Notes on Finite Differences

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February 19, 2020

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1 Time stepping and finite differences: the whole line

Recall the equation we obtained for the surface elevation on the whole line:

$$\eta_{tt} = \eta_{xx} + \mu^2 \left(\frac{1}{3}\eta_{xxxx} + \eta_{xt} \int_{-\infty}^{x} \eta_t \,\mathrm{d}x' + \eta_x^2 + \eta_t^2 + \eta\eta_{xx} + \partial_x^2 \left(\int_{-\infty}^{x} \eta_t \,\mathrm{d}x' \right)^2 \right). \tag{1}$$

To do time stepping, introduce

$$u = \eta_t. \tag{2}$$

Also, note that

$$\partial_x^2 \left(\int_{-\infty}^{x} \eta_t \,\mathrm{d}x' \right)^2 = 2(\eta_t^2 + \eta_{tx} \int_{-\infty}^{x} \eta_t \,\mathrm{d}x')$$

Then, combining (2) and (1), we obtain a two-dimensional system:

$$\partial_t \begin{bmatrix} u \\ \eta \end{bmatrix} = \begin{bmatrix} \mu^2 \left(2u_x \int_{-\infty}^{x} u \,\mathrm{d}x' + 2u^2 \right) + \eta_{xx} + \mu^2 \left(\frac{1}{3}\eta_{xxxx} + \eta_x^2 + \eta\eta_{xx} \right) \\ u \end{bmatrix}. \tag{3}$$

Now, consider (1) on a finite interval $[a, b]$, and let partition the interval into $n+1$ points $\{x_k\}_{k=0}^{n}$, with $x_0 = a$ and $x_n = b$. This means that the integral terms becomes

$$\int_{-\infty}^{x} \eta_t \, \mathrm{d}x' = \left\{ \int_{-\infty}^{a} + \int_{a}^{x} \right\} \eta_t \, \mathrm{d}x' \approx \int_{a}^{x} \eta_t \, \mathrm{d}x',$$

while assuming that

$$\int_{-\infty}^{a} \eta_t \, \mathrm{d}x'$$

is small enough. Now, we employ forward Euler time stepping. First, observe that

$$\begin{aligned} u_t(x_k, t_j) &= \frac{u(x_k, t_{j+1}) - u(x_k, t_j)}{\Delta t} = f_1(\eta, u, t) && \implies u(x_k, t_{j+1}) = u(x_k, t_j) + \Delta t f_1(\eta, u, t) \\ \eta_t(x_k, t_j) &= \frac{\eta(x_k, t_{j+1}) - \eta(x_k, t_j)}{\Delta t} = f_2(\eta, u, t) && \implies \eta(x_k, t_{j+1}) = \eta(x_k, t_j) + \Delta t f_2(\eta, u, t), \end{aligned}$$

where

$$\begin{aligned} f_1(\eta, u, t) &= \eta_{xx} + \mu^2 \left(2u_x \int_{-\infty}^{x} u \, \mathrm{d}x' + 2u^2 + \frac{1}{3}\eta_{xxxx} + \eta_x^2 + \eta\eta_{xx} \right) \\ f_2(\eta, u, t) &= u(x_k, t_j). \end{aligned}$$

Observe that the highest order derivative is 4, so we need to use five-point stencils. In particular, we use five point midpoint stencil for $x_2, \ldots, x_{n-2}$, and five point, one-sided stencils at x_0, x_1, x_{n-1}, x_n. First, consider the system for $x_2, \ldots, x_{n-2}$:

$$\begin{aligned} f_1(\eta(x_k, t_j), u(x_k, t_j), t) &= \eta(x_k, t_j)_{xx} + \frac{\mu^2}{3}\eta(x_k, t_j)_{xxxx} \\ &\quad + \mu^2 \left(2u(x_k, t_j)_x \int_{x_0}^{x_k} u \, \mathrm{d}x' + 2u(x_k, t_j)^2 + \eta(x_k, t_j)_x^2 + \eta(x_k, t_j)\eta(x_k, t_j)_{xx} \right), \end{aligned}$$

where we have separated the linear and nonlinear terms. Let $\Delta x = x_k - x_{k-1}$ and recall the finite difference formulas at x :

$$\begin{aligned} f'(x) &= \frac{f(x - 2\Delta x) - 8f(x - \Delta x) + 8f(x + \Delta x) - f(x + 2\Delta x)}{12\Delta x} + \mathcal{O}((\Delta x)^4) \\ f''(x) &= \frac{-f(x - 2\Delta x) + 16f(x - \Delta x) - 30f + 16f(x + \Delta x) - f(x + 2\Delta x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2) \\ f''''(x) &= \frac{f(x - 2\Delta x) - 4f(x - \Delta x) + 6f - 4f(x + \Delta x) + f(x + 2\Delta x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2) \end{aligned}$$

