8

Solitons and integrable equations

In this chapter we begin by finding traveling wave solutions. We then discuss the so-called Miura transformation that is used to find an infinite number of conservation laws and the associated linear scattering problem, the time-independent Schrödinger scattering problem, for the Korteweg–de Vries (KdV) equation. Lax pairs and an expansion method that shows how compatible linear problems are related to other nonlinear evolution equations are described. An evolution operator, sometimes referred to as a recursion operator, is also included. This operator can be used to find general classes of integrable nonlinear evolution equations. We point out that there are a number of texts that discuss these matters (Ablowitz and Segur, 1981; Novikov et al., 1984; Calogero and Degasperis, 1982; Dodd et al., 1984; Faddeev and Takhtajan, 1987); we often follow closely the analytical development Ablowitz and Clarkson (1991).

8.1 Traveling wave solutions of the KdV equation

We begin with the Korteweg–de Vries equation in non-dimensional form

$$u_t + 6uu_x + u_{xxx} = 0. \tag{8.1}$$

For the dimensional form of the equation we refer the reader to Chapter 1. An important property of the KdV equation is the existence of traveling wave solutions, including solitary wave solutions. The solutions are written in terms of elementary or elliptic functions. Since a discussion of this topic appears in Chapter 1 we will sometimes call on those results here.

A traveling wave solution of a PDE in one space, one time dimension, such as the KdV equation, where $t \in \mathbb{R}$, $x \in \mathbb{R}$ are temporal and spatial variables and $u \in \mathbb{R}$ the dependent variable, has the form

$$u(x,t) = w(x - Ct) = w(z). \tag{8.2}$$

A solitary wave is a special traveling wave solution, which is bounded and has constant asymptotic states as $z \to \mp\infty$.

To obtain traveling wave solutions of the KdV equation (8.1), we seek a solution in the form (8.2) that yields a third-order ordinary differential equation for w

$$\frac{d^3 w}{dz^3} + 6w\frac{dw}{dz} - C\frac{dw}{dz} = 0.$$

Integrating this once gives

$$\frac{d^2 w}{dz^2} + 3w^2 - Cw = E_1, \tag{8.3}$$

with E_1 an arbitrary constant. Multiplying (8.3) by dw/dz and integrating again yields

$$\tfrac{1}{2}\left(\frac{dw}{dz}\right)^2 = f(w) = -(w^3 - Cw^2/2 - E_1 w - E_2), \tag{8.4}$$

with E_2 another arbitrary constant. We are interested in obtaining real, bounded solutions for the KdV equation (8.1), so we require that $f(w) \geq 0$. This leads us to study the zeros of $f(w)$. There are two cases to consider: when $f(w)$ has only one real zero or when $f(w)$ has three real zeros.

If $f(w)$ has only one real zero, say $w = \alpha$, so that $f(w) = -(w-\alpha)^3$, we can integrate directly to find that

$$u(x,t) = w(x - Ct) = \alpha - \frac{2}{(x - 6\alpha t - x_0)^2}.$$

So there are no bounded solutions in this case.

On the other hand, if $f(w)$ has three real zeros, which we take without loss of generality to satisfy $\alpha \leq \beta \leq \gamma$, then

$$f(w) = -(w-\alpha)(w-\beta)(w-\gamma), \tag{8.5}$$

where $\gamma = 2(\alpha + \beta + \gamma)$, $E_1 = -2(\alpha\beta + \beta\gamma + \gamma\alpha)$, $E_2 = 2\alpha\beta\gamma$.

Suppose α, β, γ are distinct; then the solution of the KdV equation may be expressed in terms of the Jacobian elliptic function $\mathrm{cn}(z;m)$ (see also Chapter 1)

$$u(x,t) = w(x - Ct) = \beta + (\gamma - \beta)\,\mathrm{cn}^2\left\{\left[\tfrac{1}{2}(\gamma-\alpha)\right]^{1/2}[x - Ct - x_0]\,;m\right\}, \tag{8.6}$$

with

$$C = 2(\alpha + \beta + \gamma), \quad m = \frac{\gamma - \beta}{\gamma - \alpha}, \quad \text{where} \quad 0 < m < 1, \tag{8.7}$$

x_0 is a constant and m denotes the modulus of the Jacobian elliptic function. These solutions of the KdV equation were originally found by Korteweg and de Vries; in fact they called these solutions cnoidal waves. Since $\mathrm{cn}(z; m)$ has period $[z] = 4K(m)$ where K is the complete elliptic integral of the first kind, we have that the period of $u(x, t)$ is $[x] = L = 2K(m)/\left(\frac{1}{2}(\gamma - \alpha)\right)$.

Noting that $\mathrm{cn}(z; m) = \cos(z) + O(m)$ for $m \ll 1$ ($\beta \to \gamma$), it follows that

$$u(x, t) = w(x - Ct) \sim \tfrac{1}{2}(\beta + \gamma) + \frac{\gamma - \beta}{2} \cos\{2\kappa[x - Ct - x_0]\},$$

where $\kappa^2 = \frac{1}{2}(\gamma - \alpha)$, $C \approx 2(\alpha - 2\beta)$. Thus when $\gamma \to \beta$ we have the limiting case of a sinusoidal wave.

Next suppose in (8.6), (8.7) we take the limit $\beta \to \alpha$ (i.e., $m \to 1$), with $\beta \neq \gamma$; then since $\mathrm{cn}(z; 1) = \mathrm{sech}(z)$ it reduces to the solitary wave solution

$$u(x, t) = w(x - Ct) = \beta + 2\kappa^2 \,\mathrm{sech}^2 \left\{\kappa[x - (4\kappa^2 + 6\beta u_0)t + \delta_0]\right\} \tag{8.8}$$

where $\kappa^2 = \frac{1}{2}(\gamma - \beta)$.

It then follows that the speed of the wave relative to the uniform state u_0 is related to the amplitude: bigger solitary waves move faster than smaller ones. Note, if $\beta = 0$ ($\alpha = 0$), then the wave speed is twice the amplitude. It is also clear that the amplitude $2\kappa^2$ is independent of the uniform background β. Finally the width of the solitary wave is inversely proportional to κ; hence narrower solitary waves are taller and faster. We note that these solitary waves are consistent with the observations of J. Scott Russell mentioned in Chapter 1.

If $\alpha < \beta = \gamma$, then the solution of (8.6) is given by

$$\begin{aligned} u(x, t) &= w(x - Ct), \\ &= \beta - (\beta - \alpha)\sec^2\left\{\left[\tfrac{1}{2}(\beta - \alpha)\right]^{1/2}[x - 2(\alpha + 2\beta)t - x_0]\right\} \end{aligned}$$

[this can be obtained by using (8.5) and directly integrating], where x_0 is a constant. This solution is unbounded unless $\alpha = \beta$ in which case $u = u_0 = \beta$.

8.2 Solitons and the KdV equation

As discussed in Chapter 1, the physical model that motivated the recent discoveries associated with the KdV equation (1.5) was a problem of a one-dimensional anharmonic lattice of equal masses coupled by nonlinear strings that remarkably was studied numerically by Fermi, Pasta and Ulam (FPU).

