CHAPTER

8

Laminar Flow

OUTLINE

CHAPTER OBJECTIVES

- To present a variety of exact and approximate solutions to the viscous equations of fluid motion in confined and unconfined geometries.
- To introduce lubrication theory and indicate its utility.
- To define and present similarity solutions to exact and approximate viscous flow field equations.
- To develop the equations for creeping flow and illustrate their use.

8.1. INTRODUCTION

Chapters 6 and 7 covered flows in which the viscous terms in the Navier-Stokes equations were dropped because the flow was ideal (irrotational and constant density) or the effects of

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viscosity were small. For these situations, the underlying assumptions were either that 1) viscous forces and rotational flow were spatially confined to a small portion of the flow domain (thin boundary layers near solid surfaces), or that 2) fluid particle accelerations caused by fluid inertia $\sim U^2/L$ were much larger than those caused by viscosity $\sim \mu U/\rho L^2$, where U is a characteristic velocity, L is a characteristic length, ρ is the fluid's density, and μ is the fluid's kinematic viscosity. Both of these assumptions are valid if the Reynolds number is large and boundary layers stay attached to the surface on which they have formed.

However, for low values of the Reynolds number, the *entire flow* may be influenced by viscosity, and inviscid flow theory is no longer even approximately correct. The purpose of this chapter is to present some exact and approximate solutions of the Navier-Stokes equations for simple geometries and situations, retaining the viscous terms in (4.38) everywhere in the flow and applying the no-slip boundary condition at solid surfaces (see Section 4.10).

Viscous flows generically fall into two categories, *laminar* and *turbulent*, but the boundary between them is imperfectly defined. The basic difference between the two categories is phenomenological and was dramatically demonstrated in 1883 by Reynolds, who injected a thin stream of dye into the flow of water through a tube (Figure 8.1). At low flow rates, the dye stream was observed to follow a well-defined straight path, indicating that the fluid moved in parallel layers (laminae) with no unsteady macroscopic mixing or overturning motion of the layers. Such smooth orderly flow is called *laminar*. However, if the flow rate

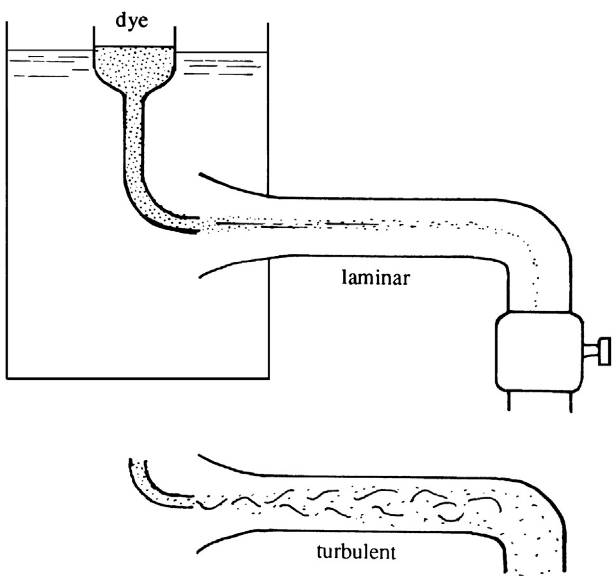


FIGURE 8.1 Reynolds's experiment to distinguish between laminar and turbulent flows. At low flow rates (the upper drawing), the pipe flow was laminar and the dye filament moved smoothly through the pipe. At high flow rates (the lower drawing), the flow became turbulent and the dye filament was mixed throughout the cross section of the pipe.

was increased beyond a certain critical value, the dye streak broke up into irregular filaments and spread throughout the cross section of the tube, indicating the presence of unsteady, apparently chaotic three-dimensional macroscopic mixing motions. Such irregular disorderly flow is called *turbulent*. Reynolds demonstrated that the transition from laminar to turbulent flow always occurred at a fixed value of the ratio that bears his name, the Reynolds number, $\mathrm{Re} = Ud/\nu \sim 2000$ to 3000 where U is the velocity averaged over the tube's cross section, d is the tube diameter, and $\nu = \mu/\rho$ is the kinematic viscosity.

As will be further verified in Section 8.4, the fluid's kinematic viscosity specifies the propensity for vorticity to diffuse through a fluid. Consider (5.13) for the z-component of vorticity in a two-dimensional flow confined to the x-y plane so that $\boldsymbol{\omega} \cdot \nabla \mathbf{u} = 0$:

$$D\omega_z/Dt = \nu \nabla^2 \omega_z.$$

This equation states that the rate of change of ω_z following a fluid particle is caused by diffusion of vorticity. Clearly, for the same initial vorticity distribution, a fluid with larger ν will produce a larger diffusion term, $\nu \nabla^2 \omega$, and more rapid changes in the vorticity. This equation is similar to the Boussinesq heat equation,

$$DT/Dt = \kappa \nabla^2 T, \tag{4.89}$$

where $\kappa \equiv k/\rho C_p$ is the *thermal diffusivity*, and this similarity suggests that vorticity diffuses in a manner analogous to heat. At a coarse level, this suggestion is correct since both ν and κ arise from molecular processes in real fluids and both have the same units (length^2/time). The similarity emphasizes that the diffusive effects are controlled by ν and κ, and not by μ (viscosity) and k (thermal conductivity). In fact, the constant-density, constant-viscosity momentum equation,

$$D\mathbf{u}/Dt = -(1/\rho)\nabla p + \nu \nabla^2 \mathbf{u}, \tag{4.85, 8.1}$$

also shows that the acceleration due to viscous diffusion is proportional to ν. Thus, at room temperature and pressure, air ($\nu = 15 \times 10^{-6}$ m^2/s) is 15 times more diffusive than water ($\nu = 1 \times 10^{-6}$ m^2/s), although μ for water is larger. Both ν and κ have the units of m^2/s; thus, the kinematic viscosity ν is sometimes called the *momentum diffusivity*, in analogy with κ, the *thermal diffusivity*. However, velocity cannot be simply regarded as being diffused and advected in a flow because of the presence of the pressure gradient in (8.1).

Laminar flows in which viscous effects are important throughout the flow are the subject of the present chapter. The primary field equations will be $\nabla \cdot \mathbf{u} = 0$ (4.10) and (8.1) or the version that includes a body force (4.39b). The velocity boundary conditions on a solid surface are:

$$\mathbf{n} \cdot \mathbf{U}_s = (\mathbf{n} \cdot \mathbf{u})_{on\ the\ surface} \text{ and } \mathbf{t} \cdot \mathbf{U}_s = (\mathbf{t} \cdot \mathbf{u})_{on\ the\ surface}, \tag{8.2, 8.3}$$

where $\mathbf{U}_s$ is the velocity of the surface, $\mathbf{n}$ is the normal to the surface, and $\mathbf{t}$ is the tangent to the surface. Here fluid density will be assumed constant, and the frame of reference will be inertial. Thus, gravity can be dropped from the momentum equation as long as no free surface is present (see Section 4.9 "Neglect of Gravity in Constant Density Flows"). Laminar flows in which frictional effects are confined to boundary layers near solid surfaces are discussed in the next chapter. Chapter 11 considers the stability of laminar flows and their transition to turbulence; fully turbulent flows are discussed in Chapter 12. Some viscous flow solutions in rotating coordinates, such as the Ekman layers, are presented in Chapter 13.

8.2. EXACT SOLUTIONS FOR STEADY INCOMPRESSIBLE VISCOUS FLOW

Because of the presence of the nonlinear acceleration term $\mathbf{u}\cdot\nabla\mathbf{u}$ in (8.1), very few exact solutions of the Navier-Stokes equations are known in closed form. An example of an exact solution is that for steady laminar flow between infinite parallel plates (Figure 8.2). Such a flow is said to be *fully developed* when its velocity profile $u(x,y)$ becomes independent of the downstream coordinate x so that $u = u(y)$ alone. The entrance length of the flow, where the velocity profile depends on the downstream distance, may be several or even many times longer than the spacing between the plates. Within the entrance length, the derivative $\partial u/\partial x$ is not zero so the continuity equation $\partial u/\partial x + \partial v/\partial y = 0$ requires that $v \neq 0$, so that the flow is *not* parallel to the walls within the entrance length. Laminar flow development is the subject of Section 8.4, and the next chapter. Here we are interested in steady, fully developed flows.

Steady Flow between Parallel Plates

Consider the situation depicted in Figure 8.3 where a viscous fluid flows between plates parallel to the x-axis with lower and upper plates at $y = 0$ and $y = h$, respectively. The flow

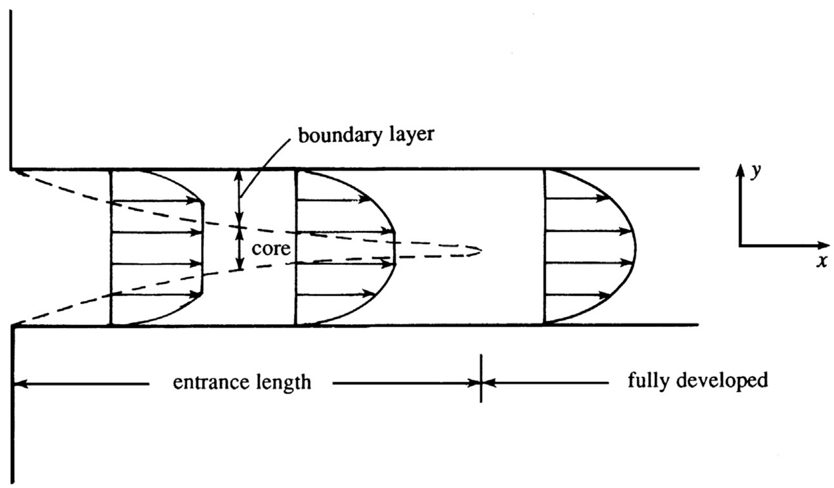


FIGURE 8.2 Developing and fully developed flows in a channel. Within the entrance length, the viscous boundary layers on the upper and lower walls are separate and the flow profile $u(x,y)$ depends on both spatial coordinates. Downstream of the point where the boundary layers merge, the flow is fully developed and its profile $u(y)$ is independent of the stream-wise coordinate x.

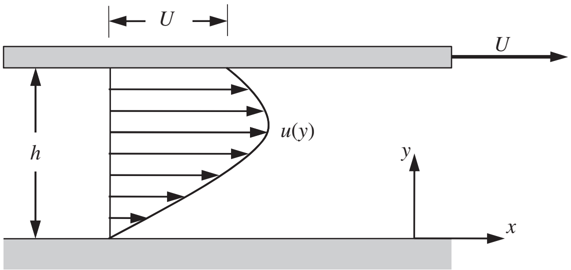


FIGURE 8.3 Flow between parallel plates when the lower plate at $y = 0$ is stationary, the upper plate at $y = h$ is moving in the positive-x direction at speed U, and a nonzero $dp/dx < 0$ leads to velocity profile curvature.

is sustained by an externally applied pressure gradient ($\partial p/\partial x \neq 0$) in the x-direction, and horizontal motion of the upper plate at speed U in the x-direction. For this situation, the flow should be independent of the z-direction so $w = 0$ and $\partial/\partial z = 0$ can be used in the equations of motion. A steady, fully developed flow will have a horizontal velocity $u(y)$ that does not depend on x so $\partial u/\partial x = 0$. Thus, the continuity equation, $\partial u/\partial x + \partial v/\partial y = 0$, requires $\partial v/\partial y = 0$, and since $v = 0$ at $y = 0$ and h, it follows that $v = 0$ everywhere, which reflects the fact that the flow is parallel to the walls. Under these circumstances, $\mathbf{u} = (u(y), 0, 0)$, and the x- and y-momentum equations reduce to:

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{d^2u}{dy^2}, \quad \text{and} \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial y}. \tag{8.4a,b}$$

The y-momentum equation shows that p is not a function of y, so $p = p(x)$. Thus, the first term in the x-momentum equation must be a function of x alone, while the second term must be a function of y alone. The only way the equation can be satisfied throughout x-y space is if both terms are constant. The *pressure gradient is therefore a constant*, which implies that the pressure varies linearly along the channel. Integrating the x-momentum equation twice, we obtain

$$0 = -\frac{y^2}{2}\frac{dp}{dx} + \mu u + Ay + B,$$

where A and B are constants and dp/dx has replaced $\partial p/\partial x$ because p is a function of x alone. The constants are determined from the boundary conditions: $u = 0$ at $y = 0$, and $u = U$ at $y = h$. The results are $B = 0$ and $A = (h/2)(dp/dx) - \mu U/h$, so the velocity profile becomes

$$u(y) = \frac{U}{h}y - \frac{1}{2\mu}\frac{dp}{dx}y(h - y), \tag{8.5}$$

which is illustrated in Figure 8.4 for various cases. The volume flow rate Q per unit width of the channel is

$$Q = \int_0^h u\,dy = U\frac{h}{2}\left[1 - \frac{h^2}{6\mu U}\frac{dp}{dx}\right],$$

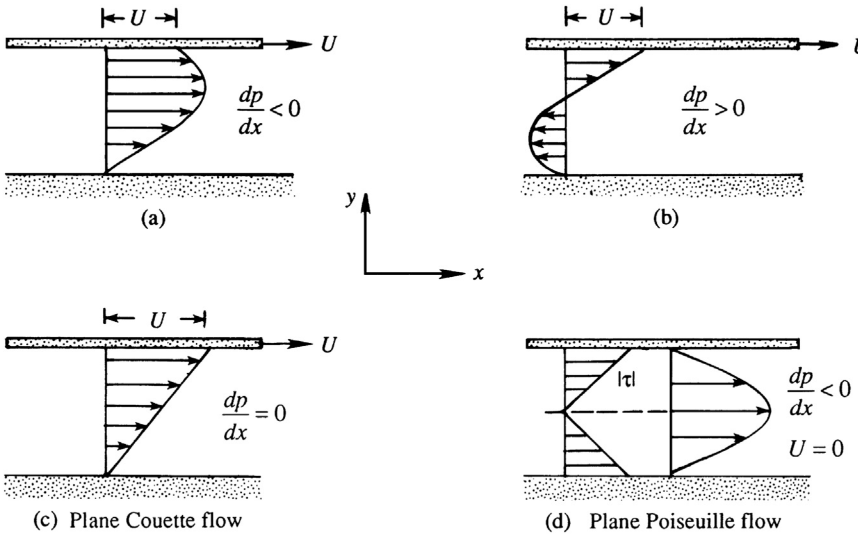


FIGURE 8.4 Various cases of parallel flow in a channel: (a) positive U and favorable $dp/dx < 0$, (b) positive U and adverse $dp/dx > 0$, (c) positive U and $dp/dx = 0$, and (d) $U = 0$ and favorable $dp/dx < 0$.

so that the average velocity is

$$V \equiv \frac{Q}{h} = \int_0^h u\, dy = \frac{U}{2}\left[1 - \frac{h^2}{6\mu U}\frac{dp}{dx}\right].$$

Here, negative and positive pressure gradients increase and decrease the flow rate, respectively.

When the flow is driven by motion of the upper plate alone, without any externally imposed pressure gradient, it is called a *plane Couette flow*. In this case (8.5) reduces to $u(y) = Uy/h$, and the magnitude of the shear stress is $\tau = \mu(du/dy) = \mu U/h$, which is uniform across the channel.

When the flow is driven by an externally imposed pressure gradient without motion of either plate, it is called a *plane Poiseuille flow*. In this case (8.5) reduces to the parabolic profile (Figure 8.4d):

$$u(y) = -\frac{1}{2\mu}\frac{dp}{dx}y(h-y).$$

The shear stress is

$$\tau = \mu\frac{du}{dy} = -\left(\frac{h}{2} - y\right)\frac{dp}{dx},$$

which is linear with a magnitude of $(h/2)(dp/dx)$ at the walls (Figure 8.4d).

Interestingly, the constancy of the pressure gradient and the linearity of the shear stress distribution are general results for a fully developed channel flow and persist for appropriate averages of these quantities when the flow is turbulent.

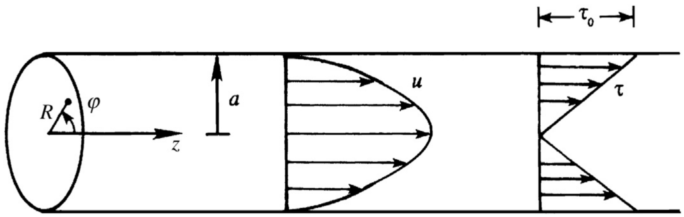


FIGURE 8.5 Laminar flow through a round tube. The flow profile is parabolic, similar to pressure-driven flow between stationary parallel plates (Figure 8.4d).

Steady Flow in a Round Tube

A second geometry for which there is an exact solution of (4.10) and (8.1) is steady, fully developed laminar flow through a round tube of constant radius a, frequently called *circular Poiseuille flow*. We employ cylindrical coordinates (R, φ, z), with the z-axis coinciding with the axis of the tube (Figure 8.5). The equations of motion in cylindrical coordinates are given in Appendix B. The only nonzero component of velocity is the axial velocity $u_z(R)$, and $\mathbf{u} = (0, 0, u_z(R))$ automatically satisfies the continuity equation. The radial and angular equations of motion reduce to

$$0 = \partial p/\partial\varphi \text{ and } 0 = \partial p/\partial R,$$

so p is a function of z alone. The z-momentum equation gives

$$0 = -\frac{dp}{dz} + \frac{\mu}{R}\frac{d}{dR}\left(R\frac{du_z}{dR}\right).$$

As for flow between parallel plates, the first term must be a function of the stream-wise coordinate, z, alone and the second term must be a function of the cross-stream coordinate, R, alone, so both terms must be constant. The pressure therefore falls linearly along the length of the tube. Integrating the stream-wise momentum equation twice produces

$$u_z(R) = \frac{R^2}{4\mu}\frac{dp}{dz} + A\ln R + B.$$

To keep u_z bounded at $R = 0$, the constant A must be zero. The no-slip condition $u_z = 0$ at $R = a$ gives $B = -(a^2/4\mu)(dp/dz)$. The velocity distribution therefore takes the parabolic shape:

$$u_z(R) = \frac{R^2 - a^2}{4\mu}\frac{dp}{dz}. \tag{8.6}$$

From Appendix B, the shear stress at any point is

$$\tau_{zR} = \mu\left(\frac{\partial u_R}{\partial z} + \frac{\partial u_z}{\partial R}\right).$$

In this case the radial velocity u_R is zero. Dropping the subscripts on τ and differentiating (8.6) yields

$$\tau = \mu \frac{\partial u_z}{\partial R} = \frac{R}{2}\frac{dp}{dz}, \tag{8.7}$$

which shows that the stress distribution is linear, having a maximum value at the wall of

$$\tau_0 = \frac{a}{2}\frac{dp}{dz}. \tag{8.8}$$

Here again, (8.8) is also valid for appropriate averages of τ_0 and p for turbulent flow in a round pipe.

