Report 2

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Contents

1 Water-wave problem on the whole line: nonlocal formulation

Recall the full water-wave problem on a whole line:

$$\phi_{xx} + \phi_{zz} = 0 \qquad -h < z < \eta(x,t) \tag{1a}$$

$$\phi_z = 0 \qquad z = -h \tag{1b}$$

$$\eta_t + \phi_x \eta_x = \phi_z \qquad z = \eta(x,t) \tag{1c}$$

$$\phi_t + g\eta + \frac{1}{2}(\phi_x^2 + \phi_z^2) = 0 \qquad z = \eta(x,t) \tag{1d}$$

where we set the surface tension to be zero. Consider the velocity potential evaluated at the surface:

$$q(x) = \phi(x, \eta(x)).$$

Combining (1c) and (1d), we obtain

$$q_t + \frac{1}{2}q_x^2 + g\eta - \frac{1}{2}\frac{(\eta_t + q_x\eta_x)^2}{1+\eta_x^2} = 0, \tag{2}$$

which is an equation for two unknowns q, η. Now, we introduce an operator that maps the normal derivative at a surface η to the tangential derivative at the surface:

$$\mathcal{H}(\eta, D)\{\nabla\phi \cdot \vec{N}\} = \nabla\phi \cdot \vec{T}, \tag{3}$$

where $D = -i\nabla$. For convenience, we drop the vector notation. Note that

$$\nabla\phi \cdot N = \begin{bmatrix}\phi_x \\ \phi_z\end{bmatrix} \cdot \begin{bmatrix}-\eta_x \\ 1\end{bmatrix} = \phi_z - \phi_x\eta_x = \eta_t$$

and

$$\nabla\phi \cdot T = \begin{bmatrix}\phi_x \\ \phi_z\end{bmatrix} \cdot \begin{bmatrix}1 \\ \eta_x\end{bmatrix} = \phi_x + \eta_x\phi_z = q_x.$$

This allows us to rewrite (3) as

$$\mathcal{H}(\eta, D)\{\eta_t\} = q_x. \tag{4}$$

Looking at the system

$$\begin{aligned} q_t + \frac{1}{2}q_x^2 + g\eta - \frac{1}{2}\frac{(\eta_t + q_x\eta_x)^2}{1+\eta_x^2} &= 0, \\ \mathcal{H}(\eta, D)\{\eta_t\} &= q_x, \end{aligned}$$

we recognise that we can rewrite the full water-wave problem in terms of the $\mathcal{H}$ operator. This is done by differentiating (2) with respect to x and (4) with respect to t :

$$\partial_t(q_x) + \partial_x\left(\frac{1}{2}q_x^2 + g\eta - \frac{1}{2}\frac{(\eta_t + q_x\eta_x)^2}{1+\eta_x^2}\right) = 0, \tag{5}$$

$$\partial_t(\mathcal{H}(\eta, D)\{\eta_t\}) = q_{xt}. \tag{6}$$

Substituting (6) into (5), we obtain

$$\partial_t\left(\mathcal{H}(\eta, D)\{\eta_t\}\right) + \partial_x\left(\frac{1}{2}\left(\mathcal{H}(\eta, D)\{\eta_t\}\right)^2 + \varepsilon\eta - \frac{1}{2}\frac{(\eta_t + \eta_x\mathcal{H}(\eta, D)\{\eta_t\})^2}{1+\eta_x^2}\right) = 0. \tag{7}$$

The utility of (7) depends on whether we can find a useful representation for the operator $\mathcal{H}(\eta, D)$. In the next section, we describe an equation that the $\mathcal{H}$ operator must satisfy.

1.0.1 Behaviour of the $\mathcal{H}$ operator: the whole line.

Consider the following boundary value problem:

$$\phi_{xx} + \phi_{zz} = 0 \qquad -h < z < \eta(x,t) \tag{8a}$$

$$\phi_z = 0 \qquad z = -h \tag{8b}$$

$$\nabla\phi \cdot N = f(x) \qquad z = \eta(x,t) \tag{8c}$$

Let φ be harmonic, so that

$$\varphi_{xx} + \varphi_{zz} = 0.$$

Clearly, φ_z is harmonic, so we have

$$\varphi_z(\phi_{xx} + \phi_{zz}) - \phi((\varphi_z)_{zz} + (\varphi_z)_{xx}) = 0.$$

Taking the integral over the domain yields

$$\int_{-\infty}^{\infty}\int_{-h}^{\eta(x)} \varphi_z(\phi_{xx} + \phi_{zz}) - \phi((\varphi_z)_{zz} + (\varphi_z))\,\mathrm{d}z\,\mathrm{d}x = 0.$$

An application of Green's theorem yields

$$\int_D \varphi_z(\nabla\phi \cdot N) - \phi(\nabla\varphi_z \cdot N)\,\mathrm{d}s = 0,$$

where D is the boundary of the domain, $\mathrm{d}s$ is the area element. Now, observe the following:

$$\begin{aligned} -\nabla\varphi_z \cdot N = -\begin{pmatrix}\varphi_{zx} \\ \varphi_{zz}\end{pmatrix} \cdot \begin{pmatrix} -\dfrac{\mathrm{d}z}{\mathrm{d}s} \\ \dfrac{\mathrm{d}x}{\mathrm{d}s}\end{pmatrix} &= -\begin{pmatrix}\varphi_{zx} \\ -\varphi_{xx}\end{pmatrix} \cdot \begin{pmatrix} -\dfrac{\mathrm{d}z}{\mathrm{d}s} \\ \dfrac{\mathrm{d}x}{\mathrm{d}s}\end{pmatrix} \\ &= \begin{pmatrix}\varphi_{zx} \\ \varphi_{xx}\end{pmatrix} \cdot \begin{pmatrix} \dfrac{\mathrm{d}z}{\mathrm{d}s} \\ \dfrac{\mathrm{d}x}{\mathrm{d}s}\end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \varphi_{xx} \\ \varphi_{xz} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}s} \\ \dfrac{\mathrm{d}z}{\mathrm{d}s} \end{pmatrix}$$
$$= \nabla \varphi_x \cdot T,$$

from which we rewrite the integral equation:

$$0 = \int_D \varphi_z (\nabla\phi \cdot N) + \phi(\nabla \varphi_z \cdot T)\, \mathrm{d}s.$$

Applying the dot product, we obtain a contour integral:

$$\int_D \varphi_z(\phi_z \,\mathrm{d}x - \phi_x \,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) = 0. \tag{9}$$

We split the contour into four segments:

$$\begin{aligned} \int_D &= \int_{-\infty}^{\infty} \Big|^{z=-h} + \int_{-h}^{\eta(x)} \Big|^{x\to\infty} + \int_{\infty}^{-\infty} \Big|^{\eta(x)} + \int_{\eta(x)}^{-h} \Big|^{x\to-\infty} \\ &= \int_{-\infty}^{\infty} \Big|^{z=-h} + \int_{-h}^{\eta(x)} \Big|^{x\to\infty} - \int_{-\infty}^{\infty} \Big|^{\eta(x)} - \int_{-h}^{\eta(x)} \Big|^{x\to-\infty}. \end{aligned}$$

Consider each segment:

- At $|x| \to \infty$, we know that ϕ and its gradient vanish, so the integral also vanishes on these segments.
- At $z = -h$, $\mathrm{d}z = 0$, so we have

$$\begin{aligned} \int_{-\infty}^{\infty} \varphi_z(\phi_z \,\mathrm{d}x - \phi_x \,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) &= \int_{-\infty}^{\infty} \varphi_z\phi_z + \phi\varphi_{xx}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \phi\varphi_{xx}\,\mathrm{d}x \qquad (\text{since } \phi_z = 0 \text{ at } z=-h) \\ &= \phi(x,-h)\varphi_x(x,-h)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \phi_x(x,-h)\varphi_x(x,-h)\,\mathrm{d}x \\ &= 0. \end{aligned}$$

  where we pick φ such that $\varphi_x(x,-h) = 0$.
- At $z = \eta$, $\mathrm{d}z = \eta_x\,\mathrm{d}x$, so we have

$$\begin{aligned} \int_{-\infty}^{\infty} \varphi_z(\phi_z - \phi_x\varepsilon\phi_x\eta_x) + \phi(\varphi_{xx} + \varphi_{xz}\eta_x)\,\mathrm{d}x &= \int_{-\infty}^{\infty} \varphi_z(\begin{pmatrix} \phi_x \\ \phi_z \end{pmatrix} \cdot \begin{pmatrix} -\eta_x \\ 1 \end{pmatrix} + \phi\frac{\mathrm{d}\varphi_x(x,\eta)}{\mathrm{d}x}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \varphi_z\nabla\phi\cdot N + \phi\frac{\mathrm{d}\varphi_x(x,\eta)}{\mathrm{d}x}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \varphi_z\nabla\phi\cdot N - \varphi_x\frac{\mathrm{d}\phi(x,\eta)}{\mathrm{d}x}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \varphi_z\nabla\phi\cdot N - \varphi_x\begin{pmatrix} \phi_x \\ \phi_z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \phi_x\eta_x \end{pmatrix}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \varphi_z\nabla\phi\cdot N - \varphi_x\nabla\phi\cdot T\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \varphi_z f(x) - \varphi_x(x,\eta)\mathcal{H}(\eta,D)\{f(x)\}\,\mathrm{d}x. \end{aligned}$$

Combining segments, we obtain

$$\int_{-\infty}^{\infty} \varphi_z f(x) - \varphi_x(x,\eta)\mathcal{H}(\eta,D)\{f(x)\}\,\mathrm{d}x = 0.$$

Note that we could choose $\varphi(x,z) = e^{-ikx}\sinh(k(z+h))$, so that the integral becomes:

$$\int_{-\infty}^{\infty} e^{-ikx}(k\cosh(k(\eta+h))f(x) + ik\sinh(k(\eta+h))\mathcal{H}(\eta,D)\{f(x)\})\,\mathrm{d}x = 0.$$

It can be shown that we can take out k in the integral, so that the below holds for all $k \in \mathbb{R}$.

$$\int_{-\infty}^{\infty} e^{-ikx}(i\cosh(k(\eta+h))f(x) - \sinh(k(\eta+h))\mathcal{H}(\eta,D)\{f(x)\})\,\mathrm{d}x = 0. \tag{10}$$

Remark 1. Even though (10) holds for all $k \in \mathbb{R}$, the case $k = 0$ still poses some challenges. Namely,the first term of $\mathcal{H}$ will contain $\coth(uk)$ term, which blows up as $k \to 0$. As will be seen, this problem should be dealt by picking η such that its Fourier transform decays faster than $\mathcal{H}$ blows up.

Remark 2. A related comment to the above is that the choice of φ is not unique. Indeed, we only required that φ is harmonic, and that $\varphi_x(x,-h) = 0$, which allowed us to cancel the contribution from the bottom. If we choose different φ, then we will end up with different version of (10). Furthermore, if we let $\varphi(x+iz)^n, n \in \mathbb{N}$, then we will end up with conservation laws, which we will exploit later.

1.0.2 Nondimensional, nonlocal formulation: the whole line.

We derive the non-dimensional version of the above work. Let

$$t^\star = \frac{t\sqrt{gh}}{L}, \qquad x^\star = \frac{x}{L}, \qquad z^\star = \frac{z}{h}, \qquad \eta^\star = \frac{\eta}{a}, \qquad k^\star = Lk, \qquad \phi = \frac{Lga}{\sqrt{gh}}\phi^\star, \qquad q^\star = \frac{\sqrt{gh}}{agL}q, \tag{11}$$

and define parameters ε and μ so that

$$\varepsilon = \frac{a}{h}, \qquad \mu = \frac{h}{L}, \qquad \varepsilon\mu = \frac{a}{L}.$$

Let $\varphi(x,z)$ be harmonic, and recall the equation (9)

$$\int_D \varphi_z(\phi_z\,\mathrm{d}x - \phi_x\,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) = 0,$$

which we obtained in the previous section. In non-dimensional coordinates,

$$\mathrm{d}x = L\,\mathrm{d}x^\star, \qquad \mathrm{d}z = h\,\mathrm{d}z^\star, \qquad \phi_z = \frac{L}{h}\frac{ga}{\sqrt{gh}}\phi^\star_{z^\star}, \qquad \phi_x = \frac{ga}{\sqrt{gh}}\phi^\star_{x^\star}.$$

Moreover, we leave φ the same but rescale its variables, which should be contrasted with that we rescaled both the function ϕ and its variables:

$$\varphi_x = \frac{1}{L}\varphi_{x^\star}, \qquad \varphi_z = \frac{1}{h}\varphi_{z^\star}, \qquad \varphi_{xx} = \frac{1}{L^2}\varphi_{x^\star x^\star}, \qquad \varphi_{xz} = \frac{1}{Lh}\varphi_{x^\star z^\star}.$$

Then, equation (9) becomes

$$\int_D \varphi_z(\frac{1}{\mu^2}\phi_z\,\mathrm{d}x - \phi_x\,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) = 0, \tag{12}$$

where we have dropped starred notation. Now, split the contour into the following segments

$$\begin{aligned}\int_D &= \int_{-\infty}^{\infty}\Big|^{z=-1} + \int_{-1}^{\varepsilon\eta(x)}\Big|^{x\to\infty} + \int_{\infty}^{-\infty}\Big|^{z=\varepsilon\eta(x)} + \int_{\varepsilon\eta(x)}^{-1}\Big|^{x\to-\infty} \\ &= \int_{-\infty}^{\infty}\Big|^{z=-1} + \int_{-1}^{\varepsilon\eta(x)}\Big|^{x\to\infty} - \int_{-\infty}^{\infty}\Big|^{z=\varepsilon\eta(x)} - \int_{-1}^{\varepsilon\eta(x)}\Big|^{x\to-\infty}.\end{aligned}$$

Consider integral on each of the segments:

- As $|x| \to \infty$, we know that ϕ and its gradient vanish, so the integral also vanishes on these segments.

