Chapter 3

Perhaps make a comment in KdV derivation in previous section that getting to equation (2.18) requires a lot of work as all the B.C. must also be expanded.

Non-local derivation on the whole line

You might want to consider a transition that bridges the gap from the previous section to this. Hint at "we are looking for a more efficent way of deriving asy. models on diff. of domains.

Recall that the equations of fluid motion are challenging to work with directly, due to the nonlinear boundary conditions and the unknown domain. In addition, these complications lead to difficulties when attempting to deal with questions of existence and well-posedness. To address these issues, reformulations of the problem have been introduced: they result in equivalent problems that are more tractable. While such reformulations can be helpful, they may suffer from other issues. Below, we give a short overview of these formulations, along with explaining the pros and cons of each. Our main goal is to look for a reformulation in the water surface η, and that generalises easily to two-dimensional problem. We chose this criteria with a view towards applications: indeed, in applications, determining the water surface η is the main interest.

For example, for one-dimensional surfaces (no y variable), conformal mappings can be used to eliminate these problems (for an overview, see Dyachenko et al., 1996). However, this approach is limited to one-dimensional surfaces. For both one- and two-dimensional surfaces, other formulations (such as the Hamiltonian formulation given in Zakharov, 1968 or the Zakharov–Craig–Sulem formulation, Craig and Sulem, 1993) reduce the Euler equations to a system of two equations, in terms of

$q(x,t) = \Phi(x,\eta(x,t),t)$ $\eta(x,t)$

surface variables $q = \phi(x,\eta)$ and η only, by introducing a Dirichlet-to-Neumann operator (DNO). A new non-local formulation is introduced

by

in → ~~in~~ Ablowitz, Fokas, and Musslimani, 2006, (henceforth referred to as the AFM formulation) that results in a system of two equations for the same variables as in the DNO formulation. Both the DNO and AFM formulations reduce the problem from the full fluid domain to a system of equations that depend on the surface elevation $\eta(x,t)$ and the velocity potential evaluated at the surface, $q(x,t) = \phi(x,\eta)$. However, these formulations involve solving for ~~an additional function~~ $q(x,t)$, which may be of little relevance in applications, and hard to measure in experiments.

Indicate that we typically are interested in wave height.

A new formulation is introduced in Oliveras and Vasan, 2013, which reduces the water waves problem to [a system of two equations,] in one variable η. This formulation allows to rigorously investigate one- and two-dimensional water waves. The computation of Stokes-wave asymptotic expansions for periodic waves justifies the use of the formulation; indeed, following Oliveras and Vasan, 2013, the computations can be performed with arguably less effort, especially for two-dimensional waves. Our goal is to further justify the use of this formulation, which we call the $\mathcal{H}$ formulation.

this may be confusing to the reader who seeing the eqn. You might just say "formally eliminate $q(x,t)$ so the resulting equation only depend on η.

In this chapter, we first rewrite the water wave problem by introducing a normal-to-tangential operator. We then perform a perturbation expansion for the operator, and proceed to obtain an expression for the surface elevation. Finally, performing asymptotics and applying time scales yields the desired approximate equations. We emphasise that it is not our intention here to further the study of the water wave problem per se, but rather to demonstrate the efficacy of the $\mathcal{H}$ formulation for doing asymptotics.

3.1 Water-wave problem on the whole line: non-local formulation

Recall the water wave problem given in (1.1) along with

$$
\begin{aligned}
\phi_{xx} + \phi_{zz} &= 0 & -h < z < \eta(x,t) && (3.1a)\\
\phi_z &= 0 & z = -h && (3.1b)\\
\eta_t + \phi_x \eta_x &= \phi_z & z = \eta(x,t) && (3.1c)\\
\phi_t + g\eta + \frac{1}{2}(\phi_x^2 + \phi_z^2) &= 0 & z = \eta(x,t) && (3.1d)
\end{aligned}
$$

and consider the velocity potential evaluated at the surface:

$$q(x,t) = \phi(x,\eta(x,t)\cancel{,t}) \qquad \text{inline equation.}$$
$$= \phi(x,\eta(x,t),t)$$

We seek to reformulate the problem ~~(3.1)~~. (1.1) Combining (3.8c) and (3.1d), evaluated at $z = \eta$, we obtain

$$q_t + \frac{1}{2}q_x^2 + g\eta - \frac{1}{2}\frac{(\eta_t + q_x\eta_x)^2}{1+\eta_x^2} = 0, \tag{3.2}$$

which is an equation for two unknowns q, η. We ~~need~~ aim to find an equation in ~~one unknown~~ $\eta(x,t)$ only.

notation issue:

Given the domain $D = \mathbb{R} \times (-h, \eta)$, let

$$\vec{N} = \begin{bmatrix} -\eta_x \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{T} = \begin{bmatrix} 1 \\ \eta_x \end{bmatrix}$$

be vectors normal and tangent to the surface $\cancel{D}$ $z=\eta(x,t)$, respectively. We introduce an operator that maps the normal derivative at a surface η to the tangential derivative at the surface:

$$\mathcal{H}(\eta, D)\{\nabla\phi \cdot \vec{N}\} = \nabla\phi \cdot \vec{T}, \quad \checkmark \tag{3.3}$$

where $D = -i\cancel{\nabla}$ $-i\partial_x$ unmatching. For convenience, we drop the vector notation. Note that by (3.8c),

$$\nabla\phi \cdot N = \underbrace{\begin{bmatrix} \phi_x \\ \phi_z \end{bmatrix} \cdot \begin{bmatrix} -\eta_x \\ 1 \end{bmatrix}}_{*} = \phi_z - \phi_x\eta_x = \eta_t,$$

skip the matrix notation and make these all inline equation

and by chain rule,

$$\nabla\phi \cdot T = \overbrace{\begin{bmatrix} \phi_x \\ \phi_z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \eta_x \end{bmatrix}}^{*} = \phi_x + \eta_x\phi_z = q_x.$$

