5

Water waves and KdV-type equations

Fluid dynamics is concerned with behavior on scales that are large compared to the distance between the molecules that they consist of. Physical quantities, such as mass, velocity, energy, etc., are usually regarded as being spread continuously throughout the region of consideration; this is often termed the continuum assumption and continuum derivations are based on conservation principles. From these concepts, the equations of fluid dynamics are derived in many books, cf. Batchelor (1967).

A microscopic description of fluids is provided by the Boltzmann equation. When the average distance between collisions of the fluid molecules becomes small relative to the macroscopic dimensions of the fluid, the Boltzmann equation can be simplified to the fluid equations (Chapman and Cowling, 1970).

In this chapter, we focus on the most important results derived from conservation laws and, in particular, how they relate to water waves.

We will use $\rho = \rho(\boldsymbol{x}, t)$ to denote the fluid mass density, $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t)$ is the fluid velocity, P is the pressure, $\boldsymbol{F}$ is a given external force, and ν_* is the kinematic viscosity that is due to frictional forces. In vector notation, the relevant equations of fluid dynamics we will consider are:

- conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0,$$

- conservation of momentum:

$$\rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} \right] = \boldsymbol{F} - \boldsymbol{\nabla} P + \nu_* \Delta \boldsymbol{v},$$

where $\Delta \equiv \nabla^2$, is the Laplacian.[1] We omit the equation of energy that describes the temperature (T) variation of the fluid, so an equation of state, such as

[1] In three-dimensional Cartesian coordinates, the Laplacian is $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

$P = P(\rho, T)$, is added to close the system of equations. These equations correspond to the first three moments of the Boltzmann equation. When $\rho = \rho_0$ is constant, the first equation then describes an incompressible fluid: $\nabla \cdot \boldsymbol{v} = 0$, also called the divergence equation. The divergence and the momentum equations are often called the incompressible Navier–Stokes equations. The energy equation is not necessary to close this system of equations, which (in three dimensions) are four equations in four unknowns: $\boldsymbol{v} = (u, v, w)$ and P.

In this chapter, we will consider the free surface water wave problem, which (interior to the fluid) is the inviscid reduction ($\nu_* = 0$) of the above equations; these equations are called the Euler equations. Supplemented with appropriate boundary conditions, we have the Euler equations with a free surface. We will derive the shallow-water or long wave limit of this system and discuss certain approximate equations: the Korteweg–de Vries (KdV) equation in 1 + 1 dimensions[2] and the Kadomtsev–Petviashvili (KP) equation in 2 + 1 dimensions.

5.1 Euler and water wave equations

For our discussion of water waves, we will use the above incompressible Navier–Stokes description with constant density $\rho = \rho_0$ and we will assume an ideal fluid: that is, a fluid with zero viscosity ($\nu_* = 0$). Thus, an ideal, incompressible fluid is described by the following Euler equations:

$$\nabla \cdot \boldsymbol{v} = 0,$$
$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v} = \frac{1}{\rho_0} (\boldsymbol{F} - \nabla P)\,.$$

Suppose now that the external force is conservative, i.e., we can write $\boldsymbol{F} = -\nabla U$, for some scalar potential U. We can then write the momentum equation as

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v} = -\nabla \left(\frac{U + P}{\rho_0} \right).$$

Using the vector identity

$$(\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v} = \frac{1}{2} \nabla (\boldsymbol{v} \cdot \boldsymbol{v}) - \boldsymbol{v} \times (\nabla \times \boldsymbol{v})\,,$$

gives

$$\frac{\partial \boldsymbol{v}}{\partial t} - \boldsymbol{v} \times (\nabla \times \boldsymbol{v}) = -\nabla \left(\frac{1}{2} \boldsymbol{v} \cdot \boldsymbol{v} + \frac{U + P}{\rho_0} \right). \tag{5.1}$$

[2] The notation "$n + 1$ dimensions" means there are n space dimensions and one time dimension.

Now define the vorticity to be $\boldsymbol{\omega} \equiv \boldsymbol{\nabla} \times \boldsymbol{v}$, which is a local measure of the degree to which the fluid is spinning; more precisely, $\frac{1}{2}\|\boldsymbol{\nabla} \times \boldsymbol{v}\|$ (note that $\|\boldsymbol{v}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v}$) is the angular speed of an infinitesimal fluid element. Taking the curl of the last equation and noting that the curl of a gradient vanishes,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{\omega}) = 0.$$

Finally, using the vector identity $\boldsymbol{\nabla} \times (\boldsymbol{F} \times \boldsymbol{G}) = (\boldsymbol{G} \cdot \boldsymbol{\nabla})\boldsymbol{F} - (\boldsymbol{F} \cdot \boldsymbol{\nabla})\boldsymbol{G} + (\boldsymbol{\nabla} \cdot \boldsymbol{G})\boldsymbol{F} - (\boldsymbol{\nabla} \cdot \boldsymbol{F})\boldsymbol{G}$ for vector functions $\boldsymbol{F}$ and $\boldsymbol{G}$ and recalling that the divergence of the curl vanishes, we arrive at the so-called vorticity equation:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} \quad \text{or} \tag{5.2a}$$

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \boldsymbol{\nabla}\boldsymbol{v}, \tag{5.2b}$$

where we have used the notation

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})$$

to signify the so-called convective or material derivative that moves with the fluid particle ($\boldsymbol{v} = (u, v, w)$). Hence $\boldsymbol{\omega} = 0$ is a solution; moreover, from (5.2b), it can be proven that if the vorticity is initially zero, then (if the solution exists) it is zero for all times. Such a flow is called irrotational. Physically, in an ideal fluid there is no mechanism that will produce "local rotation" if the fluid is initially irrotational. Often it is a good approximation to assume that a fluid is irrotational, with viscosity effects occurring only in thin regions of the fluid flow called boundary layers. In this chapter, we will consider water waves and will assume that the flow is irrotational. In such circumstances, it is convenient to introduce a velocity potential $\boldsymbol{v} = \boldsymbol{\nabla}\phi$. Notice that the vorticity equation (5.2b) is trivially satisfied since

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi) = 0.$$

The Euler equations inside the fluid region can now also be simplified:

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi = \Delta\phi = 0,$$

which is Laplace's equation; it is to be satisfied internal to the fluid, $-h < z < \eta(x, y, t)$, where we denote the height of the fluid free surface above the mean level $z = 0$ to be $\eta(x, y, t)$ and the bottom of the fluid is at $z = -h$. See Figure 5.1 on page 103.

Next we discuss the boundary conditions that lead to complications; i.e., an unknown free surface and nonlinearities. We assume a flat, impenetrable bottom at $z = -h$, so that no fluid can flow through. This results in the condition

$$w = \frac{\partial \phi}{\partial z} = 0, \quad z = -h,$$

where w represents the vertical velocity. On the free surface $z = \eta(x, y, t)$ there are two conditions. The first is obtained from (5.1). Using the fact that $\boldsymbol{\nabla}$ and $\partial/\partial t$ commute,

$$\begin{aligned} \boldsymbol{\nabla}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}||\boldsymbol{v}||^2 + \frac{U+P}{\rho_0}\right) = 0 \quad &\Rightarrow \\ \frac{\partial \phi}{\partial t} + \frac{1}{2}||\boldsymbol{v}||^2 + \frac{U+P}{\rho_0} = f(t),& \end{aligned}$$

where we recall $\boldsymbol{v} = (u, v, w)$ and $||\boldsymbol{v}||^2 = u^2 + v^2 + w^2 = \phi_x^2 + \phi_y^2 + \phi_z^2$. Since the physical quantity is $\boldsymbol{v} = \boldsymbol{\nabla}\phi$, we can add an arbitrary function of time (independent of space) to ϕ,

$$\phi \to \phi + \int_0^t f(t')\, dt',$$

to get the so-called Bernoulli, dynamic, or pressure equation,

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}||\boldsymbol{v}||^2 + \frac{U+P}{\rho_0} = 0.$$

For now, we will neglect surface tension and assume that the dominant force is the buoyancy force, $\boldsymbol{F} = -\boldsymbol{\nabla}(\rho_0 g z)$, which implies that $U = \rho_0 g z$, where g is the gravitational constant of acceleration. For convenience, we take the pressure to vanish (i.e., $P = 0$) on the free surface, yielding:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}||\boldsymbol{\nabla}\phi||^2 + g\eta = 0, \qquad z = \eta(x, y, t)$$

on the free surface.

The second equation governing the free surface is derived from the assumption that if a fluid packet is initially on the free surface, then it will stay there. Mathematically, this implies that if $F = F(x, y, z, t)$, where (x, y, z) is a point on the free surface, then

$$\frac{DF}{Dt} \equiv \frac{\partial F}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} F = 0.$$

On the surface, $F = z - \eta(x, y, t) = 0$. Then

$$\frac{Dz}{Dt} = \frac{D\eta}{Dt} \quad \Rightarrow \quad w = \frac{\partial \eta}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\eta,$$

where we have used $\boldsymbol{v} = \left(\frac{Dx}{Dt}, \frac{Dy}{Dt}, \frac{Dz}{Dt}\right) = (u, v, w)$. So, using $w = \partial\phi/\partial z$,

$$w = \frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\eta, \qquad z = \eta(x, y, t), \tag{5.3}$$

on the free surface. Equation (5.3) is often referred to as the kinematic condition. It should be noted that on a single-valued surface $z = \eta(x, y, t)$, equation (5.3) can be written as

$$\frac{\partial \phi}{\partial n} = \frac{\partial \eta}{\partial t},$$

where $\partial\phi/\partial\boldsymbol{n} = (\boldsymbol{n}\cdot\nabla)\phi$ and $\boldsymbol{n} = (-\nabla\eta, 1)$ is the outward normal. Thus, the free surface $z = \eta(x, y, t)$ moves in the direction of the normal velocity.