- At $z = -1$, $\mathrm{d}z = 0$, so we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \varphi_z(\frac{1}{\mu^2}\phi_z \,\mathrm{d}x - \phi_x \,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) &= \int_{-\infty}^{\infty} \frac{1}{\mu^2}\varphi_z\phi_z + \phi\varphi_{xx}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \phi\varphi_{xx}\,\mathrm{d}x \qquad (\text{since } \phi_z = 0 \text{ at } z = -1) \\
&= \phi(x,-h)\varphi_x(x,-h)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \phi_x(x,-h)\varphi_x(x,-h)\,\mathrm{d}x \\
&= 0,
\end{aligned}$$

where we pick φ such that $\varphi_x(x,-h) = 0$.

- At $z = \varepsilon\eta$, $\mathrm{d}z = \varepsilon\eta_x\,\mathrm{d}x$. Moreover, introduce

$$\tilde{\nabla} = \begin{pmatrix} \phi_x \\ \frac{1}{\mu}\phi_z \end{pmatrix} \qquad \tilde{N} = \begin{pmatrix} -\varepsilon\phi_x\eta_x \\ 1 \end{pmatrix}$$

we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \varphi_z(\frac{1}{\mu^2}\phi_z - \phi_x\varepsilon\phi_x\eta_x) + \phi(\varphi_{xx} + \varepsilon\varphi_{xz}\eta_x)\,\mathrm{d}x &= \int_{-\infty}^{\infty} \frac{1}{\mu}\varphi_z \begin{pmatrix} \phi_x \\ \frac{1}{\mu}\phi_z \end{pmatrix} \cdot \begin{pmatrix} -\varepsilon\phi_x\eta_x \\ 1 \end{pmatrix} + \phi\frac{\mathrm{d}\varphi_x(x,\varepsilon\eta)}{\mathrm{d}x}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{1}{\mu}\varphi_z\tilde{\nabla}\phi\cdot\tilde{N} + \phi\frac{\mathrm{d}\varphi_x(x,\varepsilon\eta)}{\mathrm{d}x}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{1}{\mu}\varphi_z\tilde{\nabla}\phi\cdot\tilde{N} - \varphi_x\frac{\mathrm{d}\phi(x,\varepsilon\eta)}{\mathrm{d}x}\,\mathrm{d}x + \varphi\phi(x,\varepsilon\eta)\Big|_{-\infty}^{\infty} \\
&= \int_{-\infty}^{\infty} \frac{1}{\mu}\varphi_z\tilde{\nabla}\phi\cdot\tilde{N} - \varphi_x\frac{\mathrm{d}\phi(x,\varepsilon\eta)}{\mathrm{d}x}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{1}{\mu}\varphi_z\tilde{\nabla}\phi\cdot\tilde{N} - \varphi_x\begin{pmatrix} \phi_x \\ \frac{1}{\mu}\phi_z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \varepsilon\phi_x\eta_x \end{pmatrix}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{1}{\mu}\varphi_z\tilde{\nabla}\phi\cdot\tilde{N} - \varphi_x\tilde{\nabla}\phi\cdot\tilde{T}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{1}{\mu}\varphi_z g(x) - \varphi_x(x,\varepsilon\eta)\mathcal{H}(\varepsilon\eta, D)\{g(x)\}\,\mathrm{d}x.
\end{aligned}$$

Observe that

$$\nabla\phi\cdot N = \frac{ga}{\sqrt{gh}}\tilde{\nabla}\phi\cdot\tilde{N} = \frac{ga}{\sqrt{gh}}g(x^\star) = f(x^\star L) = f(x),$$

so that

$$g(x^\star) = \frac{\sqrt{gh}}{ga}f(x^\star L) = \frac{\sqrt{gh}}{ga}f(x).$$

Combining segments, we obtain

$$\int_{-\infty}^{\infty} \frac{1}{\mu}\varphi_z g(x) - \varphi_x(x,\varepsilon\eta)\mathcal{H}(\varepsilon\eta, D)\{g(x)\}\,\mathrm{d}x = 0.$$

As before, we choose $\varphi(x,z) = e^{-ikx}\sinh(k(z+h)) = e^{-ik^\star x^\star}\sinh(\mu k^\star(z^\star+1)) = e^{-ikx}\sinh(\mu k(z+1))$ so that the integral becomes:

$$\int_{-\infty}^{\infty} e^{-ikx}(k\cosh(\mu k(\eta+1))g(x) + ik\sinh(\mu k(\eta+1))\mathcal{H}(\varepsilon\eta, D)\{g(x)\})\,\mathrm{d}x = 0.$$

Taking k out and multiplying by i yields an equation that relates g and the operator $\mathcal{H}$ acting on g :

$$\int_{-\infty}^{\infty} e^{-ikx}(i\cosh(\mu k(\eta+1))g(x) - \sinh(\mu k(\eta+1))\mathcal{H}(\varepsilon\eta, D)\{g(x)\})\,\mathrm{d}x = 0. \tag{13}$$

1.0.3 Perturbation expansion of the $\mathcal{H}$ operator: the whole line.

In this section, we derive a representation of the $\mathcal{H}$ operator in the leading two terms. To begin, consider (13) and expand in ε :

$$\begin{aligned}\cosh(\mu k(\eta+1)) &= \cosh(\mu k) + \mu k\varepsilon\eta\sinh(\mu k) + \ldots,\\ \sinh(\mu k(\eta+1)) &= \sinh(\mu k) + \mu k\varepsilon\eta\cosh(\mu k) + \ldots,\\ \mathcal{H}(\eta, D)\{g(x)\} &= [\mathcal{H}_0 + \varepsilon\mathcal{H}_1 + \ldots](\varepsilon\eta, D)\{g(x)\}.\end{aligned}$$

Equation (13) becomes:

$$\begin{aligned}&\int_{-\infty}^{\infty} e^{-ikx}(i\,[\cosh(\mu k) + \mu k\varepsilon\eta\sinh(\mu k) + \ldots]\,g(x)\\ &\qquad - [\sinh(\mu k) + \mu k\varepsilon\eta\cosh(\mu k) + \ldots]\,[\mathcal{H}_0 + \varepsilon\mathcal{H}_1 + \ldots](\varepsilon\eta, D)\{g(x)\})\,\mathrm{d}x = 0.\end{aligned}$$

Within $\mathcal{O}(\varepsilon^0)$: from expansions above, we have

$$\int_{-\infty}^{\infty} e^{-ikx}(i\cosh(\mu k)g(x) - \sinh(\mu k)\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\})\,\mathrm{d}x = 0.$$

Dividing by $\sinh(\mu k)$, we obtain

$$\int_{-\infty}^{\infty} e^{-ikx}(i\coth(\mu k)g(x) - \mathcal{H}_0(\varepsilon\eta, D)\{g(x)\})\,\mathrm{d}x = 0.$$

Splitting the integrand and recognizing Fourier transform yields:

$$\mathcal{F}\{\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\}\}_k = \int_{-\infty}^{\infty} e^{-ikx}\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\}\,\mathrm{d}x = \int_{-\infty}^{\infty} e^{-ikx} i\coth(\mu k)g(x)\,\mathrm{d}x = i\coth(\mu k)\mathcal{F}\{g(x)\}_k.$$

Finally, we invert Fourier transform to obtain

$$\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\} = \mathcal{F}^{-1}\{i\coth(\mu k)\mathcal{F}\{g(x)\}_k\}_k,$$

where we write out k's explicitly to keep track of transforms. Note that as $k \to 0, \mathcal{H}_0$ blows up as $\coth(\mu k)$ has a singularity of order 1 at $k = 0$.

Within $\mathcal{O}(\varepsilon^1)$: from expansions above, we have

$$\int_{-\infty}^{\infty} e^{-ikx}(i\mu k\eta\sinh(\mu k)g(x) - [\sinh(\mu k)\mathcal{H}_1 + \mu k\eta\cosh(\mu k)\mathcal{H}_0](\varepsilon\eta, D)\{g(x)\})\,\mathrm{d}x = 0.$$

Dividing by $\sinh(\mu k)$ and dropping $(\varepsilon\eta, D)$ for convenience, we have

$$\int_{-\infty}^{\infty} e^{-ikx}(i\mu k\eta g(x) - [\mathcal{H}_1 + \mu k\eta\coth(\mu k)\mathcal{H}_0]\{g(x)\})\,\mathrm{d}x = 0.$$

Rearranging and recognising Fourier transform yields:

$$\begin{aligned}\mathcal{F}\{\mathcal{H}_1\{g(x)\})\}_k &= \int_{-\infty}^{\infty} e^{-ikx}\mathcal{H}_1\{g(x)\})\,\mathrm{d}x\\ &= \int_{-\infty}^{\infty} e^{-ikx}(i\mu k\eta g - \mu k\eta\coth(\mu k)\mathcal{H}_0\{g(x)\})\,\mathrm{d}x\\ &= \mu\mathcal{F}\{ik\eta g\}_k - \mu k\coth(\mu k)\mathcal{F}\{\eta\mathcal{H}_0\{g(x)\})\}_k.\end{aligned}$$

Inverting Fourier transform, we obtain an expression for $\mathcal{H}_1$:

$$\begin{aligned}\mathcal{H}_1\{g(x)\} &= \mathcal{F}^{-1}\{\mu\eta g\}_k - \mathcal{F}^{-1}\{\mu k\coth(\mu k)\mathcal{F}\{\eta\mathcal{H}_0\{g(x)\}\})\}_k\}\\ &= \mu\partial_x(\eta g) - \mathcal{F}^{-1}\{\mu k\coth(\mu k)\mathcal{F}\{\eta\mathcal{H}_0\{g(x)\}\})\}_k\}_k\\ &= \mu\partial_x(\eta g) - \mathcal{F}^{-1}\{\mu k\coth(\mu k)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\coth(\mu l)\mathcal{F}\{g\}_l\}_l\}_k\}_k.\end{aligned}$$

In sum, we obtain

$$\begin{aligned}\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\} &= \mathcal{F}^{-1}\{i\coth(\mu k)\mathcal{F}\{g(x)\}_k\}_k,\\ \mathcal{H}_1(\varepsilon\eta, D)\{g(x)\} &= \mu\partial_x(\eta g) - \mathcal{F}^{-1}\{\mu k\coth(\mu k)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\coth(\mu l)\mathcal{F}\{g\}_l\}_l\}_k\}_k.\end{aligned}$$

1.0.4 Deriving an expression for surface elevation: the whole-line.

In this section, we would like to derive an expression for η. We can do this because the scalar equation (7) is written in terms of η. The non-dimensional version of (7) is given by

$$\partial_t \left(\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right) + \partial_x \left(\frac{1}{2}\left(\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 + \varepsilon\eta - \frac{1}{2}\varepsilon^2\mu^2 \frac{(\eta_t + \eta_x \mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\})^2}{1+\varepsilon^2\mu^2\eta_x^2}\right) = 0. \tag{14}$$

Within $\mathcal{O}(\mu^0)$. In the leading order, the equation (14) becomes

$$\partial_t \left(\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right) + \varepsilon\partial_x\eta = 0.$$

