Report 3

Sultan Aitzhan

February 2, 2020

Contents

1 Time stepping and finite differences: the whole line

Recall the equation we obtained for the surface elevation on the whole line:

$$\eta_{tt} = \eta_{xx} + \mu^2 \left(\frac{1}{3}\eta_{xxxx} + \eta_{xt} \int_{-\infty}^{x} \eta_t \, \mathrm{d}x' + \eta_x^2 + \eta_t^2 + \eta\eta_{xx} + \partial_x^2 \left(\int_{-\infty}^{x} \eta_t \, \mathrm{d}x' \right)^2 \right). \tag{1}$$

To do time stepping, introduce

$$u = \eta_t. \tag{2}$$

Also, note that

$$\partial_x^2 \left(\int_{-\infty}^{x} \eta_t \, \mathrm{d}x' \right)^2 = 2(\eta_t^2 + \eta_{tx} \int_{-\infty}^{x} \eta_t \, \mathrm{d}x')$$

Then, combining (2) and (1), we obtain a two-dimensional system:

$$\partial_t \begin{bmatrix} u \\ \eta \end{bmatrix} = \begin{bmatrix} \mu^2 \left(2u_x \int_{-\infty}^x u \, \mathrm{d}x' + 2u^2\right) + \eta_{xx} + \mu^2 \left(\frac{1}{3}\eta_{xxxx} + \eta_x^2 + \eta\eta_{xx}\right) \\ u \end{bmatrix}. \tag{3}$$

Now, consider (1) on a finite interval $[a, b]$, and let partition the interval into $n+1$ points $\{x_k\}_{k=0}^n$, with $x_0 = a$ and $x_n = b$. This means that the integral terms becomes

$$\int_{-\infty}^x \eta_t \, \mathrm{d}x' = \left\{ \int_{-\infty}^a + \int_a^x \right\} \eta_t \, \mathrm{d}x' \approx \int_a^x \eta_t \, \mathrm{d}x',$$

while assuming that

$$\int_{-\infty}^a \eta_t \, \mathrm{d}x'$$

is small enough. Now, we employ forward Euler time stepping. First, observe that

$$\begin{aligned} u_t(x_k, t_j) &= \frac{u(x_k, t_{j+1}) - u(x_k, t_j)}{\Delta t} = f_1(\eta, u, t) && \implies u(x_k, t_{j+1}) = u(x_k, t_j) + \Delta t f_1(\eta, u, t) \\ \eta_t(x_k, t_j) &= \frac{\eta(x_k, t_{j+1}) - \eta(x_k, t_j)}{\Delta t} = f_2(\eta, u, t) && \implies \eta(x_k, t_{j+1}) = \eta(x_k, t_j) + \Delta t f_2(\eta, u, t), \end{aligned}$$

where

$$\begin{aligned} f_1(\eta, u, t) &= \eta_{xx} + \mu^2 \left(2u_x \int_{-\infty}^x u \, \mathrm{d}x' + 2u^2 + \frac{1}{3}\eta_{xxxx} + \eta_x^2 + \eta\eta_{xx}\right) \\ f_2(\eta, u, t) &= u(x_k, t_j). \end{aligned}$$

Observe that the highest order derivative is 4, so we need to use five-point stencils. In particular, we use five point midpoint stencil for $x_2, \ldots, x_{n-2}$, and five point, one-sided stencils at x_0, x_1, x_{n-1}, x_n. First, consider the system for $x_2, \ldots, x_{n-2}$:

$$\begin{aligned} f_1(\eta(x_k, t_j), u(x_k, t_j), t) &= \eta(x_k, t_j)_{xx} + \frac{\mu^2}{3}\eta(x_k, t_j)_{xxxx} \\ &\quad + \mu^2 \left(2u(x_k, t_j)_x \int_{x_0}^{x_k} u \, \mathrm{d}x' + 2u(x_k, t_j)^2 + \eta(x_k, t_j)_x^2 + \eta(x_k, t_j)\eta(x_k, t_j)_{xx}\right), \end{aligned}$$

where we have separated the linear and nonlinear terms. Let $\Delta x = x_k - x_{k-1}$ and recall the finite difference formulas at x :

$$f'(x) = \frac{f(x - 2\Delta x) - 8f(x - \Delta x) + 8f(x + \Delta x) - f(x + 2\Delta x)}{12\Delta x} + \mathcal{O}((\Delta x)^4)$$

$$
\begin{aligned}
f''(x) &= \frac{-f(x-2\Delta x)+16f(x-\Delta x)-30f+16f(x+\Delta x)-f(x+2\Delta x)}{12(\Delta x)^2}+\mathcal{O}((\Delta x)^2)\\
f''''(x) &= \frac{f(x-2\Delta x)-4f(x-\Delta x)+6f-4f(x+\Delta x)+f(x+2\Delta x)}{(\Delta x)^4}+\mathcal{O}((\Delta x)^2)
\end{aligned}
$$

so that

$$
\begin{aligned}
(\eta_k)_x &= \frac{\eta_{k-2}-8\eta_{k-1}+8\eta_{k+1}-\eta_{k+2}}{12\Delta x}+\mathcal{O}((\Delta x)^4)\\
(u_k)_x &= \frac{u_{k-2}-8u_{k-1}+8u_{k+1}-u_{k+2}}{12\Delta x}+\mathcal{O}((\Delta x)^4)\\
(\eta_k)_{xx} &= \frac{-\eta_{k-2}+16\eta_{k-1}-30\eta_k+16\eta_{k+1}-\eta_{k+2}}{12(\Delta x)^2}+\mathcal{O}((\Delta x)^2)\\
(\eta_k)_{xxxx} &= \frac{\eta_{k-2}-4\eta_{k-1}+6\eta_k-4\eta_{k+1}+\eta_{k+2}}{(\Delta x)^4}+\mathcal{O}((\Delta x)^2),
\end{aligned}
$$

where it is assumed $t = t_j$. Also, by trapezoidal rule,

$$
\int_{x_0}^{x_k} u \,\mathrm{d}x' = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} u \,\mathrm{d}x = \frac{\Delta x}{2}\sum_{i=0}^{n-1} u(x_i)+u(x_{i+1}).
$$