The FPU model consisted of identical masses, m, connected by nonlinear springs with the force law $F(\Delta) = -k(\Delta + \alpha\Delta^p)$, with $k > 0$, α being spring constants:

$$\begin{aligned} m\frac{d^2 y_n}{dt^2} &= k[y_{n+1} - y_n + \alpha(y_{n+1} - y_n)^p] - k[y_n - y_{n-1} + \alpha(y_n - y_{n-1})^p] \\ &= k(y_{n+1} - 2y_n + y_{n-1}) + k\alpha[(y_{n+1} - y_n)^p - (y_n - y_{n-1})^p] \end{aligned} \tag{8.9}$$

for $n = 1, 2, \ldots, N-1$, with $y_0 = 0 = y_N$ and initial condition $y_n(0) = \sin(n\pi/N)$, $y_{n,t}(0) = 0$.

In 1965, Zabusky and Kruskal transformed the differential-difference equation (8.9) to a continuum model and studied the case $p = 2$. Calling the length between springs h, $t = \omega t$ (with $\omega = \sqrt{K/m}$), $x = x/h$ with $x = nh$ and expanding $y_{n\pm1}$ in a Taylor series, then (dropping the tildes) (8.9) reduces to

$$y_{tt} = y_{xx} + \varepsilon y_x y_{xx} + \tfrac{1}{12}h^2 y_{xxxx} + O(\varepsilon h^2, h^4), \tag{8.10}$$

where $\varepsilon = 2\alpha h$. They found (see Chapter 1) an asymptotic reduction by looking for a solution of the form (unidirectional waves)

$$y(x,t) \sim \phi(\xi, \tau), \qquad \xi = x - t, \qquad \tau = \tfrac{1}{2}\varepsilon t,$$

whereupon (8.10), with $u = \phi_\xi$, and ignoring small terms, where $\delta^2 = h^2/12\varepsilon$, reduces to the KdV equation [with different coefficients from (8.1)]

$$u_\tau + uu_\xi + \delta^2 u_{\xi\xi\xi} = 0. \tag{8.11}$$

It is also noted that when the force law is taken as $F(\Delta) = -k(\Delta + \alpha\Delta^p)$ with $p = 3$ then, after rescaling, one obtains the modified KdV (mKdV) equation (see also the next section)

$$u_\tau + u^2 u_\xi + \delta^2 u_{\xi\xi\xi} = 0.$$

We saw earlier that the KdV equation possesses a solitary wave solution; see (8.8). Although solitary wave solutions of the KdV equation had been well known for a long time, it was not until Zabusky and Kruskal, in 1965, did extensive numerical studies for the KdV equation (8.11) that the remarkable properties of the solitary waves associated with the KdV equation were discovered. Zabusky and Kruskal considered the initial value problem for the KdV equation (8.11) with ($\delta^2 \ll 1$) the initial condition

$$u(\xi, 0) = \cos(\pi\xi), \qquad 0 \le \xi \le 2,$$

where u, u_ξ, $u_{\xi\xi}$ are periodic on $[0, 2]$ for all t. As also described in Chapter 1, they found that after a short time the wave steepens and almost produces a shock, but the dispersive term $\delta^2 u_{xxx}$ then becomes significant and a balance between the nonlinear and dispersion terms ensues. Later the solution develops a train of (eight) well-defined waves, each like a solitary wave (i.e., sech^2 functions), with the faster (taller) waves catching up and overtaking the slower (smaller) waves.

From this they observed that when two solitary waves with different wave speeds are initially well separated, with the larger one to the left (the waves are traveling from left to right), then the faster, taller wave catches up and subsequently overlaps the slower, smaller one; the waves interact nonlinearly. After the interaction, they found that the waves separate, with the larger one in front and slower one behind, but both having regained the same amplitudes they had before the interactions, the only effect of the interaction being a phase shift; that is, the centers of the waves are at different positions than where they would have been had there been no interaction. Thus they deduced that these nonlinear solitary waves essentially interacted elastically. Making the analogy with particles, Zabusky and Kruskal called these special waves *solitons*.

Mathematically speaking, a *soliton* is a solitary wave that asymptotically preserves its shape and velocity upon nonlinear interaction with other solitary waves, or more generally, with another (arbitrary) localized disturbance.

In the physics literature the term soliton is often meant to be a solitary wave without the elastic property. When dealing with physical problems in order to be consistent with what is currently accepted, we frequently refer to solitons in the latter sense (i.e., solitary waves without the elastic property).

Kruskal and Zabusky's remarkable numerical discovery demanded an analytical explanation and a detailed mathematical study of the KdV equation. However the KdV equation is nonlinear and at that time no general method of solution for nonlinear equations existed.

8.3 The Miura transformation and conservation laws for the KdV equation

We have seen that the KdV equation arises in the description of physically interesting phenomena; however interest in this nonlinear evolution equation

 https://doi.org/10.1017/CBO9780511998324.009

is also due to the fact that analytical developments have led researchers to consider the KdV equation to be an *integrable* or sometimes referred to as an *exactly solvable* equation. This terminology is a consequence of the fact that the initial value problem corresponding to appropriate data (e.g., rapidly decaying initial data) for the KdV equation can be "solved exactly", or more precisely, linearized by a method that employs *inverse scattering*. The inverse scattering method was discovered by Gardner, Greene, Kruskal and Miura (1967) as a means for solving the initial value problem for the KdV equation on the infinite line, for initial values that decay sufficiently rapidly at infinity. Subsequently this method has been significantly enhanced and extended by many researchers and is usually termed the inverse scattering transform (IST) (cf. Ablowitz and Segur, 1981; Ablowitz and Clarkson, 1991; Ablowitz et al., 2004b). We also note that Lax and Levermore (1983a,b,c) and Venakides (1985) have reinvestigated (8.11) with δ small; that leads to an understanding of the weak limit of KdV.

The KdV equation has several other remarkable properties beyond the existence of solitons (discussed earlier), including the possession of an infinite number of polynomial conservation laws. Discovering this was important in the development of the general method of solution for the KdV equation.

Associated with a partial differential equation denoted by

$$F[x, t; u(x, t)] = 0, \tag{8.12}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}$ are temporal and spatial variables and $u(x, t) \in \mathbb{R}$ the dependent variable (in the argument of F additional partial derivatives of u are understood) may be a *conservation law*; that is, an equation of the form

$$\partial_t T_i + \partial_x X_i = 0, \tag{8.13}$$

that is satisfied for all solutions of (8.12). Here $T_i(x, t; u)$, called the *conserved density*, and $X_i(x, t; u)$, the *associated flux*, are, in general, functions of x, t, u and the partial derivatives of u – see also Chapter 2.

If additionally, $u \to 0$ as $|x| \to \infty$ sufficiently rapidly, then by integrating (8.13) we get

$$\frac{d}{dt}\int_{-\infty}^{\infty} T_i(x, t; u)\, dx = 0$$

in which case

$$\int_{-\infty}^{\infty} T_i(x, t; u)\, dx = c_i,$$

where c_i, constant, is called the *conserved quantity*.