To summarize, the free-surface water wave equations with a flat bottom are:

- Euler ideal flow

$$\Delta\phi = 0, \qquad -h < z < \eta(x, y, t). \tag{5.4}$$

- No flow through the bottom

$$\frac{\partial \phi}{\partial z} = 0, \qquad z = -h. \tag{5.5}$$

- Bernoulli's or the pressure equation

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\|\boldsymbol{\nabla}\phi\|^2 + g\eta = 0, \qquad z = \eta(x, y, t). \tag{5.6}$$

- Kinematic condition

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\eta, \qquad z = \eta(x, y, t). \tag{5.7}$$

These four equations constitute the equations for water waves with the unknowns $\phi(x, y, z, t)$ and $\eta(x, y, t)$. This is a free-boundary problem. In contrast to a Dirichlet or a Neumann boundary value problem where the boundary is fixed and known, in free-boundary problems part of solving the problem is to determine the dynamics of the boundary. This aspect makes the solution to the water wave equations particularly difficult. We also note that if we were given an η that satisfies Bernoulli's equation (5.6), the remaining three equations would satisfy a Neumann boundary value problem. The geometry of the problem is shown in Figure 5.1.

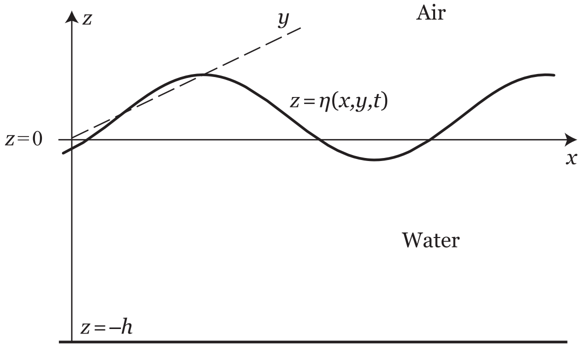

Figure 5.1 Geometry of water waves. The bottom of the water is at a constant level $z = -h$, while the free surface of the water is denoted by $z = \eta(x, y, t)$, where η is to be determined. The undisturbed fluid is at a level $z = 0$.

5.2 Linear waves

We will first look at the linear case by assuming $|\eta| \ll 1$ and $\|\nabla\phi\| \ll 1$. The first two equations remain unchanged, except that they are to be satisfied on the known surface $z = 0$. The last two equations, using

$$\begin{aligned}\phi(x, y, \eta, t) &= \phi(x, y, 0, t) + \eta\frac{\partial\phi}{\partial z}(x, y, 0, t) + \cdots ,\\ &= \phi_0(x, y, t) + \eta\phi_{0z}(x, y, t) + \cdots ,\end{aligned}$$

become

$$\begin{aligned}\frac{\partial\phi_0}{\partial t} &= -g\eta, \qquad z = 0,\\ \frac{\partial\eta}{\partial t} &= \phi_{0z}(x, y, t), \qquad z = 0.\end{aligned}$$

We now look for special solutions to the Euler and free-surface equations of the form

$$\phi_s(x, y, z, t) = A(k, l, z, t)\exp(ikx + ily),$$

i.e., we decompose the solution into Fourier modes. Substituting this ansatz into Laplace's equation (5.4),

$$A_{zz} - (k^2 + l^2)A = 0.$$

Calling $\kappa^2 = k^2 + l^2$, the solution is given by

$$A = \tilde{A}(k, l, t)\cosh[\kappa(z + h)] + \tilde{B}(k, l, t)\sinh[\kappa(z + h)];$$

note we translated the solution by h units because satisfying the bottom boundary condition (5.5) requires

$$\frac{\partial A}{\partial z} = 0, \qquad z = -h,$$

which implies that $\tilde{B} = 0$.

If we assume that the free surface has a mode of the form

$$\eta(x, y, t) = \tilde{\eta}(k, l, t) \exp(ikx + ily),$$

then, substituting the above mode into (5.6)–(5.7) yields,

$$\frac{\partial \tilde{A}}{\partial t} \cosh(\kappa h) + g\tilde{\eta} = 0,$$
$$\frac{\partial \tilde{\eta}}{\partial t} - \kappa \sinh(\kappa h)\tilde{A} = 0.$$

Taking the time derivative of the second equation and substituting into the first gives

$$\frac{\partial^2 \tilde{\eta}}{\partial t^2} + g\kappa \tanh(\kappa h)\tilde{\eta} = 0.$$

Assuming $\tilde{\eta}(k, l, t) = \tilde{\eta}(k, l, 0)e^{-i\omega t}$, we find the dispersion relationship:

$$\omega^2 = g\kappa \tanh(\kappa h), \tag{5.8}$$

which has two branches. "Adding up" all such special solutions implies that the general Fourier solution for a rapidly decaying solution for η is

$$\eta = \frac{1}{(2\pi)^2} \iint \{\tilde{\eta}_+ \exp[i(kx + ly - \omega_+ t)] + \tilde{\eta}_- \exp[i(kx + ly + \omega_- t)]\}\, dk\, dl, \tag{5.9}$$

with $\tilde{\eta}_+, \tilde{\eta}_-$ determined by the initial data that is assumed to be real and decaying sufficiently rapidly in space.

The dispersion relation has several interesting limits. The so-called deep-water dispersion relationship, i.e., when $\kappa h \gg 1$, is

$$\omega^2 \simeq g|\kappa|,$$

while for shallow water, i.e., when $\kappa h \ll 1$, it is

$$\omega^2 \simeq g\kappa \left(\kappa h - \frac{(\kappa h)^3}{3} + \cdots \right).$$

The leading term in the shallow-water dispersion relationship is $\omega^2 \simeq gh\kappa^2$. Alternatively, as discussed in earlier chapters, we have that to leading order the amplitude η satisfies the wave equation,

$$\eta_{tt} - c_0^2 \eta_{xx} = 0,$$

with a propagation speed $c_0^2 = gh$. We will come back to this point when we discuss the KdV equation later in this chapter.

5.3 Non-dimensionalization

To analyze the full equations, we will first non-dimensionalize them. By doing this, it will be easier to compare the "size" of each term. This is convenient since we will then be working with "pure" or dimensionless numbers. For example, a velocity can be large if measured in units of, say, microns per second but in units of light-years per hour it would be small. Indeed small or large in terms of coefficients in an equation are relative concepts and are meaningful only in a comparative sense. By studying non-dimensional equations we can more easily decide which terms are negligible.

For shallow-water waves it is convenient to use the following non-dimensionalization:

$$x = \lambda_x x', \quad y = \lambda_y y', \quad z = hz',$$
$$t = \frac{\lambda_x}{c_0} t', \quad \eta = a\eta', \quad \phi = \frac{\lambda_x g a}{c_0} \phi',$$

where $c_0 = \sqrt{gh}$ is the shallow-water wave speed, λ_x, λ_y are typical wavelengths of the initial data (in the x or y direction), and a is the maximum or typical amplitude of the initial data. The primed variables are dimensionless. For Laplace's equation:

$$\frac{1}{\lambda_x^2} \phi'_{x'x'} + \frac{1}{\lambda_y^2} \phi'_{y'y'} + \frac{1}{h^2} \phi'_{z'z'} = 0.$$

For the sake of notational simplicity, we will drop the primed notation so that

$$\phi_{zz} + \left(\frac{h}{\lambda_x}\right)^2 \phi_{xx} + \left(\frac{h}{\lambda_y}\right)^2 \phi_{yy} = 0, \qquad -1 < z < \frac{a\eta}{h}.$$

The no-flow condition becomes

$$\frac{\partial \phi}{\partial z} = 0, \qquad z = -1.$$

Bernoulli's equation becomes

$$\phi_t + \frac{a}{2h}\left[\phi_x^2 + \left(\frac{\lambda_x}{\lambda_y}\right)^2 \phi_y^2 + \left(\frac{\lambda_x}{h}\right)^2 \phi_z^2\right] + \eta = 0, \qquad z = \frac{a\eta}{h}.$$

And the kinematic condition becomes

$$\eta_t + \frac{a}{h}\left[\phi_x\eta_x + \left(\frac{\lambda_x}{\lambda_y}\right)^2 \phi_y\eta_y\right] = \left(\frac{\lambda_x}{h}\right)^2 \phi_z, \qquad z = \frac{a\eta}{h}.$$

Now we define the dimensionless parameters $\epsilon \equiv a/h$, $\delta \equiv \lambda_x/\lambda_y$, and $\mu \equiv h/\lambda_x$: ϵ is a measure of the nonlinearity, or amplitude, of the wave; μ is a measure of the depth relative to the characteristic wavelength, sometimes called the dispersion parameter; and δ measures the size of the transverse variations. In summary, the dimensionless equations for water waves propagating over a flat bottom are

- Euler ideal flow, Laplace's equation

$$\phi_{zz} + \mu^2\phi_{xx} + \mu^2\delta^2\phi_{yy} = 0, \quad -1 < z < \epsilon\eta.$$

- No flow through the bottom

$$\frac{\partial\phi}{\partial z} = 0, \quad z = -1.$$

- Bernoulli's or the pressure equation

$$\phi_t + \frac{\epsilon}{2}\left[\phi_x^2 + \delta^2\phi_y^2 + \frac{1}{\mu^2}\phi_z^2\right] + \eta = 0, \quad z = \epsilon\eta.$$

- Kinematic condition:

$$\mu^2\left[\eta_t + \epsilon\left(\phi_x\eta_x + \delta^2\phi_y\eta_y\right)\right] = \phi_z, \quad z = \epsilon\eta.$$

We note that the linear, or small-amplitude, case arises when $\epsilon \ll 1, \mu \sim O(1)$. Then the above equations reduce, by dropping terms of order ϵ, to the linear equations that are equivalent to those we analyzed earlier, i.e.,

$$\phi_t + \eta = 0, \quad z = 0,$$
$$\mu^2\eta_t = \phi_z, \quad z = 0,$$

but now in non-dimensional form.

