

Report 3

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1 Time stepping and finite differences: the whole line

Recall the equation we obtained for the surface elevation on the whole line:

$$\eta_{tt} = \eta_{xx} + \mu^2 \left(\frac{1}{3} \eta_{xxxx} + \eta_{xt} \int_{-\infty}^x \eta_t dx' + \eta_x^2 + \eta_t^2 + \eta \eta_{xx} + \partial_x^2 \left(\int_{-\infty}^x \eta_t dx' \right)^2 \right). \quad (1)$$

To do time stepping, introduce

$$u = \eta_t. \quad (2)$$

Also, note that

$$\partial_x^2 \left(\int_{-\infty}^x \eta_t dx' \right)^2 = 2(\eta_t^2 + \eta_{tx} \int_{-\infty}^x \eta_t dx')$$

Then, combining (2) and (1), we obtain a two-dimensional system:

$$\partial_t \begin{bmatrix} u \\ \eta \end{bmatrix} = \begin{bmatrix} \mu^2 \left(2u_x \int_{-\infty}^x u dx' + 2u^2 \right) + \eta_{xx} + \mu^2 \left(\frac{1}{3} \eta_{xxxx} + \eta_x^2 + \eta \eta_{xx} \right) \\ u \end{bmatrix}. \quad (3)$$

Now, consider (1) on a finite interval $[a, b]$, and let partition the interval into $n + 1$ points $\{x_k\}_{k=0}^n$, with $x_0 = a$ and $x_n = b$. This means that the integral terms becomes

$$\int_{-\infty}^x \eta_t dx' = \left\{ \int_{-\infty}^a + \int_a^x \right\} \eta_t dx' \approx \int_a^x \eta_t dx',$$

while assuming that

$$\int_{-\infty}^a \eta_t dx'$$

is small enough. Now, we employ forward Euler time stepping. First, observe that

$$\begin{aligned} u_t(x_k, t_j) &= \frac{u(x_k, t_{j+1}) - u(x_k, t_j)}{\Delta t} = f_1(\eta, u, t) & \implies u(x_k, t_{j+1}) &= u(x_k, t_j) + \Delta t f_1(\eta, u, t) \\ \eta_t(x_k, t_j) &= \frac{\eta(x_k, t_{j+1}) - \eta(x_k, t_j)}{\Delta t} = f_2(\eta, u, t) & \implies \eta(x_k, t_{j+1}) &= \eta(x_k, t_j) + \Delta t f_2(\eta, u, t), \end{aligned}$$

where

$$\begin{aligned} f_1(\eta, u, t) &= \eta_{xx} + \mu^2 \left(2u_x \int_{-\infty}^x u dx' + 2u^2 + \frac{1}{3} \eta_{xxxx} + \eta_x^2 + \eta \eta_{xx} \right) \\ f_2(\eta, u, t) &= u(x_k, t_j). \end{aligned}$$

Observe that the highest order derivative is 4, so we need to use five-point stencils. In particular, we use five point midpoint stencil for x_2, \dots, x_{n-2} , and five point, one-sided stencils at x_0, x_1, x_{n-1}, x_n . First, consider the system for x_2, \dots, x_{n-2} :

$$f_1(\eta(x_k, t_j), u(x_k, t_j), t) = \eta(x_k, t_j)_{xx} + \frac{\mu^2}{3} \eta(x_k, t_j)_{xxxx} \\ + \mu^2 \left(2u(x_k, t_j)_x \int_{x_0}^{x_k} u \, dx' + 2u(x_k, t_j)^2 + \eta(x_k, t_j)_x^2 + \eta(x_k, t_j) \eta(x_k, t_j)_{xx} \right),$$

where we have separated the linear and nonlinear terms. Let $\Delta x = x_k - x_{k-1}$ and recall the finite difference formulas at x :

$$f'(x) = \frac{f(x-2\Delta x) - 8f(x-\Delta x) + 8f(x+\Delta x) - f(x+2\Delta x)}{12\Delta x} + \mathcal{O}((\Delta x)^4) \\ f''(x) = \frac{-f(x-2\Delta x) + 16f(x-\Delta x) - 30f(x) + 16f(x+\Delta x) - f(x+2\Delta x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2) \\ f'''(x) = \frac{f(x-2\Delta x) - 4f(x-\Delta x) + 6f(x) - 4f(x+\Delta x) + f(x+2\Delta x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2)$$

