The Half-Line Problem:

$$\phi_{xx} + \phi_{zz} = 0 \qquad -h < z < \eta(x,t) \qquad (1.1)$$
$$\phi_z = 0 \qquad z = -h \qquad (1.2)$$
$$\phi_x = 0 \qquad x = 0 \qquad (1.3)$$

$$\eta_t + \phi_x \eta_x = \phi_z \qquad z = \eta(x,t) \qquad (1.4)$$
$$\phi_t + g\eta + \frac{1}{2}(\phi_x^2 + \phi_z^2) = 0 \qquad z = \eta(x,t) \qquad (1.5)$$

$$\phi_z(0,\eta) = \eta_t(0,t) \qquad x = 0. \qquad (1.6)$$

Linearization:

Assume the following: $\phi(x,z,t) = A(x)B(z)C(t)$, so that we can apply the separation of variables. Substituting in yields two SL problem, depending on parameter λ, which Ablowitz calls κ (kappa). The solutions of the SL problems are:

$\lambda > 0$: $A(x) = \cos(x\sqrt{\lambda})$, $B(z) = \cosh((z+h)\sqrt{\lambda})$ $\qquad$ $\lambda < 0$: $A(x) = \cosh(x\sqrt{\lambda})$, $B(z) = \cos((z+h)\sqrt{\lambda})$ $\qquad$ $\lambda = 0$: $A = \tilde{A}$, $B = \tilde{B}$

Applying BC $\phi \to 0$ as $x \to \infty$ eliminates cases $\lambda < 0$, $\lambda = 0$, so that

$$\phi(x,z,t) = \cos(x\sqrt{\lambda})\cosh((z+h)\sqrt{\lambda})\underbrace{C(t)}_{\text{TBD}}.$$

Now we linearise equations (1.4) & (1.5). Assume $\begin{cases} \eta = \eta_1 + \eta_2 \\ \phi = \phi_1 + \phi_2 \end{cases}$, where $\eta_i, \phi_i \sim O(\varepsilon^i)$. Expand the terms:

① $$\left(\eta_t + \phi_x \eta_x\right)\Big|_{z=\eta} \approx \underbrace{\left(\eta_t + \phi_x \eta_x\right)\Big|_{z=0}}_{①} + \underbrace{\eta \frac{\partial}{\partial z}\left(\eta_t + \phi_x \eta_x\right)\Big|_{z=0}}_{②}$$

① $$\left(\eta_t + \phi_x \eta_x\right)\Big|_{z=0} = \frac{\partial \eta_1}{\partial t} + \frac{\partial \eta_2}{\partial t} + \left(\frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial x}\right)\left(\frac{\partial \eta_1}{\partial x} + \frac{\partial \eta_2}{\partial x}\right)\Big|_{z=0}$$

$$= \underset{\varepsilon^1}{\underset{\uparrow}{\frac{\partial \eta_1}{\partial t}}} + \underset{\varepsilon^2}{\underset{\Uparrow}{\frac{\partial \eta_2}{\partial t}}} + \underset{\varepsilon^2}{\underset{\Uparrow}{\frac{\partial \phi_1}{\partial x}\frac{\partial \eta_1}{\partial x}}} + O(\varepsilon^3) \quad \text{at } z=0$$

② $$\eta \frac{\partial}{\partial z}\left(\eta_t + \phi_x \eta_x\right)\Big|_{z=0} = (\eta_1 + \eta_2)\frac{\partial}{\partial z}\left(\frac{\partial \eta_1}{\partial t} + \frac{\partial \eta_2}{\partial t} + \frac{\partial \phi_1}{\partial x}\frac{\partial \eta_1}{\partial x} + O(\varepsilon^3)\right)$$

$$= (\eta_1 + \eta_2)\frac{\partial^2 \phi_1}{\partial z \partial x}\frac{\partial \eta_2}{\partial x} = O(\varepsilon^4) \quad \text{at } z=0$$