5.4 Shallow-water theory

We will make some simplifying assumptions about the sizes of ϵ, μ, and δ in order to obtain some interesting limiting equations:

- We will consider shallow water waves (sometimes called long water waves). This regime corresponds to small depth relative to the water wavelength. In our system of parameters, this corresponds to $\mu = h/\lambda_x \ll 1$.
- We will also assume that the wavelength in the transverse direction is much larger than the propagation wavelength, so $\delta = \lambda_x/\lambda_y \ll 1$.
- As before in the linear regime, we assume small-amplitude waves $|\varepsilon| = a/h \ll 1$. Recall that a similar assumption was made the Fermi–Pasta–Ulam (FPU) problem involving coupled nonlinear springs studied in Chapter 1.
- We will make the assumption of "maximal balance" (the small terms, i.e., nonlinearity and dispersion, are of the same order) as was made in the continuum approximation of the FPU problem to the Boussinesq model in order to derive the KdV equation. We will do the same thing now by assuming maximal balance with $\varepsilon = \mu^2$. This reflects a balance of weak nonlinearity and weak dispersion.

5.4.1 Neglecting transverse variations

First, we will consider the special case of no transverse waves and later we will incorporate them into our model. Let us rewrite the fluid equations with the simplifying assumptions $\varepsilon = \mu^2$ – this is our "maximal balance" assumption. Let us also consider the one-dimensional case, i.e., we remove the terms involving derivatives with respect to y. The four equations then become

$$\varepsilon\phi_{xx} + \phi_{zz} = 0, \qquad -1 < z < \varepsilon\eta \tag{5.10a}$$

$$\phi_z = 0, \qquad z = -1 \tag{5.10b}$$

$$\phi_t + \frac{\varepsilon}{2}\left(\phi_x^2 + \frac{1}{\varepsilon}\phi_z^2\right) + \eta = 0, \qquad z = \varepsilon\eta \tag{5.10c}$$

$$\varepsilon(\eta_t + \varepsilon\phi_x\eta_x) = \phi_z, \qquad z = \varepsilon\eta. \tag{5.10d}$$

These are coupled, nonlinear partial differential equations in ϕ and η with a free boundary, and are very difficult to solve exactly. We will use perturbation theory to obtain equations that are more tractable. We will asymptotically expand ϕ as

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots .$$

Substituting this expansion into (5.10a) gives

$$\phi_{0zz} + \varepsilon(\phi_{0xx} + \phi_{1zz}) + \varepsilon^2(\phi_{1xx} + \phi_{2zz}) + \cdots = 0.$$

Equating terms with like powers of ε, we find that $\phi_{0zz} = 0$. This implies that $\phi_0 = A + B(z+1)$, where A, B are functions of x, t. But the boundary condition (5.10b) forces $B = 0$. Hence the leading-order solution for the velocity potential is independent of z,

$$\phi_0 = A(x,t).$$

Proceeding to the next order in ε, we see that $\phi_{1zz} = -A_{xx}$. Again using the boundary condition (5.10b),

$$\phi_1 = -A_{xx}(z+1)^2/2.$$

As before, we absorb any homogeneous solutions that arise in higher-order terms into the leading-order term (that is ϕ_0). Similarly we find $\phi_2 = A_{xxxx}(z+1)^4/4!$. Thus, we have the approximation

$$\phi = A - \frac{\varepsilon}{2}A_{xx}(z+1)^2 + \frac{\varepsilon^2}{4!}A_{xxxx}(z+1)^4 + \cdots, \tag{5.11}$$

which is valid on the interval $-1 < z < \varepsilon\eta$, in particular, right up to the free boundary $\varepsilon\eta$. This expansion can be carried out to any order in ε, but the first three terms are sufficient for our purpose.

Substituting (5.11) into Bernoulli's equation (5.10c) along the free boundary $z = \varepsilon\eta$ gives

$$\begin{aligned} A_t - \frac{\varepsilon}{2}A_{xxt}(1+\varepsilon\eta)^2 + \frac{\varepsilon^2}{4!}A_{xxxxt}(1+\varepsilon\eta)^4 + \cdots \\ + \frac{\varepsilon}{2}\left(A_x - \frac{\varepsilon}{2}A_{xxx}(1+\varepsilon\eta)^2 + \cdots\right)^2 \\ + \frac{1}{2}(-\varepsilon A_{xx}(1+\varepsilon\eta) + \cdots)^2 + \eta = 0. \end{aligned}$$

Retaining only the first two terms leads to

$$\eta = -A_t + \frac{\varepsilon}{2}\left(A_{xxt} - A_x^2\right) + \cdots. \tag{5.12}$$

Now let us do the same thing for the kinematic equation (5.10d). Substituting in our expansion for ϕ and retaining the two lowest-order terms gives

$$\varepsilon\eta_t + \varepsilon^2\eta_x A_x = -\varepsilon A_{xx}(1+\varepsilon\eta) + \frac{\varepsilon^2}{3!}A_{xxxx} + \cdots. \tag{5.13}$$

We wish to decouple the system of equations involving A and η. Since we have an expression for η in terms of A, we substitute (5.12) into (5.13) and retain only the two lowest-order terms:

$$\varepsilon\left(-A_{tt} + \frac{\varepsilon}{2}(A_{xxtt} - 2A_xA_{xt}) + \varepsilon^2(-A_{xt}A_x)\right)$$
$$= -\varepsilon A_{xx}(1 - \varepsilon A_t) + \frac{\varepsilon^2}{3!}A_{xxxx} + \cdots .$$

Dividing through by ε and keeping order ϵ terms gives

$$A_{tt} - A_{xx} = \varepsilon\left(\frac{A_{xxtt}}{2} - \frac{A_{xxxx}}{6} - 2A_xA_{xt} - A_{xx}A_t\right). \tag{5.14}$$

Within this approximation, this equation is asymptotically the same as the one Boussinesq derived in 1871, cf. Ablowitz and Segur (1981). Another form, equally valid up to order ε, is derived by noting that on the left-hand side of (5.14) we have $A_{tt} = A_{xx} + O(\varepsilon)$. Then, in particular, we can take two x derivatives to obtain $A_{xxtt} = A_{xxxx} + O(\varepsilon)$. Now we replace the $A_{xxtt}/2$ term in the above Boussinesq model to get

$$A_{tt} - A_{xx} = \varepsilon\left(\frac{A_{xxxx}}{3} - 2A_xA_{xt} - A_{xx}A_t\right), \tag{5.15}$$

which is still valid up to $O(\varepsilon)$.

Now we will make a remark about the linearized model. We have seen that there are various ways to write the equation; for example,

$$A_{tt} - A_{xx} = \varepsilon\left(\frac{1}{2}A_{xxtt} - \frac{1}{6}A_{xxxx}\right) \tag{5.16}$$

$$A_{tt} - A_{xx} = \frac{\varepsilon}{3}A_{xxxx}. \tag{5.17}$$

But only one of these two gives rise to a well-posed problem. We can see this from their dispersion relations: assume a wave solution $A(x, t) = \exp(i(kx - \omega t))$ and substitute it into (5.16) (or recalling the correspondences $k \to -i\partial_x$ and $\omega \to i\partial_t$), then

$$-\omega^2 + k^2 = \varepsilon\left(\frac{\omega^2k^2}{2} - \frac{k^4}{6}\right) \quad \Longrightarrow \quad \omega^2 = \frac{k^2 + \varepsilon k^4/6}{1 + \varepsilon k^2/2}.$$

Note $\omega^2 > 0$ and for very large k, $\omega^2 \sim k^2/3$. If we substitute this back into our wave solution, then we see that $A(x, t) = \exp(i(kx \pm\ kt/\sqrt{3}))$ is bounded for all time. But, if we do the same for (5.17), we get

$$-\omega^2 + k^2 = \varepsilon\frac{k^4}{3}.$$

The large k limit of ω in this case is $\omega_\pm \sim \pm i\sqrt{\varepsilon}k^2/\sqrt{3}$. Substituting the negative root of ω into the wave solution we end up with

$$A(x, t) \sim \exp(ikx)\exp\left(\sqrt{\tfrac{\varepsilon}{3}}k^2t\right).$$

This solution blows up arbitrarily fast as $k \to \infty$ so we do not have a convergent Fourier integral solution for "reasonable" (non-analytic) initial data (see also the earlier discussion of well-posedness). If we wish to do numerical calculations then (5.16) is preferred as it is well-posed, as opposed to (5.17). The ill-posedness of (5.17) is due to the long-wave approximation. This is a common difficulty associated with long-wave expansions; see also the discussion in Benjamin et al. (1972).

Nevertheless since there is still a small parameter, ε, in the equation, we can and should do further asymptotics on these equations. This will also remove the ill-posedness and is done in the next section.

