3

Asymptotic analysis of wave equations: Properties and analysis of Fourier-type integrals

In the previous chapter we constructed the solution of equations using Fourier transforms. This yields integrals that depend on parameters describing space and time variation. When used with asymptotic analysis the Fourier method becomes extremely powerful. Long-time asymptotic evaluation of integrals yields the notion of group velocity and the structure of the solution. In this chapter, we apply these techniques to the linear Schrödinger equation, the linear KdV equation, discrete equations, and to the Burgers equation, which is reduced to the heat equation via the Hopf–Cole transformation.

Consider the asymptotic behavior of Fourier integrals resulting from the use of Fourier transforms (see Chapter 2). These integrals take the form

$$u(x,t) = \frac{1}{2\pi}\int \widehat{u}(k)\exp\left[i(kx-\omega(k)t)\right]\,dk \tag{3.1}$$

in the limit as $t \to \infty$ with $\chi \equiv x/t$ held fixed, i.e., we will study approximations to (3.1) for large times and for large distances from the origin. We can rewrite this integral in a slightly more general form by defining $\phi(k) \equiv k\chi - \omega$ (for simplicity of notation here we suppress the dependence on χ in ϕ) and considering integrals of the form

$$I(t) = \int_a^b \widehat{u}(k)\exp\left[i\phi(k)t\right]\,dk, \tag{3.2}$$

where we assume $\phi(k)$ is smooth on $-\infty \le a < b \le \infty$. First, by the Riemann–Lebesgue lemma (cf. Ablowitz and Fokas, 2003), provided $\widehat{u}(k) \in L^1(a,b)$, we have

$$\lim_{t\to\infty} I(t) = 0.$$

But this does not provide any additional qualitative information. Insight can be obtained if we assume that both $\hat{u}(k)$ and $\phi(k)$ are sufficiently smooth. If $\phi'(k) \neq 0$ in $[a,b]$, then we can rewrite (3.2) as

$$\int_a^b \frac{\widehat{u}(k)}{i\phi'(k)t} \frac{d}{dk} e^{i\phi(k)t} dk;$$

then integrating by parts gives

$$I(t) = e^{i\phi(k)t} \frac{\widehat{u}(k)}{i\phi'(k)t}\Bigg|_a^b - \int_a^b \exp\left[i\phi(k)t\right] \frac{d}{dk}\left(\frac{\widehat{u}(k)}{i\phi'(k)t}\right) dk.$$

Repeating the process we can show that the second term is of order $1/t^2$.[1] Thus, we have

$$I(t) = i\left[\frac{\widehat{u}(a)e^{i\phi(a)t}}{\phi'(a)} - \frac{\widehat{u}(b)e^{i\phi(b)t}}{\phi'(b)}\right]\frac{1}{t} + O(1/t^2).$$

If $\widehat{u}(k)$ and $\phi'(k)$ are C^∞ (i.e., infinitely differentiable) functions, then we may continue to integrate by parts to get the asymptotic expansion

$$I(t) \sim \sum_{n=1}^{\infty} \frac{c_n(t)}{t^n},$$

where the c_n are bounded functions. Notice that if $\widehat{u} \in L^1(-\infty, \infty)$, then

$$\lim_{a\to-\infty} \widehat{u}(a) = \lim_{b\to\infty} \widehat{u}(b) = 0.$$

Therefore, if $\phi'(k) \neq 0$ for all k and if both $\widehat{u}(k), \phi(k) \in C^\infty$, then we have

$$u(x,t) = o(t^{-n}), \qquad n = 1, 2, 3, \ldots, \tag{3.3}$$

i.e., u goes to zero faster than any power of t.

It is more common, however, that there exists a point or points where $\phi'(c) = 0$. The technique used to analyze this case is called the *method of stationary phase*, which was originally developed by Kelvin (cf. Jeffreys and Jeffreys, 1956; Ablowitz and Fokas, 2003). We will explore this technique now.

3.1 Method of stationary phase

When considering the Fourier integral (3.2), we wish to determine the long-time asymptotic behavior of $I(t)$. As our previous analysis shows, intuitively, for large values of t, there is very rapid oscillation, leading to cancellation. But, the oscillation is less rapid near points where $\phi'(k) = 0$ and the cancellation will not be as complete. We will call a point, k, where $\phi'(k) = 0$, a *stationary point*.

[1] We say that $u = O(t^{-n})$ if $\lim_{t\to\infty} |u|t^n$ = constant and $u = o(t^{-n})$ if $\lim_{t\to\infty} |u|t^n = 0$. See also Section 3.2 for the definitions of big "O", little "o" and asymptotic expansions.

Suppose there is a so-called stationary point c, $a < c < b$, such that $\phi'(c) = 0$, $\phi''(c) \neq 0$ and $\widehat{u}(c)$ exists and is non-zero. In addition, suppose $\widehat{u}(k)$ and $\phi(k)$ are sufficiently smooth. Then we expect that the dominant contribution to (3.2) should come from a neighborhood of c, i.e., the leading asymptotic term is

$$\begin{aligned} I(t) &\sim \int_{c-\epsilon}^{c+\epsilon} \widehat{u}(c) \exp\left[i\left(\phi(c) + \frac{\phi''(c)}{2}(k-c)^2\right) t\right] dk, \\ &= \widehat{u}(c) \exp(i\phi(c)t) \int_{c-\epsilon}^{c+\epsilon} \exp\left[i\frac{\phi''(c)}{2}(k-c)^2 t\right] dk, \end{aligned} \tag{3.4}$$

where ϵ is small. Let

$$\xi^2 = \frac{|\phi''(c)|}{2}(k-c)^2 t, \qquad \mu \equiv \mathrm{sgn}(\phi''(c)).$$

Then (3.4) becomes

$$I(t) \sim \sqrt{\frac{2}{t|\phi''(c)|}} \widehat{u}(c) \exp(i\phi(c)t) \int_{-\epsilon\sqrt{\frac{t|\phi''(c)|}{2}}}^{\epsilon\sqrt{\frac{t|\phi''(c)|}{2}}} \exp(i\mu\xi^2)\, d\xi.$$

For $|\phi''(c)| = O(1)$, extending the limits of integration from $-\infty$ to ∞, i.e., we take $\epsilon\sqrt{t} \to \infty$, yields the dominant term. The integral

$$\int_{-\infty}^{\infty} \exp(i\mu\xi^2)\, d\xi = \sqrt{\pi} \exp\left(i\mu\frac{\pi}{4}\right),$$

is well known, hence

$$I(t) \sim \widehat{u}(c) \sqrt{\frac{2\pi}{t|\phi''(c)|}} \exp\left(i\phi(c)t + i\mu\frac{\pi}{4}\right). \tag{3.5}$$

If we are considering Fourier transforms, we have $\phi(k) = k\chi - \omega(k)$, for which $\phi'(k) = 0$ implies

$$\chi = \frac{x}{t} = \omega'(k_0) \tag{3.6}$$

for a stationary point k_0. Or, assuming ω' has an inverse, $k_0 = (\omega')^{-1}(x/t) \equiv k_0(x/t)$. In addition, if we are working with a dispersive PDE, then by definition, except at special points, $\phi''(k) \neq 0$. Substituting this into (3.5),

$$u(x,t) \sim \frac{\widehat{u}(k_0)}{\sqrt{2\pi t|\phi''(k_0)|}} \exp\left(i\phi(k_0)t + i\mu\frac{\pi}{4}\right), \tag{3.7}$$

which can be rewritten as

$$u(x,t) \sim \frac{A(x/t)}{\sqrt{t}} \exp\left(i\phi(k_0)t\right), \tag{3.8}$$

where $A(x/t)/\sqrt{t}$ is the decaying in time, slowly varying, complex amplitude. This decay rate, $O(1/\sqrt{t})$, is slow, especially when compared to those cases when the solution decays exponentially or as $O(1/t^n)$ for n large. Note that dimensionally, ω' corresponds to a speed. Stationary phase has shown that the leading-order contribution comes from a region moving with speed ω', which is termed the group velocity. This is the velocity of a slowly varying packet of waves. We will see later that this is also the speed at which energy propagates.

Example 3.1 Linear KdV equation: $u_t + u_{xxx} = 0$. The dispersion relation is $\omega = -k^3$ and so $\phi(k) = kx/t + k^3$. Thus, the stationary points, satisfying (3.6), occur at $k_0 = \pm\sqrt{-x/(3t)}$ so long as $x/t < 0$. Note that we might expect something interesting to happen as $x/t \to 0$, since $\phi'(k)$ vanishes there and implies a higher-order stationary point. We also expect different behavior when $x/t > 0$ since k_0 becomes imaginary there.

Example 3.2 Free-particle Schrödinger equation: $i\psi_t + \psi_{xx} = 0$, $|x| < \infty$, $\psi(x,0) = f(x)$, and $\psi \to 0$ as $|x| \to \infty$. This is a fundamental equation in quantum mechanics. The wavefunction, ψ, has the interpretation that $|\psi(x,t)|^2 dx$ is the probability of finding a particle in dx, a small region about x at time t. The dispersion relation $\omega = k^2$ shows that the stationary point satisfying (3.6) occurs at $k_0 = x/(2t)$.