For the KdV equation (1.5), the first three conservation laws are

$$(u)_t + (3u^2 + u_{xx})_x = 0,$$
$$(u^2)_t + \left(4u^3 + 2uu_{xx} - u_x^2\right)_x = 0,$$
$$\left(u^3 - \tfrac{1}{2}u_x^2\right)_t + \left(\tfrac{9}{2}u^4 + 3u^2u_{xx} - 6uu_x^2 - u_xu_{xxx} + \tfrac{1}{2}u_{xx}^2\right)_x = 0.$$

The first two of these correspond to conservation of mass (or sometimes momentum) and energy, respectively. The third is related to the Hamiltonian (Whitham, 1965). The fourth and fifth conservation laws for the KdV equation were found by Kruskal and Zabusky. Subsequently some additional ones were discovered by Miura and it was conjectured that there was an infinite number.

After studying these conservation laws of the KdV equation and those associated with another equation, the so-called modified KdV (mKdV) equation that we write as

$$m_t - 6m^2m_x + m_{xxx} = 0, \tag{8.14}$$

Miura (1968) discovered that with a solution m of (8.14), then u given by

$$u = -m^2 - m_x, \tag{8.15}$$

is a solution of the KdV equation (8.1). Equation (8.15) is known as the *Miura transformation*. Direct substitution of (8.15) into (1.5) yields

$$\begin{aligned}
u_t + 6uu_x + u_{xxx} &= -\left(m^2 + m_x\right)_t + 6\left(m^2 + m_x\right)\left(m^2 + m_x\right)_x - \left(m^2 + m_x\right)_{xxx} \\
&= -(2mm_t + m_{xt}) + 6\left(m^2 + m_x\right)(2mm_x + m_{xx}) \\
&\quad - (2mm_{xxx} + 6m_xm_{xx} + m_{xxxx}) \\
&= -\left(m_{xt} - 6m^2m_{xx} - 12mm_x^2 + v_{xxxx}\right) \\
&\quad - (2mm_t - 12m^3m_x + 2mm_{xxx}) \\
&= -(2m + \partial_x)(m_t - 6m^2m_x + m_{xxx}) \\
&= -(2m + \partial_x)\left(m_t - 6m^2m_x + m_{xxx}\right).
\end{aligned}$$

Note that every solution of the mKdV equation (8.14) leads, via Miura's transformation (8.15), to a solution of the KdV equation, but the converse is not true; i.e., not every solution of the KdV equation can be obtained from a solution of the mKdV equation – see, for example, Ablowitz, Kruskal and Segur (1979). Miura's transformation leads to many other important results related to the KdV equation. Initially it formed the basis of a proof that the KdV and mKdV equations have an infinite number of conserved densities (Miura et al., 1968), which is outlined below. Importantly, Miura's transformation (8.15) was also critical in the development of the *inverse scattering method* or *inverse scattering transform* (IST) for solving the initial value problem for the KdV equation (8.1) – details are in the next chapter.

To show that the KdV equation admits an infinite number of conservation laws, define a variable w by the relation

$$u = w - \varepsilon w_x - \varepsilon^2 w^2, \tag{8.16}$$

which is a generalization of Miura's transformation (8.15).[1] Then the equivalent relation is

$$u_t + 6uu_x + u_{xxx} = \left(1 - \varepsilon\partial_x - 2\varepsilon^2 w\right)(w_t + 6(w - \varepsilon^2 w^2)w_x + w_{xxx}).$$

Therefore u, as defined by (8.16), is a solution of the KdV equation provided that w is a solution of

$$w_t + 6(w - \varepsilon^2 w^2)w_x + w_{xxx} = w_t + \partial_x(3w^2 - 2w^3 + w_{xx}) = 0. \tag{8.17}$$

Since the KdV equation does not contain ε, then its solution u depends only upon x and t. However w, a solution of (8.17), depends on x, t and ε. Since ε is arbitrary we can seek a formal power series solution of (8.16), in the form

$$w(x, t; \varepsilon) = \sum_{n=0}^{\infty} w_n(x, t)\varepsilon^n. \tag{8.18}$$

The equation (8.17) is in conservation form, thus

$$\int_{-\infty}^{\infty} w(x, t; \varepsilon)\, dx = \text{constant},$$

and so

$$\int_{-\infty}^{\infty} w_n(x, t)\, dx = \text{constant},$$

for each $n = 0, 1, 2, \ldots$. Substituting (8.18) into (8.16), equating coefficients of powers of ε and solving recursively gives

$$\begin{aligned}
w_0 &= u, \\
w_1 &= w_{0,x} = u_x, \\
w_2 &= w_{1,x} + w_0^2 = u_{xx} + u^2, \\
w_3 &= w_{2,x} + 2w_0 w_1 = u_{xxx} + 4uu_x = (u_x x + 2u^2)_x\,, \\
w_4 &= w_{3,x} + 2w_0 w_2 + w_1^2 = u_{xxxx} + 6uu_{xx} + 5u_x^2 + 2u^3,
\end{aligned} \tag{8.19}$$

[1] Taking $m = \varepsilon w - 1/\varepsilon$ and performing a Galilean transformation.

etc. Continuing to all powers of ε gives an infinite number of conserved densities. The corresponding conservation laws may be found by substituting (8.18)–(8.19) into (8.17) and equating coefficients of powers of ε. We further note that odd powers of ε are trivial since they are exact derivatives. So the even powers give the non-trivial conservation laws for the KdV equation.

8.4 Time-independent Schrödinger equation and a compatible linear system

The Miura transformation (8.15) may be viewed as a Riccati equation for m in terms of u. It is well known that it may be linearized by the transformation $m = v_x/v$, which yields

$$u + m^2 + m_x = u + \frac{v_x^2}{v^2} + \left(\frac{v_{xx}}{v} - \frac{v_x^2}{v^2}\right) = 0,$$

and so $u = -v_{xx}/v$ or

$$v_{xx} + uv = 0. \tag{8.20}$$

This is reminiscent of the Cole–Hopf transformation (see Section 3.6). However, (8.20) does not yield an explicit linearization for the KdV equation. Much more must be done. Since the KdV equation is Galilean invariant – that is, it is invariant under the transformation

$$(x, t, u(x,t)) \to (x - 6\lambda t, t, u(x,t) + \lambda),$$

where λ is some constant – then it is natural to consider

$$\mathcal{L}v := v_{xx} + u(x,t)v = \lambda v. \tag{8.21}$$

This equation is the time-independent Schrödinger equation that has been extensively studied by mathematicians and physicists; we note that here t plays the role of a parameter and $u(x,t)$ the potential.

The associated time dependence of the eigenfunctions of (8.21) is given by

$$v_t = (\gamma + u_x)v - (4\lambda + 2u)v_x, \tag{8.22}$$

where γ is an arbitrary constant. This form is different from the original one found by Gardner, Greene, Kruskal and Miura (1967); see also Gardner et al. (1974): it is simpler and leads to the same results. Then if $\lambda = \lambda(t)$ from (8.21) and (8.22) we obtain

$$\begin{aligned}
v_{txx} &= \left[(\gamma + u_x)(\lambda - u) + u_{xxx} + 6uu_x\right] v - (4\lambda + 2u)(\lambda - u)v_x, \\
v_{xxt} &= \left[(\lambda - u)(\gamma + u_x) - u_t + \lambda_t\right] v - (\lambda - u)(4\lambda + 2u)v_x.
\end{aligned}$$

Therefore (8.21) and (8.22) are compatible, i.e., $v_{xxt} = v_{txx}$, if $\lambda_t = 0$ and u satisfies the KdV equation (1.5). Similarly, if (1.5) is satisfied, then necessarily the eigenvalues must be time independent (i.e., $\lambda_t = 0$).