so that

$$
\begin{aligned}
(\eta_k)_x &= \frac{\eta_{k-2} - 8\eta_{k-1} + 8\eta_{k+1} - \eta_{k+2}}{12\Delta x} + \mathcal{O}((\Delta x)^4) \\
(u_k)_x &= \frac{u_{k-2} - 8u_{k-1} + 8u_{k+1} - u_{k+2}}{12\Delta x} + \mathcal{O}((\Delta x)^4) \\
(\eta_k)_{xx} &= \frac{-\eta_{k-2} + 16\eta_{k-1} - 30\eta_k + 16\eta_{k+1} - \eta_{k+2}}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2) \\
(\eta_k)_{xxxx} &= \frac{\eta_{k-2} - 4\eta_{k-1} + 6\eta_k - 4\eta_{k+1} + \eta_{k+2}}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2),
\end{aligned}
$$

where it is assumed $t = t_j$. Also, by trapezoidal rule,

$$
\int_{x_0}^{x_k} u \,\mathrm{d}x' = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} u \,\mathrm{d}x = \frac{\Delta x}{2} \sum_{i=0}^{n-1} u(x_i) + u(x_{i+1}).
$$

At $x = x_0$, we have

$$
\begin{aligned}
f'(x) &= \frac{-25f(x) + 48f(x+\Delta x) - 36f(x+2\Delta x) + 16f(x+3\Delta x) - 3f(x+4\Delta x)}{12\Delta x} + \mathcal{O}((\Delta x)^4) \\
f''(x) &= \frac{35f(x) - 104f(x+\Delta x) + 114f(x+2\Delta x) - 56f(x+3\Delta x) + 11f(x+4\Delta x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2) \\
f''''(x) &= \frac{f(x) - 4f(x+\Delta x) + 6f(x+2\Delta x) - 4f(x+3\Delta x) + f(x+4\Delta x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2),
\end{aligned}
$$

so that

$$
\begin{aligned}
(\eta_0)_x &= \frac{-25\eta_0 + 48\eta_1 - 36\eta_2 + 16\eta_3 - 3\eta_4}{12\Delta x} \\
(u_0)_x &= \frac{-25u_0 + 48u_1 - 36u_2 + 16u_3 - 3u_4}{12\Delta x} \\
(\eta_0)_{xx} &= \frac{35\eta_0 - 104\eta_1 + 114\eta_2 - 56\eta_3 + 11\eta_4}{12\Delta x^2} \\
(\eta_0)_{xxxx} &= \frac{\eta_0 - 4\eta_1 + 6\eta_2 - 4\eta_3 + \eta_4}{\Delta x^4}
\end{aligned}
$$

At $x = x_1$, we have

$$
f'(x) = \frac{-3f(x-\Delta x) - 10f(x) + 18f(x+\Delta x) - 6f(x+2\Delta x) + f(x+3\Delta x)}{12\Delta x} + \mathcal{O}((\Delta x)^4)
$$

$$f''(x) = \frac{11f(x-\Delta x) - 20f(x) + 6f(x+\Delta x) + 4f(x+2\Delta x) - f(x+3\Delta x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2)$$
$$f''''(x) = \frac{f(x-\Delta x) - 4f(x) + 6f(x+\Delta x) - 4f(x+2\Delta x) + f(x+3\Delta x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2),$$

so that

$$(\eta_1)_x = \frac{-3\eta_0 - 10\eta_1 + 18\eta_2 - 6\eta_3 + \eta_4}{12\Delta x}$$
$$(u_1)_x = \frac{-3u_0 - 10u_1 + 18u_2 - 6u_3 + u_4}{12\Delta x}$$
$$(\eta_1)_{xx} = \frac{11\eta_0 - 20\eta_1 + 6\eta_2 + 4\eta_3 - \eta_4}{12\Delta x^2}$$
$$(\eta_1)_{xxxx} = \frac{\eta_0 - 4\eta_1 + 6\eta_2 - 4\eta_3 + \eta_4}{\Delta x^4}$$

At $x = x_{n-1}$, we have

$$f'(x) = \frac{-f(x-3\Delta x) + 6f(x-2\Delta x) - 18f(x-\Delta x) + 10f(x) + 3f(x+\Delta x)}{12\Delta x} + \mathcal{O}((\Delta x)^4)$$
$$f''(x) = \frac{-f(x-3\Delta x) + 4f(x-2\Delta x) + 6(x-\Delta x) - 20f(x) + 11f(x+\Delta x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2)$$
$$f''''(x) = \frac{f(x-3\Delta x) - 4f(x-2\Delta x) + 6f(x-\Delta x) - 4f(x) + f(x+\Delta x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2),$$

so that

$$(\eta_{n-1})_x = \frac{-\eta_{n-4} + 6\eta_{n-3} - 18\eta_{n-2} + 10\eta_{n-1} + 3\eta_n}{12\Delta x}$$
$$(u_{n-1})_x = \frac{-u_{n-4} + 6u_{n-3} - 18u_{n-2} + 10u_{n-1} + 3u_n}{12\Delta x}$$
$$(\eta_{n-1})_{xx} = \frac{-\eta_{n-4} + 4\eta_{n-3} + 6\eta_{n-2} - 20\eta_{n-1} + 11\eta_n}{12\Delta x^2}$$
$$(\eta_{n-1})_{xxxx} = \frac{\eta_{n-4} - 4\eta_{n-3} + 6\eta_{n-2} - 4\eta_{n-1} + \eta_n}{\Delta x^4}$$

