Appendix D

Answers and hints

The answer, where one is given, is designated by the prefix A; for example, the answer to Q1.1 is A1.1. In some cases a hint to the method of solution is included.

Chapter 1

A1.1 Use a subscript notation, and so consider

(a) $\dfrac{\partial}{\partial x_i}(\phi u_i)$; (b) $\varepsilon_{ijk}\dfrac{\partial}{\partial x_j}(\phi u_k)$; (c) $\varepsilon_{ijk}u_j\left(\varepsilon_{klm}\dfrac{\partial u_m}{\partial x_l}\right)$;

(d) $\varepsilon_{ijk}\dfrac{\partial}{\partial x_j}(\varepsilon_{klm}u_l v_m)$.

In (c) and (d) use $\varepsilon_{ijk}\varepsilon_{klm} = \varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$.

A1.2 In $\int_V \nabla\cdot\mathbf{a}\,\mathrm{d}v = \int_S \mathbf{a}\cdot\mathbf{n}\,\mathrm{d}s$ write $\mathbf{a} = \phi\mathbf{c}$; $\mathbf{a} = \mathbf{u}\wedge\mathbf{c}$, where in each $\mathbf{c}$ is an *arbitrary* constant vector.

A1.3 Consider $\int_S A_i\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}s = \int_S (A_i\mathbf{u})\cdot\mathbf{n}\,\mathrm{d}s$ for each i.

A1.4 Write

$$\frac{\mathrm{d}U_i}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{u_i(x_1(t), x_2(t), x_3(t), t)\right\} = \dot{x}_j\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial t}.$$

A1.5 Find $\partial F/\partial t + \mathbf{u}\cdot\nabla F$ directly; note that, for both, $\nabla\cdot\mathbf{u} = 0$.

A1.6 (a) Find $\mathbf{u} = \mathrm{d}\mathbf{x}/\mathrm{d}t$ and introduce $\mathbf{x} \equiv (x, y, z)$.

(b) Find $\mathrm{d}\mathbf{u}/\mathrm{d}t \equiv (4x + 16t^2x,\ -2y + 4t^2y,\ -2z + 4t^2z)$.

(c) Find $\partial\mathbf{u}/\partial t \equiv (4x, -2y, -2z)$.

(d) Follows directly.

A1.8 $\mathbf{u} \equiv (\alpha x, \beta y, \gamma z)$; $\nabla\cdot\mathbf{u} = \alpha + \beta + \gamma = 0$.

A1.9 $\nabla\cdot\mathbf{u} = 3f + rf'$ so $f(r) = A/r^3$ (A is an arbitrary constant).

A1.10 From

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla P + \mathbf{F}, \quad \nabla P \equiv \rho(-x - xt^2, -y - yt^2, 2z - 4zt^2 - g)$$

$$\text{so } P = -\frac{1}{2}\rho(x^2 + y^2)(1 + t^2) + \rho z^2(1 - 2t^2) - \rho g z + P_0(t)$$

A1.11 From $(1/\rho)\nabla P \equiv (0, 0, -g)$, then $P = P_a + \rho g(h_0 - z)$; $P(0) = P_a + \rho g h_0$.

A1.12 Stokes' Theorem gives

$$\oint_C \mathbf{u} \cdot \mathrm{d}\boldsymbol{\ell} = \int_S (\nabla \wedge \mathbf{u}) \cdot \mathbf{n}\mathrm{d}s \approx (\boldsymbol{\omega} \cdot \mathbf{n})\pi a^2 \text{ so } \boldsymbol{\Omega} \cdot \mathbf{n} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{n};$$

but $\mathbf{n}$ is arbitrary so $\boldsymbol{\Omega} = \frac{1}{2}\boldsymbol{\omega}$.

NB $$\oint_C \mathbf{u} \cdot \mathrm{d}\boldsymbol{\ell} = \oint_C \mathbf{U} \cdot \mathrm{d}\boldsymbol{\ell} + \oint_C (\boldsymbol{\Omega} \wedge \mathbf{r}) \cdot \mathrm{d}\boldsymbol{\ell} = \boldsymbol{\Omega} \cdot \oint_C \mathbf{r} \wedge \mathrm{d}\boldsymbol{\ell} = \boldsymbol{\Omega} \cdot \mathbf{n} \int_0^{2\pi} a^2 \mathrm{d}\theta.$$

A1.13 $\boldsymbol{\omega} \equiv (0, U'(z), (0)$; $\boldsymbol{\omega} \equiv (0, 0, -U'(y))$.

A1.14 Use $\mathbf{u} \wedge \boldsymbol{\omega} = \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\mathbf{u}$ and

$$\frac{1}{\rho}\nabla P = \nabla\left(\int \frac{\mathrm{d}P}{\rho}\right)$$

to give

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \wedge \boldsymbol{\omega} = -\nabla\left(\frac{1}{2}\mathbf{u} \cdot \mathbf{u} + \int \frac{\mathrm{d}P}{\rho} + \Omega\right);$$

$$\text{curl:} \frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \wedge (\mathbf{u} \wedge \boldsymbol{\omega}) = \mathbf{0}$$

and use Q1.1 (d) with $\nabla \cdot \mathbf{u} = 0$, $\nabla \cdot \boldsymbol{\omega} = 0$. In 2D, $\boldsymbol{\omega}$ is orthogonal to ∇ so $\boldsymbol{\omega} \cdot \nabla \equiv 0$; then $\mathrm{D}\boldsymbol{\omega}/\mathrm{D}t = \mathbf{0}$.

A1.15 As for A1.14, but with

$$\nabla \wedge \left(\frac{1}{\rho}\nabla P\right) = \frac{1}{\rho}\nabla \wedge (\nabla P) + \nabla(\rho^{-1}) \wedge (\nabla P),$$

and multiply by ρ^{-1}. For $P = P(\rho)$, then $\nabla(\rho^{-1}) \wedge (\nabla P) = \mathbf{0}$.

A1.16 $\boldsymbol{\omega} \equiv (u/r,\ -\theta u',\ u')$.

A1.17 $$\boldsymbol{\omega} \equiv \begin{cases} (0, 0, \omega), & 0 \le r < a; \\ \mathbf{0} \quad , & r > a \end{cases}$$

$$\frac{P}{\rho}+\Omega=\begin{cases} P_0/\rho+\frac{1}{8}\omega^2(r^2-2a^2), & 0\le r\le a\\ P_0/\rho-\frac{1}{8}\omega^2\frac{a^4}{r^2}, & r>a.\end{cases}$$

Must have

$$P_0>\frac{\rho}{8}\omega^2a^2.$$

A1.18 Consider

$$\frac{1}{\rho}\frac{\partial P}{\partial x_i}=\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}\rho}\frac{\partial\rho}{\partial x_i}=\frac{\partial}{\partial x_i}\left(\int\frac{\mathrm{d}P}{\rho}\right).$$

A1.19 (a) $\mathbf{x}\equiv(x_0\mathrm{e}^{ct}, y_0\mathrm{e}^{-ct}, z_0)$, $xy=$ constant;
(b) $\mathbf{x}\equiv(x_0\exp(t^2), y_0\exp(-t^2), z_0)$, $xy=$ constant;
(c) $\mathbf{x}\equiv(1+t+(x_0-1)\mathrm{e}^t,\ y_0\mathrm{e}^{-t}, z_0)$, $y(x-t)=$ constant (at fixed t);

(d) $\mathbf{x}\equiv\left(\dfrac{x_0}{1-cx_0t},\ \dfrac{y_0}{1-cy_0t},\ z_0(1-cx_0t)^2(1-cy_0t)^2\right),$

$y=x+Axy$
with $(1-Ax)^2=Bzx^4$ where A, B are arbitrary constants.

A1.20 (a) $\psi=cxy$; (b) $\psi=2xyt$; (c) $\psi=(x-t)y$.

A1.21 $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{v}{u}=-\psi_x/\psi_y$ so $\dfrac{\mathrm{d}}{\mathrm{d}x}\{\psi(x,y(x))\}=0$; $\psi=$constant.

A1.22 Write

$$u=\frac{1}{r}\psi_\theta,\ v=-\psi_r;\ \psi=rU\sin\theta.$$

A1.23 (a) $u=\dfrac{1}{r}\psi_z$, $w=-\dfrac{1}{r}\psi_r$; (b) $v=r\psi_z$, $w=-\psi_\theta$.

A1.24 (a) Write $u_k=b_ka_jx_j+a_kb_jx_j$ and form $\varepsilon_{ijk}\partial u_k/\partial x_j$. Then $u_k=\partial\phi/\partial x_k$ yields

$$\phi=(\mathbf{a}\cdot\mathbf{x})(\mathbf{b}\cdot\mathbf{x})\ (+\text{constant}).$$

(b) $\phi=\dfrac{yz}{x^2+y^2}$ (+ constant).

A1.25 $\nabla\cdot\mathbf{u}=0$ yields $u=\psi_y$, $v=-\psi_x$; $\nabla\wedge\mathbf{u}=\mathbf{0}$ so $\mathbf{u}=\nabla\phi$, then $u=\phi_x$, $v=\phi_y$: $\phi_x=\psi_y$, $\phi_y=-\psi_x$. Thus $\phi+\mathrm{i}\psi=w(z)$; then

$$\frac{\partial}{\partial x}w=\frac{\partial}{\partial x}(\phi+i\psi)\text{ gives }\frac{\mathrm{d}w}{\mathrm{d}z}=u-iv.$$

Here $w = Ue^{i\alpha} = u - iv$, a uniform flow of speed $U(t)$ at $-\alpha$ to the x-axis.

A1.27 Use $\mathbf{u} = \dfrac{d\mathbf{x}}{dt} \equiv \left(\dfrac{d\mathbf{x}_\perp}{dt}, \dfrac{dz}{dt}\right) = (\mathbf{u}_\perp, w)$.

