4

Perturbation analysis

In terms of the methods of asymptotic analysis, so far we have studied integral asymptotics associated with Fourier integrals that represent solutions of linear PDEs. Now, suppose we want to study physical problems like the propagation of waves in the ocean, or the propagation of light in optical fibers; the general equations obtained from first principles in these cases are the Euler or Navier–Stokes equations governing fluid motion on a free surface and Maxwell's electromagnetic (optical wave) equations with nonlinear induced polarization terms. These equations are too difficult to handle using linear methods or, in most situations, by direct numerical simulation. Loosely speaking, these physical equations describe "too much".

Mathematical complications often arise when one has widely separated scales in the problem, e.g., the wavelength of a typical ocean wave is small compared to the ocean's depth and the wavelength of light in a fiber is much smaller than the fiber's length or transmission distance. For example, the typical wavelength of light in an optical fiber is of the order of 10^{-6} m, whereas the length (distance) of an undersea telecommunications fiber is of the order of 10,000 km or 10^7 m, i.e., 13 orders of magnitude larger than the wavelength! Therefore, if we were to try solving the original equations numerically – and resolve both the smallest scales as well as keep the largest ones – we would require vast amounts of computer time and memory.

Yet it is imperative to retain some of the features from all of these scales, otherwise only limited information can be obtained. Therefore, before we can study the solutions of the governing equations we must first obtain useful and manageable equations. For that we need to simplify the general equations, while retaining the essential phenomena we want to study. This is the role of perturbation analysis and the reason why perturbation analysis often plays a decisive role in the physical sciences.

In this chapter we will introduce some of the perturbation methods that will be used later in physical problems. The reader can find numerous references on perturbation techniques, cf. Bender and Orszag (1999), Cole (1968), and Kevorkian and Cole (1981). An early paper with many insightful principles of asymptotic analysis ("asymptotology") is Kruskal (1963). We will begin with simpler "model" problems (ODEs) before discussing more complex physical problems.

4.1 Failure of regular perturbation analysis

Consider the ODE

$$\frac{d^2y}{dt^2} + y = \varepsilon y, \tag{4.1}$$

where ε is a small (constant) parameter, i.e., $|\varepsilon| \ll 1$. Suppose we try to expand the solution as

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots ,$$

where the y_j, $j = 0, 1, 2, \ldots$, are assumed to be $O(1)$ functions that are to be found. This is usually called regular perturbation analysis, because it is the simplest and nothing out of the ordinary is used. Substituting this expansion into (4.1) leads to an infinite number of equations, i.e., a perturbation series of equations. We group terms according to their power of ε. To $O(1)$, i.e., for those terms that have no ε before them, we get

$$O(1): \qquad y_{0,tt} + y_0 = 0.$$

This is called the leading-order equation. Its solution, assumed to be real, called the leading-order solution, is conveniently given by

$$y_0(t) = A_0 e^{it} + A_0^* e^{-it} = A_0 e^{it} + \text{c.c.},$$

where A_0 is a complex constant and c.c. denotes the complex conjugate of the terms to its left. Clearly, this solution does not have any ε in it, i.e., it completely disregards the εy term in (4.1). However, it may still be relatively close to the exact solution, because $|\varepsilon| \ll 1$. Two questions naturally arise:

(a) How can we improve the approximation?

(b) Is the solution we obtained a good approximation of the exact solution?

We can try to improve the approximation by going to the next order of the perturbation series, i.e., equating the terms that multiply ε^1. This leads to

$$y_{1,tt} + y_1 = y_0.$$

Since we already found y_0, this is an equation for y_1, the next order in the perturbation series (called the $O(\varepsilon)$ correction). Substituting in the y_0 above gives

$$O(\varepsilon): \qquad y_{1,tt} + y_1 = A_0 e^{it} + \text{c.c.} \tag{4.2}$$

Ignoring unimportant homogeneous solutions, the above equation has a solution of the form

$$y_1(t) = A_1 t e^{it} + \text{c.c.},$$

where A_1 is a constant, and substituting it into (4.2) leads to the condition

$$2i\, A_1 = A_0.$$

Therefore, we get that

$$y_1(t) = -\frac{it}{2} A_0 e^{it} + \text{c.c.},$$

where we will omit additional terms due to the homogeneous solution. We can continue in a similar manner to obtain y_2: the equation for y_2 is found to be

$$y_{2,tt} + y_2 = y_1;$$

we can guess a solution of the form $y_2 = A_2 t^2 e^{it} + A_3 t e^{it} + \text{c.c.}$ and we obtain

$$8A_2 = -A_0, 8A_3 = -iA_0.$$

Using the first condition, $2iA_1 = A_0$, we arrive at

$$y_2 = -\left(\frac{t^2}{8} + i\frac{t}{8}\right) A_0 e^{it} + \text{c.c.}$$

Let us inspect the solution we have so far obtained:

$$\begin{aligned} y &= y_0 + \varepsilon y_1 + \varepsilon^2 y_2, \\ &= (A_0 e^{it} + \text{c.c.}) - \varepsilon\left(\frac{it}{2} A_0 e^{it} + \text{c.c.}\right) - \varepsilon^2\left(\left(\frac{t^2}{8} + i\frac{t}{8}\right) A_0 e^{it} + \text{c.c.}\right), \qquad (4.3) \\ &= A_0\left(1 - \varepsilon\frac{it}{2} - \varepsilon^2\left(\frac{t^2}{8} + \frac{it}{8}\right)\right) e^{it} + \text{c.c.} \qquad (4.4) \end{aligned}$$

It is not difficult to convince ourselves that if we continue in the same fashion to include higher-order terms we will be adding higher-order monomials of t inside the parentheses. So long as t is $O(1)$, our approximate solution should be quite close to the exact one. However, when t becomes large a problem arises with our solution. Indeed, when $t = O(1/\varepsilon)$ the terms $(\varepsilon t)^n$ inside the brackets are of $O(1)$, hence our formal solution is not asymptotic (i.e., the remainder of any finite number of terms is not smaller than the previous terms). It must be realized at this stage that this problem is not a feature of the exact solution itself. Indeed, it is straightforward to obtain the exact solution of the linear equation (4.1):

$$y_{\text{exact}}(t) = Ae^{i\sqrt{1-\varepsilon}\,t} + \text{c.c.}, \tag{4.5}$$

where A is an arbitrary constant. This solution is finite and infinitely differentiable for any value of t. Moreover, from the binomial expansion $\sqrt{1-\varepsilon} = 1 - \varepsilon/2 - \varepsilon^2/8 + \cdots$ and by further expanding the exponential we find the approximation above; i.e., (4.4).