It turns out that the eigenvalues and the behavior of the eigenfunctions of the scattering problem (8.21) as $|x| \to \infty$, determine the *scattering data*, $S(\lambda, t)$, which in turn depends upon the potential $u(x, t)$. The *direct scattering problem* is to map the potential into the scattering data. The time evolution equation (8.22) takes initial scattering data $S(\lambda, t = 0)$ to data at any time t, i.e., to $S(\lambda, t)$. The *inverse scattering problem* is to reconstruct the potential from the scattering data. The details of the solution method of the KdV equation (8.1) are described in the next chapter.

8.5 Lax pairs

Recall that the operators associated with the KdV equation are

$$\mathcal{L}v = v_{xx} + u(x, t)v = \lambda v,$$
$$v_t = \mathcal{M}v = (u_x + \gamma)v - (2u + 4\lambda)v_x,$$

where γ is an arbitrary constant parameter and λ is a spectral parameter that is constant ($\lambda_t = 0$). We found that these equations are compatible (i.e., $v_{xxt} = v_{txx}$) provided that u satisfies the KdV equation (8.1).

Lax (1968) put the inverse scattering method for solving the KdV equation into a more general framework that subsequently paved the way to generalizations of the technique as a method for solving other partial differential equations. Consider two operators $\mathcal{L}$ and $\mathcal{M}$, where $\mathcal{L}$ is the operator of the spectral problem and $\mathcal{M}$ is the operator governing the associated time evolution of the eigenfunctions

$$\mathcal{L}v = \lambda v, \tag{8.23}$$

$$v_t = \mathcal{M}v. \tag{8.24}$$

Now take $\partial/\partial t$ of (8.23), giving

$$\mathcal{L}_t v + \mathcal{L}v_t = \lambda_t v + \lambda v_t;$$

hence using (8.24)

$$\begin{aligned} \mathcal{L}_t v + \mathcal{L}\mathcal{M}v &= \lambda_t v + \lambda\mathcal{M}v, \\ &= \lambda_t v + \mathcal{M}\lambda v, \\ &= \lambda_t v + \mathcal{M}\mathcal{L}v. \end{aligned}$$

Therefore we obtain

$$[\mathcal{L}_t + (\mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L})]\, v = \lambda_t v,$$

and hence in order to solve for non-trivial eigenfunctions $v(x,t)$

$$\mathcal{L}_t + [\mathcal{L}, \mathcal{M}] = 0, \tag{8.25}$$

where

$$[\mathcal{L}, \mathcal{M}] := \mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L},$$

if and only if $\lambda_t = 0$. Equation (8.25) is sometime called *Lax's equation* and $\mathcal{L}$, $\mathcal{M}$ are called the Lax pair. Furthermore, (8.25) contains a nonlinear evolution equation for suitably chosen $\mathcal{L}$ and $\mathcal{M}$. For example, if we take [suitably replacing λ by $\partial_x^2 + u$ in (8.24)]

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} + u, \tag{8.26}$$

$$\mathcal{M} = \gamma - 3u_x - 6u\frac{\partial}{\partial x} - 4\frac{\partial^3}{\partial x^3}, \tag{8.27}$$

then $\mathcal{L}$ and $\mathcal{M}$ satisfy (8.25) and u satisfies the KdV equation (8.1). Therefore, the KdV equation may be thought of as the compatibility condition of the two linear operators given by (8.26), (8.27). As we will see below, there is a general class of equations that are associated with the Schrödinger operator (8.26).

8.6 Linear scattering problems and associated nonlinear evolution equations

Following the development of the method of inverse scattering for the KdV equation by Gardner, Greene, Kruskal and Miura (1967), it was then of considerable interest to determine whether the method would be applicable to other physically important nonlinear evolution equations. The method was thought to possibly only apply to one physically important equation; perhaps it was analogous to the Cole–Hopf transformation (Cole, 1951; Hopf, 1950) that linearizes Burgers' equation (see Chapter 3)

$$u_t + 2uu_x - u_{xx} = 0.$$

Namely, if we make the transformation $u = -\phi_x/\phi$, then $\phi(x,t)$ satisfies the linear heat equation $\phi_t - \phi_{xx} = 0$. [We remark that Forsyth (1906) first pointed out the relationship between Burgers' equation and the linear heat equation]. But Burgers' equation is the only known physically significant equation linearizable by the above transformation.

However, Zakharov and Shabat (1972) proved that the method was indeed more general. They extended Lax's method and related the nonlinear Schrödinger equation

$$iu_t + u_{xx} + \kappa u^2 u^* = 0, \tag{8.28}$$

where $*$ denotes the complex conjugate and κ is a constant, to a certain linear scattering problem or Lax pairs where now $\mathcal{L}$ and $\mathcal{M}$ are 2×2 matrix operators. Using these operators, Zakharov and Shabat were able to solve (8.28), given initial data $u(x, 0) = f(x)$, assuming that $f(x)$ decays sufficiently rapidly as $|x| \to \infty$. Shortly thereafter, Wadati (1974) found the method of solution for the modified KdV equation

$$u_t - 6u^2 u_x + u_{xxx} = 0, \tag{8.29}$$

and Ablowitz, Kaup, Newell and Segur (1973a), motivated by some important observations by Kruskal, solved the sine–Gordon equation:

$$u_{xt} = \sin u. \tag{8.30}$$

Ablowitz, Kaup, Newell and Segur (1973b, 1974) then developed a general procedure, which showed that the initial value problem for a remarkably large class of physically interesting nonlinear evolution equations could be solved by this method. There is also an analogy between the Fourier transform method for solving the initial value problem for linear evolution equations and the inverse scattering method. This analogy motivated the term: *inverse scattering transform* (IST) (Ablowitz et al., 1974) for the new method.

8.6.1 Compatible equations

Next we outline a convenient method for finding "integrable" equations. Consider two linear equations

$$\mathbf{v}_x = \mathsf{X}\mathbf{v}, \qquad \mathbf{v}_t = \mathsf{T}\mathbf{v}, \tag{8.31}$$

where $\mathbf{v}$ is an n-dimensional vector and X and T are $n\times n$ matrices. If we require that (8.31) are compatible – that is, requiring that $\mathbf{v}_{xt} = \mathbf{v}_{tx}$ – then X and T must satisfy

$$\mathsf{X}_t - \mathsf{T}_x + [\mathsf{X}, \mathsf{T}] = 0. \tag{8.32}$$

Equation (8.32) and Lax's equation (8.26), (8.27) are similar; equation (8.31) is somewhat more general as it allows more general eigenvalue dependence other than $\mathcal{L}v = \lambda v$.