At $x = x_n$, we have

$$f'(x) = \frac{f(x-4\Delta x) - 4f(x-3\Delta x) + 6f(x-2\Delta x) - 4f(x-\Delta x) + f(x)}{12\Delta x} + \mathcal{O}((\Delta x)^4)$$

$$f''(x) = \frac{11f(x-4\Delta x) - 56f(x-3\Delta x) + 114f(x-2\Delta x) - 104f(x-\Delta x) + 35f(x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2)$$
$$f''''(x) = \frac{f(x-4\Delta x) - 4f(x-3\Delta x) + 6f(x-2\Delta x) - 4f(x-\Delta x) + f(x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2),$$

so that

$$(\eta_n)_x = \frac{3\eta_{n-4} - 16\eta_{n-3} + 36\eta_{n-2} - 48\eta_{n-1} + 25\eta_n}{12\Delta x}$$
$$(u_n)_x = \frac{3u_{n-4} - 16u_{n-3} + 36u_{n-2} - 48u_{n-1} + 25u_n}{12\Delta x}$$
$$(\eta_n)_{xx} = \frac{11\eta_{n-4} - 56\eta_{n-3} + 114\eta_{n-2} - 104\eta_{n-1} + 35\eta_n}{12\Delta x^2}$$
$$(\eta_n)_{xxxx} = \frac{\eta_{n-4} - 4\eta_{n-3} + 6\eta_{n-2} - 4\eta_{n-1} + \eta_n}{\Delta x^4}.$$

All in all, we obtain:

$$\begin{aligned}
f_1(\eta(x_0,t_j),u(x_0,t_j),t) &= \eta(x_0,t_j)_{xx} + \frac{\mu^2}{3}\eta(x_0,t_j)_{xxxx} + \mu^2\left(2u(x_0,t_j)^2 + \eta(x_0,t_j)_x^2 + \eta(x_0,t_j)\eta(x_0,t_j)_{xx}\right),\\
f_1(\eta(x_1,t_j),u(x_1,t_j),t) &= \eta(x_1,t_j)_{xx} + \frac{\mu^2}{3}\eta(x_1,t_j)_{xxxx}\\
&\quad + \mu^2\left(2u(x_1,t_j)_x\int_{x_0}^{x_1} u\,\mathrm{d}x' + 2u(x_1,t_j)^2 + \eta(x_1,t_j)_x^2 + \eta(x_1,t_j)\eta(x_1,t_j)_{xx}\right),\\
f_1(\eta(x_2,t_j),u(x_2,t_j),t) &= \eta(x_2,t_j)_{xx} + \frac{\mu^2}{3}\eta(x_2,t_j)_{xxxx}\\
&\quad + \mu^2\left(2u(x_2,t_j)_x\int_{x_0}^{x_2} u\,\mathrm{d}x' + 2u(x_2,t_j)^2 + \eta(x_2,t_j)_x^2 + \eta(x_2,t_j)\eta(x_2,t_j)_{xx}\right),\\
&\dots\\
f_1(\eta(x_k,t_j),u(x_k,t_j),t) &= \eta(x_k,t_j)_{xx} + \frac{\mu^2}{3}\eta(x_k,t_j)_{xxxx}\\
&\quad + \mu^2\left(2u(x_k,t_j)_x\int_{x_0}^{x_k} u\,\mathrm{d}x' + 2u(x_k,t_j)^2 + \eta(x_k,t_j)_x^2 + \eta(x_k,t_j)\eta(x_k,t_j)_{xx}\right),\\
&\dots\\
f_1(\eta(x_{n-1},t_j),u(x_{n-1},t_j),t) &= \eta(x_{n-1},t_j)_{xx} + \frac{\mu^2}{3}\eta(x_{n-1},t_j)_{xxxx}
\end{aligned}$$

$$
\begin{aligned}
&+\mu^2\left(2u(x_{n-1},t_j)_x\int_{x_0}^{x_{n-1}} u\,\mathrm{d}x' + 2u(x_{n-1},t_j)^2 + \eta(x_{n-1},t_j)_x^2 + \eta(x_{n-1},t_j)\eta(x_{n-1},t_j)_{xx}\right),\\
f_1(\eta(x_n,t_j),u(x_n,t_j),t) &= \eta(x_n,t_j)_{xx} + \frac{\mu^2}{3}\eta(x_n,t_j)_{xxxx}\\
&+\mu^2\left(2u(x_n,t_j)_x\int_{x_0}^{x_n} u\,\mathrm{d}x' + 2u(x_n,t_j)^2 + \eta(x_n,t_j)_x^2 + \eta(x_n,t_j)\eta(x_n,t_j)_{xx}\right),
\end{aligned}
$$