3.2 Linear free Schrödinger equation

Now we will consider a specific example of the method of stationary phase as applied to the linear free Schrödinger equation

$$iu_t + u_{xx} = 0. \tag{3.9}$$

First assume a wave solution of the form $u_s = e^{i(kx-\omega(k)t)}$, substitute it into the equation and derive the dispersion relation

$$\omega(k) = k^2.$$

Notice that ω is real and $\omega''(k) = 2 \neq 0$ and hence solutions to (3.9) are in the dispersive wave regime. Thus the solution, via Fourier transforms, is

$$u(x,t) = \frac{1}{2\pi}\int \hat{u}_0(k)e^{i(kx-k^2t)}\,dk,$$

where $\hat{u}_0(k)$ is the Fourier transform of the initial condition $u(x,0) = u_0(x)$. As before, to apply the method of stationary phase, we rewrite the exponential in the Fourier integral the following way

$$u(x,t) = \frac{1}{2\pi}\int \hat{u}_0(k)e^{i\phi(k)t}\,dk,$$

where $\phi(k) = k\chi - k^2$ with $\chi = x/t$. The dominant contribution to the integral occurs at a stationary point k_0 when $\phi'(k_0) = 0$; $k_0 = x/(2t)$ in this case. Noting that $\mathrm{sgn}(\phi''(k_0)) = -1$, the asymptotic estimate for $u(x,t)$ using (3.7) is

$$u(x,t) = \frac{\hat{u}_0\left(\frac{x}{2t}\right)}{\sqrt{4\pi t}}\exp\left(i\left(\frac{x}{2t}\right)^2 t - i\frac{\pi}{4}\right) + o\left(\frac{1}{\sqrt{t}}\right). \tag{3.10}$$

A more detailed analysis, using the higher-order terms arising from the Taylor series of $\hat{u}(k)$ and $\phi(k)$ near $k = k_0$, shows that if $\hat{u}_0(k) \in C^\infty$ then we have the more general result

$$u(x,t) \sim \frac{1}{\sqrt{4\pi t}}\exp\left(i\left(\frac{x}{2t}\right)^2 t - i\frac{\pi}{4}\right)\left[\hat{u}_0(\tfrac{x}{2t}) + \sum_{n=1}^{\infty}\frac{g_n(\frac{x}{2t})}{(4it)^n}\right],$$

where $g_n(y)$ are related to the higher derivatives of $\hat{u}_0$ (see Ablowitz and Segur, 1981).

Definition 3.3 The function $I(t)$ has an *asymptotic expansion* around $t = t_0$ (Ablowitz and Fokas, 2003) in terms of an ordered *asymptotic sequence* $\{\delta_j(t)\}$ if
(a) $\delta_{j+1}(t) \ll \delta_j(t)$, as $t \to t_0$, $j = 1, 2, \ldots, n$, and
(b) $I(t) = \sum_{j=1}^n a_j\delta_j(t) + R_n(t)$ where the remainder $R_n(t) = o(\delta_n(t))$ or

$$\lim_{t\to t_0}\left|\frac{R_n(t)}{\delta_n(t)}\right| = 0.$$

The notation $\delta_{j+1}(t) \ll \delta_j(t)$, means

$$\lim_{t\to t_0}\left|\frac{\delta_{j+1}(t)}{\delta_j(t)}\right| = 0.$$

We also write

$$I(t) \sim \sum_{j=1}^{n} a_j\delta_j(t),$$

which says $I(t)$ has the asymptotic expansion given on the right-hand side of the symbol "$\sim$". Notice that if $n = \infty$, then we mean $R_n(t) = o(\delta_n(t))$ for all n. Asymptotic series expansions may not (usually do not) converge.

The asymptotic solution (3.10) can be viewed as a "slowly varying" similarity or self-similar solution. In general, a similarity solution takes the form

$$u(x,t) = \frac{1}{t^p} f\left(\frac{x}{t^q}\right), \tag{3.11}$$

for suitable constants p and q. This important concept describes phenomena in terms of a reduced number of variables, when time and space are suitably rescaled. Similarity solutions find important applications in many areas of science. One well-known example is that of G.I. Taylor who in the 1940s found a self-similar solution to the problem of determining the shock-wave radius and speed after an (nuclear) explosion (Taylor, 1950). He found that the front of an approximately spherical blast wave is described self-similarly with the independent variables scaled according to a two-fifths law, $r/t^{2/5}$, where r is the radial distance from the origin.

We determine the similarity solution to the linear Schrödinger equation by assuming the form (3.11) and inserting it into (3.9); i.e., we assume the ansatz $u_{\text{sim}}(x,t) = \frac{1}{t^p} f(\eta)$ where $\eta = \frac{x}{t^q}$. Then we find the relation

$$\frac{f''}{t^{p+2q}} - i\left(\frac{pf}{t^{p+1}} + \frac{q\eta f'}{t^{p+1}}\right) = 0.$$

In order to keep all the powers of t the same, we require $2q = 1$ but p can be arbitrary. In general, linear equations do not determine both p and q but nonlinear equations usually fix both. For linear equations, p is fixed by additional information; e.g., initial values or side conditions. For the linear free Schrödinger equation we take $p = 1/2$, motivated by our previous analysis with the stationary phase technique. Then our equation takes the form

$$f'' - \frac{i}{2}(\eta f' + f) = 0,$$

which we can integrate directly to obtain

$$f' - \frac{i}{2}\eta f = c_1, c_1 \text{ const.}$$

Introducing the integrating factor $e^{-i\eta^2/4}$, we integrate this equation to find

$$f(\eta) = c_1 \int e^{i(\eta^2 - (\eta')^2)/4} d\eta' + c_2 e^{i\eta^2/4}, c_2 \text{ const.}$$

If $c_1 = 0$, we have the similarity solution

$$u_{\text{sim}}(x,t) = \frac{c_2}{\sqrt{t}} \exp\left(\frac{i}{4}\left(\frac{x}{t}\right)^2 t\right) = \frac{\tilde{c}_2}{\sqrt{4\pi t}} \exp\left(i\tilde{\eta}^2 - i\pi/4\right),$$

where $\tilde{\eta} = \frac{x}{2\sqrt{t}}$, $c_2 = \frac{\tilde{c}_2}{\sqrt{4\pi}} e^{-i\pi/4}$.

By identifying the "constant of integration" $\tilde{c}_2$, with the slowly varying function $\hat{u}_0(x/t)$, we can write the asymptotic solution, obtained earlier via the method of stationary phase, (3.10), as

$$u(x,t) \sim \hat{u}_0(x/(2t))u_{\text{sim}}(x,t) = \frac{\hat{u}_0(x/(2t))}{\sqrt{4\pi t}} e^{i(x/(2t))^2 t} e^{-i\pi/4}. \tag{3.12}$$

In the argument of the first term $\hat{u}_0$, the spatial coordinate, x, is scaled by $1/2t$ but in the second term ($e^{ix^2/(4t)} = e^{i(x/(2\sqrt{t}))^2}$), x is scaled by $1/\sqrt{t}$. For large values of t, the first term varies in x much more slowly than the second term. We refer to this as a slowly varying similarity solution. In a sense $u_{\text{sim}}(x,t)$ is an "attractor" for this linear problem. Slowly varying similarity solutions arise frequently – even for certain nonlinear problems such as the nonlinear Schrödinger equation

$$iu_t + u_{xx} \pm |u|^2 u = 0,$$

e.g., see Ablowitz and Segur (1981).

3.3 Group velocity

Recall that the "group velocity", defined to be $\omega'(k)$, enters into the method of stationary phase through the solution of the equation $\phi'(k) = 0$; i.e., through the stationary points. We have seen that, for linear problems, the dominant terms in the solution come from the region near the stationary points. The energy can be shown to be transported by the group velocity. Next we describe this using a different asymptotic procedure.

Frequently we look directly for slowly varying wave groups in order to approximate the solution; see Whitham (1974). Suppose we consider, based on our stationary phase result, that the solution to a linear dispersive wave problem [e.g., the linear Schrödinger equation (3.9)] is of the form

$$\begin{aligned} u(x,t) &\sim A(x,t)e^{i\theta(x,t)}, \qquad t \gg 1, \qquad (3.13)\\ A(x,t) &= \frac{1}{\sqrt{t}} g(x/t),\\ \theta(x,t) &= (k_0(x/t)x/t - \omega(k_0(x/t)))t = \phi(x/t)t, \end{aligned}$$

where g and ϕ are slowly varying. That is,

$$\frac{\partial g}{\partial x} = \frac{1}{t} g'\left(\tfrac{x}{t}\right) \ll 1 \qquad \text{and} \qquad \frac{\partial g}{\partial t} = -\frac{x}{t^2} g'\left(\tfrac{x}{t}\right) \ll 1;$$

also $\dfrac{\partial\phi}{\partial x} \ll 1$ and $\dfrac{\partial\phi}{\partial t} \ll 1$. Notice that θ is rapidly varying. To generalize the notion of wavenumber and frequency as defined in Section 2.2, we take

$$\frac{\partial\theta}{\partial x} = \phi'(x/t) = k_0 \tag{3.14}$$
$$\frac{\partial\theta}{\partial t} = \phi(x/t) - (x/t)\phi'(x/t) = k_0x/t - \omega(k_0) - k_0x/t = -\omega(k_0), \tag{3.15}$$

where we used (3.13). So we can define the slowly varying wavenumber and frequency by the above relations. The ansatz (3.13) with a rapidly varying phase (or the real part in cases where the solution u is real) for wave-like problems is sometimes called the WKB approximation (after Wentzel–Kramer–Brillouin) for wave problems. Many linear problems we see naturally have solutions of the WKB type. In fact there exists a wide range of nonlinear problems that admit generalizations of such rapidly varying phase solutions. In Chapter 4 we will study ODEs where such rapidly varying phase methods (i.e., WKB methods) are useful.

