$$2\cos kx \cos kt = \frac{e^{ikx}+e^{-ikx}}{2} \cdot \frac{e^{ikt}+e^{-ikt}}{2}$$

$$= \frac{e^{ik(t+x)}}{4} + e^{-ik(x+t)} + \dots$$

$T = f(t)$.

Multiply (42) (41) by $-k$:

$$\hat{\eta}_{tt}{}^{k}_{c} = k\hat{\eta}_x{}^{k}_{s} + \mu^2 \Big(\cancel{\eta_{t} \mathcal{F}_c^k \{\partial_t(\eta \int_0^x \eta_t dx')\}}$$

$$-k\mathcal{F}_c^k \{\partial_t(\eta \int_0^x \eta_t dx')\} - \frac{k}{2} \mathcal{F}_s^k \{\partial_x (\int_0^x \eta_t dx')^2\}$$

$$+ \frac{k^2}{3} \hat{\eta}_{tt}{}^{k}_{c} \Big) = 0$$

$$\Rightarrow \hat{\eta}_{tt}{}^{k}_{c} - k\hat{\eta}_x{}^{k}_{s} + \mu^2 \Big(-k\mathcal{F}_c^k \{\partial_t (\eta \int_0^x \eta_t dx')\}$$

$$- \frac{k}{2} \mathcal{F}_s^k \{\partial_x (\int_0^x \eta_t dx')^2\} + \frac{k^2}{3} \hat{\eta}_{tt}{}_c \Big].$$

$$\hat{\eta}_x{}^{k}_{s} = \int_0^\infty \sin kx \,\eta_x \,dx = \underbrace{-\frac{\cos kx}{k} \eta_x \Big|_0^\infty}_{} + \frac{1}{k} \int_0^\infty \cos kx \,\eta_{xx}\, dx$$

$$= \frac{1}{k} \cos kx \,\eta_x \Big|_\infty^0 = \frac{1}{k} \Big(\eta_x(0,t) - \lim_{x\to\infty} \cos kx \,\eta_x \Big)$$

0. $\quad\quad \eta_x \to 0$.

So, $-k\hat{\eta}_x{}^{k}_{s} = -\int_0^\infty \cos kx \,\eta_{xx}\, dx = -\hat{\eta}_{xx}{}^{k}_{c}$.

Or: $\hat{\eta}_x{}^{k}_{s} = \int_0^\infty \sin kx \,\eta_x dx = \underbrace{\sin kx \,\eta \Big|_0^\infty} - k \int_0^\infty \cos kx \,\eta \, dx$

assume $\eta \to 0$.

$= -k\hat{\eta}^k_c$

~~$vu' = vu' -$~~

$(uv)' = vu' + uv'$

$\Rightarrow \int vu' = uv| - \int uv'$

so, $\boxed{-k\hat{\eta}_x{}^{k}_{s} = k^2 \hat{\eta}_c^k}$

$$\hat{\eta}_{tt}{}^{k}_{c} + k^2 \hat{\eta}_c^k +$$

$$\mu^2 \Big(-k\mathcal{F}_c^k \{\partial_t (\eta \int_0^x \eta_t dx')\} - \frac{k}{2} \mathcal{F}_s^k \{\partial_x (\int_0^x \eta_t dx')^2\} + \frac{k^2}{3} \hat{\eta}_{tt}{}^{k}_{c} \Big).$$

$$-\frac{k}{2}\mathcal{F}_s^k\{g(x)\} = -\frac{k}{2}\int_0^\infty \sin kx\, g(x)\, dx =$$

$$= \frac{k}{2}\int_0^\infty \sin kx \; \partial_x\left(\left(\int_0^x \eta_t\, dx'\right)^2\right) dx =$$

$$= \underbrace{\frac{k}{2}\left\{\sin kx \left(\int_0^x \eta_t\, dx'\right)^2 \Big|_0^\infty\right.}_{0} + \frac{k^2}{2}\int_0^\infty \cos kx \left(\int_0^x \eta_t\, dx'\right)^2 dx.$$

$$= +\frac{k^2}{2}\mathcal{F}_c^k\left\{\left(\int_0^x \eta_t\, dx'\right)^2\right\}$$

$\cancel{-\frac{k}{}\mathcal{F}_c^k}$

Only Cosine - Transforms:

$$\hat{\eta}_{tt} + k^2\hat{\eta} + \varepsilon\left(-k\mathcal{F}\left\{\partial_t\left(\eta\int_0^x \eta_t\, dx'\right)\right\} + \frac{k^2}{2}\mathcal{F}_c\left\{\left(\int_0^x \eta_t\, dx'\right)^2\right\} + \frac{k^2}{3}\hat{\eta}_{tt}\right)$$

$$= 0.$$

Introduce: $T = f(t) \Rightarrow \frac{d}{dt} = f'(t)\frac{d}{dT}$, $\frac{d^2}{dt^2} = f''(t)\frac{d}{dT} + f'(t)^2\frac{d^2}{dT^2}$

pick $T = \varepsilon i t \partial_x$. In $\mathcal{O}(\varepsilon^0)$:

$$\hat{\eta}_{tt} + k^2\hat{\eta} = 0 \iff \hat{\eta}_{TT} + \hat{\eta} = 0.$$

$$\iff \hat{\eta} = f_1(k)\sin T + f_2(k)\cos T.$$

pick f_i to avoid secularity.