Substituting an expression for $\mathcal{H}_0$, we obtain:

$$\mathcal{F}^{-1}\{i\coth(\mu k)\mathcal{F}\{\varepsilon\mu\eta_{tt}\}_k\}_k + \varepsilon\partial_x\eta = 0,$$

where we brought the time derivative inside the transform. Inverting the Fourier transform and multiplying by $\frac{k}{i\varepsilon}$ yields

$$\mu k\coth(\mu k)\widehat{\eta_{tt}}_k + k^2\widehat{\eta}_k = 0.$$

Recall

$$\coth(\mu k) \approx \frac{1}{\mu k} + \mathcal{O}(\mu),$$

so that

$$\widehat{\eta_{tt}}_k + k^2\widehat{\eta}_k = 0.$$

Inverting the Fourier transform, we have

$$\eta_{tt} + (-i\partial_x)^2\eta = 0,$$

which is

$$\eta_{tt} - \eta_{xx} = 0.$$

This is the wave equation, as we desired.
Within $\mathcal{O}(\mu^2)$. In the second leading order, the non-dimensional equation (14) becomes

$$\partial_t \left(\mathcal{H}_0\{\varepsilon\mu\eta_t\} + \varepsilon\mathcal{H}_1\{\varepsilon\mu\eta_t\}\right) + \partial_x\left(\frac{1}{2}(\mathcal{H}_0\{\varepsilon\mu\eta_t\})^2 + \varepsilon\eta\right) = 0.$$

Note

$$\begin{aligned}
\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\} &= \varepsilon\mu\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}_k\}_k;\\
\mathcal{H}_1(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\} &= \varepsilon\mu^2(\eta\eta_t)_x - \varepsilon\mathcal{F}^{-1}\{\mu k\coth(\mu k)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\mu\coth(\mu k)\widehat{\eta_t}_l\}_l\}_k\}_k.
\end{aligned}$$

Then,

$$\frac{1}{2}\left(\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 = \frac{1}{2}\left(\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 = \frac{\varepsilon^2}{2}\left(\mathcal{F}^{-1}\{i\mu\coth(\mu j)\widehat{\eta_t}_j\}_j\right)^2,$$

and

$$\begin{aligned}
\partial_t\left(\left[\mathcal{H}_0(\varepsilon\eta, D) + \varepsilon\mathcal{H}_1(\varepsilon\eta, D)\right]\{\varepsilon\mu\eta_t\}\right) &= \varepsilon\mu\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_{tt}}_k\}_k + \varepsilon^2\mu^2(\eta\eta_t)_{tx}\\
&\quad - \varepsilon^2\mathcal{F}^{-1}\{\mu k\coth(\mu k)\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{i\mu\coth(\mu l)\widehat{\eta_t}_l\}_l\right]\}_k\}_k.
\end{aligned}$$

The single equation becomes

$$\begin{aligned}
\varepsilon\mu\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_{tt}}_k\}_k + \varepsilon^2\mu^2(\eta\eta_t)_{tx} - \varepsilon^2\mathcal{F}^{-1}\{\mu k\coth(\mu k)\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{i\mu\coth(\mu l)\widehat{\eta_t}_l\}_l\right]\}_k\}_k&\\
+ \frac{\varepsilon^2}{2}\partial_x\left(\mathcal{F}^{-1}\{i\mu\coth(\mu j)\widehat{\eta_t}_j\}_j\right)^2 + \varepsilon\partial_x\eta = 0.&
\end{aligned}$$

Application of Fourier transform yields

$$\varepsilon\mu i \coth(\mu k)\widehat{\eta_{tt}}_k + \varepsilon^2\mu^2 ik(\eta\eta_t)_t - \varepsilon^2\mu k\coth(\mu k)\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{i\mu\coth(\mu l)\widehat{\eta_t}_l\}_l\right]\}_k + \frac{\varepsilon^2}{2}ik\mathcal{F}\{\left(\mathcal{F}^{-1}\{i\mu\coth(\mu j)\widehat{\eta_t}_j\}_j\right)^2\}_k + \varepsilon ik\widehat{\eta}_k = 0.$$

Divide by $i\varepsilon$:

$$\mu\coth(\mu k)\widehat{\eta_{tt}}_k + \varepsilon\mu^2 k(\eta\eta_t)_t - \varepsilon\mu k\coth(\mu k)\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{\mu\coth(\mu l)\widehat{\eta_t}_l\}_l\right]\}_k + \frac{\varepsilon}{2}k\mathcal{F}\{\left(\mathcal{F}^{-1}\{i\mu\coth(\mu j)\widehat{\eta_t}_j\}_j\right)^2\}_k + k\widehat{\eta}_k = 0.$$

Let $\varepsilon = \mu^2$ and recall an expansion:

$$\coth(\mu k) \approx \frac{1}{\mu k} + \frac{\mu k}{3} + \mathcal{O}(\mu^3).$$

Substitution of the expansion yields:

$$\left(\frac{1}{k} + \frac{\mu^2 k}{3}\right)\widehat{\eta_{tt}}_k + \mu^4 k(\eta\eta_t)_t - \mu^2 k\left(\frac{1}{k} + \frac{\mu^2 k}{3}\right)\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{\left(\frac{1}{l} + \frac{\mu^2 l}{3}\right)\widehat{\eta_t}_l\}_l\right]\}_k - \frac{\mu^2}{2}k\mathcal{F}\{\left(\mathcal{F}^{-1}\{\left(\frac{1}{j} + \frac{\mu^2 j}{3}\right)\widehat{\eta_t}_j\}_j\right)^2\}_k + k\widehat{\eta}_k = 0.$$

Within $\mathcal{O}(\mu^4)$, the equation becomes

$$\left(\frac{1}{k} + \frac{\mu^2 k}{3}\right)\widehat{\eta_{tt}}_k - \mu^2\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{\frac{1}{l}\widehat{\eta_t}_l\}_l\right]\}_k - \frac{\mu^2}{2}k\mathcal{F}\{\left(\mathcal{F}^{-1}\{\frac{1}{j}\widehat{\eta_t}_j\}_j\right)^2\}_k + k\widehat{\eta}_k = 0,$$

or re-arranging and multiplying by k, we have

$$\widehat{\eta_{tt}}_k + k^2\widehat{\eta}_k + \mu^2\left(\frac{k^2}{3}\widehat{\eta_{tt}}_k - k\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{\frac{1}{l}\widehat{\eta_t}_l\}_l\right]\}_k - \frac{1}{2}k^2\mathcal{F}\{\left(\mathcal{F}^{-1}\{\frac{1}{j}\widehat{\eta_t}_j\}_j\right)^2\}_k\right) = 0.$$

Finally, inverting Fourier transform yields:

$$\eta_{tt} - \eta_{xx} + \mu^2\left(-\frac{\partial_x^2}{3}\eta_{tt} + i\partial_x\left(\partial_t\left[\eta\mathcal{F}^{-1}\{\frac{1}{l}\widehat{\eta_t}_l\}_l\right]\right) + \frac{1}{2}\partial_x^2\left(\mathcal{F}^{-1}\{\frac{1}{j}\widehat{\eta_t}_j\}_j\right)^2\right) = 0,$$

or more conveniently,

$$\eta_{tt} - \eta_{xx} = \mu^2\left(\frac{\partial_x^2}{3}\eta_{tt} - i\partial_x\left(\partial_t\left[\eta\mathcal{F}^{-1}\{\frac{1}{l}\widehat{\eta_t}_l\}_l\right]\right) - \frac{1}{2}\partial_x^2\left(\mathcal{F}^{-1}\{\frac{1}{j}\widehat{\eta_t}_j\}_j\right)^2\right). \tag{15}$$

Observe the following:

$$\begin{aligned}
\frac{1}{l}\widehat{\eta_t}_l &= \frac{1}{l}\frac{2}{\pi}\int_{-\infty}^{\infty} e^{-ilx}\eta_t\,\mathrm{d}x \\
&= \frac{1}{l}\frac{2}{\pi}e^{-ilx}\int_{-\infty}^{x}\eta_t(x',t)\,\mathrm{d}x'\Big|_{-\infty}^{\infty} + i\frac{2}{\pi}\int_{-\infty}^{\infty} e^{-ilx}\int_{-\infty}^{x}\eta_t(x',t)\,\mathrm{d}x'\,\mathrm{d}x \\
&= i\frac{2}{\pi}\int_{-\infty}^{\infty} e^{-ilx}\int_{-\infty}^{x}\eta_t(x',t)\,\mathrm{d}x'\,\mathrm{d}x \\
&= i\mathcal{F}\{\int_{-\infty}^{x}\eta_t(x',t)\,\mathrm{d}x'\}_l.
\end{aligned}$$

so that

$$\mathcal{F}^{-1}\{\frac{1}{l}\widehat{\eta_t}_l\}_l = \mathcal{F}^{-1}\{i\mathcal{F}\{\int_{-\infty}^{x}\eta_t(x',t)\,\mathrm{d}x'\}_l\}_l = i\int_{-\infty}^{x}\eta_t(x',t)\,\mathrm{d}x',$$

where we applied the Fourier inversion theorem. Moreover, we also have

$$\eta_{tt} = \eta_{xx} + \mathcal{O}(\mu^2).$$

Using these two facts, equation (15) becomes

$$\begin{aligned}
\eta_{tt} - \eta_{xx} &= \mu^2 \left(\frac{1}{3}\eta_{xxxx} + \partial_x \partial_t \left[\eta \left(\int_{-\infty}^{x} \eta_t \, \mathrm{d}x' \right) \right] + \frac{1}{2}\partial_x^2 \left(\int_{-\infty}^{x} \eta_t \, \mathrm{d}x' \right)^2 \right) \\
&= \mu^2 \left(\frac{1}{3}\eta_{xxxx} + \partial_x \left[\eta_t \left(\int_{-\infty}^{x} \eta_t \, \mathrm{d}x' \right) + \eta\eta_x \right] + \frac{1}{2}\partial_x^2 \left(\int_{-\infty}^{x} \eta_t \, \mathrm{d}x' \right)^2 \right) \\
&= \varepsilon \left[\frac{1}{3}\eta_{xxxx} + \partial_x^2 \left(\frac{\eta^2}{2} + \left(\int_{-\infty}^{x} \eta_t \, \mathrm{d}x' \right)^2 \right) \right].
\end{aligned}$$

For direct comparison, this equation is the same as the one in [1, p. 111], the unnumbered equation between (5.20) and (5.21). It remains to derive the wave and KdV equations.

1.0.5 Derivation of wave and KdV equations: the whole line.

In this section, we derive the approximate equations from

$$\eta_{tt} - \eta_{xx} = \varepsilon \left[\frac{1}{3}\eta_{xxxx} + \partial_x^2 \left(\frac{\eta^2}{2} + \left(\int_{-\infty}^{x} \eta_t \, \mathrm{d}x' \right)^2 \right) \right]. \tag{16}$$

As we approximate, we assume an expansion of η in ε :

$$\eta = \eta_0 + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2). \tag{17}$$

First order approximation

Substitution of (17) into equation (16) yields

$$\eta_{0tt} - \eta_{0xx} + \varepsilon(\eta_{1tt} - \eta_{1xx}) = \varepsilon \left[\frac{1}{3}\eta_{0xxxx} + \partial_x^2 \left(\frac{(\eta_0 + \varepsilon\eta_1)^2}{2} + \left(\int_{-\infty}^{x} (\eta_0 + \varepsilon\eta_1)_t \, \mathrm{d}x' \right)^2 \right) \right] + \mathcal{O}(\varepsilon^2). \tag{18}$$

In the leading order $\mathcal{O}(\varepsilon^0)$, equation (18) becomes

$$\eta_{0tt} - \eta_{0xx} = 0. \tag{19}$$

This is the wave equation with velocity 1, and whose general solution is

$$\eta_0 = F(x-t) + G(x+t),$$

where F, G are some functions.

