DETAILED NOTES: A REDUCTION OF THE EULER EQUATIONS TO A SINGLE TIME DEPENDENT EQUATION

ABSTRACT. Here, we use the results of [6] to write a single time dependent equation for a one-dimensional fluid surface without approximation. Using this single equation, we will derive various asymptotic models. Much of what follows in these notes comes directly from [6]. It would need to be re-written and consolidated in the final form.

Also, I'm typing my work as I go, so there could be typos. There may also be missing information, redundancies, better alternative arguments, etc. Basically, this is just a disclaimer that this is a work in progress.

1. Introduction

Assuming an irrotational, inviscid and incompressible flow, the governing equations for the fluid surface $\eta(x,t)$ and velocity potential $\phi(x,z,t)$ are given by a^2

$$\phi_{xx} + \phi_{zz} = 0, \qquad (x,z) \in S \times [-h,\eta], \tag{1}$$
$$\phi_z = 0, \qquad z = -h, \tag{2}$$
$$\eta_t + \eta_x \phi_x = \phi_z, \qquad z = \eta(x,t), \tag{3}$$
$$\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + g\eta = \frac{\sigma}{\rho}\frac{\eta_{xx}}{\left(1+\eta_x^2\right)^{3/2}}, \qquad z = \eta(x,t), \tag{4}$$

where S is a subset of $\mathbb{R}$, z is the vertical coordinate, g is the acceleration of gravity, h is the constant depth of the fluid when at a state of rest, σ represents the coefficient of surface tension, and ρ is the constant fluid density.

As written above, the equations of motion are challenging to work with directly: they are a free-boundary problem with nonlinear boundary conditions. Specifically, one must solve Laplace's equation inside an unknown domain while simultaneously satisfying the highly nonlinear boundary conditions applied at the unknown free surface η.

Various reformulations of (1-4) have been presented in the literature in order to simplify the equations of motion. For example, for one-dimensional surfaces (no x_2-variable), conformal mappings have been used to eliminate these problems for traveling waves (*for an overview, see* [4,5]). However, the conformal mapping approach does not generalize to two-dimensional surfaces, nor has it been used for the time dependent problem (we need to verify this statement).

For both one- and two-dimensional surfaces, other formulations (such as the Hamiltonian formulation given in [7] or the Zakharov-Craig-Sulem formulation [3]) reduce the Euler equations to a system of two equations, in terms of surface variables only, by introducing a Dirichlet-to-Neumann operator (DNO). In a similar spirit, [1] introduce a new non-local formulation of Euler's equation (henceforth referred to as the AFM formulation) that results in a system of two equations for the same variables as presented in the DNO formulation.

Both the DNO and AFM formulations reduce the problem from the full fluid domain to a system of equations that depend on the surface elevation η and the velocity potential evaluated at the surface $q(x)$ where

$$q(x) = \phi(x, \eta(x)).$$

While this simplification significantly reduces the computational domain, one could argue that the equations require solving for an additional function q, which is typically of less interest and not easily measured in experiments. The primary interest in applications is determining the surface elevation η.

For the one-dimensional case, we present a new scalar equation that determines the fully nonlinear time evolution of solutions to Euler valid for a one-dimensional surfaces.

2. Derivation of the Single Equation

We begin in a similar manner to the Hamiltonian, DNO, and AFM formulation [1,3,7]. By combining the boundary conditions at the free surface, we form a single Bernoulli-like equation in terms of the surface variables. Let q represent the velocity potential defined at the surface $q(x) = \phi(x, \eta(x))$. Equations (3) and (4) can be combined to obtain

$$q_t + \frac{1}{2}q_x^2 + g\eta - \frac{1}{2}\frac{(\eta_t + q_x\eta_x)^2}{1+\eta_x^2} = 0. \tag{5}$$

Equation (5) is an equation for the two unknowns (q, η). In order to find a single equation for the surface η, we introduce the operator $\mathcal{H}(\eta, D)$ for the Laplace equation which defines the following normal-to-tangential derivative map:

$$\mathcal{H}(\eta, D)\{\eta_t\} = q_x, \tag{6}$$

where we assume that $(q, \eta, \vec{c})$ is a solution set to Equations (1-4), and D is the typical differential operator defined as $D = -i\nabla$. The quantity η_t is the normal derivative of the potential due to equation (3).

Since the operator $\mathcal{H}(\eta, D)\{\eta_t\}$ produces q_x, we cannot directly substitute in the operator expression for q into (5). There are two different options that we can use:

(1) We could integrate both sides of (6) with respect to x. This would give

$$q = \int \mathcal{H}(\eta, D)\{\eta_t\}\, dx$$

which could be substituted directly into (5).