② $$\phi_z\Big|_{z=\eta} = \frac{\partial}{\partial z}\left(\phi\Big|_{z=0} + \eta \phi_z\Big|_{z=0}\right) = \frac{\partial}{\partial z}\left(\phi_1 + \phi_2 + (\eta_1 + \eta_2)\left(\frac{\partial \phi_1}{\partial z} + \frac{\partial \phi_2}{\partial z}\right)\right)$$

$$= \frac{\partial \phi_1}{\partial z} + \frac{\partial \phi_2}{\partial z} + \frac{\partial}{\partial z}\left(\underset{\varepsilon^2}{\eta_1 \frac{\partial \phi_1}{\partial z}} + \underset{\varepsilon^3}{\eta_1 \frac{\partial \phi_2}{\partial z}} + \underset{\varepsilon^3}{\eta_2 \frac{\partial \phi_1}{\partial t}} + \underset{\varepsilon^4}{\eta_2 \frac{\partial \phi_2}{\partial t}}\right)$$

$$= \frac{\partial \phi_1}{\partial z} + \frac{\partial \phi_3}{\partial z} + \eta_1 \frac{\partial^2 \phi_1}{\partial z^2} \quad \text{at } z = 0.$$

By these expansions, the kinematic condition (1.4) becomes

$$\underbrace{(\eta_t + \phi_x \eta_x)\Big|_{z=\eta}}_{1} = \underbrace{\phi_z\Big|_{z=\eta}}_{2} \Rightarrow$$

$$(\eta_t + \phi_x \eta_x)\Big|_{z=0} + \eta \frac{\partial}{\partial z}(\eta_t + \phi_x \eta_x)\Big|_{z=0} = \phi_z\Big|_{z=\eta} \Rightarrow$$

$$\underset{\varepsilon^1}{\frac{\partial \eta_1}{\partial t}} + \underset{\varepsilon^2}{\frac{\partial \eta_2}{\partial t}} + \underset{\varepsilon^2}{\frac{\partial \phi_1}{\partial x}\frac{\partial \eta_1}{\partial x}} = \underset{\varepsilon^1}{\frac{\partial \phi_1}{\partial z}} + \underset{\varepsilon^2}{\frac{\partial \phi_2}{\partial z}} + \underset{\varepsilon^2}{\eta_1 \frac{\partial^2 \phi_1}{\partial z^2}} + \mathcal{O}(\varepsilon^3)$$

In the leading power of ε^1, we have $\boxed{\frac{\partial \eta_1}{\partial t} = \frac{\partial \phi_1}{\partial z} \quad \text{at } z=0}$,

Now, we expand the terms in the dynamic condition:

From 2, $\phi_z\Big|_{z=\eta} = \frac{\partial \phi_1}{\partial z} + \frac{\partial \phi_2}{\partial z} + \eta_1 \frac{\partial^2 \phi_1}{\partial z^2} \Rightarrow \phi_z^2 = \left(\frac{\partial \phi_1}{\partial z}\right)^2 + \mathcal{O}(\varepsilon^3)$

$$\phi_x\Big|_{z=\eta} = \underset{\uparrow \, \varepsilon^1}{\frac{\partial \phi_1}{\partial x}} + \underset{\uparrow \, \varepsilon^2}{\frac{\partial \phi_2}{\partial x}} + \underset{\uparrow \, \varepsilon^2}{\eta_1 \frac{\partial^2 \phi_1}{\partial x^2}} \Rightarrow \phi_x^2 = \left(\frac{\partial \phi_1}{\partial x}\right)^2 + \mathcal{O}(\varepsilon^3)$$

Now, (1.5) becomes $\frac{\partial \phi}{\partial t} + g\eta + \frac{1}{2}(\phi_x^2 + \phi_z^2) = 0$ at $z=\eta \Rightarrow$

$$\underset{\varepsilon^1}{\frac{\partial \phi_1}{\partial t}} + \underset{\varepsilon^2}{\frac{\partial \phi_2}{\partial t}} + \underset{\varepsilon^1}{g(\eta_1} + \underset{\varepsilon^2}{\eta_2)} + \underset{\varepsilon^2}{\frac{1}{2}\left(\left(\frac{\partial \phi_1}{\partial z}\right)^2 + \left(\frac{\partial \phi_1}{\partial x}\right)^2\right)} = 0 \quad \text{at } z=0.$$

So, in ε^1, (1.5) becomes $\boxed{\frac{\partial \phi_1}{\partial t} = -g\eta_1 \quad z=0}$.