This allows us to rewrite (3.3) as

$$\mathcal{H}(\eta, D)\{\eta_t\} = q_{x\cancel{x}}. \tag{3.4}$$

~~and obtain a system~~ Together, (3.2) and (3.4) form a system of two eqns for the two unknowns η and q.

$$\cancel{q_t + \frac{1}{2}q_x^2 + g\eta - \frac{1}{2}\frac{(\eta_t + q_x\eta_x)^2}{1+\eta_x^2} = 0,}$$

$$\cancel{\mathcal{H}(\eta, D)\{\eta_t\} = q_x.}$$

eliminate

Differentiating (3.2) with respect to x and (3.4) with respect to t allows us to further reduce to a single equation for η

$$\cancel{\partial_t(q_x) + \partial_x\left(\frac{1}{2}q_x^2 + g\eta - \frac{1}{2}\frac{(\eta_t + q_x\eta_x)^2}{1+\eta_x^2}\right) = 0,} \tag{3.5}$$

$$\cancel{\partial_t(\mathcal{H}(\eta, D)\{\eta_t\}) = q_{xt}.} \tag{3.6}$$

~~Substituting (3.4) and (3.6) into (3.5), we obtain~~

$$\begin{aligned} &\partial_t\left(\mathcal{H}(\eta, D)\{\eta_t\}\right) \\ &\quad + \partial_x\left(\frac{1}{2}\left(\mathcal{H}(\eta, D)\{\eta_t\}\right)^2 + \varepsilon\eta - \frac{1}{2}\frac{(\eta_t + \eta_x\mathcal{H}(\eta, D)\{\eta_t\})^2}{1+\eta_x^2}\right) = 0. \end{aligned} \tag{3.7}$$

Equation (3.7) represents a scalar equation for the water wave surface η. The utility of (3.7) depends on whether we can find a useful representation for the operator $\mathcal{H}(\eta, D)$. In the next section, we proceed to find an equation that the $\mathcal{H}$ operator must satisfy.

try to find a transition to this section.

3.2 Behaviour of the $\mathcal{H}$ operator

Consider the following boundary value problem:

$$\phi_{xx} + \phi_{zz} = 0 \qquad -h < z < \eta(x,t) \tag{3.8a}$$

$$\phi_z = 0 \qquad z = -h \tag{3.8b}$$

$$\nabla\phi \cdot N = f(x) \qquad z = \eta(x,t) \tag{3.8c}$$

same D as earlier

Let φ be harmonic on D. Using (3.8a) and that φ_z is also harmonic on D, we have

$$\varphi_z(\phi_{xx} + \phi_{zz}) - \phi((\varphi_z)_{zz} + (\varphi_z)_{xx}) = 0.$$

Taking the integral over the domain yields

$$\int_{-\infty}^{\infty}\int_{-h}^{\eta(x)} \varphi_z(\phi_{xx} + \phi_{zz}) - \phi((\varphi_z)_{zz} + (\varphi_z)_{xx})\, \mathrm{d}z\, \mathrm{d}x = 0.$$

An application of Green's theorem gives

$$\int_{\partial D} \varphi_z(\nabla\phi \cdot \mathbf{n}) - \phi(\nabla\varphi_z \cdot \mathbf{n})\, \mathrm{d}s = 0, \tag{3.9}$$

where ∂D is the boundary of the domain, $\mathrm{d}s$ is the area element, and $\mathbf{n}$ is the normal vector. Now, observe that

$$-\nabla\varphi_z \cdot \mathbf{n} = \nabla\varphi_x \cdot \mathbf{t},$$

where **t** is the tangential vector. We use this to rewrite (3.9) and obtain the following contour integral:

$$
\begin{aligned}
0 &= \int_{\partial D} \varphi_z(\nabla\phi \cdot N) + \phi(\nabla\varphi_z \cdot \mathbf{t})\,\mathrm{d}s \\
&= \int_{\partial D} \varphi_z(\phi_z\,\mathrm{d}x - \phi_x\,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z)
\end{aligned}
$$

We split the contour into four segments:

people might not like this notation.

I typically use $\oint_{\partial D} (\cdot)\,ds = \int_{-\infty}^{\infty} (\cdot)\Big|_{z=-h} ds + \ldots$

$$
\begin{aligned}
\int_{\partial D} &= \int_{-\infty}^{\infty}\Big|^{z=-h} + \int_{-h}^{\eta(x)}\Big|^{x\to\infty} + \int_{\infty}^{-\infty}\Big|^{z=\eta(x)} + \int_{\eta(x)}^{-h}\Big|^{x\to-\infty} \\
&= \int_{-\infty}^{\infty}\Big|^{z=-h} + \int_{-h}^{\eta(x)}\Big|^{x\to\infty} - \int_{-\infty}^{\infty}\Big|^{z=\eta(x)} - \int_{-h}^{\eta(x)}\Big|^{x\to-\infty}.
\end{aligned}
$$

Ask Dave if he has a preference.