First we make another remark about different Boussinesq models. We have derived an approximate equation for the leading term in the expansion of the velocity potential ϕ (5.14) or (5.15). Combined with (5.11) it determines the velocity potential, accurate to $O(\epsilon)$. We can also determine an equation for the wave amplitude η. First we take an x derivative of (5.12) and then use the substitution $u = A_x$ in (5.12)–(5.13) to find

$$\text{Bernoulli:} \qquad \eta_x = -u_t + \frac{\varepsilon}{2}(u_{xxt} - 2uu_x) + \cdots, \tag{5.18}$$

$$\text{Kinematic:} \qquad \eta_t = -u_x + \varepsilon\left(-\eta_x u - \eta u_x + \frac{u_{xxx}}{6}\right) + \cdots. \tag{5.19}$$

This is called the coupled Boussinesq model for η and u.

On the other hand, by differentiating (5.19) with respect to t, and by differentiating (5.18) with respect to x, we get

$$\eta_{tt} - \eta_{xx} = \varepsilon\left(-\eta_{xt}u - \eta_x u_t - \eta_t u_x - \eta u_{xt} + \frac{u_{xxxt}}{6} - \frac{u_{xxxt}}{2} + \frac{(u^2)_{xx}}{2}\right) + O(\varepsilon).$$

Now we use the relations $u_x = -\eta_t + O(\varepsilon)$ and $u_t = -\eta_x + O(\varepsilon)$ from the kinematic and Bernoulli equations, respectively, and make the replacement $u = -\int_{-\infty}^{x} \eta_t \, dx'$. Hence we find

$$\eta_{tt} - \eta_{xx} = \varepsilon\left[\frac{1}{3}\eta_{xxxx} + \eta_{xt}\int_{-\infty}^{x} \eta_t \, dx' + \eta_x^2 + \eta_t^2 + \eta\eta_{xx} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\int_{-\infty}^{x} \eta_t \, dx'\right)^2\right] \tag{5.20}$$

$$= \varepsilon\left[\frac{\eta_{xxxx}}{3} + \frac{\partial}{\partial x}\left(\eta_t \int_{-\infty}^{x} \eta_t\, dx' + \eta\eta_x\right)\right.$$
$$\left. + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\int_{-\infty}^{x} \eta_t\, dx'\right)^2\right]$$
$$= \varepsilon\left[\frac{\eta_{xxxx}}{3} + \frac{\partial^2}{\partial x^2}\left(\frac{\eta^2}{2} + \left(\int_{-\infty}^{x} \eta_t\, dx'\right)^2\right)\right]. \tag{5.21}$$

The last equation (5.21) is the Boussinesq model for the wave amplitude. If we omit the non-local term (which can be done via a suitable transformation) (5.21) is reduced to the Boussinesq model discussed in Chapter 1 [see (1.2), where $\eta = y_x$].

5.4.2 Multiple-scale derivation of the KdV equation

Now let us return to the Boussinesq model for the velocity potential equation (5.15) to do further asymptotics

$$A_{tt} - A_{xx} = \varepsilon\left(\frac{A_{xxxx}}{3} - 2A_x A_{xt} - A_{xx}A_t\right). \tag{5.22}$$

Assume an asymptotic expansion for A:

$$A = A_0 + \varepsilon A_1 + \cdots ;$$

substituting this expansion into (5.22) gives

$$A_{0tt} + \varepsilon A_{1tt} - A_{0xx} - \varepsilon A_{1xx} + O(\varepsilon^2) = \varepsilon\left(\frac{A_{0xxxx}}{3} - \right.$$
$$\left. -2A_{0x}A_{0xt} - A_{0xx}A_{0t} + O(\varepsilon)\right).$$

Equating the leading-order terms yields the wave equation: $A_{0tt} - A_{0xx} = 0$. We solved this equation earlier and found $A_0(x,t) = F(x-t) + G(x+t)$ where F and G are determined by the initial conditions. Anticipating secular terms that will need to be removed in the next-order equation, we employ multiple scales; i.e., $A_0 = A_0(\xi, \zeta, T)$:

$$A_0 = F(\xi, T) + G(\zeta, T),$$

where

$$\xi = x - t, \quad \zeta = x + t, \quad T = \varepsilon t.$$

These new variables imply

$$\partial_t = -\partial_\xi + \partial_\zeta + \varepsilon\partial_T \quad \text{and} \quad \partial_x = \partial_\xi + \partial_\zeta.$$

Substituting the expressions for the differential operators into (5.22) leads to

$$
\begin{aligned}
&((-\partial_\xi + \partial_\zeta + \varepsilon\partial_T)^2 - (\partial_\xi + \partial_\zeta)^2)A = \\
&\qquad \varepsilon\Bigg(\frac{(\partial_\xi + \partial_\zeta)^4}{3}A - 2(\partial_\xi + \partial_\zeta)A(\partial_\xi + \partial_\zeta)(-\partial_\xi + \partial_\zeta + \varepsilon\partial_T)A \\
&\qquad\qquad - (\partial_\xi + \partial_\zeta)^2 A(-\partial_\xi + \partial_\zeta + \varepsilon\partial_T)A\Bigg).
\end{aligned} \tag{5.23}
$$

First we will assume unidirectional waves and only work with the right-moving wave. So $A_0 = F(\xi, T)$ for now. Substituting

$$A = A_0 + \varepsilon A_1 + \cdots$$

into (5.23) and keeping all the $O(\varepsilon)$ terms we get

$$-4A_{1\xi\zeta} = 2F_{\xi T} + \frac{1}{3}F_{\xi\xi\xi\xi} + 3F_{\xi\xi}F_\xi.$$

This equation can be integrated directly to give

$$A_1 \sim -\frac{1}{4}\left(2F_T + \frac{1}{3}F_{\xi\xi\xi} + \frac{3}{2}F_\xi^2\right)\zeta + \cdots,$$

remembering that we absorb homogeneous terms in the A_0 solution. In order to remove secular terms, in particular the ζ term, we require that

$$2F_T + \frac{1}{3}F_{\xi\xi\xi} + \frac{3}{2}F_\xi^2 = 0;$$

or, if we take a derivative with respect to ξ and make the substitution $U = F_\xi$ then we have the Korteweg–de Vries equation

$$2U_T + \tfrac{1}{3}U_{\xi\xi\xi} + 3UU_\xi = 0.$$

Next we discuss the more general case of waves moving to the left and right; hence $A_0 = F(\xi, T) + G(\zeta, T)$, and we get the following expression for A_1

$$
\begin{aligned}
-4A_{1\xi\zeta} &= 2(F_{\xi T} - G_{\zeta T}) + \frac{1}{3}(F_{\xi\xi\xi\xi} + +G_{\zeta\zeta\zeta\zeta}) \\
&\qquad + 3(F_\xi F_{\xi\xi} + \frac{1}{3}F_{\xi\xi}G_\zeta - \frac{1}{3}G_{\zeta\zeta}F_\xi - G_\zeta G_{\zeta\zeta}).
\end{aligned}
$$

When we integrate this expression, secular terms arise from the pieces that are functions of ξ or ζ alone, not both. Removal of the secular terms implies the following two equations

$$
\begin{aligned}
2F_{\xi T} + \frac{1}{3}F_{\xi\xi\xi\xi} + 3F_{\xi\xi}F_\xi &= 0 \\
-2G_{\zeta T} + \frac{1}{3}G_{\zeta\zeta\zeta\zeta} - 3G_{\zeta\zeta}G_\zeta &= 0,
\end{aligned}
$$

which are two uncoupled KdV equations. Simplifying things a bit, we can rewrite the above two equations in terms of $U = F_\xi$ and $V = G_\zeta$:

$$2U_T + \frac{1}{3}U_{\xi\xi\xi} + 3UU_\xi = 0 \tag{5.24}$$

$$2V_T - \frac{1}{3}V_{\zeta\zeta\zeta} + 3VV_\zeta = 0. \tag{5.25}$$

The solution A_1 can be obtained by integrating the remaining terms. Since we are only interested in the leading-order asymptotic solution we need not solve for A_1.

Hence we have the following conclusion: asymptotic analysis of the fluid equations under shallow-water conditions has given rise to two Korteweg–de Vries (KdV) equations, (5.24) and (5.25), for the right- and left-going waves. Given rapidly decaying initial conditions, we can solve these two PDEs for U and V by the inverse scattering transform (IST). The KdV equation has been studied intensely and its IST solution is described in Chapter 9. Keeping only the leading-order terms, we have an approximate solution for the velocity potential

$$\begin{aligned}\phi(x,z,t) &\sim A_0(x,t) = F(x-t,\varepsilon t) + G(x+t,\varepsilon t),\\ &= \int_{-\infty}^{x-t} U(\xi',\varepsilon t)\,d\xi' + \int_{-\infty}^{x+t} V(\zeta',\varepsilon t)\,d\zeta',\end{aligned}$$

or for the velocity

$$u = \phi_x = F_x + G_x + \cdots = U + V + \cdots .$$

Using the Bernoulli equation (5.12), we have an approximate solution for the wave amplitude of the free boundary

$$\eta(x,t) \sim -A_{0t}(x,t) \sim F_\xi(\xi,T) - G_\zeta(\zeta,T) = U(x-t,\varepsilon t) - V(x+t,\varepsilon t).$$

Thus the wave amplitude η has right- and left-going waves that satisfy the KdV equation. Alternatively, we can derive the KdV equation directly by performing an asymptotic expansion in the η equation (5.21) though we will not do that here. One can also proceed to obtain higher-order terms and corrections to the KdV equation, though there is much more in the way of details that are required. We will not discuss this here.

5.4.3 Dimensional equations

In the previous section, we made our fluid equations non-dimensional for the purpose of determining what terms are "small" in our approximation of shallow-water waves. Now that we have approximate equations (5.24) and (5.25), valid to leading order, we wish to understand the results in dimensional units. We will now transform back to a dimensional equation for (5.24).

