phase function, θ, defined by

$$\left.\begin{aligned} &\nabla\theta = \mathbf{k}(X, Y, T), \text{ that is } (\theta_x, \theta_y) = \{k(X, Y, T), l(X, Y, T)\} \\ &\text{with} \\ &\theta_t = -\omega(X, Y, T), \end{aligned}\right\} \tag{2.72}$$

which is precisely the approach adopted in Q1.54. We therefore obtain the transformation

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$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \equiv \alpha\left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}\right) + (k, l)\frac{\partial}{\partial \theta}; \quad \frac{\partial}{\partial t} \equiv \alpha\frac{\partial}{\partial T} - \omega\frac{\partial}{\partial \theta}, \tag{2.73}$$

and their use in equations (2.66), (2.69) and (2.70) yields

$$\phi_{zz} + \delta^2\{(k^2 + l^2)\phi_{\theta\theta} + 2\alpha(k\phi_{\theta X} + l\phi_{\theta Y}) \\ + \alpha(k_x + l_y)\phi_\theta + \alpha^2(\phi_{XX} + \phi_{YY})\} = 0,$$

with

$$\phi_z + \delta^2(\omega^2\phi_{\theta\theta} - 2\alpha\omega\phi_{\theta T} - \alpha\omega_T\phi_\theta + \alpha^2\phi_{TT}) = 0 \text{ on } z = 1;$$

and

$$\phi_z = \alpha\delta^2(kB_X + lB_Y)\phi_\theta + \alpha^2(B_X\phi_X + B_Y\phi_Y) \text{ on } z = B(X, Y),$$

where we now regard $\phi = \phi(\theta, X, Y, T, z; \alpha)$.

The solution that we seek takes the form of a single harmonic wave

$$\phi = a(X, Y, T, z; \alpha)\mathrm{e}^{i\theta} + \text{c.c.},$$

and so the problem for the amplitude function, a, becomes

$$a_{zz} + \delta^2\{-(k^2 + l^2)a + 2\mathrm{i}\alpha(ka_X + la_Y) \\ + \mathrm{i}\alpha(k_X + l_Y)a + \alpha^2(a_{XX} + a_{YY})\} = 0,$$

with

$$a_z + \delta^2(-\omega^2 a - 2\mathrm{i}\alpha\omega a_T - \mathrm{i}\alpha\omega_T a + \alpha^2 a_{TT}) = 0 \text{ on } z = 1,$$

and

$$a_z = \mathrm{i}\alpha\delta^2(kB_X + lB_Y)a + \alpha^2\delta^2(B_X a_X + B_Y a_Y) \text{ on } z = B(X, Y).$$

To proceed, we assume that a can be expressed as the asymptotic expansion

$$a \sim \sum_{n=0}^{\infty} \alpha^n a_n(X, Y, T, z) \quad \text{as } \alpha \to 0,$$

and then the problem for a_0 is simply

$$a_{0zz} - \delta^2(k^2 + l^2)a_0 = 0 \tag{2.74}$$

with

$$a_{0z} = \delta^2\omega^2 a_0 \text{ on } z = 1; \quad a_{0z} = 0 \text{ on } z = B. \tag{2.75}$$

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We write

$$\sigma(X, Y, T) = \delta\sqrt{k^2 + \ell^2} \quad (> 0) \tag{2.76}$$

and then the solution for a_0 is immediately

$$a_0 = A_0 \cosh\{\sigma(z - B)\}, \tag{2.77}$$

where $A_0(X, Y, T)$ is, at this stage, arbitrary, and the dispersion relation is

$$\omega^2 = \frac{\sigma}{\delta^2}\tanh\{\sigma(1 - B)\}; \tag{2.78}$$

cf. equations (2.9) and (2.13), when we set $\sigma = \delta k$ (that is, $l = 0$) and $B = 0$ (constant depth).

The problem for a_1 is obtained from the equations that arise at $O(\alpha)$; these are

$$a_{1zz} - \delta^2(k^2 + l^2)a_1 = -i\delta^2\{2(ka_{0X} + la_{0Y}) + (k_X + l_Y)a_0\}, \tag{2.79}$$

with

$$a_{1z} - \delta^2\omega^2 a_1 = -i\delta^2(2\omega a_{0T} + \omega_T a_0) \text{ on } z = 1 \tag{2.80}$$

and

$$a_{1z} = i\delta^2(kB_X + lB_Y)a_0 \text{ on } z = B(X, Y). \tag{2.81}$$

Now, the main purpose in examining the solution for a_1 is in order to determine the A_0 (the amplitude function in equation (2.77)) which ensures uniform validity of the asymptotic expansion. This could be done by simply solving for a_1 directly and examining the nature of this solution (cf. Section 2.1.2), but since we do not require a_1 itself, here we develop the necessary condition on A_0 by finding the condition that a solution for a_1 exists; see Q2.30. To accomplish this we multiply equation (2.79) by a_0 and then integate over z (for $1 \geq z \geq B$), using the boundary conditions on a_0 and a_1 as required. (That such a condition must exist is related to an important idea in the theory of differential and integral equations: the *Fredholm alternative*. The particular method that we choose to use here can be interpreted as an application of *Green's formula*. However, knowledge of these two results is not a prerequisite to an understanding of the presentation that we now give.)

Equation (2.79) is multiplied by a_0 to give

$$a_0 a_{1zz} - \sigma^2 a_0 a_1 = -i\delta^2\{2a_0(ka_{0X} + la_{0Y}) + (k_X + l_Y)a_0^2\}, \tag{2.82}$$

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(where σ is given by equation (2.76)), which is then integrated with respect to z. The first term gives

$$\int_B^1 a_0 a_{1zz}\mathrm{d}z = [a_0 a_{1z}]_B^1 - \int_B^1 a_{0z}a_{1z}\mathrm{d}z = [a_0 a_{1z} - a_{0z}a_1]_B^1 + \int_B^1 a_{0zz}a_1\mathrm{d}z,$$

so the left-hand side of equation (2.82) becomes

$$[a_0\{\delta^2\omega^2 a_1 + \mathrm{i}\delta^2(2\omega a_{0T} + \omega_T a_0)\} - \delta^2\omega^2 a_0 a_1]_{z=1}$$
$$- [a_0 \mathrm{i}\delta^2(kB_X + lB_Y)a_0]_{z=B} + \int_B^1 (a_{0zz} - \sigma^2 a_0)a_1\mathrm{d}z,$$

where the boundary conditions (2.75), (2.80), and (2.81) have been used. Since a_0 is a solution of equation (2.74), equation (2.82) now reduces to

$$[\mathrm{i}\delta^2\frac{\partial}{\partial T}(\omega a_0^2)]_{z=1} - [\mathrm{i}\delta^2(kB_X + lB_Y)a_0^2]_{z=B}$$
$$= -\mathrm{i}\delta^2\left\{k\int_B^1 \frac{\partial(a_0^2)}{\partial X}\mathrm{d}z + l\int_B^1 \frac{\partial}{\partial Y}(a_0^2)\mathrm{d}z + (k_X + l_Y)\int_B^1 a_0^2\mathrm{d}z\right\}.$$

When we introduce the technique of differentiating under the integral sign (Q1.30), we see that this equation is written far more compactly as

$$\nabla\cdot(\mathbf{k}\int_B^1 a_0^2\mathrm{d}z) + \left[\frac{\partial}{\partial T}(\omega a_0^2)\right]_{z=1} = 0, \tag{2.83}$$

where

$$\nabla \equiv \left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}\right) \quad \text{and} \quad \mathbf{k} \equiv (k, l).$$

Finally, we write a_0 from equation (2.77) so that

$$\int_B^1 a_0^2\mathrm{d}z = A_0^2\int_B^1 \cosh^2\{\sigma(z - B)\}\mathrm{d}z$$
$$= \frac{1}{2}A_0^2\int_B^1[1 + \cosh\{2\sigma(z - B)\}]\mathrm{d}z$$
$$= \frac{1}{2}A_0^2\left\{D + \frac{1}{2\sigma}\sinh(2\sigma D)\right\}$$

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where $D = 1 - B$ is the local depth. But we find from (2.76) and (2.78) – it is left as an exercise (cf. Q2.27) – that

$$\frac{\partial \omega}{\partial k} = \frac{\delta^2 k \omega}{2\sigma^2}\left(1 + \frac{2\sigma D}{\sinh 2\sigma D}\right), \tag{2.84}$$

and correspondingly for $\partial\omega/\partial l$, so

$$\mathbf{k}\int_B^1 a_0^2 \mathrm{d}z = (\omega A_0^2 \cosh^2 \sigma \mathrm{D})\mathbf{c}_g$$

where $\mathbf{c}_g \equiv (\partial\omega/\partial k, \partial\omega/\partial l)$, the group velocity; see. Q2.38. Now the amplitude of the surface wave (obtained as $(-\phi_t)$ evaluated on $z = 1$) is, to leading order as $\alpha \to 0$, $\omega A_0 \cosh \sigma D$; let us write

$$E = \frac{1}{2}\omega^2 A_0^2 \cosh^2 \sigma D \tag{2.85}$$

to denote the energy associated with the wave (cf. Q2.31). Then equation (2.83) can be written as

$$\frac{\partial}{\partial T}\left(\frac{E}{\omega}\right) + \nabla \cdot \left(\frac{E}{\omega}\mathbf{c}_g\right) = 0 \tag{2.86}$$

where the term in $\partial/\partial T$ follows directly from the expression for a_0 given in (2.77).

It is not unusual, in the study of oscillators, to call the ratio of energy to frequency the *action*; in the context of these wave-like problems, therefore, we call E/ω the *wave action*. This quantity turns out to be more fundamental than energy in that, as the wave properties slowly change, so in general E (the energy) and ω both change, but (E/ω) is conserved as it is transported at the group velocity. Equation (2.86), for the wave action, is the main result of our calculations and, as we shall see, it plays an important rôle in the development and interpretation of the properties of wave propagation. However, there is at least one other important result that we shall require in due course: an expression for the lines of constant phase – the wavefronts (or wave crests) – which are defined by $\theta = \text{constant}$.

From equations (2.72) we see, first, that (provided the appropriate derivatives exist)

$$\theta_{xt} = \alpha k_T = -\alpha\omega_X; \quad \theta_{yt} = \alpha l_T = -\alpha\omega_Y$$

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and so

$$\nabla\omega + \frac{\partial \mathbf{k}}{\partial T} = \mathbf{0} \quad \left(\nabla \equiv \left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}\right)\right), \tag{2.87}$$

which is the two-dimensional version of the consistency condition described in Q2.29. The relevant equation for θ follows directly from

$$\theta_x = k \quad \text{and} \quad \theta_y = l$$

for then

$$(\theta_x)^2 + (\theta_y)^2 = k^2 + l^2 = |\mathbf{k}|^2, \tag{2.88}$$

which is called the *eikonal* equation (from the Greek εικον, meaning *image* or *form*). This is more naturally expressed as

$$\Theta_X^2 + \Theta_Y^2 = |\mathbf{k}|^2 = \left(\frac{\sigma}{\delta}\right)^2, \tag{2.89}$$

where $\theta = \Theta/\alpha$ is one way to represent the fast phase variable as compared with the slow evolution of the wave parameters. This equation, (2.89), is an equation for Θ, given $\sigma(X, Y, T)$; its solution is a fairly standard exercise in the method of characteristics. We also have

$$\theta_{xy} = \alpha k_Y \quad \text{and} \quad \theta_{yx} = \alpha l_X, \ \text{ that is } k_Y = l_X,$$

so that the vector $\mathbf{k}$ can be treated as 'irrotational'.

Lines which everywhere have the group velocity vector, $\mathbf{c}_g$, as their tangent are called *rays*; these lines are therefore defined by

$$\frac{\mathrm{d}\mathbf{x}_\perp}{\mathrm{d}t} = \mathbf{c}_g.$$

Further, since $\mathbf{c}_g$ and $\mathbf{k}$ are parallel (see above and Q2.32), and the waves propagate in the $\mathbf{k}$-direction, we see that *rays are orthogonal to the wave-fronts*. (We shall find that this is no longer true if a current is present; see Section 2.3.3.) Also, by virtue of equation (2.86), we see that the wave action (E/ω) is conserved along rays as it propagates at the group velocity.

We now explore these ideas by examining a few specific examples which, in particular, make use of equations (2.89) and (2.86). This will enable us to describe how the surface waves refract as the depth varies and, via the wave action, how the amplitude varies along rays. However, before we present these particular calculations let us confirm that our equations recover the usual results for steady propagation over *constant* depth. In this case, equation (2.89) becomes

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$$\Theta_X^2 + \Theta_Y^2 = \left(\frac{\sigma}{\delta}\right)^2 = \text{constant},$$

and the relevant solution (at fixed T) takes the form

$$\Theta = f(X + \lambda Y), \quad \lambda = \text{constant}. \tag{2.91}$$

Thus

$$(1 + \lambda^2)(f')^2 = (\sigma/\delta)^2$$

and so

$$\Theta = f = \pm \frac{(\sigma/\delta)}{\sqrt{1+\lambda^2}}(X + \lambda Y) + G(T)$$

where $G(T)$ is arbitrary – the arbitrary 'constant' of integration; the lines $\theta =$ constant are therefore

$$\theta = \pm \frac{(\sigma/\delta)}{\sqrt{1+\lambda^2}}(x + \lambda y) + g(t) = \text{constant},$$

where $g(t) = G(T)/\alpha$. But from equations (2.72)

$$\theta_x = k \left(= \pm \frac{(\sigma/\delta)}{\sqrt{1+\lambda^2}} \right) \quad \text{and} \quad \theta_y = l \left(= \pm \frac{(\sigma/\delta)\lambda}{\sqrt{1+\lambda^2}} \right)$$

with $\theta_t = g'(t) = -\omega$ and hence, as expected,

$$\theta = kx + ly - \omega t = \text{constant}$$

describes the wavefronts; cf. Q2.7. Finally, equation (2.86) for the wave action gives $E/\omega =$ constant with all the parameters (such as ω) constant, and so the amplitude of the wave also remains constant (again, as expected).

2.3.1 Steady, oblique plane waves over variable depth

Let us consider the case of a depth variation which depends only on X: $1 - B = D(X)$. A steady, oblique plane wave is propagating on the surface. (The restriction to steady motion is in order to simplify the calculation; this assumption implies that, over constant depth, the wave parameters will remain constant.) For steady propagation,

$$\frac{\partial \mathbf{k}}{\partial T} = \mathbf{0},$$

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and then equation (2.87) shows us that

$$\nabla\omega = \mathbf{0};$$

that is, $\omega =$ constant (since $\omega = \omega(X, Y)$, at most, for steady motion), and so, from equation (2.78),

$$\sigma \tanh(\sigma D) = \text{constant}. \tag{2.92}$$

Thus, with $D = D(X)$, we have that $\sigma = \sigma(X)$ and, further, as D decreases so σ increases, and vice versa. (We shall be more precise about this relationship later; also see Q2.39.) The eikonal equation, (2.89), is therefore

$$\Theta_X^2 + \Theta_Y^2 = \left(\frac{\sigma(X)}{\delta}\right)^2, \tag{2.93}$$

which possesses the solution (cf. equation (2.91) *et seq.*)

$$\Theta = f(X) + \lambda Y - \omega T, \quad \lambda = \text{constant},$$

where

$$(f')^2 + \lambda^2 = (\sigma/\delta)^2.$$

The solution for the phase, Θ, is therefore

$$\Theta = \frac{1}{\delta}\left\{\mu Y \pm \int^X \sqrt{\sigma^2(X) - \mu^2}\,\mathrm{d}X\right\} - \omega T, \tag{2.94}$$

where we have written $\lambda = \mu/\delta$; the wavefronts are then represented by $\Theta =$ constant. Correspondingly, the rays (which are orthogonal to the wavefronts) are given (at fixed T) by

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \pm\frac{1}{\mu}\frac{1}{\sqrt{\sigma^2 - \mu^2}}$$

or

$$\mu Y \mp \int^X \frac{\mathrm{d}X}{\sqrt{\sigma^2(X) - \mu^2}} = \text{constant}. \tag{2.95}$$

As a first example, consider a plane wave which propagates from a region of constant depth ($D = D_0$ in $X \leq X_0$) into a region which contains a submerged ridge. Let the wave have a phase, where $D = D_0$, given by

$$\Theta = k_0 X + l_0 Y - \omega T,$$

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so that $\mu/\delta = l_0$ (that is, $l = l_0$ for $\forall X$) and $\sigma^2(X) = k_0^2 + \mu^2$ (in $X \leq X_0$). In this situation, equation (2.92) can be written

$$\sigma \tanh(\sigma D) = \sigma_0 \tanh(\sigma_0 D_0) \tag{2.96}$$

where $\sigma_0 = \delta\sqrt{k_0^2 + l_0^2}$ (and then $\omega^2 = (\sigma_0/\delta^2)\tanh(\sigma_0 D_0)$). The slope of the wavefronts (at fixed T) is

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = -\frac{1}{l_0}\sqrt{\sigma^2(X) - \mu^2} \quad (= -k_0/l_0 \text{ for } X \leq X_0), \tag{2.97}$$

and as the wave passes over the ridge, $D(X)$ first decreases and then increases; consequently $\sigma^2 - \mu^2$ increases and then decreases (eventually returning to its value of k_0^2, we will suppose). Thus the slope, $\mathrm{d}Y/\mathrm{d}X$, decreases and then increases, resulting in the wavefront turning more in-line with the ridge, and then away from it; see Figure 2.5, where a typical set of wavefronts and rays is depicted.

An extension of this problem arises if the depth decreases to zero, thereby producing a shoreline. In this case $\sigma \to +\infty$ as $D \to 0^+$, so $\mathrm{d}Y/\mathrm{d}X \to -\infty$: the wavefronts turn so that, in the limit (at the shoreline), they all become *parallel* to the shoreline. This explains the very familiar observation that virtually all ocean waves arrive at a beach parallel to one another and to the shoreline itself; see Figure 2.6.

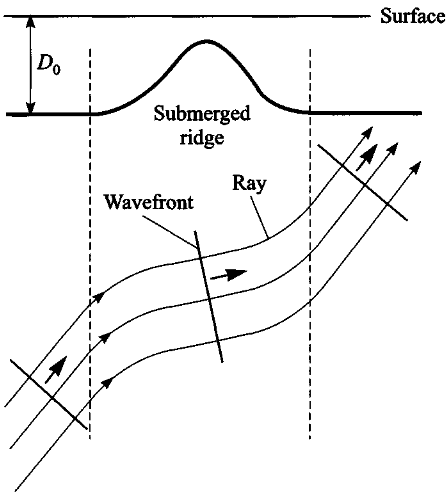

Figure 2.5. The rays and wavefronts for oblique plane waves passing over a submerged ridge; the undisturbed depth is $D = D_0$.

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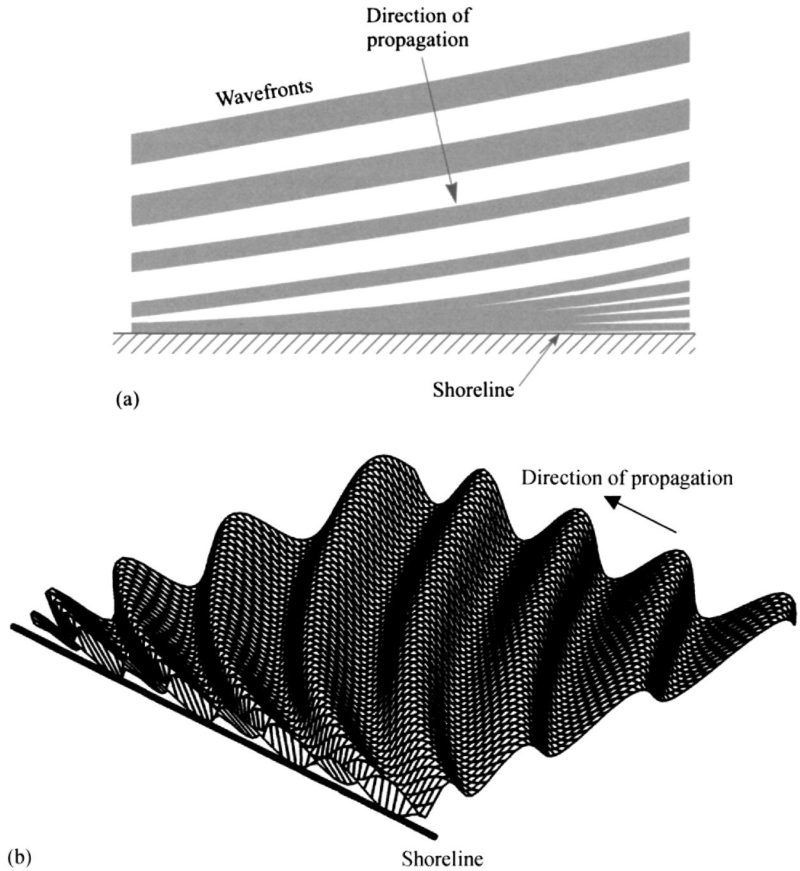

Figure 2.6. A representation of waves approaching a beach; (a) viewed from above, (b) seen as a surface in 3-space.

Further, we can also examine how the amplitude varies as the shoreline is approached. Along the rays we have, from equation (2.86),

$$\frac{\partial}{\partial X}\left(A^2\frac{\partial \omega}{\partial k}\right)+\frac{\partial}{\partial Y}\left(A^2\frac{\partial \omega}{\partial l}\right)=0,$$

and so

$$A^2\frac{\partial \omega}{\partial k}=\text{constant} \tag{2.98}$$

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since there is no variation in Y. Near the shoreline we have $D \to 0$ and $\sigma \to \infty$, but with $\sigma D \to 0$, as we see from equation (2.96), which then gives

$$\sigma^2 D \to \sigma_0 \tanh(\sigma_0 D_0) = \delta^2 \omega^2$$

and

$$\frac{\partial \omega}{\partial k} \sim \frac{\delta^2 k \omega}{\sigma^2} \sim \frac{kD}{\omega} \sim \frac{\sigma D}{\delta \omega} \sim \sqrt{D}$$

since $k \sim \sigma/\delta$ as $\sigma \to \infty$ (from $\sigma = \delta\sqrt{k^2 + l_0^2}$). Hence, using equation (2.98), we see that

$$A = \mathrm{O}(D^{-1/4}) \text{ as } D \to 0$$

which is Green's law again (see equation (2.46), *et seq.*, and Q2.34). Also, since $k \to \infty$ as $D \to 0$, the waves approaching the shoreline get shorter, as we have already discussed in Section 2.2; this phenomenon is included in Figure 2.6. (We should recall the warnings given in Section 2.2 concerning the dubious validity of the linear equations as the depth decreases to zero.)

Finally, we consider a wave which is propagating in a region where the depth is $D = D_0$, for $0 \le X \le X_0$ let us say. For $X < 0$ and $X > X_0$ the depth increases (so that $D = D_0$, $0 \le X \le X_0$, describes a submerged ridge); as before, we then have $\mu/\delta = l_0$ and $\sigma^2(X) = \delta^2 k_0^2 + \mu^2 = \delta^2(k_0^2 + l_0^2)$, given that $\mathbf{k} \equiv (k_0, l_0)$ in $0 \le X \le X_0$. As the depth increases so σ decreases, and if it drops sufficiently so that $\sigma^2 < \mu^2 = \delta^2 l_0^2$ then equation (2.97) shows that the wavefronts no longer exist. Of course, exactly the same can be said of the rays. Indeed, at the points where $\sigma^2 = \mu^2$ the slope of the rays becomes infinite and this will happen for all rays; the lines along which $\sigma^2 = \mu^2$ are called *caustics* (and are, perhaps, familiar from the theory of geometrical optics). The caustic is therefore the *envelope* of the rays. The continuation of a ray, beyond the point where $\mathrm{d}Y/\mathrm{d}X$ on it becomes infinite, is possible by switching to the other sign in the equation of the ray, (2.95), and producing it back into the region where the depth decreases. If this phenomenon occurs in both $X < 0$ and $X > X_0$, then the surface wave over depth $D = D_0$ remains *trapped* in a region containing the ridge; it is called a *trapped wave*, and this is depicted in Figure 2.7.

The caustic is where $\sigma^2 - \mu^2 = \delta^2 k^2 \to 0$, and so $\partial\omega/\partial k \to 0$; see equation (2.84) and remember that ω is constant and that both σ and D approach finite (nonzero) values at the caustic. Hence equation (2.98)

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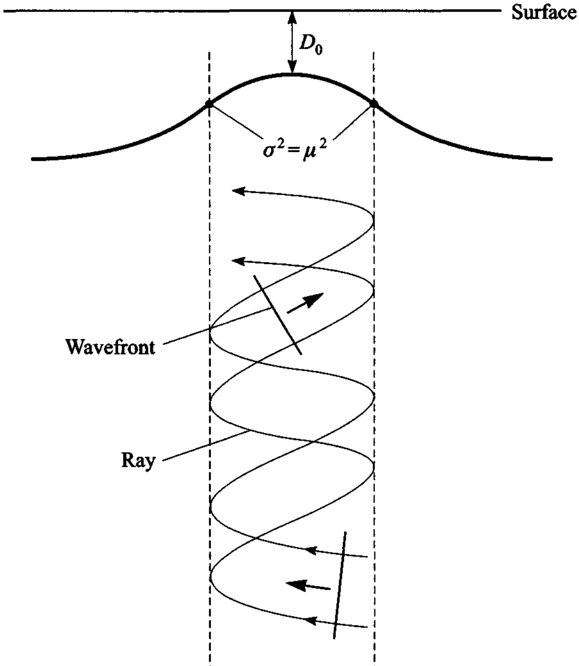

Figure 2.7. The rays and wavefronts for waves trapped between caustics (which are represented by the dashed lines).

shows that the amplitude of the wave diverges at the caustic, so our simple linear theory is no longer adequate. Some appropriate higher-order effects must be invoked in the neighbourhood of the caustic in order to produce a theory in which the wave amplitude remains finite. This more detailed discussion is not pursued here, but some further reading in this direction is mentioned at the end of the chapter.

2.3.2 Ray theory in cylindrical geometry

The equations and examples that we have presented so far have been written in rectangular Cartesian coordinates. However, problems that involve cylindrical surface waves or circular depth contours are clearly best discussed in cylindrical polar coordinates. Rather than derive the relevant equations from first principles, we follow the far simpler route of merely transforming the equations that we already have, according to

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$$X = R\cos\theta, \quad Y = R\sin\theta.$$

Here, R is the radial coordinate suitably scaled like both X and Y, that is, $R = \alpha r$; the angular variable, θ, obviously requires no scaling. The phase function then satisfies

$$\Theta_R = |\mathbf{k}|\cos(\chi - \theta), \quad \frac{1}{R}\Theta_\theta = |\mathbf{k}|\sin(\chi - \theta) \tag{2.99}$$

since, in the constant state, Θ takes the form

$$\begin{aligned}\Theta &= |\mathbf{k}|R\cos(\chi - \theta) - \omega T\\ &= (|\mathbf{k}|\cos\chi)R\cos\theta + (|\mathbf{k}|\sin\chi)R\sin\theta - \omega T,\end{aligned}$$

where χ is a constant and the phase function is written as Θ, and only this form will be used here in order to avoid the obvious confusion with the angular variable θ. The wave-number vector in cylindrical polars is written using the same notation as earlier, so

$$\mathbf{k} \equiv (k, l)\ [= |\mathbf{k}|\{\cos(\chi - \theta), \sin(\chi - \theta)\} \text{ from above}].$$

Thus we obtain

$$\Theta_R^2 + \frac{1}{R^2}\Theta_\theta^2 = k^2 + l^2 = |\mathbf{k}|^2, \tag{2.100}$$

which is obviously the polar form of equation (2.89). The corresponding equation for the action is, from equation (2.86),

$$\frac{\partial}{\partial T}\left(\frac{E}{\omega}\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(\frac{E}{\omega}Rc_{g1}\right) + \frac{1}{R}\frac{\partial}{\partial\theta}\left(\frac{E}{\omega}c_{g2}\right) = 0, \tag{2.101}$$

where the group velocity is written as $\mathbf{c}_g \equiv (c_{g1}, c_{g2})$ in cylindrical polars.