Now, we obtain the discretised problem. First, consider the column of linear terms:

$$
\begin{aligned}
&(\eta_0)_{xx} + \frac{\mu^2}{3}(\eta_0)_{xxxx}\\
&\quad= \frac{35\eta_0 - 104\eta_1 + 114\eta_2 - 56\eta_3 + 11\eta_4}{12\Delta x^2} + \frac{\mu^2}{3}\frac{\eta_0 - 4\eta_1 + 6\eta_2 - 4\eta_3 + \eta_4}{\Delta x^4}\\
&\quad= \frac{35\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_0 - \frac{104\Delta x^2 + 16\mu^2}{12\Delta x^4}\eta_1 + \frac{114\Delta x^2 + 24\mu^2}{12\Delta x^4}\eta_2 - \frac{56\Delta x^2 + 16\mu^2}{12\Delta x^4}\eta_3 + \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_4\\
&(\eta_1)_{xx} + \frac{\mu^2}{3}(\eta_1)_{xxxx}\\
&\quad= \frac{11\eta_0 - 20\eta_1 + 6\eta_2 + 4\eta_3 - \eta_4}{12\Delta x^2} + \frac{\mu^2}{3}\frac{\eta_0 - 4\eta_1 + 6\eta_2 - 4\eta_3 + \eta_4}{\Delta x^4}\\
&\quad= \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_0 - \frac{20\Delta x^2 + 16\mu^2}{12\Delta x^4}\eta_1 + \frac{6\Delta x^2 + 24\mu^2}{12\Delta x^4}\eta_2 + \frac{4\Delta x^2 - 16\mu^2}{12\Delta x^4}\eta_3 + \frac{-\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_4\\
&\quad\ldots\\
&(\eta_k)_{xx} + \frac{\mu^2}{3}(\eta_k)_{xxxx}\\
&\quad= \frac{-\eta_{k-2} + 16\eta_{k-1} - 30\eta_k + 16\eta_{k+1} - \eta_{k+2}}{12(\Delta x)^2} + \frac{\mu^2}{3}\frac{\eta_{k-2} - 4\eta_{k-1} + 6\eta_k - 4\eta_{k+1} + \eta_{k+2}}{(\Delta x)^4}\\
&\quad= \frac{-\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_{k-2} + \frac{16(\Delta x^2 - \mu^2)}{12\Delta x^4}\eta_{k-1} + \frac{-30\Delta x^2 + 24\mu^2}{12\Delta x^4}\eta_k - \frac{16(\Delta x^2 - \mu^2)}{12\Delta x^4}\eta_{k+1} + \frac{-\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_{k+2}\\
&\quad\ldots\\
&(\eta_{n-1})_{xx} + \frac{\mu^2}{3}(\eta_{n-1})_{xxxx}\\
&\quad= \frac{-\eta_{n-4} + 4\eta_{n-3} + 6\eta_{n-2} - 20\eta_{n-1} + 11\eta_n}{12\Delta x^2} + \frac{\mu^2}{3}\frac{\eta_{n-4} - 4\eta_{n-3} + 6\eta_{n-2} - 4\eta_{n-1} + \eta_n}{\Delta x^4}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_{n-4} + \frac{4\Delta x^2 - 16\mu^2}{12\Delta x^4}\eta_{n-3} + \frac{6\Delta x^2 + 24\mu^2}{12\Delta x^4}\eta_{n-2} - \frac{20\Delta x^2 + 16\mu^2}{12\Delta x^4}\eta_{n-1} + \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_n \\
(\eta_n)_{xx} &+ \frac{\mu^2}{3}(\eta_n)_{xxxx} \\
&= \frac{11\eta_{n-4} - 56\eta_{n-3} + 114\eta_{n-2} - 104\eta_{n-1} + 35\eta_n}{12\Delta x^2} + \frac{\mu^2}{3}\frac{\eta_{n-4} - 4\eta_{n-3} + 6\eta_{n-2} - 4\eta_{n-1} + \eta_n}{\Delta x^4} \\
&= \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_{n-4} - \frac{56\Delta x^2 + 16\mu^2)}{12\Delta x^4}\eta_{n-3} + \frac{114\Delta x^2 + 24\mu^2}{12\Delta x^4}\eta_{n-2} - \frac{104\Delta x^2 + 16\mu^2}{12\Delta x^4}\eta_{n-1} + \frac{35\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_n.
\end{aligned}
$$