Recall that we made two changes of variable. We first changed coordinates x, t, η into primed coordinates x', t', η' (though we immediately dropped the primes). Then the independent variable substitutions $\xi = x' - t'$ and $T = \varepsilon t'$ were made. First, we remove the ξ and T dependence. Notice that

$$\frac{\partial}{\partial t'} U(\xi, T) = -U_\xi + \varepsilon U_T$$
$$\frac{\partial}{\partial x'} U(\xi, T) = U_\xi.$$

The above two expressions imply that $\partial_T = (\partial_{t'} + \partial_{x'})/\varepsilon$. Making the above substitutions into (5.24), we get

$$2(U_{t'} + U_{x'}) + \varepsilon\left(\frac{1}{3}U_{x'x'x'} + 3UU_{x'}\right) = 0.$$

If we neglect left-traveling waves, V, then we have $\eta' \sim U$. Recalling our original non-dimensionalization $x = \lambda_x x'$, $t = \dfrac{\lambda_x}{c_0} t'$, $\epsilon = a/h$ and $\eta = a\eta'$, we use $\partial_{x'} = \lambda_x \partial_x$ and $\partial_{t'} = \dfrac{\lambda_x}{c_0}\partial_t$ to transform the previous expression in U to

$$2\left(\frac{\lambda_x}{c_0}\frac{\eta_t}{a} + \lambda_x \frac{\eta_x}{a}\right) + \varepsilon\left(\frac{\lambda_x^3}{3a}\eta_{xxx} + \frac{3\lambda_x}{a^2}\eta\eta_x\right) = 0 \Rightarrow$$
$$2\left(\frac{\eta_t}{c_0} + \eta_x\right) + \frac{a}{h}\left(\frac{\lambda_x^2}{3}\eta_{xxx} + \frac{3}{a}\eta\eta_x\right) = 0.$$

From maximal balance, we used $\varepsilon = a/h = (h/\lambda_x)^2$ so we can simplify the non-dimensional KdV equation (5.24) to the dimensional form

$$\frac{1}{c_0}\eta_t + \eta_x + \frac{h^2}{6}\eta_{xxx} + \frac{3}{2h}\eta\eta_x = 0, \tag{5.26}$$

where $c_0 = \sqrt{gh}$, g is the gravitational constant of acceleration and h is the water depth. This equation was derived by Korteweg and de Vries (1895).

Let us consider the linear part of (5.26)

$$\frac{1}{c_0}\eta_t + \eta_x + \frac{h^2}{6}\eta_{xxx} = 0.$$

We will interpret the above equation in terms of the dispersion relation for water waves [see (5.8)], noting $\kappa^2 = k^2 + l^2$ with $l = 0$, so $\kappa = k$:

$$\omega^2 = gk\tanh(kh) = gk\left[kh - \frac{1}{3}(kh)^3 + \cdots\right].$$

In shallow-water waves, $kh \ll 1$ so if we retain only the leading-order term in the Taylor series expansion above, we have

$$\omega = \pm k\sqrt{gh} = \pm kc_0.$$

This is the dispersion relation for the linear wave equation and c_0 is the wave speed. Retaining the next term in the Taylor series of $\tanh(kh)$ gives

$$\begin{aligned}\omega &\sim \pm k\sqrt{gh(1 - \tfrac{1}{3}(kh)^2)}\\ &\sim \pm k\sqrt{gh}(1 - \tfrac{1}{6}(kh)^2)\\ &= \pm(kc_0 - \tfrac{1}{6}c_0h^2k^3).\end{aligned}$$

Now, we wish to see what linear PDE gives rise to the above dispersion relation. Taking the positive root, replacing ω with $i\partial_t$ and k with $-i\partial_x$, we find

$$\begin{gathered}i\partial_t\eta = c_0(-i\partial_x)\eta - \tfrac{1}{6}c_0h^2(-i\partial_x)^3\eta \Rightarrow\\ \frac{1}{c_0}\eta_t + \eta_x + \frac{h^2}{6}\eta_{xxx} = 0.\end{gathered}$$

This is exactly the dimensional linear KdV equation (5.26)! To get the nonlinear term we used the multiple-scales method.

5.4.4 Adding surface tension

The model equations (5.4) through (5.7) do not take into account the effects of surface tension. Actually only Bernoulli's equation (5.10c) is affected by surface tension. The modification is due to an additional pressure term from surface tension effects involving the curvature at the surface. We will not go through the derivation here: rather we will just state the result. First, recall Bernoulli's equation used so far

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0, \tag{5.27}$$

on $z = \eta$. Adding the surface tension term gives (cf. Lamb 1945 or Ablowitz and Segur 1981)

$$\phi_t + \tfrac{1}{2}|\nabla\phi|^2 + g\eta = \frac{T}{\rho}\nabla\cdot\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right)$$
$$= \frac{T}{\rho}\frac{\left(\eta_{xx}\left(1+\eta_y^2\right)+\eta_{yy}\left(1+\eta_x^2\right)-2\eta_{xy}\eta_x\eta_y\right)}{\left(1+\eta_x^2+\eta_y^2\right)^{\frac{3}{2}}} \tag{5.28}$$

on $z = \eta$ where T is the surface tension coefficient. Retaining dimensions and keeping linear terms that will affect the previous result to $O(\varepsilon)$, we have:

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta - \frac{T}{\rho}(\eta_{xx}+\eta_{yy}) = 0 \qquad z = \eta. \tag{5.29}$$

Using (5.29) and the same asymptotic procedure as before, the corresponding leading-order asymptotic equation for the free surface is found to be

$$\frac{1}{c_0}\eta_t + \eta_x + \gamma\eta_{xxx} + \frac{3}{2h}\eta\eta_x = 0. \tag{5.30}$$

The only difference between this equation and (5.26) is the coefficient γ of the third-derivative term. This term incorporates the surface tension

$$\gamma = \frac{h^2}{6} - \frac{T}{2\rho g} = \frac{h^2}{6}\left(1 - \frac{3T}{\rho g h^2}\right) = \frac{h^2}{6}(1-3\hat{T})$$
$$\hat{T} = \frac{T}{\rho g h^2}.$$

Note (5.30) was also derived by Korteweg and deVries [see Chapter 1, equation (1.4)], and we can see that, depending on the relative sizes of the parameters (in particular $\hat{T} < 1/3$ or $\hat{T} > 1/3$), γ will be positive or negative. This affects the types of solutions and behavior allowed by the equation and is discussed later. In particular, when we include transverse waves, we will discover the Kadomtsev–Petviashvili (KP) equation.

5.4.5 Including transverse waves: The KP equations

Our investigation so far has been for waves in shallow water without transverse modulations. We will now relax our assumptions and include weak transverse variation.

First, let us look at the dispersion relation for water waves. Using the same ideas as previously [see the discussion leading to (5.8)] the reader can verify that the dispersion relation in multidimensions (two space, one time) is given by

$$\omega^2 = \left(g\kappa + \frac{T}{\rho}\kappa^3\right)\tanh(\kappa h), \tag{5.31}$$

where $\kappa^2 = k^2 + l^2$; here k, l are the wavenumbers that correspond to the x- and y-directions, respectively. Recall that the wavelengths of a wave solution in the form $\alpha e^{i(kx+ly-\omega t)}$ are $\lambda_x = 2\pi/k$ and $\lambda_y = 2\pi/l$. We further assume that (recall our earlier definitions of scales)

$$\delta = \frac{\lambda_x}{\lambda_y} = \frac{l}{k} \ll 1.$$

The above relation says that the wavelength in the y-direction is much larger than the wavelength in the x-direction. This is what is referred to as weak transverse variation.

We now derive the linear PDE associated with the dispersion relation (5.31) with weak transverse variation. Since we are still in the shallow-water regime, we have $|hk| \propto |h/\lambda_x| \ll 1$ and we assume that $\kappa h = kh\sqrt{1 + l^2/k^2} \ll 1$. Then we expand the hyperbolic tangent term in a Taylor series to find

$$\begin{aligned}
\omega^2 &\approx \left(g\kappa + \frac{T}{\rho}\kappa^3\right)\left(\kappa h - \frac{1}{3}(\kappa h)^3\right) + \cdots \\
&= gh\kappa^2\left(1 + \frac{T}{g\rho}\kappa^2\right)\left(1 - \frac{1}{3}(\kappa h)^2\right) + \cdots \\
\omega &= \sqrt{gh}\,k\left(1 + \frac{l^2}{k^2}\right)^{\frac{1}{2}}\left(1 + \frac{T}{g\rho}k^2 + O(l^2)\right)^{\frac{1}{2}} \\
&\quad \times\left(1 - \frac{1}{3}(hk)^2 + O(l^2)\right)^{\frac{1}{2}}.
\end{aligned}$$

Use of the binomial expansion and assuming $|l/\kappa| \ll 1$ and the maximal balance relation $l^2 \sim O(k^4)$, we end up with ($c_0 = \sqrt{gh}$),

$$\omega k \approx c_0\left(k^2 + k^4\left(\frac{T}{2g\rho} - \tfrac{1}{6}h^2\right) + \tfrac{1}{2}l^2 + O(k^6)\right).$$

Now we can write down the linear PDE associated with this dispersion relation using $\omega \to i\partial_t$, $k \to -i\partial_x$:

$$\frac{1}{c_0}\eta_{tx} + \eta_{xx} + \tfrac{1}{2}\eta_{yy} + \frac{h^2}{6}(1 - 3\hat{T})\eta_{xxxx} = 0,$$

where we recall $\hat{T} = T/\rho g h^2$. This is the linear Kadomtsev–Petviashvili (KP) equation. To derive the full nonlinear version, we must resort to multiple scales