The volume flow rate in the tube is:

$$Q = \int_0^a u(R) 2\pi R dR = -\frac{\pi a^4}{8\mu}\frac{dp}{dz},$$

where the negative sign offsets the negative value of dp/dz. The average velocity over the cross section is

$$V = \frac{Q}{\pi a^2} = -\frac{a^2}{8\mu}\frac{dp}{dz}.$$

Steady Flow between Concentric Rotating Cylinders

A third example in which the nonlinear advection terms drop out of the equations of motion is steady flow between two concentric, rotating cylinders, also know as *circular Couette flow*. Let the radius and angular velocity of the inner cylinder be R_1 and Ω_1 and those for the outer cylinder be R_2 and Ω_2 (Figure 8.6). Using cylindrical coordinates and assuming that $\mathbf{u} = (0, u_\varphi(R), 0)$, the continuity equation is automatically satisfied, and the momentum equations for the radial and tangential directions are

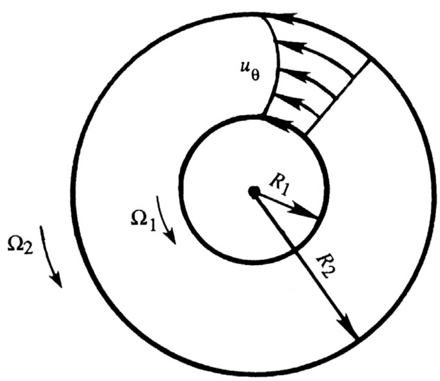


FIGURE 8.6 Circular Couette flow. The viscous fluid flows in the gap between an inner cylinder with radius R_1 that rotates at angular speed Ω_1 and an outer cylinder with radius R_2 that rotates at angular speed Ω_2.

$$-\frac{u_\varphi^2}{R} = -\frac{1}{\rho}\frac{dp}{dR}, \quad \text{and} \quad 0 = \mu\frac{d}{dR}\left[\frac{1}{R}\frac{d}{dR}(Ru_\varphi)\right].$$

The R-momentum equation shows that the pressure increases radially outward due to the centrifugal acceleration. The pressure distribution can therefore be determined once $u_\varphi(R)$ has been found. Integrating the φ-momentum equation twice produces

$$u_\varphi(R) = AR + B/R. \tag{8.9}$$

Using the boundary conditions $u_\varphi = \Omega_1 R_1$ at $R = R_1$, and $u_\varphi = \Omega_2 R_2$ at $R = R_2$, A and B are found to be

$$A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}, \quad \text{and} \quad B = -\frac{(\Omega_2 - \Omega_1)R_1^2 R_2^2}{R_2^2 - R_1^2}.$$

Substitution of these into (8.9) produces the velocity distribution,

$$u_\varphi(R) = \frac{1}{R_2^2 - R_1^2}\left\{\left[\Omega_2 R_2^2 - \Omega_1 R_1^2\right]R - \left[\Omega_2 - \Omega_1\right]\frac{R_1^2 R_2^2}{R}\right\}, \tag{8.10}$$

which has interesting limiting cases when $R_2 \to \infty$ with $\Omega_2 = 0$, and when $R_1 \to 0$ with $\Omega_1 = 0$.

The first limiting case produces the flow outside a long circular cylinder with radius R_1 rotating with angular velocity Ω_1 in an infinite bath of viscous fluid (Figure 8.7). By direct simplification of (8.10), the velocity distribution is

$$u_\varphi(R) = \frac{\Omega_1 R_1^2}{R}, \tag{8.11}$$

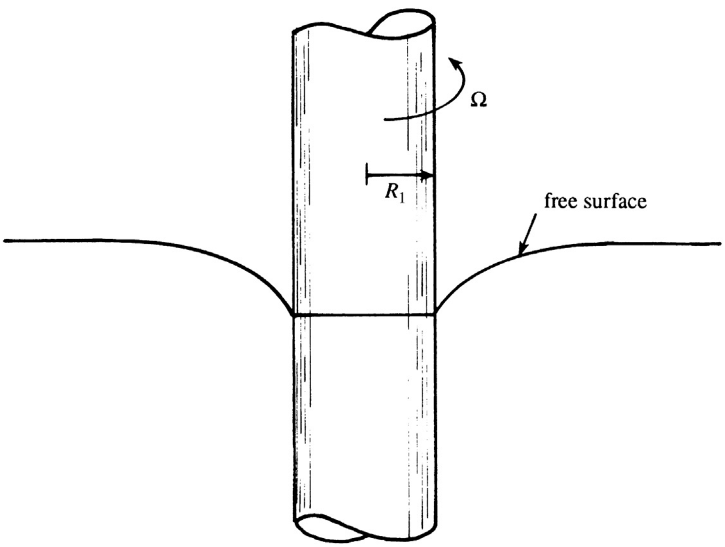


FIGURE 8.7 Rotation of a solid cylinder of radius R_1 in an infinite body of viscous fluid. If gravity points downward along the cylinder's axis, the shape of a free surface pierced by the cylinder is also indicated. The flow field is viscous but irrotational.

which is identical to that of an ideal vortex, see (5.2), for $R > R_1$ when $\Gamma = 2\pi\Omega_1 R_1^2$. This is the only example of a viscous solution that is completely irrotational. As described in Section 5.1, shear stresses do exist in this flow, but there is no *net* viscous force on a fluid element. The viscous shear stress at any point is given by

$$\sigma_{R\varphi} = \mu\left[\frac{1}{R}\frac{\partial u_R}{\partial\varphi} + R\frac{\partial}{\partial R}\left(\frac{u_\varphi}{R}\right)\right] = -\frac{2\mu\Omega_1 R_1^2}{R^2}.$$

The mechanical power supplied to the fluid (per unit length of cylinder) is $(2\pi R_1)\tau_{R\varphi}u_\varphi$, and it can be shown that this power equals the integrated viscous dissipation of the flow field (Exercise 8.12).

The second limiting case of (8.10) produces steady viscous flow within a cylindrical tank of radius R_2 rotating at rate Ω_2. Setting R_1 and Ω_1 equal to zero in (8.10) leads to

$$u\phi(R) = \Omega_2 R, \tag{8.12}$$

which is the velocity field of solid body rotation, see (5.1) and Section 5.1.

The three exact solutions of the incompressible viscous flow equations (4.10) and (8.1) described in this section are all known as internal or confined flows. In each case, the velocity field was confined between solid walls and the symmetry of each situation eliminated the nonlinear advective acceleration term from the equations. Other exact solutions of the incompressible viscous flow equations for confined and unconfined flows are described in other fine texts (Sherman, 1990; White, 2006), in Section 8.4, and in the Exercises of this chapter. However, before proceeding to these, a short diversion into elementary lubrication theory is provided in the next section.

8.3. ELEMENTARY LUBRICATION THEORY

The exact viscous flow solutions for ideal geometries presented in the prior section indicate that a linear or simply varying velocity profile is a robust solution for flow within a confined space. This observation has been developed into the theory of lubrication, which provides approximate solutions to the viscous flow equations when the geometry is not ideal but at least one transverse flow dimension is small. The elementary features of lubrication theory are presented here because of its connection to the exact solutions described in Section 8.2, especially the Couette and Poiseuille flow solutions. Plus, the development of approximate equations in this section parallels that necessary for the boundary layer approximation (see Section 9.1).

The economic importance of lubrication with viscous fluids is hard to overestimate, and lubrication theory covers the mathematical formulation and analysis of such flows. The purpose of this section is to develop the most elementary equations of lubrication theory and illustrate some interesting phenomena that occur in viscous constant-density flows where the flow's boundaries or confining walls are close together, but not precisely parallel, and their motion is mildly unsteady. For simplicity consider two spatial dimensions, x and y, where the primary flow direction, x, lies along the narrow flow passage with gap height $h(x,t)$ (see Figure 8.8). The length L of this passage is presumed to be large

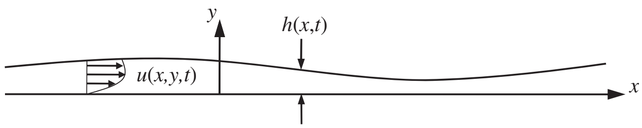


FIGURE 8.8 Nearly parallel flow of a viscous fluid having a film thickness of $h(x,t)$ above a flat stationary surface.

compared to h so that viscous and pressure forces are the primary terms in any fluid-momentum balance. If the passage is curved, this will not influence the analysis as long as the radius of curvature is much larger than the gap height h. The field equations are (4.10) and (8.1) for the horizontal u, and vertical v velocity components, and the pressure p in the fluid:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{6.2}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),\text{ and }\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right). \tag{8.13a, 8.13b}$$

Here, the boundary conditions are $u = U_0(t)$ on $y = 0$ and $u = U_h(t)$ on $y = h(x, t)$, and the pressure is presumed to be time dependent as well.

To determine which terms are important and which may be neglected when the passage is narrow, recast these equations in terms of dimensionless variables:

$$x^* = x/L,\ y^* = y/h = y/\varepsilon L,\ \ t^* = Ut/L,\ \ u^* = u/U,\ \ v^* = v/\varepsilon U,\text{ and } p^* = p/P_a, \tag{8.14}$$

where U is a characteristic velocity of the flow, P_a is atmospheric pressure, and $\varepsilon = h/L$ is the passage's *fineness ratio* (the inverse of its aspect ratio). The goal of this effort is to find a set of approximate equations that are valid for common lubrication geometries where $\varepsilon \ll 1$ and the flow is unsteady. Because of the passage geometry, the magnitude of v is expected to be much less than the magnitude of u and gradients along the passage, $\partial/\partial x \sim 1/L$, are expected to be much smaller than gradients across it, $\partial/\partial y \sim 1/h$. These expectations have been incorporated into the dimensionless scaling (8.14). Combining (6.2), (8.13), and (8.14) leads to the following dimensionless equations:

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0, \tag{8.15}$$

$$\begin{aligned}
&\varepsilon^2\,\mathrm{Re}_L\left(\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right)=-\frac{1}{\Lambda}\frac{\partial p^*}{\partial x^*}+\varepsilon^2\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}},\text{ and}\\
&\varepsilon^4\,\mathrm{Re}_L\left(\frac{\partial v^*}{\partial t^*}+u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}\right)=-\frac{1}{\Lambda}\frac{\partial p^*}{\partial y^*}+\varepsilon^4\frac{\partial^2 v^*}{\partial x^{*2}}+\varepsilon^2\frac{\partial^2 v^*}{\partial y^{*2}},
\end{aligned} \tag{8.16a, 8.16b}$$

where $\mathrm{Re}_L = \rho UL/\mu$, and $\Lambda = \mu UL/P_a h^2$ is the ratio of the viscous and pressure forces on a fluid element; it is sometimes called the *bearing number*. All the dimensionless derivative terms should be of order unity when the scaling (8.14) is correct. Thus, possible simplifying approximations are based on the size of the dimensionless coefficients of the various terms. The scaled two-dimensional continuity equation (8.15) does not contain any dimensionless coefficients so mass must be conserved without approximation. The two-scaled momentum equations (8.16) contain ε, Re_L, and Λ. For the present purposes, Λ must be considered to be near unity, Re_L must be finite, and $\varepsilon \ll 1$. When $\varepsilon^2\mathrm{Re}_L \ll 1$, the left side and the middle term on the right side of (8.16a) may be ignored. In (8.16b) the pressure derivative is the only term not multiplied by ε. Therefore, the momentum equations can be approximately simplified to:

$$0 \cong -\frac{1}{\mu}\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad 0 \cong -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{8.17a, 8.17b}$$

when $\varepsilon^2\mathrm{Re}_L \rightarrow 0$. As a numerical example of this approximation, $\varepsilon^2\mathrm{Re}_L = 0.001$ for room temperature flow of common 30-weight oil with $\nu \approx 4 \times 10^{-4}$ m^2/s within a 0.1 mm gap between two 25-cm-long surfaces moving with a differential speed of 10 m/s. When combined with a statement of conservation of mass, the equations (8.17) are the simplest form of the lubrication approximation (the *zeroth*-order approximation), and these equations are readily extended to two-dimensional gap-thickness variations (see Exercise 8.19). Interestingly, the approximations leading to (8.17) eliminated both the unsteady and the advective fluid acceleration terms from (8.16); a steady-flow approximation was not made. Therefore, time is still an independent variable in (8.17) even though it does not explicitly appear.

A generic solution to (8.17) is readily produced by following the steps used to solve (8.4a, 8.4b). Equation (8.17b) implies that p is not a function of y, so (8.16a) can be integrated twice to produce:

$$u(x, y, t) \cong \frac{1}{\mu}\frac{\partial p(x,t)}{\partial x}\frac{y^2}{2} + Ay + B, \tag{8.18}$$

where A and B might be functions of x and t but not y. Applying the boundary conditions mentioned earlier allows A and B to be evaluated, and the fluid velocity within the gap is found to be:

$$u(x, y, t) \cong -\frac{h^2(x,t)}{2\mu}\frac{\partial p(x,t)}{\partial x}\frac{y}{h(x,t)}\left(1 - \frac{y}{h(x,t)}\right) + U_h(t)\frac{y}{h(x,t)} + U_0(t). \tag{8.19}$$

The basic result here is that balancing viscous and pressure forces leads to a velocity profile that is parabolic in the cross-stream direction. While (8.19) represents a significant simplification of the two momentum equations (8.13), it is not a complete solution because the pressure $p(x, t)$ within the gap has not yet been determined. The complete solution to an elementary lubrication flow problem is typically obtained by combining (8.19), or an appropriate equivalent, with a differential or integral form of (4.10) or (6.2), and pressure boundary conditions. Such solutions are illustrated in the following examples.

EXAMPLE 8.1

A sloped bearing pad of width B into the page moves horizontally at a steady speed U on a thin layer of oil with density ρ and viscosity μ. The gap between the bearing pad and a stationary hard, flat surface located at $y = 0$ is $h(x) = h_o(1 + \alpha x/L)$ where $\alpha \ll 1$. If p_e is the exterior pressure and $p(x)$ is the pressure in the oil under the bearing pad, determine the load W (per unit width into the page) that the bearing can support.

Solution

The solution plan is to conserve mass exactly using (4.5), a control volume (CV) that is attached to the bearing pad, and the generic velocity profile (8.19). Then, pressure boundary conditions at the ends of the bearing pad should allow the pressure distribution under the pad to be found. Finally, W can be determined by integrating this pressure distribution.

Use the fixed-shape, but moving CV shown in Figure 8.9 that lies between x_1 and x_2 at the moment of interest. The mass of fluid in the CV is constant so the unsteady term in (4.5) is zero, and the control surface velocity is $\mathbf{b} = U\mathbf{e}_x$. Denote the fluid velocity as $\mathbf{u} = u(x, y)\mathbf{e}_x$, and recognize $\mathbf{n}dA = -\mathbf{e}_x B dy$ on the vertical CV surface at x_1 and $\mathbf{n}dA = +\mathbf{e}_x B dy$ on the vertical CV surface at x_2. Thus, (4.5) simplifies to:

$$\rho B\left[-\int_0^{h(x_1)}(u(x_1,y)-U)dy + \int_0^{h(x_2)}(u(x_2,y)-U)dy\right] = 0.$$

Dividing this equation by $\rho B(x_2 - x_1)$ and taking the limit as $(x_2 - x_1) \to 0$ produces:

$$\frac{d}{dx}\left[\int_0^{h(x)}(u(x,y)-U)dy\right] = 0, \text{ or } \int_0^{h(x)}(u(x,y)-U)dy = C_1,$$

where C_1 is a constant. For the flow geometry and situation in Figure 8.9, (8.19) simplifies to:

$$u(x,y,t) \cong -\frac{h^2(x)}{2\mu}\frac{dp(x)}{dx}\frac{y}{h(x)}\left(1 - \frac{y}{h(x)}\right) + U\frac{y}{h(x)},$$

which can substituted into with the conservation of mass result and integrated to determine C_1 in terms of dp/dx and $h(x)$:

$$C_1 = -\frac{h^3(x)}{12\mu}\frac{dp(x)}{dx} - \frac{Uh(x)}{2}, \text{ or } \frac{dp(x)}{dx} = -\frac{12\mu}{h^3(x)}C_1 - \frac{6\mu U}{h^2(x)}.$$

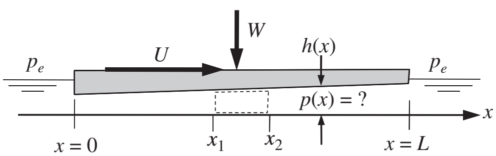


FIGURE 8.9 Schematic drawing of a bearing pad with load W moving above a stationary flat surface coated with viscous oil. The gap below the pad has a mild slope and the pressure ahead, behind, and on top of the bearing pad is p_e.

