① $\eta_{tt} - \eta_{xx} = 0, \quad \eta_x(0,t) = 0, \quad x > 0.$

Introduce time scale $T = f(t)$. Then,

$$\frac{d}{dt} = f'(t)\frac{d}{dT} \quad \& \quad \frac{d^2}{dT^2} = f''(t)\frac{d}{dT} + (f'(t))^2 \frac{d^3}{dT^2}$$

① becomes ②: $f'(t)^2 \frac{d^2}{dT^2}\eta + f''(t)\frac{d}{dT}\eta - \eta_{xx} = 0, \quad \eta_x(0,t) = 0.$

Apply Fourier Cosine Transform to ②:

$$③ \quad f'(t)^2 \frac{d^2}{dT^2}\hat{\eta} + f''(t)\frac{d}{dT}\hat{\eta} - \hat{\eta}_{xx} = 0$$

$$\Rightarrow \quad f'(t)^2 \frac{d^2}{dT^2}\hat{\eta} + f''(t)\frac{d}{dT}\hat{\eta} + k^2\hat{\eta} = 0$$

$$\Rightarrow \quad ④ \quad \frac{d^2}{dT^2}\hat{\eta} + \frac{f''(t)}{f'(t)^2}\frac{d}{dT}\hat{\eta} + \frac{k^2}{f'(t)^2}\hat{\eta} = 0.$$

<u>Want</u>: $k^2 = f'(t)^2 \Rightarrow f'(t) = \pm|k| = \pm k$, since $k > 0$.

$\Rightarrow f(t) = \pm tk + C$, set $C = 0$

SL problem on unbounded domain $\Rightarrow f(t) = \pm it\partial_x$, since $k \to i\partial_x$

④ becomes: $\frac{d^2}{dT^2}\hat{\eta} + \hat{\eta} = 0$, $k > 0$. + (some BC)

the general solution is $\hat{\eta} = f_1(k)\sin T + f_2(k)\cos T$. ⑤

To do: given $\eta_{tt} - \eta_{xx} =$

$$\varepsilon\left(\frac{1}{3}\eta_{xxxx} + \partial_x\left\{\mathcal{F}_s^k\right\}^{-1}\left\{\mathcal{F}_c^k\left\{\partial_t\left(\eta\int_0^x \eta_t\,dx'\right)\right\}\right\} + \frac{1}{2}\partial_x^2\left(\int_0^x \eta_t\,dx'\right)^2\right)$$

⑥

- Transform to k-space
- Transform the equation via T
- Use (5)

FYI: (6) $\Leftrightarrow$ Eq 42 in Report 2.

$$\mathcal{F}_s^k\left\{\int_0^x \eta_{tt}\, dx'\right\} - \hat{\eta}_x{}^k_s = \varepsilon\left(\mathcal{F}_c^k\left\{\partial_t\left(\eta\int_0^x \eta_t\, dx'\right)\right\} + \frac{1}{2}\mathcal{F}_s^k\left\{\partial_x\left(\int_0^x \eta_t\, dx'\right)^2\right\} + \frac{1}{3}\mathcal{F}_s^k\left\{\eta_{ttx}\right\}\right)$$

need to change appropriately

looks hard to deal with directly, so maybe use Eq 41 in Report 2

$\Leftrightarrow$ Eq 41 in Report 2.

Note: $\partial_t = \dfrac{dT}{dt}\dfrac{d}{dT} = \pm i\, \partial_x \dfrac{d}{dT}$???

$\partial_x =$???