The method is sometimes referred to as the AKNS method (Ablowitz et al., 1974). As an example, consider the 2×2 scattering problem [this is a generalization of the Lax pair studied by Zakharov and Shabat (1972)] given by (with the eigenvalue denoted as k)

$$\begin{aligned} v_{1,x} &= -ikv_1 + q(x,t)v_2, \\ v_{2,x} &= ikv_2 + r(x,t)v_1, \end{aligned} \tag{8.33}$$

with the linear time dependence given by

$$\begin{aligned} v_{1,t} &= Av_1 + Bv_2, \\ v_{2,t} &= Cv_1 + Dv_2, \end{aligned} \tag{8.34}$$

where A, B, C and D are scalar functions of $q(x,t)$, $r(x,t)$, and their derivatives, and k. In terms of X and T, we specify that

$$\mathsf{X} = \begin{pmatrix} -\mathrm{i}k & q \\ r & \mathrm{i}k \end{pmatrix}, \qquad \mathsf{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Note that if there were any x-derivatives on the right-hand side of (8.34) then they could be eliminated by use of (8.33). Furthermore, when $r = -1$, then (8.33) reduces to the Schrödinger scattering problem

$$v_{2,xx} + (k^2 + q)v_2 = 0. \tag{8.35}$$

It is interesting to note that physically interesting nonlinear evolution equations arise from this procedure when either $r = -1$ or $r = q^*$ (or $r = q$ if q is real).

This method provides an elementary technique that allows us to find nonlinear evolution equations. The compatibility of (8.33)–(8.34) – that is, requiring that $v_{j,xt} = v_{j,tx}$, for $j = 1, 2$ – and the assumption that the eigenvalue k is time-independent – that is, $dk/dt = 0$ – imposes a set of conditions that A, B, C and D must satisfy. Sometimes the nonlinear evolution equations obtained this way are referred to as *isospectral* flows. Namely,

$$\begin{aligned} v_{1,xt} &= -ikv_{1,t} + q_t v_2 + qv_{2,t}, \\ &= -ik(Av_1 + Bv_2) + q_t v_2 + q(Cv_1 + Dv_2), \\ v_{1,tx} &= A_x v_1 + Av_{1,x} + B_x v_2 + Bv_{2,x}, \\ &= A_x v_1 + A(-ikv_1 + qv_2) + B_x v_2 + B(ikv_2 + rv_1). \end{aligned}$$

Hence by equating the coefficients of v_1 and v_2, we obtain

$$\begin{aligned} A_x &= qC - rB, \\ B_x + 2ikB &= q_t - (A - D)q, \end{aligned} \tag{8.36}$$

respectively. Similarly

$$\begin{aligned}
v_{2,xt} &= ikv_{2,t} + r_t v_1 + rv_{1,t},\\
&= ik(Cv_1 + Dv_2) + r_t v_1 + a(A_1 + Bv_2),\\
v_{2,tx} &= C_x v_1 + Cv_{1,x} + D_x v_2 + Dv_{2,x},\\
&= C_x v_1 + C(-ikv_1 + qv_2) + D_x v_2 + D(ikv_2 + rv_1),
\end{aligned}$$

and equating the coefficients of v_1 and v_2 we obtain

$$\begin{aligned}
C_x - 2ikC &= r_t + (A - D)r,\\
(-D)_x &= qC - rB.
\end{aligned} \tag{8.37}$$

Therefore, from (8.36), without loss of generality, we may assume $D = -A$, and hence it is seen that A, B and C necessarily satisfy the compatibility conditions

$$\begin{aligned}
A_x &= qC - rB,\\
B_x + 2ikB &= q_t - 2Aq,\\
C_x - 2ikC &= r_t + 2Ar.
\end{aligned} \tag{8.38}$$

We next solve (8.38) for A, B and C, thus ensuring that (8.33) and (8.34) are compatible. In general, this can only be done if another condition (on r and q) is satisfied, this condition being the evolution equation. Since k, the eigenvalue, is a free parameter, we may find solvable evolution equations by seeking finite power series expansions for A, B and C:

$$A = \sum_{j=0}^{n} A_j k^j, \qquad B = \sum_{j=0}^{n} B_j k^j, \qquad C = \sum_{j=0}^{n} C_j k^j. \tag{8.39}$$

Substituting (8.39) into (8.38) and equating coefficients of powers of k, we obtain $3n+5$ equations. There are $3n+3$ unknowns, A_j, B_j, C_j, $j = 0, 1, \dots, n$, and so we also obtain two nonlinear evolution equations for r and q. Now let us consider some examples.

Example 8.1 $n = 2$. Suppose that A, B and C are quadratic polynomials, that is

$$\begin{aligned}
A &= A_2 k^2 + A_1 k + A_0,\\
B &= B_2 k^2 + B_1 k + B_0,\\
C &= C_2 k^2 + C_1 k + C_0.
\end{aligned} \tag{8.40}$$

Substitute (8.40) into (8.38) and equate powers of k. The coefficients of k^3 immediately give $B_2 = C_2 = 0$. At order k^2, we obtain $A_2 = a$, a constant (we could have allowed a to be a function of time, but for simplicity we do not do so), $B_1 = iaq$, $C_1 = iar$. At order k^1, we obtain $A_1 = b$, constant, and for simplicity we set $b = 0$ (if $b \neq 0$ then a more general evolution equation is obtained); then $B_0 = -\frac{1}{2}aq_x$ and $C_0 = \frac{1}{2}ar_x$. Finally, at order k^0, we obtain $A_0 = \frac{1}{2}arq + c$, with c a constant (again for simplicity we set $c = 0$). Therefore we obtain the following evolution equations

$$\begin{aligned} -\tfrac{1}{2}aq_{xx} &= q_t - aq^2r, \\ \tfrac{1}{2}ar_{xx} &= r_t + aqr^2. \end{aligned} \tag{8.41}$$

If in (8.41) we set $r = \mp q^*$ and $a = 2i$, then we obtain the nonlinear Schrödinger equation

$$iq_t = q_{xx} \pm 2q^2q^*. \tag{8.42}$$

for both focusing (+) and defocusing (−) cases. In summary, setting $r = \mp q^*$, we find that

$$\begin{aligned} A &= 2ik^2 \mp iqq^*, \\ B &= 2qk + iq_x, \\ C &= \pm 2q^*k \mp iq_x^* \end{aligned} \tag{8.43}$$

satisfy (8.38) provided that $q(x, t)$ satisfies the nonlinear Schrödinger equation (8.42).