From relations (3.14) and (3.15), we find that (dropping the subscript on k)

$$k_t = \frac{\partial^2\theta}{\partial t\partial x} = \frac{\partial^2\theta}{\partial x\partial t} = -\omega_x \tag{3.16a}$$
$$k_t + \omega_x = 0 \tag{3.16b}$$
$$k_t + \omega'(k)k_x = 0. \tag{3.16c}$$

Equation (3.16b), which follows from (3.16a), is the so-called conservation of wave equation. Assuming $\omega = \omega(k)$, equation (3.16c) is a first-order hyperbolic PDE that, as we have seen, can be solved via the method of characteristics. It follows (see Chapter 2) that $k(x,t)$ is constant along group lines (curves) in the (x,t)-plane defined by $dx/dt = \omega'(k(x,t))$ for any slowly varying wave packet.

Now we will show that the energy in the asymptotic solution to (3.13) propagates with the group velocity. Recall that the energy in the solution of a PDE is proportional to the L^2-norm in space. We will limit ourselves to the interval $[x_1(t), x_2(t)]$ as the region of integration to find the energy in a wave packet. For this approximation, we have

$$Q(t) = \int_{x_1(t)}^{x_2(t)} |A|^2\,dx.$$

Differentiating this quantity with respect to t we find

$$\begin{aligned}\frac{dQ}{dt} &= \int_{x_1(t)}^{x_2(t)} \frac{\partial}{\partial t}|A|^2\, dx + |A|^2(x_2)\frac{dx_2}{dt} - |A|^2(x_1)\frac{dx_1}{dt} \\ &= \int_{x_1(t)}^{x_2(t)} \left\{\frac{\partial}{\partial t}|A|^2 + \frac{\partial}{\partial x}(|A|^2\omega'(k))\right\} dx, \end{aligned} \tag{3.17}$$

where we used the characteristic curves $\dfrac{dx_2}{dt} = \omega'(k_2)$ and $\dfrac{dx_1}{dt} = \omega'(k_1)$ with k remaining constant along them.

Notice in our notation that $A = g(x/t)/\sqrt{t}$ is associated with the long-time asymptotic result (3.7)–(3.8). So let us substitute this into the expression for $Q(t)$ to get

$$Q(t) = \int_{x_1(t)}^{x_2(t)} \frac{1}{t}\left|g\left(\tfrac{x}{t}\right)\right|^2 dx.$$

Now make the change of variable $u = x/t$ and, along with the relations from the characteristic curves, $\dfrac{dx_2}{dt} = \omega'(k_2)$ and $\dfrac{dx_1}{dt} = \omega'(k_1)$, we have

$$Q(t) = \int_{\omega'(k_1)+c_1/t}^{\omega'(k_2)+c_2/t} |g(u)|^2 du,$$

where c_1, c_2 are constants, and that for $t \gg 1$ is asymptotically constant in time. Thus to leading order the quantity $Q(t)$ is conserved, i.e., $dQ/dt = 0$. Thus returning to (3.17), since x_1, x_2 are arbitrary, we find the conservation equation

$$\frac{\partial}{\partial t}|A|^2 + \frac{\partial}{\partial x}(\omega'(k)|A|^2) = 0. \tag{3.18}$$

The energy density, $|A|^2$, shows that the group velocity $\omega'(k)$, associated with the energy flux term $\omega'(k)|A|^2$, is the velocity with which the energy is transported. We also note that this equation is of the same form as the conservation of mass equation from fluid dynamics

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(u\rho) = 0,$$

which is discussed later in this book. Here, ρ is the fluid density and u is the velocity of the fluid.

Another way to derive (3.18) is directly from the PDE itself. For the linear free Schrödinger equation, using

$$\begin{aligned} u^*(iu_t + u_{xx}) &= 0, \\ u\,(-iu_t^* + u_{xx}^*) &= 0, \end{aligned}$$

where u^* is the complex conjugate, and then subtracting these two equations, we find

$$\begin{aligned} i\,(u_t u^* + u u_t^*) + u^* u_{xx} - u u_{xx}^* &= 0, \\ i\frac{\partial}{\partial t}|u|^2 + \frac{\partial}{\partial x}\,(u^* u_x - u u_x^*) &= 0, \\ \frac{\partial}{\partial t}T + \frac{\partial}{\partial x}F &= 0, \end{aligned}$$

which we have seen is a conservation law with density T and flux F. Assuming $u = Ae^{i\theta}$, we have $T = |A|^2$, as the conserved energy, and the term F is

$$\begin{aligned} F &= -i\,(u^* u_x - u u_x^*)\,, \\ &= -i\left(A^* e^{-i\theta}(iA\theta_x + A_x)e^{i\theta} - Ae^{i\theta}\,(-iA^*\theta_x + A_x^*)\,e^{-i\theta}\right). \end{aligned}$$

Recall that A is slowly varying so $|A_x| = \left|A_x^*\right| \ll 1$; dropping these terms and substituting the generalized wavenumber relation (3.14), $\theta_x \sim k$ and $\omega' \sim 2k$, the flux turns out to be, at leading order for $t \gg 1$,

$$F = 2k|A|^2 = \omega'(k)|A|^2.$$

Thus we have again found (3.18) for large time.

3.4 Linear KdV equation

The linear Korteweg–de Vries (KdV) equation (which models the unidirectional propagation of small-amplitude long water waves or shallow-water waves) is given by

$$u_t + u_{xxx} = 0. \tag{3.19}$$

In this section we will use multiple levels of asymptotics, the method of stationary phase as discussed before, the method of steepest descent, and self-similarity, to find the long-time behavior of the solution. There are three asymptotic regions to consider:

- $x/t < 0$;
- $x/t > 0$;
- $x/t \to 0$.

The asymptotic solutions found in each of these regions will be "connected" via the Airy function, which is the relevant self-similar solution for this problem.

The Fourier solution to (3.19) is given by

$$u(x,t) = \frac{1}{2\pi}\int \hat{u}_0(k)e^{i(kx-\omega(k)t)}\,dk.$$

Substituting this into (3.19), the dispersion relation is found to be $\omega(k) = -k^3$, which implies

$$u(x,t) = \frac{1}{2\pi}\int \widehat{u_0}(k)e^{i(kx+k^3t)}\,dk.$$

By the convolution theorem, see Ablowitz and Fokas (2003) this can also be written in the form

$$u(x,t) = \frac{1}{(3t)^{1/3}}\int \mathrm{Ai}\left(\frac{x-x'}{(3t)^{1/3}}\right)u_0(x')\,dx',$$

where Ai denotes the the Airy function; see (3.22)–(3.23).

3.4.1 Stationary phase, $x/t < 0$

The method of stationary phase uses the Fourier integral in the form

$$u(x,t) = \frac{1}{2\pi}\int \hat{u}_0(k)e^{i\phi(k)t}\,dk,$$

where $\phi = k\chi + k^3$ and $\chi = x/t$. The important quantities are $\phi' = \chi + 3k^2 = 0$, $k_{0\pm} = \pm\sqrt{\dfrac{-x}{3t}}$, and $\phi'' = 6k$. Stationary phase is applied only for real stationary points. In this case, when $x/t < 0$, $k_{0\pm} = \pm\sqrt{|x/(3t)|}$ are real.

Since we have two stationary points, we add the contributions from both of them to form our asymptotic estimate. Recalling that the general form of the stationary phase result is

$$u(x,t) \sim \sum_j \frac{\hat{u}_0(k_{0j})}{\sqrt{2\pi t|\phi''(k_{0j})|}}e^{i\phi(k_{0j})t+i\mu_j\pi/4},$$

where the sum is over all stationary points, we have, in our particular case,

$$\phi(k_{0\pm}) = \mp 2\left|\frac{x}{3t}\right|^{3/2},$$

$$\phi''(k_{0\pm}) = \pm 2\sqrt{3}\sqrt{\left|\frac{x}{t}\right|},$$

$$\mu_\pm = \mathrm{sgn}(\phi''(k_{0\pm})) = \pm 1,$$

$$u(x,t) \sim \frac{\left|\hat{u}_0\left(\sqrt{\left|\frac{x}{3t}\right|}\right)\right|}{\sqrt{\pi t}\left|\frac{3x}{t}\right|^{1/4}}\cos\left[2\left|\frac{x}{3t}\right|^{3/2}t - \frac{\pi}{4} - \widehat{\psi_0}\left(\sqrt{\left|\frac{x}{3t}\right|}\right)\right], \tag{3.20}$$

where we define $\hat{u}_0(k) \equiv |\hat{u}_0(k)|e^{i\widehat{\psi_0(k)}}$ and use the relation $\hat{u}_0(k) = \hat{u}_0^*(-k)$. Equation (3.20) is the leading asymptotic estimate for $t \gg 1$, when $x/t < 0$.