(2) We could differentiate (5) with respect to x and (6) with respect to t. This would give the following expressions:

$$\partial_t (q_x) + \partial_x\left(\frac{1}{2}q_x^2 + g\eta - \frac{1}{2}\frac{(\eta_t + q_x\eta_x)^2}{1+\eta_x^2}\right) = 0 \tag{7}$$

$$q_{xt} = \partial_t\left(\mathcal{H}(\eta, D)\{\eta_t\}\right) \tag{8}$$

We choose to go with the second scenario for multiple reasons which I won't go into here (some of the reasons are highly suggestive). Using this operator and (5) we can write a single equation for the water-wave surface as

$$\partial_t\left(\mathcal{H}(\eta, D)\{\eta_t\}\right) + \partial_x\left(\frac{1}{2}\left(\mathcal{H}(\eta, D)\{\eta_t\}\right)^2 + g\eta - \frac{1}{2}\frac{\left(\eta_t + \eta_x\mathcal{H}(\eta, D)\{\eta_t\}\right)^2}{1+\eta_x^2}\right) = 0. \tag{9}$$

The above equation only depends on the surface elevation $\eta(x)$ and is completely independent of the velocity potential q. Thus, Equation (9) represents a scalar equation for the water-wave surface expressed in terms of η.

The usefulness of (9) is dependent on finding a useful representation for the operator $\mathcal{H}(\eta, D)$. In the next section, we describe a method for determining $\mathcal{H}(\eta, D)$ based on the work of Ablowitz & Haut [2]. Using this representation for $\mathcal{H}(\eta, D)$, we demonstrate it's usefulness in determining various asymptotic models for the water-wave problem.

3. Determining the operator $\mathcal{H}$

In this section we discuss the relationship between $\mathcal{H}(\eta, D)$ and the Dirichlet-to-Neumann operator (DNO) given by [3]. Let $\mathcal{G}(\eta)$ represent the DNO for the water-wave problem. As described in [3], the equations of motion for η and q implied by equations (1-4) are

$$\eta_t = \mathcal{G}(\eta)q, \tag{10}$$

$$q_t + \frac{1}{2}q_x^2 + g\eta - \frac{1}{2}\frac{(\eta_t + \eta_x q_x)^2}{1+\eta_x^2} = 0. \tag{11}$$

One could reduce the above system of equations to a single equation for η if we could find the inverse of the operator $\mathcal{G}(\eta)$. This would allow us to write q in terms of the inverse operator $\mathcal{G}^{-1}(\eta)$, η, and η_t. However, it is readily seen that $\mathcal{G}(\eta)$ does not have a unique inverse since solutions to the Neumann problem for the Laplace equation are unique up to a constant.

In light of the way in which the Dirichlet data appears in the non-local equation of the AFM formulation (equation I of [1]), a map from the Neumann data to the gradient of the Dirichlet data at the surface of the fluid *i.e.* a normal-to-tangential derivative operator is reasonable, particularly when we recognize that q appears in equation (11) through its x and t derivatives alone. The operator $\mathcal{H}(\eta, D)$ has the following formal relationship with the DNO: $\mathcal{H}(\eta, D)\mathcal{G}(\eta, D) \equiv \partial_x$.

3.1. Taylor Series of $\mathcal{H}(\eta)$.

In order to compute wave surfaces, one would like to compute the operator $\mathcal{H}(\eta, D)$. There are several ways to find a representation for this operator including, but not limited to, the Taylor series approach taken for the DNO by [3], or the different approach taken for the DNO given by [2]. We choose to follow the latter as it leads to an easier and more straightforward interpretation.

To determine an expression for $\mathcal{H}(\eta, D)$, we assume that $\mathcal{H}(\eta, D)$ has a Taylor series representation in η of the form

$$\mathcal{H}(\eta, D)\{f\} = \sum_{j=0}^{\infty} \mathcal{H}_j(\eta, D)\{f\},$$

where each $\mathcal{H}_j(\eta, D)$ is homogeneous of order j in η, *i.e.* $\mathcal{H}_j(\lambda\eta, D) = \lambda^j \mathcal{H}_j(\eta, D)$.

Consider the following boundary value problem:

$$\Delta\phi = 0, \quad (x, z) \in S \times [-h, \eta] \tag{12}$$

$$\frac{\partial\phi}{\partial n}(x, \eta) = f(x), \tag{13}$$

$$\frac{\partial\phi}{\partial z}(x, -h) = 0, \tag{14}$$

where $\frac{\partial}{\partial n}$ represents the normal derivative and S is a subset of $\mathbb{R}$. In the horizontal direction, we consider either periodic boundary conditions or decay at infinity. For the infinite domain case we assume f is suitably smooth and has appropriate decay.