Similar to our discussion in Section 2.3.1, let us consider steady wave propagation over a depth variation which depends only on R, so that $D = D(R)$. The dispersion relation is unchanged:

$$\omega^2 = \frac{\sigma}{\delta^2}\tanh(\sigma D) \quad \text{with} \quad \sigma = \delta\sqrt{k^2 + l^2},$$

since the derivation leading to these (given earlier as equations (2.78) and (2.76)) does not involve derivatives with respect to the slow scales. (Remember that we have used the same notation here for the wave-number vector, and so $|\mathbf{k}|^2 = k^2 + l^2$ is the relevant expression.) For the analogue of a submerged ridge (which was discussed above), we now have a shoal with cylindrical symmetry (with the origin of coordinates chosen so that $R = 0$ is the centre of the shoal); the minimum depth

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occurs at the centre of the shoal. The wavefronts are described by $\Theta =$ constant, where

$$\Theta = \frac{1}{\delta}\left\{\mu\theta \pm \int^{R} \sqrt{R^2\sigma^2(R) - \mu^2}\, \frac{\mathrm{d}R}{R}\right\} - \omega T; \tag{2.102}$$

cf. equation (2.94). Correspondingly, the equations for the rays (at fixed T) become

$$\mu\theta \mp \int^{R} \frac{\mathrm{d}R}{R\sqrt{R^2\sigma^2(R) - \mu^2}} = \text{constant}; \tag{2.103}$$

cf. equation (2.95), and remember that the orthogonality of two curves $(r = f(\theta), r = g(\theta))$, written in polar coordinates, requires that

$$f'(\theta)g'(\theta) = -r^2.$$

Let us suppose that $R\sigma(R)$ is monotonic in $R \geq 0$ (which will certainly describe a class of circular shoals). On the rays we have

$$\frac{\mathrm{d}R}{\mathrm{d}\theta} = \pm\mu R\sqrt{R^2\sigma^2 - \mu^2} \tag{2.104}$$

and a ray approaching the shoal must have either $\mathrm{d}R/\mathrm{d}\theta > 0$ or $\mathrm{d}R/\mathrm{d}\theta < 0$; then R decreases as either θ decreases or increases, respectively. This obtains until the ray reaches a minimum distance from $R = 0$, which occurs where

$$\frac{\mathrm{d}R}{\mathrm{d}\theta} = 0 \quad \text{or} \quad R^2\sigma^2 = \mu^2$$

on the ray; thereafter R increases, which is accommodated by switching to the other sign in equation (2.104). This type of solution, for two different rays (one with $\mathrm{d}R/\mathrm{d}\theta > 0$ initially, that is, to the left, the other with $\mathrm{d}R/\mathrm{d}\theta < 0$) is shown in Figure 2.8. Two important observations can now be made: first, as the depth decreases, so the rays turn towards the centre of the shoal until they reach a minimum distance from $R = 0$, and then they turn away. This general description is what we should expect, based on the corresponding problems with $D = D(X)$ given in Section 2.3.1. Second, we see that a consequence of the bending of the rays is that, in the lee of the shoal (that is, 'behind'), the rays – and wavefronts, of course – cross. Where these waves intersect there may be either a constructive or a destructive interaction; a peak plus a peak (or trough plus trough) is constructive, but a peak plus a trough will – at least

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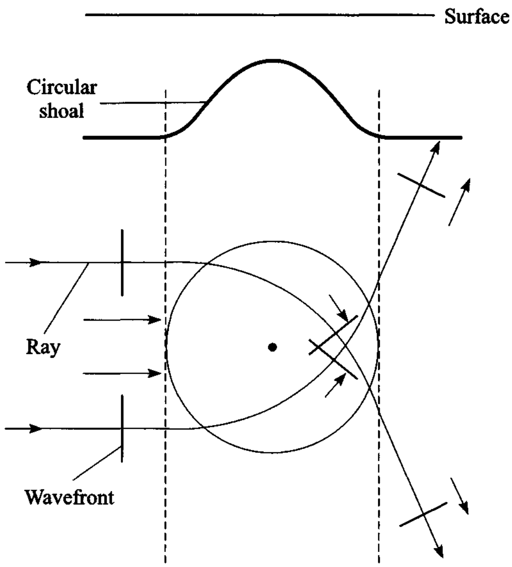

Figure 2.8. Two typical rays and wavefronts for propagation over a circular shoal.

in part – cancel. Obviously, which case arises will depend on the phases of the individual waves.

Other calculations for different choices of $D(R)$ are clearly possible. Indeed, for example, corresponding to our discussion in Section 2.3.1 for straight contours, we may construct a shoreline problem; this becomes a circular island when $D = D(R)$. Similarly, waves trapped on a straight ridge translates into the problem of a circular ridge, for which it is then possible to find conditions which ensure that the waves remain trapped on the ridge. Simple examples of these types of problem, and others, will be found in the exercises (Q2.47, Q2.48).

2.3.3 Steady plane waves on a current

Our second example of a slowly varying environment arises when the surfaces waves propagate in the presence of a (slowly varying) current. (Of course, the effect of both variable depth and a varying current could be studied together, but we opt for the simplification which treats these two problems separately.) A current is the movement, in horizontal

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directions, of a body of water (but, in order to maintain the condition of mass conservation, some vertical motion may also be present); we shall treat the current as a prescribed ambient state which is then perturbed by the surface waves. These motions are, in general, rotational, and so we must use the Euler equation (rather than Laplace's equation). In Cartesian geometry, we therefore start with the problem described by the equations (1.57) and (1.63)–(1.65), namely

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x}, \quad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{\partial p}{\partial y}, \quad \delta^2 \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z},$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

with

$$w = H_t + uH_x + vH_y \quad \text{and} \quad p = H \text{ on } z = 1 + H;$$

and

$$w = 0 \text{ on } z = 0.$$

Here we have chosen to consider only gravity waves (so that the Weber number, W, is set to zero) and the bottom is $z = b = 0$; we have written $\varepsilon\eta = H$. It is seen that we have not quoted the corresponding scaled equations ((1.67), (1.69), (1.70)) for these cannot accommodate the imposed current; cf. Q2.11 and Q2.12.

The relevant form of the governing equations required here – that is, linearised about the ambient state – is obtained (cf. Section 1.3.3) by transforming

$$\mathbf{u}_\perp \to \mathbf{U}_\perp + \varepsilon \mathbf{u}_\perp, \quad w \to W + \varepsilon w$$

where $(\mathbf{U}_\perp, W)$ represents the current; this state must satisfy the equations with $\varepsilon = 0$, so we also require

$$p \to P + \varepsilon p, \quad H \to H + \varepsilon\eta.$$

Thus, with $\mathbf{u}_\perp \equiv (U, V)$, we have

$$\left.\begin{aligned} &\frac{\mathrm{D}U}{\mathrm{D}t} = -\frac{\partial P}{\partial x}, \quad \frac{\mathrm{D}V}{\mathrm{D}t} = -\frac{\partial P}{\partial y}, \quad \delta^2 \frac{\mathrm{D}W}{\mathrm{D}t} = -\frac{\partial P}{\partial z}, \\ &\text{where} \\ &\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y} + W\frac{\partial}{\partial z}, \\ &\text{with} \end{aligned}\right\} \quad (2.105)$$

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$$\left.\begin{aligned}\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}=0,\end{aligned}\right.$$

and

$$\left.\begin{aligned}W=H_t+UH_x+VH_y,\quad P=H \text{ on } z=1+H;\\ W=0 \text{ on } z=0,\end{aligned}\right\}$$

for the current alone. Note that, in general, in the presence of a current, the undisturbed surface is not a $z=$ constant surface. We now restrict consideration to a current which is steady but which slowly varies in the horizontal directions; thus we regard $\mathbf{U}_\perp$, W, P, and H as functions of $(\alpha x, \alpha y)$, and z as appropriate, with $\alpha \to 0$ (which corresponds to the choice made in Section 2.3.1). It is clear from the equation of mass conservation (in (2.105)) that $W = \mathrm{O}(\alpha)$; consequently any upwelling or down-currents are weak (although necessarily present, in general).

The linearised equations for the surface wave are now obtained by collecting the leading-order terms as $\varepsilon \to 0$ (from the governing equations), but after we have satisfied equations (2.105) for the current. This leads to the set of equations

$$\frac{\mathrm{D}u}{\mathrm{D}t}+(\mathbf{u}\cdot\nabla)U=-\frac{\partial p}{\partial x},\quad \frac{\mathrm{D}v}{\mathrm{D}t}+(\mathbf{u}\cdot\nabla)V=-\frac{\partial p}{\partial y},$$

$$\delta^2\left\{\frac{\mathrm{D}w}{\mathrm{D}t}+(\mathbf{u}\cdot\nabla)W\right\}=-\frac{\partial p}{\partial z},$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t}\equiv\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y}+W\frac{\partial}{\partial z}$$

with

$$\left.\begin{aligned}w+\eta W_z&=\eta_t+U\eta_x+V\eta_y+uH_x+vH_y\\ p+\eta P_z&=\eta\end{aligned}\right\}\text{ on } z=1+H$$

and

$$w=0 \text{ on } z=0.$$

We seek a solution of this linearised problem in the form of asymptotic expansions valid as $\alpha \to 0$, just as we did for the case of variable depth. To this end (cf. equations (2.71)–(2.73)) we write

$$X=\alpha x,\quad Y=\alpha y,\quad T=\alpha t,$$

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and

$$(\theta_x, \theta_y) = \{k(X, Y, T),\ l(X, Y, T)\}, \quad \theta_t = -\omega(X, Y, T)$$

which produces the set

$$\begin{aligned}
&\alpha u_T - \omega u_\theta + U(\alpha u_X + ku_\theta) + V(\alpha u_Y + lu_\theta)\\
&\qquad\qquad + Wu_z + \alpha(uU_X + vU_Y) + wU_z = -(\alpha p_X + kp_\theta);\\
&\alpha v_T - \omega v_\theta + U(\alpha v_X + kv_\theta) + V(\alpha v_Y + lv_\theta)\\
&\qquad\qquad + Wv_z + \alpha(uV_X + vV_Y) + wV_z = -(\alpha p_Y + lp_\theta);\\
&\delta^2\{\alpha w_T - \omega w_\theta + U(\alpha w_X + kw_\theta) + V(\alpha w_Y + lw_\theta)\\
&\qquad + Ww_z + \alpha(uW_X + vW_Y) + wW_z\} = -p_z;\\
&\qquad\qquad ku_\theta + lv_\theta + w_z + \alpha(u_X + v_Y) = 0
\end{aligned}$$

with

$$\left.\begin{aligned}
w + \eta W_z &= \alpha\eta_T - \omega\eta_\theta + U(\alpha\eta_X + k\eta_\theta)\\
&\qquad + V(\alpha\eta_Y + l\eta_\theta) + \alpha(uH_X + vH_Y)\\
p &= \eta - \eta P_z
\end{aligned}\right\} \text{ on } z = 1 + H$$

and

$$w = 0 \text{ on } z = 0.$$

Further, in order to make the problem a little more manageable, we shall assume that the horizontal components of the current depend only on (X, Y), at least to $O(\alpha^2)$, and therefore not z. Thus, from equations (2.105), we obtain

$$W = -\alpha z(U_X + V_Y) + O(\alpha^3)$$

so that $H(X, Y; \alpha)$ satisfies

$$\{(1 + H)U\}_X + \{(1 + H)V\}_Y = O(\alpha^2)$$

and then

$$P = H + O(\alpha^2) \text{ for } 0 \le z \le 1 + H.$$

The expression for W describes the upwelling (or down current) associated with the current; it is absent at this order only if the current satisfies $U_X + V_Y = 0$.

The solution that we seek comprises a single harmonic component, and so we write

$$Q \sim (Q_0 + \alpha Q_1)e^{i\theta} + \text{c.c.} + O(\alpha^2),$$

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where Q represents each of u, v, w, p, and η. The leading-order problem then becomes

$$\begin{aligned} -\omega u_0 + Uku_0 + Vlu_0 &= -kp_0; \\ -\omega v_0 + Ukv_0 + Vlv_0 &= -lp_0; \\ \mathrm{i}\delta^2(-\omega w_0 + Ukw_0 + Vlw_0) &= -p_{0z}; \\ \mathrm{i}(ku_0 + lv_0) + w_{0z} &= 0, \end{aligned}$$

with

$$\left.\begin{aligned} w_0 &= -\mathrm{i}\omega\eta_0 + \mathrm{i}kU\eta_0 + \mathrm{i}lV\eta_0 \\ p_0 &= \eta_0 \end{aligned}\right\} \text{ on } z = 1 + H$$

and

$$w_0 = 0 \text{ on } z = 0.$$

Thus

$$(k^2 + l^2)p_0 = \Omega(ku_0 + lv_0),$$

where we have written

$$\Omega = \omega - kU - lV \tag{2.106}$$

and then

$$\frac{\mathrm{i}}{\Omega}(k^2 + l^2)p_0 + w_{0z} = 0; \quad \mathrm{i}\delta^2\Omega w_0 = p_{0z}$$

so that

$$w_{0zz} - \delta^2(k^2 + l^2)w_0 = 0;$$

cf. equation (2.74). The boundary conditions for w_0 are

$$w_0 = -\mathrm{i}\Omega\eta_0 \text{ on } z = 1 + H; \quad w_0 = 0 \text{ on } z = 0$$

which require

$$w_0 = -\mathrm{i}\Omega\eta_0 \frac{\sinh(\sigma z)}{\sinh\{\sigma(1 + H)\}}$$

where

$$\sigma(X, Y, T) = \delta\sqrt{k^2 + l^2},$$

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exactly as before (equations (2.76)). Finally, from

$$p_0 = \frac{i\Omega}{k^2 + l^2} w_{0z} = \frac{\Omega^2 \delta \eta_0}{\sqrt{k^2 + l^2}} \frac{\cosh(\alpha z)}{\sinh\{\sigma(1 + H)\}}$$

and the boundary condition on p_0, we obtain the familiar dispersion relation

$$\Omega^2 = (\omega - kU - lV)^2 = \frac{\sigma}{\delta^2} \tanh\{\sigma(1 + H)\}, \tag{2.107}$$

where Ω replaces ω; see equation (2.78) and Q2.12.

We now proceed to the next order, but we shall describe the calculation in outline only. The technique follows precisely that presented for the case of slowly varying depth (given in equations (2.79)–(2.81) *et seq.*). Furthermore, the results are essentially identical to those obtained previously, the difference arising from the appearance of Ω (for ω), for example. The equations at O(α) take the form

$$kp_1 = \Omega u_1 + F_1; \quad lp_1 = \Omega v_1 + F_2; \quad p_{1z} = \mathrm{i}\delta^2 \Omega w_1 + F_3;$$
$$w_{1z} + \mathrm{i}(ku_1 + lv_1) = F_4,$$

with

$$w_1 = -\mathrm{i}\Omega\eta_1 + F_5 \text{ and } p_1 = \eta_1 \text{ on } z = 1 + H,$$

and

$$w_1 = 0 \text{ on } z = 0,$$

where the forcing terms, F_i ($i = 1, \ldots, 5$), depend on the leading-order solution. These produce an equation for w_1, of the form

$$w_{1zz} - \delta^2(k^2 + l^2)w_1 = G,$$

where G depends on the F_i. Rather than solve for w_1, we multiply by w_0 and integrate with respect to z, $0 \leq z \leq 1 + H$; cf. equation (2.82) *et seq.*, to see how this will produce the equation for $\eta_0(X, Y, T)$ (which is the first term in the asymptotic expansion of the (complex) amplitude of the surface wave).

As before, this equation for η_0 is far more usefully written in terms of the energy, E; cf. equation (2.85). We then find directly (although the details are rather tiresome and are left as an exercise) that, corresponding to equation (2.86), E satisfies

$$\frac{\partial}{\partial T}\left(\frac{E}{\Omega}\right) + \nabla \cdot \left((\mathbf{U}_\perp + \mathbf{c}_g)\frac{E}{\Omega}\right) = 0. \tag{2.108}$$

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Here, $\mathbf{c}_g$ is the group velocity relative to the current, so $\mathbf{c}_g \equiv (\partial\Omega/\partial k, \partial\Omega/\partial l)$. In other words, equation (2.108) does indeed correspond precisely to equation (2.86), provided due account is taken of motion *relative* to the current; thus $\omega \to \Omega$ and $\mathbf{c}_g \to \mathbf{U}_\perp + \mathbf{c}_g$. The eikonal equation for the phase function, θ (or $\Theta = \alpha\theta$), is exactly as before (equation (2.89)):

$$\Theta_X^2 + \Theta_Y^2 = |\mathbf{k}|^2.$$

It is, perhaps, no surprise to learn that equation (2.108) for the wave action (E/Ω) also arises when both a variable depth and a varying current occur together (defined on the same slow scales, of course). Indeed, the conservation of wave action as it is transported at the group velocity (relative to the current and then plus the current velocity here) is a very general result. It appears in many (nondissipative) physical systems that incorporate a slowly varying background on which small amplitude waves are superimposed. (We have shown how this equation comes about in a very direct manner, but a far more elegant approach is to use the average-Lagrangian methods developed by Whitham; these ideas are beyond the scope of this text, but additional reading in this direction is indicated at the end of the chapter.)

Finally, we describe a few consequences for waves that propagate on a slowly varying current. First a general result: the rays are now defined by

$$\frac{\mathrm{d}\mathbf{x}_\perp}{\mathrm{d}t} = \mathbf{U}_\perp + \mathbf{c}_g$$

and only $\mathbf{c}_g$ is in the direction of the wave-number vector, $\mathbf{k}$. Hence the wavefronts, which are orthogonal to $\mathbf{k}$, are *no longer orthogonal* to the rays (in the presence of a general current); cf. variable depth only, as discussed in Section 2.3. The eikonal equation for θ (or $\Theta = \alpha\theta$) is unchanged; see equations (2.88) and (2.89). Thus the methods for finding the wave fronts, $\Theta =$ constant, are the same no matter whether we have a slowly varying depth or current (or, indeed, the two phenomena combined). Now let us briefly examine two particular examples of waves on a current.

First we consider one of the simplest problems of this type: a steady wave propagating in the x-direction (so $\mathbf{k} \equiv (k, 0)$), with a current $\mathbf{U}_\perp \equiv (U(X), 0)$. (We note that $W \neq 0$ for this solution; see equations (2.105).) As we saw in Section 2.3.1, for steady propagation we obtain

$$\omega = \text{constant},$$

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and so

$$\Omega + kU = \text{constant } (= \omega) \tag{2.109}$$

(since, here, $V = 0$ and $l = 0$), where Ω is determined from

$$\Omega^2 = \frac{\sigma}{\delta^2}\tanh(\sigma \mathrm{D}), \quad \sigma = \delta k,$$

and D $(= 1 + H)$ is the local depth. Just to make the calculation a little more transparent – but this does not alter the essential character of the problem – let us suppose that we have short waves so that $\delta k \to \infty$ (which, as we have seen in Section 2.1, is equivalent to having deep water). In this case we may write $\Omega \sim \sqrt{k/\delta}$, and then the speed of the wave relative to the current in $\Omega/k \sim \sqrt{1/\delta k}$. Hereafter, we therefore choose to write

$$c = \frac{\Omega}{k} = \frac{1}{\sqrt{\delta k}} \quad \text{or} \quad k = \frac{1}{\delta c^2} \tag{2.110}$$

so equation (2.109) becomes

$$\omega = kc + kU = \frac{1}{\delta c} = \frac{U}{\delta c^2}$$

or

$$(\delta\omega)c^2 - c - U = 0,$$

a quadratic equation for the speed c, given the constant $\delta\omega$, and the current $U(X)$. It is convenient to introduce the phase speed, c_p, of the waves in the absence of any current; that is, $c_p = 1/\delta\omega$, and then

$$c = \frac{1}{2}c_p(1 \pm \sqrt{1 + 4U/c_p}); \tag{2.111}$$

clearly only the positive sign is meaningful, for then $c = c_p$ when $U = 0$ (as we have just prescribed). The negative sign yields $c = 0$ when $U = 0$, which is plainly inconsistent.

This surprisingly simple solution, (2.111), yields important results, only some of which might have been anticipated. For example, a current moving in the same direction as the wave (that is, $U > 0$), produces a (local) phase speed $c > c_p$ with a decreased k (from equations (2.111)): the waves travel faster in the presence of a current, but are longer. On the other hand, if the current opposes the waves, so that $U < 0$, then $c < c_p$ and the waves are now shorter. However, the significant prediction from equation (2.111) is that c does not exist (as a meaningful wave speed) if

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$U < -c_p/4$. What has happened? The explanation is readily obtained from the equation for the wave action, (2.108).

For our problem, equation (2.108) reduces to

$$\left(U + \frac{\partial\Omega}{\partial k}\right)\frac{E}{\Omega} = \text{constant},$$

where E is proportional to the square of the wave amplitude, A say, and $c_g = \partial\Omega/\partial k$ is the group speed relative to the current. For $\delta k \to \infty$ (or deep water) we have that $c_g = c/2$ (see Q2.27) and thus we obtain

$$\left(U + \frac{1}{2}c\right)cA^2 = \text{constant};$$

consequently, as $U \to -c_p/4$, we then have $c \to c_p/2$ and so $U + c/2 \to 0$; that is, $A \to \infty$. This is exactly the phenomenon associated with a caustic, as described in Section 2.3.1; our solution is inadmissible close to any region where the current is such that $U \to -c_p/4$. As our theory stands, the caustic constitutes a line across which the wave cannot cross; the waves approaching the caustic line produce a build-up of energy (and amplitude) there.

For our second example we describe another classical problem, namely that of a steadily propagating oblique wave moving across the current $\mathbf{U}_\perp \equiv (0, V(X))$. In this case $U_X + V_Y = 0$, so no up-welling is required to maintain the ambient state (and, indeed, there then exists a solution $H = 0$; see equation (2.105)). As above, for steady propagation, we have

$$\omega = \text{constant}$$

so

$$\Omega + lV = \text{constant } (= \omega)$$

where the wave number is $\mathbf{k} \equiv (k, l)$. Further, since there is no variation of $\mathbf{k}$ in Y, we have that $l_X = 0$ (from the irrotationality of $\mathbf{k}$), so $l = \text{constant}$ $(= l_0$, say). Thus we obtain

$$l_0 V + \frac{1}{\delta}\sqrt{\sigma \tanh(\sigma \mathrm{D})} = \omega, \quad \sigma = \delta\sqrt{k^2 + l_0^2},$$

which becomes the equation for $k(X)$, given $V(X)$; here, we may write $D = 1$ if $H = 0$. The short-wave approximation (as used above) then leads to the simplified equation

$$\omega = \frac{1}{\delta c} + l_0 V,$$

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where $c = \Omega/|\mathbf{k}|$ and $|\mathbf{k}| = 1/\delta c^2$. This time we have a linear equation for c, where

$$c = \frac{1}{\delta(\omega - l_0 V)},$$

with

$$|\mathbf{k}| = \sqrt{k^2 + l_0^2} = \delta(\omega - l_0 V)^2, \tag{2.112}$$

and (essentially as before) the conservation of wave action moving with the group yields

$$\frac{\partial}{\partial X}\left(\frac{\partial \Omega}{\partial k}\frac{E}{\Omega}\right) = 0 \quad \text{or} \quad \frac{\partial \Omega}{\partial k}\frac{E}{\Omega} = \text{constant},$$

where the group speed is

$$\frac{\partial \Omega}{\partial k} = \frac{1}{2}\frac{k}{\sqrt{\delta}}(k^2 + l_0^2)^{-3/4} = \frac{1}{2}c\frac{k}{|\mathbf{k}|}.$$

Now from equation (2.112) we see that, as $V(X)$ increases, so $k(X)$ decreases (since we take $\omega > 0$ and $l_0 > 0$). Eventually we shall reach the condition $k = 0$; the wavefront has turned so that it is perpendicular to the direction of the current. From the equation for the wave action we have that

$$c\frac{k}{|\mathbf{k}|}\frac{E}{\Omega} = \frac{k}{|\mathbf{k}|^2}E = \text{constant},$$

and as $k \to 0$ so $E \to \infty$: again, the amplitude of the wave grows without bound (in this theory) when the caustic associated with $k = 0$ is encountered. Typical of other results like this – some of which we described earlier – we must expect that waves cannot cross the caustic (but a reflection may occur). This situation is represented in Figure 2.9.

2.4 The ship-wave pattern

One of the most intriguing, and often spectacular sights when viewed from a distance, is the pattern produced by a moving object on the surface of water. Surprisingly, this pattern is essentially the same no matter whether it is a moorhen or an aircraft carrier that is the source of the disturbance. (However, even from a photograph, the scale can often be

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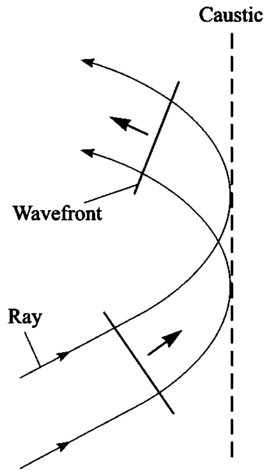

Figure 2.9. The reflection of rays and wavefronts at a caustic formed by the presence of a current.

judged – in the absence of the source – if capillary waves are also present.) A typical wave pattern is depicted in Figure 2.10.

It is our intention here to give an explanation and description of this pattern, an explanation which was first presented by Lord Kelvin. In fact, it was the solution of this problem (in 1887) which led him, first, to develop his method of stationary phase; see Section 2.1.3 and Q2.16. He realised that the salient features of the pattern can be extracted from an otherwise intractable integral (which itself is based on an idealised model of the phenomenon) provided that a suitable limit is taken. Of course, the fine detail of the wave pattern, and particularly a precise estimate of the energy lost in generating the waves, are very significant problems in naval design. The sophisticated analysis required to accomplish this is far beyond the scope of the material presented here; we shall concern ourselves only with the classical – and simple – problem posed by the idealisation introduced by Kelvin.

This method of solution proposes that the disturbance caused by the object (be it the moorhen or the ship) is replaced by a moving *point impulse* at the surface. This impulse – an impulsive pressure, analogous to the impulse in elementary mechanics – at the instant it is applied, causes no displacement, but does impart a vertical velocity to the surface. That is, the surface wave (η) satisfies

$$\eta = 0 \quad \text{with} \quad \eta_t \neq 0$$

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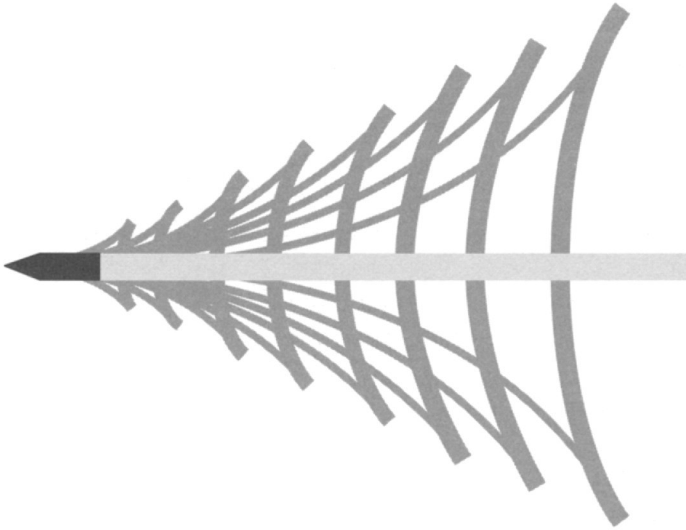

Figure 2.10. A representation of the ship-wave pattern generated by an object moving on the surface of water.

at $t = 0$, when the impulse is applied at time $t = 0$ (cf. Q2.18). Of course, our model here requires this process to occur continuously as the point (the ship) moves and so continuously disturbs the surface; this is the essence of Kelvin's wonderfully perceptive view of the problem. Further, the restriction to a point disturbance indicates that the results we obtain are valid at, perhaps, moderate and, probably, large distances from the centre of the disturbance. We must not expect to produce a description of the waves which is accurate close to a specific object. Indeed, the picture that we shall present corresponds closely to the typical observer's view: the waves are seen, and are well-defined, a reasonable distance from the ship (or whatever) and extend to far distances. Kelvin's approximation to a moving point models the (finite) dimensions of the initiating disturbance, when viewed on a scale that is large; that is, from far away. It is for this reason, primarily, that Kelvin's theory for the ship-wave pattern is independent of the scale – bird or ship – of the moving object.

We shall first present a development based on Kelvin's approach (but within the framework of our earlier discussions, and we shall also need to recall some of the exercises). Then, second, we shall recover some of the

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main features of the wave pattern by a far simpler approach: we shall invoke the ideas of ray theory.

2.4.1 Kelvin's theory

Before we describe the details of the solution, which follows that obtained by Kelvin, we first demonstrate that the region of the wave pattern is easily characterised by the application of simple principles. Indeed, this was one part of the important explanations given by Kelvin.

We consider, in order readily to familiarise ourselves with the idea, a ship (or any object) moving at a constant speed, U, in a straight line on stationary water. The only ingredient that we require which is especially pertinent to this problem is the observation that the waves, when viewed relative to the ship, are stationary; this we shall assume is a given property here. Because of this, the natural way to present the wave field is also relative to the ship; this requires the water to be regarded as moving at speed U opposite to the ship. Let the ship be at P, with the water flowing from left to right. We examine the contribution to the wave profile which was generated at point P' at a time t earlier; see Figure 2.11(a). It is clear that the distance $P'P$ is Ut. Now consider a wave front (at W) which travels at a (constant) speed c_p away from P' in the direction θ (measured with respect to $P'P$); let this wave have wave number k (so that $c_p = c_p(k)$).