The failure is therefore on the part of the method we used. Inspecting the solution (4.4) we see that the problem arises when $t = O(1/\varepsilon)$ because the terms inside the parentheses become of the same order when $t = O(1/\varepsilon)$. Hence the perturbation method breaks down because the higher-order terms are not small, violating our assumption. The violation is due to the terms that *grow arbitrarily large with t*. Such terms are referred to as "secular" or "resonant". Thus $t^n e^{it}$ is a secular term for any $n > 1$. If we trace our steps back we see that the secular terms arise because the right-hand side of the non-homogeneous equations we obtain are in the kernels of the left-hand sides, e.g., in (4.2) the left-hand side Ae^{it} is a solution of the homogeneous equation for y_1. This problem is inevitable when using the regular perturbation method as outlined above. So how can we resolve it? This is where so-called "singular" perturbation methods enter the picture.

4.2 Stokes–Poincaré frequency-shift method

We would like to find an approximate solution to (4.1) that is valid for large values of t, e.g., up to $t = O(1/\varepsilon)$. One perturbative approach is what we will call the Stokes–Poincaré or frequency-shift method; the essential ideas go back to Stokes (see Chapter 5) and later Poincaré. Although it does not work in every case, it is often the most efficient method at hand. The idea of the frequency-shift method is explained below using (4.1) as an example. We define a new time variable $\tau = \omega t$, where ω is called the "frequency". Since $d/dt = \omega d/d\tau$, the equation takes the form

$$\omega^2 \frac{d^2y}{d\tau^2} + y = \varepsilon y.$$

We now expand the solution as before

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots,$$

but also expand ω perturbatively as

$$\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots .$$

Substituting these expansions into the ODE and collecting the same powers of ε gives the leading-order equation

$$O(1): \qquad y_{0,\tau\tau} + y_0 = 0,$$

which is the same as we got using regular perturbation analysis. The leading-order solution is therefore

$$y_0(\tau) = Ae^{i\tau} + \text{c.c.}$$

The next-order equation is given by

$$O(\varepsilon): \qquad y_{1,\tau\tau} + y_1 = -2\omega_1 y_{0,\tau\tau} + y_0.$$

This equation has a term containing ω_1, which is a modification of the regular perturbation analysis. Substituting the leading-order solution into the right-hand side leads to

$$y_{1,\tau\tau} + y_1 = (1 + 2\omega_1)Ae^{i\tau} + \text{c.c.}$$

By choosing $\omega_1 = -1/2$, we eliminate the right-hand side and therefore avoid the secular term in the solution. Without loss of generality, we choose the trivial homogeneous solution, i.e., $y_1 = 0$ (since we can "incorporate them" in the leading-order solution). We can continue in this fashion to obtain ω_2; the next-order equation is given by

$$y_{2,\tau\tau} + y_2 = -\left(2\omega_2 + \omega_1^2\right) y_{0,\tau\tau} + y_1 - 2\omega_1 y_1.$$

Using $y_1 = 0$ gives

$$y_{2,\tau\tau} + y_2 = -\left(2\omega_2 + \omega_1^2\right) y_{0,\tau\tau},$$

and choosing $\omega_2 = -\omega_1^2/2 = -1/8$ eliminates the secular terms; finally, we choose $y_2 = 0$.

Let us inspect our approximate solution: the only non-zero part of y is y_0, but it contains ω to order $O(\varepsilon^2)$:

$$\begin{aligned} y(t) &= Ae^{i\tau} + \text{c.c.} \ = Ae^{i\omega t} + \text{c.c.} \\ &= Ae^{i(1+\varepsilon\omega_1+\varepsilon^2\omega_2+\cdots)t} + \text{c.c.} \ = Ae^{i\left[1-\frac{\varepsilon}{2}-\frac{\varepsilon^2}{8}+\cdots\right]t} + \text{c.c.} \end{aligned}$$

Hence we expect the solution to be valid for $t = O(1/\varepsilon^2)$. To verify this, let us expand the square-root in the exact solution (4.5) in powers of ε:

$$y_{\text{exact}}(t) = Ae^{i\sqrt{1-\varepsilon}\,t} + \text{c.c.} \;= Ae^{i\left[1-\frac{\varepsilon}{2}-\frac{\varepsilon^2}{8}+O(\varepsilon^3)\right]t} + \text{c.c.},$$

so our exact and approximate solution agree to the first three terms in the exponent. Note that the remainder term in the exponent is $O(\varepsilon^3 t)$, which is small for $t = o(1/\varepsilon^3)$. Furthermore, the approximate solution can be made valid for larger values of t by finding a sufficient number of terms in the frequency expansion. Note that the higher-order homogeneous solutions are chosen as zero (i.e, $y_k = 0$ for $k \geq 1$) and the higher non-homogeneities only change the amplitude, not the frequency, of the leading-order solution.

This example demonstrates the ease and efficiency of the frequency-shift method. Unfortunately this method does not always work. For example, let us consider the equation with a damping term

$$\frac{d^2y}{dt^2} + y = -\varepsilon\frac{dy}{dt}$$

and repeat the analysis using the frequency-shift method. Defining $\tau = \omega t$ as before leads to the equation

$$\omega^2\frac{d^2y}{d\tau^2} + y = -\varepsilon\omega\frac{dy}{d\tau}.$$

Expanding the solution and ω in powers of ε we get

$$(1+\varepsilon\omega_1+\varepsilon^2\omega_2+\cdots)^2\frac{d^2y}{d\tau^2} + y = -\varepsilon(1+\varepsilon\omega_1+\varepsilon^2\omega_2+\cdots)\frac{dy}{d\tau}.$$

The leading-order solution is found to be

$$y_0(\tau) = Ae^{i\tau} + \text{c.c.}$$

The next-order equation is then

$$O(\varepsilon): \qquad y_{1,\tau\tau} + y_1 = -2\omega_1 y_{0,\tau\tau} - y_{0,\tau}.$$

Substituting the leading-order solution into the right-hand side gives that

$$\begin{aligned} y_{1,\tau\tau} + y_1 &= 2\omega_1(Ae^{i\tau} + \text{c.c.}) - (iAe^{i\tau} + \text{c.c.}) \\ &= (2\omega_1 - i)Ae^{i\tau} + (2\omega_1 + i)A^*e^{-i\tau}. \end{aligned}$$

It may appear at first glance that we can avoid the secularity by choosing $\omega_1 = i/2$; however, this is not so, because that only eliminates the terms that multiply $e^{i\tau}$. Indeed, to eliminate the terms that multiply $e^{-i\tau}$ we would need to choose $\omega_1 = -i/2$. In other words, we cannot choose ω_1 consistently to eliminate all the secular terms. Without eliminating the secular terms we are back to the original problem, i.e., the solution will not be valid for $t \sim O(1/\varepsilon)$. Another method is required.