Then, the matrix becomes

$$
\begin{bmatrix}
(\eta_0)_{xx} + \frac{\mu^2}{3}(\eta_0)_{xxxx} \\
(\eta_1)_{xx} + \frac{\mu^2}{3}(\eta_1)_{xxxx} \\
(\eta_2)_{xx} + \frac{\mu^2}{3}(\eta_2)_{xxxx} \\
\vdots \\
(\eta_k)_{xx} + \frac{\mu^2}{3}(\eta_k)_{xxxx} \\
\vdots \\
(\eta_{n-2})_{xx} + \frac{\mu^2}{3}(\eta_{n-2})_{xxxx} \\
(\eta_{n-1})_{xx} + \frac{\mu^2}{3}(\eta_{n-1})_{xxxx} \\
(\eta_n)_{xx} + \frac{\mu^2}{3}(\eta_n)_{xxxx}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\frac{35\Delta x^2+4\mu^2}{12\Delta x^4} & -\frac{104\Delta x^2+16\mu^2}{12\Delta x^4} & \frac{114\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{56\Delta x^2+16\mu^2}{12\Delta x^4} & \frac{11\Delta x^2+4\mu^2}{12\Delta x^4} & 0 & 0 & 0 & \cdots & \\
\frac{11\Delta x^2+4\mu^2}{12\Delta x^4} & -\frac{20\Delta x^2+16\mu^2}{12\Delta x^4} & \frac{6\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{4\Delta x^2-16\mu^2}{12\Delta x^4} & \frac{-\Delta x^2+4\mu^2}{12\Delta x^4} & 0 & 0 & 0 & \cdots & \\
\frac{-\Delta x^2+4\mu^2}{12\Delta x^4} & \frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & \frac{-30\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & \frac{-\Delta x^2+4\mu^2}{12\Delta x^4} & 0 & 0 & 0 & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \\
0 & \cdots & 0 & \frac{-\Delta x^2+4\mu^2}{12\Delta x^4} & \frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & \frac{-30\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & \frac{-\Delta x^2+4\mu^2}{12\Delta x^4} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \\
\cdots & 0 & 0 & 0 & \frac{-\Delta x^2+4\mu^2}{12\Delta x^4} & \frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & \frac{-30\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & \frac{-\Delta x^2+4\mu^2}{12\Delta x^4} & \\
\cdots & 0 & 0 & 0 & -\frac{\Delta x^2+4\mu^2}{12\Delta x^4} & \frac{4\Delta x^2-16\mu^2)}{12\Delta x^4} & \frac{6\Delta x^2+24\mu^2}{12\Delta x^4} & \frac{-20\Delta x^2-16\mu^2}{12\Delta x^4} & \frac{11\Delta x^2+4\mu^2}{12\Delta x^4} & \\
\cdots & 0 & 0 & 0 & \frac{11\Delta x^2+4\mu^2}{12\Delta x^4} & -\frac{56\Delta x^2+16\mu^2)}{12\Delta x^4} & \frac{114\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{104\Delta x^2+16\mu^2}{12\Delta x^4} & \frac{35\Delta x^2+4\mu^2}{12\Delta x^4} &
\end{bmatrix}
\begin{bmatrix}
\eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_k \\ \vdots \\ \eta_{n-2} \\ \eta_{n-1} \\ \eta_n
\end{bmatrix}
$$

For simplicity, let $\mathcal{A}$ represent the above matrix. Now, recall the system:

$$\begin{aligned}
u(x_k, t_{j+1}) &= u(x_k, t_j) + \Delta t f_1(\eta(x_k, t_j), u(x_k, t_j), t),\\
\eta(x_k, t_{j+1}) &= \eta(x_k, t_j) + \Delta t u(x_k, t_j),
\end{aligned}$$

where

$$\begin{aligned}
f_1(\eta(x_k, t_j), u(x_k, t_j), t) &= \eta(x_k, t_j)_{xx} + \frac{\mu^2}{3}\eta(x_k, t_j)_{xxxx}\\
&+ \mu^2 \left(2u(x_k, t_j)_x \int_{x_0}^{x_k} u \,\mathrm{d}x' + 2u(x_k, t_j)^2 + \eta(x_k, t_j)_x^2 + \eta(x_k, t_j)\eta(x_k, t_j)_{xx}\right).
\end{aligned}$$

For convenience, let

$$B_k = \left(2u(x_k, t_j)_x \int_{x_0}^{x_k} u \,\mathrm{d}x' + 2u(x_k, t_j)^2 + \eta(x_k, t_j)_x^2 + \eta(x_k, t_j)\eta(x_k, t_j)_{xx}\right),$$

Let $\mathcal{B}$ represent the column vector of B_k's. Then, we can write the system

$$\begin{aligned}
u(x_0, t_{j+1}) &= u(x_0, t_j) + \Delta t f_1(\eta(x_0, t_j), u(x_0, t_j), t)\\
u(x_1, t_{j+1}) &= u(x_1, t_j) + \Delta t f_1(\eta(x_1, t_j), u(x_1, t_j), t)\\
u(x_2, t_{j+1}) &= u(x_2, t_j) + \Delta t f_1(\eta(x_2, t_j), u(x_2, t_j), t)\\
&\vdots\\
u(x_{n-2}, t_{j+1}) &= u(x_{n-2}, t_j) + \Delta t f_1(\eta(x_{n-2}, t_j), u(x_{n-2}, t_j), t)\\
u(x_{n-1}, t_{j+1}) &= u(x_{n-1}, t_j) + \Delta t f_1(\eta(x_{n-1}, t_j), u(x_{n-1}, t_j), t)\\
u(x_n, t_{j+1}) &= u(x_n, t_j) + \Delta t f_1(\eta(x_n, t_j), u(x_n, t_j), t)
\end{aligned}$$