Second order approximation

As in the velocity potential case, we employ multiple scales. First, we find an expression for η_0. We introduce

$$\tau_0 = t, \qquad \tau_1 = \varepsilon t, \qquad \tau_2 = \varepsilon^2 t, \ldots,$$

so that

$$\eta(x,t) = \eta(x, \tau_0, \tau_1, \ldots).$$

With this in mind, the expansion (17) becomes

$$\eta(x, \tau_0, \tau_1, \ldots) = \eta_0(x, \tau_0, \tau_1, \ldots) + \mathcal{O}(\varepsilon^1). \tag{20}$$

Substituting (20) into (16), within $\mathcal{O}(\varepsilon^0)$, we obtain

$$\eta_{0\tau_0\tau_0} - \eta_{0xx} = 0, \tag{21}$$

so that the general solution is

$$\eta_0(x,\tau_0,\tau_1,\dots) = F(x-\tau_0,\tau_1,\dots) + G(x+\tau_0,\tau_1,\dots).$$

Now, although we have found an expression for η_0, the functions F, G used are still general functions. To determine F, G, we proceed to the next order, i.e. $\mathcal{O}(\varepsilon^1)$. We introduce

$$\xi = x - \tau_0 \qquad \zeta = x + \tau_0$$

so that

$$\begin{aligned}
\partial_x &= \partial_\xi \frac{\mathrm{d}\xi}{\mathrm{d}x} + \partial_\zeta \frac{\mathrm{d}\zeta}{\mathrm{d}x} = \partial_\xi + \partial_\zeta,\\
\partial_t &= \partial_\xi \frac{\mathrm{d}\xi}{\mathrm{d}t} + \partial_\zeta \frac{\mathrm{d}\zeta}{\mathrm{d}t} + \partial_{\tau_1}\frac{\mathrm{d}\tau_1}{\mathrm{d}t} = -\partial_\xi + \partial_\zeta + \varepsilon\partial_{\tau_1}.
\end{aligned}$$

We can rewrite (20) as follows

$$\begin{aligned}
\eta &= \eta_0 + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2)\\
&= F(x-t,\varepsilon t,\dots) + G(x+t,\varepsilon t,\dots) + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2)\\
&= F(\xi,\tau_1,\dots) + G(\zeta,\tau_1,\dots) + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2)\\
&= F + G + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

For ease of writing, we suppressed explicit dependence on variables, though the reader should bear in mind that function F (G) depend on ξ $(\zeta), \tau_1, \tau_2$, etc. In addition, observe that

$$\begin{aligned}
(\partial_t^2 - \partial_x^2) &= \left((-\partial_\xi + \partial_\zeta + \varepsilon\partial_{\tau_1})^2 - (\partial_\xi + \partial_\zeta)^2\right)\\
&= \left(\partial_\xi^2 - 2\partial_\xi\partial_\zeta + \partial_\zeta^2 + 2\varepsilon(\partial_\zeta\partial_{\tau_1} - \partial_\xi\partial_{\tau_1}) + \varepsilon^2\partial_{\tau_1}^2 - \partial_\xi^2 - 2\partial_\xi\partial_\zeta - \partial_\zeta^2\right)\\
&= \left(-4\partial_\xi\partial_\zeta + 2\varepsilon(\partial_\zeta\partial_{\tau_1} - \partial_\xi\partial_{\tau_1}) + \varepsilon^2\partial_{\tau_1}^2\right),
\end{aligned}$$

so that the LHS of (16) becomes

$$\begin{aligned}
(\partial_t^2 - \partial_x^2)\eta &= \left(-4\partial_\xi\partial_\zeta + 2\varepsilon(\partial_\zeta\partial_{\tau_1} - \partial_\xi\partial_{\tau_1}) + \varepsilon^2\partial_{\tau_1}^2\right)(F + G + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2))\\
&= -4\partial_\xi\partial_\zeta(F + G + \varepsilon\eta_1) + 2\varepsilon(\partial_\zeta\partial_{\tau_1} - \partial_\xi\partial_{\tau_1})(F+G) + \mathcal{O}(\varepsilon^2)\\
&= \varepsilon\left(-4\eta_{1\xi\zeta} - 2F_{\tau_1\xi} + 2G_{\tau_1\zeta}\right) + \mathcal{O}(\varepsilon^2). \qquad (22)
\end{aligned}$$

Now, we deal with the RHS of (16). By appropriate substitutions, the terms become:

$$\begin{aligned}
\frac{1}{3}\eta_{xxxx} &= \frac{1}{3}(\partial_x^2)^2\eta\\
&= \frac{1}{3}(\partial_\xi^2 + 2\partial_\xi\partial_\zeta + \partial_\zeta^2)^2\eta\\
&= \frac{1}{3}(\partial_\xi^4 + \partial_\zeta^4 + 4\partial_\xi^3\partial_\zeta + 2\partial_\xi\partial_\zeta^3 + 6\partial_\xi\partial_\zeta)(F + G + \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2))\\
&= \frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta} + \varepsilon(\partial_\xi + \partial_\zeta)^4\eta_1 + \mathcal{O}(\varepsilon^2))\\
&= \frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta} + \mathcal{O}(\varepsilon));\\
\frac{1}{2}\eta^2 &= \frac{1}{2}\left(F + G + \varepsilon\eta_1\right)^2\\
&= \frac{1}{2}\left((F+G)^2 + 2\varepsilon(F+G)\eta_1 + \varepsilon^2\eta_1^2\right)\\
&= \frac{1}{2}(F^2 + 2FG + G^2) + \varepsilon(F+G)\eta_1 + \mathcal{O}(\varepsilon^2)\\
&= \frac{1}{2}(F^2 + 2FG + G^2) + \mathcal{O}(\varepsilon);\\
\left(\int_{-\infty}^x \eta_t \,\mathrm{d}x'\right)^2 &= \left(\int_{-\infty}^x \eta_{0t}\,\mathrm{d}x' + \varepsilon\int_{-\infty}^x \eta_{1t}\,\mathrm{d}x'\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_{-\infty}^{x} \eta_{0t} \,\mathrm{d}x' + \varepsilon \int_{-\infty}^{x} \eta_{1t} \,\mathrm{d}x' \right)^2 \\
&= \left(\int_{-\infty}^{x} \eta_{0t} \,\mathrm{d}x' \right)^2 + \mathcal{O}(\varepsilon) \\
&= \left(\int_{-\infty}^{x} (-\partial_\xi + \partial_\zeta + \varepsilon \partial_{\tau_1})(F+G) \,\mathrm{d}x' \right)^2 + \mathcal{O}(\varepsilon) \\
&= \left(\int_{-\infty}^{x} -F_\xi + G_\zeta \,\mathrm{d}x' + \varepsilon \int_{-\infty}^{x} \partial_{\tau_1}(F+G) \,\mathrm{d}x' \right)^2 + \mathcal{O}(\varepsilon) \\
&= \left(\int_{-\infty}^{x} -F_\xi + G_\zeta \,\mathrm{d}x' \right)^2 + \mathcal{O}(\varepsilon) \\
&= \left(\int_{-\infty}^{x} F_\xi \,\mathrm{d}x' \right)^2 - 2 \left(\int_{-\infty}^{x} F_\xi \,\mathrm{d}x' \right) \left(\int_{-\infty}^{x} G_\zeta \,\mathrm{d}x' \right) + \left(\int_{-\infty}^{x} G_\zeta \,\mathrm{d}x' \right)^2 + \mathcal{O}(\varepsilon) \\
&= F^2 - 2FG + G^2 + \mathcal{O}(\varepsilon),
\end{aligned}
$$

where for the last line we translate $\xi' = x' - t, \zeta' = x' + t$ to obtain

$$
\begin{aligned}
\int_{-\infty}^{x} F_\xi \,\mathrm{d}x' = \lim_{a \to -\infty} \int_{a}^{x} F_{\xi'}(x' - t, \tau_1) \,\mathrm{d}x' &= \lim_{a \to -\infty} \int_{a-t}^{x-t} F_{\xi'}(\xi', \tau_1) \,\mathrm{d}\xi' \\
&= \lim_{a \to -\infty} \int_{a-t}^{\xi} F_{\xi'}(\xi', \tau_1) \,\mathrm{d}\xi' \\
&= \int_{-\infty}^{\xi} F_{\xi'}(\xi', \tau_1) \,\mathrm{d}\xi' = F(\xi, \tau_1), \\
\int_{-\infty}^{x} G'_\zeta \,\mathrm{d}x' = \lim_{a \to -\infty} \int_{a}^{x} F_{\zeta'}(x' - t, \tau_1) \,\mathrm{d}x' &= \lim_{a \to -\infty} \int_{a+t}^{x+t} G_{\zeta'}(\zeta', \tau_1) \,\mathrm{d}\zeta' \\
&= \lim_{a \to -\infty} \int_{a-t}^{\zeta} G_{\zeta'}(\zeta', \tau_1) \,\mathrm{d}\zeta' \\
&= \int_{-\infty}^{\zeta} G_{\zeta'}(\zeta', \tau_1) \,\mathrm{d}\zeta' = G(\zeta, \tau_1).
\end{aligned}
$$

Note we assumed F, G vanish as $\xi, \zeta \to -\infty$. Substitution of terms into the RHS of (16) leads to:

$$
\begin{aligned}
&\varepsilon \left[\frac{1}{3} \eta_{xxxx} + \partial_x^2 \left(\frac{\eta^2}{2} + \left(\int_{-\infty}^{x} \eta_t \,\mathrm{d}x' \right)^2 \right) \right] \\
&= \varepsilon \left[\frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + (\partial_\xi^2 + 2\partial_\xi\partial_\zeta + \partial_\zeta^2) \left(\frac{1}{2}(F^2 + 2FG + G^2) + F^2 - 2FG + G^2 \right) \right] + \mathcal{O}(\varepsilon^2) \\
&= \varepsilon \left[\frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + (\partial_\xi^2 + 2\partial_\xi\partial_\zeta + \partial_\zeta^2) \left(\frac{3}{2}F^2 + \frac{3}{2}G^2 - FG \right) \right] + \mathcal{O}(\varepsilon^2). \qquad (23)
\end{aligned}
$$

Combining (22) and (23), in $\mathcal{O}(\varepsilon^1)$ we have

$$
-4\eta_{1\xi\zeta} = 2F_{\tau_1\xi} - 2G_{\tau_1\zeta} + \frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + (\partial_\xi^2 + 2\partial_\xi\partial_\zeta + \partial_\zeta^2) \left(\frac{3}{2}F^2 + \frac{3}{2}G^2 - FG \right). \qquad (24)
$$

In the last term of (24), differentiation yields:

$$
(\partial_\xi^2 + 2\partial_\xi\partial_\zeta + \partial_\zeta^2) \left(\frac{3}{2}F^2 + \frac{3}{2}G^2 - FG \right) = \partial_\xi(3FF_\xi - GF_\xi) + \partial_\zeta(3GG_\zeta - FG_\zeta) - 2F_\xi G_\zeta,
$$

so that equation (24) becomes

$$
-4\eta_{1\xi\zeta} = 2F_{\tau_1\xi} - 2G_{\tau_1\zeta} + \frac{1}{3}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + \partial_\xi(3FF_\xi - GF_\xi) + \partial_\zeta(3GG_\zeta - FG_\zeta) - 2F_\xi G_\zeta
$$

$$= \partial_\xi(2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + 3FF_\xi) + \partial_\zeta(-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + 3GG_\zeta) - (GF_\xi + FG_\zeta). \tag{25}$$

Integration of (25) with respect to ζ yields

$$-4\eta_{1\xi} = \partial_\xi(2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + 3FF_\xi)\zeta + (-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + 3GG_\zeta) - \left(F_\xi \int G\,\mathrm{d}\zeta + GF\right),$$

and further integration with respect to ξ leads to

$$-4\eta_1 = (2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + 3FF_\xi)\zeta + (-2G_{\tau_1} + \frac{1}{3}G_{\zeta\zeta\zeta} + 3GG_\zeta)\xi - \left(F \int G\,\mathrm{d}\zeta + G\int F\,\mathrm{d}\xi\right).$$

Since η_1 must be bounded, we must have

$$2F_{\tau_1} + \frac{1}{3}F_{\xi\xi\xi} + 3FF_\xi = 0 \tag{26}$$

$$2G_{\tau_1} - \frac{1}{3}G_{\zeta\zeta\zeta} - 3GG_\zeta = 0. \tag{27}$$

In other words, we have obtained two KdV equations, (26) and (27), whose solutions describe behaviour of the surface elevation in the leading order. The derivation is complete.

2 Water-wave problem on a half line: nonlocal formulation

The (tentative) half line problem is given by the following system:

$$\phi_{xx} + \phi_{zz} = 0, \qquad -h < z < \eta(x,t), \tag{28a}$$

$$\phi_z = 0, \qquad z = -h, \tag{28b}$$

$$\phi_x = 0, \qquad x = 0, \tag{28c}$$

$$\eta_t + \phi_x\eta_x = \phi_z, \qquad z = \eta(x,t), \tag{28d}$$

$$\phi_t + g\eta + \frac{1}{2}(\phi_x^2 + \phi_z^2) = 0, \qquad z = \eta(x,t), \tag{28e}$$

$$\phi_z(0,\eta,t) = \eta_t(0,t), \qquad (x,z) = (0,\eta), \tag{28f}$$

and $\eta, \phi, \nabla\phi \to 0$ as $x \to \infty$. Introducing the nondimensional variables as before yields the nondimensional problem:

$$\varepsilon\phi_{xx} + \phi_{zz} = 0 \qquad -1 < z < \varepsilon\eta \tag{29a}$$

$$\phi_z = 0 \qquad z = -1 \tag{29b}$$

$$\phi_x = 0 \qquad x = 0 \tag{29c}$$

$$\varepsilon\eta_t + \varepsilon^2\phi_x\eta_x = \phi_z \qquad z = \varepsilon\eta \tag{29d}$$

$$\phi_t + \eta + \frac{1}{2}(\varepsilon\phi_x^2 + \phi_z^2) = 0 \qquad z = \varepsilon\eta \tag{29e}$$

$$\phi_z(0,\varepsilon\eta,t) = \varepsilon\eta_t(0,t) \qquad (x,z) = (0,\varepsilon\eta), \tag{29f}$$

and the conditions on decay of ϕ and η remain the same, except there is only the right side. We seek a nonlocal formulation of the problem.