Dispension:

In sum, we have derived the following:

$$\begin{cases} \dfrac{\partial \phi_1}{\partial t} + g\eta_1 = 0 & @\ z=0 \\ \dfrac{\partial \eta_1}{\partial t} - \dfrac{\partial \phi_1}{\partial z} = 0 & @\ z=0 \end{cases} \quad 3$$

Now, plug-in the special function into one of the equations to determine do determine the form of $\eta(t)$:

$$\phi(x,z,t) = \cos(x\sqrt{\lambda})\cosh((z+h)\sqrt{\lambda})\,C(t) \Rightarrow \frac{\partial \phi}{\partial t} = \cos(x\sqrt{\lambda})\cosh((z+h)\sqrt{\lambda})\,C'(t)$$

$$\Rightarrow \frac{\partial \phi}{\partial t} + g\eta = \cos(x\sqrt{\lambda})\cosh((z+h)\sqrt{\lambda})\,C'(t) + g\eta = 0 \quad \text{at } z=0$$

$$\Rightarrow \eta(x,t) = -\underbrace{\frac{\cosh((z+h)\sqrt{\lambda})}{g}}_{\text{some constant}}\ \underbrace{\cos(x\sqrt{\lambda})}_{\text{varies with } x.}\ \underbrace{C'(t)}_{\text{varies with } t} \quad \text{at } z=0.$$

$$\Rightarrow \eta(x,t) = \cos(x\sqrt{\lambda})\,\tilde{\eta}(t)$$

(absorb the constant into $\tilde{\eta}(t)$)

✱ $\tilde{\eta}(t)$ is also a function of λ but Sultan's being lazy.

Sub $\phi(x,z,t)$ & $\eta(x,t)$ into 3:

$$0 = \frac{\partial \phi}{\partial t} + g\eta = \cos(x\sqrt{\lambda})\cosh((z+h)\sqrt{\lambda})\,C'(t) + g\cos(x\sqrt{\lambda})\,\tilde{\eta}(t) \quad \text{at } z=0$$

$$= \cos(x\sqrt{\lambda})\left(\cosh((z+h)\sqrt{\lambda})\,C'(t) + g\,\tilde{\eta}(t)\right)$$

$$\Rightarrow \tilde{\eta}(t) = -\frac{1}{g}\cosh((z+h)\sqrt{\lambda})\,C'(t) \quad \text{at } z=0.$$

$$0 = \frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial z} = \frac{\partial \tilde{\eta}}{\partial t}\cos(x\sqrt{\lambda}) - \frac{\partial \phi}{\partial z} \quad \text{at } z=0$$

$$= -\frac{1}{g}\cosh((z+h)\sqrt{\lambda})\,C''(t)\cos(x\sqrt{\lambda}) - \sqrt{\lambda}\cos(x\sqrt{\lambda})\sinh((z+h)\sqrt{\lambda})\,C(t)$$

$$= -\left(\frac{1}{g}\cosh(h\sqrt{\lambda})\,C''(t) + \sqrt{\lambda}\sinh(h\sqrt{\lambda})\,C(t)\right)\cos(x\sqrt{\lambda})$$

$$\Rightarrow C''(t)\cosh(h\sqrt{\lambda}) + C(t)\,g\sqrt{\lambda}\sinh(h\sqrt{\lambda}) = 0.$$ ✱

Finally, assume $\tilde{\eta}(t) = \tilde{\eta}(\lambda, t) = \tilde{\eta}(\lambda, 0)\, e^{-i\omega t}$ (why?)

Then, direct substitution into ☆ yields

$$\omega^2 + g\sqrt{\lambda}\tanh(h\sqrt{\lambda}) = 0,$$

Dispersion Relation.

which is the same as what Ablowitz derive.

Sanity Check: consider the condition (1.6) $\phi_z(x,z,t)\Big|_{\substack{x=0\\ z=\eta}} = \eta_t(x,t)\Big|_{x=0}$

Expand $\phi_z(x,z,t)\Big|_{\substack{x=0\\ z=\eta}}$ around $z=0$:

$$\phi_z(x,z,t)\Big|_{\substack{x=0\\ z=\eta}} = \frac{\partial}{\partial z}\left(\phi(x,z,t)\right)\Big|_{\substack{x=0\\ z=\eta}}$$

$$= \frac{\partial}{\partial z}\left(\phi\Big|_{z=0} + \eta\, \phi_z\Big|_{z=0}\right)\Big|_{x=0}$$

$$= \frac{\partial}{\partial z}\left(\{\phi_1 + \phi_2\}\Big|_{z=0} + \left[(\eta_1+\eta_2)\left(\frac{\partial}{\partial z}\phi_1 + \frac{\partial}{\partial z}\phi_2\right)\right]\Big|_{z=0}\right)\Big|_{x=0}$$