3.4.2 Steepest descent, $x/t > 0$

In this section we introduce the asymptotic method of steepest descent by working through an example with the linear KdV equation (3.19). The steepest descent technique is similar to that of stationary phase except we define our Fourier integral with complex-valued ϕ as follows

$$I(t) = \int_C f(z)e^{t\phi(z)}\,dz,$$

where C is, in general, a specified contour, which for Fourier integrals is $(-\infty, \infty)$. In contrast, the stationary phase method uses the exponential in the form $e^{i\phi(k)t}$ where $\phi(k) \in \mathbb{R}$ and the contour C is the interval $(-\infty, \infty)$. We are interested in the long-time asymptotic behavior, $t \gg 1$. We can find the dominant contribution to $I(t)$ whenever we can deform (by Cauchy's theorem if deformations are allowed) on to a new contour C' called the steepest descent contour, defined to be the contour where $\phi(z)$ has a constant imaginary part: $\operatorname{Im}\phi(\chi) =$ constant. The theory of complex variables then shows that $\operatorname{Re}\{\phi\}$ has its most rapid change along the steepest descent path (Ablowitz and Fokas, 2003; Erdelyi, 1956; Copson, 1965; Bleistein and Handelsman, 1986). Hence, this asymptotic scheme will incorporate contributions from points along the steepest descent contour with the most rapid change, namely at saddle points, $\phi'(z_0) = 0$ corresponding to a local maxima. The method takes the following general steps:

(a) Find the saddle points $\phi'(z_0) = 0$; they can be complex.
(b) Use Cauchy's theorem to deform the contour C to C', the steepest descent contour through a saddle point z_0. The steepest descent contour is defined by $\operatorname{Im}\{\phi(z)\} = \operatorname{Im}\{\phi(z_0)\}$. Use Taylor series to expand about the saddle point and keep only the low-order terms when finding the dominant contribution.

Let us now study the asymptotic solution for the KdV equation (3.19) with $x/t > 0$. The Fourier solution takes the form

$$u(x,t) = \frac{1}{2\pi}\int_C \hat{u}_0(k)e^{t\phi(k)}\,dk$$

where $\phi(k) = i(k\,x\,/\,t + k^3)$ and C is the real axis: $-\infty < k < \infty$. In this regime, $x/t > 0$, $\phi'(k) \neq 0$ for all $k \in C$, and we can perform integration by parts, as in (3.3), to prove that $u(x,t) \sim o(t^{-n})$ for all n. But this does not give us the exact rate of decay, though we expect it is exponential. Steepest descent gives the exponential rate of decay under suitable assumptions. First we find the saddle points: $\phi'(k) = i(x/t + 3k^2) = 0$ implies that we have two saddle points, $k_\pm = \pm i\sqrt{\frac{x}{3t}}$.

Next, we determine the steepest descent contour: we wish to find the curves defined by the relations $\text{Im}\{\phi(k) - \phi(k_\pm)\} = 0$ and choose one that "makes sense", namely that the integral will converge along our choice of contour. In practice, one can expand $\phi(k)$ in a Taylor series near a saddle point, keep the second-order terms, and use them to determine the steepest descent contour. In our case, this gives

$$\begin{aligned}\text{Im}\{\phi(k) - \phi(k_\pm)\} &= \text{Im}\left\{\frac{\phi''(k_\pm)}{2}(k - k_\pm)^2\right\} \\ &= \text{Im}\left\{\mp 3\left(\frac{x}{3t}\right)^{1/2} r_\pm^2 e^{2i\theta_\pm}\right\} = 0\end{aligned}$$

where we have defined $k - k_\pm = r_\pm e^{i\theta_\pm}$.

Let us first look at the case k_+: We wish to make $\text{Im}\left\{-3\left(\frac{x}{3t}\right)^{1/2} r^2 e^{2i\theta}\right\} = 0$ for all $x/t > 0$. This requires that $\text{Im}\{e^{2i\theta}\} = 0$ or $\sin(2\theta) = 0$. This is true when $\theta = 0, \pi/2, \pi, 3\pi/2, 2\pi, \ldots$ In order to choose a specific contour, we must get a convergent result when integrating along this contour. In particular, we need $\text{Re}\left\{-3(x/3t)^{1/2} r^2 e^{2i\theta}\right\} < 0$ or $-\cos(2\theta) < 0$ so that we have a decaying exponential. Therefore, we choose $\theta = 0, \pi$.

Now, for the k_- case, the choices for θ are the same as before, namely $\text{Im}\{\phi(k) - \phi(k_-)\} = 0$ when $\theta = 0, \pi/2, \pi, 3\pi/2, 2\pi, \ldots$ But this time, the convergent integral relation for the real part requires that $\cos(2\theta) < 0$, so θ must be $\pi/2$ or $3\pi/2$. We can rule out these choices by considering the contour that passes through the point $k_- = -i\sqrt{\frac{x}{3t}}$. Starting at $-\infty$, it is impossible to pass through the point k_- with local angle $\pi/2$ or $3\pi/2$ and continue on to $+\infty$. There is a "turn" at k_- that sends the contour down to $-i\infty$ and we cannot reach $+\infty$ as required.

Our contour locally (near k_+) follows the curve $C' = \left\{re^{i\theta} \in \mathbb{C} \,\middle|\, re^{i\theta} = k - i\sqrt{\frac{x}{3t}},\ k \in (-\infty, \infty)\right\}$. Of the two possible choices for θ, we take $\theta = 0$ so that we have a positively oriented contour. In order to satisfy Cauchy's theorem, we must smoothly transition from $-\infty$ up to the curve defined near k_+ and then back down to $+\infty$. This is not a problem for us as we are only interested in what happens on the contour near k_+.

Now that we have found our steepest descent contour, we evaluate the Fourier integral along this contour using Cauchy's theorem. Assuming no poles between the contours C and C', we deform the contour C to C' and find the asymptotic estimate, with the dominant contribution from the saddle point k_+, to be

$$\begin{aligned}
u(x,t) &\sim \frac{1}{2\pi}\int_{C'} \hat{u}_0(k_+)e^{t\phi(k_+)}e^{\frac{t}{2}\phi''(k_+)(z-k_+)^2}\,dz,\\
&= \frac{1}{2\pi}\hat{u}_0\left(i\sqrt{\tfrac{x}{3t}}\right)e^{-2\left(\frac{x}{3t}\right)^{3/2}t}\int_{-\infty}^{\infty} e^{-3\sqrt{\frac{x}{3t}}r^2t}\,dr,\\
&= \frac{\hat{u}_0\left(i\sqrt{\frac{x}{3t}}\right)}{2\pi\sqrt{t}\left(\frac{3x}{t}\right)^{1/4}}e^{-2\left(\frac{x}{3t}\right)^{3/2}t}\int_{-\infty}^{\infty} e^{-s^2}\,ds,\\
&= \frac{\hat{u}_0\left(i\sqrt{\frac{x}{3t}}\right)}{2\sqrt{\pi t}\left(\frac{3x}{t}\right)^{1/4}}e^{-2\left(\frac{x}{3t}\right)^{3/2}t}. \qquad (3.21)
\end{aligned}$$

In going from the first line to the second line, we made the change of variable $z-k_+ = r$. From the second line to the third, the substitution $r = s/\sqrt{3t\left(\frac{x}{3t}\right)^{1/2}}$ was made. The last line results from the evaluation of the Gaussian integral $\int e^{-s^2}ds = \sqrt{\pi}$. We also assumed that $\hat{u}_0(k)$ is analytically extendable in the upper half-plane. If $\hat{u}_0(k)$ had poles between C and C', then these contributions would need to be included; indeed they would be larger than the steepest descent contribution.

