Chapter 4

Discussion

In this section, we discuss and justify the relevance of the derivation. Taking the water wave problem on the whole line as our starting point, we obtain wave and KdV equations in the shallow water limit. This result is not novel; indeed, the AFM formulation, given in Ablowitz, Fokas, and Musslimani, 2006, also arrives at the same equations, though the unknown variable there is $q(x) = \phi(x, \eta(x))$, the velocity potential evaluated at the surface. Rather, what is novel by our result is that we have derived the expected equations from a different, non-local formulation, thereby further justifying the use of this formulation when doing asymptotics.

In addition, although our goal is to approximate the solution of the water wave problem, the derivation of the KdV equations deserves a special consideration. Here, we obtain the KdV as an equation needed to remove secular terms. In literature, another approach to arrive at KdV is given in Benjamin, Bona, and Mahony, 1972. In the paper, the authors begin by considering a first-order wave equation,

↳ advection

$$u_t + c_0 u_x = 0, \tag{4.1}$$

moved to a separate area.

which is a model for small-amplitude, long waves, propagating in $+x$ direction, with speed c_0. The model (4.1) has limited utility, since the non-linear and dispersive effects accumulate and cause the model to lose its validity over large times. One can correct for these effects by considering each separately. For non-linearity, this involves approximating the characteristic velocity by making it dependent on u :

$$\frac{1}{c_0}\frac{\mathrm{d}x}{\mathrm{d}t} = 1 + \varepsilon u, \qquad \varepsilon \ll 1,$$

so that (4.1) becomes

$$u_t + c_0(1+\varepsilon u)u_x = u_t + c_0 u_x + c_0 \varepsilon u u_x = 0. \tag{4.2}$$

The validity of (4.2) relies on the condition that the amplitude parameter ε is sufficiently small, and the implicit error is $\mathcal{O}(\varepsilon)$. As such, the model (4.2) can be regarded as an improvement over (4.1), accounting for non-linear effects, within $\mathcal{O}(\varepsilon)$.

Similarly, one can account for the dispersion by considering a linear transformation

$$Lu = u + \varepsilon\alpha^2 u_{xx}, \qquad \varepsilon \ll 1.$$

Substituting into (4.1) yields

$$u_t + c_0(Lu)_x = u_t + c_0 u_x + c_0 \varepsilon \alpha^2 u_{xxx} = 0, \tag{4.3}$$

which can be thought of as an improvement over (4.1), accounting for dispersive effects, to $\mathcal{O}(\varepsilon)$.

We obtain (4.2) and (4.3) as the respective first-order approximations by allowing for weak nonlinearity and dispersive effects. The authors then argue that an approximation accounting for both effects can be anticipated by simply combining the ε terms:

$$u_t + u_x + \varepsilon(uu_x + \alpha^2 u_{xxx}) = 0, \tag{4.4}$$

where we set $c_0 = 1$. In nondimensional variables, we transform (4.4) to obtain

$$u_t + u_x + uu_x + u_{xxx} = 0.$$

Galilean transformations yield the usual form of the KdV

$$u_t + uu_x + u_{xxx} = 0.$$

The derivation is indeed elegant, and certainly much shorter, than the one presented here. Note that the use of (4.1) as the starting point is not problematic: indeed, upon a closer look, one obtains the equation directly from the dynamic boundary condition of the water wave problem, by imposing the shallow water limit. The issue is addition of the ε terms in (4.2) and (4.3): by doing so, the authors already presuppose a certain balance between nonlinearity and dispersion. However, there is no reason to assume this choice of balance; indeed, for a self-consistent theory we must account for the nonlinear and dispersive effects simultaneously.

Chapter 5

Water waves on the half-line

In the previous section, we use the $\mathcal{H}$-formulation to obtain expected, well-known results. In this section, we use the formulation to study a slightly different problem: the water waves problem, but on the half-line. Physically, we put up a tall, impenetrable barrier at $x = 0$. This requires imposing several conditions on both η, ϕ at $x = 0$. As such, the problem we consider is the following: Equations (1.1) along with the new ~~equations~~ boundary conditions listed below

Emphasize that this hasn't been done for either formulation before.