Consider each segment:

sketch the domain and consider a formal principle value $\lim_{R\to\infty} \int_{-R}^{R} \cdots \, d\cdots$.

- As $|x| \to \infty$, we know that ϕ and its gradient vanish, so the integrals

$$
\int_{-h}^{\eta(x)}\Big|^{x\to\infty}, \int_{-h}^{\eta(x)}\Big|^{x\to-\infty}
$$

ok. definitely don't use this here.

vanish.

You could simple state "there are no contributions as $|x|\to\infty$".

you may be able to skip this section and simply jump to equation (3.10) where you should note that $\mathcal{H}(\eta, D)\{f(x)\}$ is defined as $\nabla\phi \cdot T = \mathcal{H}(\eta, D)\{f(x)\}$

- At $z = -h$, $\mathrm{d}z = 0$, so we have

$$
\begin{aligned}
\int_{-\infty}^{\infty} &\varphi_z(\phi_z\,\mathrm{d}x - \phi_x\,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) \\
&= \int_{-\infty}^{\infty} \phi\varphi_{xx}\,\mathrm{d}x \qquad (\text{since } \phi_z = 0 \text{ at } z = -h) \\
&= \phi(x,-h)\varphi_x(x,-h)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \phi_x(x,-h)\varphi_x(x,-h)\,\mathrm{d}x \\
&= 0,
\end{aligned}
$$

where we pick φ such that $\varphi_x(x,-h) = 0$.

will resume tomorrow.

- At $z = \eta$, $\mathrm{d}z = \eta_x\,\mathrm{d}x$, so we have

$$
\begin{aligned}
\int_{-\infty}^{\infty} &\varphi_z(\phi_z - \phi_x\eta_x) + \phi(\varphi_{xx} + \varphi_{xz}\eta_x)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \varphi_z(\phi_z - \phi_x\eta_x) + \phi\frac{\mathrm{d}\varphi_x(x,\eta)}{\mathrm{d}x}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \varphi_z(\phi_z - \phi_x\eta_x) - \varphi_x\frac{\mathrm{d}\phi(x,\eta)}{\mathrm{d}x}\,\mathrm{d}x \qquad \text{(integration by parts)} \\
&= \int_{-\infty}^{\infty} \varphi_z\begin{pmatrix}\phi_x\\ \phi_z\end{pmatrix}\cdot\begin{pmatrix}-\eta_x\\ 1\end{pmatrix} - \varphi_x\begin{pmatrix}\phi_x\\ \phi_z\end{pmatrix}\cdot\begin{pmatrix}1\\ \phi_x\eta_x\end{pmatrix}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \varphi_z\nabla\phi\cdot N - \varphi_x\nabla\phi\cdot T\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \varphi_z f(x) - \varphi_x(x,\eta)\mathcal{H}(\eta,D)\{f(x)\}\,\mathrm{d}x \qquad \text{(recall (3.8c) and (3.3)).}
\end{aligned}
$$

Combining segments, we obtain:

skip to here where you indicate that $\mathcal{H}(\eta,D)\{f(x)\} = \nabla\phi\cdot T$

$$\int_{-\infty}^{\infty} \varphi_z f(x) - \varphi_x(x,\eta)\mathcal{H}(\eta,D)\{f(x)\}\,\mathrm{d}x = 0. \tag{3.10}$$

Note that $\varphi(x,z) = e^{-ikx}\sinh(k(z+h)), k \in \mathbb{R}$ is one solution of

$$\Delta\varphi = 0, \qquad \varphi_z(-h,z) = 0.$$

Then, (3.10) becomes

$$\int_{-\infty}^{\infty} e^{-ikx}(k\cosh(k(\eta+h))f(x) + ik\sinh(k(\eta+h))\mathcal{H}(\eta,D)\{f(x)\})\,\mathrm{d}x = 0.$$

It can be shown that we can take out k in the integral, so that the below holds for all $k \in \mathbb{R}$:

$$\int_{-\infty}^{\infty} e^{-ikx}(i\cosh(k(\eta+h))f(x) - \sinh(k(\eta+h))\mathcal{H}(\eta,D)\{f(x)\})\,\mathrm{d}x = 0. \tag{3.11}$$

* see footnote

The equation (3.11) gives a description of how the operator $\mathcal{H}(\eta, D)$ behaves, in dimensional coordinates.

In summary, introduction of the normal-to-tangential operator $\mathcal{H}(\eta, D)$ allows to reduce the water waves problem (3.1) to a scalar equation for η :

$$\partial_t\left(\mathcal{H}(\eta,D)\{\eta_t\}\right) + \partial_x\left(\frac{1}{2}\left(\mathcal{H}(\eta,D)\{\eta_t\}\right)^2 + \varepsilon\eta - \frac{1}{2}\frac{(\eta_t + \eta_x\mathcal{H}(\eta,D)\{\eta_t\})^2}{1+\eta_x^2}\right) = 0, \tag{3.12}$$

where the operator H is described via

$$\int_{-\infty}^{\infty} e^{-ikx}(i\cosh(k(\eta+h))f(x) - \sinh(k(\eta+h))\mathcal{H}(\eta,D)\{f(x)\})\,\mathrm{d}x = 0, \qquad k \in \mathbb{R}. \tag{3.13}$$

In doing so, we go from solving for two unknowns ϕ and η to solving for one unknown η. Now, the utility of (3.12) depends on whether we can find a useful representation of the operator $\mathcal{H}(\eta, D)$.