3.4.3 Similarity solution, $x/t \to 0$

Motivated by the formulas (3.20) and (3.21), let us look for a similarity solution of the linear KdV equation (3.19) with the form

$$u_{\text{sim}}(x,t) = \frac{1}{(3t)^p}f\left(\tfrac{x}{(3t)^q}\right).$$

Substituting this ansatz into (3.19) and calling $\eta = x/(3t)^q$, we obtain the equation

$$\frac{f'''}{(3t)^{p+3q}} - \frac{1}{(3t)^{p+1}}(3pf + 3q\eta f') = 0.$$

To obtain an ODE independent of t requires $q = 1/3$; for a linear problem; p is free and is fixed depending on additional (e.g., initial) conditions. The above asymptotic analysis to the IVP implies (based on asymptotic matching of the solution – see below) that $p = 1/3$ as well. Integrating the above equation once gives

$$f'' - \eta f = c_1 = 0, \qquad (3.22)$$

where we take $f \to 0$ as $\eta \to \infty$. This equation is known as Airy's equation. The bounded solution to this ODE is a well-known special function of mathematical physics and is denoted by $\mathrm{Ai}(\eta)$ (up to a multiplicative constant), which has the integral representation (Ablowitz and Fokas, 2003)

$$\mathrm{Ai}(\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(s\eta + s^3/3)}\, ds. \tag{3.23}$$

As with the linear Schrödinger equation (3.12), we can now write the self-similar asymptotic solution to the linear KdV equation as

$$u(x,t) \sim \hat{u}_0(0)u_{\mathrm{sim}}(x,t) = \frac{\hat{u}_0(0)}{(3t)^{1/3}}\,\mathrm{Ai}\left(\tfrac{x}{(3t)^{1/3}}\right).$$

An alternative, and in this case improved, way of doing this is starting with the Fourier integral solution of this equation (as we have done before)

$$u(x,t) = \frac{1}{2\pi}\int \widehat{u_0}(k)e^{i(kx+k^3t)}\, dk$$

and changing the integration variable to s defined by $k = s/(3t)^{1/3}$. This gives

$$u(x,t) = \frac{1}{2\pi(3t)^{1/3}}\int \widehat{u_0}\left(\frac{s}{(3t)^{1/3}}\right)\exp\left\{i\left(\frac{sx}{(3t)^{1/3}} + \frac{s^3}{3}\right)\right\} ds.$$

Note that when $x/(3t)^{1/3} = O(1)$ (recall x/t is small) the exponent is not rapidly varying. We assume $\hat{u}(k)$ is smooth. Therefore, as $t \to \infty$ we can expand u_0 inside the integral in a Taylor series, which gives

$$u(x,t) = \frac{\widehat{u_0}(0)}{2\pi(3t)^{1/3}}\int e^{i(s\eta+s^3/3)}\, ds + \frac{\widehat{u_0'}(0)}{2\pi(3t)^{2/3}}\int s e^{i(s\eta+s^3/3)}\, ds + \cdots,$$

where $\eta = x/(3t)^{1/3}$. Using

$$\mathrm{Ai}'(\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} i s e^{i(s\eta+s^3/3)} ds,$$

we arrive at the asymptotic expansion for $x/(3t)^{1/3} = O(1)$,

$$u(x,t) \sim \frac{\widehat{u_0}(0)\,\mathrm{Ai}(\eta)}{(3t)^{1/3}} - \frac{i\widehat{u_0'}(0)\mathrm{Ai}'(\eta)}{(3t)^{2/3}} + \cdots. \tag{3.24}$$

3.4.4 Matched asymptotic expansion

We have succeeded in determining the asymptotic behavior of the solution to the linear KdV equation (3.19) in three different regions: $x/t < 0$, $x/t \to 0$, and $x/t > 0$. Each region required a different asymptotic technique. Now we will see how the Airy function matches the seemingly disparate solutions together.

Either applying the method of steepest descent directly on the integral solution (3.23) of the Airy equation (Ablowitz and Fokas, 2003) or consulting a reference such as the *Handbook of Mathematical Functions* (Abramowitz and Stegun, 1972) reveals that

$$\mathrm{Ai}(\eta) \sim \frac{e^{-\frac{2}{3}\eta^{3/2}}}{2\sqrt{\pi}\eta^{1/4}} \qquad \text{as} \qquad \eta \to +\infty,$$
$$\mathrm{Ai}(\eta) \sim \frac{\cos\left(\frac{2}{3}\eta^{3/2} - \pi/4\right)}{\sqrt{\pi}|\eta|^{1/4}} \qquad \text{as} \qquad \eta \to -\infty.$$

Then we rewrite (3.20) and (3.21) in the following form:

$$u(x,t) \sim \frac{\left|\hat{u}_0\left(i\left(\frac{x}{3t}\right)^{1/2}\right)\right|}{2\sqrt{\pi}} \frac{e^{-\frac{2}{3}\left(\frac{x}{(3t)^{1/3}}\right)^{3/2}}}{(3t)^{1/3}\left(\frac{x}{(3t)^{1/3}}\right)^{1/4}}, \qquad \frac{x}{t} > 0$$
$$u(x,t) \sim \frac{\left|\hat{u}_0\left(\left|\frac{x}{3t}\right|^{1/2}\right)\right|}{\sqrt{\pi}} \frac{\cos\left(\frac{2}{3}\left|\frac{x}{(3t)^{1/3}}\right|^{3/2} - \frac{\pi}{4} - \widehat{\psi}_0\left(\left|\frac{x}{3t}\right|^{1/2}\right)\right)}{(3t)^{1/3}\left|\frac{x}{(3t)^{1/3}}\right|^{1/4}}, \qquad \frac{x}{t} < 0.$$

Comparing the previous two sets of equations, we will see that this can be interpreted as a slowly varying similarity solution involving the Airy function that matches the asymptotic behavior in all three regions. Calling $\eta = x/(3t)^{1/3}$, we see that the regime $x/t < 0$, $\eta \to -\infty$, exhibits oscillatory behavior and decays like $|\eta|^{-1/4}$ just like the Airy function. When $x/t > 0$, $\eta \to \infty$, we have exponential decay that behaves like $e^{-\frac{2}{3}\eta^{3/2}}/\eta^{1/4}$. We further notice that the formulas (3.20) and (3.21) are valid for $x/t = O(1)$ and $|\eta| \gg 1$. As $x/t \to 0$, they match to the solution (3.24), which is valid for $\eta = O(1)$. We also note that further analysis shows that there is a uniform asymptotic expansion that is valid for all three regions of $\left(x/(3t)^{1/3}\right)$:

$$u(x,t) \sim \frac{\hat{u}_0(k_0) + \hat{u}_0(-k_0)}{2(3t)^{1/3}}\,\mathrm{Ai}(\eta) - \frac{\hat{u}_0(k_0) - \widehat{u}_0(-k_0)}{2ik_0(3t)^{2/3}}\mathrm{Ai}'(\eta) + \ldots,$$

where $k_0 = \sqrt{-x/(3t)}$ (see Figure 3.1). There is, in fact, a systematic method of finding uniform asymptotic expansions (Chester et al., 1957).

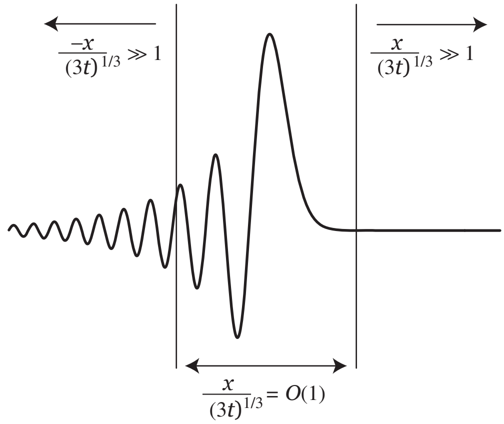

Figure 3.1 Asymptotic expansion for the linear KdV equation (3.19).

3.5 Discrete equations

It is well known that discrete equations arise in the study of numerical methods for differential equations. However, discrete equations also arise in important physical applications, cf. Ablowitz and Musslimani (2003) and Christodoulides and Joseph (1998). For example, consider the semidiscrete linear free Schrödinger equation for the function $u_n(t) = u(nh, t)$

$$i\frac{\partial u_n}{\partial t} + \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} = 0, \tag{3.25}$$

with $h > 0$ constant and given initial values $u_n(t = 0) = f_n$, with $f_n \to 0$ sufficiently rapidly as $|n| \to \infty$. This equation can be used to solve the continuous free Schrödinger equation numerically, in which case the parameter h is taken to be sufficiently small. However, as described in the references, this semidiscrete equation also arises in the study of waveguide arrays in optics, in which case h can be $O(1)$. In either case, we can study this equation using integral asymptotics (Ablowitz and Segur, 1979, 1981).

3.5.1 Continuous limit

Before deriving the exact solutions of (3.25) and their long-time asymptotics, it is important to understand the "continuous limit" of this equation as $h \to 0$. To do that we imagine that $x = nh$ is a continuous variable with n large and $h = \Delta x$ small; that is, $n \to \infty$, $h \to 0$, and $nh \to x$. We can then formally expand the solution in a Taylor series around x in the following way:

$$u_{n\pm1}(t) = u(nh \pm h, t) = u(x \pm h, t),$$
$$\approx u(x,t) \pm hu_x(x,t) + \frac{h^2}{2}u_{xx}(x,t) + \cdots.$$

Substituting this expansion into (3.25) and keeping $O(1)$ terms leads to the continuous free Schrödinger equation,

$$iu_t + u_{xx} = O(h^2).$$

The initial conditions can be approximated in a similar manner, i.e., if $u_n(t = 0) = f_n$ then in the continuous limit $u(x, 0) = f(x)$.

3.5.2 Discrete dispersion relation

In analogy to the continuous case, we can find wave solutions to (3.25) that serve as the basic building blocks for the solution. Similar to the constant coefficient continuous case, we can look for wave solutions (or special solutions) in the form

$$u_{n\,s}(t) = Z^n e^{-i\omega t},$$

where $Z = e^{ikh}$. Note that in the continuous limit i.e., as $nh \to x$, we have $Z^n = e^{iknh} \to e^{ikx}$, so this solution becomes $e^{i(kx-\omega t)}$, i.e., the continuous wave solution. Substituting $u_{n\,s}$ into (3.25) leads to the discrete dispersion relation

$$\omega + \frac{Z - 2 + 1/Z}{h^2} = 0.$$

It is convenient to use Euler's formula $(Z + 1/Z)/2 = \cos(kh)$ to rewrite the dispersion relation as

$$\omega(k) = \frac{2[1 - \cos(kh)]}{h^2}. \tag{3.26}$$

This is the dispersion relation for (3.25). Taking the continuous limit $h \to 0$, $k = O(1)$ fixed, and using a Taylor expansion leads to

$$\omega(k) = k^2,$$

which is exactly the dispersion relation for the continuous Schrödinger equation. But it is important to note that the dispersion relations for the discrete and continuous problems are markedly different for large values of k.