$$\phi_{xx} + \phi_{zz} = 0, \qquad -h < z < \eta(x,t), \tag{5.1a}$$

$$\phi_z = 0, \qquad z = -h, \tag{5.1b}$$

$$\phi_x = 0, \qquad x = 0, \tag{5.1c}$$

$$\eta_t + \phi_x \eta_x = \phi_z, \qquad z = \eta(x,t), \tag{5.1d}$$

$$\phi_t + g\eta + \frac{1}{2}(\phi_x^2 + \phi_z^2) = 0, \qquad z = \eta(x,t), \tag{5.1e}$$

$$\phi_z(0,\eta,t) = \eta_t(0,t), \qquad (x,z) = (0,\eta), \tag{5.1f}$$

only new

where $x \in [0,\infty)$ ~~and (5.1c), (5.1f) are the new boundary conditions.~~ In particular, (5.1c) implies that the fluid does not leak through the barrier at $x = 0$, and (5.1f) governs an interaction between the fluid and the surface at $x = 0$. Approximate equations are conjectured to be the wave and KdV-like equations. While the wave equation can be justified, there is no reason to expect that we can derive KdV equations. Indeed, a literature review reveals that ~~the~~ a KdV-like equation has not been derived on the half-line, in the way that we derive the equation on the whole line.

Using the $\mathcal{H}$-formulation, we derive the approximate equations on the half-line, note the main differences, and discuss the difficulties that arise. To begin, we observe that the scalar equation (3.12) for η and $\mathcal{H}$ remains the same, while the non-local equation (3.13) changes to:

$$\int_0^\infty \cos(kx)\cosh(k(\eta+h))f(x)+\sin(kx)\sinh(k(\eta+h))\mathcal{H}(\eta,D)\{f(x)\}\,\mathrm{d}x=0, \tag{5.2}$$

and the nondimensional version (3.14) becomes

$$\int_0^\infty \cos(kx)\cosh(\mu k(\eta+1))f(x)+\sin(kx)\sinh(\mu k(\eta+1))\mathcal{H}(\varepsilon\eta,D)\{f(x)\}\,\mathrm{d}x=0. \tag{5.3}$$

It is worth noting that taking the real part of the whole-line equations and restricting integrals to $[0,\infty)$ yields the half-line, non-local equations (??), (??).

By the same procedure, the expansion for $\mathcal{H}$ operator is given by

$$\begin{aligned}
\mathcal{H}_0(\varepsilon\eta,D)\{f(x)\} &= -\{\mathcal{F}_s^k\}^{-1}\{\coth(\mu k)\widehat{f_c^k}\},\\
\mathcal{H}_1(\varepsilon\eta,D)\{f(x)\} &= -\{\mathcal{F}_s^k\}^{-1}\{\mu k\widehat{(\eta f(x))}_c^k+\mu k\coth(\mu k)\widehat{(\eta\mathcal{H}_0\{f(x)\})}_c^k\}.
\end{aligned}$$

- introduce the notation. f_s etc. f_c

The notable difference from the whole-line is the presence of Fourier cosine and sine transforms, in place of Fourier transform.

We apply the expansion to the scalar equation equation. In the leading order, this yields

\left\lbrace \right

$$-\mathcal{F}_s^k\{\int_0^x \eta_{tt}\,\mathrm{d}x'\} + \widehat{\eta_x}_s^k = 0.$$

Inverting Fourier sine transform and differentiating with respect to x yields the wave equation on the half-line.

The next order approximation yields the equivalent of (3.24):

$$\eta_{tt} - \eta_{xx} = \mu^2\left(\frac{1}{3}\eta_{xxxx} + \partial_x(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\}\} + \frac{1}{2}\partial_x^2\left(\int_0^x \eta_t\,\mathrm{d}x'\right)^2\right). + \mathcal{O}(\mu^4) \tag{5.4}$$

The notable difference between the equations on two domains is the presence of the inverse sine transform of the cosine transform. Anticipating ~~secularity~~, we introduce the same time scales

Now you seek solutions and are repeating the derivation in Section 3.5.

secularities as before

$$\tau_0 = t, \qquad \tau_1 = \varepsilon t.$$

Along with an expansion $\eta = \eta_0 + \varepsilon\eta_1$, within $\mathcal{O}(\varepsilon^0)$, we obtain

$$\eta_{0\tau_0\tau_0} - \eta_{0xx} = 0,$$

which is the wave equation on the half line. The general solution on the half-line is

$$\eta_0(x,\tau_0,\tau_1,\ldots) = \begin{cases} F_2(x-\tau_0,\tau_1,\ldots) + G_2(x+\tau_0,\tau_1,\ldots) & x \geqslant \tau_0 \\ F_1(\tau_0 - x,\tau_1,\ldots) + G_1(x+\tau_0,\tau_1,\ldots) & x < \tau_0 \end{cases},$$

where we emphasise that the difference between F_i and G_i. Even though

incomplete sentence

Indicate that you are omitting the parallel calculations of a similar (3.31) equation as it is really long. By just jumping to the system on the next page, you are hiding your hard work.