as follows:

$$\begin{bmatrix} u(x_0, t_{j+1})\\ u(x_1, t_{j+1})\\ u(x_2, t_{j+1})\\ \vdots\\ u(x_{n-2}, t_{j+1})\\ u(x_{n-1}, t_{j+1})\\ u(x_n, t_{j+1}) \end{bmatrix} = \begin{bmatrix} u(x_0, t_j)\\ u(x_1, t_j)\\ u(x_2, t_j)\\ \vdots\\ u(x_{n-2}, t_j)\\ u(x_{n-1}, t_j)\\ u(x_n, t_j) \end{bmatrix} + \Delta t \begin{bmatrix} f_1(\eta(x_0, t_j), u(x_0, t_j), t)\\ f_1(\eta(x_1, t_j), u(x_1, t_j), t)\\ f_1(\eta(x_2, t_j), u(x_2, t_j), t)\\ \vdots\\ f_1(\eta(x_{n-2}, t_j), u(x_{n-2}, t_j), t)\\ f_1(\eta(x_{n-1}, t_j), u(x_{n-1}, t_j), t)\\ f_1(\eta(x_n, t_j), u(x_n, t_j), t) \end{bmatrix}$$

$$
= \begin{bmatrix} u(x_0, t_j) \\ u(x_1, t_j) \\ u(x_2, t_j) \\ \vdots \\ u(x_{n-2}, t_j) \\ u(x_{n-1}, t_j) \\ u(x_n, t_j) \end{bmatrix} + \Delta t \mathcal{A} \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-2} \\ \eta_{n-1} \\ \eta_n \end{bmatrix} + \Delta t \mathcal{B}
$$

Now, let us see how one would perform time-stepping. As such, impose initial conditions

$$
\eta(x, t_0) = f(x), \qquad u(x, t_0) = \eta_t(x, t_0) = g(x).
$$

Let $j = 0$, and for simplicity, pick $k \in [0, n]$. The system is

$$
\begin{aligned}
u(x_k, t_1) &= u(x_k, t_0) + \Delta t f_1(\eta(x_k, t_0), u(x_k, t_0)), \\
\eta(x_k, t_1) &= \eta(x_k, t_0) + \Delta t u(x_k, t_0),
\end{aligned}
$$

where

$$
\begin{aligned}
f_1(\eta(x_k, t_0), u(x_k, t_0)) &= \eta(x_k, t_0)_{xx} + \frac{\mu^2}{3}\eta(x_k, t_0)_{xxxx} \\
&+ \mu^2 \left(2u(x_k, t_0)_x \int_{x_0}^{x_k} u \, \mathrm{d}x' + 2u(x_k, t_0)^2 + \eta(x_k, t_0)_x^2 + \eta(x_k, t_0)\eta(x_k, t_0)_{xx} \right) \\
&= \frac{-\eta(x_{k-2}, t_0) + 16\eta(x_{k-1}, t_0) - 30\eta(x_k, t_0) + 16\eta(x_{k+1}, t_0) - \eta(x_{k+2}, t_0)}{12(\Delta x)^2} \\
&+ \frac{\mu^2}{3} \frac{\eta(x_{k-2}, t_0) - 4\eta(x_{k-1}, t_0) + 6\eta(x_k, t_0) - 4\eta(x_{k+1}, t_0) + \eta(x_{k+2}, t_0)}{(\Delta x)^4} \\
&+ \mu^2 \left(2u(x_k, t_0)_x \int_{x_0}^{x_k} u(x', t_0) \, \mathrm{d}x' + 2u(x_k, t_0)^2 + \eta(x_k, t_0)_x^2 + \eta(x_k, t_0)\eta(x_k, t_0)_{xx} \right)
\end{aligned}
$$

Note that all the terms on the last line can be computed via finite differences and both initial conditions. With this, we obtain the values of u, η at point x_k and time t_1. Performing this calculation for all k, we move on to compute u, η at time t_2, and so on.

2 Time stepping and finite differences: the half line

On the half line, the equation for the surface elevation is given by:

$$\eta_{tt} = \eta_{xx} + \mu^2 \left(\frac{1}{3}\eta_{xxxx} - \frac{1}{\pi} \int_0^\infty \partial_t \left(\eta \int_0^y \eta_t \, \mathrm{d}y' \right) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] \mathrm{d}y + \frac{1}{2} \partial_x^2 \left(\int_0^x \eta_t \, \mathrm{d}x' \right)^2 \right) \tag{4}$$