$$= \frac{\partial}{\partial z}\phi_1 + \frac{\partial}{\partial z}\phi_2 + (\eta_1+\eta_2)\left(\frac{\partial^2}{\partial z^2}\phi_1 + \frac{\partial^2}{\partial z^2}\phi_2\right)\Big|_{\substack{z=0\\ x=0}}$$

$$= \frac{\partial}{\partial z}\phi_1 + \frac{\partial}{\partial z}\phi_2 + \eta_1\frac{\partial^2\phi_1}{\partial z^2} + \mathcal{O}(\varepsilon^3)\Big|_{z=0,\, x=0}$$

Similarly $\eta_t(x,t)\Big|_{x=0} = \frac{\partial}{\partial t}(\eta_1+\eta_2)\Big|_{x=0} = \frac{\partial}{\partial t}\eta_1 + \frac{\partial}{\partial t}\eta_2\Big|_{x=0}$

Thus, $\phi_z(x,z,t)\Big|_{\substack{x=0\\ z=\eta}} = \eta_t(x,t)\Big|_{x=0} \Rightarrow \underbrace{\frac{\partial}{\partial z}\phi_1}_{\varepsilon^1} + \underbrace{\frac{\partial}{\partial z}\phi_2}_{\varepsilon^2} + \underbrace{\eta_1\frac{\partial^2\phi_1}{\partial z^2}}_{\varepsilon^1\cdot\,\varepsilon^1}\Big|_{z=0,\, x=0} = \underbrace{\frac{\partial}{\partial t}\eta_1}_{\varepsilon^1} + \underbrace{\frac{\partial}{\partial t}\eta_2}_{\varepsilon^2}\Big|_{x=0}$

$$\Rightarrow \varepsilon^1:\ \frac{\partial}{\partial z}\phi_1 = \frac{\partial}{\partial t}\eta_1 \quad \text{at } (x,z) = (0,0)$$

Note that the equation we derived is the same as before, except that we have evaluated at $x=0$.

Nondimensionalisation:

Recall the problem:

$$\phi_{xx} + \phi_{zz} = 0 \qquad -h < z < \eta(x,t) \qquad (1.1)$$

$$\phi_z = 0 \qquad z = -h \qquad (1.2)$$

$$\phi_x = 0 \qquad x = 0 \qquad (1.3)$$

$$\eta_t + \phi_x \eta_x = \phi_z \qquad z = \eta(x,t) \qquad (1.4)$$

$$\phi_t + g\eta + \frac{1}{2}(\phi_x^2 + \phi_z^2) = 0 \qquad z = \eta(x,t) \qquad (1.5)$$

$$\phi_z(0,\eta) = \eta_t(0,t) \qquad x = 0. \qquad (1.6)$$

Introduce the nondimensional variables as in Prof. Bernard's book:

$$z^* = z/h \qquad t^* = \sqrt{\frac{\varepsilon g}{h}}\, t \qquad \phi = h\sqrt{\varepsilon g h}\, \phi^*$$

$$x^* = \sqrt{\varepsilon}\, \frac{x}{h} \qquad \eta = \varepsilon h \eta^*$$

Then, equations (1.1) - (1.5) transform as before, and (1.6) becomes:

$$x = 0 \Rightarrow x^\circ = 0 \qquad z = \eta \Rightarrow z^\circ = \frac{z}{h} = \frac{1}{h}\eta = \frac{\varepsilon h}{h}\eta^\circ = \varepsilon\eta^\circ$$

$$\phi(x,z,t) = h\sqrt{\varepsilon g h}\, \phi^\circ(x^\circ, z^\circ, t^\circ) \Rightarrow \phi_z = h\sqrt{\varepsilon g h}\, \phi^\circ_{z^\circ}(x^\circ, z^\circ, t^\circ)\, \frac{\partial z^\circ}{\partial z}$$

$$\Rightarrow \phi_z = \sqrt{\varepsilon g h}\, \phi^\circ_{z^\circ}(x^\circ, z^\circ, t^\circ)$$

$$\text{at } x = 0,\ z = \eta \qquad \Rightarrow \phi_z(0,\eta,t) = \sqrt{\varepsilon g h}\, \phi^\circ_{z^\circ}(0, \varepsilon\eta^\circ, t^\circ)$$

$$\eta(x,t) = \varepsilon h \eta^\circ(x^\circ, t^\circ) \Rightarrow \eta_t(x,t) = \varepsilon h\, \eta^\circ_{t^\circ}(x^\circ, t^\circ)\, \frac{\partial t^\circ}{\partial t} = \varepsilon h \sqrt{\frac{\varepsilon g}{h}}\, \eta^\circ_{t^\circ}(x^\circ, t^\circ)$$

$$\Rightarrow \eta_t(0,t) = \varepsilon\sqrt{\varepsilon h g}\, \eta^\circ_{t^\circ}(x^\circ, t^\circ)$$