2.0.1 Derivation of the nonlocal formulation: a half-line

First, we begin with a dimensional system. As previously, let φ be harmonic. Then, after some manipulation, we have the following contour integral (in dimensional variables):

$$\int_D \varphi_z(\phi_z\,\mathrm{d}x - \phi_x\,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) = 0 \tag{30}$$

Break the contour D in the following segments:

$$\begin{aligned}\int_D &= \int_{-\infty}^{\infty}\Big|^{z=-h} + \int_{-h}^{\eta(x)}\Big|^{x\to\infty} + \int_{\infty}^{-\infty}\Big|^{z=\eta(x)} + \int_{\eta}^{-h}\Big|^{x=0} \\ &= \int_{-\infty}^{\infty}\Big|^{z=-h} + \int_{-h}^{\eta(x)}\Big|^{x\to\infty} - \int_{-\infty}^{\infty}\Big|^{z=\eta(x)} - \int_{-h}^{\eta}\Big|^{x=0}\end{aligned}$$

We consider each of the segments:

- As $x \to \infty$, the integral vanishes due to behaviour of ϕ and its gradient.
- At $x = 0$, $\mathrm{d}x = 0$, so we have

$$\begin{aligned}\int_{-h}^{\eta} -\varphi_z\phi_x + \phi\varphi_{xz}\,\mathrm{d}z &= \int_{-h}^{\eta} \phi\varphi_{xz} - \varphi_z\phi_x\,\mathrm{d}z \\ &= \int_{-h}^{\eta} \phi\varphi_{xz}\,\mathrm{d}z \qquad (\text{since } \phi_x = 0 \text{ at } x = 0) \\ &= \phi\varphi_x\Big|_{-h}^{\eta} - \int_{-h}^{\eta}\phi_z\varphi_x\,\mathrm{d}z \\ &= \phi(0,\eta)\varphi_x(0,\eta) - \phi(0,-h)\varphi_x(0,-h) - \int_{-h}^{\eta}\phi_z(0,z)\varphi_x(0,z)\,\mathrm{d}z.\end{aligned}$$

- At $z = -h$, $\mathrm{d}z = 0$, so we have

$$\begin{aligned}\int_0^{\infty} \varphi_z\phi_z + \phi\varphi_{xx}\,\mathrm{d}x &= \int_0^{\infty}\phi\varphi_{xx}\,\mathrm{d}x \qquad (\text{since } \phi_z(x,-h) = 0) \\ &= \phi(x,-h)\varphi_x(x,-h)\ |_0^{\infty} - \int_0^{\infty}\phi_x(x,-h)\varphi_x(x,-h)\,\mathrm{d}x \\ &= -\phi(0,-h)\varphi_x(0,-h) - \int_0^{\infty}\phi_x(x,-h)\varphi_x(x,-h)\,\mathrm{d}x.\end{aligned}$$

- At $z = \eta(x)$, $\mathrm{d}z = \eta_x\,\mathrm{d}x$, so we have

$$\begin{aligned}\int_0^{\infty}\varphi_z(\phi_x - \phi_x\eta_x) + \phi(\varphi_{xx} + \varphi_{xz}\eta_x)\,\mathrm{d}x &= \int_0^{\infty}\varphi_z\frac{\partial\phi}{\partial N} + \phi\frac{\mathrm{d}\varphi_x(x,\eta(x))}{\mathrm{d}x}\,\mathrm{d}x \\ &= \int_0^{\infty}\varphi_z\frac{\partial\phi}{\partial N}\,\mathrm{d}x + \phi(x,\eta)\varphi_x(x,\eta)\Big|_0^{\infty} - \int_0^{\infty}\phi_x(x,\eta)\varphi_x(x,\eta)\,\mathrm{d}x \\ &= \int_0^{\infty}\varphi_z(x,\eta)\frac{\partial\phi}{\partial N}(x,\eta)\,\mathrm{d}x - \phi(0,\eta)\varphi_x(0,\eta) - \int_0^{\infty}\phi_x(x,\eta)\varphi_x(x,\eta)\,\mathrm{d}x.\end{aligned}$$

Combine the segments:

$$\begin{aligned}0 &= \int_D \varphi_z(\phi_x\,\mathrm{d}x - \phi_x\,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) \\ &= \left\{\int_{-\infty}^{\infty}\Big|^{z=-h} + \int_{-h}^{\eta(x)}\Big|^{x\to\infty} - \int_{-\infty}^{\infty}\Big|^{z=\eta(x)} - \int_{-h}^{\eta}\Big|^{x=0}\right\}\varphi_z(\phi_x\,\mathrm{d}x - \phi_x\,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) \\ &= -\phi(0,-h)\varphi_x(0,-h) - \int_0^{\infty}\phi_x(x,-h)\varphi_x(x,-h)\,\mathrm{d}x \\ &\quad - \int_0^{\infty}\varphi_z(x,\eta)\frac{\partial\phi}{\partial N}(x,\eta)\,\mathrm{d}x + \phi(0,\eta)\varphi_x(0,\eta) + \int_0^{\infty}\phi_x(x,\eta)\varphi_x(x,\eta)\,\mathrm{d}x \\ &\quad - \phi(0,\eta)\varphi_x(0,\eta) + \phi(0,-h)\varphi_x(0,-h) + \int_{-h}^{\eta}\phi_z(0,z)\varphi_x(0,z)\,\mathrm{d}z \\ &= -\int_0^{\infty}\phi_x(x,-h)\varphi_x(x,-h)\,\mathrm{d}x - \int_0^{\infty}\varphi_z(x,\eta)\frac{\partial\phi}{\partial N}(x,\eta)\,\mathrm{d}x + \int_0^{\infty}\phi_x(x,\eta)\varphi_x(x,\eta)\,\mathrm{d}x + \int_{-h}^{\eta}\phi_z(0,z)\varphi_x(0,z)\,\mathrm{d}z\end{aligned}$$

$$= \int_0^\infty \phi_x(x,\eta)\varphi_x(x,\eta) - \phi_x(x,-h)\varphi_x(x,-h) - \varphi_z(x,\eta)\frac{\partial\phi}{\partial N}(x,\eta)\,\mathrm{d}x + \int_{-h}^{\eta} \phi_z(0,z)\varphi_x(0,z)\,\mathrm{d}z$$

Force $\varphi_x(0,z) = 0$, so that we are left with:

$$\int_0^\infty \varphi_z(x,\eta)\frac{\partial\phi}{\partial N}(x,\eta) + \phi_x(x,-h)\varphi_x(x,-h) - \phi_x(x,\eta)\varphi_x(x,\eta)\,\mathrm{d}x = 0. \tag{31}$$

Let $\varphi = \cos(kx)\sinh(k(z+h))$, and note that $\phi_x(x,\eta) = \dfrac{\partial\phi}{\partial T}$ is the tangential derivative at $z = \eta$. The equation (31) becomes:

$$\int_0^\infty k\cos(kx)\cosh(k(\eta+h))\frac{\partial\phi}{\partial N}(x,\eta) + k\sin(kx)\sinh(k(\eta+h))\frac{\partial\phi}{\partial T}(x,\eta)\,\mathrm{d}x = 0, \tag{32}$$

since $\varphi_x(x,-h) = -k\sin(kx)\sinh(k(h-h)) = 0$. Let $\dfrac{\partial\phi}{\partial N}(x,\eta) = f(x)$, $\dfrac{\partial\phi}{\partial T}(x,\eta) = \mathcal{H}(\eta,D)\{f(x)\}$ and assume $k \neq 0$, so that we obtain

$$\int_0^\infty \cos(kx)\cosh(k(\eta+h))f(x) + \sin(kx)\sinh(k(\eta+h))\mathcal{H}(\eta,D)\{f(x)\}\,\mathrm{d}x = 0. \tag{33}$$

Observe that (36) is kinda like the real part of the original equation but on a half-line (maybe the other half-line is the imaginary part).

2.0.2 Nondimensional, nonlocal formulation: a half-line

As previously, let ϕ be harmonic. Then, after some manipulation, we have the following contour integral (in non-dimensional variables):

$$\int_D \varphi_z(\frac{1}{\mu^2}\phi_z\,\mathrm{d}x - \phi_x\,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) = 0. \tag{34}$$

Break the contour D in the following segments:

$$\{x\to\infty, -1 < z < \varepsilon\eta\}, \qquad \{x = 0, -1 < z < \varepsilon\eta\}, \qquad \{z = -1, 0 < x < \infty\}, \qquad \{z = \varepsilon\eta, 0 < x < \infty\}.$$

We consider each of the segments.

- At $x \to \infty$, the integral vanishes due to behaviour of ϕ and its gradient.
- At $x = 0$, $\mathrm{d}x = 0$, so we have

$$\int_{-1}^{\varepsilon\eta} -\varphi_z\phi_x + \phi\varphi_{xz}\,\mathrm{d}z = \phi(0,\varepsilon\eta)\varphi_x(0,\varepsilon\eta) - \phi(0,-1)\varphi_x(0,-1) - \int_{-1}^{\varepsilon\eta}\phi_z(0,z)\varphi_x(0,z)\,\mathrm{d}z.$$

- At $z = -1$, $\mathrm{d}z = 0$, so we have

$$\int_0^\infty \varphi_z\phi_z + \phi\varphi_{xx}\,\mathrm{d}x = -\phi(0,-1)\varphi_x(0,-1) - \int_0^\infty \phi_x(x,-1)\varphi_x(x,-1)\,\mathrm{d}x.$$

- At $z = \varepsilon\eta(x)$, $\mathrm{d}z = \varepsilon\eta_x\,\mathrm{d}x$, so we have

$$\begin{aligned}\int_0^\infty &\frac{1}{\mu^2}\varphi_z(\phi_x - \varepsilon\phi_x\eta_x) + \phi(\varphi_{xx} + \varepsilon\varphi_{xz}\eta_x)\,\mathrm{d}x \\ &= \int_0^\infty \frac{1}{\mu}\varphi_z\tilde{\nabla}\phi\cdot\tilde{N} + \frac{\mathrm{d}\phi_x(x,\varepsilon\eta(x))}{\mathrm{d}x}\,\mathrm{d}x \\ &= \int_0^\infty \frac{1}{\mu}\varphi_z\tilde{\nabla}\phi\cdot\tilde{N}\,\mathrm{d}x + \phi(x,\varepsilon\eta)\varphi_x(x,\varepsilon\eta)\,\Big|_0^\infty - \int_0^\infty \phi_x(x,\varepsilon\eta)\varphi_x(x,\varepsilon\eta)\,\mathrm{d}x \\ &= \int_0^\infty \frac{1}{\mu}\varphi_z\tilde{\nabla}\phi\cdot\tilde{N}\,\mathrm{d}x - \phi(0,\varepsilon\eta)\varphi_x(0,\varepsilon\eta) - \int_0^\infty \phi_x(x,\varepsilon\eta)\varphi_x(x,\varepsilon\eta)\,\mathrm{d}x\end{aligned}$$

Combine the segments:

$$
\begin{aligned}
0 &= \int_D \varphi_z(\frac{1}{\mu^2}\phi_x \,\mathrm{d}x - \phi_x \,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) \\
&= \phi(0,\varepsilon\eta)\varphi_x(0,\varepsilon\eta) - \phi(0,-1)\varphi_x(0,-1) - \int_{-1}^{\varepsilon\eta} \phi_z(0,z)\varphi_x(0,z)\,\mathrm{d}z \\
&- \phi(0,-1)\varphi_x(0,-1) - \int_0^\infty \phi_x(x,-1)\varphi_x(x,-1)\,\mathrm{d}x \\
&+ \int_0^\infty \frac{1}{\mu}\varphi_z\tilde{\nabla}\phi\cdot\tilde{N}\,\mathrm{d}x - \phi(0,\varepsilon\eta)\varphi_x(0,\varepsilon\eta) - \int_0^\infty \phi_x(x,\varepsilon\eta)\varphi_x(x,\varepsilon\eta)\,\mathrm{d}x \\
&= -2\phi(0,-h)\varphi_x(0,-1) - \int_{-1}^{\varepsilon\eta} \phi_z(0,z)\varphi_x(0,z)\,\mathrm{d}z \\
&+ \int_0^\infty \frac{1}{\mu}\varphi_z\tilde{\nabla}\phi\cdot\tilde{N}\,\mathrm{d}x - \int_0^\infty \phi_x(x,\varepsilon\eta)\varphi_x(x,\varepsilon\eta)\,\mathrm{d}x - \int_0^\infty \phi_x(x,-1)\varphi_x(x,-1)\,\mathrm{d}x.
\end{aligned}
$$

Force $\varphi_x(0,z) = 0$, so that we are left with:

$$
\int_0^\infty \frac{1}{\mu}\varphi_z\tilde{\nabla}\phi\cdot\tilde{N}\,\mathrm{d}x - \phi_x(x,\varepsilon\eta)\varphi_x(x,\varepsilon\eta)\,\mathrm{d}x - \phi_x(x,-1)\varphi_x(x,-1)\,\mathrm{d}x.
$$

Let $\varphi = \cos(kx)\sinh(\mu k(z+1))$, and note that $\phi_x(x,\varepsilon\eta) = \dfrac{\partial\phi}{\partial T}$ is the tangential derivative at $z = \varepsilon\eta$. The above becomes:

$$
\int_0^\infty k\cos(kx)\cosh(\mu k(\eta+1))\tilde{\nabla}\phi\cdot\tilde{N} + k\sin(kx)\sinh(\mu k(\eta+1))\tilde{\nabla}\phi\cdot\tilde{T}\,\mathrm{d}x = 0, \tag{35}
$$

since $\varphi_x(x,-1) = -k\sin(kx)\sinh(k(1-1)) = 0$. Let

$$
\tilde{\nabla}\phi\cdot\tilde{T} = \frac{\partial\phi}{\partial N}(x,\varepsilon\eta) = f(x), \qquad \frac{\partial\phi}{\partial T}(x,\varepsilon\eta) = \mathcal{H}(\varepsilon\eta, D)\{f(x)\},
$$

so that we obtain

$$
\int_0^\infty \cos(kx)\cosh(\mu k(\eta+1))f(x) + \sin(kx)\sinh(\mu k(\eta+1))\mathcal{H}(\varepsilon\eta,D)\{f(x)\}\,\mathrm{d}x = 0, \tag{36}
$$

where we took k out of integral.