To do time stepping, introduce

$$u = \eta_t. \tag{5}$$

Also, note that

$$\frac{1}{2}\partial_x^2 \left(\int_0^x \eta_t \, \mathrm{d}x' \right)^2 = \eta_t^2 + \eta_{tx} \int_0^x \eta_t \, \mathrm{d}x' = u^2 + u_x \int_0^x u \, \mathrm{d}x',$$

and

$$\partial_t \left(\eta \int_0^y \eta_t \, \mathrm{d}y' \right) = \eta_t \int_0^x \eta_t \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)) = u \int_0^x u \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)).$$

Then, combining (5) and (4), we obtain a two-dimensional system:

$$\partial_t \begin{bmatrix} u \\ \eta \end{bmatrix} = \begin{bmatrix} \eta_{xx} + \mu^2 \left(\frac{1}{3}\eta_{xxxx} + u^2 + u_x \int_0^x u \, \mathrm{d}x' - \frac{1}{\pi} \int_0^\infty u \int_0^x u \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] \mathrm{d}y \right) \\ u \end{bmatrix}. \tag{6}$$

Now, consider (4) on a finite interval $[a, b]$, and let partition the interval into $n+1$ points $\{x_k\}_{k=0}^n$, with $x_0 = a$ and $x_n = b$. Note that we need to pick the partition such that

$$\int_0^\infty u \int_0^x u \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] \mathrm{d}y \approx \int_{x_0}^{x_n} u \int_0^x u \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] \mathrm{d}y.$$

We proceed to forward Euler time stepping. First, observe that

$$\begin{aligned} u_t(x_k, t_j) &= \frac{u(x_k, t_{j+1}) - u(x_k, t_j)}{\Delta t} = f_1(\eta, u, t) && \Longrightarrow u(x_k, t_{j+1}) = u(x_k, t_j) + \Delta t f_1(\eta, u, t) \\ \eta_t(x_k, t_j) &= \frac{\eta(x_k, t_{j+1}) - \eta(x_k, t_j)}{\Delta t} = f_2(\eta, u, t) && \Longrightarrow \eta(x_k, t_{j+1}) = \eta(x_k, t_j) + \Delta t f_2(\eta, u, t), \end{aligned}$$

where

$$f_1(\eta, u, t) = \eta_{xx} + \mu^2 \left(\frac{1}{3}\eta_{xxxx} + u^2 + u_x \int_0^x u \, \mathrm{d}x' - \frac{1}{\pi} \int_0^\infty u \int_0^x u \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] \mathrm{d}y \right)$$

$$f_2(\eta, u, t) = u(x_k, t_j).$$

Observe that the highest order derivative is 4, so we need to use five-point stencils. In particular, we use five point midpoint stencil for $x_2, \dots, x_{n-2}$, and five point, one-sided stencils at x_0, x_1, x_{n-1}, x_n. First, consider the system for $x_2, \dots, x_{n-2}$:

$$\begin{aligned} f_1(\eta(x_k, t_j), u(x_k, t_j), t) &= \eta(x_k, t_j)_{xx} + \frac{\mu^2}{3}\eta(x_k, t_j)_{xxxx} \\ &+ \mu^2 \left(u^2(x_k) + u_x(x_k) \int_0^{x_k} u \, \mathrm{d}x' - \frac{1}{\pi} \int_0^{x_n} u \int_0^y u \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] \mathrm{d}y \right), \end{aligned}$$

where we have separated the linear and nonlinear terms. The only difference between this system and the whole-line system is the non-linear term; in other words, we can reuse our prior work on the linear term, and only deal with the non-linear term. Let

$$C_k = u^2(x_k) + u_x(x_k) \int_0^{x_k} u \, \mathrm{d}x' - \frac{1}{\pi} \int_0^{x_n} \left(u \int_0^y u \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)) \right) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] \mathrm{d}y,$$

and let $\mathcal{C}$ be the column vector of C_k; thus, $\mathcal{C}$ represents the non-linear part of the system. To discretise C_k, note that

$$\int_0^{x_k} u \, \mathrm{d}x' = \frac{\Delta x}{2} \sum_{i=0}^{k-1} u(x_{i+1}) + u(x_i)$$

$$\begin{aligned} \int_0^{x_n} u \int_0^y u \, \mathrm{d}x' \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] \mathrm{d}y &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \left(u \int_0^y u \, \mathrm{d}x' \right) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] \mathrm{d}y \\ &= \frac{\Delta x}{2} \sum_{i=0}^{n-1} u(x_{i+1}) \int_0^{x_{i+1}} u \, \mathrm{d}x' \left[\frac{1}{(x_k - x_{i+1})^2} + \frac{1}{(x_k + x_{i+1})^2} \right] + u(x_i) \int_0^{x_i} u \, \mathrm{d}x' \left[\frac{1}{(x_k - x_i)^2} + \frac{1}{(x_k + x_i)^2} \right] \\ &= \frac{\Delta x}{2} \sum_{i=0}^{n-1} \frac{\Delta x}{2} u(x_{i+1}) \left(\sum_{j=0}^{i} u(x_j) + u(x_{j+1}) \right) \left[\frac{1}{(x_k - x_{i+1})^2} + \frac{1}{(x_k + x_{i+1})^2} \right] \\ &+ \frac{\Delta x}{2} u(x_i) \left(\sum_{j=0}^{i-1} u(x_j) + u(x_{j+1}) \right) \left[\frac{1}{(x_k - x_i)^2} + \frac{1}{(x_k + x_i)^2} \right] \end{aligned}$$