The frequency-shift method usually works for conservative systems. The addition of the damping term results in a non-conservative (non-energy conserving) problem.

4.3 Method of multiple scales: Linear example

Since the frequency-shift method works in some cases but not in others, we introduce the method of multiple scales, cf. Bender and Orszag (1999), Cole (1968) and Kevorkian and Cole (1981). The method of multiple scales has its roots in the method of averaging, which have "rapid phases" and the other terms are "slowly varying", cf. Krylov and Bogoliubov (1949), Bogoliubov and Mitropolsky (1961); see also Sanders et al. (2009a).

The method of multiple scales is much more robust than the frequency-shift method. It works for a wide range of problems and, in particular, for the problems we will tackle here and for many problems that arise in nonlinear waves, as we will see later in this book. The idea is to introduce *fast* and *slow* variables into the equation. Let us consider the damped example above, where the frequency-shift method failed,

$$\frac{d^2y}{dt^2} + y = -\varepsilon\frac{dy}{dt}. \tag{4.6}$$

We can introduce the slow variable $T = \varepsilon t$ and consider y to be a function of *two variables*: the "slow" variable T and the "fast" one t. We can then denote y as

$$y = y(t, T; \varepsilon),$$

where the ε is added here in order to stress that we will expand the solutions in powers of ε. To be precise, we introduce two variables $\tilde{t} = t$ and $T = \varepsilon t$, however, we will omit the "~" from $\tilde{t}$ hereafter for notational convenience.

Using the chain rule for differentiation gives

$$\frac{dy}{dt} \to \frac{\partial y}{\partial t} + \varepsilon\frac{\partial y}{\partial T}$$

or in operator form

$$\frac{d}{dt} \to \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T}$$

and the equation is transformed into

$$\left(\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial T}\right)^2 y + y = -\varepsilon\left(\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial T}\right)y,$$

or

$$y_{tt} + 2\varepsilon y_{tT} + \varepsilon^2 y_{TT} + y = -\varepsilon(y_t + \varepsilon y_T). \tag{4.7}$$

It is important to note here that we have transformed an ODE into a PDE! Thus, formally, we have complicated the problem by this transformation, but this complication is valuable, as we will now see. We now expand the solution as

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots,$$

substitute into the PDE (4.7), and collect like powers of ε. This leads to the leading-order equation

$$O(1): \qquad y_{0,tt} + y_0 = 0.$$

Strictly speaking, this is a PDE since y depends on T as well. So the general solution is given by

$$y_0(t, T) = A(T)e^{it} + \text{c.c.},$$

where $A(T)$ is an arbitrary function of T. The next-order equation is found to be

$$O(\varepsilon): \qquad y_{1,tt} + y_1 = -2y_{0,tT} - y_{0t}.$$

Substituting the leading-order solution into the right-hand side gives that

$$y_{1,tt} + y_1 = -i(2A_T + A)e^{it} + \text{c.c.} \tag{4.8}$$

As in the frequency-shift method, we would like to eliminate the secular terms. So we require that

$$2A_T + A = 0. \tag{4.9}$$

This ODE for $A(T)$ gives

$$A(T) = A_0 e^{-T/2}, \tag{4.10}$$

where A_0 is constant.

Now we can choose the homogeneous solution $y_1 = 0$, as in the frequency-shift method. So far, our approximate solution is

$$y(t, T) \sim A(T)e^{it} + \text{c.c.} = A_0 e^{it-T/2} + \text{c.c.}$$

In terms of the original variable t, this reads

$$y(t) \sim A_0 e^{it-\varepsilon t/2} + \text{c.c.} = A_0 e^{\left(i-\frac{\varepsilon}{2}\right)t} + \text{c.c.} \tag{4.11}$$

We can continue to obtain higher orders by allowing $A(T)$ to vary perturbatively, in such a way as to avoid secular terms in the expansion of y_k for $k \geq 1$. Our method is to expand the equation for $A(T)$ *without* including additional time-scales. Finding higher-order terms in the *equation* governing the slowly varying amplitude is usually what one desires in physical problems. However, *care must be taken* here, as we will see below. In the example above, we perturb (4.9) as

$$2A_T + A = \varepsilon r_1 + \varepsilon^2 r_2 + \cdots . \tag{4.12}$$

Substituting this into (4.8) shows that

$$y_{1,tt} + y_1 = -i(2A_T + A)e^{it} + \text{c.c.} = \underbrace{-i\varepsilon r_1 e^{it} + \text{c.c.}}_{\varepsilon R_1} + O(\varepsilon^2). \tag{4.13}$$

Notice that there is an $O(\varepsilon)$ residual term on the right-hand side. That term must be added to the right-hand side of the next-order equation. When doing so we can choose $y_1 = 0$ in this case (but see the next example!). Therefore, the $O(\varepsilon^2)$ equation corresponding to (4.7) is

$$\begin{aligned} y_{2,tt} + y_2 &= -y_{0,TT} - y_{0,T} - 2y_{1,tT} - y_{1,t} - R_1, \\ &= -(A_{TT} + A_T)e^{it} - ir_1 e^{it} + \text{c.c.}, \end{aligned}$$

where we have used $y_1 = 0$ and $R_1 = (-ir_1 e^{it} + \text{c.c.})$, the coefficient of the $O(\varepsilon)$ residual term from (4.13). In order to eliminate the secular terms in this equation one requires that

$$r_1 = i(A_{TT} + A_T).$$

Substituting r_1 in (4.12) leads to

$$(2A_T + A) = i\varepsilon(A_{TT} + A_T). \tag{4.14}$$

This is a higher-order equation, which at first glance may be a concern. The method for solving this equation, for $\varepsilon \ll 1$, is to solve (4.14) recursively; i.e., replace the A_{TT} term with lower-order derivatives of A by using the equation itself, while maintaining $O(\varepsilon)$ accuracy. Let us see how that works in this example. Equation (4.14) implies that

$$A_T = -\frac{1}{2}A + O(\varepsilon). \tag{4.15}$$

Differentiating with respect to T and using (4.15) to replace A_T gives

$$A_{TT} = -\frac{1}{2}A_T + O(\varepsilon) = -\frac{1}{2}\left[-\frac{1}{2}A + O(\varepsilon)\right] + O(\varepsilon) = \frac{1}{4}A + O(\varepsilon).$$