3.5.3 Z transforms and discrete Fourier transforms

The next step in the analysis is to represent the general solution as an integral of the special (wave) solutions, in analogy to the Fourier transform. This kind of representation is sometimes referred to as the Z transform. To understand it better, let us recall some properties of the Fourier transform. In the continuous case, recall we have the pair

$$\begin{aligned} u(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{u}(k)e^{ikx}\,dk, \\ \widehat{u}(k) &= \int_{-\infty}^{\infty} u(x)e^{-ikx}\,dx. \end{aligned}$$

Substituting the second into the first and interchanging the integrals yields

$$\begin{aligned} u(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} u(x')e^{-ikx'}\,dx'\right]e^{ikx}\,dk, \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x')\left(\int_{-\infty}^{\infty} e^{ik(x-x')}\,dk\right)dx', \\ &= \int_{-\infty}^{\infty} u(x')\delta(x-x')\,dx', \\ &= u(x), \end{aligned} \tag{3.27}$$

where we have used that the delta function satisfies

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk;$$

see Ablowitz and Fokas (2003) for more details.

Now, the Z transform (ZT) and its inverse (IZT) are given by

$$u_n = \frac{1}{2\pi i}\oint_C \tilde{u}(z)z^{n-1}\,dz, \tag{3.28}$$

$$\tilde{u}(z) = \sum_{m=-\infty}^{\infty} u_m z^{-m}, \tag{3.29}$$

where the contour C is the unit circle in the complex domain. In analogy with the Fourier transform (FT) pair, we substitute (3.29) into (3.28) and use complex analysis to get that

$$u_n = \frac{1}{2\pi i}\oint_C \Big[\sum_{m=-\infty}^{\infty} u_m z^{-m}\Big] z^{n-1}\,dz$$
$$= \sum_{m=-\infty}^{\infty} u_m \frac{1}{2\pi i}\oint_C z^{n-m-1}\,dz$$
$$= \sum_{m=-\infty}^{\infty} u_m \delta_{n,m} = u_n, \tag{3.30}$$

where the Kronecker delta $\delta_{n,m}$ is 1 when $n = m$ and zero otherwise. Here we have relied on the fact that $\frac{1}{2\pi i}\oint z^\alpha\,dz$, α integer, is equal to 1 only for a simple pole, i.e., $\alpha = -1$, and is otherwise equal to zero. Equation (3.30) using complex analysis is the analog to (3.27) for the ZT.

In discrete problems it is convenient to put the ZT in a "Fourier dress". We can do that by substituting z with k according to $z = e^{ikh}$. Since the contour of integration is the (counterclockwise) unit circle, k ranges from $-\pi/h$ to π/h and z goes from -1 to 1. Thus, the ZT is transformed into the discrete Fourier transform (DFT)

$$u_n = \frac{1}{2\pi i}\int_{-\pi/h}^{\pi/h} \tilde{u}(z) e^{ik(nh-h)} ihe^{ikh}\,dk = \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} \tilde{u}(z)e^{iknh}\,dk, \tag{3.31}$$

using $z(k) = e^{ikh}$. In the continuous limit as $h \to 0$, we have $h\tilde{u}(z) = h\tilde{u}(e^{ikh}) \to \widehat{u}(k)$, $nh \to x$, $u_n = u(nh) \to u(x)$ and we get the continuous FT

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{u}(k)e^{ikx}\,dk.$$

For the IZT there is a corresponding inverse DFT (IDFT), which can also be obtained using $z(k) = e^{ikh}$:

$$h\tilde{u}(z) = h\sum_{m=-\infty}^{\infty} u_m z^{-m} = \sum_{m=-\infty}^{\infty} u(mh)e^{-i(mh)k}h,$$

or, with $mh \to x$, and noting that the Riemann sum tends to an integral with $h = \Delta x$:

$$\widehat{u}(k) = \int_{-\infty}^{\infty} u(x)e^{-ixk}\,dx.$$

Thus, in the continuous limit one obtains the continuous IFT.

3.5.4 Asymptotics of the discrete Schrödinger equation

Using the DFT (3.31) replacing $\tilde{u}(z)$ by $\tilde{u}(z,t) = \tilde{u}_0(z)e^{-i\omega t}$ on the semidiscrete linear free Schrödinger equation (3.25) leads to the solution represented as

$$u_n(t) = \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} \tilde{u}_0(e^{ikh})e^{iknh}e^{-i\omega(k)t}\,dk,$$

where from (3.26) we have that $\omega(k) = 2[1-\cos(kh)]/h^2$ and the initial condition is $\tilde{u}_0(e^{ikh}) = \sum_{m=-\infty}^{\infty} u_m(0)e^{-ikmh}$. Therefore, we get that

$$u_n(t) = \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} \tilde{u}_0(e^{ikh})e^{i\phi(k)t}\,dk,$$

where

$$\phi(k) = k\chi_n - \frac{2[1-\cos(kh)]}{h^2}$$

and $\chi_n = nh/t$. Using the method of stationary phase we have that

$$\phi'(k) = \frac{nh}{t} - \frac{2\sin(kh)}{h} = 0,$$

from which the stationary points, k_0, satisfy

$$\sin(k_0 h) = \frac{nh^2}{2t}. \tag{3.32}$$

Note that the group velocity for the discrete equation is

$$v_g = \omega'(k) = 2\sin(kh)/h \to 2k$$

as $h \to 0$ (k fixed), which is the group velocity corresponding to the continuous Schrödinger equation. In contrast to the continuous case, where $v_g(k)$ is unbounded, in the discrete case we now have that the group velocity is bounded

$$|v_g| = \left|\frac{2\sin(kh)}{h}\right| \le \frac{2}{h}, \tag{3.33}$$

where equality holds only when $kh = \pm\pi/2$ (the degenerate case). This means that in contrast to the continuous Schrödinger equation, where information can travel with an arbitrarily large speed, in the semidiscrete Schrödinger equation information can travel at most with a finite speed. This is typical in discrete equations. Only in the continuous limit $h \to 0$, k fixed, does v_g become unbounded.

We can further obtain that

$$-\phi''(k) = 2\cos(kh).$$

Using (3.32) gives that at the stationary points

$$\begin{aligned} -\phi''(k_0) &= 2\cos(k_0 h), \\ &= \pm 2\sqrt{1-\sin^2(k_0 h)}, \\ &= \pm 2\sqrt{1-\left(-\frac{nh^2}{2t}\right)^2}, \\ &= \pm\frac{\sqrt{(2t)^2-(nh^2)^2}}{t}. \end{aligned}$$

Thus, using the stationary phase method, we obtain that

$$u_n(t) \sim \sum_{k_0} \frac{h\widehat{u_0}(k_0)e^{i\phi(k_0)t+i\mu\pi/4}}{\sqrt{2\pi t|\phi''(k_0)|}},$$

where $\mu = \mathrm{sgn}(\phi''(k_0))$. Note also that from (3.32), k_0 is a function of n and from (3.32) there are generically two values of k_0 for each n. Note that in the degenerate case $kh = \pm\pi/2$, or $(nh)^2 = (2t)^2$, one gets that $\phi'' = 0$, which means that we need to take higher-order terms in the stationary phase method.

There are three regions that, in principle, can be studied: $|nh^2/2t| < 1$, $|nh^2/2t| > 1$, and near $nh^2/2t = \pm 1$. The first, which is the most important region in this problem, is obtained by the stationary phase method given above. It gives rise to oscillations decaying like $O(1/\sqrt{t})$. The "outside" region has exponential decay (similar to the KdV equation as $x/t \to 0$ and $x/t > 0$). The matching region has the somewhat slower $O(t^{-1/3})$ decay rate (this corresponds to $kh \approx \pi/2$).

This discrete problem has behavior that is similar to the Klein–Gordon equation studied in Chapter 2,

$$u_{tt} - u_{xx} + m^2 u = 0,$$

where the dispersion relation is given by $\omega(k) = \sqrt{k^2+m^2}$ and the group velocity is bounded,

$$|v_g| = |\omega'(k)| = \left|\frac{k}{\sqrt{k^2+m^2}}\right| \le 1,$$

cf. equation (3.33).

In a similar way one can also study the asymptotic analysis of the solution to the doubly discrete Schrödinger equation,

$$i\frac{u_n^{m+1}-u_n^{m-1}}{\Delta t} + \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{h^2} = 0,$$

but we leave this as an exercise to the interested reader (see Exercise 3.5).