At $x = x_0$, we have

$$
\begin{aligned}
f'(x) &= \frac{-25f(x)+48f(x+\Delta x)-36f(x+2\Delta x)+16f(x+3\Delta x)-3f(x+4\Delta x)}{12\Delta x}+\mathcal{O}((\Delta x)^4)\\
f''(x) &= \frac{35f(x)-104f(x+\Delta x)+114f(x+2\Delta x)-56f(x+3\Delta x)+11f(x+4\Delta x)}{12(\Delta x)^2}+\mathcal{O}((\Delta x)^2)\\
f''''(x) &= \frac{f(x)-4f(x+\Delta x)+6f(x+2\Delta x)-4f(x+3\Delta x)+f(x+4\Delta x)}{(\Delta x)^4}+\mathcal{O}((\Delta x)^2),
\end{aligned}
$$

so that

$$
\begin{aligned}
(\eta_0)_x &= \frac{-25\eta_0+48\eta_1-36\eta_2+16\eta_3-3\eta_4}{12\Delta x}\\
(u_0)_x &= \frac{-25u_0+48u_1-36u_2+16u_3-3u_4}{12\Delta x}\\
(\eta_0)_{xx} &= \frac{35\eta_0-104\eta_1+114\eta_2-56\eta_3+11\eta_4}{12\Delta x^2}
\end{aligned}
$$

$$(\eta_0)_{xxxx} = \frac{\eta_0 - 4\eta_1 + 6\eta_2 - 4\eta_3 + \eta_4}{\Delta x^4}$$

At $x = x_1$, we have

$$\begin{aligned}
f'(x) &= \frac{-3f(x-\Delta x) - 10f(x) + 18f(x+\Delta x) - 6f(x+2\Delta x) + f(x+3\Delta x)}{12\Delta x} + \mathcal{O}((\Delta x)^4) \\
f''(x) &= \frac{11f(x-\Delta x) - 20f(x) + 6f(x+\Delta x) + 4f(x+2\Delta x) - f(x+3\Delta x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2) \\
f''''(x) &= \frac{f(x-\Delta x) - 4f(x) + 6f(x+\Delta x) - 4f(x+2\Delta x) + f(x+3\Delta x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2),
\end{aligned}$$

so that

$$\begin{aligned}
(\eta_1)_x &= \frac{-3\eta_0 - 10\eta_1 + 18\eta_2 - 6\eta_3 + \eta_4}{12\Delta x} \\
(u_1)_x &= \frac{-3u_0 - 10u_1 + 18u_2 - 6u_3 + u_4}{12\Delta x} \\
(\eta_1)_{xx} &= \frac{11\eta_0 - 20\eta_1 + 6\eta_2 + 4\eta_3 - \eta_4}{12\Delta x^2} \\
(\eta_1)_{xxxx} &= \frac{\eta_0 - 4\eta_1 + 6\eta_2 - 4\eta_3 + \eta_4}{\Delta x^4}
\end{aligned}$$

At $x = x_{n-1}$, we have

$$\begin{aligned}
f'(x) &= \frac{-f(x-3\Delta x) + 6f(x-2\Delta x) - 18f(x-\Delta x) + 10f(x) + 3f(x+\Delta x)}{12\Delta x} + \mathcal{O}((\Delta x)^4) \\
f''(x) &= \frac{-f(x-3\Delta x) + 4f(x-2\Delta x) + 6(x-\Delta x) - 20f(x) + 11f(x+\Delta x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2) \\
f''''(x) &= \frac{f(x-3\Delta x) - 4f(x-2\Delta x) + 6f(x-\Delta x) - 4f(x) + f(x+\Delta x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2),
\end{aligned}$$

so that

$$\begin{aligned}
(\eta_{n-1})_x &= \frac{-\eta_{n-4} + 6\eta_{n-3} - 18\eta_{n-2} + 10\eta_{n-1} + 3\eta_n}{12\Delta x} \\
(u_{n-1})_x &= \frac{-u_{n-4} + 6u_{n-3} - 18u_{n-2} + 10u_{n-1} + 3u_n}{12\Delta x}
\end{aligned}$$

$$
\begin{aligned}
(\eta_{n-1})_{xx} &= \frac{-\eta_{n-4} + 4\eta_{n-3} + 6\eta_{n-2} - 20\eta_{n-1} + 11\eta_n}{12\Delta x^2} \\
(\eta_{n-1})_{xxxx} &= \frac{\eta_{n-4} - 4\eta_{n-3} + 6\eta_{n-2} - 4\eta_{n-1} + \eta_n}{\Delta x^4}
\end{aligned}
$$

At $x = x_n$, we have

$$
\begin{aligned}
f'(x) &= \frac{f(x-4\Delta x) - 4f(x-3\Delta x) + 6f(x-2\Delta x) - 4f(x-\Delta x) + f(x)}{12\Delta x} + \mathcal{O}((\Delta x)^4) \\
f''(x) &= \frac{11f(x-4\Delta x) - 56f(x-3\Delta x) + 114f(x-2\Delta x) - 104f(x-\Delta x) + 35f(x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2) \\
f''''(x) &= \frac{f(x-4\Delta x) - 4f(x-3\Delta x) + 6f(x-2\Delta x) - 4f(x-\Delta x) + f(x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2),
\end{aligned}
$$

so that

$$
\begin{aligned}
(\eta_n)_x &= \frac{3\eta_{n-4} - 16\eta_{n-3} + 36\eta_{n-2} - 48\eta_{n-1} + 25\eta_n}{12\Delta x} \\
(u_n)_x &= \frac{3u_{n-4} - 16u_{n-3} + 36u_{n-2} - 48u_{n-1} + 25u_n}{12\Delta x} \\
(\eta_n)_{xx} &= \frac{11\eta_{n-4} - 56\eta_{n-3} + 114\eta_{n-2} - 104\eta_{n-1} + 35\eta_n}{12\Delta x^2} \\
(\eta_n)_{xxxx} &= \frac{\eta_{n-4} - 4\eta_{n-3} + 6\eta_{n-2} - 4\eta_{n-1} + \eta_n}{\Delta x^4}.
\end{aligned}
$$