A1.28 With $P_s - P_a = \Gamma h'' = (1 + h'^2)^{3/2}$ where $P = P_s$ on $z = h(x)$ and $P_s = P_b - \rho g h$ ($P = P_b(x)$ on $z = 0$).
For equilibrium, P_b = constant. Thus

$$\Gamma h'' = (P_b - P_a - \rho g h)(1 + h'^2)^{3/2}, -x_0 \le x \le x_0.$$

A1.29 Similar to A1.28:

$$P_b - P_a - \rho g h = \Gamma\left\{\frac{h''}{(1 + h'^2)^{3/2}} + \frac{h'}{r(1 + h'^2)^{1/2}}\right\},$$

and then for $\varepsilon \to 0$:

$$H'' + H'/R = \beta - \alpha H, \quad \beta = r_0^2(P_b - P_a)/\Gamma h_0.$$

Hence $H = \beta/\alpha + AJ_0(\sqrt{\alpha}R)$ so

$$A + \beta/\alpha = 1; \quad \beta/\alpha + (1 - \beta/\alpha)\, J_0(\sqrt{\alpha}) = 0;$$
$$H'(1) = (\sqrt{\alpha} - \beta/\sqrt{\alpha})J_0'(\sqrt{\alpha}).$$

This solution requires $0 < \beta/\alpha < 1$ with $J_0(\sqrt{\alpha}) < 0$, $J_0'(\sqrt{\alpha}) < 0$ which gives $\alpha_0 < \alpha < \alpha_1$.

A1.31 (a) $I'(x) = x^{-1}\{3\exp(x^3) - 2\exp(x^2)\}$; (b) $n = 4$.

A1.32 Euler's equation leads to

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}\right) + \rho\mathbf{u}\cdot\{(\mathbf{u}\cdot\nabla)\mathbf{u}\} = -\mathbf{u}\cdot\nabla(P + \rho\Omega)$$

with

$$\rho\mathbf{u}\cdot\{(\mathbf{u}\cdot\nabla)\mathbf{u}\} = \nabla\cdot\left\{\frac{1}{2}(\rho\mathbf{u}\cdot\mathbf{u})\mathbf{u}\right\},$$
$$\mathbf{u}\cdot\nabla(P + \rho\Omega) = \nabla\cdot\{(P + \rho\Omega)\mathbf{u}\}$$

since $\nabla\cdot\mathbf{u} = 0$.

A1.33 (a) Write $T = \frac{1}{2}\int_V \rho\mathbf{u}\cdot\nabla\phi dv = \frac{1}{2}\int_V \nabla\cdot(\rho\phi\mathbf{u})dv$ since $\nabla\cdot\mathbf{u} = 0$.
(b) Since the conditions on S are given, either $\Phi = 0$ or $\mathbf{U} = \mathbf{0}$ on S. Thus

$$\int_V |\mathbf{U}|^2 dv = 0 \Rightarrow \mathbf{U} = \mathbf{0} \text{ in V}; \text{ that is, } \mathbf{u}_1 \equiv \mathbf{u}_2.$$

A1.34 $\psi \to ch\psi$; $\phi \to c\lambda\phi$; $w \to \dfrac{ch}{\lambda} w$; $w = \phi_z \to w = \left(\dfrac{\lambda}{h}\right)^2 \phi_z$.

A1.35 The Reynolds number is $\rho\lambda\sqrt{gh_0}/\mu$.

A1.36
$$p - \varepsilon\eta = -\varepsilon\delta^2 W\Big\{[(1+\varepsilon^2\delta^2\eta_\theta^2/r^2)\eta_{rr} + (1+\varepsilon^2\delta^2\eta_r^2)(r\eta_r + \eta_{\theta\theta})/r^2 - 2\varepsilon^2\delta^2\left(\eta_{r\theta} - \frac{1}{r}\eta_\theta\right)\eta_r\eta_\theta/r^2]/(1+\varepsilon^2\delta^2\eta_r^2 + \varepsilon^2\delta^2\eta_\theta^2/r^2)^{3/2}\Big\}$$

and then $p \to \varepsilon p$ with $\varepsilon \to 0$ yields

$$p - \eta = -\delta^2 W\left(\eta_{rr} + \frac{1}{r}\eta_r + \frac{1}{r^2}\eta_{\theta\theta}\right).$$

A1.37 $\phi_t + \eta + \frac{1}{2}\varepsilon(u^2 + v^2 + \delta^2 w^2) = 0$.

A1.38 $\phi_{zz} + \delta^2\nabla_\perp^2\phi = 0$; $\phi_z = \delta^2\{\eta_t + \varepsilon(\mathbf{u}_\perp \cdot \nabla_\perp)\eta\}$ on $z = 1 + \varepsilon\eta$;

$$\phi_t + \eta + \frac{1}{2}\varepsilon\left\{(\nabla_\perp\phi)^2 + \frac{1}{\delta^2}\phi_z^2\right\} = 0 \text{ on } z = 1 + \varepsilon\eta;$$

$$\phi_z = \delta^2(\mathbf{u}_\perp \cdot \nabla)b \text{ on } z = b. \text{ NB } \mathbf{u}_\perp = \nabla_\perp\phi.$$

For $\varepsilon \to 0 : \phi_{zz} + \delta^2\nabla_\perp^2\phi = 0$; $\phi_z = \delta^2\eta_t$ and $\phi_t + \eta = 0$ (or $p = \eta$) on $z = 1$; $\phi_z = \delta^2(\mathbf{u}_\perp \cdot \nabla)b$ on $z = b$. With surface tension; $p - \eta = -\delta^2 W\nabla_\perp^2\eta$ on $z = 1$.

A1.39 $u = f(x - ct) + g(x + ct)$;

$$u = \tfrac{1}{2}\{p(x - ct) + p(x + ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} q(y)\,\mathrm{d}y.$$

A1.40 The right- and left-going waves no longer overlap.

A1.41 Dispersion relation is $\omega = k - k^3 - \mathrm{i}k^2$, which is dispersive ($\mathscr{R}(\omega/k) = 1 - k^2$) and dissipative (decaying as $\exp(-k^2 t)$).

A1.42 First equation gives $\omega = k - k^3$; second gives $\omega = k/(1+k^2)$ so $\omega = k - k^3 + \mathrm{O}(k^5)$ as $k \to 0$ (long waves) but $\omega \sim 1/k$ as $k \to \infty$; note that $\omega > 0$ for $\forall k > 0$ in the second case, but not in the first.

A1.43

$$u(x,t) = \begin{cases} \alpha(x-t)/(1+\alpha t), & 0 \le (x-t)/(1+\alpha t) \le 1, \\ \alpha(2 - x + t)/(1 - \alpha t), & 1 \le (x - t - 2\alpha t)/(1 - \alpha t) \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

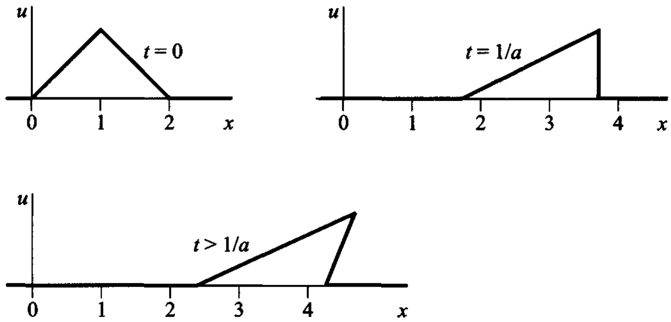

Figure A.1.

A1.44 $u = \cos\{\pi(x - ut)\}$ for $u(x, t)$;
$u_x = -\pi \sin\{\pi(x - ut)\}/[1 - \pi t \sin\{\pi(x - ut)\}]$.
The solution becomes multi-valued for $t > \pi^{-1}$.

A1.45 (a) $x = \mathrm{O}(1)$: $f \sim 1 + \varepsilon(x^{-1} - x)$;

$$x = \varepsilon X: f \sim \left(1 - \frac{1}{1+X} + \mathrm{e}^{-X}\right)^{-1};$$

$x = \chi/\varepsilon$: $f \sim (1 + \chi)^{-1}$.
The first and second match: $1 + 1/x$; the first and third match: $1 - \chi$.

(b) $x = \mathrm{O}(1)$: $f \sim 1 - \frac{1}{2}\varepsilon x + \varepsilon^2(\frac{3}{8}x^2 - \frac{1}{2}x^4)$;
$x = \varepsilon^{-1/3}X$: $f \sim 1 - \frac{1}{2}\varepsilon^{2/3}(X + X^4)$;
$X = \varepsilon^{-1/6}\chi$: $f \sim (1 + \chi^4)^{-1/2} - \frac{1}{2}\varepsilon^{1/2}\chi(1 + \chi^4)^{-3/2}$.
The first and second match: $1 - \frac{1}{2}\varepsilon^{2/3}(X + X^4)$; the second and third match: $1 - \frac{1}{2}\chi^4 - \frac{1}{2}\varepsilon^{1/2}\chi$.

A1.46 (a) $f \sim 1 - \frac{1}{2}\varepsilon x - \frac{1}{2}\mathrm{e}^{-x/\varepsilon}$
(b) $x = \varepsilon X$: $f \sim (1 - e^{-X})^{1/2} - \frac{1}{2}\varepsilon^2 X(1 - e^{-X})^{-1/2}$
(c) $x = \chi/\varepsilon$: $f \sim (1 - \chi)^{1/2} - \frac{1}{2}\varepsilon\chi^3(1 - \chi)^{-1/2}$. The first and second match: $1 - \frac{1}{2}\varepsilon^2 X - \frac{1}{2}\mathrm{e}^{-X}$; the first and third match: $1 - \frac{1}{2}\chi$. From (c), $\chi \le \chi_0(\varepsilon)$ where $\chi_0 \sim 1$ (since $f = 0$ at $\chi = \chi_0$); try $\chi_0 \sim 1 + \varepsilon\alpha$ then $\alpha = -1$; that is, $x_0 \sim \varepsilon^{-1} - 1$.

A1.47 $2u_X + 2uu_\xi + u_{\xi\xi\xi} = 0$.

A1.48 $2u_\tau - 2uu_\zeta - u_{\zeta\zeta\zeta} = 0$.