Example 8.2 $n = 3$. If we substitute the following third-order polynomials in k

$$\begin{aligned} A &= a_3k^3 + a_2k^2 + \tfrac{1}{2}(a_3qr + a_1)k + \tfrac{1}{2}a_2qr - \tfrac{1}{4}ia_3(qr_x - rq_x) + a_0, \\ B &= ia_3qk^2 + \left(ia_2q - \tfrac{1}{2}a_3q_x\right)k + \left[ia_1q - \tfrac{1}{2}a_2q_x + \tfrac{1}{4}ia_3(2q^2r - q_{xx})\right], \\ C &= ia_3rk^2 + \left(ia_2r + \tfrac{1}{2}a_3r_x\right)k + \left[ia_1r + \tfrac{1}{2}a_2r_x + \tfrac{1}{4}ia_3(2r^2q - r_{xx})\right], \end{aligned}$$

into (8.38), with a_3, a_2, a_1 and a_0 constants, then we find that $q(x, t)$ and $r(x, t)$ satisfy the evolution equations

$$\begin{aligned} q_t + \tfrac{1}{4}ia_3(q_{xxx} - 6qrq_x) + \tfrac{1}{2}a_2(q_{xx} - 2q^2r) - ia_1q_x - 2a_0q = 0, \\ r_t + \tfrac{1}{4}ia_3(r_{xxx} - 6qrr_x) - \tfrac{1}{2}a_2(r_{xx} - 2qr^2) - ia_1r_x + 2a_0r = 0. \end{aligned} \tag{8.44}$$

For special choices of the constants a_3, a_2, a_1 and a_0 in (8.44) we find physically interesting evolution equations. If $a_0 = a_1 = a_2 = 0$, $a_3 = -4i$ and $r = -1$, then we obtain the KdV equation

$$q_t + 6qq_x + q_{xxx} = 0.$$

If $a_0 = a_1 = a_2 = 0$, $a_3 = -4i$ and $r = q$, then we obtain the mKdV equation

$$q_t - 6q^2q_x + q_{xxx} = 0.$$

[Note that if $a_0 = a_1 = a_3 = 0$, $a_2 = -2i$ and $r = -q^*$, then we obtain the nonlinear Schrödinger equation (8.42).]

We can also consider expansions of A, B and C in inverse powers of k.

Example 8.3 $n = -1$. Suppose that

$$A = \frac{a(x,t)}{k}, \qquad B = \frac{b(x,t)}{k}, \qquad C = \frac{c(x,t)}{k};$$

then the compatibility conditions (8.38) are satisfied if

$$a_x = \tfrac{i}{2}(qr)_t, \qquad q_{xt} = -4iaq, \qquad r_{xt} = -4iar.$$

Special cases of these are

(i)

$$a = \tfrac{1}{4}i\cos u, \qquad b = -c = \tfrac{1}{4}i\sin u, \qquad q = -r = -\tfrac{1}{2}u_x :$$

then u satisfies the sine–Gordon equation

$$u_{xt} = \sin u.$$

(ii)

$$a = \tfrac{1}{4}i\cosh u, \qquad b = -c = -\tfrac{1}{4}i\sinh u, \qquad q = r = \tfrac{1}{2}u_x,$$

where u satisfies the sinh–Gordon equation:

$$u_{xt} = \sinh u.$$

These examples only show a few of the many nonlinear evolution equations that may be obtained by this procedure. We saw above that when $r = -1$, the scattering problem (8.33) reduced to the Schrödinger equation (8.35). In this case we can take an alternative associated time dependence

$$v_t = Av + Bv_x.$$

By requiring that this and ($\lambda = k^2$)

$$v_{xx} + (\lambda + q)v = 0$$

are compatible and assuming that $d\lambda/dt = 0$, yields equations for A and B analogous to (8.38), then by expanding in powers of λ, we obtain a general class of equations.

8.7 More general classes of nonlinear evolution equations

A natural question is: *what is a general class of "solvable" nonlinear evolution equations*? Here we use the 2×2 scattering problem

$$v_{1,x} = -ikv_1 + q(x,t)v_2, \tag{8.45a}$$

$$v_{2,x} = -ikv_2 + r(x,t)v_1, \tag{8.45b}$$

to investigate this issue. Ablowitz, Kaup, Newell and Segur (1974) answered this question by considering the equations (8.45) associated with the functions A, B and C in (8.34). Under certain restrictions, a relation was found that gives a class of solvable nonlinear equations. This relationship depends upon the dispersion relation of the linearized form of the nonlinear equations and a certain integro-differential operator. Suppose that q and r vanish sufficiently rapidly as $|x| \to \infty$: then a general evolution equation can be shown to be

$$\begin{pmatrix} r \\ -q \end{pmatrix}_t + 2A_0(L)\begin{pmatrix} r \\ q \end{pmatrix} = 0 \tag{8.46a}$$

where $A_0(k) = \lim_{|x|\to\infty} A(x,t,k)$ (we may think of $A_0(k)$ as the ratio of two entire functions), and L is the integro-differential operator given by

$$L = \frac{1}{2i}\begin{pmatrix} \partial_x - 2r(I_-q) & 2r(I_-r) \\ -2q(I_-q) & -\partial_x + 2q(I_-r) \end{pmatrix}, \tag{8.46b}$$

where $\partial_x \equiv \partial/\partial x$ and

$$(I_-f)(x) \equiv \int_{-\infty}^{x} f(y)dy. \tag{8.46c}$$

Note that L operates on (r,q), and I_- operates both on the functions immediately to its right and also on the functions to which L is applied. Equation (8.46) may be written in matrix form

$$\sigma_3\mathbf{u}_t + 2A_0(L)\mathbf{u} = 0,$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix} r \\ q \end{pmatrix}.$$

It is significant that $A_0(k)$ is closely related to the dispersion relation of the associated *linearized* problem. If $f(x)$ and $g(x)$ vanish sufficiently rapidly as $|x| \to \infty$, then in the limit $|x| \to -\infty$,

$$f(x)(I_-g)(x) \equiv f(x)\int_{-\infty}^{x} g(y)dy \to 0,$$

and so for infinitesimal q and r, we have that L is the diagonal differential operator

$$L = \frac{1}{2i}\frac{\partial}{\partial x}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence in this limit (8.46) yields

$$r_t + 2A_0\left(-\frac{1}{2}i\partial_x\right)r = 0,$$
$$-q_t + 2A_0\left(\frac{1}{2}i\partial_x\right)q = 0.$$

The above equations are linear (decoupled) partial differential equations solvable by Fourier transform methods. Substituting the wave solutions $r(x,t) = \exp(i(kx - \omega_r(2k)t))$, $q(x,t) = \exp(i(kx - \omega_q(-2k)t))$, into the above equations, we obtain the relationship

$$A_0(k) = \frac{1}{2}i\omega_r(2k) = -\frac{1}{2}i\omega_q(-2k). \tag{8.47}$$

Therefore a general evolution equation is given by

$$\begin{pmatrix} r \\ -q \end{pmatrix} = -i\omega(2L)\begin{pmatrix} r \\ q \end{pmatrix} \tag{8.48}$$

with $\omega_r(k) = \omega(k)$; in matrix form this equation takes the form

$$\sigma_3\mathbf{u}_t + i\omega(2L)\mathbf{u} = 0.$$

Hence, the form of the nonlinear evolution equation is characterized by the dispersion relation of its associated linearized equations and an integro-differential operator; further details can be found in Ablowitz et al. (1974).