Now, this is to be stationary in our frame of reference, and thus we must have

$$c_p = U\cos\theta, \tag{2.113}$$

which therefore describes how k must vary with θ (for fixed U, and a known dispersion relation yielding $c_p = \omega(k)/k$). But equation (2.113) implies that $P'WP$ is a right-angled triangle, with the angle $\pi/2$ at W. Thus, for fixed P' but different angles θ, all wavefronts emanating from P' must lie on a semicircle with diameter $P'P$; see Figure 2.11(b). In Figure 2.11(c) we show the result of including some waves that have been generated *after* the ship has passed P'. Presumably the complete picture is now obtained by combining all such waves (which is, in essence, what we shall do later), and then the envelope of these waves will describe the region inside which the wave pattern is observed. This is, in principle, correct, but an important property of these waves has so far been omitted.

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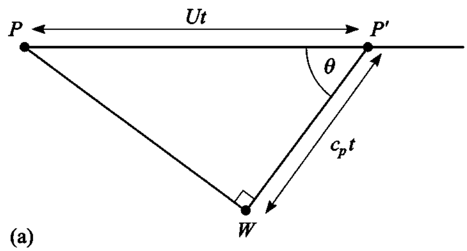

Figure 2.11(a). The ship is at P, which was at P' at a time t earlier; the wavefront has reached the point W.

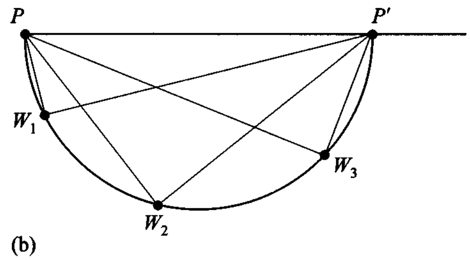

Figure 2.11(b). Three different wavefronts (W_1, W_2, W_3) all emanating from P'.

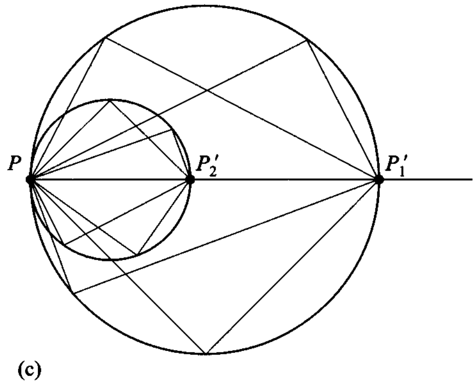

Figure 2.11(c). Wavefronts generated when the ship was at P_1' and then at P_2'.

We know that the energy of water waves does not propagate at the phase speed, c_p, but at the group speed c_g. Waves that are observed can have an appreciable amplitude only near where the energy has reached (see Section 2.1.2). For gravity waves we found that $c_g/c_p < 1$ and in

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particular, for infinitely deep water, $c_g = c_p/2$; let us suppose that our ship creates gravity waves and is moving in deep water (because this is the simplest choice, and it will correspond to most – but, of course, not all – wave patterns that are observed). The propagation of the relevant (energy-carrying) fronts is now only half as far as that supposed earlier; we now have Figure 2.12 where $U = c_p$ on $\theta = 0$. We see that the waves are restricted to a wedge-shaped region, with the ship at the vertex. The semi-angle of the wedge, Θ, is then $\arcsin(1/3)$ ($\approx \pi/9$) (which some readers may recognise as the Mach angle associated with a supersonic flow at Mach number 3).

Of course, this simple analysis cannot supply any predictions for the wave pattern itself; it tells us only where to expect to see the main disturbance (and this is easily confirmed, by observation, to be essentially correct). Also, we have described the case for deep water; as the depth decreases, so c_g approaches c_p and the wedge angle increases (see Q2.49). Now we turn to a far more detailed and careful analysis, following the route laid down by Kelvin.

We consider stationary water of infinite depth, over which a point moves on a prescribed path (which need not be a straight line). Since (as for many of our calculations) we are concerned only with the generation of gravity waves, we set the Weber number to zero. (Of course, we can always retain the effects of surface tension; indeed, capillary ship-waves – no gravity at all – provide an amusing exercise; see Q2.54. Generalisations to finite depth, as we mentioned earlier, are also possible.)

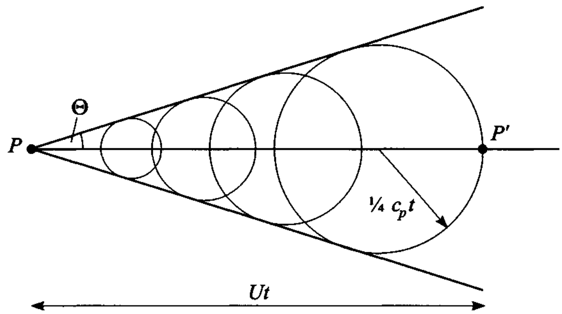

Figure 2.12. The wedge-shaped region inside which the ship-wave pattern is evident, for the case of deep water.

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The first stage in this calculation is to obtain the relevant concentric surface wave, $\eta(r, t)$ which is produced by a point impulse. To this end, we recall the analysis for concentric waves on deep water (Section 2.1.3); here, however, we require the solution (written via the Hankel transform) which satisfies

$$\eta(r, 0) = 0 \quad \text{and} \quad \eta_t(r, 0) \neq 0.$$

However, it is not clear what form we should choose for $\eta_t(r, 0)$. Of course, the main idea here – Kelvin's – is to impose a point impulse, so that is what we use in order to make headway.

To see how the impulse is introduced, it is convenient to call upon the pressure equation evaluated at the surface, where $P = P_s$ on $z = h$:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} + \frac{P_s}{\rho} + gh = f(t),$$

which is written here in physical (dimensional) variables; see Section 1.2.2. As before, let us suppose that somewhere $h = h_0$ (= constant) and $P_s = P_a$ (= constant atmospheric pressure) with no motion, then

$$f(t) = \frac{P_a}{\rho} + gh_0,$$

and hence

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} + \frac{1}{\rho}(P_s - P_a) + g(h - h_0) = 0.$$

In the present context, we are analysing a certain class of linear waves, so it is the linearised version of this equation that we require: we have, approximately,

$$\frac{\partial \phi}{\partial t} + \frac{1}{\rho}(P_s - P_a) + g\eta = 0 \text{ on } z = h_0,$$

where $h - h_0 = \eta$. The impulse is obtained by integrating this equation over the time interval $(0, T)$ and then letting $T \to 0$. Performing this integration yields

$$\phi(\mathbf{x}_\perp, h_0, T) + \frac{1}{\rho}\int_0^T (P_s - P_a)\mathrm{d}t + g\int_0^T \eta \,\mathrm{d}t = 0$$

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where we have set $\phi(\mathbf{x}_\perp, h_0, 0) = 0$, as we may always do. Now, for a finite-amplitude surface wave we must have

$$\int_0^T \eta \,\mathrm{d}t \to 0 \quad \text{as} \quad T \to 0^+,$$

but for an impulse we require

$$\int_0^T (P_s - P_a)\mathrm{d}t$$

to have a finite and nonzero limit as $T \to 0^+$; this is the *impulsive pressure*. Hence the required initial condition (for the concentric wave) is that $\phi(\mathbf{x}_\perp, h_0, 0)$ is to be specified. This condition is to be incorporated into the determination of $\eta(r, t)$ (where $\mathbf{x}_\perp \equiv (r, \theta)$, and there is no dependence here on θ).

It is clear that $\phi(r, 1, t)$ satisfies the same wave equation as $\eta(r, t)$, equation (2.14), essentially by virtue of the boundary condition

$$\phi_t + \eta = 0 \quad \text{on} \quad z = 1;$$

see equation (2.67) (and note that we have reverted to our non-dimensional variables). Thus, immediately, we have the appropriate solution for ϕ (cf. equation (2.30)) as

$$\phi(r, 1, t) = \int_0^\infty p\hat{f}(p)\cos\left(t\sqrt{\frac{p}{\delta}}\right)J_0(rp)\,\mathrm{d}p$$

where

$$\phi(r, 1, 0) = \int_0^\infty p\hat{f}(p)J_0(rp)\,\mathrm{d}p = f(r),$$

say, and $\eta(r, 0) = -\phi_t(r, 1, 0) = 0$. The impulse that we use (a point impulse) is modelled by

$$f(r) = \begin{cases} I, & 0 \le r \le a \\ 0, & r > a \end{cases}$$

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with $a \to 0$, so that

$$\hat{f}(p) = I \int_0^a rJ_0(pr)\,\mathrm{d}r$$

$$\to Ia^2 \int_0^1 y\mathrm{d}y = \frac{1}{2}Ia^2 \quad \text{as} \quad a \to 0,$$

if Ia^2 is fixed. (We have written $r = ay$ here and used the familiar result $J_0(x) \to 1$ as $x \to 0$.) Thus, with $\hat{f}(p) = \frac{1}{2}Ia^2 = \beta$, say, we obtain

$$\eta(r, t) = -\phi_t(r, 1, t) = \frac{\beta}{\sqrt{\delta}} \int_0^\infty p^{3/2} \sin\left(t\sqrt{\frac{p}{\delta}}\right) J_0(rp)\mathrm{d}p \tag{2.114}$$

(from which we can determine the corresponding form taken by $\eta_t(r, 0)$; cf. Q2.19)). It is this solution, (2.114), for concentric waves on deep water generated by a point impulse, which we now examine.

This first stage of the calculation follows precisely that presented in Section 2.1.3. We introduce the integral representation of J_0 (see equation (2.31)) and then write η as the real part of the sum of two integrals (as in equation (2.32). This is transformed according to

$$p = \frac{t^2}{\delta r^2}q^2, \quad \sigma = \frac{t^2}{\delta r},$$

and then we use Kelvin's method of stationary phase for $\sigma \to \infty$ to give

$$\eta(r, t) \sim \frac{\beta}{8\sqrt{2}} \frac{1}{\delta^2} \frac{t^3}{r^4} \sin\left(\frac{1}{4} \frac{t^2}{\delta r}\right); \tag{2.115}$$

cf. equation (2.36). This calculation, which parallels that described in Section 2.1.3, is left as an exercise (Q2.50). Our task now is to incorporate this result into a description of the waves generated when the point impulse moves along a prescribed path.

The point impulse – a ship, perhaps – moves on the surface of the water along a path Γ, described in Cartesian coordinates by

$$\mathbf{x}_\perp \equiv (X(t), Y(t)),$$

where t is the time elapsed since the ship (let us call it that) was at the point P', namely at (X, Y); the ship is now at P, the origin of coordinates. The path is assumed smooth (so that both $X(t)$ and $Y(t)$ are (at least) once-differentiable functions) and then the X-axis is chosen to be tangent

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to Γ at P; all this is summarised in Figure 2.13. The ship, as it passes through P', initiates a disturbance there that propagates outwards and, in the direction defined by θ and at distance r from P', which has reached W. (The angle θ is measured relative to the (backwards) tangent to Γ at P'.) The disturbance at W is a distance r from P' (in the direction θ), and it has taken a time t to reach there; we shall assume, for the purposes of the following discussion, that the elevation of the wave at the time t, and distance r from P', where the impulse was applied at $t = 0$, is given by

$$\eta(r, t) = A\frac{t^3}{r^4}\sin\left(\frac{1}{4}\frac{t^2}{\delta r}\right), \tag{2.116}$$

where A is a constant; cf. equation (2.115). But the total disturbance at W will have contributions from all points along the path, to a greater or lesser extent (depending on the position of W). We therefore require the sum of all contributions like (2.116) over all time; however, it is clear that the integral of (2.116) in t over $[0, \infty)$ does not exist. We circumvent this difficulty by positing that the ship has been moving only for a finite time, T, say. Thus the total effect of all impulses along the path produces the amplitude

$$H(x, y) = A\int_0^T \frac{t^3}{r^4}\sin\left(\frac{1}{4}\frac{t^2}{\delta r}\right)\mathrm{d}t \tag{2.117}$$

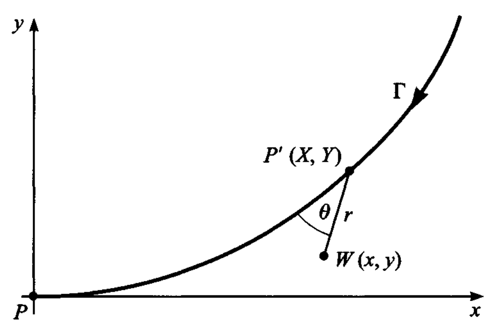

Figure 2.13. The path of the ship is Γ; the ship is now at P (the origin) and it was at P' at a time t earlier.

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at W, where $r^2 = (x - X(t))^2 + (y - Y(t))^2$. (It is clear that this integral also does not exist for points *on* the ship's path, where $r = 0$; however, the method of stationary phase that led to (2.115) has already been interpreted only for points away from $r = 0$: we are seeking the wave pattern as seen some distance from the ship.)

To proceed, we express (2.117) in the form

$$H(x, y) = \mathscr{I}\left\{ A \int_0^T \frac{t^3}{r^4} \exp(\mathrm{i}t^2/4\delta r)\mathrm{d}t \right\}, \tag{2.118}$$

and we have previously used $\sigma = t^2/\delta r \to \infty$; thus we may apply the method of stationary phase yet once more! The point(s) of stationary phase occur where

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^2}{r}\right) = 0 \quad \text{for} \quad r = r(t),$$

at fixed x, y: thus

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{2r}{t}. \tag{2.119}$$

(The fact that we are treating $\sigma = \sigma(t)$, and $\sigma \to \infty$ is required for the method of stationary phase, is irrelevant in the application of the method.) But from

$$r^2 = (x - X(t))^2 + (y - Y(t))^2$$

we have (at fixed (x, y))

$$\begin{aligned} r\frac{\mathrm{d}r}{\mathrm{d}t} &= -\left\{(x - X)\frac{\mathrm{d}X}{\mathrm{d}t} + (y - Y)\frac{\mathrm{d}Y}{\mathrm{d}t}\right\} \\ &= -(x - X, y - Y)\cdot\left(\frac{\mathrm{d}X}{\mathrm{d}t}, \frac{\mathrm{d}Y}{\mathrm{d}t}\right) \\ &= rU(t)\cos\theta, \end{aligned}$$

where $U(t)$ $(= \sqrt{(\mathrm{d}X/\mathrm{d}t)^2 + (\mathrm{d}Y/\mathrm{d}t)^2})$ is the speed of the ship, so (since $r \neq 0$)

$$\frac{\mathrm{d}r}{\mathrm{d}t} = U\cos\theta. \tag{2.120}$$

Thus the condition of stationary phase, (2.119), becomes

$$r = \frac{1}{2}Ut\cos\theta, \tag{2.121}$$

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which represents (in plane polar coordinates) a circle of diameter $\frac{1}{2}Ut$, with the end of a diameter tangent to Γ at P' and the circle orientated from P$'$ towards P; see Figure 2.14. (If the ship is moving on a straight-line path – the x-axis – at constant speed, then this construction immediately recovers Figure 2.12.) It is only points on this circle that correspond to the points of stationary phase and therefore provide the dominant contribution to the wave amplitude. All points P$'$ which contribute, in this sense, to the disturbance at a given point W are usually called the *influence points* of W. A question that we might pose at this stage is: how many influence points are there, for a given W? For example, for constant speed, straight-line motion, it is a simple exercise to show that there are just two influence points (in general); see Q2.51 and Q2.52. This suggests that there are two families of curves that contribute to the ship-wave pattern.

The wave pattern, whose determination is our main goal, is obtained by constructing the lines of constant phase, consistent with the condition of stationary phase. We shall describe this calculation for the simple case of constant speed, straight-line motion; the corresponding problem for a circular course is set as an exercise (see Q2.53). The phase (see (2.118)) is proportional to t^2/r; it is convenient to introduce

$$\lambda = \frac{U^2t^2}{2r},$$

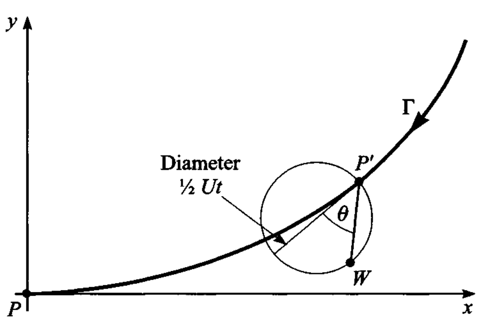

Figure 2.14. The position of the points of stationary phase (points W on the circle) for the disturbance initiated at P' at a time t earlier; the ship is now at P.

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where U is the constant speed of the ship, and then $\lambda =$ constant yields the curves of constant phase. But the condition of stationary phase, from (2.121), yields

$$rUt = \frac{1}{2}U^2t^2\cos\theta = \lambda r\cos\theta$$

so

$$\left.\begin{aligned} &Ut = \lambda\cos\theta, \\ &\text{and then} \\ &r = \frac{1}{2}Ut\cos\theta = \frac{1}{2}\lambda\cos^2\theta. \end{aligned}\right\} \qquad (2.122)$$

(In fact, the equations (2.122) are valid for any path and, indeed, in the construction of these equations we may allow $U = U(t)$.) The path here is simply

$$X = Ut, \quad Y = 0,$$

and then any point W is

$$x = Ut - r\cos\theta, \quad y = -r\sin\theta,$$

where r and θ are shown in Figure 2.13. Thus, using equations (2.122), we obtain directly

$$x = \lambda(\cos\theta - \frac{1}{2}\cos^3\theta), \quad y = -\frac{1}{2}\lambda\cos^2\theta\sin\theta, \qquad (2.123)$$

which are the parametric equations (parameter θ) for the dominant contribution to the wave pattern, each wave crest/trough being associated with a fixed value of λ. The pattern of wave crests (or troughs) is shown in Figure 2.15, which closely resembles the wave pattern produced in nature; compare this figure with Figure 2.10. Note that in this figure we have included points $r = 0$ (which do exist on the curve (2.123)) for completeness only.

Before we leave our discussion of this pattern, we comment that Figure 2.15 plainly shows two families of curves (exactly as we observe) which meet on the boundary of the region. Where these two families meet is quite significant; consider the derivatives obtained from equations (2.123):

$$\frac{dx}{d\theta} = \lambda(-\sin\theta + \frac{3}{2}\cos^2\theta\sin\theta) = \frac{1}{2}\lambda(1 - 3\sin^2\theta)\sin\theta$$

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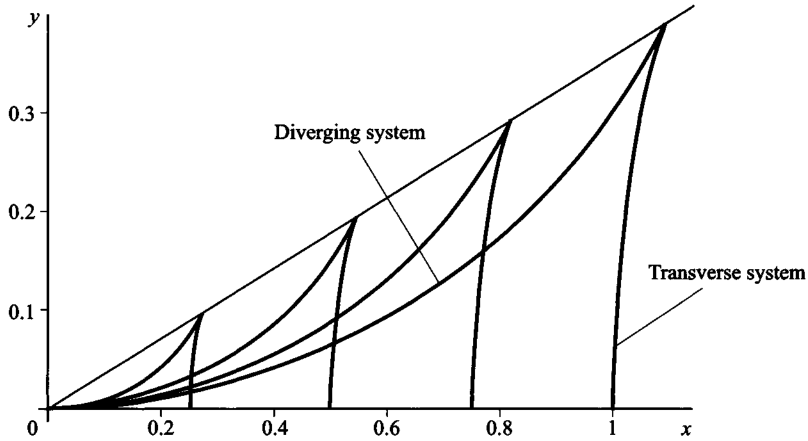

Figure 2.15. The ship-wave pattern as obtained from equations (2.123), for various values of λ (= 0.5, 1, 1.5, 2); the fine line denotes the boundary of the wedge which contains the dominant contributions.

and

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = -\frac{1}{2}\lambda(\cos^3\theta - 2\cos\theta\sin^2\theta) = -\frac{1}{2}\lambda(1 - 3\sin^2\theta)\cos\theta.$$

It is clear that $\mathrm{d}y/\mathrm{d}x$ is singular at $\theta = 0$ (which is where all curves meet at P) and also where $3\sin^2\theta - 1 = 0$; this defines the angle θ_0 that is attained where the two influence points coincide (and we note that this is the same for all waves, since it is independent of λ). This and other relevant points are included in Figure 2.15; in particular we see that the two families are defined by $0 \le \theta \le \theta_0$ and $\theta_0 \le \theta \le \pi/2$, respectively.

Finally, we make full use of Kelvin's method of stationary phase in order to provide an estimate for the wave amplitude along the lines of constant phase where the dominant contributions occur. To this end, we use the general result given in equation (2.35) and apply it to the integral in (2.118). Thus we require

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{t^2}{r}\right)$$

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evaluated at the points of stationary phase; first we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{t^2}{r}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2t}{r}-\frac{t^2}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t}\right)=\frac{2}{r}-\frac{4t}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t}+\frac{2t^2}{r^3}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2-\frac{t^2}{r^2}\frac{\mathrm{d}^2r}{\mathrm{d}t^2}$$

which, on lines $\mathrm{d}r/\mathrm{d}t = 2r/t$ (equation (2.119), for stationary phase), yields the expression

$$\frac{1}{r}\left(2-\frac{t^2}{r}\frac{\mathrm{d}^2r}{\mathrm{d}t^2}\right). \tag{2.124}$$

But on these lines we also have, (2.120),

$$\frac{\mathrm{d}r}{\mathrm{d}t}=U\cos\theta,$$

which gives

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2}=-U\frac{\mathrm{d}\theta}{\mathrm{d}t}\sin\theta$$

since $U = \text{constant}$; now we must find $\mathrm{d}\theta/\mathrm{d}t$.

For the straight-line course (along $y = Y = 0$), at constant speed U, we see that (cf. Figure 2.13)

$$\theta+\arctan\left(\frac{y}{Ut-x}\right)=\pi$$

with $X = Ut$. Thus, at fixed (x, y), we obtain

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}-\frac{yU}{(x-Ut)^2+y^2}=0$$

and we also have $\sin(\pi-\theta) = -y/r$ where $r^2 = (x-Ut)^2 + y^2$; hence

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=-\frac{U}{r}\sin\theta$$

and so

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2}=\frac{U^2}{r}\sin^2\theta.$$

The expression (2.124) therefore becomes

$$\begin{aligned}\frac{1}{r}\left(2-\frac{U^2t^2}{r^2}\sin^2\theta\right)&=\frac{2}{r}(1-2\tan^2\theta)\\&=\frac{2}{r}(1-3\sin^2\theta)/\cos^2\theta,\end{aligned} \tag{2.125}$$

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which we observe is zero at $\theta = \theta_0$ $(= \pm\arcsin(1/\sqrt{3}))$ where the two families of wave crests/troughs meet. Further, we must include two dominant contributions to the wave amplitude – one from each family (although, perhaps, we may find that one of these dominates the other). Since the two families are generated (in $y > 0$) by $0 \le \theta \le \theta_0$ and $\theta_0 \le \theta \le \pi/2$, respectively, we see from (2.125) that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{t^2}{r}\right) > 0 \quad \text{for} \quad 0 \le \theta < \theta_0$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{t^2}{r}\right) < 0 \quad \text{for} \quad \theta_0 < \theta < \pi/2.$$

For a given point, W, we let the contribution in the range $0 \le \theta < \theta_0$ – usually called the *transverse* wave system – be designated by the subscript t, and for the other contribution, usually called the *diverging* system, we shall write the subscript d; this terminology is used in Figure 2.15.

The two terms that provide the dominant asymptotic behaviour (as $t^2/\delta r \to \infty$), according to Kelvin's result (2.35), therefore yield (after a little manipulation)

$$H \sim \mathscr{I}\left\{A\sqrt{\frac{\pi}{r}}\frac{\cos\theta_t \exp\{\mathrm{i}(r/\delta\alpha_t^2 + \pi/4)\}}{\alpha_t^3\sqrt{1 - 3\sin^2\theta_t}} + A\sqrt{\frac{\pi}{r}}\frac{\cos\theta_d \exp\{\mathrm{i}(r/\delta\alpha_d^2 - \pi/4)\}}{\alpha_d^3\sqrt{3\sin^2\theta_d - 1}}\right\}. \qquad (2.126)$$

Here, we have substituted for time t from (2.121) and written

$$\alpha_q = \frac{1}{2}U\cos\theta_q \quad (q \equiv t, d).$$

The solution expressed by (2.126) is the final result that we present in this section. We see that both contributions are of the same order, that the amplitude decays like $r^{-1/2}$ away from the ship's path, but that the amplitude is undefined where the two families meet (at $\theta_t = \theta_d = \theta_0 = \arcsin(1/\sqrt{3})$). (The amplitude is also undefined where $\theta_d = \pi/2$, but this is at P, the origin, and is to be expected because of the nature of our point impulse model.) An analysis can be performed, by taking Kelvin's method of stationary phase to the next order, for the case $\theta_t = \theta_d = \theta_0$;

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the wave amplitude can then be shown to be finite but now it behaves like $r^{-1/3}$ away from the ship's path. This goes some way to explaining why the waves near the edge of the wedge-shaped region are observed so readily: they are larger than those nearby and on each of the families separately. Finally, we note that the two systems of waves, transverse and diverging, have a phase difference of $\pi/2$ even when $\alpha_t \approx \alpha_d$. Thus we anticipate that near the edge of the wedge, where α_t and α_d are nearly equal, a phase difference will be evident; this is, indeed, seen in well-defined ship waves (and just hinted at in our Figure 2.10). The effect of the phase difference is to produce transverse and diverging systems that do not meet with a common tangent at $\theta = \theta_0$; this phenomenon is shown in Figure 2.16.

In summary, we have seen how the application of Kelvin's method of stationary phase – ultimately three times – enables us to provide a surprisingly accurate description of the ship-wave pattern. This approach is based on the point impulse model for the moving object (bird or ship) and, importantly, on the limiting process $t^2/r \to \infty$ (provided $r \neq 0$). We conclude by observing that this requires, first, that we are not at points on the ship's path. (Indeed, in practice, this region is the one that is significantly disturbed by the propulsion system, be it a screw-propeller or paddling feet.) Second, since $r \neq 0$, the limit must be interpreted as

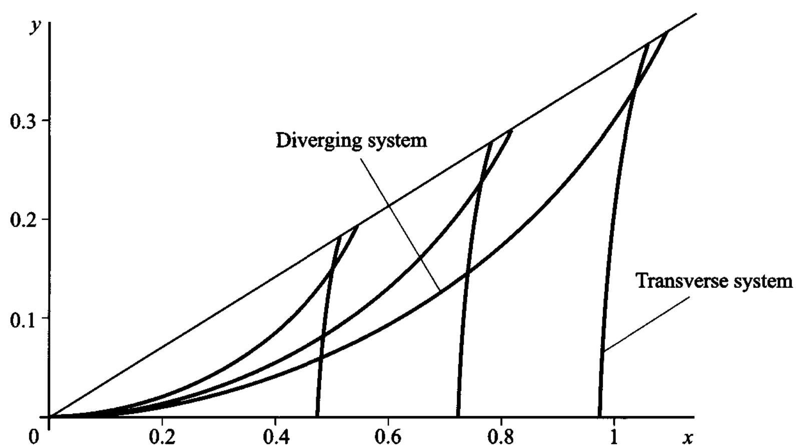

Figure 2.16. A more accurate version of the ship-wave pattern, with the phase difference between the transverse and diverging wave systems now evident (particularly near the edge of the region).

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$t \to \infty$ sufficiently rapidly; that is, we are seeing the pattern well after the passing of the ship (and consequently well behind and far away from the ship). Of course this means, as mentioned earlier, that the precise nature of the object producing the waves will play no part in this theory.

2.4.2 Ray theory

In the previous section we described, with some care and in some detail, the important predictions first developed by Lord Kelvin. We now demonstrate how the salient features can be obtained directly from ray theory (Section 2.3).

We invoke ray theory by treating the problem as a stationary object (the ship), at the origin of the horizontal coordinate system, in the presence of a current. The current is, of course, just that required to bring the ship to a halt (and, as before, we suppose that the ship is moving in stationary water). Let the (steady) current be

$$\mathbf{U}_\perp \equiv (U(X, Y),\; V(X, Y)),$$

at least to $O(\alpha^2)$; cf. the discussion in Section 2.3.3. We seek waves that are steady, so $\omega =$ constant, where

$$\Omega + kU + lV = \omega$$

with

$$\Omega = -\frac{1}{\delta}\sqrt{\sigma \tan\mathrm{h}\{\sigma(1+H)\}}, \quad \sigma = \delta|\mathbf{k}|;$$

see equation (2.107). (We have chosen the sign of the square root to correspond to waves behind the ship in $X > 0$.)