Following [1,2], it can be shown that $f(x)$ satisfies the relationship

$$\int_S e^{-ikx}\left[i\cosh(|k|(\eta+h))f(x) - \sinh(k(\eta+h))\mathcal{H}(\eta, D)\{f(x)\}\right]dx = 0, \tag{15}$$

where k is determined by the boundary conditions in x. For the infinite-line case where (12-14) decay as $|\vec{x}| \to \infty$, $k \in \mathbb{R}$. However, if we consider Equations (12-14) with periodic boundary conditions (where $f(x) = f(x+L)$), then the vector k is restricted to the dual lattice Λ' of the problem's period lattice Λ.

A calculation similar to the one presented in [2] allows us to determine the following recursive relationship for $\mathcal{H}_j(\eta, D)$ in terms of lower-order terms:

$$\begin{aligned}\int_S e^{-ikx}\mathcal{H}_j(\eta,D)\{f\}\,d\vec{x} = i\int_S e^{-ikx}\frac{(k\eta)^j}{j!}\left[\begin{array}{cc}\coth(kh); & j \text{ even}\\ 1; & j \text{ odd}\end{array}\right] f\,dx\\ -\int_S e^{-ikx}\sum_{m=1}^{j}\left(\mathcal{H}_{j-m}(\eta,D)\{f\}\frac{(k\eta)^m}{m!}\left[\begin{array}{cc}1; & m \text{ even}\\ \coth(kh); & m \text{ odd}\end{array}\right]\right)dx.\end{aligned} \tag{16}$$

In the above, we have used the brackets [ ] as a conditional multiplier at the appropriate index of summation. The only difference between infinite domain and periodic boundary conditions is the allowable values of the k.

We proceed to find $\mathcal{H}_0(\eta, D)$ by equating $j = 0$ in (16) and evaluating the Fourier transform of the above expression to show that

$$\begin{aligned}\mathcal{H}_0(\eta,D)\{f(x)\} &= \int_{-\infty}^{\infty} e^{ikx} i\coth(kh)\left(\int_S e^{-ik\xi}f(\xi)\,d\xi\right)dk\\ &= \int_{-\infty}^{\infty} dk\int_S d\xi\; e^{ik(x-\xi)} i\coth(kh)f(\xi)\end{aligned}$$

in the case of the whole-line, and

$$\begin{aligned}\mathcal{H}_0(\eta,D)\{f(x)\} &= \sum_{k\in\Lambda} e^{ikx} i\coth(kh)\left(\int_S e^{-ik\xi}f(\xi)\,d\xi\right)dk\\ &= \sum_{k\in\Lambda}\int_S d\xi\; e^{ik(x-\xi)} i\coth(kh)f(\xi)\end{aligned}$$

for the case of periodic boundary conditions. Since the above equation must be true for arbitrary $f(x)$ which satisfies our boundary conditions, we see that the symbol of the pseudo-differential operator $\mathcal{H}_0(\eta, D)$ is given by

$$\mathcal{H}_0(\eta,k) = i\coth(kh).$$

We may now continue to first order to find $\mathcal{H}_1(\eta, D)$ in terms η, $\mathcal{H}_0(\eta)$ and D which gives

$$\begin{aligned}\mathcal{H}_0(\eta,D) &= i\coth(hD),\\ \mathcal{H}_1(\eta,D) &= i\left[D\eta - \coth(hD)D\eta\coth(hD)\right],\\ \mathcal{H}_2(\eta,D) &= \frac{1}{2}\left[iD^2\coth(hD)\eta^2 - D^2\eta^2\mathcal{H}_0(\eta) - 2D\coth(hD)\eta\mathcal{H}_1(\eta)\right],\end{aligned}$$

where we have written the operators in a form suitable to comparison with the DNO of [3]. The symbol of each operator has a singularity at $k = 0$; this singularity is always of order one. Since the operator $\mathcal{H}$ acts on the normal derivative of a function the singularity is cancelled. Note that there was originally a typo above as pointed out by Vishal for the $\mathcal{H}_2(\eta, D)$ term. It has since been corrected.

4. Asymptotic Models

We introduce the nondimensional quantities

$$t^* = t\sqrt{gh}/L,\quad x^* = x/L,\quad z^* = z/h,\quad \eta^* = \eta/a,\quad k^* = Lk,\quad q^* = \frac{\sqrt{gh}}{agL} \tag{17}$$

into (5) and (6) so that we can explore various asymptotic expansions of the equations. We also introduce the parameters ε and μ

$$\varepsilon = \frac{a}{h},\quad \mu = \frac{h}{L},\quad \text{such that } \varepsilon\mu = \frac{a}{L}$$

The non-dimensional version of (5) is given by

$$\frac{agL}{\sqrt{gh}}\frac{\sqrt{gh}}{L}q_t+\frac{1}{2}\frac{a^2g^2L^2}{gh}\frac{1}{L^2}q_x^2+ga\eta-\frac{1}{2}\frac{\left(\frac{a\sqrt{gh}}{L}\eta_t+\frac{agL}{L\sqrt{gh}}q_x\frac{a}{L}\eta_x\right)^2}{1+\frac{a^2}{L^2}\eta_x^2}=0. \tag{18}$$

