Chapter 3

Perhaps make a comment in KdV derivation in previous section that getting to equation (2.18) requires a lot of work as all the B.C. must also be expanded.

Non-local derivation on the whole line

you might want to consider a transition that bridges the gap from the previous section to this. Hint at "we are looking for a more efficent way of deriving asy. models on diff. & domains".

Recall that the equations of fluid motion are challenging to work with directly, due to the nonlinear boundary conditions and the unknown domain. In addition, these complications lead to difficulties when attempting to deal with questions of existence and well-posedness. To address these issues, reformulations of the problem have been introduced: they result in equivalent problems that are more tractable. While such reformulations can be helpful, they may suffer from other issues. Below, we give a short overview of these formulations, along with explaining the pros and cons of each. Our main goal is to look for a reformulation in the water surface η, and that generalises easily to two-dimensional problem. We chose this criteria with a view towards applications: indeed, in applications, determining the water surface η is the main interest.

For example, for one-dimensional surfaces (no y variable), conformal mappings can be used to eliminate these problems (for an overview, see Dyachenko et al., 1996). However, this approach is limited to one-dimensional surfaces. For both one- and two-dimensional surfaces, other formulations (such as the Hamiltonian formulation given in Zakharov, 1968 or the Zakharov–Craig–Sulem formulation, Craig and Sulem, 1993) reduce the Euler equations to a system of two equations, in terms of surface variables $q = \phi(x, \eta)$ and η only, by introducing a Dirichlet-to-Neumann operator (DNO). A new non-local formulation is introduced ~~in~~ Ablowitz, Fokas, and Musslimani, 2006, (henceforth referred to as the AFM formulation) that results in a system of two equations for the same variables as in the DNO formulation. Both the DNO and AFM formulations reduce the problem from the full fluid domain to a system of equations that depend on the surface elevation $\eta(x)$ and the velocity potential evaluated at the surface, $q(x) = \phi(x, \eta)$. However, these formulations involve solving for ~~an additional function~~ $q(x)$, which may be of little relevance in applications, and hard to measure in experiments.

$q(x,t) = \Phi(x, \eta(x,t), t)$ $\eta(x,t)$

in →

$\eta(x,t)$ $q(x,t)$ $,t$

Indicate that we typically are interested in wave height.

A new formulation is introduced in Oliveras and Vasan, 2013, which reduces the water waves problem to [a system of two equations,] in one variable η. This formulation allows to rigorously investigate one- and two-dimensional water waves. The computation of Stokes-wave asymptotic expansions for periodic waves justifies the use of the formulation; indeed, following Oliveras and Vasan, 2013, the computations can be performed with arguably less effort, especially for two-dimensional waves. Our goal is to further justify the use of this formulation, which we call the $\mathcal{H}$ formulation.

this may be confusing to the reader w/o seeing the eqn. You might just say "formally eliminate $q(x,t)$ so the resulting equation only depend on η.

In this chapter, we first rewrite the water wave problem by introducing a normal-to-tangential operator. We then perform a perturbation expansion for the operator, and proceed to obtain an expression for the surface elevation. Finally, performing asymptotics and applying time scales yields the desired approximate equations. We emphasise that it is not our intention here to further the study of the water wave problem per se, but rather to demonstrate the efficacy of the $\mathcal{H}$ formulation for doing asymptotics.

3.1 Water-wave problem on the whole line: non-local formulation

Recall the water wave problem given in (1.1) along with

$$\phi_{xx} + \phi_{zz} = 0 \qquad -h < z < \eta(x,t) \tag{3.1a}$$

$$\phi_z = 0 \qquad z = -h \tag{3.1b}$$

$$\eta_t + \phi_x \eta_x = \phi_z \qquad z = \eta(x,t) \tag{3.1c}$$

$$\phi_t + g\eta + \frac{1}{2}(\phi_x^2 + \phi_z^2) = 0 \qquad z = \eta(x,t) \tag{3.1d}$$

and consider the velocity potential evaluated at the surface:

$$q(x,t) = \phi(x, \eta(x,t)\cancel{,t}) \qquad \text{inline equation.}$$

$$= \phi(x, \eta(x,t), t)$$

We seek to reformulate the problem ~~(3.1)~~. (1.1) Combining (3.8c) and (3.1d), evaluated at $z = \eta$, we obtain

$$q_t + \frac{1}{2}q_x^2 + g\eta - \frac{1}{2}\frac{(\eta_t + q_x\eta_x)^2}{1+\eta_x^2} = 0, \tag{3.2}$$

which is an equation for two unknowns q, η. We ~~need~~ aim to find an equation in ~~one unknown~~ $\eta(x,t)$ only.

notation issue:

Given the domain $D = \mathbb{R} \times (-h, \eta)$, let

$$\vec{N} = \begin{bmatrix} -\eta_x \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{T} = \begin{bmatrix} 1 \\ \eta_x \end{bmatrix}$$

be vectors normal and tangent to the surface $\cancel{D}$ $z=\eta(x,t)$, respectively. We introduce an operator that maps the normal derivative at a surface η to the tangential derivative at the surface:

$$\mathcal{H}(\eta, D)\{\nabla\phi \cdot \vec{N}\} = \nabla\phi \cdot \vec{T}, \quad \checkmark \tag{3.3}$$

$-i\partial_x$ [illegible]

where $D = -i\cancel{\nabla}$. For convenience, we drop the vector notation. Note that by (3.8c),

skip the matrix notation and make these all inline equation

$$\nabla\phi \cdot N = \underbrace{\begin{bmatrix} \phi_x \\ \phi_z \end{bmatrix} \cdot \begin{bmatrix} -\eta_x \\ 1 \end{bmatrix}}_{*} = \phi_z - \phi_x\eta_x = \eta_t,$$

and by chain rule,

$$\nabla\phi \cdot T = \overbrace{\begin{bmatrix} \phi_x \\ \phi_z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \eta_x \end{bmatrix}}^{*} = \phi_x + \eta_x\phi_z = q_x.$$

This allows us to rewrite (3.3) as

$$\mathcal{H}(\eta, D)\{\eta_t\} = q_{x\cancel{x}}. \tag{3.4}$$

~~and obtain a system~~ Together, (3.2) and (3.4) form a system of two eqns for the two unknowns η and q.

$$\cancel{q_t + \frac{1}{2}q_x^2 + g\eta - \frac{1}{2}\frac{(\eta_t + q_x\eta_x)^2}{1+\eta_x^2} = 0,}$$

$$\cancel{\mathcal{H}(\eta, D)\{\eta_t\} = q_x.}$$

eliminate

~~Differentiate~~ Differentiating (3.2) with respect to x and (3.4) with respect to t allows us to further reduce to a single equation for η

$$\cancel{\partial_t(q_x) + \partial_x\left(\frac{1}{2}q_x^2 + g\eta - \frac{1}{2}\frac{(\eta_t + q_x\eta_x)^2}{1+\eta_x^2}\right) = 0,} \tag{3.5}$$

$$\cancel{\partial_t(\mathcal{H}(\eta, D)\{\eta_t\}) = q_{xt}.} \tag{3.6}$$

~~Substituting (3.4) and (3.6) into (3.5), we obtain~~

$$\begin{aligned} &\partial_t\left(\mathcal{H}(\eta, D)\{\eta_t\}\right) \\ &+ \partial_x\left(\frac{1}{2}\left(\mathcal{H}(\eta, D)\{\eta_t\}\right)^2 + \varepsilon\eta - \frac{1}{2}\frac{(\eta_t + \eta_x\mathcal{H}(\eta, D)\{\eta_t\})^2}{1+\eta_x^2}\right) = 0. \end{aligned} \tag{3.7}$$

Equation (3.7) represents a scalar equation for the water wave surface η. The utility of (3.7) depends on whether we can find a useful representation for the operator $\mathcal{H}(\eta, D)$. In the next section, we proceed to find an equation that the $\mathcal{H}$ operator must satisfy.

try to find a transition to this section.

3.2 Behaviour of the $\mathcal{H}$ operator

Consider the following boundary value problem:

$$\phi_{xx} + \phi_{zz} = 0 \qquad -h < z < \eta(x,t) \tag{3.8a}$$

$$\phi_z = 0 \qquad z = -h \tag{3.8b}$$

$$\nabla\phi \cdot N = f(x) \qquad z = \eta(x,t) \tag{3.8c}$$

same D as earlier

Let φ be harmonic on D. Using (3.8a) and that φ_z is also harmonic on D, we have

$$\varphi_z(\phi_{xx} + \phi_{zz}) - \phi((\varphi_z)_{zz} + (\varphi_z)_{xx}) = 0.$$

Taking the integral over the domain yields

$$\int_{-\infty}^{\infty}\int_{-h}^{\eta(x)} \varphi_z(\phi_{xx} + \phi_{zz}) - \phi((\varphi_z)_{zz} + (\varphi_z)_{xx})\,\mathrm{d}z\,\mathrm{d}x = 0.$$

An application of Green's theorem gives

$$\int_{\partial D} \varphi_z(\nabla\phi \cdot \mathbf{n}) - \phi(\nabla\varphi_z \cdot \mathbf{n})\,\mathrm{d}s = 0, \tag{3.9}$$

where ∂D is the boundary of the domain, $\mathrm{d}s$ is the area element, and $\mathbf{n}$ is the normal vector. Now, observe that

$$-\nabla\varphi_z \cdot \mathbf{n} = \nabla\varphi_x \cdot \mathbf{t},$$

where $\mathbf{t}$ is the tangential vector. We use this to rewrite (3.9) and obtain the following contour integral:

$$\begin{aligned} 0 &= \int_{\partial D} \varphi_z(\nabla\phi \cdot N) + \phi(\nabla\varphi_z \cdot \mathbf{t})\, \mathrm{d}s \\ &= \int_{\partial D} \varphi_z(\phi_z \,\mathrm{d}x - \phi_x \,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) \end{aligned}$$