We restrict the calculation to the case of deep water (equivalently, that is, for short waves), and so hereafter we write

$$\Omega = -\sqrt{|\mathbf{k}|/\delta}.$$

Further, since the ship waves are stationary – do not change with time – in the frame of reference fixed relative to the ship, we have $\omega = 0$. Thus

$$\sqrt{|\mathbf{k}|/\delta} = kU + lV,$$

which describes the relation between k and l (given U and V) for stationary waves to exist. Indeed, for $U =$ constant and $V = 0$, we obtain

$$c_p = \frac{1}{\sqrt{\delta|\mathbf{k}|}} = \frac{k}{|\mathbf{k}|}U = U\cos\theta \tag{2.127}$$

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exactly as in Section 2.4.1 (equation (2.113) and Figure 2.11(a)). The rays are described by

$$\begin{aligned}\frac{\mathrm{d}\mathbf{x}_\perp}{\mathrm{d}t} &= \mathbf{U} + \mathbf{c}_g \\ &\equiv (U + \partial\Omega/\partial k, V + \partial\Omega/\partial l) \\ &= \left(U - \frac{1}{2}c_p\frac{k}{|\mathbf{k}|}, V - \frac{1}{2}c_p\frac{l}{|\mathbf{k}|}\right)\end{aligned}$$

or

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{V - \frac{1}{2}c_p l/|\mathbf{k}|}{U - \frac{1}{2}c_p k/|\mathbf{k}|};$$

see Figure 2.17. But from equation (2.127) we see that for $U =$ constant and $V = 0$ we may write this in the form

$$\tan\phi = \frac{\frac{1}{2}\sin\theta\cos\theta}{1 - \frac{1}{2}\cos^2\theta}, \qquad (2.128)$$

which determines ϕ in terms of θ; conversely, this equation can be rewritten as

$$2\tan\phi\tan^2\theta - \tan\theta + \tan\phi = 0.$$

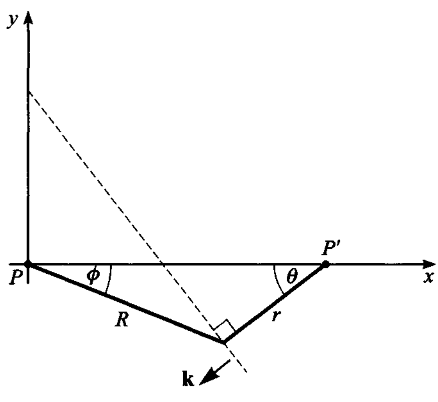

Figure 2.17. A wavefront, with wave number vector **k**, emanating from the point P'; the ray measured from P is at a distance R from P, and at an angle ϕ to the X-axis (measured in the negative sense, to be consistent with the direction in which θ is measured).

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Thus the disturbance at any point on a given ray is contributed to by two waves, in general, determined by two values of θ – which, of course, correspond to the two influence points that we introduced earlier.

The solution for $\tan\theta$ is immediately

$$\tan\theta = \frac{1}{4}\cot\phi\left(1 \pm \sqrt{1 - 8\tan^2\phi}\right),$$

so two solutions exist for $\tan^2\phi < 1/8$, but no (real) solutions exist for $\tan^2\phi > 1/8$. The two wave systems (to use the terminology of the previous analysis) coincide where

$$\tan^2\phi = 1/8 \quad \text{or} \quad \sin^2\phi = 1/9,$$

which recovers our result for the angle of the wedge inside which the dominant disturbance occurs.

We now turn to the determination of the lines of constant phase, $\Theta =$ constant. This requires us to find the appropriate solution of the eikonal equation

$$\Theta_X^2 + \Theta_Y^2 = |\mathbf{k}|^2,$$

where $|\mathbf{k}|^2 = \sec^4\theta/\delta^2 U^4$ (from equation (2.127) with $U =$ constant). It is convenient to express this equation in polar coordinates defined at the origin of (X, Y), which here we write as (R, ϕ); see Figure 2.17. Thus we have

$$\Theta_R^2 + \frac{1}{R^2}\Theta_\phi^2 = \frac{\sec^4\theta}{\delta^2 U^4};$$

cf. equation (2.100). The relevant wavefronts are obtained by mapping the rays that correspond to the lines of constant phase, and the rays are radial lines ($\phi =$ constant) out from the origin. But, on the rays, θ and ϕ are related by equation (2.128) and so we seek a solution

$$\Theta = Rf(\theta);$$

thus

$$f^2 + (f')^2\left(\frac{d\theta}{d\phi}\right)^2 = \frac{\sec^4\theta}{\delta^2 U^4} \tag{2.129}$$

where

$$\frac{d\theta}{d\phi} = \frac{4 - 3\cos^2\theta}{3\cos^2\theta - 2}$$

(which follows directly from equation (2.128)).

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It is a fairly straightforward exercise to show that equation (2.129) has a solution which is proportional to

$$\left(\cos\theta\sqrt{4-3\cos^2\theta}\right)^{-1};$$

the verification of this result is left as an exercise (and you may wish to find the constant of proportionality in this solution, but its precise form is irrelevant here). Thus the lines $\Theta = \text{constant}$ become

$$\frac{R}{\cos\theta\sqrt{4-3\cos^2\theta}} = \text{constant} = \frac{1}{2}\lambda, \text{ say.} \tag{2.130}$$

We revert to Cartesian coordinates in order to present the lines of constant phase, where we use

$$X = R\cos\phi = \frac{R(2-\cos^2\theta)}{\sqrt{4-3\cos^2\theta}} \quad (>0)$$

and

$$Y = -R\sin\phi = -\frac{R\sin\theta\cos\theta}{\sqrt{4-3\cos^2\theta}} \qquad (<0 \text{ for } 0<\phi<\pi).$$

(Again, these follow directly from equation (2.128), and we have chosen the signs of the square roots to be consistent with our definitions.) Inserting the expression for R from equation (2.130), we obtain

$$X = \lambda\cos\theta(1-\frac{1}{2}\cos^2\theta), \quad Y = -\frac{1}{2}\lambda\cos^2\theta\sin\theta$$

which is precisely the parametric form obtained in Section 2.4.1 (equation (2.123)). It is clear, however, that ray theory does not contain sufficient information to describe the phase difference along the edge of the wedge (which Kelvin's more complete wave theory produced). Finally, we comment that the equation for the wave action can be used to show that the amplitude of the dominant wave decays like $r^{-1/2}$ away from the ship's path (as previously given in equation (2.126)).

This concludes our presentations of various linear problems in the theory of water waves. As we have mentioned earlier, the exercises may be used to discover and investigate other interesting problems – but even these do not claim to be exhaustive. Additional material can be found in the books listed in the further reading at the end of this chapter.

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II Nonlinear problems

A higher height, a deeper deep.

In Memoriam A.H.H. LXII

Our discussion thus far has been restricted to various problems in linear theory. These have been chosen for their mathematical content, and to give a flavour of the breadth of results that is available. We now turn to the more demanding arena that is the study of nonlinear wave propagation. As before, we shall continue our philosophy of selecting problems which contain interesting mathematical elements and which, for the most part, lay the foundations for our later presentations.

Most – but by no means all – of our earlier analyses have considered the case of gravity waves (which are, after all, the most relevant waves for the engineer involved in the design of ships, offshore platforms, or sea walls, to mention but three). Here, for all our work on nonlinear phenomena, we shall limit ourselves to the description of gravity waves. Thus, for the inviscid model with no surface tension ($W = 0$), we have (equations (1.67), (1.69) and (1.70))

$$\left.\begin{aligned}
&\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x}, \quad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{\partial p}{\partial y}, \quad \delta^2 \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z}\\
&\text{where}\\
&\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \varepsilon\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\\
&\text{and}\\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\\
&\text{with}\\
&w = \eta_t + \varepsilon(u\eta_x + v\eta_y) \quad \text{and} \quad p = \eta \quad \text{on } z = 1 + \varepsilon\eta;
\end{aligned}\right\} \qquad (2.131)$$

$$w = ub_x + vb_y \quad \text{on } z = b,$$

written in Cartesian coordinates. Correspondingly, for irrotational flow (Q1.38), we have

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$$\left.\begin{array}{l} \phi_{zz} + \delta^2(\phi_{xx} + \phi_{yy}) = 0 \\ \text{with} \\ \left.\begin{array}{l} \phi_z = \delta^2\{\eta_t + \varepsilon(\phi_x\eta_x + \phi_y\eta_y)\}; \\ \phi_t + \eta + \dfrac{1}{2}\varepsilon\left(\dfrac{1}{\delta^2}\phi_z^2 + \phi_x^2 + \phi_y^2\right) = 0 \end{array}\right\} \text{ on } z = 1 + \varepsilon\eta, \\ \text{and} \\ \phi_z = \delta^2(\phi_x b_x + \phi_y b_y) \text{ on } z = b. \end{array}\right\} \quad (2.132)$$

(In both these sets of equations we have taken the bed of the flow to be steady; that is, $z = b(x, y)$.)

For what we present here, we shall no longer consider the approximate equations obtained by setting $\varepsilon = 0$: of course, the inclusion of nonlinearity requires $\varepsilon \neq 0$. 'Full' nonlinearity occurs for ε fixed, that is, O(1), even if $\delta \to 0$; indeed, in this case, we may just as well set $\varepsilon = 1$. (This is equivalent to scaling the wave on the typical undisturbed depth of the water.) The consequences of allowing the strongest possible contribution from the nonlinearity will be the basis for much of what we shall present here, but we start with a simpler problem: $\varepsilon \to 0$. This is the problem first discussed by G. G. Stokes in 1847, and aims to produce higher approximations to the (linear) oscillatory wave (given in Section 2.1). This approach, as we shall see, is in the spirit of much that we shall present in later chapters.

2.5 The Stokes wave

The problem that we present here is that of determining the solution of equations (2.132) (which describe irrotational flow) as an asymptotic solution for $\varepsilon \to 0$ (at fixed δ). This is to be compared with the analysis given in Section 2.1 where only the first approximation was obtained (and there we worked from Euler's equation with the effects of surface tension retained). Here, we shall seek a solution in the form

$$Q \sim \sum_{n=0}^{\infty} \varepsilon^n Q_n,$$

where Q (and correspondingly Q_n) represents each of ϕ and η; these expansions, or more precisely, those for the velocity components (ϕ_x, ϕ_z), and for η, are to be uniformly valid as $t \to \infty$ and as $|x| \to \infty$. We shall restrict the discussion to plane harmonic waves that travel in the x-direction. The undisturbed water is stationary and the depth is constant (so we set $b = 0$).

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The procedure is, in principle, altogether straightforward, but complications do arise, not least because of the nature of the surface boundary conditions. To be consistent with our assumed form of solution (in powers of ε) we must expand these boundary conditions about $z = 1$. (Strictly, any requirement on the convergence of the series that we generate is unnecessary: our solution need only satisfy the conditions laid down for *asymptotic* validity as $\varepsilon \to 0$.) In addition, to describe the harmonic wave we introduce the phase variable

$$\theta = kx - \omega t$$

where we shall regard the wave number, k, as prescribed. Equations (2.132) now become

$$\phi_{zz} + \delta^2 k^2 \phi_{\theta\theta} = 0,$$

with

$$\phi_z + \varepsilon\eta\phi_{zz} + \frac{1}{2}\varepsilon^2\eta^2\phi_{zzz} = -\delta^2\omega\eta_\theta + \varepsilon\delta^2 k^2 \eta_\theta(\phi_\theta + \varepsilon\eta\phi_{\theta z}) + \mathrm{O}(\varepsilon^3);$$

$$\eta + \alpha - \omega(\phi_\theta + \varepsilon\eta\phi_{\theta z} + \frac{1}{2}\varepsilon^2\eta^2\phi_{\theta zz})$$
$$+ \frac{1}{2}\varepsilon\left\{\frac{1}{\delta^2}(\phi_z^2 + 2\varepsilon\eta\phi_z\phi_{zz}) + k^2(\phi_\theta^2 + 2\varepsilon\eta\phi_\theta\phi_{\theta z})\right\} = \mathrm{O}(\varepsilon^3)$$

both on $z = 1$, and

$$\phi_z = 0 \quad \text{on} \quad z = 0,$$

where we have incorporated the convenient shift $\phi \to \alpha t + \phi$ (which we shall discuss below; also cf. equation (1.23)). It will transpire that, in order to find a uniform solution, we must expand the frequency in terms of ε; thus we write

$$\omega \sim \sum_{n=0}^{\infty} \varepsilon^n \omega_n,$$

where the ω_n are constants, which may depend on k. This dependence of ω on ε, essentially the amplitude, is a very significant result, as we shall see.

The expansions for ϕ, η, and the constant α (treated like ω) are used in the above equations; the leading-order problem is then

$$\phi_{0zz} + \delta^2 k^2 \phi_{0\theta\theta} = 0$$

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with

$$\phi_{0z} = -\omega_0 \delta^2 \eta_{0\theta} \quad \text{and} \quad \eta_0 + \alpha_0 - \omega_0 \phi_{0\theta} = 0 \quad \text{on} \quad z = 1$$

and

$$\phi_{0z} = 0 \quad \text{on} \quad z = 0.$$

This is the familiar and standard problem; see Section 2.1 and Q2.5. The solution for a single harmonic wave is

$$\left.\begin{aligned} \eta_0 &= AE + \text{c.c.}, \quad \alpha_0 = 0, \\ \phi_0 &= -\frac{\mathrm{i}A}{\omega_0}\frac{\cosh(\delta k z)}{\cosh(\delta k)}E + \text{c.c.}, \end{aligned}\right\} \tag{2.133}$$

where $E = \exp(\mathrm{i}\theta)$, A is a complex constant and c.c. denotes the complex conjugate. We see that this solution does not require a contribution from α, but it exists only if

$$\omega_0^2 = \frac{k}{\delta}\tanh(\delta k); \tag{2.134}$$

cf. equations (2.9) and (2.13).

At the next order we obtain the equations

$$\phi_{1zz} + \delta^2 k^2 \phi_{1\theta\theta} = 0$$

with

$$\left.\begin{aligned} \phi_{1z} + \eta_0 \phi_{0zz} &= -\delta^2(\omega_1 \eta_{0\theta} + \omega_0 \eta_{1\theta}) + \delta^2 k^2 \eta_{0\theta} \phi_{0\theta}; \\ \eta_1 + \alpha_1 - (\omega_1 \phi_{0\theta} + \omega_0 \phi_{1\theta}) - \omega_0 \eta_0 \phi_{0\theta z} &+ \frac{1}{2}\left(\frac{1}{\delta^2}\phi_{0z}^2 + k^2 \phi_{0\theta}^2\right) = 0 \end{aligned}\right\} \text{ on } z = 1$$

and

$$\phi_{1z} = 0 \quad \text{on} \quad z = 0.$$

We could include a term in the solution of this problem which contributes to the first harmonic ($E^{\pm 1}$), but we choose not to do so; the amplitude of the first harmonic (to this order) is taken to be A. However, the surface boundary conditions *do* include terms $E^{\pm 1}$, but these are completely eliminated if we choose $\omega_1 = 0$; thus we set $\omega_1 = 0$. Now we seek a solution

$$\eta_1 = A_1 E^2 + \text{c.c.}, \quad \phi_1 = B_1 \cosh(2\delta k z) E^2 + \text{c.c.},$$

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and α_1 (a real constant) will be required here to remove the non-periodic term E^0 (which is generated by the product E^1E^{-1}). The corresponding terms in the first boundary condition exactly cancel.

Our solution for ϕ_1 satisfies Laplace's equation and the bottom boundary condition; the other two boundary conditions (on $z = 1$) yield

$$kB_1 \sinh(2\delta k) + \mathrm{i}\delta\omega_0 A_1 = \mathrm{i}\frac{\delta k^2}{\omega_0}A^2;$$

$$A_1 - 2\mathrm{i}\omega_0 B_1 \cosh(2\delta k) = \delta k A^2 \tanh(\delta k) - \delta k A^2 \operatorname{cosech}(2\delta k),$$

with

$$\alpha_1 = -2\delta k|A|^2 \operatorname{cosech}(2\delta k).$$

We see that the term α_1 is needed here; it can be associated with the arbitrary function, $f(t)$, that appears in the pressure equation, (1.23). It might be thought that such a term could not appear after we have introduced appropriate conditions at infinity; see equation (1.29). However, once we have fixed the undisturbed surface level at $\eta = 0$, the constant pressure condition has to be maintained in this way if the nonlinearity is also included. There is, nevertheless, an alternative which allows $\alpha_1 = 0$: this is to redefine the undisturbed water level as

$$\eta \sim -2\varepsilon\delta k|A|^2 \operatorname{cosech}(2\delta k),$$

which hydraulic engineers usually call the *set-down*. In any event, we see that the term αt does not contribute to the velocity components (ϕ_x, ϕ_z).

To proceed, we solve for A_1 and B_1 and simplify, to give

$$A_1 = A^2\delta k \coth(\delta k)\left\{1 + \frac{3}{2}\operatorname{cosech}^2(\delta k)\right\}, \quad B_1 = -\mathrm{i}A^2\frac{3}{4}\delta^2\omega_0 \operatorname{cosech}^4(\delta k).$$

Thus we have, so far, the asymptotic solution

$$\eta \sim AE + \varepsilon A^2 E^2 \delta k \coth(\delta k)\left\{1 + \frac{3}{2}\operatorname{cosech}^2(\delta k)\right\} + \text{c.c.} \tag{2.135}$$

and

$$\begin{aligned}\phi \sim -\frac{\mathrm{i}A}{\omega_0}E \operatorname{sech}(\delta k)\cosh(\delta kz) - 2\varepsilon\delta k|A|^2 t \operatorname{cosech}(2\delta k)\\ - \mathrm{i}\varepsilon A^2\frac{3}{4}\delta^2\omega_0 E^2 \operatorname{cosech}^2(\delta k)\cosh(2\delta kz) + \text{c.c.},\end{aligned} \tag{2.136}$$

both as $\varepsilon \to 0$. The non-uniformity implied by the contribution from αt, as $t \to \infty$, appears only in the expansion of ϕ; the relevant asymptotic

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expansions are for ϕ_x (that is, ϕ_θ) and ϕ_z, which do not contain this term. We now examine the next order, but only to demonstrate the rôle of ω_2 and how we determine its value.

The terms at $O(\varepsilon^2)$ yield the equations

$$\phi_{2zz} + \delta^2 k^2 \phi_{2\theta\theta} = 0,$$

with

$$\begin{aligned}\phi_{2z} + \eta_0\phi_{1zz} + \eta_1\phi_{0zz} + \frac{1}{2}\eta_0^2\phi_{0zzz}& \\ = -\delta^2(\omega_0\eta_{2\theta} + \omega_2\eta_{0\theta}) &+ \delta^2k^2(\eta_{0\theta}\phi_{1\theta} + \eta_{1\theta}\phi_{0\theta} + \eta_0\eta_{0\theta}\phi_{0\theta z});\\ \eta_2 + \alpha_2 - \omega_0\Big(\phi_{2\theta} + \eta_0\phi_{1\theta z} + \eta_1\phi_{0\theta z} + \frac{1}{2}\eta_0^2\phi_{0\theta zz}\Big) &- \omega_2\phi_{0\theta}\\ + \frac{1}{\delta^2}(\phi_{0z}\phi_{1z} + \eta_0\phi_{0z}\phi_{0zz}) &+ k^2(\phi_{0\theta}\phi_{1\theta} + \eta_0\phi_{0\theta}\phi_{0\theta z}) = 0\end{aligned}$$

both on $z = 1$, and

$$\phi_{2z} = 0 \quad \text{on} \quad z = 0.$$

To find ω_2 we must be more circumspect in our treatment of these equations than we were for ω_1. Here, the boundary conditions on $z = 1$ include terms $E^{\pm 1}$ which cannot be eliminated; thus our solution for ϕ and η must include these terms. But we know that the combinations $\phi_{2z} + \delta^2\omega_0\eta_{2\theta}$ and $\eta_2 - \omega_0\phi_{2\theta}$ (evaluated on $z = 1$) are essentially identical when evaluated from terms in $E^{\pm 1}$ and the expression for ω_0 is invoked. (This was how we determined ω_0 in the first place.) To be consistent, the same property must obtain for all the terms in $E^{\pm 1}$; this is possible only for one choice of ω_2. Let us now fill in some of the details in this calculation.

If we write

$$\eta_2 = A_2E + \text{c.c.}, \quad \phi_2 = B_2\cosh(\delta kz)E + \text{c.c.}$$

(which would constitute one part of the complete solution for η_2 and ϕ_2), then, on $z = 1$,

$$\begin{aligned}\phi_{2z} + \delta^2\omega_0\eta_{2\theta} &= \delta kB_2\sinh(\delta k)E + \mathrm{i}A_2\delta^2\omega_0E + \text{c.c.}\\ &= \mathrm{i}\delta^2\omega_0\Big(A_2E - \mathrm{i}\frac{k}{\delta}\frac{B_2}{\omega_0}\sinh(\delta k)E\Big) + \text{c.c.}\\ &= \mathrm{i}\delta^2\omega_0(A_2E - \mathrm{i}\omega_0B_2\cosh(\delta k)E) + \text{c.c.}\\ &= \mathrm{i}\delta^2\omega_0(\eta_2 - \omega_0\phi_{2\theta})\end{aligned}$$

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where we have used equation (2.134) for ω_0^2. Therefore we form $\mathrm{i}\,\delta^2\omega_0 \times$ (second boundary condition on $z = 1$) and subtract the first boundary condition, but we retain only the terms in E^1 (which can arise here from the products E^2E^{-1} and E^1E^0); these terms are to be absent from the combined boundary conditions, thereby fixing ω_2. After some rather tedious algebra, we find that the appropriate choice is

$$\omega_2 = \frac{1}{4}\delta^2k^2\omega_0|A|^2\{8\coth^2(\delta k) + 9\,\mathrm{cosech}^4(\delta k)\},$$

so the dispersion function becomes

$$\omega \sim \omega_0 + \frac{\varepsilon^2}{4}\delta^2k^2\omega_0|A|^2\{8\coth^2(\delta k) + 9\,\mathrm{cosech}^4(\delta k)\} \tag{2.137}$$

where ω_0 is obtained from equation (2.134).

The significant result embodied in equation (2.137), and first described by Stokes, is that the frequency (and hence the phase speed) now depends on the amplitude of the wave. This is a fundamental property of nonlinear waves, and has no counterpart in linear theory (but remember that, in linear theory, water waves are dispersive, so their speed does still depend on the wave number). In particular we see that

$$c_p \sim c_{p0}\left\{1 + \frac{\varepsilon^2}{4}\delta^3k^2|A|^2[8\coth^2(\delta k) + 9\,\mathrm{cosech}^4(\delta k)]\right\},$$

where $c_{p0} = \omega_0/k$ is the speed of linear waves; here, waves of larger amplitude travel faster (although we are still restricted by the small-amplitude assumption implied by $\varepsilon \to 0$).

Furthermore, the inclusion of higher-order terms in the representation of the surface profile (equation (2.135)) distorts its shape away from the (linear) sinusoidal curve. The effects of the nonlinearity are to make peaks narrower (sharper) and the troughs flatter; this tendency is depicted in Figure 2.18. The resulting profile more accurately portrays the gravity waves that are observed in nature. Later (Section 2.9) we shall describe more fully the characteristics of certain nonlinear waves for which the Stokes expansion can give only a hint.

Before we leave the Stokes expansion, we make two observations. First, we have presented the results for arbitrary wavelength (or depth); clearly, we may approximate further for long waves (or shallow water) and for short waves (or deep water). For example, from (2.137) and (2.135), we obtain

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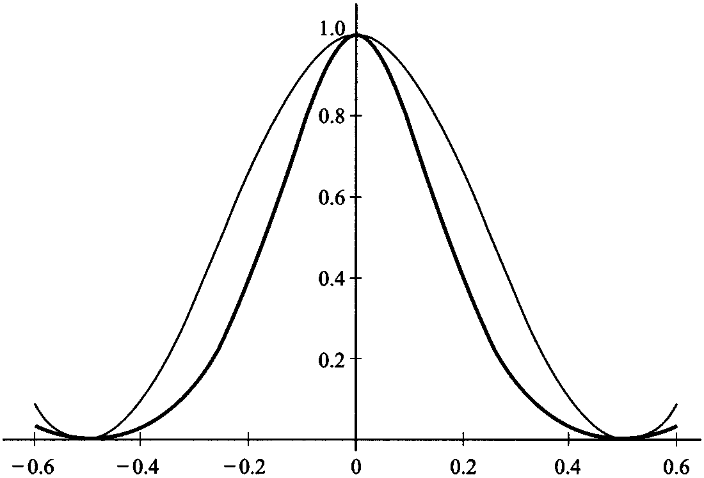

Figure 2.18. A nonlinear wave (——) and a corresponding linear wave (——) for comparison; the waves have been drawn with the same amplitude and the same period.

$$\omega \sim k\left\{1 - \frac{1}{6}\delta^2 k^2 + \frac{9}{4}\frac{\varepsilon^2 |A|^2}{\delta^2 k^2}\right\} \quad \text{as} \quad \delta \to 0,$$

and

$$\omega \sim \sqrt{\frac{k}{\delta}}\left\{1 + \frac{1}{2}\varepsilon^2 |A| \delta^2 k^2\right\} \quad \text{as} \quad \delta \to \infty,$$

provided we also have $(\varepsilon/\delta) \to 0$ in the former, and $(\varepsilon\delta) \to 0$ in the latter. (These simple derivations are left as an exercise.) Now, second, we may use our more complete results to compute, for example, the correct average mass flux in the water as the wave propagates; see Q2.32. Previously we calculated as far as $O(\varepsilon^2)$, yet the solution to this order was unknown. We have

$$\mathscr{F} = \frac{\varepsilon}{2\pi}\int_0^{2\pi}\int_0^{1+\varepsilon\eta} u \, dz \, d\theta$$

where

$$\eta \sim AE + \varepsilon A_1 E^2 + \text{c.c.}$$

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and

$$u \sim \frac{AE}{\omega_0}\frac{\cosh(\delta kz)}{\cosh(\delta k)} + \varepsilon 2\mathrm{i}B_1E^2\cosh(2\delta kz) + \text{c.c.};$$

see equations (2.133), (2.135) and (2.136). Thus we obtain

$$\mathscr{F} \sim \frac{\varepsilon}{2\pi}\int_0^{2\pi}\left\{\frac{AE}{\delta k\omega_0}\frac{\sinh[\delta k(1+\varepsilon AE)]}{\cosh(\delta k)} + \frac{\varepsilon \mathrm{i}B_1E^2}{\delta k}\sinh(2\delta k) + \text{c.c.}\right\}\mathrm{d}\theta$$

from which it is clear that the term at $O(\varepsilon)$ in the expression for u does not contribute at $O(\varepsilon^2)$ (because it is periodic in θ). The non-periodic term arising from the expansion of $\sinh[\delta k(1+\varepsilon AE)]$, exactly as in Q2.32, provides the $O(\varepsilon^2)$ term in $\mathscr{F}$. The conclusion we reached in Q2.32, it turns out, is correct: there is a mass flux of $O(\varepsilon^2)$ generated by the passage of the $O(\varepsilon)$ surface wave. (This is discussed further in Q4.4.)

2.6 Nonlinear long waves

We now undertake our first examination of a set of equations that describe fully nonlinear wave propagation. To simplify matters, we restrict the discussion to waves that are propagating in only one (spatial) dimension and, most importantly, we shall invoke the condition for long waves. From equations (2.131), for propagation in the x-direction and with the bed fixed at $z = 0$, we obtain

$$u_t + \varepsilon(uu_x + wu_z) = -p_x;$$
$$\delta^2\{w_t + \varepsilon(uw_x + ww_z)\} = -p_z;$$
$$u_x + w_z = 0,$$

with

$$w = \eta_t + \varepsilon u\eta_x \quad \text{and} \quad p = \eta \quad \text{on} \quad z = 1 + \varepsilon\eta$$

and

$$w = 0 \quad \text{on} \quad z = 0.$$

Then for long waves (or shallow water) we impose the condition $\delta \to 0$, so

$$p_z = O(\delta^2),$$

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and the first approximation for p requires that

$$p = \eta,$$

everywhere. The equations are now reduced to

$$u_t + \varepsilon(uu_x + wu_z) = -\eta_x; \quad u_x + w_z = 0, \tag{2.138}$$

with

$$w = \eta_t + \varepsilon u\eta_x \text{ on } z = 1 + \varepsilon\eta; \quad w = 0 \text{ on } z = 0,$$

to leading order as $\delta \to 0$.