3.3 ~~Perturbation expansion~~ of the $\mathcal{H}$ operator

ok

In this section, we derive a perturbation series of the $\mathcal{H}$ operator. First, observe that (3.13) is written in dimensional coordinates. To obtain the

* you should include a remark that states that "technically" this is only valid for $k \neq 0$. However, for the case where $f(x) = \eta_t$ as in the water wave problem, then in the limit as $k \to 0$, (3.11) reduces to $\int_{-\infty}^{\infty} \eta_t \, dx = 0$ which is known to be true as it is conservation of mass. Cite Benjamin & Olver JFM paper from 80's

non-dimensional version~~, introduce new variables~~: we

$$t^\star = \frac{t\sqrt{gh}}{L}, \qquad x^\star = \frac{x}{L}, \qquad z^\star = \frac{z}{h}, \qquad \eta^\star = \frac{\eta}{a}, \qquad k^\star = Lk,$$

indicate that this is the same scaling used in (2.15) where $\lambda = \frac{1}{L}$.

skip

rescale functions via

$$\phi = \frac{Lga}{\sqrt{gh}}\phi^\star, \qquad q^\star = \frac{\sqrt{gh}}{agL}q,$$

and define parameters ε and μ so that

$$\varepsilon = \frac{a}{h}, \qquad \mu = \frac{h}{L}, \qquad \varepsilon\mu = \frac{a}{L}.$$

Starting with

$$\int_{\partial D} \varphi_z(\phi_z \, \mathrm{d}x - \phi_x \, \mathrm{d}z) + \phi(\varphi_{xx} \, \mathrm{d}x + \varphi_{xz} \, \mathrm{d}z) = 0,$$

one may continue the same procedure to obtain the nondimensional ver-
pick up here
sion of (3.11):

$$\int_{-\infty}^{\infty} e^{-ikx}(i\cosh(\mu k(\eta+1))g(x) - \sinh(\mu k(\eta+1))\mathcal{H}(\varepsilon\eta, D)\{g(x)\})\, \mathrm{d}x = 0, \tag{3.14}$$

where $k \in \mathbb{R}$ and we dropped the starred notation for convenience. In addition, note that g and f are related by

$$g(x^\star) = \cancel{\frac{\sqrt{gh}}{ga}g(x^\star L)} = \frac{\sqrt{gh}}{ga}f(x).$$

$$g(x^\star) = \frac{\sqrt{gh}}{ga}f(x)$$

just state this

Make an important point that (3.14) gives an implicit relationship that can be solved for $\mathcal{H}(\eta, D)\{f(x)\}$ as explored in the next section

New section. Solving for $\mathcal{H}(\eta, D)\{f(x)\}$

Short intro ¶ stating that following Craig & Sulem, you will find a formal series expansion for $\mathcal{H}(\eta, D)\{f(x)\}$

Since $\varepsilon \ll 1$, we expand the hyperbolic functions as a Taylor series in ε :

remind that $\varepsilon \ll 1$.

$$\cosh(\mu k(\eta + 1)) = \cosh(\mu k) + \mu k \varepsilon \eta \sinh(\mu k) + \mathcal{O}(\varepsilon^2),$$

$$\sinh(\mu k(\eta + 1)) = \sinh(\mu k) + \mu k \varepsilon \eta \cosh(\mu k) + \mathcal{O}(\varepsilon^2).$$

Now, we note that the idea of a regular perturbation series applies not only to classical functions but also to operators. Therefore, we expand

"formal expan" this is important

$$\mathcal{H}(\eta, D)\{g(x)\} = \left[\mathcal{H}_0 + \varepsilon \mathcal{H}_1 + \mathcal{O}(\varepsilon^2)\right] (\varepsilon\eta, D)\{g(x)\}.$$

Re read Craig/Sulem for some terminology / defining this series.

Equation (3.14) becomes:

$$\int_{-\infty}^{\infty} e^{-ikx}\Big(i \left[\cosh(\mu k) + \mu k \varepsilon \eta \sinh(\mu k) + \mathcal{O}(\varepsilon^2)\right] g(x) \\ - \left[\sinh(\mu k) + \mu k \varepsilon \eta \cosh(\mu k) + \mathcal{O}(\varepsilon^2)\right] \left[\mathcal{H}_0 + \varepsilon \mathcal{H}_1 + \mathcal{O}(\varepsilon^2)\right] (\varepsilon\eta, D)\{g(x)\}\Big) \, \mathrm{d}x = 0. \tag{3.15}$$

At leading order ~~**Within**~~ $\mathcal{O}(\varepsilon^0)$: Using (3.15), we obtain

$$\int_{-\infty}^{\infty} e^{-ikx}(i \cosh(\mu k) g(x) - \sinh(\mu k) \mathcal{H}_0(\varepsilon\eta, D)\{g(x)\}) \, \mathrm{d}x = 0.$$

Dividing by $\sinh(\mu k)$ yields

make a statement that this is only valid for $k \neq 0$.

$$\int_{-\infty}^{\infty} e^{-ikx}(i \coth(\mu k) g(x) - \mathcal{H}_0(\varepsilon\eta, D)\{g(x)\}) \, \mathrm{d}x = 0.$$

Splitting the integrand and recognizing the Fourier transform yields:

$$
\begin{aligned}
\mathcal{F}_k\{\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\}\} &= \int_{-\infty}^{\infty} e^{-ikx}\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} e^{-ikx} i\coth(\mu k) g(x)\,\mathrm{d}x \\
&= i\coth(\mu k)\mathcal{F}_k\{g(x)\}.
\end{aligned}
$$

faster than $\mathcal{O}(\mu k)$

Provided that $\mathcal{F}_k\{g(x)\} \to 0$, ~~faster than~~ ← then this is defined for all k.