$\phi_z(0,y) = \eta_t(0,t)$ @ $x=0$. $\Rightarrow \sqrt{\varepsilon g h}\, \phi^\circ_{z^\circ}(0, \varepsilon\eta^\circ, t^\circ) = \varepsilon\sqrt{\varepsilon h g}\, \eta^\circ_{t^\circ}(x^\circ, t^\circ)$

$\Rightarrow \phi^\circ_{z^\circ}(0, \varepsilon\eta^\circ, t^\circ) = \varepsilon\, \eta^\circ_{t^\circ}(x^\circ, t^\circ)$

$\Rightarrow \phi_z(0, \varepsilon\eta, t) = \varepsilon\, \eta_t(x,t)$, $x=0$.

Thus, the nondimensional equations are:

$\phi_{zz} + \varepsilon\phi_{xx} = 0$ $\qquad -1 < z < \varepsilon\eta$

$\phi_z = 0$ $\qquad z = -1$

$\phi_x = 0$ $\qquad x = 0$

$\varepsilon\eta_t + \varepsilon^2\phi_x\eta_x = \phi_z$ $\qquad z = \varepsilon\eta$

$\phi_t + \eta + \frac{1}{2}\phi_z^2 + \frac{\varepsilon}{2}\phi_x^2 = 0$ $\qquad z = \varepsilon\eta$

$\phi_z(0, \varepsilon\eta, t) = \varepsilon\,\eta_t(x,t)$ $\qquad x=0,\ z = \varepsilon\eta$

Determining the dependence on z.

Suppose $\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots$, ϕ_i is a function of x, y, t.

$$\phi_{zz} + \varepsilon\phi_{xx} = \phi_{0zz} + \varepsilon\phi_{1zz} + \varepsilon^2\phi_{2zz} + \varepsilon(\phi_{0xx} + \varepsilon\phi_{1xx} + \varepsilon^2\phi_{2xx}) + \dots$$

$$= \phi_{0zz} + \varepsilon(\phi_{1zz} + \phi_{0xx}) + \varepsilon^2(\phi_{2zz} + \phi_{1xx}) + \dots = 0$$

$$\Rightarrow \begin{cases} \varepsilon^0: & \phi_{0zz} = 0 & (1) \\ \varepsilon^1: & \phi_{1zz} + \phi_{0xx} = 0 & (2) \\ \varepsilon^2: & \phi_{2zz} + \phi_{1xx} = 0 & (3) \end{cases}$$

must find ϕ_i s.t.

$(\phi_i)_z = 0$, $z = -1$.

$(\phi_i)_x = 0$, $x = 0$.

(1) $\phi_{0zz} = 0 \Rightarrow \phi_0 = A(x,t)$ s.t. $\partial_x \phi_0(x,t)\big|_{x=0} = \partial_x A(x,t)\big|_{x=0} = 0$

(2) $\phi_{1zz} + \phi_{0xx} = 0 \Rightarrow \phi_1 = -\partial_x^2 A(x,t)\frac{(z+1)^2}{2}$ s.t. $\partial_x^3 A(x,t)\big|_{x=0} = 0$

(3) $\phi_{2zz} + \phi_{1xx} = 0 \Rightarrow \phi_2 = \partial_x^4 A(x,t)\frac{(z+1)^4}{4!}$ s.t. $\partial_x^5 A(x,t)\big|_{x=0} = 0$

The solution therefore has the following expansion

$$\phi(x,z,t) = \phi_0(x,t) - \varepsilon \frac{(z+1)^2}{2!}\phi_{0xx} + \varepsilon^2 \frac{(z+1)^4}{4!}\phi_{0xxxx} - \dots = \sum_{k=0}^{\infty} (-1)^k \varepsilon^k \frac{(z+1)^{2k}}{(2k)!} \partial_x^{2k} \phi_0(x,t) \quad (\star)$$

In addition, we also require that $\partial_x^n \phi_0(x,t)\big|_{x=0} = 0$ for n odd.