These equations admit a solution for which $u = u(x, t)$ (which is the only solution if, somewhere, u is independent of z, for then it will remain so); thus w_z $(= -u_x)$ is independent of z, so

$$w = \left(\frac{\eta_t + \varepsilon u\eta_x}{1 + \varepsilon\eta}\right)z,$$

where the boundary conditions have been used. The two equations in (2.138) therefore become

$$u_t + \varepsilon uu_x + \eta_x = 0;$$
$$(1 + \varepsilon\eta)u_x + \eta_t + \varepsilon u\eta_x = 0,$$

where we have made no assumption about the size of ε. We wish to retain 'full' nonlinearity, so that $\varepsilon = O(1)$ as $\delta \to 0$; it is therefore convenient to set $\varepsilon = 1$ and to write the surface as

$$1 + \eta(x, t) = h(x, t).$$

Our pair of equations are then expressed as

$$u_t + uu_x + h_x = 0; \quad h_t + (hu)_x = 0; \tag{2.139}$$

these are often called the *shallow water* equations (for obvious reasons). The important simplifying assumption that leads to these equations is, of course, $\delta \to 0$; this, in turn, implies that $p = \eta$ (to leading order), which means that the pressure is everywhere dominated by the hydrostatic pressure distribution (see Q1.11). The higher-order corrections to the pressure, as the wave propagates, are ignored in this model.

An interesting observation about our equations (2.139) is made if we write

$$h(u_t + uu_x) = -hh_x = -\left(\frac{1}{2}h^2\right)_x,$$

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for then the shallow water equations take the form

$$u_t + uu_x = -\frac{1}{\rho}p_x, \quad \rho_t + (\rho u)_x = 0, \quad p = \frac{1}{2}\rho^2, \tag{2.140}$$

where ρ is written for h. Equations (2.140) are the equations of one-dimensional gas dynamics, for the *adiabatic law* $P \propto \rho^2$, that is, $p \propto \rho^\gamma$, $\gamma = 2$; see equations (1.2) and (1.12). Of course, for a real gas, $\gamma = 2$ cannot be realised; nevertheless, our equations (2.139) are identical in structure to the appropriate equations of gas dynamics (and, as such, are the basis for demonstrating some steady gas-dynamic flow phenomena on a water table). All this means that we may take over much of the analysis and discussion pertinent to gas dynamics. This we shall now do, at least in part, but we shall provide all the relevant information and derivations.

2.6.1 The method of characteristics

We have already found (Section 2.1), for long waves, that the speed of propagation of small amplitude waves is $\sqrt{gh_0}$ (in dimensional variables; see equation (2.11)). Here we are also working with long waves, so we might hope that a similar result obtains. Of course, our equations (2.139) are fully nonlinear, which could lead to some doubt about the validity of this proposition. To see that there is a connection, we introduce the definition

$$c(x, t) = \sqrt{h}, \tag{2.141}$$

which is the nondimensional equivalent of $\sqrt{gh_0}$ (and which also avoids the restriction to small amplitude waves). We note that h is the total depth, and so $h \geq 0$. Equations (2.139) then become

$$u_t + uu_x + 2cc_x = 0;$$

$$2cc_t + c^2u_x + 2ucc_x = 0 \quad \text{or} \quad (2c)_t + u(2c)_x + cu_x = 0,$$

which are added to give

$$(u + 2c)_t + u(u + 2c)_x + 2cc_x + cu_x = 0$$

and subtracted to give

$$(u - 2c)_t + u(u - 2c)_x + 2cc_x - cu_x = 0.$$

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This pair of equations is rewritten in the form

$$\left\{\frac{\partial}{\partial t}+(u+c)\frac{\partial}{\partial x}\right\}(u+2c)=0;$$
$$\left\{\frac{\partial}{\partial t}+(u-c)\frac{\partial}{\partial x}\right\}(u-2c)=0,$$

which can be solved directly (cf. equation (1.84)) to give

$$\left.\begin{aligned} u+2c = \text{ constant on lines } C^{+}\colon \frac{\mathrm{d}x}{\mathrm{d}t}=u+c;\\ u-2c = \text{ constant on lines } C^{-}\colon \frac{\mathrm{d}x}{\mathrm{d}t}=u-c,\end{aligned}\right\} \tag{2.142}$$

by the method of characteristics.

The lines (C^{+}, C^{-}) are the two families of characteristic lines, and the functions $(u \pm 2c)$, which are constant on their respective lines, are usually called the *Riemann invariants*. We see that these characteristic lines describe propagation at a speed $(\mathrm{d}x/\mathrm{d}t)$ that is either upstream or downstream $(\mp c)$ relative to the flow speed (u). The (implicit) solution can be expressed in the form

$$\left.\begin{aligned} u+2c=f(\alpha),\quad \alpha \text{ constant on lines } \frac{\mathrm{d}x}{\mathrm{d}t}=u+c;\\ u-2c=g(\beta),\quad \beta \text{ constant on lines } \frac{\mathrm{d}x}{\mathrm{d}t}=u-c,\end{aligned}\right\} \tag{2.143}$$

where f and g are arbitrary functions. The problem is then completely described if we are given, for example, the initial $(t=0)$ distribution (as a function of x) of both u and c (that is, h); this will prescribe both $f(\cdot)$ and $g(\cdot)$.

A particularly important and special class of solutions is obtained when one of the Riemann invariants (f or g) is constant *everywhere* (or at least constant where we seek a solution). These special types of solution are called *simple waves*. As an example, let us consider the propagation of a wave moving only rightwards into stationary water of constant depth $h=h_0$. All the C^{-} characteristics emanate from the undisturbed region (see Figure 2.19), so

$$u-2c=g=-2c_0,$$

since $u=0$ here and we have written $c_0=\sqrt{h_0}$. Now, $u-2c$ is constant everywhere and $u+2c$ is constant on C^{+} characteristics, so u and c are constant on these C^{+} lines; hence

$$x-(u+c)t=\alpha \quad \text{and then} \quad u+2c=f\{x-(u+c)t\}.$$

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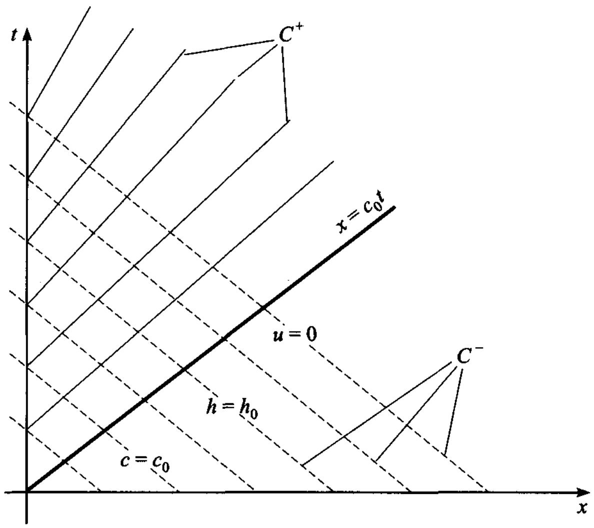

Figure 2.19. The characteristic lines, C^+ and C^-, for a wave moving rightwards into stationary water ($u = 0$) of constant depth ($h = h_0$) in $x > 0$.

On $t = 0$ we prescribe

$$h = H(x)$$

and so

$$f(x) = u + 2c = 4c - 2c_0 = 4\sqrt{H(x)} - 2\sqrt{h_0}.$$

Thus we have

$$u + 2c = 4\sqrt{H\{x - (u + c)t\}} - 2\sqrt{h_0},$$

and so

$$h(x, t) = H\{x - (u + \sqrt{h})t\}$$

where

$$u(x, t) = 3\{\sqrt{h(x, t)} - \sqrt{h_0}\}$$

which means that we can write, finally,

$$h(x, t) = H\{x - (3\sqrt{h} - 2\sqrt{h_0})t\} \tag{2.144}$$

the implicit solution for $h(x, t)$, given $H(x)$ and h_0. If the initial profile, $H(x)$, incorporates any wave of elevation (that is, $H(x) > 0$ for some x),

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then the solution given by (2.144) will eventually 'break' (in the sense that the characteristic lines then cross; cf. equation (1.85) *et seq.*, and Figures 1.5 and 1.6).

A second example, which is very much a classical one, is the problem of the 'dam break'. Although much is lost in the use of our shallow water equations in modelling this situation, these equations do capture the essential features of the resulting flow. Furthermore, this does prove to be an interesting – and surprisingly simple – application of equations (2.142) (or (2.143)). At time $t = 0$ the dam is broken, and therefore at this instant we suppose that $u = 0$ everywhere and that

$$h(x) = \begin{cases} h_0, & x < 0 \\ 0, & x > 0, \end{cases}$$

where h_0 (> 0) is constant. This represents (at $t = 0$) a vertical wall of water behind which the water is at rest at a constant depth. Our problem is therefore modelled by the instantaneous removal of the vertical retaining wall: hence the dam break problem.

Now, on the C^+ characteristics which emanate from the region $x < 0$ (where the water is situated at $t = 0$), we have that $u = 0$ and $c = \sqrt{h_0}$ there, and so

$$u + 2c = 2\sqrt{h_0} = \text{constant}$$

everywhere in the flow. Further, it is clear that an infinity of characteristic lines, each of different slope, will emerge from the origin $x = t = 0$ because of the step in $h(x)$ at $t = 0$. (That is, at $x = t = 0$, h must take all values $0 \leq h \leq h_0$ and each h determines the slope of a characteristic line.) To accommodate this phenomenon we require a degenerate form of the characteristic solution.

The C^- characteristics are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u - c$$

on which $u - 2c = $ constant; but $u + 2c$ is the same constant ($= 2\sqrt{h_0}$) everywhere (the simple wave condition), and so, corresponding to the first example, on C^- lines u, c, and then $u - c$ are constant. Hence the C^- characteristics are

$$x = (u - c)t + \text{constant} = (u - c)t$$

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since all these lines pass through $(0, 0)$; this pattern of characteristic lines is usually called an *expansion fan* (see Figure 2.20). Thus we have

$$u + 2c = 2\sqrt{h_0}, \quad u - c = x/t, \quad (c = \sqrt{h}),$$

which is the solution, for we now obtain

$$\sqrt{h} = \frac{1}{3}(2\sqrt{h_0} - x/t); \quad u = \frac{2}{3}(\sqrt{h_0} + x/t). \tag{2.145}$$

This solution is defined in the wedge (in (x, t)-space) from where $h = h_0$ to where $h = 0$, namely

$$-\sqrt{h_0} \le x/t \le 2\sqrt{h_0},$$

since

$$x/t = u - \sqrt{h} = 2\sqrt{h_0} - 3\sqrt{h}.$$

This solution describes an evolving surface profile, which is represented by the parabola

$$h(x, t) = \frac{1}{9}(2\sqrt{h_0} - \frac{x}{t})^2, \quad -\sqrt{h_0} \le \frac{x}{t} \le 2\sqrt{h_0},$$

at any fixed $t > 0$. In particular, at $x/t = 2\sqrt{h_0}$, we have $h = 0$: the wave front moves forward at a speed $2\sqrt{h_0}$. Correspondingly at $x/t = -\sqrt{h_0}$, where $h = h_0$, the uppermost point of the collapsing wall of water moves

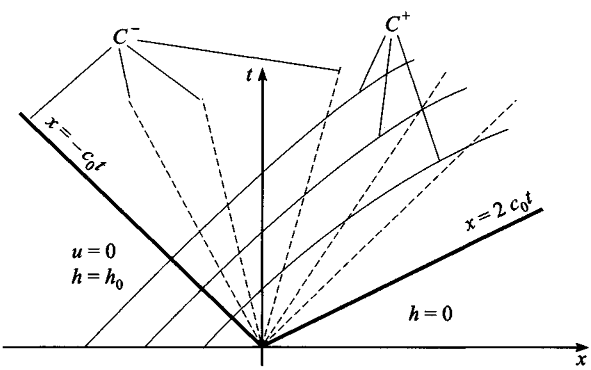

Figure 2.20. The characteristic lines, C^+ and C^-, for the dam-break problem; at $t = 0$ the water exists only in $x < 0$, where it is stationary ($u = 0$) and of constant depth ($h = h_0$).

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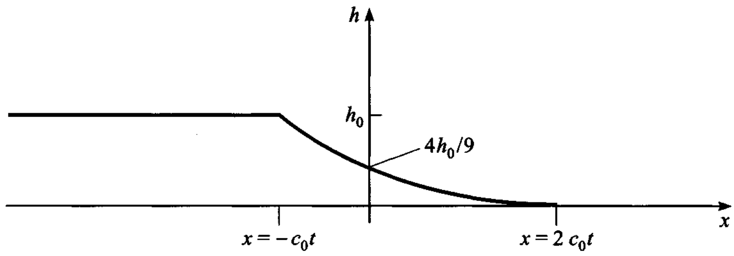

Figure 2.21. The surface profile at a time t after the dam has broken.

backwards at a speed $\sqrt{h_0}$; the profile is shown in Figure 2.21. Finally, we observe from solution (2.145) that, at $x = 0$ (which marks the initial position of the dam wall), the depth of the water remains at the constant value $4h_0/9$ for $t > 0$; indeed, as $t \to \infty$, the depth approaches this same constant value $(4h_0/9)$ everywhere.

These two examples that we have described are particularly straightforward because we have been able to incorporate the idea of a simple wave; some other related problems will be found in Q2.55–2.57. Of course, not all problems can be treated in this manner; certainly, if nontrivial (that is, variable) information is carried by both sets of characteristics then a more general approach must be adopted. This is what we now describe.

2.6.2 The hodograph transformation

A technique that is sometimes employed in the solution of ordinary differential equations is to interchange the rôles of the dependent and independent variables. So, for example, the equation

$$\{xf(y) + g(y)\}\frac{\mathrm{d}y}{\mathrm{d}x} = 1,$$

which is, in general, nonlinear, nonseparable, and nonhomogeneous, can be rewritten as

$$\frac{\mathrm{d}x}{\mathrm{d}y} - xf(y) = g(y).$$

This equation is linear in x; thus standard methods can be employed to find the solution $x = x(y)$. This same idea – to interchange the dependent

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and independent variables – provides a powerful method in the solution of certain types of partial differential equation. A particular example is our pair of shallow water equations, (2.139).

The method was first developed for the corresponding problem in gas dynamics, and it has retained its name used in this context: the *hodograph transformation*. (The word 'hodograph' is based on the Greek ὁδος, which means *way* or *road*, and is used to describe the (graphical) representation of a motion which uses as coordinates the components of the velocity vector rather than of the position vector.) As before, it is convenient to introduce $c = \sqrt{h}$, so we obtain from equations (2.139)

$$\left.\begin{aligned} u_t + uu_x + 2cc_x &= 0; \\ c_t + uc_x + \frac{1}{2}cu_x &= 0, \end{aligned}\right\} \tag{2.146}$$

where the coefficients of the derivative terms depend only on u and c, and otherwise all terms are first partial derivatives. We introduce the hodograph transformation

$$x = x(u, c), \quad t = t(u, c);$$

differentiating each of these with respect to x yields

$$1 = x_u u_x + x_c c_x; \quad 0 = t_u u_x + t_c c_x$$

and so

$$u_x = t_c/J, \quad c_x = -t_u/J \tag{2.147}$$

where

$$J = \frac{\partial(x, t)}{\partial(u, c)} = x_u t_c - x_c t_u \tag{2.148}$$

is the *Jacobian* of the transformation. Similarly, by differentiating with respect to t, we obtain two equations for u_t and c_t which yield

$$u_t = -x_c/J, \quad c_t = x_u/J, \tag{2.149}$$

and clearly these transformations of the derivatives require $J \neq 0$.

We now substitute from equations (2.147) and (2.149) into equations (2.146), to obtain

$$\begin{aligned} x_c - ut_c + 2ct_u &= 0; \\ x_u - ut_u + \frac{1}{2}ct_c &= 0, \end{aligned}$$

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which are *linear* equations in x and t. Furthermore, the two equations involve only either x_c or x_u; thus we may form x_{uc} from both and thereby eliminate x. Thus we have

$$\frac{\partial}{\partial u}(ut_c - 2ct_u) = \frac{\partial}{\partial c}(ut_u - \frac{1}{2}ct_c)$$

which simplifies to give

$$4ct_{uu} - ct_{cc} = 3t_c, \tag{2.150}$$

a linear second-order partial differential equation which can be solved by standard methods. Indeed, the characteristic variables for this equation, (2.150), are

$$\xi = u - 2c, \quad \eta = u + 2c$$

(combinations that we recognise from equations (2.142)), and then we obtain

$$2(\eta - \xi)t_{\xi\eta} = 3(t_\eta - t_\xi). \tag{2.151}$$

The solution is then completely determined by imposing appropriate boundary conditions, but these must (for equation (2.151)) describe t in the (ξ, η)–plane, a prescription that may not be straightforward. This is a difficulty that is often encountered in the hodograph method: interchanging the dependent and independent variables simplifies the governing equation(s), but complicates the boundary/initial conditions. A further inconvenience is that the simple-wave solutions cannot be accessed through the hodograph method, since the transformation is singular in this case. We can see this directly if we calculate

$$J^{-1} = u_x c_t - u_t c_x;$$

a simple wave exists when $u - 2c$ or $u + 2c$ is constant, and then clearly $J^{-1} = 0$. The transformation from $\{u(x, t), c(x, t)\}$ to $\{x(u, c), t(u, c)\}$, and back again, requires J (and therefore J^{-1}) to be finite and nonzero everywhere. Nevertheless, because equation (2.151) is linear, its solution can be approached by standard techniques (such as the separation of variables or integral transforms). Indeed, a more useful result in this respect is obtained from equation (2.150) by writing

$$t = \frac{1}{c}\frac{\partial T}{\partial c} \quad \text{where} \quad T = T(u/2, c),$$

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for then (2.150) becomes

$$T_{cvv} - T_{ccc} + \frac{2}{c}T_{cc} - \frac{2}{c^2}T_c = 3\left(\frac{1}{c}T_{cc} - \frac{1}{c^2}T_c\right),$$

where $v = u/2$. This equation is clearly

$$(T_{vv} - T_{cc})_c - (T_c/c)_c = 0$$

and so

$$T_{vv} = T_{cc} + \frac{1}{c}T_c + F(v)$$

which is the (inhomogeneous) cylindrical wave equation. If, finally, we map $T \to T + G(v)$ where $G'' = F$, we are left with

$$T_{vv} = T_{cc} + \frac{1}{c}T_c,$$

for which the application of the method of separation of variables, for example, is a familiar exercise; see Section 2.1, Q2.20 and Q2.21. The problem of finding solutions of the equation for T is addressed in Q2.58 and Q2.59.

2.7 Hydraulic jump and bore

A familiar phenomenon, observed particularly below weirs or dams, is the *hydraulic jump*. This is a relatively rapid increase in the depth of the water (essentially across the whole width of the river). The depth increase is often associated with a very turbulent mixing of the water, producing a significant energy loss there. (A similar jump can be seen when water from a tap hits a horizontal surface. In this case there is a (roughly) circular region of fast-flowing water moving radially outwards in a thin layer. This region is terminated by a sudden increase in depth: the circular hydraulic jump, Q2.62.) The hydraulic jump is stationary with respect to the riverbank; when this same phenomenon moves along a river it is called a *bore*. The most famous bore in Britain is the one that appears periodically on the River Severn, athough there are other rivers in other parts of the world that can boast much larger bores with depth changes of many feet.

For either the hydraulic jump or the bore, the change in depth can be a few metres but this will occur in, typically, a distance of only a metre or two. In other words, it might be reasonable to model this change as an abrupt jump or *discontinuity*; this is what we shall now investigate. A

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sketch of a section through an hydraulic jump (or bore) is shown in Figure 2.22. We have already mentioned the analogy between the water-wave equations and the equations of gas dynamics; the corresponding jump in gas dynamics is, of course, the shock wave associated with supersonic flow. In this case the jump is far narrower, and far more dramatic, and in consequence is more readily modelled as a discontinuity.

The hydraulic jump is formed when a wave has fully broken, and as such can also be observed at a shoreline after a wave has completely broken and is in the final stage of its run-up. The jump is what replaces the breaking of our nonlinear waves, which in that context corresponds to the crossing of the characteristics. The mathematical device we then adopt is to replace the region where the solution is multivalued by a line which separates the two sets of characteristics and, therefore, across which there will be a jump in value; see Section 1.4.1. This certainly enables us to produce a solution that is meaningful after breaking has occurred. (We recall that the accurate representation of a real breaking wave – at a shoreline, for example – requires a far more sophisticated theory than we are working with here.) However, as we mention in Section 1.4.1, a discontinuity cannot be regarded as a solution of our partial differential equations (for which continuity and some differentiability is needed). Our main task now is to describe how to overcome this mathematical difficulty, and then we shall be able to present some properties of the hydraulic jump (or bore) as based on our model.

The differential equations (the Euler equation and mass conservation equation) are not valid for discontinuous solutions, but the integral form from which they have been obtained (Sections 1.1.1 and 1.1.2) do admit

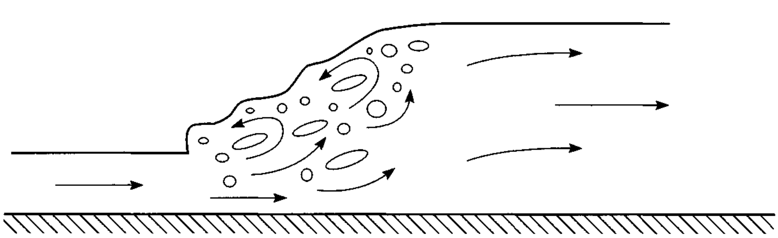

Figure 2.22. Sketch of an hydraulic jump, where the flow is from left to right. (The equivalent bore over stationary water moves to the left at the speed of the oncoming flow.)

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such solutions. Indeed, it is the integral form of the governing equations which should be regarded as the fundamental equations, and it is to these that we must turn. Now, rather than quote the general equations from Chapter 1, we choose to construct the appropriate forms from the equations (2.131). But to simplify the problem still further we shall incorporate, *ab initio*, the long-wave assumption, so that p and u (for one-dimensional motion) are independent of z. Thus we start from equations (2.139),

$$h_t + (hu)_x = 0; \quad u_t + uu_x + h_x = 0. \tag{2.152}$$

The first of these, the equation that describes the conservation of mass, is already in the form that we obtain by integrating in z, namely

$$\int_0^h u_x \mathrm{d}z + [w]_0^h = 0$$

so

$$\int_0^h u_x \mathrm{d}z + h_t + uh_x = 0,$$

which immediately gives the above equation (since $u = u(x, t)$). Thus the integral form of the equation we require is recovered if we integrate in x, between constants a and b, say; thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_a^b h\mathrm{d}x\right\} + [hu]_a^b = 0. \tag{2.153}$$

Let h (and u) be discontinuous at $x = X(t)$, so that we may accommodate either the hydraulic jump or the bore, and such that $a < X < b$. Then we may write (2.153) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_a^{X^-} h\,\mathrm{d}x + \int_{X^+}^b h\,\mathrm{d}x\right\} + [hu]_a^b = 0$$

where the superscripts $-/+$ denote evaluation as $x \to X^-/x \to X^+$, in the usual way. Upon differentiating under the integral signs (Q1.30), we obtain

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$$\int_a^b h_t \mathrm{d}x + h^- \frac{\mathrm{d}X}{\mathrm{d}t} - h^+ \frac{\mathrm{d}X}{\mathrm{d}t} + [hu]_a^b = 0,$$

where we have assumed that the *path* of the discontinuity, $x = X(t)$, is differentiable. Finally, we find the *jump condition* that must be satisfied across the discontinuity by taking the limit $a \to b$, which yields

$$-U[\![h]\!] + [\![hu]\!] = 0, \tag{2.154}$$

where $U(t) = \mathrm{d}X/\mathrm{d}t$ and $[\![y]\!] = y^+ - y^-$, the jump in value across $x = X(t)$. Equation (2.154) is the first jump condition, which, particularly in the context of the gas-dynamic shock wave, is usually called a *Rankine–Hugoniot condition*. We see that, if the discontinuity is stationary (the hydraulic jump), then $U = 0$, and so hu is conserved across the discontinuity. Indeed, we can write (2.154) as

$$[\![h(u - U)]\!] = 0,$$

since $U(t)$ is continuous, which states the otherwise obvious condition that mass (volume per unit width here) is conserved relative to the jump: what goes in from one side must come out the other.

The second equation in (2.152) is clearly the appropriate x-momentum equation based on Euler's equation, and hence this must be integrated in both z and x. First we have

$$\int_0^h (u_t + uu_x + h_x)\mathrm{d}z = 0$$

which yields immediately

$$hu_t + huu_x + hh_x = 0;$$

we rewrite this as

$$hu_t + uh_t + (huu)_x + hh_x = 0$$

by incorporating equation (2.152a). This is integrated in x, from a to b as above, to give

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_a^b hu \,\mathrm{d}x\right\} + \left[hu^2 + \frac{1}{2}h^2\right]_a^b = 0$$

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and then

$$\int_a^b (hu)_t \mathrm{d}x + (hu)^- \frac{\mathrm{d}X}{\mathrm{d}t} - (hu)^+ \frac{\mathrm{d}X}{\mathrm{d}t} + \left[hu^2 + \frac{1}{2}h^2 \right]_a^b = 0.$$

When we take $a \to b$ we obtain the second jump condition

$$-U[\![hu]\!] + [\![hu^2 + \frac{1}{2}h^2]\!] = 0, \tag{2.155}$$

which describes the conservation of momentum across $x = X(t)$. This is obviously interpreted as: the total momentum change across the moving front $(-U[\![hu]\!])$ is produced by the difference in momentum on either side $([\![hu^2]\!])$ plus the difference in the pressure forces $([\![\frac{1}{2}h^2]\!])$.

In summary, we have the pair of jump (Rankine–Hugoniot) conditions

$$-U[\![h]\!] + [\![hu]\!] = 0; \quad -U[\![hu]\!] + \left[\!\!\left[hu^2 + \frac{1}{2}h^2 \right]\!\!\right] = 0 \tag{2.156}$$

which can be regarded as two equations for h^- and u^-, say, given h^+, u^+ and U. That is, given the speed of the bore (which may be zero – the hydraulic jump), and the conditions on one side, equations (2.156) determine the conditions on the other side. However, it is reasonable to ask whether there is a third jump condition that has been overlooked, namely an energy condition. This possibility we shall now investigate.

The appropriate energy integral (an integration in z) is equation (1.47), which here becomes

$$\frac{\partial}{\partial t}\left\{ \int_0^h \left(\frac{1}{2}u^2 + z \right) \mathrm{d}z \right\} + \frac{\partial}{\partial x}\left\{ \int_0^h u\left(\frac{1}{2}u^2 + h \right) \mathrm{d}z \right\} = 0 \tag{2.157}$$

with $P = P_a + \rho g(h - z)$ and we have used our familiar nondimensionalisation (with $\varepsilon = 1$). Thus, since $u = u(x, t)$, we obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{2}hu^2 + \frac{1}{2}h^2 \right) + \frac{\partial}{\partial x}\left(\frac{1}{2}hu^3 + uh^2 \right) = 0,$$

and integrating in x across the jump $x = X(t)$ we find that

$$-U\left[\!\!\left[\frac{1}{2}hu^2 + \frac{1}{2}h^2 \right]\!\!\right] + \left[\!\!\left[\frac{1}{2}hu^3 + uh^2 \right]\!\!\right] = 0. \tag{2.158}$$

(This can be written down directly if we observe that, to obtain equations (2.156), we merely use the correspondence

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$$\frac{\partial}{\partial x}(\alpha) \to [\![\alpha]\!], \quad \frac{\partial}{\partial t}(\beta) \to -U[\![\beta]\!].)$$

We now have, apparently, a third equation relating $h^{\pm}, u^{\pm}$, and U, but this is unreasonable, since we would expect to be able to determine the conditions on one side given the conditions on the other (and given U) – and two equations are sufficient for this. In order to investigate the rôle of equation (2.158), let us consider the simple case of $U = 0$ (the hydraulic jump); then we obtain

$$\left[\!\!\left[\frac{1}{2}hu^3 + uh^2\right]\!\!\right] = 0.$$

But equations (2.156) imply, after a little manipulation, that

$$\begin{aligned}\left[\!\!\left[\frac{1}{2}hu^3 + uh^2\right]\!\!\right] &= \frac{1}{2}m[\![u^2]\!] + m[\![h]\!] \\ &= \frac{1}{4}(u^+ + u^-)[\![-h^2]\!] + m[\![h]\!] \\ &= \frac{\mathrm{m}}{4h^+h^-}(h^+ - h^-)^3, \end{aligned} \qquad (2.159)$$

where $m = (uh)^+ = (uh)^-$. Clearly the expression in (2.159) will be zero only if $h^+ = h^-$ (since $m \neq 0$): there is no jump. Consequently, if there *is* a jump, then we cannot impose the energy conservation condition, (2.158). Indeed, solving equations (2.156) for a jump, we may use the expressions in equation (2.158) (that is, (2.159) if $U = 0$) to determine the appropriate sense of the transition. The flow through the jump is taken to correspond to energy *loss*, this loss normally occurring (as we mentioned at the outset) because of the turbulent nature of the conditions in the neighbourhood of the jump. When we impose this energy-loss condition we find that, relative to the jump, the flow must enter from the faster and shallower side (that is, $u^- > u^+$ and $h^- < h^+$), and then the expression in (2.159) is negative (energy loss). (The alternative ($h^- > h^+$) requires an energy input and no mechanism in nature exists for providing an energy source.)