3.5.5 Fully discrete wave equation and the CFL condition

Consider the wave equation

$$u_{tt} = c^2 u_{xx},$$

where c is a constant. Suppose we want to solve this equation using a standard second-order explicit centered finite-difference scheme, i.e.,

$$\frac{u_n^{m+1} - 2u_n^m + u_n^{m-1}}{\Delta t^2} = c^2\left(\frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{\Delta x^2}\right), \tag{3.34}$$

where $\Delta x > 0, \Delta t > 0$ are constant and $u_n^m = u(n\Delta x, m\Delta t)$. When using the scheme (3.34), we are faced with the problem of numerical stability/instability. We can use the ZT in order to understand the origin of this issue as follows. Let us define the special (wave) solutions

$$u_{n\,s}^m = Z^n \Omega^m,$$

where $Z = e^{ikh}$ and $\Omega = e^{-i\omega t}$. Note that as in the continuous limit, substituting $u_{n\,s}^m$ into the scheme above leads to the "dispersion relation"

$$\frac{\Omega - 2 + \dfrac{1}{\Omega}}{\Delta t^2} = c^2 \frac{Z - 2 + \dfrac{1}{Z}}{\Delta x^2}.$$

It is convenient to define

$$p = \frac{\Delta t}{\Delta x} c$$

and rewrite the dispersion relation as a quadratic polynomial in Ω (or in Z) as

$$\Omega^2 - \left[2 + p^2\left(Z - 2 + \frac{1}{Z}\right)\right]\Omega + 1 = 0. \tag{3.35}$$

In this way it is easier to spot the features of the dispersion relation. Indeed, since this relation is quadratic in Ω it has two roots $\Omega_{1,2}$ and can be written as

$$(\Omega - \Omega_1)(\Omega - \Omega_2) = \Omega^2 - (\Omega_1 + \Omega_2)\Omega + \Omega_1\Omega_2 = 0.$$

Thus the sum of the two roots must be equal to the square brackets in (3.35) and the product of the two roots must be equal to 1. Hence, there are two cases to consider: either one root is greater than 1 and the other smaller, i.e., $|\Omega_1| < 1$ and $|\Omega_2| > 1$ (the distinct-real root case), or both roots are equal to 1 in magnitude. Defining $b = 1 + p^2(Z - 2 + 1/Z)/2 = 1 + p^2[\cos(kh) - 1]$, where $Z = e^{ikh}$, we see that the discriminant, Δ, of (3.35) is given by $\Delta/4 = b^2 - 1$. Hence when $|b| > 1$ there are two distinct real-valued solutions and when $|b| < 1$ there are two complex roots. The double-root case corresponds to $b = 1$ or $kh = \pm\pi/2$.

When $|b| > 1$ the solutions are unstable, because one of the roots is larger than 1. From a numerical perspective this results in massively unstable growth, or "numerical ill-posedness"; note, the growth rate becomes arbitrarily large as $\Delta t \to 0, m \to \infty$ for fixed $m\Delta t$. The condition for stability $|b| \le 1$ corresponds to $|1 + p^2[\cos(k\Delta x) - 1]| \le 1$, which is satisfied by $-2 \le p^2[\cos(k\Delta x) - 1] \le 0$, or we can guarantee this inequality for all k if $p^2 \le 1$, i.e.,

$$0 \le \frac{\Delta t}{\Delta x} c \le 1 \quad \text{or} \quad 0 \le \frac{\Delta t}{\Delta x} \le \frac{1}{c}.$$

This condition was first discovered and named after Courant, Friedrichs, and Lewy (Courant et al., 1928) and is often referred to as the CFL condition.

3.6 Burgers' equation and its solution: Cole–Hopf transformation

In this section we study the solution of the viscous Burgers equation

$$u_t + uu_x = \nu u_{xx}, \tag{3.36}$$

where ν is a constant, with the initial condition

$$u(x, 0) = f(x),$$

where $f(x)$ is a sufficiently decaying function as $|x| \to \infty$ and $t > 0$. It is convenient to transform (3.36) using $u = \partial\psi/\partial x$ to get

$$\psi_t + \frac{1}{2}\psi_x^2 = \nu\psi_{xx},$$

where we assume $\psi \to$ constant as $|x| \to \infty$. Now let

$$\psi(x, t) = -2\nu \log \phi(x, t), \tag{3.37}$$

so

$$u(x, t) = -2\nu \frac{\phi_x}{\phi}(x, t), \tag{3.38}$$

which is called the *Cole–Hopf* transformation (Cole, 1951; Hopf, 1950). This yields

$$(-2\nu)\frac{\phi_t}{\phi} + \frac{1}{2}(-2\nu)^2\left(\frac{\phi_x}{\phi}\right)^2 = \nu(-2\nu)\left[\frac{\phi_{xx}}{\phi} - \left(\frac{\phi_x}{\phi}\right)^2\right],$$

which simplifies to

$$\phi_t = \nu\phi_{xx}, \tag{3.39}$$

the linear heat equation!

Later in this book we will discuss the linearization and complete solution of some other important equations, such as the KdV equation, by the method of the *inverse scattering transform* (IST).

For Burgers' equation, the solution is obtained by solving (3.39) and transforming back to $u(x,t)$ via (3.38). To solve (3.39) we need the initial data $\phi(x,0)$, which are found from

$$-2\nu\frac{\phi_x}{\phi}(x,0) = f(x)$$

or

$$\phi(x,0) = h(x) = C\exp\left(-\frac{1}{2\nu}\int_0^x f(x')\,dx'\right). \tag{3.40}$$

The constant C does not appear in (3.37) so without loss of generality we take $C = 1$.

The solution to the heat equation with $\phi(x,0) = h(x)$ can be obtained from the fundamental solution or Green's function $G(x,t)$:

$$\phi(x,t) = \int_{-\infty}^{\infty} h(x')G(x-x',t)\,dx',$$

where it is well known that

$$G(x,t) = \frac{1}{\sqrt{4\pi\nu}}e^{-x^2/4\nu t}. \tag{3.41}$$

We note that $G(x,t)$ satisfies

$$G_t = \nu G_{xx},$$
$$G(x,0) = \delta(x).$$

Further, $G(x,t)$ can be obtained by Fourier transforms

$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{G}(k,t)e^{ikx}\,dk,$$
$$\hat{G}(k,t) = \int_{-\infty}^{\infty} G(x,t)e^{-ikx}\,dx.$$

Then

$$\hat{G}_t = -\nu k^2\hat{G}$$

and with $\hat{G}(k,0) = 1$ we have

$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2\nu t}\,dk. \tag{3.42}$$

The integral equation (3.42) can be evaluated by completing the square

$$G(x,t) = e^{-x^2/4\nu t}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(k-\frac{x}{2\nu t}\right)^2}\,dk = \frac{e^{-x^2/4\nu t}}{2\sqrt{\pi\nu t}},$$

which yields (3.41). Thus the solution to the heat equation is

$$\phi(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} h(x') e^{-(x-x')^2/4\nu t}\, dx',$$

with $\phi(x,0) = h(x)$ given by (3.40). Then the solution of the Burgers equation takes the form

$$u(x,t) = -2\nu \frac{\phi_x}{\phi} = \frac{\int_{-\infty}^{\infty} \frac{x-x'}{t} h(x') e^{-(x-x')^2/4\nu t}\, dx'}{\int_{-\infty}^{\infty} h(x') e^{-(x-x')^2/4\nu t}\, dx'}$$

or using (3.40)

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \frac{x-x'}{t} e^{-\Gamma(x,t,x')/2\nu}\, dx'}{\int_{-\infty}^{\infty} e^{-\Gamma(x,t,x')/2\nu}\, dx'}, \tag{3.43}$$

with

$$\Gamma(x,t,x') = \frac{(x-x')^2}{2t} + \int_0^{x'} f(\zeta)\, d\zeta.$$

In principle we can determine the solution at any time t by carrying out the quadrature in (3.43). But for general integrable functions $f(x)$ the task is still formidable and little qualitative information is obtained from (3.43) by itself. However, in the limit $\nu \to 0$ we can recover valuable information. When $\nu \to 0$ we can use Laplace's method (see Ablowitz and Fokas, 2003) to determine the leading-order contributions of the integral. We note that the minimum (in x') of $\Gamma(x,t,x')/2\nu$, determined from

$$\frac{\partial \Gamma}{\partial x'} = \Gamma'(x') = f(x') - \frac{x-x'}{t} = 0,$$

gives the dominant contribution. An integral of the form

$$I(\nu) = \int_{-\infty}^{\infty} g(x') e^{-\Gamma(x,t,x')/2\nu}\, dx',$$

can be evaluated as $\nu \to 0$ by expanding Γ near its minimum $x' = \xi$,

$$\Gamma(x,t,x') = \Gamma(x,t,\xi) + \frac{1}{2}(x'-\xi)^2 \Gamma''(x,t,\xi) + \dots,$$

and

$$I(\nu) \underset{\nu\to 0}{\sim} \sqrt{\frac{4\pi\nu}{\Gamma''(x,t,x')}}\, e^{-\Gamma(x,t,\xi)/2\nu}.$$

Thus the solution $u(x,t)$ from (3.43) is given, as $\nu \to 0$, by

$$u(x,t) \sim \frac{\frac{x-\xi}{t} e^{-\Gamma(x,t,\xi)/2\nu} \sqrt{\frac{4\pi\nu}{\Gamma''(x,t,\xi)}}}{e^{-\Gamma(x,t,\xi)/2\nu} \sqrt{\frac{4\pi\nu}{\Gamma''(x,t,\xi)}}}$$

or

$$u(x,t) \sim \frac{x-\xi}{t} = f(\xi).$$

We recall that this is the exact solution to the inviscid Burgers equation

$$u_t + uu_x = 0,$$

with $u(x,0) = f(x)$ in Section 2.6; we found $u(x,t) = f(\xi)$ along $x = \xi + f(\xi)t$ in regions where the solution is smooth. We remark that shocks can occur when there are multiple minima of equal value to $\Gamma(x,t,\nu)$, but we will not go into further details of that solution here.