Substituting $(\star)$ into kinematic & dynamic conditions, & keeping the terms in the lowest power of ε, we obtain the following system of equations:

$$\left.\begin{aligned} &\eta_t + \partial_x \phi_{0x} = \mathcal{O}(\varepsilon) \\ &(\phi_{0x})_t + \partial_x \eta = \mathcal{O}(\varepsilon) \end{aligned}\right\} \text{ at } z = \varepsilon\eta$$

$$\left.\begin{aligned} &\eta_t + \partial_x \phi_{0x} = \mathcal{O}(\varepsilon) \\ &\phi_{0x} = 0 \text{ at } x = 0 \end{aligned}\right\} \text{ at } (x,z) = (0, \varepsilon\eta)$$

Next, perform perturbation expansion: $\begin{aligned} \phi_{0x} &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots \\ \eta &= \eta_0 + \varepsilon \eta_1 + \varepsilon^2 \eta_2 + \dots \end{aligned}$ $\triangle$ where u_i, η_i are sufficiently nice.

Introduce the slow-time scales: $\tau_0 = t$, $\tau_1 = \varepsilon t$, $\tau_2 = \varepsilon^2 t$ $\Rightarrow$ $\partial_t = \partial_{\tau_0} + \varepsilon \partial_{\tau_1} + \varepsilon^2 \partial_{\tau_2} + \dots$

Lowest Order Results:

Substituting the expansion $\triangle$ into the above system yields

$\eta_t + \partial_x \phi_{0x} = \mathcal{O}(\varepsilon)$	$z = \varepsilon\eta$	$\Rightarrow$	$\eta_{0\tau_0} + u_{0x} = 0$	at $z = \varepsilon\eta$	(1)
$(\phi_{0x})_t + \partial_x \eta = \mathcal{O}(\varepsilon)$	$z = \varepsilon\eta$	$\Rightarrow$	$u_{0\tau_0} + \eta_{0x} = 0$	at $z = \varepsilon\eta$	(2)
$\eta_t + \partial_x \phi_{0x} = \mathcal{O}(\varepsilon)$	$x = 0,\ z = \varepsilon\eta$	$\Rightarrow$	$\eta_{0\tau_0} + u_{0x} = 0$	at $x = 0,\ z = \varepsilon\eta$	(3)
$\phi_{0x} = 0$	$x = 0$	$\Rightarrow$	$u_0 = 0$	at $x = 0$	(4)

Note (4) doesn't yield any constraints; combining (1) & (2) gives two wave equations

$$\frac{\partial^2 \eta_0}{\partial \tau_0^2} - \frac{\partial^2 \eta_0}{\partial x^2} = 0 \qquad \frac{\partial^2 u_0}{\partial \tau_0^2} - \frac{\partial^2 u_0}{\partial x^2} = 0 \qquad x>0, \quad z=\varepsilon\eta$$

The general solutions are

$$\eta_0 = F(x-\tau_0,\tau_1,\dots) + G(x+\tau_0,\tau_1,\dots) =: F(x-\tau_0) + G(x+\tau_0)$$

$$u_0 = F(x-\tau_0,\tau_1,\dots) - G(x+\tau_0,\tau_1,\dots) =: F(x-\tau_0) - G(x+\tau_0)$$

Since $x>0$, the domain of u_0, η_0 is $[0,\infty)$, so F & G must also be defined on $[0,\infty)$. Thus, whenever $x<\tau_0$, the solutions above lose their validity.

Suppose $u_0 = \tilde{F}(x-\tau_0) - G(x+\tau_0)$, where $\tilde{F}$ is unknown. Note $u_0(0,\tau_0) = \tilde{F}(-\tau_0) - G(\tau_0) = 0 \Rightarrow \tilde{F}(-\tau_0) = G(\tau_0)$.
Thus, if we let $F(z) = G(-z)$ $\forall z<0$, we have a valid representation $u_0 = G(\tau_0 - x) - G(x+\tau_0)$, if $x<\tau_0$.
This means that u_0 is piecewise:

$$u_0 = \begin{cases} F(x-\tau_0) + G(x+\tau_0) & x>\tau_0 \\ G(\tau_0 - x) - G(x+\tau_0) & x<\tau_0 \end{cases}$$