Finally, we briefly examine the consequences of using equations (2.156) for the hydraulic jump (so again $U = 0$). Suppose that we are given the conditions to the left, u^- and h^-; then we write

$$u^+ = \frac{u^-h^-}{h^+} \quad \text{and} \quad \frac{1}{2}(h^{+2} - h^{-2}) = h^-u^{-2} - h^+u^{+2};$$

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thus

$$\frac{1}{2}(h^{+2} - h^{-2}) = h^{-}u^{-2}\left(1 - \frac{h^{-}}{h^{+}}\right).$$

It is convenient to introduce

$$H = \frac{h^{+}}{h^{-}} \quad \text{and} \quad F = \frac{u^{-}}{\sqrt{h^{-}}},$$

then we obtain

$$H^2 - 1 = 2F^2\left(1 - \frac{1}{H}\right),$$

which has a root $H = 1$ (of no interest since this corresponds to no change) and otherwise

$$H = \frac{1}{2}\left(-1 \pm \sqrt{1 + 8F^2}\right).$$

A physically meaningful solution is possible only for the positive sign, and then $H > 1$ only if $F > 1$. The parameter F is called the *Froude number* (which in dimensional variables is usually written $u/\sqrt{gh_0}$); this parameter corresponds to the *Mach number* for the flow of a compressible gas. There can be a jump in water depth only if the flow upstream is *supercritical* ($F > 1$) (sometimes called *shooting flow*); if the flow is *subcritical* or *tranquil* ($F < 1$) then no hydraulic jump is possible.

We have commented that the energy loss at the hydraulic jump or bore is by virtue of the dissipation of this energy through the turbulent motion in the neighbourhood of the jump; see Figure 2.22. However, if the energy loss is not too great (typically, if $1 < F \lesssim 1.2$) then the required energy loss can be *transported away* by a train of waves on the downstream side of the jump. This gives rise to the so-called *undular bore*, which is a form of the bore that sometimes occurs on the River Severn. A more detailed discussion of this phenomenon, together with descriptions of how it may be modelled, will be given in Chapter 5.

2.8 Nonlinear waves on a sloping beach

In Section 2.2 we presented the theory of linearised long waves moving over a bed of constant slope, and in Section 2.5 we developed some of the ideas involved in the theory of nonlinear long waves. We now turn to a brief discussion of a mathematically interesting problem that combines

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these two phenomena, namely nonlinearity and variable depth. From Section 2.5, and following that development, we consider long waves ($\delta \to 0$) and 'full' nonlinearity ($\varepsilon = 1$) so that the governing equations are

$$u_t + uu_x + wu_z = -p_x, \quad p_z = 0, \quad u_x + w_z = 0$$

with

$$w = \eta_t + u\eta_x \quad \text{and} \quad p = \eta \quad \text{on} \quad z = 1 + \eta$$

and

$$w = ub'(x) \quad \text{on} \quad z = b(x).$$

Thus $p = \eta$ for all z and, as before, we take $u = u(x, t)$ so that

$$u_t + uu_x + \eta_x = 0 \quad \text{and} \quad w = \left(\frac{\eta_t + u\eta_x - ub'}{1 + \eta - b}\right)(z - b) + ub'$$

and then $u_x + w_z = 0$ yields

$$(1 + \eta - b)u_x + \eta_t + u\eta_x - ub' = 0.$$

It is convenient to introduce

$$d(x, t) = 1 + \eta(x, t) - b(x),$$

the local depth of the water, to give

$$u_t + uu_x + d_x - b'(x) = 0; \quad d_t + (du)_x = 0, \tag{2.160}$$

which are to be compared with equations (2.139). The important difference is, of course, the appearance of the term in $b'(x)$ in equations (2.160); for general $b(x)$ this makes the methods used earlier essentially inapplicable. However, one special case can be successfully explored, as Carrier and Greenspan (1958) first showed.

We choose $b'(x)$ to be a constant, so the bed is of constant slope; following equation (2.41) we write

$$b(x) = 1 - \alpha(x_0 - x), \quad \alpha > 0,$$

so that $b'(x) = \alpha$. Our equations (2.160) therefore become

$$u_t + uu_x + d_x - \alpha = 0; \quad d_t + (du)_x = 0. \tag{2.161}$$

We saw in Section 2.5.1 that $c = \sqrt{h}$ was a useful change of variable, and the same applies here; we introduce $c = \sqrt{d}$ to give

$$u_t + uu_x + 2cc_x - \alpha = 0$$

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and

$$2c_t + 2uc_x + cu_x = 0.$$

Again, we combine these to produce the equations written in characteristic form

$$(u + 2c)_t + u(u + 2c)_x + c(u + 2c)_x - \alpha = 0$$

and

$$(u - 2c)_t + u(u - 2c)_x - c(u - 2c)_x - \alpha = 0,$$

which can be expressed as

$$\left.\begin{aligned}\left\{\frac{\partial}{\partial t} + (u + c)\frac{\partial}{\partial x}\right\}(u + 2c - \alpha t) = 0;\\ \left\{\frac{\partial}{\partial t} + (u - c)\frac{\partial}{\partial x}\right\}(u - 2c - \alpha t) = 0,\end{aligned}\right\} \tag{2.162}$$

by interpreting α as $\partial(\alpha t)/\partial t$. Thus (cf. equations (2.142)) we have

$$\left.\begin{aligned}u + 2c - \alpha t = \text{ constant on lines } C^+\text{: } \frac{\mathrm{d}x}{\mathrm{d}t} = u + c;\\ u - 2c - \alpha t = \text{ constant on lines } C^-\text{: } \frac{\mathrm{d}x}{\mathrm{d}t} = u - c,\end{aligned}\right\} \tag{2.163}$$

so the method of characteristics again results in a particularly simple structure.

The important realisation described by Carrier and Greenspan was that this problem, like that with $\alpha = 0$, can be linearised by an appropriate hodograph transformation. It is far from obvious that this is a possibility, since the method described earlier certainly requires the cancellation of the Jacobian (J) throughout the equation, which apparently cannot happen with $\alpha \neq 0$. To proceed, we recall that the neatest form of our earlier calculation involved $\xi = u - 2c$ and $\eta = u + 2c$, which led to equation (2.151). Here, we define corresponding variables

$$\xi = u - 2c - \alpha t, \quad \eta = u + 2c - \alpha t,$$

for use in the hodograph method, and transform

$$(x, t) \rightarrow (\xi, \eta).$$

This gives, after differentiating with respect to x,

$$1 = x_\xi(u_x - 2c_x) + x_\eta(u_x + 2c_x); \quad 0 = t_\xi(u_x - 2c_x) + t_\eta(u_x + 2c_x)$$

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and so

$$u_x = \frac{1}{2}(t_\eta - t_\xi)/J, \quad c_x = -\frac{1}{4}(t_\xi + t_\eta)/J$$

where, here, the Jacobian is

$$J = \frac{\partial(x, t)}{\partial(\xi, \eta)} = x_\xi t_\eta - x_\eta t_\xi.$$

Similarly, the derivatives with respect to t yield

$$u_t = \alpha + \frac{1}{2}(x_\xi - x_\eta)/J, \quad c_t = \frac{1}{4}(x_\xi + x_\eta)/J.$$

Equations (2.162) therefore become

$$x_\xi - \frac{1}{4}(\xi + 3\eta + 4\alpha t)t_\xi = 0;$$
$$x_\eta - \frac{1}{4}(3\xi + \eta + 4\alpha t)t_\eta = 0,$$

which are *nonlinear* in t; this is bad news but exactly what we would have expected. However, when we form $x_{\xi\eta}$, and eliminate this term between these two equations, we also eliminate the nonlinear term – this is the crucial observation presented in Carrier and Greenspan (1958). Thus we finally obtain

$$2(\eta - \xi)t_{\xi\eta} = 3(t_\eta - t_\xi),$$

the same linear equation for $t(\xi, \eta)$ that we found for the nonlinear problem with *constant* depth, equation (2.151). The reduction of this equation to the cylindrical wave equation then follows (much as described in Section 2.6.2). Simple solutions of this standard equation can now be used to describe the behaviour of a nonlinear wave as it runs up a beach, for example; cf. Section 2.2. This particular application is addressed through the exercises (Q2.58 and Q2.59).

2.9 The solitary wave

At this stage in our investigations it would not be unreasonable to suppose that the fully nonlinear (inviscid) equations of motion admit travelling-wave solutions of permanent form: that is, waves that propagate at constant speed without change of shape (see Q1.55). We have previously (Section 2.4) obtained approximations to the periodic waves of this type – the Stokes wave – where the wave profile is a distortion of the sine wave

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and the speed is dependent on both the wave number and the amplitude. The appearance of the amplitude here is indicative of the rôle of the nonlinear terms, and also suggests that waves of larger amplitude might be possible (even if we cannot express them in closed form).

It is a matter of observation that gravity waves of permanent form, and of considerable amplitude, can propagate on the surface of water. Indeed, this can occur whether the water is stationary or is moving with some velocity distribution below the surface. In particular, it is sometimes observed that single waves can be generated. These have a profile which is a symmetrical hump of water which drops smoothly back to the undisturbed surface level far ahead and far behind the wave; the wave propagates at a constant speed. This wave was first observed and described by J. Scott Russell, an engineer, naval architect and Victorian man of affairs. In 1834 he was observing the motion of a boat on the Edinburgh–Glasgow canal and the waves that it generated. Russell's description of what he saw is now much-quoted, but it still evokes the era and the man; we make no apologies for reproducing it here. In his 'Report on Waves' to the British Association meeting (at York) in 1844, he writes:

> I believe I shall best introduce the phaenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel.

After his initial observations, Russell performed a number of laboratory experiments to investigate the nature of what he called 'the great wave of translation', but which soon came to be known as the *solitary wave*. The most significant experiment involved the dropping of a weight at one end of a water channel (see Figure 2.23). He found that the volume of water displaced was the volume of water in the wave and, by careful measurement, that the wave moved at a speed, c, where

$$c^2 = g(h_0 + a),$$

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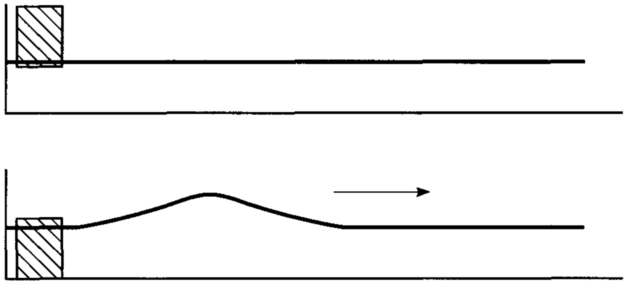

Figure 2.23. J. Scott Russell's experiment in which a weight is dropped at one end of the channel; the displaced water is propagated away as a solitary wave.

where h_0 is the undisturbed depth of the water and a is the amplitude of the wave. We see that we recover the wave speed of small-amplitude long waves ($c = \pm\sqrt{gh_0}$, equation (2.11)). Furthemore, it is clear that higher waves (that is, larger a) travel faster (cf. equation (2.137) *et seq.*). Here we have described a wave of *elevation*; the corresponding wave of depression does not exist, for it immediately collapses into a train of oscillatory waves.

Early attempts were made by Boussinesq (1871) and Rayleigh (1876) to find a mathematical description of the solitary wave. On the basis that the wave is long ($\delta \to 0$ in our terminology), they were able to confirm Russell's formula for the speed of the wave, and also to show that the profile is accurately represented by the sech^2 function (although this requires the additional assumption of small amplitude). (In the early days, the existence of this wave excited some controversy; in fact, both Airy and Stokes were initially of the opinion that it could not exist.) Much of the mathematical detail in this description will be developed here and in the later chapters. Indeed, it is the mathematical investigations that were initiated by Russell's observations that have eventually led to the extensive and modern ideas in nonlinear wave propagation, and in water-wave theory in particular, that we shall describe in the following chapters.

We begin our study of the solitary wave by treating the flow as irrotational, with the wave propagating (as a plane wave) in the x-direction. Thus, from equations (2.132), we have

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$$\left.\begin{array}{l}\phi_{zz} + \delta^2\phi_{xx} = 0, \\ \text{with} \\ \left.\begin{array}{l}\phi_z = \delta^2(\eta_t + \varepsilon\phi_x\eta_x); \\ \phi_t + \eta + \dfrac{1}{2}\varepsilon\left(\dfrac{1}{\delta^2}\phi_z^2 + \phi_x^2\right) = 0,\end{array}\right\} \text{ on } z = 1 + \varepsilon\eta \\ \text{and} \\ \phi_z = 0 \quad \text{on} \quad z = 0,\end{array}\right\} \qquad (2.164)$$

where the bed is taken to be fixed and horizontal ($b = 0$). The general solitary-wave solution is associated with arbitrary values of ε and δ; they are not assumed to be small. It is convenient (as we have done previously) to set $\varepsilon = 1$, but retain the parameter δ in our formulation. We are seeking a travelling-wave solution, and so we treat $\phi = \phi(\xi, z)$ and $\eta = \eta(\xi)$ where $\xi = x - ct$, and c is the (nondimensional) speed of the wave. Then from equations (2.164) we obtain

$$\left.\begin{array}{l}\phi_{zz} + \delta^2\phi_{\xi\xi} = 0, \\ \text{with} \\ \left.\begin{array}{l}\phi_z = \delta^2(\phi_\xi - c)\eta_\xi; \\ -c\phi_\xi + \eta + \dfrac{1}{2}\left(\dfrac{1}{\delta^2}\phi_z^2 + \phi_\xi^2\right) = 0,\end{array}\right\} \text{ on } z = 1 + \eta \\ \text{and} \\ \phi_z = 0 \quad \text{on} \quad z = 0.\end{array}\right\} \qquad (2.165)$$

We first see if these equations admit a solution that represents a profile which decays exponentially as $|\xi| \to \infty$. Thus we write

$$\eta \sim a\mathrm{e}^{-\alpha|\xi|}, \quad \phi \sim \psi(z)\mathrm{e}^{-\alpha|\xi|}, \quad |\xi| \to \infty,$$

where α (> 0) is the exponent; it is clear that both η and ϕ must have the same exponential behaviour in order to satisfy the surface boundary conditions. Laplace's equation (in (2.165)) then requires that

$$\psi'' + \alpha^2\delta^2\psi = 0$$

so

$$\psi = A\cos(\alpha\delta z),$$

when the boundary condition on $z = 0$ is invoked; A is an arbitrary constant. The leading-order balance from the boundary conditions on $z \sim 1$ gives (for $\xi > 0$, say)

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$$-A\alpha\delta\sin(\alpha\delta) = c a \alpha\delta^2, \quad cA\alpha\cos(\alpha\delta) + a = 0$$

so

$$c^2 = \frac{\tan(\alpha\delta)}{\alpha\delta}. \tag{2.166}$$

A solution with the required behaviour does therefore exist provided c (the speed, which here is the same as the Froude number since the non-dimensionalisation uses $\sqrt{gh_0}$; see Section 2.7) and α (the exponent) are related by equation (2.166), a result first found by Stokes (1880). All solitary waves exhibit exponential decay in their tails and all satisfy the relation (2.166).

Another important and general question addressed by Stokes in 1880 concerned the notion of a *highest wave*; that is, to examine what might limit the amplitude of the solitary wave and then what conditions obtain when this occurs. We consider a wave of permanent form travelling at the speed c in the positive x-direction over water of constant depth which is stationary at infinity. It is convenient to use a coordinate which is moving at the speed c, so that in this frame the wave is stationary; see Figure 2.24. We introduce $\Phi(\xi, z)$

$$\Phi = \phi - c\xi \quad \text{so that} \quad \Phi_\xi = \phi_\xi - c;$$

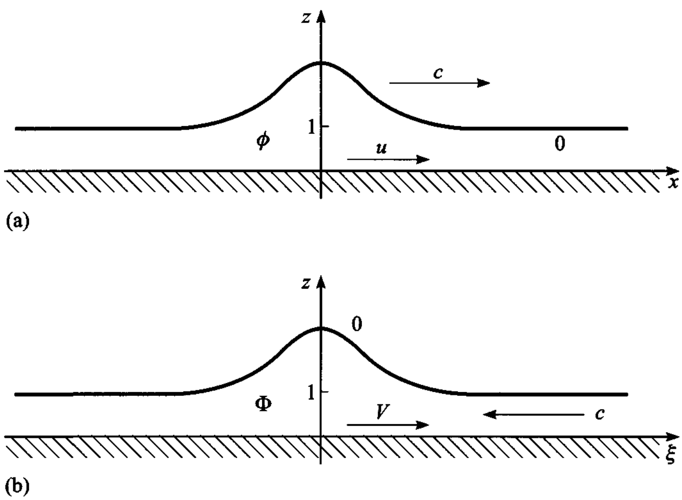

Figure 2.24. (a) The physical frame of reference for the solitary wave moving at speed c to the right into stationary water. (b) The frame of reference moving at speed c to the right.

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that is, $V = u - c$, since $u = \phi_x = \phi_\xi$, where $\xi = x - ct$. Equations (2.165) then become

$$\left.\begin{aligned} \Phi_{zz} + \delta^2\Phi_{\xi\xi} &= 0; \\ \Phi_z = \delta^2\Phi_\xi\eta_\xi \text{ and } 2\eta - c^2 + \frac{1}{\delta^2}\Phi_z^2 + \Phi_\xi^2 &= 0 \text{ on } z = 1 + \eta; \\ \Phi_z = 0 \text{ on } z = 0. & \end{aligned}\right\} \quad (2.167)$$

Stokes argued that the highest wave will be attained when the fluid-particle speed at the peak of the wave is equal to the speed of the wave. For waves of small amplitude, the particle speed is certainly less than the wave speed; if the particle speed exceeds the wave speed then the wave will be breaking and so cannot be steady; that is, not of permanent form. Thus $V = \Phi_\xi = 0$ at the peak, where $\eta = \eta_0$ say (so $c^2 = 2\eta_0$); further, we shift the origin of the (z, ξ)-coordinates to the peak of the wave, so that the peak is now at $z = 0 = \xi$ (which is how we have presented Figure 2.24).

The most direct route is to invoke the approach based on analytic functions of a complex variable; we therefore write

$$\Phi + \mathrm{i}\Psi = F(Z), \quad Z = \xi + \mathrm{i}\delta z,$$

where $\Psi(\xi, z)$ is the stream function for the flow (see Q1.25). In the neighbourhood of $Z = 0$ we seek a solution in the form

$$F(Z) \sim AZ^m, \quad |Z| \to 0,$$

and at the surface

$$\eta \sim -H|\xi|^n, \quad \xi \to 0, \ (H > 0)$$

where this η is relative to the peak at $\eta = \eta_0$; we expect $n > 0$ and $m > 1$ for physically reasonable behaviours near the peak. The kinematic surface condition in equations (2.167) then yields

$$\mathscr{R}\{\mathrm{i}\delta AmZ_0^{m-1}\} \sim -\delta^2 nH\xi^{n-1}\mathscr{R}\{AmZ_0^{m-1}\}, \quad Z_0 \sim \xi - \mathrm{i}\delta H\xi^n,$$

(for $\xi > 0$, say) which requires that $n = 1$. Thus the surface, in this limiting case, has a peak which is not smooth: it has a sharp crest. The dynamic (pressure) boundary condition now gives (again for $\xi > 0$)

$$-2H\xi \sim \xi^{2(m-1)}\left(\frac{1}{\delta^2}\left[\mathscr{R}\{\mathrm{i}\delta Am(1 - \mathrm{i}\delta H)^{m-1}\}\right]^2 + \left[\mathscr{R}\{Am(1 - \mathrm{i}\delta H)^{m-1}\}\right]^2\right)$$

which requires $m = 3/2$.

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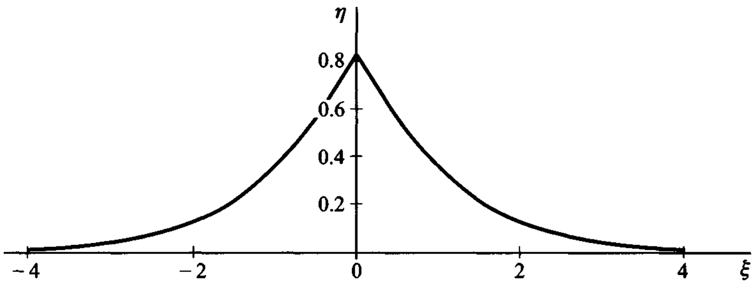

Figure 2.25. The highest wave of Stokes, based on the calculations described in Q2.63; here we have approximated the wave by two exponentials.

Finally, the angle of the wedge that forms the sharp crest is determined directly by this value of m. The solution near the crest is described by

$$\Phi + \mathrm{i}\Psi = F(Z) \sim AZ^{3/2}, \quad |Z| \to 0,$$

and this represents flow in a wedge of angle $\theta = 2\pi/3$; that is, $Z^{3/2}$ is the complex potential for a flow with boundaries $\theta = 0$, $\theta = 2\pi/3$ (and also $\theta = 4\pi/3$). For our problem, the requirement for symmetry about $\xi = 0$ leads to a choice of the complex constant A that, when combined with $Z^{3/2}$, implies a rotation of these boundaries. Thus, near $Z = 0$, the sharp crest is represented by the lines $\theta = 7\pi/6$, $\theta = 11\pi/6$: the crest includes an angle of 120°, the result first found by Stokes. We must emphasise that neither large-amplitude solitary waves, nor the sharp-crested highest wave, can be represented by a mathematical expression of closed form. This wave, based on a numerical approximation for its shape, is shown in Figure 2.25; see Q2.63. Nevertheless, the work initiated by Longuet-Higgins has enabled very accurate numerical representations of these waves and their properties to be obtained; see the section on Further Reading at the end of this chapter (and also Section 2.9.2).

2.9.1 The sech^2 solitary wave

In our discussion of the solitary wave thus far we have described various exact results that provide some useful information about the nature of this wave. What we cannot do is to present a complete solution of the governing equations, for arbitrary amplitude, which would then give us a mathematical representation of the solitary wave. Nevertheless, much that we have described can be incorporated in very accurate numerical

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solutions of these equations (and employed with great success by Longuet-Higgins and his co-workers). We therefore return to the approach that was first developed by Boussinesq and Rayleigh, which we mentioned earlier. We shall now see how we can proceed with an appropriate approximation of the equations; this eventually leads to a fundamental equation that provides the starting point for the work in the next chapter.

The equations are those given in (2.164), and we examine these for the case of long waves and small amplitude. The solitary wave extends from $-\infty$ to $+\infty$, so its length scale is certainly much greater than any (finite) depth of water. The assumption of long waves ($\delta \to 0$) should therefore be appropriate for the solitary wave. The restriction to small amplitude ($\varepsilon \to 0$) is necessary because we cannot otherwise make headway. In the initial stages of the calculation we shall treat these two parameters as independent.

Laplace's equation, from equation (2.164), is

$$\phi_{zz} + \delta^2 \phi_{xx} = 0$$

which, for small δ, clearly has the asymptotic solution

$$\phi(x, t, z; \delta) \sim \sum_{n=0}^{\infty} \delta^{2n} \phi_n(x, t, z), \quad \delta \to 0,$$

where

$$\phi_0 = \theta_0(x, t)$$

in order to satisfy the bottom boundary condition; θ_0 is an arbitrary function. The higher-order terms are given by

$$\phi_{n+1zz} = -\phi_{nxx}, \quad n = 0, 1, 2, \ldots .$$

We therefore obtain

$$\phi_1 = -\frac{1}{2} z^2 \theta_{0xx} + \theta_1(x, t);$$

$$\phi_2 = \frac{1}{24} z^4 \theta_{0xxxx} - \frac{1}{2} z^2 \theta_{1xx} + \theta_2(x, t),$$

and so on, where each θ_n is an arbitrary function and each ϕ_n satisfies the boundary condition

$$\phi_{nz} = 0 \quad \text{on} \quad z = 0.$$

The expansion for ϕ is used in the two surface boundary conditions (2.164), which involve evaluation on $z = 1 + \varepsilon\eta$. The first of these gives

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$$-(1+\varepsilon\eta)\theta_{0xx}+\delta^2\left\{\frac{1}{6}(1+\varepsilon\eta)^3\theta_{0xxxx}-(1+\varepsilon\eta)\theta_{1xx}\right\}+\dots$$
$$\sim\eta_t+\varepsilon\eta_x\left\{\theta_{0x}+\delta^2\left[\theta_{1x}-\frac{1}{2}(1+\varepsilon\eta)^2\theta_{0xxx}\right]+\dots\right\}, \quad (2.168)$$

and the second becomes

$$\theta_{0t}+\delta^2\left\{-\frac{1}{2}(1+\varepsilon\eta)^2\theta_{0xxt}+\theta_{1t}\right\}+\dots+\eta$$
$$\sim-\frac{1}{2}\varepsilon\delta^2\{-(1+\varepsilon\eta)\theta_{0xx}+\dots\}^2$$
$$-\frac{1}{2}\varepsilon\left\{\theta_{0x}+\delta^2\left[\theta_{1x}-\frac{1}{2}(1+\varepsilon\eta)^2\theta_{0xxx}\right]+\dots\right\}^2. \quad (2.169)$$

Now, for $\varepsilon\to 0$ and $\delta\to 0$, we see that the leading order terms yield

$$-\theta_{0xx}\sim\eta_t \quad \text{and} \quad \theta_{0t}\sim-\eta \quad (2.170)$$

and so

$$\theta_{0xx}\sim\theta_{ott}.$$

Thus we seek a solution which depends on $\xi=x-t$ (for right-running waves). This means that the wave will propagate, at this order of approximation, at the (nondimensional) speed of unity, which is completely consistent with our earlier work on long waves (see equations (2.10), (2.137), *et seq.*). We therefore treat both θ and η, in their dependence on x and t, as functions of $\xi=x-t$ and t; equations (2.170) then become

$$-\theta_{0zz}\sim\eta_t-\eta_\xi \quad \text{and} \quad \theta_{0t}-\theta_{0\xi}\sim-\eta$$

which imply

$$2\theta_{0t\xi}\sim\theta_{0tt}.$$

But when these terms are balanced against the others in the boundary conditions (2.168) and (2.169) we see that derivatives in t are small; cf. equation (1.99) and Q1.47–Q1.54. Thus we proceed with

$$\xi=x-t \quad \text{and} \quad \tau=\Delta t, \quad \Delta\to 0,$$

and we shall choose Δ later.