The second equation is just an algebraic rearrangement of the first, and is a simple first-order differential equation for the pressure that can be integrated using the known $h(x)$ from the problem statement:

$$p(x) = -\frac{12\mu C_1}{h_o^3}\int\frac{dx}{(1-\alpha x/L)^3} - \frac{6\mu U}{h_o^2}\int\frac{dx}{(1-\alpha x/L)^2} + C_2 = \frac{6\mu L}{h_o^2\alpha}\left[\frac{C_1/h_o}{(1-\alpha x/L)^2} + \frac{U}{1-\alpha x/L}\right] + C_2.$$

Using the two ends of this extended equality and the pressure conditions $p(x=0) = p(x=L) = p_e$ produces two algebraic equations that can be solved simultaneously for the constants C_1 and C_2:

$$C_1 = -\left(\frac{1+\alpha}{2+\alpha}\right)Uh_o, \text{ and } C_2 = p_e - \frac{6\mu LU}{h_o^2\alpha}\left(\frac{1}{2+\alpha}\right).$$

Thus, after some algebra the following pressure distribution is found:

$$p(x) - p_e = \frac{6\mu LU}{h_o^2}\left[\frac{\alpha(x/L)(1-x/L)}{(2+\alpha)(1+\alpha x/L)}\right].$$

However, this distribution may contain some superfluous dependence on α and x, because no approximations have been made regarding the size of α while (8.19) is only valid when $\alpha \ll 1$. Thus, keeping only the linear term in α produces:

$$p(x) - p_e \cong \frac{3\alpha\mu LU}{h_o^2}\left(\frac{x}{L}\right)\left(1-\frac{x}{L}\right) \text{ for } \alpha \ll 1.$$

The bearing load per unit depth into the page is

$$W = \int_0^L (p(x) - p_e)\,dx = \frac{3\alpha\mu LU}{h_o^2}\int_0^L \left(\frac{x}{L}\right)\left(1-\frac{x}{L}\right)dx = \frac{\alpha\mu L^2 U}{2h_o^2}.$$

This result shows that larger loads may be carried when the bearing slope, the fluid viscosity, the bearing size, and/or the bearing speed are larger, or the oil passage is smaller. Thus, the lubrication action of this bearing pad as described in this example is stable to load perturbations when the other parameters are held constant; an increase in load will lead to a smaller h_o where the bearing's load-carrying capacity is higher. However, the load-carrying capacity of this bearing goes to zero when α, μ, L, or U go to zero, and this bearing design fails (i.e., W becomes negative so the pad and surface are drawn into contact) when either α or U are negative. Thus, the bearing only works when it moves in the correct direction. A more detailed analysis of this bearing flow is provided in Sherman (1990).

EXAMPLE 8.2: VISCOUS FLOW BETWEEN PARALLEL PLATES (HELE-SHAW 1898)

A viscous fluid flows with velocity $\mathbf{u} = (u, v, w)$ in a narrow gap between stationary parallel plates lying at $z = 0$ and $z = h$ as shown in Figure 8.10. Nonzero x- and y-directed pressure gradients are maintained at the plates' edges, and obstacles or objects of various sizes may be placed between the plates. Using the continuity equation (4.10) and the two horizontal (x, y) and one vertical (z) momentum equations (deduced in Exercise 8.19),

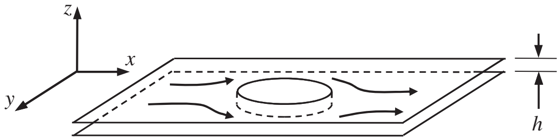


FIGURE 8.10 Pressure-driven viscous flow between parallel plates that trap an obstacle. The gap height h is small compared to the extent of the plates and the extent of the obstacle, shown here as a round disk.

$$0 \cong -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial z^2}, \quad 0 \cong -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial^2 v}{\partial z^2}, \text{ and } 0 \cong -\frac{1}{\rho}\frac{\partial p}{\partial z},$$

show that the two in-plane velocity components parallel to the plates, u and v, can be determined from the equations for two-dimensional potential flow:

$$u = \frac{\partial \phi}{\partial x} \text{ and } v = \frac{\partial \phi}{\partial y} \text{ with } \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \qquad (6.10,\ 6.12)$$

for an appropriate choice of ϕ.

Solution

The solution plan is to use the two horizontal momemtum equations given above to determine the functional forms of u and v. Then ϕ can be determined via integration of (6.10). Combining these results into (4.10) should produce (6.12), the two-dimensional Laplace equation for ϕ. Integrating the two horizontal momemtum equations twice in the z-direction produces:

$$u \cong \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{z^2}{2} + Az + B, \quad \text{and} \quad v \cong \frac{1}{\mu}\frac{\partial p}{\partial y}\frac{z^2}{2} + Cz + D,$$

where A, B, C, and D are constants that can be determined from the boundary conditions on $y = 0$, $u = v = 0$ which produces $B = D = 0$, and on $y = h$, $u = v = 0$, which produces $A = -(h/2\mu)(\partial p/\partial x)$ and $C = -(h/2\mu)(\partial p/\partial y)$. Thus, the two in-plane velocity components are:

$$u \cong -\frac{1}{2\mu}\frac{\partial p}{\partial x}z(h - z) = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v \cong -\frac{1}{2\mu}\frac{\partial p}{\partial y}z(h - z) = \frac{\partial \phi}{\partial y}.$$

Integrating the second equality in each case produces:

$$\phi = -\frac{z(h - z)}{2\mu}p + E(y) \quad \text{and} \quad \phi = -\frac{z(h - z)}{2\mu}p + F(x).$$

These equations are consistent when $E = F = const.$, and this constant can be set to zero without loss of generality because it does not influence u and v, which are determined from derivatives of ϕ. Therefore, the velocity field requires a potential ϕ of the form:

$$\phi = -\frac{z(h - z)}{2\mu}p.$$

To determine the equation satisfied by p or ϕ, place the results for u and v into the continuity equation (4.10) and integrate in the z-direction from $z = 0$ to h to find:

$$\int_0^h \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz = -\int_0^h \frac{\partial w}{\partial z} dz = -(w)_{z=0}^{z=h} = 0 \rightarrow -\frac{1}{2\mu}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) \int_0^h z(h-z)dz = 0,$$

where the no-through-flow boundary condition ensures $w = 0$ on $y = 0$ and h. The vertical momemtum equation given above requires p to be independent of z, that is, $p = p(x, y, t)$, so p may be taken outside the z integration. The integral of $z(h - z)$ from $z = 0$ to h is not zero, so it and $-1/2\mu$ can be divided out of the last equation to achieve:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0, \text{ or } \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

where the final equation follows from the form of ϕ determined from the velocity field.

This is a rather unusual and unexpected result because it requires viscous flow between closely spaced parallel plates to produce the same potential-line and streamline patterns as two-dimensional ideal flow. Interestingly, this suggestion is correct, except in thin layers having a thickness of order h near the surface of obstacles where the no-slip boundary condition on the obstacle prevents the tangential-flow slip that occurs in ideal flow. (Hele-Shaw flow near the surface of an obstacle is considered in Exercise 8.34). Thus, two-dimensional, ideal-flow streamlines past an object or obstacle may be visualized by injecting dye into pressure-driven viscous flow between closely spaced glass plates that trap a cross-sectional slice of the object or obstacle. Hele-Shaw flow has practical applications, too. Much, if not all, of the manufacturing design analysis done to create molds and tooling for plastic-forming operations is based on Hele-Shaw flow.

The basic balance of pressure and viscous stresses underlying lubrication theory can be extended to gravity-driven viscous flows by appropriately revising the meaning of the pressure gradient and evaluating the constants A and B in (8.18) for different boundary conditions. Such an extension is illustrated in the next example in two dimensions for gravity-driven flow of magma, paint, or viscous oil over a horizontal surface. Gravity re-enters the formulation here because there is a large density change across the free surface of the viscous fluid (see Section 4.9 "Neglect of Gravity in Constant Density Flows").

EXAMPLE 8.3

A two-dimensional bead of a viscous fluid with density ρ and viscosity μ spreads slowly on a smooth horizontal surface under the action of gravity. Ignoring surface tension and fluid acceleration, determine a differential equation for the thickness $h(x, t)$ of the spreading bead as a function of time.

Solution

The solution plan is to conserve mass exactly using (4.5), a stationary control volume (CV) of thickness dx, height h, and unit depth into the page (see Figure 8.11), and the generic velocity profile (8.18) when the pressure gradient is recast in terms of the thickness gradient $\partial h/\partial x$. The constants A and B in (8.18) can be determined from the no-slip condition at $y = 0$, and a stress-free condition on

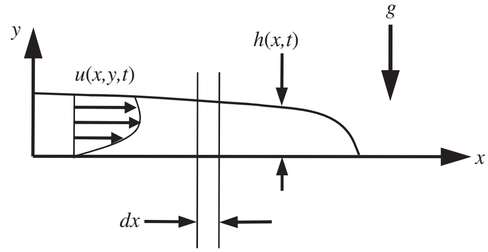


FIGURE 8.11 Gravity-driven spreading of a two-dimensional drop or bead on a flat, stationary surface. The fluid is not confined from above. Hydrostatic pressure forces cause the fluid to move but it is impeded by the viscous shear stress at $y = 0$. The flow is assumed to be symmetric about $x = 0$ so only half of it is shown.

$y = h$. When this refined version of (8.18) is put into the conservation of mass statement, the result is the differential equation that is sought.

By conserving mass between the two vertical lines in Figure 8.11, (4.5) becomes:

$$\rho \frac{\partial h}{\partial t} dx - \int_0^{h(x,t)} \rho u(x,y,t)\, dy + \int_0^{h(x+dx,t)} \rho u(x+dx,y,t)\, dy = 0.$$

When rearranged and the limit $dx \to 0$ is taken, this becomes:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\int_0^{h(x,t)} u(x,y,t)\, dy \right) = 0.$$

At any location within the spreading bead, the pressure p is hydrostatic: $p(x, y, t) = \rho g(h(x, t) - y)$ when fluid acceleration is ignored. Thus, the horizontal pressure gradient in the viscous fluid is

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left[\rho g(h(x,t) - y)\right] = \rho g \frac{\partial h}{\partial x},$$

which is independent of y, so (8.18) becomes:

$$u(x,y,t) \cong \frac{\rho g}{2\mu} \frac{\partial h(x,t)}{\partial x} y^2 + Ay + B.$$

The no-slip condition at $y = 0$ implies that $B = 0$, and the no-stress condition at $y = h$ implies:

$$0 = \mu \left(\frac{\partial u}{\partial y} \right)_{y=h(x,t)} = \rho g \frac{\partial h(x,t)}{\partial x} h(x,t) + \mu A, \text{ so } A = -\frac{\rho g}{\mu} h \frac{\partial h}{\partial x}.$$

So, the velocity profile within the bead is:

$$u \cong -\frac{\rho g}{2\mu} \frac{\partial h}{\partial x} y(2h - y),$$

and its integral is:

$$\int_0^h u(x,y,t)\, dy \cong -\frac{\rho g}{2\mu} \frac{\partial h}{\partial x} \int_0^h y(2h - y)\, dy = -\frac{\rho g}{3\mu} h^3 \frac{\partial h}{\partial x}.$$

When this result is combined with the integrated conservation of mass statement, the final equation is:

$$\frac{\partial h}{\partial t} = \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right).$$

This is a single nonlinear partial differential equation for $h(x,t)$ that in principle can be solved if a bead's initial thickness, $h(x,0)$, is known. Although this completes the effort for this example, a *similarity solution* to this equation does exist.

Similarity solutions to partial differential equations are possible when a variable transformation exists that allows the partial differential equation to be rewritten as an ordinary differential equation, and several such solutions for the Navier-Stokes equations are presented in the next section.

8.4. SIMILARITY SOLUTIONS FOR UNSTEADY INCOMPRESSIBLE VISCOUS FLOW

So far, we have considered steady flows with parallel, or nearly parallel, streamlines. In this situation, the nonlinear advective acceleration is zero, or small, and the stream-wise velocity reduces to a function of one spatial coordinate, and time. When a viscous flow with parallel or nearly parallel streamlines is impulsively started from rest, the flow depends on the spatial coordinate and time. For such unsteady flows, exact solutions still exist because the nonlinear advective acceleration drops out again (see Exercise 8.31). In this section, several simple and physically revealing unsteady flow problems are presented and solved. The first is the flow due to impulsive motion of a flat plate parallel to itself, commonly known as *Stokes' first problem*. (The flow is sometimes unfairly associated with the name of Rayleigh, who used Stokes' solution to predict the thickness of a developing boundary layer on a semi-infinite plate.)

A similarity solution is one of several ways to solve Stokes' first problem. The geometry of this problem is shown in Figure 8.12. An infinite flat plate lies along $y = 0$, surrounded by an initially quiescent fluid (with constant ρ and μ) for $y > 0$. The plate is impulsively given a velocity U at $t = 0$ and constant pressure is maintained at $x = \pm\infty$. At first, only the fluid near the plate will be drawn into motion, but as time progresses the thickness of this moving region will increase. Since the resulting flow at any time is invariant in the x direction ($\partial/\partial x = 0$), the continuity equation $\partial u/\partial x + \partial v/\partial y = 0$ requires $\partial v/\partial y = 0$. Thus, it follows that $v = 0$ everywhere since it is zero at $y = 0$. Therefore, the simplified horizontal and vertical momentum equations are:

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}, \quad \text{and} \quad 0 = -\frac{\partial p}{\partial y}.$$

Just before $t = 0$, all the fluid is at rest so $p =$ constant. For $t > 0$, the vertical momentum equation only allows the fluid pressure to depend on x and t. However, at any finite time, there will be a vertical distance from the plate where the fluid velocity is still zero, and, at this

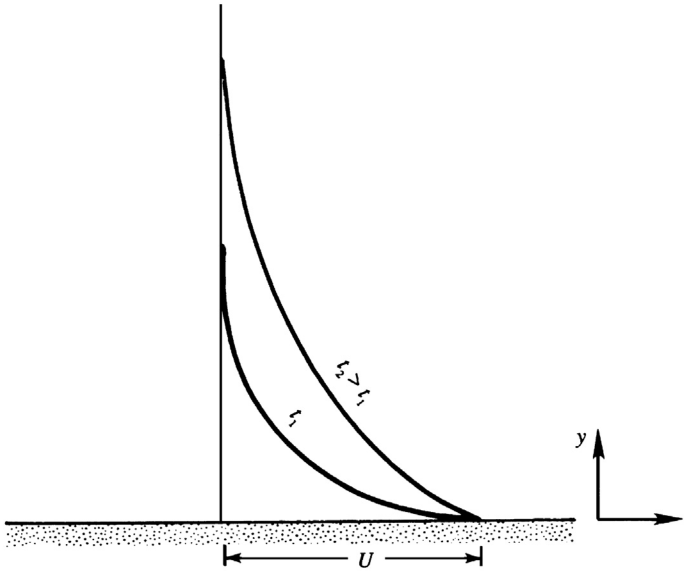


FIGURE 8.12 Laminar flow due to a flat plate that starts moving parallel to itself at speed U at $t = 0$. Before $t = 0$, the entire fluid half-space ($y > 0$) was quiescent. As time progresses, more and more of the viscous fluid above the plate is drawn into motion. Thus, the flow profile with greater vertical extent corresponds to the later time.

vertical distance from the plate, $\partial p/\partial x$ is zero. However, if $\partial p/\partial x = 0$ far from the plate, then $\partial p/\partial x = 0$ on the plate because $\partial p/\partial y = 0$. Thus, the horizontal momentum equation reduces to

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{8.20}$$

subject to the boundary and initial conditions:

$$u(y, t = 0) = 0, u(y = 0, t) = \left\{ \begin{matrix} 0 \text{ for } t < 0 \\ U \text{ for } t \geq 0 \end{matrix} \right\}, \text{ and } u(y \to \infty, t) = 0. \qquad \text{(8.21, 8.22, 8.23)}$$

The problem is well posed because (8.22) and (8.23) are conditions at two values of y, and (8.21) is a condition at one value of t; this is consistent with (8.20), which involves a first derivative in t and a second derivative in y.

The partial differential equation (8.20) can be transformed into an ordinary differential equation by switching to a similarity variable. The reason for this is the absence of enough other parameters in this problem to render y and t dimensionless without combining them. Based on dimensional analysis (see Section 1.11), the functional form of the solution to (8.20) can be written:

$$u/U = f(y/\sqrt{\nu t}, y/Ut). \tag{8.24}$$

where f is an undetermined function. However, (8.20) is a linear equation, so u must be proportional to U. This means that the final dimensionless group in (8.24) must be dropped, leaving:

$$u/U = F(y/\sqrt{\nu t}) \equiv F(\eta), \tag{8.25}$$

where F is an undetermined function, but this time it is a function of only one dimensionless group and this dimensionless group $\eta = y/(\nu t)^{1/2}$ combines both independent variables. This reduces the dimensionality of the solution space from two to one, an enormous simplification.

Equation (8.25) is the similarity form for the fluid velocity in Stokes's first problem. The similarity variable η could have been defined differently, such as $\nu t/y^2$, but different choices for η merely change F, not the final answer. The chosen η allows F to be interpreted as a velocity profile function with y appearing to the first power in the numerator of η. At any fixed $t > 0$, y and η are proportional.

Using (8.25) to form the derivatives in (8.20) leads to

$$\frac{\partial u}{\partial t} = U\frac{dF}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{Uy}{2\sqrt{\nu t^3}}\frac{dF}{d\eta} = -\frac{U\eta}{2t}\frac{dF}{d\eta} \text{ and}$$
$$U\frac{\partial^2 F}{\partial y^2} = U\frac{\partial}{\partial y}\left(\frac{dF}{d\eta}\frac{\partial \eta}{\partial y}\right) = U\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{\nu t}}\frac{dF}{d\eta}\right) = \frac{U}{\sqrt{\nu t}}\frac{d}{d\eta}\left(\frac{dF}{d\eta}\right)\frac{\partial \eta}{\partial y} = \frac{U}{\nu t}\frac{d}{d\eta}\left(\frac{dF}{d\eta}\right),$$

and these can be combined to provide the equivalent of (8.20) in similarity form:

$$-\frac{\eta}{2}\frac{dF}{d\eta} = \frac{d}{d\eta}\left(\frac{dF}{d\eta}\right). \tag{8.26}$$

The initial and boundary conditions (8.21) through (8.23) for F reduce to

$$F(\eta = 0) = 1, \text{ and } F(\eta \rightarrow \infty = 0), \tag{8.27, 8.28}$$

because (8.21) and (8.23) reduce to the same condition in terms of η. This reduction is expected because (8.20) was a partial differential equation and needed two conditions in y and one condition in t. In contrast, (8.26) is a second-order ordinary differential equation and needs only two boundary conditions.