Substituting u_0 into $\begin{bmatrix} \eta_{0\tau_0} + u_{0x} = 0 \\ u_{0\tau_0} + \eta_{0x} = 0 \end{bmatrix}$ means: $\eta_0 = F(x-\tau_0) + G(x+\tau_0)$, $x>\tau_0$ and for $x<\tau_0$,

$$u_{0x} = -G'(\tau_0 - x) - G'(x+\tau_0) \Rightarrow \eta_{0\tau_0} = G'(\tau_0 - x) + G'(x+\tau_0) \Rightarrow \eta_0 = G(\tau_0 - x) + G(x+\tau_0)$$

$$u_{0\tau_0} = +G'(\tau_0 - x) - G'(x+\tau_0) \Rightarrow \eta_{0x} = -G'(\tau_0 - x) + G'(x+\tau_0) \Rightarrow \eta_0 = G(\tau_0 - x) + G(x+\tau_0)$$

so that

$$\frac{\partial^2 \eta_0}{\partial \tau_0^2} - \frac{\partial^2 \eta_0}{\partial x^2} = \frac{\partial^2}{\partial \tau_0^2} G(\tau_0 - x) + G(x+\tau_0) - \frac{\partial^2}{\partial x^2} G(\tau_0 - x) + G(x+\tau_0)$$

$$= G''(\tau_0 - x) + G''(x+\tau_0) - G''(\tau_0 - x) - G''(x+\tau_0) = 0 .$$

Thus, the general solutions are:

$$u_0 = \begin{cases} F(x-\tau_0) + G(x+\tau_0) & x>\tau_0 \\ G(\tau_0 - x) - G(x+\tau_0) & x<\tau_0 \end{cases} \qquad \eta_0 = \begin{cases} F(x-\tau_0) + G(x+\tau_0) & x>\tau_0 \\ G(\tau_0 - x) + G(x+\tau_0) & x<\tau_0 \end{cases}$$

Higher Order Results:

Following Prof. Bernard's procedure on page 71-72, dynamic & kinematic conditions yield the above wave equations in the lowest order, & the following system in the second lowest order:

$$\begin{cases} \eta_{1\tau_0} + u_{1x} = -\left(\eta_{0\tau_1} + u_0\,\eta_{0x} + \eta_0\,u_{0x} - \frac{1}{6}\,u_{0xxx}\right) & (5) \\ u_{0\tau_1} + \eta_{1x} = -\left(u_{0\tau_1} - \frac{1}{2}\,u_{0xx\tau_0} + u_0\,u_{0x}\right) & (6) \end{cases}$$

Also, recall the conditions $\phi_{0x} = 0$ at $x=0$, and $\phi_z(0, \varepsilon\eta, t) = \varepsilon\,\eta_t(x,t)$ at $x=0$, $z=\varepsilon\eta$. Clearly, $\phi_{0x} = 0 \Rightarrow u_0 = u_1 = 0$ at $x=0$. Also, from page 71, we have

$$\phi_z(x, \varepsilon\eta, t)\Big|_{x=0} = -\varepsilon u_{0x} + \varepsilon^2\left(-\eta_0 u_{0x} - u_{1x} + \frac{1}{6}u_{0xxx}\right) + O(\varepsilon^3)\Big|_{x=0}$$

and

$$\varepsilon\eta_t(x,t)\Big|_{x=0} = \varepsilon\,\eta_{0\tau_0} + \varepsilon^2\left(\eta_{1\tau_0} + \eta_{0\tau_1}\right) + O(\varepsilon^3)\Big|_{x=0},$$

so that $\phi_z(x, \varepsilon\eta, t) = \varepsilon\eta_t(x,t)$ at $x=0$ $\Leftrightarrow$

$$-\varepsilon u_{0x} + \varepsilon^2\left(-\eta_0 u_{0x} - u_{1x} + \frac{1}{6}u_{0xxx}\right) + O(\varepsilon^3)$$
$$= \varepsilon\,\eta_{0\tau_0} + \varepsilon^2\left(\eta_{1\tau_0} + \eta_{0\tau_1}\right) + O(\varepsilon^3) \quad \text{at } x=0 \quad \Leftrightarrow$$

$$-u_{0x} + \varepsilon\left(-\eta_0 u_{0x} - u_{1x} + \frac{1}{6}u_{0xxx}\right) + O(\varepsilon^2)$$
$$= \eta_{0\tau_0} + \varepsilon\left(\eta_{1\tau_0} + \eta_{0\tau_1}\right) + O(\varepsilon^2) \quad \text{at } x=0 \quad \Leftrightarrow$$

$$\begin{cases} \varepsilon^0: & \eta_{0\tau_0} + u_{0x} = 0 & \text{at } x=0 \\ \varepsilon^1: & u_{1x} + \eta_{1\tau_0} = -\left(\eta_{0\tau_1} + \eta_0 u_{0x} - \frac{1}{6}u_{0xxx}\right) & \text{at } x=0 \end{cases}$$