A1.49
$$2f_\tau + 2ff_\xi + f_{\xi\xi\xi} = 0; \quad 2g_\tau - 2gg_\zeta - g_{\zeta\zeta\zeta} = 0;$$
$$\phi_{\xi\zeta} = -f_\xi g_\zeta - \tfrac{1}{2}(fg_{\zeta\zeta} + gf_{\xi\xi}).$$

Then

$$\phi = -\frac{1}{2}\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\zeta}\right)^2\left(\int f\mathrm{d}\xi\right)\left(\int g\mathrm{d}\zeta\right) \quad (= 0 \text{ if } f \equiv 0 \text{ or } g \equiv 0).$$

A1.50 $c_p^2 = 1 - k^{-2}$; $c_g = 1/c_p$; $\omega^2 = k^2 - 1$;
so $c_g = \mathrm{d}\omega/\mathrm{d}k = k/\omega = 1/c_p$;
$A_{10} = -4k^2|A_{01}|^2$ and $A_{12} = 0$.

A1.51 Introduce $\xi = x - c_1 t$, $\tau = \varepsilon^2 t$; then $u \sim u_0(\xi, \tau)$ satisfies

$$u_{0\tau} + u_0 u_{0\xi} = -\lambda u_0, \ \ \lambda = (c - c_1)/(c_2 - c_1), \ \ u_0 \to 0 \text{ as } \xi \to +\infty.$$

Thus $u_0 = \mathrm{e}^{-\lambda\tau} f(\xi + u_0/\lambda)$; exponential decay requires $\lambda > 0$.
Similarly, with $\zeta = x - c_2 t$, $\tau = \varepsilon^2 t$:

$$u_{0\tau} + u_0 u_{0\zeta} = -\mu u_0, \quad \mu = (c_2 - c)/(c_2 - c_1).$$

Thus $\lambda > 0$, $\mu > 0$ if $c_1 < c < c_2$.

A1.52 $\lambda = (c_2 - c)(c - c_1) > 0$; $\phi = \frac{1}{2}\left\{1 - \tanh\left(\dfrac{X - \frac{1}{2}T - X_0}{4\lambda}\right)\right\}$
where X_0 is an arbitrary constant.

A1.53 $2G_{\tau\eta} + G_\eta^2 + 2GG_{\eta\eta} - G_{\eta\eta\eta\eta} = 0$;
$f = \frac{1}{2}\int G(\eta, \tau)\mathrm{d}\eta$, $g = -\frac{1}{2}\int F(\xi, \tau)\mathrm{d}\xi$.

A1.54 $\omega^2 U_{0\theta\theta} + k^2 U_{0\theta\theta} + k^4 U_{0\theta\theta\theta\theta} + U_0 = 0$; then $\omega^2 = k^4 - k^2 + 1$.
U_1 is periodic if

$$A\omega_T + 2\omega A_T + 4k^3 A_X + 6k^2 k_X A - k_X A - 2kA_X - 3\mathrm{i}A|A|^2 = 0,$$

and use $k_T + k_X\omega'(k) = 0$. Finally

$$(\alpha^2)_T + (\omega'\alpha^2)_X = 0; \quad \beta_T + \omega'\beta_X = \frac{3}{2\omega}\alpha^2.$$

A1.55 (a) $u = -\frac{1}{2}c\,\mathrm{sech}^2\{\frac{1}{2}\sqrt{c}(x - ct - x_0)\}$;
(b) $u = \frac{1}{2}u_0\{1 - \tanh[\frac{1}{4}u_0(x - \frac{1}{2}u_0 t - x_0)]\}$ so that $c = \frac{1}{2}u_0$ (see A1.52); in both, x_0 is an arbitrary constant.

Chapter 2

A2.1 For example, find $\mathrm{d}c_p^2/\mathrm{d}\lambda$ and then sketch $y = W(t + \lambda s^2)$, $y = (t - \lambda s^2)/\lambda^2$ (where $t \equiv \tanh\lambda$, $s \equiv \mathrm{sech}\,\lambda$). Show that one point of intersection exists for $\lambda \in (0, \infty)$ provided $W < \frac{1}{3}$. As $\lambda \to \infty$, $c_p \sim \pm\sqrt{\lambda W}$.

A2.2 From A2.1, obtain

$$\left(\frac{c_p}{c_m}\right)^2 = \left(\frac{t}{t_m}\right)\left\{\frac{1}{2}(l^{-1}+l) + \frac{\lambda_m}{s_m}(l^{-1}-l)\right\};$$

for moderate λ_m (and λ), then $t/t_m \approx 1$, $\lambda_m/s_m \ll 1$, so

$$\left(\frac{c_p}{c_m}\right)^2 \approx \frac{1}{2}(l^{-1}+l).$$

The minimum is, of course, at $l = 1$, where $\lambda = \lambda_m = 1/\sqrt{W}$.

A2.3 $U(z) = A\delta\omega \cosh \delta kz / \sinh \delta k$; $P(z) = A\delta\omega^2 \cosh \delta kz/(k \sinh \delta k)$.

A2.4 In physical coords, $\hat{X}, \hat{Z}$:

$$\left(\frac{\hat{X}}{\cosh \delta kz_0}\right)^2 + \left(\frac{\hat{Z}}{\sinh \delta kz_0}\right)^2 = \text{constant}.$$

Hence approaches a circular path as $\delta k \to \infty$ (short waves).

A2.5 The problem is $\phi_{zz} + \delta^2\phi_{xx} = 0$; $\phi_z = 0$ on $z = 0$ with $\phi_z = \delta^2\eta_t$ and $\phi_t + \eta = 0$ both on $z = 1$. Write $\phi = X(x, t)Z(z)$.

A2.6 From example, $A(t) \equiv 0$, $B(t) = B_0 \sin \omega t$; then

$$\eta \propto \sin \omega t \sin kx \left(= \frac{1}{2}\cos(kx - \omega t) - \frac{1}{2}\cos(kx + \omega t)\right).$$

A2.7 It follows that $W'' - \delta^2(k^2 + l^2)W = 0$, so k^2 is replaced by $k^2 + l^2$; wavefront is $kx + ly =$ const. or $\mathbf{n} \cdot \mathbf{x} =$ constant, where $\mathbf{n} \propto \mathbf{k}$.

A2.8 Use Laplace's equation: $\phi_{zz} + \delta^2(\phi_{xx} + \phi_{yy}) = 0$ with boundary conditions as in A2.5 plus $\phi_y = 0$ on $y = 0, l$. Then $\alpha = n\pi/l$ and $\omega^2 = (\sigma/\delta^2) \tanh \sigma$, where $\sigma^2 = \delta^2(k^2 + \alpha^2)$.

A2.9 Cf. A2.8; $\alpha = n\pi/l$, $\beta = m\pi/L$, $\omega^2 = (\sigma/\delta^2) \tanh \sigma$, $\sigma^2 = \delta^2(\alpha^2 + \beta^2)$.

A2.10 $\omega^2 = (\sigma/\delta^2) \tanh \sigma$, $\sigma^2 = \delta^2(k^2 + l^2 + m^2 + n^2)$ with $mk + nl = 0$; so $(m, n) \cdot (k, l) = 0$: the wave-number vectors are perpendicular. Wave propagates in the direction (k, l) at a speed $\omega/|\mathbf{k}|$; that is, $\delta\sqrt{k^2 + l^2 + m^2 + n^2}/\sqrt{k^2 + l^2}$, which is *faster* than in the absence of waves along the crests (for which $m = n = 0$).

A2.11 Equations are

$$u_t + u_0 u_x = -p_x; \quad \delta^2(w_t + u_0 w_x) = -p_z; \quad u_x + w_z = 0$$

with

$$w = 0 \quad \text{on} \quad z = 0$$

and

$$w = \eta_t + u_0\eta_x, \quad p = \eta - \delta^2 W \eta_{xx} \quad \text{on} \quad z = 1.$$

Then $(\omega - u_0 k)^2 = (1 + \delta^2 k^2 W)(\tanh \delta k)/\delta k = \sigma^2$; speed of the waves is $u_0 + \sigma/k$.

A2.12 Cf. A2.11, but with $u_t + u_0 u_x + v_0 u_y = -p_x$, etc.; then $\Omega^2 = (\omega - u_0 k - v_0 l)^2 = (1 + \sigma^2 W)(\tanh \sigma)/\sigma, \quad \sigma^2 = \delta^2(k^2 + l^2)$. Also

$$kx + ly - \omega t = \mathbf{k} \cdot \mathbf{x} - \left\{\mathbf{U} + \frac{\Omega \mathbf{k}}{|\mathbf{k}|^2}\right\} \cdot \mathbf{k}t, \;\; \mathbf{U} \equiv (u_0, v_0),$$

so velocity as required. Stationary implies independent of time (t), so $\mathbf{U} = -\Omega\mathbf{k}/|\mathbf{k}|^2$.

A2.13 To be valid for all t, the solution takes the form $\eta = A\exp\{\mathrm{i}(kx - \omega t)\} + R\exp\{-\mathrm{i}(k_- x + \omega t)\} + \text{c.c. in } x < 0$ and $\eta = T\exp\{\mathrm{i}(k_+ x - \omega t)\} + \text{c.c. in } x > 0$, where $\omega^2 = (k/\delta)\tanh(\delta k)$ for (k, δ_-), (k_-, δ_-), (k_+, δ_+) with $\delta_\pm = h_\pm/\lambda$; that is, $k_- = k$. Continuity of η gives $A + R = T$; continuity of mass flux is $u_- h_- = u_+ h_+$, where $u \propto \eta_x$, so $kh_-(A - R) = k_+ h_+ T$.
Thus $R = A(kh_- - k_+ h_+)/(kh_- + k_+ h_+)$
and $T = 2Akh_-/\,(kh_- + k_+ h_+)$.

A2.14 Stable only if ω is real, from which condition follows. The minimum is at $k = \sqrt{(1-\lambda)/W}$ (for $k > 0$).