Equation (8.26) is readily separated:

$$-\frac{\eta}{2}d\eta = \frac{d(dF/d\eta)}{dF/d\eta},$$

and integrated:

$$-\frac{\eta^2}{4} = \ln(dF/d\eta) + const.$$

Exponentiating produces:

$$dF/d\eta = A\exp(-\eta^2/4),$$

where A is a constant. Integrating again leads to:

$$F(\eta) = A\int_0^{\eta}\exp(-\xi^2/4)d\xi + B, \tag{8.29}$$

where ξ is just an integration variable and B is another constant. The condition (8.27) sets $B = 1$, while condition (8.28) gives

$$0 = A\int_0^\infty \exp\left(-\xi^2/4\right)d\xi + 1, \text{ or } 0 = 2A\int_0^\infty \exp\left(-\zeta^2\right)d\zeta + 1, \text{ so}$$
$$0 = 2A\left(\sqrt{\pi}/2\right) + 1, \text{thus } A = -1/\sqrt{\pi},$$

where the tabulated integral $\int_{-\infty}^{+\infty} \exp(-\zeta^2)d\zeta = \sqrt{\pi}$ has been used. The final solution for u then becomes:

$$\frac{u(y,t)}{U} = 1 - \mathrm{erf}\left(\frac{y}{2\sqrt{\nu t}}\right), \text{ where } \mathrm{erf}(\zeta) = \frac{2}{\sqrt{\pi}}\int_0^\zeta \exp\left(-\xi^2\right)d\xi \tag{8.30}$$

is the error function and again ξ is just an integration variable. The error function is a standard tabulated function (see Abramowitz & Stegun, 1972). It is apparent that *the solutions at different times all collapse into a single curve of u/U vs* η, as shown in Figure 8.13.

The nature of the variation of u/U with y for various values of t is sketched in Figure 8.12. The solution clearly has a diffusive nature. At $t = 0$, a vortex sheet (that is, a velocity discontinuity) is created at the plate surface. The initial vorticity is in the form of a delta function, which is infinite at the plate surface and zero elsewhere. The integral $\int_0^\infty \omega dy = \int_0^\infty (-\partial u/\partial y)dy = -U$ is independent of time, so *no new vorticity is generated after the initial time*. The flow given by (8.30) occurs as the initial vorticity diffuses away from the wall. The situation is analogous to a heat conduction problem in a semi-infinite solid extending from $y = 0$ to $y = \infty$. Initially, the solid has a uniform temperature, and at $t = 0$ the face at $y = 0$ is suddenly brought to a different temperature. The temperature distribution for this heat conduction problem is given by an equation similar to (8.30).

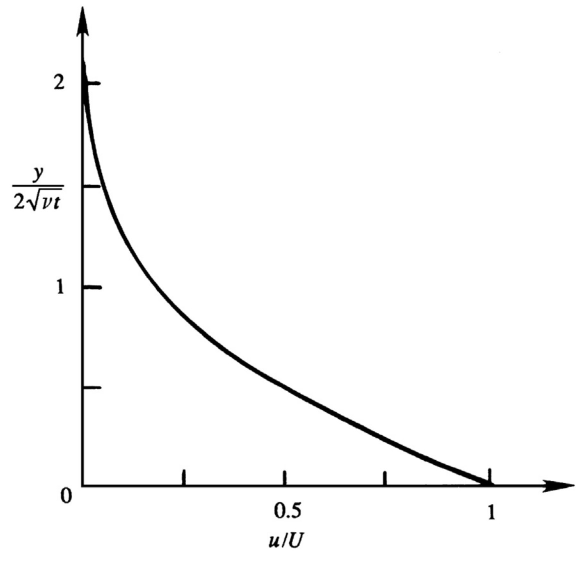


FIGURE 8.13 Similarity solution of laminar flow due to an impulsively started flat plate. Using these scaled coordinates, all flow profiles like those shown on Figure 8.12 will collapse to the same curve. The factor of two in the scaling of the vertical axis follows from (8.30).

We may arbitrarily define the thickness of the diffusive layer as the distance at which u falls to 1% of U. From Figure 8.13, $u/U = 0.01$ corresponds to $y/(\nu t)^{1/2} = 3.64$. Therefore, in time t the diffusive effects propagate to a distance of

$$\delta_{99} \sim 3.64\sqrt{\nu t}, \tag{8.31}$$

which defines the 99% thickness of the layer of moving fluid and this layer's thickness increases as $t^{1/2}$. Obviously, the factor of 3.64 is somewhat arbitrary and can be changed by choosing a different ratio of u/U as the definition for the edge of the diffusive layer. However, 99% thicknesses are commonly considered in boundary layer theory (see Chapter 9).

Stokes' first problem illustrates an important class of fluid mechanical problems that have *similarity solutions*. Because of the absence of suitable scales to render the independent variables dimensionless, the only possibility was a combination of variables that resulted in a reduction in the number of independent variables required to describe the problem. In this case the reduction was from two (y, t) to one (η) so that the formulation reduced a partial differential equation in y and t to an ordinary differential equation in η.

The solution (8.30) for $u(y, t)$ is *self-similar* in the sense that at different times t_1, t_2, t_3, … the various velocity profiles $u(y, t_1)$, $u(y, t_2)$, $u(y, t_3)$, … all collapse into a single curve if the velocity is scaled by U and y is scaled by the thickness $(\nu t)^{1/2}$. Moreover, such a collapse will occur for different values of U and for fluids having different ν.

Similarity solutions arise in situations in which there are no imposed length or time scales provided by the initial or boundary conditions (or the field equation). A similarity solution would not be possible if, for example, the boundary conditions were changed after a certain time t_1 since this introduces a time scale into the problem (see Exercise 8.30). Likewise, if the flow in Stokes' first problem was bounded above by a second parallel plate, there could be no similarity solution because the distance to the second plate introduces a length scale into the problem.

Similarity solutions are often ideal for developing an understanding of flow phenomena, so they are sought wherever possible. A method for finding similarity solutions starts from a presumed form for the solution:

$$\gamma = At^{-n}F(\xi/\delta(t)) \equiv At^{-n}F(\eta) \text{ or } \gamma = A\xi^{-n}F(\xi/\delta(t)) \equiv A\xi^{-n}F(\eta), \tag{8.32a,b}$$

where γ is the dependent field variable of interest, A is a constant (units = $[\gamma] \times [\text{time}]^n$ or $[\gamma] \times [\text{length}^n]$), ξ is the independent spatial coordinate, t is time, and $\eta = \xi/\delta$ is the similarity variable, and $\delta(t)$ is a time-dependent length scale. The factor of At^{-n} or $A\xi^{-n}$ that multiplies F in (8.32) is sometimes needed for similarity solutions that are infinite (or zero) at $t = 0$ or $\xi = 0$. Use of (8.32) is illustrated in the following examples.

EXAMPLE 8.4

Use (8.32a) to find the similarity solution to Stokes' first problem.

Solution

The solution plan is to populate (8.32a) with the appropriate variables, substitute it into the field equation (8.20), and then require that the coefficients all have the same time dependence. For Stokes' first problem $\gamma = u/U$, and the independent spatial variable is y. For this flow, the coefficient At^{-n} is not needed since $u/U = 1$ at $\eta = 0$ for all $t > 0$ and this can only happen when $A = 1$ and $n = 0$. Thus, the dimensional analysis result (8.25) may be replaced by (8.32a) with $A = 1$, $n = 0$, and $\xi = y$:

$$u/U = F(y/\delta(t)) \equiv F(\eta).$$

A time derivative produces:

$$\frac{\partial u}{\partial t} = U\frac{dF}{d\eta}\left(-\frac{y}{\delta^2}\right)\frac{d\delta}{dt},$$

while two y-derivatives produce:

$$\frac{\partial^2 u}{\partial y^2} = U\frac{d^2F}{d\eta^2}\frac{1}{\delta^2}.$$

Reconstructing (8.20) with these replacements yields:

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} \rightarrow U\frac{dF}{d\eta}\left(-\frac{y}{\delta^2}\right)\frac{d\delta}{dt} = \nu U\frac{d^2F}{d\eta^2}\frac{1}{\delta^2},$$

which can be rearranged to find:

$$-\left[\frac{1}{\delta}\frac{d\delta}{dt}\right]\eta\frac{dF}{d\eta} = \left[\frac{\nu}{\delta^2}\right]\frac{d^2F}{d\eta^2}.$$

For this equation to be in similarity form, the coefficients in [,]-brackets must both have the same time dependence so that division by this common time dependence will leave an ordinary differential equation for $F(\eta)$ and t will no longer appear. Thus, we require the two coefficients to be proportional:

$$\frac{1}{\delta}\frac{d\delta}{dt} = C_1\frac{\nu}{\delta^2},$$

where C_1 is the constant of proportionality. This is a simple differential equation for $\delta(t)$ that is readily rearranged and solved:

$$\delta\frac{d\delta}{dt} = C_1\nu \rightarrow \frac{\delta^2}{2} = C_1\nu t + C_2 \rightarrow \delta = \sqrt{2C_1\nu t},$$

where the condition $\delta(0) = 0$ has been used to determine that the constant of integration $C_2 = 0$. When $C_1 = ½$, the prior definition of η in (8.25) is recovered, and the solution for u proceeds as before (see (8.26) through (8.30)).

EXAMPLE 8.5

At $t = 0$ an infinitely thin vortex sheet in a fluid with density ρ and viscosity μ coincides with the plane defined by $y = 0$, so that the fluid velocity is U for $y > 0$ and $-U$ for $y < 0$. The coordinate axes are aligned so that only the z-component of vorticity is nonzero. Determine the similarity solution for $\omega_z(y, t)$ for $t > 0$.

Solution

The solution plan is the same as for Example 8.4, except here the coefficient At^{-n} must be included. In this circumstance, there will be only one component of the fluid velocity, $\mathbf{u} = u(y, t)\mathbf{e}_x$, so $\omega_z(y, t) = -\partial u/\partial y$. The independent coordinate y does not appear in the initial condition, so (8.32a) is the preferred choice. Its appropriate form is:

$$\omega_z(y,t) = At^{-n}F(y/\delta(t)) \equiv At^{-n}F(\eta),$$

and the field equation,

$$\frac{\partial \omega_z}{\partial t} = \nu \frac{\partial^2 \omega_z}{\partial y^2},$$

is obtained by applying $\partial/\partial y$ to (8.20). Here, the derivatives of the similarity solution are:

$$\frac{\partial \omega_z}{\partial t} = -nAt^{-n-1}F(\eta) + At^{-n}\frac{dF}{d\eta}\left(-\frac{y}{\delta^2}\right)\frac{d\delta}{dt}, \text{ and } \frac{\partial^2 \omega_z}{\partial y^2} = At^{-n}\frac{d^2F}{d\eta^2}\frac{1}{\delta^2}.$$

Reassembling the field equation and canceling common factors produces:

$$-\left[\frac{n}{t}\right]F(\eta) - \left[\frac{1}{\delta}\frac{d\delta}{dt}\right]\eta\frac{dF}{d\eta} = \left[\frac{\nu}{\delta^2}\right]\frac{d^2F}{d\eta^2}.$$

From Example 8.4, we know that requiring the second and third coefficients in [,]-brackets to be proportional with a proportionality constant of ½ produces $\delta = (\nu t)^{1/2}$. With this choice for δ, each of the coefficients in [,]-brackets is proportional to $1/t$ so, the similarity equation becomes:

$$-nF(\eta) - \frac{1}{2}\eta\frac{dF}{d\eta} = \frac{d^2F}{d\eta^2}.$$

The boundary conditions are: 1) at any finite time the vorticity must go to zero infinitely far from the initial location of the vortex sheet, $F(\eta) \to 0$ for $\eta \to \infty$, and 2) the velocity difference across the diffusing vortex sheet is constant and equal to $2U$:

$$-\int_{-\infty}^{+\infty} \omega_z\, dy = \int_{-\infty}^{+\infty} \frac{\partial u}{\partial y}\, dy = [u(y,t)]_{-\infty}^{+\infty} = U - (-U) = 2U.$$

Substituting the similarity solution into this second requirement leads to:

$$-\int_{-\infty}^{+\infty} \omega_z\, dy = -\int_{-\infty}^{+\infty} At^{-n}F(\eta)\, dy = -At^{-n}\delta\int_{-\infty}^{+\infty} F(\eta)d(y/\delta) = -At^{-n}\delta\int_{-\infty}^{+\infty} F(\eta)d\eta = 2U.$$

The final integral is just a number so $t^{-n}\delta(t)$ must be constant, and this implies $n =$ ½ so the similarity equation may be rewritten, and integrated:

$$-\frac{1}{2}\left(F(\eta) + \eta\frac{dF}{d\eta}\right) = -\frac{1}{2}\frac{d}{d\eta}(\eta F) = \frac{d}{d\eta}\left(\frac{dF}{d\eta}\right) \to \frac{dF}{d\eta} + \frac{1}{2}\eta F = C.$$

The first boundary condition implies that both F and $dF/d\eta \rightarrow 0$ when η is large enough. Therefore, assume that $\eta F \rightarrow 0$ when $\eta \rightarrow \infty$ so that the constant of integration C can be set to zero (this assumption can be checked once F is found). When $C = 0$, the last equation can be separated and integrated to find:

$$F(\eta) = D\exp(-\eta^2/4),$$

where D is a constant, and the assumed limit, $\eta F \rightarrow 0$ when $\eta \rightarrow \infty$, is verified so C is indeed zero. The velocity-difference constraint and the tabulated integral used to reach (8.30) allow the product AD to be evaluated. Thus, the similarity solutions for the vorticity $\omega_z = -\partial u/\partial y$ and velocity u are:

$$\omega_z(y,t) = -\frac{U}{\sqrt{\pi\nu t}}\exp\left\{-\frac{y^2}{4\nu t}\right\}, \text{ and } u(y,t) = U\mathrm{erf}\left\{\frac{y}{2\sqrt{\nu t}}\right\}.$$

Schematic plots of the vorticity and velocity distributions are shown in Figure 8.14. If we define the width of the velocity transition layer as the distance between the points where $u = \pm 0.95U$, then the corresponding values of η are ± 2.76 and consequently the width of the transition layer is $5.54(\nu t)^{1/2}$.

The results of this example are closely related to Stokes' first problem, and to the laminar boundary layer flows discussed in the next chapter, for several reasons. First of all, this flow is essentially the same as that in Stokes' first problem. The velocity field in the upper half of Figure 8.14 is identical to that in Figure 8.13 after a Galilean transformation to a coordinate system

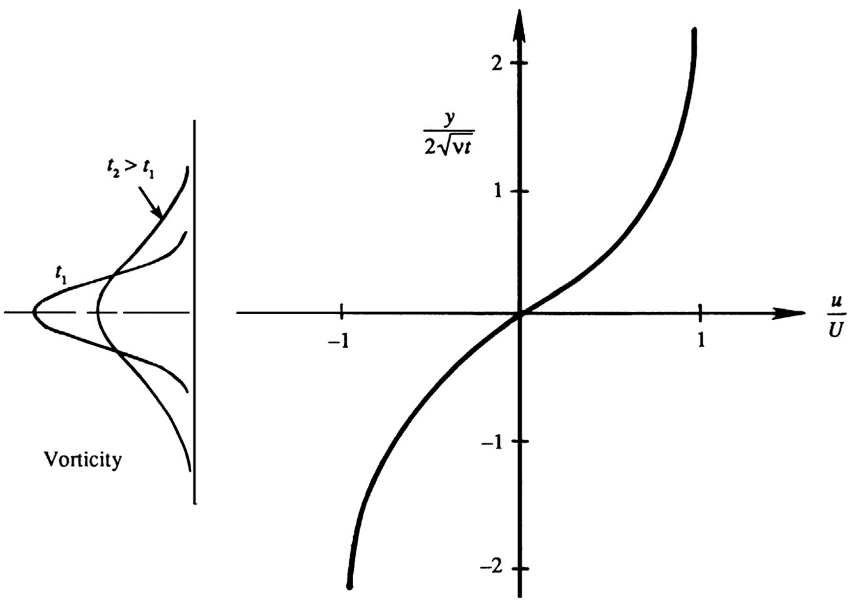


FIGURE 8.14 Viscous thickening of a vortex sheet. The left panel indicates the vorticity distribution at two times, while the right panel shows the velocity field solution in similarity coordinates. The upper half of this flow is equivalent to a temporally developing boundary layer.

moving at speed $+U$ followed by a sign change. In addition, the flow for $y > 0$ represents a *temporally developing* boundary layer that begins at $t = 0$. The velocity far from the surface is irrotational and uniform at speed U while the no-slip condition ($u = 0$) is satisfied at $y = 0$. Here, the wall shear stress and skin friction coefficient C_f are time dependent:

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\mu U}{\sqrt{\pi \nu t}}, \text{ or } C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{2}{\sqrt{\pi}}\sqrt{\frac{\nu}{U^2 t}}.$$

When Ut is interpreted as a surrogate for the downstream distance, x, in a *spatially developing* boundary layer, the last square-root factor above becomes $(\nu/Ux)^{1/2} = \mathrm{Re}_x^{-1/2}$, and this is the correct parametric dependence for C_f in a laminar boundary layer that develops on a smooth, flat surface below a steady uniform flow.

EXAMPLE 8.6

A thin, rapidly spinning cylinder produces the two-dimensional flow field, $u_\theta = \Gamma/2\pi r$, of an ideal vortex of strength Γ located at $r = 0$. At $t = 0$, the cylinder stops spinning. Use (8.32) to determine $u_\theta(r, t)$ for $t > 0$.