so that

$$(\eta_k)_x = \frac{\eta_{k-2} - 8\eta_{k-1} + 8\eta_{k+1} - \eta_{k+2}}{12\Delta x} + \mathcal{O}((\Delta x)^4) \\ (u_k)_x = \frac{u_{k-2} - 8u_{k-1} + 8u_{k+1} - u_{k+2}}{12\Delta x} + \mathcal{O}((\Delta x)^4) \\ (\eta_k)_{xx} = \frac{-\eta_{k-2} + 16\eta_{k-1} - 30\eta_k + 16\eta_{k+1} - \eta_{k+2}}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2) \\ (\eta_k)_{xxx} = \frac{\eta_{k-2} - 4\eta_{k-1} + 6\eta_k - 4\eta_{k+1} + \eta_{k+2}}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2),$$

where it is assumed $t = t_j$. Also, by trapezoidal rule,

$$\int_{x_0}^{x_k} u \, dx' = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} u \, dx = \frac{\Delta x}{2} \sum_{i=0}^{n-1} u(x_i) + u(x_{i+1}).$$

At $x = x_0$, we have

$$f'(x) = \frac{-25f(x) + 48f(x+\Delta x) - 36f(x+2\Delta x) + 16f(x+3\Delta x) - 3f(x+4\Delta x)}{12\Delta x} + \mathcal{O}((\Delta x)^4) \\ f''(x) = \frac{35f(x) - 104f(x+\Delta x) + 114f(x+2\Delta x) - 56f(x+3\Delta x) + 11f(x+4\Delta x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2) \\ f'''(x) = \frac{f(x) - 4f(x+\Delta x) + 6f(x+2\Delta x) - 4f(x+3\Delta x) + f(x+4\Delta x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2),$$

so that

$$(\eta_0)_x = \frac{-25\eta_0 + 48\eta_1 - 36\eta_2 + 16\eta_3 - 3\eta_4}{12\Delta x} \\ (u_0)_x = \frac{-25u_0 + 48u_1 - 36u_2 + 16u_3 - 3u_4}{12\Delta x} \\ (\eta_0)_{xx} = \frac{35\eta_0 - 104\eta_1 + 114\eta_2 - 56\eta_3 + 11\eta_4}{12\Delta x^2} \\ (\eta_0)_{xxx} = \frac{\eta_0 - 4\eta_1 + 6\eta_2 - 4\eta_3 + \eta_4}{\Delta x^4}$$

At $x = x_1$, we have

$$f'(x) = \frac{-3f(x-\Delta x) - 10f(x) + 18f(x+\Delta x) - 6f(x+2\Delta x) + f(x+3\Delta x)}{12\Delta x} + \mathcal{O}((\Delta x)^4)$$

$$f''(x) = \frac{11f(x - \Delta x) - 20f(x) + 6f(x + \Delta x) + 4f(x + 2\Delta x) - f(x + 3\Delta x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2)$$

$$f'''(x) = \frac{f(x - \Delta x) - 4f(x) + 6f(x + \Delta x) - 4f(x + 2\Delta x) + f(x + 3\Delta x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2),$$

so that

$$(\eta_1)_x = \frac{-3\eta_0 - 10\eta_1 + 18\eta_2 - 6\eta_3 + \eta_4}{12\Delta x}$$

$$(u_1)_x = \frac{-3u_0 - 10u_1 + 18u_2 - 6u_3 + u_4}{12\Delta x}$$

$$(\eta_1)_{xx} = \frac{11\eta_0 - 20\eta_1 + 6\eta_2 + 4\eta_3 - \eta_4}{12\Delta x^2}$$

$$(\eta_1)_{xxxx} = \frac{\eta_0 - 4\eta_1 + 6\eta_2 - 4\eta_3 + \eta_4}{\Delta x^4}$$