We can now substitute A_{TT} using this equation and substitute A_T using (4.15) in (4.14) to find

$$(2A_T + A) = -i\varepsilon\left[\frac{1}{4}A + O(\varepsilon) - \frac{1}{2}A + O(\varepsilon)\right].$$

Simplifying leads to the equation

$$A_T = -\frac{1}{2}\left(1 + \frac{i\varepsilon}{4}\right)A + O(\varepsilon^2),$$

whose solution to $O(\varepsilon)$ is

$$A(T) = A(0)e^{-\left(\frac{1}{2}+\frac{i\varepsilon}{8}\right)T}.$$

The above equation is a higher-order improvement of (4.10). Substituting $A(T)$ and using $T = \varepsilon t$ in (4.11) gives

$$y \sim A(0)e^{(1-\varepsilon^2/8)it}e^{-\varepsilon t/2} + \text{c.c.} \tag{4.16}$$

This approximate solution is valid for times $t = O(1/\varepsilon^2)$.

We can now compare our result to the exact solution that can be found by substituting $y_s = e^{rt}$ into (4.6). Doing so gives $r^2 + \varepsilon r + 1 = 0$ and therefore

$$y_{\text{exact}}(t) = A(0)e^{\left(-\frac{\varepsilon}{2}+\frac{i}{2}\sqrt{4-\varepsilon^2}\right)t} + \text{c.c.}$$

If we now expand the exponent in the exact solution in powers of ε, we find that it agrees with the exponent in (4.16) to $O(\varepsilon^2)$, which is the order of accuracy we kept in the perturbation analysis:

$$-\frac{\varepsilon}{2} + \frac{i}{2}\sqrt{4-\varepsilon^2} = -\frac{\varepsilon}{2} + i\left(1 - \frac{\varepsilon^2}{8}\right) + O(\varepsilon^3).$$

This agrees with our approximate solution in (4.16).

4.4 Method of multiple scales: Nonlinear example

Below we give an example of using the method of multiple scales for the nonlinear ODE

$$\frac{d^2y}{dt^2} + y - \varepsilon y^3 = 0.$$

The first steps are similar to the previous example. We set $T = \varepsilon t$ and obtain the equation

$$y_{tt} + y + \varepsilon(2y_{tT} - y^3) + \varepsilon^2 y_{TT} = 0.$$

The leading-order equation and solution are the same as in the previous example,

$$y_0(t, T) = A(T)e^{it} + \text{c.c.}$$

The next-order equation is

$$O(\varepsilon): \qquad y_{1,tt} + y_1 = -2y_{0,tT} - y_0^3.$$

Substituting the leading-order solution in the right-hand side and expanding the nonlinear term leads to

$$\begin{aligned} y_{1,tt} + y_1 &= -(2iA_T e^{it} + \text{c.c.}) + (Ae^{it} + A^* e^{-it})^3 \\ &= -(2iA_T e^{it} + \text{c.c.}) + (A^3 e^{3it} + 3A^2 A^* e^{it} + \text{c.c.}) \\ &= \underbrace{-(2iA_T - 3|A|^2 A)e^{it}}_{\text{secular}} + \underbrace{A^3 e^{3it}}_{\text{non-secular}} + \text{c.c.} \end{aligned}$$

It is important to distinguish the terms that multiply e^{it} from those that multiply e^{3it}, because the former is a secular term, i.e., it satisfies the homogeneous solution for y_1, whereas the second one does not. Therefore, we must remove the secular terms, which multiply e^{it}, but keep the non-secular terms, which multiply e^{3it}, in the equation for y_1 – the non-secular term results in a bounded contribution to y_1. To remove secularity, we require that

$$2iA_T = 3|A|^2 A \tag{4.17}$$

and the equation for y_1 becomes

$$y_{1,tt} + y_1 = A^3 e^{3it} + \text{c.c.} \tag{4.18}$$

The solution of the (complex-valued) ODE (4.17) can be found by noting that $|A|^2$ is a conserved quantity in time. One way to see this is by multiplying (4.17) by A^*, subtracting the complex conjugate, and integrating:

$$2iA_T A^* + 2iA_T^* A = 3|A|^2 AA^* - 3|A|^2 A^* A$$

and therefore $i(|A|^2)_T = 0$. Hence $|A|^2(T) = |A|^2(0) = |A_0|^2$, where A_0 is our arbitrary constant. Using this conservation relation we can rewrite (4.17) as

$$2iA_T = 3|A_0|^2 A.$$

This is now a *linear equation* for A and its solution is given by

$$A(T) = A_0 e^{-\frac{3i}{2}|A_0|^2 T} = A_0 e^{-\frac{3i}{2}|A_0|^2 \varepsilon t}.$$

Note $A(T)$ depends on $T = \varepsilon t$, so we can assume that $A(T)$ is constant when integrating the equation for y_1 in (4.18). Assuming that $y_1 = Be^{3it}$ + c.c. gives $B = -A^3/8$, which is bounded, and therefore

$$y_1(t) = -\frac{1}{8}A^3(T)e^{3it} + \text{c.c.} + O(\varepsilon) = -\frac{1}{8}A_0^3 e^{-\frac{9i}{2}|A_0|^2\varepsilon t}e^{3it} + \text{c.c.} + O(\varepsilon).$$

Finally, the perturbed solution we find is

$$\begin{aligned} y(t) &= A_0 e^{\left(1-\frac{3\varepsilon}{2}|A_0|^2\right)it} + \varepsilon y_1(t) + \text{c.c.}, \\ &= A_0 e^{\left(1-\frac{3\varepsilon}{2}|A_0|^2\right)it} - \frac{\varepsilon}{8}A_0^3 e^{-\frac{9i}{2}|A_0|^2\varepsilon t}e^{3it} + \text{c.c.} \end{aligned}$$

Inspecting our solution we note that the second term (the one multiplied by ε) plays a minor role compared with the first term, since the amplitude of the second term is bounded and so remains $O(\varepsilon)$ small relative to the first term for all values of t. We therefore focus our attention on the first term: its effective frequency is given by

$$\Omega = 1 - \frac{3\varepsilon}{2}|A_0|^2. \tag{4.19}$$

Therefore, when $\varepsilon > 0$, the additional frequency contribution decreases with amplitude $|A_0|$ (beyond the linear solution) and the period of oscillations increases (since $T = 2\pi/\Omega$). This is sometimes called a “soft spring”, in analogy with a spring whose period is elongated compared with a linear spring. Conversely, when $\varepsilon < 0$ the frequency increases and the period of oscillations decreases. This is called a “hard spring”.