A2.16 (b) $I(\sigma) \sim \displaystyle\int_a^{a+\varepsilon} f(x)\mathrm{e}^{\mathrm{i}\delta\alpha(x)}\mathrm{d}x = \mathrm{e}^{\mathrm{i}\sigma\alpha(a)} \int_0^{\hat{u}} \mathrm{e}^{\mathrm{i}\sigma u^2} f(x)\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u$

$$= \mathrm{e}^{\mathrm{i}\sigma\alpha(a)} \int_0^{\hat{u}} \mathrm{e}^{\mathrm{i}\sigma u^2}\{c_0 + uF(u)\}\mathrm{d}u = \mathrm{e}^{\mathrm{i}\sigma\alpha(a)}(I_1 + I_2).$$

Then

$$I_1 = \frac{1}{2}c_0\sqrt{\frac{\pi}{\sigma}}\mathrm{e}^{\mathrm{i}\pi/4} + \mathrm{O}(\sigma^{-1}); \;\; I_2 = \mathrm{O}(\sigma^{-1}),$$

with $c_0 = b_1 f(a)$ and $\frac{1}{2}b_1^2\alpha''(a) = 1$; required result follows.

A2.18 $\eta(r, t) = \int_0^\infty p\bar{f}(p)\cos(tp)J_0(rp)\mathrm{d}p; \quad f(r) = \int_0^\infty p\bar{f}(p)J_0(rp)\mathrm{d}p.$

A2.19 $\eta(r, t) = \int_0^\infty p\bar{f}(p)\cos(t\sqrt{p/\delta})J_0(rp)\mathrm{d}p.$ NB: $\bar{w} = \bar{\eta}_t\mathrm{e}^{\delta p(z-1)}$.

A2.20 $\omega^2 = (\sigma\tanh\sigma)/\delta^2; \quad J_n'(\sigma a) = 0.$
If $n = 0$, then solution is independent of θ.

A2.22
$$\eta(x,t)+\int_{-\infty}^{\infty}F(k)\{\exp[\mathrm{i}k(x-c_pt)]+\exp[\mathrm{i}k(x+c_pt)]\}\mathrm{d}k$$

where

$$F(x)=\frac{1}{4\pi}\int_{-\infty}^{\infty}f(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x.$$

For $f(x)=A\delta(x)$ then

$$\eta(x,t)=\frac{A}{\pi}\int_{0}^{\infty}\cos kx\cos\omega t\,\mathrm{d}k\quad(\text{where } c_p(k)=\omega(k)/k).$$

A2.23 $\omega\sim k(1-\frac{1}{6}\delta^2k^2)$: $u_t+u_x+\frac{1}{6}\delta^2u_{xxx}=0.$

A2.24
$$\eta(x,t)\sim\frac{A_0}{2\pi}\int_{-\infty}^{\infty}\exp\{\mathrm{i}[k(x-t)+\frac{1}{6}\delta^2k^3t]\}\mathrm{d}k,$$

$$F(k)\sim\frac{1}{4\pi}\int_{-\infty}^{\infty}\eta_0(x)\mathrm{d}x=\frac{A_0}{2\pi}.$$

Then

$$\eta\sim A_0\left(\frac{2}{\delta^2t}\right)^{1/3}\mathrm{Ai}\left\{\left(\frac{2}{\delta^2t}\right)^{1/3}(x-t)\right\}$$

where

$$\mathrm{Ai}(y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left\{\mathrm{i}\left(ly+\frac{l^3}{3}\right)\right\}\mathrm{d}l;$$

exponential decay ahead of $x=t$, oscillatory behind; amplitude decays like $t^{-1/3}$ at $x=t$.

A2.25 Since $\psi=0$ on $z=0$, then $W(\cdot,t)$ is real; that is, $\overline{W(Z,t)}=W(\bar{Z},t)$.

A2.27 $\omega^2=k(1+\delta^2k^2W_e)/\delta$;

$$c_g=\frac{1}{2}\left(\frac{1+3\delta^2k^2W_e}{1+\delta^2k^2W_e}\right).$$

A2.28 $A(X,T)=A_0\{1+\exp[\mathrm{i}X(X-\omega'T)]\}.$

A2.29 Waves in $(0, x)$:

$$\int_0^x k\mathrm{d}x, \text{ so } \frac{\partial}{\partial t}\int_0^x k\mathrm{d}x = \omega_0 - \omega = \text{net waves/unit time entering } (0, x).$$

$E(k), k = k(x, t)$, yields $E_t + \omega'(k)E_x = 0$.

A2.30

$$[W_0 W_{1z} - W_{0z} W_1]_0^1 + \int_0^1 W_1(W_{0zz} - \delta^2 k^2 W_0)\mathrm{d}z$$
$$= -2k\omega\delta^2 A_{0X}\int_0^1 W_0 \frac{\sinh \delta kz}{\sinh \delta k}\mathrm{d}z;$$

and

$$W_0(1) = -\mathrm{i}\omega A_0, \quad W_1(1) = -\mathrm{i}\omega A_1 + A_{0T}, \quad W_{0z}(1) = \frac{\mathrm{i}k^2}{\omega}A_0,$$
$$W_{1z}(1) = -\frac{k}{\omega}\left\{\mathrm{i}kA_1 + A_{0X} + \frac{k}{\omega}A_{0T}\right\} - \frac{k}{\omega}A_{0X},$$
$$\int_0^1 W_0 \frac{\sinh \delta kz}{\sinh \delta k}\mathrm{d}z = -\frac{-\mathrm{i}\omega A_0}{2\sinh^2 \delta k}\left\{\frac{\sinh 2\delta k}{2\delta k} - 1\right\}.$$

Finally:

$$A_{0T} + \frac{\omega}{2k}\{1 + \delta k(\coth \delta k - \tanh \delta k)\}A_{0X} = 0.$$

A2.31

$$\mathscr{E} = \int_0^{1+\varepsilon\eta} \left\{\frac{1}{2}\varepsilon^2(u^2 + v^2 + \delta^2 w^2) + z\right\}\mathrm{d}z; \quad \bar{\mathscr{E}} \sim \frac{1}{2} + \frac{1}{2}\varepsilon^2 A^2;$$

$$\mathscr{F} = \varepsilon \int_0^{1+\varepsilon\eta} (u, v)\left\{\frac{P_a}{\rho g h_0} + 1 - \varepsilon\phi_t\right\}\mathrm{d}z$$

and $$c_g = \frac{1}{2}\frac{\omega}{|\mathbf{k}|}\left(1 + \frac{2\sigma}{\sinh 2\sigma}\right), \quad \sigma^2 = \delta^2(k^2 + l^2).$$

A2.32 $\mathscr{E}_0 = \frac{1}{2}, \quad \mathscr{E}_{\mathrm{w}} = \frac{1}{2}\varepsilon^2 A^2, \quad \mathscr{F}_0 = \frac{1}{2}\frac{\varepsilon^2 A^2}{\omega}\mathbf{k}, \quad \mathscr{F}_{\mathrm{w}} = \frac{1}{2}\varepsilon^2 A^2 c_g \frac{\mathbf{k}}{|\mathbf{k}|}.$

A2.33 $8\eta_{\xi\zeta} + \left(\dfrac{d'}{\sqrt{d}}\right)(\eta_\xi + \eta_\zeta) = 0;$

$$\frac{2}{\sqrt{\alpha}}\{\sqrt{x_0} - \sqrt{x_0 - x}\} \pm t \quad (x \le x_0).$$

A2.34 $16H_{\xi\zeta} - d^{1/4}(d'/d^{1/4})'H = 0;$
(a) $H_{\xi\zeta} = 0$; (b) $16H_{\xi\zeta} - \alpha^2 H = 0.$

A2.35 $\phi_{zz} + \delta^2\phi_{xx} = 0$; $\phi_z + \delta^2\phi_{tt} = 0$ on $z = 1$; $\phi_z = \alpha\delta^2\phi_x$ on $z = \alpha x$ (and $\eta = -\phi_t$ on $z = 1$). Set $\phi = F(x, z)\mathrm{e}^{-\mathrm{i}\omega t}$ with
$F = (A_1\mathrm{e}^{\mathrm{i}kx} + B_1\mathrm{e}^{-\mathrm{i}kx})\mathrm{e}^{\delta kz} + (A_2\mathrm{e}^{\mathrm{i}\delta kz} + B_2\mathrm{e}^{-\mathrm{i}\delta kz})\mathrm{e}^{kx}$
and $k = \delta\omega^2$, $\alpha\delta = 1$.

A2.36 As in A2.35, but with

$$F = \left(A_1\mathrm{e}^{\mathrm{i}kx} + B_1\mathrm{e}^{-\mathrm{i}kx}\right)\mathrm{e}^{\delta kz} + \left(A_2\mathrm{e}^{\mathrm{i}\delta lz} + B_2\mathrm{e}^{-\mathrm{i}\delta lz}\right)\mathrm{e}^{lx} + \left(A_3\mathrm{e}^{\mathrm{i}\delta mz} + B_3\mathrm{e}^{-\mathrm{i}\delta mz}\right)\mathrm{e}^{mx}$$

where $l, m = \frac{1}{2}(\sqrt{3} \pm \mathrm{i})k$.

A2.37 Write $F = A\mathrm{e}^{\mathrm{i}kx + \delta mz} + B\mathrm{e}^{mx + \mathrm{i}\delta kz} + \text{c.c.}$
where $m = \sqrt{k^2 + l^2}$, $\delta\omega^2 = m$; cf. A2.35.

A2.38
$$\mathbf{c}_g \equiv \frac{\delta^2\omega}{2\sigma^2}\left(1 + \frac{2\sigma D}{\sinh 2\sigma D}\right)(k, l);$$
$$\omega^2 = \frac{\sigma}{\delta^2}\tanh(\sigma D), \qquad \sigma = \delta\sqrt{k^2 + l^2}.$$

A2.39 $\sigma \sim c/\sqrt{D}$ as $D \to 0$; $\sigma \to c$ as $D \to \infty$ ($c =$ constant).