Finally, we invert Fourier transform to obtain

$$
\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\} = \mathcal{F}_k^{-1}\{i\coth(\mu k)\mathcal{F}_k\{g(x)\}\}, \tag{3.16}
$$

where we write out k's explicitly to keep track of transforms.

Within $\mathcal{O}(\varepsilon^1)$: from (3.15), we obtain

$$
\int_{-\infty}^{\infty} e^{-ikx}(i\mu k\eta\sinh(\mu k)g(x) - [\sinh(\mu k)\mathcal{H}_1 + \mu k\eta\cosh(\mu k)\mathcal{H}_0]\,(\varepsilon\eta, D)\{g(x)\})\,\mathrm{d}x = 0.
$$

Dividing by sinh(μk) and dropping $(\varepsilon\eta, D)$ for the notational convenience, we have

↳ again $k \neq 0$...

$$
\int_{-\infty}^{\infty} e^{-ikx}(i\mu k\eta g(x) - [\mathcal{H}_1 + \mu k\eta\coth(\mu k)\mathcal{H}_0]\,\{g(x)\})\,\mathrm{d}x = 0.
$$

Splitting the integral and recognising the Fourier transform yields:

$$
\begin{aligned}
\mathcal{F}_k\{\mathcal{H}_1\{g(x)\})\} &= \int_{-\infty}^{\infty} e^{-ikx}\mathcal{H}_1\{g(x)\})\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} e^{-ikx}(i\mu k\eta g - \mu k\eta\coth(\mu k)\mathcal{H}_0\{g(x)\})\,\mathrm{d}x \\
&= \mu\mathcal{F}_k\{ik\eta g\} - \mu k\coth(\mu k)\mathcal{F}_k\{\eta\mathcal{H}_0\{g(x)\}\})\}.
\end{aligned}
$$

Inverting Fourier transform and using (3.16), we obtain an expression for $\mathcal{H}_1$:

$$
\begin{aligned}
\mathcal{H}_1\{g(x)\} &= \mathcal{F}_k^{-1}\{\mu\eta g\} - \mathcal{F}_k^{-1}\{\mu k \coth(\mu k)\mathcal{F}_k\{\eta\mathcal{H}_0\{g(x)\}\}\} \\
&= \mu\partial_x(\eta g) - \mathcal{F}_k^{-1}\{\mu k \coth(\mu k)\mathcal{F}_k\{\eta\mathcal{H}_0\{g(x)\}\}\} \\
&= \mu\partial_x(\eta g) - \mathcal{F}_k^{-1}\{\mu k \coth(\mu k)\mathcal{F}_k\{\eta\mathcal{F}_l^{-1}\{i\coth(\mu l)\mathcal{F}_l\{g\}\}\}\}.
\end{aligned}
$$

In sum, we find a representation for the $\mathcal{H}$ operator within two orders:

$$\mathcal{H}(\varepsilon\eta, D)\{g(x)\} = [\mathcal{H}_0 + \varepsilon\mathcal{H}_1](\varepsilon\eta, D)\{g(x)\} + \mathcal{O}(\varepsilon^2),$$

where

$$
\begin{aligned}
\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\} &= \mathcal{F}_k^{-1}\{i\coth(\mu k)\mathcal{F}_k\{g(x)\}\}, \\
\mathcal{H}_1(\varepsilon\eta, D)\{g(x)\} &= \mu\partial_x(\eta g) - \mathcal{F}_k^{-1}\{\mu k \coth(\mu k)\mathcal{F}_k\{\eta\mathcal{F}_l^{-1}\{i\coth(\mu l)\mathcal{F}_l\{g\}\}\}\}.
\end{aligned}
$$

Remark 3. As we proceed to use the operator $\mathcal{H}$, we must exercise caution. Recall the expression for $\mathcal{H}_0$:

$$\mathcal{H}_0(\varepsilon\eta, D)\{g(x)\} = \mathcal{F}_k^{-1}\{i\coth(\mu k)\mathcal{F}_k\{g(x)\}\}.$$

Expanding $\coth(\mu k)$ via its Laurent series in μk gives

$$\coth(\mu k) = \frac{1}{\mu k} + \frac{\mu k}{3} + \mathcal{O}(\mu^3).$$

It is readily seen that since $\coth(\mu k)$ has a simple pole as $k \to 0$, so do $\mathcal{H}_0, \mathcal{H}_1$ and $\mathcal{H}$.

$\hookrightarrow$ good. tie this back to needing $\mathcal{F}_k\{g(x)\} \to 0$ as $k \to 0$ at a rate such that $\mathcal{F}_k\{\mathcal{H}_i\{g(x)\}\} \Rightarrow 0$ as $k \to 0$.

you show all the derivation for the $\mathcal{H}$ formulation but not ~~velocity~~ velocity potential.
You should comment
① why you made this choice
② how the compare.