All in all, we obtain:

$$
\begin{aligned}
f_1(\eta(x_0,t_j), u(x_0,t_j), t) &= \eta(x_0,t_j)_{xx} + \frac{\mu^2}{3}\eta(x_0,t_j)_{xxxx} + \mu^2\left(2u(x_0,t_j)^2 + \eta(x_0,t_j)_x^2 + \eta(x_0,t_j)\eta(x_0,t_j)_{xx}\right), \\
f_1(\eta(x_1,t_j), u(x_1,t_j), t) &= \eta(x_1,t_j)_{xx} + \frac{\mu^2}{3}\eta(x_1,t_j)_{xxxx} \\
&\quad + \mu^2\left(2u(x_1,t_j)_x\int_{x_0}^{x_1} u\,\mathrm{d}x' + 2u(x_1,t_j)^2 + \eta(x_1,t_j)_x^2 + \eta(x_1,t_j)\eta(x_1,t_j)_{xx}\right), \\
f_1(\eta(x_2,t_j), u(x_2,t_j), t) &= \eta(x_2,t_j)_{xx} + \frac{\mu^2}{3}\eta(x_2,t_j)_{xxxx} \\
&\quad + \mu^2\left(2u(x_2,t_j)_x\int_{x_0}^{x_2} u\,\mathrm{d}x' + 2u(x_2,t_j)^2 + \eta(x_2,t_j)_x^2 + \eta(x_2,t_j)\eta(x_2,t_j)_{xx}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\dots\\
f_1(\eta(x_k,t_j),u(x_k,t_j),t) &= \eta(x_k,t_j)_{xx} + \frac{\mu^2}{3}\eta(x_k,t_j)_{xxxx}\\
&\quad + \mu^2\left(2u(x_k,t_j)_x\int_{x_0}^{x_k} u\,\mathrm{d}x' + 2u(x_k,t_j)^2 + \eta(x_k,t_j)_x^2 + \eta(x_k,t_j)\eta(x_k,t_j)_{xx}\right),\\
&\dots\\
f_1(\eta(x_{n-1},t_j),u(x_{n-1},t_j),t) &= \eta(x_{n-1},t_j)_{xx} + \frac{\mu^2}{3}\eta(x_{n-1},t_j)_{xxxx}\\
&\quad + \mu^2\left(2u(x_{n-1},t_j)_x\int_{x_0}^{x_{n-1}} u\,\mathrm{d}x' + 2u(x_{n-1},t_j)^2 + \eta(x_{n-1},t_j)_x^2 + \eta(x_{n-1},t_j)\eta(x_{n-1},t_j)_{xx}\right),\\
f_1(\eta(x_n,t_j),u(x_n,t_j),t) &= \eta(x_n,t_j)_{xx} + \frac{\mu^2}{3}\eta(x_n,t_j)_{xxxx}\\
&\quad + \mu^2\left(2u(x_n,t_j)_x\int_{x_0}^{x_n} u\,\mathrm{d}x' + 2u(x_n,t_j)^2 + \eta(x_n,t_j)_x^2 + \eta(x_n,t_j)\eta(x_n,t_j)_{xx}\right),
\end{aligned}
$$

Now, we obtain the discretised problem. First, consider the column of linear terms:

$$
\begin{aligned}
&(\eta_0)_{xx} + \frac{\mu^2}{3}(\eta_0)_{xxxx}\\
&\quad= \frac{35\eta_0 - 104\eta_1 + 114\eta_2 - 56\eta_3 + 11\eta_4}{12\Delta x^2} + \frac{\mu^2}{3}\frac{\eta_0 - 4\eta_1 + 6\eta_2 - 4\eta_3 + \eta_4}{\Delta x^4}\\
&\quad= \frac{35\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_0 - \frac{104\Delta x^2 + 16\mu^2}{12\Delta x^4}\eta_1 + \frac{114\Delta x^2 + 24\mu^2}{12\Delta x^4}\eta_2 - \frac{56\Delta x^2 + 16\mu^2}{12\Delta x^4}\eta_3 + \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_4\\
&(\eta_1)_{xx} + \frac{\mu^2}{3}(\eta_1)_{xxxx}\\
&\quad= \frac{11\eta_0 - 20\eta_1 + 6\eta_2 + 4\eta_3 - \eta_4}{12\Delta x^2} + \frac{\mu^2}{3}\frac{\eta_0 - 4\eta_1 + 6\eta_2 - 4\eta_3 + \eta_4}{\Delta x^4}\\
&\quad= \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_0 - \frac{20\Delta x^2 + 16\mu^2}{12\Delta x^4}\eta_1 + \frac{6\Delta x^2 + 24\mu^2}{12\Delta x^4}\eta_2 + \frac{4\Delta x^2 - 16\mu^2}{12\Delta x^4}\eta_3 + \frac{-\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_4\\
&\quad\dots\\
&(\eta_k)_{xx} + \frac{\mu^2}{3}(\eta_k)_{xxxx}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-\eta_{k-2} + 16\eta_{k-1} - 30\eta_k + 16\eta_{k+1} - \eta_{k+2}}{12(\Delta x)^2} + \frac{\mu^2}{3}\frac{\eta_{k-2} - 4\eta_{k-1} + 6\eta_k - 4\eta_{k+1} + \eta_{k+2}}{(\Delta x)^4} \\
&= \frac{-\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_{k-2} + \frac{16(\Delta x^2 - \mu^2)}{12\Delta x^4}\eta_{k-1} + \frac{-30\Delta x^2 + 24\mu^2}{12\Delta x^4}\eta_k - \frac{16(\Delta x^2 - \mu^2)}{12\Delta x^4}\eta_{k+1} + \frac{-\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_{k+2} \\
&\dots \\
(\eta_{n-1})_{xx} + \frac{\mu^2}{3}(\eta_{n-1})_{xxxx} & \\
&= \frac{-\eta_{n-4} + 4\eta_{n-3} + 6\eta_{n-2} - 20\eta_{n-1} + 11\eta_n}{12\Delta x^2} + \frac{\mu^2}{3}\frac{\eta_{n-4} - 4\eta_{n-3} + 6\eta_{n-2} - 4\eta_{n-1} + \eta_n}{\Delta x^4} \\
&= \frac{-\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_{n-4} + \frac{4\Delta x^2 - 16\mu^2}{12\Delta x^4}\eta_{n-3} + \frac{6\Delta x^2 + 24\mu^2}{12\Delta x^4}\eta_{n-2} - \frac{20\Delta x^2 + 16\mu^2}{12\Delta x^4}\eta_{n-1} + \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_n \\
(\eta_n)_{xx} + \frac{\mu^2}{3}(\eta_n)_{xxxx} & \\
&= \frac{11\eta_{n-4} - 56\eta_{n-3} + 114\eta_{n-2} - 104\eta_{n-1} + 35\eta_n}{12\Delta x^2} + \frac{\mu^2}{3}\frac{\eta_{n-4} - 4\eta_{n-3} + 6\eta_{n-2} - 4\eta_{n-1} + \eta_n}{\Delta x^4} \\
&= \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_{n-4} - \frac{56\Delta x^2 + 16\mu^2)}{12\Delta x^4}\eta_{n-3} + \frac{114\Delta x^2 + 24\mu^2}{12\Delta x^4}\eta_{n-2} - \frac{104\Delta x^2 + 16\mu^2}{12\Delta x^4}\eta_{n-1} + \frac{35\Delta x^2 + 4\mu^2}{12\Delta x^4}\eta_n.
\end{aligned}
$$