A2.40 (a) $\Theta = c(X\cos\theta + Y\sin\theta)$ where $\cos\theta + \sin\theta = k$.
(b) Singular solution, but still of the form

$$\Theta = \frac{c}{\sqrt{2}}(X + Y).$$

A2.41 Wavefront: $l_0 Y \pm \dfrac{\sigma_0}{\delta\beta}\ln(\cosh\beta X) - \omega T = \text{constant}.$

Ray: $\mu Y \mp \dfrac{1}{\beta\sigma_0}\ln|\sinh\beta X| = \text{constant}.$

A2.42 Wavefront: $l_0 Y \pm \dfrac{2}{\delta}\sqrt{-\beta X} - \omega T = \text{constant}.$

Ray: $\mu Y \mp \dfrac{2}{3\sqrt{\beta}}(-X)^{3/2} = \text{constant}.$

Amplitude: form $A^2\dfrac{\partial\omega}{\partial k} = \text{constant},$
where $\omega^2 = (\sigma\tanh(\sigma D))/\delta^2$ and $\sigma = \delta\sqrt{k^2 + (\mu/\delta)^2}$.

A2.43 Rays: $X = \frac{1}{2}X_0\{1 - \sin(\text{constant} \pm \mu\sqrt{\beta}Y\}$; periodic, trapped.

A2.44 Start from

$$\frac{\mathrm{d}\Theta}{\mathrm{d}s} = 2(p^2 + q^2), \quad \frac{\mathrm{d}X}{\mathrm{d}s} = 2p, \quad \frac{\mathrm{d}Y}{\mathrm{d}s} = 2q,$$
$$\frac{\mathrm{d}p}{\mathrm{d}s} = 2cc_X, \quad \frac{\mathrm{d}q}{\mathrm{d}s} = 2cc_Y$$

where $p = \Theta_X$, $q = \Theta_Y$ (so $p^2 + q^2 = c^2$). Then

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{p}{q} \text{ and so } Y'' = \frac{c}{p^2}(c_Y - Y'c_X) \text{ and } \frac{c^2}{p^2} = 1 + (Y')^2.$$

(a) Straight path;

(b) Becomes $(cY'\big/\sqrt{1+(Y')^2})' = 0$: $cY' \propto \sqrt{1+(Y')^2}$.

A2.45 Time $= \int_a^b c(X, Y)\sqrt{1+(Y')^2}\mathrm{d}X = \int_a^b F(X, Y)\mathrm{d}X$; Euler–Lagrange is

$$\frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{\partial F}{\partial Y'}\right) - \frac{\partial F}{\partial Y} = 0$$

which is the required equation.

A2.46 Immediately, $Y'\big/\sqrt{1+(Y')^2} = \sin\alpha$, so the result follows.

A2.47 Rays: $R = \dfrac{\mu^2}{\beta}\left\{1 + \tan^2\left[\dfrac{1}{2}\pi \pm \dfrac{1}{2}\mu^2(\theta - \theta_0)\right]\right\}$
where $R \to \infty$ as $\theta \to \theta_0$; closest approach to $R = 0$ is $R = \mu^2/\beta$ where $\theta = \theta_0 \mp \pi/\mu^2$.

A2.48 Rays: $\dfrac{1}{\sqrt{R_0}}\arctan\left(\sqrt{\dfrac{R}{R_0} - 1}\right) - \sqrt{R - R_0} = \pm\mu\sqrt{\beta}\theta + \text{constant}$;

rays cease to exist at $R = R_0$, at which point $\mathrm{d}R/\mathrm{d}\theta$ is infinite; the ray is perpendicular to the circle $R = R_0$.

A2.49 Let $c_g = \lambda c_p$; then $\sin\Theta = 1/(2/\lambda - 1)$ which increases as λ increases from $\frac{1}{2}$ to 1 (depth decreasing): the wedge angle increases; $\lambda = 3/4$ yields $\Theta = \arcsin(3/5)$.

A2.51 Let circle intersect course at Q_i $(i = 1, 2)$; limiting case is when circle touches. Join WQ_i; draw the perpendicular to WQ_i at W to intersect course at P_i'. Then, by similar triangles, $|PQ_i| = \frac{1}{2}|PP_i'|$, so circle through W, diameter Q_iP_i': two $(i = 1, 2)$ influence points for a given W.

A2.52 Let W be at (a,b), then W lies on the circle $(a-\frac{3}{4}Ut)^2+b^2 = \frac{1}{4}U^2t^2$: quadratic in t, given a, b, U.

A2.53 Circular path: circle of radius R, centre at $(0, R)$, ship at origin. Write P' as $(X, Y) = R(\sin\alpha,\ 1-\cos\alpha)$, $\alpha = Ut/R$, then W is $(x, y) = \{X - r\cos(\alpha+\theta),\ Y - r\sin(\alpha+\theta\}$; cf. Figure 2.13. Condition of stationary phase is $r = \frac{1}{2}\lambda\cos^2\theta$ (equation 2.122) which gives (x, y). Often written as

$$x/R = \sin(\mu\cos\theta) - \frac{1}{2}\mu\cos^2\theta\cos(\theta+\mu\cos\theta)$$
$$y/R = 1 - \cos(\mu\cos\theta) - \frac{1}{2}\mu\cos^2\theta\sin(\theta+\mu\cos\theta)$$

where $\mu = \lambda/R$, $\alpha = \mu\cos\theta$ (equation 2.122). Straight-line course is $R \to \infty$, $R\mu = \lambda$ (fixed).

A2.54 Use $\Omega = -\delta\sqrt{W|\mathbf{k}|^3}$ and follow Section 2.4.2; roots for $\tan\theta$ always real.

A2.55 $h = H\{t - x/(3\sqrt{h} - 2c_0)\}$; $u = U\{t - x/(3u/2 + c_0)\}$ $(c_0 = \sqrt{h_0})$.

A2.56 $u =$ constant on lines $\mathrm{d}x/\mathrm{d}t = 3u/2 + c_0$ $(c_0 = \sqrt{h_0})$; consider characteristic through $t = \alpha$, $x = X(\alpha)$, then $u = X'(\alpha)$ on lines

$$x - (3X'(\alpha)/2 + c_0)(t-\alpha) = X(\alpha); \quad \text{also } h = (X'(\alpha)/2 + c_0)^2.$$

A2.57 $I = kZ$, with $c'\sqrt{1+H} = \mp\frac{3}{2}$, gives $-k = 2k(k - 3\sqrt{k/2})$, which has the solution $k = 1$; this is the no-shear case.

A2.58 Set
$X = \xi + \eta = 2(u - \alpha t)$, $Y = \eta - \xi = 4c$, $t = (-\frac{1}{2}X + T_Y/Y)/\alpha$:

$$T_{XX} = T_{YY} + \frac{1}{Y}T_Y$$

(after one integration + decay conditions). Then $c = Y/4$, $u = T_Y/Y$, and $x = (X\hat{T}_Y/Y + \frac{1}{2}(\hat{T}_Y/Y)^2 - \frac{1}{2}\hat{T}_X)/2\alpha$ where $\hat{T} = T - \frac{1}{4}XY^2$; $T = AJ_0(\omega Y)\cos(\omega X)$, say, since shoreline is at $Y = 0$. Maximum run-up is where $u = 0$; which determines X and hence x. Far from the shoreline is $Y \to \infty$.

A2.59 First show that $J = 0$ can be written as $t_Y^2 - t_X^2 = 0$, then that $t_Y \pm t_X = A\omega^2\{J_2(\omega Y)\cos(\omega X) \pm J_1(\omega Y)\sin(\omega X)\}/Y \mp \frac{1}{2}$.
So $J = 0$, provided $A\omega^3 \geq 1$, first on $Y = 0$.

A2.60 $u^+/u^- = 2/(\sqrt{1+8F^2} - 1) < 1$ for $F > 1$; form $u^{+2}/h^+ = \alpha/(\sqrt{1+\alpha} - 1)^3$, where $\alpha = 8F^2$ (> 8) where $\alpha < (\sqrt{1+\alpha} - 1)^3$ (from, for example, $4 + \alpha > 4\sqrt{1+\alpha}$, $\alpha > 8$). For the bore, move in the frame which brings the flow ahead of the hydraulic jump (u^-) to rest; then the speed of the bore is $U = u^-$.

A2.61 Behind approaching bore, let the depth be h_1 speed u_1; thus $u_1 = U(1 - h_0/h_1)$. After reflection, let the bore move away at speed V, depth h_2 behind; that is, in contact with the wall. Hence $V = U(h_1 - h_0)/(h_2 - h_1)$ and $h_2/h_1 = (\sqrt{1+8F^2} - 1)/2$, where $F = (V + u_1)/\sqrt{h_1}$; thus $(H^2 - 1)(H - 1) = 2HF_1^2$, where $H = h_2/h_1$ and $F_1^2 = U^2(1 - h_0/h_1)^2/h_1$.

A2.62 $R\,[\![h]\!] = [\![uh]\!]$; $\dot{R}\,[\![uh]\!] = [\![hu^2 + \frac{1}{2}h^2]\!]$.

A2.63 Requires $c^2 = 2a$, $\eta' = -1/(\delta\sqrt{3})$ (Stokes' highest wave) $= -\alpha a$, $c^2 = \tan(\alpha\delta)/\alpha\delta$; this yields $c^2 \approx 1.347$. Now

$$c^2 = 2(a+b) = \tan(\alpha\delta)/\alpha\delta, \quad a + 2b = 1/(\alpha\delta\sqrt{3}),$$
$$3(\tfrac{1}{2}a^2 + \tfrac{2}{3}ab + \tfrac{1}{4}b^2) = (c^2 - 1)(2a + b);$$

so $c^2 \approx 1.665$.

A2.64 Remember, speed of solitary wave here is $1 + \varepsilon c$; speed from general result is $\sqrt{\tan\alpha\delta/\alpha\delta}$ with $\delta \to 0$ and $K = \delta^2/\varepsilon$, which agrees at $O(\varepsilon)$. Simply write $\mathrm{i}\alpha$ for α.

A2.65 (b) $m = 1$ by direct integration, $u = \operatorname{arcsech}(\cos\phi)$.