At $x = x_{n-1}$, we have

$$f'(x) = \frac{-f(x - 3\Delta x) + 6f(x - 2\Delta x) - 18f(x - \Delta x) + 10f(x) + 3f(x + \Delta x)}{12\Delta x} + \mathcal{O}((\Delta x)^4)$$

$$f''(x) = \frac{-f(x - 3\Delta x) + 4f(x - 2\Delta x) + 6f(x - \Delta x) - 20f(x) + 11f(x + \Delta x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2)$$

$$f'''(x) = \frac{f(x - 3\Delta x) - 4f(x - 2\Delta x) + 6f(x - \Delta x) - 4f(x) + f(x + \Delta x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2),$$

so that

$$(\eta_{n-1})_x = \frac{-\eta_{n-4} + 6\eta_{n-3} - 18\eta_{n-2} + 10\eta_{n-1} + 3\eta_n}{12\Delta x}$$

$$(u_{n-1})_x = \frac{-u_{n-4} + 6u_{n-3} - 18u_{n-2} + 10u_{n-1} + 3u_n}{12\Delta x}$$

$$(\eta_{n-1})_{xx} = \frac{-\eta_{n-4} + 4\eta_{n-3} + 6\eta_{n-2} - 20\eta_{n-1} + 11\eta_n}{12\Delta x^2}$$

$$(\eta_{n-1})_{xxxx} = \frac{\eta_{n-4} - 4\eta_{n-3} + 6\eta_{n-2} - 4\eta_{n-1} + \eta_n}{\Delta x^4}$$

At $x = x_n$, we have

$$f'(x) = \frac{f(x - 4\Delta x) - 4f(x - 3\Delta x) + 6f(x - 2\Delta x) - 4f(x - \Delta x) + f(x)}{12\Delta x} + \mathcal{O}((\Delta x)^4)$$

$$f''(x) = \frac{11f(x - 4\Delta x) - 56f(x - 3\Delta x) + 114f(x - 2\Delta x) - 104f(x - \Delta x) + 35f(x)}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^2)$$

$$f'''(x) = \frac{f(x - 4\Delta x) - 4f(x - 3\Delta x) + 6f(x - 2\Delta x) - 4f(x - \Delta x) + f(x)}{(\Delta x)^4} + \mathcal{O}((\Delta x)^2),$$

so that

$$(\eta_n)_x = \frac{3\eta_{n-4} - 16\eta_{n-3} + 36\eta_{n-2} - 48\eta_{n-1} + 25\eta_n}{12\Delta x}$$

$$(u_n)_x = \frac{3u_{n-4} - 16u_{n-3} + 36u_{n-2} - 48u_{n-1} + 25u_n}{12\Delta x}$$

$$(\eta_n)_{xx} = \frac{11\eta_{n-4} - 56\eta_{n-3} + 114\eta_{n-2} - 104\eta_{n-1} + 35\eta_n}{12\Delta x^2}$$

$$(\eta_n)_{xxxx} = \frac{\eta_{n-4} - 4\eta_{n-3} + 6\eta_{n-2} - 4\eta_{n-1} + \eta_n}{\Delta x^4}.$$

All in all, we obtain:

$$f_1(\eta(x_0, t_j), u(x_0, t_j), t) = \eta(x_0, t_j)_{xx} + \frac{\mu^2}{3}\eta(x_0, t_j)_{xxxx} + \mu^2 (2u(x_0, t_j)^2 + \eta(x_0, t_j)_x^2 + \eta(x_0, t_j)\eta(x_0, t_j)_{xx}),$$