Then, the matrix becomes

$$
\begin{bmatrix}
(\eta_0)_{xx} + \frac{\mu^2}{3}(\eta_0)_{xxxx} \\
(\eta_1)_{xx} + \frac{\mu^2}{3}(\eta_1)_{xxxx} \\
(\eta_2)_{xx} + \frac{\mu^2}{3}(\eta_2)_{xxxx} \\
\vdots \\
(\eta_k)_{xx} + \frac{\mu^2}{3}(\eta_k)_{xxxx} \\
\vdots \\
(\eta_{n-2})_{xx} + \frac{\mu^2}{3}(\eta_{n-2})_{xxxx} \\
(\eta_{n-1})_{xx} + \frac{\mu^2}{3}(\eta_{n-1})_{xxxx} \\
(\eta_n)_{xx} + \frac{\mu^2}{3}(\eta_n)_{xxxx}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\frac{35\Delta x^2+4\mu^2}{12\Delta x^4} & -\frac{104\Delta x^2+16\mu^2}{12\Delta x^4} & \frac{114\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{56\Delta x^2+16\mu^2}{12\Delta x^4} & \frac{11\Delta x^2+4\mu^2}{12\Delta x^4} & 0 & 0 & 0 & \cdots & \\
\frac{11\Delta x^2+4\mu^2}{12\Delta x^4} & -\frac{20\Delta x^2+16\mu^2}{12\Delta x^4} & \frac{6\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{4\Delta x^2-16\mu^2}{12\Delta x^4} & \frac{-\Delta x^2+4\mu^2}{12\Delta x^4} & 0 & 0 & 0 & \cdots & \\
\frac{-\Delta x^2+4\mu^2}{12\Delta x^4} & \frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & \frac{-30\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & \frac{-\Delta x^2+4\mu^2}{12\Delta x^4} & 0 & 0 & 0 & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \\
0 & \cdots & 0 & \frac{-\Delta x^2+4\mu^2}{12\Delta x^4} & \frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & \frac{-30\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & \frac{-\Delta x^2+4\mu^2}{12\Delta x^4} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \\
\cdots & 0 & 0 & 0 & \frac{-\Delta x^2+4\mu^2}{12\Delta x^4} & \frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & \frac{-30\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{16(\Delta x^2-\mu^2)}{12\Delta x^4} & \frac{-\Delta x^2+4\mu^2}{12\Delta x^4} & \\
\cdots & 0 & 0 & 0 & -\frac{\Delta x^2+4\mu^2}{12\Delta x^4} & \frac{4\Delta x^2-16\mu^2)}{12\Delta x^4} & \frac{6\Delta x^2+24\mu^2}{12\Delta x^4} & \frac{-20\Delta x^2-16\mu^2}{12\Delta x^4} & \frac{11\Delta x^2+4\mu^2}{12\Delta x^4} & \\
\cdots & 0 & 0 & 0 & \frac{11\Delta x^2+4\mu^2}{12\Delta x^4} & -\frac{56\Delta x^2+16\mu^2)}{12\Delta x^4} & \frac{114\Delta x^2+24\mu^2}{12\Delta x^4} & -\frac{104\Delta x^2+16\mu^2}{12\Delta x^4} & \frac{35\Delta x^2+4\mu^2}{12\Delta x^4} &
\end{bmatrix}
\begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_k \\ \vdots \\ \eta_{n-2} \\ \eta_{n-1} \\ \eta_n \end{bmatrix}
$$

For simplicity, let $\mathcal{A}$ represent the above matrix. Now, recall the system:

$$
\begin{aligned}
u(x_k, t_{j+1}) &= u(x_k, t_j) + \Delta t f_1(\eta(x_k, t_j), u(x_k, t_j), t), \\
\eta(x_k, t_{j+1}) &= \eta(x_k, t_j) + \Delta t u(x_k, t_j),
\end{aligned}
$$

where

$$
\begin{aligned}
f_1(\eta(x_k, t_j), u(x_k, t_j), t) = \eta(x_k, t_j)_{xx} + \frac{\mu^2}{3}\eta(x_k, t_j)_{xxxx} \\
+ \mu^2 \left(2u(x_k, t_j)_x \int_{x_0}^{x_k} u \, \mathrm{d}x' + 2u(x_k, t_j)^2 + \eta(x_k, t_j)_x^2 + \eta(x_k, t_j)\eta(x_k, t_j)_{xx} \right).
\end{aligned}
$$

For convenience, let

$$
B_k = \left(2u(x_k, t_j)_x \int_{x_0}^{x_k} u \, \mathrm{d}x' + 2u(x_k, t_j)^2 + \eta(x_k, t_j)_x^2 + \eta(x_k, t_j)\eta(x_k, t_j)_{xx} \right),
$$

Let $\mathcal{B}$ represent the column vector of B_k's. Then, we can write the system

$$
\begin{aligned}
u(x_0, t_{j+1}) &= u(x_0, t_j) + \Delta t f_1(\eta(x_0, t_j), u(x_0, t_j), t) \\
u(x_1, t_{j+1}) &= u(x_1, t_j) + \Delta t f_1(\eta(x_1, t_j), u(x_1, t_j), t) \\
u(x_2, t_{j+1}) &= u(x_2, t_j) + \Delta t f_1(\eta(x_2, t_j), u(x_2, t_j), t) \\
&\vdots \\
u(x_{n-2}, t_{j+1}) &= u(x_{n-2}, t_j) + \Delta t f_1(\eta(x_{n-2}, t_j), u(x_{n-2}, t_j), t)
\end{aligned}
$$

$$\begin{aligned}
u(x_{n-1}, t_{j+1}) &= u(x_{n-1}, t_j) + \Delta t f_1(\eta(x_{n-1}, t_j), u(x_{n-1}, t_j), t) \\
u(x_n, t_{j+1}) &= u(x_n, t_j) + \Delta t f_1(\eta(x_n, t_j), u(x_n, t_j), t)
\end{aligned}$$

as follows:

$$\begin{aligned}
\begin{bmatrix} u(x_0, t_{j+1}) \\ u(x_1, t_{j+1}) \\ u(x_2, t_{j+1}) \\ \vdots \\ u(x_{n-2}, t_{j+1}) \\ u(x_{n-1}, t_{j+1}) \\ u(x_n, t_{j+1}) \end{bmatrix} &= \begin{bmatrix} u(x_0, t_j) \\ u(x_1, t_j) \\ u(x_2, t_j) \\ \vdots \\ u(x_{n-2}, t_j) \\ u(x_{n-1}, t_j) \\ u(x_n, t_j) \end{bmatrix} + \Delta t \begin{bmatrix} f_1(\eta(x_0, t_j), u(x_0, t_j), t) \\ f_1(\eta(x_1, t_j), u(x_1, t_j), t) \\ f_1(\eta(x_2, t_j), u(x_2, t_j), t) \\ \vdots \\ f_1(\eta(x_{n-2}, t_j), u(x_{n-2}, t_j), t) \\ f_1(\eta(x_{n-1}, t_j), u(x_{n-1}, t_j), t) \\ f_1(\eta(x_n, t_j), u(x_n, t_j), t) \end{bmatrix} \\
&= \begin{bmatrix} u(x_0, t_j) \\ u(x_1, t_j) \\ u(x_2, t_j) \\ \vdots \\ u(x_{n-2}, t_j) \\ u(x_{n-1}, t_j) \\ u(x_n, t_j) \end{bmatrix} + \Delta t \mathcal{A} \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-2} \\ \eta_{n-1} \\ \eta_n \end{bmatrix} + \Delta t \mathcal{B}
\end{aligned}$$

Now, let us see how one would perform time-stepping. As such, impose initial conditions

$$\eta(x, t_0) = f(x), \qquad u(x, t_0) = \eta_t(x, t_0) = g(x).$$

Let $j = 0$, and for simplicity, pick $k \in [0, n]$. The system is

$$\begin{aligned}
u(x_k, t_1) &= u(x_k, t_0) + \Delta t f_1(\eta(x_k, t_0), u(x_k, t_0)), \\
\eta(x_k, t_1) &= \eta(x_k, t_0) + \Delta t u(x_k, t_0),
\end{aligned}$$

where

$$\begin{aligned}
f_1(\eta(x_k, t_0), u(x_k, t_0)) &= \eta(x_k, t_0)_{xx} + \frac{\mu^2}{3} \eta(x_k, t_0)_{xxxx} \\
&+ \mu^2 \left(2u(x_k, t_0)_x \int_{x_0}^{x_k} u \, \mathrm{d}x' + 2u(x_k, t_0)^2 + \eta(x_k, t_0)_x^2 + \eta(x_k, t_0)\eta(x_k, t_0)_{xx} \right)
\end{aligned}$$

$$= \frac{-\eta(x_{k-2},t_0) + 16\eta(x_{k-1},t_0) - 30\eta(x_k,t_0) + 16\eta(x_{k+1},t_0) - \eta(x_{k+2},t_0)}{12(\Delta x)^2}$$
$$+ \frac{\mu^2}{3}\frac{\eta(x_{k-2},t_0) - 4\eta(x_{k-1},t_0) + 6\eta(x_k,t_0) - 4\eta(x_{k+1},t_0) + \eta(x_{k+2},t_0)}{(\Delta x)^4}$$
$$+ \mu^2\left(2u(x_k,t_0)_x \int_{x_0}^{x_k} u(x',t_0)\,\mathrm{d}x' + 2u(x_k,t_0)^2 + \eta(x_k,t_0)_x^2 + \eta(x_k,t_0)\eta(x_k,t_0)_{xx}\right)$$

Note that all the terms on the last line can be computed via finite differences and both initial conditions. With this, we obtain the values of u, η at point x_k and time t_1. Performing this calculation for all k, we move on to compute u, η at time t_2, and so on.

2 The half line problem

In this section, we deal with this term

$$\partial_x(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\}\}$$

More generally, we have the following result:

Theorem 1. *For nice enough f defined on $x \geqslant 0$, we have*

$$(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} = \frac{1}{\pi}\int_0^\infty f(y)\left(\frac{1}{x-y} + \frac{1}{x+y}\right)\mathrm{d}y.$$

Before we begin, recall the Riemann-Lebesgue lemma:

Lemma 2 (Theorem 11.6, [1])**.** *Assume that $f \in L(I)$. Then, for each real β, we have*

$$\lim_{\alpha\to\infty}\int_I f(t)\sin(\alpha t+\beta)\,\mathrm{d}t = 0.$$

Proof of Theorem 1. Consider

$$(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\}.$$

For generality, we consider $(\mathcal{F}_s^k)^{-1}\{G(k)\}$, where G is a function of k defined on $k \geqslant 0$. Expanding the integral, we obtain:

$$(\mathcal{F}_s^k)^{-1}\{G(k)\} = \int_0^\infty \sin(kx)G(k)\,\mathrm{d}k$$

$$\begin{aligned}
&= \frac{1}{2i}\int_0^\infty (e^{ikx} - e^{-ikx}) G(k)\,\mathrm{d}k \\
&= \frac{1}{2i}\left[\int_0^\infty e^{ikx} G(k)\,\mathrm{d}k - \int_0^\infty e^{-ikx} G(k)\,\mathrm{d}k\right] \\
&= \frac{1}{2i}\left[\int_0^\infty e^{ikx} G(k)\,\mathrm{d}k + \int_0^{-\infty} e^{ikx} G(-k)\,\mathrm{d}k\right] && \text{(apply } k \mapsto -k \text{ in the 2nd term)} \\
&= \frac{1}{2i}\left[\int_0^\infty e^{ikx} G(k)\,\mathrm{d}k + \int_{-\infty}^{0} e^{ikx} (-G(-k))\,\mathrm{d}k\right],
\end{aligned}$$

where $-G(-k)$ is an odd extension to $k < 0$. Now, observe the following:

$$\begin{aligned}
\frac{2}{\pi}\int_0^\infty \cos(kx) f(x)\,\mathrm{d}x &= \frac{1}{\pi}\int_0^\infty (e^{ikx} + e^{-ikx}) f(x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\int_0^\infty e^{ikx} f(x)\,\mathrm{d}x + \int_0^\infty e^{-ikx} f(x)\,\mathrm{d}x\right] \\
&= \frac{1}{\pi}\left[-\int_0^{-\infty} e^{-ikx} f(-x)\,\mathrm{d}x + \int_0^\infty e^{-ikx} f(x)\,\mathrm{d}x\right] && \text{(apply } x \mapsto -x \text{ in the 1st term)} \\
&= \frac{1}{\pi}\left[\int_{-\infty}^0 e^{-ikx} f(-x)\,\mathrm{d}x + \int_0^\infty e^{-ikx} f(x)\,\mathrm{d}x\right] \\
&= \frac{1}{\pi}\int_{-\infty}^\infty e^{-ikx} F(x)\,\mathrm{d}x,
\end{aligned}$$

where we used an even extension to $x < 0$ and defined

$$F(x) = \begin{cases} f(x) & x > 0 \\ f(-x) & x < 0 \end{cases}.$$

For $k > 0$, we have

$$G(k) = \mathcal{F}_c^k\{f\} = \frac{2}{\pi}\int_0^\infty \cos(kx) f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_{-\infty}^\infty e^{-ikx} F(x)\,\mathrm{d}x. \tag{4}$$