Solution

Follow the approach specified for the Example 8.4 but this time use (8.32b) because r appears in the initial condition. Here u_θ is the dependent field variable and r is the independent spatial variable, so the appropriate form of (8.32b) is:

$$u_\theta(r,t) = Ar^{-n}F(r/\delta(t)) \equiv Ar^{-n}F(\eta).$$

The initial and boundary conditions are: $u_\theta(r, 0) = \Gamma/2\pi r = u_\theta(r \to \infty, t)$, and $u_\theta(0, t) = 0$ for $t > 0$, which are simplified to $F(\eta \to \infty) = 1$ and $F(0) = 0$ when Ar^{-n} is set equal to $\Gamma/2\pi r$. In this case, the field equation for u_θ (see Appendix B) is

$$\frac{\partial u_\theta}{\partial t} = \nu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right).$$

Inserting $u_\theta = (\Gamma/2\pi r)F(\eta)$ produces:

$$-\frac{\Gamma}{2\pi r}\left(\frac{1}{\delta}\frac{d\delta}{dt}\right)\eta\frac{dF}{d\eta} = \nu\frac{\Gamma}{2\pi}\left(-\frac{1}{\delta r^2}\frac{dF}{d\eta} + \frac{1}{\delta^2 r}\frac{d^2F}{d\eta^2}\right) \to -\left[\frac{r^2}{\nu\delta}\frac{d\delta}{dt}\right]\eta\frac{dF}{d\eta} = -\eta\frac{dF}{d\eta} + \eta^2\frac{d^2F}{d\eta^2}.$$

For a similarity solution, the coefficient in [,]-brackets must depend on η alone, not on r or t. Here, this coefficient reduces to $\eta^2/2$ when $\delta = (\nu t)^{1/2}$ (as in the prior examples). With this replacement, the similarity equation can be integrated twice:

$$\left(\frac{1}{\eta} - \frac{\eta}{2}\right)\frac{dF}{d\eta} = \frac{d}{d\eta}\left(\frac{dF}{d\eta}\right) \to \ln\eta - \frac{\eta^2}{4} + const. = \ln\left(\frac{dF}{d\eta}\right) \to C\int \eta\exp\{-\eta^2/4\}d\eta + D = F(\eta).$$

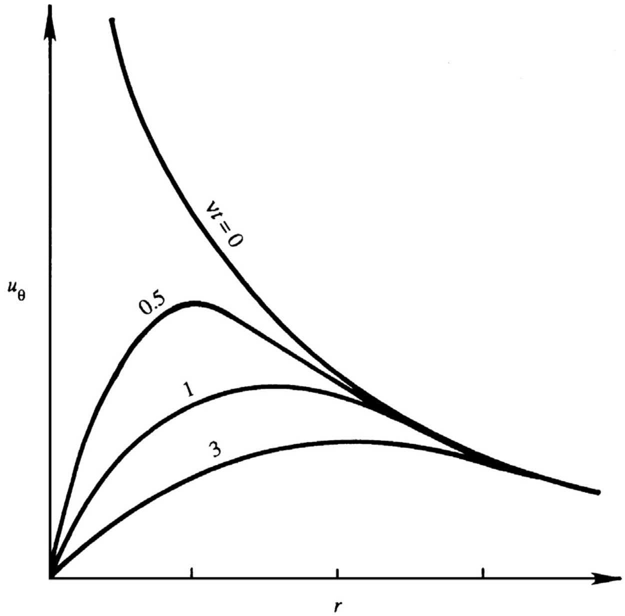


FIGURE 8.15 Viscous decay of a line vortex showing the tangential velocity u_θ at different times. The velocity field nearest to the axis of rotation changes the most quickly. At large radii, flow alterations occur more slowly.

The remaining integral is elementary, and the boundary conditions given above for F allow the constants C and D to be evaluated. The final result is $F(\eta) = 1 - \exp\{-\eta^2/4\}$, so the velocity distribution is:

$$u_\theta(r,t) = \frac{\Gamma}{2\pi r}\left[1 - \exp\left\{-\frac{r^2}{4\nu t}\right\}\right],$$

which is identical to the Gaussian vortex of (3.29) when $\sigma^2 = 4\nu t$. A sketch of the velocity distribution for various values of t is given in Figure 8.15. Near the center, $r \ll (\nu t)^{1/2}$, the flow has the form of rigid-body rotation, while in the outer region, $r \gg (\nu t)^{1/2}$, the motion has the form of an ideal vortex.

The foregoing presentation applies to the *decay* of a line vortex. The case where a line vortex is suddenly *introduced* into a fluid at rest leads to the velocity distribution,

$$u_\theta(r,t) = \frac{\Gamma}{2\pi r}\exp\left\{-\frac{r^2}{4\nu t}\right\}$$

(see Exercise 8.26). This situation is equivalent to the impulsive rotational start of an infinitely thin and quickly rotating cylinder.

EXAMPLE 8.7

Use (8.32a) and an appropriate constraint on the total volume of fluid to determine the form of the similarity solution to the two-dimensional, viscous, drop-spreading equation of Example 8.3.

Solution

The solution plan is to populate (8.32a) with the appropriate variables,

$$h = At^{-n}F(x/\delta(t)) \equiv At^{-n}F(\eta),$$

substitute it into the equation from Example 8.3, and require that: 1) the coefficients all have the same time dependence, and 2) the total fluid volume per unit depth into the page, $\int_{-\infty}^{+\infty} h(x,t)dx$, is independent of time. The starting point is the evaluation of derivatives:

$$\frac{\partial h}{\partial t} = -nAt^{-n-1}F(\eta) + At^{-n}\frac{dF}{d\eta}\left(-\frac{x}{\delta^2}\right)\frac{d\delta}{dt}, \text{ and } \frac{\partial h}{\partial x} = At^{-n}\frac{dF}{d\eta}\left(\frac{1}{\delta}\right),$$

which, when inserted in the final equation of Example 8.3, produces:

$$\frac{\partial h}{\partial t} = -\left[nAt^{-n-1}\right] F(\eta) - \left[At^{-n}\frac{1}{\delta}\frac{d\delta}{dt}\right]\eta\frac{dF}{d\eta} = \left[\frac{\rho g}{3\mu}A^4t^{-4n}\frac{1}{\delta^2}\right]\left(3F^2\left(\frac{dF}{d\eta}\right)^2 + F^3\frac{d^2F}{d\eta^2}\right) = \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right).$$

Requiring proportionality between the first two coefficients in [,]-brackets with C as the constant of proportionality yields:

$$CnAt^{-n-1} = At^{-n}\frac{1}{\delta}\frac{d\delta}{dt} \rightarrow C\frac{n}{t} = \frac{1}{\delta}\frac{d\delta}{dt}.$$

The second equation is satisfied when $\delta = Dt^m$ where D is another constant and $m = Cn$. Requiring proportionality between the second and third coefficients and using $\delta = Dt^m$ produces:

$$EAt^{-n}\frac{1}{\delta}\frac{d\delta}{dt} = \frac{\rho g}{3\mu}A^4t^{-4n}\frac{1}{\delta^2} \rightarrow -E\frac{m}{t} = \frac{\rho g}{3\mu}A^3t^{-3n}D^2t^{-2m} \rightarrow -1 = -3n - 2m,$$

where E is another constant of proportionality; the final equation for the exponents follows from equating powers of t in the second equation. These results set the form of $\delta(t)$ and specify one relationship between n and m. A second relationship between m and n comes from conserving the volume per unit depth into the page:

$$\int_{-\infty}^{+\infty} h(x,t)dx = \int_{-\infty}^{+\infty} At^{-n}F(\eta)dx = At^{-n}Dt^m\int_{-\infty}^{+\infty} F(\eta)d\eta = const.$$

The final integral is just a number so the exponents of t outside this integral must sum to zero for the volume to be constant. This implies: $-n + m = 0$. Taken together, the two equations for m and n imply: $n = m = {}^1/_5$. Thus, the form of the similarity solution of the final equation of Example 8.2 is:

$$h(x,t) = At^{-1/5}F(x/Dt^{1/5}).$$

Determining the constants A and D requires solution of the equation for F and knowledge of the bead's volume per unit depth, and is beyond the scope of this example.

After reviewing these examples, it should be clear that diffusive length scales in unsteady viscous flow are proportional to $(\nu t)^{1/2}$. The viscous bead-spreading example produces a length scale with a different power, but this is not a diffusion time scale. Instead it is an advection time scale that specifies how far fluid elements travel in the direction of the flow.

8.5. FLOW DUE TO AN OSCILLATING PLATE

The unsteady flows discussed in the preceding sections have similarity solutions, because there were no imposed or specified length or time scales. The flow discussed here is an unsteady viscous flow that includes an imposed time scale.

Consider an infinite flat plate lying at $y = 0$ that executes sinusoidal oscillations parallel to itself. (This is sometimes called *Stokes' second problem*.) Here, only the steady periodic solution after the starting transients have died out is considered, thus there are no initial conditions to satisfy. The governing equation (8.20) is the same as that for Stokes' first problem. The boundary conditions are:

$$u(y=0,t) = U\cos(\omega t), \text{ and } u(y\to\infty,t) = \text{bounded}, \qquad (8.33,\ 8.34)$$

where ω is the oscillation frequency (rad./s). In the steady state, the flow variables must have a periodicity equal to the periodicity of the boundary motion. Consequently, a complex separable solution of the form,

$$u(y,t) = \mathrm{Re}\left\{e^{i\omega t} f(y)\right\}, \qquad (8.35)$$

is used here, and the specification of the real part is dropped until the final equation for u is reached. Substitution of (8.35) into (8.20) produces:

$$i\omega f = \nu(d^2 f/dy^2), \qquad (8.36)$$

which is an ordinary differential equation with constant coefficients. It has exponential solutions of the form: $f = \exp(ky)$ where $k = (i\omega/\nu)^{1/2} = \pm(i+1)(\omega/2\nu)^{1/2}$. Thus, the solution of (8.36) is

$$f(y) = A\exp\left\{-(i+1)y\sqrt{\omega/2\nu}\right\} + B\exp\left\{+(i+1)y\sqrt{\omega/2\nu}\right\}. \qquad (8.37)$$

The condition (8.34) requires that the solution must remain bounded as $y \to \infty$, so $B = 0$, and the complex solution only involves the first term in (8.37). The surface boundary condition (8.33) requires $A = U$. Thus, after taking the real part as in (8.35), the final velocity distribution for Stokes' second problem is

$$u(y,t) = U\exp\left\{-y\sqrt{\frac{\omega}{2\nu}}\right\}\cos\left(\omega t - y\sqrt{\frac{\omega}{2\nu}}\right). \qquad (8.38)$$

The cosine factor in (8.38) represents a dispersive wave traveling in the positive-y direction, while the exponential term represents amplitude decay with increasing y. The flow therefore

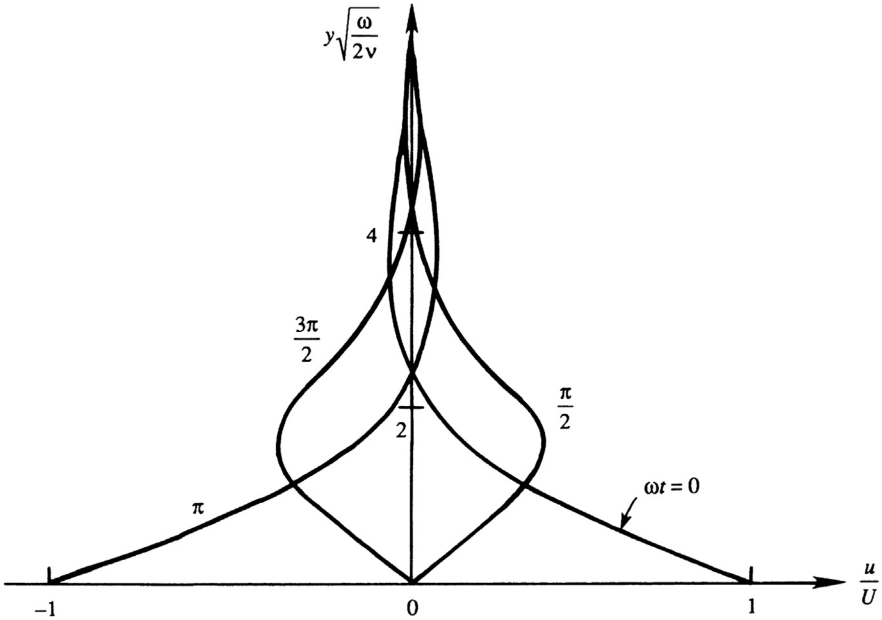


FIGURE 8.16 Velocity distribution in laminar flow near an oscillating plate. The distributions at $\omega t = 0$, $\pi/2$, π, and $3\pi/2$ are shown. The diffusive distance is of order $\delta \sim 4(\nu/\omega)^{1/2}$.

resembles a highly damped transverse wave (Figure 8.16). However, this is a diffusion problem and *not* a wave-propagation problem because there are no restoring forces involved here. The apparent propagation is merely a result of the oscillating boundary condition. For $y = 4(\nu/\omega)^{1/2}$, the amplitude of u is $U\exp\{-4/\sqrt{2}\} = 0.06U$, which means that the influence of the wall is confined within a distance of order $\delta \sim 4(\nu/\omega)^{1/2}$, which decreases with increasing frequency.

The solution (8.38) has several interesting features. First of all, it cannot be represented by a single curve in terms of dimensionless variables. A dimensional analysis of Stokes' second problem produces three dimensionless groups: u/U, ωt, and $y(\omega/\nu)^{1/2}$. Here the independent spatial variable y can be fully separated from the independent time variable t. Self-similar solutions exist only when the independent spatial and temporal variables must be combined in the absence of imposed time or length scales. However, the fundamental concept associated with viscous diffusion holds true, the spatial extent of the solution is parameterized by $(\nu/\omega)^{1/2}$, the square root of the product of the kinematic viscosity, and the imposed time scale $1/\omega$. In addition, (8.38) can be used to predict the weak absorption of sound at solid flat surfaces.

8.6. LOW REYNOLDS NUMBER VISCOUS FLOW PAST A SPHERE

Many physical problems can be described by the behavior of a system when a certain parameter is either very small or very large. Consider the problem of steady constant-density flow of a viscous fluid at speed U around an object of size L. The governing equations will be (4.10) and the steady flow version of (8.1):

$$\rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mu\nabla^2\mathbf{u}. \tag{8.39}$$

As described in Chapter 4.11, this equation can be scaled to determine which terms are most important. The purpose of such a scaling is to generate dimensionless terms that are of order unity in the flow field. For example, when the flow speeds are high and the viscosity is small, the pressure and inertia forces dominate the momentum balance, showing that pressure changes are of order ρU^2. Consequently, for high Reynolds number, the scaling (4.100) is appropriate for nondimensionalizing (8.39) to obtain

$$\mathbf{u}^*\cdot\nabla^*\mathbf{u}^* + \nabla^* p^* = \frac{1}{\mathrm{Re}}\nabla^{*2}\mathbf{u}^*, \tag{8.40}$$

where $\mathrm{Re} = \rho UL/\mu$ is the Reynolds number. For $\mathrm{Re} \gg 1$, (8.40) may be solved by treating $1/\mathrm{Re}$ as a small parameter, and as a first approximation, $1/\mathrm{Re}$ may be set to zero everywhere in the flow, which reduces (8.40) to the inviscid Euler equation without a body force.

However, viscous effects may still be felt at high Re because a single length scale is typically inadequate to describe all regions of high-Re flows. For example, complete omission of the viscous term cannot be valid near a solid surface because the inviscid flow cannot satisfy the no-slip condition at the body surface. Viscous forces are important near solid surfaces because of the high shear in the boundary layer near the body surface. The scaling (4.100), which assumes that velocity gradients are proportional to U/L, is invalid in such boundary layers. Thus, there is a region of *nonuniformity* near the body where a perturbation expansion in terms of the small parameter $1/\mathrm{Re}$ becomes *singular*. The proper scaling in the *boundary layer* and a procedure for analyzing wall-bounded high Reynolds number flows will be discussed in Chapter 9. A hint of what is to come is provided by the scaling (8.14), which leads to the lubrication approximation and involves different-length scales for the stream-wise and cross-stream flow directions.

Now consider flows in the opposite limit of very low Reynolds numbers, $\mathrm{Re} \to 0$. Clearly such flows should have negligible inertia forces, with pressure and viscous forces providing the dominant balance. Therefore, multiply (8.40) by Re to obtain

$$\mathrm{Re}(\mathbf{u}^*\cdot\nabla^*\mathbf{u}^* + \nabla^* p^*) = \nabla^{*2}\mathbf{u}^*. \tag{8.41}$$

Although this equation does have negligible inertia terms as $\mathrm{Re} \to 0$, it does not lead to a balance of pressure and viscous forces as $\mathrm{Re} \to 0$ since it reduces to $0 = \mu\nabla^2\mathbf{u}$, which is not the proper governing equation for low Reynolds number flows. The source of the inadequacy is the scaling of the pressure term specified by (4.100). For low Reynolds number flows, pressure is *not* of order ρU^2. Instead, at low Re, pressure differences should be scaled with a generic viscous stress such as $\mu\partial u/\partial y \sim \mu U/L$. Thus, the pressure scaling $p^* = (p - p_\infty)/\rho U^2$ in (4.100) should be replaced by $p^* = (p - p_\infty)L/\mu U$, and this leads to a correctly revised version of (8.41)

$$\mathrm{Re}(\mathbf{u}^*\cdot\nabla^*\mathbf{u}^*) = -\nabla^* p^* + \nabla^{*2}\mathbf{u}^*, \tag{8.42}$$

which does exhibit the proper balance of terms as $\mathrm{Re} \to 0$ and becomes the linear (dimensional) equation

$$\nabla p = \mu \nabla^2 \mathbf{u}, \tag{8.43}$$

when this limit is taken.