Since the equation is conservative and there is only a (nonlinear) frequency shift, this problem can also be done by the frequency-shift method. This is left as an exercise.

4.5 Method of multiple scales: Linear and nonlinear pendulum

For our final application of multiple scales to ODEs, we will look at a nonlinear pendulum with a slowly varying length; see Figure 4.1. Newton’s second law of motion gives

$$ml\ddot{y} + mg\sin(y) = 0$$

as the equation of motion, where m is the pendulum mass, g the gravitational constant of acceleration, and l the slowly varying length. This implies that

$$\ddot{y} + \rho^2(\varepsilon t)\sin(y) = 0, \tag{4.20}$$

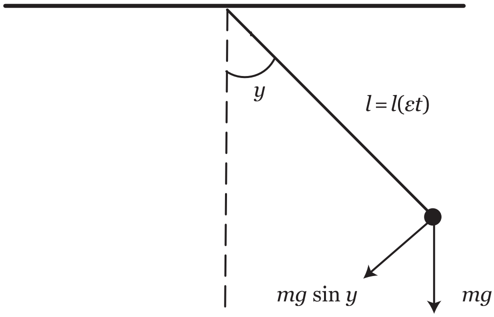

Figure 4.1 Nonlinear pendulum.

where for convenience we denote $\rho^2(\varepsilon t) \equiv g/l(\varepsilon t)$, or more simply $\rho^2(T)$ with $T \equiv \varepsilon t$, which we assume is smooth. If the length were constant, an exact solution would be available in terms of elliptic functions.

Associated with this pendulum is what is usually called an *adiabatic invariant*, i.e., a quantity in the system that is asymptotically invariant when a system parameter is slowly (adiabatically) changed. While analyzing the pendulum problem, we will look for adiabatic invariants. Adiabatic invariants received much attention during the development of the theory of quantum mechanics, cf. Goldstein (1980) and Landau and Lifshitz (1981), and more recently in such areas as plasma physics, particle accelerators, and galactic dynamics, etc.

4.5.1 Linear pendulum

To start, we will consider the linear problem

$$\ddot{y} + \rho^2(\varepsilon t)y = 0. \tag{4.21}$$

Suppose we assume a "standard" multi-scale solution in the form $y(t) = y(t, T; \varepsilon)$. Expand d/dt and y as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T},$$
$$y(t) = y_0(t, T) + \varepsilon y_1(t, T) + \varepsilon^2 y_2(t, T) + \cdots$$

and substitute them into (4.21). Collecting like powers of ε gives

$O(1)$:

$$Ly_0 \equiv \partial_t^2 y_0 + \rho^2(T)y_0 = 0.$$

$O(\varepsilon)$:

$$Ly_1 = -2\frac{\partial^2 y_0}{\partial t \partial T}.$$

The leading-order solution is

$$y_0 = A(T)\exp(i\rho t) + \text{c.c.}$$

Notice that

$$\partial_T y_0 = (A_T + it\rho_T)\exp(i\rho t) + \text{c.c.}$$

grows as $t \to \infty$; i.e., it is secular! The problem is that we chose the wrong fast time-scale. We modify the fast scale in y as follows:

$$y(t) = y(\Theta(t), T; \varepsilon),$$

where $\Theta_t = \omega(\varepsilon t)$ and ω will be chosen so that the leading-order solution is not secular. Expand d/dt as

$$\frac{d}{dt} = \Theta_t \frac{\partial}{\partial \Theta} + \varepsilon \frac{\partial}{\partial T} = \omega(T)\frac{\partial}{\partial \Theta} + \varepsilon \frac{\partial}{\partial T}.$$

A little care must be taken in calculating d^2/dt^2, since ω is a function of the slow variable: we get

$$\begin{aligned}\frac{d^2}{dt^2} &= \left(\omega(T)\frac{\partial}{\partial \Theta} + \varepsilon \frac{\partial}{\partial T}\right)\left(\omega(T)\frac{\partial}{\partial \Theta} + \varepsilon \frac{\partial}{\partial T}\right)\\ &= \omega^2 \frac{\partial^2}{\partial \Theta^2} + \varepsilon\left[2\omega \frac{\partial^2}{\partial \Theta \partial T} + \omega_T \frac{\partial}{\partial \Theta}\right] + \varepsilon^2 \frac{\partial^2}{\partial T^2}.\end{aligned} \tag{4.22}$$

Substituting this into (4.21) and expanding $y = y_0 + \varepsilon y_1 + \cdots$ we get the first two equations

$O(1)$:

$$\omega^2(T)\frac{\partial^2 y_0}{\partial \Theta^2} + \rho^2(T) y_0 = 0.$$

$O(\varepsilon)$:

$$\omega^2(T)\frac{\partial^2 y_1}{\partial \Theta^2} + \rho^2(T) y_1 = -\left(\omega_T \frac{\partial y_0}{\partial \Theta} + 2\omega \frac{\partial^2 y_0}{\partial T \partial \Theta}\right).$$

The leading-order solution is

$$y_0 = A(T)\exp\left(i\frac{\rho}{\omega}\Theta\right) + \text{c.c.}$$

To prevent secularity at this order we require that

$$\frac{\partial y_0}{\partial T} = (A_T + i(\rho/\omega)_T \theta A)e^{i(\rho/\omega)\theta}$$

is bounded in θ. Hence we must take ω/ρ to be constant in order to remove the secular term. The choice of constant does not affect the final result, so for convenience we take $\omega = \rho$. The order ε equation then becomes, after substituting in the expression for y_0,

$$\rho^2(T)\left(\frac{\partial^2 y_1}{\partial\Theta^2} + y_1\right) = -\left(i\rho_T A e^{i\Theta} + 2iA_T\rho e^{i\Theta} + \text{c.c.}\right).$$