We split the contour into four segments:

I typically use $\oint_{\partial D} (\cdot)\, ds = \int_{-\infty}^{\infty} (\cdot)\Big|_{z=-h} ds + \ldots$

$$\begin{aligned} \int_{\partial D} &= \int_{-\infty}^{\infty} \Big|^{z=-h} + \int_{-h}^{\eta(x)} \Big|^{x\to\infty} + \int_{\infty}^{-\infty} \Big|^{z=\eta(x)} + \int_{\eta(x)}^{-h} \Big|^{x\to-\infty} \\ &= \int_{-\infty}^{\infty} \Big|^{z=-h} + \int_{-h}^{\eta(x)} \Big|^{x\to\infty} - \int_{-\infty}^{\infty} \Big|^{z=\eta(x)} - \int_{-h}^{\eta(x)} \Big|^{x\to-\infty}. \end{aligned}$$

people would might not like this notation.

Ask Dave if he has a preference.

Consider each segment:

sketch the domain and consider a formal principle value $\lim_{R\to\infty} \int_{-R}^{R} \sim dx$.

- As $|x| \to \infty$, we know that ϕ and its gradient vanish, so the integrals

$$\int_{-h}^{\eta(x)} \Big|^{x\to\infty}, \int_{-h}^{\eta(x)} \Big|^{x\to-\infty}$$

ok. definitely don't use this here.

You could simple state "there are no contributions as $|x|\to\infty$". ~~[illegible]~~

vanish.

- At $z = -h$, $\mathrm{d}z = 0$, so we have

$$\begin{aligned} \int_{-\infty}^{\infty} &\varphi_z(\phi_z \,\mathrm{d}x - \phi_x \,\mathrm{d}z) + \phi(\varphi_{xx}\,\mathrm{d}x + \varphi_{xz}\,\mathrm{d}z) \\ &= \int_{-\infty}^{\infty} \phi\varphi_{xx}\,\mathrm{d}x \qquad (\text{since } \phi_z = 0 \text{ at } z = -h) \\ &= \phi(x,-h)\varphi_x(x,-h)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \phi_x(x,-h)\varphi_x(x,-h)\,\mathrm{d}x \\ &= 0, \end{aligned}$$

where we pick φ such that $\varphi_x(x,-h) = 0$.

will resume tomorrow.

- At $z = \eta$, $\mathrm{d}z = \eta_x \, \mathrm{d}x$, so we have

$$
\begin{aligned}
\int_{-\infty}^{\infty} & \varphi_z(\phi_z - \phi_x\eta_x) + \phi(\varphi_{xx} + \varphi_{xz}\eta_x) \, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \varphi_z(\phi_z - \phi_x\eta_x) + \phi \frac{\mathrm{d}\varphi_x(x,\eta)}{\mathrm{d}x} \, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \varphi_z(\phi_z - \phi_x\eta_x) - \varphi_x \frac{\mathrm{d}\phi(x,\eta)}{\mathrm{d}x} \, \mathrm{d}x \qquad \text{(integration by parts)} \\
&= \int_{-\infty}^{\infty} \varphi_z \begin{pmatrix} \phi_x \\ \phi_z \end{pmatrix} \cdot \begin{pmatrix} -\eta_x \\ 1 \end{pmatrix} - \varphi_x \begin{pmatrix} \phi_x \\ \phi_z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \phi_x\eta_x \end{pmatrix} \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \varphi_z \nabla\phi \cdot N - \varphi_x \nabla\phi \cdot T \, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \varphi_z f(x) - \varphi_x(x,\eta)\mathcal{H}(\eta, D)\{f(x)\} \, \mathrm{d}x \qquad \text{(recall (3.8c) and (3.3)).}
\end{aligned}
$$

Combining segments, we obtain:

$$
\int_{-\infty}^{\infty} \varphi_z f(x) - \varphi_x(x,\eta)\mathcal{H}(\eta, D)\{f(x)\} \, \mathrm{d}x = 0. \tag{3.10}
$$

Note that $\varphi(x,z) = e^{-ikx}\sinh(k(z+h)), k \in \mathbb{R}$ is one solution of

$$
\Delta\varphi = 0, \qquad \varphi_z(-h,z) = 0.
$$

Then, (3.10) becomes

$$
\int_{-\infty}^{\infty} e^{-ikx}(k\cosh(k(\eta+h))f(x) + ik\sinh(k(\eta+h))\mathcal{H}(\eta, D)\{f(x)\}) \, \mathrm{d}x = 0.
$$