For $k < 0$, we have

$$-G(-k) = -\mathcal{F}_c^{-k}\{f\} = -\frac{2}{\pi}\int_0^\infty \cos(-kx) f(x)\,\mathrm{d}x = -\frac{2}{\pi}\int_0^\infty \cos(kx) f(x)\,\mathrm{d}x = -\frac{1}{\pi}\int_{-\infty}^\infty e^{-ikx} F(x)\,\mathrm{d}x, \tag{5}$$

since cosine is an even function. Thus, using (4) and (5), we obtain

$$
\begin{aligned}
(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} &= \frac{1}{2i}\left[\int_0^\infty e^{ikx}\mathcal{F}_c^k\{f\}\,\mathrm{d}k + \int_{-\infty}^0 e^{ikx}(-\mathcal{F}^(-k)_c\{f\})\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty e^{ikx}\int_{-\infty}^\infty e^{-iky}F(y)\,\mathrm{d}y\,\mathrm{d}k - \int_{-\infty}^0 e^{ikx}\int_{-\infty}^\infty e^{-iky}F(y)\,\mathrm{d}y\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty e^{ikx}\int_{-\infty}^\infty e^{ik(x-y)}F(y)\,\mathrm{d}y\,\mathrm{d}k - \int_{-\infty}^0 \int_{-\infty}^\infty e^{ik(x-y)}F(y)\,\mathrm{d}y\,\mathrm{d}k\right].
\end{aligned} \tag{6}
$$

Let

$$
\begin{aligned}
V(k) &= \int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y = -V(-k), \\
U(k) &= \int_{-\infty}^\infty \cos(k(x-y))F(y)\,\mathrm{d}y = U(-k),
\end{aligned}
$$

so that V is odd and U is even. This allows to rewrite (6) as:

$$
\begin{aligned}
(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} &= \frac{1}{2\pi i}\left[\int_0^\infty e^{ikx}\int_{-\infty}^\infty e^{ik(x-y)}F(y)\,\mathrm{d}y\,\mathrm{d}k - \int_{-\infty}^0 \int_{-\infty}^\infty e^{ik(x-y)}F(y)\,\mathrm{d}y\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty U(k)+iV(k)\,\mathrm{d}k - \int_{-\infty}^0 U(k)+iV(k)\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty U(k)+iV(k)\,\mathrm{d}k + \int_{\infty}^0 U(-k)+iV(-k)\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty U(k)+iV(k)\,\mathrm{d}k + \int_0^\infty -U(-k)+i(-V(-k))\,\mathrm{d}k\right] \\
&= \frac{1}{2\pi i}\left[\int_0^\infty U(k)+iV(k)\,\mathrm{d}k + \int_0^\infty -U(k)+iV(k)\,\mathrm{d}k\right] \\
&= \frac{1}{\pi}\int_0^\infty V(k)\,\mathrm{d}k,
\end{aligned}
$$

where on the third last line, we flipped the bounds of integration and brought the minus sign inside the integral, and on the second last line, we used that U is even and V is odd. Thus, we obtain

$$
(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} = \frac{1}{\pi}\int_0^\infty V(k)\,\mathrm{d}k = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y\,\mathrm{d}k.
$$

Note that the integral in k is an improper integral, so

$$\int_0^\infty \int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y\,\mathrm{d}k = \lim_{\alpha\to\infty}\int_0^\alpha \int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y\,\mathrm{d}k.$$

Now, interchanging the order of integration, we have

$$\begin{aligned}\int_0^\alpha \int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y\,\mathrm{d}k &= \int_{-\infty}^\infty F(y)\int_0^\alpha \sin(k(x-y))\,\mathrm{d}k\,\mathrm{d}y\\ &= \int_{-\infty}^\infty F(y)\left[-\frac{\cos(k(x-y))}{x-y}\,\Big|_0^\alpha\right]\mathrm{d}y\\ &= \int_{-\infty}^\infty F(y)\left[\frac{1}{x-y}-\frac{\cos(\alpha(x-y))}{x-y}\right]\mathrm{d}y\\ &= \int_{-\infty}^\infty F(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y.\end{aligned}$$

The interchange is justified, since sine is bounded and differentiable on $\mathbb{R}$. Finally, we use the Riemann-Lebesgue lemma to deal with the last term:

$$\begin{aligned}\int_{-\infty}^\infty F(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y &= \int_0^\infty f(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y + \int_{-\infty}^0 f(-y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y\\ &= \int_0^\infty f(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y - \int_\infty^0 f(y)\frac{1-\cos(\alpha(x+y))}{x+y}\,\mathrm{d}y\\ &= \int_0^\infty f(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y + \int_0^\infty f(y)\frac{1-\cos(\alpha(x+y))}{x+y}\,\mathrm{d}y\\ &= \int_0^\infty f(y)\frac{1}{x-y}\,\mathrm{d}y - \int_0^\infty f(y)\frac{\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y\\ &+ \int_0^\infty f(y)\frac{1}{x+y}\,\mathrm{d}y - \int_0^\infty f(y)\frac{\cos(\alpha(x+y))}{x+y}\,\mathrm{d}y.\end{aligned}$$

As $\alpha\to\infty$, the terms

$$\int_0^\infty f(y)\frac{\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y,\qquad \int_0^\infty f(y)\frac{\cos(\alpha(x+y))}{x+y}\,\mathrm{d}y \to 0$$

by the Riemann-Lebesgue lemma with $\beta=\pi/2$, so that

$$\int_{-\infty}^\infty F(y)\frac{1-\cos(\alpha(x-y))}{x-y}\,\mathrm{d}y = \int_0^\infty f(y)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y.$$

Thus,

$$(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{f\}\} = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty \sin(k(x-y))F(y)\,\mathrm{d}y\,\mathrm{d}k = \frac{1}{\pi}\int_0^\infty f(y)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y.$$

The proof is complete. □

Remark 3. Note that the integral

$$\frac{1}{\pi}\int_0^\infty f(y)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y =$$

looks like a convolution-type transform. In fact, the term with $1/(x-y)$ is pretty much the Hilbert transform, but on a half-line.