(c) Use $\mathrm{d}/\mathrm{d}u \equiv (\mathrm{d}\phi/\mathrm{d}u)\mathrm{d}/\mathrm{d}\phi$; $\mathrm{d}(\mathrm{sn}u)/\mathrm{d}u = \mathrm{cn}u\,\mathrm{dn}u$; $\mathrm{d}(\mathrm{dn}u)/\mathrm{d}u = -m\,\mathrm{sn}u\,\mathrm{cn}u$.

A2.66 Period of $\sin\phi$, $\cos\phi$ is 2π, hence period of Jacobian elliptic functions is

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{\sqrt{1 - \mathrm{m}\sin^2\theta}} = 4\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}} = 4K(m).$$

(b) Compare the terms in the series expansion in powers of m on each side of the equation.

(c) Or use properties of F from (b).

A2.67 $3b = 8K\alpha^2 m,\ 6a + \left(5 - \dfrac{1}{m}\right)b = 4c.$

A2.68 (a) On $z = 0$, $\mathbf{u} \equiv (\phi_\xi, 0)$ and $\mathrm{d}\boldsymbol{\ell} = (\mathbf{u}/|\mathbf{u}|)\mathrm{d}\xi$, so $C = \int_{-\infty}^{\infty} \phi_\xi \mathrm{d}\xi$.

(b) Use construction given in Figure 2.27; then by Stokes' theorem the integral all around the path is zero (since $\nabla \wedge \mathbf{u} = \mathbf{0}$). But $\mathbf{u}$ on $\xi = \pm\xi_0$ approaches zero as $\xi_0 \to \infty$, so the integral (from r to l) on the surface = integral in (a).

A2.69 With $\eta = \varepsilon\,\mathrm{sech}^2(\frac{1}{2}\sqrt{3\varepsilon}\xi)$, then $M \approx I \approx C \approx 4\sqrt{\varepsilon/3}$ and $V \approx T \approx 4/(3\varepsilon\sqrt{3\varepsilon})$.

A2.70 Variation, with integration by parts in z, yields

$$\int\!\!\int_D \left\{ \left[\phi_t + \frac{1}{2}(\nabla\phi)^2 + z\right]_{z=\eta} \delta\eta + \frac{\partial}{\partial t}\int_b^\eta \delta\phi \mathrm{d}z + \frac{\partial}{\partial x}\int_b^\eta \phi_x \delta\phi \mathrm{d}z \right.$$
$$+ \frac{\partial}{\partial y}\int_b^\eta \phi_y\delta\phi\mathrm{d}z - \int_b^\eta [\phi_{xx} + \phi_{yy} + \phi_{zz}]\delta\phi\mathrm{d}z$$
$$- \left[(\eta_t + \phi_x\eta_x + \phi_y\eta_y - \phi_z)\delta\phi\right]_{z=\eta}$$
$$\left. + \left[(\phi_x\eta_x + \phi_y\eta_y + \phi_z)\delta\phi\right]_{z=b} \right\} \mathrm{d}x\mathrm{d}t,$$

from which all the equations follow.

Chapter 3

A3.1 Requires

$$\frac{\gamma}{A\beta} = \frac{\beta^2}{C} = -\frac{6}{\alpha B}.$$

A3.2 $2\eta_{0\tau} - 3\eta_0\eta_{0\zeta} - \frac{1}{3}\eta_{0\zeta\zeta\zeta} = 0.$

A3.3 $2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \left(\frac{1}{3} - W\right)\eta_{0\xi\xi\xi} = 0$, after writing $\delta^2 = \varepsilon$.

A3.4 $$2\eta_{1\tau} + 3(\eta_0\eta_1)_\xi + \frac{1}{3}\eta_{1\xi\xi\xi} = \frac{21}{4}\eta_0^2\eta_{0\xi} + \frac{31}{12}\eta_{0\xi}\eta_{0\xi\xi} + \frac{7}{6}\eta_0\eta_{0\xi\xi\xi} + \frac{1}{36}\eta_{0\xi\xi\xi\xi\xi},$$

where the KdV equation for η_0 has been used; set r.h.s. $= G'(\xi - c\tau)$, then $\left(\eta_0'^2 F'\right)' = 3\eta_0' G$, etc.

A3.5 $m = -\frac{2}{3}$; $n = -\frac{1}{3}$; $\lambda^2 = 1$.

A3.6 Try setting $u = 6t^{-2/3} f\left(xt^{-1/3}\right)$ and observing that $f = -\frac{1}{3}(\log F)''$ where $F = \eta^3 + 12$ $\left(\eta = xt^{-1/3}\right)$.

A3.7 $2H_{0\tau} + \dfrac{1}{\tau}H_0 + 3H_0H_{0\xi} + \dfrac{1}{3}H_{0\xi\xi\xi} = 0.$

A3.8 $a = -2b^2$; $c^2 = 1 + 4b^2$ (and so $c > 0$ or $c < 0$).

A3.9 Leading order becomes $\left(\pm 2\eta_\tau - \frac{3}{2}(\eta^2)_\xi - \frac{1}{3}\eta_{\xi\xi\xi}\right)_\xi = 0$, where $\xi = x \pm t$, $\tau = \varepsilon t$.

A3.11 $a = -2k^2$; $\omega = 4k^3 + 3l^2/k$.

A3.15 $u \sim -2k_n^2 \text{sech}^2\{k_n\xi_n \mp x_n\}$ as $t \to \pm\infty$ where $\xi_n = x - 4k_n^2 t$ $(n = 1, 2)$ and

$$\exp(2x_1) = \left|\frac{k_1 + k_2}{k_1 - k_2}\right|, \quad \exp(2x_2) = \left|\frac{k_1 - k_2}{k_1 + k_2}\right| \quad (k_1 \neq k_2).$$

A3.16 Let $k_1 < k_2$, then sech^2 at $t = 0$ only if $k_1 = 1$, $k_2 = 2$; two maxima if $\sqrt{3} > k_2/k_1 > 1$; one maximum if $k_2/k_1 \geq \sqrt{3}$.

A3.21 $a = -2k^2$; $\omega^2 = k^2 + k^4 + l^2$ (so $\omega > 0$ or $\omega < 0$).

A3.22 $$u = -2(12t)^{-2/3} \frac{\partial^2}{\partial \xi^2} \log\left(1 + kt^{-1/3} \int_{\xi}^{\infty} A_i^2(s)\mathrm{d}s\right), \quad \xi = x/(12t)^{1/3}.$$

A3.25 See Q3.17.

A3.29 Q3.26: $f = 1 + \mathrm{e}^{\theta}$, $\theta = kx + ly - \omega t + \alpha$, $\omega = k^3 + 3l/k$;
Q3.27: $f = 1 + \mathrm{e}^{\theta}$, $\theta = kx - \omega t + \alpha$, $\omega^2 = k^2 + k^4$;

Q3.28: $f = 1 + kt^{-1/3} \displaystyle\int_{\xi}^{\infty} A_i^2(s)\mathrm{d}s$, $\xi = x/(12t)^{1/3}$, k constant.

A3.31 Set $f = 1 + E_1 + E_2 + AE_1E_2$, $E_i = \exp(2k_i x - \varepsilon_i \omega_i t)$, $\varepsilon_i = \pm 1$; then

$$A = -\frac{(\omega_1 - \omega_2)^2 - (k_1 - k_2)^2 - (k_1 - k_2)^4}{(\omega_1 + \omega_2)^2 - (k_1 + k_2)^2 - (k_1 + k_2)^4}.$$

A3.32 $f \sim 1 + \mathrm{e}^{\theta_2}$ (a θ_2 solitary wave at infinity); $f \sim 1 + \mathrm{e}^{\theta_1}$ (a θ_1 solitary wave at infinity); $f \sim 1 + \mathrm{e}^{\theta_1 - \theta_2}$ (a $\theta_1 - \theta_2$ solitary wave at infinity). NB

$$A = \frac{(m_1 - m_2)(n_1 - n_2)}{(m_1 + m_2)(n_1 + n_2)}.$$

A3.33 For $\mathscr{E}$, remember that $\int_{-\infty}^{\infty} \eta dx =$ constant.

A3.34 First, from energy conservation

$$\int_{-\infty}^{\infty} \left\{2\eta_0^2 + \varepsilon\left(4\eta_0\eta_1 - \frac{1}{2}\eta_0^3\right) + \mathrm{O}(\varepsilon^2)\right\}\mathrm{d}x = \text{const.};$$

second, use the equation for η_1 (A3.4) and the KdV equation η_0 to find

$$\int_{-\infty}^{\infty} \left(2\eta_0\eta_1 - \frac{1}{12}\eta_{0\xi}^2\right)\mathrm{d}\xi = \text{const.},$$

when the required result follows.

A3.36 Form

$$(xu+3tu^2)_t = \left(12tu^2 - 6tuu_{xx} + 3tu_x^2 + 3xu^2 - xu_{xx} + u_x\right)_x.$$

A3.37 Use Q3.36; the centre of mass has the x-coordinate $(\int_{-\infty}^{\infty} xu\mathrm{d}x)/(\int_{-\infty}^{\infty} u\mathrm{d}x)$. Consider two solitons (u_1, u_2) far apart, and then write $u = u_1 + u_2$.

A3.38 Write

$$u \sim -\sum_{n=1}^{N} k_n^2 \operatorname{sech}^2\{k_n(x - 4k_n^2 t - x_n)\} \text{ as } t \to +\infty,$$

then

$$\int_{-\infty}^{\infty} u\mathrm{d}x = -2\sum_{n=1}^{N} k_n; \quad \int_{-\infty}^{\infty} u^2\mathrm{d}x = \frac{4}{3}\sum_{n=1}^{N} k_n^3, \quad \text{etc.}$$

A3.40 For the third law, add $H\times$ (first equation) to $U\times$ (second equation).

A3.44 $c = \pm 1$; $I_{31} = \mp 1$; $I_{41} = 1$; $J_1 = \frac{1}{3}$.

$$c = \frac{1}{2}\left\{U_0 + U_1 \pm \sqrt{4 + (U_1 - U_0)^2}\right\}; \quad I_{31} = \frac{U_1 + U_0 - 2c}{2(U_1 - c)^2(U_0 - c)^2};$$

A3.45

$$I_{41} = \frac{U_1^2 + U_1U_0 + U_0^2 + 3c(U_0 + U_1) + 3c^2}{3(U_1 - c)^3(U_0 - c)^3}, \quad \text{etc.}$$

A3.46 (a) With critical level, c given in A3.45 is recovered for which $c > U_1$ or $c < U_0$: no critical level.