2.0.3 Perturbation expansion of the $\mathcal{H}$ operator: a half line.

Suppose

$$
\mathcal{H}(\varepsilon\eta,D)\{f(x)\} = \sum_{j=0}^\infty \varepsilon^j\mathcal{H}_0(\varepsilon\eta,D)\{f(x)\}.
$$

For notational convenience, we assume throughout that the $\mathcal{H}$ operator is evaluated at $(\varepsilon\eta, D)$, so that we drop this term in writing. Expand in ε :

$$
\begin{aligned}
\cosh(\mu k(\varepsilon\eta+1)) &= \cosh(\mu k) + \varepsilon\mu k\eta\sinh(\mu k) + \frac{(\varepsilon\mu k\eta)^2}{2}\cosh(\mu k) + \ldots, \\
\sinh(\mu k(\varepsilon\eta+1) &= \sinh(\mu k) + \varepsilon\mu k\eta\cosh(\mu k) + \frac{(\varepsilon\mu k\eta)^2}{2}\sinh(\mu k) + \ldots,
\end{aligned}
$$

so that (36) becomes

$$
\begin{aligned}
\int_0^\infty &\cos(kx)(\cosh(\mu k) + \varepsilon\mu k\eta\sinh(\mu k) + \ldots)f(x) \\
&\qquad + \sin(kx)(\sinh(\mu k) + \varepsilon\mu k\eta\cosh(\mu k) + \ldots)(\mathcal{H}_0 + \varepsilon\mathcal{H}_1 + \ldots)\{f(x)\}\,\mathrm{d}x = 0.
\end{aligned} \tag{37}
$$

Within $\mathcal{O}(\varepsilon^0)$, we obtain

$$\int_0^\infty \cos(kx)\cosh(\mu k)f(x) + \sin(kx)\sinh(\mu k)\mathcal{H}_0\{f(x)\}\,\mathrm{d}x = 0.$$

Let $\mathcal{F}_c^k$ indicate the Fourier cosine transform, and similarly for the Fourier sine transform. Then, we have

$$\begin{aligned}
\mathcal{F}_s^k\{\mathcal{H}_0\{f(x)\}\} &= -\mathcal{F}_c^k\{\coth(\mu k)f(x)\} \\
&\implies \mathcal{H}_0\{f(x)\} = -(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\coth(\mu k)f(x)\}\} \\
&\implies \mathcal{H}_0\{f(x)\} = -\int_0^\infty \sin(kx)\coth(\mu k)\left(\frac{2}{\pi}\int_0^\infty \cos(kx)f(x)\,\mathrm{d}x\right)\mathrm{d}k \\
&\implies \mathcal{H}_0\{f(x)\} = -\int_0^\infty \sin(kx)\coth(\mu k)\widehat{f_c^k}\,\mathrm{d}k = -\{\mathcal{F}_s^k\}^{-1}\{\coth(\mu k)\widehat{f_c^k}\}.
\end{aligned}$$

Within $\mathcal{O}(\varepsilon^1)$, the equation (37) is

$$\int_0^\infty \cos(kx)\mu k\eta f(x) + \sin(kx)(\mathcal{H}_1\{f(x)\} + \mu k\eta\coth(\mu k)\mathcal{H}_0\{f(x)\})\,\mathrm{d}x = 0.$$

Then,

$$\begin{aligned}
\int_0^\infty \sin(kx)\mathcal{H}_1\{f(x)\}\,\mathrm{d}x &= -\mu k\left[\int_0^\infty \cos(kx)\eta f(x)\,\mathrm{d}x + \coth(\mu k)\int_0^\infty \sin(kx)\eta\mathcal{H}_0\{f(x)\}\,\mathrm{d}x\right] \\
&= -\mu k\left[\widehat{(\eta f(x))}_c^k + \coth(\mu k)\widehat{(\eta\mathcal{H}_0\{f(x)\})}_c^k\right],
\end{aligned}$$

where Fourier sine transform is inverted to obtain

$$\mathcal{H}_1\{f(x)\} = -\{\mathcal{F}_s^k\}^{-1}\{\mu k\widehat{(\eta f(x))}_c^k + \mu k\coth(\mu k)\widehat{(\eta\mathcal{H}_0\{f(x)\})}_c^k\}.$$

In sum, we obtain

$$\begin{aligned}
\mathcal{H}_0(\varepsilon\eta, D)\{f(x)\} &= -\{\mathcal{F}_s^k\}^{-1}\{\coth(\mu k)\widehat{f_c^k}\}, \\
\mathcal{H}_1(\varepsilon\eta, D)\{f(x)\} &= -\{\mathcal{F}_s^k\}^{-1}\{\mu k\widehat{(\eta f(x))}_c^k + \mu k\coth(\mu k)\widehat{(\eta\mathcal{H}_0\{f(x)\})}_c^k\}.
\end{aligned}$$

2.0.4 Deriving an expression for surface elevation: a half-line.

First, we'd like to approximate. Recall the non-dimensional single equation:

$$\partial_t\left(\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right) + \partial_x\left(\frac{1}{2}\left(\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 + \varepsilon\eta - \frac{1}{2}\varepsilon\mu^2\frac{(\eta_t + \eta_x\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\})^2}{1 + \varepsilon^2\mu^2\eta_x^2}\right) = 0. \quad (38)$$

Within $\mathcal{O}(\mu^0)$:

we have $\mathcal{H} \approx \mathcal{H}_0$, and the single equation becomes:

$$\partial_t\left(\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right) + \varepsilon\partial_x\eta = 0. \quad (39)$$

Note

$$\begin{aligned}
\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\} &= -\int_0^\infty \sin(kx)\coth(\mu k)\widehat{(\varepsilon\mu\eta_t)}_c^k\,\mathrm{d}k \\
&= -\varepsilon\int_0^\infty \sin(kx)\mu\coth(\mu k)\widehat{(\eta_t)}_c^k\,\mathrm{d}k \\
&= -\varepsilon\int_0^\infty \sin(kx)\left(\frac{1}{k} + \frac{\mu^2 k}{3} + \dots\right)\widehat{(\eta_t)}_c^k\,\mathrm{d}k \\
&\approx -\varepsilon\int_0^\infty \sin(kx)\frac{1}{k}\widehat{(\eta_t)}_c^k\,\mathrm{d}k.
\end{aligned}$$

Substituting into (39) yields

$$\partial_t(-\varepsilon\int_0^\infty \sin(kx)\frac{1}{k}\widehat{(\eta_t)}_c^k\,\mathrm{d}k)+\varepsilon\eta_x=0 \implies -\int_0^\infty \sin(kx)\frac{1}{k}\widehat{(\eta_{tt})}_c^k\,\mathrm{d}k+\eta_x=0$$
$$\implies -\frac{1}{k}\widehat{(\eta_{tt})}_c^k+\widehat{\eta_x}_s^k=0.$$

Note that via integration by parts and using a conservation law,

$$-\frac{1}{k}\widehat{(\eta_{tt})}_c^k=-\mathcal{F}_s^k\{\int_0^x \eta_{tt}\,\mathrm{d}x'\},$$

so that

$$-\frac{1}{k}\widehat{(\eta_{tt})}_c^k+\widehat{\eta_x}_s^k=-\mathcal{F}_s^k\{\int_0^x \eta_{tt}\,\mathrm{d}x'\}+\widehat{\eta_x}_s^k=0.$$

Inverting the Sine transform yields:

$$-\int_0^x \eta_{tt}\,\mathrm{d}x'+\eta_x=0,$$

and differentiating with respect to x yields the wave equation.

Within $\mathcal{O}(\mu^2)$:

We have $\mathcal{H}\approx\mathcal{H}_0+\varepsilon\mathcal{H}_1$, and the single equation becomes:

$$\partial_t\left(\mathcal{H}_0\{\varepsilon\mu\eta_t\}+\varepsilon\mathcal{H}_1\{\varepsilon\mu\eta_t\}\right)+\partial_x\left(\frac{1}{2}(\mathcal{H}_0\{\varepsilon\mu\eta_t\})^2+\varepsilon\eta\right)=0. \tag{40}$$

Note

$$\begin{aligned}
\mathcal{H}_0\{\varepsilon\mu\eta_t\}&=-\{\mathcal{F}_s^k\}^{-1}\{\coth(\mu k)\widehat{\varepsilon\mu(\eta_t)}_c^k\}\\
&=-\varepsilon\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_t)}_c^k\},\\
\mathcal{H}_1\{\varepsilon\mu\eta_t\}&=-\{\mathcal{F}_s^k\}^{-1}\{k\mu\widehat{(\eta\varepsilon\mu\eta_t)}_c^k+k\mu\coth(\mu k)\widehat{(\eta\mathcal{H}_0\{\varepsilon\mu\eta_t\})}_c^k\}\\
&=-\varepsilon\mu\{\mathcal{F}_s^k\}^{-1}\{\mu k\widehat{(\eta\eta_t)}_c^k+\mu k\coth(\mu k)\widehat{(\eta\mathcal{H}_0\{\eta_t\})}_c^k\}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\partial_t\left(\mathcal{H}_0\{\varepsilon\mu\eta_t\}+\varepsilon\mathcal{H}_1\{\varepsilon\mu\eta_t\}\right)\\
&=-\partial_t\left(\varepsilon\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_t)}_c^k\}+\varepsilon^2\mu\{\mathcal{F}_s^k\}^{-1}\{\mu k\widehat{(\eta\eta_t)}_c^k+\mu k\coth(\mu k)\widehat{(\eta\mathcal{H}_0\{\eta_t\})}_c^k\}\right)\\
&=-\varepsilon\left(\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_{tt})}_c^k\}+\varepsilon\mu\{\mathcal{F}_s^k\}^{-1}\{\mu k\partial_t\widehat{(\eta\eta_t)}_c^k+\mu k\coth(\mu k)\partial_t\widehat{(\eta\mathcal{H}_0\{\eta_t\})}_c^k\}\right)
\end{aligned}$$

Moreover,

$$\begin{aligned}
\frac{1}{2}(\mathcal{H}_0\{\varepsilon\mu\eta_t\})^2&=\frac{1}{2}(-\varepsilon\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_t)}_c^k\})^2\\
&=\frac{\varepsilon^2}{2}(\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_t)}_c^k\})^2.
\end{aligned}$$

When combining the terms, the equation (40) becomes

$$\begin{aligned}
&\partial_t(\mathcal{H}_0\{\varepsilon\mu\eta_t\}+\varepsilon\mathcal{H}_1\{\varepsilon\mu\eta_t\})+\partial_x\left(\frac{1}{2}(\mathcal{H}_0\{\varepsilon\mu\eta_t\})^2+\varepsilon\eta\right)=0\\
\implies&-\varepsilon(\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_{tt})}_c^k\}+\varepsilon\mu\{\mathcal{F}_s^k\}^{-1}\{\mu k\partial_t\widehat{(\eta\eta_t)}_c^k+\mu k\coth(\mu k)\partial_t\widehat{(\eta\mathcal{H}_0\{\eta_t\})}_c^k\})
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\varepsilon^2}{2}\partial_x(\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_t)_c^k}\})^2+\varepsilon\eta_x=0.\\
\implies &-(\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_{tt})_c^k}\}+\varepsilon\mu\{\mathcal{F}_s^k\}^{-1}\{\mu k\partial_t\widehat{(\eta\eta_t)}_c^k+\mu k\coth(\mu k)\partial_t\widehat{(\eta\mathcal{H}_0\{\eta_t\})}_c^k\})\\
&+\frac{\varepsilon}{2}\partial_x(\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_t)_c^k}\})^2+\eta_x=0.
\end{aligned}
$$