$$\begin{aligned}
f_1(\eta(x_1, t_j), u(x_1, t_j), t) &= \eta(x_1, t_j)_{xx} + \frac{\mu^2}{3} \eta(x_1, t_j)_{xxxx} \\
&\quad + \mu^2 \left(2u(x_1, t_j)_x \int_{x_0}^{x_1} u \, dx' + 2u(x_1, t_j)^2 + \eta(x_1, t_j)_x^2 + \eta(x_1, t_j) \eta(x_1, t_j)_{xx} \right), \\
f_1(\eta(x_2, t_j), u(x_2, t_j), t) &= \eta(x_2, t_j)_{xx} + \frac{\mu^2}{3} \eta(x_2, t_j)_{xxxx} \\
&\quad + \mu^2 \left(2u(x_2, t_j)_x \int_{x_0}^{x_2} u \, dx' + 2u(x_2, t_j)^2 + \eta(x_2, t_j)_x^2 + \eta(x_2, t_j) \eta(x_2, t_j)_{xx} \right), \\
&\dots \\
f_1(\eta(x_k, t_j), u(x_k, t_j), t) &= \eta(x_k, t_j)_{xx} + \frac{\mu^2}{3} \eta(x_k, t_j)_{xxxx} \\
&\quad + \mu^2 \left(2u(x_k, t_j)_x \int_{x_0}^{x_k} u \, dx' + 2u(x_k, t_j)^2 + \eta(x_k, t_j)_x^2 + \eta(x_k, t_j) \eta(x_k, t_j)_{xx} \right), \\
&\dots \\
f_1(\eta(x_{n-1}, t_j), u(x_{n-1}, t_j), t) &= \eta(x_{n-1}, t_j)_{xx} + \frac{\mu^2}{3} \eta(x_{n-1}, t_j)_{xxxx} \\
&\quad + \mu^2 \left(2u(x_{n-1}, t_j)_x \int_{x_0}^{x_{n-1}} u \, dx' + 2u(x_{n-1}, t_j)^2 + \eta(x_{n-1}, t_j)_x^2 + \eta(x_{n-1}, t_j) \eta(x_{n-1}, t_j)_{xx} \right), \\
f_1(\eta(x_n, t_j), u(x_n, t_j), t) &= \eta(x_n, t_j)_{xx} + \frac{\mu^2}{3} \eta(x_n, t_j)_{xxxx} \\
&\quad + \mu^2 \left(2u(x_n, t_j)_x \int_{x_0}^{x_n} u \, dx' + 2u(x_n, t_j)^2 + \eta(x_n, t_j)_x^2 + \eta(x_n, t_j) \eta(x_n, t_j)_{xx} \right),
\end{aligned}$$

Now, we obtain the discretised problem. First, consider the column of linear terms:

$$\begin{aligned}
&(\eta_0)_{xx} + \frac{\mu^2}{3} (\eta_0)_{xxxx} \\
&= \frac{35\eta_0 - 104\eta_1 + 114\eta_2 - 56\eta_3 + 11\eta_4}{12\Delta x^2} + \frac{\mu^2}{3} \frac{\eta_0 - 4\eta_1 + 6\eta_2 - 4\eta_3 + \eta_4}{\Delta x^4} \\
&= \frac{35\Delta x^2 + 4\mu^2}{12\Delta x^4} \eta_0 - \frac{104\Delta x^2 + 16\mu^2}{12\Delta x^4} \eta_1 + \frac{114\Delta x^2 + 24\mu^2}{12\Delta x^4} \eta_2 - \frac{56\Delta x^2 + 16\mu^2}{12\Delta x^4} \eta_3 + \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4} \eta_4 \\
&(\eta_1)_{xx} + \frac{\mu^2}{3} (\eta_1)_{xxxx} \\
&= \frac{11\eta_0 - 20\eta_1 + 6\eta_2 + 4\eta_3 - \eta_4}{12\Delta x^2} + \frac{\mu^2}{3} \frac{\eta_0 - 4\eta_1 + 6\eta_2 - 4\eta_3 + \eta_4}{\Delta x^4} \\
&= \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4} \eta_0 - \frac{20\Delta x^2 + 16\mu^2}{12\Delta x^4} \eta_1 + \frac{6\Delta x^2 + 24\mu^2}{12\Delta x^4} \eta_2 - \frac{4\Delta x^2 - 16\mu^2}{12\Delta x^4} \eta_3 + \frac{-\Delta x^2 + 4\mu^2}{12\Delta x^4} \eta_4 \\
&\dots \\
&(\eta_k)_{xx} + \frac{\mu^2}{3} (\eta_k)_{xxxx} \\
&= \frac{-\eta_{k-2} + 16\eta_{k-1} - 30\eta_k + 16\eta_{k+1} - \eta_{k+2}}{12(\Delta x)^2} + \frac{\mu^2}{3} \frac{\eta_{k-2} - 4\eta_{k-1} + 6\eta_k - 4\eta_{k+1} + \eta_{k+2}}{(\Delta x)^4} \\
&= \frac{-\Delta x^2 + 4\mu^2}{12\Delta x^4} \eta_{k-2} + \frac{16(\Delta x^2 - \mu^2)}{12\Delta x^4} \eta_{k-1} + \frac{-30\Delta x^2 + 24\mu^2}{12\Delta x^4} \eta_k - \frac{16(\Delta x^2 - \mu^2)}{12\Delta x^4} \eta_{k+1} + \frac{-\Delta x^2 + 4\mu^2}{12\Delta x^4} \eta_{k+2} \\
&\dots \\
&(\eta_{n-1})_{xx} + \frac{\mu^2}{3} (\eta_{n-1})_{xxxx} \\
&= \frac{-\eta_{n-4} + 4\eta_{n-3} + 6\eta_{n-2} - 20\eta_{n-1} + 11\eta_n}{12\Delta x^2} + \frac{\mu^2}{3} \frac{\eta_{n-4} - 4\eta_{n-3} + 6\eta_{n-2} - 4\eta_{n-1} + \eta_n}{\Delta x^4} \\
&= \frac{-\Delta x^2 + 4\mu^2}{12\Delta x^4} \eta_{n-4} + \frac{4\Delta x^2 - 16\mu^2}{12\Delta x^4} \eta_{n-3} + \frac{6\Delta x^2 + 24\mu^2}{12\Delta x^4} \eta_{n-2} - \frac{20\Delta x^2 + 16\mu^2}{12\Delta x^4} \eta_{n-1} + \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4} \eta_n
\end{aligned}$$