The theorem yields

$$\partial_x(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\}\} = \partial_x\left(\frac{1}{\pi}\int_0^\infty \partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y\right).$$

For generality, let $f(y)=\partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)$. Note the following:

$$\begin{aligned}\partial_x\left(\frac{1}{\pi}\int_0^\infty f(y)\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y\right) &= \frac{1}{\pi}\int_0^\infty f(y)\partial_x\left[\frac{1}{x-y}+\frac{1}{x+y}\right]\mathrm{d}y\\ &= -\frac{1}{\pi}\int_0^\infty f(y)\left[\frac{1}{(x-y)^2}+\frac{1}{(x+y)^2}\right]\mathrm{d}y,\end{aligned}$$

so that

$$\partial_x(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\}\} = -\frac{1}{\pi}\int_0^\infty \partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)\left[\frac{1}{(x-y)^2}+\frac{1}{(x+y)^2}\right]\mathrm{d}y. \tag{7}$$

As can be seen, the integral (7) is singular whenever $x=y$ or $x=-y$, over y. To deal with this issue, one may need to use a Residue theorem. To conclude, the surface expression on a half-line becomes:

$$\begin{aligned}\eta_{tt}-\eta_{xx} &= \mu^2\left(\frac{1}{3}\eta_{xxxx} + \partial_x(\mathcal{F}_s^k)^{-1}\{\mathcal{F}_c^k\{\partial_t\left(\eta\int_0^x \eta_t\,\mathrm{d}x'\right)\}\} + \frac{1}{2}\partial_x^2\left(\int_0^x \eta_t\,\mathrm{d}x'\right)^2\right)\\ &= \mu^2\left(\frac{1}{3}\eta_{xxxx} - \frac{1}{\pi}\int_0^\infty \partial_t\left(\eta\int_0^y \eta_t\,\mathrm{d}y'\right)\left[\frac{1}{(x-y)^2}+\frac{1}{(x+y)^2}\right]\mathrm{d}y + \frac{1}{2}\partial_x^2\left(\int_0^x \eta_t\,\mathrm{d}x'\right)^2\right).\end{aligned}$$

3 Time stepping and finite differences: the half line

On the half line, the equation for the surface elevation is given by:

$$\eta_{tt} = \eta_{xx} + \mu^2 \left(\frac{1}{3}\eta_{xxxx} - \frac{1}{\pi} \int_0^\infty \partial_t \left(\eta \int_0^y \eta_t \, \mathrm{d}y' \right) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] \mathrm{d}y + \frac{1}{2}\partial_x^2 \left(\int_0^x \eta_t \, \mathrm{d}x' \right)^2 \right) \tag{8}$$

To do time stepping, introduce

$$u = \eta_t. \tag{9}$$

Also, note that

$$\frac{1}{2}\partial_x^2 \left(\int_0^x \eta_t \, \mathrm{d}x' \right)^2 = \eta_t^2 + \eta_{tx} \int_0^x \eta_t \, \mathrm{d}x' = u^2 + u_x \int_0^x u \, \mathrm{d}x',$$

and

$$\partial_t \left(\eta \int_0^y \eta_t \, \mathrm{d}y' \right) = \eta_t \int_0^x \eta_t \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)) = u \int_0^x u \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)).$$

Then, combining (9) and (8), we obtain a two-dimensional system:

$$\partial_t \begin{bmatrix} u \\ \eta \end{bmatrix} = \begin{bmatrix} \eta_{xx} + \mu^2 \left(\frac{1}{3}\eta_{xxxx} + u^2 + u_x \int_0^x u \, \mathrm{d}x' - \frac{1}{\pi} \int_0^\infty u \int_0^x u \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] \mathrm{d}y \right) \\ u \end{bmatrix}. \tag{10}$$

Now, consider (8) on a finite interval $[a, b]$, and let partition the interval into $n+1$ points $\{x_k\}_{k=0}^n$, with $x_0 = a$ and $x_n = b$. Note that we need to pick the partition such that

$$\int_0^\infty u \int_0^x u \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] \mathrm{d}y \approx \int_{x_0}^{x_n} u \int_0^x u \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] \mathrm{d}y.$$

We proceed to forward Euler time stepping. First, observe that

$$\begin{aligned} u_t(x_k, t_j) &= \frac{u(x_k, t_{j+1}) - u(x_k, t_j)}{\Delta t} = f_1(\eta, u, t) & &\implies u(x_k, t_{j+1}) = u(x_k, t_j) + \Delta t f_1(\eta, u, t) \\ \eta_t(x_k, t_j) &= \frac{\eta(x_k, t_{j+1}) - \eta(x_k, t_j)}{\Delta t} = f_2(\eta, u, t) & &\implies \eta(x_k, t_{j+1}) = \eta(x_k, t_j) + \Delta t f_2(\eta, u, t), \end{aligned}$$

where

$$f_1(\eta, u, t) = \eta_{xx} + \mu^2 \left(\frac{1}{3}\eta_{xxxx} + u^2 + u_x \int_0^x u \, \mathrm{d}x' - \frac{1}{\pi} \int_0^\infty u \int_0^x u \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} \right] \mathrm{d}y \right)$$

$$f_2(\eta, u, t) = u(x_k, t_j).$$

Observe that the highest order derivative is 4, so we need to use five-point stencils. In particular, we use five point midpoint stencil for $x_2, \ldots, x_{n-2}$, and five point, one-sided stencils at x_0, x_1, x_{n-1}, x_n. First, consider the system for $x_2, \ldots, x_{n-2}$:

$$\begin{aligned} f_1(\eta(x_k, t_j), u(x_k, t_j), t) &= \eta(x_k, t_j)_{xx} + \frac{\mu^2}{3}\eta(x_k, t_j)_{xxxx} \\ &+ \mu^2 \left(u^2(x_k) + u_x(x_k) \int_0^{x_k} u \, \mathrm{d}x' - \frac{1}{\pi} \int_0^{x_n} u \int_0^{y} u \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] \mathrm{d}y \right), \end{aligned}$$

where we have separated the linear and nonlinear terms. The only difference between this system and the whole-line system is the non-linear term; in other words, we can reuse our prior work on the linear term, and only deal with the non-linear term. Let

$$C_k = u^2(x_k) + u_x(x_k) \int_0^{x_k} u \, \mathrm{d}x' - \frac{1}{\pi} \int_0^{x_n} \left(u \int_0^{y} u \, \mathrm{d}x' + \eta(\eta_x - \eta_x(0)) \right) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] \mathrm{d}y,$$