3.7 Burgers' equation on the semi-infinite interval

In this section we will discuss Burgers' equation, (3.36), on the semi-infinite interval $[0,\infty)$.[2] We assume $u(x,t) \to 0$ rapidly as $x \to \infty$ and at $x = 0$ the mixed boundary condition

$$\alpha(t)u(0,t) + \beta(t)u_x(0,t) = \gamma(t), \tag{3.44}$$

where α, β, γ are prescribed functions of t and we take for convenience $u(x,0) = 0$. It will be helpful to employ the Dirichlet–Neumann (DN) relationship (often termed the DN map) associated with the heat equation (3.39)

$$\begin{aligned} \phi_x(0,t) &= -\frac{1}{\sqrt{\nu\pi}} \int_0^t \frac{\phi_\tau(0,\tau)}{\sqrt{t-\tau}}, \\ &= -\frac{1}{\sqrt{\nu}} D_t^{1/2}\phi_0(t), \end{aligned} \tag{3.45}$$

where $\phi_0(t) = \phi(0,t)$ and the "half-derivative" is defined as

$$D_t^{1/2}\phi_0(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{\phi_\tau(0,\tau)}{\sqrt{t-\tau}}.$$

Thus if we know $\phi(0,t)$ associated with (3.39) then (3.45) determines $\phi_x(0,t)$. Equation (3.45) can be found in many ways, e.g., via Laplace transforms (Ablowitz and Fokas, 2003): define the Laplace transform of $\phi(x,t)$ as

$$\hat{\Phi}(x,s) = \mathcal{L}\{\phi\} = \int_0^\infty \phi(x,t)e^{-st}\,dt;$$

then (3.39) implies

$$\nu\hat{\Phi}_{xx} - s\hat{\Phi} + 1 = 0,$$

[2] The analysis here follows the work of M. Hoefer developed for the "Nonlinear Waves" course taught by M.J. Ablowitz in 2003.

where we take $\phi(x,0) = 1$ with no loss of generality. Thus

$$\Phi(x,s) = A(s)e^{-\sqrt{\frac{s}{\nu}}x} + \frac{1}{s},$$

where we omitted the exponentially growing part. Then using (3.39) at $x = 0$,

$$\hat{\Phi}_{xx}(0,s) = \mathcal{L}\left\{\frac{1}{\nu}\phi_t(0,t)\right\},$$

implies

$$A(s) = \frac{1}{\sqrt{\nu s}}\mathcal{L}\{\phi_t(0,t)\}.$$

Hence

$$\hat{\Phi}_x(0,s) = -\frac{1}{\sqrt{\nu s}}\mathcal{L}\{\phi_t(0,t)\}$$

or by taking the inverse Laplace transform, using the convolution theorem, and $\mathcal{L}\{1/\sqrt{s}\} = 1/\sqrt{\pi t}$, we find (3.45).

The boundary condition (3.44) is converted to

$$\alpha(t)\frac{\phi_x}{\phi}(0,t) + \beta(t)\Big[\underbrace{\frac{\phi_{xx}}{\phi}}_{\frac{1}{\nu}\frac{\phi_t}{\phi}} - \left(\frac{\phi_x}{\phi}\right)^2\Big](0,t) + \frac{\gamma(t)}{2\nu} = 0$$

via the Hopf–Cole transformation (3.38). Or calling $\phi_0(t) = \phi(0,t)$ and using (3.45) we have

$$\alpha(t)\frac{D_t^{1/2}\phi_0}{\phi_0} - \frac{\beta(t)}{\sqrt{\nu}}\left[\frac{\phi_{0t}}{\phi_0} - \left(\frac{D_t^{1/2}\phi_0}{\phi_0}\right)^2\right](0,t) - \frac{\gamma(t)}{2\sqrt{\nu}} = 0. \tag{3.46}$$

We see that, in general, (3.46) is a nonlinear equation. However, in the case of Dirichlet boundary conditions for Burgers' equation, $\beta(t) = 0$, we find from (3.46) a linear integral equation of the Abel type for ϕ_0:

$$D_t^{1/2}\phi_0 = \frac{g(t)}{2\sqrt{\nu}}\phi_0,$$

where $g(t) = \gamma(t)/\alpha(t)$. Using $D_t^{1/2}D_t^{1/2}\phi_0(t) = \phi_0(t) - 1$, i.e.,

$$\frac{1}{\sqrt{\pi}}\int_0^t \frac{d\tau_1}{\sqrt{t-\tau_1}} \cdot \frac{1}{\sqrt{\pi}}\int_0^t \frac{\phi_{0,\tau_2}d\tau_2}{\sqrt{t-\tau_2}}$$
$$= \frac{1}{\pi}\int_0^t d\tau_2\ \phi_{0,\tau_2}\ \underbrace{\int_{\tau_2}^t \frac{d\tau_1}{\sqrt{t-\tau_1}\sqrt{\tau_1-\tau_2}}}_{=\pi} = \phi_0(t) - 1$$

(recall $\phi_0(0) = 1$), we have

$$\phi_0(t) = 1 + \frac{1}{2\sqrt{\nu\pi}} \int_0^t \frac{g(t')\phi_0(t')}{\sqrt{t-t'}}\, dt', \tag{3.47}$$

which is an inhomogeneous weakly singular *linear* integral equation. This equation was obtained by a different procedure by Calogero and DeLillo (1989). Note, solving (3.47) gives $\phi(0,t)$, which combined with (3.40) allows Burgers' equation to be linearized via (3.39) with the Cole–Hopf transformation (3.38). However, in general, the Hopf–Cole transformation does not linearize the equation. Rather, we see that one has a nonlinear integral equation (3.46) to study; but the number of variables has been reduced, from (x,t) to t.

Exercises

3.1 Analyze the long-time asymptotic solution of
   (a) $iu_t + u_{4x} = 0$,
   (b) $u_t + u_{5x} = 0$, and
   (c) $u_{tt} - u_{xx} + m^2u = 0, m > 0$ constant.

3.2 Consider the equation

$$u_t + cu_x = 0,$$

with $c > 0$ constant and $u(x,0) = f(x)$. Show that the solution is $u = f(x - ct)$ and that this does not decay as $t \to \infty$. Reconcile this with the Riemann–Lesbegue lemma.

3.3 Consider the linear KdV equation

$$u_t + u_{xxx} = 0.$$

Use the asymptotic analysis discussed in this chapter to analyze the behavior of the energy inside each of the three asymptotic regions as $t \to \infty$. At what speed does each propagate?

3.4 For the differential–difference, free Schrödinger equation,

$$i\frac{\partial u_n}{\partial t} + \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} = 0,$$

with $h > 0$ constant, find the long-time solution in the regions $|nh^2/2t| > 1$ and near $nh^2/2t = \pm 1$.

3.5 Consider the partial-difference equation,

$$i\frac{u_n^{m+1} - u_n^{m-1}}{\Delta t} + \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{\Delta x^2} = 0,$$

with $\Delta x > 0, \Delta t > 0$ constant, assuming $u_n^m = u(n, m)$ and $u(n, 0)$ and $u(n, 1)$ are given as initial conditions. Investigate the "long-time", i.e., $m \gg 1$, asymptotic solution.

3.6 Find a nonlinear PDE that is third order in space, first order in time that is linearized by the Cole–Hopf transformation.

3.7 Use the Fourier transform method to solve the boundary value problem

$$u_{xx} + u_{yy} = -x \exp(-x^2),$$
$$u(x, 0) = 0,$$
$$-\infty < x < \infty, \quad 0 < y < \infty,$$

where u and its derivative vanish as $y \to \infty$. Hint: It is convenient to use the inverse Fourier transform result: $\mathcal{F}^{-1}\left\{\frac{1}{k}\hat{f}(k)\, dk\right\} = \int f(x)\, dx$.

3.8 Solve the inhomogeneous partial differential equation

$$u_{xt} = -\omega \sin(\omega t), \qquad t > 0,$$
$$u(x, 0) = x, \qquad u(0, t) = 0.$$

3.9 Find the solution to Burgers' equation (3.36) on the semi-infinite interval $0 < x < \infty$ with $u(x, 0) = f(x)$ where $f(x)$ decays rapidly as $|x| \to \infty$ and $u(0, t) = g(t)$.

3.10 Solve the linear one-dimensional linear Schrödinger equation with quadratic potential (the "simple harmonic oscillator")

$$iu_t = u_{xx} - V_0 x^2 u,$$

with $V_0 > 0$ constant and $u(x, 0) = f(x)$ where $f(x)$ decays rapidly as $|x| \to \infty$. In what sense is the "ground state" (i.e., the lowest eigenvalue) the most important solution in the long-time limit? Hint: Use separation of variables.