Our two surface boundary conditions (2.168, 2.169) now become, upon retaining only terms as small as $O(\varepsilon)$, $O(\delta^2)$ and $O(\Delta)$,

$$-(1+\varepsilon\eta)\theta_{0\xi\xi}+\delta^2\left(\frac{1}{6}\theta_{0\xi\xi\xi\xi}-\theta_{1\xi\xi}\right)\sim\Delta\eta_\tau-\eta_\xi+\varepsilon\eta\theta_{0\xi}$$

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and

$$\Delta\theta_{0\tau} - \theta_{0\xi} + \delta^2\left(\frac{1}{2}\theta_{0\xi\xi\xi} - \theta_{1\xi}\right) + \eta \sim -\frac{1}{2}\varepsilon(\theta_{0\xi})^2.$$

The second of these is differentiated with respect to ξ and subtracted from the first, thereby eliminating the terms $-\theta_{0\xi\xi} + \eta_\xi$; this produces

$$-\varepsilon\eta\theta_{0\xi\xi} + \delta^2\left(\frac{1}{6}\theta_{0\xi\xi\xi\xi} - \theta_{1\xi\xi}\right) - \Delta\theta_{0\tau\xi} - \delta^2\left(\frac{1}{2}\theta_{0\xi\xi\xi\xi} - \theta_{1\xi\xi}\right)$$
$$\sim \Delta\eta_\tau + \varepsilon\eta\theta_{0\xi} + \varepsilon\theta_{0\xi}\theta_{0\xi\xi}, \qquad (2.171)$$

and we see that the terms in θ_1 cancel identically. Finally, from the second equation in (2.170), we have that

$$\eta = \theta_{0\xi} + \mathrm{O}(\Delta)$$

so (2.171) is rewritten as

$$2\Delta\eta_\tau + 3\varepsilon\eta\eta_\xi \sim -\frac{\delta^2}{3}\eta_{\xi\xi\xi}.$$

Let us choose $\varepsilon = \mathrm{O}(\delta^2)$ and $\Delta = \varepsilon$, and write $\delta^2 = K\varepsilon$, then the leading-order equation for the surface profile is

$$2\eta_\tau + 3\eta\eta_\xi + \frac{K}{3}\eta_{\xi\xi\xi} = 0, \qquad (2.172)$$

the Korteweg–de Vries (KdV) equation (Korteweg and de Vries, 1895); cf. equation (1.102), Q1.47–Q1.49 and Q1.55. This equation describes a balance between nonlinearity ($\eta\eta_\xi$), which tends to steepen the wave profile, and dispersion (by virtue of $\eta_{\xi\xi\xi}$) which works the other way. The solitary wave is that wave of permanent form for which this balance is precisely maintained. To see how this happens we seek the travelling-wave solution of equation (2.172) by writing $\eta = f(\xi - c\tau)$, for some constant c, then

$$-2cf' + 3ff' + \frac{K}{3}f''' = 0; \qquad (2.173)$$

see Q1.55. The solution of this equation (see Q2.64) which satisfies

$$f, f', f'' \to 0 \quad \text{as} \quad |\xi - c\tau| \to \infty$$

is

$$f = 2c\,\mathrm{sech}^2\left\{\sqrt{\frac{3c}{2K}}(\xi - c\tau)\right\}$$

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or

$$\varepsilon\eta \sim \varepsilon a \operatorname{sech}^2\left[\sqrt{\frac{3a}{4K}}\{x - (1 + \frac{1}{2}\varepsilon a)t\}\right], \qquad (2.174)$$

where $\varepsilon\eta$ is the surface wave and its amplitude is εa $(= 2\varepsilon c)$. This is the *sech² solitary wave*, which is the small-amplitude version of the classical solitary wave. We see that the speed of the wave $(1 + \frac{1}{2}\varepsilon a)$ increases as εa increases; indeed, solution (2.174) is defined for all $\varepsilon a > 0$ (but remember that it is a solution of the governing equations only for small ε, since we have used $\varepsilon = O(\delta^2)$ and $\delta \to 0$). The wave speed agrees with the early observations of Russell for, in nondimensional variables, the speed is

$$\sqrt{1 + \varepsilon a} \sim 1 + \frac{1}{2}\varepsilon a \quad \text{as} \quad \varepsilon \to 0.$$

Finally, we observe that the 'width' of this solitary wave (defined as the distance between points of height $\frac{1}{2}\varepsilon a$, say) is inversely proportional to $\sqrt{a}$. This means that taller waves not only travel faster but are also narrower; see Figure 2.26. The behaviour of the exponential tails should also satisfy the general result given by equation (2.166); see Q2.64.

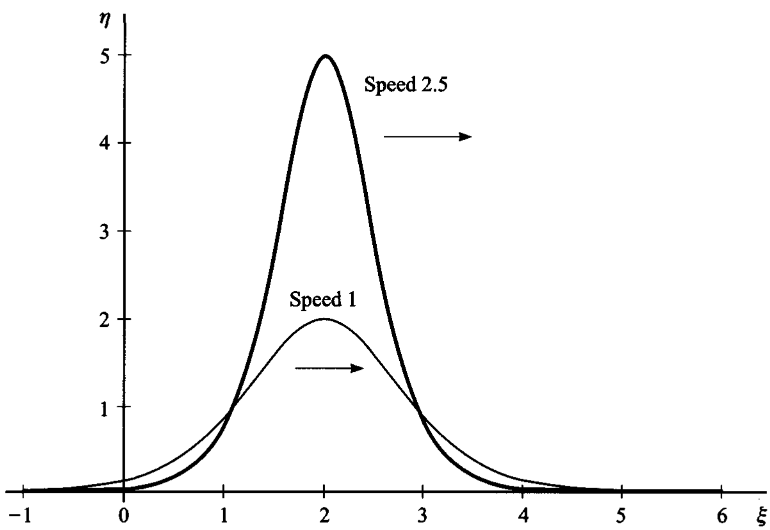

Figure 2.26. Two sech² solitary waves, each drawn in the frame $\xi = x - ct$ for $c = 1$ and $c = 2.5$.

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In conclusion, two comments: the first addresses a general observation about a crucial assumption underlying the calculation that we have just presented. It would appear that we can obtain the sech^2 solitary wave (via the KdV equation) only if a special balance of parameter values arises, namely $\varepsilon = \mathrm{O}(\delta^2)$. (The choice of the time-scale, Δ, is at our disposal; this merely tells us when and where to look for the wave.) This requirement for the balance would suggest that the solitary wave is a rare occurrence, rather than a familiar object. Certainly single such waves may be rather rare, but their counterparts in many-wave interactions, or perhaps as periodic waves, are often observed. It will be shown in the next chapter that a minor adjustment to our formulation enables us to show that the results described here are more widely applicable.

The second point picks up the comment just made about periodic solutions. The KdV equation for travelling waves, (2.173), admits periodic solutions of permanent form. That such solutions exist is easily demonstrated by integrating this equation twice, but without the use of decay conditions at infinity; this gives

$$\frac{K}{6}(f')^2 = cf^2 - \frac{1}{2}f^3 + Af + B = F(f),$$

where A and B are arbitrary constants. In the case where the cubic $F(f)$ has three distinct zeros, the solution can be expressed in terms of the Jacobian elliptic function, *cn*, giving rise to the Korteweg and de Vries *cnoidal wave*, which they first named. This description, and some related properties of the Jacobian elliptic functions, are explored through Q2.65–Q2.67.

2.9.2 Integral relations for the solitary wave

We conclude this chapter of classical results by briefly returning to the general solitary wave (Section 2.8). In work that dates back to McCowan (1891), and taken much further by Longuet-Higgins over the last twenty years or so, some exact identities for the solitary wave have been obtained. In recent times these have proved very powerful in the development of numerical methods for describing large-amplitude solitary waves (including the highest wave) and for laying the foundations for calculations that allow a study of breaking waves; much of this work has been pioneered by Longuet-Higgins and his co-workers. Here we shall give a brief introduction to these ideas, and a few are taken further in the

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exercises. The interested reader may also explore this material through the references given in the further reading at the end of this chapter.

We consider a wave of permanent form, moving at the speed c, which decays for $|\xi| \to \infty$; this is described by the equations (2.165):

$$\phi_{zz} + \delta^2 \phi_{\xi\xi} = 0$$

with

$$\left.\begin{aligned} \phi_z &= \delta^2(\phi_\xi - c)\eta_\xi; \\ -c\phi_\xi + \eta + \frac{1}{2}\left(\frac{1}{\delta^2}\phi_z^2 + \phi_\xi^2\right) &= 0 \end{aligned}\right\} \text{ on } z = 1 + \eta$$

and

$$\phi_z = 0 \quad \text{on} \quad z = 0.$$

We define a number of properties of the wave and its motion. These are the mass associated with the wave

$$M = \int_{-\infty}^{\infty} \eta \, \mathrm{d}\xi, \tag{2.175}$$

the total momentum (or impulse) of the motion of the fluid

$$I = \int_{-\infty}^{\infty} \int_{0}^{1+\eta} \phi_\xi \, \mathrm{d}z \, \mathrm{d}\xi, \tag{2.176}$$

the total kinetic energy of the motion

$$T = \frac{1}{2} \int_{-\infty}^{\infty} \int_{0}^{1+\eta} \left(\frac{1}{\delta^2}\phi_z^2 + \phi_\xi^2\right) \mathrm{d}z \, \mathrm{d}\xi \tag{2.177}$$

and the potential energy of the wave

$$V = \frac{1}{2} \int_{-\infty}^{\infty} \eta^2 \, \mathrm{d}\xi. \tag{2.178}$$

In addition we define a *circulation* for the motion,

$$C = \int_{-\infty}^{\infty} \mathbf{u} \cdot \mathrm{d}\mathbf{s} = [\phi]_{-\infty}^{\infty}, \tag{2.179}$$

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where the integral is taken along any streamline. These forms of these fundamental quantities are all defined here as the nondimensional counterparts of their physical equivalents.

First, from the equation of mass conservation,

$$u_\xi + w_z = 0,$$

and, in particular, since we are in the frame moving with the wave we write

$$(u - c)_\xi + w_z = 0,$$

and then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left\{\int_0^{1+\eta}(u-c)\mathrm{d}z\right\} = 0;$$

cf. equation (1.40). Thus

$$\int_0^{1+\eta}(u-c)\mathrm{d}z = \text{constant} = \int_0^1(-c)\mathrm{d}z = -c$$

since both $u = \phi_\xi$ and η tend to zero as $|\xi| \to \infty$; hence

$$\int_0^{1+\eta} u\mathrm{d}z\left(= \int_0^{1+\eta}\phi_\xi\mathrm{d}z\right) = c\eta,$$

and then

$$\int_{-\infty}^{\infty}\int_0^{1+\eta}\phi_\xi\mathrm{d}z\,\mathrm{d}\xi = c\int_{-\infty}^{\infty}\eta\mathrm{d}\xi$$

or

$$I = cM. \tag{2.180}$$

This is an identity first obtained by Starr (1947).

Next we use Green's theorem in the form

$$\int_V\{(\nabla u)\cdot(\nabla v) + u\nabla^2 v\}\,\mathrm{d}V = \int_S u(\nabla v)\cdot\mathrm{d}\mathbf{S},$$

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per unit length in the y-direction (so $\mathrm{d}V = 1 \times \mathrm{d}s$, $\mathrm{d}\mathbf{S} = \mathbf{n}(1 \times \mathrm{d}l)$, and choose

$$u = v = \Phi = \phi - c\xi \quad \text{and} \quad \nabla \equiv \left(\frac{\partial}{\partial \xi}, \frac{1}{\delta}\frac{\partial}{\partial z}\right).$$

The resulting plane region for the integration is bounded by a curve (Γ) which is taken to be

$$z = 1 + \eta \quad \text{and} \quad z = 0 \quad \text{for} \quad -\xi_0 \le \xi \le \xi_0$$

and $\xi = \pm\xi_0$, $\xi_0 > 0$; see Figure 2.27. We may think of ξ_0 as large for, eventually, we shall impose $\xi_0 \to \infty$. Now, since

$$\nabla^2 \Phi = \nabla^2 \phi = \phi_{\xi\xi} + \frac{1}{\delta^2}\phi_{zz} = 0,$$

we obtain Green's theorem in the form

$$\int_{-\xi_0}^{\xi_0} \int_0^{1+\eta} \left\{ \frac{1}{\delta^2}\phi_z^2 + (\phi_\xi - c)^2 \right\} \mathrm{d}z\, \mathrm{d}\xi = \int_\Gamma \Phi\left(m\Phi_\xi + \frac{n}{\delta}\Phi_z\right) \mathrm{d}l, \qquad (2.181)$$

where $\mathbf{n} \equiv (m, n)$ is the outward unit normal vector on Γ. Note that, because we are using the coordinates (ξ, z), the surface wave is stationary in our frame; also, across $\xi = \pm\xi_0$, there is (approximately) a uniform stream of speed c in the negative ξ-direction.

To proceed, we evaluate the various contributions in equation (2.181). The left-hand side becomes

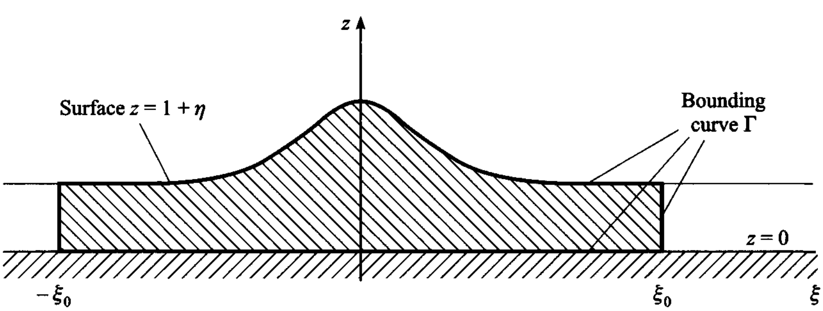

Figure 2.27. The region (whose boundary is designated by Γ) used in the application of Green's theorem to find one of the identities satisfied by the solitary wave.

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$$2\hat{T} - 2c \int_{-\xi_0}^{\xi_0} \int_0^{1+\eta} \phi_\xi \, dz \, d\xi + c^2 \int_{-\xi_0}^{\xi_0} \int_0^{1+\eta} dz \, d\xi = 2\hat{T} - 2c\hat{I} + 2c^2\xi_0 + c^2\hat{M}$$

where $\hat{T} \to T$, $\hat{I} \to I$ and $\hat{M} \to M$ as $\xi_0 \to \infty$; see equations (2.177), (2.176) and (2.175). For the right-hand side we find that

$$\begin{array}{lll} \text{on } z = 0: & m = 0, n = -1, & \Phi_z = \phi_z = 0; \\ \text{on } \xi = \xi_0: & m = 1, n = 0, & \Phi_\xi = -\hat{c}; \\ \text{on } \xi = -\xi_0: & m = -1, n = 0, & \Phi_\xi = -\hat{c}; \end{array}$$

and on $z = 1 + \eta$, which is a streamline (or, rather, a stream surface), $\mathbf{n}$ is here normal to $\nabla\Phi$; we have introduced $\hat{c}$ where $\hat{c} \to c$ as $\xi_0 \to \infty$. Thus equation (2.181) becomes

$$2\hat{T} - 2c\hat{I} + 2c^2\xi_0 + c^2\hat{M} = -\int_0^{1+\eta} \Phi_+ \hat{c} \, dz + \int_0^{1+\eta} \Phi_- \hat{c} \, dz \qquad (2.182)$$

where $\Phi_\pm$ denotes Φ evaluated on $\xi = \pm\xi_0$. It is simplest, at this stage, to allow $\xi_0 \to \infty$ (so that $\hat{c} \to c$ and $\eta \to 0$) and hence obtain for the right-hand side of (2.182)

$$-\int_0^{1+\eta} \Phi_+ \hat{c} \, dz + \int_0^{1+\eta} \Phi_- \hat{c} \, dz \sim -c\Phi_+ + c\Phi_- \sim -c[\phi]_{-\infty}^{\infty} + 2c^2\xi_0.$$

Thus equation (2.182) produces, in the limit $\xi_0 \to \infty$, the identity

$$2T - 2cI + c^2 M = -cC$$

or

$$2T = c(I - C) \qquad (2.183)$$

after we introduce equation (2.180). The relation (2.183) was first derived by McCowan (1891).

A third useful identity introduces the potential energy, V, and takes the form

$$3V = (c^2 - 1)M;$$

a derivation of this result can be found in Longuet-Higgins (1974). Other identities (involving these quantities or for the surface profile itself) have been obtained by Longuet-Higgins, and used very successfully in numerical investigations of the large-amplitude solitary wave. These three

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integral identities are examined, for the approximate sech^2 profile, in Q2.69.

Further reading

This chapter has introduced a number of classical problems in both linear and nonlinear water-wave theory. Similar material will be found in many of the classical texts and, in some cases, the presentation in these will go beyond the topics developed here or use a different approach to that adopted here. General texts that the reader may find useful are Stoker (1957), Crapper (1984), Mei (1989) and the more recent publication Debnath (1994). In addition, some important aspects of water-wave theory are developed in Whitham (1974). A more engineering-oriented approach is to be found in Dean & Dalrymple (1984). All these references are particularly relevant to the fundamental ideas described in Sections 2.1–2.1.3.

2.1.3 The method of stationary phase, and of steepest descents, is nicely described in Copson (1967). A far more thorough and expansive treatment will be found in the excellent text by Olver (1974).

2.2 A neat discussion of waves over variable depth, and in particular building on the work of Hanson (1926), will be found in Whitham (1979). This monograph also includes some work on edge waves, as does the text by Mei (1989).

2.3 A fairly complete description of ray theory, with some applications to variable depth and to variable currents, is given by Mei (1989). Ray theory is also mentioned in Crapper (1984) and in Whitham (1974), and an introduction to Whitham's averaged Lagrangian will also be found in this latter text.

2.4 Stoker (1957) provides an extensive presentation of many aspects of ship waves; the elements can also be found in Crapper (1984). A text which incorporates more practical aspects of ship waves and ship hydrodynamics is Timman, Hermans & Hsiao (1985).

2.5 A description of the Stokes wave can be found in many texts on fluid mechanics. In the context of books on water waves, the reader is directed to Mei (1989), Dean & Dalrymple (1984), Whitham (1974) and Crapper (1984).

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2.6 and 2.7 Excellent descriptions of the method of characteristics, Riemann invariants, discontinuous solutions and the hodograph transformation can be found in Stoker (1957) and Courant & Friedrichs (1967). Presented from the viewpoint of the theory of partial differential equations, there is no better text than Garabedian (1964).

2.8 The work that was first described by Carrier & Greenspan (1958) is given a careful treatment in Whitham (1979), and is also mentioned in Mei (1989) and Debnath (1994).

2.9 The classical (small-amplitude) solitary wave is described in Stoker (1957), as well as in numerous other texts on fluids or nonlinear waves (especially those that touch on 'soliton' theory, for example Drazin & Johnson (1993)). The more modern treatments on the large-amplitude solitary wave, and on breaking waves, are best addressed through some of Longuet-Higgins' papers, which are listed in the references, in particular Longuet–Higgins (1974, 1975), Longuet-Higgins & Fenton (1974) and Longuet-Higgins & Cokelet (1976).

Our text does not incorporate photographs of surface waves. Although the quality of some of the pictures does vary considerably, the readers who wish to add to their own observations are directed, for example, to Stoker (1957) and Crapper (1984); a few useful pictures appear in Lighthill (1978). A fine collection of early photographs, with extensive descriptions, will be found in Cornish (1910).

Exercises

Q2.1 *Minimum of c_p*. Write $\delta k = \lambda$ in the expression for c_p^2 (equation (2.9)), and show that c_p has a single minimum (in $0 < \lambda < \infty$). Also describe the behaviour of $c_p(\lambda)$ as $\lambda \to \infty$.

Q2.2 *Simplified form of c_p*. Show, for moderate values of $\lambda = \delta k$, that c_p^2 may be written (approximately) as a linear combination of λ and λ^{-1}; cf. Q2.27. Hence find the minimum of c_p ($= c_m$), at $\lambda = \lambda_m$ ($0 < \lambda_m < \infty$), and find the expression for $(c_p/c_m)^2$ in terms of $l = \lambda/\lambda_m$.

[All these results are to be compared with those obtained in Q2.1; it is clear that this simplified (approximate) form of c_p is much easier to work with, and it is often used because of this.]

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Q2.3 *Plane harmonic wave.* Extend the problem described in Section 2.1, to obtain the functions $U(z)$ and $P(z)$ that correspond to $W(z)$ (given by solution (2.8)).

Q2.4 *Particle paths.* Show, when written in original physical variables, that the particle paths described by equation (2.16) are circles in the short-wave limit.

Q2.5 *Laplace's equation and separation of variables.* Recover the results presented in Section 2.1, for the case of gravity waves only, by first formulating the problem in terms of the velocity potential, ϕ; see Q1.38. To proceed, construct the solution of Laplace's equation (for ϕ) by the method of separation of variables; in particular show that ϕ takes the form

$$\phi = \{A(t)\cos kx + B(t)\sin kx\}\cosh \delta kz,$$

for any given value of $k\,(\neq 0)$, where A and B are both general solutions of

$$\frac{\mathrm{d}^2F}{\mathrm{d}t^2} + \omega^2 F = 0, \quad \omega^2 = \frac{k}{\delta}\tanh \delta k.$$

Q2.6 *Standing waves.* Take, as a special case of the result obtained from Q2.5, a choice of $A(t)$ and $B(t)$ which describes a solution for $\eta(x, t)$ which is a single separable function of x and t. In this solution, at a given position (x), the surface oscillates vertically between its maximum and minimum values; the maximum (or minimum) value does not propagate. This is therefore a standing wave. Use your solution to show how this wave can be interpreted as two propagating waves.

Q2.7 *Oblique plane waves.* Follow the presentation given in Section 2.1, but for a surface wave described by

$$\eta = A\mathrm{e}^{\mathrm{i}(kx+ly-\omega t)} + \mathrm{c.c.};$$

see equation (2.4). Find the dispersion relation, and show that this is equation (2.9) with k^2 replaced by $k^2 + l^2$. Confirm that the wave propagates in the direction of the wave-number vector $\mathbf{k} \equiv (k, l)$.

Q2.8 *Waves along a rectangular channel.* A channel, $-\infty < x < \infty$ with $0 \le y \le l$, contains water ($0 \le z \le 1$ when undisturbed) on the surface of which a gravity wave propagates in the (positive) x-direction. Show that there is a solution of the governing linear equations (cf. Q2.5) for which

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$$\eta = A\cos(\alpha y)\cos(kx - \omega t).$$

Determine the constant α and the dispersion function ω.

Q2.9 *Sloshing in a rectangular container*. A rectangular tank, $0 \le x \le l$ and $0 \le y \le L$, contains a liquid ($0 \le z \le 1$ when undisturbed) whose surface is described by the standing gravity wave

$$\eta = A\cos(\alpha x)\cos(\beta y)\cos(\omega t),$$

for suitable constants α and β; A is the fixed amplitude of the wave. Seek an appropriate solution of Laplace's equation, (2.66), which satisfies the surface (2.67) and bottom conditions (2.68) (with $b = 0$), as well as the conditions on the side walls; that is, $\phi_x = 0$ on $x = 0, l$; $\phi_y = 0$ on $y = 0, L$. (Note that our formulation here is in terms of nondimensional variables.) Find α, β and the dispersion function ω.

[It is the *standing wave* which constitutes the *sloshing mode* in a container.]

Q2.10 *Short-crested waves*. Follow the formulation described in Q2.5, but retain the dependence on y; cf. Q2.7. Seek a solution for ϕ, by using an appropriate separation variables, that will allow the surface wave to take the form

$$\eta(x, y, t) = A\cos(mx + ny)\cos(kx + ly - \omega t),$$

where A, m, n, k and l are constants; ω is the dispersion function. Find ω and the relation that must exist between the wave numbers m, n, k and l for this type of solution to exist. Interpret this condition geometrically. Describe your solution, and find the speed and direction of propagation of the wave.

[These waves are called *short-crested* to differentiate them from plane waves, which are *non-oscillatory* along their wavefronts.]

Q2.11 *Waves on a uniform stream*. Consider the propagation of plane harmonic waves in the x-direction, on the surface of a fluid (of constant depth, $b = 0$) which moves at constant speed, $u = u_0$, also in the x-direction. (The relevant equations are obtained from (1.57), (1.58), (1.63), (1.64) and (1.65), with $\varepsilon u \to u_0 + \varepsilon u$; otherwise follow the method that leads to equations (2.1).) Show that the dispersion relation corresponding to equation (2.9) is exactly (2.9), but with ω replaced by $\omega - u_0 k$ (where k is the wave number).

[This result describes the familiar *Doppler shift*.]

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Q2.12 *Oblique waves on a uniform stream.* See Q2.11; repeat this calculation for the constant uniform flow $\mathbf{u} \equiv (u_0, v_0)$ and for a plane wave with a wave-number vector $\mathbf{k} \equiv (k, l)$. Show that, now, ω is replaced by $\omega - u_0 k - v_0 l = \Omega$, and that the wave crests move forward at the velocity $(u_0 + \Omega\hat{k}, v_0 + \Omega\hat{l})$, where $(\hat{k}, \hat{l}) = (k, l)/(k^2 + l^2)$. Hence describe the condition for which the waves are stationary in the physical frame of reference.

Q2.13 *Gravity waves over a step.* Stationary water of constant depth, h_-, is in $x < 0$, and of constant depth, h_+, in $x > 0$; there is a step at $x = 0$. Small amplitude gravity waves ($W_e = 0$), of wave number k and amplitude A, approach the step from $-\infty$. The step generates, in general, a transmitted wave which propagates towards $+\infty$ and a reflected wave which moves back to $-\infty$. Follow the development given in Section 2.1 and, at $x = 0$, impose the conditions of (a) continuity of wave amplitude; (b) conservation of mass flux across $x = 0$. Find the amplitudes of the transmitted and reflected waves.

Q2.14 *Kelvin–Helmholtz instability.* An incompressible fluid of density $\lambda\,(< 1)$ exists in $z > 0$ and is moving at a constant speed, U, in the (positive) x-direction. Another incompressible fluid of density 1, which is stationary in its undisturbed state, is in $z < 0$; at $z = 0$ there exists an interface on which a small amplitude harmonic wave propagates (also in the x-direction). (This problem is presented in nondimensional variables, using the properties of the lower fluid for the purposes of nondimensionalisation; thus λ = (density of upper fluid)/(density of lower fluid). Because the fluids are of infinite depth, it is convenient to choose the vertical scale to be the same as the horizintal; thus set $\delta = 1$.)

Formulate the problem in terms of Laplace's equation for each of the fluids with an interfacial wave

$$\eta = A\mathrm{e}^{\mathrm{i}(kx - \omega t)} + \text{c.c.},$$

where A is a complex constant. Use the kinematic condition on $z = 0$ for both fluids, and impose the continuity of pressure across $z = 0$ (and include the effects of surface tension). Show that the dispersion function is

$$(1 + \lambda)\omega^2 - 2\lambda k U \omega - k(1 - \lambda) + \lambda k^2 U^2 - k^3 W = 0,$$

and hence deduce that harmonic waves are *stable* if

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$$\lambda U^2 \leq (1+\lambda)\{(1-\lambda)/k + kW\}$$

for all $k\,(>0)$.

[This exercise describes the simplest model for wind blowing over the surface of water. Notice that the expression on the right in the inequality has a minimum in $k > 0$; what is it?]

Q2.15 *Rayleigh–Taylor instability*. See Q2.14; now set $U = 0$ (so that both fluids are stationary in the undisturbed state) but consider $\lambda > 1$, so that the *heavier* fluid is above the lighter. Show that the wave with number k is *stable* if

$$k^2 W \geq \lambda - 1.$$

[This demonstrates the property that surface tension tends to stabilise the system: it is certainly *unstable* if W is small enough, for any $k \neq 0$.]

Q2.16 *Method of stationary phase*. Consider the integral

$$I(\sigma) = \int_a^b f(x)\mathrm{e}^{\mathrm{i}\sigma\alpha(x)}\mathrm{d}x,$$

for $\sigma \to \infty$, where the path of integration is taken along the real axis with a and b independent of the parameter σ.

(a) Suppose that $\alpha'(x)$ does not vanish for any $x \in [a, b]$; then show by integration by parts that

$$I(\sigma) = \frac{\mathrm{i}}{\sigma}\left\{\frac{f(a)}{\alpha'(a)}\mathrm{e}^{\mathrm{i}\sigma\alpha(a)} - \frac{f(b)}{\alpha'(b)}\mathrm{e}^{\mathrm{i}\sigma\alpha(b)}\right\} + \mathrm{O}(\sigma^{-2}),$$

provided that $f(a)$ and $f(b)$ are not both zero.

(b) This time suppose that $\alpha'(a) = 0$, with $\alpha''(a) > 0$, and $\alpha'(x) \neq 0$ for all $x \in (a, b]$. Write the interval (a, b) as $(a, a+\varepsilon)$ plus $(a+\varepsilon, b)$, where we can use the calculation in (a) for the latter interval. In the former interval, write

$$\alpha(x) = \alpha(a) + u^2, \quad x = \alpha + \sum_{n=1}^{\infty} b_n u^n, \quad u \in [0, \hat{u}]$$

where $\hat{u} = \sqrt{\alpha(a+\varepsilon) - \alpha(a)}$; further, it is convenient to introduce

$$f(x)\frac{\mathrm{d}x}{\mathrm{d}u} = \sum_{n=0}^{\infty} c_n u^n = c_0 + uF(u), \quad c_0 = b_1 f(a),$$

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where $F(u)$ is regular for $u \in [0, \hat{u}]$. Hence show that

$$I(\sigma) = \left\{\frac{\pi}{2\sigma\alpha''(a)}\right\}^{1/2} f(\alpha)\exp\left\{\mathrm{i}[\sigma\alpha(a) + \pi/4]\right\} + \mathrm{O}(\sigma^{-1}).$$

[More details of this calculation, and of related problems, can be found in Copson (1967), Olver (1974).]

Q2.17 *Cylindrical coordinates*. Use equations (2.2) to obtain, for long waves ($\delta \to 0$) and with $b = 0$, the equation (2.14) for the surface waves written in cylindrical coordinates.

Q2.18 *Concentric waves I*. See Q2.17; now consider waves that are purely concentric (so that $\eta = \eta(r, t)$ only). Use the Hankel transform to obtain the solution which satisfies

$$\eta(r, 0) = f(r), \quad \eta_t(r, 0) = 0,$$

for which $\eta(0, t)$ and $\eta_r(0, t)$ are bounded and $\eta(r, t) \to 0$ as $r \to \infty$ for $0 < t < \infty$.