3.4 Deriving an expression for surface elevation

We proceed to derive approximate equations for the surface η. Recall the scalar equation (3.7):

$$\partial_t \left(\mathcal{H}(\eta, D)\{\eta_t\}\right) + \partial_x \left(\frac{1}{2}\left(\mathcal{H}(\eta, D)\{\eta_t\}\right)^2 + \varepsilon\eta - \frac{1}{2}\frac{(\eta_t + \eta_x \mathcal{H}(\eta, D)\{\eta_t\})^2}{1+\eta_x^2}\right) = 0. \tag{3.17}$$

The non-dimensional version of (3.17) is given by

$$\partial_t \left(\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right) + \partial_x \left(\frac{1}{2}\Big(\mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\Big)^2 + \varepsilon\eta - \frac{1}{2}\varepsilon^2\mu^2\frac{(\eta_t + \eta_x \mathcal{H}(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\})^2}{1+\varepsilon^2\mu^2\eta_x^2}\right) = 0. \tag{3.18}$$

Also, recall our maximal balance assumption $\varepsilon = \mu^2$, and recall the first-order and second order expansions for $\coth(\mu k)$:

$$\coth(\mu k) = \frac{1}{\mu k} + \mathcal{O}(\mu) = \frac{1}{\mu k} + \frac{\mu k}{3} + \mathcal{O}(\mu^3).$$

Within $\mathcal{O}(\mu^0)$: In the leading order, the equation (3.18) becomes

$$\partial_t \left(\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right) + \varepsilon\partial_x\eta = 0.$$

Bringing ∂_t inside $\mathcal{H}_0$ and substituting the expression for $\mathcal{H}_0$, we obtain:

$$\mathcal{F}_k^{-1}\{i\coth(\mu k)\mathcal{F}_k\{\varepsilon\mu\eta_{tt}\}\} + \varepsilon\partial_x\eta = 0.$$

Inverting the Fourier transform and multiplying by $k/(i\varepsilon)$ yields

$$\mu k\coth(\mu k)\widehat{\eta_{tt}}_k + k^2\widehat{\eta}_k = 0.$$

Expanding $\coth(\mu k)$ in the leading order gives

$$\widehat{\eta_{tt}}_k + k^2\widehat{\eta}_k = 0.$$

Inverting the Fourier transform, we have

$$\eta_{tt} + (-i\partial_x)^2\eta = 0,$$

which is

$$\eta_{tt} - \eta_{xx} = 0.$$

This is the wave equation, as we desired.

Within $\mathcal{O}(\mu^2)$: The non-dimensional equation (3.18) becomes

$$\partial_t\left(\mathcal{H}_0\{\varepsilon\mu\eta_t\} + \varepsilon\mathcal{H}_1\{\varepsilon\mu\eta_t\}\right) + \partial_x\left(\frac{1}{2}(\mathcal{H}_0\{\varepsilon\mu\eta_t\})^2 + \varepsilon\eta\right) = 0. \quad (3.19)$$

Recall

$$\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\} = \varepsilon\mu\mathcal{F}_k^{-1}\{i\coth(\mu k)\widehat{\eta_t}_k\};$$
$$\mathcal{H}_1(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\} = \varepsilon\mu^2(\eta\eta_t)_x - \varepsilon\mathcal{F}_k^{-1}\{\mu k\coth(\mu k)\mathcal{F}_k\{\eta\mathcal{F}_l^{-1}\{i\mu\coth(\mu l)\widehat{\eta_t}_l\}\}\}.$$

Observe that

$$\frac{1}{2}\left(\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 = \frac{1}{2}\left(\mathcal{H}_0(\varepsilon\eta, D)\{\varepsilon\mu\eta_t\}\right)^2 = \frac{\varepsilon^2}{2}\left(\mathcal{F}_k^{-1}\{i\mu\coth(\mu j)\widehat{\eta_t}_k\}\right)^2,$$

and

$$\partial_t\Big(\Big[\mathcal{H}_0(\varepsilon\eta, D) + \varepsilon\mathcal{H}_1(\varepsilon\eta, D)\Big]\{\varepsilon\mu\eta_t\}\Big)$$
$$= \varepsilon\mu\mathcal{F}_k^{-1}\{i\coth(\mu k)\widehat{\eta_{tt}}_k\} + \varepsilon^2\mu^2(\eta\eta_t)_{tx}$$
$$- \varepsilon^2\mathcal{F}_k^{-1}\{\mu k\coth(\mu k)\mathcal{F}_k\{\partial_t\left[\eta\mathcal{F}_l^{-1}\{i\mu\coth(\mu l)\widehat{\eta_t}_l\}\right]\}\}.$$

The single equation (3.19) becomes

$$\varepsilon\mu\mathcal{F}_k^{-1}\{i\coth(\mu k)\widehat{\eta_{tt}}_k\} + \varepsilon^2\mu^2(\eta\eta_t)_{tx}$$
$$- \varepsilon^2\mathcal{F}_k^{-1}\{\mu k\coth(\mu k)\mathcal{F}_k\{\partial_t\left[\eta\mathcal{F}_l^{-1}\{i\mu\coth(\mu l)\widehat{\eta_t}_l\}\right]\}\}$$
$$+ \frac{\varepsilon^2}{2}\partial_x\left(\mathcal{F}_j^{-1}\{i\mu\coth(\mu j)\widehat{\eta_t}_j\}\right)^2 + \varepsilon\partial_x\eta = 0.$$