Apply Fourier sine transform:

$$
\begin{aligned}
-(\mu\coth(\mu k)\widehat{(\eta_{tt})_c^k}+\varepsilon\mu^2k\partial_t\widehat{(\eta\eta_t)}_c^k+\varepsilon\mu^2k\coth(\mu k)\partial_t\widehat{(\eta\mathcal{H}_0\{\eta_t\})}_c^k)&\\
+\frac{\varepsilon}{2}\mathcal{F}_s^k\{\partial_x(\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_t)_c^k}\})^2\}+\widehat{(\eta_x)_s^k}=0.&
\end{aligned}
$$

Then, letting $\varepsilon=\mu^2$ and expanding $\mathcal{H}_0$ in

$$
\begin{aligned}
-(\mu\coth(\mu k)\widehat{(\eta_{tt})_c^k}+\varepsilon\mu^2k\partial_t\widehat{(\eta\eta_t)}_c^k+\varepsilon\mu^2k\coth(\mu k)\partial_t\widehat{(\eta\mathcal{H}_0\{\eta_t\})}_c^k)&\\
+\frac{\varepsilon}{2}\mathcal{F}_s^k\{\partial_x(\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_t)_c^k}\})^2\}+\widehat{(\eta_x)_s^k}=0,&
\end{aligned}
$$

we obtain

$$
\begin{aligned}
-\mu\coth(\mu k)\widehat{(\eta_{tt})_c^k}-\mu^4k\partial_t\widehat{(\eta\eta_t)_c^k}+\mu^3k\coth(\mu k)\partial_t\mathcal{F}_c^k\{\Big(\eta\{\mathcal{F}_s^l\}^{-1}\{\mu\coth(\mu l)\widehat{(\eta_t)_c^l}\}\Big)\}&\\
+\frac{\mu^2}{2}\mathcal{F}_s^k\{\partial_x(\{\mathcal{F}_s^k\}^{-1}\{\mu\coth(\mu k)\widehat{(\eta_t)_c^k}\})^2\}+\widehat{(\eta_x)_s^k}=0.&
\end{aligned}
$$

Expanding $\coth(\mu k)$ yields:

$$
\begin{aligned}
-\mu\left(\frac{1}{\mu k}+\frac{\mu k}{3}\right)\widehat{(\eta_{tt})_c^k}-\mu^4k\partial_t\widehat{(\eta\eta_t)_c^k}+\mu^3k\left(\frac{1}{\mu k}+\frac{\mu k}{3}\right)\partial_t\mathcal{F}_c^k\{\left(\eta\{\mathcal{F}_s^l\}^{-1}\{\mu\left(\frac{1}{\mu l}+\frac{\mu l}{3}\right)\widehat{(\eta_t)_c^l}\}\right)\}&\\
+\frac{\mu^2}{2}\mathcal{F}_s^k\{\partial_x(\{\mathcal{F}_s^k\}^{-1}\{\mu\left(\frac{1}{\mu k}+\frac{\mu k}{3}\right)\widehat{(\eta_t)_c^k}\})^2\}+\widehat{(\eta_x)_s^k}=0,&
\end{aligned}
$$

and so

$$
\begin{aligned}
-\left(\frac{1}{k}+\frac{\mu^2k}{3}\right)\widehat{(\eta_{tt})_c^k}-\mu^4k\partial_t\widehat{(\eta\eta_t)}_c^k+\left(\mu^2+\frac{\mu^4k^2}{3}\right)\partial_t\mathcal{F}_c^k\{\left(\eta\{\mathcal{F}_s^l\}^{-1}\{\left(\frac{1}{l}+\frac{\mu^2 l}{3}\right)\widehat{(\eta_t)_c^l}\}\right)\}&\\
+\frac{\mu^2}{2}\mathcal{F}_s^k\{\partial_x(\{\mathcal{F}_s^k\}^{-1}\{\left(\frac{1}{k}+\frac{\mu^2k}{3}\right)\widehat{(\eta_t)_c^k}\})^2\}+\widehat{(\eta_x)_s^k}=0.&
\end{aligned}
$$

Removing the terms of order $\mathcal{O}(\mu^4)$ and rearranging, we obtain:

$$
-\frac{1}{k}\widehat{(\eta_{tt})_c^k}+\widehat{(\eta_x)_s^k}+\mu^2\left(\partial_t\mathcal{F}_c^k\{\left(\eta\{\mathcal{F}_s^l\}^{-1}\{\frac{1}{l}\widehat{(\eta_t)_c^l}\}\right)\}+\frac{1}{2}\mathcal{F}_s^k\{\partial_x(\{\mathcal{F}_s^k\}^{-1}\{\frac{1}{k}\widehat{(\eta_t)_c^k}\})^2\}-\frac{k}{3}\widehat{(\eta_{tt})_c^k}\right)=0. \tag{41}
$$

Now, we would like to manipulate (41) so that we can apply inverse Fourier sine transform. First, note

$$
\begin{aligned}
-\frac{1}{k}\widehat{(\eta_{tt})_c^k}&=-\frac{1}{k}\frac{1}{2\pi}\int_0^\infty\cos(kx)\eta_{tt}\,\mathrm{d}x\\
&=-\frac{1}{2\pi}\frac{\cos(kx)}{k}\left(\int_0^x\eta_{tt}\,\mathrm{d}x'\right)\Big|_0^\infty-\frac{1}{2\pi}\int_0^\infty\sin(kx)\left(\int_0^x\eta_{tt}\,\mathrm{d}x'\right)\mathrm{d}x\\
&=-\mathcal{F}_s^k\{\int_0^x\eta_{tt}\,\mathrm{d}x'\},
\end{aligned}
$$

where the final line follows since $\displaystyle\int_0^\infty\eta_{tt}\,\mathrm{d}x'=0$ is a conservation law (to be verified). Second, observe that

$$
\frac{1}{l}\widehat{(\eta_t)_c^l}=\frac{1}{l}\frac{1}{2\pi}\int_0^\infty\cos(lx)\eta_t\,\mathrm{d}x
$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\frac{\cos(lx)}{l}\left(\int_0^x \eta_t \,\mathrm{d}x'\right)\Big|_0^\infty + \frac{1}{2\pi}\int_0^\infty \sin(lx)\left(\int_0^x \eta_t\,\mathrm{d}x'\right)\mathrm{d}x \\
&= \mathcal{F}_s^l\{\int_0^x \eta_t\,\mathrm{d}x'\},
\end{aligned}
$$

where similarly the last line follows since $\displaystyle\int_0^\infty \eta_t\,\mathrm{d}x' = 0$, a conservation law (to be verified). This identity yields:

$$
\begin{aligned}
\partial_t \mathcal{F}_c^k\{\eta\{\mathcal{F}_s^l\}^{-1}\{\frac{1}{l}\widehat{(\eta_t)_c^l}\} &= \mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\}, \\
\frac{1}{2}\mathcal{F}_s^k\{\partial_x(\{\mathcal{F}_s^k\}^{-1}\{\frac{1}{k}\widehat{(\eta_t)_c^k}\})^2\} &= \frac{1}{2}\mathcal{F}_s^k\{\partial_x\left(\int_0^x \eta_t\,\mathrm{d}x'\right)^2\}.
\end{aligned}
$$

Thirdly, we have

$$
\begin{aligned}
-\frac{k}{3}\widehat{(\eta_{tt})_c^k} &= -\frac{k}{3}\frac{1}{2\pi}\int_0^\infty \cos(kx)\eta_{tt}\,\mathrm{d}x \\
&= -\frac{k}{3}\frac{1}{2\pi}\frac{\sin(kx)}{k}\eta_{tt}\Big|_0^\infty + \frac{1}{3}\frac{1}{2\pi}\int_0^\infty \sin(kx)\eta_{ttx}\,\mathrm{d}x \\
&= \frac{1}{3}\mathcal{F}_s^k\{\eta_{ttx}\},
\end{aligned}
$$

where the last line follows by the assumption $\lim_{x\to\infty}\eta_{tt} = 0$. With these manipulations in mind, the equation (41) becomes

$$
-\mathcal{F}_s^k\{\int_0^x \eta_{tt}\,\mathrm{d}x'\} + \widehat{(\eta_x)_s^k} + \mu^2\left(\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\} + \frac{1}{2}\mathcal{F}_s^k\{\partial_x\left(\int_0^x \eta_t\,\mathrm{d}x'\right)^2\} + \frac{1}{3}\mathcal{F}_s^k\{\eta_{ttx}\}\right) = 0. \tag{42}
$$

Inverting the Fourier sine transform, we obtain

$$
-\int_0^x \eta_{tt}\,\mathrm{d}x' + \eta_x + \mu^2\left((\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\}\} + \frac{1}{2}\partial_x\left(\int_0^x \eta_t\,\mathrm{d}x'\right)^2 + \frac{1}{3}\eta_{ttx}\right) = 0. \tag{43}
$$

Take the derivative with respect to x :

$$
-\eta_{tt} + \eta_{xx} + \mu^2\left(\partial_x(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\}\} + \frac{1}{2}\partial_x^2\left(\int_0^x \eta_t\,\mathrm{d}x'\right)^2 + \frac{1}{3}\eta_{ttxx}\right) = 0. \tag{44}
$$

Rearranging and using $\eta_{tt} = \eta_{xx} + \mathcal{O}(\mu^2)$, we obtain the equation

$$
\eta_{tt} - \eta_{xx} = \mu^2\left(\frac{1}{3}\eta_{xxxx} + \partial_x(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\}\} + \frac{1}{2}\partial_x^2\left(\int_0^x \eta_t\,\mathrm{d}x'\right)^2\right). \tag{45}
$$

Clearly, the presence of the term with mixed transforms complicates things: we cannot apply integration by parts like we did for other terms, because doing so results in a multiple of k in the new term, which is exactly what we want to avoid. For comparison, the whole line equation for the surface is given by

$$
\eta_{tt} - \eta_{xx} = \mu^2\left(\frac{1}{3}\eta_{xxxx} + \partial_x\left[\eta_t\left(\int_{-\infty}^x \eta_t\,\mathrm{d}x'\right) + \eta\eta_x\right] + \frac{1}{2}\partial_x^2\left(\int_{-\infty}^x \eta_t\,\mathrm{d}x'\right)^2\right).
$$

Sultan's calculations

Consider the term

$$
\varepsilon\mu^2\frac{(\eta_t + \eta_x\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\})^2}{1+\varepsilon^2\mu^2\eta_x^2}.
$$

Note that $|\varepsilon\mu\eta_x| < 1$, so

$$\frac{1}{1+\varepsilon^2\mu^2\eta_x^2} = \frac{1}{1-(-\varepsilon^2\mu^2\eta_x^2)} \approx 1+\varepsilon^2\mu^2\eta_x^2,$$

by geometric series argument. Furthermore,

$$\begin{aligned}(\eta_t + \eta_x\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\})^2 &\approx \eta_t^2 + 2\eta_t\eta_x\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\\ &\approx \eta_t^2 + 2\eta_t\eta_x\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\},\end{aligned}$$

so we can assume

$$\varepsilon\mu^2\frac{(\eta_t + \eta_x\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\})^2}{1+\varepsilon^2\mu^2\eta_x^2} \approx \varepsilon\mu^2(\eta_t^2 + 2\eta_t\eta_x\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\})$$

Then,

$$\begin{aligned}\frac{1}{2}\left(\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 &\approx \frac{1}{2}\left([\mathcal{H}_0(\varepsilon\eta, D) + \varepsilon\mathcal{H}_1(\varepsilon\eta, D)]\{\varepsilon\mu\eta_t\}\right)^2\\ &= \frac{1}{2}\left(\left[\mathcal{H}_0(\varepsilon\eta, D)^2 + 2\varepsilon\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\mathcal{H}_1(\varepsilon\eta, D)\right]\{\varepsilon\mu\eta_t\}\right)\\ &= \frac{1}{2}\left(\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 + \varepsilon\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\mathcal{H}_1(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\\ &= \frac{1}{2}\left(\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 + \varepsilon^3\mu^2\mathcal{H}_0(\varepsilon\eta, D)\{\eta_t\}\mathcal{H}_1(\varepsilon\eta, D)\{\eta_t\}\\ &= \frac{(\varepsilon\mu)^2}{2}\left[\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]^2\\ &+ \varepsilon^3\mu^2\left[\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right](\mu(\eta f)_x - \mathcal{F}^{-1}\{\mu k\coth(\mu k)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\}\}).\end{aligned}$$