(Ablowitz and Segur, 1979, 1981). Though we will not do that here, if we take a hint from our work on the KdV equation, we can expect that the nonlinear part of our equation will not change when we add in slow transverse (y) dependence. This is true and the full KP equation, first derived in 1970 (Kadomtsev and Petviashvili, 1970), in dimensional form is

$$\begin{gathered}\partial_x\left(\frac{1}{c_0}\eta_t + \eta_x + \frac{3}{2h}\eta\eta_x + \gamma\eta_{xxx}\right) + \frac{1}{2}\eta_{yy} = 0, \; or \\ \frac{1}{c_0}\eta_t + \eta_x + \frac{3}{2h}\eta\eta_x + \gamma\eta_{xxx} = -\frac{1}{2}\int_{-\infty}^{x}\eta_{yy}\,dx',\end{gathered} \tag{5.32}$$

where $\gamma = \dfrac{h^2}{6}(1-3\hat{T})$. Usually in water waves surface tension is small and we have $\hat{T} < 1/3$ which gives rise to the so-called KPII equation

$$\frac{1}{c_0}\eta_{xt} + \eta_{xx} + \frac{3}{2h}(\eta\eta_x)_x + \frac{h^2}{6}\eta_{xxxx} = -\frac{1}{2}\eta_{yy}.$$

The equation termed the KPI equation arises when $\hat{T} = T/\rho g h^2 > 1/3$; i.e., when surface tension effects are large. Alternatively, we can rescale the equation into non-dimensional form

$$\partial_x(u_t + 6uu_x + u_{xxx}) + 3\sigma u_{yy} = 0, \tag{5.33}$$

where σ has the following meaning

- $\sigma = +1 \implies$ KPII: typical water waves, small surface tension,
- $\sigma = -1 \implies$ KPI: water waves, large surface tension.

We note that if $\eta_{yy} = 0$ in (5.32) and we rescale, the resulting KP equation can be reduced to the KdV equation in standard form

$$u_t \pm 6uu_x + u_{xxx} = 0. \tag{5.34}$$

The "+" corresponds to $\hat{T} < 1/3$ whereas the "–" arises when $\hat{T} > 1/3$.

5.5 Solitary wave solutions

As discussed in Chapter 1, the KdV and KP equations admit special, exact solutions known as solitary waves. We also mentioned in Chapter 1 that a solitary wave was noted by John Scott Russell in 1834 when he observed a wave detach itself from the front of a boat brought to rest. This wave evolved into a localized rounded hump of water that Russell termed the Great Wave of Translation. He followed this solitary wave on horseback as it moved along the Union Canal between Edinburgh and Glasgow. He noted that it hardly changed its shape or lost speed for over two miles; see Russell (1844); Ablowitz and Segur (1981); Remoissenet (1999) and www.ma.hw.uk/solitons. Today, scientists often use the term soliton instead of solitary wave for localized solutions of many equations despite the fact that the original definition of a soliton reflected the fact that two solitary waves interacted elastically.

5.5.1 A soliton in dimensional form for KdV

Recall from above that the KdV equation in standard, non-dimensional form, (5.34), can be written

$$u_t \pm 6uu_x + u_{xxx} = 0, \quad \text{and } \pm \text{ when } \gamma \gtrless 0,$$

where

$$\gamma = \frac{h^2}{6}(1 - 3\hat{T}).$$

A soliton solution admitted by the non-dimensional equation (5.34) is given by

$$u(x,t) = \pm 2\beta^2 \operatorname{sech}^2(\beta(x \mp 4\beta^2 t - x_0)). \tag{5.35}$$

Notice that the speed of the wave, $c = 4\beta^2$, is twice the amplitude of the wave. Also, the "+" corresponds to $\gamma > 0$ that is physically a positive elevation wave traveling on the surface like the one first observed by Russell. The "–" case results from $\gamma < 0$ and corresponds to a dip in the surface of the water. In fact, an experiment done only recently with high surface tension produced just such a "depression" wave (Falcon et al., 2002).

The solitary wave or soliton solution equation (5.35) is in non-dimensional form. We can convert this solution of the non-dimensional equation (5.34) directly to a solution of the dimensional equation (5.26). We will only consider here $\gamma > 0$; we leave it as an exercise to find the dimensional soliton when $\gamma < 0$. First consider (5.24) in the form

$$2U'_{T'} + \frac{1}{3}U'_{\xi'\xi'\xi'} + 3U'U'_{\xi'},$$

where we denote all variables with a prime. We will rescale the variables appropriately. Assume the following transformation of coordinates

$$\xi' = l_1\xi, \quad T' = l_2T, \quad U' = l_3U.$$

Then (5.24) becomes

$$\frac{2}{l_2}U_T + \frac{1}{3l_1^3}U_{\xi\xi\xi} + 3\frac{l_3}{l_1}UU_\xi = 0.$$

If we multiply the whole equation by l_1/l_3 then we get

$$\frac{2l_1}{l_2 l_3} U_T + \frac{1}{3l_1^2 l_3} U_{\xi\xi\xi} + 3UU_\xi = 0.$$

To get (5.24) into standard form, (5.34), we set

$$\frac{2l_1}{l_2 l_3} = \frac{1}{2} \quad \text{and} \quad \frac{1}{3l_1^2 l_3} = \frac{1}{2}.$$

One solution is $(l_1, l_2, l_3) = (1, 6, 2/3)$. Now our soliton solution takes the form

$$U'(\xi, T) = \frac{4\beta^2}{3} \operatorname{sech}^2 \left(\beta(\xi - \tfrac{2}{3}\beta^2 T - x_0)\right).$$

Next use, $\xi' = x' - t'$ and $T' = \varepsilon t'$ to find

$$U'(x', t') = \frac{4\beta^2}{3} \operatorname{sech}^2 \left(\beta(x' - (1 + \tfrac{2}{3}\beta^2\varepsilon)t' - x_0\right).$$

The dimensional solution employs the following change of variables

$$\eta = aU', \quad x = \lambda_x x', \quad t = \frac{\lambda_x}{c_0} t'.$$

Substituting this into our solution, we have

$$\eta(x, t) = \frac{4\beta^2 a}{3} \operatorname{sech}^2 \left(\beta\left(\frac{x}{\lambda_x} - \left(1 + \tfrac{2}{3}\beta^2\varepsilon\right)\frac{c_0}{\lambda_x} t - \frac{x_0}{\lambda_x}\right)\right).$$

Now we use the relations $a/h = \varepsilon$, $\mu = h/\lambda_x$, and $\mu^2 = \varepsilon$ to write down the dimensional solution to the KdV equation in the form

$$\frac{\eta}{h} = \tfrac{4}{3}\varepsilon\beta^2 \operatorname{sech}^2 \left(\frac{\beta}{h}\sqrt{\varepsilon}(x - (1 + \tfrac{2}{3}\beta^2\varepsilon)c_0 t - x_0)\right).$$

Note that both β and ε are non-dimensional numbers, and ϵ is related to the size of a typical wave amplitude. Recall that $a = |\eta_{\max}|$ and $c_0^2 = gh$. The excess speed beyond the long wave speed is $\dfrac{2\beta^2}{3} c_0\varepsilon$. Russell observed this as well! Later, in 1871, Boussinesq found this relationship from the more general point of view of weakly nonlinear waves moving in two directions. Boussinesq also found KdV-type equations (Boussinesq, 1877). Korteweg and de Vries found this result in 1895 concentrating on unidirectional water waves. They also found a class of periodic solutions in terms of elliptic functions and called them "cnoidal" functions (see also the discussion in Chapter 1).

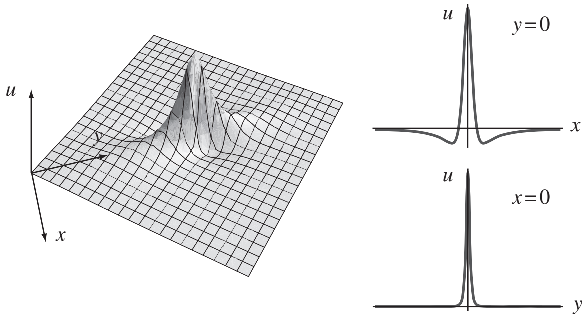

Figure 5.2 Lump solution of KPI, on the left. To better illustrate, we also plot slices along the x- and y-directions on the right.

5.5.2 Solitons in the KP equation

Recall the KPI equation (with "large" surface tension) in non-dimensional form, (5.33), is

$$\partial_x(u_t + 6uu_x + u_{xxx}) - 3u_{yy} = 0.$$

This equation admits the following non-singular traveling "lump" soliton solution (cf. Ablowitz and Clarkson, 1991) (see Figure 5.2)

$$u(x, y, t) = 2\partial_x^2 \left[\log\left((\hat{x} - 2k_R\hat{y})^2 + 4k_I^2\hat{y}^2 + \frac{1}{4k_I^2}\right)\right]$$
$$\hat{x} = x - 12\left(k_R^2 + k_I^2\right)t - x_0$$
$$\hat{y} = y - 12k_Rt - y_0.$$

Note that this lump only moves in the positive x-direction. Though the one-dimensional solitons in the previous section have been observed, this lump solution has not yet been seen in the laboratory. Presumably, this is because of the difficulties of working with large surface tension fluids. But, as mentioned above, only recently has the one-dimensional KdV "depression" solitary wave been observed for large surface tension (Falcon et al., 2002).

