Appendix A

The equations for a viscous fluid

The representation of a viscous fluid requires a change only to the form of the local (short-range) force; the equation of mass conservation is unaltered. The local force is now described through the (Cartesian) *stress tensor*, σ_{ij} $(i, j = 1, 2, 3)$, which represents the i-component of the stress (force/unit area) on the surface whose outward normal is in the j-direction. If $i = j$ then σ_{ij} is a *normal stress*, and for $i \neq j$ it is a *tangential* or *shearing stress*. In order that the local forces give rise only to finite accelerations of a fluid particle, it is necessary that σ_{ij} be symmetric; that is, $\sigma_{ij} = \sigma_{ji}$. Now, symmetric tensors possess the property that, in a certain coordinate system (the *principal coordinates* or *axes*), they may be written with diagonal elements only. (Indeed, as the coordinates are transformed under rotations, the sum of the diagonal elements is unchanged.) All this leads to the choice of stress tensor for a fluid as

$$\sigma_{ij} = -P\delta_{ij} + d_{ij},$$

where P is the pressure in the fluid and δ_{ij} is the *Kronecker delta*; d_{ij} is called the *deviatoric stress tensor* and it is absent for a stationary fluid. It is this contribution which is ignored in the derivation of Euler's equation, (1.12).

It is well established empirically (and supported by arguments based on molecular transport) that, for most common fluids, d_{ij} is proportional to the velocity gradients at a point in the fluid. Thus we write

$$d_{ij} = A_{ijkl}\frac{\partial u_k}{\partial x_l},$$

where A_{ijkl} is a rank four Cartesian tensor, and the *summation convention* is employed; the position vector is written as $\mathbf{x} \equiv (x_1, x_2, x_3)$ and the corresponding velocity vector is $\mathbf{u} \equiv (u_1, u_2, u_3)$. We require d_{ij} to be symmetric (because σ_{ij} is) and, further, we assume that the fluid is *isotropic*; that is, the properties are the same in all directions. These considerations lead to

$$d_{ij} = 2\mu(e_{ij} - \frac{1}{3}\delta_{ij}e_{kk})$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

is the *rate of strain tensor*, and μ is the *coefficient of* (*Newtonian*) *viscosity*. The term $e_{kk} = \partial u_k/\partial x_k$ is called the *dilatation*, and it is zero for an incompressible fluid (that is, $\nabla \cdot \mathbf{u} = 0$); we note that $d_{ii} = 0$ (since $\delta_{ii} = 3$).

The application of Newton's second law to the fluid now yields

$$\int_V \left(\rho \frac{\mathrm{D}u_i}{\mathrm{D}t} - \frac{\partial \sigma_{ij}}{\partial x_j} - \rho F_i \right) \mathrm{d}v = 0$$

where $\mathbf{F} \equiv (F_1, F_2, F_3)$, and so

$$\frac{\mathrm{D}u_i}{\mathrm{D}t} = \frac{1}{\rho} \frac{\partial \sigma_{ij}}{\partial x_j} + F_i;$$

cf. equation (1.12). For an incompressible fluid with constant viscosity, this becomes

$$\frac{\mathrm{D}u_i}{\mathrm{D}t} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + F_i + \frac{\mu}{\rho} \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

with

$$\frac{\partial u_i}{\partial x_i} = 0.$$

Written in a vector notation, these equations are expressed as

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{1}{\rho} \nabla P + \mathbf{F} + \nu \nabla^2 \mathbf{u}; \quad \nabla \cdot \mathbf{u} = 0, \tag{A.1}$$

where $\nu = \mu/\rho$ is the *kinematic viscosity* and $\nabla^2 \equiv \nabla \cdot \nabla$ is the *Laplace operator*. The first of these equations is the *Navier–Stokes equation* for a (classical) viscous fluid; this equation clearly reduces to Euler's equation, (1.12), for an inviscid fluid: $\mu = 0$ (so $\nu = 0$).

Finally, we write these equations in rectangular Cartesian coordinates, $\mathbf{x} \equiv (x, y, z)$, with $\mathbf{u} \equiv (u, v, w)$ and $\mathbf{F} \equiv (0, 0, -g)$, to give

$$\left.\begin{gathered}
\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \nabla^2 u, \quad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \nabla^2 v, \\
\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho} \frac{\partial P}{\partial z} - g + \nu \nabla^2 w, \\
\text{where} \\
\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \\
\text{and} \\
\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \\
\text{with} \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.
\end{gathered}\right\} \tag{A.2}$$

These same equations, written in cylindrical coordinates, $\mathbf{x} \equiv (r, \theta, z)$ and $\mathbf{u} \equiv (u, v, w)$, are

$$\left.\begin{aligned}
&\frac{\mathrm{D}u}{\mathrm{D}t} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right),\\
&\frac{\mathrm{D}v}{\mathrm{D}t} + \frac{uv}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial P}{\partial \theta} + \nu\left(\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2}\right),\\
&\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g + \nu\nabla^2 w,\\
&\text{where}\\
&\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}\\
&\text{and}\\
&\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2},\\
&\text{with}\\
&\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0.
\end{aligned}\right\} \qquad \text{(A.3)}$$