$$\begin{aligned} \int_0^{x_n} \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] \mathrm{d}y &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] \mathrm{d}y \\ &= \frac{\Delta x}{2} \sum_{i=0}^{n-1} \eta(x_{i+1})(\eta_x(x_{i+1}) - \eta_x(0)) \left[\frac{1}{(x_k - x_{i+1})^2} + \frac{1}{(x_k + x_{i+1})^2} \right] + \eta(x_i)(\eta_x(x_i) - \eta_x(0)) \left[\frac{1}{(x_k - x_i)^2} + \frac{1}{(x_k + x_i)^2} \right]. \end{aligned}$$

Therefore, we obtain that

$$
\begin{aligned}
C_k &= u^2(x_k) + u_x(x_k)\int_0^{x_k} u\,\mathrm{d}x' - \frac{1}{\pi}\int_0^{x_n}\left(u\int_0^{y} u\,\mathrm{d}x' + \eta(\eta_x - \eta_x(0))\right)\left[\frac{1}{(x_k-y)^2} + \frac{1}{(x_k+y)^2}\right]\mathrm{d}y \\
&= u^2(x_k) + \frac{\Delta x}{2}u_x(x_k)\sum_{i=0}^{k-1} u(x_{i+1}) + u(x_i) \\
&- \frac{1}{\pi}\frac{\Delta x}{2}\sum_{i=0}^{n-1}\left(\frac{\Delta x}{2}u(x_{i+1})\left(\sum_{j=0}^{i} u(x_j) + u(x_{j+1})\right) + \eta(x_{i+1})(\eta_x(x_{i+1}) - \eta_x(0))\right)\left[\frac{1}{(x_k - x_{i+1})^2} + \frac{1}{(x_k + x_{i+1})^2}\right] \\
&+ \left(\frac{\Delta x}{2}u(x_i)\left(\sum_{j=0}^{i-1} u(x_j) + u(x_{j+1})\right) + \eta(x_i)(\eta_x(x_i) - \eta_x(0))\right)\left[\frac{1}{(x_k - x_i)^2} + \frac{1}{(x_k + x_i)^2}\right],
\end{aligned}
$$

where the expressions for derivatives depend on the point x_k. Let

$$
F_i = \frac{\Delta x}{2}u(x_i)\left(\sum_{j=0}^{i-1} u(x_j) + u(x_{j+1})\right) + \eta(x_i)(\eta_x(x_i) - \eta_x(0)), \qquad i = 0, \dots n,
$$

which we will store as an array. This simplification, we obtain

$$
C_k = u^2(x_k) + \frac{\Delta x}{2}u_x(x_k)\sum_{i=0}^{k-1} u(x_{i+1}) + u(x_i) - \frac{\Delta x}{2\pi}\sum_{i=0}^{n-1} F_{i+1}\left[\frac{1}{(x_k - x_{i+1})^2} + \frac{1}{(x_k + x_{i+1})^2}\right] + F_i\left[\frac{1}{(x_k - x_i)^2} + \frac{1}{(x_k + x_i)^2}\right],
$$

With this in mind, we obtain the finite differences system:

$$
\begin{bmatrix} u(x_0, t_{j+1}) \\ u(x_1, t_{j+1}) \\ u(x_2, t_{j+1}) \\ \vdots \\ u(x_{n-2}, t_{j+1}) \\ u(x_{n-1}, t_{j+1}) \\ u(x_n, t_{j+1}) \end{bmatrix} = \begin{bmatrix} u(x_0, t_j) \\ u(x_1, t_j) \\ u(x_2, t_j) \\ \vdots \\ u(x_{n-2}, t_j) \\ u(x_{n-1}, t_j) \\ u(x_n, t_j) \end{bmatrix} + \Delta t \begin{bmatrix} f_1(\eta(x_0, t_j), u(x_0, t_j), t) \\ f_1(\eta(x_1, t_j), u(x_1, t_j), t) \\ f_1(\eta(x_2, t_j), u(x_2, t_j), t) \\ \vdots \\ f_1(\eta(x_{n-2}, t_j), u(x_{n-2}, t_j), t) \\ f_1(\eta(x_{n-1}, t_j), u(x_{n-1}, t_j), t) \\ f_1(\eta(x_n, t_j), u(x_n, t_j), t) \end{bmatrix}
$$

$$= \begin{bmatrix} u(x_0, t_j) \\ u(x_1, t_j) \\ u(x_2, t_j) \\ \vdots \\ u(x_{n-2}, t_j) \\ u(x_{n-1}, t_j) \\ u(x_n, t_j) \end{bmatrix} + \Delta t \mathcal{A} \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-2} \\ \eta_{n-1} \\ \eta_n \end{bmatrix} + \Delta t \mathcal{C}$$