To remove secular terms, we require

$$2\rho A_T + A\rho_T = 0,$$
$$2\rho A_T^* + A^*\rho_T = 0.$$

Multiplying the first equation by A^*, the second by A, and then adding, we find that

$$\frac{\partial}{\partial T}\left(\rho|A|^2\right) = 0 \quad \Rightarrow \quad \rho(T)|A(T)|^2 = \rho(0)|A(0)|^2 = \frac{E}{\rho},$$

where $E = \rho^2 A^2$ is related to the unperturbed energy $E = \dot{y}^2/2 + \rho^2 y^2/2$, from (4.21); thus $\rho(T)|A(T)|^2 = E/\rho$ is constant in time. This is usually called the adiabatic invariant. Notice that $\rho(T)|A(T)|^2$ is constant on the same time-scale as the length is being varied. Also, using separation of variables on the equation for A_T,

$$\frac{A_T}{A} = \frac{-\rho_T}{2\rho} \quad \Rightarrow \quad \log(A\rho^{1/2}) = \text{constant} \quad \Rightarrow \quad A(T) = \frac{C}{\rho^{1/2}(T)}$$

where C is constant. Since $\omega = \rho$,

$$\Theta(t) = \int_0^t \rho(\varepsilon t')\, dt' = \frac{1}{\varepsilon}\int_0^{\varepsilon t} \rho(s)\, ds.$$

The leading-order solution is then

$$y(t) \sim \frac{C}{\sqrt{\rho(\varepsilon t)}} \exp\left\{\frac{i}{\varepsilon}\int_0^{\varepsilon t} \rho(s)\, ds\right\} + \text{c.c.}$$

Alternatively, we can arrive at the same approximate solution using the so-called WKB method,[1] cf. Bender and Orszag (1999). Instead of introducing multiple time-scales, let $T = \varepsilon t$ and simply change variables in (4.21) to get

$$\varepsilon^2 \frac{d^2 y}{dT^2} + \rho^2(T) y(T) = 0.$$

[1] The WKB method is named after Wentzel, Kramers and Brillouin who used the method extensively. However, these ideas were used by others including Jeffries and so is sometimes referred to as the WKBJ method.

If ρ were constant, the solution would be $y = e^{i\rho/\varepsilon} + \text{c.c.}$ This suggests that we look for a solution in the form $y \sim e^{i\phi(T;\varepsilon)/\varepsilon} + \text{c.c.}$ Using this ansatz, we find

$$-\phi_T^2 + i\varepsilon\phi_{TT} + \rho^2 = 0.$$

We now expand ϕ as $\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots$ to get the leading-order equation

$$\frac{\partial\phi_0}{\partial T} = \pm\rho(T) \Rightarrow \phi_0 = \pm\int_0^T \rho(s)\,ds + \mu_0,$$

where μ_0 is the constant of integration. The $O(\varepsilon)$ equation is $-2\phi_{0T}\phi_{1T} + i\phi_{0TT} = 0$, so

$$\frac{\partial\phi_1}{\partial T} = \frac{i}{2}\frac{\phi_{0TT}}{\phi_{0T}} = \frac{i}{2}\frac{\partial}{\partial T}\log|\phi_{0T}|,$$

which gives

$$\phi_1 = \frac{i}{2}\log|\phi_{0T}| = \frac{i}{2}\log(\rho).$$

Thus, in terms of the original variables:

$$\begin{aligned} y(t) &\sim \exp\left[i\left(\phi_0 + \varepsilon\phi_1\right)/\varepsilon\right], \\ &= \exp\left[\frac{i}{\varepsilon}\left(\int_0^{\varepsilon t}\rho(s)\,ds + \mu_0\right) - \frac{1}{2}\log(\rho(\varepsilon t))\right] + \text{c.c.} \end{aligned}$$

Setting $C = \exp(i\mu_0/\varepsilon)$,

$$y(t) \sim \frac{C}{\sqrt{\rho(\varepsilon t)}}\exp\left[\frac{i}{\varepsilon}\int_0^{\varepsilon t}\rho(s)\,ds\right] + \text{c.c.},$$

which is identical to the multiple-scales result.

4.5.2 Nonlinear pendulum

Now we will analyze the nonlinear equation, (4.20),

$$\frac{d^2y}{dt^2} + \rho^2(T)\sin(y) = 0.$$

These and more general problems were analyzed by Kuzmak (1959), see also Luke (1966). We will see that this problem is considerably more complicated than the linear problem: multiple scales are required. As in the linear problem, assuming $y = y(\Theta, T; \varepsilon) = y_0 + \varepsilon y_1 + \cdots$, $T = \varepsilon t$ and using (4.22) to expand d^2/dt^2, the leading- and first-order equations are, respectively,

$$\omega^2 y_{0\Theta\Theta} + \rho^2 \sin(y_0) = 0,$$
$$L(y_1) = \omega^2 y_{1\Theta\Theta} + \rho^2 \cos(y_0) y_1 = -(\omega_T y_{0\Theta} + 2\omega y_{0\Theta T}) = F_1.$$

The crucial part of the perturbation analysis is to understand the leading-order equation. Multiplying the leading-order equation by $y_{0\Theta}$, we find the "energy integral":

$$\frac{\partial}{\partial\Theta}\left(\frac{\omega^2}{2} y_{0\Theta}^2 - \rho^2 \cos(y_0)\right) = 0 \quad \Rightarrow \tag{4.23a}$$
$$\frac{\omega^2}{2} y_{0\Theta}^2 - \rho^2 \cos(y_0) = E(T) \quad \Rightarrow \tag{4.23b}$$
$$\omega^2 y_{0\Theta}^2 = 2(E + \rho^2 \cos(y_0)). \tag{4.23c}$$

Notice (i) that the coefficient ρ is constant with respect to Θ and so is E; (ii) the left-hand side of the integral equation (4.23b) is not necessarily positive. If we redefine $\tilde{E} = E + \rho^2$ so that $\omega^2 y_{0\Theta}^2/2 + \rho^2(1 - \cos y_0) = \tilde{E}$, then $\tilde{E} \geq 0$ and $\tilde{E}$ is an energy. However, we use E to simplfy our notation. Solving for $dy_0/d\Theta$ and separating variables gives

$$\int \frac{dy_0}{\sqrt{2\,(E + \rho^2 \cos y_0)}} = \int \frac{d\Theta}{\omega}.$$

We will be concerned with periodic solutions in Θ; see also the phase plane in Figure 4.2. Periodic solutions are obtained for values $|y| \leq y_*$ obtained from $\cos y_* = -E/\rho^2$.