If there exists a solution to (5.33) such that $u, u_x, u_t, u_{xxx} \to 0$ as $x \to -\infty$ then we have

$$u_t + 6uu_x + u_{xxx} = -3\sigma \int_{-\infty}^{x} u_{yy}\, dx'.$$

In particular, if $u, u_x, u_{xxx} \to 0$ and $u_t \to 0$ as $x \to \infty$, we also have

$$\int_{-\infty}^{\infty} u_{yy}(x, y, t)\, dx = 0.$$

But if we give a general initial condition $u(x, y, t = 0) = u_0(x, y)$, it need not satisfy this constraint at the initial instant. For the linearized version of the KP equation, one can show (Ablowitz and Villarroel, 1991) that if the initial condition satisfies

$$\int_{-\infty}^{\infty} u_0(x, y)\, dx \neq 0,$$

nevertheless the solution obeys

$$\int_{-\infty}^{\infty} u(x, y, t)\, dx = 0, \quad \text{for all } t \neq 0$$

and the function $\partial u/\partial t$ is discontinuous at $t = 0$. Ablowitz and Wang (1997) show why KP models of water waves lead to such constraints asymptotically and how a two-dimensional Boussinesq equation has an "initial value layer" that "smooths" the discontinuity in $\int u_{yy}(x, y, t)\, dx$.

5.5.3 KdV and related models

Suppose we consider the KdV equation in the form

$$u_t + u_x + \varepsilon(u_{xxx} + \alpha u u_x) = 0 \tag{5.36}$$

where α is constant. First look at the linear problem and assume the wave solution $u \sim \exp(i(kx - \omega t)$. The dispersion relation is then

$$\omega = k - \varepsilon k^3,$$

which, for $|k| \gg 1$, implies $\omega \sim -\varepsilon k^3$. In particular, note that $\omega \to -\infty$, which can present some numerical difficulties. One way around this issue is to use the relation $u_t = -u_x + O(\varepsilon)$ to alter the equation slightly (Benjamin et al., 1972)

$$u_t + u_x + \varepsilon(-u_{xxt} + \alpha u u_x) = 0. \tag{5.37}$$

Note that (5.36)–(5.37) are asymptotically equivalent to $O(\epsilon)$. But for large k, the dispersion relation for the linear problem is now

$$\omega = \frac{k}{1 + \varepsilon k^2} \sim \frac{1}{\varepsilon k}$$

for $|k \gg 1$. Numerically, (5.37) has some advantages over (5.36). But, we still have a "small" parameter, ε, in both equations so they are not asymptotically reduced. Alternatively, if we use the following transformation in (5.36), as indicated by multiple-scale asymptotics,

$$\xi = x - t, \quad T = \varepsilon t,$$
$$\partial_x = \partial_\xi, \quad \partial_t = -\partial_\xi + \varepsilon \partial_T,$$

then we get the equation

$$u_T + u_{\xi\xi\xi} + \alpha u u_\xi = 0, \tag{5.38}$$

which has no direct dependence on the small parameter ε. Equation (5.38) is also useful for numerical computation since $T = O(1)$ corresponds to $t = O(1/\varepsilon)$.

Finally, recall the Fermi–Pasta–Ulam (FPU) model of nonlinear coupled springs,

$$m\ddot{y}_i = k(y_{i+1} + y_{i-1} - 2y_i) + \alpha((y_{i+1} - y_i)^2 - (y_i - y_{i-1})^2) \tag{5.39}$$

discussed in Chapter 1. In the continuum limit, use $y_{i+1} = y(x) + hy'(x) + h^2y''(x)/2 + \cdots$, $x = ih$, to find

$$\omega^2 y_{tt} - h^2 y_{xx} = \frac{h^4}{12} y_{xxxx} + 2\alpha h^3 y_x y_{xx} + \cdots,$$

where we have used a dot to denote $\dfrac{d}{dt}$. Making the following substitutions $t = \tau/h\omega$, $\omega^2 = k/m$, $\varepsilon = 2\alpha h$, and $\delta^2\varepsilon = h^2/12$, where the last two equalities arise from the stipulation of maximal balance, we have

$$y_{\tau\tau} - y_{xx} = \varepsilon(\delta^2 y_{xxxx} + y_x y_{xx}). \tag{5.40}$$

This is a Boussinesq-type equation and is also known to be integrable (Ablowitz and Segur, 1981; Ablowitz and Clarkson, 1991). The term integrable has several interpretations. We will use the notion that if we can exactly linearize (as opposed to solving the equation via a perturbation expansion) the equation then we consider the equation to be integrable (see Chapters 8 and 9). Linearization or direct methods when applied to an integrable equation allow us to find wide classes of solutions.

Performing asymptotic analysis, in particular multiple scales on (5.40) with the expansion

$$y \sim y_0 + \varepsilon y_1 + \cdots$$
$$y_0 = \phi(X; T), \qquad X = x - \tau,\ T = \varepsilon\tau,$$

we find that, after removing secular terms (see also Chapter 1),

$$2\phi_{XT} + \phi_X \phi_{XX} + \delta^2 \phi_{XXXX} = 0.$$

Or, if we make the substitution $\phi_X = u$ then we end up with the integrable KdV equation

$$u_T + \tfrac{1}{2} u u_X + \frac{\delta^2}{2} u_{XXX} = 0.$$

Another lattice equation, called the Toda lattice that is known to be integrable, is given by

$$m\ddot{y}_i = e^{k(y_{i+1}-y_i)} - e^{k(y_i-y_{i-1})}.$$

If we expand the exponential and keep quadratic nonlinear terms, we get the FPU model (5.39), which in turn asymptotically yields the KdV equation. This further shows that the KdV equation (5.34) arises widely in applied mathematics and the FPU problem itself is very "close" to being integrable.

5.5.4 Non-local system and the Benney–Luke equations

In a related development, a recent reformulation (see Ablowitz et al., 2006) of the fully nonlinear water wave equations with surface tension leads to two equations for two unknowns: η and $q = q(x, y, t) = \phi(x, y, \eta(x, y, t))$. The equations are given by the following "simple looking" system

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdye^{i(kx+ly)}\Big(i\eta_t \cosh[\kappa(\eta+h)] + \frac{1}{\kappa}(k,l)\cdot\nabla q \sinh\left[\kappa(\eta+h)\right]\Big) = 0 \qquad \text{(I)}$$

$$q_t + \frac{1}{2}|\nabla q|^2 + g\eta - \frac{(\eta_t + \nabla q\cdot\nabla\eta)^2}{2(1+|\nabla\eta|^2)} = \sigma\nabla\cdot\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right) \qquad \text{(II)}$$

where $(k, l)\cdot\nabla q = kq_x + lq_y$, $\kappa^2 = k^2 + l^2$, $\sigma = T/\rho$ where T is the surface tension and ρ is the density; here η and derivatives of q are assumed to decay rapidly at infinity. The first of the above two equations is non-local and is written in a "spectral" form.

The non-local equation satisfies the Laplace equation, the kinematic condition and the bottom boundary condition. The second equation is Bernoulli's equation rewritten in terms of the new variable q. These variables were introduced by Zakharov (1968) in his study of the Hamiltonian formulation of the water wave problem. Subsequently they were used by Craig and co-workers (Craig and Sulem, 1993; Craig and Groves, 1994) in their discussion of the Dirichlet–Neumann (DN) map methods in water waves. In this regard we note:

$$\eta_t = \phi_z - \nabla\phi\cdot\nabla\eta = \nabla\phi\cdot\vec{n}$$

where $\vec{n} = (-\nabla\eta, 1)$ is the normal on $y - \eta = 0$. Thus finding $q(x, y, t)$, $\eta(x, y, t)$ yields η_t, which in turn leads to the normal derivative $\partial\phi/\partial n$; hence the DN map is contained in the solution of the non-local system (I–II). Also, solving equations (I)–(II) for η, q reduces to solving the remaining water wave equation, Laplace's equation (5.4), which is a linear problem due to the now fixed boundary conditions.

The non-local equation can be derived from Laplace's equation, the kinematic condition and the bottom boundary condition via Green's identity (cf. Haut and Ablowitz, 2009). The essential ideas are outlined below. We also mention that the reader will find valuable discussions of non-local systems with auxiliary parameters in Fokas (2008). Fokas has made numerous important contributions in the study of such non-local nonlinear equations. We define an associated potential $\psi(x, y, z)$ satisfying

$$\Delta\psi(x, y, z) = 0, \quad \text{in } D, \quad \psi_y(x, y, z = -h) = 0. \tag{5.41}$$

Then, from Green's identity,

$$\begin{aligned} 0 &= \int_{D(\eta)} ((\Delta\psi)\,\phi - (\Delta\phi)\,\psi)\; dV \\ &= \int_{\partial D(\eta)} (\phi(\nabla\psi \cdot \hat{n}) - \psi(\nabla\phi \cdot \hat{n}))\, dS \end{aligned}$$

where dV, dS are the volume and surface elements respectively, and $\hat{n}$ is the unit normal. It then follows that

$$\int \psi(x, y, \eta)\eta_t\, dxdy = \int q\left(\psi_z(x, \eta) - \nabla_{x,y}\psi(x, y, \eta) \cdot \nabla\eta\right) dx\, dy, \tag{5.42}$$

where we have used $\eta_t = \nabla\phi \cdot \vec{n}$, $\vec{n} = (-\nabla\eta, 1)$ on the free surface as well as the bottom boundary condition and the condition $|\eta|$ decaying at infinity.