Flows with Re → 0 are called *creeping* flows, and they occur at low flow speeds of viscous fluids past small objects or through narrow passages. Examples of such flows are the motion of a thin film of oil in the bearing of a shaft, settling of sediment particles in nominally quiescent water, the fall of mist droplets in the atmosphere, or the flow of molten plastic during a molding process. A variety of other creeping flow examples are presented in Sherman (1990).

From this discussion of scaling, we conclude that *the proper length and time scales depend on the nature and the region of the flow, and are obtained by balancing the terms that are most important in the region of the flow field under consideration*. Identifying the proper length and time scales is commonly the goal of experimental and numerical investigations of viscous flows, so that the most appropriate simplified versions of the full equations for fluid motion may be analyzed. The remainder of this section presents a solution for the creeping flow past a sphere, first given by Stokes in 1851. This is a flow where different field equations should be used in regions close to and far from the sphere.

We begin by considering the near-field flow around a stationary sphere of radius a placed in a uniform stream of speed U (Figure 8.17) with Re → 0. The problem is axisymmetric, that is, the flow patterns are identical in all planes parallel to U and passing through the center of

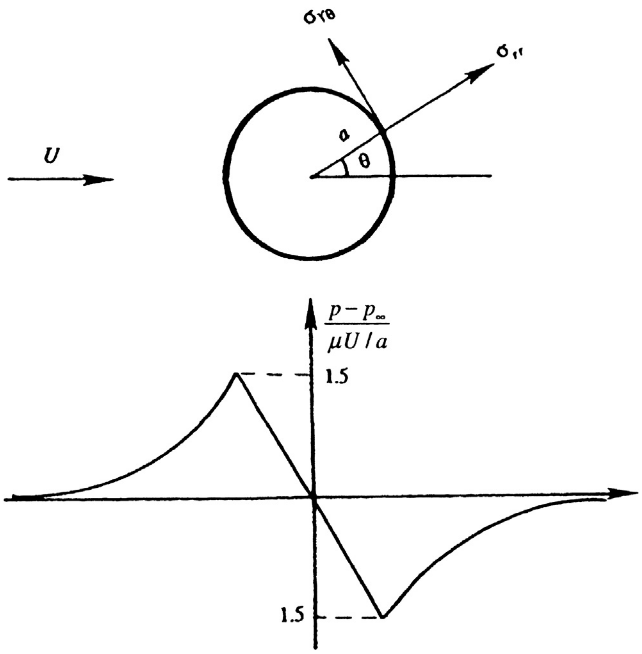


FIGURE 8.17 Creeping flow over a sphere. The upper panel shows the viscous stress components at the surface. The lower panel shows the pressure distribution in an axial (φ = const.) plane.

the sphere. Since Re → 0, as a first approximation, neglect the inertia forces altogether and seek a solution to (8.43). Taking the curl of (8.43) produces an equation for the vorticity alone

$$\nabla^2 \boldsymbol{\omega} = 0,^1$$

because $\nabla \times \nabla p = 0$ and the order of the operators curl and ∇^2 can be interchanged. (The reader may verify this using indicial notation.) The only component of vorticity in this axisymmetric problem is ω_φ, the component perpendicular to $\varphi =$ constant planes in Figure 8.17, and it is given by

$$\omega_\varphi = \frac{1}{r}\left[\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right].$$

This is an axisymmetric flow, so the r and θ velocity components can be found from an axisymmetric stream function ψ:

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}, \text{ and } u_\theta = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}. \tag{6.83}$$

In terms of this stream function, the vorticity becomes

$$\omega_\varphi = -\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right],$$

which is governed by:

$$\nabla^2 \omega_\varphi = 0.^1$$

Combining the last two equations, we obtain

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2 \psi = 0.^2 \tag{8.44}$$

The boundary conditions on the preceding equation are

$$\psi(r = a, \theta) = 0 \; [u_r = 0 \text{ at surface}], \tag{8.45}$$

$$\partial\psi(r = a, \theta)/\partial r = 0 \; [u_\theta = 0 \text{ at surface}], \text{ and} \tag{8.46}$$

$$\psi(r \to \infty, \theta) = \frac{1}{2} U r^2 \sin^2\theta \; [\text{uniform flow far from the sphere}]. \tag{8.47}$$

The last condition follows from the fact that the stream function for a uniform flow is $\frac{1}{2} U r^2 \sin^2\theta$ in spherical coordinates (see (6.86)).

The far-field condition (8.47) suggests a separable solution of the form

$$\psi(r, \theta) = f(r)\sin^2\theta.$$

[1]In spherical polar coordinates, the operator in the footnoted equations is actually $-\nabla \times \nabla \times$ (−curl curl_), which is different from the Laplace operator defined in Appendix B.

[2]Equation (8.44) is the square of the operator, and not the biharmonic.

Substitution of this into the governing equation (8.44) gives

$$f^{iv} - \frac{4f''}{r^2} + \frac{8f'}{r^3} - \frac{8f}{r^4} = 0,$$

whose solution is

$$f = Ar^4 + Br^2 + Cr + D/r.$$

The far-field boundary condition (8.47) requires that $A = 0$ and $B = U/2$. The surface boundary conditions then give $C = -3Ua/4$ and $D = Ua^3/4$. The solution then reduces to

$$\psi = Ur^2 \sin^2\theta\left(\frac{1}{2} - \frac{3a}{4r} + \frac{a^3}{4r^3}\right). \tag{8.48}$$

The velocity components are found from (8.48) using (6.83):

$$u_r = U\cos\theta\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right), \text{ and } u_\theta = -U\sin\theta\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right). \tag{8.49}$$

The pressure is found by integrating the momentum equation $\nabla p = \mu\nabla^2\mathbf{u}$. The result is

$$p - p_\infty = -\frac{3\mu aU\cos\theta}{2r^2}, \tag{8.50}$$

which is sketched in Figure 8.17. The maximum $p - p_\infty = 3\mu U/2a$ occurs at the forward stagnation point ($\theta = \pi$), while the minimum $p - p_\infty = 3\mu U/2a$ occurs at the rear stagnation point ($\theta = 0$).

The drag force D on the sphere can be determined by integrating its surface pressure and shear stress distributions (see Exercise 8.35) to find:

$$D = 6\pi\mu aU, \tag{8.51}$$

of which one-third is pressure drag and two-thirds is skin friction drag. It follows that drag in a creeping flow is proportional to the velocity; this is known as *Stokes' law of resistance*.

In a well-known experiment to measure the charge of an electron, Millikan (1911) used (8.51) to estimate the radius of an oil droplet falling through air. Suppose ρ' is the density of a spherical falling particle and ρ is the density of the surrounding fluid. Then the effective weight of the sphere is $4\pi a^3 g(\rho' - \rho)/3$, which is the weight of the sphere minus the weight of the displaced fluid. The falling body reaches its terminal velocity when it no longer accelerates, at which point the viscous drag equals the effective weight. Then,

$$(4/3)\pi a^3 g(\rho' - \rho) = 6\pi\mu aU,$$

from which the radius a can be estimated.

Millikan (1911) was able to deduce the charge on an electron (and win a Nobel prize) making use of Stokes' drag formula by the following experiment. Two horizontal parallel plates can be charged by a battery (see Figure 8.18). Oil is sprayed through a very fine hole in the upper plate and develops static charge (+) by losing a few (n) electrons in passing through the small hole. If the plates are charged, then an electric force neE will act on each of the drops. Now n is not known but $E = -V_b/L$, where V_b is the battery voltage and L is the gap between the plates, provided that the charge density in the gap is very low. With the

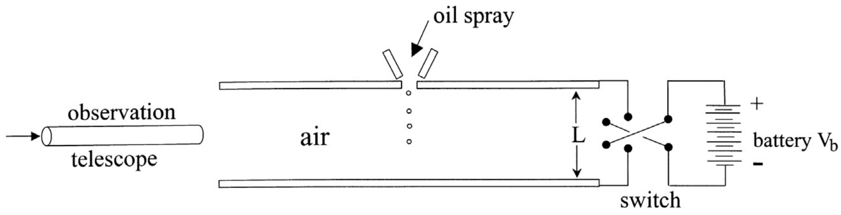


FIGURE 8.18 Simplified schematic of the Millikan oil drop experiment where observations of charged droplet motion and Stokes' drag law were used to determine the charge on an electron.

plates uncharged, measurement of the downward terminal velocity allowed the radius of a drop to be calculated assuming that the viscosity of the drop is much larger than the viscosity of the air. The switch is thrown to charge the upper plate negatively. The same droplet then reverses direction and is forced upward. It quickly achieves its terminal velocity U_u by virtue of the balance of upward forces (electric + buoyancy) and downward forces (weight + drag). This gives

$$6\pi\mu U_u a + (4/3)\pi a^3 g(\rho' - \rho) = neE,$$

where U_u is measured by the observation telescope and the radius of the particle is now known. The data then allow for the calculation of ne. As n must be an integer, data from many droplets may be differenced to identify the minimum difference that must be e, the charge of a single electron.

The drag coefficient, C_D, defined by (4.107) with $A = \pi a^2$, for Stokes' sphere is

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 \pi a^2} = \frac{24}{\mathrm{Re}}, \tag{8.52}$$

where $\mathrm{Re} = 2aU/\nu$ is the Reynolds number based on the diameter of the sphere. This dependence on the Reynolds number can be predicted from dimensional analysis when fluid inertia, represented by ρ, is not a parameter (see Exercise 4.60). Without fluid density, the drag force on a slowly moving sphere may only depend on the other parameters of the problem:

$$D = f(\mu, U, a).$$

Here there are four variables and the three basic dimensions of mass, length, and time. Therefore, only one dimensionless parameter, $D/\mu Ua$, can be formed. Hence, it must be a constant, and this leads to $C_D \propto 1/\mathrm{Re}$.

The flow pattern in a reference frame fixed to the fluid at infinity can be found by superposing a uniform velocity U to the left. This cancels out the first term in (8.48), giving

$$\psi = Ur^2 \sin^2\theta\left(-\frac{3a}{4r} + \frac{a^3}{4r^3}\right),$$

which gives the streamline pattern for a sphere moving from right to left in front of an observer (Figure 8.19). The pattern is symmetric between the upstream and the downstream directions, which is a result of the linearity of the governing equation (9.63); reversing the direction of the free-stream velocity merely changes **u** to −**u** and $p - p_\infty$ to $-p + p_\infty$. The flow therefore does not leave a velocity-field wake behind the sphere.

In spite of its fame and success, the Stokes solution is not valid at large distances from the sphere because the advective terms are not negligible compared to the viscous terms at these distances. At large distances, the viscous terms are of the order

$$\text{viscous force/volume} = \text{stress gradient} \sim \frac{\mu U a}{r^3} \text{ as } r \to \infty,$$

while from (8.49), the largest inertia term is:

$$\text{inertia force/volume} \sim \rho u_r \frac{\partial u_\theta}{\partial r} \sim \frac{\rho U^2 a}{r^2} \text{ as } r \to \infty;$$

therefore,

$$\text{inertia force/viscous force} \sim \frac{\rho U a}{\mu}\frac{r}{a} \sim \mathrm{Re}\frac{r}{a} \text{ as } r \to \infty,$$

which shows that the inertia forces are not negligible for distances larger than $r/a \sim 1/\mathrm{Re}$.

Solutions of problems involving a small parameter can be developed in terms of a perturbation series in which the higher-order terms act as corrections on the lower-order terms. If

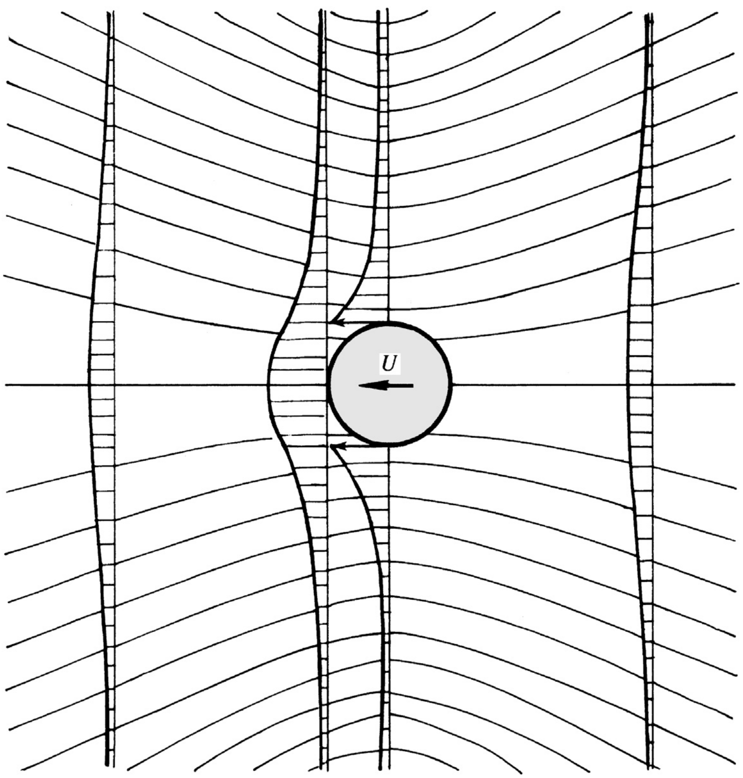


FIGURE 8.19 Streamlines and velocity distributions in Stokes' solution of creeping flow due to a moving sphere. Note the upstream and downstream symmetry, which is a result of complete neglect of nonlinearity.

we regard the Stokes solution as the first term of a series expansion in the small parameter Re, then the expansion is not uniformly valid because it breaks down as $r \to \infty$. If we tried to calculate the next term (to order Re) of the perturbation series, we would find that the velocity corresponding to the higher-order term becomes unbounded compared to that of the first term as $r \to \infty$.

The situation becomes worse for two-dimensional objects such as the circular cylinder. In this case, the Stokes balance, $\nabla p = \mu \nabla^2 \mathbf{u}$, has *no solution at all* that can satisfy the uniform-flow boundary condition at infinity. From this, Stokes concluded that steady, slow flows around cylinders cannot exist in nature. It has now been realized that the nonexistence of a first approximation of the Stokes flow around a cylinder is due to the *singular* nature of low Reynolds number flows in which there is a region of *nonuniformity* for $r \to \infty$. The nonexistence of the second approximation for flow around a sphere is due to the same reason. In a different (and more familiar) class of singular perturbation problems, the region of nonuniformity is a thin layer (the boundary layer) near the surface of an object. This is the class of flows with Re $\to \infty$, that are discussed in the next chapter. For these high Reynolds number flows the small parameter 1/Re multiplies the *highest*-order derivative in the governing equations, so that the solution with 1/Re identically set to zero cannot satisfy all the boundary conditions. In low Reynolds number flows this classic symptom of the loss of the highest derivative is absent, but it is a singular perturbation problem nevertheless.

Oseen (1910) provided an improvement to Stokes' solution by partly accounting for the inertia terms at large distances. He made the substitutions,

$$u = U + u', v = v', \text{ and } w = w',$$

where u', v', and w' are the Cartesian components of the perturbation velocity, and are small at large distances. Substituting these, the advection term of the x-momentum equation becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = U\frac{\partial u'}{\partial x} + \left[u'\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y} + w'\frac{\partial u'}{\partial z}\right].$$

Neglecting the quadratic terms, a revised version of the equation of motion (8.43) becomes

$$\rho U\frac{\partial u_i'}{\partial x} = \frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i',$$

where u'_i represents u', v', or w'. This is called *Oseen's equation*, and the approximation involved is called *Oseen's approximation*. In essence, the Oseen approximation linearizes the advective acceleration term $\mathbf{u}\cdot\nabla\mathbf{u}$ to $U(\partial \mathbf{u}/\partial x)$, whereas the Stokes approximation drops advection altogether. Near the body both approximations have the same order of accuracy. However, the Oseen approximation is better in the far field where the velocity is only slightly different from U. The Oseen equations provide a lowest-order solution that is uniformly valid everywhere in the flow field.

The boundary conditions for a stationary sphere with the fluid moving past it at velocity $U\mathbf{e}_x$ are:

$$u', v', w' \to 0 \text{ as } r \to \infty, \text{ and } u' = -U \text{ and } v', w' = 0$$

on the sphere's surface. The solution found by Oseen is:

$$\frac{\psi}{Ua^2} = \left[\frac{r^2}{2a^2} + \frac{a}{4r}\right] \sin^2\theta - \frac{3}{\mathrm{Re}}(1+\cos\theta)\left\{1 - \exp\left[-\frac{\mathrm{Re}}{4}\frac{r}{a}(1-\cos\theta)\right]\right\}, \tag{8.53}$$

where $\mathrm{Re} = 2aU/\nu$. Near the surface $r/a \approx 1$, a series expansion of the exponential term shows that Oseen's solution is identical to the Stokes solution (9.68) to the lowest order. The Oseen approximation predicts that the drag coefficient is

$$C_D = \frac{24}{\mathrm{Re}}\left(1 + \frac{3}{16}\mathrm{Re}\right),$$

which should be compared with the Stokes formula (8.52). Experimental results show that the Oseen and the Stokes formulas for C_D are both fairly accurate for $\mathrm{Re} < 5$ (experimental results fall between them), an impressive range of validity for a theory developed for $\mathrm{Re} \to 0$.