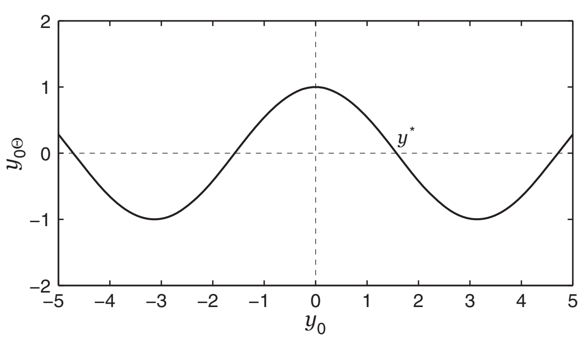

Figure 4.2 Phase plane.

The period of motion, P, is

$$P = \omega \oint \frac{dy_0}{\sqrt{2\left(E + \rho^2 \cos y_0\right)}},$$

where the loop integral is taken over one period of motion. Since $\cos(y_0)$ is even and symmetric, we can also calculate the period from

$$P = 4\omega \int_0^{y_*} \frac{dy_0}{\sqrt{2\left(E + \rho^2 \cos y_0\right)}}. \tag{4.24}$$

Since the motion is periodic, for any integer n

$$y_0(\Theta + nP, T) = y_0(\Theta, T).$$

At this stage P is a function of T, thus

$$\frac{\partial y_0(\Theta + nP, T)}{\partial T} = n \frac{\partial P}{\partial T} \frac{\partial y_0(\Theta, T)}{\partial \Theta} + \frac{\partial y_0(\Theta, T)}{\partial T}. \tag{4.25}$$

For a well-ordered perturbation expansion, the first derivative $\partial y_0(\Theta + nP, T)/\partial T$ must be bounded. However, from (4.25), $\partial y_0(\Theta + nP, T)/\partial T \to \infty$ as $n \to \infty$. We can eliminate this divergence if we set $\partial P/\partial T = 0$, i.e., we require the period to be constant. The choice of constant does not affect the final result, so we can take $P = 2\pi$ and then we have from (4.24),

$$\omega = \omega(E) = \frac{\pi}{2 \displaystyle\int_0^{y_*} \frac{dy_0}{\sqrt{2\left(E + \rho^2 \cos y_0\right)}}}. \tag{4.26}$$

To find out how E varies, we must go to the next-order equation, namely,

$$L y_1 = \left[\omega^2 \partial_\Theta^2 + \rho^2(T) \cos(y_0)\right] y_1 = F_1 = -\left(\omega_T y_{0\Theta} + 2\omega y_{0\Theta T}\right). \tag{4.27}$$

We will not actually solve the $O(\varepsilon)$ equation here, but instead we only need to derive a so-called *solvability condition* using a Fredholm alternative. We start by examining the homogeneous equation adjoint to (4.27): $L^* W = 0$, where L^* is the adjoint to the operator L, with respect to the inner product

$$\langle Ly, W \rangle = \int_0^{2\pi} (Ly)\, W \, d\Theta = \langle y, L^* W \rangle;$$

integration by parts shows that $L = L^*$, i.e., the operator L is self-adjoint. Now consider

$$\begin{aligned} L y_1 &= \left[\omega^2 \partial_\Theta^2 + \rho^2(T) \cos(y_0)\right] y_1 = F_1, \\ L^* W &= LW = 0, \end{aligned}$$

where W is the periodic solution of the adjoint equation. There is another, non-periodic solution to the adjoint problem (obtained later for completeness); but since we are only interested in periodic solutions, we do not need to consider it in the secularity analysis. Multiply the first equation by W, the second by y_1, and subtract to find

$$\omega^2 \frac{\partial}{\partial\Theta}\left(W y_{1\Theta} - W_\Theta y_1\right) = W F_1.$$

Integrating this over one period and using the periodicity of the solution y_1 gives

$$\int_0^{2\pi} W F_1 \, d\Theta = 0 \tag{4.28}$$

as a necessary condition for a periodic solution of the $O(\varepsilon)$ equation to exist. This orthogonality condition must hold for the periodic solution, W, of the homogeneous problem, which we will now determine. Operate on the leading-order equation with $\partial/\partial\Theta$ to find

$$\omega^2 y_{0\Theta\Theta\Theta} + \rho^2 \cos(y_0) y_{0\Theta} = 0,$$

i.e., $y_{0\Theta}$ is a solution of $LW = 0$; hence we set $W_1 = y_{0\Theta}$. This is the periodic, homogeneous solution. To find the non-periodic homogeneous solution operate on the leading-order equation with $\partial/\partial E$ to find

$$L y_{0E} = \omega^2 y_{0E\Theta\Theta} + \rho^2 \cos(y_0) y_{0E} = -(\omega^2)_E y_{0\Theta\Theta}.$$

On the other hand, for any constant α,

$$\begin{aligned} L(\alpha\Theta y_{0\Theta}) &= \omega^2 \frac{\partial^2}{\partial\Theta^2}(\alpha\Theta y_{0\Theta}) + \rho^2 \cos(y_0)\alpha\Theta y_{0\Theta} \\ &= \alpha\left[2\omega^2 y_{0\Theta\Theta} + \Theta\left(\omega^2 y_{0\Theta\Theta\Theta} + \rho^2\cos(y_0) y_{0\Theta}\right)\right]. \end{aligned}$$

The term in parentheses vanishes and we have

$$L(\alpha\Theta y_{0\Theta}) = 2\alpha\omega^2 y_{0\Theta\Theta}.$$

Hence for the combination, $y_{0E} + \alpha\Theta y_{0\Theta}$:

$$\begin{aligned} L(y_{0E} + \alpha\Theta y_{0\Theta}) &= L(y_{0E}) + L(\alpha\Theta y_{0\Theta}) \\ &= -\left(\omega^2\right)_E y_{0\Theta\Theta} + 2\alpha\omega^2 y_{0\Theta\Theta}. \end{aligned}$$

Thus, if we set $\alpha = \omega_E/\omega$,

$$L(y_{0E} + \alpha\Theta y_{0\Theta}) = 0$$

and we have the second solution:

$$W_2 = y_{0E} + \frac{\omega_E}{\omega}\Theta y_{0\Theta},$$

cf. Luke (1966). This solution, however, is not generally periodic since $\omega_E \neq 0$ in the nonlinear problem. We therefore need only one homogeneous solution W_1 in the Fredholm alternative (4.28). But the non-periodic solutions are useful if one wishes to find the higher-order solutions. The solution y_1 to $Ly_1 = F_1$ can be obtained by using the method of variation of parameters. The solvability condition (4.28) now becomes

$$\int_0^{2\pi} \frac{\partial y_0}{\partial \Theta} F_1 \, d\Theta = 0,$$

$$\int_0^{2\pi} \frac{\partial y_0}{\partial \Theta} \left(\omega_T y_{0\Theta} + 2\omega y_{0\Theta T}\right) d\Theta = 0,$$

$$\frac{\partial}{\partial T} \int_0^{2\pi} \omega \left(\frac{\partial y_0}{\partial \Theta}\right)^2 d\Theta = 0.$$