Note that the equation in ε^0 is the same as (1), and the equation in ε^1 is the same as (5), since at $x=0$, we require that $u_0 = 0$. Thus, the boundary condition at $x=0$ is compatible with the original kinematic condition.

Characteristic Variables

Now, consider eq-ns (5) + (6). Let $L = x+\tau_0$, $r = x-\tau_0$, so that $\partial_x = \partial_r + \partial_L$, $\partial_{\tau_0} = \partial_L - \partial_r$, so that

$$u_0 = \begin{cases} F(r) + G(L) & x > \tau_0 \\ G(-r) - G(L) =: G(-r) - G & x < \tau_0 \end{cases} \qquad \eta_0 = \begin{cases} F(r) + G(L) & x > \tau_0 \\ G(-r) + G(L) =: G(-r) + G & x < \tau_0 \end{cases}$$

where on the last line, we let G be a function of L, for the ease of notation. Also, keep in mind that F, G are functions in $\tau_0, \tau_1, \tau_2, \dots$ etc.

The transformed equations are, in the case $x < \tau_0$:

$$\eta_{1L} - \eta_{1r} + u_{1r} + u_{1L} = -\Big(G(-r)_{\tau_1} + G_{\tau_1} + (G(-r) - G)(-G(-r)_r + G_L) - (G(-r) + G)(G(-r)_r + G_L) + \tfrac{1}{6}(G(-r)_{rrr} + G_{LLL})\Big)$$
$$= -\Big(G(-r)_{\tau_1} + G_{\tau_1} - 2G(-r)G(-r)_r - 2GG_L + \tfrac{1}{6}G(-r)_{rrr} + \tfrac{1}{6}G_{LLL}\Big)$$

$$u_{1L} - u_{1r} + \eta_{1r} + \eta_{1L} = -\Big(G(-r)_{\tau_1} - G_{\tau_1} - \tfrac{1}{2}(G(-r)_{rrr} - G_{LLL}) - (G(-r) - G)(G(-r)_r + G_L)\Big)$$
$$= -\Big(G(-r)_{\tau_1} - G_{\tau_1} - \tfrac{1}{2}G(-r)_{rrr} + \tfrac{1}{2}G_{LLL} - G(-r)G(-r)_r - G(-r)G_L + GG(-r)_r + GG_L\Big)$$

Add & Subtract:

$$2(\eta_{1L} + u_{1L}) = -\Big(2G(-r)_{\tau_1} - 3G(-r)G(-r)_r - \tfrac{1}{3}G(-r)_{rrr}\Big) + GG_L + G\,G(-r)_r - G(-r)G_L - \tfrac{2}{3}G_{LLL}$$

$$2(u_{1r} - \eta_{1r}) = -\Big(2G_{\tau_1} - 3GG_L - \tfrac{1}{3}G_{LLL}\Big) + G(-r)G(-r)_r - G(-r)G_L + GG(-r)_r + \tfrac{2}{3}G(-r)_{rrr}$$

Integrate the first w.r.t. L & the second w.r.t. r:

$$2(\eta_1 + u_1) = -\Big(2G(-r)_{\tau_1} - 3G(-r)G(-r)_r - \tfrac{1}{3}G(-r)_{rrr}\Big)L$$
$$+ \tfrac{1}{2}G^2 + G(-r)_r \int G\, dL - G(-r)G - \tfrac{2}{3}G_{LL} + C_1$$

KdV-like terms

Additional Secular Terms

$$2(u_1 - \eta_1) = -\Big(2G_{\tau_1} - 3GG_L - \tfrac{1}{3}G_{LLL}\Big)r$$
$$-(G(-r))^2 - G_L \int G(-r)\, dr - GG(-r) - \tfrac{2}{3}G(-r)_{rr} + C_2$$