The streamlines corresponding to the Oseen solution (8.53) are shown in Figure 8.20, where a uniform flow of U is added to the left to generate the pattern of flow due to a sphere moving in front of a stationary observer. It is seen that the flow is no longer symmetric, but has a wake where the streamlines are closer together than in the Stokes flow. The velocities in

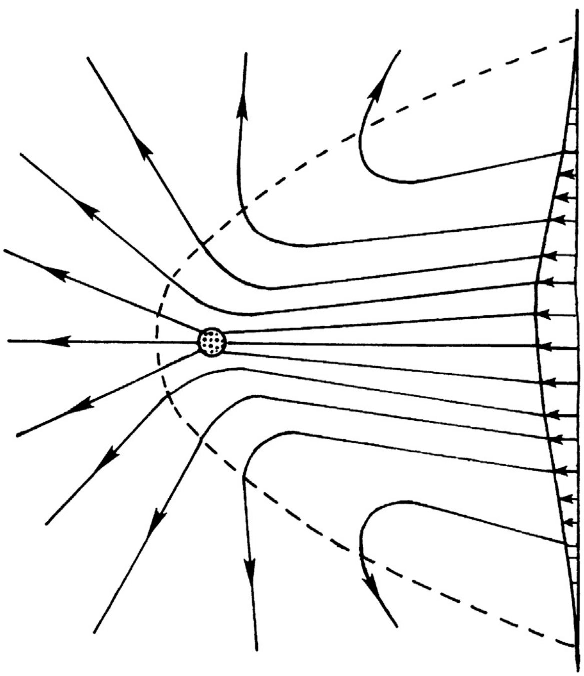


FIGURE 8.20 Streamlines and velocity distribution in Oseen's solution of creeping flow due to a moving sphere. Note the upstream and downstream asymmetry, which is a result of partial accounting for advection in the far field.

the wake are larger than in front of the sphere. Relative to the sphere, the flow is slower in the wake than in front of the sphere.

In 1957, Oseen's correction to Stokes' solution was rationalized independently by Kaplun (1957), and Proudman and Pearson (1957) in terms of matched asymptotic expansions. Higher-order corrections were obtained by Chester and Breach (1969).

8.7. FINAL REMARKS

As in other fields, analytical methods in fluid flow problems are useful in understanding the physics of fluid flows and in making generalizations. However, it is probably fair to say that most of the analytically tractable problems in ordinary laminar flow have already been solved, and approximate methods are now necessary for further advancing our knowledge. One of these approximate techniques is the perturbation method, where the flow is assumed to deviate slightly from a basic linear state. Another method that is playing an increasingly important role is that of solving the Navier-Stokes equations numerically using a computer. A proper application of such techniques requires considerable care and familiarity with various iterative techniques and their limitations. It is hoped that the reader will have the opportunity to learn numerical methods in a separate study. However, for completeness, Chapter 10 introduces several basic methods for computational fluid dynamics.

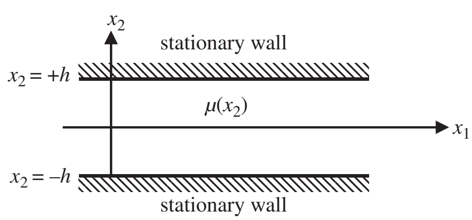
EXERCISES

8.1. **a)** Write out the three components of (8.1) in x-y-z Cartesian coordinates.
b) Set $\mathbf{u} = (u(y), 0, 0)$, and show that the x- and y-momentum equations reduce to:

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{d^2u}{dy^2}, \text{ and } 0 = -\frac{1}{\rho}\frac{\partial p}{\partial y}.$$

8.2. For steady pressure-driven flow between parallel plates (see Figure 8.3), there are 7 parameters: $u(y)$, U, y, h, ρ, μ, and dp/dx. Determine a dimensionless scaling law for $u(y)$, and rewrite the flow-field solution (8.5) in dimensionless form.

8.3. An incompressible viscous liquid with density ρ fills the gap between two large, smooth parallel walls that are both stationary. The upper and lower walls are located at $x_2 = \pm h$, respectively. An additive in the liquid causes its viscosity to vary in the x_2 direction. Here the flow is driven by a constant nonzero pressure gradient: $\partial p/\partial x_1 = const.$



a) Assume steady flow, ignore the body force, set $\mathbf{u} = (u_1(x_2), 0, 0)$ and use

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0,\ \rho\frac{\partial u_j}{\partial t} + \rho u_i \frac{\partial u_j}{\partial x_i}$$
$$= -\frac{\partial p}{\partial x_j} + \rho g_j + \frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] + \frac{\partial}{\partial x_j}\left[\left(\mu_v - \frac{2}{3}\mu\right)\frac{\partial u_i}{\partial x_i}\right]$$

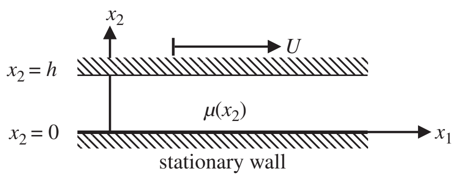
to determine $u_1(x_2)$ when $\mu = \mu_o(1 + \gamma(x_2/h)^2)$

b) What shear stress is felt on the lower wall?

c) What is the volume flow rate (per unit depth into the page) in the gap when $\gamma = 0$?

d) If $-1 < \gamma < 0$, will the volume flux be higher or lower than the case when $\gamma = 0$?

8.4. An incompressible viscous liquid with density ρ fills the gap between two large, smooth parallel plates. The upper plate at $x_2 = h$ moves in the positive x_1-direction at speed U. The lower plate at $x_2 = 0$ is stationary. An additive in the liquid causes its viscosity to vary in the x_2 direction.

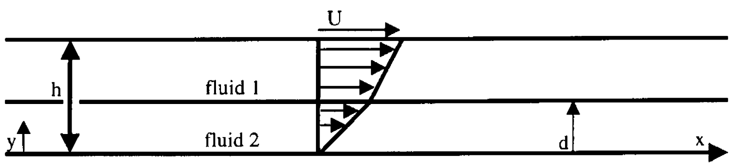


a) Assume steady flow, ignore the body force, set $\mathbf{u} = (u_1(x_2), 0, 0)$ and $\partial p/\partial x_1 = 0$, and use the equations specified in Exercise 8.3 to determine $u_1(x_2)$ when $\mu = \mu_o(1 + \gamma x_2/h)$.

b) What shear stress is felt on the lower plate?

c) Are there any physical limits on γ? If, so specify them.

8.5. Planar Couette flow is generated by placing a viscous fluid between two infinite parallel plates and moving one plate (say, the upper one) at a velocity U with respect to the other one. The plates are a distance h apart. Two immiscible viscous liquids are placed between the plates as shown in the diagram. The lower fluid layer has thickness d. Solve for the velocity distributions in the two fluids.



8.6. Consider the laminar flow of a fluid layer falling down a plane inclined at an angle θ with respect to the horizontal. If h is the thickness of the layer in the fully developed stage, show that the velocity distribution is $u(y) = (g/2\nu)(h^2 - y^2)\sin\theta$, where the x-axis points in the direction of flow along the free surface, and the y-axis points toward the plane. Show that the volume flow rate per unit width is $Q = (gh^3/3\nu)\sin\theta$, and that the frictional stress on the wall is $\tau_0 = \rho gh\sin\theta$.

8.7. Room temperature water drains through a round vertical tube with diameter d. The length of the tube is L. The pressure at the tube's inlet and outlet is atmospheric, the flow is steady, and $L \gg d$.

a) Using dimensional analysis, write a physical law for the mass flow rate $\dot{m}$ through the tube.

b) Assume that the velocity profile in the tube is independent of the vertical coordinate, determine a formula for $\dot{m}$, and put it in dimensionless form.

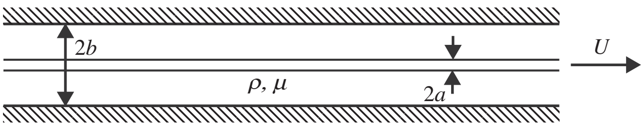
c) What is the change in $\dot{m}$ if the temperature is raised and the water's viscosity drops by a factor of two?

8.8. Consider steady laminar flow through the annular space formed by two coaxial tubes aligned with the z-axis. The flow is along the axis of the tubes and is maintained by a pressure gradient dp/dz. Show that the axial velocity at any radius R is

$$u_z(R) = \frac{1}{4\mu}\frac{dp}{dz}\left[R^2 - a^2 - \frac{b^2 - a^2}{\ln(b/a)}\ln\frac{R}{a}\right],$$

where a is the radius of the inner tube and b is the radius of the outer tube. Find the radius at which the maximum velocity is reached, the volume flow rate, and the stress distribution.

8.9. A long, round wire with radius a is pulled at a steady speed U, along the axis of a long round tube of radius b that is filled with a viscous fluid. Assuming laminar, fully developed axial flow with $\partial p/\partial z = 0$ in cylindrical coordinates (R, φ, z) with $\mathbf{u} = (0, 0, w(R))$, determine $w(R)$ assuming constant fluid density ρ and viscosity μ with no body force.



a) What force per unit length of the wire is needed to maintain the motion of the wire?

b) Explain what happens to $w(R)$ when $b \to \infty$. Is this situation physically meaningful? What additional term(s) from the equations of motion need to be retained to correct this situation?

8.10. [1]Consider steady unidirectional incompressible viscous flow in Cartesian coordinates, $u = v = 0$ with $w = w(x,y)$ without body forces.

a) Starting from the steady version of (8.1), derive a single equation for w assuming that $\partial p/\partial z$ is nonzero and constant.

b) Guess $w(x,y)$ for a tube with elliptical cross section $(x/a)^2 + (y/b)^2 = 1$.

[1]Inspired by problem 2 on page 383 in Yih (1979).

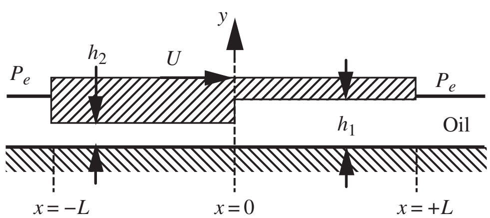
c) Determine $w(x,y)$ for a tube of rectangular cross section: $-a/2 \le x \le +a/2, -b/2 \le y \le +b/2$. [*Hint*: find particular (a polynomial) and homogeneous (a Fourier series) solutions for w.]

8.11. A long vertical cylinder of radius b rotates with angular velocity Ω concentrically outside a smaller stationary cylinder of radius a. The annular space is filled with fluid of viscosity μ. Show that the steady velocity distribution is: $u_\varphi = \dfrac{R^2 - a^2}{b^2 - a^2}\dfrac{b^2\Omega}{R}$, and that the torque exerted on either cylinder, per unit length, equals $4\pi\mu\Omega a^2 b^2/(b^2 - a^2)$.

8.12. Consider a solid cylinder of radius a, steadily rotating at angular speed Ω in an infinite viscous fluid. The steady solution is irrotational: $u_\theta = \Omega a^2/R$. Show that the work done by the external agent in maintaining the flow (namely, the value of $2\pi\, R u_\theta \tau_{r\theta}$ at $R = a$) equals the viscous dissipation rate of fluid kinetic energy in the flow field.

8.13. For lubrication flow under the sloped bearing of Example 8.1, the assumed velocity profile was $u(x,y) = -(1/2\mu)(dP/dx)y(h(x) - y)) + Uy/h(x)$, the derived pressure was $P(x) = P_e + (3\mu U\alpha/h_o^2 L)x(L - x)$, and the load (per unit depth) carried by the bearing was $W = \mu U\alpha L^2/2h_o^2$. Use these equations to determine the frictional force (per unit depth), F_f, applied to the lower (flat) stationary surface in terms of W, h_o/L, and α. What is the spatially averaged coefficient of friction under the bearing?

8.14. A bearing pad of total length $2L$ moves to the right at constant speed U above a thin film of incompressible oil with viscosity μ and density ρ. There is a step change in the gap thickness (from h_1 to h_2) below the bearing as shown. Assume that the oil flow under the bearing pad follows: $u(y) = -\dfrac{y(h_j - y)}{2\mu}\dfrac{dP(x)}{dx} + \dfrac{Uy}{h_j}$, where $j = 1$ or 2. The pad is instantaneously aligned above the coordinate system shown. The pressure in the oil ahead and behind the bearing is P_e.



a) By conserving mass for the oil flow, find a relationship between μ, U, h_j, dP/dx, and an unknown constant C.

b) Use the result of part a) and continuity of the pressure at $x = 0$, to determine

$$P(0) - P_e = \frac{6\mu UL(h_1 - h_2)}{h_2^3 + h_1^3}.$$

c) Can this bearing support an externally applied downward load when $h_1 < h_2$?

8.15. A flat disk of radius a rotates above a solid boundary at a steady rotational speed of Ω. The gap, h ($\ll a$), between the disk and the boundary is filled with an incompressible Newtonian fluid with viscosity μ and density ρ. The pressure at the edge of the disk is $p(a)$.

a) Using cylindrical coordinates and assuming that the only nonzero velocity component is $u_\varphi(R,z)$, determine the torque necessary to keep the disk turning.

b) If $p(a)$ acts on the exposed (upper) surface of the disk, will the pressure distribution on the disk's wetted surface tend to pull the disk *toward* or *away from* the solid boundary?

c) If the gap is increased, eventually the assumption of part a) breaks down. What happens? Explain why and where u_R and u_z might be nonzero when the gap is no longer narrow.

8.16. A circular block with radius a and weight W is released at $t = 0$ on a thin layer of an incompressible fluid with viscosity μ that is supported by a smooth horizontal motionless surface. The fluid layer's initial thickness is h_o. Assume that flow in the gap between the block and the surface is quasi-steady with a parabolic velocity profile:

$$u_R(R,z,t) = -(dP(R)/dR)z(h(t) - z))/2\mu,$$

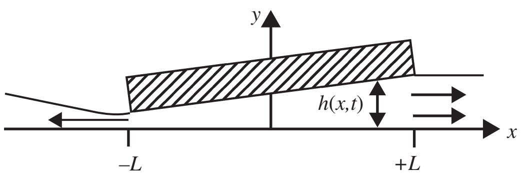
where R is the distance from the center of the block, $P(R)$ is the pressure at R, z is the vertical coordinate from the smooth surface, $h(t)$ is the gap thickness, and t is time.

a) By considering conservation of mass, show that: $dh/dt = (h^3/6\mu R)(dP/dR)$.

b) If W is known, determine $h(t)$ and note how long it takes for $h(t)$ to reach zero.

8.17. Consider the inverse of the previous exercise. A block and a smooth surface are separated by a thin layer of a viscous fluid with thickness h_o. At $t = 0$, a force, F, is applied to separate them. If h_o is arbitrarily small, can the block and plate be separated easily? Perform some tests in your kitchen. Use maple syrup, creamy peanut butter, liquid soap, pudding, etc. for the viscous liquid. The flat top side of a metal jar lid or the flat bottom of a drinking glass makes a good circular block. (Lids with raised edges and cups and glasses with ridges or sloped bottoms do not work well). A flat countertop or the flat portion of a dinner plate can be the motionless smooth surface. Can the item used for the block be more easily separated from the surface when tilted relative to the surface? Describe your experiments and try to explain your results.

8.18. A rectangular slab of width $2L$ (and depth B into the page) moves *vertically* on a thin layer of oil that flows horizontally as shown. Assume $u(y,t) = -(h^2/2\mu)(dP/dx)(y/h)(1 - y/h)$, where $h(x,t)$ is the instantaneous gap between the slab and the surface, μ is the oil's viscosity, and $P(x,t)$ is the pressure in the oil below the slab. The slab is slightly misaligned with the surface so that $h(x,t) = h_o(1 + \alpha x/L) + \dot{h}_o t$ where $\alpha \ll 1$ and $\dot{h}_o$ is the vertical velocity of the slab. The pressure in the oil outside the slab is P_o. Consider the instant $t = 0$ in your work below.



a) Conserve mass in an appropriate CV to show that: $\dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}\left(\displaystyle\int_0^{h(x,t)} u(y,t)dy\right) = 0.$

b) Keeping only linear terms in α, and noting that C and D are constants, show that:

$$P(x, t=0) = \frac{12\mu}{h_o^3}\left(\frac{\dot{h}_o x^2}{2}\left(1 - \frac{2\alpha x}{L}\right) + Cx\left(1 - \frac{3\alpha x}{2L}\right)\right) + D.$$

c) State the boundary conditions necessary to evaluate the constants C and D.

d) Evaluate the constants to show that the pressure distribution below the slab is:

$$P(x,t) - P_o = -\frac{6\mu \dot{h}_o L^2}{h_o^3(t)}\left(1 - (x/L)^2\right)\left(1 - 2\alpha\frac{x}{L}\right).$$

e) Does this pressure distribution act to increase or decrease alignment between the slab and surface when the slab is moving downward? Answer the same question for upward slab motion.

8.19. Show that the lubrication approximation can be extended to viscous flow within narrow gaps $h(x,y,t)$ that depend on two spatial coordinates. Start from (4.10) and (8.1), and use Cartesian coordinates oriented so that x-y plane is locally tangent to the center-plane of the gap. Scale the equations using a direct extension of (8.14):

$$x^* = \frac{x}{L},\ y^* = \frac{y}{L},\ z^* = \frac{z}{h} = \frac{y}{\varepsilon L},\ t^* = \frac{Ut}{L},\ u^* = \frac{u}{U},\ v^* = \frac{v}{U},\ w^* = \frac{w}{\varepsilon U},\ \text{and } p^* = \frac{p}{Pa},$$

where L is the characteristic distance for the gap thickness to change in either the x or y direction, and $\varepsilon = h/L$. Simplify these equation when $\varepsilon^2 \mathrm{Re}_L \rightarrow 0$, but $\mu UL/P_a h^2$ remains of order unity to find:

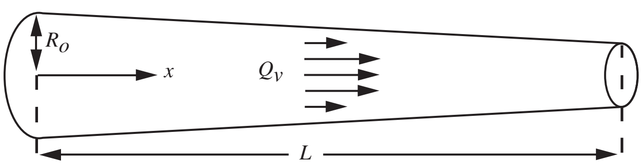
$$0 \cong -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial z^2},\ 0 \cong -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial^2 v}{\partial z^2},\ \text{and } 0 \cong -\frac{1}{\rho}\frac{\partial p}{\partial z}.$$

8.20. A thin film of viscous fluid is bounded below by a flat stationary plate at $z = 0$. If the in-plane velocity at the upper film surface, $z = h(x, y, t)$, is $\mathbf{U} = U(x, y, t)\mathbf{e}_x + V(x, y, t)\mathbf{e}_y$, use the equations derived in Exercise 8.19 to produce the Reynolds equation for constant-density, thin-film lubrication:

$$\nabla \cdot \left[\left(\frac{h^3}{\mu}\right)\nabla p\right] = 12\frac{\partial h}{\partial t} + 6\nabla \cdot (h\mathbf{U}).$$

where $\nabla = \mathbf{e}_x(\partial/\partial x) + \mathbf{e}_y(\partial/\partial y)$ merely involves the two in-plane dimensions.