$$\begin{aligned}
& (\eta_n)_{xx} + \frac{\mu^2}{3}(\eta_n)_{xxxx} \\
&= \frac{11\eta_{n-4} - 56\eta_{n-3} + 114\eta_{n-2} - 104\eta_{n-1} + 35\eta_n}{12\Delta x^2} + \frac{\mu^2}{3} \frac{\eta_{n-4} - 4\eta_{n-3} + 6\eta_{n-2} - 4\eta_{n-1} + \eta_n}{\Delta x^4} \\
&= \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4} \eta_{n-4} - \frac{56\Delta x^2 + 16\mu^2}{12\Delta x^4} \eta_{n-3} + \frac{114\Delta x^2 + 24\mu^2}{12\Delta x^4} \eta_{n-2} - \frac{104\Delta x^2 + 16\mu^2}{12\Delta x^4} \eta_{n-1} + \frac{35\Delta x^2 + 4\mu^2}{12\Delta x^4} \eta_n.
\end{aligned}$$

Then, the matrix becomes

$$\begin{aligned}
& \begin{bmatrix} (\eta_0)_{xx} + \frac{\mu^2}{3}(\eta_0)_{xxxx} \\ (\eta_1)_{xx} + \frac{\mu^2}{3}(\eta_1)_{xxxx} \\ (\eta_2)_{xx} + \frac{\mu^2}{3}(\eta_2)_{xxxx} \\ \vdots \\ (\eta_k)_{xx} + \frac{\mu^2}{3}(\eta_k)_{xxxx} \\ \vdots \\ (\eta_{n-2})_{xx} + \frac{\mu^2}{3}(\eta_{n-2})_{xxxx} \\ (\eta_{n-1})_{xx} + \frac{\mu^2}{3}(\eta_{n-1})_{xxxx} \\ (\eta_n)_{xx} + \frac{\mu^2}{3}(\eta_n)_{xxxx} \end{bmatrix} \\
&= \begin{bmatrix} \frac{35\Delta x^2 + 4\mu^2}{12\Delta x^4} & -\frac{104\Delta x^2 + 16\mu^2}{12\Delta x^4} & \frac{114\Delta x^2 + 24\mu^2}{12\Delta x^4} & -\frac{56\Delta x^2 + 16\mu^2}{12\Delta x^4} & \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4} & 0 & 0 & 0 \\ \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4} & -\frac{20\Delta x^2 + 16\mu^2}{12\Delta x^4} & \frac{6\Delta x^2 + 24\mu^2}{12\Delta x^4} & -\frac{4\Delta x^2 - 16\mu^2}{12\Delta x^4} & -\frac{\Delta x^2 + 4\mu^2}{12\Delta x^4} & 0 & 0 & 0 \\ \frac{\Delta x^2 + 4\mu^2}{12\Delta x^4} & \frac{16(\Delta x^2 - \mu^2)}{12\Delta x^4} & -\frac{30\Delta x^2 + 24\mu^2}{12\Delta x^4} & -\frac{16(\Delta x^2 - \mu^2)}{12\Delta x^4} & -\frac{\Delta x^2 + 4\mu^2}{12\Delta x^4} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & -\frac{\Delta x^2 + 4\mu^2}{12\Delta x^4} & \frac{16(\Delta x^2 - \mu^2)}{12\Delta x^4} & -\frac{30\Delta x^2 + 