We assume a solution of (5.41) can be written in the form

$$\psi(x, y, z) = \int \hat{\xi}(k, l)\hat{\psi}_{k,l}(x, y, z)dkdl, \;\; \hat{\psi}_{k,l} = e^{i(kx+ly)}\cosh[\kappa(z + h)].$$

Inserting $\psi_{k,l}(x, y, z)$ into (5.42) yields

$$\begin{aligned} &\int_{\mathbb{R}^2} e^{i(kx+ly)}\cosh(\kappa(\eta + h)\eta_t\, dxdy \\ &\qquad = \int_{\mathbb{R}^2} q\Big(e^{i(kx+ly)}\kappa\sinh(\kappa(\eta + h) - ie^{i(kx+ly)}\cosh(\kappa(\eta + h)\,(k \cdot \nabla)\,\eta\Big)\, dx\, dy. \end{aligned} \tag{5.43}$$

Then, using

$$e^{i(kx+ly)}\kappa \sinh(\kappa(\eta + h) - ie^{i(kx+ly)} \cosh(\kappa(\eta + h)\,((k,l)\cdot\nabla)\,\eta$$
$$= -i\nabla \cdot \left(e^{ikx} \frac{\sinh(\kappa(\eta + h)}{\kappa}(k,l)\right)$$

in (5.43) and integrating by parts yields the non-local equation (I).

The second equation is essentially a change of variables. Differentiating $q(x,y,t) = \phi(x,y,\eta(x,y,t),t)$ leads to

$$q_x + \phi_x + \phi_z\eta_x, \quad q_y = \phi_y + \phi_z\eta_y.$$

These equations and

$$\eta_t + \nabla\phi\cdot\nabla\eta = \phi_y$$

yield ϕ_x, ϕ_y, ϕ_z in terms of derivatives of η, q. Then from Bernoulli's equation with surface tension included, (5.28), we obtain equation (II).

If $|\eta|$, $|\nabla q|$ are small then equations (I) and (II) simplify to the linearized water wave equations. Calling the Fourier transform $\hat{\eta} = \int dx\,dy e^{i(kx+ly)}\eta$, and similarly for the derivatives of q, we find from equations (I) and (II) respectively

$$i\hat{\eta}_t \cosh\kappa h + \frac{k\cdot\widehat{\nabla q}}{\kappa}\sinh\kappa h = 0$$
$$\widehat{q_t} + (g + \sigma\kappa^2)\hat{\eta} = 0.$$

Then from these two equations we get

$$\hat{\eta}_{tt} = -(g\kappa + \sigma\kappa^3)\tanh\kappa h\,\hat{\eta}$$

which is the linearized water wave equation for η in Fourier space.

We also remark that one can find integral relations by taking $k,l \to 0$ in equations (I) and (II) above. The first three, corresponding to powers k^0, l^0, k,l, are given by

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\,\eta(x,y,t) = 0 \quad \text{(Mass)}$$
$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy(x\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\,q_x(\eta + h) \quad (\mathrm{COM}_x)$$
$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy(y\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\,q_y(\eta + h) \quad (\mathrm{COM}_y).$$

They correspond to conservation of mass and the motion of the center of mass in x,y respectively. The right-hand sides of (COM_{x}, y) are related to the x,y-momentum respectively. Higher powers of k,l lead to "virial" type identities.

From the above system (I) and (II), one can also derive both deep- and shallow-water approximations to the water wave equations. In deep water, nonlinear Schrödinger equation (NLS) systems result (see Ablowitz et al., 2006; Ablowitz and Haut, 2009b). In shallow water the Benney–Luke (BL) equation (Benney and Luke, 1964), properly modified to take into account surface tension, can be obtained.

To do this it is convenient to make all variables non-dimensional:

$$x_1' = \frac{x_1}{\lambda}\,,\ x_2' = \gamma\frac{x_2}{\lambda}\,,\ t' = \frac{c_0}{\lambda}t\,,\ a\eta' = \eta\,, q' = \frac{a\lambda g}{c_0}q\,,\ \sigma' = \frac{\sigma}{gh^2}$$

where $c_0 = \sqrt{gh}$ and λ, a are the characteristic horizontal length and amplitude, and γ is a non-dimensional transverse length parameter. The equations are written in terms of non-dimensional variables $\epsilon = a/h \ll 1$, small-amplitude, $\mu = h/\lambda \ll 1$ long waves; $\gamma \ll 1$, slow transverse variation. Dropping $'$ we find

$$\int dxdye^{i(kx+ly)}\left(i\eta_t \cosh[\tilde{\kappa}\mu(1+\varepsilon\eta)] + \sinh[\tilde{\kappa}\mu(1+\varepsilon\eta)]\frac{(k,l)\cdot\tilde{\nabla}q}{\tilde{\kappa}\mu}\right) = 0 \quad (5.44)$$

where $(k,l)\cdot\tilde{\nabla} = kq_x + \gamma^2 lq_y,\ \tilde{\kappa}^2 = k^2 + \gamma^2 l^2$.

In (5.44) we use

$$\cosh[\tilde{\kappa}\mu(1+\varepsilon\eta)] \sim 1 + \frac{\mu^2}{2}\tilde{\kappa}^2, \quad \sinh[\tilde{\kappa}\mu(1+\varepsilon\eta)] \sim \mu\tilde{\kappa} + \frac{\mu^3}{6}\tilde{\kappa}^3 + \varepsilon\mu\eta\tilde{\kappa}.$$

Then, after inverse Fourier transforming, and with $(k,l) \to (i\partial x, i\partial y)$ we find from equations (I) and (II)

$$\left(1 - \frac{\mu^2}{2}\tilde{\Delta}\right)\eta_t + \left(\tilde{\Delta} - \frac{\mu^2}{6}\tilde{\Delta}^2\right)q + \varepsilon\left(\tilde{\nabla}\eta\cdot\tilde{\nabla}q\right) + \varepsilon\eta\tilde{\Delta}q = 0 \qquad \text{(Ia)}$$

$$\eta = -q_t - \frac{\varepsilon}{2}|\tilde{\nabla}q|^2 + \tilde{\sigma}\mu^2\tilde{\Delta}\eta \qquad \text{(IIa)}$$

where $\tilde{\Delta} = \partial_x^2 + \delta^2\partial_y^2$, $\tilde{\nabla}\eta\cdot\tilde{\nabla}q = \eta_x q_x + \delta^2\eta_y q_y$, $|\tilde{\nabla}q|^2 = \left(q_x^2 + \delta^2 q_y^2\right)$, and $\tilde{\sigma} = \sigma - 1/3$.

Eliminating the variable η we find, when $\epsilon, \mu, \delta \ll 1$, the Benney–Luke (BL) equation with surface tension included:

$$q_{tt} - \tilde{\Delta}q + \tilde{\sigma}\mu^2\tilde{\Delta}^2 q + \varepsilon(\partial_t|\tilde{\nabla}q|^2 + q_t\tilde{\Delta}q) = 0, \qquad (5.45)$$

where $\tilde{\sigma} = \sigma - 1/3$. Alternatively, one can derive the BL system from the classical water wave equations via asymptotic expansions as we have done previously for the Boussinesq models. The equations look somewhat different due to different representations of the velocity potential (see also the exercises).

We remark that if we take the maximal balance $\epsilon = \mu^2 = \delta^2$ then using multiple scales the BL equation can be reduced to the unidirectional KP equation, which we have shown can be further reduced to the KdV equation if there is no transverse variation. Namely letting $\xi = x - t$, $T = \epsilon t/2$, $w = q_\xi$ we find the KP equation in the form

$$\partial_\xi (w_T - \tilde{\sigma} w_{\xi\xi\xi} + 3(ww_\xi)) + w_{yy} = 0.$$

Hence the non-local system (I)–(II) contains the well-known integrable water wave reductions. We also note that higher-order asymptotic expansions of solitary waves and lumps can be obtained from equations (I) and (II) using the same non-dimensionalization as we used for the BL equation. These expansions yield solitary waves close to their maximal amplitude (Ablowitz and Haut, 2009a, 2010)

Exercises

5.1 Derive the KP equation, with surface tension included, from the Benney–Luke (BL) equation (5.45): i.e., letting $w = q_\xi, \varepsilon = \mu^2 = \delta^2$, find

$$w_{T\xi} + \left(1/3 - \hat{T}\right) w_{\xi\xi\xi\xi} + w_{yy} + (3ww_\xi)_\xi = 0 \qquad (KP)$$

where $T = \epsilon t, \hat{T} = T/\rho g h^2$, which is the well-known Kadomtsev–Petviashvili (KP) equation.

5.2 From the non-dimensional KP equation derived in Exercise 5.1, show that the equation in dimensional form is given by

$$\frac{1}{c_0}\eta_{xt} + \frac{3}{2h}(\eta\eta_x)_x + \gamma\eta_{xxxx} = -\frac{1}{2}\eta_{yy}$$

where $\gamma = \frac{h^2}{6}(1 - 3\hat{T})$.

5.3 Find the dimensional form of a soliton solution to the KdV equation when $\hat{T} > 1/3$, i.e., $\gamma < 0$.

5.4 Derive the equivalent BL equation for the velocity potential $\phi(x, y, 0, t)$ from the classical water wave equations with surface tension included.

5.5 Rewrite the lump solution given in this section in dimensional form.

5.6 Given the "Boussinesq" model

$$u_{tt} - \Delta u + \Delta^2 u + ||\nabla u||^2 = 0,$$

make suitable assumptions about the size of terms, then rescale, and derive a corresponding KP-type equation.

5.7 Given a "modified Boussinesq" model

$$u_{tt} - \Delta u + \Delta^2 u + ||\nabla u||^{2+n} = 0$$

with n a positive integer, make suitable assumptions about the size of terms, then rescale, and derive a corresponding "modified KP"-type equation.

5.8 Beginning with (5.22) assume right-going waves and obtain the KdV equation with a higher-order correction.