Application of Fourier transform yields

$$\varepsilon\mu i\coth(\mu k)\widehat{\eta_{tt}}_k + \varepsilon^2\mu^2 ik(\eta\eta_t)_t - \varepsilon^2\mu k\coth(\mu k)\mathcal{F}_k\{\partial_t\left[\eta\mathcal{F}_l^{-1}\{i\mu\coth(\mu l)\widehat{\eta_t}_l\}\right]\}$$
$$+ \frac{\varepsilon^2}{2}ik\mathcal{F}_k\{\left(\mathcal{F}_j^{-1}\{i\mu\coth(\mu j)\widehat{\eta_t}_j\}\right)^2\} + \varepsilon ik\widehat{\eta}_k = 0.$$

Divide by $i\varepsilon$ and recall $\varepsilon = \mu^2$ to obtain

$$\mu\coth(\mu k)\widehat{\eta_{tt}}_k + \mu^4 k(\eta\eta_t)_t - \mu^3 k\coth(\mu k)\mathcal{F}\{\partial_t\left[\eta\mathcal{F}^{-1}\{\mu\coth(\mu l)\widehat{\eta_t}_l\}_l\right]\}_k$$
$$+ \frac{\mu^2}{2}k\mathcal{F}\{\left(\mathcal{F}^{-1}\{i\mu\coth(\mu j)\widehat{\eta_t}_j\}_j\right)^2\}_k + k\widehat{\eta}_k = 0.$$

Expanding $\coth(\mu k)$-like terms yields

$$\left(\frac{1}{k}+\frac{\mu^2 k}{3}\right)\widehat{\eta_{tt}}_k+\mu^4 k(\eta\eta_t)_t-\mu^2 k\left(\frac{1}{k}+\frac{\mu^2 k}{3}\right)\mathcal{F}_k\{\partial_t\left[\eta\mathcal{F}^{-1}\{\left(\frac{1}{l}+\frac{\mu^2 l}{3}\right)\widehat{\eta_t}_l\}_l\right]\}$$
$$-\frac{\mu^2}{2}k\mathcal{F}_k\{\left(\mathcal{F}_j^{-1}\{\left(\frac{1}{j}+\frac{\mu^2 j}{3}\right)\widehat{\eta_t}_j\}\right)^2\}+k\widehat{\eta}_k=0. \tag{3.20}$$

Within $\mathcal{O}(\mu^2)$, the equation (3.20) becomes

$$\left(\frac{1}{k}+\frac{\mu^2 k}{3}\right)\widehat{\eta_{tt}}_k-\mu^2\mathcal{F}_k\{\partial_t\left[\eta\mathcal{F}_l^{-1}\{\frac{1}{l}\widehat{\eta_t}_l\}\right]\}-\frac{\mu^2}{2}k\mathcal{F}_k\{\left(\mathcal{F}_j^{-1}\{\frac{1}{j}\widehat{\eta_t}_j\}\right)^2\}+k\widehat{\eta}_k=0,$$

or rearranging and multiplying by k, we have

$$\widehat{\eta_{tt}}_k+k^2\widehat{\eta}_k+\mu^2\left(\frac{k^2}{3}\widehat{\eta_{tt}}_k-k\mathcal{F}_k\{\partial_t\left[\eta\mathcal{F}_l^{-1}\{\frac{1}{l}\widehat{\eta_t}_l\}\right]\}-\frac{1}{2}k^2\mathcal{F}_k\{\left(\mathcal{F}_j^{-1}\{\frac{1}{j}\widehat{\eta_t}_j\}\right)^2\}\right)=0.$$

Finally, inverting the Fourier transform yields:

$$\eta_{tt}-\eta_{xx}+\mu^2\left(-\frac{\partial_x^2}{3}\eta_{tt}+i\partial_x\left(\partial_t\left[\eta\mathcal{F}_l^{-1}\{\frac{1}{l}\widehat{\eta_t}_l\}\right]\right)+\frac{1}{2}\partial_x^2\left(\mathcal{F}_j^{-1}\{\frac{1}{j}\widehat{\eta_t}_j\}\right)^2\right)=0,$$

or more conveniently,

$$\eta_{tt}-\eta_{xx}=\mu^2\left(\frac{\partial_x^2}{3}\eta_{tt}-i\partial_x\left(\partial_t\left[\eta\mathcal{F}_l^{-1}\{\frac{1}{l}\widehat{\eta_t}_l\}\right]\right)-\frac{1}{2}\partial_x^2\left(\mathcal{F}_j^{-1}\{\frac{1}{j}\widehat{\eta_t}_j\}\right)^2\right). \tag{3.21}$$