F_1 and F_2 are both right-going waves, they have different domains, and hence are different functions. Within $\mathcal{O}(\varepsilon)$, a careful calculation yields the following system of 4 equations in four unknowns F_1, F_2, G_1, G_2 :

Maybe use a ~~header~~ text line

for $\xi \geq 0$ . . . for $\xi < 0$ . . .

$$
\begin{aligned}
&2\partial_{\tau_1}F_1 + \frac{1}{3}\partial_\xi^3 F_1 + (F_1 - A)\partial_\xi F_1 + \frac{1}{\pi}\Big(\int_{-\tau_0}^{0}(2F_1 - A)\partial_{\xi'}F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' \\
&\qquad + \int_0^\infty (2F_2 - (A+B))\partial_{\xi'}F_2\frac{1}{\xi-\xi'}\,\mathrm{d}\xi'\Big) = 0, \quad \xi. \\
&2\partial_{\tau_1}F_2 + \frac{1}{3}\partial_\xi^3 F_2 + (F_2 - A - B)\partial_\xi F_2 + \frac{1}{\pi}\Big(\int_{-\tau_0}^{0}(2F_1 - A)\partial_{\xi'}F_1\frac{1}{\xi-\xi'}\,\mathrm{d}\xi' \\
&\qquad + \int_0^\infty (2F_2 - (A+B))\partial_{\xi'}F_2\frac{1}{\xi-\xi'}\,\mathrm{d}\xi'\Big) = 0, \quad \xi. \\
&-2\partial_{\tau_1}G_1 + \frac{1}{3}\partial_\zeta^3 G_1 + (G_1 + A)\partial_\zeta G_1 + \frac{1}{\pi}\Big(\int_{\tau_0}^{2\tau_0}(2G_1 + A)\partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta' \\
&\qquad + \int_{2\tau_0}^\infty (2G_2 + (A+B))\partial_{\zeta'}G_2\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\Big) = 0, \quad \xi. \\
&-2\partial_{\tau_1}G_2 + \frac{1}{3}\partial_\zeta^3 G_2 + (G_2 + A + B)\partial_\zeta G_2 + \frac{1}{\pi}\Big(\int_{\tau_0}^{2\tau_0}(2G_1 + A)\partial_{\zeta'}G_1\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta' \\
&\qquad + \int_{2\tau_0}^\infty (2G_2 + A + B)\partial_{\zeta'}G_2\frac{1}{\zeta-\zeta'}\,\mathrm{d}\zeta'\Big) = 0, \quad \xi;
\end{aligned}
\tag{5.5}
$$

include an appendix for this.

State that these "Hilbert-like" transforms require care and are derived in an appendix. By not stating this, you are discounting your hard work. Now is the time to brag ☺

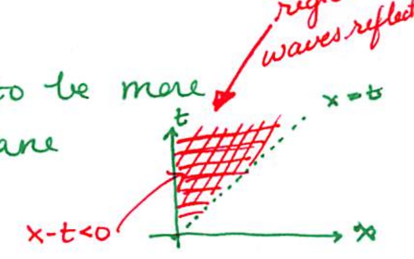
where

$$
A = F_1(\tau_0) - G_1(\tau_0), \qquad B = F_2(0) - F_1(0) + G_1(2\tau_0) - G_2(2\tau_0).
$$

indicate where these relationships come from.

Each equation in the system (5.5) is somewhat similar to KdV: the time derivative and dispersion term are preserved, whereas nonlinear terms are drastically different. Further, a careful look at the system reveals that unlike on the whole line case, the approximate equations for F_i and G_i are still dependent on the time scale τ_0. This is an issue, as the reason behind the time scales is to separate the dependence on different time scales. As of now, it is not clear why this issue appears. One possible reason is that

explain physically why we expect this to be more complicated by considering the x-t plane



the linear time scales

$$\tau_0 = t, \qquad \tau_1 = \varepsilon t$$

should be replaced with different time scales. Another reason could be that that $\mathcal{H}$ formulation does not provide sufficient information to do asymptotics. ← can you expand on this

In summary, although we do not obtain the approximate equations on the half-line, we see the utility of the $\mathcal{H}$ formulation in aiding to understand the physical and mathematical difficulties associated with the half-line problem. This should not be taken for granted: for example, if one conducts asymptotic expansions via the velocity potential formulation, one obtains 4 KdV equations for $F_i, G_i, i = 1, 2$, which clearly does not agree with the results of this section. As such, the $\mathcal{H}$ formulation shows that the half-line problem has several subtleties, which may not be readily seen in other formulations.

you might also indicate that if you make ~~you~~ certain simplifications, you can recover KdV like equations that ~~exist on~~ are valid when $x - t \gg 0$. ← away from the interaction region.

You do need an overall conclusion

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