Second Order approximation:

$$\hat{\eta} = \hat{\eta}_0 + \varepsilon\hat{\eta}_1 \iff \eta = \eta_0 + \varepsilon\eta_1.$$

$$\Rightarrow \hat{\eta}_{0tt} + \varepsilon\hat{\eta}_{1tt} + k^2(\hat{\eta}_0 + \varepsilon\hat{\eta}_1) -$$

$$+ \varepsilon\left(-k\mathcal{F}\left\{\partial_t\left(\eta_0\int_0^x \eta_{0t}\, dx'\right)\right\} + \frac{k^2}{2}\mathcal{F}\left\{\left(\int_0^x \eta_{0t}\, dx'\right)^2\right\} + \frac{k^2}{3}\hat{\eta}_{0tt}\right)$$

$O(\varepsilon^0)$: $\hat{\eta}_{0tt} + k^2 \hat{\eta}_0 = 0$

$O(\varepsilon^1)$: $\hat{\eta}_{1tt} + k^2 \hat{\eta}_1 = + k \mathcal{F}\left\{\partial_t \left(\eta_0 \int_0^x \eta_{0t}\, dx' \right)\right\}$

$$+ \frac{k^2}{2} \mathcal{F}\left\{\left(\int_0^x \eta_{0t}\, dx'\right)^2\right\} - \frac{k^2}{3} \hat{\eta}_{0tt} \qquad (*)$$

let $T = it\partial_x \Rightarrow \hat{\eta}_{0TT} + \hat{\eta}_0 = 0.$

$\Rightarrow \hat{\eta}_0 = f_1(k) \sin T + f_2(k) \cos T.$

- Note $\frac{d}{dt} = f'(t) \frac{d}{dT} = \underbrace{k \frac{d}{dT}}_{} = i\partial_x \frac{d}{dT}$

$\eta_{0t} = i\partial_x \eta_{0T}$

$\hat{\eta}_{0t} = k \hat{\eta}_{0t}$

$\frac{d^2}{dt^2} = k^2 \frac{d^2}{dT^2}.$

$\Rightarrow \hat{\eta}_{1tt} + k^2 \hat{\eta}_1 = k^2 \hat{\eta}_{1TT} + k^2 \hat{\eta}_1$

$\Rightarrow (*)$ becomes: $\hat{\eta}_{1TT} + \hat{\eta}_1 = \mathcal{F}\left\{ \eta_0 \int_0^x i\partial_{x'} \eta_{0T}\, dx' \right\}$

$$- \frac{T}{2} \mathcal{F}\left\{\left(\int_0^x i\partial_x \eta_{0T}\, dx'\right)^2\right\} - \frac{k^2}{3} \hat{\eta}_{0TT}$$

$\Rightarrow \hat{\eta}_{1TT} + \hat{\eta}_1 = \mathcal{F}\left\{ i\eta_0 \eta_{0T} \right\} - \frac{T}{2} \mathcal{F}\left\{ - \eta_{0T}^2 \right\} - \frac{k^2}{3} \hat{\eta}_{0TT}$

$= \cancel{\mathcal{F}\left\{ i\eta_0\eta_{0T}\right\} - \frac{T}{2}}$

$= \mathcal{F}\left\{ i\eta_0\eta_{0T} + \frac{T}{2}\eta_{0T}^2 \right\} - \frac{k^2}{3} \hat{\eta}_{0TT}$

$k^2 \hat{\eta}_0 = \hat{\eta}_{0xx}$

$\hat{\eta}_{0TT} = -f_1 \sin T - f_2 \cos T = -\hat{\eta}_0.$

$= \mathcal{F}\left\{ i\eta_0\eta_{0T} + \frac{T}{2}\eta_{0T}^2 \right\} + \frac{k^2}{3} \hat{\eta}_0$

$= \mathcal{F}\left\{ \frac{i}{2}\partial_T(\eta_0^2) + \frac{T}{2}\eta_{0T}^2 + \frac{1}{3}\eta_{0xx} \right\}.$

← nonlinearity

So, we obtain $\hat{\eta}_{1\tau\tau} + \hat{\eta}_1 = F\left\{ \frac{i}{2}\partial_x(\eta_0^2) + \frac{\tau}{2}\eta_{0\tau}^2 + \frac{\tau}{3}\eta_{0xx} \right\}$

dissipation

general solution: $\hat{\eta}_1 = F_1(k)\sin\tau + F_2(k)\cos\tau$

$+$ particular solution.

→ nice form

$\eta_\tau^2 = \partial_x\left(\frac{\eta^3}{3}\right)\ldots$