In the leading two orders, we have

$$\partial_t\left(\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right) + \partial_x\left(\frac{1}{2}\left(\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 + \varepsilon\eta - \frac{1}{2}\varepsilon\mu^2\frac{(\eta_t + \eta_x\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\})^2}{1+\varepsilon^2\mu^2\eta_x^2}\right) = 0 \implies$$

$$\begin{aligned}&\partial_t([\mathcal{H}_0(\varepsilon\eta, D) + \varepsilon\mathcal{H}_1(\varepsilon\eta, D)]\{\varepsilon\mu\eta_t\} + \partial_x\Bigg(\frac{1}{2}\left(\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 + \varepsilon\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\mathcal{H}_1(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\\ &\qquad + \varepsilon\eta - \frac{1}{2}\varepsilon\mu^2\eta_t^2\Bigg) = 0\\ &\partial_t([\mathcal{H}_0(\varepsilon\eta, D) + \varepsilon\mathcal{H}_1(\varepsilon\eta, D)]\{\varepsilon\mu\eta_t\} + \partial_x\Bigg(\frac{1}{2}\left(\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 + \varepsilon^3\mu^2\mathcal{H}_0(\varepsilon\eta, D)\{\eta_t\}\mathcal{H}_1(\varepsilon\eta, D)\{\eta_t\}\\ &\qquad + \varepsilon\eta - \frac{1}{2}\varepsilon\mu^2\eta_t^2\Bigg) = 0.\end{aligned}$$

Consider each term:

$$\begin{aligned}\partial_t\left([\mathcal{H}_0(\varepsilon\eta, D) + \varepsilon\mathcal{H}_1(\varepsilon\eta, D)]\{\varepsilon\mu\eta_t\}\right) &= \varepsilon\mu\partial_t\left([\mathcal{H}_0(\varepsilon\eta, D) + \varepsilon\mathcal{H}_1(\varepsilon\eta, D)]\{\eta_t\}\right)\\ &= \varepsilon\mu\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_{tt}}\} + \varepsilon^2\mu^2(\eta\eta_t)_{tx}\\ &- \varepsilon^2\mu\mathcal{F}^{-1}\{\mu k\coth(\mu k)\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]\}.\end{aligned}$$

Then,

$$\begin{aligned}&\varepsilon\mu\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_{tt}}\} + \varepsilon^2\mu^2(\eta\eta_t)_{tx} - \varepsilon^2\mu\mathcal{F}^{-1}\{\mu k\coth(\mu k)\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]\}\\ &\quad + \frac{(\varepsilon\mu)^2}{2}\partial_x\left[\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]^2\\ &\quad + \varepsilon^3\mu^2\partial_x\left[\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]\left[\mu(\eta f)_x - \mathcal{F}^{-1}\{\mu l\coth(\mu l)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\coth(\mu j)\widehat{\eta_t}\}_k\}\right]\\ &\quad + \varepsilon\partial_x\eta - \frac{1}{2}\varepsilon\mu^2\partial_x(\eta_t^2) = 0.\end{aligned}$$

Invert Fourier transform:

$$\varepsilon\mu i\coth(\mu k)\widehat{\eta_{tt}} + \varepsilon^2\mu^2 ik\widehat{(\eta\eta_t)_t} - \varepsilon^2\mu(\mu k\coth(\mu k)\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right])\}$$

$$
\begin{aligned}
&+ \frac{(\varepsilon\mu)^2}{2} ik\mathcal{F}\{\left[\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]^2\} \\
&+ \varepsilon^3\mu^2 ik\mathcal{F}\left(\left[\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]\left[\mu(\eta f)_x - \mathcal{F}^{-1}\{\mu l\coth(\mu l)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\coth(\mu j)\widehat{\eta_t}\}_k\}\right]\right) \\
&+ \varepsilon ik\widehat{\eta} - \frac{1}{2}\varepsilon\mu^2 ik\widehat{\eta_t^2} = 0.
\end{aligned}
$$

Divide by $i\varepsilon$:

$$
\begin{aligned}
\mu\coth(\mu k)\widehat{\eta_{tt}} &+ \varepsilon\mu^2 k\widehat{(\eta\eta_t)_t} - \varepsilon\mu(\mu k\coth(\mu k)\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{\coth(\mu k)\widehat{\eta_t}\}\}\right]) \\
&+ \frac{\varepsilon\mu^2}{2} k\mathcal{F}\{\left[\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]^2\} \\
&+ \varepsilon^2\mu^2 k\mathcal{F}\left(\left[\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]\left[\mu(\eta f)_x - \mathcal{F}^{-1}\{\mu l\coth(\mu l)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\coth(\mu j)\widehat{\eta_t}\}_k\}\right]\right) \\
&+ k\widehat{\eta} - \frac{1}{2}\mu^2 k\widehat{\eta_t^2} = 0.
\end{aligned}
$$

Divide by $\coth(\mu k)$:

$$
\begin{aligned}
\mu\widehat{\eta_{tt}} &+ \varepsilon\mu^2 k\tanh(\mu k)\widehat{(\eta\eta_t)_t} - \varepsilon\mu(\mu k\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{\coth(\mu k)\widehat{\eta_t}\}\right]\}) \\
&+ \frac{\varepsilon\mu^2}{2} k\tanh(\mu k)\mathcal{F}\{\left[\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]^2\} \\
&+ \varepsilon^2\mu^2 k\tanh(\mu k)\mathcal{F}\left(\left[\mathcal{F}^{-1}\{i\coth(\mu k)\widehat{\eta_t}\}\right]\left[\mu(\eta f)_x - \mathcal{F}^{-1}\{\mu l\coth(\mu l)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\coth(\mu j)\widehat{\eta_t}\}_k\}\right]\right) \\
&+ \tanh(\mu k)k\widehat{\eta} - \tanh(\mu k)\frac{1}{2}\mu^2 k\widehat{\eta_t^2} = 0.
\end{aligned}
$$

Let $\varepsilon = \mu^2$ and recall expansions:

$$
\tanh(\mu k) \approx \mu k - \frac{(\mu k)^3}{3} + \mathcal{O}(\mu^5), \qquad \coth(\mu k) \approx \frac{1}{\mu k} + \frac{\mu k}{3} + \mathcal{O}(\mu^3).
$$

Substitute the expansions appropriately:

$$
\begin{aligned}
\mu\widehat{\eta_{tt}} &+ \mu^4 k\left(\mu k - \frac{(\mu k)^3}{3}\right)\widehat{(\eta\eta_t)_t} - i\mu^3 k(\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{\left(\frac{1}{k} + \frac{\mu^2 k}{3}\right)\widehat{\eta_t}\}\right]) \\
&- \frac{\varepsilon\mu^2}{2} k\left(\mu k - \frac{(\mu k)^3}{3}\right)\mathcal{F}\{\left[\mathcal{F}^{-1}\{\left(\frac{1}{\mu l} + \frac{\mu l}{3}\right)\widehat{\eta_t}\}\right]^2\} \\
&+ \varepsilon^2\mu^3 k^2\left(1 - \frac{(\mu k)^2}{3}\right)\mathcal{F}\left(\left[\mathcal{F}^{-1}\{i\left(\frac{1}{\mu k} + \frac{\mu k}{3}\right)\widehat{\eta_t}\}\right]\left[\mu(\eta f)_x - \mathcal{F}^{-1}\{\left(1 + \frac{(\mu l)^2}{3}\right)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\left(\frac{1}{\mu j} + \frac{\mu j}{3}\right)\widehat{\eta_t}\}\}\right]\right) \\
&+ \left(\mu k - \frac{(\mu k)^3}{3}\right)k\widehat{\eta} - \left(\mu k - \frac{(\mu k)^3}{3}\right)\frac{1}{2}\mu^2 k\widehat{\eta_t^2} = 0.
\end{aligned}
$$

Consider the term on the third line: we divide by μ, and eliminate terms within $\mathcal{O}(\mu^4)$:

$$
\begin{aligned}
&\varepsilon^2\mu^3 k^2\left(1 - \frac{(\mu k)^2}{3}\right)\mathcal{F}\left(\left[\mathcal{F}^{-1}\{i\left(\frac{1}{\mu k} + \frac{\mu k}{3}\right)\widehat{\eta_t}\}\right]\left[\mu(\eta f)_x - \mathcal{F}^{-1}\{\left(1 + \frac{(\mu l)^2}{3}\right)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\left(\frac{1}{\mu j} + \frac{\mu j}{3}\right)\widehat{\eta_t}\}\}\right]\right) = \\
&\varepsilon^2\mu k^2\left(1 - \frac{(\mu k)^2}{3}\right)\mathcal{F}\left(\left[\mathcal{F}^{-1}\{i\left(\frac{\mu}{\mu k} + \frac{\mu^2 k}{3}\right)\widehat{\eta_t}\}\right]\left[\mu^2(\eta f)_x - \mathcal{F}^{-1}\{\left(1 + \frac{(\mu l)^2}{3}\right)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\left(\frac{\mu}{\mu j} + \frac{\mu^2 j}{3}\right)\widehat{\eta_t}\}\}\right]\right) = \\
&\varepsilon^2\mu k^2\left(1 - \frac{(\mu k)^2}{3}\right)\mathcal{F}\left(\left[\mathcal{F}^{-1}\{i\left(\frac{1}{k} + \frac{\mu^2 k}{3}\right)\widehat{\eta_t}\}\right]\left[\mu^2(\eta f)_x - \mathcal{F}^{-1}\{\left(1 + \frac{(\mu l)^2}{3}\right)\mathcal{F}\{\eta\mathcal{F}^{-1}\{i\left(\frac{1}{j} + \frac{\mu^2 j}{3}\right)\widehat{\eta_t}\}\}\right]\right) \approx \\
&\varepsilon^2\mu k^2\mathcal{F}\left(\eta\left[\mathcal{F}^{-1}\{\frac{1}{k}\widehat{\eta_t}\}\right]\right]\mathcal{F}^{-1}\{\frac{1}{j}\widehat{\eta_t}\}\left.\vphantom{\frac{1}{j}}\right]\Big),
\end{aligned}
$$

divide by μ to obtain

$$
\varepsilon^2 k^2\mathcal{F}\left(\eta\left[\mathcal{F}^{-1}\{\frac{1}{k}\widehat{\eta_t}\}\right]\left[\mathcal{F}^{-1}\{\frac{1}{j}\widehat{\eta_t}\}\}\right]\right) = \varepsilon^2 k^2\mathcal{F}\left(\eta\left[\int_{-\infty}^{x}\eta_t\,\mathrm{d}x'\right]^2\right),
$$

Divide through by μ, eliminate terms of order $\mathcal{O}(\mu^4)$ and rearrange:

$$\widehat{\eta_{tt}} + k^2\hat{\eta} + \mu^2 \left(-ik(\mathcal{F}\{\partial_t \left[\eta \int_{-\infty}^{x} \eta_t \, \mathrm{d}x\}\right]) - \frac{1}{2}k^2\mathcal{F}\{\left[\mathcal{F}^{-1}\{\frac{1}{l}\widehat{\eta_t}\}\right]^2\} - \frac{k^4}{3}\hat{\eta} - \frac{k^2}{2}\widehat{\eta_t^2} \right) = 0.$$

Finally, invert Fourier transform:

$$\eta_{tt} - \eta_{xx} + \mu^2 \left(-\partial_x\partial_t \left[\eta \int_{-\infty}^{x} \eta_t \, \mathrm{d}x\}\right] + \frac{1}{2}\partial_x^2 \left[\mathcal{F}^{-1}\{\frac{1}{l}\widehat{\eta_t}\}\right]^2 - \frac{1}{3}\eta_{xxxx} + \frac{1}{2}\partial_x^2\eta_t^2 \right) = 0,$$

or more conveniently,

$$\eta_{tt} - \eta_{xx} = \mu^2 \left(\frac{1}{3}\eta_{xxxx} - \frac{1}{2}\partial_x^2\eta_t^2 + \partial_x\partial_t \left[\eta \int_{-\infty}^{x} \eta_t \, \mathrm{d}x\}\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2} \left[\int_{-\infty}^{x} \eta_t \, \mathrm{d}x'\right]^2 \right)$$

For direct comparison, in [1, p. 110], the corresponding equation is the equation (5.20)

$$\eta_{tt} - \eta_{xx} = \mu^2 \left(\frac{1}{3}\eta_{xxxx} + \eta_{xt}\int_{-\infty}^{x} \eta_t \, \mathrm{d}x' + \eta_x^2 + \eta_t^2 + \eta\eta_{xx} + \frac{1}{2}\frac{\partial^2}{\partial x^2} \left[\int_{-\infty}^{x} \eta_t \, \mathrm{d}x'\right]^2 \right),$$

References

[1] Mark J. Ablowitz, *Nonlinear dispersive waves: Asymptotic analysis and solitons*, Cambridge University Press, 2011.