(b) With critical level, then $\alpha = c/U$ satisfies (for $\alpha < 1$)

$$1 - \frac{\alpha}{2\sqrt{1-\alpha}}\ln\left|\frac{1+\sqrt{1-\alpha}}{1-\sqrt{1-\alpha}}\right| = 2U_1^2\alpha(\alpha - 1),$$

which has one solution only, namely in $\alpha < 0$: no critical level.

A3.47 $c(c - U_1)^2 = c - dU_1$: three real roots (for $0 < d < 1$) with one satisfying $0 < c < U_1$: critical level exists.

A3.48 $\alpha = c/U_1$ satisfies (cf. A3.46)

$$1 - \frac{\alpha}{2\sqrt{1-\alpha}}\ln\left|\frac{1+\sqrt{1-\alpha}}{1-\sqrt{1-\alpha}}\right| = \frac{2}{d}\ U_1^2\alpha(\alpha - 1) + \frac{2\alpha(1-d)}{d(1-\alpha)},$$

and one solution satisfies $0 < \alpha < 1$: critical level exists. NB Interesting exercise: examine this equation for $d \to 1$; $d \to 0$.

A3.49 See equation (3.139).

A3.50 $1 + c\cos\theta_0 = U_0 \pm 1$.

A3.51 Choose $c = U_1$; $k = a\cos\theta + b(a)\sin\theta$ where $b^2 = 1 + a(U_1 - U_0) - a^2$.

A3.52 Set

$$h(p) = a(p)\cos p + b(p)\sin p$$

and

$$h'(p) = -a(p)\sin p + b(p)\cos p,$$

then $\mathrm{d}h/\mathrm{d}p \equiv h'(p)$ requires $a'\cos p + b'\sin p = 0$ (and a, b are related by equation (3.139b)). Required solution is described by the set: $a' = -b'\tan p$, $h = a\cos p + b\sin p$, equation (3.139), p derivative of equation (3.139).

A3.53 $D = (aX + b)^{9/4}$, a, b constants.

A3.54 Takes the form $\left(H_0^3\right)_X - \frac{1}{3}D^{9/4}\left(H_{0\xi}^2\right)_X + Q_\xi = 0$, for some Q.

A3.56 $\left(2F_\tau + \frac{3}{2}F_\xi^2 + \frac{1}{3}F_{\xi\xi\xi}\right)_\xi = \mathrm{O}(\varepsilon)$ and $H \sim F_\xi$.

Chapter 4

A4.1 $A(\zeta, \tau) \sim \int_{-\infty}^{\infty} f(\kappa; 0)\exp\left\{\kappa\zeta - \frac{1}{2}\kappa^2\omega''(\kappa_0)\tau\right\}\mathrm{d}\kappa$.

A4.2 (a) $F = A\cosh(\omega z) + B\sinh(\omega z) + \dfrac{z}{2\omega}\sinh(\omega z)$;

(b) $$F = A\cosh(\omega z) + B\sinh(\omega z) + \frac{1}{4\omega^2}\left\{\omega z^2\cosh(\omega z) - z\sinh(\omega z)\right\}.$$

A4.4 $u = \phi_x = \phi_\xi + \varepsilon\phi_\zeta = \varepsilon f_{0\zeta} +$ periodic terms.

A4.5 See equations (4.43).

A4.6 For $\delta \to 0$:

$$-2\mathrm{i}k\delta^2 A_{0\tau} + \delta^4 k^2 A_{0\zeta\zeta} - \delta^2 A_{0YY} + \frac{9}{2}A_0|A_0|^2 + 3\delta^2 k^2 A_0 f_{0\zeta} = 0,$$

$$\delta^2 k^2 f_{0\zeta\zeta} + f_{0YY} = -3\left(|A_0|^2\right)_\zeta;$$

for $\delta \to \infty$: with $c_p \sim 1/\sqrt{\delta k}$ then

$$-2i\sqrt{\frac{k}{\delta}}A_{0\tau} + \frac{1}{4\delta k}A_{0\zeta\zeta} - \frac{1}{2\delta k}A_{0YY} + 4k^3\delta A_0|A_0|^2 + 2k\sqrt{\frac{k}{\delta}}A_0 f_{0\zeta} = 0,$$

$$f_{0\zeta\zeta} + f_{0YY} = -2\sqrt{\delta k}\left(|A_0|^2\right)_\zeta.$$

For $\delta \to 0$:

$$-2ik\delta^2 A_{0\tau} + \delta^4 k^2 A_{0\zeta\zeta} - \frac{9}{2} A_0 |A_0|^2 = 0;$$

for $\delta \to \infty$:

$$-2\mathrm{i}\sqrt{\frac{k}{\delta}}\, A_{0\tau} + \frac{1}{4\delta k}\, A_{0\zeta\zeta} + 4k^3 \delta A_0 |A_0|^2 = 0.$$

A4.7 $c_{p1} = k^2/6$, $c_{g1} = k^2/2$, $A_{12} = 3A_{01}^2/2k^2$; then

$$-2ikA_{01T} + k^2 A_{01ZZ} - A_{01YY} + \frac{9}{2k^2} A_{01}|A_{01}|^2 + 3A_{01} f_{0Z} = 0,$$

$$2c_{g1} f_{0ZZ} + f_{0YY} + 3\big(|A_{01}|^2\big)_Z = 0.$$

A4.8 $t \to At$, $x \to Bx$, $u \to Cu$: $AC^2 = \alpha/\gamma$, $B^2 C^2 = \beta/\gamma$.

A4.10 Oscillates like e^{int}, with amplitude $\sqrt{-n}$.

A4.11 Try $u(x, t) = a\exp(\mathrm{i}a^2 t)(1 + f + \mathrm{i}g)$ where $f(x, t)$ and $g(x, t)$ are both real. Oscillates like $\exp(\mathrm{i}a^2 t)$, with amplitude a.

A4.12 Use same approach as adopted for Q4.11.

A4.13 $u \sim 2am\,\mathrm{sech}\,(am\sqrt{2}x)\exp\{\mathrm{i}a^2(1 + 2m^2)t\}$; cf. Q4.9.

A4.16 $u(x, t) = t^{-1/2} f(\eta)$, $f'' - \frac{1}{2}\mathrm{i}(\eta f)' + \varepsilon f |f|^2 = 0$, $\eta = xt^{-1/2}$.

A4.18
$$c + g_0 \exp\{\mu(lx - mz) + \mathrm{i}\mu^2(m^2 - l^2)t/\alpha\}$$
$$+ \int_x^\infty d g_0 \exp\{\mu(ly - mz) + \mathrm{i}\mu^2(m^2 - l^2)t/\alpha\}\mathrm{d}y = 0;$$

$$d + \int_x^\infty c f_0 \exp\{\lambda(my - lz) + \mathrm{i}\lambda^2(l^2 - m^2)t/\alpha\}\mathrm{d}y = 0.$$

For example,

$$c = -g_0 \mathrm{e}^{3\mu(x - 3\mathrm{i}\mu t)} \Big/ \left\{1 + \mathrm{e}^{3(\mu - 1)x + 9\mathrm{i}(\lambda^2 - \mu^2)t}\right\}.$$

A4.19 For $c = u^*$ then $g_0 = f_0^*$; but $f_0 g_0 = -8k^2$, so $|f_0|^2 = -8k^2$, which is impossible.

A4.20 Show that

$$\{(\mathrm{i}\mathrm{D}_t + \mathrm{D}_x^2)(g \cdot f)\}/f^2 + g\{\varepsilon|g|^2 - \mathrm{D}_x^2(f \cdot f)\}/f^3 = 0.$$

A4.21 Show that

$$\left\{(\mathrm{iD}_t + \beta \mathrm{D}_x^2 + \mathrm{i}\gamma \mathrm{D}_x^3)(g \cdot f)\right\}/f^2 + 3\mathrm{i}\left\{\delta|g|^2 - \gamma \mathrm{D}_x^2(f \cdot f)\right\}(fg_x - gf_x)/f^4 + g\left\{\varepsilon|g|^2 - \beta \mathrm{D}_x^2(f \cdot f)\right\}/f^3 = 0;$$

$\gamma = 0,\ \delta = 0,\ \beta = 1.$

A4.22 $g = \mathrm{e}^{\theta},\ f = 1 + (\delta/2\gamma)(k + k^*)^{-2}\exp(\theta + \theta^*),$
$\theta = kx + (\mathrm{i}\beta k^2 - \gamma k^3)t + \alpha.$

A4.23 $g_3 = 4\sqrt{2}\left(\mathrm{e}^{\mathrm{i}t+7x} + 3\mathrm{e}^{9\mathrm{i}t+5x}\right);$
$f_2 = 4\left(\mathrm{e}^{2x} + \mathrm{e}^{6x}\right) + 3\mathrm{e}^{4x}\left(\mathrm{e}^{8\mathrm{i}t} + \mathrm{e}^{-8\mathrm{i}t}\right);$
$f_4 = \mathrm{e}^{8x}$ and rest are zero.

A4.24 Set $X = l\zeta + mY$, then

$$-2\mathrm{i}kc_pA_{0\tau} + \left(\alpha l^2 - m^2c_pc_g\right)A_{0XX} + \left\{\beta + \frac{m^2k^2\gamma^2}{(1-c_g^2)c_p^2\left\{m^2 + (1-c_g^2)l^2\right\}}\right\}A_0|A_0|^2 = 0.$$

A4.25 $-2\mathrm{i}k\delta^2A_{0\tau} + \delta^4k^2l^2A_{0XX} + \frac{9}{2}A_0|A_0|^2 = 0$ (retaining only the dominant contribution to each coefficient, but see Section 4.2.3).