We seek to simplify (3.21). First, integration by parts gives

$$
\begin{aligned}
\frac{1}{l}\widehat{\eta_t}_l &= \frac{1}{l}\frac{2}{\pi}\int_{-\infty}^{\infty} e^{-ilx}\eta_t \,\mathrm{d}x \\
&= \frac{1}{l}\frac{2}{\pi}e^{-ilx}\int_{-\infty}^{x}\eta_t(x',t)\,\mathrm{d}x'\Big|_{-\infty}^{\infty} + i\frac{2}{\pi}\int_{-\infty}^{\infty} e^{-ilx}\int_{-\infty}^{x}\eta_t(x',t)\,\mathrm{d}x'\,\mathrm{d}x \\
&= i\frac{2}{\pi}\int_{-\infty}^{\infty} e^{-ilx}\int_{-\infty}^{x}\eta_t(x',t)\,\mathrm{d}x'\,\mathrm{d}x \\
&= i\mathcal{F}_l\{\int_{-\infty}^{x}\eta_t(x',t)\,\mathrm{d}x'\},
\end{aligned}
$$

so that

$$
\mathcal{F}_l^{-1}\{\frac{1}{l}\widehat{\eta_t}_l\} = \mathcal{F}_l^{-1}\{i\mathcal{F}_l\{\int_{-\infty}^{x}\eta_t(x',t)\,\mathrm{d}x'\}\} = i\int_{-\infty}^{x}\eta_t(x',t)\,\mathrm{d}x', \tag{3.22}
$$

where we applied the Fourier inversion theorem. Using (3.22) and that $\eta_{tt} = \eta_{xx} + \mathcal{O}(\mu^2)$, the equation (3.21) becomes

$$
\begin{aligned}
\eta_{tt} - \eta_{xx} &= \mu^2\left(\frac{1}{3}\eta_{xxxx} + \partial_x\partial_t\left[\eta\left(\int_{-\infty}^{x}\eta_t\,\mathrm{d}x'\right)\right] + \frac{1}{2}\partial_x^2\left(\left(\int_{-\infty}^{x}\eta_t\,\mathrm{d}x'\right)^2\right)\right) \\
&= \mu^2\left(\frac{1}{3}\eta_{xxxx} + \partial_x\left[\eta_t\left(\int_{-\infty}^{x}\eta_t\,\mathrm{d}x'\right) + \eta\eta_x\right] + \frac{1}{2}\partial_x^2\left(\left(\int_{-\infty}^{x}\eta_t\,\mathrm{d}x'\right)^2\right)\right) \\
&= \varepsilon\left[\frac{1}{3}\eta_{xxxx} + \partial_x^2\left(\frac{\eta^2}{2} + \left(\int_{-\infty}^{x}\eta_t\,\mathrm{d}x'\right)^2\right)\right]
\end{aligned} \tag{3.23}
$$

In summary, the second order approximation of a scalar equation for η resulted into equation (3.23).

Remark 4. At the end of Section 2.4, we mentioned secular terms in the next order. We now show directly by examining the dispersion relation (3.23). Assume a plane wave solution of the form $\tilde{\eta}(x,t) = \exp(i(kx -$

$\omega t)$). Substituting $\tilde{\eta}$ into the linearised equation

$$\tilde{\eta}_{tt} - \tilde{\eta}_{xx} = \varepsilon \frac{1}{3} \tilde{\eta}_{xxxx},$$

leads to the following relation

$$-\omega^2 + k^2 = \varepsilon \frac{k^4}{3}.$$

$\omega = \sqrt{k^2 - \varepsilon \frac{k^4}{3}}$

$e^{|k|}$

Substituting the negative root of ω into the wave solution gives

$$\eta(x,t) \approx \exp(ikx) \exp\left(\sqrt{\frac{\varepsilon}{3}} k^2 t\right). \; .$$

??

As is seen, this solution is unbounded in time as $k \to \infty$. This phenomenon suggests that (3.33) contains secularity and therefore warrants an application of time scales.

reference the equation?

3.5 Derivation of wave and KdV equations

We derive the approximate equations from

$$\eta_{tt} - \eta_{xx} = \varepsilon \left[\frac{1}{3} \eta_{xxxx} + \partial_x^2 \left(\frac{\eta^2}{2} + \left(\int_{-\infty}^{x} \eta_t \, dx' \right)^2 \right) \right]. \tag{3.24}$$

We assume an expansion of η in ε :

$$\eta = \eta_0 + \varepsilon \eta_1 + \mathcal{O}(\varepsilon^2). \tag{3.25}$$

First order approximation

Substitution of (3.25) into equation (3.24) yields

$$\eta_{0tt} - \eta_{0xx} + \varepsilon(\eta_{1tt} - \eta_{1xx}) \\ = \varepsilon\left[\frac{1}{3}\eta_{0xxxx} + \partial_x^2\left(\frac{(\eta_0 + \varepsilon\eta_1)^2}{2} + \left(\int_{-\infty}^{x}(\eta_0 + \varepsilon\eta_1)_t\,\mathrm{d}x'\right)^2\right)\right] + \mathcal{O}(\varepsilon^2). \tag{3.26}$$

In the leading order $\mathcal{O}(\varepsilon^0)$, equation (3.26) becomes

$$\eta_{0tt} - \eta_{0xx} = 0.$$

This is the wave equation with velocity 1, and whose general solution is

$$\eta_0 = F(x - t) + G(x + t),$$

where F, G are general functions.

Second order approximation

As was discussed in Remark (4), anticipating the secular terms, we introduce slow time scales

$$\tau_0 = t, \qquad \tau_1 = \varepsilon t.$$

so that

$$\eta(x, t) = \eta(x, \tau_0, \tau_1).$$

The expansion (3.25) becomes

$$\eta(x, \tau_0, \tau_1) = \eta_0(x, \tau_0, \tau_1) + \mathcal{O}(\varepsilon^1). \tag{3.27}$$