Simplifying the above expression, we find

$$\varepsilon q_t+\frac{\varepsilon^2}{2}q_x^2+\varepsilon\eta-\frac{1}{2}\frac{\varepsilon^2\mu^2\left(\eta_t+\varepsilon q_x\eta_x\right)^2}{1+\varepsilon^2\mu^2\eta_x^2}=0. \tag{19}$$

Similarly, we look at the definition of the operator $\mathcal{H}$ as given by

$$\mathcal{H}(\eta,D)\left\{\frac{a\sqrt{gh}}{L}\eta_t\right\}=\frac{agL}{L\sqrt{gh}}q_x, \tag{20}$$

Using the fact that the operator $\mathcal{H}$ is linear,

$$\mathcal{H}(\eta,D)\{\varepsilon\mu\eta_t\}=\varepsilon q_x. \tag{21}$$

If we consider a similar problem to the one described earlier, we find

$$\int_S e^{-ikx}\left[i\cosh(\mu k(\varepsilon\eta+1))f(x)-\sinh(\mu k(\varepsilon\eta+1))\mathcal{H}(\eta,D)\{f(x)\}\right]dx=0, \tag{22}$$

where we can recursively define the following

$$\begin{aligned}\int_S e^{-ikx}\mathcal{H}_j(\eta,D)\{f\}\,dx=&\,i\int_S e^{-ikx}\frac{(\varepsilon\mu k\eta)^j}{j!}\begin{bmatrix}\coth(\mu k); & j \text{ even}\\ 1; & j \text{ odd}\end{bmatrix}f\,dx\\ &-\int_S e^{-ikx}\sum_{m=1}^{j}\left(\mathcal{H}_{j-m}(\eta,D)\{f\}\frac{(\varepsilon\mu k\eta)^m}{m!}\begin{bmatrix}1; & m \text{ even}\\ \coth(\mu k); & m \text{ odd}\end{bmatrix}\right)dx.\end{aligned} \tag{23}$$

Since the above equation must be true for arbitrary $f(x)$ which satisfies our boundary conditions, we see that the symbol of the pseudo-differential operator $\mathcal{H}_0(\eta,D)$ is given by

$$\int_S e^{-ikx}\mathcal{H}_0(\eta,D)\{f\}\;dx=i\int_S e^{-ikx}\coth(\mu k)f(x)\,dx.$$

Summing up the Fourier coefficients we find that

$$\mathcal{H}_0(\eta,D)\{f\}=i\sum_k e^{ikx}\coth(\mu k)\hat{f}_k$$

where $\hat{f}_k$ represents the k-th Fourier coefficient of $f(x)$.

We may now continue to first order to find $\mathcal{H}_1(\eta,D)$ in terms η, $\mathcal{H}_0(\eta)$ and D to find

$$\int_S e^{-ikx}\mathcal{H}_1(\eta,D)\{f\}\;dx=i\int_S e^{-ikx}\varepsilon\mu k\eta(x)f(x)\,dx-\int_S e^{-ikx}\mathcal{H}_0(\eta,D)\{f\}\coth(\mu k)\varepsilon\mu k\eta\,dx$$

Inverting the Fourier transforms, we find that

$$\mathcal{H}_1(\eta,D)\{f\}=\varepsilon\mu\left(\sum_k e^{ikx}ik\mathcal{F}\{\eta f\}-\sum_k e^{ikx}k\coth(\mu k)\mathcal{F}\left\{\eta\,\mathcal{H}_0(\eta,D)\{f\}\right\}_k\right)$$

Simplifying the above expression, we find

$$\mathcal{H}_1(\eta,D)\{f\}=\varepsilon\mu\left(\partial_x(\eta f)-\sum_k e^{ikx}k\coth(\mu k)\mathcal{F}\left\{\eta\,\mathcal{H}_0(\eta,D)\{f\}\right\}_k\right)$$

So, in summary, we find the following operators:

$$
\begin{aligned}
\mathcal{H}_0(\eta, D)\{f\} &= i\sum_k e^{ikx}\coth(\mu k)\hat{f}_k, \\
\mathcal{H}_1(\eta, D)\{f\} &= \varepsilon\mu\left(\partial_x\left(\eta f\right) - \sum_k e^{ikx} k\coth(\mu k)\mathcal{F}\left\{\eta\,\mathcal{H}_0(\eta, D)\left\{f\right\}\right\}_k\right),
\end{aligned}
$$

where we have written the operators in a form suitable to comparison with the DNO of [3]. The symbol of each operator has a singularity at $k = 0$; this singularity is always of order one. Since the operator $\mathcal{H}$ acts on the normal derivative of a function the singularity is cancelled.

References

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