24\mu^2}{12\Delta x^4} & -\frac{16(\Delta x^2 - \mu^2)}{12\Delta x^4} & -\frac{\Delta x^2 + 4\mu^2}{12\Delta x^4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & 0 & 0 & -\frac{\Delta x^2 + 4\mu^2}{12\Delta x^4} & \frac{16(\Delta x^2 - \mu^2)}{12\Delta x^4} & -\frac{30\Delta x^2 + 24\mu^2}{12\Delta x^4} & -\frac{16(\Delta x^2 - \mu^2)}{12\Delta x^4} \\ \dots & 0 & 0 & 0 & -\frac{\Delta x^2 + 4\mu^2}{12\Delta x^4} & \frac{4\Delta x^2 - 16\mu^2}{12\Delta x^4} & \frac{6\Delta x^2 + 24\mu^2}{12\Delta x^4} & -\frac{20\Delta x^2 - 16\mu^2}{12\Delta x^4} \\ \dots & 0 & 0 & 0 & \frac{11\Delta x^2 + 4\mu^2}{12\Delta x^4} & -\frac{56\Delta x^2 + 16\mu^2}{12\Delta x^4} & \frac{114\Delta x^2 + 24\mu^2}{12\Delta x^4} & -\frac{104\Delta x^2 + 16\mu^2}{12\Delta x^4} \end{bmatrix}
\end{aligned}$$

For simplicity, let \mathcal{A} represent the above matrix. Now, recall the system:

$$\begin{aligned}
u(x_k, t_{j+1}) &= u(x_k, t_j) + \Delta t f_1(\eta(x_k, t_j), u(x_k, t_j), t), \\
\eta(x_k, t_{j+1}) &= \eta(x_k, t_j) + \Delta t u(x_k, t_j),
\end{aligned}$$

where

$$\begin{aligned}
f_1(\eta(x_k, t_j), u(x_k, t_j), t) &= \eta(x_k, t_j)_{xx} + \frac{\mu^2}{3} \eta(x_k, t_j)_{xxxx} \\
&+ \mu^2 \left(2u(x_k, t_j)_x \int_{x_0}^{x_k} u \, dx' + 2u(x_k, t_j)^2 + \eta(x_k, t_j)_x^2 + \eta(x_k, t_j) \eta(x_k, t_j)_{xx} \right).
\end{aligned}$$

For convenience, let

$$B_k = \left(2u(x_k, t_j)_x \int_{x_0}^{x_k} u \, dx' + 2u(x_k, t_j)^2 + \eta(x_k, t_j)_x^2 + \eta(x_k, t_j) \eta(x_k, t_j)_{xx} \right),$$

Let \mathcal{B} represent the column vector of B_k 's. Then, we can write the system

$$\begin{aligned}
u(x_0, t_{j+1}) &= u(x_0, t_j) + \Delta t f_1(\eta(x_0, t_j), u(x_0, t_j), t) \\
u(x_1, t_{j+1}) &= u(x_1, t_j) + \Delta t f_1(\eta(x_1, t_j), u(x_1, t_j), t) \\
u(x_2, t_{j+1}) &= u(x_2, t_j) + \Delta t f_1(\eta(x_2, t_j), u(x_2, t_j), t) \\
&\vdots \\
u(x_{n-2}, t_{j+1}) &= u(x_{n-2}, t_j) + \Delta t f_1(\eta(x_{n-2}, t_j), u(x_{n-2}, t_j), t)
\end{aligned}$$