A4.26
$$A_0 = \frac{a\sqrt{2}}{3}\exp\left\{\mathrm{i}\left[\frac{c}{2k\delta^2}\left(\frac{X}{l} + \frac{c\tau}{2}\right) - \frac{n\tau}{2k\delta^2}\right]\right\} \times \operatorname{sech}\left\{\frac{a}{k\delta^2}\left(\frac{X}{l} + \frac{c\tau}{2}\right)\Big/\sqrt{2}\right\}$$

where $a^2 = 2\left(n - \frac{1}{4}c^2\right) > 0.$

A4.29 Structure is evident if we write $\theta = \phi + \mathrm{i}\psi$ (ϕ, ψ real); then equation (4.90) gives

$$\left(\mathrm{e}^{\mathrm{i}\psi}/\sqrt{\lambda}\right)\Big/\left\{\sqrt{\lambda}\mathrm{e}^{\phi} + \mathrm{e}^{-\phi}/\sqrt{\lambda}\right\} = \left\{\mathrm{e}^{\mathrm{i}\psi}/(2\sqrt{\lambda})\right\}\operatorname{sech}(\phi + \phi_0) \text{ where } \mathrm{e}^{\phi_0} = \sqrt{\lambda}.$$

A4.31

$$f_m = 1/4(a_m^2); \quad c = 1/(a_1 + a_2)^2; \quad b_m = \frac{(a_1 - a_2)^2}{4a_m^2(a_1 + a_2)^2};$$

$$d = \frac{(a_1 - a_2)^4}{16a_1^2a_2^2(a_1 + a_2)^2}.$$

A4.32 For the first, form $u_{xt}u_x^* + u_x u_{xt}^*$; for the second form $u_t u_{xxx}^* + uu_{xxxt}^*$.

A4.33 Form

$$i\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} x|u|^2\mathrm{d}x - \int_{-\infty}^{\infty}(u^*u_x - uu_x^*)\,\mathrm{d}x = 0.$$

A4.34 Integrate over one period to give

$$\int_{-\infty}^{\infty}\overline{\left(\int_0^{1+\varepsilon\eta} u\mathrm{d}z\right)}\mathrm{d}\zeta = \text{const.};$$

then with $u = \phi_\xi + \varepsilon\phi_\zeta$, the oscillatory part of ϕ_ζ yields

$$\int_{-\infty}^{\infty}(A_0^* A_{0\zeta} - A_0 A_{0\zeta}^*)\mathrm{d}\zeta = \text{const.}$$

A4.35 (a) Each a function of t only.
(b) If somewhere independent of t.
(c) $\{A + g(t)\}\zeta$ where both A (= const.) and g are arbitrary.
(d) If, as $\zeta \to +\infty$ or $-\infty$, the integral in (c) approaches a constant, then $g(t) = -A$ and the integral is zero.

A4.37 Write $u = f(kx + ly, t)$ to give $\mathrm{i}f_t + (k^2 + l^2)f_{\xi\xi} + f|f|^2 = 0$; then set $\xi = X\sqrt{k^2 + l^2}$ and follow Q4.9.

A4.38 For the first, form $\mathrm{i}(u^*u_t + uu_t^*)$; for the second form

$$\frac{\partial}{\partial t}\left(u_x u_x^* + u_y u_y^* - \frac{1}{2}u^2 u^{*2}\right).$$

A4.39 First form $\mathrm{i}\left(x^2 uu^*\right)_{tt} + x^2(u^*u_x - uu_x^*)_{xt} = 0$ and also obtain

$$\mathrm{i}(u^*u_x - uu_x^*)_t = \left\{4u_x u_x^* - (u^*u_x + uu_x^*)_x - \varepsilon|u|^4\right\}_x.$$

A4.40 (a) $A \to 0$ as $|x| \to \infty$; $\omega = a^2/2 > 0$; $A = a\,\mathrm{sech}(ax/\sqrt{2})$.
(b) $A \to \pm a$ as $x \to \pm\infty$; $\omega = -a^2 < 0$; $A = a\tanh(ax/\sqrt{2})$.

A4.41 $$(A^2)_t + 2(kA^2)_x = 0 \text{ so } \int_{-\infty}^{\infty} A^2\mathrm{d}x\left(= \int_{-\infty}^{\infty}|u|^2\mathrm{d}x\right) = \text{const.};$$

use this first equation in the second to give

$$\left\{\frac{1}{2}A_x^2+\frac{1}{2}k^2A^2+\frac{1}{4}\varepsilon A^4-\left(k_xA^2+2kAA_x\right)\int^x k\mathrm{d}x\right\}_x - \left(AA_x\int^x k\mathrm{d}x\right)_t=0.$$

A4.42 Set-down is $\dfrac{-2\delta k}{\sinh 2\delta k}|A_0|^2$; mean drift is $c_g f_{0\zeta}$.

A4.43 $(2K\omega c_p)^2=\alpha k^2(\alpha k^2-2\beta|A|^2)$.

A4.44 $P=(\cosh\delta kz)/\cosh\delta k$; $c_p^2=(\tanh\delta k)/\delta k$.

A4.45 Use

$$\frac{\partial}{\partial z}\left(W^{-2}P_{zk}\right)-(\delta k/W)^2P_k = 2\delta^2kW^{-2}P+2\delta^2kc_p'W^{-3}P-2c_p\frac{\partial}{\partial z}\left(W^{-3}P_z\right),$$

integrate in z, with the boundary conditions for P, and write

$$P_{zk}(0;k)=0;\quad P_k(1;k)=0;\quad P_{zk}(1;k)=2\delta^2kW_1^2-2\delta^2k^2c_p'W_1.$$

A4.46 $P\sim 1-\delta^2k^2\int_z^1 W^2\left\{\int_0^z W^{-2}\mathrm{d}z\right\}\mathrm{d}z$.

A4.48 Coefficients of NLS equation now functions of $\hat{X}=\sigma X$; NLS then gives $B(\zeta,X;\hat{X})$. Use Q4.8, Q4.9.

Chapter 5

A5.2 See Q5.1; terms are the same size when $\delta kR=\mathrm{O}(1)$. Set $1/R=\alpha\delta k$, then $\delta k\to 0$ for α fixed, yields

$$\mu-\tan\mathrm{h}\mu+\alpha^2\mu^5=0 \quad\text{with}\quad \mu^2=-\frac{\mathrm{i}}{\alpha}\left(\frac{\omega}{k}\right).$$

A5.4 See equation (5.21); first term (not involving R) is multiplied by $(1+\delta^2k^2W_\mathrm{e})$.

A5.5 $y=\left(\mathrm{e}^{-x}-\mathrm{e}^{-x/\varepsilon}\right)/\left(\mathrm{e}^{-1}-\mathrm{e}^{-1/\varepsilon}\right)$;
(a) $y\sim\mathrm{e}^{1-x}$; (b) $y\sim\mathrm{e}\left(1-e^{-X}\right)$.

A5.6 To within exponentially small terms:
(a) $y_0=\mathrm{e}^{1-x}$, $y_n=0$, $n\geq 1$; (b) $Y_0=\mathrm{e}(1-\mathrm{e}^{-X})$, $Y_1=-X\mathrm{e}$.

A5.8 $u_{1z}=-z\eta_{0\xi\xi}$.

A5.9 (b) $\hat{u} = \dfrac{x}{2\sqrt{\pi}} \displaystyle\int_{-\infty}^{t} f(t')(t-t')^{-3/2} \exp\{-x^2/4(t-t')\}\mathrm{d}t'$.

A5.10 Change the order of the integration and introduce $x + z^2/4y^2 = x'$; integral $= \mathrm{e}^{-\alpha x}\sqrt{\pi/\alpha}$.

A5.11
$$\frac{\mathrm{d}}{\mathrm{d}T}\int_{-\infty}^{\infty} \eta_0^2 \mathrm{d}\xi = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \eta_0 \left\{\int_{\xi}^{\infty} \eta_{0\xi'} \frac{\mathrm{d}\xi'}{\sqrt{\xi'-\xi}}\right\}\mathrm{d}\xi,$$
$$\eta_0 = 2c\,\mathrm{sech}^2\left(\xi\sqrt{\frac{3c}{2}}\right).$$

A5.12 $\xi = x - t + \dfrac{c_0}{3\alpha\Delta}\{(1+\varepsilon\Delta\alpha t)^{-3} - 1\}$ $(\sim x - t - \varepsilon c_0 t$ as $\varepsilon\Delta \to 0)$.

A5.13 $c = \dfrac{1}{2}\left(\dfrac{1}{3}\alpha^2 - \lambda\sqrt{\pi/\alpha}\right)$; $\dfrac{3}{2}\dfrac{a^2}{b} = -\alpha^2 - \lambda\left(1 - \dfrac{1}{\sqrt{2}}\right)\sqrt{\pi/\alpha}$.

A5.14 Form
$$p_{1z} = (2U_0 - U)w_{0\xi} + \frac{1}{\mathscr{R}}(w_{1zz} + w_{0zz})$$
with
$$p_1 + \eta_0 p_{0z} - \eta_1 - \frac{2}{\mathscr{R}}\{w_{1z} + \eta_0 w_{0z} - \eta_{0\xi} u_{0z} - \eta_{1\xi} U'\} = 0 \text{ on } z = 1$$
and $U = U_0(2z - z^2)$; u_1 and w_1 as given.

A5.15 $\alpha = \frac{1}{2}\left(\sqrt{\lambda^2+4} - \lambda\right) > 0$; $\beta = \frac{1}{2}\left(\lambda \pm \sqrt{\lambda^2 - 4}\right)$.

A5.16 Saddle at $(0,0)$; stable node at $(1,0)$ for $\lambda \geq 2$; stable spiral point for $0 < \lambda < 2$; a focus at $(1,0)$ for $\lambda = 0$.

A5.17 Stable node at $(1,0)$ for $0 \leq 1/\lambda \leq \frac{1}{2}$.

A5.18 $\eta_0 = \frac{1}{2}\{1 + \tanh(-X/2)\}$,
$\eta_1 = -\frac{1}{4}\mathrm{sech}^3(-X/2)\ln\{\mathrm{sech}^2(-X/2)\}$.