For the nonlinear Schrödinger equation

$$iq_t = q_{xx} \pm 2q^2q^*, \tag{8.49}$$

the associated linear equation is

$$iq_t = q_{xx}.$$

Therefore the dispersion relation is $\omega_q(k) = -k^2$, and so from (8.47), $A_0(k)$ is given by

$$A_0(k) = 2ik^2. \tag{8.50}$$

The evolution is obtained from either (8.46) or (8.48); therefore we have

$$\begin{pmatrix} r \\ -q \end{pmatrix}_t = -4iL^2\begin{pmatrix} r \\ q \end{pmatrix} = -2L\begin{pmatrix} r_x \\ q_x \end{pmatrix} = i\begin{pmatrix} r_{xx} - 2r^2q \\ q_{xx} - 2q^2r \end{pmatrix}.$$

When $r = \mp q^*$, both of these equations are equivalent to the nonlinear Schrödinger equation (8.49). Note that (8.50) is in agreement with (8.41) with $a = 2i$ (that is $\lim_{|x|\to\infty} A = 2ik^2$, since $\lim_{|x|\to\infty} r(x,t) = 0$). This explains *á posteriori* why the expansion of A, B and C in powers of k are related so closely to the dispersion relation. Indeed, now in retrospect, the fact that the nonlinear Schrödinger equation is related to (8.50) implies that an expansion starting at k^2 will be a judicious choice. Similarly, the modified KdV equation and sine–Gordon equation can be obtained from the operator equation (8.48) using the dispersion relations $\omega(k) = -k^3$ and $\omega(k) = k^{-1}$, respectively. These dispersion relations suggest expansions commencing in powers of k^3 and k^{-1}, respectively, that indeed we saw to be the case in the earlier section.

The derivation of (8.48) required that q and r tend to 0 as $|x| \to 0$ and therefore we cannot simply set $r = -1$ in order to obtain the equivalent result for the Schrödinger scattering problem

$$v_{xx} + (k^2 + q)v = 0. \tag{8.51}$$

However the essential ideas are similar and Ablowitz, Kaup, Newell and Segur (1974) also showed that a general evolution equation in this case is

$$q_t + \gamma(L)q_x = 0, \tag{8.52a}$$

where

$$L \equiv -\frac{1}{4}\partial_x^2 - q + \frac{1}{2}q_x I_+, \tag{8.52b}$$

$$I_+ f(x) \equiv \int_x^\infty f(y)dy. \tag{8.52c}$$

$$\gamma(k^2) = \frac{\omega(2k)}{2k}, \tag{8.52d}$$

with $\partial_x = \partial/\partial x$ and $\omega(k)$, is the dispersion relation of the associated linear equation with $q = exp(i(kx - \omega(k)t))$.

For the KdV equation

$$q_t + 6qq_x + q_{xxx} = 0,$$

the associated linear equation is

$$q_t + q_{xxx} = 0.$$

Therefore $\omega(k) = -k^3$ and so $\gamma(k^2) = -4k^2$, thus $\gamma(L) = -4L$. Hence (8.52) yields

$$q_t - 4Lq_x = 0,$$

thus

$$q_t - 4\left(-\frac{1}{4}\partial_x^2 - q + \frac{1}{2}q_x I_+\right)q_x = 0,$$

which is the KdV equation!

Associated with the operator L there is a hierarchy of equations (sometimes referred to as the Lenard hierarchy) given by

$$u_t + L^k u_x = 0, \qquad k = 1, 2, \ldots$$

The first two subsequent higher-order equations in the KdV hierarchy are given by

$$u_t - \frac{1}{4}\left(u_{5x} + 10uu_{3x} + 20u_x u_{xx} + 30u^2 u_x\right) = 0,$$

$$u_t + \frac{1}{16}(u_{7x} + 14uu_{5x} + 42u_x u_{4x} + 70u_{xx}u_{3x} + 70u^2 u_{3x}$$
$$+280uu_x u_{xx} + 70u_x^3 + 140u^3 n_x\big) = 0$$

where $u_{nx} = \partial^n u/\partial x^n$.

In the above we studied two different scattering problems, namely the classical time-independent Scrödinger equation

$$v_{xx} + (u + k^2)v = 0, \tag{8.53}$$

which can be used to linearize the KdV equation, and the 2×2 scattering problem

$$\mathbf{v}_x = k\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\mathbf{v} + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}\mathbf{v}, \tag{8.54}$$

which can linearize the nonlinear Schrödinger, mKdV and Sine–Gordon equations. While (8.53) can be interpreted as a special case of (8.54), from the point of view of possible generalizations, however, we regard them as different scattering problems.

There have been numerous applications and generalizations of this method only a few of which we will discuss below. One generalization includes that of Wadati et al. (1979a,b) (see also Shimizu and Wadati, 1980; Ishimori, 1981, 1982; Wadati and Sogo, 1983; Konno and Jeffrey, 1983). For example, instead of (8.33), suppose we consider the scattering problem

$$\begin{aligned} v_{1,x} &= -f(k)v_1 + g(k)q(x,t)v_2, \\ v_{2,x} &= f(k)v_2 + g(k)r(x,t)v_1, \end{aligned} \tag{8.55}$$

where $f(k)$ and $g(k)$ are polynomial functions of the eigenvalue k, and the time dependence is given (as previously) by

$$\begin{aligned} v_{1,t} &= Av_1 + Bv_2, \\ v_{2,t} &= Cv_1 + Dv_2. \end{aligned} \tag{8.56}$$

The compatibility of (8.55) and (8.56) requires that A, B, C, D satisfy linear equations that generalize equations (8.36)–(8.38).

As earlier, expanding A, B, C, f and g in finite power series expansions in k (where the expansions for f and g have constant coefficients), then one obtains a variety of physically interesting evolution equations (cf. Ablowitz and Clarkson, 1991, for further details).

Extensions to higher-order scattering problems have been considered by several authors. There are two important classes of one-dimensional linear scattering problems, associated with the solvable nonlinear evolution equations: scalar and systems of equations. Some generalizations are now mentioned.

Ablowitz and Haberman (1975) studied an $N \times N$ matrix generalization of the scattering problem (8.54):

$$\frac{\partial v}{\partial x} = ik\mathbf{J}\mathbf{v} + \mathbf{Q}\mathbf{v}, \tag{8.57a}$$

where the matrix of potentials $\mathbf{Q}(x) \in M_N(\mathbb{C})$ (the space of $N \times N$ matrices over $\mathbb{C}$) with $Q^{ii} = 0$, $\mathbf{J} = \mathrm{diag}(J^2, J^2, \ldots, J^N)$, where $J^i \neq J^j$ for $i \neq j$, $i, j = 1, 2, \ldots, N$, and $\mathbf{v}(x,t)$ is an N-dimensional vector eigenfunction. In this case, to obtain evolution equations, the associated time dependence is chosen to be

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{T}\mathbf{v}, \tag{8.57b}$$

where $\mathbf{T}$ is also an $N \times N$ matrix. The compatibility of (8.57) yields

$$\mathbf{T}_x = \mathbf{Q}_t + ik[\mathbf{J}, \mathbf{T}] + [\mathbf{Q}, \mathbf{T}].$$

In the same way for the 2×2 case discussed earlier in the chapter, associated nonlinear evolution equations may be found by assuming an expansion for $\mathbf{T}$ in powers or inverse powers of the eigenvalue k

$$\mathbf{T} = \sum_{j=0}^{n} k^j \mathbf{T}_j. \tag{8.58}$$

Ablowitz and Haberman showed that this scattering problem can be used to solve several physically interesting equations such as the three-wave interaction equations in one spatial dimension (with $N = 3$ and $n = 1$) and the

Boussinesq equation (with $N = 3$ and $n = 2$). The associated recursion operators have been obtained by various authors, cf. Newell (1978) and Fokas and Anderson (1982). The inverse problem associated with the scattering problem (8.57) in the general $N \times N$ case has been rigorously studied by Beals and Coifman (1984, 1985), Caudrey (1982), Mikhailov (1979, 1981) and Zhou (1989).