$$\begin{aligned}
u(x_{n-1}, t_{j+1}) &= u(x_{n-1}, t_j) + \Delta t f_1(\eta(x_{n-1}, t_j), u(x_{n-1}, t_j), t) \\
u(x_n, t_{j+1}) &= u(x_n, t_j) + \Delta t f_1(\eta(x_n, t_j), u(x_n, t_j), t)
\end{aligned}$$

as follows:

$$\begin{aligned}
\begin{bmatrix} u(x_0, t_{j+1}) \\ u(x_1, t_{j+1}) \\ u(x_2, t_{j+1}) \\ \vdots \\ u(x_{n-2}, t_{j+1}) \\ u(x_{n-1}, t_{j+1}) \\ u(x_n, t_{j+1}) \end{bmatrix} &= \begin{bmatrix} u(x_0, t_j) \\ u(x_1, t_j) \\ u(x_2, t_j) \\ \vdots \\ u(x_{n-2}, t_j) \\ u(x_{n-1}, t_j) \\ u(x_n, t_j) \end{bmatrix} + \Delta t \begin{bmatrix} f_1(\eta(x_0, t_j), u(x_0, t_j), t) \\ f_1(\eta(x_1, t_j), u(x_1, t_j), t) \\ f_1(\eta(x_2, t_j), u(x_2, t_j), t) \\ \vdots \\ f_1(\eta(x_{n-2}, t_j), u(x_{n-2}, t_j), t) \\ f_1(\eta(x_{n-1}, t_j), u(x_{n-1}, t_j), t) \\ f_1(\eta(x_n, t_j), u(x_n, t_j), t) \end{bmatrix} \\
&= \begin{bmatrix} u(x_0, t_j) \\ u(x_1, t_j) \\ u(x_2, t_j) \\ \vdots \\ u(x_{n-2}, t_j) \\ u(x_{n-1}, t_j) \\ u(x_n, t_j) \end{bmatrix} + \Delta t \mathcal{A} \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-2} \\ \eta_{n-1} \\ \eta_n \end{bmatrix} + \Delta t \mathcal{B}
\end{aligned}$$

Now, let us see how one would perform time-stepping. As such, impose initial conditions

$$\eta(x, t_0) = f(x), \quad u(x, t_0) = \eta_t(x, t_0) = g(x).$$

Let $j = 0$, and for simplicity, pick $k \in [0, n]$. The system is

$$\begin{aligned}
u(x_k, t_1) &= u(x_k, t_0) + \Delta t f_1(\eta(x_k, t_0), u(x_k, t_0)), \\
\eta(x_k, t_1) &= \eta(x_k, t_0) + \Delta t u(x_k, t_0),
\end{aligned}$$

where

$$\begin{aligned}
f_1(\eta(x_k, t_0), u(x_k, t_0)) &= \eta(x_k, t_0)_{xx} + \frac{\mu^2}{3} \eta(x_k, t_0)_{xxxx} \\
&+ \mu^2 \left(2u(x_k, t_0)_x \int_{x_0}^{x_k} u \, dx' + 2u(x_k, t_0)^2 + \eta(x_k, t_0)_x^2 + \eta(x_k, t_0) \eta(x_k, t_0)_{xx} \right) \\
&= \frac{-\eta(x_{k-2}, t_0) + 16\eta(x_{k-1}, t_0) - 30\eta(x_k, t_0) + 16\eta(x_{k+1}, t_0) - \eta(x_{k+2}, t_0)}{12(\Delta x)^2} \\
&+ \frac{\mu^2}{3} \frac{\eta(x_{k-2}, t_0) - 4\eta(x_{k-1}, t_0) + 6\eta(x_k, t_0) - 4\eta(x_{k+1}, t_0) + \eta(x_{k+2}, t_0)}{(\Delta x)^4} \\
&+ \mu^2 \left(2u(x_k, t_0)_x \int_{x_0}^{x_k} u(x', t_0) \, dx' + 2u(x_k, t_0)^2 + \eta(x_k, t_0)_x^2 + \eta(x_k, t_0) \eta(x_k, t_0)_{xx} \right)
\end{aligned}$$

Note that all the terms on the last line can be computed via finite differences and both initial conditions. With this, we obtain the values of u, η at point x_k and time t_1 . Performing this calculation for all k , we move on to compute u, η at time t_2 , and so on.