and let $\mathcal{C}$ be the column vector of C_k; thus, $\mathcal{C}$ represents the non-linear part of the system. To discretise C_k, note that

$$\int_0^{x_k} u \, \mathrm{d}x' = \frac{\Delta x}{2} \sum_{i=0}^{k-1} u(x_{i+1}) + u(x_i)$$

$$\begin{aligned} \int_0^{x_n} u \int_0^{y} u \, \mathrm{d}x' \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] \mathrm{d}y &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \left(u \int_0^{y} u \, \mathrm{d}x' \right) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] \mathrm{d}y \\ &= \frac{\Delta x}{2} \sum_{i=0}^{n-1} u(x_{i+1}) \int_0^{x_{i+1}} u \, \mathrm{d}x' \left[\frac{1}{(x_k - x_{i+1})^2} + \frac{1}{(x_k + x_{i+1})^2} \right] + u(x_i) \int_0^{x_i} u \, \mathrm{d}x' \left[\frac{1}{(x_k - x_i)^2} + \frac{1}{(x_k + x_i)^2} \right] \\ &= \frac{\Delta x}{2} \sum_{i=0}^{n-1} \frac{\Delta x}{2} u(x_{i+1}) \left(\sum_{j=0}^{i} u(x_j) + u(x_{j+1}) \right) \left[\frac{1}{(x_k - x_{i+1})^2} + \frac{1}{(x_k + x_{i+1})^2} \right] \\ &+ \frac{\Delta x}{2} u(x_i) \left(\sum_{j=0}^{i-1} u(x_j) + u(x_{j+1}) \right) \left[\frac{1}{(x_k - x_i)^2} + \frac{1}{(x_k + x_i)^2} \right] \end{aligned}$$

$$\begin{aligned} \int_0^{x_n} \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] \mathrm{d}y &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \eta(\eta_x - \eta_x(0)) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] \mathrm{d}y \\ &= \frac{\Delta x}{2} \sum_{i=0}^{n-1} \eta(x_{i+1})(\eta_x(x_{i+1}) - \eta_x(0)) \left[\frac{1}{(x_k - x_{i+1})^2} + \frac{1}{(x_k + x_{i+1})^2} \right] + \eta(x_i)(\eta_x(x_i) - \eta_x(0)) \left[\frac{1}{(x_k - x_i)^2} + \frac{1}{(x_k + x_i)^2} \right]. \end{aligned}$$

Therefore, we obtain that

$$
\begin{aligned}
C_k &= u^2(x_k) + u_x(x_k) \int_0^{x_k} u \,\mathrm{d}x' - \frac{1}{\pi} \int_0^{x_n} \left(u \int_0^{y} u \,\mathrm{d}x' + \eta(\eta_x - \eta_x(0)) \right) \left[\frac{1}{(x_k - y)^2} + \frac{1}{(x_k + y)^2} \right] \mathrm{d}y \\
&= u^2(x_k) + \frac{\Delta x}{2} u_x(x_k) \sum_{i=0}^{k-1} u(x_{i+1}) + u(x_i) \\
&- \frac{1}{\pi} \frac{\Delta x}{2} \sum_{i=0}^{n-1} \left(\frac{\Delta x}{2} u(x_{i+1}) \left(\sum_{j=0}^{i} u(x_j) + u(x_{j+1}) \right) + \eta(x_{i+1})(\eta_x(x_{i+1}) - \eta_x(0)) \right) \left[\frac{1}{(x_k - x_{i+1})^2} + \frac{1}{(x_k + x_{i+1})^2} \right] \\
&+ \left(\frac{\Delta x}{2} u(x_i) \left(\sum_{j=0}^{i-1} u(x_j) + u(x_{j+1}) \right) + \eta(x_i)(\eta_x(x_i) - \eta_x(0)) \right) \left[\frac{1}{(x_k - x_i)^2} + \frac{1}{(x_k + x_i)^2} \right],
\end{aligned}
$$

where the expressions for derivatives depend on the point x_k. Let

$$
F_i = \frac{\Delta x}{2} u(x_i) \left(\sum_{j=0}^{i-1} u(x_j) + u(x_{j+1}) \right) + \eta(x_i)(\eta_x(x_i) - \eta_x(0)), \qquad i = 0, \ldots n,
$$

which we will store as an array. This simplification, we obtain

$$
C_k = u^2(x_k) + \frac{\Delta x}{2} u_x(x_k) \sum_{i=0}^{k-1} u(x_{i+1}) + u(x_i) - \frac{\Delta x}{2\pi} \sum_{i=0}^{n-1} F_{i+1} \left[\frac{1}{(x_k - x_{i+1})^2} + \frac{1}{(x_k + x_{i+1})^2} \right] + F_i \left[\frac{1}{(x_k - x_i)^2} + \frac{1}{(x_k + x_i)^2} \right],
$$

With this in mind, we obtain the finite differences system:

$$
\begin{bmatrix} u(x_0, t_{j+1}) \\ u(x_1, t_{j+1}) \\ u(x_2, t_{j+1}) \\ \vdots \\ u(x_{n-2}, t_{j+1}) \\ u(x_{n-1}, t_{j+1}) \\ u(x_n, t_{j+1}) \end{bmatrix}
=
\begin{bmatrix} u(x_0, t_j) \\ u(x_1, t_j) \\ u(x_2, t_j) \\ \vdots \\ u(x_{n-2}, t_j) \\ u(x_{n-1}, t_j) \\ u(x_n, t_j) \end{bmatrix}
+ \Delta t
\begin{bmatrix} f_1(\eta(x_0, t_j), u(x_0, t_j), t) \\ f_1(\eta(x_1, t_j), u(x_1, t_j), t) \\ f_1(\eta(x_2, t_j), u(x_2, t_j), t) \\ \vdots \\ f_1(\eta(x_{n-2}, t_j), u(x_{n-2}, t_j), t) \\ f_1(\eta(x_{n-1}, t_j), u(x_{n-1}, t_j), t) \\ f_1(\eta(x_n, t_j), u(x_n, t_j), t) \end{bmatrix}
$$

$$= \begin{bmatrix} u(x_0, t_j) \\ u(x_1, t_j) \\ u(x_2, t_j) \\ \vdots \\ u(x_{n-2}, t_j) \\ u(x_{n-1}, t_j) \\ u(x_n, t_j) \end{bmatrix} + \Delta t \mathcal{A} \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-2} \\ \eta_{n-1} \\ \eta_n \end{bmatrix} + \Delta t \mathcal{C}$$

References

[1] Tom M. Apostol, *Mathematical analysis*, Pearson, 1974.