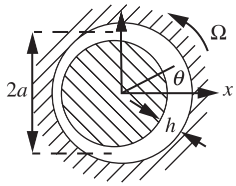
8.21. Fluid of density ρ and viscosity μ flows inside a long tapered tube of length L and radius $R(x) = (1 - \alpha x/L)R_o$, where $\alpha < 1$ and $R_o \ll L$.



a) Estimate the volume discharge rate Q_v through the tube, for a given pressure difference Δp sustained between the inlet and the outlet.

b) Discuss the range of validity of your solution in terms of the parameters of the problem.

8.22. A circular lubricated bearing of radius a holds a stationary round shaft. The bearing hub rotates at angular rate Ω as shown. A load per unit depth on the shaft, **W**, causes the center of the shaft to be displaced from the center of the rotating hub by a distance εh_o, where h_o is the average gap thickness and $h_o \ll a$. The gap is filled with an incompressible oil of viscosity μ.

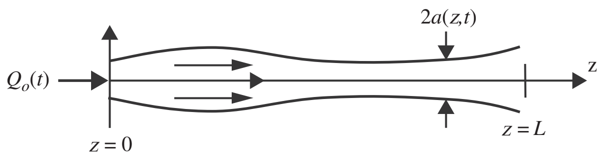


a) Determine a dimensionless scaling law for $|\mathbf{W}|$.

b) Determine **W** by assuming a lubrication flow profile in the gap and $h(\theta) = h_o(1 + \varepsilon \cos\theta)$ with $\varepsilon \ll 1$.

c) If **W** is increased a little bit, is the lubrication action stabilizing?

8.23. As a simple model of small-artery blood flow, consider slowly varying viscous flow through a round flexible tube with inlet at $z = 0$ and outlet at $z = L$. At $z = 0$, the volume flux entering the tube is $Q_o(t)$. At $z = L$, the pressure equals the exterior pressure p_e. The radius of the tube, $a(z,t)$, expands and contracts in proportion to pressure variations within the tube so that: 1) $a - a_e = \gamma(p - p_e)$, where a_e is the tube radius when the pressure, $p(z,t)$, in the tube is equal to p_e, and γ is a positive constant. Assume the local volume flux, $Q(z,t)$, is related to $\partial p/\partial z$ by 2) $Q = -(\pi a^4/8\mu)(\partial p/\partial z)$.



a) By conserving mass, find a partial differential equation that relates Q and a.

b) Combine 1), 2), and the result of part a) into one partial differential equation for $a(z,t)$.

c) Determine $a(z)$ when Q_o is a constant and the flow is perfectly steady.

8.24. Consider a simple model of flow from a tube of toothpaste. A liquid with viscosity μ and density ρ is squeezed out of a round horizontal tube having radius $a(t)$. In your work, assume that a is decreasing and use cylindrical coordinates with the z-axis coincident with the centerline of the tube. The tube is closed at $z = 0$, but is open to the atmosphere at $z = L$. Ignore gravity.

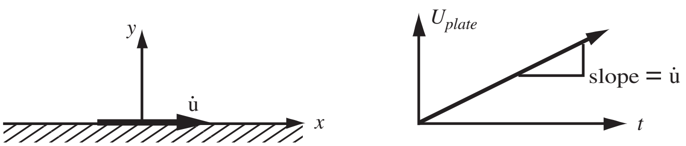
a) If w is the fluid velocity along the z-axis, show that: $za\dfrac{da}{dt} + \displaystyle\int_0^a w(z,R,t)R\,dR = 0$.

b) Determine the pressure distribution, $p(z) - p(L)$, by assuming the flow in the tube can be treated within the lubrication approximation by setting $w(z,R,t) = -\frac{1}{4\mu}\frac{dp}{dz}(a^2(t) - R^2)$.

c) Find the cross-section-average flow velocity $w_{ave}(z,t)$ in terms of z, a, and da/dt.

d) If the pressure difference between $z = 0$ and $z = L$ is ΔP, what is the volume flux exiting the tube as a function of time. Does this answer partially explain why fully emptying a toothpaste tube by squeezing it is essentially impossible?

8.25. A large flat plate below an infinite stationary incompressible viscous fluid is set in motion with a constant acceleration, $\dot{u}$, at $t = 0$. A prediction for the subsequent fluid motion, $u(y,t)$, is sought.



a) Use dimensional analysis to write a physical law for $u(y,t)$ in this flow.

b) Starting from the x-component of (8.1) determine a linear partial differential equation for $u(y,t)$.

c) The linearity of the equation obtained for part c) suggests that $u(y,t)$ must be directly proportional to $\dot{u}$. Simplify your dimensional analysis to incorporate this requirement.

d) Let $\eta = y/(\nu t)^{1/2}$ be the independent variable, and derive a second-order ordinary linear differential equation for the unknown function $f(\eta)$ left from the dimensional analysis.

e) From an analogy between fluid acceleration in this problem and fluid velocity in Stokes' first problem, deduce the solution $u(y,t) = \dot{u}\int_0^t [1 - erf(y/2\sqrt{\nu t'})]dt'$ and show that it solves the equation of part b).

f) Determine $f(\eta)$ and—if your patience holds out—show that it solves the equation found in part d).

g) Sketch the expected velocity profile shapes for several different times. Note the direction of increasing time on your sketch.

8.26. Suppose a line vortex of circulation Γ is suddenly introduced into a fluid at rest at $t = 0$. Show that the solution is $u_\theta(r,t) = (\Gamma/2\pi r)\exp\{-r^2/4\nu t\}$. Sketch the velocity distribution at different times. Calculate and plot the vorticity, and observe how it diffuses outward.

8.27. Obtain several liquids of differing viscosity (water, cooking oil, pancake syrup, shampoo, etc.). Using an eyedropper, a small spoon, or your finger, place a drop of each on a smooth vertical surface (a bathroom mirror perhaps) and measure how far the drops have moved or extended in a known period of time (perhaps a minute or two). Try to make the mass of all the drops equal. Using dimensional analysis, determine how the drop-sliding distance depends on the other parameters. Does this match your experimental results?

8.28. A drop of an incompressible viscous liquid is allowed to spread on a flat horizontal surface under the action of gravity. Assume the drop spreads in an axisymmetric fashion and use cylindrical coordinates (R, φ, z). Ignore the effects of surface tension.

a) Show that conservation of mass implies: $\frac{\partial h}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\int_o^h udz) = 0$, where $u = u(R, z, t)$ is the horizontal velocity within the drop, and $h = h(R, t)$ is the thickness of the spreading drop.

b) Assume that the lubrication approximation applies to the horizontal velocity profile, that is, $u(R,z,t) = a(R,t) + b(R,t)z + c(R,t)z^2$, apply the appropriate boundary conditions on the upper and lower drop surfaces, and require a pressure and shear-stress force balance within a differential control volume $h(R,t)RdRd\theta$ to show that: $u(R,z,t) = -\frac{g}{2\nu}\frac{\partial h}{\partial R}z(2h - z)$.

c) Combine the results of a) and b) to find $\frac{\partial h}{\partial t} = \frac{g}{3\nu R}\frac{\partial}{\partial R}\left(Rh^3\frac{\partial h}{\partial R}\right)$.

d) Assume a similarity solution: $h(R,t) = \frac{A}{t^n}f(\eta)$ with $\eta = \frac{BR}{t^m}$, use the result of part c) and $2\pi\int_0^{R_{\max}(t)} h(R,t)RdR = V$, where $R_{\max}(t)$ is the radius of the spreading drop and V is the initial volume of the drop to determine $m = 1/8$, $n = 1/4$, and a single nonlinear ordinary differential equation for $f(\eta)$ involving only A, B, g/ν, and η. You need not solve this equation for f. [Given that $f \to 0$ as $\eta \to \infty$, there will be a finite value of η for which f is effectively zero. If this value of η is $\eta_{\max}$ then the radius of the spreading drop, $R(t)$, will be: $R_{\max}(t) = \eta_{\max}t^m/B$.]

8.29. Obtain a clean, flat glass plate, a watch, a ruler, and some nonvolatile oil that is more viscous than water. The plate and oil should be at room temperature. Dip the tip of one of your fingers in the oil and smear it over the center of the plate so that a thin bubble-free oil film covers a circular area ~10 to 15 cm in diameter. Set the plate on a horizontal surface and place a single drop of oil at the center of the oil-film area and observe how the drop spreads. Measure the spreading drop's diameter 1, 10, 10^2, 10^3, and 10^4 seconds after the drop is placed on the plate. Plot your results and determine if the spreading drop diameter grows as $t^{1/8}$ (the predicted drop-diameter time dependence from the prior problem) to within experimental error.

8.30. An infinite flat plate located at $y = 0$ is stationary until $t = 0$ when it begins moving horizontally in the positive x-direction at a constant speed U. This motion continues until $t = T$ when the plate suddenly stops moving.

a) Determine the fluid velocity field, $u(y, t)$ for $t > T$. At what height above the plate does the peak velocity occur for $t > T$? [*Hint*: the governing equation is linear so superposition of solutions is possible.]

b) Determine the mechanical impulse I (per unit depth and length) imparted to the fluid while the plate is moving: $I = \int_0^T \tau_w dt$.

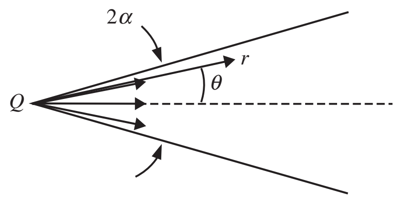
c) As $t \to \infty$, the fluid slows down and eventually stops moving. How and where was the mechanical impulse dissipated? What is t/T when 99% of the initial impulse has been lost?

8.31. Consider the development from rest of plane Couette flow. The flow is bounded by two rigid boundaries at $y = 0$ and $y = h$, and the motion is started from rest by suddenly accelerating the lower plate to a steady velocity U. The upper plate is held stationary. Here a similarity solution cannot exist because of the appearance of the parameter h. Show that the velocity distribution is given by

$$u(y,t) = U\left(1 - \frac{y}{h}\right) - \frac{2U}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-n^2\pi^2\frac{\nu t}{h^2}\right)\sin\left(\frac{n\pi y}{h}\right).$$

Sketch the flow pattern at various times, and observe how the velocity reaches the linear distribution for large times.

8.32. [2]Two-dimensional flow between flat nonparallel plates can be formulated in terms of a normalized angular coordinate, $\eta = \theta/\alpha$, where α is the half angle between the plates, and a normalized radial velocity, $u_r(r,\theta) = u_{\max}(r)f(\eta)$, where $\eta = \theta/\alpha$ for $|\theta| \le \alpha$. Here, $u_\theta = 0$, the Reynolds number is $Re = u_{\max}r\alpha/\nu$, and Q is the volume flux (per unit width perpendicular to the page).



a) Using the appropriate versions of (4.10) and (8.1), show that $f'' + \mathrm{Re}\,\alpha f^2 + 4\alpha^2 f = const.$

b) Find $f(\eta)$ for symmetric creeping flow, that is, $Re = 0 = f(+1) = f(-1)$, and $f(0) = 1$.

c) Above what value of the channel half-angle will backflow always occur?

8.33. Consider steady viscous flow inside a cone of constant angle θ_o. The flow has constant volume flux $= Q$, and the fluid has constant density $= \rho$ and constant kinematic viscosity $= \nu$. Use spherical coordinates, and assume that the flow only has a radial component, $\mathbf{u} = (u_r(r,\theta), 0, 0)$, which is independent of the azimuthal angle φ, so that the equations of motion are:

Conservation of mass,

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) = 0,$$

Conservation of radial momentum,

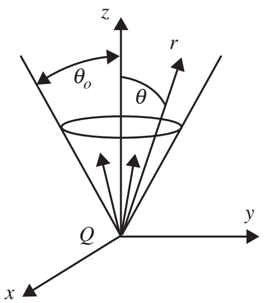
$$u_r\frac{\partial u_r}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u_r}{\partial \theta}\right) - \frac{2}{r^2}u_r\right), \text{ and}$$

Conservation of θ-momerntum,

$$0 = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right).$$

[2]Rephrased from White (2006) p. 211, problem 3.32.

For the following items, assume the radial velocity can be determined using: $u_r(r,\theta) = QR(r)\Theta(\theta)$. Define the Reynolds number of this flow as: $Re = Q/(\pi\nu r)$.



a) Use the continuity equation to determine $R(r)$.
b) Integrate the θ-momentum equation, assume the constant of integration is zero, and combine the result with the radial momentum equation to determine a single differential equation for $\Theta(\theta)$ in terms of θ and Re.
c) State the matching and/or boundary conditions that $\Theta(\theta)$ must satisfy.

8.34. The boundary conditions on obstacles in Hele-Shaw flow were not considered in Example 8.2. Therefore, consider them here by examining Hele-Shaw flow parallel to a flat obstacle surface at $y = 0$. The Hele-Shaw potential in this case is:

$$\phi = Ux\frac{z}{h}\left(1 - \frac{z}{h}\right),$$

where (x, y, z) are Cartesian coordinates and the flow is confined to $0 < z < h$ and $y > 0$.
a) Show that this potential leads to a slip velocity of $u(x, y \rightarrow 0) = U(z/h)(1 - z/h)$, and determine the pressure distribution implied by this potential.
b) Since this is a viscous flow, the slip velocity must be corrected to match the genuine no-slip condition on the obstacle's surface at $y = 0$. The analysis of Example (8.2) did not contain the correct scaling for this situation near $y = 0$. Therefore, rescale the x-component of (8.1) using:

$$x^* = x/L,\ \ y^* = y/h = y/\varepsilon L,\ \ z^* = z/h = z/\varepsilon L,\ \ t^* = Ut/L,\ \ u^* = u/U,\ \ v^* = v/\varepsilon U,$$
$$w^* = w/\varepsilon U, \text{ and } p^* = p/Pa,$$

and then take the limit as $\varepsilon^2 \mathrm{Re}_L \rightarrow 0$, with $\mu UL/P_a h^2$ remaining of order unity, to simplify the resulting dimensionless equation that has

$$0 \cong \frac{dp}{dx} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

as its dimensional counterpart.
c) Using boundary conditions of $u = 0$ on $y = 0$, and $u = U(z/h)(1 - z/h)$ for $y \gg h$. Show that

$$u(x, y, z) = U\frac{z}{h}\left(1 - \frac{z}{h}\right) + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{h}z\right) \exp\left(-\frac{n\pi}{h}y\right), \text{ where}$$

$$A_n = -\frac{2U}{h}\int_0^h \frac{z}{h}\left(1 - \frac{z}{h}\right)\sin\left(\frac{n\pi}{h}z\right)dz.$$

[The results here are directly applicable to the surfaces of curved obstacles in Hele-Shaw flow when the obstacle's radius of curvature is much greater than h.]

8.35. Using the velocity field (8.49), determine the drag on Stokes' sphere from the surface pressure and the viscous surface stresses σ_{rr} and $\sigma_{r\theta}$.

8.36. Calculate the drag on a spherical droplet of radius $r = a$, density ρ' and viscosity μ' moving with velocity U in an infinite fluid of density ρ and viscosity μ. Assume $\mathrm{Re} = \rho Ua/\mu \ll 1$. Neglect surface tension.

8.37. Consider a very low Reynolds number flow over a circular cylinder of radius $r = a$. For $r/a = O(1)$ in the $\mathrm{Re} = Ua/\nu \to 0$ limit, find the equation governing the stream function $\psi(r, \theta)$ and solve for ψ with the least singular behavior for large r. There will be one remaining constant of integration to be determined by asymptotic matching with the large r solution (which is not part of this problem). Find the domain of validity of your solution.

8.38. A small, neutrally buoyant sphere is centered at the origin of coordinates in a deep bath of a quiescent viscous fluid with density ρ and viscosity μ. The sphere has radius a and is initially at rest. It begins rotating about the z-axis with a constant angular velocity Ω at $t = 0$. The relevant equations for the fluid velocity, $\mathbf{u} = (u_r, u_\theta, u_\varphi)$, in spherical coordinates (r, θ, φ) are:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(u_\varphi) = 0, \text{ and}$$

$$\frac{\partial u_\varphi}{\partial t} + u_r\frac{\partial u_\varphi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\varphi}{\partial\theta} + \frac{u_\varphi}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} + \frac{1}{r}(u_r u_\varphi + u_\theta u_\varphi \cot\theta)$$

$$= -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\varphi} + \nu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\varphi}{\partial\theta}\right)\right.$$

$$\left. + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\varphi}{\partial\varphi^2} - \frac{u_\varphi}{r^2\sin^2\theta} + \frac{2}{r^2\sin^2\theta}\frac{\partial u_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi}\right).$$

a) Assume $\mathbf{u} = (0, 0, u_\varphi)$ and reduce these equations to:

$$\frac{\partial u_\varphi}{\partial t} = -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\varphi} + \nu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\varphi}{\partial\theta}\right) - \frac{u_\varphi}{r^2\sin^2\theta}\right)$$

b) Set $u_\varphi(r,\theta,t) = \Omega a F(r,t)\sin\theta$, make an appropriate assumption about the pressure field, and derive the following equation for F: $\dfrac{\partial F}{\partial t} = \nu\left(\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial F}{\partial r}\right) - 2\dfrac{F}{r^2}\right)$.

c) Determine F for $t\to\infty$ for boundary conditions $F = 1$ at $r = a$, and $F\to 0$ as $r\to\infty$.

d) Find the surface shear stress and torque on the sphere.

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