The following Nth order generalization of the scattering problem (8.53) was considered by Gel'fand and Dickii (1977)

$$\frac{d^N v}{dx^N} + \sum_{j=2}^{N} u_j(x) \frac{d^{N-j} v}{dx^{N-j}} = \lambda v, \tag{8.59}$$

where $u_j(x)$ are considered as potentials; they investigated many of the *algebraic* properties of the nonlinear evolution equations solvable through this scattering problem. The general inverse problem associated with this scattering problem is treated in detail in the monograph of Beals, Deift, and Tomei (1988).

The scattering problems (8.53) and (8.54), and their generalizations (8.59) and (8.57), are only some of the examples of scattering problems used in connection with the solution of nonlinear partial differential equations in (1+1)-dimensions; i.e., one spatial and one temporal dimension. The reader can find many more interesting scattering problems related to integrable systems in the literature (Camassa and Holm, 1993a; Konopelchenko, 1993; Rogers and Schief, 2002; Matveev and Salle, 1991; Dickey, 2003; Newell, 1985).

Exercises

8.1 Use (8.17) to find two conserved quantities additional to (8.19); i.e., w_5 and w_6. Then also obtain the conserved fluxes.

8.2 Use the method explained in Section 8.6 associated with (8.33)–(8.34) to find the nonlinear evolution equation corresponding to

$$A = a_4 k^4 + a_3 k^3 + a_2 k^2 + a_1 k + a_0.$$

Show that this agrees with the nonlinear equations obtained via the operator approach in Section 8.7, i.e., equations (8.46a)–(8.48).

8.3 Use the pair of equations

$$v_{xx} + (\lambda + q)v = 0$$
$$v_t = Av + Bv_x$$

and the method of Section 8.6 to find an "integrable" nonlinear evolution equation of order 5 in space, i.e., of the form $u_t + u_{xxxxx} + \cdots = 0$. Show that this agrees with the the nonlinear equations obtained via the operator approach in Section 8.7, i.e., equation (8.52).

8.4 Suppose we wish to find a real integrable equation associated with (8.51) having a dispersion relation

$$\omega = \frac{1}{1+k^2}.$$

Find this equation via (8.52).

8.5 Consider the scattering problem (8.57a) of order 3×3 with time dependence (8.57) with T given by (8.58) with $n = 1$. Find the nonlinear evolution equation. Hint: see Ablowitz and Haberman (1975). Extend this to scattering problems of order $N \times N$.

8.6 Consider the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0.$$

(a) Show that $u(x,t) = -2k^2 \operatorname{csech}^2[k(x - 4k^2 t)]$ is a singular solution.
(b) Obtain the rational solution $u(x,t) = -2/x^2$ by letting $k \to 0$.
(c) Show that the singular solution can also be obtained from the classical soliton solution $u(x,t) = 2k^2 \operatorname{sech}^2[k(x - 4k^2 t) - x_0]$ by setting $\exp(2x_0) = -1$.

8.7 Show that the concentric KdV equation

$$u_t + \frac{1}{2t}u + 6uu_x + u_{xxx} = 0$$

has the following conservation laws:

(i) $\displaystyle\int_{-\infty}^{\infty} \sqrt{t}u \, dx = \text{constant},$
(ii) $\displaystyle\int_{-\infty}^{\infty} tu^2 \, dx = \text{constant},$
(iii) $\displaystyle\int_{-\infty}^{\infty} \left(\sqrt{t}xu - 6t^{3/2}u^2\right) dx = \text{constant}.$

8.8 Consider the KP equation

$$(u_t + 6uu_x + u_{xxx}) + 3\sigma u_{yy} = 0$$

where $\sigma = \pm 1$. Show that the KP equation has the conserved quantities:

(i) $\displaystyle I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u \, dxdy = \text{constant};$
(ii) $\displaystyle I_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^2 \, dxdy = \text{constant};$

(iii) $\displaystyle\int_{-\infty}^{\infty} u_{yy}\, dx = \text{constant}$

and the integral relation

(iv) $\displaystyle\frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xu\, dxdy = \alpha I_2$

where α is constant.

8.9 Show that the NLS equation

$$iu_t + u_{xx} + |u|^2 u = 0$$

has the combined rational and oscillatory solution

$$u(x,t) = \left[1 - \frac{4(1+2it)}{1+2x^2+4t^2}\right]\exp(it).$$

8.10 (a) Use the pair of equations

$$v_{xx} + (\lambda q)v = 0$$
$$v_t = Av + Bv_x$$

and the method of Section 8.6, with A, B taken to be linear in λ with the "isospectral" condition $\lambda_t = 0$, to find the nonlinear evolution

$$q_t = (q^{-1/2})_{xxx}.$$

This equation is usually referred to as the Harry–Dym equation, which can be shown to be integrable via "hodograph transformations" cf. Ablowitz and Clarkson (1991).

(b) Show that the method of Section 6.8 with

$$v_{xx} + \lambda(m+\alpha)v = 0, \quad \alpha = \kappa + 1/4$$
$$v_t = Av + Bv_x, \quad A = -\frac{u_x}{2}, \; B = u - \frac{1}{2\lambda}$$

and $m = u_{xx} - u$ results in the "isospectral" equation

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0.$$

The above equation was shown to have an infinite number of symmetries by Fokas and Fuchssteiner (1981); the inverse scattering problem, solutions and relation to shallow-water waves was considered by Camassa and Holm (1993b). When $\kappa = 0$, the equation has traveling wave "peakon" solutions, which have cusps; they have a shape of the form $e^{-|x-ct|}$. The above equation, which is usually referred to as the Camassa–Holm equation, has been studied extensively due to its many interesting properties.

8.11 Consider the Harry–Dym equation

$$u_t + 2(u^{-1/2})_{xxx}.$$

(a) Let $u = v_x^2$ and show that v satisfies

$$v_t = \frac{v_{xxx}}{v_x^3} - \frac{3v_{xx}^2}{2v_x^4}.$$

(b) Show that the "hodograph" transformation $\tau = t, \xi = v, \eta = x$, yields the relations

$$\eta_\xi = \frac{1}{v_x}, \quad \eta_\tau = \frac{-v_t}{v_x}.$$

(c) Using the above relations find the equation

$$w_\tau = w_{\xi\xi\xi} - \left(\frac{3w_\xi^2}{2w}\right)_\xi$$

where $w = \eta_\xi$, which is known to be "integrable" (linearizable). Hint: see Clarkson et al. (1989).