Q2.19 *Concentric waves II*. Repeat the calculations described in Q2.18, but now for the linear water-wave problem which represents propagation on infinitely deep water in the absence of surface-tension effects. (This requires starting from equations (2.2), with $W = 0$, $\partial/\partial\theta \equiv 0$ and $w \to 0$ as $z \to -\infty$.) What is the corresponding solution which satisfies the initial data

$$\eta(r, 0) = 0, \quad \eta_t(r, 0) = f(t)?$$

Q2.20 *Sloshing in a cylindrical tank*. A cylindrical tank, $0 \le r \le a$, contains a liquid ($0 \le z \le 1$ when undisturbed) which is in motion due to the presence of a small-amplitude standing gravity wave. Show that there is a solution which takes the form

$$\eta = AJ_n(\sigma r)\cos(n\theta)\sin(\omega t)$$

for suitable σ and ω; n (≥ 0) is an integer and J_n is the Bessel function of the first kind, of order n. (See equations (2.66)–(2.68), Q1.38 and Q2.17.) What is special about the choice $n = 0$?

Q2.21 *Wave propagation in a cylindrical tank*. See Q2.20; now seek a solution

$$\eta = AJ_n(\sigma r)\sin(n\theta - \omega t),$$

which describes a wave propagating around the tank. Find σ and ω, and compare all your results with those obtained in Q2.20 (and, in particular, check agreement for $n = 0$).

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Q2.22 *General initial-value problem.* Use the Fourier transform to write down the solution described in Section 2.1 which satisfies $\eta(x, 0) = f(x)$ and $\eta_t(x, 0) = 0$; to do this you must allow the possibility that waves may propagate in both directions. In the special case where $f(x) = A\delta(x)$, where $\delta(x)$ is the Dirac delta function and A is a constant, find a wholly real expression (that is, $\mathrm{i} = \sqrt{-1}$ appears nowhere) for $\eta(x, t)$.

Q2.23 *Simple linear dispersion.* Write down the dispersion relation for gravity waves moving over water of arbitrary depth. Consider waves propagating only to the right and approximate $\omega(k)$ as $\delta k \to 0$, retaining terms as far as $\mathrm{O}(\delta^2 k^2)$. Hence write down a simple linear partial differential equation which has your approximate dispersion relation as its (exact) dispersion relation; cf. equation (1.78).

Q2.24 *Behaviour near a wavefront.* See Q2.22; consider the component of $\eta(x, t)$ (for a general initial profile) which is propagating to the right, and examine the approximate form of this solution for long gravity waves. (This is the relevant approximation near a wavefront.) To accomplish this, retain terms as far as $\mathrm{O}(k^3)$ in $\omega(k)$ (cf. Q2.23) and retain just the first term in the expansion (as $\delta k \to 0$) of the Fourier transform of $\eta(x, 0)$. (You should assume that $\int_{-\infty}^{\infty} f(x)\mathrm{d}x$ is finite and nonzero.) Now express this solution in terms of the Airy function, Ai, and hence describe the behaviour of $\eta(x, t)$ (a) ahead of the wavefront; (b) behind the wavefront; (c) at the wavefront, as a function of t.

Q2.25 *Complex variable method.* Consider the problem described by equations (2.3) and Q2.5 (but include the Weber number, labelled W_e here, in this latter problem). Introduce the complex potential

$$W(Z, t) = \phi + \mathrm{i}\psi,$$

in the usual notation, where $Z = x + \mathrm{i}\delta z$. Let the bottom, $z = 0$, correspond to the streamline $\psi = 0$ and hence deduce that

$$\bar{W} = W(\bar{Z}, t) = \phi - \mathrm{i}\psi,$$

where the overbar denotes the complex conjugate. Show that the problem reduces to finding an appropriate solution of

$$\delta\frac{\partial^2}{\partial t^2}(W + \bar{W}) + \mathrm{i}\left(\frac{\partial}{\partial x} - \delta^2 W_\mathrm{e}\frac{\partial^3}{\partial x^3}\right)(W - \bar{W}) = 0$$

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on $z = 1$. Confirm that the dispersion relation (2.9) is recovered if we seek a solution $W = A\cos(kZ - \omega t)$, where A, k, and ω are real constants.

Q2.26 *Group speed for general water waves.* Repeat the calculation described in Section 2.1.2, but now retain the effects of surface tension. Show that the amplitude of the wave, which is prescribed as a function of $X = \alpha x$ at $t = 0$, propagates at the group speed

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} \quad \text{where} \quad \omega^2 = \left(\frac{k}{\delta} + \delta k^3 W_\mathrm{e}\right)\tanh \delta k.$$

Further, show that c_g may be written as

$$c_g = \frac{1}{2}c_p\left\{\frac{1 + 3\delta^2 k^2 W_\mathrm{e}}{1 + \delta^2 k^2 W_\mathrm{e}} + \frac{2\delta k}{\sinh 2\delta k}\right\} \text{ where } c_p = \frac{\omega}{k}.$$

Q2.27 *Propagation over infinitely deep water.* A plane wave propagates in the x-direction over water of infinite depth. Follow the calculation described in Section 2.1, starting from equations (2.1) but with $w \to 0$ as $z \to -\infty$, and hence obtain the dispersion relation; cf. Q2.2. What is the group speed?

[Observe that this solution describes a disturbance which decays exponentially with depth.]

Q2.28 *Group speed: general argument I.* A wave motion is described by the sum of two components

$$\eta(x, t) = A_0 \exp\{\mathrm{i}(kx - \omega(k)t\} + A_0 \exp\{\mathrm{i}(lx - \omega(l)t)\} + \text{c.c.},$$

based on two different wave numbers (k and l), but one dispersion relation, $\omega = \omega(k)$; both components have the same amplitude, A_0. Now suppose that $l = k(1 + \alpha)$ with $\alpha \to 0$ (so that the wave numbers differ by $O(\alpha)$); for x and t fixed, as $\alpha \to 0$, show that

$$\eta \sim A(X, T)\exp\{\mathrm{i}(kx - \omega(k)t)\},$$

where $\alpha x = X$, $\alpha t = T$. Further, confirm that A is a wave which propagates at the speed $\omega'(k) = c_g$.

Q2.29 *Group speed: general argument II.* A wave motion depends on the phase variable θ, such that

$$\frac{\partial\theta}{\partial x} = k, \quad \frac{\partial\theta}{\partial t} = -\omega.$$

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Confirm that $\theta = kx - \omega t +$ constant if both k and ω are constants. We now suppose that the wave evolves so that both k and ω change; deduce that

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0,$$

and explain how this can be interpreted as a conservation of waves. (It is usual to regard k and ω as *slowly* evolving, in the sense that they are functions of $X = \alpha x$ and $T = \alpha t$ as $\alpha \to 0$; see Section 2.1.1 and Q2.28.) Given, further, that $\omega = \omega(k)$ and that the energy is represented by $E = E(k)$, deduce that E propagates at the group speed, $\omega'(k)$.

Q2.30 *Group speed: orthogonality approach.* Derive equation (2.29), for $A_0(X, T)$, directly from the equation for W_1, (2.24). To accomplish this, multiply this equation by W_0 and then integrate it in z, from $z = 0$ to $z = 1$. Use integration by parts to form the term W_{0zz}, and use the equation defining W_0 together with boundary conditions for both W_0 and W_1.

[Since the equation for W_1 is an inhomogeneous version of W_0, with corresponding boundary conditions, a solution for W_1 exists only if an *orthogonality condition* is satisfied. This condition is the equation for A_0.]

Q2.31 *Energy and energy flux I.* Consider the solution developed in Q2.10, and choose the case of plane oblique waves (that is, $m = n = 0$). Use the details derived in this calculation to find the energy, $\mathscr{E}$, of the plane waves; see equation (1.48) and Section 2.1.2. Write your expression for $\mathscr{E}$ with error $O(\varepsilon^3)$ as $\varepsilon \to 0$. Now obtain the corresponding expression for the energy flux, $\mathscr{F}$ (given by equation (1.49) with P measured relative to P_a), but written in nondimensional form.

As mentioned in Section 2.1.2, it is convenient to compute the average values of $\mathscr{E}$ and $\mathscr{F}$ taken over one period; do this by integrating in θ from 0 to 2π (where $\theta = kx + ly - \omega t$) and dividing by 2π. Show that these average values (denoted by the overbar) are

$$\bar{\mathscr{E}} = \bar{\mathscr{E}}_0 + \frac{1}{2}\varepsilon^2 A^2 + O(\varepsilon^3),$$

where $\mathscr{E}_0$ is the contribution to the potential energy in the absence of the wave, and

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$$\bar{\mathscr{F}} \equiv \frac{1}{2}\varepsilon^2 A^2\left(\frac{1}{\omega}+\frac{c_g}{|\mathbf{k}|}\right)\mathbf{k}+\mathrm{O}(\varepsilon^3)$$

where $\mathbf{k} \equiv (k, l)$. (You may also confirm that the contribution from the wave to $\bar{\mathscr{E}}, \frac{1}{2}\varepsilon^2 A^2$, is comprised of two equal parts: the average kinetic and potential energies of the wave; this equality always arises in linear problems.)

Q2.32 *Energy and energy flux II.* See Q2.31; now show that the first term in $\bar{\mathscr{F}}, \frac{1}{2}(\varepsilon^2 A^2/\omega)\mathbf{k}$, arises from a contribution from the mass flux $\int_0^h \rho\mathbf{u}_\perp \mathrm{d}z$ (written here in dimensional variables). We write this contribution as $\mathscr{F}_0$, and then set

$$\bar{\mathscr{E}} = \mathscr{E}_0 + \mathscr{E}_\mathrm{w} + \mathrm{O}(\varepsilon^3), \quad \bar{\mathscr{F}} = \mathscr{F}_0 + \mathscr{F}_\mathrm{w} + \mathrm{O}(\varepsilon^3),$$

where the subscript w denotes the contribution from the wave motion. Hence deduce that

$$\mathscr{F}_\mathrm{w} = c_g \mathscr{E}_\mathrm{w}\mathbf{k}/|\mathbf{k}|,$$

and so the energy flux is in the direction of the wave-number vector, and the energy moves at the speed c_g.

[The term $\mathscr{F}_0$ shows that there is a mass flux of $\mathrm{O}(\varepsilon^2)$, even for particle paths that are *closed* at $\mathrm{O}(\varepsilon)$; this is usually called the *Stokes mean drift*, and it is explored further in Q4.4.]

Q2.33 *Characteristics for variable depth.* The equation for variable depth, (2.40), is

$$\eta_{tt} - (d\eta_x)_x = 0;$$

rewrite this equation in terms of the characteristic variables $\xi = \int_0^x \mathrm{d}x/\sqrt{d} - t$ and $\zeta = \int_0^x \mathrm{d}x/\sqrt{d} + t$. Sketch the characteristic lines for $d(x) = \alpha(x_0 - x)$, where $\alpha > 0$ is a constant and x_0 is fixed.

Q2.34 *Green's law.* See Q2.33; now seek a solution in the form

$$\eta = d^{-1/4}H(\xi, \zeta),$$

and obtain the equation for H (which will include coefficients that depend on $d(x)$; these could be written in terms of ξ and ζ, but there is no need to do that here). Describe the special forms that H takes in the two cases

$$\text{(a) } d(x) = (\alpha x + \beta)^{4/3}; \quad \text{(b) } d(x) = (\alpha x + \beta)^2,$$

where α and β are arbitrary constants.

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[The amplitude factor, $d^{-1/4}$, is the property usually associated with Green's law, although a more precise statement of the law usually includes the requirement that the horizontal velocity component also be proportional to $d^{-3/4}$.]

Q2.35 *Laplace's equation and waves over a constant slope*. State the problem described by equations (2.48), with $b(x) = \alpha x$, in terms of Laplace's equation; cf. Q2.5. For the choice $\alpha\delta = 1$, recover Hanson's solution given by equation (2.61). [Hint: see also equations (2.66)–(2.69).]

Q2.36 *Waves over a constant slope with $\alpha\delta = 1/\sqrt{3}$*. Repeat the calculation of Q2.35, but now for the case $\alpha\delta = 1/\sqrt{3}$. Show that a consistent solution is obtained, following the approach introduced by Hanson, if *three* sets of terms are now introduced: one oscillatory in x, with wave number k (> 0), and two oscillatory in z with (complex) wave numbers $\frac{1}{2}(\sqrt{3} \pm \mathrm{i})k$.

[See Whitham (1979) for a further exploration of these ideas, and Hanson (1926) for applications to a variety of wave problems.]

Q2.37 *Oblique-cum-edge waves*. See Q2.35 and Q2.7; seek a solution of these equations with $b(x) = \alpha x$ and $\alpha\delta = 1$, in the form

$$\phi = F(x, z)\mathrm{e}^{\mathrm{i}(ly-\omega t)} + \text{c.c.},$$

which is bounded as $x \to -\infty$, $z \to -\infty$. Show that your solution represents an oblique wave at infinity (with both incoming and outgoing components), as well as an edge-wave structure in a neighbourhood of the shoreline.

Q2.38 *Group velocity for slow depth change*. From equations (2.76) and (2.78), with $D = 1 - B$, obtain an expression for the group velocity $\mathbf{c}_g \equiv (\partial\omega/\partial k, \partial\omega/\partial l)$; see equation (2.84).

Q2.39 *Dispersion relation for steady waves*. Describe the variation of σ with D (for $0 < D < \infty$), as given by $\sigma \tanh(\sigma D) = \text{constant}$; see equation (2.92).

Q2.40 *Eikonal equation*. Use the method of characteristics to obtain the solutions of

$$\Theta_X^2 + \Theta_Y^2 = c^2,$$

where $c > 0$ is a constant, in the two cases

(a) $\Theta = kcs$ on the line $X = s$, $Y = s$ (where $k \neq \pm\sqrt{2}$ is a constant);

(b) $\Theta = \sqrt{2}cs$ on the line $X = s$, $Y = s$.

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Q2.41 *Ray theory for propagation over a ridge.* See equations (2.95) and (2.97); obtain the equations, for both the rays and the wavefronts, for a depth variation which gives $\sigma^2(X) - \mu^2 = \sigma_0^2 \tanh^2 \beta X$, where σ_0 and β are positive constants.

Q2.42 *Ray theory with a shoreline.* See Q2.41; repeat this calculation for a depth variation which gives rise to $\sigma^2(X) - \mu^2 = -\beta/X$ for $X < 0$, where $\beta > 0$ is a constant. Also determine how the amplitude, $A(X)$, varies (cf. equation (2.98) *et seq.*).

Q2.43 *Trapped waves.* Obtain the equation for the rays in the case where the depth variation is such that $\sigma^2(X) - \mu^2 = \beta X(X_0 - X)$, where β and X_0 are positive constants.

Q2.44 *Differential equation for the rays.* Consider the eikonal equation given in Q2.40, but now with $c = c(X, Y)$. Write down the equations that define the solution using the method of characteristics. (These are equations for X, Y, Θ, Θ_X, Θ_Y in terms of a parameter.) Treat the ray as a curve $Y = Y(X)$ and, by eliminating Θ_X and Θ_Y between your equations, show that $Y(X)$ satisfies

$$c\frac{\mathrm{d}^2 Y}{\mathrm{d}X^2} + \left(c_X \frac{\mathrm{d}Y}{\mathrm{d}X} - c_Y\right)\left\{1 + \left(\frac{\mathrm{d}Y}{\mathrm{d}X}\right)^2\right\} = 0.$$

Hence describe the rays for
(a) $c =$ constant; (b) $c = c(X)$ only.

Q2.45 *Fermat's principle.* This states that light travels between any two points along a path which minimises the time. If the path is represented by $Y = Y(X)$, and the speed of light at any point is $1/c(X, Y)$, show that the Euler–Lagrange equation for this problem in the calculus of variations recovers the equation given in Q2.44. (The speed is written in this form, rather than simply $c(X, Y)$, in order to correspond to the particular choice of eikonal equation used in Q2.40.)

Q2.46 *Snell's Law.* Suppose that $c = c(X)$; show that the equation for the rays may be integrated once to yield

$$cY'/\sqrt{1 + (Y')^2} = \text{ constant},$$

where $Y' = \mathrm{d}Y/\mathrm{d}X$; see Q2.44(b). On the ray, let $Y'(X) = \tan \alpha(X)$ and deduce that

$$c(X) \sin \alpha(X) = \text{ constant},$$

which is Snell's law of refraction.

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Q2.47 *A circular shoal.* In cylindrical geometry, suppose that the depth varies so that $R^2\sigma^2(R) = \beta R$, where β is a positive constant; see equation (2.104). Obtain the equation for the rays that approach from infinity, and describe their behaviour.

Q2.48 *A circular island.* See Q2.47; the depth now varies so that $R^2\sigma^2(R) - \mu^2 = \beta R^2/(R - R_0)$, where β and R_0 are positive constants. Find the equation for the rays and describe their behaviour as $R \to R_0$ (which is the shoreline).

Q2.49 *Ship waves: the wedge angle.* Describe the behaviour of the angle of the wedge inside which the dominant ship-wave pattern is observed as the depth is decreased. (You should consider only constant speed, straight-line motion.) What is the wedge angle if $c_g = 3c_p/4$?

Q2.50 *Ship waves: method of stationary phase.* Use Kelvin's method of stationary phase to show that the dominant asymptotic behaviour of

$$\int_0^\infty p^{3/2}\sin\left(t\sqrt{\frac{p}{\delta}}\right)J_0(rp)\mathrm{d}p,$$

as $t^2/\delta r \to \infty$, is

$$\frac{1}{8\delta\sqrt{2\delta}}\,\frac{t^3}{r^4}\sin\left(\frac{1}{4}\,\frac{t^2}{\delta r}\right).$$

Q2.51 *Influence points I.* A simple geometrical construction enables us to show that there are just two influence points. Consider the motion of a point (ship) moving at constant speed in a straight line; the ship is at P, and W is any point behind the ship and off the ship's path. Draw PW, the mid-point of PW at M and the circle with diameter MW; identify the points (where they exist) where this circle intersects the path of the ship. Hence deduce that, at most, only two influence points exist.

Q2.52 *Influence points II.* Reconstruct the argument used in Q2.51 by an algebraic method. (For example, show that there are, at most, only two instants in time before $t = 0$ at which disturbances could have been initiated and which contribute to any given point W). Repeat this calculation for a ship moving on a circular course at constant speed.

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Q2.53 *Ship on a circular path.* A ship is moving on a circular path, of radius R_0, at a constant speed U. Find the parametric representation of the curves that describe the dominant wave pattern, equivalent to equations (2.123). [You may also show how equations (2.123) are recovered from your equations derived here.]

Q2.54 *Ship waves: capillary-wave limit.* Repeat the analysis described in Section 2.4.2, but now take the dispersion relation for Ω to be that which describes capillary waves in the absence of gravity waves (and approximated for long waves). Show that

$$2\tan\phi\tan^2\theta - 3\tan\theta - \tan\phi = 0,$$

and deduce that solutions exist for all ϕ (and so the dominant waves are no longer confined to a wedge-shaped region).

Q2.55 *Simple waves: wave-maker problem.* Use the method of characteristics (Section 2.6.1) to solve the problem of flow in $x > 0$, over constant depth, given $h(0, t) = H(t)$ for $t > 0$ with $u(x, 0) = 0$ and $h(x, 0) = h_0 =$ constant. What is the corresponding solution if $u(0, t) = U(t)$, $t > 0$, is given?

Q2.56 *Simple waves: piston problem.* See Q2.55; repeat this calculation but now in $x > X(t)$, $t > 0$; that is, the 'end wall' is moved according to $x = X(t)$. The water in $x > 0$ is of constant depth (h_0), and it is stationary, at $t = 0$.

[If $X'(t) < 0$, then we generate an expansion fan.]

Q2.57 *Simple waves with a shear structure.* Use the governing equations discussed in Section 2.8, but for constant depth $b(x) = 0$, and show that we may seek a solution in the form

$$\eta = H(\xi), \quad u = U(\xi, z), \quad w = \hat{W}(\xi, z)\frac{\partial \xi}{\partial x},$$

for suitable functions H, U, and $\hat{W}$, where $\xi = x - ct$ and $c = c(H)$. Further, simplify the problem by writing $U = U(H, z)$ and $\hat{W} = H'W(H, z)$; show that

$$\int_0^{1+H} \frac{\mathrm{d}z}{(U-c)^2} = 1 \quad \text{(the Burns condition)}$$

and that

$$I_{zH} + II_{zz} = 2I_z(I_z \pm c'\sqrt{|I_z|})$$

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with $I(H, 1+H) = 1$, $I(H, 0) = 0$, where

$$I(H, z) = \int_0^z \frac{\mathrm{d}z}{(U-c)^2}.$$

(The complete formulation of this problem requires the 'initial' condition: $I(H, z)$ given at some H; $c(H)$ also must be known, which is determined from the Burns condition, with either $U - c > 0$ or $U - c < 0$.)

Finally, rewrite this problem in terms of the similarity variable $Z = z/(1+H)$, so that now $I = I(H, Z)$. Confirm that the choice $I = kZ$ and $c'\sqrt{1+H} = \mp 3/2$ recovers the simple-wave solution (in the absence of shear) described in Section 2.6.1.

[More information about this problem can be found in Freeman (1972) and Blythe, Kazakia & Varley (1972); the Burns condition is described in Burns (1953) and in Thompson (1949). We shall provide a discussion involving some properties of the Burns condition in Chapter 3.]

Q2.58 *Nonlinear wave run-up*. See Section 2.8; reduce the equation for $t(\xi, \eta)$ to the cylindrical wave equation in $T(\xi+\eta, \eta-\xi)$ (cf. Section 2.6.2), and find expressions for u, c, t and x in terms of T. Hence use the method of separation of variables to find a solution for T which is bounded at the shoreline. Use your results to find: (a) the maximum run-up; (b) the behaviour of the solution far from the shoreline (cf. Section 2.2).

Q2.59 *Wave breaking*. See Q2.58; the condition for the breaking of the wave corresponds to where the Jacobian in the hodograph transformation is first zero. Use the results obtained in Q2.58 to show that breaking first occurs at the shoreline (as we would expect).

[This problem requires the introduction of some identities involving the Bessel functions J_0, J_1 and J_2; a description of this problem is to be found in Whitham (1979) and Mei (1989).]

Q2.60 *Hydraulic jump and bore*. Extend the analysis of Section 2.7 for the case of the hydraulic jump ($U = 0$), and find the speed of the flow behind the jump (u^+) and verify that $u^+ < u^-$ if $F > 1$. In this case, show that the Froude number for the flow behind the jump is less than unity.

Use the results obtained here, and in Section 2.7, to describe the characteristics of the flow associated with a bore which moves at a speed U into stationary water.

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Q2.61 *Reflection of a bore from a wall.* Stationary water of depth $h = h_0$ is in $x < 0$, and there is a vertical wall at $x = 0$. A bore moving at a constant speed U approaches the wall from $-\infty$ and is reflected by it and returns to $-\infty$. Find expressions for the speed of the returning bore and the depth of water behind it.

[The solution of this problem reduces to the solution of a cubic equation, a problem which need not be pursued.]

Q2.62 *Circular hydraulic jump.* It is readily observed that water flowing from a tap into a sink almost always spreads out in a thin, fast-moving layer over the surface of the sink. Furthermore, this layer is often approximately circular in shape being terminated by a narrow region (a jump) where the depth and speed change dramatically; thereafter the water makes its way down the plughole.

Consider the problem of the circular hydraulic jump and, from equations (2.2) without the effects of surface tension and without any variation in θ, follow the methods of Section 2.7 to find the jump conditions across the circular hydraulic jump.

[Incorporate the long-wave assumption, exactly as in Section 2.7; a discussion of this problem, with the inclusion of many realistic physical properties, will be found in Watson (1964).]

Q2.63 *Modelling the highest wave.* Stokes' highest wave can be modelled by satisfying some appropriate conditions, but not all of them. The simplest model is obtained by writing

$$\eta = a\mathrm{e}^{-\alpha|\xi|}$$

and then satisfying the conditions that prescribe $\eta(0)$ and $\eta'(0)$, together with equation (2.166). What is the value of c in this case?

An improvement is to write

$$\eta = a\mathrm{e}^{-\alpha|\xi|} + b\mathrm{e}^{-2\alpha|\xi|},$$

to impose the same conditions as above and, in addition, to ensure that

$$3V = (c^2 - 1)M$$

is satisfied. What now is the value of c?

[The value of c for the highest wave is, based on numerical evidence, about 1.286; see Longuet-Higgins & Fenton (1974) for more details, where it is shown that a wave exists for which $c \approx 1.294$!]

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Q2.64 *The* $sech^2$ *solitary wave*. Verify, or by direct integration show, that

$$f = 2c\,\mathrm{sech}^2\left(\sqrt{\frac{3c}{2K}}\zeta\right)$$

is a solution of equation (2.173), where $\zeta = \xi - c\tau$. Confirm that the behaviour of this solution, as $|\zeta| \to \infty$, satisfies the condition (2.166). Explain the connection between the two expressions for c^2: (2.13) and (2.166).

Q2.65 *Jacobian elliptic functions*. Define the integral

$$u = \int_0^{\phi} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}},$$

where m $(0 \leq m \leq 1)$ is a parameter (called the *modulus*). Then we write

$$\mathrm{sn}\,u = \sin\phi, \quad \mathrm{cn}\,u = \cos\phi,$$

the Jacobian elliptic functions; these are sometimes written as $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$. Show that

(a) $\mathrm{sn}^2u + \mathrm{cn}^2u = 1$;

(b) $\mathrm{cn}\,u = \cos u$ if $m = 0$; $\mathrm{cn}\,u = \mathrm{sech}\,u$ if $m = 1$;

(c) $\dfrac{\mathrm{d}}{\mathrm{d}u}(\mathrm{cn}\,u) = -\mathrm{sn}\,u\,\mathrm{dn}\,u$ where $\mathrm{dn}\,u = \sqrt{1 - m\sin^2\phi}$, and find the corresponding results for the derivatives of $\mathrm{sn}\,u$ and $\mathrm{dn}\,u$.

Q2.66 *Complete elliptic integral*. Define the integral

$$K(m) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - \mathrm{m}\sin^2\theta}},$$

the complete elliptic integral of the first kind. Deduce that the period of the elliptic functions sn and cn is $4K(m)$, $0 \leq m < 1$. Show that

(a) $K(0) = \pi/2$;

(b) $K(m) = \dfrac{\pi}{2}F(\dfrac{1}{2}, \dfrac{1}{2};\ 1;\ m)$, where $F(a, b; c; z)$ is the hypergeometric function;

(c) $K(m) \sim \frac{1}{2}\log\{16/(1 - m)\}$ as $m \to 1^-$.

[Hint: in (c) write $\mathrm{d}\theta = (1 - \sqrt{m}\sin\theta + \sqrt{m}\sin\theta)\,\mathrm{d}\theta$.]

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Q2.67 *Cnoidal-wave solution.* Verify that

$$f(\zeta) = a + b\,\mathrm{cn}^2\{\alpha(\zeta - \zeta_0)|m\},$$

where $\zeta = \xi - c\tau$, is a solution of equation (2.173) for suitable relations between the constants a, b, α and m; the phase shift, ζ_0, is an arbitrary constant.

[For this solution to exist, the cubic, $F(f)$, in Section 2.9.1 has three real, distinct zeros; indeed, it is convenient to write

$$F(f) = -\frac{1}{2}(f - f_1)(f - f_2)(f - f_3),$$

where f_i, $i = 1, 2, 3$, are the three roots.]

Q2.68 *Circulation associated with a solitary wave.* Show that

$$C = \int_{-\infty}^{\infty} \mathbf{u} \cdot \mathbf{dl} = [\phi]_{-\infty}^{\infty}$$

by evaluating (a) along the bottom streamline; (b) along the surface streamline (and Stokes' theorem may be invoked).

Q2.69 *Integral identities for the sech^2 profile.* Examine the three integral identities, relating T, V, I, C and M (discussed in Section 2.9.2), when the solitary wave is approximated by the sech^2 profile of small amplitude; that is, η is written as $\varepsilon\eta$, $\varepsilon \to 0$ and $\eta \propto \mathrm{sech}^2$ (see Section 2.9.1).

Q2.70 *Variational principle for water waves.* Show that the equations for gravity waves on stationary water over a rigid impermeable surface (Q1.38) are obtained from the variational principle

$$\delta \iint_D L\,\mathrm{d}x\,\mathrm{d}t = 0,$$

where the Lagrangian is

$$L = \int_{b(x,y)}^{\eta(x,y,t)} [\phi_t + \frac{1}{2}(\nabla\phi)^2 + z]\mathrm{d}z.$$

The region D, assumed to contain fluid, is arbitrary; the variations of ϕ and η are zero on the boundaries of D. (The nondimensional parameters, ε and δ, are not included in this formulation.)

[These ideas are developed in Luke (1967) and Whitham (1965, 1974).]

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