We have therefore found an adiabatic invariant for this nonlinear problem:

$$\mathcal{A}(T) \equiv \omega(T) \int_0^{2\pi} \left(\frac{\partial y_0}{\partial \Theta}\right)^2 d\Theta = \mathcal{A}(0).$$

The invariant can also be written as

$$\mathcal{A}(0) = \omega(T) \int_0^{2\pi} \left(\frac{\partial y_0}{\partial \Theta}\right)^2 d\Theta = \omega(T) \oint \left(\frac{\partial y_0}{\partial \Theta}\right)^2 \frac{dy_0}{\frac{dy_0}{d\Theta}},$$

$$\mathcal{A}(0) = 4\omega(T) \int_0^{y_*} \frac{\partial y_0}{\partial \Theta} \, dy_0,$$

$$= 4 \int_0^{y_*} \sqrt{2(E + \rho^2 \cos(y_0))} \, dy_0 \equiv F(E, \rho^2). \tag{4.29}$$

Thus $E \equiv E(T)$ is a function of $\rho \equiv \rho(T)$ in terms of the const $\mathcal{A}(0)$. Recall that (4.26) gives us $\omega \equiv \omega(E(T))$, in particular this leads to the convenient formula

$$\omega = \pi \Big/ \frac{\partial}{\partial E} F(E, \rho^2).$$

Hence the solution of the problem is determined in principle; i.e., these relations give $E = E(\rho(T)) = E(T)$ and $\omega = \omega(E(\rho(T))) = \omega(T)$. So, we have determined the leading-order solution: $y \sim y_0(\Theta, E(T), \omega(T))$.

As an aside, the leading-problem can be solved explicitly in terms of elliptic functions. For completeness we now outline the method to do this. First, consider

$$\left(\frac{du}{dx}\right)^2 = P(u),$$

where P is a polynomial. If P is of second degree, we can express the solution of the above equation in terms of trigonometric functions. If P is of third or fourth degree, the solution can be expressed in terms of elliptic functions (Byrd and Friedman, 1971).

Now recall the energy integral:

$$\frac{\omega^2}{2}y_{0\Theta}^2 = E(T) + \rho^2\cos(y_0).$$

To show that y_0 can be related to elliptic problems, let $z = \tan(y_0/2)$ so $dz/dy_0 = \frac{1}{2}\sec^2(y_0/2)$. Then

$$\frac{dy_0}{d\Theta} = \frac{dy_0}{dz}\frac{dz}{d\Theta} = 2\cos^2(y_0/2)\frac{dz}{d\Theta}.$$

Using

$$\cos(y_0/2) = \frac{1}{\sqrt{1+z^2}},$$
$$\cos(y_0) = 2\cos(y_0/2)^2 - 1 = \frac{1-z^2}{1+z^2},$$

we substitute these results into the energy integral yielding

$$\frac{4\omega^2}{2(1+z^2)^2}\left(\frac{dz}{d\Theta}\right)^2 = \left(E + \rho^2\frac{1-z^2}{1+z^2}\right),$$

and hence

$$2\omega^2\left(\frac{dz}{d\Theta}\right)^2 = (1+z^2)^2E + \rho^2(1-z^2)(1+z^2).$$

The right-hand side is a fourth-order polynomial and thus the solutions are elliptic functions. One can also transform the above integrals for ω and E into standard elliptic integrals, see Exercise 4.6.

Exercises

4.1 (a) Obtain the nonlinear change in the frequency given by (4.19) by applying the frequency-shift method to the equation

$$\frac{d^2y}{dt^2} + y - \varepsilon y^3 = 0, \quad |\varepsilon| \ll 1.$$

(b) Use the multiple-scales method to find the leading-order approximation to the solution of

$$\frac{d^2y}{dt^2} + y - \varepsilon\left(y^3 + \frac{dy}{dt}\right) = 0, \quad 0 < \varepsilon \ll 1.$$

(c) Find the next-order (first-order) approximation, valid for times $t = o(1/\varepsilon^2)$, to the solution of part (b).

4.2 Apply both the frequency-shift method and the method of multiple scales ($|\varepsilon| \ll 1$) to find the solution of:

(a) $\dfrac{dA}{dt} - iA + \varepsilon A = 0,$

(b) $\dfrac{dA}{dt} - iA + \varepsilon |A|^2 A = 0.$

4.3 Use the method of multiple scales ($|\varepsilon| \ll 1$) to find an approximation to the periodic solutions of:

(a) $\dfrac{d^4y}{dt^4} - y - \varepsilon y = 0,$

(b) $\dfrac{d^4y}{dt^4} - y - \varepsilon y^3 = 0.$

4.4 Use the WKB method to find the leading-order approximation to the solutions of

$$\frac{d^4y}{dt^4} - \rho^4(\varepsilon t)y = 0, \quad |\varepsilon| \ll 1.$$

4.5 Use the multiple-scale method to find:

(a) the adiabatic invariant and leading-order solution of

$$\frac{d^2y}{dt^2} - \rho^2(\varepsilon t)(y + y^3) = 0, \quad |\varepsilon| \ll 1;$$

(b) the leading-order approximation to the solution of

$$\frac{d^2y}{dt^2} - \rho^2(\varepsilon t)(y + y^3) = \varepsilon \frac{dy}{dt}, \quad 0 < \varepsilon \ll 1.$$

4.6 Consider

$$\frac{d^2y}{dt^2} + \omega^2 \sin y = 0,$$

where $\omega > 0$ is a constant.

(a) Find the energy integral by multiplying the equation with dy/dt and integrating.

(b) Let $y = 2\tan^{-1}(z)$ (inverse tan-function) and find an equation for z. Solve the equation in terms of the Jacobi elliptic functions.

(c) Use part (b) to find all bounded solutions y.

4.7 Suppose we are given

$$\frac{d^2y}{dt^2} + \rho^2(\varepsilon t)\sin(y) = \varepsilon\frac{dy}{dt}, \quad 0 < \varepsilon \ll 1$$

Find the leading-order solution and contrast it with the asymptotic solution for the pendulum discussed in Section 4.5.